HW1 High Dimensional Probability

Kiarash Gheisari Pour 402102302

Q1)

1.1) $\mathbb{E}[\varphi(|X|)]$: since φ is non-decreasing and derivable $\rightarrow$ $\varphi(x) - \varphi(0) \geq 0$

we know that if $X \geq 0$ then $\mathbb{E}X = \int_0^\infty P(X \geq t)\,dt$

therefore we can write: $\mathbb{E}(\varphi|X| - \varphi_{(0)}) = \int_0^\infty \mathbb{P}(\varphi|X| - \varphi_0 > t)\,dt$

if $P(\varphi|X| - \varphi_0 > t) = \mathbb{P}(|X| \geq \varphi^{-1}(\varphi_0 + t))$

say $\varphi^{-1}(\varphi_0 + t) = t'$ $\rightarrow$ $\frac{dt}{dt'} = \frac{d\varphi(t')}{dt'}$

$\rightarrow \mathbb{E}[\varphi|X| - \varphi_0] = \int_0^\infty \varphi'(t')\,\mathbb{P}(|X| > t')\,dt'$

note that $\mathbb{E}[\varphi_0] = \varphi_0$. so that completes the Proof. $\square$.

1.2) since $X \sim subg(\sigma^2)$

then $M_X(\lambda) = \mathbb{E}[e^{\lambda X}] \leq e^{\frac{\lambda^2\sigma^2}{2}}$.

Q1) continued:

1.2) so we want to bound

$$P(X \geq t) = P(e^{\lambda X} > e^{\lambda t}) \underset{\text{markov}}{\leq} \frac{M_X(\lambda)}{e^{\lambda t}} \leq e^{\frac{\lambda^2 \sigma^2}{2} - \lambda t}$$

$$\rightarrow P(X \geq t) \leq \inf_{\lambda > 0} e^{\frac{\lambda^2\sigma^2}{2} - \lambda t} = e^{\frac{-t^2}{2\sigma^2}}$$

similiarly we have: $\mathbb{P}(X \leq -t) = P(e^{\lambda X} \geq e^{-\lambda t})$

$$\leq \frac{M_X(\lambda)}{e^{-\lambda t}} \leq e^{\frac{\lambda^2\sigma^2}{2}} e^{\lambda t} \xrightarrow{\text{minimize}} P(X \leq -t) \leq e^{\frac{-t^2}{2\sigma^2}}$$

therefore: $\mathbb{P}(|X| \geq t) \leq P(X > t) + P(X \leq -t) \leq 2 e^{\frac{-t^2}{2\sigma^2}}$ $\square$.

1.3) $$E(e^{\frac{X^2}{6\sigma^2}}) \leq 1 + \int_0^\infty \frac{t}{3\sigma^2} e^{\frac{t^2}{6\sigma^2}} 2 e^{\frac{-t^2}{2\sigma^2}} dt$$

$$= 1 + 2\int_0^\infty \frac{t}{3\sigma^2} e^{\frac{-t^2}{3\sigma^2}} dt = 1 + 2\int_0^\infty t e^{-t^2} dt$$

$(t = \sqrt{3}\sigma t')$

$$= 1 + 2\left(\tfrac{1}{2}\right) = 2. \quad \square.$$

1.4)

$$e^x \leqslant 1 + x + \frac{x^2}{2} e^{|x|} \quad \text{therefor} \quad \mathbb{E}[e^{\lambda X}] \leqslant 1 + E[\lambda X] + E\,\lambda^2 X^2 e^{|\lambda X|}$$

$$\left(\begin{array}{l}\text{note that} \\ EX = 0\end{array}\right) \to M_X(\lambda) \leqslant 1 + \lambda^2 E X^2 e^{|\lambda X|}$$

$$2|\lambda X| \leqslant \frac{X^2}{6\sigma^2} + 6\sigma^2\lambda^2 \to e^{|\lambda X|} \leqslant e^{\frac{X^2}{12\sigma^2}} e^{3\sigma^2\lambda^2}$$

