Brief Notes on Differential Geometry

Lim Kian Hwee

Last updated: May 23, 2020, 3:57am

Contents

Foreword

This set of notes is a summary of the NUS Physics module "Mathematical Methods in Physics 3" (or MM3 for short) taken in my fourth year of study in NUS. Naturally, the references for this set of notes would be:

1. PC4274 Lecture Notes and tutorials, Kuldip

2. PC4274 Lecture Notes, Edward Teo (to a small extent)

The purpose of this set of notes is to summarise the key ideas of Differential Geometry as I understand it, so that in the future I can quickly reference this to get a brief idea of the topic. For more detailed treatments, there are textbooks like Frankel, Nakahara, or just the lecture notes themselves that are listed above. Unless absolutely necessary, no examples will be given.

Also, Einsten summation convention will be used throughout. I.e, for tensors, repeated upper and lower indices are summed over. Sometimes we will sum over repeated indices for things that are not tensors, in which case we might sometimes sum over two repeated upper indices, or two repeated lower indices (sloppy notation, I know).

Lastly, most importantly, *Soli Deo Gloria!*.

Chapter 1

Basic Topology

1.1 Quick recap on Sets, Maps, and Equivalence classes

Set theory

A set is a well-defined collection of objects. The cardinality of a set A, denoted as $|A$, is the number of elements in the set. The elements in a set are unique, and the ordering of elements in a set doesn't matter. The set with no elements at all is known as the null set, denoted as ϕ or $\{\}$. Given a set A, we also have A^c as another set that contains all other elements that are not in A. A^c is known as "A complement".

Given two sets A and B, we can produce new sets from A and B using the following operations

$$\cap \quad \cup \quad \backslash$$

known as the intersection, the union and the complement respectively.

1. $A \cap B$: is a new set that contains elements that are in both A and B.
2. $A \cup B$: is a new set that contains elements that are in either A or B.
3. $A \backslash B$: is a new set that contains elements that are in A but not in B. The complement is also sometimes written as $A - B$.

These operations obey De Morgan's laws. The two more important ones are: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. There is also a very important theorem known as the "inclusion-exclusion" principle, which is only stated in name here but can be found in any introductory mathematics books.

Suppose we have two sets A and B. If $\forall b \in B$, $b \in A$, we say that $B \subset A$, or that B is a subset of A. For a set A, the set of all its subsets is known as the power set of A, denoted as $\mathcal{P}(A)$. E.g, if $A = \{1, 2, 3\}$, then $\mathcal{P}(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$. We can easily prove that $|\mathcal{P}(A)| = 2^{|A|}$.

Two sets A and B are defined to be equal if $A \subset B$ and $B \subset A$.

Relationships between sets can be illustrated very easily by a Venn diagram, as shown in Figure 1.1.

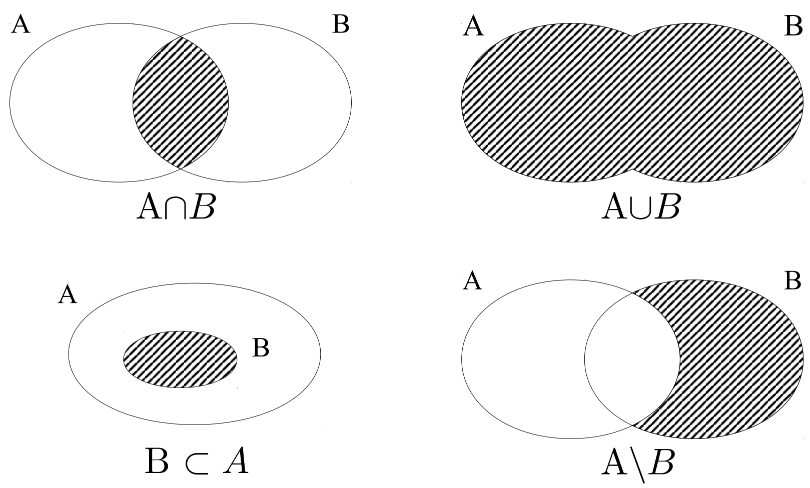

Figure 1.1:

Maps

Definition 1.1 (Maps between sets)**.** Let A and B be two sets. A map f is a many-to-one or one-to-one relation between elements in set A and elements in set B. I.e, $\forall a \in A$, we have a corresponding element $f(a) \in B$. We denote a map in one of these two ways:

$$A \xrightarrow{f} B \qquad f : A \to B$$

Definition 1.2 (Images and pre-images)**.** Suppose we have $f : A \to B$. We say that $f(A)$ is the image of A under f.
We can also define a concept known as the pre-image of a map f, which we denote as f^{-1}. I.e, $\forall b \in B$, we have: $f^{-1}(b) = \{a \in A \,|\, f(a) = b\}$. Note that f^{-1} is not necessarily a map, because f^{-1} can be one-to-many.

Definition 1.3. (Injective, Surjective and Bijective)

- A map $f : A \to B$ is one-to-one, or injective, if $\forall a, b \in A$, $f(a) = f(b) \Leftrightarrow a = b$.
- A map $f : A \to B$ is onto, or surjective, if $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$. In other words, $f(A) = B$.
- A map $f : A \to B$ is bijective if and only if it is both surjective and injective.

Chapter 2

Manifolds and Coordinates

Definition 2.1 (C^k Functions)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be in C^k if all its partial derivatives up to and including order k exist and are continuous.

Remark. When $k \to \infty$, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be smooth, or in C^∞.

2.1 Topological and Differentiable Manifolds

Definition 2.2 (Topological Manifold[2.1])**.** A topological manifold is a Hausdorff space such that every point has a neighbourhood homeomorphic to $\mathbb{R}^n$.

Remark. A manifold that is locally homeomorphic to $\mathbb{R}^n$ is said to have dimension n.

Since every point on the manifold has a neighbourhood homeomorphic to $\mathbb{R}^n$, locally the neighbourhood on the manifold "inherits" the metric topology of $\mathbb{R}^n$. I.e, the local homeomorphism to $\mathbb{R}^n$ implies the existence of open sets on the manifold, and because a homeomorphism is by definition bijective, there is a one-to-one correspondence between open sets in the neighbourhood and open sets in $\mathbb{R}^n$.

In short, "locally" (in the neighbourhood of a point), the manifold has the same topology as $\mathbb{R}^n$.

By the way, the manifold "inheriting" the metric topology of $\mathbb{R}^n$ locally is a very different concept from the idea of a metric tensor. So far, no metric has been defined on the manifold yet!

2.2 Charts

Definition 2.3 (Chart)**.** A chart (U, ϕ) of a manifold $\mathcal{M}$ is an open set U of $\mathcal{M}$, called the domain of the chart, together with a homeomorphism $\phi : U \to V$ of U onto an open set V in $\mathbb{R}^n$

I.e, for an arbitrary $p \in U \subset \mathcal{M}$, we have $\phi(p) = \left(x^1(p), x^2(p), ..., x^n(p)\right) \in \mathbb{R}^n$. We say that $p \in U$ has coordinates $\phi(p)$ in the chart (U, ϕ).

[2.1] For a precise meaning of the terms "Hausdorff","Neighbourhood", "Homeomorphic", wait for chapter 1 to be done.

In Physics terms, we would call $\{x^i(p)\}_{i=1,...n}$ the coordinate functions, and we would call the chart (U, ϕ) a local coordinate system.

2.3 Atlas

Definition 2.4 (Atlas). An atlas of class C^k on a manifold $\mathcal{M}$ is a set (i.e family) $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$[2.2] of charts of $\mathcal{M}$ such that the following holds:

1. M is covered by the family in the sense that

$$\mathcal{M} = \bigcup_{\alpha \in I} U_\alpha$$

2. The maps $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \phi_\beta(U_\alpha \cap U_\beta)$ are maps of open sets of $\mathbb{R}^n$ into $\mathbb{R}^n$ of class C^k.

Let's spend more time looking at property 2 of defintion 2.4. With reference to Figure 2.1,

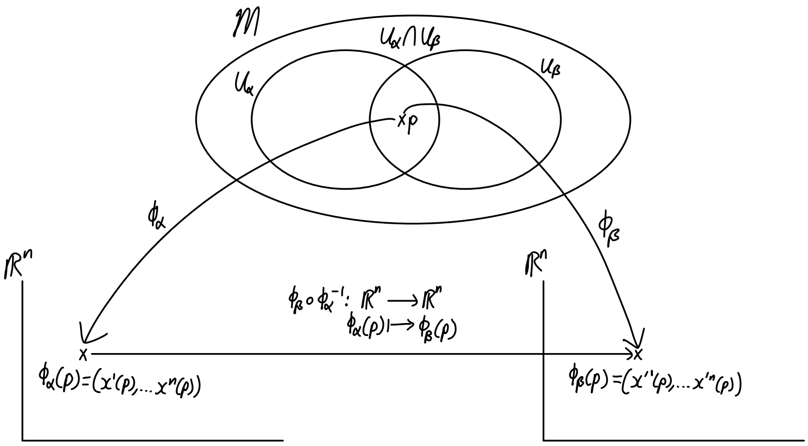

Figure 2.1:

consider a point $p \in U_\alpha \cap U_\beta$. p has coordinates $\phi_\alpha(p)$ in the chart (U_α, ϕ_α), but has coordinates $\phi_\beta(p)$ in the chart (U_β, ϕ_β). These two charts induce a map[2.3] $\phi_\beta \circ \phi_\alpha^{-1}$ from $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ to $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$. Now, the map $\phi_\beta \circ \phi_\alpha^{-1}$ is actually n coordinate transformation functions $\{x'^i = x'^i(x^1, ..., x^n)\}_{i=1,...n}$. Thus, when we say that the map $\phi_\beta \circ \phi_\alpha^{-1}$ is of class C^k, we are saying that those n coordinate transformation functions are also of class C^k.

Two charts (U_α, ϕ_α), (U_β, ϕ_β) that satisfy the property are said to be C^k compatible.

Definition 2.5 (Equivalent Atlases). Two C^k atlases $\mathcal{F} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ and $\mathcal{G} = \{(U_{\alpha'}, \phi_{\alpha'})\}_{\alpha' \in I}$ are equivalent[2.4] if every pair of charts $(U_\alpha, \phi_\alpha) \in \mathcal{F}$ and $(U_{\alpha'}, \phi_{\alpha'}) \in \mathcal{G}$ are C^k compatible.

[2.2] Note that $I \subset \mathbb{R}$

[2.3] ϕ_α^{-1} exists because ϕ_α is a homeomorphism.

[2.4] "Equivalent" here refers to equivalence relation, so we can form equivalence classes.

Remark. An equivalent definiton (can be proved) is: Two C^k atlases $\mathcal{F} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ and $\mathcal{G} = \{(U_{\alpha'}, \phi_{\alpha'})\}_{\alpha' \in I}$ are equivalent if their union $\mathcal{F} \cup \mathcal{G}$ is a C^k atlas.

2.4 Differentiable Manifold

Definition 2.6 (Differentiable Manifold)**.** A topological manifold $\mathcal{M}$ together with an equivalence class of C^k atlases is a C^k structure on $\mathcal{M}$; we say that $\mathcal{M}$ is a C^k manifold or a differentiable manifold.

Remark. Very often, we want $k \to \infty$, so we can define tensor fields on $\mathcal{M}$. In this case, the manifold is called a smooth manifold.

2.5 Key Idea Behind Differential Geometry

In Physics, most of the time the variables that we are concerned with such as time, position, etc are all either elements of $\mathbb{R}$ or $\mathbb{C}$, and our physical laws themselves are usually differential equations. Yet we know that the laws of Physics are independent of the choice of coordinates used. Thus, differential geometry is a natural language for expressing the laws of Physics. When we model our physical system with a differentiable manifolds, we can do calculus because locally the manifold looks like $\mathbb{R}^n$. But our results are also coordinate independent, because we can freely switch between coordinates due to the C^k structure; we just have to use the chain rule to do so.

By the way, from here on, we deal strictly with differentiable or smooth manifolds.

2.6 Diffeomorphisms

Definition 2.7 (Differentiable Maps between Manifolds)**.** Let $\mathcal{M}, \mathcal{N}$ be two C^k differentiable manifolds of dimensions m and n respectively. Let $\mathcal{M}$ have an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$[2.5], and $\mathcal{N}$ have an atlas $\{(V_\beta, \psi_\beta)\}_{\beta \in I}$. Let f be a map from $\mathcal{M}$ to $\mathcal{N}$.

The map f is said to be C^r differentiable at $p \in \mathcal{M}$, $r \leq k$ if the n coordinate transforms induced by $F \equiv \psi \circ f \circ \phi^{-1}$ are C^r differentiable at $\phi(p)$, where $(U, \phi) \in \{(U_\alpha, \phi_\alpha)\}$ is a chart such that $p \in U$ and $(V, \psi) \in \{(V_\beta, \psi_\beta)\}$ is a chart such that $f(p) \in V$.

If f is differentiable $\forall p \in \mathcal{M}$, then f is a differentiable map.

Remark. In definition 2.7, two particular charts (U, ϕ), (V, ψ) are chosen. Yet it doesn't matter which charts we choose. Consider two other charts (U', ϕ') and (V', ψ') such that $p \in U'$, $f(p) \in V'$. Consider $F' = \psi' \circ f \circ \phi'^{-1}$. Note that we can write[2.6] $F' = \psi' \circ (\psi^{-1} \circ \psi) \circ f \circ (\phi^{-1} \circ \phi) \circ \phi'^{-1} = (\psi' \circ \psi^{-1}) \circ F \circ (\phi \circ \phi'^{-1})$, and because of the C^k structure of the differentiable manifold, both $(\psi' \circ \psi^{-1})$ and $(\phi \circ \phi'^{-1})$ are both C^k functions. Thus, if F is a C^r function, then F' is also a C^r function.

[2.5] We will slowly drop the "$\alpha \in I$" label for ease of notation...

[2.6] This trick of inserting an identity map $(\psi^{-1} \circ \psi)$ is a very useful trick, especially in proofs.

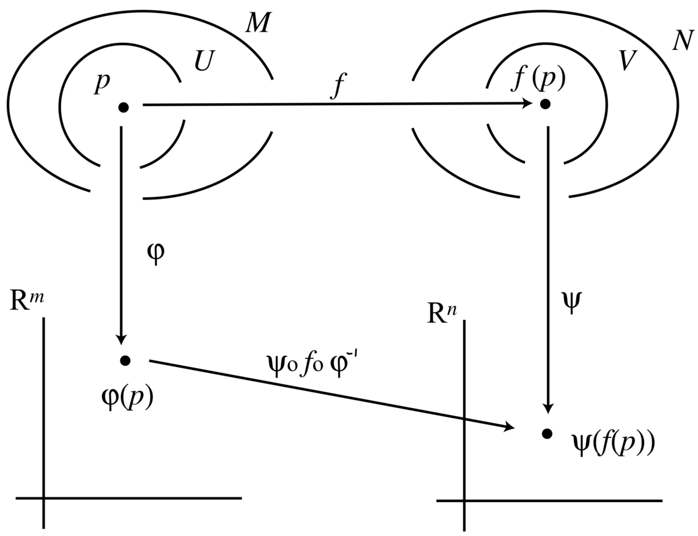

Figure 2.2: Figure copied from Nakahara

Let's try to make sense of the above definition. With reference to Figure 2.2, consider a point $p \in \mathcal{M}$ and its image under f, i.e $f(p) \in \mathcal{N}$. Consider two charts, $(U, \phi) \in \{(U_\alpha, \phi_\alpha)\}$ and $(V, \psi) \in \{(V_\beta, \psi_\beta)\}$ such that $p \in U$ and $f(p) \in V$. Then, $F = \psi \circ f \circ \phi^{-1}$ is a map from $\phi(U) \subset \mathbb{R}^n$ to $\psi(V) \subset \mathbb{R}^m$. In fact, we see that F induces n coordinate transformation functions; if we let $\phi(p) = \left(x^1, ..., x^m\right)$, then $\psi(f(p)) = \left(x'^1, ...x'^n\right) = \left(x'^1(x^1, ..., x^m), ...x'^n(x^1, ..., x^m)\right)$. Then, for f to be a differentiable map, these n coordinate functions have to be C^r.

Definition 2.8 (Diffeomorphism)**.** A map $f : \mathcal{M} \to \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are C^k-differentiable manifolds of dimensions m and n respectively, is said to be a diffeomorphism if:

1. f is bijective ($\implies m = n$).
2. f and f^{-1} are both C^r functions.

Chapter 3

Tangent Spaces

3.1 Curves and Functions on a Manifold

Definition 3.1 (Curve on a Manifold)**.** A (parametrized) curve σ on a m-dimensional manifold $\mathcal{M}$ is an injective map from $I \subset \mathbb{R}$ into $\mathcal{M}$ by:

$$t \in I \to \sigma(t) \in \mathcal{M}$$

The curve is also said to be differentiable if the map σ is differentiable.

