일차논리 메타정리

임기정

1   Generalized Weakening

Let η be a variable renaming. Then,

$$\frac{\Gamma \subseteq \Gamma' \qquad \Gamma \vdash \varphi}{\Gamma'\,[\eta] \vdash \varphi\,[\eta]}$$

1.1 $\forall$-case

(1) $y \notin \mathrm{FV}(\Gamma)$

(2) $y \notin \mathrm{FV}((\forall x)\,\varphi)$

(3) $\Gamma \vdash \varphi\,[x := y]$

(4) $\Gamma \subseteq \Gamma'$

(G) $\Gamma'\,[\eta] \vdash (\forall x \varphi)\,[\eta]$

Proof. Let z be a fresh variable. Since $\Gamma\,[y := z] \subseteq \Gamma'\,[y := z]$,

$$
\begin{aligned}
&\Gamma'\,[\eta] \vdash (\forall x \varphi)\,[\eta] \\
&\quad \Longleftarrow\ \Gamma'\,[\eta] \vdash \forall x\,(\varphi\,[\eta]) \\
&\quad \Longleftarrow\ \Gamma'\,[y := z]\,[z := \eta\,(y)\,;\eta] \vdash \forall x\,(\varphi\,[\eta]) \\
&\quad \Longleftarrow\ \Gamma'\,[y := z]\,[z := \eta\,(y)\,;\eta] \vdash \forall x\,(\varphi\,[z := \eta\,(y)\,;\eta]) \\
&\quad \Longleftarrow\ \Gamma'\,[y := z]\,[z := \eta\,(y)\,;\eta] \vdash \varphi\,[z := \eta\,(y)\,;\eta]\,[x := z] \\
&\quad \Longleftarrow\ \Gamma'\,[y := z]\,[z := \eta\,(y)\,;\eta] \vdash \varphi\,[y := z; z := \eta\,(y)\,;\eta]\,[x := z] \\
&\quad \Longleftarrow\ \Gamma'\,[y := z]\,[z := \eta\,(y)\,;\eta] \vdash \varphi\,[y := z; z := \eta\,(y)\,;\eta]\,[x := y\,[y := z; z := \eta\,(y)\,;\eta]] \\
&\quad \Longleftarrow\ \Gamma'\,[y := z]\,[z := \eta\,(y)\,;\eta] \vdash \varphi\,[x := y]\,[y := z; z := \eta\,(y)\,;\eta] \\
&\quad \Longleftarrow\ \Gamma'\,[y := z]\,[y := z; z := \eta\,(y)\,;\eta] \vdash \varphi\,[x := y]\,[y := z; z := \eta\,(y)\,;\eta] \\
&\quad \Longleftarrow\ \begin{cases} \Gamma \subseteq \Gamma'\,[y := z], \\ \Gamma \vdash \varphi\,[x := y]. \end{cases}
\end{aligned}
$$

$\square$

Discussion 1. Does it hold $\varphi\,[x := x;\eta]\,[x := z] \equiv \varphi\,[x := y]\,[y := z; z := \eta\,(y)\,;\eta]$?
I think it does... – 2024/07/15 09:46 $\dashv$

Discussion 2. Consider $\varphi\left[x := x; \eta\right]\left[x := z\right]$.

$$\varphi\left[x := x; \eta\right]\left[x := z\right]$$
$$\equiv \varphi\left[x := x; z := \eta\left(y\right); \eta\right]\left[x := z\right]$$
$$\equiv \varphi\left[x := z; z := \left(y\right); \eta\right]\left[x := z\right]$$
$$\equiv \varphi\left[x := z; y := z; z := \eta\left(y\right); \eta\right]\left[x := z\right]$$
$$\equiv \varphi\left[x := z; y := z; z := \eta\left(y\right); \eta\right]\left[x := y\left[x := z; y := z; z := \eta\left(y\right); \eta\right]\right]$$
$$\equiv \varphi\left[x := y\right]\left[x := z; y := z; z := \eta\left(y\right); \eta\right]$$
$$\equiv \varphi\left[x := y\right]\left[y := z; z := \eta\left(y\right); \eta\right]$$

Did I make a mistake? $\dashv$