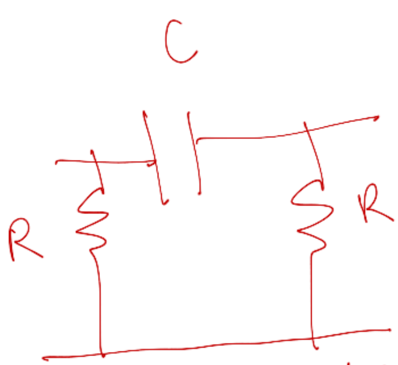
Chapter 9 R – L – C circuits

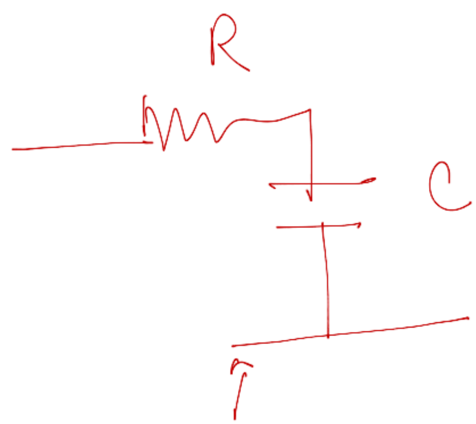
- Series connection
- parallel connection

Applications

1) Radio tuners (series ckt.)
2) Freq. multiplexers
3) Harmonic suppression filters.

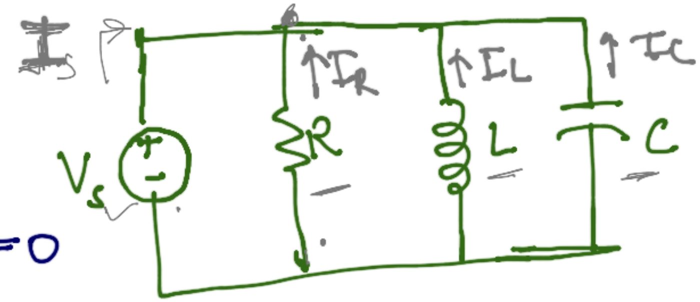


High pass freq.



L.P filter

With Source R||L||C circuit



KCL (flowing into the node)

$$I_s + \frac{V_s}{R} + i_L + C\frac{dV_s}{dt} = 0$$

$$i_L = \frac{1}{L}\int V_s\,dt$$

⇒ Taking derivative of natural part (leave out I_s, V_s forcing function)

$$\frac{1}{R}\frac{dV_s}{dt} + \frac{V_s}{L} + C\frac{d^2V_s}{dt^2} = 0 \quad \text{2nd ODE}$$

const

If $V_s = Ae^{st}$ (General Solution)

$$\frac{1}{R}Ase^{st} + \frac{Ae^{st}}{L} + Cs^2Ae^{st} = 0$$

Quadratic Equation

$$\boxed{s^2 + \frac{s}{RC} + \frac{1}{LC} = 0}$$

Characteristic Equation

2 solution

Solution for 's': $s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\frac{1}{4(RC)^2} - \frac{1}{LC}}$ ← s Negative changing.

2nd ODE

⇒ Solution (natural response)

$$\boxed{V_n = A_1e^{s_1t} + A_2e^{s_2t}}$$

$V_f = V_f$ (found by $L \to s.c$ & $C \to o.c$ & solving)

Solution

$$\therefore \quad V_s = V_f + v_n = \boxed{\underset{\text{forcing}}{V_f} + \underbrace{A_1 e^{s_1 t} + A_2 e^{s_2 t}}} = V_s$$

Finding A_1 & A_2 (A_1 & $A_2 \to$ constants)

$\boxed{V_s(t=0)}$ = Initial Condition 1 = $\boxed{V_f + A_1 + A_2 = V_0}$ — (1)

($V_s = V_0$ at $t=0$)

$$\frac{dv}{dt} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}$$

$\boxed{\left.\frac{dv}{dt}\right|_{t=0}}$ = Initial Condition 2 = $\boxed{A_1 s_1 + A_2 s_2 = \left.\frac{dv}{dt}\right|_{t=0}}$ — (2)