$$, \; x^2 \leqslant e^{\frac{x^2}{2}} \quad \to \quad X^2 \leqslant \sigma^2 e^{\frac{X^2}{12\sigma^2}}$$

$$\to \quad \mathbb{E} X^2 e^{|\lambda X|} \leqslant e^{(3\sigma^2\lambda^2)} 6\sigma^2 \underbrace{\mathbb{E} e^{\frac{X^2}{6\sigma^2}}}_{\leqslant 2} \leqslant 6\sigma^2 e^{3\sigma^2\lambda^2}$$

$$\to M_X(\lambda) \leqslant 1 + \lambda^2 6\sigma^2 e^{3\sigma^2\lambda^2} \leqslant (1 + \lambda^2)e^{3\sigma^2\lambda^2}$$

$$\leqslant e^{9\sigma^2\lambda^2} = e^{\frac{18\sigma^2\lambda^2}{2}} \quad \text{therefore: } X \sim \text{subg}(\sqrt{18}\sigma)$$

$(1 + u \leqslant e^u)$

1.5) let's try to bound the moments of $X^2 - \mathbb{E}X^2$

by question "6" we know that: $E X^{2K} \leqslant 2(2\sigma^2)^K K!$

$$\mathbb{E}\,|X^2 - EX^2|^K \leqslant 2^{K-1}\left(E X^{2K} + (\mathbb{E}X^2)^K\right)$$

$$(|a-b|^K \leqslant 2^{K-1}(|a|^K, |b|^K)$$

$$(\|X\|_{L_2} \leqslant \|X\|_{L_{2K}}) \to \quad \leqslant 2^K\left(\mathbb{E} X^{2K}\right) = 2(4\sigma^2)^K K!$$

1.5) continued:

now let's try and bound the MGF

lets state the moment bound like this:

$$\mathbb{E}|X^2 - \mathbb{E}X^2|^k \leq 2(4\sigma^2)^k k! = \frac{1}{2}(\underbrace{64\sigma^4}_{\sigma_x^2})\underbrace{(4\sigma^2)^{k-2}}_{b} k!$$

$$\mathbb{E}\, e^{s(X^2 - \mathbb{E}X^2)} \underset{\text{taylor expansion}}{=} 1 + s\mathbb{E}(X^2 - \mathbb{E}X^2) + \frac{s^2}{2}\mathbb{E}(X^2 - \mathbb{E}X^2)^2 + \sum_3^\infty s^k \frac{\mathbb{E}(X^2 - \mathbb{E}X^2)^k}{k!}$$

$$\leq 1 + \frac{s^2\sigma_x^2}{2} + \frac{s^2\sigma_x^2}{2}\sum_3^\infty (|s|b)^{k-2}$$

→ Geometric Sum (converges for $|s| \leq \frac{1}{4\sigma^2}$

$$= 1 + \frac{\frac{s^2\sigma_x^2}{2}}{1 - b|s|} \underset{1+x\leq e^x}{\leq} \exp\left(\frac{32s^2\sigma^4}{1 - 4|s|\sigma^2}\right)$$

$\square$.

note that the above is the def. of subexp that fits for $X^2 - \mathbb{E}X^2$ → $\mathbb{E}X^2 \leq \sigma^2$, $|s| \leq \frac{1}{8\sigma^2}$

→ $e^{s\mathbb{E}X^2} \leq \exp\frac{1}{8}$

→ therefore for $\mathbb{E}e^{sX^2} \leq \exp\frac{1}{8}\exp\left(\frac{32s^2\sigma^4}{1 - 4|s|\sigma^2}\right)$

we can tweak the constants to get:

$$\mathbb{E}e^{sX^2} \leq \exp\left(\frac{cs^2}{1 - 4|s|\sigma^2}\right)$$

$\square$.

Q2) we want to Calculate the distribution and expectation for d_i. (let's name the Cluster set C.

