일차논리 메타정리

임기정

1 Generalized Weakening

Let η be a variable renaming. Then,

$$\frac{\Gamma \subseteq \Gamma' \qquad\qquad \Gamma \vdash \varphi}{\Gamma'\left[\eta\right] \vdash \varphi\left[\eta\right]}$$

1.1 $\forall$-case

(1) $y \notin \mathrm{FV}\left(\Gamma\right)$

(2) $y \notin \mathrm{FV}\left(\left(\forall x\right)\varphi\right)$

(3) $\Gamma \vdash \varphi[\![x := y]\!]$

(4) $\Gamma \subseteq \Gamma'$

(G) $\Gamma'\left[\eta\right] \vdash \left(\forall x \varphi\right)\left[\eta\right]$

Proof. Let z be a fresh variable. Since $\Gamma\left[y := z\right] \subseteq \Gamma'\left[y := z\right]$,

$$\begin{aligned}
&\Gamma'\left[\eta\right] \vdash \left(\forall x \varphi\right)\left[\eta\right] \\
&\quad\Longleftarrow\ \Gamma'\left[\eta\right] \vdash \forall x\left(\varphi\left[\eta\right]\right) \\
&\quad\Longleftarrow\ \Gamma'\left[y := z\right]\left[z := \eta\left(y\right); \eta\right] \vdash \forall x\left(\varphi\left[\eta\right]\right) \\
&\quad\Longleftarrow\ \Gamma'\left[y := z\right]\left[z := \eta\left(y\right); \eta\right] \vdash \forall x\left(\varphi\left[z := \eta\left(y\right); \eta\right]\right) \\
&\quad\Longleftarrow\ \Gamma'\left[y := z\right]\left[z := \eta\left(y\right); \eta\right] \vdash \varphi\left[z := \eta\left(y\right); \eta\right][\![x := z]\!] \\
&\quad\Longleftarrow\ \Gamma'\left[y := z\right]\left[z := \eta\left(y\right); \eta\right] \vdash \varphi\left[y := z; z := \eta\left(y\right); \eta\right][\![x := z]\!] \\
&\quad\Longleftarrow\ \Gamma'\left[y := z\right]\left[z := \eta\left(y\right); \eta\right] \vdash \varphi\left[y := z; z := \eta\left(y\right); \eta\right][\![x := y\left[y := z; z := \eta\left(y\right); \eta\right]]\!] \\
&\quad\Longleftarrow\ \Gamma'\left[y := z\right]\left[z := \eta\left(y\right); \eta\right] \vdash \varphi[\![x := y]\!]\left[y := z; z := \eta\left(y\right); \eta\right] \\
&\quad\Longleftarrow\ \Gamma'\left[y := z\right]\left[y := z; z := \eta\left(y\right); \eta\right] \vdash \varphi[\![x := y]\!]\left[y := z; z := \eta\left(y\right); \eta\right] \\
&\quad\Longleftarrow\ \begin{cases} \Gamma \subseteq \Gamma'\left[y := z\right], \\ \Gamma \vdash \varphi[\![x := y]\!]. \end{cases}
\end{aligned}$$

□

Discussion 1. Does it hold $\varphi\left[x := x; \eta\right][\![x := z]\!] \equiv \varphi[\![x := y]\!]\left[y := z; z := \eta\left(y\right); \eta\right]$?
I hope so... – 2024/07/15 09:46 ⊣

Discussion 2. Consider $\varphi\left[x := x; \eta\right] \llbracket x := z \rrbracket$.

$$
\begin{aligned}
&\varphi\left[x := x; \eta\right] \llbracket x := z \rrbracket \\
&\equiv \varphi\left[x := x; z := \eta\left(y\right); \eta\right] \llbracket x := z \rrbracket \\
&\equiv \varphi\left[x := z; z := \left(y\right); \eta\right] \llbracket x := z \rrbracket \\
&\equiv \varphi\left[x := z; y := z; z := \eta\left(y\right); \eta\right] \llbracket x := z \rrbracket \\
&\equiv \varphi\left[x := z; y := z; z := \eta\left(y\right); \eta\right] \llbracket x := y\left[x := z; y := z; z := \eta\left(y\right); \eta\right] \rrbracket \\
&\equiv \varphi\llbracket x := y \rrbracket \left[x := z; y := z; z := \eta\left(y\right); \eta\right] \\
&\equiv \varphi\llbracket x := y \rrbracket \left[y := z; z := \eta\left(y\right); \eta\right]
\end{aligned}
$$

Did I make a mistake? To check it out, let

(a) $\varphi :\equiv \varphi\left(x, u\right)$ with $u \not\equiv x$;

(b) $y \equiv x \vee \left(y \not\equiv x \wedge y \not\equiv u\right)$ – i.e., $y \notin \mathrm{FV}\left(\forall x \varphi\right)$;

(c) η be a variable renaming – i.e., $\eta \in \mathbb{L}\left(\mathbb{V} \times \mathbb{V}\right)$; and

(d) z be a fresh – i.e., $z \not\equiv x \wedge z \not\equiv y \wedge z \not\equiv u \wedge z \notin \mathrm{Dom}\left(\eta\right) \cup \mathrm{Cod}\left(\eta\right)$.

Then, the L.H.S. is

$$
\begin{aligned}
\varphi\left[x := x; \eta\right]\left[x := z\right] &\equiv \varphi\left(x, u\right)\left[x := x; \eta\right]\left[x := z\right] \\
&\equiv \varphi\left(x, \eta\left(u\right)\right)\left[x := z\right] \\
&\equiv \varphi\left(z, \left(\eta\left(u\right)\right)\left[x := z\right]\right).
\end{aligned}
$$

On the other hand, the R.H.S. is $\varphi\left(z, \eta\left(u\right)\right) \equiv \varphi\left[x := z; \eta\right]$. ⊣