2

Notes for Information and Communication

Signals are crucial for signal processing & communications. Hence we study them.

Signals:

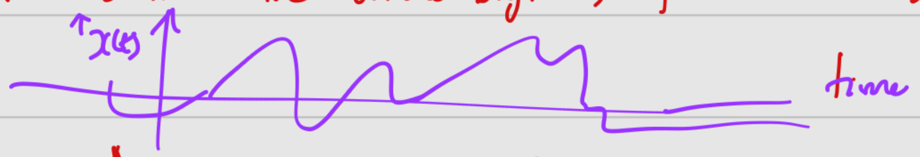
↳ functions of time. We will consider only real-valued signals in this course

A continuous signal: Denoted by $\{x(t) : t \in \mathbb{R}\}$

Signal is 'real valued' $\Rightarrow x(t) \in \mathbb{R}, \forall t \in \mathbb{R}$.

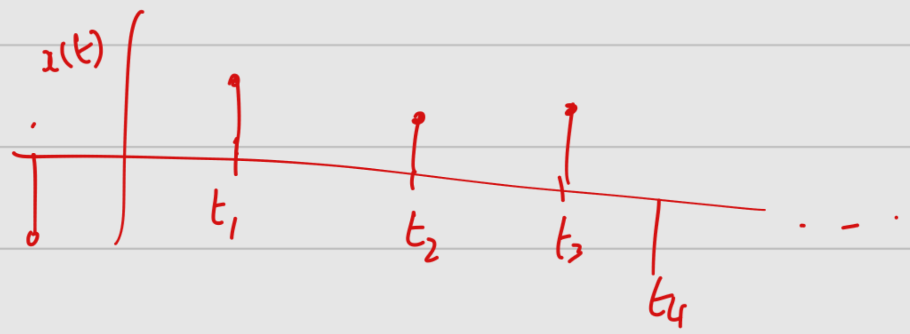
↳ '$x(t)$' denotes the value of the signal at time t.

$x(t)\ \forall t \in \mathbb{R}$ denotes the whole signal, for all values of t.



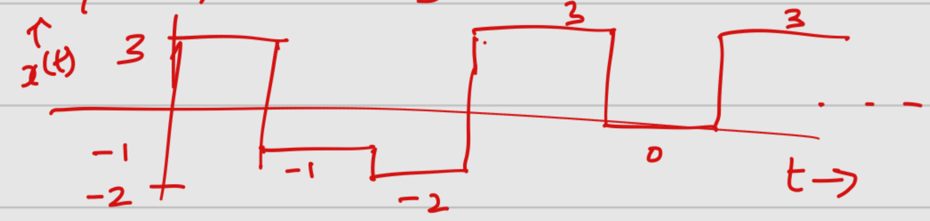
A discrete-time signal: A discrete-time signal is one which takes zero value, except at some discrete points of time.

For example



A discrete-valued signal is one whose value at any instant of time is some value from a finite set. For example suppose the set is $\{-2, -1, 0, 3\}$.

Consider this signal



A few important signals of interest in this case.

Definition: Dirac Delta function: (or) Impulse at $t=0$.

→ This is the function denoted by $\delta(t)$, where $\delta(t)$ satisfies

$$\delta(t) = 0 \;\; \forall t \neq 0, \quad \text{and} \quad \int_{t=-\infty}^{\infty} \delta(t)\, dt = 1.$$

The impulse function satisfies the SIFTING PROPERTY: For any $x(t)$,

$$\int_{t=-\infty}^{\infty} x(t)\,\delta(t-t_0)\, dt = \int x(t_0)\,\delta(t-t_0)\, dt = x(t_0)$$

Sinusoidal functions.

$\cos 2\pi f_0 t$ → Sinusoidal cosine signal with time period $= 1/f_0$.

(also called) cos sinusoid with freq. f_0

IIIly $\sin 2\pi f_0 t$ → sinusoidal sine signal

& $e^{j2\pi f_0 t}$ → At times we will deal with the complex sinusoid.

$$e^{j2\pi f_0 t} = \cos 2\pi f_0 t + j \sin 2\pi f_0 t.$$

PERIODIC SIGNALS: A signal is said to be periodic if it repeats itself in some time.

