Calculus and Probability
Assignment 5

x
x
Group 6

September 1, 2018

Exercise 6

a)

$$\begin{aligned}\int \sin(x)\cos(x)dx &= \sin(x)\int \cos(x)dx - \int \cos(x) * (\int \cos(x)dx)dx \\ &= \sin(x) * \sin(x) - \int \sin(x)\cos(x)dx \\ &= \sin^2(x) - \int \sin(x)\cos(x)dx\end{aligned}$$

$$\begin{aligned}\int \sin(x)\cos(x) + \int \sin(x)\cos(x) &= \sin^2(x) \\ 2\int \sin(x)\cos(x) &= \sin^2(x) \\ \int \sin(x)\cos(x) &= \frac{1}{2}\sin^2(x) + C\end{aligned}$$

$\int \sin(x)\cos(x) = \frac{1}{2}\sin^2(x) + C$.

b)

$$\begin{aligned}\int \ln(ax)dx &= \frac{1}{a}\int \ln(ax) * adx \\ &= \frac{1}{a}\int \ln(u)du \\ &= \frac{1}{a}(u\ln(u) - u) \\ &= \frac{1}{a}(ax\ln(ax) - ax) \\ &= x\ln(x) - x + C\end{aligned}$$

$\int \ln(ax)dx = x\ln(x) - x + C$.

Exercise 7

a)

$$
\begin{aligned}
\int_{-1}^{1} \sqrt{1+(f'(x))^2} &= \int_{-1}^{1} \sqrt{1+(\frac{-x}{\sqrt{1-x^2}})^2} \\
&= \int_{-1}^{1} \sqrt{1+(\frac{x^2}{1-x^2})} \\
&= \int_{-1}^{1} \sqrt{\frac{1-x^2}{1-x^2}+\frac{x^2}{1-x^2}} \\
&= \int_{-1}^{1} \sqrt{\frac{1-x^2+x^2}{1-x^2}} \\
&= \int_{-1}^{1} \sqrt{\frac{1}{1-x^2}} \\
&= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \\
&= sin^{-1}(x)]_{-1}^{1} \\
&= sin^{-1}(1) - sin^{-1}(-1) \\
&= sin^{-1}(1) - sin^{-1}(-1) \\
&= \frac{1}{2}\pi + \frac{1}{2}\pi \\
&= \pi
\end{aligned}
$$

$\int_{-1}^{1} \sqrt{1+(f'(x))^2} = \pi$.

b) The graph that is drawn from the formula forms a circle with radius one (entering 0 in the formula gives a radius of 1). This means that the length of the line is equal to π. The graph forms half a circle.

Exercise 8

a)

$$
\begin{aligned}
\int_{0}^{\infty} e^{-x} dx &= -e^{-x}]_{0}^{\infty} \\
&= \lim_{x\to\infty} -e^{-x} - (-e^{-0}) \\
&= \lim_{x\to\infty} -e^{-x} - (-e^{0}) \\
&= \lim_{x\to\infty} -e^{-x} - (-1) \\
&= \lim_{x\to\infty} -e^{-x} + 1 \\
&= \lim_{x\to\infty} -\frac{1}{e^x} + 1 \\
&= 0 + 1 \\
&= 1
\end{aligned}
$$

$\int_{0}^{\infty} e^{-x} dx = 1$.

b)

$$\begin{aligned}\int_0^\infty xe^{-x}dx &= x\int e^{-x}dx - \int 1*(\int e^{-x}dx)dx]_0^\infty \\ &= -xe^{-x} - e^{-x}]_0^\infty \\ &= \lim_{x\to\infty} -xe^{-x} - e^{-x} - (0e^{-0} - e^{-0}) \\ &= \lim_{x\to\infty} -xe^{-x} - e^{-x} - (-e^0) \\ &= \lim_{x\to\infty} -xe^{-x} - e^{-x} - (-1) \\ &= \lim_{x\to\infty} -xe^{-x} - e^{-x} + 1 \\ &= \lim_{x\to\infty} \frac{-x}{e^x} - \frac{1}{e^x} + 1 \\ &= \lim_{x\to\infty} \frac{-1-x}{e^x} + 1 \\ &= 1 + \lim_{x\to\infty} \frac{-1-x}{e^x}\end{aligned}$$

We can apply L'Hopital on $\lim_{x\to\infty} \frac{-1-x}{e^x}$ because:

$$\lim_{x\to\infty} -1 - x = -\infty$$
$$\lim_{x\to\infty} e^x = \infty$$

Thus:

$$\begin{aligned}\lim_{x\to\infty} \frac{-1-x}{e^x} &= \lim_{x\to\infty} \frac{-1}{e^x} \\ &= 0\end{aligned}$$

Thus:

$$\begin{aligned}1 + \lim_{x\to\infty} \frac{-1-x}{e^x} &= 1 + 0 \\ &= 1\end{aligned}$$