$X_j \sim Ber(\frac{1}{2})$, $d_i \mid i \in C \sim K-1 + \sum^{n-K} X_j$

$d_i \mid i \notin C \sim \sum_1^{n-1} X_j$ and $\mathbb{P}(i \in C) = \frac{\binom{n-1}{K-1}}{\binom{n}{K}} = \frac{K}{n}$

So $P(d_i = D) = \frac{K}{n} \mathbb{P}(Y_1 = D-K-1) + \left(\frac{n-K}{n}\right) \mathbb{P}(Y_2 = D)$

because $\left(P(\omega) = \sum P(\omega \mid \Omega_i) P(\Omega_i)\right)$ $\quad Y_1 \sim \text{Binomial}(n-K, \frac{1}{2})$

Ω_i is a Partition on Ω $\quad Y_2 \sim \text{Binomial}(n-1, \frac{1}{2})$

$$\mathbb{E}(d) = \frac{K}{n}\left(K-1 + \frac{n-K}{2}\right) + \frac{n-K}{n}\left(\frac{n-1}{2}\right)$$

we want to show that the maximum (top) K degrees are most likely the ones that make up the Cluster. in order to do so we'll bound the Probability of the error outcome: $\mathbb{P}\left(\max_{i \notin C} d_i > \min_{j \in C} d_j\right)$

CamScanner

Q2) continued:

$$\left(\frac{K-1}{4}\right) = \frac{\mu_c - \mu_{c'}}{2}$$

$$\mathbb{P}\left(\max_{i \notin C} d_i > \min_{j \in C} d_j\right) \leq \mathbb{P}\left(\exists i \in C' : d_i > \mu_{c'} + \frac{K-1}{4}, \exists j \in C : d_j < \mu_c - \frac{K-1}{4}\right)$$

$$\leq \mathbb{P}\left(\exists i \in C' : d_i > \mu_{c'} + \frac{K-1}{4}\right) + \mathbb{P}\left(\exists j \in C : d_j < \mu_c - \frac{K-1}{4}\right)$$

$P(A \cap B) \leq P(A) + P(B)$

by the first Part of the question we know that d_i have a "Binomial like" Behaviour we can obtain a Hoeffding bound by taking them as Sub-g R.V.s

for $v \in C$ $\quad \deg(v) \sim \mathrm{subg}\left(\sigma^2 = \frac{(n-K)}{4}\right)$

$u \in C'$ $\quad \deg(u) \sim \mathrm{subg}\left(\sigma^2 = \frac{n-1}{4}\right)$

Ber : $\mathrm{subg}\left(\frac{1}{4}\right)$

therefore... $\leq n-K \, \mathbb{P}\left(d_i > \mu_{c'} + \frac{K-1}{4}\right) + K \, \mathbb{P}\left(d_j < \mu_c - \frac{K-1}{4}\right)$

Hoeffding: $\leq (n-K) \, e^{\frac{-2\left(\frac{K-1}{4}\right)^2}{n-1}} + K \, e^{\frac{-2\left(\frac{K-1}{4}\right)^2}{n-k}}$

$\leq n \, e^{\frac{-2\left(\frac{K-1}{4}\right)^2}{n-1}}$, $K = O(\sqrt{n \log n}$

$= O(1)$ we can take c so that this goes to zero

Q3) X_i iid ; $X_i \sim Ber(\frac{1}{2}+\delta)$

$S_N = \sum_1^N X_i$, we want to bound $\mathbb{P}(S_N < \frac{N}{2})$

by applying chernoff inequality

to sums of bernoulli random variables

we can achieve the following bound:

$\mathbb{P}(S_N \leq t) \leq e^{-\mu}\left(\frac{e\mu}{t}\right)^t$ for $0<t\leq\mu$ (vershynin P.30)

$\mu = \mathbb{E} S_N = n(\frac{1}{2}+\delta)$

$\rightarrow \mathbb{P}(S_N < \frac{N}{2}) \leq e^{-\frac{N}{2}} e^{-N\delta} \left(\frac{eN(\frac{1}{2}+\delta)}{\frac{N}{2}}\right)^{\frac{N}{2}}$

