Calculus and Probability
Assignment 5

x
x
Group 6

September 1, 2018

Exercise 5

a) The domain of $f(x)$ is equal to all numbers. When looking at the positive limit of e^{x^3-x}, we see that is approaches infinity. When looking at the negative limit however, we will notice the following:

$$\begin{aligned}\lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} e^{x^3-x} - 1 \\ &= -1 + \lim_{x\to-\infty} e^{x^3-x} \\ &= -1 + 0 \\ &= -1\end{aligned}$$

This means that at minus infinity, $f(x)$ approaches -1. The domain thus is equal to $(-1, \infty)$. $D(f) = \mathbb{R}$, $R(f) = (-1, \infty)$.

b) The roots of f can be calculated like this:

$$\begin{aligned}f(x) &= 0 \\ e^{x^3-x} - 1 &= 0 \\ e^{x^3-x} &= 1 \\ x^3 - x &= 0 \\ x(x^2-1) &= 0 \\ x &= 0 \\ x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= -1 \\ x &= 1\end{aligned}$$

There are three roots of f, namely: $x = 0$, $x = -1$ and $x = 1$. $x = 0$, $x = -1$ and $x = 1$.

c) We need the derivative to calculate the local minima and maxima:

$$\begin{aligned} f(x) &= e^{x^3-x} - 1 \\ f'(x) &= e^{x^3-x} * (3x^2 - 1) \\ &= 3x^2 * e^{x^3-x} - e^{x^3-x} \end{aligned}$$

This derivative must be equal to zero to find all minima and maxima:

$$\begin{aligned} f'(x) &= 0 \\ 3x^2 * e^{x^3-x} - e^{x^3-x} &= 0 \\ 3x^2 * e^{x^3-x} &= e^{x^3-x} \\ 3x^2 &= 1 \\ x^2 &= \frac{1}{3} \\ x &= -\frac{1}{\sqrt{3}} \\ x &= \frac{1}{\sqrt{3}} \end{aligned}$$

Now we will fill in these points by filling them in:

$$\begin{aligned} f(-\frac{1}{\sqrt{3}}) &= e^{-(\frac{1}{\sqrt{3}})^3 + \frac{1}{\sqrt{3}}} - 1 \\ &= e^{-(\frac{1}{3\sqrt{3}}) + \frac{1}{\sqrt{3}}} - 1 \\ &= e^{\frac{2}{3\sqrt{3}}} - 1 \\ f(\frac{1}{\sqrt{3}}) &= e^{(\frac{1}{\sqrt{3}})^3 - \frac{1}{\sqrt{3}}} - 1 \\ &= e^{(\frac{1}{3\sqrt{3}}) - \frac{1}{\sqrt{3}}} - 1 \\ &= e^{-(\frac{2}{3\sqrt{3}})} - 1 \end{aligned}$$

We thus have a minimum at $(\frac{1}{\sqrt{3}}, e^{-(\frac{2}{3\sqrt{3}})} - 1)$ and a maximum at $(-\frac{1}{\sqrt{3}}, e^{\frac{2}{3\sqrt{3}}} - 1)$. Minimum: $(\frac{1}{\sqrt{3}}, e^{-(\frac{2}{3\sqrt{3}})} - 1)$, maximum: $(-\frac{1}{\sqrt{3}}, e^{\frac{2}{3\sqrt{3}}} - 1)$.

d)

$$\begin{aligned} \lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} e^{x^3-x} - 1 \\ &= -1 + \lim_{x\to-\infty} e^{x^3-x} \\ &= -1 + 0 \\ &= -1 \\ \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} e^{x^3-x} - 1 \\ &= -1 + \lim_{x\to\infty} e^{x^3-x} \\ &= -1 + \infty \\ &= \infty \end{aligned}$$

$\lim_{x\to-\infty} f(x) = -1$, $\lim_{x\to\infty} f(x) = \infty$.