Simultaneously solve. (1) & (2) to find A_1 & A_2

$$V_s = V_f + A e^{s_1 t} + A_2 e^{s_2 t}$$

Analysis of solutions s_1 & s_2

$$s_1 , s_2 = -\underbrace{\frac{1}{2RC}}_{\alpha} \pm \sqrt{\underbrace{\frac{1}{(2RC)^2}}_{\alpha^2} - \underbrace{\frac{1}{LC}}_{\omega_0^2}}$$ → units should be 1/seconds

Let, $\boxed{\omega_0 = \frac{1}{\sqrt{LC}}}$: Resonant frequency

$\boxed{\alpha = \frac{1}{2RC}}$: Damping Coefficient

s_1 , s_2 → complex frequencies

→ $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ & $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

Case I $\alpha > \omega_0$ s_1 & s_2 are Real no.s & Negative Over damped response

Case II $\alpha = \omega_0$ $s_1 = s_2 = -\alpha$ Critically damped response

Case III $\alpha < \omega_0$ s_1 , s_2 are complex no.s Under damped response

$\alpha > \omega_0$: **Overdamped response**

$\alpha^2 > \omega_0^2$

$\Rightarrow \frac{1}{(2RC)^2} > \frac{1}{LC} \Rightarrow LC > 4R^2C^2 \Rightarrow \boxed{L > 4R^2C}$ s_1, s_2 { Negative Real

Time Constants: $s_1 = -\alpha + \underbrace{\sqrt{\alpha^2 - \omega_0^2}}_{\text{negative}} > s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$

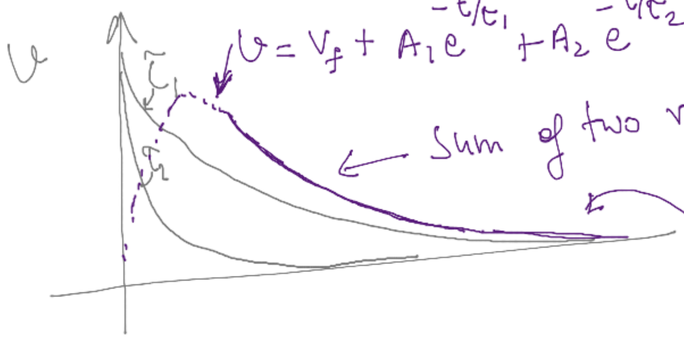
$\tau_1 = \frac{1}{s_1}$; $\tau_2 = \frac{1}{s_2}$

$|s| \rightarrow$ magnitude

$|s_1| < |s_2| \Rightarrow \left|\frac{1}{\tau_1}\right| < \left|\frac{1}{\tau_2}\right| \Rightarrow |\tau_2| < |\tau_1|$

$|\tau_1| \rightarrow$ slower decay

$|\tau_2| \rightarrow$ faster decay



$v = V_f + A_1 e^{-t/\tau_1} + A_2 e^{-t/\tau_2}$

$\leftarrow$ Sum of two responses

Damped $\Rightarrow v \rightarrow 0$ fast.

$\alpha = \omega_0$: **Critically damped** $LC = 4R^2C^2 \Rightarrow \boxed{L = 4R^2C}$

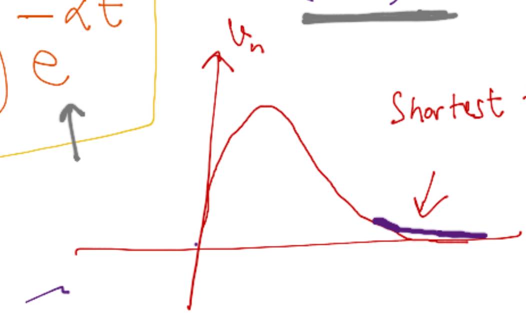
$s_1 = s_2 = -\alpha$

$\therefore$ Characteristic equation $\rightarrow s^2 + 2\alpha s + \alpha^2 = 0$ $(s+\alpha)^2 = 0$

