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* Assignment :-1 *

Name :- Anmol
Roll no :- 01914803120.
Section :- 4I123

Ques-1

(i) $r = a^2 t$

We know that $r = \frac{\partial^2 z}{\partial x^2}$ and $t = \frac{\partial^2 z}{\partial y^2}$

Substitute in the given equation :-

$$\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}.$$

Next put $\frac{\partial^2 z}{\partial x^2} = D^2 z$ and $\frac{\partial^2 z}{\partial y^2} = D'^2 z$.

The equation is :- $D^2 z = a^2 D'^2 z$

Now Solution = C.f + P.I.

Cf :- $D^2 z = a^2 D'^2 z$

$z(D^2 - a^2 D'^2) = 0.$

Put $D = m$ and $D' = 1$:-

AE :- $m^2 - a^2 = 0$

$(m+a)(m-a) = 0$

$m = -a, a \Rightarrow m_1 = -a$ and $m_2 = a$.

Cf :- $\phi_1(y + m_1 x) + \phi_2(y + m_2 x)$

Substitute m_1 and m_2 :-

Cf $\phi_1(y - ax) + \phi_2(y + ax)$

$PI = 0$

Thus, Complete Solution = $\phi_1(y + ax) + \phi_2(y - ax)$ Ans.

(ii) $(D^4 - D'^4)z = 0$

Auxilliary Equation :-

Put $D = m$ and $D' = 1$

$\Rightarrow m^4 - 1 = 0$

$(m^2+1)(m^2-1) = 0$

$(m^2+1)(m+1)(m-1)=0$

$m_1=1,\ m_2=-1,\ m_3=i,\ m_4=-i$

$\Rightarrow$ Cf :- $\phi_1(y+m_1x)+\phi_2(y+m_2x)+\phi_3(y+m_3x)+\phi_4(y+m_4x)$

$\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix)$

$\Rightarrow$ PI $=0$. (R.H.S is zero).

Complete Solution = Cf + PI

$CS=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix)$ Ans

(iii) $(D^4+D'^4)z=0$.

Auxilliary equation :-

$(m^4+1)\cancel{z}=0$.

Roots of $m^4+1=0$.

$m^4=-1 \Rightarrow m=(-1)^{1/4}$

By Demorgon's Theorem :-

$$m=\left[\cos\left(\frac{2n\pi+\pi}{4}\right)+i\sin\left(\frac{2n\pi+\pi}{4}\right)\right]$$

$m_1=\frac{1}{\sqrt2}+i\frac{1}{\sqrt2},\ m_2=-\frac{1}{\sqrt2}+i\frac{1}{\sqrt2},\ m_3=\frac{-1}{\sqrt2}-i\frac{1}{\sqrt2},\ m_4=\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}$.

Cf $\phi_1(y+m_1x)+\phi_2(y+m_2x)+\phi_3(y+m_3x)+\phi_4(y+m_4x)$

Cf :- $\phi_1\left[y+\left(\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}\right)x\right]+\phi_2\left[y+\left(\frac{-1}{\sqrt2}+i\frac{1}{\sqrt2}\right)x\right]+\phi_3\left[y+\left(\frac{-1}{\sqrt2}-i\frac{1}{\sqrt2}\right)x\right]+\phi_4\left[y+\left(\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}\right)x\right]$

PI $=0$.

So, Complete Solution = Cf + 0.

$CS=\phi_1\left[y+\left(\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}\right)x\right]+\phi_2\left[y+\left(\frac{-1}{\sqrt2}+i\frac{1}{\sqrt2}\right)x\right]+\phi_3\left[y+\left(\frac{-1}{\sqrt2}-i\frac{1}{\sqrt2}\right)x\right]+\phi_4\left[y+\left(\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}\right)x\right]$

(iv) $(D^3 - 2D^2D')z = 0$

Auxilliary equation :-

$m^3 - 2m^2 = 0.$

$m^2(m-2) = 0.$

$m_1 = 0,\ m_2 = 0,\ m_3 = 2.$

Cf $\Rightarrow \phi_1(y + m_1x) + \phi_2(y + m_2x)x + \phi_3(y + m_3x)$

$\phi_1(y) + \phi_2 x(y) + \phi_3(y + 2x)$

PI $= 0$ (R.H.S is zero)

Complete Solution = Cf + PI $\Rightarrow$ Cf + 0.