Exercise 6

a)

$$\begin{aligned}
f(x) &= x^x \\
&= e^{ln\ x^x} \\
&= e^{xln\ x} \\
f'(x) &= e^{xln\ x} * ((ln\ x) + (x * \frac{1}{x})) \\
&= e^{xln\ x} * (ln\ x + 1) \\
&= x^x * ln\ x + x^x
\end{aligned}$$

$f'(x) = x^x * ln\ x + x^x$.

b) We need to invert $f^{-1}(x)$:

$$\begin{aligned}
y &= sin(x^2) \\
x &= sin(y^2) \\
arcsin(x) &= arcsin(sin(y^2)) \\
arcsin(x) &= y^2 \\
\sqrt{arcsin(x)} &= y \\
y &= \sqrt{arcsin(x)}
\end{aligned}$$

So $f(x) = \sqrt{arcsin(x)}$. We now only need to compute $f'(x)$:

$$\begin{aligned}
f(x) &= \sqrt{arcsin(x)} \\
&= arcsin(x)^{-\frac{1}{2}} \\
f'(x) &= \frac{1}{2\sqrt{arcsin(x)}} * \frac{1}{\sqrt{1-x^2}} \\
&= \frac{1}{2\sqrt{arcsin(x) * (1-x^2)}}
\end{aligned}$$

$f'(x) = \frac{1}{2\sqrt{arcsin(x)*(1-x^2)}}$.

Exercise 7

a) We will first apply L'Hopitals rule because:

$$\begin{aligned}
\lim_{x\to-3} sin(\pi x) &= sin(-3\pi) \\
&= 0 \\
\lim_{x\to-3} x^2 - 9 &= -9 + \lim_{x\to 3} x^2 \\
&= -9 + 9 \\
&= 0
\end{aligned}$$

We apply L'Hopital:

$$\lim_{x\to 3} \frac{sin(\pi x)}{x^2 - 9} = \lim_{x\to -3} \frac{cos(\pi x) * \pi}{2x}$$
$$= \frac{cos(-3\pi) * \pi}{2 * -3}$$
$$= \frac{\pi}{-6}$$
$$= -\frac{1}{6}\pi$$

$\lim_{x\to 3} \frac{sin(\pi x)}{x^2-9} = -\frac{1}{6}\pi$.

b) We will first apply L'Hopitals rule because:

$$\lim_{x\to -\infty} e^{3-x} = \lim_{x\to \infty} e^{3+x}$$
$$= \infty$$
$$\lim_{x\to -\infty} 7x^2 = \lim_{x\to \infty} 7x^2$$
$$= \infty$$

We apply L'Hopital:

$$\lim_{x\to -\infty} \frac{e^{3-x}}{7x^2} = \lim_{x\to -\infty} \frac{-e^{3-x}}{14x}$$

We need to apply L'Hopital again, but first we check if we can do that:

$$\lim_{x\to -\infty} -e^{3-x} = \lim_{x\to \infty} -e^{3+x}$$
$$= -\infty$$
$$\lim_{x\to -\infty} 14x = -\infty$$

We apply L'Hopital:

$$\lim_{x\to -\infty} \frac{-e^{3-x}}{14x} = \lim_{x\to -\infty} \frac{e^{3-x}}{14}$$
$$= \lim_{x\to \infty} \frac{e^{3+x}}{14}$$
$$= \infty$$

$\lim_{x\to -\infty} \frac{e^{3-x}}{7x^2} = \infty$.

Exercise 8

a)

$$
\begin{aligned}
f(x) &= 2sin(x)cos(x) \\
F(x) &= 2sin(x) * sin(x) - \int 2cos(x) * sin(x) \\
\int 2cos(x) * sin(x) &= 2sin(x) * sin(x) - \int 2cos(x) * sin(x) \\
2\int 2cos(x) * sin(x) &= 2sin^2(x) \\
F(x) &= sin^2(x)
\end{aligned}
$$

$F(x) = sin^2(x)$.

b)

$$
\begin{aligned}
f(x) &= \frac{2}{1+4x^2} \\
&= \frac{1}{\frac{1}{2}+2x^2} \\
F(x) &= tan^{-1}(2x)
\end{aligned}
$$

$F(x) = tan^{-1}(2x)$.

Exercise 9

a) We first have to determine the vertexes between these points.

$$\begin{aligned} x+2 &= -x+6 \\ 2x+2 &= 6 \\ 2x &= 4 \\ x &= 2 \\ y &= 2+2 \\ &= 4 \end{aligned}$$

$$\begin{aligned} x+2 &= 2x-3 \\ -x+2 &= -3 \\ -x &= -5 \\ x &= 5 \\ y &= 5+2 \\ &= 7 \end{aligned}$$

$$\begin{aligned} -x+6 &= 2x-3 \\ -3x+6 &= -3 \\ -3x &= -9 \\ -x &= -3 \\ x &= 3 \\ y &= -3+6 \\ &= 3 \end{aligned}$$