In order words, $\exists$ a smallest constant value T such that (this symbol denotes "There exists")

$$x(t+T) = x(t), \quad \forall t \in \mathbb{R}.$$

In other words, periodic signal content looks identical in intervals of length T. Here T is called the fundamental period of $x(t)$, or sometimes simply as 'period of $x(t)$'.

Idea of Fourier transform:

In a previous course, you have encountered the idea of a FOURIER SERIES REPRESENTATION for a periodic signal $x(t)$, with period T. It is given as

$$\boxed{x(t) = \sum_{k \in \mathbb{Z}} c_k \, \underbrace{e^{j2\pi \cdot \frac{k}{T} \cdot t}}}$$

(complex sinusoid with freq k/T)

periodic signal with period T

$k \in \mathbb{Z}$: integers

Fourier series coefficient of k^{th} sinusoid with frequency $\frac{k}{T}$.

Fourier series representation

Here, the value of c_k can be obtained as $\frac{1}{T}\int_{t=0}^{T} x(t)\, e^{-j2\pi k/T\, t}\, dt$

→ Observe that c_k is the coefficient of the k^{th} sinusoid with frequency k/T.

→ Here, since the signal is periodic with period T, the Fourier series has coefficients only at frequency intervals of $1/T$. ie $c_k : k \in \{\ldots, -2/T, -1/T, 0, 1/T, 2/T, \ldots\}$

→ If the signal is NOT periodic, then, the above expression for $x(t)$ does NOT WORK, (ie) the equality is NOT true. So instead, we have a 'continuous-frequency' generalization of the Fourier series representation, which we call the Inverse - Fourier transform. (IFT).

$$\text{IFT}: \quad x(t) = \int_{f=-\infty}^{\infty} \tilde{x}(f)\, e^{j2\pi f t}\, dt, \quad \rightarrow (1)$$

this is an equation NOT a definition

here $\tilde{x}(f)$ is calculated/defined as

$$\tilde{x}(f) \triangleq \int_{t=-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt. \quad \rightarrow (2)$$

$\tilde{x}(f) : f \in \mathbb{R}$ is the FOURIER TRANSFORM of the signal $x(t)$.

We then say that $x(t)$ & $\tilde{x}(f)$ form a Fourier transform pair. This is denoted as $\boxed{x(t) \longleftrightarrow \tilde{x}(f).}$

<u>Note</u>: $\tilde{x}(f)$, $f \in \mathbb{R}$ is a 'signal' of its own right, in the frequency domain. Normally if $x(t)$ can be written as RHS of (1), then $\tilde{x}(f)$ can be written as RHS of (2).

That is, $FT(x(t))$ gives $\tilde{x}(f)$. AND $IFT(\tilde{x}(f))$ gives $x(t)$.

In some signals, it may happen that

$IFT(\tilde{x}(f))$ is NOT exactly equal to $x(t)$ for all t.

We DO NOT BOTHER ABT SUCH signals here. In other words, we always assume in this course that $IFT\left[FT(x(t))\right] = x(t)$.

Therefore, in this course, $x(t)$ and $\tilde{x}(f)$ represent the SAME SIGNAL, only in two different domains, one in time, another in frequency

Following the note above,

we now check if ① holds, given that ② is the definition of the FT. $\tilde{x}(f)$

Calculating RHS of ①

Using defn of $\tilde{x}(f)$ as in ②, but using τ to avoid mix up with t.

$$\underbrace{\int_{f=-\infty}^{\infty} \tilde{x}(f)\, e^{j2\pi f t}\, dt}_{\text{(Observe that this is a function of } t\text{)}} \overset{(\text{using ②})}{=} \int_{f=-\infty}^{\infty} \left(\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j2\pi f \tau}\, d\tau \right) e^{j2\pi f t}\, dt$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \left(\int_{f=-\infty}^{\infty} e^{-j2\pi f \tau} \cdot e^{j2\pi f t}\, dt \right) d\tau$$

Integral order was swapped

this is the IFT of $e^{-j2\pi f \tau}$, which is $\delta(t-\tau)$ as we will show later)

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

$$= x(t) = \text{LHS of } (1).$$

Remark: Note that we have used the idea that $IFT(e^{-j2\pi f \tau}) = \delta(t-\tau)$. for showing this. Further integral order was swapped. We have not formally proved these so far. We just accept this, at the current point.

