Combinatorics
Assignment 4

November 11, 2017

Exercise 9

a) Let a_n be the number of ways the bus driver can pay a toll of n cents.
Nickel: 5 cents
Dime: 10 cents
Then we can make these rows:
5 cents: Nickel
10 cents: Nickel Nickel
10 cents: Dime
15 cents: Nickel Nickel Nickel
15 cents: Nickel Dime
15 cents: Dime Nickel
20 cents: Nickel Nickel Nickel Nickel
20 cents: Nickel Nickel Dime
20 cents: Nickel Dime Nickel
20 cents: Dime Nickel Nickel
20 cents: Dime Dime

We see that in either step, a nickel or a dime can be inserted first, after a nickel, we can insert n-5 cents. After a dime, we can insert n-10 cents. We can then create a recurrence relation for a_n:
$a_n = a_{(n-5)} + a_{(n-10)}$ with $a_5 = 1$ and $a_{10} = 2$ and $n > 10$ and n $mod(5) = 0$

b) The driver has $a_{45} = a_{40} + a_{35} = a_{35} + a_{30} + a_{30} + a_{25} = 21 + 13 + 13 + 8 = 55$ ways to pay 45 cents.

Exercise 10

a) Let a_n be the number of ways to lay out a path of red, green or gray slate tiles such that no two red tiles are adjacent and tiles of the same color are considered indistinguishable.
Then we can make these rows:
1 tile: red
1 tile: green
1 tile: gray
2 tiles: red green
2 tiles: red gray
2 tiles: green red
2 tiles: green gray
2 tiles: green green
2 tiles: gray red
2 tiles: gray green
2 tiles: gray gray

3 tiles: red green red
3 tiles: red green green
3 tiles: red green gray
3 tiles: red gray red
3 tiles: red gray green
3 tiles: red gray gray
3 tiles: green red green
3 tiles: green red gray
...
Then we can split the possible endings of a tile walkway in four different possibilities:
A walkway ending with green red, this will make n-2 possible options
A walkway ending with gray red, this will make n-2 possible options
A walkway ending with green, this will make n-1 possible options
A walkway ending with gray, this will make n-1 possible options
In total, this makes: $a_n = 2 * a_{n-1} + 2 * a_{n-2}$ with $a_0 = 1$ and $a_1 = 3$ and $a > 1$

b) The initial conditions are:

- $a_0 = 1$, because there is only one way of making a walkway with 0 tiles.
- $a_1 = 3$, because there are three ways of making a walkway with 1 tile.

c) There are 1224 ways to lay out a path of seven tiles, because:
$a_0 = 1$
$a_1 = 3$
$a_2 = 8$
$a_3 = 22$
$a_4 = 60$
$a_5 = 164$
$a_6 = 448$
$a_7 = 1224$

Exercise 11

a) Let a_n be the number of moves required to solve the puzzle for n disks with the added restriction.
$a_1 = 2$
$a_2 = 8$
$a_3 = 26$
$a_4 = 80$
We need n-1 moves to move the top of the disks to the most left pillars, then, we need 1 move to move the largest disk to the second pillar. After that, we need n-1 moves to move the top of the disks to the left most pillar, then we need 1 move to move the largest disk to the right most pillar. The last step is to move the other disks to the right most pillar, this costs another n-1 moves.
In total, this makes: $a_n = 3 * a_{n-1} + 2$ moves.

b) We try to find the solution by computing the first few values $a_1, a_2, a_3, a_4, \ldots$ and use induction to prove that our guess is indeed a good solution.
$a_1 = 2$
$a_2 = 8$
$a_3 = 26$
$a_4 = 80$
We can guess that the closed formula is: $a_n = 3^n - 1$ because this is correct for the first 4 values of n. Now we only need to prove this:
$3 * a_{n-1} + 2 = 3^n - 1$
This equation holds for the base case $n = 1$. We assume that $3 * a_{k-1} + 2 = 3^k - 1$. Now we

only need to prove:
$3 * a_{k+1-1} + 2 = 3^{k+1} - 1$, then:
$3 * a_k + 2 = 3^{k+1} - 1$
$3 * a_k + 2 = 3^k * 3^1 - 1$
$3 * a_k + 3 = 3^k * 3^1$
$a_k + 1 = 3^k$
$a_k = 3^k - 1$

c) There are $3 * a_{n-1} + 2$ different arrangements of the n disks on three pegs so that no disk is on top of a smaller disk, because when solving the Hanoi puzzle, we cycle through every possible combination of disk arrangement.

d) When trying this out on smaller numbers like $n = 1$ or $n = 2$, we see that this is true. When trying out $n = 3$, we see that the largest disk will be on peg 1, 2 or 3. We also see that when moving the other two disks around, the two smaller disks will take every possible position while the third disk is switching to the other pegs.