$= e^{-N\delta}(1+2\delta)^{\frac{N}{2}} \leq \varepsilon$

$\rightarrow \dfrac{\ln(\frac{1}{\varepsilon})}{\delta - \frac{\ln(1+2\delta)}{2}} \leq N$ (taylor series: $\ln(1+2\delta) = 2\delta - 2\delta^2 + o(\delta^2)$)

$\rightarrow N \geq \dfrac{\ln(\frac{1}{\varepsilon})}{\delta^2 + o(\delta^2)} \geq \dfrac{\ln(\frac{1}{\varepsilon})}{2\delta^2}$

Q4)

it's evident that if $X \sim subg(\sigma^2)$

$$: \mathbb{P}(X - \mathbb{E}[X] > t) \leq e^{-\frac{t^2}{2\sigma^2}}$$

$$P\left(\max_{i \in \{1:n\}} \{X_i - EX_i\} > t\right) \underset{\text{(union bound)}}{\leq} n \, e^{-\frac{t^2}{2\sigma^2}}$$

$$t = (1+\varepsilon)\sigma\sqrt{2 \log n}$$

$$\leq n \, e^{-\frac{(1+\varepsilon)^2 \sigma^2 2 \log n}{2\sigma^2}} = n \times n^{-(1+\varepsilon^2)}$$

$= n^{-(\varepsilon^2 + 2\varepsilon)}$. for every $\varepsilon > 0$ this bound goes to zero

Q5)

5.1) $Z_n = \sum X_i \quad , X_i \sim Bernoulli(\alpha) \quad , iid$

$$\mathbb{P}(Z_n \leq n\delta) = \mathbb{P}(e^{\lambda Z_n} \geq e^{n\lambda\delta}) \leq \prod M_x(\lambda) \times e^{-n\lambda\delta}$$

$$(M_x(\lambda) = \mathbb{E}(e^{\lambda X_i}) = (1-\alpha) + \alpha e^{\lambda}) \qquad = (1+\alpha(e^{\lambda}-1))^n e^{-n\lambda\delta}$$

$$= \underbrace{\left(e^{-\lambda\delta}(1+\alpha(e^{\lambda}-1))\right)^n}_{L(\lambda)}$$

$L(\lambda)$ we want to minimize this w.r.t. λ $\rightarrow \log(L(\lambda)) = -\lambda\delta + \log(1+\alpha(1-e^{\lambda}))$

$$\frac{d}{d\lambda}(\log loss) = -\delta + \frac{\alpha e^{\lambda}}{(1-\alpha)+\alpha e^{\lambda}} \rightarrow e^{\lambda^*} = \frac{\delta}{\alpha}\,\frac{1-\alpha}{1-\delta}$$

$$L(\lambda^*) = -\left(\delta \log\frac{\delta}{\alpha} + (1-\delta)\log\frac{1-\delta}{1-\alpha}\right) = -D(\delta \| \alpha)$$

$$\rightarrow P(Z_n \leq n\delta) \leq e^{-nD(\delta\|\alpha)}$$

to show that this bound is strictly better than Hoeffding bound we take the following steps. we know that Hoeffding come from Chernoff and a rather crude estimation of MGF at least in the case of the Bernoulli random var.

(5) continued:

therefore: $\mathbb{P}(Z_n \leq \delta n) \leq \frac{M_x(\lambda)}{e^{\lambda\delta n}} < e^{\frac{n\lambda^2}{8}} e^{-\lambda\delta n}$

KL-Bound $\qquad$ Hoeff. Bound $\qquad$ $F_1(\lambda)$ $\qquad$ $F_2(\lambda)$

$\rightarrow \inf_\lambda F_1(\lambda) < \inf_\lambda F_2(\lambda)$

5.2) lower bound:

$$P(Z_n \leq n\delta) \geq P(Z_n = \lfloor n\delta \rfloor)$$

$$\mathbb{P}(Z_n = m) = \binom{n}{m} \alpha^m (1-\alpha)^{n-m}$$

$m = \lfloor n\delta \rfloor$

$$\rightarrow \frac{1}{n} \log P(Z_n \leq n\delta) \geq \frac{1}{n} \log\binom{n}{m} + \tilde{\delta} \log \alpha + (1-\tilde{\delta}) \log 1-\alpha$$

$\square$.

