Probability Theory

Mutually Ex. Events

- A & B are M.E. if they cannot occur together
i.e. $A \cap B = \emptyset$.

let $S = \{H, T\}$. $A = \{H\}$, $B = \{T\}$.
Then $A \cap B = \emptyset$. A & B are M.E

Probability: let $\mathcal{E}$ be an exp. let S be a S.S. associated with $\mathcal{E}$. With each event A, we associate a real no, $P(A) = p$, called the probability of A s.t.

(i). $0 \leq P(A) \leq 1$
(ii). $P(S) = 1$
(iii). If A & B are M.E. Then $P(A \cup B) = P(A) + P(B)$.

or $P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$

In general if A, B, C are not pairwise ME then

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Equally likely Outcomes

Eg. A coin is tossed twice. let $A = \{$ one head appears $\}$.

Then $S_1 = \{0, 1, 2\}$ & $P(A) = 1/3$ ✗

Because 0 head appears is not as likely as 1 " " .

Rather $S_2 = \{TT, HT, TH, HH\}$

$A = \{HT, TH\}$

$\therefore P(A) = 1/2$.

Remember, each point in S.S (or outcome in S.S) in this case is "equally likely".

i.e. TT is equally likely to occur as HT etc.

The S.S. S_1 could be used as follows

There are 3 possible outcomes but outcomes '0' & '2' are equally likely whereas outcome '1' is more likely to occur.

So if 'p' is probability for '0'

Then $p + p + 2p = 1$

or $4p = 1$

or $\boxed{p = 1/4}$ $\Rightarrow$ probability of 1H $= 1/2$

Conditional Probability

Eg

80 ND
20 D
100

$A = \{\text{First item is def}\}$

$B = \{\text{second item is def}\}$.

Choose two items

(a). with replacement (b) without replacement.

Then

For (a) $P(A) = \frac{20}{100} = 1/5$.

& $P(B) = 20/100 = 1/5$.

For (b) : $P(A) = 20/100 = 1/5$.

$P(B) = ?$

Case (i) First item was Defective.

Then

80 ND
19 D
99

$\therefore P(B) = 19/99$

$= P(B|A)$

Reduced S.S.

~~P(B|A)~~

Case (ii) : First item was ND.

79 ND
20 D
99

$P(B) = 20/99$.

$= P(B|A')$

Eg Rolling of two fair dice

$$S = \begin{Bmatrix} (1,1), & (1,2), & \cdots & (1,6) \\ (2,1), & (2,2), & \cdots & (2,6) \\ \vdots & & & \\ (6,1), & (6,2), & \cdots & (6,6) \end{Bmatrix}$$

let $A = \{ (x_1, x_2) \mid x_1 + x_2 = 10 \}$

$B = \{ (x_1, x_2) \mid x_1 > x_2 \}$

$A = \{ (5,5), (6,4), (4,6) \}$

$B = \{ (2,1), (3,2), (3,1),$
$(4,3), (4,2), (4,1),$
$(5,4), (5,3), (5,2), (5,1),$
$(6,5), (6,4), (6,3), (6,2), (6,1) \}$

$\boxed{P(A) = 3/36 \quad P(B) = 15/36}$

$\boxed{(A \cap B) = \{ (6,4) \}}$

$\boxed{P(A \cap B) = 1/36}$

$P(B/A) = ?$

Reduced S.S. $A = \{ (5,5), (6,4), (4,6) \}$
where $(6,4) \in B$

$\therefore \boxed{P(B/A) = 1/3}$

$P(A|B) = ?$

Reduced S.S. $B = \{(2,1),$
$(3,2), (3,1),$
$\vdots$
$(6,5), (6,4), \cdots (6,1)\}$

where $(6,4) \in A$

$\therefore \boxed{P(A|B) = 1/15}$

We observe that

$$P(A \cap B) = \frac{1}{15} \cdot \frac{15}{36} = \frac{1}{36}$$

$$= P(A|B) \cdot P(B)$$

$$\therefore \boxed{P(A \cap B) = P(A|B) \cdot P(B)}$$

Also $P(A \cap B) = \frac{1}{36} = \frac{1}{3} \cdot \frac{3}{36}$

$$= P(B|A) \cdot P(A)$$

$$\therefore \boxed{P(A \cap B) = P(B|A) \cdot P(A)}$$

or $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

& $P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Q^n What is the difference between $P(A \cap B)$ & $P(B|A)$?

