Calculus and Probability
Assignment 7

x
x
Group 6

September 1, 2018

Exercise 6

a) We have four possibilities thus $|S| = 4$. $S = \{HH, HT, TH, TT\}$.

b) A: $\{HT, TH\}$.
B: $\{HH, HT, TH\}$.
C: $\{HH, HT\}$.
A: $\{HT, TH\}$. B: $\{HH, HT, TH\}$. C: $\{HH, HT\}$.

c)

$$
\begin{aligned}
P(\emptyset) &= 0 \\
P(u|u \in S) &= \frac{1}{4} \\
P(u, v|u, v \in S \wedge u \neq v) &= \frac{1}{2} \\
P(u, v, w|u, v, w \in S \wedge u \neq v \neq w) &= \frac{3}{4} \\
P(\{HH, HT, TH, TT\}) &= 1
\end{aligned}
$$

We define $P(S') = p$ as $p = |S'| * \frac{1}{4}$.

Exercise 7

a) We will prove this with induction. Base case ($n = 2$): $P(A_1 \cup A_2) = P(A_1) + P(A_2)$. This is Axiom 2 so immediately proven. As inductive case we take $P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$. We will prove the following:

$$
\begin{aligned}
P(A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}) &= P(\{A_1 \cup A_2 \cup \cdots \cup A_k\} \cup \{A_{k+1}\}) \\
&= P(\{A_1 \cup A_2 \cup \cdots \cup A_k\}) + P(\{A_{k+1}\}) \\
&= P(A_1) + P(A_2) + \cdots + P(A_k) + P(A_{k+1})
\end{aligned}
$$

See explaination.

Exercise 8

a) The probability that student A succeeds every exercise is $(0, 5)^8$. This holds for every possible combination of passes/fails. We can thus consider a pass/fail sequence as a bit string with 0 for fail and 1 for success. We can then say that all the bit strings with five ones are exactly the probability that student A succeeds five exercises. This means that we need to choose five positions out of eight where we can put a one. Thus $P(X_A = 5) = (0,5)^8 * \binom{8}{5}$. $P(X_A = 5) = (0,5)^8 * \binom{8}{5} = 0,21875$.

b) Now we can choose either 5,6,7 or 8 out of eight positions to put ones in the bit string. This thus means that $P(X_A \geq 5) = (0,5)^8 * (\binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8})$. $P(X_A \geq 5) = 0,36328125$.

c) This is equal to the previous one only we swap out $0,5$ for $0,8$ and we need to include the other chance as well.

$$\begin{aligned}
P(X_B = 5) &= (0,8)^5 * (0,2)^3 * \tbinom{8}{5} && = 0,14680064 \\
P(X_B = 6) &= (0,8)^6 * (0,2)^2 * \tbinom{8}{6} && = 0,29360128 \\
P(X_B = 7) &= (0,8)^7 * (0,2)^1 * \tbinom{8}{7} && = 0,33554432 \\
P(X_B = 8) &= (0,8)^8 * (0,2)^0 * \tbinom{8}{8} && = 0,16777216
\end{aligned}$$

Now we can add them all together. $P(X_A \geq 5) = 0,14680064 + 0,29360128 + 0,33554432 + 0,16777216 = 0,9437184$. $P(X_A \geq 5) \approx 0,94$.

Exercise 9

a) We need to make sure that $\int_{-\infty}^{\infty} f(x)dx = 1$. Because for x out of the range $(-\frac{1}{2}, \frac{1}{2})$, $y = 0$ we only need to make sure $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx = 1$. We thus need to make sure that:

$$\begin{aligned}
\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)dx &= 1 \\
\int_{-\frac{1}{2}}^{\frac{1}{2}} a * (1 - 4x^2) &= 1 \\
a * \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 - 4x^2 &= 1 \\
a * x - \frac{4}{3}x^3]_{-\frac{1}{2}}^{\frac{1}{2}} &= 1 \\
a * ((\frac{1}{2} - \frac{4}{3}(\frac{1}{2})^3) - (-\frac{1}{2} - \frac{4}{3}(-\frac{1}{2})^3)) &= 1 \\
a * ((\frac{1}{2} - \frac{4}{3} * \frac{1}{8}) - (-\frac{1}{2} - \frac{4}{3} * -\frac{1}{8}) &= 1 \\
a * ((\frac{1}{2} - \frac{4}{24}) - (-\frac{1}{2} + \frac{4}{24}) &= 1 \\
a * ((\frac{1}{2} - \frac{1}{6}) - (-\frac{1}{2} + \frac{1}{6}) &= 1 \\
a * (\frac{1}{3} + \frac{1}{3}) &= 1 \\
a * \frac{2}{3} &= 1 \\
a &= 1\frac{1}{2}
\end{aligned}$$

