Cambridge (CIE) A Level Maths: Probability & Statistics 2

Your notes

Hypothesis Testing (Binomial & Poisson Distributions)

Contents

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Binomial Hypothesis Testing

How is a hypothesis test carried out with the binomial distribution?

- The **population parameter** being tested will be the **probability, *p*** in a **binomial distribution** $B(n, p)$
- A hypothesis test is used when the assumed probability is questioned
- The **null hypothesis**, H_0 and **alternative hypothesis**, H_1 will always be given in terms of p
  - Make sure you clearly define p before writing the hypotheses
  - The null hypothesis will always be $\mathbf{H_0 : p = \ldots}$
  - The alternative hypothesis will depend on if it is a one-tailed or two-tailed test
  - A one-tailed test would test to see if the **value of *p*** has **either increased or decreased**
    - The **alternative hypothesis, H_1 will be $H_1 : p > \ldots$ or $H_1 : p < \ldots$**
  - A two-tailed test would test to see if the **value of *p*** has **changed**
    - The **alternative hypothesis, H_1 will be $H_1 : p \neq \ldots$**
- To carry out a hypothesis test with the binomial distribution, the **random variable** used in the testwill be the number of **successes** in a defined number of **trials**
- When defining the distribution, remember that the value of p is being tested, so this should be written as p in the original definition, followed by the null hypothesis stating the assumed value of p
- The binomial distribution will be used to **calculate the probability** of the test statistic taking the observed value **or a more extreme value**
- The hypothesis test can be carried out by
  - either calculating the probability of the test statistic taking the observed or a more extreme value (**p – value**) and comparing this with the significance level
  - or by finding the critical region and seeing whether the observed value of the test statistic lies within it
    - Finding the critical region can be more useful for considering more than one observed value or for further testing

How is the critical value found in a hypothesis test with the binomial distribution?

- The **critical value** will be the **first value** to fall within the **critical region**

- The binomial distribution is a **discrete** distribution so the probability of the observed value being within the critical region, given a true null hypothesis may be less than the **significance level**
- This is the **actual significance level** and is the probability of **incorrectly** rejecting the null hypothesis
- For a **one-tailed test** use your calculator to find the first value for which the probability of that **or a more extreme value** is less than the given significance level
  - Check that the next value would cause this probability to be greater than the significance level
    - For **H$_1$:** p < ... if $P(X \le c) \le \alpha\%$ and $P(X \le c+1) \le \alpha\%$ then c is the critical value
    - For **H$_1$:** p > ... if $P(X \ge c) \le \alpha\%$ and $P(X \ge c-1) > \alpha\%$ then c is the critical value
- For a **two-tailed test** you will need to find **both critical values,** one at each end of the distribution
  - Find the first value for which the probability of that **or a more extreme value** is less than **half** of the given significance level in both the upper and lower tails
    - Often one of the critical regions will be much bigger than the other
    - If the probability in the null hypothesis is 0.5 the critical regions will have an equal size

What steps should I follow when carrying out a hypothesis test with the binomial distribution?

- **Step 1.** Define the probability, p
- **Step 2.** Write the null and alternative hypotheses clearly using the form

$$H_0 : p = ...$$

$$H_1 : p = ...$$

- **Step 3.** Define the distribution, usually $X \sim B(n,p)$ where n is a defined number of trials and p is the population parameter to be tested
- **Step 4.** Calculate either the **critical value(s)** or the **necessary probability** for the test
- **Step 5.** Compare the observed value of the test statistic with the critical value(s) or the probability with the significance level
- **Step 6.** Decide whether there is enough evidence to reject H$_0$ or whether it has to be accepted
- **Step 7.** Write a conclusion in context

Worked Example

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Jacques, a breadmaker, claims that more than 60% of people that shop in a particular supermarket buy his brand of bread. Jacques takes a random sample of 12 customers that have purchased bread and asks them which brand of bread they have purchased. He records that 10 of them had purchased his brand of bread. Test, at the 10% level of significance, whether Jacques' claim is justified.