$\int_0^\infty xe^{-x}dx = 1.$

c)

$$
\begin{aligned}
\int_0^\infty x^0 * e^{-x} dx &= 1 \\
\int_0^\infty x^1 * e^{-x} dx &= 1 \\
\int_0^\infty x^2 * e^{-x} dx &= x^2 \int e^{-x} dx - \int 2x * (\int e^{-x} dx) dx]_0^\infty \\
&= -x^2 e^{-x} - \int -2xe^{-x} dx]_0^\infty \\
&= -x^2 e^{-x} - (-2x * \int e^{-x} dx - \int 2 * (\int e^{-x} dx) dx)]_0^\infty \\
&= -x^2 e^{-x} - (-2x * -e^{-x} - \int 2 * (-e^{-x}) dx)]_0^\infty \\
&= -x^2 e^{-x} - (2xe^{-x} - \int -2e^{-x} dx)]_0^\infty \\
&= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}]_0^\infty \\
&= \lim_{x \to \infty} -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} - (-0^2 e^{-0} - 2xe^{-0} - 2e^{-0}) \\
&= \lim_{x \to \infty} -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} - (-2) \\
&= 2 + \lim_{x \to \infty} -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \\
&= 2 + \lim_{x \to \infty} e^{-x}(-x^2 - 2x - 2) \\
&= 2 + \lim_{x \to \infty} \frac{-x^2 - 2x - 2}{e^x}
\end{aligned}
$$

We can again apply L'Hopital because:

$$
\begin{aligned}
\lim_{x \to \infty} -x^2 - 2x - 2 &= -\infty \\
\lim_{x \to \infty} e^x &= \infty \\
\lim_{x \to \infty} \frac{-x^2 - 2x - 2}{e^x} &= \lim_{x \to \infty} \frac{-2x - 2}{e^x}
\end{aligned}
$$

We can apply L'Hopital again because:

$$
\begin{aligned}
\lim x \to \infty - 2x - 2 &= -\infty \\
\lim_{x \to \infty} \frac{-2x - 2}{e^x} &= \lim_{x \to \infty} \frac{-2}{e^x} \\
&= 0
\end{aligned}
$$

Thus:

$$
\begin{aligned}
2 + \lim_{x \to \infty} \frac{-x^2 - 2x - 2}{e^x} &= 2 + 0 \\
&= 2
\end{aligned}
$$

Now we calculate for $n = 3$:

$$\begin{aligned}
\int_0^\infty x^3 * e^{-x} dx &= x^3 \int e^{-x} dx - \int 3x^2 * (\int e^{-x} dx) dx]_0^\infty \\
&= -x^3 e^{-x} dx - (\int -3x^2 e^{-x} dx)]_0^\infty \\
&= -x^3 e^{-x} dx - (-3x^2 \int e^{-x} dx - \int -6x(\int e^{-x} dx) dx)]_0^\infty \\
&= -x^3 e^{-x} dx - (3x^2 e^{-x} - \int 6x e^{-x} dx)]_0^\infty \\
&= -x^3 e^{-x} dx - (3x^2 e^{-x} - (6x \int e^{-x} dx - \int 6(\int e^{-x} dx) dx))]_0^\infty \\
&= -x^3 e^{-x} dx - (3x^2 e^{-x} - (-6x e^{-x} - \int -6 e^{-x} dx))]_0^\infty \\
&= -x^3 e^{-x} dx - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x}]_0^\infty
\end{aligned}$$

We can now see the pattern in the equation. The limit always equals zero because L'Hopital can be applied many times and an equation of the following form will be created from it: $\frac{a}{e^x}$. We also know that the terms that are created from the integral are always of the form $-x^n e^{-x} - a x^{n-1} e^{-x} - \cdots - b e^{-x}$. When filling in $x = 0$ we can see that all terms disappear but one, namely: $b e^{-x}$. This term will be equal to b if $x = 0$ is put into the formula. This means we only need to know the value of b to know the value of the complete integral. The value of b is equal to $n!$ because of the integration by parts rule. The derivative in the integration by parts rule is causing b to be equal to $n!$. $\int_0^\infty x^n e^{-x} = n!$.

Exercise 9

a) First we will calculate the intersections:

$$\begin{aligned}
(x-1)^3 &= (x-1)^2 \\
(x-1)(x-1)(x-1) &= (x-1)(x-1) \\
(x^2 - 2x + 1)(x-1) &= x^2 - 2x + 1 \\
x^3 - 2x^2 + x - x^2 + 2x - 1 &= x^2 - 2x + 1 \\
x^3 - 3x^2 + 3x - 1 &= x^2 - 2x + 1 \\
x^3 - 4x^2 + 5x - 2 &= 0 \\
(x-2)(x^2 - 2x + 1) &= 0 \\
(x-2) &= 0 \\
x &= 2 \\
x^2 - 2x + 1 &= 0 \\
(x-1)(x-1) &= 0 \\
x &= 1
\end{aligned}$$