Exercise 12

a) If we complete the table we get:

n	$J(n)$	order of death
1	1	$-$
2	1	2
3	3	$2-1$
4	1	$2-4-3$
5	3	$2-4-1-5$
6	5	$2-4-6-3-1$
7	7	$2-4-6-1-5-3$
8	1	$2-4-6-8-3-7-5$
9	3	$2-4-6-8-1-5-9-7$
10	5	$2-4-6-8-10-3-7-1-9$
11	7	$2-4-6-8-10-1-5-9-3-11$
12	9	$2-4-6-8-10-12-3-7-11-5-1$
13	11	$2-4-6-8-10-12-1-5-9-13-7-3$
14	13	$2-4-6-8-10-12-14-3-7-11-1-9-5$
15	15	$2-4-6-8-10-12-14-1-5-9-13-3-11-7$
16	1	$2-4-6-8-10-12-14-16-3-7-11-15-5-13-9$

b) We create a formula for $J(n)$: $n = 2^m + k$.

Exercise 13

a) We apply the default algorithm:

Step 1 $a_n = 1 * a_{n-1}$

Step 2 $r - 1 = 0$

Step 3 $r = 1$

Step 4 $a_n = \alpha_1 * 1^n$

Step 5 $a_n = 2 * 1^n$

So the specific solution is

$$a_n = 2$$

b) We apply the default algorithm:

Step 1 $a_n = -4a_{n-1} - 4a_{n-2}$

Step 2 $r^2 + 4r + 4 = 0$

Step 3 $r = -2$

Step 4 $a_n = \alpha_1 * -2^n + \alpha_2 * n * -2^n = (\alpha_1 + \alpha_2 * n) * -2^n$

Step 5 $a_0 : \alpha_1 = 0$ and
$a_1 : 1 = (\alpha_1 + \alpha_2 * 1) * -2$
$a_1 : -1/2 = \alpha_1 + \alpha_2 * 1$
$a_1 : -1/2 = \alpha_1 + \alpha_2$
In total:
$\alpha_1 = 0$ and $\alpha_2 = -1/2$

So the specific solution is

$$a_n = (-1/2 * n) * -2^n$$

c) We apply the default algorithm:

Step 1 $a_n = 1/4 * a_{n-2} = 0 * a_{n-1} + 1/4 * a_{n-2}$

Step 2 $r^2 - 0r - 1/4 = 0$

Step 3 $r = 1/2$ of $r = -1/2$

Step 4 $a_n = \alpha_1 * (1/2)^n + \alpha_2 * (-1/2)^n$

Step 5 $a_0 : \alpha_1 * (1/2)^0 + \alpha_2 * (-1/2)^0 = 1$ and
$a_1 : \alpha_1 * (1/2)^1 + \alpha_2 * (-1/2)^1 = 0$
In total:
$\alpha_1 = 1/2$ and $\alpha_2 = 1/2$

So the specific solution is

$$a_n = (1/2) * (1/2)^n + (1/2) * (-1/2)^n$$

Exercise 14

Let s_n be the number of lucky numbers less than 10^n.
Note that there are $s_n = 9 * (n-2) + 9 * (n-1)$ types of lucky numbers less than 10^n, namely because we can make 9 combinations of string ending with ...x3 where $x = 2, 3, 4, 5, 6, 7, 8, 9, 0$ and 9 combinations of string ending with ...x where $x = 1, 2, 4, 5, 6, 7, 8, 9, 0$
So the general solution is

$$s_n = 9 * (n-1) + 9 * (n-2)$$

with $s_1 = 10$ and $s_2 = 99$ and $n > 2$.
Now we apply the default algorithm:

Step 1 $s_n = 9 * (n-1) + 9 * (n-2)$

Step 2 $r^2 - 9r - 9 = 0$

Step 3 $r_1 = 9,908$ and $r_2 = -0,908$

Step 4 $a_n = \alpha_1 * 9,908^n + \alpha_2 * -0,908^n$

Step 5 $a_0 : 10 = \alpha_1 * 1 + \alpha_2 * 1 =$
$a_1 : 99 = \alpha_1 * 9,908 + \alpha_2 * -0,908$

When solving this system of equations, we see that α_1 approaches 10 and α_2 approaches 0.

So the specific solution is

$$s_n = 10 * 9,908^n$$

So there are $10 * 9,908^2 3$ lucky numbers that are smaller than or equal to 10^{23}.