Remark: With the note above in mind, understand that $x(t)$ is the time signal that is observed, accessed, manipulated. $\tilde{x}(f)$ is NOT A DIFF signal, nor is it observed or manipulated or accessed separately from $x(t)$. The $FT(x(t)) = \tilde{x}(f)$ is just a A CONCEPTUAL REPRESENTATION that helps us understand certain aspects in a useful way.

Some basic properties of FT:

1. CONJUGATE SYMMETRY: We are generally dealing with real signals, ie $x(t) \in \mathbb{R}, \forall t$. In other words $x(t) = x^*(t), \forall t \in \mathbb{R}$.

For such signals, we observe

$$\underbrace{\tilde{x}(-f_0)}_{\text{Value of } FT(x(t)) \text{ at } -f_0} = \int_{t=-\infty}^{\infty} x(t)\, e^{-j2\pi(-f_0)t}\, dt$$

$$= \int_{t=-\infty}^{\infty} \left(x^*(t)\, e^{-j2\pi f_0 t}\right)^* dt$$

(Using the fact that $a^* \cdot b^* = (a \cdot b)^*$)

$$= \Bigg(\underbrace{\int_{t=-\infty}^{\infty} x(t)\, e^{-j2\pi f_0 t}\, dt}_{\tilde{x}(f_0)}\Bigg)^*$$

(pull out conjugate from integral, and use the fact that $x(t) = x^*(t), \forall t$ as it is real.)

Thus, $\boxed{\tilde{x}(-f_0) = \tilde{x}^*(f_0)}$, $\forall f_0$. for real $x(t)$

2. TIME SHIFT: If $x(t) \leftrightarrow \tilde{x}(f)$ are a FT pair, then, for any constant z,

$x(t-z) \leftrightarrow \tilde{x}(f)\, e^{-j2\pi f z}$ is another FT pair

Proof: $$FT(x(t-z)) = \int_{t=-\infty}^{\infty} x(t-z)\, e^{-j2\pi f t}\, dt$$

$$\left(\text{by change of variable } t' = t-\tau\right) = \int_{t'=-\infty}^{\infty} x(t') e^{-j2\pi f(t'+\tau)} dt'$$

$$= e^{-j2\pi f\tau} \cdot \underbrace{\int_{t'=-\infty}^{\infty} x(t') e^{-j2\pi f t'} dt'}_{=\tilde{x}(f)}$$

$$= \tilde{x}(f) e^{-j2\pi f\tau}.$$

MODULATION PROPERTY:

(3) Let $x(t) \leftrightarrow \tilde{x}(f)$ be a FT pair. Then, for any

$$x(t)\, e^{j2\pi f_0 t} \leftrightarrow \tilde{x}(f-f_0)$$

Proof:

$$FT\left(x(t) e^{j2\pi f_0 t}\right) = \int_{t=-\infty}^{\infty} x(t) e^{j2\pi f_0 t} \cdot e^{-j2\pi f t} dt$$

$$= \underbrace{\int_{t=-\infty}^{\infty} x(t) e^{-j2\pi (f-f_0)t} dt}_{\text{exactly the value of the FT at } f-f_0.}$$

$$= \tilde{x}(f-f_0).$$

(4) Duality:

If $x(t) \leftrightarrow \tilde{x}(f)$ is a FT pair, then so is

$$\tilde{x}(t) \leftrightarrow x(-f).$$

Remark: Before we write a proof of this, lets understand what this property means. The idea is NOT to look at this property as some magical signal processing property, but purely as a mathematical one. Firstly remember that x & $\tilde{x}$ are just functions with domain $\mathbb{R}$. Here $\tilde{x}(t)$ is just viewing the function $\tilde{x}$ as a function of t, by simply replacing the 'f' variable in $\tilde{x}(f)$ with the 't' variable. Similarly $x(-f)$ is just a function of f ($y(f) = x(-f)$) whose value at f happens to be equal to the value of the fn x at $-f$. This property is ONLY useful to quickly compute FTs of some new signals.

