Combinatorics
Assignment 6

November 11, 2017

Exercise 8

We define three sets. Let A denote all students. Let S denote the sets of students that like Brussels sprouts. Let B denote the sets of students that like broccoli. Let C denote the sets of students that like cauliflower.

Then

$$
\begin{aligned}
|A| &= 270 \\
|S| &= 64 \\
|B| &= 94 \\
|C| &= 58 \\
|S \cap B| &= 26 \\
|S \cap C| &= 28 \\
|B \cap C| &= 22 \\
|B \cap C \cap S| &= 14 \\
|S \cup B \cup C| &= |S| + |B| + |C| - |S \cap B| - |S \cap C| - |B \cap C| + |B \cap C \cap S| \\
|S \cup B \cup C| &= 64 + 94 + 58 - 26 - 28 - 22 + 14 \\
|S \cup B \cup C| &= 154 \\
|A| - |S \cup B \cup C| &= 270 - 154 \\
|A| - |S \cup B \cup C| &= 116
\end{aligned}
$$

Exercise 9

The general principle of inclusion and exclusion for four sets tells us that:

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| = \; & |A_1| + |A_2| + |A_3| + |A_4| \\
& - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4| \\
& + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4| \\
& - |A_1 \cap A_2 \cap A_3 \cap A_4|
\end{aligned}
$$

However, if we apply the facts that are given we get

$$
\begin{aligned}
|A_1 \cup A_2 \cup A_3 \cup A_4| &= 100 + 100 + 100 + 100 \\
& \quad - 50 - 50 - 50 - 50 - 50 - 50 \\
& \quad + 25 + 25 + 25 + 25 \\
& \quad - 5 \\
|A_1 \cup A_2 \cup A_3 \cup A_4| &= 195
\end{aligned}
$$

Exercise 10

$$
\begin{aligned}
p(E_1 \cup E_2 \cup E_3 \cup E_4) \quad := \quad & p(E_1) + p(E_2) + p(E_3) + p(E_4) \\
& - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_3 \cap E_4) \\
& + p(E_1 \cap E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_4) + p(E_2 \cap E_3 \cap E_4)
\end{aligned}
$$

Exercise 11

Since we are looking for squarefree positive integers less than 100, we can solve this problem by defining only four properties, because there are only four prime numbers smaller than 10.

$$
\begin{aligned}
P_1(x) &:= 4 \\
P_2(x) &:= 9 \\
P_3(x) &:= 25 \\
P_4(x) &:= 49
\end{aligned}
$$

Note that a number is squarefree if and only if none of these properties hold. Hence we can start counting in the usual way:

- $N = 99$ because there are 99 positive integers below 100.
- $N(P_1) = \lfloor \frac{99}{4} \rfloor = 24$
- $N(P_2) = \lfloor \frac{99}{9} \rfloor = 11$
- $N(P_3) = \lfloor \frac{99}{25} \rfloor = 3$
- $N(P_4) = \lfloor \frac{99}{49} \rfloor = 2$
- $N(P_1) \cap N(P_2) = \lfloor \frac{99}{36} \rfloor = 2$
- $N(P_1) \cap N(P_3) = \lfloor \frac{99}{100} \rfloor = 0$
- $N(P_1) \cap N(P_4) = \lfloor \frac{99}{196} \rfloor = 0$
- $N(P_2) \cap N(P_3) = \lfloor \frac{99}{225} \rfloor = 0$
- $N(P_2) \cap N(P_4) = \lfloor \frac{99}{441} \rfloor = 0$
- $N(P_3) \cap N(P_4) = \lfloor \frac{99}{1225} \rfloor = 0$
- $N(P_1) \cap N(P_2) \cap N(P_3) = \lfloor \frac{99}{900} \rfloor = 0$
- $N(P_1) \cap N(P_2) \cap N(P_4) = \lfloor \frac{99}{1764} \rfloor = 0$
- $N(P_2) \cap N(P_3) \cap N(P_4) = \lfloor \frac{99}{11025} \rfloor = 0$
- $N(P_1) \cap N(P_2) \cap N(P_3) \cap N(P_4) = \lfloor \frac{99}{44100} \rfloor = 0$

And hence $N(P_1' P_2' P_3' P_4') = 99 - 24 - 11 - 3 - 2 + 2 - 0 - \cdots = 99 - 38 = 61$.

