산술적인 함수들의 구문론과 의미론

임기정

Arith : $\mathbb{N} \to \mathbf{Set}$ is defined inductively by:

$$\frac{i < n}{\dot{x}_i \in \mathrm{Arith}_n}$$

$$\frac{e_1 \in \mathrm{Arith}_n \quad e_2 \in \mathrm{Arith}_n}{e_1 \dot{+} e_2 \in \mathrm{Arith}_n}$$

$$\frac{e_1 \in \mathrm{Arith}_n \quad e_2 \in \mathrm{Arith}_n}{e_1 \dot{\times} e_2 \in \mathrm{Arith}_n}$$

$$\frac{e_1 \in \mathrm{Arith}_n \quad e_2 \in \mathrm{Arith}_n}{e_1 \dot{<} e_2 \in \mathrm{Arith}_n}$$

$$\frac{e_1 \in \mathrm{Arith}_{n+1}}{\dot{\mu} e_1 \in \mathrm{Arith}_n}$$

$\mathbf{Eval}\,[\cdot]\cdot \rightsquigarrow \cdot : \prod_{n \in \mathbb{N}} \left(\mathrm{Arith}_n \to (\omega^n \to \omega) \to \mathbf{Prop}\right)$ is defined inductively by:

$$\frac{}{\mathbf{Eval}_n\,[\dot{x}_i] \rightsquigarrow (x_0, \cdots, x_{n-1}) \mapsto x_i}$$

$$\frac{\mathbf{Eval}_n\,[e_1] \rightsquigarrow f \quad \mathbf{Eval}_n\,[e_2] \rightsquigarrow g}{\mathbf{Eval}_n\,\left[e_1 \dot{+} e_2\right] \rightsquigarrow (x_0, \cdots, x_{n-1}) \mapsto f\,(x_0, \cdots, x_{n-1}) + g\,(x_0, \cdots, x_{n-1})}$$

$$\frac{\mathbf{Eval}_n\,[e_1] \rightsquigarrow f \quad \mathbf{Eval}_n\,[e_2] \rightsquigarrow g}{\mathbf{Eval}_n\,\left[e_1 \dot{\times} e_2\right] \rightsquigarrow (x_0, \cdots, x_{n-1}) \mapsto f\,(x_0, \cdots, x_{n-1}) \times g\,(x_0, \cdots, x_{n-1})}$$

$$\frac{\mathbf{Eval}_n\,[e_1] \rightsquigarrow f \quad \mathbf{Eval}_n\,[e_2] \rightsquigarrow g}{\mathbf{Eval}_n\,[e_1 \dot{<} e_2] \rightsquigarrow (x_0, \cdots, x_{n-1}) \mapsto \begin{cases} 0, & f\,(x_0, \cdots, x_{n-1}) < g\,(x_0, \cdots, x_{n-1}) \\ 1, & \text{otherwise} \end{cases}}$$

$$\frac{\mathbf{Eval}_{n+1}\,[e_1] \rightsquigarrow f \quad (\forall x_0 \in \omega) \cdots (\forall x_{n-1} \in \omega)\,(\exists x_n \in \omega)\,(f\,(x_0, \cdots, x_n) = 0)}{\mathbf{Eval}_n\,[\dot{\mu} e_1] \rightsquigarrow (x_0, \cdots, x_{n-1}) \mapsto \mu x_n\,(f\,(x_0, \cdots, x_n) = 0)}$$