Maths

Assignment

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1. $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

A.E. $\Rightarrow m^2 - m - 2 = 0$

$\Rightarrow (m-2)(m+1) = 0$

$m = 2, -1$

$C.F = \phi(y+2x) + \phi_2(y-x)$

$$P.I. = \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x$$

$$= e^x \frac{1}{(D+1)^2 - (D+1)D' - 2D'^2}(y-1)$$

$$= e^x \frac{1}{D^2 - DD' - 2D'^2 + 2D' - D' + 1}(y-1)$$

$$= e^x [1 + (2D - D' - 2D'^2 - DD' + D^2)]^{-1}(y-1)$$

Using binomial theorem

$$= e^x [1 - 2D + D' + \dots](y-1)$$

$$= e^x[y - 1 + 1] = ye^x$$

$$z = \phi(y+2x) + \phi_2(y-x) + ye^x$$

2. $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$

A.E. $= m^3 - 7m - 6 = 0$

$= (m+1)(m^2 - m - 6) = 0$

$\Rightarrow m = -2, 3, -1$

$C.F. = \phi_1(y-2x) + \phi_2(y+3x) + \phi_3(y-x)$

$$P.I. = \frac{1}{D^3 - 7DD'^2 - 6D'^3}(\sin(x+2y) + e^{2x+y})$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{2x+y}$$

$$= \frac{((-1) - 38D')\sin(x+2y)}{(-D + 38D')(-D - 38D')} + \frac{1}{-12}e^{2x+y}$$

$$= \frac{(-1)-38D')}{5775} \sin(x+2y) - \frac{1}{12}e^{2x+y}$$

$$= \frac{-\cos(x+2y)-76\cos(x+2y)}{5775} - \frac{1}{12}e^{2x+y}$$

$$= -\frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$

$$z = \phi_1(y-2x)+\phi_2(y+3x)+\phi_3(y-x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$

3. $(D^2+2DD'+D'^2-2D-2D')z = \sin(x+2y)$

$$A.E. = m^2+2m+1-2m-2=0$$
$$= m^2-1=0$$
$$= m = \pm 1$$

$$C.F. = \phi_1(y-x)+\phi_2(y+x)$$

$$P.I. = \frac{1}{D^2+2DD'+D'^2-2D-2D'}\sin(x+2y)$$

$$= \frac{1}{-1+2(-2)+(-4)-2D-2D'}\sin(x+2y)$$

$$= \frac{-1}{2D+2D'+9}\sin(x+2y)$$

$$= \frac{-(2D+2D'-9)}{(2D+2D'+9)(2D+2D'-9)}\sin(x+2y)$$

$$= \frac{(2D+2D'-9)}{117}\sin(x+2y)$$

$$= \frac{1}{39}[2\cos(x+2y)-3\sin(x+2y)]$$

$$z = e^{2x}\phi_1(y-x)+\phi_2(y-x)+\frac{1}{39}[2\cos(x+2y)-3\sin(x+2y)]$$

4. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy$

$\Rightarrow (D+D')(D-2D'+2)z = e^{2x+y} + xy$

$C.F. = e^{-2x}\phi_1(y+2x) + \phi_2(y-x)$

$P.I = \dfrac{1}{(D^2-DD'-2D'^2+2D'+2D')} e^{2x+3y} + \dfrac{1}{(D^2-DD'-2D^2+2D+2D')} xy$

$\Rightarrow \dfrac{-e^{2x+3y}}{10} + \dfrac{1}{(D+D')(D-2D'+2)}$

$\Rightarrow \dfrac{e^{2x+3y}}{10} + \dfrac{1}{2D}\left(1 - \dfrac{D}{2} + D' - DD' - D' + \dfrac{D'}{2} - \dfrac{D'^2}{D} - D'^2\right)xy$

$\Rightarrow \dfrac{-e^{2x+3y}}{10} - \dfrac{1}{24}(2x^3 - 6x^2y - 9x^3 + 6x^3 + 12x)$

$z = e^{-2x}\phi_1(y+2x) + \phi_2(y-x) - \dfrac{e^{2x+3y}}{10} - \dfrac{1}{24}(2x^3 - 6x^2y - 9x^2 + 6xy + 12x)$