Definition 3.2 (Differentiable Curve on a Manifold)**.** The idea of σ being differentiable is defined as follows: Consider a chart (U, ϕ) that covers the entire curve, i.e $\sigma(I) \subset U$. Then, σ being differentiable is defined as the map $\bar{\sigma} \equiv \phi \circ \sigma$ from I to $\mathbb{R}^m$ being differentiable.

Note that $\bar{\sigma}(t) = \big(x^1(t), ...x^m(t)\big)$, thus $\bar{\sigma}$ being differentiable means that the m coordinate functions $x^i(t)$ are differentiable with respect to t.

Note also that the differentiability of σ is coordinate independent, as can be easily proved by considering another chart (V, ψ) that covers the entire curve, and noting that if $\phi \circ \sigma$ is differentiable, then so is $\psi \circ \sigma$ because:

$$\psi \circ \sigma = \underbrace{\psi \circ \phi^{-1}}_{\in C^k} \circ\ \phi \circ \sigma \tag{3.1}$$

Remark. Equation 3.1 is an example of a really useful trick, where we insert an identity map $\phi^{-1} \circ \phi$ or $\phi \circ \phi^{-1}$ to show that certain things are independent of the chart chosen. In fact, we have seen something like this in section 2.6. We will use this trick frequently, but from now on, we will not show the explicit calculation, but only make a brief comment.

Definition 3.3 (Function on a Manifold)**.** A function f on a manifold $\mathcal{M}$ is the map $f : \mathcal{M} \to \mathbb{R}$.

Definition 3.4 (C^rFunction on a Manifold)**.** A function f on a manifold $\mathcal{M}$ is said to be C^r at $p \in \mathcal{M}$ if in a chart (U, ϕ) such that $p \in U$, $\bar{f} \equiv f \circ \phi^{-1}$ is a C^r function from $\mathbb{R}^>$ to $\mathbb{R}$.

Note that the above defintion can be shown to be independent of the choice of chart (U, ϕ); we can consider another chart (V, ψ) and do the insertion of identity trick as was done in equation 3.1.

If f is C^r $\forall p \in \mathcal{M}$, then we say that $f \in C^r(\mathcal{M})$, or that f is a C^r function on the manifold $\mathcal{M}$.

3.1.1 Useful notation: Overbar

The astute reader might have noticed by now that we have been quite consistent in our notation, in where we place an overbar. Specifically, for some chart (U, ϕ) we have done:

- $f : \mathcal{M} \to \mathbb{R}$ leads to $\bar{f} \equiv f \circ \phi : \mathbb{R}^m \to \mathbb{R}$
- $\sigma : I \subset \mathbb{R} \to \mathcal{M}$ leads to $\bar{\sigma} \equiv \phi \circ \sigma : \mathbb{R} \to \mathbb{R}^m$

Thus, for any map that involves a manifold $\mathcal{M}$, we shall use the overbar to denote its representation in some local coordinate system (i.e, in some chart). Some examples (including those which we will see later):

- $f : \mathcal{M} \to \mathcal{N}$ leads to $\bar{f} = \phi^{-1} \circ f \circ \psi : \mathbb{R}^m \to \mathbb{R}^n$ where (U, ϕ) is some chart in M, (V, ψ) is some chart in N.
- $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ leads to $\bar{X} : C^\infty(\mathbb{R}^m) \to C^\infty(\mathbb{R}^m)$

3.2 Tangent Vectors

Definition 3.5 (Tangent Vector to a Curve)**.** The tangent vector to a curve σ at some point $p \in \mathcal{M}$ is defined:

$$V_p^\sigma(f) = \frac{d}{dt}(f \circ \sigma)(t)\Big|_{t=0}, \quad \sigma(0) = p$$

$\forall f \in C^k(\mathcal{M})$.

Note that $V_p^\sigma(f)$ is a number. Now, there could be multiple curves with the same tangent vector; for example in $\mathbb{R}^2$, the curves $\sigma_1(t) = (t, e^t)$ and $\sigma_2(t) = (t, 1 + t)$ are tangent at $p = (1, 1) \in \mathbb{R}^2$. Let's extend this idea to a general manifold $\mathcal{M}$.

Definition 3.6 (Two curves being tangent at $p \in \mathcal{M}$)**.** Two curves on the manifold $\mathcal{M}$, $\sigma_1(t)$ and $\sigma_2(t)$, are said to be tangent at $\sigma_1(0) = \sigma_2(0) = p \in \mathcal{M}$ if for all $f \in C^k(\mathcal{M})$, the following holds:

$$\frac{d}{dt}(f \circ \sigma_1)(t)\Big|_{t=0} = \frac{d}{dt}(f \circ \sigma_2)(t)\Big|_{t=0}$$

Remark. The $\sigma_1(0) = \sigma_2(0) = p \in \mathcal{M}$ condition is just saying that the two curves must intersect. Without loss of generality, if $\sigma_1(t_1) = \sigma_2(t_1) = p \in \mathcal{M}$ and $\frac{d}{dt}(f \circ \sigma_1)(t)|_{t=t_1} = \frac{d}{dt}(f \circ \sigma_2)(t)|_{t=t_1}$ for some $t_1 \in \mathbb{R}$, then we can just do a change of variables $t = t + t_1$ to recover our original definition. I.e, definition 3.6 is general enough.

Remark. In other words, two curves $\sigma_1(t)$ and $\sigma_2(t)$ are said to be tangent at $p \in \mathcal{M}$ if they have the same tangent vector at point $p \in \mathcal{M}$, i.e $V_p^{\sigma_1} = V_p^{\sigma_2}$. We will see what this means in more detail when we go into a local chart (U, ϕ).

3.2.1 Tangent vectors to a curve in a local chart (U, ϕ)

This concept is very important and thus deserves a section of its own. Consider $p \in \mathcal{M}$, and a local chart (U, ϕ) around p, i.e $p \in U$. Consider also a curve σ such that $\sigma(0) = p$. Then, we have:

$$\begin{aligned} V_p^\sigma(f) &= \frac{d}{dt}(f \circ \sigma)(t)\Big|_{t=0} \\ &= \frac{d}{dt}(f \circ \phi^{-1} \circ \phi \circ \sigma)(t)\Big|_{t=0} \\ &= \frac{d}{dt}(\bar{f} \circ \bar{\sigma})(t)\Big|_{t=0} \end{aligned}$$

where $\bar{\sigma}(t) = \big(x^1(t), ..., x^m(t)\big)$ is a curve in $\mathbb{R}^m$, and $\bar{f}$ is a map from $\mathbb{R}^m$ to $\mathbb{R}$. Continuing with our calculation, we have:

$$\begin{aligned} V_p^\sigma(f) &= \frac{d}{dt}(\bar{f} \circ \bar{\sigma})(t)\Big|_{t=0} \\ &= \frac{d}{dt}\bar{f}(x^1(t), ..., x^m(t))\Big|_{t=0} \\ &= \frac{dx^i}{dt}\Big|_{t=0}\frac{\partial \bar{f}}{\partial x^i}\Big|_{\phi(p)} \\ &= \frac{dx^i}{dt}\Big|_{t=0}\frac{\partial}{\partial x^i}(\bar{f}) \end{aligned}$$

where in the last line, we write $\frac{\partial \bar{f}}{\partial x^i}\Big|_{\phi(p)}$ as $\frac{\partial \bar{f}}{\partial x^i}$ for ease of notation, as we will sometimes do in the future. Now, since the above calculations are true for all $f \in C^k(\mathcal{M})$, we have:

$$V_p^\sigma \xrightarrow{\text{local chart}} \frac{dx^i}{dt}\Big|_{t=0}\frac{\partial}{\partial x^i} \tag{3.2}$$

Equation 3.2 tells us what it means for two curves $\sigma_1(t)$ and $\sigma_2(t)$ to be tangent at $p \in \mathcal{M}$. In a local chart (U, ϕ), $\sigma_1(t)$ and $\sigma_2(t)$ are tangent at $p \in \mathcal{M}$ if and only if these two conditions hold:

1. $\sigma_1(0) = \sigma_2(0) = p$
2. If we write $\bar{\sigma_1}(t) = \big(x^1(t), ..., x^m(t)\big)$ and $\bar{\sigma_2}(t) = \big(x'^1(t), ..., x'^m(t)\big)$, then

$$\frac{dx^i}{dt}\Big|_{t=0} = \frac{dx'^i}{dt}\Big|_{t=0}$$

   for all $i = 1, ..., m$.

By the way, it can be proven[3.1] that if condition 2 is true in one local chart (U, ϕ), then it is true for all charts.

[3.1] Refer to Kuldip's tutorials, or Nakahara/Frankel

3.2.2 Motivating the definition of a tangent vector at a point

So far, we have defined a tangent vector to a curve. However, the above calculation explicitly shows us that we cannot identify tangent vectors uniquely with curves, since there could be many different curves that give the same tangent vector.

However, if we take the set of all curves that pass through a point $p \in \mathcal{M}$, and separate them into equivalence classes where the equivalence relation between two curves is that they are tangent at the point $p \in \mathcal{M}$, then we can uniquely identify different tangent vectors with different equivalence classes of curves. This is what we will do.

Definition 3.7 (Tangent Vector at a point $p \in \mathcal{M}$)**.** A tangent vector at $p \in \mathcal{M}$, V_p^σ is defined as

$$V_p^\sigma(f) = \frac{d}{dt}(f \circ \sigma)(t)\Big|_{t=0}, \quad \sigma(0) = p$$

$\forall f \in C^k(\mathcal{M})$, where σ is any representative of an equivalence class of curves, denoted $[\sigma]_p$. The equivalence relation between two curves is that they are tangent at the point $p \in \mathcal{M}$.

Remark. Sometimes, we write $V_p^\sigma = [\sigma]_p$, i.e we identify the tangent vector directly with the equivalence class of curves that it is defined with. Recall: different equivalence class of curves $\implies$ different tangent vectors!

In fact, we will do this a lot in this set of notes; this is a convention inherited from Kuldip.

3.2.3 Tangent vectors at a point $p \in \mathcal{M}$ in a local chart (U, ϕ)

The local representation of a tangent vector at a point, V_p^σ, is exactly the same the local representation of a tangent vector to a curve, given in equation 3.2. The only difference is that the term $\frac{dx^i}{dt}\Big|_{t=0}$ is now uniquely identified with an equivalence class of curves $[\sigma]_p$. Let's reproduce equation 3.2 here for convenience:

$$V_p^\sigma \xrightarrow{\text{local chart}} \frac{dx^i}{dt}\Big|_{t=0} \frac{\partial}{\partial x^i}$$

Now, we can write $V^i \equiv \frac{dx^i}{dt}\Big|_{t=0}$, and call V^i the components of the vector V_p^σ in a local chart (U, ϕ). Then, we have:

$$V_p^\sigma \xrightarrow{\text{local chart}} V^i \frac{\partial}{\partial x^i} = V^i \partial_i \tag{3.3}$$

From equation 3.3, we also see that V_p^σ is associated with a directional derivative in a local coordinate chart. This is the breakthrough of modern differential geometry, to associate tangent vectors to a point in the manifold with a directional derivative in $\mathbb{R}^m$. Note that because V_p^σ is associated with a directional derivative, we have the Leibniz product rule. We will revisit this idea again when we talk about derivations.

Lastly, we shall call the m objects $\{\partial_i\}_{i=1,\ldots,m}$ the coordinate basis vectors to the tangent space $T_p(\mathcal{M})$. But let's not get ahead of ourselves, and revisit this idea again later when we formally develop the tangent space.

Note that from here on, whenever we say "tangent vector", we are referring to tangent vectors at a point. Also, from definiton 3.7, we see that a local chart is unnecessary to define a tangent vector; local charts are used just for our convenience, when we want to do computations.

3.2.4 Transformation properties of the components of a tangent vector when we switch local charts

Since tangent vectors exist independently of any local charts, lets see how the components of the tangent vector V_p^σ in a local chart (U, ϕ) are related to the components in another local chart (U', ϕ').

Consider $f \in C^k(\mathcal{M})$. Then, in the local chart (U, ϕ), we have:

$$V_p^\sigma(f) = V^i \left(\frac{\partial}{\partial x^i} \bar{f}(x^1, ..., x^m) \right)\Big|_{\phi(p)}$$

where $\bar{f} = f \circ \phi^{-1}$. On the other hand, in the other local chart (U', ϕ'), we have:

$$V_p^\sigma(f) = V'^i \left(\frac{\partial}{\partial x'^i} \bar{f}'(x'^1, ..., x'^m) \right)\Big|_{\phi'(p)}$$

where $\bar{f}' = f \circ \phi'^{-1}$. Note that $\bar{f}(x^1, ..., x^m) = \bar{f}'(x'^1, ..., x'^m) = f(p)$.

But $\phi' \circ \phi^{-1}$, a map from $\phi(U \cap U') \subset \mathbb{R}^m$ to $\phi'(U \cap U') \subset \mathbb{R}^m$, induces m coordinate transformation functions $\left\{x'^i(x^1, ..., x^m)\right\}_{i=1,...,m}$. Thus, using the chain rule, we have:

$$\begin{aligned}
\left(\frac{\partial}{\partial x^i} \bar{f}(x^1, ..., x^m) \right)\Big|_{\phi(p)} &= \left(\frac{\partial}{\partial x^i} \bar{f}'(x'^1(x^1, ..., x^m), ..., x'^m(x^1, ..., x^m)) \right)\Big|_{\phi(p)} \\
&= \frac{\partial x'^j}{\partial x^i}\Big|_{\phi(p)} \left(\frac{\partial}{\partial x'^j} \bar{f}'(x'^1, ..., x'^m) \right)\Big|_{\phi'(p)}
\end{aligned}$$

Thus, substituting the chain rule calculation above into the expression for $V_p^\sigma(f)$ in the local chart (U, ϕ), we have:

$$V_p^\sigma(f) = V^i \frac{\partial x'^j}{\partial x^i}\Big|_{\phi(p)} \left(\frac{\partial}{\partial x'^j} \bar{f}'(x'^1, ..., x'^m) \right)\Big|_{\phi'(p)}$$

which, when we compare with the expression for $V_p^\sigma(f)$ in the local chart (U', ϕ'), we get:

$$V'^i = V^i \frac{\partial x'^j}{\partial x^i}\Big|_{\phi(p)} \tag{3.4}$$

Thus, we note that the components of the tangent vector in a local coordinate chart transforms contravariantly under a change of coordinates! This contravariant transformation is a direct consequence of the identification of tangent vectors with directional derivatives in a local coordinate chart.

3.2.5 Tangent vectors as derivations

From definiton 3.7, we see that V_p^{σ} is really a map from $C^k(\mathcal{M}) \to \mathbb{R}$. In fact, since

$$V_p^{\sigma}(f) = V^i \frac{\partial \bar{f}}{\partial x^i}$$

in a local chart (U, ϕ), where $\bar{f} = f \circ \phi^{-1}$,we have the following few properties:

1. $V_p^{\sigma}(f + g) = V_p^{\sigma}(f) + V_p^{\sigma}(g) \quad \forall f \in C^k(\mathcal{M})$
2. $V_p^{\sigma}(\alpha f) = \alpha V_p^{\sigma}(f) \quad \forall \alpha \in \mathbb{R}$ and $\forall f \in C^k(\mathcal{M})$
3. $V_p^{\sigma}(f \cdot g) = g(p)V_p^{\sigma}(f) + f(p)V_p^{\sigma}(g) \quad \forall f, g \in C^k(\mathcal{M})$

Thus, V_p^{σ} is something we call a derivation, which is a fancy name for a generalised derivative operator that satisfies the three properties above (and especially Leibniz's product rule).

3.3 The Tangent Space $T_p(\mathcal{M})$

3.3.1 Constructing the Tangent Space $T_p(\mathcal{M})$

Here, we want to show that the set of all tangent vectors at a point $p \in \mathcal{M}$, which we denote as $\{V_p^{\sigma}\}$, has a vector space structure.

To do so, we first need to define the addition between two tangent vectors $V_p^{\sigma_1} = [\sigma_1]$ and $V_p^{\sigma_2} = [\sigma_2]$. Since tangent vectors are identified uniquely with equivalence classes of curves, if we could take the two equivalence classes of curves and produce a third equivalence class of curves, then we would have produced another tangent vector. Note that it is possible for $[\sigma_1] = [\sigma_2]$, in which case our result will still be in the same equivalence class as $[\sigma_1]$ (i.e, adding of two parallel vectors will give a third parallel vector).