Eg

80 ND
20 D

$A = \{\text{First item def}\}$

$B = \{\text{Second item def}\}$

$P(B|A) = 19/99$

$\equiv$ Probability that

"Second item was defective given that the first was defective".

Reduced S.S.

$P(A \cap B) \equiv$ Probability that

"Both items are defective".

Original S.S. because nothing is given

$P(A \cap B) = P(B|A) \cdot P(A)$

$= \frac{19}{99} \cdot \frac{20}{100}$

$$\boxed{P(A \cap B) = 19/495}$$

or ~~$P(A \cap B) = P$~~

For $P(A \cap B)$ directly.

Total no. of possible outcomes $= 100 \times 99$

Of these the outcomes where both items are defective i.e. $(A \cap B)$ will have.

$20 \times 19 = 380$ outcomes.

$\therefore |S| = 100 \times 99$

& $|(A \cap B)| = 380$

$$\therefore P(A \cap B) = \frac{38\cancel{0}}{10\cancel{0} \times 99} = \frac{38}{10 \times 99}$$

$$= \frac{19}{5 \times 99} = \frac{19}{495}$$

$\therefore \boxed{P(A \cap B) = 19/495}$

$\therefore$ $P(A \cap B)$ means probability that both items are defective. ~~and its~~ over the entire S.S.

$P(B/A)$ means probability that 2nd item is defective in the S.S. of A.

Multiplication theorem of Probability

$$\boxed{P(A \cap B) = P(B|A) \cdot P(A)}$$

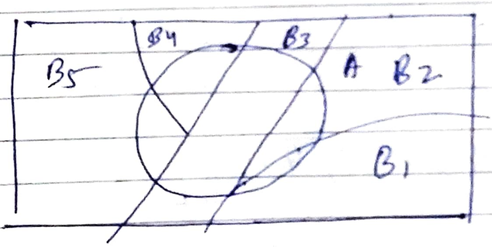
In general:

$\cancel{P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_2|A_1) \cdot P(A_1) \cdot P(A_3|A_2) \dots}$

$\cancel{P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_2|A_1) \, P(A_1) \cdot P(A_3|A_1, A_2)}$

$$\boxed{P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_2|A_1) \, P(A_3|A_1, A_2) \cdot P(A_4|A_1, A_2, A_3) \dots \cdot P(A_n|A_1, A_2, \dots, A_{n-1}) \cdot P(A_1)}$$

Partition of S.S.



Events $B_1, B_2, \dots, B_k$ generates a partition on S if

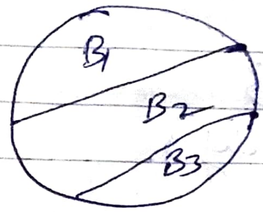
(i) $\bigcup_{i=1}^{k} B_i = S$.

(ii) $B_i \cap B_j = \emptyset \;\forall\; i \neq j$

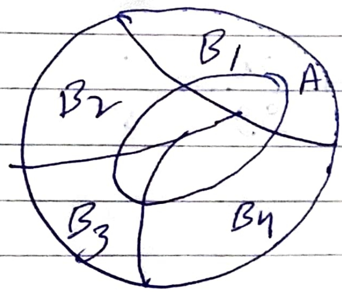
(iii) $P(B_i) > 0 \;\forall\; i$

Eg Tossing of a dice

$B_1 = \{1, 2\}$ $B_2 = \{3, 4, 5\}$, $B_3 = \{6\}$



The Total Probability Th



$$A = (A \cap B_1) \cup (A \cap B_2) \cup (A \cap B_3) \cup (A \cap B_4)$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + P(A \cap B_3) + P(A \cap B_4)$$

[Pairwise ME]

$$P(A) = P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2) + P(A|B_3)\,P(B_3) + P(A|B_4)\,P(B_4)$$

$$\therefore \boxed{P(A) = P(A|B_1)\,P(B_1) + P(A|B_2)\,P(B_2) + \dots + P(A|B_n)\,P(B_n)}$$

Total Prob Th

Eg

20 D

80 ND.

$A = \{ \text{First item is df} \}$

$B = \{ \text{Second item is df} \}$.

