Ques Derive PDF for Bivariate Normal Distribution.

Solv We know that

For d dimensions

$$P(x^{\rightarrow}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{\{-1/2 [x-\mu]^T \Sigma^{-1} [x-\mu]\}} \quad \cdots (1)$$

Covarience Matrix , $\Sigma = \begin{bmatrix} \sigma_1\sigma_1 & \sigma_1\sigma_2 \\ \sigma_2\sigma_1 & \sigma_2\sigma_2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$

$$|\Sigma| = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$$

$$= \sigma_1^2\sigma_2^2\left(1 - \frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right)$$

$$= \sigma_1^2\sigma_2^2(1-\rho^2) \cdots (2)$$

Putting (2) in equation 1

$$P(x^{\rightarrow}) = \frac{1}{(2\pi)^{2/2}\sqrt{\sigma_1^2\sigma_2^2(1-\rho^2)}} e^{\{-1/2 (x-\mu)^T \Sigma^{-1} (x-\mu)\}}$$

Now

$$\Sigma^{-1} = \frac{Adj\,\Sigma}{|\Sigma|} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\begin{bmatrix} \sigma_2^2 & \sigma_{12}^2 \\ -\sigma_{12}^2 & \sigma_1^2 \end{bmatrix}$$

Solving the exponential part

$$e^{-\{-1/2 [x-\mu_1 \;\; y-\mu_2] \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\begin{bmatrix} \sigma_2^2 & -\sigma_{12}^2 \\ -\sigma_{12}^2 & \sigma_1^2 \end{bmatrix}\begin{bmatrix} x-\mu_1 \\ y-\mu_2 \end{bmatrix}\}}$$

$$= e^{-\{-1/2 [x-\mu_1 \;\; y-\mu_2] \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}\begin{bmatrix} \sigma_2^2(x-\mu_1) - \sigma_{12}(y-\mu_2) \\ -\sigma_{12}^2(x-\mu_1) - \sigma_1^2(y-\mu_2) \end{bmatrix}\}}$$

$$= e^{\{-1/2 \times \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}[\sigma_2^2(x-\mu_1)^2 - \sigma_{12}(y-\mu_2)(x-\mu_1) - \sigma_{12}(x-\mu_1)(y-\mu_2) + \sigma_1^2(y-\mu_2)^2\}}$$

$$= e^{\left\{-\frac{1}{2\sigma_1^2\sigma_2^2 - 2\sigma_{12}^2}\left(\sigma_2^2(x-\mu_1)^2 - 2\sigma_{12}^2(y-\mu_2)(x-\mu_1) + \sigma_1^2(y-\mu_2)^2\right)\right\}}$$

Multiplying & dividing by $\sigma_1^2\sigma_2^2$

$$= e^{\left\{\frac{-\sigma_1^2\sigma_2^2}{2\sigma_1^2\sigma_2^2 - 2\sigma_{12}^2}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}(y-\mu_2)(x-\mu_1) + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right\}}$$

$$= e^{\left\{-\frac{1}{2}\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2\sigma_2^2\left(1-\frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\sigma_{12}}{\sigma_1\sigma_2}\left(\frac{y-\mu_2}{\sigma_2}\right)\left(\frac{x-\mu_1}{\sigma_1}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)\right\}}$$

$$= e^{\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}}$$

Hence

$P(\vec{x})$ for Bivarite Normal Distribution is

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$