5.2.2) $1 = \sum_{l}^{n} P(Y = l) \leq n+1 \binom{n}{m} (\tilde{\delta})^m$

$Y \sim \text{Binomial}(n, \tilde{\delta})$

$$\rightarrow 0 \leq \log(n+1) + \log\binom{n}{m} + \frac{m}{} \log \tilde{\delta} + n-m \log 1-\tilde{\delta}$$

$$\rightarrow \frac{1}{n} \log\binom{n}{m} \geq -\left(\tilde{\delta} \log \tilde{\delta} + (1-\tilde{\delta}) \log(1-\tilde{\delta})\right) - \frac{\log n+1}{n}$$

5.2.3) $\frac{1}{n} \log P(Z_n < \delta n) \geq -\frac{\log n+1}{n} + \tilde{\delta} \log \alpha - \tilde{\delta} \log \tilde{\delta}$ $\square$.

$+ (1-\tilde{\delta}) \log 1-\alpha - (1-\tilde{\delta}) \log(1-\tilde{\delta})$

Q5) continued:

$$= -\frac{\log n+1}{n} + \tilde{\delta} \log \frac{\alpha}{\tilde{\delta}} + (1-\tilde{\delta}) \log\left(\frac{1-\alpha}{1-\tilde{\delta}}\right)$$

$$m = n\tilde{\delta} = \lfloor n\delta \rfloor \rightarrow \quad n\tilde{\delta} = n\delta - c$$

$$0 \leqslant c < 1$$

$$\rightarrow \tilde{\delta} = \delta - \frac{c}{n}$$

$$\rightarrow = -\frac{\log n+1}{n} - D(\delta \| \alpha) \underbrace{- \frac{c}{n} \Big(\underbrace{\log \overset{>1}{\frac{\alpha}{\delta}} + \log \overset{>1}{\frac{1-\delta}{1-\alpha}}}_{>0}}_{<0}$$

neglecting this small term yields:

$$P(Z_n \leqslant n\delta) \geqslant \frac{1}{n+1} e^{nD(\delta \| \alpha)}$$

Q6)

6.1) $\mathbb{E}[X^{2q}] = \int_0^\infty P(X^{2q} > t)\,dt = \int_0^\infty P(|X| > u)\, 2q u^{2q-1} du$

$u^{2q} = t$

$P(|X| > u) \leqslant e^{\frac{-u^2}{2\sigma^2}} \rightarrow \mathbb{E}[X^{2q}] \leqslant \int_0 e^{\frac{-u^2}{2\sigma^2}} 2q u^{2q-1} du$

$X \sim sub\text{-}g(\sigma^2)$

$u^2 = 2\sigma^2 t \rightarrow \quad = 2q \int_0^\infty e^{-t} (2\sigma^2 t)^{q-\frac{1}{2}} \frac{\sqrt{2\sigma^2}}{2} \frac{1}{\sqrt{t}} dt$

$= 2q (2\sigma^2)^q \int_0^\infty e^{-t} t^{q-1} dt = 2(2\sigma^2)^q\, q\, \Gamma(q)$

$(q \in \mathbb{N})$

$= 2(2\sigma^2)^q\, q!$

6.2) $\max |X_i| \leqslant \left(\sum |X_i|^p\right)^{1/p}$

$\rightarrow \mathbb{E}[\max |X_i|] \leqslant \mathbb{E}[(\sum |X_i|^p)^{1/p}] \leqslant \mathbb{E}[\sum |X_i|^p]^{\frac{1}{p}}$

Jensen ($1/p$ is a concave function)

from 6.1) $X \sim subg(\sigma^2) \rightarrow \mathbb{E}[|X_i|^p]^{1/p} \leqslant C\sigma\sqrt{p}$

