Q: Suppose a radio tube is tested.
Let probability of positive test = 3/4
& Neg test = 1/4

Let a large supply till a positive tube appears.

X = No. of tests required to terminate the exp.

$S = \{+, -+, --+, ---+, \ldots\}$

$\therefore \; p(n) = P(X=n) = \left(\frac{1}{4}\right)^{n-1}\left(\frac{3}{4}\right)$

Check $\quad \sum p(n) = \frac{3}{4}\left(1+\frac{1}{4}+\frac{1}{16}+\cdots\right)$

$= \frac{3}{4} \cdot \frac{1}{1-1/4} = 1.$

Expected value of a RV

$E(X) = \sum_{i=1}^{\infty} x_i \, p(x_i) \quad \Big| \quad E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$

Variance of a R.V.

$V(X) = E[X - E(X)]^2$

$+\sqrt{V(X)}$ is called standard dev of X

Binomial distribution

D = 0.2
N = 0.8
choose 3 items.

S = {DDD, DDN, DND, DNN, NDD,
NDN, NND, NNN}.

or $S_1 \times S_2 \times S_3$
$S_i = \{D, N\}$.

How many defective items were found?

DDD	$\longrightarrow$	$(.2)^3$	$P[X=3]$
DDN	$\longrightarrow$	$(.2)^2(.8)$	$P[X=2]$
DND	$\longrightarrow$	$(.2)^2(.8)$	$P[X=2]$
DNN	$\longrightarrow$	$(.2)(.8)^2$	$P[X=1]$
NDD	$\longrightarrow$	$(.8)(.2)^2$	$P[X=2]$
NDN	$\longrightarrow$	$(.8)^2(.2)$	$P[X=1]$
NND	$\longrightarrow$	$(.8)^2(.2)$	$P[X=1]$
NNN	$\longrightarrow$	$(.8)^3$	$P[X=0]$

$\therefore \; p(0) = (.8)^3 \quad = .512$

$p(1) = (.2)(.8)^2 \quad = .128 \times 3$

$p(2) = (.2)^2(.8) \quad = .032 \times 3$

$p(3) = (.2)^3 \quad = .008$

0

$\Sigma p(0) = 1.$

Also $(.8+.2)^3 = 1$

Binomial r.v $\boxed{E(X) = np}$

let $\mathcal{E}$ be an expt. Let A be an event of A.S.S. let $P(A) = p$ and $p(\bar{A}) = 1-p$. Consider n independent repititions of $\mathcal{E}$ st $p(A) = p$ for all repititions..

let the r.v 'X' be:

X = No. of times event 'A' occured.

Then 'X' is Bin r.v and individual rep. is called Bernoulli trial

Then $P(X=k) = p(k) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k}$

$k = 0, 1, 2, \ldots \ldots n$

The corresponding distr is called Bin. dist.

Ex. A radio tube is inserted in a set. Prob. of a tube functioning > 500 hrs. $= 0.2$. 20 tubes are tested. What is the probability that exactly 'k' of these will function for more than 500 hrs.

Ans $P(X=k) = {}^{20}C_{k}\,(0.2)^{k}\,(0.8)^{n-k}$

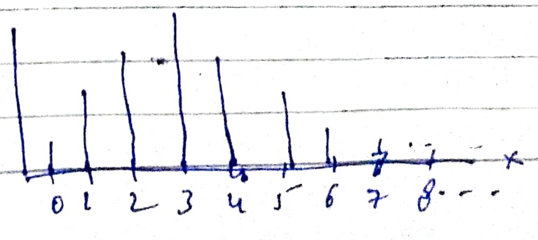
Work out: $P(X=0) = 0.012$

$P(X=1) = 0.058$

$\vdots$

$P(X=10) = 0.002$

etc



Continuous R.V

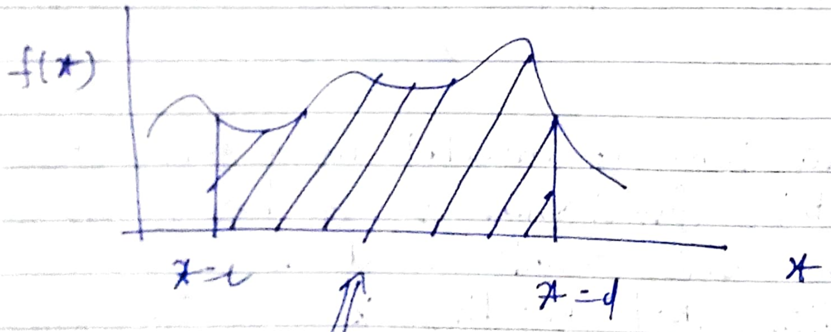
X is said to be a continuous R.V if $\exists$ a function 'f', called the probability density f^n (pdf) of X, satisfying the following conditions:

(i) $f(x) \geq 0 \quad \forall x$ &

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ &

(iii) $\forall a, b \in (-\infty, \infty)$ & $a < b$

$$P(a \leq X \leq b) = \int_a^b f(x)\,dx.$$



$\int_c^d f(x)\,dx$ = Area under pdf

Note $\int_c^c f(x)\,dx = 0 \quad \forall c$

Ex: Let pdf of X ~~is~~ f(x) is given as:

$f(x) = 2x, \quad 0 < x < 1$

$\quad = 0$ Elsewhere.