C.S $\Rightarrow \phi_1(y) + x\phi_2(y) + \phi_3(y+2x)$ Ans

Ques-2 Solve

(i) $(D^2 + 3DD' + 2D'^2)z = x+y$

Auxilliary equation = $m^2 + 3m + 2 = 0.$

$(m+2)(m+1) = 0.$

$m = -2, -1 \Rightarrow m_1 = -2,\ m_2 = -1$

Cf $\phi_1(y + m_1x) + \phi_2(y + m_2x)$

$\phi_1(y - 2x) + \phi_2(y - 1x)$

PI :- $f(D, D')z = f(x, y)$

$$z = \frac{f(x,y)}{f(z,z')}$$

$$PI = \frac{x+y}{D^2 + 3DD' + 2D'^2}$$

Binomial expansion :- $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \text{------}$

$$PI\ \frac{x+y}{D^2\left(1 + 3\frac{D'}{D} + \frac{2D'^2}{D^2}\right)} \Rightarrow D^{-2}\left[1 + \frac{3D'}{D} + \frac{2D'^2}{D^2}\right]^{-1}(x+y)$$

Apply Binomial on it :-

PI: $D^{-2}\left[1+(-1)\left(\frac{3D'}{D}+2\left(\frac{D'}{D}\right)^2\right)\right](x+y)$

$D^{-2}\left[x+y-\frac{3D'}{D}x-\frac{3D'}{D}y-2\left(\frac{D'}{D}\right)^2x-2\left(\frac{D'}{D}\right)^2y\right]$

$D^{-2}\left[x+y-3(0)-3D^{-1}-2(0)-2(0)\right]$

$\Rightarrow D^{-2}(x+y-3D^{-1})$

$\Rightarrow D^{-2}(x+y-3x) \Rightarrow D^{-2}(y-2x)$

$= \iint (y-2x)\,dx$

PI $= \int yx-x^2 \Rightarrow y\frac{x^2}{2}-\frac{x^3}{3}$

C.S = Cf + PI

CS $= \phi_1(y-x)+\phi_2(y-2x)+y\frac{x^2}{2}-\frac{x^3}{3}$ Ans.

Ques 2 (ii) $(D^2+D'^2)z = \sin mx \cdot \sin my$.

Putting $D=m$ and $D'=1$, we get

$AE = (m^2+1)=0$

$m^2=-1 \Rightarrow m=\pm i$

Cf $\phi_1(y+ix)+\phi_2(y-xi)$

PI: $z = \frac{f(x,y)}{f(D,D')} = \frac{\sin mx \cdot \sin my}{(D^2+D'^2)}$

We know that $2\sin A\cdot\sin B = \cos(A-B)-\cos(A+B)$

Multiplying and Dividing PI by 2 we get

$PI = \frac{\cos(mx-my)-\cos(mx+my)}{2(D^2+D'^2)} = \frac{\cos(mx-my)}{2(D^2+D'^2)} - \frac{\cos(mx+my)}{2(D^2+D'^2)}$

for $\cos(mx-my) \Rightarrow -a^2=-m^2, -b^2=-(-m)^2=-m^2$

$\cos(mx+my) \Rightarrow -a^2=-m^2, -b^2=-m^2$.

$PI = \frac{\cos(mx-my)}{2(-m^2-m^2)} - \frac{\cos(mx+my)}{2(-m^2-m^2)} \Rightarrow -\frac{1}{2(m^2+m^2)}\left[\cos(mx-my)-\cos(mx+my)\right]$

CS = Cf + PI

$CS = \phi_1(y+xi)+\phi_2(y-xi)-\frac{1\left[\cos(mx-my)-\cos(mx+my)\right]}{2(m^2+m^2)}$ Ans.

(iii) $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x+y}$.

We know that $D_x = \frac{\partial z}{\partial x} = D$, $\frac{\partial z}{\partial y} = D_y = D'$

$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{3x+y}$.

<u>AE</u>:- $m^3 - 7m - 6 = 0$

$(m-3)(m+1)(m+2) = 0$

$m = -1, -2, 3 \Rightarrow m_1 = -1,\ m_2 = -2,\ m_3 = 3$

<u>Cf</u> $\Rightarrow \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$

<u>PI</u> $\Rightarrow z = \frac{f(x,y)}{f(D,D')}$

$PI \Rightarrow \frac{\sin(x+2y) + e^{3x+y}}{D^3 - 7DD'^2 - 6D'^3} = \frac{\sin(x+2y)}{D^3 - 7DD'^2 - 6D'^3} + \frac{e^{3x+y}}{D^3 - 7DD'^2 - 6D'^3}$

<u>PI</u> $\Rightarrow \frac{\sin(x+2y)}{[(-1)D - 7D(-4) - 6D'(-4)]} + \frac{e^{3x+y}}{(3)^3 - 7(3)(1)^2 - 6(1)^3}$

Denominator for e^{3x+y} is zero, thus :

$\Rightarrow \frac{\sin(x+2y)}{27D + 24D'} + \frac{e^{3x+y}.x}{3D^2 - 7D'^2}$.