Step 1: Define the probability, p

Let p be the proportion of people who purchase Jacques' brand of bread

Step 2: Write the hypotheses

$$H_0 : p = 0.6$$
$$H_1 : p > 0.6$$

Step 3: Distribution different

For $X \sim B(12, p)$

$n = 12$

p is being tested so do not use 0.6 here

Step 4: Calculate the probability

Assuming H_0 is true, then

$$P(X \geqslant 10) = \binom{12}{10}(0.6)^{10}(0.4)^2 + \binom{12}{11}(0.6)^{11}(0.4) + 0.6^{12}$$

$$= 0.08344...$$

Step 5: Compare the probability and calculation different with the given significance level

$$0.08344... < 10\%$$

Step 6: Decide whether there is enough evidence to reject H_0

The probability calculated is less than the significance level so there is enough evidence to reject H_0

Step 7: Write a conclusion in context

There is sufficient evidence to reject the null hypothesis at the 10% level of significance

The test supports Jacques' claim that more than 80% of people who shop in the supermarket buy his brand of bread

* Note: If finding the critical region, steps 4 and 5 would be:

Step 4: Calculate the critical value

Assuming H_0 is true, then $P(X \geqslant 10) = 0.08344... < 10\%$
and $P(X \geqslant 9) = 0.22533... > 10\%$

So the critical value is 10% and the critical region is $X \geqslant 10\%$

Step 5: Compare the observed value with the critical region:

10 is inside the critical region

Examiner Tips and Tricks

- If the question doesn't tell you which method to use then you can choose whether to calculate the probability or find the critical region. Choose the method that will require the least calculations, this will usually be finding the probability and comparing with the significance level.

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Poisson Hypothesis Testing

How is a hypothesis test carried out for the mean of a Poisson distribution?

- The **population parameter** being tested will be the **mean, λ**, in a **Poisson distribution**
  - As it is the population mean, sometimes μ will be used instead
- A hypothesis test is used when the mean is questioned
- The **null hypothesis**, H_0 and **alternative hypothesis**, H_1 will be given in terms of λ (or μ)
  - Make sure you clearly define λ before writing the hypotheses
  - The null hypothesis will always be $\mathbf{H_0}: \lambda = \ldots$
  - The alternative hypothesis will depend on if it is a one-tailed or two-tailed test
  - A one-tailed test would test to see if the **value of λ** has **either increased or decreased**
    - The **alternative hypothesis, will be H_1 will be $H_1: \lambda > \ldots$ or $H_1: \lambda < \ldots$**
  - A two-tailed test would test to see if the **value of λ** has **changed**
    - The **alternative hypothesis, H_1 will be $H_1: \lambda \neq \ldots$**
- To carry out a hypothesis test with the Poisson distribution, the **random variable** will be the mean number of occurrences of the event within the given time/space interval
  - Remember you may need to change the mean to fit the interval of time or space for your observed value
- When defining the distribution, remember that the value of λ is being tested, so this should be written as λ in the original definition, followed by the null hypothesis stating the assumed value of λ
- The Poisson distribution will be used to **calculate the probability** of the random variable taking the observed value **or a more extreme value**
- The hypothesis test can be carried out by
  - either calculating the probability of the random variable taking the observed or a more extreme value and comparing this with the significance level
  - or by finding the critical region and seeing whether the observed value of the test statistic lies within it
    - Finding the critical region can be more useful for considering more than one observed value or for further testing

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How is the critical value found in a hypothesis test with the Poisson distribution?