In other words, given a two curves that pass through σ_1 and σ_2 that pass through $p \in \mathcal{M}$, we want a third curve σ_3 that passes through $p \in \mathcal{M}$ too. We will do so like this:

1. Firstly, choose a local chart (U, ϕ) such that $p \in U$ and $\phi(p) = \vec{0}$, where $\vec{0} \in \mathbb{R}^m$.
2. Secondly, make sure also that $\bar{\sigma_1}(0) = \bar{\sigma_2}(0) = \vec{0}$, where $\bar{\sigma_i} \equiv \phi \circ \sigma_i$.
3. Since $\bar{\sigma_1}(t)$ and $\bar{\sigma_2}(t)$ are two curves in $\mathbb{R}^m$, and because $\mathbb{R}^m$ has a vector space structure, $(\bar{\sigma_1}(t) + \bar{\sigma_2}(t))$ is another curve in $\mathbb{R}^m$. Note that since $(\bar{\sigma_1}(0) + \bar{\sigma_2}(0)) = \vec{0} = \phi(p)$, this curve passes through $\phi(p)$.
4. Since ϕ is a homeomorphism, $\phi^{-1} \circ (\bar{\sigma_1}(t) + \bar{\sigma_2}(t))$ would be another curve in $\mathcal{M}$ that passes through p.

Thus, we shall define the addition operator between $V_p^{\sigma_1}$ and $V_p^{\sigma_2}$ as follows:

$$V_p^{\sigma_1} + V_p^{\sigma_2} \equiv [\phi^{-1} \circ (\bar{\sigma_1}(t) + \bar{\sigma_2}(t))] \tag{3.5}$$

Next, we need to define the scalar multiplication of a tangent vector V_p^σ. We shall do so via:

$$\alpha V_p^\sigma \equiv [\phi^{-1} \circ (\alpha\bar{\sigma}(t))] \tag{3.6}$$

In the cases of scalar multiplication, we have basiaclly mapped $\sigma(t)$ into $\mathbb{R}^m$ to give us a vector $\bar{\sigma}(t) \in \mathbb{R}^m$, then we multiplied $\alpha \in \mathbb{R}$ to $\bar{\sigma}(t)$ to give us another vector in $\mathbb{R}^m$, before mapping the result back to $\mathcal{M}$ through ϕ^{-1}. This gives us another curve in $\mathcal{M}$ that passes through the point p.

Now, to show that the equations 3.5,3.6 do indeed endow the set $\{V_p^\sigma\}$ with a vector space structure, it suffices to show that:

$$\left(\alpha V_p^{\sigma_1} + \beta V_p^{\sigma_2}\right)(f) = \alpha V_p^{\sigma_1}(f) + \beta V_p^{\sigma_2}(f)$$

$\forall f \in C^k(\mathcal{M}),\ \forall \alpha, \beta \in \mathbb{R}$.

Proof. Let $\phi \circ \sigma_1 = \bar{\sigma}_1(t) = \left(x^1(t), ..., x^m(t)\right)$, $\phi \circ \sigma_2 = \bar{\sigma}_2(t) = \left(y^1(t), ..., y^m(t)\right)$ and $\bar{f} = f \circ \phi^{-1}$, where clearly $\bar{f}$ is a map from $\mathbb{R}^m$ to $\mathbb{R}$. By construction, $\bar{\sigma}_1(0) = \bar{\sigma}_2(0) = \phi(p) = \vec{0}$. We shall also define:

$$\begin{aligned}\alpha\bar{\sigma}_1(t) + \beta\bar{\sigma}_2(t) &= \alpha\left(x^1(t), ..., x^m(t)\right) + \beta\left(y^1(t), ..., y^m(t)\right)\\ &\equiv (z^1(t), ..., z^m(t))\end{aligned}$$

Now,

$$\begin{aligned}\left(\alpha V_p^{\sigma_1} + \beta V_p^{\sigma_2}\right)(f) &= \frac{d}{dt}\Big[\underbrace{f \circ \phi^{-1}}_{\bar{f}} \circ (\alpha\bar{\sigma}_1(t) + \beta\bar{\sigma}_2(t))\Big]\Big|_{t=0}\\ &= \frac{d}{dt}\left[\bar{f}(z^1(t), ..., z^m(t))\right]\\ &= \frac{\partial}{\partial z^i}\bar{f}(z^1, ..., z^m)\Big|_{\phi(p)}\frac{dz^i}{dt}\Big|_{t=0}\\ &= \frac{\partial}{\partial z^i}\bar{f}(z^1, ..., z^m)\Big|_{\vec{0}}\left(\alpha\frac{dx^i}{dt}\Big|_{t=0} + \beta\frac{dy^i}{dt}\Big|_{t=0}\right)\\ &= \alpha\frac{dx^i}{dt}\Big|_{t=0}\frac{\partial}{\partial z^i}\bar{f}(z^1, ..., z^m)\Big|_{\vec{0}} + \beta\frac{dy^i}{dt}\Big|_{t=0}\frac{\partial}{\partial z^i}\bar{f}(z^1, ..., z^m)\Big|_{\vec{0}}\\ &= \alpha\frac{dx^i}{dt}\Big|_{t=0}\frac{\partial}{\partial x^i}\bar{f}(x^1, ..., x^m)\Big|_{\vec{0}} + \beta\frac{dy^i}{dt}\Big|_{t=0}\frac{\partial}{\partial y^i}\bar{f}(y^1, ..., y^m)\Big|_{\vec{0}}\\ &= \alpha\frac{d}{dt}\left[f \circ \sigma_1\right]\Big|_{t=0} + \beta\frac{d}{dt}\left[f \circ \sigma_1\right]\Big|_{t=0}\\ &= \alpha V_p^{\sigma_1}(f) + \beta V_p^{\sigma_2}(f)\end{aligned}$$

where to get from the 5th line of the proof to the 6th line, we do a renaming of variables. □

Thus, we see that the set $\{V_p^\sigma\}$ with the addition operator and the scalar multiplication operation defined in equations 3.5, 3.6 does indeed have a (real) vector space structure, which we denote as $T_p(\mathcal{M})$.

3.3.2 Basis vectors of the Tangent Space $T_p(\mathcal{M})$

In a local coordinate chart (U, ϕ), we recall that we have:

$$V_p^\sigma \xrightarrow{\text{local chart}} V^i \frac{\partial}{\partial x^i}$$

Now, since $T_p(\mathcal{M})$ has a vector space structure, we call the set of m vectors $\{\frac{\partial}{\partial x^i}\}$ the coordinate basis[3.2] of the Tangent Space $T_p(\mathcal{M})$ (in a local chart). Coordinate basis means that these basis vectors $\frac{\partial}{\partial x^i}$ are directly derived from the coordinates x^i.

We can use a non-coordinate basis too; i.e our basis can be of the form:

$$\underbrace{\left\{a^i \frac{\partial}{\partial x^i}, b^i \frac{\partial}{\partial x^i}, \dots\right\}}_{m \text{ linearly independent vectors}}$$

However, using a coordinate basis has many benefits[3.3], which is why we will stick with a coordinate basis unless explicitly stated.

3.4 The push-forward map between Tangent Spaces

If we have a differentiable map between two manifolds

$$\mathcal{F} : \mathcal{M} \to \mathcal{N}$$

at the point $p \in \mathcal{M}$, then this map induces a map between the tangent spaces $T_p(\mathcal{M})$ and $T_{\mathcal{F}(p)}(\mathcal{N})$

$$\mathcal{F}_* : T_p(\mathcal{M}) \to T_{\mathcal{F}(p)}(\mathcal{N})$$

The origins of this "push-forward" map will be obvious when we look at its definition.

Definition 3.8 (The Push-forward of a tangent vector)**.** If $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is a differentiable map at $p \in \mathcal{M}$ and $V_p^\sigma \equiv [\sigma]_p \in T_p(\mathcal{M})$, then the push-forward $\mathcal{F}_*(V_p^\sigma)$ in $T_{\mathcal{F}(p)}(\mathcal{N})$ is defined by

$$\mathcal{F}_*(V_p^\sigma) = [\mathcal{F} \circ \sigma]_{\mathcal{F}(p)}$$

We can also write

$$\mathcal{F}_*(V_p^\sigma) = W_{\mathcal{F}(p)}^{\sigma'}$$

where $\sigma' = \mathcal{F} \circ \sigma$.

Remark. We can also consider a map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$. In this case, $\mathcal{F}_*$ would be a map between the two tangent spaces of $\mathcal{M}$, i.e $\mathcal{F}_* : T_p(\mathcal{M}) \to T_{\mathcal{F}(p)}(\mathcal{N})$. We can also consider the case where $\mathcal{F}$ is a homeomorphism ϕ.[3.4]

[3.2] Since the coordinates x^i are independent, the vectors in $\{\frac{\partial}{\partial x^i}\}$ are linearly independent

[3.3] Kuldip didn't talk much about this, but Edward Teo talked about this more.

[3.4] A homeomorphism ϕ to $U \subset \mathbb{R}^n$ is a differentiable map. To see why, consider another chart (V, ψ). Then, the condition for ϕ to be a differentiable map is that $\phi \circ \psi^{-1}$ is C^k, which is naturally satisfied by the definition of differentiable manifold.

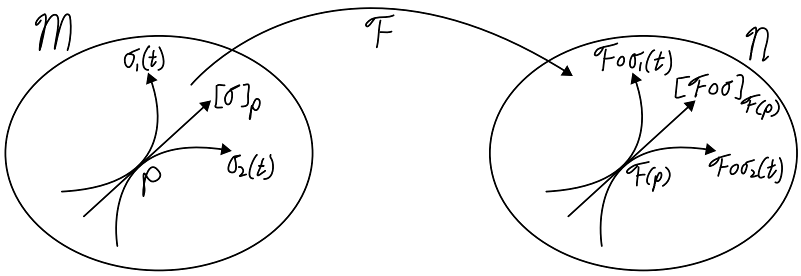

Figure 3.1: The push forward of an equivalence class of curves $[\sigma]_p$ induced by the map $\mathcal{F}$.

To understand definiton 3.8, suppose we have a curve $\sigma \subset \mathcal{M}$ passing through $p \in \mathcal{M}$. Then, the map $\mathcal{F}$ maps every point of σ to a curve $\mathcal{F} \circ \sigma \subset \mathcal{N}$, where $\mathcal{F} \circ \sigma$ is a curve passing through $\mathcal{F}(p) \in \mathcal{N}$. We can do this not only for σ, but for an entire equivalence class of curves [3.5] $[\sigma]_p$. Since equivalence classes of curves are uniquely identified with tangent vectors, $\mathcal{F}$ mapping an equivalence class of curves $[\sigma]_p$ in $\mathcal{M}$ to another equivalence class of curves $[\mathcal{F} \circ \sigma]_{\mathcal{F}(p)}$ in $\mathcal{N}$ induces a map between $T_p(\mathcal{M})$ and $T_{\mathcal{F}(p)}(\mathcal{N})$. The situation is shown in Figure 3.1.

By the way, it can be proven[3.6] that the push-forward map is linear. I.e,

$$\mathcal{F}_* \left(\alpha V_p^{\sigma_1} + \beta V_p^{\sigma_2} \right) = \alpha \mathcal{F}_*(V_p^{\sigma_1}) + \beta \mathcal{F}_*(V_p^{\sigma_2})$$

3.4.1 Calculating a push-forward of a tangent vector

From definition 3.8, we have

$$\mathcal{F}_*(V_p^{\sigma}) = [\mathcal{F} \circ \sigma]_{\mathcal{F}(p)} = W_{\mathcal{F}(p)}^{\sigma'}$$

where $W_{\mathcal{F}(p)}^{\sigma'} \in T_{\mathcal{F}(p)}(\mathcal{M})$. Now, for all $f \in C^k(\mathcal{N})$,

$$\begin{aligned} W_{\mathcal{F}(p)}^{\sigma'}(f) &= \left[\mathcal{F}_*(V_p^{\sigma})\right](f) \\ &= \frac{d}{dt} \big[\underbrace{f \circ \mathcal{F}}_{\substack{\equiv f', \\ f' \in C^k(\mathcal{M})}} \circ \sigma \big](t) \Big|_{t=0} \\ &= \frac{d}{dt} \big[f' \circ \sigma \big](t) \Big|_{t=0} \\ &= V_p^{\sigma}(f') \\ &= V_p^{\sigma}(f \circ \mathcal{F}) \end{aligned} \tag{3.7}$$

[3.5] This is proven in one of Kuldip's tutorials. Essentially, we want to show that if two curves $\sigma_1(t)$ and $\sigma_2(t)$ belong in the same equivalence class $[\sigma]_p$ at the point $p \in \mathcal{M}$, then the two curves $\mathcal{F} \circ \sigma_1(t)$ and $\mathcal{F} \circ \sigma_2(t)$ also belong to the same equivalence class in $\mathcal{N}$ at $\mathcal{F}(p)$.

[3.6] Again, done in Kuldip's tutorial.

Now, in local charts (U, ϕ) and (V, ψ) for $\mathcal{M}$ and $\mathcal{N}$ respectively, where for $p \in \mathcal{M}$ we define $\phi(p) \equiv x$ and $\psi(\mathcal{F}(p)) \equiv y$, equation 3.7 just reduces to:

$$\left(W^{\sigma'}_{\mathcal{F}(p)}\right)^i \left(\frac{\bar{f}(\mathcal{F}(p))}{\partial y^i}\right) = \left(V^{\sigma}_p\right)^i \left(\frac{\bar{f}(\mathcal{F}(p))}{\partial x^i}\right) \tag{3.8}$$

$$\implies \left(W^{\sigma'}_{\mathcal{F}(p)}\right)^i \left(\frac{\partial \bar{f}(y^1, \dots, y^m)}{\partial y^i}\right) = \left(V^{\sigma}_p\right)^i \left(\frac{\partial \bar{f}(y^1(x^1, \dots, x^m), \dots, y^m(x^1, \dots, x^m))}{\partial x^i}\right) \tag{3.9}$$

where $\bar{f} = f \circ \psi^{-1}$. This finally gives us:

$$\left(W^{\sigma'}_{\mathcal{F}(p)}\right)^i \left(\frac{\partial \bar{f}(y^1, \dots, y^m)}{\partial y^i}\right) = \left(V^{\sigma}_p\right)^i \frac{\partial y^j}{\partial x^i} \left(\frac{\partial \bar{f}(y^1, \dots, y^m)}{\partial y^j}\right) \tag{3.10}$$

The detailed working to fully derive equation 3.10 is done in the context of fields in corollary 4.1.1, but the working in corollary 4.1.1 can be easily adapted to quickly properly derive equation 3.10.

The hope is that the reader can see equation 3.8 as intuitive from equation 3.7 (if not, stare at it until it becomes intuitive).

We will use the calculation here when we talk about $\mathcal{F}$-related vector fields in section 4.2 to derive some cool results for vector fields.

3.4.2 Push forward map induced by a curve

Recall that a curve σ on a manifold is an injective map from $I \subset \mathbb{R} \to \mathcal{M}$. Now, we can treat $I \subset \mathbb{R}$ as an open set of the one-dimensional manifold $\mathbb{R}$. This manifold, which admits a global chart, has a one dimensional tangent space $T_s(\mathbb{R})$ for all $s \in \mathbb{R}$. We can characterise the basis vector of the tangent space as $\frac{d}{dt}\Big|_{t=s}$ (here the coordinates of the manifold are labelled by t).

Then, a tangent vector on $\mathcal{M}$ at the point $p = \sigma(s)$ can be defined as the push forward of the tangent vector $\frac{d}{dt}\Big|_{t=s}$ induced by σ. I.e,

$$\sigma_* \left(\frac{d}{dt}\right)_{t=s} = V_{\sigma(s)} \in T_{\sigma(s)}(\mathcal{M}) \tag{3.11}$$

We will see this again when we talk about integral curves in section 4.3.

3.4.3 Push forward map induced by a homeomorphism to $\mathbb{R}^m$

At a point $p \in \mathcal{M}$, under a local chart (U, ϕ), we recall that $T_p(\mathcal{M})$ has a coordinate basis $\left\{\frac{\partial}{\partial x^i}\right\}_{i=1,\dots,m}$. Then, the map ϕ^{-1} induces a push-forward map between the coordinate basis in the local chart, and the actual basis vectors of $T_p(\mathcal{M})$, which we denote as: $\{e_i\}_{i=1,\dots,m}$.

Chapter 4

Vector Fields and Integral Curves

A rough explanation of vector fields and integral curves

So far we have only dealt with $T_p(\mathcal{M})$, the vector space of all tangent vectors at a point $p \in \mathcal{M}$. In this section, we introduce another mathematical object known as a vector field X, which roughly speaking, assigns **one** tangent vector to each point p on a C^∞ (i.e smooth) manifold $\mathcal{M}$ in a **smooth** manner[4.1]. Different vector fields X_1, X_2 ... etc are different assignments of tangent vectors to each point $p \in \mathcal{M}$.