We have two intersections at $x = 1$ and $x = 2$. Now, we only need to calculate both areas:

$$
\begin{aligned}
\int_1^2 (x-1)^2 dx - \int_1^2 (x-1)^3 dx &= \int_1^2 x^2 - 2x + 1dx - \int_1^2 x^3 - 3x^2 + 3x - 1dx \\
&= (\frac{1}{3}x^3 - x^2 + x]_1^2) - (\frac{1}{4}x^4 - x^3 + 1\frac{1}{2}x^2 - x]_1^2) \\
&= ((\frac{1}{3}2^3 - 2^2 + 2) - (\frac{1}{3}1^3 - 1^2 + 1)) - ((\frac{1}{4}2^4 - 2^3 + 1\frac{1}{2}2^2 - 2) - (\frac{1}{4}1^4 - 1^3 + 1\frac{1}{2}1^2 \\
&= ((\frac{8}{3} - 4 + 2) - (\frac{1}{3} - 1 + 1)) - ((\frac{16}{4} - 8 + 1\frac{1}{2} * 4 - 2) - (\frac{1}{4} - 1 + 1\frac{1}{2} - 1)) \\
&= ((2\frac{2}{3} - 2) - \frac{1}{3}) - ((4 - 8 + 6 - 2) - (\frac{1}{4} - \frac{1}{2})) \\
&= (\frac{2}{3} - \frac{1}{3}) - (0 - (-\frac{1}{4})) \\
&= \frac{1}{3} - \frac{1}{4}) \\
&= \frac{4}{12} - \frac{3}{12}) \\
&= \frac{1}{12}
\end{aligned}
$$

The area of the region bounded by the two formulas is $\frac{1}{12}$.

Exercise 10

a)

$$
\begin{aligned}
\int \frac{1}{1+e^{2x}} dx &= \frac{1}{2} \int \frac{1}{1+e^{2x}} * 2dx \\
&= \frac{1}{2} \int \frac{1}{1+e^{u}} du \\
&= \frac{1}{2} \int \frac{1}{1+e^{u}} du
\end{aligned}
$$

Answer 10a

b)

$$\sqrt{4 - \sqrt{x}} dx =$$

Answer 10b

c)

$$
\begin{aligned}
\int_0^{\pi^2} \sin(\sqrt{x})dx &= 2\int_0^{\pi^2} usin(u)du \\
\int usin(u)du &= u*\int sin(u)*udu - \int cos(u)(\int sin(u)du)du \\
&= -ucos(u) - \int -cos(u)du \\
&= -ucos(u) + \int cos(u)du \\
&= -ucos(u) + sin(u) \\
&= sin(u) - ucos(u) \\
2\int_0^{\pi^2} usin(u)du &= 2*(sin(u) - ucos(u))]_0^{\pi^2} \\
&= 2sin(u) - u2cos(u)]_0^{\pi^2} \\
&= 2sin(\sqrt{x}) - 2\sqrt{x}cos(\sqrt{x})]_0^{\pi^2} \\
&= 2sin(\sqrt{\pi^2}) - 2\sqrt{\pi^2}cos(\sqrt{\pi^2}) - 2sin(\sqrt{0}) - 2\sqrt{0}cos(\sqrt{0}) \\
&= 2sin(\pi) - 2\pi\cos(\pi) - 0 - 0 \\
&= 2\pi
\end{aligned}
$$

$\int_0^{\pi^2} \sin(\sqrt{x})dx = 2\pi$.

d) ... Answer 10d

e)

$$
\begin{aligned}
\int \frac{1}{1+sin(x)}dx &= \int \frac{1}{1+\frac{2t}{1+t^2}} * \frac{2}{1+t^2}dt \\
&= \int \frac{2}{1+t^2+2t}dt \\
&= 2\int \frac{1}{1+t^2+2t}dt \\
&= 2\int \frac{1}{(t+1)^2}dt \\
&= 2\int (t+1)^{-2}dt \\
&= -2(t+1)^{-1} \\
&= -\frac{2}{t+1}
\end{aligned}
$$

$\int \frac{1}{1+sin(x)}dx = -\frac{2}{t+1}$.

Answer Form Assignment 5

Name	x
Student Number	x
Group	Group 6

Question	**Answer**
6a (1pt)	$\int \sin(x)\cos(x) = \frac{1}{2}\sin^2(x) + C.$
6b (1pt)	$\int \ln(ax)dx = x\ln(x) - x + C.$
6a (2pt)	$\int \sin(x)\cos(x) = \frac{1}{2}\sin^2(x) + C.$
6b (2pt)	$\int \ln(ax)dx = x\ln(x) - x + C.$
7a (2pt)	$\int_{-1}^{1} \sqrt{1 + (f'(x))^2} = \pi.$
7b (2pt)	The graph forms half a circle.
8a (2pt)	$\int_0^\infty e^{-x}dx = 1.$
8b (2pt)	$\int_0^\infty xe^{-x}dx = 1.$
8c (2pt)	$\int_0^\infty x^n e^{-x} = n!.$
9 (2pt)	The area of the region bounded by the two formulas is $\frac{1}{12}$.
10a (2pt)	Answer 10a
10b (2pt)	Answer 10b
10c (2pt)	$\int_0^{\pi^2} \sin(\sqrt{x})dx = 2\pi.$
10d (2pt)	Answer 10d
10e (2pt)	$\int \frac{1}{1+sin(x)}dx = -\frac{2}{t+1}.$