$\Rightarrow \frac{\sin(x+2y) \times (27D - 24D')}{(27D + 24D') \times (27D - 24D')} + \frac{x.e^{3x+y}}{3\times(3)^2 - 7(1)^2}$.

$\Rightarrow \frac{(27D - 24D')\sin(x+2y)}{(27D)^2 - (24D')^2} + \frac{x.e^{3x+y}}{27-7}$

$\Rightarrow \frac{27D.\sin(x+2y) - 24D'\sin(x+2y)}{729(-1) - 576(-4)} + \frac{x.e^{3x+y}}{20}$.

$\Rightarrow \frac{1}{525}[9D\sin(x+2y) - 8D'\sin(x+2y)] + \frac{x.e^{3x+y}}{20}$

$\Rightarrow \frac{9\cos(x+2y) - 16\cos(x+2y)}{525} + \frac{x.e^{3x+y}}{20}$.

<u>CS</u> :- $\phi_1(y-x) + \phi_2(y-2x) + \phi_3(y-3x) - \frac{\cos(x+2y)}{75} + \frac{x.e^{3x+y}}{20}$ <u>Ans</u>

(iv) $(D^2+DD'-6D'^2)z = y\cos x$

<u>AE</u> :- $m^2+m-6=0$

$m_1=2,\ m_2=-3$

<u>Cf</u> $\Rightarrow \phi_1(y+2x)+\phi_2(y-3x)$

<u>PI</u> $\Rightarrow z = \dfrac{f(x,y)}{f(D,D')}$

<u>PI</u> $\Rightarrow \dfrac{y\cos x}{(D^2+DD'-6D'^2)} = \dfrac{y\cos x}{(D+3D')(D-2D')}$

$\Rightarrow \dfrac{1}{(D+3D')}\int (a-2x)\cos x\,dx$ where $(a-2x)=y$.

$\Rightarrow \dfrac{1}{D+3D'}[((a-2x)\sin x)-(-2)(-\cos x)]$

$\Rightarrow \dfrac{1}{(D+3D')}[y\sin x-2\cos x]$

$\Rightarrow \int[(b+3x)\sin x-2\cos x]\,dx$ Where $y=b+3x$

$\Rightarrow [(b+3x)(-\cos x)-(3)(-\sin x)-2\sin x]$

$\Rightarrow -y\cos x+3\sin x-2\sin x = -y\cos x+\sin x$

<u>Complete Solution</u> :- $\phi_1(y+2x)+\phi_2(y+3x)-y\cos x+\sin x$ Ans

(v) $(r+5s+6t) = \dfrac{1}{(y-2x)}$

We know that $r=\dfrac{\partial^2 z}{\partial x^2}=D^2$, $s=\dfrac{\partial^2 z}{\partial x\cdot\partial y}$ and $t=\dfrac{\partial^2 z}{\partial y^2}$

<u>Substitute</u> :- $D^2+5DD'+6D'^2 = \dfrac{1}{(y-2x)}$

Put $D=m$ and $D'=1$, we get

AE :- $m^2+5m+6=0$

$(m+2)(m+3)=0 \Rightarrow m_1=-2$ and $m_2=-3$

<u>Cf</u> $\phi_1(y+m_1x)+\phi_2(y+m_2x)$

$\phi_1(y-2x)+\phi_2(y-3x)$

<u>PI</u> :- $z=\dfrac{f(x,y)}{f(D,D')}$

PI :- $\dfrac{1}{D^2+5DD'+6D'^2}(y-2x)^{-1} = \dfrac{1}{(D+2D')}\left[\dfrac{1}{(D+3D')}(y-2x)^{-1}\right]$

$\Rightarrow \frac{1}{(D+2D')} \times \frac{1}{(-2) * (3\times1)} \int u^{-1} dv$ where $v = y-2x$

$= \frac{1}{(D+2D')} \log N = \frac{1}{(D+2D')} \log(y-2x)$

$= \frac{1}{[1\times D - (-2)\times D']} \log(y-2x)$

$= \frac{x}{(1)^1 \times 1!} \log(y-2x)$ by formula (ii) with $a=-2$, $b=1$, $m=1$

CS :- $\phi_1(y-2x) + \phi_2(y-3x) + x\log(y-2x)$ Ans.

Ques-3 Solve

(i) $(DD'+D-D'-1)z = xy$.

$(D'+1)(D-1)z = xy$.