A familiar example would be the electric field in $\mathbb{R}^3$; to each point $p \in \mathbb{R}^3$, we have an electric field vector, which is an element of $T_p(\mathbb{R}^3)$ (which is a 3 dimensional real vector space). Different electric fields lead to different assignments of electric field vectors to all points in $\mathbb{R}^3$.

Another way of thinking about vector fields would be to imagine a family of **non-intersecting** smooth curves that fill $\mathcal{M}$. Then, for each point on the manifold, we assign to it a vector that is tangent to the curve that is passing through that point. Such curves are known as integral curves.

An example would be consider a magnetic field pointing straight up and changing with time. We know from undergraduate EM that the equipotential curves are just concentric circles, and the electric field at a point is tangent to the concentric circle passing through that point. The concentric circles are the integral curves, and the electric field is the vector field for those integral curves.

Now, let's see the precise meaning of all of the above.

The precise definition of a vector field

Definition 4.1 (Vector field)**.** A vector field X on a C^∞ manifold is a smooth assignment of a tangent vector $X_p \in T_p(\mathcal{M})$ at each point $p \in \mathcal{M}$ where "smooth" is defined to mean that, for all $f \in C^\infty(\mathcal{M})$, the function

$$Xf : \mathcal{M} \to \mathbb{R}$$

defined by:

$$p \to (Xf)(p) = X_p(f)$$

is infinitely differentiable.

[4.1] The meaning of smooth assignment will be explained more precisely later

Remark. There is a lot to unpack in definition 4.1. Points 3, 4 are especially important, because definition 4.1 will make a lot more sense once we go into a local chart.

1. Notice how we wrote X_p instead of X_p^σ. We don't write σ here because in the past, σ referenced an entire equivalence class of curves, where different σ would give different tangent vectors at the point p, but now there is just **one** tangent vector at the point p, which is assigned by the vector field. If you really want to think about curves, then X_p is the vector tangent to the integral curve at the point p.

2. X is a map from $C^\infty(\mathcal{M})$ to $C^\infty(\mathcal{M})$. Notice how $f \in C^\infty(\mathcal{M})$ and $Xf \in C^\infty(\mathcal{M})$. To easily see why $Xf \in C^\infty(\mathcal{M})$, we can write $Xf = X_{(\ )}(f)$ such that:
$$\begin{aligned} Xf : \mathcal{M} &\to \mathbb{R} \\ p &\mapsto X_{(p)}(f) \end{aligned}$$

3. Consider a local chart (U, ϕ). Then, in this local chart, with $x = \phi(p) = (x^1, \dots, x^m)$, we have:
$$\begin{aligned} (Xf)(p) &= X_p(f) \\ &= X^i(x)\frac{\partial}{\partial x^i}\Big(\bar{f}(x)\Big)\Big|_x \end{aligned} \tag{4.1}$$
where $\bar{f} = f \circ \phi^{-1}$, and where we write $X^i(x)$ to remind ourselves that the components of X_p in the local chart depend on $\phi(p) = x$, and we also write $\frac{\partial}{\partial x^i}\big(\bar{f}(x)\big)\big|_x$ to show that this partial derivative is evaluated at the point $\phi(p) = x$.

   Thus, "smooth assignment of of a tangent vector $X_p \in T_p(\mathcal{M})$ at each point $p \in \mathcal{M}$" just means that the last line in equation 4.1 is a smooth function of x when we vary x. Varying x simply means moving to another point p on the manifold, because $x = \phi(p)$. In other words, we want
$$X^i(x)\frac{\partial}{\partial x^i}\Big(\bar{f}(x)\Big)\Big|_x : \mathbb{R}^m \to \mathbb{R}$$
to be an infinitely differentiable function of x.

4. Again, consider a local chart (U, ϕ), where $x = \phi(p)$. If we write
$$(Xf)(p) = (Xf) \circ \phi^{-1} \circ \phi(p) \equiv \overline{Xf}(x)$$
we have, comparing with our work from point 3,
$$\overline{Xf}(x) = X^i(x)\frac{\partial}{\partial x^i}\Big(\bar{f}(x)\Big)\Big|_x$$
Thus, we can write:
$$\begin{aligned} \overline{X} : C^\infty(\mathbb{R}^m) &\to C^\infty(\mathbb{R}^m) \\ \bar{f} &\mapsto \overline{Xf} = X^i(\ )\frac{\partial}{\partial x^i}\Big(\bar{f}(\ )\Big)\Big|_{(\ )} \end{aligned}$$

where

$$\begin{aligned}\overline{Xf} : \mathbb{R}^m &\to \mathbb{R}\\ x &\mapsto X^i(x)\frac{\partial}{\partial x^i}\left(\bar{f}(x)\right)\Big|_x\end{aligned}$$

Thus, we see that the local representation of X in a coordinate chart is

$$\overline{X} = X^i(\ )\frac{\partial}{\partial x^i}$$

I.e, just like how X takes in a point $p \in \mathcal{M}$ and returns a tangent vector X_p at the point $p \in \mathcal{M}$, $\overline{X}$ takes in $x = \phi(p) \in \mathbb{R}^m$ and returns a tangent vector $X^i(x)\frac{\partial}{\partial x^i}$ at the point $x \in \mathbb{R}^m$, where $X^i(x)\frac{\partial}{\partial x^i}$ is a smooth function of x.

The vector space of all vector fields $\mathcal{X}(\mathcal{M})$

Let $\mathcal{X}(\mathcal{M})$ be the set of all vector fields on the manifold $\mathcal{M}$. For $X, Y \in \mathcal{X}(\mathcal{M})$ and $\alpha \in \mathbb{R}$, we define the addition operator between two vector fields and the scalar multiplication operation as:

$$(X+Y)f = X(f) + Y(f) \tag{4.2a}$$

$$(\alpha X)(f) = \alpha(X(f)) \tag{4.2b}$$

$\forall f \in C^\infty(\mathcal{M})$. From definition 4.1, since the addition of two tangent vectors at a point give a third tangent vector at a point, it is clear that $\mathcal{X}(\mathcal{M})$ together with the operations defined in equations 4.2 has a vector space structure.

Thus, we shall denote $\mathcal{X}(\mathcal{M})$ as the vector space of all vector fields on the manifold $\mathcal{M}$.

It is also meaningful to define the multiplication of a function $f \in C^\infty(\mathcal{M})$ with a vector field $X \in \mathcal{X}(\mathcal{M})$. I.e:

$$\begin{aligned}f \cdot X : C^\infty(\mathcal{M}) &\to C^\infty(\mathcal{M})\\ g &\mapsto (f \cdot X)(g)\\ &\equiv f \cdot X(g)\end{aligned}$$

where $f \cdot X(g) \in C^\infty(\mathcal{M})$ is a map from $\mathcal{M}$ to $\mathbb{R}$, i.e

$$(f \cdot X(g))\,(p) = f(p) \cdot (Xg)(p)$$

Vector fields as derivations

From definition 4.1, we see that the map $X : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ has all the properties of a derivation, since each X_p is a derivation. These properties are:

$$X(f+g) = X(f) + X(g) \tag{4.3a}$$

$$X(rf) = rX(f) \tag{4.3b}$$

$$X(f \cdot g) = f \cdot X(g) + g \cdot X(f) \tag{4.3c}$$

where $f, g \in C^\infty(\mathcal{M})$, $r \in \mathbb{R}$.

4.1 Lie algebra structure

Now, we ask ourselves: can two vector fields be multiplied together to give a third field? Since each $X \in \mathcal{X}(M)$ is a map from $C^{\infty}(\mathcal{M})$ to $C^{\infty}(\mathcal{M})$, it seems reasonable to define the multiplication of two fields X, Y as $X \cdot Y \equiv X \circ Y$, where $\forall f \in C^{\infty}(\mathcal{M})$, we have:

$$X \circ Y(f) = X(Y(f)) \tag{4.4}$$

However, equation 4.4 is a bad definition, because $X \circ Y$ is not a vector field; $X \circ Y$ fails to satisfy one of the properties of derivations, namely equation 4.3c. I.e, if $X \circ Y$ were a vector field, we should expect:

$$[X \circ Y](f \cdot g) = f \cdot [X \circ Y](g) + g \cdot [X \circ Y](f)$$

But instead, we have:

$$\begin{aligned} [X \circ Y](f \cdot g) &= X(Y(f \cdot g)) \\ &= X(f \cdot Y(g) + g \cdot Y(f)) \\ &= X(f \cdot Y(g)) + X(g \cdot Y(f)) \\ &= f \cdot X(Y(g)) + Y(g) \cdot X(f) + g \cdot X(Y(f)) + X(g) \cdot Y(f) \\ &= f \cdot [X \circ Y](g) + g \cdot [X \circ Y](f) + \Big\{ Y(g) \cdot X(f) + X(g) \cdot Y(f) \Big\} \end{aligned} \tag{4.5}$$

We see the two extra terms at the end $Y(g) \cdot X(f)$ and $X(g) \cdot Y(f)$ prevent $[X \circ Y]$ from being a vector field.

However, we can easily get rid of the last two terms; if we repeat the same calculations we did to arrive at equation 4.5, but for $[Y \circ X]$, we would arrive at:

$$[Y \circ X](f \cdot g) = f \cdot [Y \circ X](g) + g \cdot [Y \circ X](f) + \Big\{ Y(g) \cdot X(f) + X(g) \cdot Y(f) \Big\} \tag{4.6}$$

Then, subtracting equation 4.6 from equation 4.5, we obtain:

$$[X, Y](f.g) = f[X, Y](g) + g[X, Y](f) \tag{4.7}$$

where we have defined

$$[X, Y] \equiv X \circ Y - Y \circ X \tag{4.8}$$

$[X, Y]$ is known as the commutator of the two vector fields X, Y. It is also called the Lie Bracket, for reasons that will become more clear when we talk about the Lie Derivative.

Thus, we see from equation 4.7 that while $X \circ Y$ and $Y \circ X$ aren't vector fields because they don't satisfy equation 4.3c, the commutator $[X, Y]$ is a vector field because it does (we can also easily show that the commutator satisfies all the other properties of a derivation). Now, let's see what the commutator looks like in a local chart (U, ϕ).

4.1.1 The commutator $[X,Y]$ in a local chart (U,ϕ)

Consider a local chart (U,ϕ). Then, $\forall f \in C^\infty(\mathcal{M})$, $\forall p \in M$, we can write

$$\Big([X,Y](f)\Big)(p) = \Big([X,Y](f)\Big)\circ\phi^{-1}\circ\phi(p) \\ = \overline{[X,Y](f)}(x)$$

where $x=\phi(p)$ and $\overline{[X,Y](f)} = \big([X,Y](f)\big)\circ\phi^{-1}$ Now, we have:

$$\begin{aligned}
\overline{[X,Y](f)}(x) =& \overline{[X(Y(f)) - Y(X(f))]}(x) \\
=& \overline{X(\underbrace{Y(f)}_{\bar{k}\in C^\infty(\mathbb{R}^m)})}(x) - \overline{Y(\underbrace{X(f)}_{\bar{h}\in C^\infty(\mathbb{R}^m)})}(x) \\
=& \overline{X(k)}(x) - \overline{Y(h)}(x) \\
=& X^i(x)\frac{\partial}{\partial x^i}(\bar{k}) - Y^i(x)\frac{\partial}{\partial x^i}(\bar{h}) \\
=& X^i(x)\frac{\partial}{\partial x^i}\left(Y^j(x)\frac{\partial \bar{f}(x)}{\partial x^j}\right) - Y^i(x)\frac{\partial}{\partial x^i}\left(X^j(x)\frac{\partial \bar{f}(x)}{\partial x^j}\right) \\
=& X^i(x)\left(\frac{\partial \bar{f}(x)}{\partial x^j}\frac{\partial Y^j(x)}{\partial x^i} + Y^j(x)\frac{\partial^2 \bar{f}(x)}{\partial x^i \partial x^j}\right) - \\
& Y^i(x)\left(\frac{\partial \bar{f}(x)}{\partial x^j}\frac{\partial X^j(x)}{\partial x^i} + X^j(x)\frac{\partial^2 \bar{f}(x)}{\partial x^i \partial x^j}\right) \\
=& \left(X^i(x)\frac{\partial Y^j(x)}{\partial x^i}\frac{\partial}{\partial x^j} - Y^i(x)\frac{\partial X^j(x)}{\partial x^i}\frac{\partial}{\partial x^j}\right)\bar{f}(x)
\end{aligned}$$

Thus, we see that $\overline{[X,Y]}$ can be written as:

$$\left(X^i\frac{\partial}{\partial x^i}Y^j - Y^i\frac{\partial}{\partial x^i}X^j\right)\frac{\partial}{\partial x^j} \tag{4.9}$$

4.1.2 Properties of the commutator $[X,Y]$

Clearly, from the definiton of the commutator in equation 4.8, we have:

$$[X,Y] = -[Y,X] \tag{4.10a}$$
$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \tag{4.10b}$$

where $X,Y,Z \in \mathcal{X}(\mathcal{M})$. By the way, equation 4.10b is known as the Jacobi identity.[4.2]

Now, the above two properties are essential elements in the structure of a Lie algebra. Formally, a Lie algebra is defined to be a real vector space $\mathcal{L}$ with a bilinear map $\mathcal{L}\times\mathcal{L}\to\mathcal{L}$ denoted by $(A,B)\mapsto[A,B]$ which satisfies the following conditions:

[4.2] Equation 4.10b is proved in the tutorials.

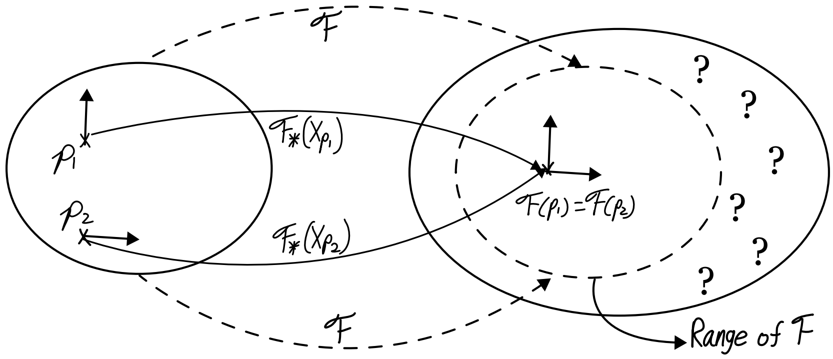

Figure 4.1:

1. $\forall A, B \in \mathcal{L}$,
$$[A, B] = -[B, A]$$
2. $\forall A, B, C \in \mathcal{L}$,
$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$$

We will come back to this when we discuss Lie groups and their associated tangent spaces.

4.2 Mapping of Vector Fields

In section 3.4, we saw how a differentiable map $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ induces a map $\mathcal{F}_* : T_p(\mathcal{M}) \to T_p(\mathcal{N})$ known as the push-forward map.

Question: does $\mathcal{F}$ also induce a map from $\mathcal{X}(\mathcal{M})$ to $\mathcal{X}(\mathcal{N})$?

A natural way to define this induced map would be:

$$(\mathcal{F}_* X)_{\mathcal{F}(p)} = \mathcal{F}_*(X_p) \tag{4.11}$$

I.e, we define a vector field $Y = \mathcal{F}_* X$ in $\mathcal{N}$ such that at the point $\mathcal{F}(p) \in \mathcal{N}$, there is a vector $\mathcal{F}_*(X_p) \in T_{\mathcal{F}(p)}(\mathcal{N})$. Now, equation 4.11 if a good attempt at defining this induced map, but it fails for two reasons:

1. If $\mathcal{F}$ is not injective, then there might be two different points, $p_1, p_2 \in \mathcal{M}$ that map to the same point in $\mathcal{N}$, i.e $\mathcal{F}(p_1) = \mathcal{F}(p_2)$. In this case, should the induced vector at $\mathcal{F}(p_1) = \mathcal{F}(p_2)$ be $\mathcal{F}_*(X_{p_1})$, or $\mathcal{F}_*(X_{p_2})$?
2. If $\mathcal{F}$ is not surjective, then $\exists p \in \mathcal{N}$ that don't have a pre-image in $\mathcal{M}$. How then should we assign a vector to $p \in \mathcal{N}$?

These two reasons are summarised in Figure 4.1. In summary, if $\mathcal{F}$ is not bijective (i.e if $\mathcal{F}$ is not a diffeomorphism), then $\mathcal{F}$ does not induce a map between $\mathcal{X}(\mathcal{M})$ and $\mathcal{X}(\mathcal{N})$.

However, if $\mathcal{F}$ is a diffeomorphism, then the definition in equation 4.11 would be good, and $\mathcal{F}$ does indeed induce a map between $\mathcal{X}(\mathcal{M})$ and $\mathcal{X}(\mathcal{N})$. Let's put this in precise terms.