We now prove the result:

Observe that

$$FT(\tilde{x}(t)) = \int_{t=-\infty}^{\infty} \tilde{x}(t)\, e^{-j2\pi f t}\, dt$$

(Plug $f' = t$)

$$= \int_{f'=-\infty}^{\infty} \tilde{x}(f')\, e^{-2\pi f f'}\, df'$$

$$= \underbrace{\int_{f'=-\infty}^{\infty} \tilde{x}(f')\, e^{2\pi f'(-f)}\, df'}_{\text{This looks exactly like IFT of } \tilde{x} \text{ evaluated at } -f.}$$

$= x(-f)$. Hence proved.

⑤ Convolution property:

The convolution of two functions $x(t)$ and $h(t)$ leads to another function $y(t)$, defined as

$$y(t) = x(t) * h(t) \triangleq \int_{z=-\infty}^{\infty} x(z)\, h(t-z)\, dz$$

(Note that the convolution is generally denoted by '$*$')

Value of the result of convolution of x & h at time instant t.

Linear Time invariant systems are those having such a convolution relationship between input $x(t)$ and output $y(t)$.

$$\underset{\text{I/p}}{x(t)} \longrightarrow \boxed{\text{LTI System}} \longrightarrow \underset{\text{O/p}}{y(t)}$$

If $h(t)$ represents the response to an impulse input $\delta(t)$ to the LTI system above, then for any general input $x(t)$, we can write the output as

$$y(t) = x(t) * h(t).$$

[Note that we are NOT proving this result here in this course. Just take it as true].

Now, the Convolution property of the FT is as follows.

$$FT\big(x(t) * h(t)\big) = FT\big(x(t)\big)\, FT\big(h(t)\big)$$

<u>Proof</u>:

$$FT\big(x(t) * h(t)\big) = \int_{t=-\infty}^{\infty} \left(\int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \right) e^{-j2\pi f t}\, dt$$

Multiplying & dividing the expression in integral by $e^{j2\pi f \tau}$, & cha

$$= \int_{z=-\infty}^{\infty} x(z) e^{-j2\pi f z} \left(\int_{t=-\infty}^{\infty} h(t-z) e^{-j2\pi f (t-z)} dt \right) dz$$

Using substitution $t' = t-z$ in inner integral

$$= \int_{z=-\infty}^{\infty} x(z) \, e^{-j2\pi f z} \left(\int_{t'=-\infty}^{\infty} h(t') \, e^{-j2\pi f t'} dt' \right) dz$$

$$= \int_{z=-\infty}^{\infty} x(z) e^{-j2\pi f z} \, \tilde{h}(f) \, dz$$

$$= \tilde{h}(f) \int_{z=-\infty}^{\infty} x(z) \, e^{-j2\pi f z} dz = \tilde{h}(f) \, \tilde{x}(f).$$

Thus proved.

Some important examples of Fourier transform pairs:

① $\delta(t) \longleftrightarrow 1$

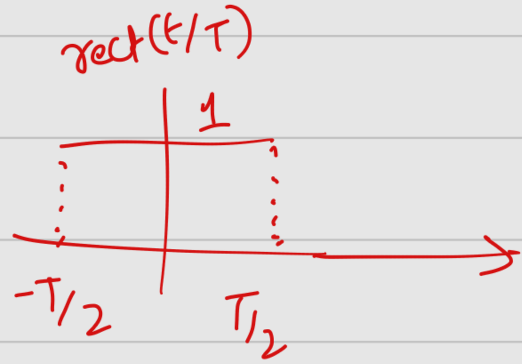
constant function that takes value 1 $\forall f \in \mathbb{R}$.

$$FT(\delta(t)) = \int_{t=-\infty}^{\infty} \delta(t) \, e^{-j2\pi f t} dt = \int_{t=-\infty}^{\infty} \delta(t-0) \, e^{-j2\pi f t} dt$$

$$= e^{-j2\pi f (0)} = 1.$$

by sifting property.

(2) Consider the function $\text{rect}(t/T)$ given by



(T is some positive constant).