Cf $\sum_{i=1}^{m} \phi_i(a_i y - b_i x) e^{-(c_i x)/a_i}$

PI $z = \frac{f(x,y)}{f(D,D')}$

$\Rightarrow \frac{-1}{(1-D)(1+D')} xy \Rightarrow -(1-D)^{-1}(1+D')^{-1} xy$.

$\Rightarrow -[1+D+D^2+ \cdots][1-D'+D'^2- \cdots]xy$.

$\Rightarrow -[1+D+D^2+\cdots][xy - D'(xy) + D'^2(xy) - \cdots]$

$\Rightarrow -[1+D+D^2+\cdots](xy-x)$

$\Rightarrow -[(xy-x) + D(xy-x) + D^2(xy-x) + \cdots]$

$\Rightarrow -[(xy-x) + (y-1)]$

$\Rightarrow -xy + x - y + 1$

(ii) $(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$

$(D-1)(D-D'+1)z = \cos(x+2y) + e^y$

Cf ⇒ $\sum_{i=1}^{m} \phi_i (a_i y - b_i x) e^{-c_i x/a_i}$

Cf $= \phi_1[(1)y - (0)x]e^{-(-1)x/1} + \phi_2[(1)y - (-1)x]e^{-(1)x/1}$

$= \phi_1 (y) e^{x} + \phi_2 (y+x) e^{-x}$

PI :- $z = \dfrac{f(x,y)}{f(D,D')}$

PI $= \dfrac{\cos(x+2y) + e^y}{D^2 - DD' + D' + 1} = \dfrac{\cos(x+2y)}{D^2 - DD' + D' + 1} + \dfrac{e^y}{D^2 - DD' + D' + 1}$

$\Rightarrow \dfrac{\cos(x+2y)}{(-1)-(-2)-1+D'} + \dfrac{e^y}{0-0+1-1}$

$\Rightarrow \dfrac{\cos(x+2y)}{D'} \times \dfrac{-D'}{-D'} + \dfrac{e^y}{0} = \dfrac{\cos(x+2y)}{-(D')^2} + \dfrac{e^y x}{2D - D'}$

$\Rightarrow \dfrac{-D'\cos(x+2y)}{4} + \dfrac{e^y x}{-1} = \dfrac{\sin(x+2y)\cdot 2}{4} - e^y x$

$\Rightarrow \dfrac{\sin(x+2y)}{2} - e^y x$

CS :- $\phi_1 y e^{x} + \phi_2 (y+x) e^{-x} + \dfrac{\sin(x+2y)}{2} - e^y x$ Ans

(iii) $[(D-3D')-2]^2 z = 2e^{+2x} \sin(y+3x)$

$(D-3D'-2)(D-3D'-2)z = 2e^{2x} \cdot \sin(y+3x)$

Cf. $e^{-c_i x/a_i}[\phi_1(a_1 y - b_1 x) + x\phi_2(a_1 y - b_1 x)]$

$e^{2x}[\phi_1(y+3x) + \phi_2 \cdot x \cdot (y+3x)]$

PI :- $z = \dfrac{f(x,y)}{f(D,D')}$

$\Rightarrow \dfrac{1}{(D-3D'-2)^2} \cdot 2 \cdot e^{2x} \cdot \sin(y+3x)$

$\Rightarrow 2e^{2x} \times \dfrac{1}{[(D+2)-3(D'+0)-2]^2} \times \sin(y+3x)$

$\Rightarrow 2e^{2x} \times \dfrac{1}{(D-3D')^2} \sin(y+3x) \Rightarrow 2e^{2x} \times \dfrac{x^2}{1^2 \times 2!} \sin(y+3x)$

$CS = e^{2x}[\phi_1(y+3x) + x\phi_2(y+3x)] + 2e^{2x}\cdot\frac{x^2}{1^2\cdot 2!}\sin(y+3x)$ Ans

Ques-4 (i) $(D^2 - D')z = 2y - x^2$

Given equation can not be written as linear factors. Hence, its Cf is taken as

$$z = \sum A e^{hx+ky}$$

$$\frac{\partial z}{\partial x} = \sum A\cdot h\cdot e^{hx+ky},\quad \frac{\partial^2 z}{\partial x^2} = \sum A h^2 e^{hx+ky},\quad \frac{\partial z}{\partial y} = \sum A\cdot k\cdot e^{hx+ky}$$

Substitute the values in given equation, we have

$(D^2 - D')z = 0 \Rightarrow \sum A h^2 e^{hx+ky} - \sum A k e^{hx+ky} = 0$

$\sum A(h^2 - k)e^{hx+ky} = 0$

$z(h^2 - k) = 0 \Rightarrow h^2 = k;\ z \neq 0$

Hence, $Cf = \sum A e^{hx+h^2y}$.