4.2.1 $\mathcal{F}$ related fields

Definition 4.2 ($\mathcal{F}$ related fields[4.3]). Let $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism, and let $X \in \mathcal{X}(\mathcal{M})$. Then, the diffeomorphism $\mathcal{F}$ induces a $\mathcal{F}$- related field $Y \in \mathcal{X}(\mathcal{N})$ by the following prescription:

$$Y_{\mathcal{F}(p)} = \mathcal{F}_*(X_p)$$

$\forall p \in \mathcal{M}$.

We can write, $Y = \mathcal{F}_* (X)$ for the two $\mathcal{F}$-related vector fields X, Y.

Remark. Informally, we shall say that the vector field Y is the push-forward of the vector field X.

4.2.2 Some properties of $\mathcal{F}$ related fields

Theorem 4.1. *Let $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism. If $X \in \mathcal{X}(\mathcal{M})$ and $Y \in \mathcal{X}(\mathcal{N})$ are $\mathcal{F}$-related, then $\forall f \in C^\infty(\mathcal{N})$,*

$$X (f \circ \mathcal{F}) = Y(f) \circ \mathcal{F}$$

Remark. Given $p \in \mathcal{M}$, we can write theorem 4.1 in the following form:

$$[X (f \circ \mathcal{F})](p) = [Y(f)] (\mathcal{F}(p))$$

Remember that $f \circ \mathcal{F} \in C^\infty(\mathcal{M})$, and thus X maps $f \circ \mathcal{F}$ to another function in $C^\infty(\mathcal{M})$.

Remark. Theorem 4.1 is very important. Given $X \in \mathcal{M}$, we can use theorem 4.1 to actually calculate Y. I.e, we can express the components of Y in some local chart (V, ψ) of $\mathcal{N}$ in terms of the components of X in some local chart (U, ϕ) of $\mathcal{M}$. We will see how this works in corollary 4.1.1.

Remark. Theorem 4.1 is the most important and the most powerful result of this section. All the other theorems in this section can be very easily proved by using this theorem.

Proof of Theorem 4.1. We begin by restating equation 3.7 which was derived in subsection 3.4.1. For a diffeomorphism $\mathcal{F} : \mathcal{M} \to \mathcal{N}$, we have, $\forall f \in C^\infty(\mathcal{N})$,

$$\begin{aligned} W^{\sigma'}_{\mathcal{F}(p)}(f) &\equiv [\mathcal{F}_* \left(V^\sigma_p\right)](f) \\ &= V^\sigma_p(f \circ \mathcal{F}) \end{aligned}$$

where $W^{\sigma'}_{\mathcal{F}(p)} \in T_{\mathcal{F}(p)}(\mathcal{N})$, $\sigma' = \sigma \circ \mathcal{F}$. Applying this result to the the $\mathcal{F}$-related fields Y and X, we have, $\forall p \in \mathcal{M}$ and $\forall f \in C^\infty(\mathcal{N})$,

$$\begin{aligned} & Y_{\mathcal{F}(p)}(f) = \mathcal{F}_*(X_p)(f) \\ \implies & Y_{\mathcal{F}(p)}(f) = X_p(f \circ \mathcal{F}) \\ \implies & [Y(f)](\mathcal{F}(p)) = [X(f \circ \mathcal{F})](p) \\ \implies & [Y(f) \circ \mathcal{F}](p) = [X(f \circ \mathcal{F})](p) \end{aligned}$$

which immediately gives us the desired result. □

[4.3] This definition is different from Kuldip's and is orignal...there might be some bugs.

Corollary 4.1.1 (Theorem 4.1, but in local coordinates.).
Let (U, ϕ) be a local chart in $\mathcal{M}$ and (V, ψ) be a local chart in $\mathcal{N}$. Then, $\overline{\mathcal{F}} = \psi \circ \mathcal{F} \circ \phi^{-1}$ is a map from $\phi(U) \subset \mathbb{R}^M$ to $\psi(V) \subset \mathbb{R}^M$. The situation is depicted in figure 2.2. For $p \in \mathcal{M}$, let $\phi(p) = (x^1, ..., x^m) \equiv x$, and $\psi(\mathcal{F}(p)) = \overline{\mathcal{F}}(x) = (y^1, ..., y^m) \equiv y$. Then, we have:

$$
\begin{aligned}
&[X(\underbrace{f \circ \mathcal{F}}_{\substack{\equiv f', \\ f' \in C^k(\mathcal{M})}})](p) = [Y(f)]\,(\mathcal{F}(p)) \\
\implies\ & [X(f')] \circ \phi^{-1} \circ \phi(p) = [Y(f)] \circ \psi^{-1} \circ \psi \circ \mathcal{F} \circ \phi^{-1} \circ \phi(p) \\
\implies\ & X^i(x) \frac{\partial}{\partial x^i} \bar{f}'(x)\bigg|_x = Y^j(y) \frac{\partial}{\partial y^j} \bar{f}(y)\bigg|_y
\end{aligned}
$$

where in the last line above, we used the fact that $\psi \circ \mathcal{F} \circ \phi^{-1} \circ \phi(p) = \overline{\mathcal{F}}(x) = y$. We also defined $\bar{f}' \equiv f \circ \mathcal{F} \circ \phi^{-1}$, and $\bar{f} \equiv f \circ \psi^{-1}$. Now, we realise that:

$$
\begin{aligned}
\bar{f}' &\equiv f \circ \mathcal{F} \circ \phi^{-1} \\
&= f \circ \psi^{-1} \circ \psi \circ \mathcal{F} \circ \phi^{-1} \\
&= \bar{f} \circ \overline{\mathcal{F}}
\end{aligned}
$$

which means that

$$
\bar{f}'(x) = \bar{f}(y) = \bar{f}(y^1(x^1, ..., x^m), .., y^m(x^1, ..., x^m))
$$

where in the last equality, we realise that $\overline{\mathcal{F}}(x^1, ..., x^m) = (y^1, ..., y^m)$ defines m coordinate transformation equations $\{y^i = y^i(x^1, \ldots, x^m)\}_{i=1,\ldots,m}$

Putting everything together, we have:

$$
\begin{aligned}
X^i(x) \left(\frac{\partial}{\partial x^i} \bar{f}(y^1(x^1, ..., x^m), .., y^m(x^1, ..., x^m)) \right)\bigg|_x &= Y^j(y) \frac{\partial}{\partial y^j} \bar{f}(y)\bigg|_y \\
\implies X^i(x) \frac{\partial y^j}{\partial x^i}\bigg|_x \frac{\partial}{\partial y^j} \bar{f}(y)\bigg|_y &= Y^j(y) \frac{\partial}{\partial y^j} \bar{f}(y)\bigg|_y
\end{aligned}
$$

which gives us:

$$
Y^j(y) = X^i(x) \frac{\partial y^j}{\partial x^i}\bigg|_x \tag{4.12}
$$

Equation 4.12 is super powerful. Given $X^i(x)$, which is the components of X in some local chart (U, ϕ) of $\mathcal{M}$, we are able to determine $Y^j(y)$, which are the components of the $\mathcal{F}$-related field in some local chart (V, ψ) of $\mathcal{N}$. Note that here, $x = \phi(p)$, $y = \psi(p)$, and y is related to x via: $y = \psi \circ \mathcal{F} \circ \phi^{-1}(x) \equiv \overline{\mathcal{F}}(x)$.

Note that

$$
\frac{\partial y^j}{\partial x^i}\bigg|_x
$$

are just elements of the Jacobian matrix of the m coordinate transformation equations $\{y^i = y^i(x^1, ..., x^m)\}_{i=1,...,m}$ induced by $y = \overline{\mathcal{F}}(x)$. Thus, if $\mathcal{M}$ and $\mathcal{N}$ are related by a diffeomorphism $\mathcal{F}$, then tangent vectors at $\mathcal{F}(p) \in \mathcal{N}$ are related to tangent vectors at $p \in \mathcal{M}$ via a linear transformation, where the linear transformation is given by the Jacobian matrix of the coordinate transformation equations. Pretty nifty result, eh.

Theorem 4.2. *If $\mathcal{F}_* X_1 = Y_1$ and $\mathcal{F}_* X_2 = Y_2$, then $\forall f \in C^\infty(\mathcal{N})$ we have:*

$$([Y_1, Y_2](f)) \circ \mathcal{F} = [X_1, X_2](f \circ \mathcal{F})$$

Proof.

$$\begin{aligned}
\text{LHS} &= ([Y_1, Y_2](f)) \circ \mathcal{F} \\
&= (Y_1[Y_2(f)] - Y_1[Y_2(f)]) \circ \mathcal{F} \\
&= Y_1[Y_2(f)] \circ \mathcal{F} - Y_2[Y_1(f)] \circ \mathcal{F} \\
&= X_1[Y_2(f) \circ \mathcal{F}] - X_2[Y_1(f) \circ \mathcal{F}] \\
&= X_1[X_2(f \circ \mathcal{F})] - X_2[X_1(f \circ \mathcal{F})] \\
&= [X_1, X_2](f \circ \mathcal{F}) \\
&= \text{RHS}
\end{aligned}$$

□

Theorem 4.3.

$$\mathcal{F}_*[X_1, X_2] = [\mathcal{F}_* X_1, \mathcal{F}_* X_2]$$

Proof. Let $\mathcal{F}_*[X_1, X_2] = K$, where K is to be determined. Then, K and $[X_1, X_2]$ are $\mathcal{F}$-related fields. Thus, $\forall f \in C^\infty(\mathcal{N})$,

$$\begin{aligned}
K(f) \circ \mathcal{F} &= [X_1, X_2](f \circ \mathcal{F}) \\
&= ([\mathcal{F}_* X_1, \mathcal{F}_* X_1](f)) \circ \mathcal{F}
\end{aligned}$$

Note that to get the last line, we used theorem 4.2. Thus, we have $K = [\mathcal{F}_* X_1, \mathcal{F}_* X_1]$. □

4.3 Integral Curves

Definition 4.3 (Integral Curve). Let V be a vector field on a manifold $\mathcal{M}$ and let p be a point on $\mathcal{M}$. Then an integral curve of V passing through the point p is a curve $t \mapsto \gamma_p(t)$ such that

$$\gamma_p(0) = p \tag{4.13a}$$

$$\gamma_{p*}\left(\frac{d}{dt}\right)\bigg|_{t=s} = V_{\gamma_p(s)} \tag{4.13b}$$

for all s in some open interval $I = (-\epsilon, \epsilon)$ of $\mathbb{R}$.

Remark. Recall that a curve is an injective map from $\mathbb{R}$ to $\mathcal{M}$. Essentially, what this definition is saying is that for $\gamma_p(t)$ to be an integral curve of the vector field V, the push-forward map induced[4.4] by $\gamma_p(t)$ must push-forward the basis vector of $T_s(\mathbb{R})$ to the vector assigned by the vector field at the point $\sigma(s) \in \mathcal{M}$. The other condition, $\gamma_p(0) = 0$ is just to make sure that the curve passes through $p \in \mathcal{M}$.

4.3.1 Local representation of an integral curve

Let's see what definition 4.3 means in a local coordinate chart (U, ϕ). For all $f \in C^\infty(\mathcal{M})$, we shall evaluate the LHS and the RHS of equation 4.13b. The LHS is:

$$\begin{aligned}
\text{LHS} &= \left(\gamma_{p*}\left(\frac{d}{dt}\right)\bigg|_{t=s}\right)(f) \\
&= \frac{d}{dt}\Big(f \circ \gamma_p(t)\Big)\Big|_{t=s} \\
&= \frac{d}{dt}\Big(f \circ \phi^{-1} \circ \phi \circ \gamma_p(t)\Big)\Big|_{t=s} \\
&= \frac{d}{dt}\Big(\bar{f}[\phi(\gamma_p(t))]\Big)\Big|_{t=s} \\
&= \frac{d}{dt}\Big(\bar{f}(x_p^1(t), x_p^2(t), \ldots, x_p^m(t))\Big)\Big|_{t=s} \\
&= \frac{\partial \bar{f}(x_p^1, \ldots, x_p^m)}{\partial x_p^i}\bigg|_{\phi(\gamma_p(s))} \frac{dx_p^i(t)}{dt}\bigg|_{t=s}
\end{aligned}$$

where the subscript p reminds us that $(x_p^1(t), x_p^2(t), \ldots, x_p^m(t))$ is a local representation of the integral curve passing through p. Note that this calculation, though it looks long, is nothing more than finding the tangent vector to the curve $\gamma(t)$ at point p in a local chart (U, ϕ).

Now, the RHS is:

$$\begin{aligned}
\text{RHS} &= V_{\gamma_p(s)}(f) \\
&= [V(f)](\gamma_p(s)) \\
&= [V(f)] \circ \phi^{-1} \circ \phi \circ (\gamma_p(s)) \\
&= [\overline{V(f)}](\overline{\gamma_p}(s)) \\
&= V^i(x_p^1(s), \ldots, x_p^m(s)) \frac{\partial \bar{f}(x_p^1, \ldots, x_p^m)}{\partial x_p^i}\bigg|_{\phi(\gamma_p(s))}
\end{aligned}$$

where $\overline{\gamma_p}(s) = (x^1(s), x^2(s), \ldots, x^m(s))$. Comparing the LHS and the RHS, we have:

$$\begin{aligned}
V^i(x_p^1(s), \ldots, x_p^m(s)) &= \frac{dx_p^i(t)}{dt}\bigg|_{t=s} \\
\Longrightarrow V^i(x_p^1(s), \ldots, x_p^m(s)) &= \frac{dx_p^i(s)}{ds} \qquad (4.14)
\end{aligned}$$

[4.4] See subsection 3.4.2 to see why a curve induces this push-forward map.

for $i = 1, ...m$. Equation 4.14 is a first order differential equation, with the initial condition at $s = 0$, $(x^1(0), x^2(0), ..., x^m(0)) = \phi(p)$. This is the equation which we will use to calculate, in some coordinate chart, the integral curve to the vector field passing through the point $p \in \mathcal{M}$.

The existence and uniqueness theorem of ordinary differential equations guarantees that there is a unique solution to equation 4.14, at least locally in a coordinate chart. It may be the cases that the integral curves is defined only on a setset of $\mathbb{R}$, in which case we have to pay attention so that the parameter s does not exceed the given interval.

It is known that[4.5] if $\mathcal{M}$ is a compact manifold, the integral curves exist for all $s \in \mathbb{R}$. This motivates the important definition of a complete vector field:

Definition 4.4 (Complete vector field). A vector field V on a manifold $\mathcal{M}$ is complete if, at every point $p \in \mathcal{M}$, the integral curve that passes through p can be extended to an integral curve for V that is defined for all $s \in \mathbb{R}$.

[4.5] Not proven in Kuldip's notes, though the proof probably exists somewhere.

Chapter 5

Cotangent Spaces and One Forms

5.1 Linear algebra recap: The dual of a vector space

This section is merely a recap of linear algebra. For more information, one can refer to textbooks on linear algebra...

Consider two vector spaces V and W, and the set of all linear transformations from V to W, denoted as $\mathcal{L}(V, W)$. On this set $\mathcal{L}(V, W)$, we define the binary addition operation as:

$$(T_1 + T_2)(\vec{v}) = T_1(\vec{v}) + T_2(\vec{v})$$

where $T_1, T_2 \in \mathcal{L}(V, W)$ and $\vec{v} \in V$. We also define the scalar multiplication operation as:

$$(cT)(\vec{v}) = cT(\vec{v})$$

where $T \in \mathcal{L}(V, W)$, $\vec{v} \in V$ and $c \in \mathbb{F}$ where $\mathbb{F}$ is any field (e.g $\mathbb{R}$, $\mathbb{C}$).

Theorem 5.1 ($\mathcal{L}(V, W)$ with the binary addition and scalar multiplication defined above is a vector space over $\mathbb{F}$).

Proof. Left as an exercise, just verify that all the axioms of vector spaces are fulfilled. Note that the zero vector in $\mathcal{L}(V, W)$ is the zero map from V to W, and the additive inverse of $T \in \mathcal{L}(V, W)$ is $-T \in \mathcal{L}(V, W)$, defined by $(-T)(\vec{v}) = -T(\vec{v})$ where $\vec{v} \in V$. □

For the rest of this section, we shall assume that the field $\mathbb{F}$ is $\mathbb{R}$. Now, we define the dual space of V as $V^* \equiv \mathcal{L}(V, \mathbb{R})$[5.1]. From theorem 5.1, V^* has a vector space structure. We say that V^* is the set of all linear functionals from V to $\mathbb{R}$, i.e elements of V^* map $\vec{v} \in V$ to $\mathbb{R}$.