$\rightarrow \leqslant C\sigma \sqrt{p}\, n^{\frac{1}{p}} \rightarrow \inf \sqrt{p}\, n^{\frac{1}{p}} \rightarrow \frac{d}{dp}\left(\frac{\log p}{2} + \frac{\log n}{p}\right)$

$= \frac{1}{2p} - \frac{\log n}{p^2} \rightarrow p = 2\log n \rightarrow C\sigma \sqrt{2\log n}\; \underbrace{n^{\frac{1}{2\log n}}}_{\sqrt{e}} = C'\sigma\sqrt{\log n}$

$\square$.

Q7) first note that $\|a^T X\|_{L_p} = (\mathbb{E}\,|a^T X|^p)^{1/p}$

now we show that the L_p norm is

monotonic. meaning if $p > q \Leftrightarrow \|X\|_{L_p} > \|X\|_{L_q}$

Jensen: $\phi[E[X^q]] \leqslant E[\phi(X^q)] \rightarrow (\mathbb{E}\,X^q)^{p/q} \leqslant EX^p$

$\phi(x) = x^{\frac{p}{q}}$
(convex)

$\rightarrow (\mathbb{E}\,X^q)^{1/q} \leqslant (\mathbb{E}\,X^p)^{1/p}$

$\|X\|_{L_q} \leqslant \|X\|_{L_p}$

$$\|a^T X\|_{L_2} = \Big(\sum a_i^2 \underbrace{\mathbb{E}X^2}_{1}\Big)^{\frac{1}{2}} = \sqrt{a^T a}$$

note that $E[X_i X_j] = 0$ because of ind.

$\sqrt{a^T a} \Rightarrow \|a^T X\|_{L_2} \leqslant \|a^T X\|_{L_p} \rightarrow$ left ineq. is proven.

$$E e^{t(a^T X)} = \prod e^{t a_i X_i} \leqslant \prod e^{\frac{a_i^2 t^2 \sigma_i^2}{2}} \leqslant e^{\frac{t^2 (\max \sigma_i^2)\, a^T a}{2}}$$

by (Q6) and vershynin 2.6.6 we know: $\|X\|_{L_p} \leqslant C \|X\|_{\psi_2} \sqrt{p}$

therefore we have $\|a^T X\|_{L_p} \leqslant C \underbrace{6 \max \sigma_i^2}_{\max \|X\|_{\psi_2}} \sqrt{p}\, \sqrt{a^T a}$

from Q1.3!

therefore the right ineq

is also proven $\square$.

Q8) Lemma : $\|X\|_{L_2} \leq \|X\|_{L_1}^{\frac{1}{4}} \|X\|_{L_3}^{\frac{3}{4}}$

Proof: $(\mathbb{E}\, X^{\frac{1}{2}} X^{\frac{3}{2}})^2 \leq \mathbb{E} X \; \mathbb{E} X^3$ Cauchy-Schwartz

$$\mathbb{E}|a^T X| = \|a^T X\|_{L_1} \leq \|a^T X\|_{L_2} = \sqrt{a^T a}$$

monotonicity of norm

from Q7

so the right ineq. is proven

for the left side we have:

$$\frac{\|a^T X\|_{L_2}^4}{\|a^T X\|_{L_3}^3} \leq \|a^T X\|_{L_1} = \mathbb{E}|a^T X|$$

$$\|a^T X\|_{L_3} \leq C K \sqrt{a^T a}\, \sqrt{3} \quad \text{(from Q7)}$$

$$\rightarrow \quad \|a^T X\|_{L_1} \geq \frac{(\sqrt{a^T a})^4}{C' K^3 (\sqrt{a^T a})^3} = C'' K^{-3} \sqrt{a^T a} \quad \square.$$

Q9)