PI $z = \frac{f(x,y)}{f(D,D')}$

$PI = \frac{2y-x^2}{(D^2-D')} = \frac{2y-x^2}{D^2\left[1-\frac{D'}{D^2}\right]} = D^{-2}\left[1-\frac{D'}{D^2}\right]^{-1}(2y-x^2)$

$\Rightarrow D^{-2}\left[1+(-1)\left(-\frac{D'}{D^2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{D'}{D^2}\right)^2+\cdots\right](2y-x^2)$

$\Rightarrow D^{-2}\left[(2y-x^2)+(1)\frac{D'}{D^2}(2y-x^2)+0\right]$

$\Rightarrow D^{-2}\left[2y-x^2+\frac{2}{D^2}\right] = D^{-2}\left[2y-x^2+\int 2x\right] = D^{-2}\left[2y-x^2+x^2\right]$

$\Rightarrow D^{-2}(2y) = \iint 2y\,dx\cdot dx = \int 2yx\cdot dx = yx^2$

Complete Solution :- Cf + PI

$= \sum A e^{hx+h^2y} + yx^2$.

Ques-5 Using the method of separation of variables, solve

(i) $3u_x + 2u_y = 0,\ u(x,0) = 4e^{-x}$

We know that $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$, on Putting in given equation

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

let $u(x,y) = X(x)\cdot Y(y)$ ---(i)

$\frac{\partial u}{\partial x} = X'Y$, $\frac{\partial u}{\partial y} = XY'$ --- (ii)

On Putting equations from (ii) in the given equation.

$$3X'Y + 2XY' = 0$$
$$3X'Y = -2XY'$$
$$\frac{3X'}{X} = \frac{-2Y'}{Y} = k$$

Integrating on Both sides:-

$\int \frac{3X'}{X} dx = \int k dx$

$3\log|x| = kx + \log|c_1|$

$\log\left|\frac{x}{c_1}\right| = \frac{kx}{3}$

$X = c_1 e^{kx/3}$

$\int -\frac{2Y'}{Y} dy = \int k dy$

$-2\log|Y| = ky + \log|c_2|$

$\log\left|\frac{Y}{c_2}\right| = \frac{-ky}{2}$

$Y = c_2 e^{-ky/2}$

Now, we have assume that $U = XY = e^{kx/3} \cdot e^{-ky/2} \cdot c'$ $[c' = c_1 + c_2]$ --- (iii)

Now, using $U(x,0) = 4e^{-x}$ and putting $y=0$ in (iii)

$$u = c' e^{kx/3} \cdot e^{-ky/2} = 4e^{-x}$$
$$c' e^{kx/3} \cdot e^{-k(0)/2} = 4e^{-x}$$
$$c' e^{kx/3} = 4e^{-x}$$
$$c_1 = 4 \text{ , } k = -3$$

Putting values of $c_1 = 4$ and $k = -3$ in (iii)

$$u = 4e^{-x} e^{-3y/2} \quad \text{Ans.}$$

(ii) $4u_x + u_y = 3u$, $u(0,y) = 3e^{-y} - e^{-5y}$

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$$

let $u(x,y) = X(x) \cdot Y(y)$

$\frac{\partial u}{\partial x} = X'Y$, $\frac{\partial u}{\partial y} = XY'$, $u = XY$ --- (i)

Putting values of equation from (i) in given equation:-

$$4X'Y + XY' = 3XY$$
$$4X'Y - 3XY = -XY'$$

$(4X' - 3X)Y = -XY'$

$\frac{4X' - 3X}{X} = \frac{-Y'}{Y} = k$

$\frac{4X' - 3X}{X} = k \Rightarrow \frac{X'}{X} = \frac{k+3}{4} \quad , \quad \frac{-Y'}{Y} = k$

Integrating on both sides :-

$\int \frac{X'}{X} = \int \frac{k+3}{4} dx$

$\log|X| = \left(\frac{k+3}{4}\right)x + \log|c_1|$

$\log\left|\frac{x}{c_1}\right| = \left(\frac{k+3}{4}\right)x$

$X = c_1 e^{\left(\frac{k+3}{4}\right)x}$

$\int \frac{-Y'}{Y} dy = \int k\, dy$

$\log|Y| = ky + \log|c_2|$

$\log\left|\frac{Y}{c_2}\right| = ky$

$Y = c_2 e^{ky}$

$U = XY = c_1 e^{\left(\frac{k+3}{4}\right)x} \cdot c_2 e^{ky} = c' e^{\left(\frac{k+3}{4}\right)x} \cdot e^{ky}$