Suppose that V has a basis $B = \{e_1, e_2, ...e_n\}$. Then, V^* has a basis $B^* = \{f_1, f_2, ...f_n\}$ defined by:

$$f_i(e_j) = \delta_{ij}$$

Note that we can also write $V = \mathcal{L}(V^*, \mathbb{R})$, i.e we can consider V the elements of V as linear maps from V^* to $\mathbb{R}$. In this case, we can write:

$$e_i(f_j) = \delta_{ij}$$

[5.1] $\mathbb{R}$ has a vector space structure, so we can do this.

where $B = \{e_1, e_2, ...e_n\}$ is a basis of V and $B^* = \{f_1, f_2, ...f_n\}$ is a basis of V^*.

To put both views on equal footing, we shall sometimes write:

$$\langle v, k\rangle \equiv v(k) = k(v)$$

where $v \in V$ and $k \in V^*$.

5.2 Cotangent Spaces

Now, we recall that $T_p(\mathcal{M})$ has a vector space structure. Thus, from section 5.1, we can define the another vector space dual to $T_p(\mathcal{M})$, which we shall denote as the cotangent space $T_p^*(\mathcal{M})$. Let's formalise this idea.

Definition 5.1 (Cotangent Space). The cotangent space at $p \in \mathcal{M}$ is defined as $T_p^*(\mathcal{M}) \equiv \mathcal{L}(T_p(\mathcal{M}), \mathbb{R})$.

Remark. I.e, $k \in T_p^*(\mathcal{M})$ is a real linear map from $T_p(\mathcal{M})$ into $\mathbb{R}$, defined by:

$$\begin{aligned} k : T_p(\mathcal{M}) &\to \mathbb{R} \\ v &\mapsto \langle k, v\rangle_p \end{aligned}$$

where v is any vector in $T_p(\mathcal{M})$, and the subscript p reminds us that all this happens only at a specific point $p \in \mathcal{M}$.

As per discussed in section 5.1, if $T_p(\mathcal{M})$ has a basis $\{e_1, ..., e_m\}$, then the dual basis for $T_p^*(\mathcal{M})$, which we denote as $\{f^1, ..., f^m\}$ can be uniquely determined by requiring that:

$$\langle f^i, e_j\rangle_p = \delta^i_j \tag{5.1}$$

for all $i, j = 1, ..., m$.

5.2.1 Note: positioning of indices

As the astute reader might have noticed, components of tangent vectors are labelled with upper indices, whereas basis vectors are labelled with lower indices. Similarly, components of cotangent vectors will be labelled with lower indices, and basis cotangent vectors will be labelled with upper indices. The position of the indices remind us of the transformation rules under a change of basis (i.e, when we move from one local chart to another). Things with upper indices transform contravariantly, like:

$$v'^i = \frac{\partial x'^i}{\partial x^j} v^j$$

whereas things with lower indices transform covariantly, like:

$$k'_i = \frac{\partial x^j}{\partial x'_i} k_j$$

It has been shown in the derivation of equation 3.4 that the components of a tangent vector do indeed transform contravariantly, and it will be shown in subsection 5.3.2 that components of a cotangent vector transform covariantly.

5.2.2 $T_p^*(\mathcal{M})$ in a local chart (U, ϕ)

Recall that geometrically, tangent vectors are constructed from curves on a manifold. In a local chart (U, ϕ), a tangent vector V_p^σ can be written as:

$$V_p^\sigma \xrightarrow[\text{chart}]{\text{Local}} V^i \frac{\partial}{\partial x^i}$$

Two important concepts to recall:

1. $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}\}$ form a basis for $T_p(\mathcal{M})$ in a local chart. We can regard $\frac{\partial}{\partial x^i}$ as a directional derivative in the direction of increasing x^i.

2. V_p^σ in a local chart is just a linear combination of $\frac{\partial}{\partial x^i}$, where the components V^i serve to produce a new directional derivative in the direction specified by $[\sigma]_p$.

Now, in a local chart, we want the dual basis of $T_p^*(\mathcal{M})$ to be dual to $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}\}$, i.e we want to find a basis of $T_p^*(\mathcal{M})$ such that equation 5.1 holds. Since in a local chart the i-th basis vector of $T_p(\mathcal{M})$ is a directional derivative in the direction of increasing x^i, a natural way to define the dual basis of $T_p^*(\mathcal{M})$ would be to define it as $\{dx^1, ..., dx^m\}$, i.e we define the the j-th element of the dual basis to be a small change in the x^j direction. The result is that we have:

$$\left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle_{\phi(p)} = \frac{\partial}{\partial x^i}(dx^j) = \delta_i^j$$

which automatically fulfils equation 5.1.

Thus, for an arbitrary tangent vector $V^i \frac{\partial}{\partial x^i}$, we have:

$$\left\langle V^i \frac{\partial}{\partial x^i}, dx^j \right\rangle_{\phi(p)} = V^j$$

I.e, the basis cotangent vector dx^j acting on an arbitrary tangent vector $V^i \frac{\partial}{\partial x^i}$ gives the length of that tangent vector in the x^j direction.

Anyway, since in a local chart $\{dx^1, ..., dx^m\}$ is a basis for $T_p^*(\mathcal{M})$, we see that for $k \in T_p^*(\mathcal{M})$, k has local representation $k_i dx^i$.

5.3 The pull-back map between cotangent spaces

Recall from section 3.4 that under a mapping $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ from a manifold $\mathcal{M}$ into a manifold $\mathcal{N}$, we could define the notion of a push-forward map between the tangent spaces of the two manifolds

$$\mathcal{F}_* : T_p(\mathcal{M}) \to T_{\mathcal{F}(p)}(\mathcal{N})$$

In the sense of cotangent spaces the map $\mathcal{F}$ induces a "pull-back" map between the two cotangent spaces:

$$\mathcal{F}^* : T^*_{\mathcal{F}(p)}(\mathcal{N}) \to T_p^*(\mathcal{M})$$

defined through

$$\langle \mathcal{F}^* k, v \rangle_p = \langle k, \mathcal{F}_* v \rangle_{\mathcal{F}(p)}$$

for all $k \in T^*_{\mathcal{F}(p)}(N)$ and $v \in T_p(\mathcal{M})$. Now, at this juncture we shall introduce a theorem regarding this pull-back map.

Theorem 5.2 (Composition of pull-back maps). *Suppose that $\mathcal{M}$,$\mathcal{N}$,$\mathcal{P}$ are three differentiable manifolds with differentiable maps:*

$$\mathcal{M} \xrightarrow{\mathcal{F}_1} \mathcal{N} \xrightarrow{\mathcal{F}_2} \mathcal{P}$$

Then, we have:

$$(\mathcal{F}_2 \circ \mathcal{F}_1)^* = \mathcal{F}_1^* \circ \mathcal{F}_2^*$$

Rough sketch of a proof. First, we can easily prove that $(\mathcal{F}_2 \circ \mathcal{F}_1)_* = \mathcal{F}_{2*} \circ \mathcal{F}_{1*}$. Then, we just use the definition of the pull-back map twice. □

The pull-back map defined above is very useful, and allows us to determine many properties of cotangent vectors based on the properties of tangent vectors. We shall see two applications below.

5.3.1 Application one: Constructing the basis for $T_p^*(\mathcal{M})$

Recall from subsection 3.4.3 that if $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^1}\}$ is a basis of $T_p(\mathcal{M})$ in a local chart (U, ϕ), then the push forward map ϕ_*^{-1} allows us to determine the basis $\{e_1, ..., e_m\}$ of $T_p(\mathcal{M})$. We can construct the basis for $T_p^*(\mathcal{M})$ in a similar way. We have:

$$\begin{aligned}
\left\langle \phi^* dx^i, e_j \right\rangle_p &= \left\langle dx^i, \phi_* e_j \right\rangle_{\phi(p)} \\
&= \left\langle dx^i, \frac{\partial}{\partial x^j} \right\rangle_{\phi(p)} \\
&= \delta^i_j
\end{aligned}$$

which tells if we define $f^i = \phi^* dx^i$, then $\{f^1, ..., f^m\}$ forms the dual basis for $T_p^*(\mathcal{M})$.

5.3.2 Application two: Transformation property of cotangent vectors under a change of coords

Suppose that (U, ϕ) and (V, ψ) are two charts of a manifold $\mathcal{M}$. Consider a point $p \in U \cap V$. Let $k \in T_p^*(\mathcal{M})$ and $v \in T_p(\mathcal{M})$. Also, let $\bar{k}, \bar{v}$ and $\bar{k}', \bar{v}'$ be the coordinate representations of k, v in the local charts (U, ϕ), (V, ψ) respectively. Aim: determine how the components of $\bar{k}'$ are related to the components of $\bar{k}$.
Since we have:

$$\begin{aligned}
\left\langle \bar{k}, \bar{v} \right\rangle_{\phi(p)} &= \left\langle \bar{k}, \phi_* v \right\rangle_{\phi(p)} \\
&= \left\langle \phi^* \bar{k}, v \right\rangle_p \\
&= \langle k, v \rangle_p
\end{aligned}$$

and we can similarly show:

$$\left\langle \bar{k}', \bar{v}' \right\rangle_{\psi(p)} = \langle k, v \rangle_p$$

we arrive at this result:

$$\left\langle \bar{k}', \bar{v}' \right\rangle_{\psi(p)} = \left\langle \bar{k}, \bar{v} \right\rangle_{\phi(p)} \tag{5.2}$$

First, we evaluate the LHS of equation 5.2 to give us:

$$\begin{aligned} \left\langle \bar{k}', \bar{v}' \right\rangle_{\psi(p)} &= \bar{k}'_i \bar{v}'^j \left\langle dx'^i, \frac{\partial}{\partial x'^j} \right\rangle \\ &= \bar{k}'_i \bar{v}'^j \delta^i_j \\ &= \bar{k}'_i \bar{v}'^i \end{aligned} \tag{5.3}$$

Then, we can simiarly evaluate the RHS of equation 5.2 to give us:

$$\left\langle \bar{k}, \bar{v} \right\rangle_{\phi(p)} = \bar{k}_j \bar{v}^j \tag{5.4}$$

Now, we know that $\overline{\mathcal{F}_*}\bar{v} = \bar{v}'$, and we also know that $\bar{v}'^i = \frac{\partial x'^i}{\partial x^j} \bar{v}^j$. Thus, using this as well as equations 5.3, 5.4 we have:

$$\begin{aligned} \left\langle \bar{k}', \bar{v}' \right\rangle_{\psi(p)} &= \left\langle \bar{k}, \bar{v} \right\rangle_{\phi(p)} \\ \implies \bar{k}'_i \bar{v}'^i &= \bar{k}_j \bar{v}^j \\ \implies \bar{k}'_i \left(\frac{\partial x'^i}{\partial x^j} \bar{v}^j \right) &= \bar{k}_j \bar{v}^j \\ \implies \bar{k}'_i \left(\frac{\partial x'^i}{\partial x^j} \right) &= \bar{k}_j \end{aligned} \tag{5.5}$$

We can then easily invert[5.2] equation 5.5 to give us:

$$\bar{k}'_i = \left(\frac{\partial x^j}{\partial x'^i} \right) \bar{k}_j \tag{5.6}$$

which tells us how the components of a cotangent vector transform under coordinate transformation. As can be seen, the components transform covariantly.

5.4 One-forms

Definition 5.2 (1-form). A one-form ω on $\mathcal{M}$ is a smooth assignment of a cotangent vector $\omega_p \in T^*_p(\mathcal{M})$ to each point $p \in \mathcal{M}$. Here, "smooth" means that for any vector field $X \in \mathcal{X}(\mathcal{M})$, the real-valued function

$$\langle \omega, \mathcal{X} \rangle (p) = \left\langle \omega_p, X_p \right\rangle$$

is smooth.

[5.2] We do this inversion either by swapping the primed and unprimed variables, or we write the equation in matrix form and then invert. When we do the latter approach, we realise that $\left(\frac{\partial x^j}{\partial x'^i}\right)$ is nothing more than just entries of the Jacobian matrix of the coordinate transform.

Remark. To better understand the notion of smoothness, we first go to a local coordinate chart (U, ϕ). We note that:

$$\begin{aligned}\left\langle \omega_p, X_p \right\rangle_p &= \left\langle \phi^* \bar{\omega}_p, X_p \right\rangle_p \\ &= \left\langle \bar{\omega}_p, \phi_* X_p \right\rangle_{\phi(p)} \\ &= \left\langle \bar{\omega}_p, \bar{X}_p \right\rangle_x\end{aligned}$$

where in the last line, we have defined $x = \phi(p)$. Now, since $\bar{X}_p = \bar{X}^i(x)\frac{\partial}{\partial x^i}$ and $\bar{\omega}_p = \bar{\omega}_j(x)dx^j$, we have:

$$\left\langle \omega_p, X_p \right\rangle_p = \bar{\omega}_i(x)\bar{X}^i(x) \tag{5.7}$$

Thus, "smooth assignment of a cotangent vector ω_p" in definition 5.2 just means that for any chart (U, ϕ) of $\mathcal{M}$, and for any $X \in \mathcal{X}(\mathcal{M})$, the expression $\bar{\omega}(x)_i \bar{X}^i(x)$ is a smooth function of x.

In fact, since $X^i(x)$ is by definition already a smooth function of x, we just require $\omega_i(x)$ to be a smooth function of x.

TL:DR Let (U, ϕ) is an arbitrary chart of the manifold, and let $x = \phi(p)$ for arbitrary $p \in \mathcal{M}$. A one-form ω has a local coord representation of $\omega_i(x)dx^i$. The function $\omega_i(x)$ is a smooth function of x.

5.4.1 The pull-back of a one-form

The pull-back of a one-form is very similar to the pull-back of cotangent vector. Just note that since a one-form is a smooth assignment of cotangent vectors to all points in the manifold, we have to pull-back at every point in the manifold.

Definition 5.3 (Pull-back of a 1-form). Let $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ be a differentiable map[5.3] between two manifolds $\mathcal{M}$ and $\mathcal{N}$. If ω is a one-form on $\mathcal{N}$ then the pull-back of ω is the one-form $\mathcal{F}^*\omega$ on $\mathcal{M}$ defined by:

$$\langle \mathcal{F}^*\omega, v\rangle_p = \langle \omega, \mathcal{F}_* v\rangle_{\mathcal{F}(p)}$$

for all points $p \in \mathcal{M}$ and all tangent vectors $v \in T_p(\mathcal{M})$.

[5.3] I'm not sure if differentiable map is good enough...do we actually require a diffeomorphism? I'm not sure.

Chapter 6

Tensors and Tensor fields

6.1 Preliminaries: Linear Algebra recap

6.1.1 Multilinear maps

Definition 6.1 (Multilinear maps). Let $V_1, V_2, ..., V_n; W$ be vector spaces. Let $V_1 \times V_2, ..., \times V_n$ be set[6.1] of all ordered n-tuples $(v_1, v_2, ..., v_n)$, where $v_i \in V_i$. A mapping

$$\phi : V_1 \times V_2 \times ... \times V_n \to W$$

is called multilinear if it satisfies the condition

$$\begin{aligned}\phi(v_1, v_2, ..., (\alpha v_i + \beta v_i'), v_{i+1}, ..., v_n) = \\ \alpha\phi(v_1, v_2, ..., v_i, v_{i+1}, ..., v_n) + \beta\phi(v_1, v_2, ..., v_i', v_{i+1}, ..., v_n)\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$, for $i = 1, 2, ..., n$.

Remark. Roughly speaking, a map ϕ is said to be multilinear if it is linear in each of its "variables" separately. We will denote the set of all multilinear maps of $V_1 \times V_2, ..., \times V_n$ into W as $\mathcal{L}(V_1, V_2, ..., V_n; W)$. Then, the set $\mathcal{L}(V_1, V_2, ..., V_n; W)$ becomes a vector space in a natural way. First, we define the binary addition operator as:

$$(\phi_1 + \phi_2)(v_1, v_2, ..., v_n) = \phi_1(v_1, v_2, ..., v_n) + \phi_2(v_1, v_2, ..., v_n)$$

and the scalar multiplication operator as:

$$(\alpha\phi)(v_1, v_2, ..., v_n) = \alpha\phi(v_1, v_2, ..., v_n)$$

where $\phi_1, \phi_2, \phi \in \mathcal{L}(V_1, V_2, ..., V_n; W)$ and $\alpha \in \mathbb{R}$. Then we can easily show that the set $\mathcal{L}(V_1, V_2, ..., V_n; W)$ with the binary addition and scalar multiplication operators fulfil the axioms of a vector space.

It is important to note that the vector spaces $V_1, V_2, ..., V_n$ can be either the tangent spaces $T_p(\mathcal{M})$ or the cotangent spaces $T_p^*(\mathcal{M})$, since both of these are vector spaces. The vector space W which appears in $\mathcal{L}(V_1, V_2, ..., V_n; W)$ can also be $\mathbb{R}$ (which also has a vector space structure).

