Combinatorics
Assignment 5

November 11, 2017

Exercise 8

This is theorem 2:

Theorem. *Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \ldots,$ where α_1, α_2 are constants.*

Proof. We have to prove two things.

1. First we show that $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ is indeed a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, if r_0 is the only root of the corresponding characteristic equation $r^2 - c_1 r - c_2$.
2. And then we show that each solution of $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ can be written as $\alpha_1 r_0^n + \alpha_2 n r_0^n$ for certain α_1 and α_2, if r_0 is the only root of the corresponding characteristic equation $r^2 - c_1 r - c_2$.

The proof of 1: This is indeed true because we have only one solution with multiplicity 2. This means that: $r^2 = c_1 r + c_2$.
So $c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 * r^{n-1} + \alpha_2 * (n-1) * r^{n-1}) + c_2(\alpha_1 * r^{n-2} + \alpha_2 * (n-2) * r^{n-2})$
So $c_1 a_{n-1} + c_2 a_{n-2} = c_1 * r^{n-1} * (\alpha_1 + \alpha_2 * (n-1)) + c_2 * r^{n-2} * (\alpha_1 + \alpha_2 * (n-2))$
So $c_1 a_{n-1} + c_2 a_{n-2} = c_1 * r^{n-2} * r * (\alpha_1 + \alpha_2 * (n-1)) + c_2 * r^{n-2} * (\alpha_1 + \alpha_2 * (n-2))$
This equals a_n.

The proof of 2: Deze is best moeilijk ja...

□

Exercise 9

a) We need to find A and B such that $a_n = An + B$ is a solution of $a_n = 2a_{n-1} + n + 5$. We try $a_n = An + B$:
Then $a_n - 2a_{n-1} = n + 5$
So $An + B - 2 * (A(n-1) + B) = n + 5$
So $An + B - 2An + 2A - 2B = n + 5$
So $2A - An - B = n + 5$
So $A = -1$ and $B = -7$
A particular solution of this system is: $a_n = -n - 7$

b) The homogeneous part of the equation is: $a_n = 2a_{n-1}$. The equation is already in default form. The characteristic equation now equals: $r - 2 = 0$, so $r = 2$ and the general solution is: $a_n = \alpha_1 * 2^n$. All solutions can be found by adding the general solution and the particular solution: $a_n = \alpha_1 * 2^n - n - 7$

c) We have $a_0 = \alpha_1 * 2^0 - 0 - 7$:
So $4 = \alpha_1 - 7$
So $\alpha = 11$
The solution to this recurrence relation is:
$a_n = 11 * 2^n - n - 7$.

Exercise 10

The characteristic equation of the associated homogeneous recurrence relation is $r^4 - 8r^2 + 16 = 0$. This can be rewritten to: $(r^2 - 4)(r^2 - 4) = 0$ This equation has the roots $r_1 = 2$ and $r_2 = -2$ with multiplicities $m_1 = 2$ and $m_2 = 2$. The sequence will now be: $a_n = (\alpha_{1,0} + \alpha_{1,1} * n) * 2^n + (\alpha_{2,0} + \alpha_{2,1} * n) * (-2)^n$.

a) We can write $F(n) = (-2)^n = (b_0) \cdot s^n$ where

$$\begin{aligned} s &= -2 \\ t &= 0 \\ b_0 &= 1 \end{aligned}$$

And because s is a root. Theorem 6 tells us that we get as particular solution $a_n^{(p)} = n^2 * p_0 * (-2)^n$

b) We can write $F(n) = n^2 * 4^n = (b_2 * n^2 + b_1 * n + b_0) \cdot s^n$ where

$$\begin{aligned} s &= 4 \\ t &= 2 \\ b_0 &= 0 \\ b_1 &= 0 \\ b_2 &= 1 \end{aligned}$$

And because s is not a root. Theorem 6 tells us that we get as particular solution $a_n^{(p)} = (p_2 * n^2 + p_1 * n + p_0) * 4^n$

c) We can write $F(n) = n^4 * 2^n = (b_4 * n^4 + b_3 * n^3 + b_2 * n^2 + b_1 * n + b_0) \cdot s^n$ where

$$\begin{aligned} s &= 2 \\ t &= 4 \\ b_0 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= 0 \\ b_4 &= 1 \end{aligned}$$

And because s is a root. Theorem 6 tells us that we get as particular solution $a_n^{(p)} = n^2 * (p_4 * n^4 + p_3 * n^3 + p_2 * n^2 + p_1 * n + p_0) * 2^n$

Exercise 11

a) We apply the first steps of the algorithm:

Step 1 $a_n = -5a_{n-1} - 6a_{n-2} + 42 * 4^n$. So $F(n) = 42 * 4^n$.

