Your notes

# Cambridge (CIE) A Level Maths: Probability & Statistics 2

## Linear Combinations of Random Variables

### Contents

Your notes

# aX+b

## How are the mean and variance of X related to the mean and variance of aX + b?

- If $a$ and $b$ are **constants** then the following results are true
  - $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$
  - $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$
- Note that the mean is affected by multiplication and addition whereas **addition does not change the variance**
- The factor of $a^2$ includes the squared because the values of X are squared in the calculation
  - You could try and use the first result and the formula for variance to verify the second result
- Remember a subtraction can be written as an addition
  - $X - b$ can be written as $X + (-b)$
- And division can be written as a multiplication
  - $\frac{X}{a}$ can be written as $\frac{1}{a}X$

## What does the distribution of aX + b look like?

- A linear function is applied to each value of $X$
- The graphical representation of $\boldsymbol{aX + b}$ is a linear transformation (a translation and a stretch) of the graphical representation of $\boldsymbol{X}$
- If $\boldsymbol{X}$ follows a **normal** distribution then $\boldsymbol{aX + b}$ will also follow a **normal** distribution
  - If $X \sim \mathrm{N}(\mu, \sigma^2)$ then $aX + b \sim \mathrm{N}(a\mu + b, a^2\sigma^2)$
- If $X$ follows a binomial, geometric or Poisson distribution then $aX + b$ will no longer follow the same type of distribution

### Worked Example

$X$ is a random variable such that $E(X) = 5$ and $\mathrm{Var}(X) = 4$.

Find the value of:

(i) $\mathrm{E}(3X + 5)$

(ii) $\text{Var}(3X+5)$

(iii) $\text{Var}(2-X)$

(i) Use "$E(aX + b) = aE(X) + b$"

$E(3X + 5) = 3E(X) + 5$

$= 3(5) + 5$

$E(3X + 5) = 20$

(ii) Use "$\text{Var}(aX + b) = a^2\text{Var}(X)$"

$\text{Var}(3X + 5) = 3^2\text{Var}(X)$

$= 9(4)$

$\text{Var}(3X + 5) = 36$

(iii) Rewrite in form $aX + b$

$2 - X = (-1)X + 2$

$\text{Var}((-1)X + 2) = (-1)^2\text{Var}(X)$

$= 1(4)$

$\text{Var}(2 - X) = 4$



Your notes

# aX + bY

## How are the means and variances of X and Y related to the mean and variance of X + Y ?

- If *X* and *Y* are two **random variables** then *X* + *Y* is the random variable whose values are the sums of each pair containing one value of *X* and one value of *Y*
- **E(*X* + *Y*) = E(*X*) + E(*Y*)**
  - this is true for **any** random variables *X* and *Y*
  - Note that **E(*X* – *Y*) = E(*X*) – E(*Y*)** (see below for more information)
- **Var(*X* + *Y*) = Var(*X*) + Var(*Y*)**
  - this is true if *X* and *Y* are **independent**
  - Note that **Var(*X* – *Y*) = Var(*X*) + Var(*Y*)** (see below for more information)

## What does the distribution of X + Y look like?

- If ***X*** and ***Y*** are **two independent Poisson** distributions then ***X* + *Y*** is also a **Poisson** distribution
  - If $X \sim Po(\lambda)$ and $Y \sim Po(\mu)$ then $X + Y \sim Po(\lambda + \mu)$
- If ***X*** and ***Y*** are **two independent normal** distributions then ***X* + *Y*** is also a **normal** distribution

- If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ then $X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

## What does the distribution of aX + bY look like?

- If $X$ and $Y$ are random variables and $a$ and $b$ are two constants we can combine the results for $aX + b$ and $X + Y$
- **E($aX + bY$) = $a$E($X$) + $b$E($Y$)**
  - this is true for **any** random variables $X$ and $Y$
- **Var($aX + bY$) = $a^2$Var($X$) + $b^2$Var($Y$)**
  - this is true if $X$ and $Y$ are **independent**
- Note that b is squared for the variance so we have
  - **E($aX - bY$) = $a$E($X$) - $b$E($Y$)**
  - **Var($aX - bY$) = $a^2$Var($X$) + $b^2$Var($Y$)**
    - Notice that the variances of $\boldsymbol{aX + bY}$ and $\boldsymbol{aX - bY}$ are the same
- If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are **two independent normal** distributions then $\boldsymbol{aX + bY}$ is also a **normal** distribution
  - If $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ then $aX \pm bY \sim N(a\mu_1 \pm b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$
- Note that $aX + bY$ is **no longer Poisson** even if $X$ and $Y$ are Poisson
  - This holds provided $a$ and $b$ are not 0 or 1

### Worked Example

$X$ and $Y$ are independent random variable such that

$\mathrm{E}(X) = 5$ and $\mathrm{Var}(X) = 3$

$\mathrm{E}(Y) = -2$ and $\mathrm{Var}(Y) = 4$.