Ques-6 (i)

The solution is $u = (c_1 \cos mx + c_2 \sin mx)(c_3 \cos mt + c_4 \sin mt)$

The initial and bounday conditions are:-

$$u(0,t) = 0 \;,\; u(\pi,t) = 0 \;,\; u(x,0) = 2(\sin x + \sin 3x) \;,\; \frac{\partial u}{\partial t}(x,0) = 0$$

Since $u(0,t) = 0$ we get $c_1 = 0$, since $u(\pi,t) = 0$, we have

$$c_2 \sin m\pi \,(c_3 \cos mt + c_4 \sin mt) = 0$$

$$\sin m\pi = 0$$

Thus, $m = m \; (m \in I)$ and

$$u = \sin mx \,(c_3 \cos mt + c_4 \sin mt)$$

$$\therefore \frac{\partial u}{\partial t} = \sin mx \,(-c_3 m \sin mt + c_4 m \cos mt)$$

$\because \left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$, we have $c_4 = 0$

$$\therefore u = c_3 \sin mx \cdot \cos mt = b_n \sin nx \cdot \cos nt$$

Most general solution is $u = \sum_{n=1}^{\infty} b_n \sin nx \cdot \cos nt$

Since $u(x,0) = 2\sin x + \sin 3x$, we have

$$2\sin x + 2\sin 3x = \sum b_n \sin nx$$

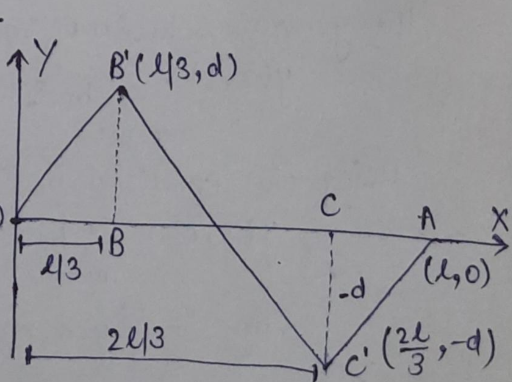
$$\Rightarrow b_1 = 2 \,,\; b_2 = 0 \,,\; b_3 = 2 \,,\; b_4 = b_5 = \dots = 0$$

Therefore, The solution is $y = 2\sin x \cos t + 2\sin 3x \cdot \cos 3t$ Ans

(ii) let the string OA be trisected at B and C.

Let the equation of vibrating string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{----(i)}$$



The solution for eq(i) is

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \quad \text{----(ii)}$$

Now using the boundary conditions:-

(i) $y(0,t) = 0$ (ii) $y(l,t) = 0$

Now use (i) and (ii) in equation (2), we get

$$y(x,t) = c_2 \sin px \,(c_3 \cos cpt + c_4 \sin cpt)$$

Where $p = \frac{m\pi}{l}$ ----(iii)

Next equation of OB' is $y = \frac{dx}{l/3} \Rightarrow y = \frac{3dx}{l}$

Equation of B'C' is $y-d = \dfrac{d+d}{\frac{l}{3}-\frac{2l}{3}}\left(x-\frac{l}{3}\right)$ | $y-y_1 = \dfrac{y_2-y_1}{x_2-x_1}(x-x_1)$

$\Rightarrow y = \dfrac{3d}{l}(l-2x)$

Equation of C'A is $y-0 = \dfrac{-d-0}{\frac{2l}{3}-l}(x-l)$

$\Rightarrow y = \dfrac{3d}{l}(x-l)$

The initial conditions of the given problem are

(iii) $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ (iv) $y(x,0) = \begin{cases} \dfrac{3dx}{l}, & 0 \le x \le l/3 \\ \dfrac{3d}{l}(l-2x), & l/3 \le x \le 2l/3 \\ \dfrac{3d}{l}(x-l), & 2l/3 \le x \le l \end{cases}$

from equation (iii) we get, $\dfrac{\partial y}{\partial t} = C_2 \sin px\,(-C_3 cp \sin cpt + C_4 cp \cos cpt)$

using (iii) initial condition in above, we obtain

$$0 = C_2 \sin px\,(C_4 cp) \Rightarrow C_4 = 0$$

Again from (3), we have

$$y(x,t) = C_2 C_3 \sin \frac{m\pi x}{l} \cos \frac{m\pi ct}{l}$$

The general solution of equation (1) is :-

$$y(x,t) = \sum_{m=1}^{\infty} b_n \sin \frac{m\pi x}{l} \cos \frac{m\pi ct}{l} \quad \text{------} (4)$$

using (iv) condition in equation (4), we get

$$y(x,0) = \sum_{m=1}^{\infty} b_n \sin \frac{m\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l y(x,0) \cdot \sin \frac{m\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^{l/3} \frac{3dx}{l} \sin \frac{m\pi x}{l}\, dx + \int_{l/3}^{2l/3} \frac{3d}{l}(l-2x) \sin \frac{m\pi x}{l}\, dx + \int_{2l/3}^{l} \frac{3d}{l}(x-l) \sin \frac{m\pi x}{l}\, dx\right]$$