Step 2 $r^2 + 5r + 6 = 0$

Step 3 $(r+3)(r+2) = 0$. So $r_1 = -2$ and $r_2 = -3$ with multiplicities $m_1 = 1$ and $m_2 = 1$.

Step 4 $a_n = \alpha_1 * (-2)^n + \alpha_2 * (-3)^n$

Step 5 $a_n + 5a_{n-1} + 6a_{n-2} = F(n)$. The particular solution is of the form: $A * 4^n$. When inserting this into the main formula we get: $(A*4^n)+5(A*4^{n-1})+6(A*4^{n-2}) = 42*4^n$.
This can be rewritten as:
$(A * 4^{n-2} * 4^2) + 5(A * 4^{n-2} * 4) + 6(A * 4^{n-2}) = 42 * 4^n$
$(A * 4^{n-2} * 4^2) + 5(A * 4^{n-2} * 4) + 6(A * 4^{n-2}) = 42 * 4^{n-2} * 4^2$
$4^{n-2} * (16A + 20A + 6A) = 42 * 4^{n-2} * 4^2$
$16A + 20A + 6A = 42 * 16$
$42A = 42 * 16$
$A = 16$ and $B = 4$ and $C = 0$ in the equation: $A * B^n + C$.

Step 6 $a_n = \alpha_1 * (-2)^n + \alpha_2 * (-3)^n + 16 * 4^n$.

b) We apply the last step of the algorithm:

Step 7 $a_n = \alpha_1 * (-2)^n + \alpha_2 * (-3)^n + 16 * 4^n$ with $a_1 = 56$ and $a_2 = 278$.
So $a_1 = \alpha_1 * (-2)^1 + \alpha_2 * (-3)^1 + 16 * 4^1$.
So $56 = \alpha_1 * -2 + \alpha_2 * -3 + 64$.
So $0 = -2\alpha_1 + -3\alpha_2 + 8$.
So $2\alpha_1 = -3\alpha_2 + 8$.
So $\alpha_1 = -1,5\alpha_2 + 4$.
We can now insert this in a_2:
$a_2 = \alpha_1 * (-2)^2 + \alpha_2 * (-3)^2 + 16 * 4^2$
$278 = \alpha_1 * 4 + \alpha_2 * 9 + 16 * 16$
$278 = 4\alpha_1 + 9\alpha_2 + 256$
$22 = 4\alpha_1 + 9\alpha_2$
$0 = 4\alpha_1 + 9\alpha_2 - 22$
$0 = 4(-1,5\alpha_2 + 4) + 9\alpha_2 - 22$
$0 = -6\alpha_2 + 16 + 9\alpha_2 - 22$
$0 = 3\alpha_2 - 6$
$\alpha_2 = 2$
Now we can calculate α_1 by:
$\alpha_1 = -1,5\alpha_2 + 4$
$\alpha_1 = 4 - 3$
$\alpha_1 = 1$
Now the general solution becomes:
$a_n = (-2)^n + 2 * (-3)^n + 16 * 4^n$

Exercise 12

It is not possible to find s, t, $b_0, \ldots, b_t$ such that $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots b_1 n + b_0) \cdot s^n$, but we can split $F(n)$ into 2^n and $n + 3$. We apply the steps of the algorithm:

Step 1 $a_n = 4a_{n-1} - 3a_{n-2} + 2^n + n + 3$ and $F(n) = 2^n + n + 3$

Step 2 $r^2 - 4r + 3 = 0$

Step 3 $(r-3)(r-1) = 0$. So $r_1 = 3$ and $r_2 = 1$ with multiplicities $m_1 = 1$ and $m_2 = 1$.

Step 4 $a_n = \alpha_1 * 3^n + \alpha_2 * 1^n = \alpha_1 * 3^n + \alpha_2$

Step 5 $a_n - 4a_{n-1} + 3a_{n-2} = 2^n + n + 3$

We take $a_n = A * 2^n + Cn + D$.

So: $A*2^n+Cn+D-4*(A*2^{n-1}+C(n-1)+D)+3*(A*2^{n-2}+C(n-2)+D) = 2^n+n+3$

So: $A*2^{n-2}*2^2+Cn+D-4*(A*2^{n-2}*2+C(n-1)+D)+3*(A*2^{n-2}+C(n-2)+D) = 2^n+n+3$