Exercise 12

First we can make a partitions, this can be done in: $S(8,3) = 966$. Now we multiply by 3!: 966 * 3! = 5796.

Exercise 13

a) Lorem ipsum dolor sit amet, consectetur adipiscing elit. Nunc nec ultricies urna. Phasellus bibendum maximus auctor. Etiam quis orci eget lorem rhoncus sollicitudin a sed est. Duis eu ultricies nisi. Nulla mattis sem a quam eleifend, a posuere nisl consequat. Sed convallis est a sem imperdiet rhoncus. Morbi nec egestas erat. Praesent ut suscipit lorem. In sit amet velit id augue feugiat pretium at sit amet risus. Mauris convallis lorem non lectus vulputate ultrices ac nec risus. Cras consectetur dolor orci, eu lacinia lorem consectetur vel. Integer eros arcu, finibus vitae mattis ut, ornare in diam. Aenean non accumsan risus.

b) Nullam porttitor velit vitae erat convallis, eu fermentum ipsum lobortis. Mauris eu neque mattis, pellentesque purus vitae, condimentum mauris. Integer molestie id eros eget finibus. Proin blandit purus eros, non varius tortor interdum sit amet. Vivamus nunc urna, auctor ac quam eget, consequat laoreet quam. Curabitur imperdiet orci ante, non consectetur metus volutpat eget. In hac habitasse platea dictumst. Proin et dapibus magna. Phasellus rhoncus cursus venenatis. Pellentesque ac leo luctus orci laoreet aliquet. Praesent tempus tincidunt enim a lacinia.

Exercise 14

There are 36 derangements of $\{1, 2, 3, 4, 5, 6\}$ ending with the integers 1, 2, and 3, in some order. We can create such a derangement by this algorithm:

Task 1 First we place 1, 2 and 3 at the end of the row, this can be done in 3! ways.

Task 2 Then we place 4, 5 and 6 at the beginning of the row, this can be done in 3! ways

So: $3! * 3! = 36$

Exercise 15

a) (i) The probability that exactly two letters end up in the proper envelope is $\frac{3}{16}$. This is because we first shuffle all letters, this can be done in 6! ways. This is the total amount of possible rows. Now we take two of the envelopes and say that these are the ones that need to be properly distributed. Then for each combination, the other letters can be deranged exactly D(4) ways. In total this makes:
$C(6,2) * D(4) = 15 * 9 = 135$.
The total amount of possible rows is: $6! = 720$.
The probability now is: $\frac{135}{720} = \frac{3}{16}$.

(ii) The probability that exactly five letters end up in the proper envelope is 0. This is because when picking 5 letters to be correct, the last one must also be correct.

b) This is an example of distributing indistinguishable objects into distinguishable boxes. The formula for this problem is: $\binom{n-1+r}{r} = \binom{n-1+r}{n-1}$ with $n = 3$ and $r = 14$. This equals: $\binom{16}{2}$.

c) We define the following three properties:

$$\begin{aligned} P_1 &:= x >= 4 \\ P_2 &:= y >= 5 \\ P_3 &:= z >= 8 \end{aligned}$$

A solution of $x + y + z = 14$ where $x, y, z \in \mathbb{N}$ complies to the requirements $x < 4$, $y < 5$ and $z < 8$ if none of these three properties hold.

So

- $N = \binom{16}{2} = 120$ because of exercise b.
- $N(P_1) = \binom{12}{2} = 66$ because now x has a spot higher than or equal to 4.
- $N(P_2) = \binom{11}{2} = 55$
- $N(P_3) = \binom{8}{2} = 28$
- $N(P_1) \cap N(P_2) = \binom{7}{2} = 21$
- $N(P_1) \cap N(P_3) = \binom{4}{2} = 6$
- $N(P_2) \cap N(P_3) = \binom{3}{2} = 3$
- $N(P_1) \cap N(P_2) \cap N(P_3) = 0$ because 4 + 5 + 8 ¿ 16.

Then $N(P_1'P_2'P_3') = 120 - 66 - 55 - 28 + 21 + 6 + 3 = 1$.