9.1) in order to show that there exists a vertex with degree higher than $10\bar{d}$ we are going to use a method named the secondorder method namely: $X \geq 0$ than $P(X > 0) \geq \frac{(EX)^2}{EX^2}$

note that $\deg(v_i) = \sum_{1}^{n-1} Y_i$ $\quad Y_i \sim \text{Bernoulli}(p)$

in this Problem, we're gonna define our graph like so:

$$G(n, p), \quad p = \frac{d}{n-1}, \quad d = O(\log n)$$

$$A = \{1, \dots, \lfloor n/2 \rfloor\}, \quad B = \{n, \dots, \lfloor n/2 \rfloor + 1\}$$

$$\forall i \in A \quad X_i := \#\{j \in B : \{i, j\} \text{ is an edge}\}$$

its evident that if $\exists i \in A : \deg(v_i) \geq 10\bar{d}$ then the condition for our original Problem is also satisfied.

define $I_i := \mathbb{1}_{\{X_i > 10\bar{d}\}}$

$$Y := \sum_{i \in A} I_i$$

Q9) continued: let's calculate

the $\begin{pmatrix}\text{second} \\ \text{first, and}\end{pmatrix}$ moments of Y

$$EY = \lfloor \tfrac{n}{2} \rfloor P(X_i \geq 10d) \geq P(X_i = \lceil 10d \rceil) \lfloor \tfrac{n}{2} \rfloor$$

$(X_i \sim Bin(m, p))$ $\quad \lceil 10d \rceil = k$

$m = |B|$

$$P(X_i = k) = \binom{m}{k} p^k (1-p)^{m-k}$$

$$(1-p)^{m-k} \geq \exp(-(m-k)\,2p) = \exp\left(-(O(n) - O(\lg n))\, O\tfrac{\lg n}{n}\right)$$

$$= \exp - O \lg n = \exp(-(O(d)))$$

$$\binom{m}{k} \geq \left(\frac{m}{k}\right)^k$$

they cancel out

$$\lg P(X_i = k) \geq \overbrace{k \lg \frac{m}{k} + k \lg p}^{\text{they cancel out}} + -O(d)$$

$(\lg n - \lg d + O1)$ $\quad (\lg d - \lg n + O(1))$

$$= -O(d)$$

$$\rightarrow P(X_i = k) \geq e^{-cd} \quad \text{for some } c > 0$$

$$\rightarrow EY \geq \frac{n}{2} e^{-cd} \text{ - diverges}$$

because $d = o \lg n$ we can set it $d \; \cancel{\geq} ((1+\varepsilon)/c) \lg n$

Q9) continued:

the beauty of partitioning the graph comes in now when we don't have to deal with correlation betwen the degrees

$$EY^2 = \text{Var}\,Y + (EY)^2 = |A|q(1-q) + |A|^2 q^2$$

$$\left(q = P(X_i \geq 10d) \geq e^{-Cd} \right)$$

$$P(Y > 0) \geq \frac{|A|^2 q^2}{|A|^2 q^2 + |A| q (1-q)} = \frac{|A| q}{|A| q + (1-q)}$$

$$\lim_{n \to \infty} \frac{|A| q}{|A| q + (1-q)} = 1$$ (because $|A| q$ diverges)

$\square$.

9.2) we'll use same exact argument here just with $d = O(1)$

and our treshold: $K_n = \lfloor \alpha \frac{\log n}{\log\log n} \rfloor$

$q_n = \mathbb{P}(X_i \geq K_n) \geq \mathbb{P}(X_i = K_n)$

Poisson approximation

$$= \frac{(\mu')^{K_n} e^{-\mu'}}{K_n!} \simeq \exp(-K_n \log K_n + O(K_n))$$

$$K_n \log K_n \simeq \alpha \log n$$

$$\rightarrow \geq e^{-\alpha \log n + O(\log n)} = n^{-\alpha + o(1)}$$

so we can tweak α to make the bound diverge

$(n \simeq n^{(1-\alpha)})$

therefore by a similar argument to last question

$$P(Y > 0) > \frac{(EY)^2}{\operatorname{Var} Y + (EY)^2} \longrightarrow 1$$