5. $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$

Let $u(x,t) = X(x).T(t)$

$\dfrac{\partial u}{\partial x} = X'T$ & $\dfrac{\partial u}{\partial t} = XT'$

$\Rightarrow X'T = 2XT' + XT$

$\Rightarrow \dfrac{X'}{X} = \dfrac{2T' + T}{T} = \lambda$

$\Rightarrow X' - \lambda X = 0 \quad | \quad T' - \left(\dfrac{\lambda - 1}{2}\right)T = 0$

$\Rightarrow X = Ae^{\lambda x}, \quad T = Be^{\left(\frac{\lambda-1}{2}\right)T}$

$\Rightarrow u = Ce^{\lambda x}e^{\left(\frac{\lambda-1}{2}\right)t}, \quad (A.B = C)$

$\Rightarrow Ce^{\lambda x} = 6e^{-3x}$

$\boxed{C = 6}, \quad \boxed{\lambda = -3}$

$\Rightarrow \boxed{u(x,t) = 6e^{-(3x+2t)}}$ Ans

6. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

$\Rightarrow u(x,y) = X(x).Y(y)$

$\frac{\partial u}{\partial x} = X'Y, \frac{\partial u}{\partial y} = XY'$

$\Rightarrow 4X' + XY' = 3XY$

$\frac{4X'}{X} = \frac{-Y'+3Y}{Y} = \lambda$

$\Rightarrow X' - \frac{\lambda X}{4} = 0, Y' - (3-\lambda)Y = 0$

$X = Ae^{\lambda/4 x}, Y = Be^{(3-\lambda)y}$

$\Rightarrow u = Ce^{\lambda/4 x + (3-\lambda)y}$

$\Rightarrow u(0,y) = 3e^{-y} - e^{-3y}$

$\therefore u(x,y)$ is sum of two solution

$\Rightarrow u = C_1 e^{\frac{\lambda_1}{4}x} e^{(3-\lambda_1)y} + C_2 e^{\lambda_2/4 x} e^{(3-\lambda_2)y}$

$\Rightarrow (0,y) = C_1 e^{(3-\lambda_1)y} + C_2 e^{(x-\lambda_2)y}$

$\Rightarrow C_1 = 3, \lambda_1 = 4\ C_2 = -1, \lambda = 8$

$\Rightarrow u(x,y) = 3e^{x-y} - e^{2x-3y}$

7. $y(0,t) = y(l,t) = 0$

Sol^n of wave eq. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ is given by:-

$y(x,t) = (A\cos px + B\sin px)(C\cos cpt + D\sin cpt)$

$= 0$

$\Rightarrow y(0,t) = 0 \Rightarrow A(C\cos cpt + D\sin cpt) = 0$

$\Rightarrow A = 0$

$y(x,t) = [E\cos cpt + F\sin cpt]\sin px$ — (1)

$BC = E, BD = F$

$y(l,t) = 0 \Rightarrow (E\cos cpt + F\sin cpt)\sin px = 0$

$\sin pl = 0$

$p = \frac{n\pi}{l}$

$$\Rightarrow y(x,t) = \left[E_n \cos\left(\frac{n\pi c}{l}\right)t + F_n \sin\left(\frac{n\pi c}{l}\right)t\right] \sin\frac{n\pi x}{l}$$

$x = 1, 2, \ldots$

$$\Rightarrow y(x,0) = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

from eq (3)

$$\sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} = \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]$$

$\Rightarrow$ Comparing both sides

$$E_1 = \frac{3y_0}{4}, \; E_2 = 0, \; E_3 = -\frac{y_0}{4}, \; E_n = 0$$

$$\Rightarrow y(x,t) = \frac{y_0}{4}\left[3\cos\left(\frac{\pi c t}{l}\right)\sin\left(\frac{\pi x}{l}\right) - \cos\left(\frac{3\pi c t}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\right] + \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi c t}{l}\right)\sin\left(\frac{n\pi x}{l}\right) \quad - (4)$$

Now, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ (Given)

$$\Rightarrow \sum_{n=1}^{\infty} F_n \left(\frac{n\pi x}{l}\right) \sin\frac{n\pi x}{l} = 0 \Rightarrow F_n = 0 \; \forall n$$

$$\Rightarrow y(x,t) = \frac{y}{4}\left[3\cos\left(\frac{\pi c t}{l}\right)\sin\left(\frac{\pi x}{l}\right) - \cos\left(\frac{3\pi c t}{l}\right)\sin\left(\frac{3\pi x}{l}\right)\right]$$

