Central Limit Theorem

Standard Normal Distribution (N(0,1)) revisited

Normal Dist:

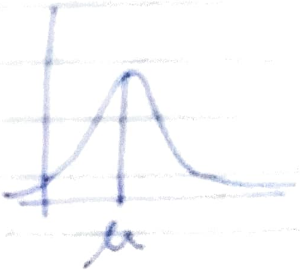
$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$

$$= N(\mu, \sigma^2), \quad -\infty < x < \infty$$

Standard N.D. $N(0,1)$

$$\cancel{f_z(z) = \frac{1}{\sqrt{2\pi}}}$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}}$$



Note: (1)

$\because$ The function $f(x)$ is not closed

$\therefore$ the probability

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

is evaluated numerically

& is tabulated in a table.

Such tables are standard Normal Distribution tables ($Z \sim N(0,1)$).

The table has entries of

$$\boxed{\underset{\text{CDF}}{F_Z(z)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\, dt} = P(Z \le z)$$

z-score or Zee score

Ex a For a non-standardized normal distribution the variable 'z' is given by

$$z = (x - \mu)/\sigma$$

(2) $P(a \le Z \le b) = F(b) - F(a)$

Ex

(3). We have Ex $F_Z(-z) = 1 - F_Z(z)$

Symmetric Property.

$\therefore$ Table is only made for $z \ge 0$.

(4). For non-standard dist.

$\because z = (x - \mu)\sigma$

$$F_Z(z) = P(Z \le z)$$
$$= P\left(\frac{X - \mu}{\sigma} \le z\right)$$
$$= P(X \le \mu + z\sigma)$$

$\Rightarrow F_Z(3) = F_X(\mu + 3\sigma)$.

or $\boxed{F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right)}$

for $\mu = 0$ & $\sigma = 1$

$$F_X(x) = F_Z(x)$$

Eg An analog signal received at a detector is a Gaussian r.v. ~~with~~ $N(200, 256)$ at a fixed point in time.

Q^n what is the probability that the signal will exceed 240 mv
what is the probability that the signal is larger than 240 mV, given that it is larger than 210 mv?

Soln (i) $P(X > 240) = 1 - P(X \leq 240)$

$$= 1 - F_Z\left(\frac{240-200}{16}\right)$$

$$= 1 - F_Z(2.5) = 1 - 0.9938$$

$$\simeq 0.00621.$$

(ii) $P(X \geq 240 \mid X \geq 210) = \frac{P(X \geq 240)}{P(X \geq 210)}$

$\left[P(A|B) = \frac{P(A \cap B)}{P(B)} \right]$

$$= \frac{1 - F_Z\left[\frac{240-200}{16}\right]}{1 - F_Z\left[\frac{\cancel{240}\,210 - 200}{16}\right]} = \frac{1 - F_Z(2.5)}{1 - F_Z(0.625)}$$

$$= \frac{\cancel{0.00621}}{\cancel{0.26599}} = \frac{1 - 0.9938}{1 -}$$

$$= 0.02335.$$

The Central Limit Theorem:

Let $X_1, X_2, \ldots, X_n$ be independent r.v. with finite $E[X_i]$ and a finite Var, $V[X_i] = \sigma_i^2$; $i = 1, 2, \ldots, n$.

~~Cond~~ Consider the normalized R.V.

$$Z_n = \frac{\left[\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu_i\right]}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}} \quad \text{--- (1)}$$

So that $E[Z_n] = 0$ & $V[Z_n] = 1$.

Then the R.V. Z_n, with certain conditions, is $Z_n \sim N(0, 1)$.

i.e. $$\boxed{\lim_{n \to \infty} F_{Z_n}(t) = P(Z_n \le t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy}$$

Ex: let $X_1, X_2, \dots X_n$ be s.t.

$$\mu_i = E[X_i] = \mu \quad \&$$

$$\sigma_i^2 = V[X_i] = \sigma^2$$

Then

$$Z_n = \frac{\left[\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} \mu\right]}{\sqrt{\sum_{i=1}^{n} \sigma^2}}$$

$$\Rightarrow Z_n = \frac{n\bar{x} - n\mu}{\sqrt{n\sigma^2}}$$

$$= \frac{n(\bar{x} - \mu)}{\sqrt{n} \cdot \sigma}$$

$$\Rightarrow \boxed{Z_n = \frac{(\bar{x} - \mu)\sqrt{n}}{\sigma}}$$

where $\bar{x}$ is Sample mean

Note: In ① let $X = \sum_{i=1}^{n} X_i$.

Then

$$Z_n = \frac{X - \sum_{i=1}^{n} \mu_i}{\left[\sum_{i=1}^{n} \sigma_i^2\right]^{1/2}}$$

This essentially represents the r.v.
~~of~~ "Z score".

This means that no matter what is the type of distribution of $X_1, X_2, \ldots X_n$, the r.v. $X = X_1 + X_2 + \cdots + X_n$ can be approximated by a standardized normal distribution.