- The **critical value** will be the **first value** to fall within the **critical region**
  - The Poisson distribution is a **discrete** distribution so the probability of the observed value being within the critical region, given a true null hypothesis may be less than the **significance level**
  - This is the **actual significance level** and is the probability of **incorrectly** rejecting the null hypothesis (a Type I error)
- For a **one-tailed test** use the formula to find the first value for which the probability of that **or a more extreme value** is less than the given significance level
  - Check that the next value would cause this probability to be greater than the significance level
    - For $H_1: \lambda < ...$ if $P(X \le c) \le \alpha\%$ and $P(X \le c+1) > \alpha\%$ then c is the critical value
    - For $H_1: \lambda > ...$ if $P(X \ge c) \le \alpha\%$ and $P(X \ge c-1) > \alpha\%$ then c is the critical value
  - Using the formula for this can be time consuming so only use this method if you need to
    - otherwise compare the probability of the random variable being at least as extreme as the observed value with the significance level
- For a **two-tailed test** you will need to find **both critical values,** one at each end of the distribution
- Take extra care when finding the critical region in the upper tail, you will have to find the probabilities for less than and subtract from one

What steps should I follow when carrying out a hypothesis test with the Poisson distribution?

Step 1. Define the mean, λ

Step 2. Write the null and alternative hypotheses clearly using the form

$$H_0 : \lambda = ...$$

$$H_1 : \lambda = ...$$

Step 3. Define the distribution, usually $X \sim \mathrm{Po}(\lambda)$ where λ is the mean to be tested

Step 4. Calculate the probability of the random variable being at least as extreme as the observed value

- Or if told to find the critical region

Step 5. Compare this probability with the significance level

- Or compare the observed value with the critical region

Step 6. Decide whether there is enough evidence to reject H_0 or whether it has to be accepted

Step 7. Write a conclusion in context

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Worked Example

Mr Viajo believes that his travel blog receives an average of 8 likes per day (24 hour period). He tries a new advertising campaign and carries out a hypothesis test at the 5% level of significance to see if there is an increase in the number of likes he gets. Over a 6-hour period chosen at random Mr Viajo's travel blog receives 5 likes.

(i) State null and alternative hypotheses for Mr Viajo's test.

(ii) Find the rejection region for the test.

(iii) Find the probability of a Type I error.

(iv) Carry out the hypothesis test, writing your conclusion clearly.

(i) Let λ be the average number of likes per day

$H_0 : \lambda = 8 \quad H_1 : \lambda > 8$

Increase so use >

(ii) 8 likes per 24 hours = 2 likes per 6 hours

For $X \sim P_o(2)$

The observed value is for 6 hours so change the mean to match

Rejection region will be the value of x for which:

$P(X \geqslant x) < 0.05$ given H_0 true

$P(X \geqslant x) = 1 - P(X < x)$

$1 - P(X < x) < 0.05$

$P(X < x) > 0.95$

It is easier to look for x such that $P(X > x) > 0.95$

$P(X < 5) = P(X \leqslant 4) = e^{-2}\left(1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!}\right) = 0.9473... < 0.95$

$P(X < 6) = P(X \leqslant 5) = 0.9473... + \frac{e^{-2}(2^5)}{5!} = 0.9834... \quad > 0.95$

This may be easier to see as:

$P(X \geqslant 5) = 1 - P(X < 5) = 1 - 0.9473 \quad > 5\%$ outside rejection region

$P(X \geqslant 6) = 1 - P(X < 6) = 1 - 0.9834... < 5\%$ inside rejection region

Rejection region is $X \geqslant 6$

(iii) P(Type I error) = P(reject H_0 given H_0 true)

$= P(X \geqslant 6 \mid \lambda = 2)$

$= 1 - 0.9834... = 0.01656...$

P(Type I error) = 0.0166 (3sf)

(iv) $5 < 6$ so 5 is outside the rejection region
Do not reject H_0

At the 5% significance level there is insufficient evidence to reject the null hypothesis

The results of the test do not support Mr Viajo's belief that the number of likes have increased

Examiner Tips and Tricks

- Take extra careful when working in the upper tail in Poisson distribution questions, this is where its easy to make mistakes.

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