So: $4A*2^{n-2}+Cn+D-8A*2^{n-2}-4C(n-1)-4D+3A*2^{n-2}+3C(n-2)+3D = 2^n+n+3$

So: $-A*2^{n-2} + Cn + D - 4Cn - 4C - 4D + 3Cn - 2C + 3D = 2^n + n + 3$

So: $-A*2^{n-2} - 2Cn - 6C = 2^n + n + 3$

So: $-A*2^{n-2} - 2Cn - 6C = 2^{n-2} * 2^2 + n + 3$

So: $-A*2^{n-2} - 2Cn - 6C = 8 * 2^{n-2} + n + 3$

This means that $A = -8$ and $C = -\frac{1}{2}$ and $D = 0$

Step 6 $a_n = \alpha_1 * 3^n + \alpha_2 - 8 * 2^n - \frac{1}{2}n$

Step 7 $a_0 = 1$ and $a_1 = 4$:

First we fill in a_0:

$a_0 = \alpha_1 * 3^0 + \alpha_2 - 8 * 2^0 - \frac{1}{2} * 0$

Then: $1 = \alpha_1 + \alpha_2 - 8$

So: $\alpha_1 = 9 - \alpha_2$

We can now fill this in into a_1:

Then: $a_1 = \alpha_1 * 3^1 + \alpha_2 - 8 * 2^1 - \frac{1}{2} * 1$

So: $4 = 3\alpha_1 + \alpha_2 - 16\frac{1}{2}$

Now we fill in α_1:

Then: $4 = 3 * (9 - \alpha_2) + \alpha_2 - 16\frac{1}{2}$

So: $4 = 27 - 3\alpha_2 + \alpha_2 - 16\frac{1}{2}$

So: $4 = 10\frac{1}{2} - 2\alpha_2$

So: $0 = 6\frac{1}{2} - 2\alpha_2$

So: $2\alpha_2 = 6\frac{1}{2}$

So: $\alpha_2 = 3\frac{1}{4}$

Then we can fill this into the above formula:

$\alpha_1 = 9 - \alpha_2$

So: $\alpha_1 = 9 - 3\frac{1}{4}$

So: $\alpha_1 = 5\frac{3}{4}$

We can now create the general solution: $a_n = 5\frac{3}{4} * 3^n + 3\frac{1}{4} - 8 * 2^n - \frac{1}{2}n$

Exercise 13

a) Note that

- $s_1 = 2$, because we can fill a bag with one red marble and a bag with one yellow marble.
- $s_2 = 4$, because we can fill one bag with two blue marbles and then fill the other bags with combinations of red and yellow marbles

b) $s_n = s_{n-2} + n + 1$ because we can either pick s_{n-2} marbles and add two blue ones, or we only pick red and yellow marbles and determine how many red ones we pick, this means n+1 possible solutions.

c) We apply the steps of the algorithm:

Step 1 $s_n = s_{n-2} + n + 1$ with $F(n) = n + 1$.

Step 2 $r^2 - 1 = 0$

Step 3 $r = 1$ or $r = -1$

Step 4 $s_n = \alpha_1 * 1^n + \alpha_2 * (-1)^n$

Step 5 $s_n - s_{n-2} = n + 1$

We try $s_n = An^2 + Bn$:

$An^2 + Bn - A(n-2)^2 - B(n-2) = n+1$
$An^2 + Bn - A(n-2)(n-2) - B(n-2) = n+1$
$An^2 + Bn - An^2 + 4An - 4A - B(n-2) = n+1$
$An^2 + Bn - An^2 + 4An - 4A - Bn + 2B = n+1$
$4An - 4A + 2B = n+1$
This means that: $A = \frac{1}{4}$ and $B = 1$.

Step 6 $s_n = \alpha_1 + \alpha_2 * (-1)^n + \frac{1}{4}n^2 + n$

Step 7 We can now fill in s_0:
$s_0 = \alpha_1 + \alpha_2 * (-1)^0 + \frac{1}{4}0^2 + 0$
$2 = \alpha_1 + \alpha_2$
$\alpha_1 = 2 - \alpha_2$
Let's fill this in into s_1:
$s_1 = \alpha_1 + \alpha_2 * (-1)^1 + \frac{1}{4}1^2 + 1$
$4 = \alpha_1 - \alpha_2 + 1\frac{1}{4}$
$4 = 2 - \alpha_2 - \alpha_2 + 1\frac{1}{4}$
$4 = 2 - 2\alpha_2 + 1\frac{1}{4}$
$4 = -2\alpha_2 + 3\frac{1}{4}$
$4 + 2\alpha_2 = 3\frac{1}{4}$
$2\alpha_2 = -\frac{3}{4}$
$\alpha_2 = -\frac{3}{8}$
This means that $\alpha_1 = 2\frac{3}{8}$
The general solution now is:
$t_0 = 2$
$t_1 = 4$
$t_n = 2\frac{3}{8} - \frac{3}{8} * (-1)^n + \frac{1}{4}n^2 + n$