$P(B) = ?$

$$P(B) = (B \mid A)\, P(A) + P(B \mid \bar{A})\, P(\bar{A})$$

$$= \frac{19}{99} \cdot \frac{1}{5} + \frac{20}{99} \cdot \frac{4}{5}$$

$$\Rightarrow \boxed{P(B) = 1/5}$$

Bayes' Theorem

Consider agent

20 D
80 ND

A = { First item is def }

B = { second item is def }

We know $P(B|A) = 19/99$

But what is $P(A|B)$?

i.e. Given that second item is defective what is the probability that first was def

So $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

$$\Rightarrow \boxed{P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B)}}$$

$\Uparrow$

Bayes' Th.

Expanding $P(B)$

$$\boxed{P(A|B) = \dfrac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\bar{A})\, P(\bar{A})}}$$

In general if 1st event partitions the S.S. 'S' in $B_1, B_2, \dots B_k$.

and 2nd event 'A' is contained partially in these then, probability of occurrence of 'B_i' given that 'A' has occurred is

$$\boxed{P(B_i|A) = \frac{P(A|B_i).\,P(B_i)}{\sum_{j=1}^{k} P(A|B_j)\,P(B_j)}} \quad — (1)$$

Bayes' Th.

→ Formula for probability of ~~cau~~ cause, given the effect

→ $P(B_i|A)$ is called Posterior probability & ~~$P(A|B_j)$ is prior probability~~ $P(B_i)$ is prior probability

(1) can be w/a

$$\boxed{P(B_i|A) = \frac{P(A|B_i).P(B_i)}{P(A)}}$$

← Likelihood

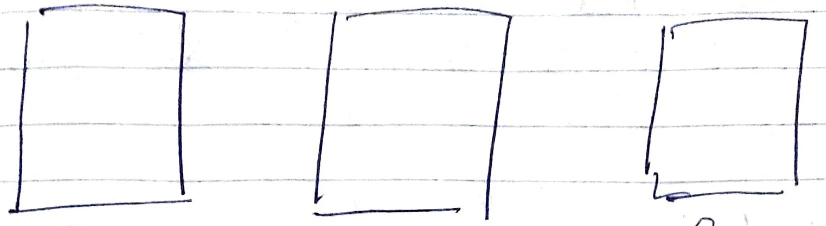
← Prior

$\left.\frac{P(A|B_i)}{P(A)}\right\}$ Likelihood Ratio

Posterior : Marginalization (Evidence)

$$\text{Posterior} = \frac{\text{Likelihood} \times \text{Prior}}{\text{Evidence}}$$

Eg



F_1	F_2	F_3
2% def	2% def	4% def
$2x$ items produced	x items produced	x items produced.

What is the prob that an item is def

Sol $A = \{\text{item is def}\}$

$B_1 = \{\text{item} \in F_1\}$

$B_2 = \{\text{item} \in F_2\}$

$B_3 = \{\text{item} \in F_3\}$

Find $P(A)$.

$$P(A) = P(A|B_1).P(B_1) + P(A|B_2)P(B_2) + P(A|B_3).P(B_3)$$

$2x + 2x = 1 \Rightarrow x = 1/4 \Rightarrow P(B_1) = 1/2$

$P(B_2) = P(B_3) = 1/4$

$$P(A) = \frac{\cancel{2}}{100}\cdot\frac{1}{\cancel{2}} + \frac{\cancel{2}}{100}\cdot\frac{1}{\cancel{4}_2} + \frac{\cancel{4}}{100}\cdot\frac{1}{\cancel{4}}$$

$$= \frac{1}{100} + \frac{0.5}{100} + \frac{1}{100} = \frac{2.5}{100} = .025$$

$P(A) = .025$.