Clearly $f(x) \geq 0$ & $\int_0^1 2x\,dx$

$= \left[2 \cdot \frac{x^2}{2}\right]_0^1$

$= [x^2]_0^1 = 1.$

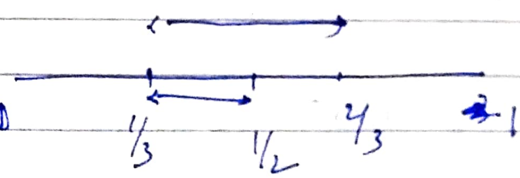
$\therefore f(x)$ is a pdf.

Find $P(X \leq 1/2)$. $= \int_0^{1/2} 2x\,dx = [x^2]_0^{1/2}$

$= 1/4.$

Ex Find $P(X \leq 1/2 \mid 1/3 \leq X \leq 2/3)$

=



$= \dfrac{\int_{1/3}^{1/2} 2x\,dx}{\int_{1/3}^{2/3} 2x\,dx} = \dfrac{5/36}{1/3} = 5/12$

Cumulative Distribution Fn (CDF)

Let X be a r.v. CDF of X, $F(x)$ is defined as

$$F(x) = P(X \leq x)$$

for Disc. R.v.

$$F(x) = \sum_j p(x_j) \; ; \quad x_j \leq x$$

for Con R.v.

$$F(x) = \int_{-\infty}^{x} f(x)\, dx .$$

Ex: $f(x) = 2x \; ; \; 0 < x < 1 .$

$$F(x) = \int_0^x 2x\, dx = x^2 \quad 0 < x \leq 1$$

$$= 1 \text{ if } x > 1$$

Note: It follows that

$$\boxed{f(x) = \frac{d}{dx} F(x)}$$

Uniform Distribution

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

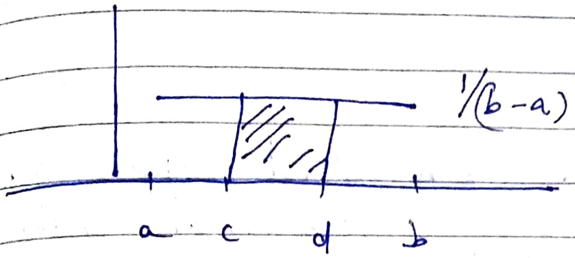
$$0 \qquad \text{otherwise.}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{1}{(b-a)} \int x\,dx$$

$$= \frac{1}{(b-a)} \frac{(b-a)(b+a)}{2}$$

$$= (a+b)/2$$



$$P(c \le x \le d) = \int_c^d \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} [x]_c^d$$

$$= \frac{d-c}{b-a}$$

If $d = b$ & $a = c$ Then $P(a \le x \le b) = 1$

CDF of U.D.

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{a}^{x} \frac{1}{b-a}\,dx = \frac{1}{b-a} (x)_a^x$$

$$= \frac{x-a}{b-a} \; ; \quad a \le x \le b.$$

Normal or Gaussian Distⁿ

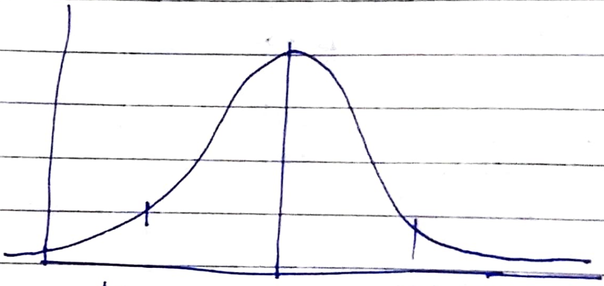
$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

$$= N(\mu, \sigma^2)$$

μ = mean of x

σ = SD of x

It serves as an excellent approximation to a large class of distributions which have great practical importance.



~~At A = μ~~

Case $\sigma = 1$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$$

At $\mu = 0$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}} = N(0,1)$$

Points of inflection

When ~~der~~ derivative changes sign

~~$x \pm \sigma$~~. $x = \mu \pm \sigma$

$\therefore$ Large $\sigma \Rightarrow$ Flat graph

Expected value

$$E(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $z = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma\, dz$.

$$\therefore E(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x \cdot e^{-\frac{1}{2} z^2} \sigma \, dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu)\, e^{-z^2/2} dz$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} \sigma \int z \cdot e^{-z^2/2} dz}_{=\,0} + \underbrace{\frac{\mu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz}_{=\,1}$$

$$\therefore \boxed{E(x) = \mu}$$

CDF of Normal Dist (standardized)

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt.$$

Also $P(a \le x \le b) = \phi(b) - \phi(a)$

$\phi(x)$ is tabulated and hence can be used to compute $P(a \le x \le b)$.