$a = 1\frac{1}{2}$

b) The cumulative distribution function $F(x) = a * x - \frac{4}{3}ax^3 = 1\frac{1}{2}x - 2x^3$. $F(x) = 1\frac{1}{2}x - 2x^3$.

c) We thus need to compute $P(\frac{1}{4} \leq x \leq \frac{1}{4}$. But this would mean that we would need to calculate $P(X \leq \frac{1}{4}) - P(X \leq \frac{1}{4})$. This equals 0. $P(X = \frac{1}{4}) = 0$.

d) We need to compute:

$$\begin{aligned} P(0 < X < \frac{1}{4}) &= P(0 \leq X \leq \frac{1}{4}) \\ &= P(X \leq \frac{1}{4}) - P(X \leq 0) \\ &= \int_0^{\frac{1}{4}} f(x)dx \\ &= 1\frac{1}{2}x - 2x^3]_0^{\frac{1}{4}} \\ &= 1\frac{1}{2} * \frac{1}{4} - 2(\frac{1}{4})^3 - 1\frac{1}{2} * 0 - 2 * 0^3 \\ &= \frac{3}{8} - \frac{1}{32} \\ &= \frac{12}{32} - \frac{1}{32} \\ &= \frac{11}{32} \end{aligned}$$

$P(0 < X < \frac{1}{4}) = \frac{11}{32}$.

Exercise 10

a) We must solve:

$$\frac{1}{3}e^{-\frac{\pi}{9}(x^2-4x+4)} = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

Thus:

$$\begin{aligned} \frac{1}{3} &= \frac{1}{\sigma\sqrt{2\pi}} \\ 3 &= \sigma\sqrt{2\pi} \\ \sigma &= \frac{3}{\sqrt{2\pi}} \end{aligned}$$

So:

$$\begin{aligned} e^{-\frac{\pi}{9}(x^2-4x+4)} &= e^{-\frac{1}{2}(\frac{x-\mu}{\frac{3}{\sqrt{2\pi}}})^2} \\ &= e^{-\frac{1}{2}((x-\mu)*\frac{\sqrt{2\pi}}{3})^2} \\ &= e^{-\frac{1}{2}((x-\mu)*\frac{1}{3}\sqrt{2\pi})^2} \\ &= e^{-\frac{1}{2}(\frac{1}{3}x\sqrt{2\pi}-\frac{1}{3}\mu\sqrt{2\pi})^2} \\ &= e^{-\frac{1}{2}(\frac{1}{3}x\sqrt{2\pi}^2-\frac{1}{3}\mu\sqrt{2\pi}^2-2*(\frac{1}{3}x\sqrt{2\pi}*\frac{1}{3}\mu\sqrt{2\pi}))} \\ &= e^{-\frac{1}{2}(\frac{1}{9}x^2*2\pi-\frac{1}{9}\mu^2*2\pi-2*\frac{1}{9}x*2\pi*\mu)} \\ &= e^{-\frac{1}{2}(\frac{2\pi}{9}x^2-\frac{2\pi}{9}\mu^2-2*\frac{2\pi}{9}x*\mu)} \\ &= e^{-\frac{\pi}{9}(x^2+\mu^2-2*x*\mu)} \end{aligned}$$

From here we can easily see that $\mu = 2$. $\mu = 2$ and $\sigma = \frac{3}{\sqrt{2\pi}}$.

b) Sorry Gijs... Answer 10b

Answer Form Assignment 7

Name	x
Student Number	x

Question	**Answer**
6a (1pt)	$S = \{HH, HT, TH, TT\}$.
6b (1pt)	A: $\{HT, TH\}$. B: $\{HH, HT, TH\}$. C: $\{HH, HT\}$.
6c (1pt)	We define $P(S') = p$ as $p = \lvert S' \rvert * \frac{1}{4}$.
7 (1pt)	See explaination.
8a (1pt)	$P(X_A = 5) = (0,5)^8 * \binom{8}{5} = 0,21875$.
8b (1pt)	$P(X_A \geq 5) = 0,36328125$.
8c (1pt)	$P(X_A \geq 5) \approx 0,94$.
9a (1pt)	$a = 1\frac{1}{2}$
9b (2pt)	$F(x) = 1\frac{1}{2}x - 2x^3$.
9c (1pt)	$P(X = \frac{1}{4}) = 0$.
9d (1pt)	$P(0 < X < \frac{1}{4}) = \frac{11}{32}$.
10a (1pt)	$\mu = 2$ and $\sigma = \frac{3}{\sqrt{2\pi}}$.
10b (1pt)	Answer 10b