Question 2: Derivation of the Generalized Black Scholes PDE

Consider an underlying asset S_t following

$$dS_t = \mu S_t \, dt + \sqrt{v_t} \, S_t \, dZ_t,$$

where Z_t is a standard Brownian motion and $v_t = \sigma_t^2$ is the variance process. Assume v_t follows a square-root diffusion of the form

$$dv_t = \kappa(\theta - v_t) \, dt + \gamma \sqrt{v_t} \, dW_t,$$

where W_t is another Brownian motion such that

$$\mathrm{Cov}(dZ_t, dW_t) = \rho \, dt.$$

Solution
Let $U(S, v, t)$ denote the price of a derivative contingent on S_t and v_t.
By Itô's Lemma,

$$dU = U_t \, dt + U_S \, dS + U_v \, dv + \tfrac{1}{2} U_{SS} (dS)^2 + \tfrac{1}{2} U_{vv} (dv)^2 + U_{Sv} \, dS \, dv.$$

We compute the second-order terms:

$$\begin{aligned}
(dS)^2 &= vS^2 \, dt, \\
(dv)^2 &= \gamma^2 v \, dt, \\
dS \, dv &= \sqrt{v} S \, dZ \; \cdot \; \gamma \sqrt{v} \, dW = \gamma v S \, \rho \, dt.
\end{aligned}$$

Substituting into dU *by* dS_t *and* dv_t

$$\begin{aligned}
dU = &\Big[U_t + \mu S U_S + \kappa(\theta - v) U_v + \tfrac{1}{2} v S^2 U_{SS} + \rho \gamma v S U_{Sv} + \tfrac{1}{2} \gamma^2 v U_{vv} \Big] dt \\
&+ \sqrt{v} S U_S \, dZ + \gamma \sqrt{v} U_v \, dW.
\end{aligned}$$

Under the risk-neutral measure $\mathbb{Q}$:

- the asset drift becomes r, i.e. $\mu = r$,
- the variance drift includes a risk-premium adjustment, so

  $$dv_t = \big[\kappa(\theta - v_t) - \lambda v_t \big] \, dt + \gamma \sqrt{v_t} \, dW_t,$$

  where λ is the market price of variance risk.

Thus the drift of U under $\mathbb{Q}$ is

$$\mathrm{drift}(dU) = U_t + rSU_S + \big[\kappa(\theta - v) - \lambda v \big] U_v + \tfrac{1}{2} v S^2 U_{SS} + \rho \gamma v S U_{Sv} + \tfrac{1}{2} \gamma^2 v U_{vv}.$$

Given the No-arbitrage condition:
The discounted derivative price $e^{-rt} U(S_t, v_t, t)$ must be a martingale.
Hence the drift of U must equal rU. Setting drift $-rU = 0$ yields the PDE:

$$\tfrac{1}{2} v S^2 U_{SS} + \rho \gamma v S U_{Sv} + \tfrac{1}{2} \gamma^2 v U_{vv} + rSU_S + \big[\kappa(\theta - v) - \lambda v \big] U_v - rU + U_t = 0.$$