Differential Equation solution

(General Solution)

$$\boxed{v_n = (A_1 t + A_2)\, e^{-\alpha t}}$$



$\alpha < \omega_0$ Underdamped Response

$$\frac{1}{2^2R^2C^2} < \frac{1}{LC} \Rightarrow LC < 4R^2C^2 \text{ or } L < 4R^2C$$

$$S_1 = -\alpha + \underbrace{\sqrt{\alpha^2 - \omega_0^2}}_{\text{Imaginary}}$$

Complex

$$= -\alpha + j\underbrace{\sqrt{\omega_0^2 - \alpha^2}}_{\omega_d}$$

$$= -\alpha + j\omega_d$$

$$S_2 = -\alpha - \underbrace{\sqrt{\alpha^2 - \omega_0^2}}_{\text{Imaginary}}$$

Complex

$$S_2 = -\alpha - j\sqrt{\omega_0^2 - \alpha^2}$$

$$= -\alpha - j\omega_d$$

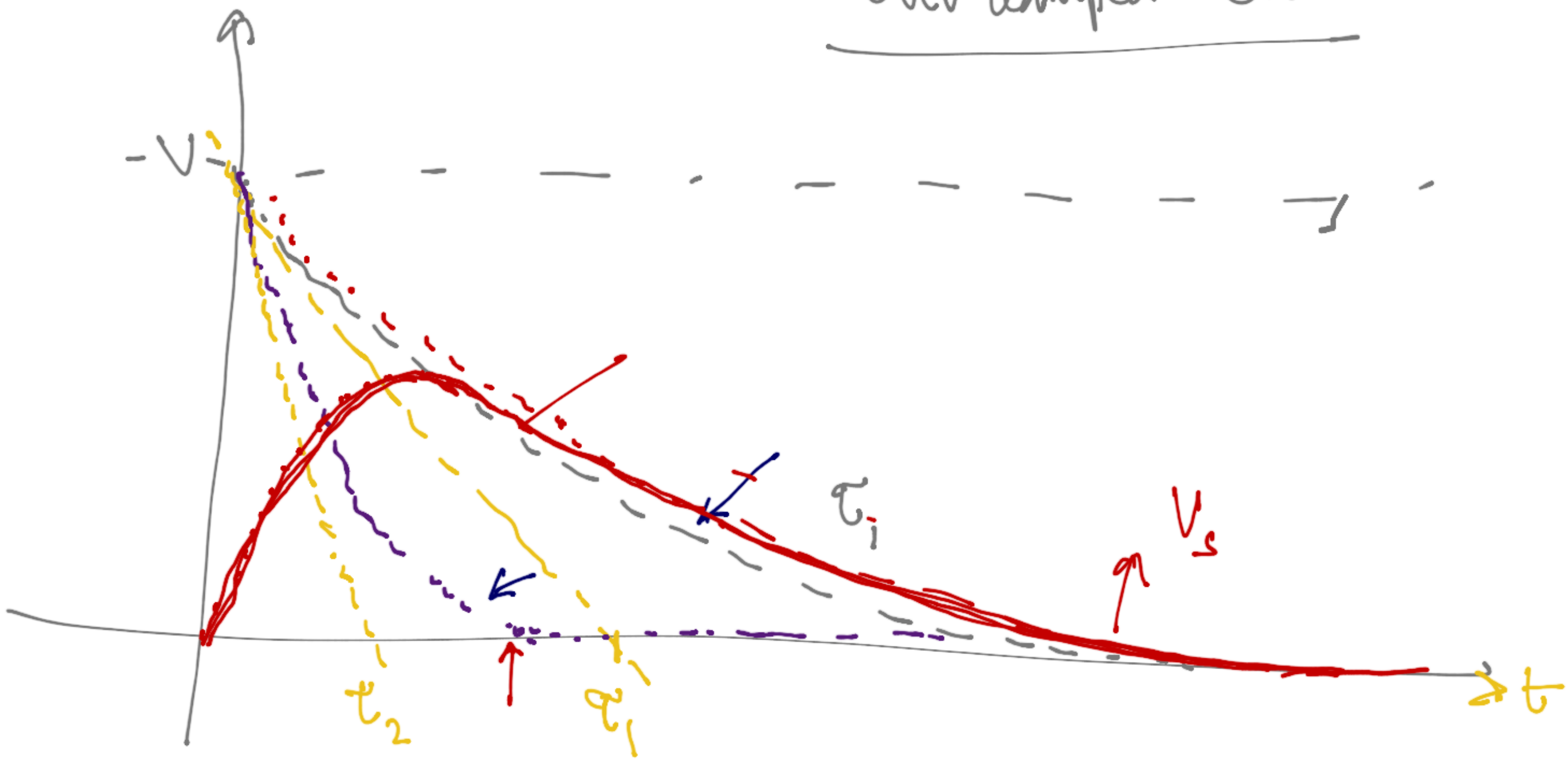
$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Natural response ∴ $v_n = A_1 e^{-\alpha t} \underbrace{e^{+j\omega_d t}}_{\text{sinusoid}} + A_2 e^{-\alpha t} \underbrace{e^{-j\omega_d t}}_{\text{sinusoid}}$