Q-7- (i) A rod of length 'l' with insulated sides is initially at a uniform temperature u. Its ends are suddenly cooled to 0°C and are kept at that temperature. Prove that the temperature function $u(x,t) = \sum_{n=1} b_n \sin \frac{n\pi x}{l} e^{-\frac{c^2\pi^2 n^2 t}{l^2}}$ where b_n is determined from the equation $u(x,0) = u_0$

The temperature function $u(x,t)$ satisfies $\frac{\delta u}{\delta t} = c^2 \frac{\delta^2 u}{\delta x^2}$ whose suitable solution is $u(x,t) = (c_1 \cos px + c_2 \sin px) e^{-c^2p^2t}$ — (1)

As per boundary conditions $u(0,t) = 0 = u(l,t)$

we have $c_1 e^{-c^2p^2t} = 0 \Rightarrow c_1 = 0$ and

$c_2 \sin pl \, e^{-c^2p^2t} = 0 \Rightarrow \sin pl = 0$ or $pl = n\pi$ or $p = \frac{n\pi}{l}$

$\therefore u = c_2 \sin \frac{n\pi x}{l} e^{-\frac{c^2n^2\pi^2t}{l^2}}$ where $n = 1, 2, 3, \dots$

hence c_2 has to be replaced by b_n then adding all such solutions, we have $u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2\pi^2n^2t/l^2}$

since $u(x,0) = u_0$ we have $u_0 = \sum b_n \sin \frac{n\pi x}{l}$

which is a half range Fourier sine series of u_0, hence

$$b_n = \frac{2}{l}\int_0^l u_0 \sin \frac{n\pi x}{l} dx = -\frac{2u_0}{l}\left[\cos \frac{n\pi x}{l}\right]_0^l \cdot \frac{l}{n\pi}$$

$$= \frac{2u_0}{n\pi}(1 - \cos n\pi) = \begin{cases} 0 & \text{when } n \text{ is even} \\ \frac{4u_0}{n\pi} & \text{when } n \text{ is odd.} \end{cases}$$

Therefore $u(x,t) = \frac{4u_0}{\pi} \sum \frac{1}{n} \sin \frac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2}$, $n = 1, 3, 5, \dots$

or $u(x,t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{(2n-1)\pi x}{l} e^{-c^2(2n-1)\pi^2t/l^2}$

Ans

(ii) A bar AB of length 20cm has its ends A and B kept at 30° and 80° temperatures respectively until steady state condition is reached. Then the temperature at A is lowered to 40°C and at B to 60°C, these temperatures are maintained. Find the subsequent temperature distribution in the bar.

At time $t = 0$ steady state prevails

$\frac{\delta^2 u}{\delta x^2}(x,0) = 0$, for $0 \leq x \leq 20$

$u(0,0) = 30$ and $u(20,0) = 80$

General Solution $\Rightarrow$ $u(x) = Ax + B$

$u(0) = B = 30$ and $u(20) = A.20 + B = 80$

Hence, $B = 30$ and $A = (80-30)/20 = 5/2$

Steady State initial temperature is $u(x,0) = \frac{5}{2}x + 30$.

On solving the initial value problem, we get

$u(0,t) = 40$ and $u(20,t) = 60$ for $t > 0$

$u(x,0) = \frac{5}{2}x + 30$ for $0 \leq x \leq 20$

(for $u(x,t) = k u_{xx}(x,t)$ for $t > 0$ and $0 < x < 20$)

We first find the steady-state temp. $\Rightarrow u_t = 0$; $u_{xx} = 0$

Hence we are looking for a function $u_s(x)$ defined for $0 \leq x \leq 20$ such that $\frac{\delta^2 u_s(x)}{\delta x^2} = 0$, for $0 \leq x < 20$

$u_s(0,t) = 40$ and $u_s(20,t) = 60$ for $t > 0$

Now, the solution to this boundary problem is easily found.

General solution $\Rightarrow u_s(x) = Ax + B$

$u_s(0) = B = 40$ and $u_s(20) = A.20 + B = 60$

Hence, $B = 40$ and $A = ((60-40)/20) = 1$

so the steady state temp is $u_s(x,0) = x + 40$

We need to calculate $N = u - u_s$ now, and it must satisfy $N_t(x,t) = k V_{xx}(x,t)$ for $t > 0$ and $0 < x < 20$.