The points are $(2,4)$, $(5,7)$, $(3,3)$. To calculate the area of the triangle, we use the 90 degree angle at the point $(2,4)$. This corner is 90 degrees because both lines cross the point $(2,4)$ and mimic the lines $y = x$ (which is 45 degree) and $y = -x$ (which also is 45 degree). This means we only need to know the length of $(2,4)$ to $(3,3)$ and from $(2,4)$ to $(5,7)$. We will calculate the line length of the first difference:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 1^2 + 1^2 &= c^2 \\ c &= \sqrt{2} \end{aligned}$$

The line from $(2,4)$ to $(5,7)$ is exactly three times the size of the line we calculated, this means that this line has a size of $3\sqrt{2}$. We only need to calculate the area:

$$\begin{aligned} \frac{1}{2}ab &= \frac{1}{2}\sqrt{2}3\sqrt{2} \\ &= \frac{1}{2} * 2 * 3 \\ &= 3 \end{aligned}$$

The points are $(2,4)$, $(5,7)$, $(3,3)$. The area of the triangle is 3.

b) We know that the inner angles of a triangle added together are 180 degree (or π). We also know that the opposite corner at each point is $\textit{opposite corner} = \pi - \textit{corner}$ because two straight lines cross at each point. This means that, if we take for the corner $(2,4) = \alpha$, $(5,7) = b$ and $(3,3) = c$ we can write down the equation as follows:

$$
\begin{aligned}
180 &= 180 - \alpha + 180 - b + 180 - c \\
\pi &= \pi - \alpha + \pi - b + \pi - c \\
\alpha + b + c &= 2\pi
\end{aligned}
$$

We thus know that all angles added together form a complete circle with a radius of A. We now only need to compute the other areas but therefore we need the length of the line from $(3,3)$ to $(5,7)$:

$$
\begin{aligned}
\sqrt{2}^2 + (3\sqrt{2})2 &= c^2 \\
c &= \sqrt{20}
\end{aligned}
$$

We then have:

$$
\begin{aligned}
A &= (\sqrt{2} + 3\sqrt{2} + \sqrt{20}) * a + \pi * a^2 \\
&= (4\sqrt{2} + \sqrt{20}) * a + \pi * a^2
\end{aligned}
$$

And for B we have:

$$B = 2\pi a + 4\sqrt{2} + \sqrt{20}$$

$A = (4\sqrt{2} + \sqrt{20}) * a + \pi * a^2$, $B = 2\pi a + 4\sqrt{2} + \sqrt{20}$.

c) We can just equal them:

$$
\begin{aligned}
A &= B \\
(4\sqrt{2} + \sqrt{20}) * a + \pi * a^2 &= 2\pi a + 4\sqrt{2} + \sqrt{20}
\end{aligned}
$$

In order for $A = B$, we need a to be larger than 1 because at $a = 1$, $B > A$ and A grows much faster than B because of the a^2. In order for $A = B$, we need a to be larger than 1.

Answer Form Assignment 5

Name	x
Student Number	x
Group	Group 6

Question	**Answer**
5a (1pt)	$D(f) = \mathbb{R}$, $R(f) = (-1, \infty)$.
5b (1pt)	$x = 0$, $x = -1$ and $x = 1$.
5c (1pt)	Minimum: $(\frac{1}{\sqrt{3}}, e^{-(\frac{2}{3\sqrt{3}})} - 1)$, maximum: $(-\frac{1}{\sqrt{3}}, e^{\frac{2}{3\sqrt{3}}} - 1)$.
5d (1pt)	$\lim_{x\to-\infty} f(x) = -1$, $\lim_{x\to\infty} f(x) = \infty$.
6a (2pt)	$f'(x) = x^x * ln\ x + x^x$.
6b (2pt)	$f'(x) = \frac{1}{2\sqrt{arcsin(x)*(1-x^2)}}$.
7a (2pt)	$\lim_{x\to 3} \frac{sin(\pi x)}{x^2-9} = -\frac{1}{6}\pi$.
7b (2pt)	$\lim_{x\to-\infty} \frac{e^{3-x}}{7x^2} = \infty$.
8a (2pt)	$F(x) = sin^2(x)$.
8b (2pt)	$F(x) = tan^{-1}(2x)$.
9a (2pt)	The points are $(2, 4)$, $(5, 7)$, $(3, 3)$. The area of the triangle is 3.
9b (2pt)	$A = (4\sqrt{2} + \sqrt{20}) * a + \pi * a^2$, $B = 2\pi a + 4\sqrt{2} + \sqrt{20}$.
9c (2pt)	In order for $A = B$, we need a to be larger than 1.