Recall
$$\begin{cases} e^{j\omega_d t} = \cos\omega_d t + j\sin\omega_d t \\ e^{-j\omega_d t} = \cos\omega_d t - j\sin\omega_d t \end{cases}$$

$$\boxed{v_n = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)} \leftarrow$$

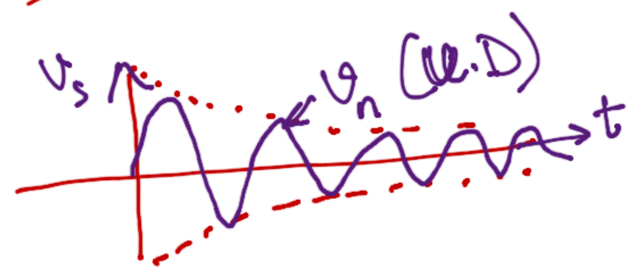
Overdamped Case



$$V_s = V_f + A_1 e^{-\frac{t}{|\tau_1|}} + A_2 e^{-\frac{t}{|\tau_2|}}$$

Critically Damped Case

$$V_s = V_f + (A_1 t + A_2)\, e^{-t/\tau}$$



Under damped case

Natural solution $\quad v_s = A_1 e^{-(\alpha + j\omega_d)t} + A_2 e^{-(\alpha - j\omega_d)t}$

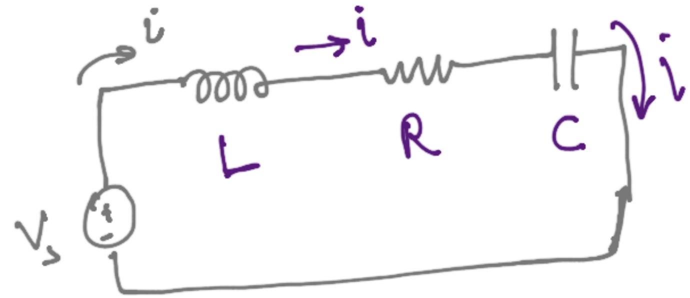
$$v_s = A_1 e^{-\alpha t} \underbrace{e^{-j\omega_d t}}_{} + A_2 e^{-\alpha t} \underbrace{e^{+j\omega_d t}}_{}$$

$$= A_1 e^{-\alpha t}(\cos\omega_d t - j\sin\omega_d t) + A_2 e^{-\alpha t}(\cos\omega_d t + j\sin\omega_d t)$$

$$= e^{-\alpha t}\left[\underline{(A_1 + A_2)}\cos\omega_d t + \underline{(-A_1 j + A_2 j)}\sin\omega_d t\right]$$

$$\boxed{v_s = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)}$$

Driven (with Source) R-C-L series



KVL

$$V_s + V_R + V_C + V_L = 0$$

$$\underbrace{V_s}_{\text{Forcing}} + \underbrace{iR + \frac{1}{C}\int i\,dt + L\frac{di}{dt} = 0}_{\text{Natural ckt.}}$$