$u(0,t) = 0$ and $u(20,t) = 0$ for $t > 0$

$N(x,0) = u(x,0) - u_s = \frac{5}{2}x + 30 - (x+40) = \frac{3}{2}x - 10, 0 \leq x \leq 20$

Having found the steady state temperature u_S and the temperature v, the solution to the original problem is
$u(x,t) = v(x,t) + u_S(x)$

Let $v(x,t) = X(x)T(t)$
When we insert $v(x,t) = X(x)T(t)$ into the heat equation $v_t(x,t) = k\, v_{xx}(x,t)$, we get,

$$X(x)T'(t) = k\, X''(x)T(t)$$

$$\Rightarrow \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

$$\Rightarrow T'(t) + \lambda k T(t) = 0 \text{ and } X''(x) + X(x)\lambda = 0$$

The first equation has the general solution
$$T(t) = Ce^{+\lambda k t}$$
now, suppose that $\lambda > 0$ and $\lambda = w^2$, then equation is $X''(x) + w^2 X(x) = 0$ which has general solution
$$X(x) = a\cos wx + b\sin wx$$
for this $X(0) = 0$ becomes $a = 0$; $X(20) = 0$ becomes $b\sin 20w = 0$
We consider $\sin 20w = 0$ (non zero solution).
This occurs if $20w = m\pi$ (for +ve integer m). When this is true, we have eigenvalue $\lambda = w^2 = \frac{m^2\pi^2}{400}$.
$X(x) = b\sin\left(\frac{m\pi x}{20}\right)$ is an eigen function.

$$v_m(x,t) = e^{-\frac{m^2\pi^2 k t}{400}} \sin\left(\frac{m\pi x}{20}\right), \text{ for } m = 1,2,3,\dots$$

$$v_m(x,t) = \sum_{m=1}^{\infty} b_n v_m(x,t) = \sum_{m=1}^{\infty} b_m e^{-\frac{m^2\pi^2 k t}{400}} \sin\left(\frac{m\pi x}{20}\right)$$

The initial condition is $v(x,0) = \frac{3}{2}x - 10 = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{20}\right)$,
where

$$b_m = \frac{2}{20}\int_0^{20}\left(\frac{3x}{2}-10\right)\sin\left(\frac{m\pi x}{20}\right)dx = \frac{3}{20}\int_0^{20} x\sin\left(\frac{m\pi x}{20}\right)dx - \int_0^{20}\sin\left(\frac{m\pi x}{20}\right)dx$$

$$= \frac{20}{\pi m}(1-(-1)^m).$$

$$b_m = \frac{-3}{20}\frac{400}{\pi m}(-1)^m + \frac{20}{\pi m}(1-(-1)^m) = \frac{20}{\pi m}(1-4(-1)^m)$$

$$v(x,t) = \sum_{m=1}^{\infty}\frac{20}{m\pi}(1-4(-1)^m)\sin\left(\frac{m\pi x}{20}\right)$$

The Temperature distribution in the rod at time t is

$$u(x,t) = x+40+\sum_{m=1}^{\infty}\frac{20}{\pi m}(1-4(-1)^m)\sin\left(\frac{m\pi x}{20}\right)$$

Q-4-(ii) $(D-2D'-1)(D-2D'^2-1)Z=0$

Complete Solution = CF + PI

CF: Here $(D-2D'-1)$ is a linear factor. Therefore its $CF_1 = \phi_1(y+2x).e^x$

for $(D-2D'^2-1)$, let the Trial solution of this factor is

$Z = \sum A e^{hx+ky}$

$DZ = Ahe^{hx+ky}$, $D'^2Z = Ak^2e^{hx+ky}$

Substituting these values in $(D-2D'^2-1)Z$, we get

$$Ahe^{hx+ky} - 2Ak^2e^{hx+ky} - \sum Ae^{hx+ky} = 0$$

$$A(h-2k^2-1)e^{hx+ky} = 0 \Rightarrow h-2k^2-1=0$$

$$\Rightarrow h = 2k^2+1, \text{ replacing } h \text{ by } 2k^2+1$$

$$CF_2 = \sum A e^{(2k^2+1)x+ky}$$

$$CF = CF_1 + CF_2 = \phi_1(y+2x)e^x + \sum Ae^{(2k^2+1)x+ky}$$

$PI = 0$; since RHS = 0

$$CS = \phi_1(y+2x)e^x + \sum A^{(2k^2+1)x+ky}$$