Ex: Given that there is dog, what is the prob that it came from

$$P(B_1|A) = \frac{P(A|B_1) \cdot P(B_1)}{P(A)}$$

$$= \frac{\frac{2}{100} \cdot \frac{1}{2}}{\frac{2.5}{100}}$$

$$= \frac{\frac{1}{100}}{\frac{2.5}{100}} = \frac{1}{100} * \frac{100}{2.5}$$

$$= \frac{1}{2.5} = .4$$

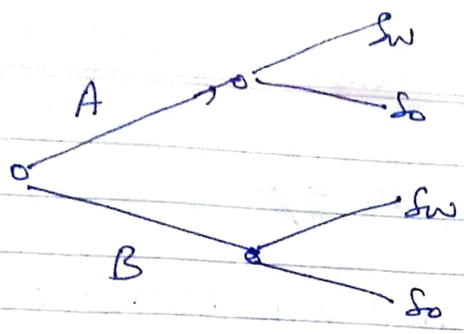
Ex: There are two types of containers. A & B.

50% A → 70% Sweet candy (Sw)
→ 30% Sour " (So)

50% B → 30% Sw
→ 70% So

An unknown jar is taken at random. One candy is taken out at random. Based on the type of candy chosen, to decide the jar picked

Soln



$P(A) = 0.6 \qquad P(B) = 0.4$

$P(Sw/A) = 0.7 \qquad P(So/A) = 0.3.$

$P(Sw/B) = 0.3 \qquad P(So/B) = 0.7$

Find $P(A/Sw)$, $P(A/So)$, $P(B/Sw)$ & $P(B/So)$.

$$P(A/Sw) = \frac{P(Sw/A) \cdot P(A)}{P(Sw)}$$

$$= \frac{P(Sw/A) \cdot P(A)}{P(Sw/A) \cdot P(A) + P(Sw/B) \cdot P(B)}$$

$$= \frac{0.7 \times 0.6}{0.7 \times 0.6 + 0.3 \times 0.4}$$

$$= 7/9.$$

$\therefore P(B/Sw) = 2/9.$

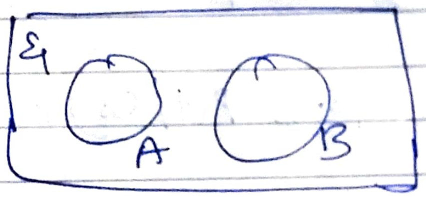
Independent Events

Note: M.E. $\Rightarrow A \cap B = \phi$

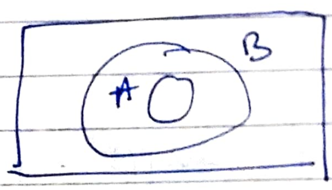
for, $P(A \cap B) = P(B|A) \cdot P(A)$
& $P(B|A) = 0$ for A & B to be ms

$\therefore$ ME $\Rightarrow P(B|A) = 0$

$$\boxed{\text{ME} \Rightarrow P(A \cap B) = 0}$$

1) 

$\Rightarrow P(A|B) = P(B|A) = 0$
$\Rightarrow P(A \cap B) = 0$

Also
2) 

$P(B|A) = 1$
& $P(A \cap B) = P(A)$

&
1) means $A \Rightarrow \neg B$
2) means $A \Rightarrow B$

These are dependent Events.

There are events when
$A \not\Rightarrow B$
& $A \not\Rightarrow \neg B$
Independent Events

Eg Let a die (fair) be tossed twice

Let $A = \{$ First toss shows even no $\}$

$B = \{$ Second toss shows 5 or 6 $\}$

S.S. $S = \{ (1,1) \; \dots \; (1,6)$

$\vdots$

$(6,1) \; \dots \; (6,6) \}$

$|S| = 36.$

$P(A) = 18/36.$

$P(B) = 12/36.$

$P(A \cap B) = 6/36$

$\therefore P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{6}{36}}{\frac{12}{36}} = 1/2$

or By reduced S.S. $P(A|B) = 6/12 = 1/2$

$\therefore \boxed{P(A|B) = 1/2 = P(A)}$

$P(B|A) = \dfrac{6}{18} = 1/3.$

$\boxed{P(B|A) = 1/3 = P(B)}$

$\therefore$ A & B are ind cy

$\boxed{P(A|B) = P(A) \; \& \; P(B|A) = P(B)}$

If ind. Events?