Taking derivative $R\frac{di}{dt} + \frac{1}{C}i + L\frac{d^2i}{dt} = 0 \rightarrow$ 2^{nd} order D.E

General solution $i = Ae^{st}$

$\therefore \Rightarrow R\,s + \frac{1}{C} + Ls^2 = 0 \Rightarrow \boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$ Characteristic Equation

$$\Rightarrow s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad : \text{solutions.}$$

$\frac{R}{2L} \rightarrow \alpha$, $\frac{1}{LC} \rightarrow \omega_0^2$

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Natural Response $\therefore \Rightarrow i_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

$i_f \rightarrow$ forced response

$$i = i_f + i_n = \boxed{i_f + A_1 e^{s_1 t} + A_2 e^{s_2 t} = i}$$

Solve A_1 & A_2 using $i(t=0)$ and $\frac{di(t=0)}{dt}$.

Char. eq^n $s^2 + 2\alpha s + \omega_0^2 = 0$

where $\alpha = \frac{R}{2L}$, $\omega_0^2 = \frac{1}{LC}$

α → Damping coefficient

ω_0 → Resonant frequency.

Over damping

$\alpha > \omega_0$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

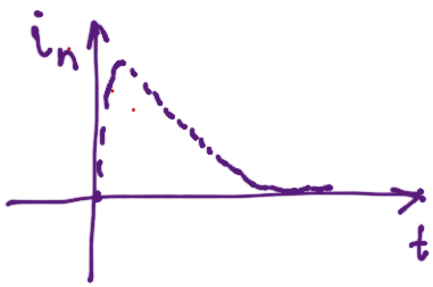
$$i_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Critically Damped

$\alpha = \omega_0$

$$s_1 = s_2 = -\alpha$$

$$i_n = e^{-\alpha t}(A_1 t + A_2)$$



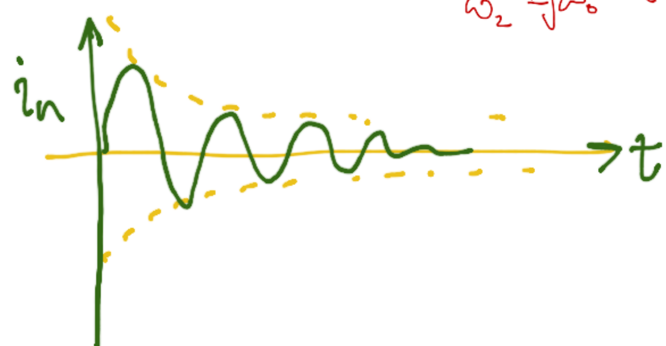
Under damped

$\alpha < \omega_0$

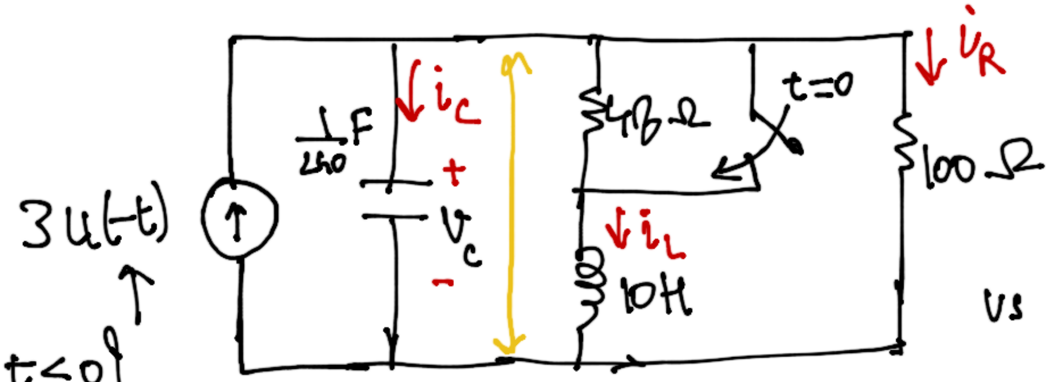
$$s_1 = s_2 = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