Ans

8. Since boundary conditions at $x=0$ are trignometric function therefor sol^n of P.D.E $\frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial t^2}$

$$y(x,t) = (A\cos px + B\sin px)(C\cos(2pt) + D\sin 2pt) \quad - (1)$$

$\Rightarrow u(0,t) = \sin t$, Given

By ①, $A(C\cos 2pt + D\sin 2pt) = \sin t$

$AC = 0$, $AD = 1$, $2p = 1$

$\Rightarrow C = 0$, $AD = 1$, $p = \frac{1}{2}$

$\Rightarrow u(x,t) = \cos\frac{x}{2}\sin t + E\sin\frac{x}{2}\sin t$ — ②

$\therefore \frac{\partial u}{\partial x} = -\frac{1}{2}\sin\frac{t}{2}\sin t + \frac{1}{2}E\cos\frac{x}{2}\sin t$

Given $\left(\frac{\partial u}{\partial x}\right)_{x=0} = \sin t$

$\Rightarrow E = 2$

$\Rightarrow u(x,t) = \cos\frac{x}{2}\sin t + 2\sin\frac{x}{2}\sin t$

9. the temp $u(x,t)$ is given by the sol^n of 1D heat eq.

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad \text{— ①}$$

$\Rightarrow$ Since $u(0,t) = u(l,t) = 0$, therefore the sol^n of equation ① is given by $u(x,t)$

$= (A\cos px + B\sin px)e^{-p^2c^2t}$; $p > 0$ — ②

$\Rightarrow u(0,t) = 0 \Rightarrow Ae^{-p^2c^2t} = 0$

$\Rightarrow A = 0$

$\Rightarrow u(x,t) = B\sin px\, e^{-p^2c^2t}$ — ③

Now, $u(l,t) = 0 \Rightarrow B\sin pl\, e^{-p^2c^2t} = 0$

$\Rightarrow \sin pl = 0$

$pl = \frac{n\pi}{l}$; $n = 1, 2 \ldots$

By (3), $u(x,t) = B_n \sin\frac{n\pi x}{l} e^{\frac{-n^2\pi^2 c^2}{l^2}t}$

$n = 1, 2, 3 \ldots$

By principle of superposition

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} e^{\frac{-n^2 c^2 \pi^2 t}{l^2}} \quad -(4)$$

$$\Rightarrow u(x,0) = u_0 \Rightarrow \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = u_0$$

which is half range fourier sine series of u_0 in $(0, l)$.

$$\therefore B_n = \frac{2}{l}\int_0^l u_0 \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2u_0}{n\pi}[1-(-1)^n]$$

$$\left.\begin{aligned} \therefore B_{2n} &= 0 \\ B_{2n-1} &= \frac{4u_0}{(2n-1)\pi} \end{aligned}\right\} n = 1, 2, \ldots$$

By eq (4)

$$u(x,t) = \frac{4u}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin\frac{(2n-1)\pi x}{l} e^{\frac{-l^2(2n-1)^2\pi^2}{l}t}$$

$\Rightarrow$

10. Since $u \to 0$ as $y \to \infty$ for all x

$$u(x,y) = e^{-py}(A\cos px + B\sin px) \; ; \; A > 0 \quad -(1)$$

$\Rightarrow u(0,y) = 0 \; \forall \; y$ given

$$Ae^{-py} = 0$$

$$A = 0$$

$$u(x,y) = Be^{-py}\sin px \; , \; p > 0 \quad -(2)$$

Now, $u(x,0) = lx = x^2 \; \forall \; x \in (0,l)$

$$\sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} = lx - x^2$$

which is half range series in $(0,l)$

$$B_n = \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \Big[(lx - x^2)\left(\frac{-l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l-2x)\left(\frac{-l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) - 2\left(\frac{l^2 \cos \frac{n\pi x}{l}}{n^3\pi^3}\right) \Big]_0^l$$

$$B_n = \frac{-4}{n^3\pi^3} l^2 \left[(-1)^n - 1 \right]$$

$$\left.\begin{aligned} B_{2n} &= 0 \\ B_{2n-1} &= \frac{8l^2}{(2n-1)^3 \pi^3} \end{aligned}\right\} \quad n = 1, 2, 3 \ldots$$

$$\Rightarrow u(x,y) = \frac{8l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{\frac{-(2n-1)\pi y}{l}} \sin \frac{(2n-1)\pi x}{l}$$

Ans