Formally
$$\text{rect}(t/T) = \begin{cases} 1, & \text{if } t \in (-T/2, T/2) \\ 0, & \text{otherwise} \end{cases}$$

$$FT\left(\text{rect}(t/T)\right) = \int_{t=-T/2}^{T/2} 1 \cdot e^{-j2\pi f t}\, dt$$

$$= \left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-T/2}^{T/2} = \left[\frac{e^{j\pi f T} - e^{-j\pi f T}}{j2\pi f}\right]$$

$$= \frac{\sin(\pi f T)}{\pi f}$$

(Recall $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$)

$$= T\left(\frac{\sin(\pi f T)}{\pi f T}\right).$$

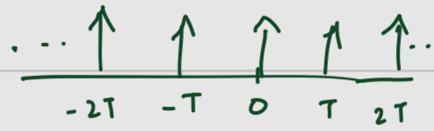
Now, the function $\frac{\sin \pi x}{\pi x}$ has a special name, $\text{sinc}(x)$.

Using this definition $\text{sinc}(x) \triangleq \frac{\sin \pi x}{\pi x}$, one get

$$\boxed{FT\left(\text{rect}(t/T)\right) = T\, \text{sinc}(fT).}$$

③ Impulse train:

Consider a train of impulses with period T.



$x(t) \triangleq \sum_{k\in\mathbb{Z}} \delta(t-kT)$. We want to find a 'useful' way to express the FT $\tilde{x}(f)$.

Now, as $x(t)$ is periodic with time period T, we can write a Fourier Series $x(t) = \sum_{n\in\mathbb{Z}} c_n e^{j2\pi\frac{n}{T}t}$

Where $c_n = \frac{1}{T}\int_{t=-T/2}^{T/2} x(t)\, e^{-j2\pi\frac{nt}{T}}\,dt = \frac{1}{T}\int_{t=-T/2}^{T/2} \sum_{k\in\mathbb{Z}} \delta(t-kT)\, e^{-j2\pi\frac{nt}{T}}\,dt$

$$= \frac{1}{T}\sum_{k\in\mathbb{Z}} \underbrace{\left(\int_{t=-T/2}^{T/2} \delta(t-kT)\, e^{-j2\pi\frac{nt}{T}}\,dt\right)}$$

Now, except in the case when $k=0$, this integral is 0 for all other k, as the curve $\delta(t-kT)e^{-j2\pi nt/T}$ is 0 in the interval $[-T/2, T/2]$ if $k\neq 0$.

When $k=0$, then observe that we can write $\int_{t=-T/2}^{T/2}\delta(t)\, e^{-j2\pi nt/T}\,dt = \int_{t=-\infty}^{\infty}\delta(t)\, e^{-j\frac{2\pi n}{T}t}\,dt$

using this

$$= \frac{1}{T}\int_{t=-\infty}^{\infty} \delta(t)\, e^{-j\frac{2\pi n}{T}t}\,dt$$

(by sifting property)

$$= \frac{1}{T}\left[e^{-j\frac{2\pi n}{T}(0)}\right]$$

$$= 1/T.$$

Thus; $c_n = 1/T \;\; \forall n \in \mathbb{Z}$.

Hence we can write $x(t) = \sum_{k \in \mathbb{Z}} \delta(t - kT) = \frac{1}{T} \sum_{n \in \mathbb{Z}} e^{j 2\pi n t/T}$

We converted $x(t)$ into this form because we want to use this to express the FT in a way we wish. We thus write

$$\tilde{x}(f) = \int_{t=-\infty}^{\infty} x(t) e^{-j 2\pi f t} \, dt$$

$$= \frac{1}{T} \sum_{n \in \mathbb{Z}} \left(\int_{t=-\infty}^{\infty} e^{j 2\pi n t/T} e^{-j 2\pi f t} \, dt \right)$$

this is exactly $\delta(f - n/T)$

$$= 1/T \sum_{n \in \mathbb{Z}} \delta(f - n/T)$$

$$= 1/T \sum_{k \in \mathbb{Z}} \delta(f - k/T)$$

[Just changing n to k for uniformity]

Thus, the FT of an impulse train is another impulse train. This will be useful to show the 'sampling theorem'; which will follow.