Find the value of:

(i) $\mathrm{E}(2X + 5Y)$

(ii) $\mathrm{Var}(2X + 5Y)$

(iii) $\mathrm{Var}(4X - Y)$

Your notes

(i) Use "$E(aX + bY) = aE(X) + bE(Y)$"

$E(2X + 5Y) = 2E(X) + 5E(Y)$

$= 2(5) + 5(-2)$

$E(2X + 5Y) = 0$

(ii) Use "$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y)$"

$Var(2X + 5Y) = 2^2 Var(X) + 5^2 Var(Y)$

$= 4(3) + 25(4)$

$Var(2X - 5Y) = 112$

(ii) Rewrite in form $aX + bY$

$4X - Y = 4X + (-1)Y$

Or you could remember and use $Var(aX - bY) = a^2 Var(X) + b^2 Var(Y)$

$Var(4X + (-1)Y) = 4^2 Var(X) + (-1)^2 Var(Y)$

$= 16(3) + 1(4)$

$Var(4X - Y) = 52$



Your notes

# Linear Combinations

## For a given random variable X, what is the difference between 2X and $X_1 + X_2$?

- **$2X$** means **one observation** of $X$ is taken and **then doubled**
- **$X_1 + X_2$** means **two observations** of $X$ are taken and **added together**
- **$2X$** and **$X_1 + X_2$** both have the **same expected value** of **$2E(X)$**
- **$2X$** and **$X_1 + X_2$** have **different variances**
  - **$Var(2X) = 2^2 Var(X) = 4Var(X)$**
  - **$Var(X_1 + X_2) = 2Var(X)$**
- Imagine $X$ could take the values 0 and 1
  - $2X$ could then take the values 0 and 2 ($2 \times 0 = 0$ and $2 \times 1 = 2$)
  - $X_1 + X_2$ could then take the values 0, 1 and 2 ($0 + 0 = 0$, $0 + 1 = 1$, $1 + 1 = 2$)
- Sometimes questions may describe the variables in context
  - The mass of a carton of half a dozen eggs is the mass of the carton plus the mass of the 6 **individual** eggs and can be modelled using the random variable
  - $C + E_1 + E_2 + E_3 + E_4 + E_5 + E_6$ where
    - $C$ is the mass of a carton
    - $E$ is the mass of an egg
  - It is **not** $C + 6E$ because the masses of the 6 eggs could be different

## How do I use linear combinations of normal random variables to find probabilities?

Your notes

- If the random variables are **normally distributed** and **independent** you might be asked to find probabilities such as
  - $P(X_1 + X_2 + X_3 > 2Y + 5)$
  - This could be given in words
    - Find the probability that the mass of three chickens ($X$) is more than 5 kg heavier than double the mass of a turkey ($Y$)
- To solve these problems:
  - **STEP 1**: **Rearrange** the inequality to get all the **random variables on one side**
    - $P(X_1 + X_2 + X_3 - 2Y > 5)$
  - **STEP 2**: Find the **mean and variance** of the **combined normal random variable**
    - $\mu = E(X_1 + X_2 + X_3 - 2Y) = E(X_1) + E(X_2) + E(X_3) - 2E(Y)$
    - $\sigma^2 = Var(X_1 + X_2 + X_3 - 2Y) = Var(X_1) + Var(X_2) + Var(X_3) + 2^2\,Var(X_1)$
  - **STEP 3**: Find the **required probability** using the **combined normal distribution**
    - $X_1 + X_2 + X_3 - 2Y \sim N(\mu, \sigma^2)$
    - Use z-values and the table of values

### Worked Example

$X \sim N(10,\, 4^2)$ and $Y \sim N(-5,\, 8^2)$

Find $P(3X > 2Y + 50)$

Step 1: Rearrange

$P(3X > 2Y + 50) = P(3X - 2Y > 50)$

Step 2: Find $\mu$ and $\sigma^2$ of the combined variable

Using "$E(aX + bY) = aE(X) + bE(Y)$"

$\mu = E(3X - 2Y) = 3(10) - 2(-5) = 40$

Using $Var(aX - bY) = a^2Var(X) + b^2Var(Y)$

$\sigma^2 = Var(3X - 2Y) = 3^2(4^2) + 2^2(8^2) = 400$

Step 3: Find the probability using the normal distribution

$3X - 2Y \sim N(40, 400)$

$\sigma = \sqrt{400} = 20$

$$P(3X - 2Y > 50) = P\left(Z > \frac{50 - 40}{20}\right)$$

$= P(Z > 0.5)$

$= 0.3085$

$0.309$ (3sf)



Your notes

### Examiner Tips and Tricks

- Be careful with negatives!