Th if X has distribution $N(\mu, \sigma^2)$ &

$Y = aX + b$ then Y is dist $N(a\mu + b, a^2\sigma^2)$

if $Y = \frac{X - \mu}{\sigma} = \frac{1}{\sigma} X - \frac{\mu}{\sigma}$ then

$$Y = N\left(\frac{1}{\sigma} \cdot \mu - \frac{\mu}{\sigma}, \left(\frac{1}{\sigma}\right)^2 \sigma^2\right)$$

$$= N\left(\frac{\mu}{\sigma} - \frac{\mu}{\sigma}, 1\right)$$

$$= N(0, 1)$$

$$\therefore P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Y \le \frac{b-\mu}{\sigma}\right)$$

$$= \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right)$$

Exponential Distribution

$$f(x) = \begin{cases} \alpha e^{-\alpha x}, & x > 0, \ \alpha > 0 \\ 0, & \text{otherwise.} \end{cases}$$

CDF

$$F(x) = P(X \leq x) = \int_0^x \alpha e^{-\alpha t} \, dt$$

$$= \begin{cases} 1 - e^{-\alpha x}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore P(X > x) = 1 - F(x)$$

$$= 1 - (1 - e^{-\alpha x})$$

$$= e^{-\alpha x}$$

Expected Value

$$E(x) = \int_0^\infty x \cdot \alpha e^{-\alpha x} \, dx$$

Let $\alpha e^{-\alpha x} dx = dv$ & $x = u$.

Solving $\boxed{E(x) = \frac{1}{\alpha}}$

$\therefore$ The reciprocal of 'α' is the mean of x

Ex. Find $P(x > 1/\alpha)$.

$$\Rightarrow e^{-\alpha x} \longrightarrow \frac{1}{\alpha}$$

We know $P(x > x) = e^{-\alpha x}$

$$\therefore \; P(x > 1/\alpha) = e^{-\alpha \cdot 1/\alpha} = e^{-1}$$

$$= 1/e < 1/2.$$

The Gamma Distribution

$$f(x) = \begin{cases} \dfrac{\alpha}{\Gamma(r)} (\alpha x)^{r-1} e^{-\alpha x}, & x \geq 0,\ r > 0,\ \alpha > 0 \\ 0 & \text{otherwise} \end{cases}$$

There are 2 parameters
(i) r &
(ii) α

$\Gamma(r)$ is a Gamma function in 'r'. defined as.

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} \, dx \qquad p > 0$$

Integrating by parts

$$\boxed{\Gamma(p) = (p-1)\,\Gamma(p-1)}$$

$$\therefore \Gamma(n) = (n-1)\,\Gamma(n-1)$$
$$= (n-1)(n-2)\,\Gamma(n-2)$$
$$\vdots$$
$$= (n-1)(n-2)\ldots\Gamma(1)$$

$$\boxed{\Gamma(1) = \int_0^\infty e^{-x} = 1}$$

$$\therefore \boxed{\Gamma(n) = (n-1)!}$$

Some Properties :

At $r=1$

$$f(x) = \frac{\alpha}{\Gamma(1)} (\alpha x)^{1-1} e^{-\alpha x}$$

$$= \alpha e^{-\alpha x}$$

= Exponential distribution.

Probability of R.V.

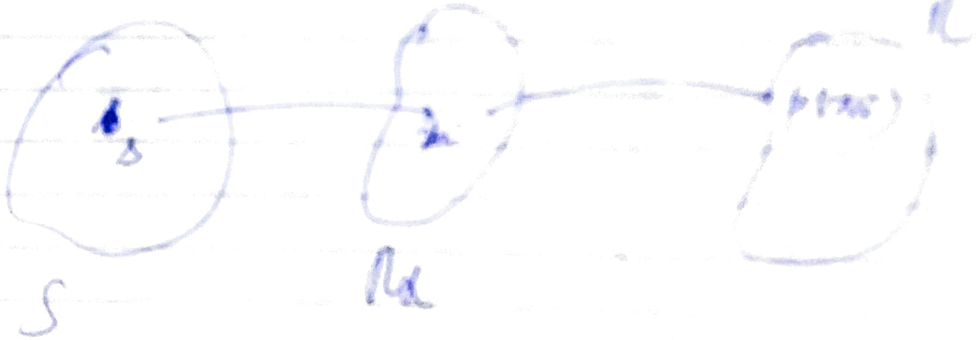
Let X be a dis. R.V. & x_i be associated with a no. $p(x_i) = P(X = x_i)$, called the probability of x_i. The numbers $p(x_i)$ must satisfy

(i) $p(x_i) \geq 0 \ \forall i$

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

'p' is called the Probability fn of X.

The collection of ordered pairs $(x_i, p(x_i))$ is called the probability distribution of X.



Probability distribution can be plotted as graph.