

# 일차논리 메타정리

## 임기정

### 1 Generalized Weakening

Let  $\eta$  be a variable renaming. Then,

$$\frac{\Gamma \subseteq \Gamma' \quad \Gamma \vdash \varphi}{\Gamma' [\eta] \vdash \varphi [\eta]}$$

#### 1.1 $\forall$ -case

- (1)  $y \notin \text{FV}(\Gamma)$
- (2)  $y \notin \text{FV}((\forall x) \varphi)$
- (3)  $\Gamma \vdash \varphi[x := y]$
- (4)  $\Gamma \subseteq \Gamma'$
- (G)  $\Gamma' [\eta] \vdash (\forall x \varphi) [\eta]$

*Proof.* Let  $z$  be a fresh variable. Since  $\Gamma[y := z] \subseteq \Gamma'[y := z]$ ,

$$\begin{aligned} & \Gamma' [\eta] \vdash (\forall x \varphi) [\eta] \\ \iff & \Gamma' [\eta] \vdash \forall x (\varphi [\eta]) \\ \iff & \Gamma' [y := z] [z := \eta(y); \eta] \vdash \forall x (\varphi [\eta]) \\ \iff & \Gamma' [y := z] [z := \eta(y); \eta] \vdash \forall x (\varphi [z := \eta(y); \eta]) \\ \iff & \Gamma' [y := z] [z := \eta(y); \eta] \vdash \varphi [z := \eta(y); \eta] [x := z] \\ \iff & \Gamma' [y := z] [z := \eta(y); \eta] \vdash \varphi [y := z; z := \eta(y); \eta] [x := z] \\ \iff & \Gamma' [y := z] [z := \eta(y); \eta] \vdash \varphi [y := z; z := \eta(y); \eta] [x := y [y := z; z := \eta(y); \eta]] \\ \iff & \Gamma' [y := z] [z := \eta(y); \eta] \vdash \varphi [x := y] [y := z; z := \eta(y); \eta] \\ \iff & \Gamma' [y := z] [y := z; z := \eta(y); \eta] \vdash \varphi [x := y] [y := z; z := \eta(y); \eta] \\ \iff & \begin{cases} \Gamma \subseteq \Gamma' [y := z], \\ \Gamma \vdash \varphi [x := y]. \end{cases} \end{aligned}$$

□

**Discussion 1.** Does it hold  $\varphi [x := x; \eta] [x := z] \equiv \varphi [x := y] [y := z; z := \eta(y); \eta]$ ?

I hope so... – 2024/07/15 09:46

⊢

**Discussion 2.** Consider  $\varphi [x := x; \eta] \llbracket x := z \rrbracket$ .

$$\begin{aligned}
& \varphi [x := x; \eta] \llbracket x := z \rrbracket \\
& \equiv \varphi [x := x; z := \eta(y); \eta] \llbracket x := z \rrbracket \\
& \equiv \varphi [x := z; z := \eta(y); \eta] \llbracket x := z \rrbracket \\
& \equiv \varphi [x := z; y := z; z := \eta(y); \eta] \llbracket x := z \rrbracket \\
& \equiv \varphi [x := z; y := z; z := \eta(y); \eta] \llbracket x := y [x := z; y := z; z := \eta(y); \eta] \rrbracket \\
& \equiv \varphi \llbracket x := y \rrbracket [x := z; y := z; z := \eta(y); \eta] \\
& \equiv \varphi \llbracket x := y \rrbracket [y := z; z := \eta(y); \eta]
\end{aligned}$$

Did I make a mistake? To check it out, let

- (a)  $\varphi \equiv \varphi(x, u)$  with  $u \neq x$ ;
- (b)  $y \equiv x \vee (y \neq x \wedge y \neq u)$  – i.e.,  $y \notin \text{FV}(\forall x \varphi)$ ;
- (c)  $\eta$  be a variable renaming – i.e.,  $\eta \in \mathbb{L}(\mathbb{V} \times \mathbb{V})$ ; and
- (d)  $z$  be a fresh – i.e.,  $z \neq x \wedge z \neq y \wedge z \neq u \wedge z \notin \text{Dom}(\eta) \cup \text{Cod}(\eta)$ .

Then, the L.H.S. is

$$\begin{aligned}
\varphi [x := x; \eta] \llbracket x := z \rrbracket & \equiv \varphi(x, u) [x := x; \eta] \llbracket x := z \rrbracket \\
& \equiv \varphi(x, \eta(u)) \llbracket x := z \rrbracket \\
& \equiv \varphi(z, (\eta(u)) \llbracket x := z \rrbracket).
\end{aligned}$$

On the other hand, the R.H.S. is  $\varphi(z, \eta(u)) \equiv \varphi [x := z; \eta]$ .

⊢