일차논리 메타정리

임기정

1 Generalized Weakening

Let η be a variable renaming. Then,

$$\frac{\Gamma \subseteq \Gamma' \qquad\qquad \Gamma \vdash \varphi}{\Gamma'[\eta] \vdash \varphi[\eta]}$$

1.1 $\forall$-case

(1) $y \notin \mathrm{FV}(\Gamma)$

(2) $y \notin \mathrm{FV}((\forall x)\varphi)$

(3) $\Gamma \vdash \varphi[x := y]$

(4) $\Gamma \subseteq \Gamma'$

(G) $\Gamma'[\eta] \vdash (\forall x \varphi)[\eta]$

Proof. Let z be a fresh variable. Since $\Gamma[y := z] \subseteq \Gamma'[y := z]$,

$$\begin{aligned}
&\Gamma'[\eta] \vdash (\forall x \varphi)[\eta] \\
&\quad\Longleftarrow\ \Gamma'[\eta] \vdash \forall x(\varphi[\eta]) \\
&\quad\Longleftarrow\ \Gamma'[y := z][z := \eta(y); \eta] \vdash \forall x(\varphi[\eta]) \\
&\quad\Longleftarrow\ \Gamma'[y := z][z := \eta(y); \eta] \vdash \forall x(\varphi[z := \eta(y); \eta]) \\
&\quad\Longleftarrow\ \Gamma'[y := z][z := \eta(y); \eta] \vdash \varphi[z := \eta(y); \eta][x := z] \\
&\quad\Longleftarrow\ \Gamma'[y := z][z := \eta(y); \eta] \vdash \varphi[y := z; z := \eta(y); \eta][x := z] \\
&\quad\Longleftarrow\ \Gamma'[y := z][z := \eta(y); \eta] \vdash \varphi[y := z; z := \eta(y); \eta][x := y[y := z; z := \eta(y); \eta]] \\
&\quad\Longleftarrow\ \Gamma'[y := z][z := \eta(y); \eta] \vdash \varphi[x := y][y := z; z := \eta(y); \eta] \\
&\quad\Longleftarrow\ \Gamma'[y := z][y := z; z := \eta(y); \eta] \vdash \varphi[x := y][y := z; z := \eta(y); \eta] \\
&\quad\Longleftarrow\ \begin{cases} \Gamma \subseteq \Gamma'[y := z], \\ \Gamma \vdash \varphi[x := y]. \end{cases}
\end{aligned}$$

$\square$

Discussion 1. Does it hold $\varphi[x := x; \eta][x := z] \equiv \varphi[x := y][y := z; z := \eta(y); \eta]$?
I think it does... – 2024/07/15 09:46 $\dashv$

Discussion 2. Consider $\varphi\left[x := x; \eta\right]\left[x := z\right]$.

$$\begin{aligned}
&\varphi\left[x := x; \eta\right]\left[x := z\right] \\
&\equiv \varphi\left[x := x; z := \eta\left(y\right); \eta\right]\left[x := z\right] \\
&\equiv \varphi\left[x := z; z := \left(y\right); \eta\right]\left[x := z\right] \\
&\equiv \varphi\left[x := z; y := z; z := \eta\left(y\right); \eta\right]\left[x := z\right] \\
&\equiv \varphi\left[x := z; y := z; z := \eta\left(y\right); \eta\right]\left[x := y\left[x := z; y := z; z := \eta\left(y\right); \eta\right]\right] \\
&\equiv \varphi\left[x := y\right]\left[x := z; y := z; z := \eta\left(y\right); \eta\right] \\
&\equiv \varphi\left[x := y\right]\left[y := z; z := \eta\left(y\right); \eta\right]
\end{aligned}$$

Did I make a mistake? To check it out, let

(a) $\varphi :\equiv \varphi\left(x, u\right)$ with $u \not\equiv x$;

(b) $y \equiv x \vee \left(y \not\equiv x \wedge y \not\equiv u\right)$ – i.e., $y \notin \mathrm{FV}\left(\forall x \varphi\right)$;

(c) η be a variable renaming – i.e., $\eta \in \mathbb{L}\left(\mathbb{V} \times \mathbb{V}\right)$; and

(d) z be a fresh – i.e., $z \not\equiv x \wedge z \not\equiv y \wedge z \not\equiv u \wedge z \notin \mathrm{Dom}\left(\eta\right) \cup \mathrm{Cod}\left(\eta\right)$.

Then, the L.H.S. is

$$\begin{aligned}
\varphi\left[x := x; \eta\right]\left[x := z\right] &\equiv \varphi\left(x, u\right)\left[x := x; \eta\right]\left[x := z\right] \\
&\equiv \varphi\left(x, \eta\left(u\right)\right)\left[x := z\right] \\
&\equiv \varphi\left(z, \left(\eta\left(u\right)\right)\left[x := z\right]\right).
\end{aligned}$$

On the other hand, the R.H.S. is $\varphi\left(z, \eta\left(u\right)\right) \equiv \varphi\left[x := z; \eta\right]$. $\dashv$