$$i_n = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$



Example 9.6



$i_L(t) = ?$

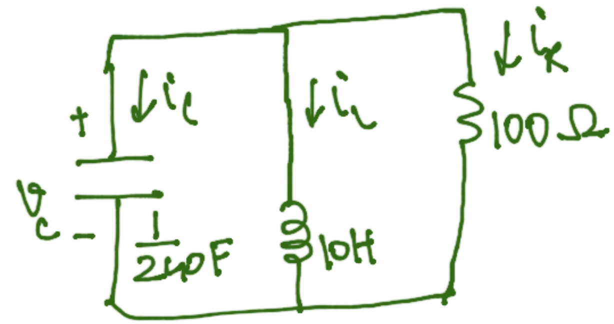
↓

finding i_L

vs using

$$3u(-t) = \begin{cases} 3A & t<0 \\ 0 & t>0 \end{cases}$$

$t>0$



Step Response 1) Nat Resp.

$L = 10 \qquad C = \frac{1}{240} \qquad R = 100$

$R \| L \| C$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 100 \times \frac{1}{240}} = \underline{1.2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 * \frac{1}{240}}} = \sqrt{24} \simeq \underline{4.9}$$

$\omega_0 > \alpha$

$\Downarrow$

Under damped

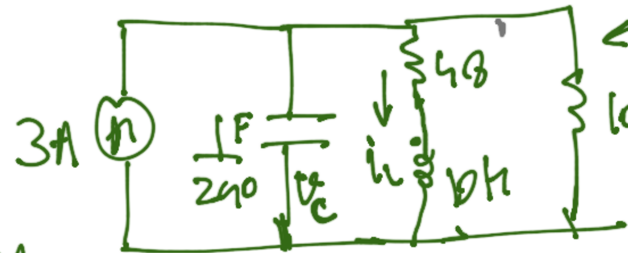
$$\boxed{i_L = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \simeq \sqrt{24 - 1.2^2} \simeq 4.8$$

$\rightarrow B_1, B_2 = ?$

$i_L(t=0)$; $\frac{di_L}{dt}(t=0)$.

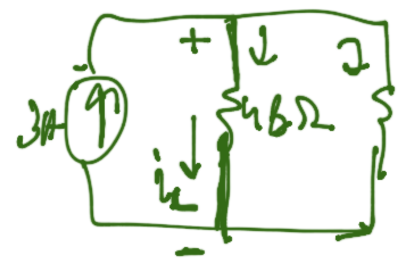
$t=0 \quad v_L = v_C$.



$(t<0)$

O.C Cap
S.C L

S.S



(S.S at $t<0$)

$i_{R_{48}} = i_L$

Step 2 Finding Initial conditions

$$i_L(t=0^-) = \frac{100}{148} * 3A = 2.027A$$

$$v_C(t=0^-) = 3 * \frac{4800}{148} = 97.3V = v_C(0^+)$$

$t<0$

$$i_L(0^-) = i_L(0^+)$$

$$v_C(0^-) = v_C(0^+)$$

i_L, $\frac{di_L}{dt}$ $(t=0)$ $\rightarrow v_L(t=0)$

$$i_L = e^{-\alpha t}(B_1 \cos w_d t + B_2 \sin w_d t)$$

$$\frac{di_L}{dt} = (-\alpha)e^{-\alpha t}(\quad) + e^{-\alpha t}(-B w_d \sin w_d t + B_2 w_d \cos w_d t)$$

$$L\frac{di}{dt} = v_L = v_C(t=0^+) = 97.3 = -\alpha(B_1 + 0) + 1(-0 + B_2 w_d) \rightarrow ①$$

$$i_L = e^{-\alpha t}\left(B_1 \cos\omega_d t + B_2 \sin\omega_d t\right)$$

$$i_L(0^-) = i_L(0^+) = B_1 \qquad \text{— (2)}$$

Solve (1) & (2) $\quad \alpha,\ \omega_d$

$$\boxed{i_L = e^{-1.2t}\left(2.027 \cos 4.7t + 2.56 \sin 4.75t\right)}$$