[6.1] This is just the Cartesian product btw.

6.1.2 Tensor product of vector spaces

Recall from section 5.1 that if we have a vector space V_1, then $V_1^* \equiv \mathcal{L}(V_1, \mathbb{R})$ is the set of all linear maps from V_1 to $\mathbb{R}$. We can also similarly write $V_1 = \mathcal{L}(V_1^*; \mathbb{R})$, i.e we can also regard V_1 as the set of all linear maps from V_1^* to $\mathbb{R}$. We shall use this idea to define the concept of a tensor product of two vector spaces.

Illustration with two vector spaces V_1, V_2

Suppose that V_1 has a basis $\{e_1, e_2, ..., e_n\}$ and V_1^* has a corresponding dual basis $\{f_1, ..., f_n\}$. Also, suppose we have another vector space $V_2 \equiv \mathcal{L}(V_2, \mathbb{R})$, with its corresponding dual V_2^*. Let the basis of V_2 be $\{e'_1, e'_2, ..., e'_n\}$, and let the basis of V_2^* be $\{f'_1, f'_2, ..., f'_n\}$.
Then, we define the tensor product of V_1^* and V_2^* as:

$$V_1^* \otimes V_2^* \equiv \mathcal{L}(V_1, V_2; \mathbb{R}) \tag{6.1}$$

From subsection 6.1.1, we see that $V_1^* \otimes V_2^*$ has a vector space structure. We shall define the basis of $V_1^* \otimes V_2^*$ as a set of n^2 bilinear maps from $V_1 \times V_2 \to \mathbb{R}$

$$\{f_i \otimes f'_j \ |i, j = 1, 2, ...n\}$$

such that

$$[f_i \otimes f'_j](e_k, e'_h) = \delta_{ik}\delta_{jh}$$

In other words, for $a = a_i e_i \in V_1$ and $b = b'_j e'_j \in V_2$, we have:

$$\begin{aligned}
[f_i \otimes f'_j](a, b) &= [f_i \otimes f'_j](a_k e_k, b'_h e'_h) \\
&= a_k b_h [f_i \otimes f'_j](e_k, e'_h) \\
&= a_i b_j
\end{aligned}$$

But anyway, since $\{f_i \otimes f'_j \ |i, j = 1, 2, ...n\}$ form a basis for $V_1^* \otimes V_2^* \equiv \mathcal{L}(V_1, V_2; \mathbb{R})$, we see that we can write $p = p_{ij} f_i \otimes f'_j$ as an arbitrary element of $V_1^* \otimes V_2^* \equiv \mathcal{L}(V_1, V_2; \mathbb{R})$, where p_{ij} are n^2 real numbers.

In fact from all the discussion above, we can similarly define:

1. $V_1 \otimes V_2 \equiv \mathcal{L}(V_1^*, V_2^*; \mathbb{R})$ where $q \in V_1 \otimes V_2$ can be written as $q_{ij} e_i \otimes e'_j$ and $[e_i \otimes e'_j](f_k, f'_h) = \delta_{ik}\delta_{jh}$.
2. $V_1 \otimes V_2^* \equiv \mathcal{L}(V_1^*, V_2; \mathbb{R})$ where $r \in V_1 \otimes V_2^*$ can be written as $r_{ij} e_i \otimes f'_j$ and $[e_i \otimes f'_j](f_k, e'_h) = \delta_{ik}\delta_{jh}$.
3. $V_1^* \otimes V_2 \equiv \mathcal{L}(V_1, V_2^*; \mathbb{R})$ where $s \in V_1^* \otimes V_2$ can be written as $s_{ij} f_i \otimes e'_j$ and $[f_i \otimes e'_j](e_k, f'_h) = \delta_{ik}\delta_{jh}$.

Of course, everything here can be trivially generalised; we can have $V_1 \otimes V_2 \otimes V_3 \equiv \mathcal{L}(V_1^*, V_2^*, V_3^*; \mathbb{R})$ etc etc. We shall call elements of a tensor product space tensors. From our discussion, we note that tensors are nothing more than multilinear maps.

Tensor product of two vectors

We can also define the notion of a tensor product of two vectors. Consider two vectors $v_1, v_2 \in V$. Then, $v_1 \otimes v_2$ is an element of $V \otimes V \equiv \mathcal{L}(V^*, V^*; \mathbb{R})$.

Theorem 6.1 (Properties of the tensor product of vectors). *Suppose v_1, v_2, w_1, w_2 are all elements of some vector space V, and suppose $\alpha \in \mathbb{R}$. Then we have:*

1. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$
2. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$
3. $(\alpha v) \otimes w = \alpha(v \otimes w)$
4. $v \otimes (\alpha w) = \alpha(v \otimes w)$

Remark. The above theorem is also true when $v_1, v_2, w_1, w_2 \in V^*$ instead too; we just have to change the proof method below slightly.

Rough sketch of a proof. We shall prove item 1, the rest can be proven in a similar fashion. Let (k_1, k_2) be an arbitrary element in $V^* \times V^*$, and let $\{e_1, ..., e_n\}$ be a basis for V. For item 1, we have:

$$\begin{aligned}
[(v_1 + v_2) \otimes w](k_1, k_2) &= [(v_1 + v_2)_i e_i \otimes w_h e_h](k_1, k_2) \\
&= [(v_1 + v_2)_i w_h (e_i \otimes e_h)](k_1, k_2) \\
&= [(v_{1i} + v_{2i}) w_h (e_i \otimes e_h)](k_1, k_2) \\
&= (v_{1i} + v_{2i}) w_h [(e_i \otimes e_h)(k_1, k_2)] \\
&= (v_{1i} w_h + v_{2i} w_h)[(e_i \otimes e_h)(k_1, k_2)] \\
&= v_{1i} w_h [(e_i \otimes e_h)(k_1, k_2)] + v_{2i} w_h [(e_i \otimes e_h)(k_1, k_2)] \\
&= v_{1i} w_h [(e_i \otimes e_h)(k_1, k_2)] + v_{2i} w_h [(e_i \otimes e_h)(k_1, k_2)] \\
&= v_1 \otimes w + v_2 \otimes w
\end{aligned}$$

□

6.2 Tensors in Differential Geometry

Now, we shall apply everything derived in section 6.1 to differential geometry. Recall that elements of $T_p(\mathcal{M})$ can be regarded as linear maps from $T_p^*(\mathcal{M}) \to \mathbb{R}$, and elements of $T_p^*(\mathcal{M})$ can be regarded as linear maps from $T_p(\mathcal{M}) \to \mathbb{R}$.

Definition 6.2 (Tensor product of cotangent and tangent spaces). The tensor product

$$\bigotimes^r T_p^*(\mathcal{M}) \bigotimes^s T_p(\mathcal{M})$$

of r cotangent spaces and s tangent spaces at $p \in \mathcal{M}$, called the space of r-covariant s-contravariant tensors $T_p^{r,s}(\mathcal{M})$ is the vector space of all multilinear maps on the cartesian product:

$$\underbrace{T_p(\mathcal{M}) \times T_p(\mathcal{M}) \times \ldots \times T_p(\mathcal{M})}_{r} \times \underbrace{T_p^*(\mathcal{M}) \times T_p^*(\mathcal{M}) \times \ldots \times T_p^*(\mathcal{M})}_{s}$$

to the real space $\mathbb{R}$.

Tl;Dr of definition 6.2

$$\bigotimes^{r} T_p^*(\mathcal{M}) \bigotimes^{s} T_p(\mathcal{M}) \equiv \mathcal{L}(\underbrace{T_p(\mathcal{M}), T_p(\mathcal{M}), \ldots, T_p(\mathcal{M})}_{r}, \underbrace{T_p^*(\mathcal{M}), T_p^*(\mathcal{M}), \ldots, T_p^*(\mathcal{M})}_{s}; \mathbb{R})$$

Remark. Let $\{e_1, \ldots, e_n\}$ be a basis for $T_p(\mathcal{M})$, and let $\{f^1, \ldots, f^n\}$ be a basis for $T_p^*(\mathcal{M})$. Then, we can write an arbitrary element of $\bigotimes^{r} T_p^*(\mathcal{M}) \bigotimes^{s} T_p(\mathcal{M})$ as:

$$a \in \bigotimes^{r} T_p^*(\mathcal{M}) \bigotimes^{s} T_p(\mathcal{M}) = a^{i_1, i_2, \ldots, i_s}_{j_1, j_2, \ldots, j_r} f^{j_1} \otimes f^{j_2} \otimes \ldots \otimes f^{j_r} \otimes e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_s} \tag{6.2}$$

In a local coordinate chart (U, ϕ), just make the substitutions:

$$e_j \to \frac{\partial}{\partial x^j}, \quad f^i \to dx^i$$

in equation 6.2.

6.2.1 Transformation properties of tensors

We first derive the transformation properties of the basis vectors $\frac{\partial}{\partial x^i}$ and dx^i.

Consider two charts (U, ϕ) and (V, ψ) of a manifold $\mathcal{M}$. For $p \in \mathcal{M}$, let $\phi(p) = (x^1, \ldots, x^m)$ and let $\psi(p) = (x'^1, \ldots, x'^m)$. Let $v \in T_p(\mathcal{M})$. Then, we have:

$$v \xrightarrow[(U,\phi)]{\text{Local chart}} v^j \frac{\partial}{\partial x^j}$$

and

$$v \xrightarrow[(V,\psi)]{\text{Local chart}} v'^i \frac{\partial}{\partial x'^i}$$

We recall that for a tangent vector $v \in T_p(\mathcal{M})$, under a change of coordinates[6.2], we have:

$$v'^i = \frac{\partial x'^i}{\partial x^j} v^j$$

[6.2] I.e, when we go from one local chart to another.

Thus, we have:

$$v \xrightarrow[(V,\psi)]{\text{Local chart}} v^j \frac{\partial x'^i}{\partial x^j} \frac{\partial}{\partial x'^i}$$

Comparing this result with the expressions above, we see that when we go from the (U, ϕ) chart to the (V, ψ) chart, we can consider the basis vector has having undergone the following transformation:

$$\frac{\partial}{\partial x^j} = \frac{\partial x'^i}{\partial x^j} \frac{\partial}{\partial x'^i} \tag{6.3}$$

which tbh is not that impressive of a result; it is nothing but the chain rule.

Similarly, for $\omega \in T_p^*(\mathcal{M})$, we have:

$$\omega \xrightarrow[(U,\phi)]{\text{Local chart}} \omega_i dx^i$$

and

$$\omega \xrightarrow[(V,\psi)]{\text{Local chart}} \omega'_j dx'^j$$

Recall again that for a cotangent vector $\omega \in T_p^*(\mathcal{M})$, under a change of coordinates, we have:

$$\omega'_j = \omega_i \frac{\partial x^i}{\partial x'^j}$$

Thus, we have:

$$\omega \xrightarrow[(V,\psi)]{\text{Local chart}} \omega_i \frac{\partial x^i}{\partial x'^j} dx'^j$$

Comparing this result with the expressions above, we see that when we go from the (U, ϕ) chart to the (V, ψ) chart, we can consider the basis vector has having undergone the following transformation:

$$dx^i = \frac{\partial x^i}{\partial x'^j} dx'^j \tag{6.4}$$

which tbh is not that impressive of a result again; it is nothing but the chain rule.

Now, consider $a \in T_p^{r,s}(\mathcal{M})$. We have:

$$a \xrightarrow[(U,\phi)]{\text{Local chart}} a^{i_1,i_2,\ldots,i_s}_{j_1,j_2,\ldots,j_r} dx^{j_1} \otimes dx^{j_2} \otimes \ldots \otimes dx^{j_r} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_s}} \tag{6.5}$$

and

$$a \xrightarrow[(V,\psi)]{\text{Local chart}} a'^{k_1,k_2,\ldots,k_s}_{h_1,h_2,\ldots,h_r} dx'^{h_1} \otimes dx'^{h_2} \otimes \ldots \otimes dx'^{h_r} \otimes \frac{\partial}{\partial x'^{k_1}} \otimes \frac{\partial}{\partial x'^{k_2}} \otimes \ldots \otimes \frac{\partial}{\partial x'^{k_s}} \tag{6.6}$$

Now, putting in the results from equation 6.3 and equation 6.4 in equation 6.5, and then comparing with equation 6.6 we have:

$$a'^{k_1,k_2,\ldots,k_s}_{h_1,h_2,\ldots,h_r} = a^{i_1,i_2,\ldots,i_s}_{j_1,j_2,\ldots,j_r} \left(\frac{\partial x^{j_1}}{\partial x'^{h_1}} \frac{\partial x^{j_2}}{\partial x'^{h_2}}, \ldots, \frac{\partial x^{j_r}}{\partial x'^{h_r}} \right) \left(\frac{\partial x'^{k_1}}{\partial x^{i_1}} \frac{\partial x'^{k_2}}{\partial x^{i_2}}, \ldots, \frac{\partial x'^{k_s}}{\partial x^{i_s}} \right) \tag{6.7}$$

which is the transformation rule for the components of the tensor $a \in T_p^{r,s}(\mathcal{M})$ under a change of coordinates.

6.2.2 Tensor contraction

To do!

6.3 Tensor fields on a manifold

Definition 6.3 (Tensor field). An $(r-s)$ tensor field on a manifold is smooth assignment of a r-covariant, s-contravariant tensor to each point $p \in \mathcal{M}$.

Remark. In a chart (U, ϕ), with $x = \phi(p)$, the local representation of a tensor field is:

$$a^{i_1,i_2,\ldots,i_s}_{j_1,j_2,\ldots,j_r}(x) dx^{j_1} \otimes dx^{j_2} \otimes \ldots \otimes dx^{j_r} \otimes \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_2}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_s}}$$

Smooth just means that the function $a^{i_1,i_2,\ldots,i_s}_{j_1,j_2,\ldots,j_r}(x)$ is a smooth function of x.

6.3.1 Transformation properties of tensor fields

Under a change of coordinates, tensor fields inherit their transformation properties directly from how tensors transform. I.e, we have:

$$a'^{k_1,k_2,\ldots,k_s}_{h_1,h_2,\ldots,h_r}(x') = a^{i_1,i_2,\ldots,i_s}_{j_1,j_2,\ldots,j_r}(x) \left(\frac{\partial x^{j_1}}{\partial x'^{h_1}} \frac{\partial x^{j_2}}{\partial x'^{h_2}}, \ldots, \frac{\partial x^{j_r}}{\partial x'^{h_r}} \right) \left(\frac{\partial x'^{k_1}}{\partial x^{i_1}} \frac{\partial x'^{k_2}}{\partial x^{i_2}}, \ldots, \frac{\partial x'^{k_s}}{\partial x^{i_s}} \right) \tag{6.8}$$

Chapter 7

Local flows and the Lie Derivative

7.1 Local One Parameter Group of Local Diffeomorphisms

Definition 7.1 (Local One Parameter Group of Local Diffeomorphisms). A **L**ocal **O**ne **P**arameter **G**roup of **L**ocal **D**iffeomorphisms (henceforth LOPGOLD for short) at a point $p \in \mathcal{M}$ consists of:

1. An open neighbourhood U of p
2. An interval $I \subset \mathbb{R}$
3. A family $\{\gamma_t \,|t \in I\}$ of diffeomorphisms of from U onto open sets $\gamma_t(U) \subset \mathcal{M}$ with the following properties

   a) The map

   $$\begin{aligned} I \times U &\to \mathcal{M} \\ (t, p) &\mapsto \gamma(t, p) \equiv \gamma_t(p) \end{aligned}$$

   is a smooth function of both t and p

   b) If $t, s, t + s \in I$ and if $p, \gamma_t(p) \in U$ then

   $$\gamma_s \circ \gamma_t(p) = \gamma_{s+t}(p)$$

   c) For each $p \in U$,

   $$\gamma_0(p) = p$$

   d) Lastly, the inverse map is obtained by setting $t \to -t$, i.e:

   $$(\gamma_t)^{-1} = \gamma_{-t}$$

Remark. A few remarks are necessary here.

1. The word "local" in the definition appears twice. The first implies that the maps γ_t are defined only for $t \in I \subset \mathbb{R}$. The second implies that the diffeomorphisms are defined only for open subsets U of $\mathcal{M}$. If $I = \mathcal{R}$, then we would have a one parameter group of local diffeomorphisms.

2. There is a group structure within the family $\{\gamma_t \,|t \in I\}$ of diffeomorphisms. $\gamma_s \circ \gamma_t(p) = \gamma_{s+t}(p)$ defines the group binary multiplication operation which is clearly associative, i.e

$$\gamma_s \circ (\gamma_t \circ \gamma_r) = \gamma_{s+t+r} = (\gamma_s \circ \gamma_t) \circ \gamma_r$$

Moreover, the identity element γ_0 is well defined, as well as is the inverse element $(\gamma_t)^{-1} = \gamma_{-t}$. Note that in this case, the group is an abelian group.

7.1.1 Obtaining a LOPGOLD from the integral curves of a vector field

Recall from definition 4.3 that an integral curve of a vector field V, we fix a point $p \in \mathcal{M}$, and put in various values of $t \in I$ to get another point $\gamma_t(p) \in \mathcal{M}$.

But what if instead we fix the value of t in the integral curve, and put in various values of $p \in U$, where U is an open subset of $\mathcal{M}$? I.e, from the definition of an integral curve, we are free to define a map $\gamma_t : U \to \mathcal{M}$ where $t \in I \subset \mathbb{R}$, and $\gamma_t(p) \in \mathcal{M}$. We say that γ_t is a family of maps parametrized by t. Or in fact, we can even define a map γ as follows:

$$\begin{aligned} &I \times U \to \mathcal{M} \\ &(t, p) \mapsto \gamma(t, p) \equiv \gamma_t(p) = \gamma_p(t) \end{aligned} \tag{7.1}$$

Now, we shall prove that the map γ defined above in equation 7.1 satisfies all the requirements of a LOPGOLD.

Theorem 7.1. *γ as defined above in equation 7.1 satisfies:*

$$\gamma_{t+s}(p) = \gamma_t \circ \gamma_s(p) \tag{7.2}$$

Proof. We shall proceed by showing that in a local chart (U, ϕ), both the LHS and the RHS of equation 7.2 are solutions of the following differential equation:

$$V^i(x^1_p(t), ..., x^m_p(t)) = \frac{dx^i_p(t)}{dt} \tag{7.3}$$

with the initial condition:

$$\left(x^1_p(0), ..., x^m_p(0)\right) = (x^1_p, ..., x^m_p) \tag{7.4}$$

where $(x^1_p(t), ..., x^m_p(t)) = \phi \circ \gamma_p(t)$ is the representation of the integral curve in the coordinate chart.

Then, since the solutions of a differential equation are unique, if both the LHS and RHS of equation 7.2 are solutions of equation 7.3 with the same initial condition 7.4, then the LHS must equal to the RHS. First, we note that the LHS of equation 7.2 satisfies the ODE with the initial condition in this way:

$$\frac{d}{dt}x^i_p(t+s) = \frac{d}{d(t+s)}x^i_p(t+s) = V^i(x^1_p(s+t), ..., x^m_p(s+t))$$
$$\left(x^1_p(0+s), ..., x^m_p(0+s)\right) = (x^1_p(s), ..., x^m_p(s))$$

Note that the first equation above is satisfied because γ is an integral curve. On the other hand, the RHS of equation 7.2 satisfies the ODE with the initial condition in this way:

$$\frac{d}{dt}x^i_{\gamma_s(p)}(t) = V^i(x^1_{\gamma_s(p)}(t), ..., x^m_{\gamma_s(p)}(t))$$
$$\left(x^1_{\gamma_s(p)}(0), ..., x^m_{\gamma_s(p)}(0)\right) = (x^1_{\gamma_s(p)}, ..., x^m_{\gamma_s(p)}) = (x^1_p(s), ..., x^m_p(s))$$

Note also that the first equation above is satisfied because γ is an integral curve.

Since both the LHS and RHS of equation 7.2 are solutions of equation 7.3 with the same initial condition 7.4, (i.e, the expression for $t = 0$ is the same), we can conclude that the LHS and RHS of equation 7.2 are equal. □

Theorem 7.2. *γ_t is family of diffeomorphisms from an open set U of $\mathcal{M}$ to open sets $\gamma_t(U) \subset \mathcal{M}$*

Proof. Since $\gamma_t(p) = \gamma_p(t)$ are integral curves of vector fields, and vector fields are smooth, this means that γ_t is a smooth function of both p and t. Since we can also choose[7.1] the map γ_t to be bijective, γ_t is a diffeomorphism. □

Theorem 7.3. *The family $\{\gamma_t \,|t \in I\}$ of diffeomorphisms of from U onto open sets $\gamma_t(U) \subset \mathcal{M}$, together with the binary multiplication operation*

$$\gamma_s \circ \gamma_t = \gamma_{s+t}$$

and γ_0 as the identity element and

$$(\gamma_t)^{-1} = \gamma_{-t}$$

as the inverse element of γ_t, has a group structure.

Proof. This should be self evident especially after equation 7.4 was proven. □

Thus, from all that was discussed here, we see that the integral curves of a vector field V naturally gives rise to a LOPGOLD on a manifold $\mathcal{M}$.

[7.1] Okay this is kinda handwavy, please forgive me lol I am just a student..:(

7.1.2 Obtaining a vector field from a LOPGOLD

Suppose that at a point p in an open subset U, we have a LOPGOLD. I.e, we have a map $p \to \gamma_t(p)$. This allows us to define a curve $\gamma_p(t)$ passing through the point p. Then, we can obtain a vector field on U by taking the tangents to this family of curves at each point $p \in U$. The resulting vector field V^γ is said to be induced by the family of local diffeomorphisms, i.e we have

$$V_p^\gamma(f) = \frac{d}{dt}\left[f \circ \gamma_t(p)\right]\Big|_{t=0}$$

Let's provide a proof for the above idea and make it more precise:

Theorem 7.4 (Induced vector fields). *If $\gamma_t : U \to \mathcal{M}$ is a LOPGOLD, then $\gamma_p : I \subset \mathbb{R} \to \mathcal{M}$ where $\gamma_p(t) = \gamma_t(p)$, is an integral curve associated with the vector field V^γ induced by γ_t.*

Proof. If $\gamma_p : I \to \mathcal{M}$ is an integral curve associated with V^γ then we have to show that:

$$\gamma_{p*}\left(\frac{d}{dt}\right)_{t=s} = V^\gamma_{\gamma_s(p)}$$

Now, for $f \in C^\infty(\mathcal{M})$, we have:

$$\begin{aligned}\gamma_{p*}\left(\frac{d}{dt}\right)_{t=s}(f) &= \frac{d}{dt}\left[f \circ \gamma_p(t)\right]\Big|_{t=s} \\ &= \frac{d}{dt}\left[f \circ \gamma_t(p)\right]\Big|_{t=s}\end{aligned}$$

Let $t = s + v$ so that the RHS becomes:

$$\frac{d}{dt}\left[f \circ \gamma_t(p)\right]\Big|_{t=s} = \frac{d}{dv}\left[f \circ \gamma_{s+v}(p)\right]\Big|_{v=0}$$

since s is just a constant. Then, continuing, we have:

$$\begin{aligned}\frac{d}{d(v)}\left[f \circ \gamma_{s+v}(p)\right]\Big|_{v=0} &= \frac{d}{dv}\left[f \circ \gamma_v \circ \gamma_s(p)\right]\Big|_{v=0} \\ &= \frac{d}{dv}\left[f \circ \gamma_v(\gamma_s(p))\right]\Big|_{v=0} \\ &= V^\gamma_{\gamma_s(p)}(f)\end{aligned}$$

This shows that γ_p is indeed the integral curve associated with a vector field V^γ. □

So far, we have seen that if we have a vector field, the integral curves of the vector field can be used to define a LOPGOLD on the manifold. Next, we have also seen that if we have a LOPGOLD, it automatically induces a vector field for us.

7.1.3 Local flows and summary of concepts thus far

Definition 7.2 (Local flows). Let W be a vector field defined on an open subset $U \in \mathcal{M}$ and let $p \in U \subset \mathcal{M}$. Then a local flow of W at p is a LOPGOLD defined on some open subset V of U and sucht that the vector field induced by this family is equal to the given field W.

Remark. In short, this is saying that if we have a vector field V, the corresponding LOPGOLD produced by its integral curves is called a local flow. Because the local flow is produced by the integral curves of V, the induced vector field of the local flow is V.

We say that a vector field generates a flow, and we say that a flow induces a vector field.

Also, we may imagine a flow as a (steady) stream flow. If a particle is observed at a point p at time $t = 0$, it will be found at $\gamma_t(p)$ at a later time t.

All of the concepts thus far are summarised in Figure 7.1.

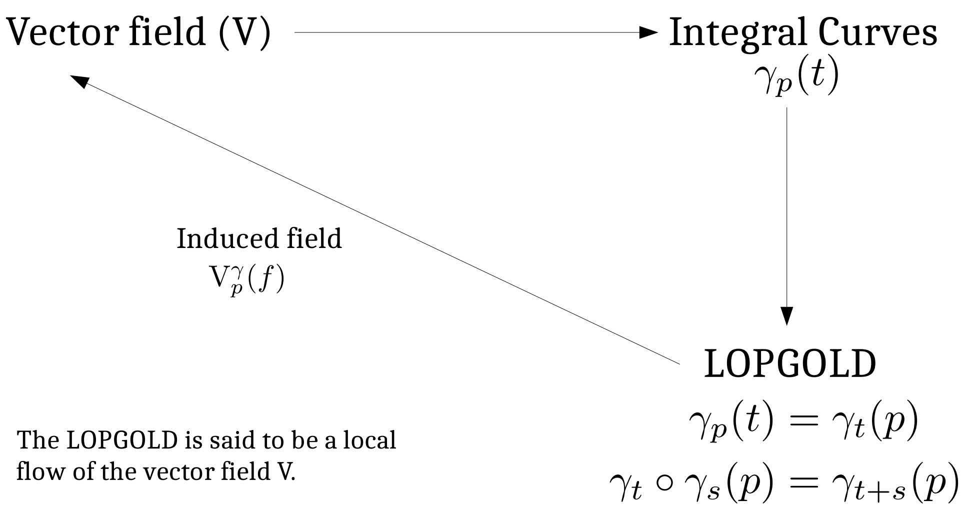

Figure 7.1:

7.2 The Lie Derivative

Suppose that we have a given vector field V, then we know from our discussion above that there are local flows associated with this vector field. Question: if we are given another vector field W, then what is the rate of change of W along the flow generated by V? To answer this question, we need the concept of the Lie derivative of a vector field:

Definition 7.3 (Lie derivative of a vector field). Suppose we have two vector fields V and W. Then, we define the Lie derivative of W with respect to V by:

$$\mathcal{L}_V W\big|_p = \lim_{t\to 0} \frac{1}{t}\left[\gamma_{-t*} W_{p'} - W_p\right] \tag{7.5}$$

where γ_t is a local flow assocaited with V and $p' = \gamma_t(p)$.

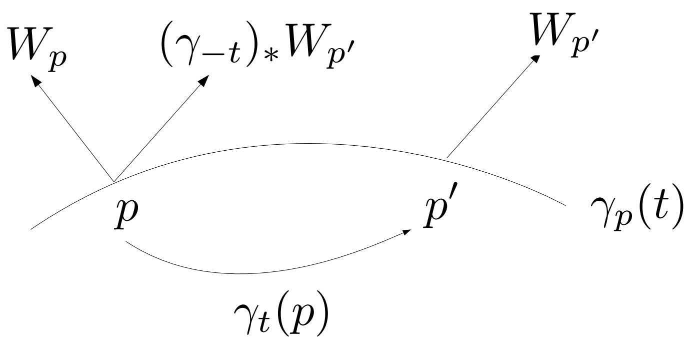

Figure 7.2:

Remark. We can visualise equation 7.5 as shown in Figure 7.2. Essentially, equation 7.5 is motivated by the fact that we want to compare $W_{p'}$ (the vector field W at the point p') with W_p (the vector field W at the point p), where p' is a point along the integral curve of the flow $\gamma_p(t)$. However, beacuse we can't compare tangent vectors belonging to different tangent spaces, we have to push forward $W_{p'}$ to $T_p(\mathcal{M})$, and that is done using the push forward map γ_{-t*}. The Lie derivative is thus the limit where $t \to 0$.

We can similarly define the Lie derivative of a one-form, but with the pull-back map rather than the push-forward map.

Definition 7.4 (Lie derivative of a one-form).

$$\mathcal{L}_V \theta\Big|_p = \lim_{t \to 0} \frac{1}{t} \left[\gamma_t^* \theta_{p'} - \theta_p\right] \tag{7.6}$$

Remark. Here, we have pulled-back the one-form at p', to the point p and subtracted θ_p from it.

The notion of a Lie derivative can be extended to any tensor field in a natural way. For instance, if $\mathcal{R} \in T^{(1,1)}$, with $\mathcal{R} = W \otimes \theta$, where W is a vector field and θ is a one-form, we define the Lie derivative of $\mathcal{R}$ as:

$$\mathcal{L}_V \mathcal{R}\Big|_p = \lim_{t \to 0} \frac{1}{t} \left[\gamma_{-t*} W_{p'} \otimes \gamma_t^* \theta_{p'} - W_p \otimes \theta_p\right] \tag{7.7}$$

In a similar way, we can extend the notion to any tensor field; just pull-back the basis one-forms, and push-forward the basis vectors.

We note in theorem 7.5 that the Lie derivative can be defined in other ways.

Theorem 7.5 (Alternate definitions of the Lie derivative). *Other than equation 7.5, the Lie derivative can be defined as:*

$$\mathcal{L}_V W\Big|_p = \lim_{t \to 0} [W_p - \gamma_{t*} W_{\gamma_{-t}(p)}] \tag{7.8}$$

$$= \lim_{t \to 0} [W_{\gamma_t(p)} - \gamma_{t*} W_p] \tag{7.9}$$

Remark. To understand equations 7.8,7.9, it is helpful to draw figures analogous to Figure 7.2. Note to self: These figures will be included in a later edition of the notes if I'm free to draw the figures...

7.2.1 Properties of Lie derivatives (part 1)

Here we use the definition of the Lie derivative to prove some properties of the Lie derivative.

1. Given two tensor fields R, S of the same type then:

$$\mathcal{L}_V(R+S) = \mathcal{L}_V(R) + \mathcal{L}_V(S) \tag{7.10}$$

Proof. This can be shown easily through the linearity of both the push-forward and pull-back maps. Using $\gamma_{\pm t*}$ to denote both push-forward and pull-back operators respectively, we have:

$$\begin{aligned}
\mathcal{L}_V(R+S)\Big|_p &= \lim_{t\to 0}\frac{1}{t}\left[\gamma_{\pm t*}(R+S)_{p'} - (R+S)_p\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\gamma_{\pm t*}R_{p'} + \gamma_{\pm t*}S_{p'} - R_p + S_p\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\gamma_{\pm t*}R_{p'} - R_p\right] + \lim_{t\to 0}\frac{1}{t}\left[\gamma_{\pm t*}S_{p'} - S_p\right] \\
&= \mathcal{L}_V(R) + \mathcal{L}_V(S)
\end{aligned}$$

□

2. For any two tensor fields X and Y, we have:

$$\mathcal{L}_V(X\otimes Y) = \mathcal{L}_V(X)\otimes Y + X\otimes\mathcal{L}_V(Y) \tag{7.11}$$

Proof. The proof proceeds as follows:

$$\mathcal{L}_V(X\otimes Y) = \lim_{t\to 0}\frac{1}{t}\left[\gamma_{\pm t*}X_{p'}\otimes\gamma_{\pm t*}Y_{p'} - X_p\otimes Y_p\right]$$

Subtracting and adding $X_p\otimes\gamma_{\pm t*}Y_{p'}$, we have:

$$\begin{aligned}
\mathcal{L}_V(X\otimes Y)_p &= \lim_{t\to 0}\frac{1}{t}\left[\gamma_{\pm t*}X_{p'}\otimes\gamma_{\pm t*}Y_{p'} - X_p\otimes\gamma_{\pm t*}Y_{p'} + X_p\otimes\gamma_{\pm t*}Y_{p'} - X_p\otimes Y_p\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\left(\gamma_{\pm t*}X_{p'} - X_p\right)\otimes\gamma_{\pm t*}Y_{p'}\right] + \lim_{t\to 0}\frac{1}{t}\left[X_p\otimes\left(\gamma_{\pm t*}Y_{p'} - Y_p\right)\right] \\
&= \mathcal{L}_V(X)\otimes Y\Big|_p + X\otimes\mathcal{L}_V(Y)\Big|_p
\end{aligned}$$

which gives the desired result. □

7.2.2 Explicit evaluation of Lie derivatives

Lie derivative of a 0-form (or a function)

First, we recall that for a general differentiable map $\mathcal{F}:\mathcal{M}\to\mathcal{N}$ where $\mathcal{M}$ and $\mathcal{N}$ are two smooth manifolds, the pull-back of $f'\in C^\infty(\mathcal{N})$ is:

$$\mathcal{F}^*f' = f'\circ\mathcal{F} \tag{7.12}$$

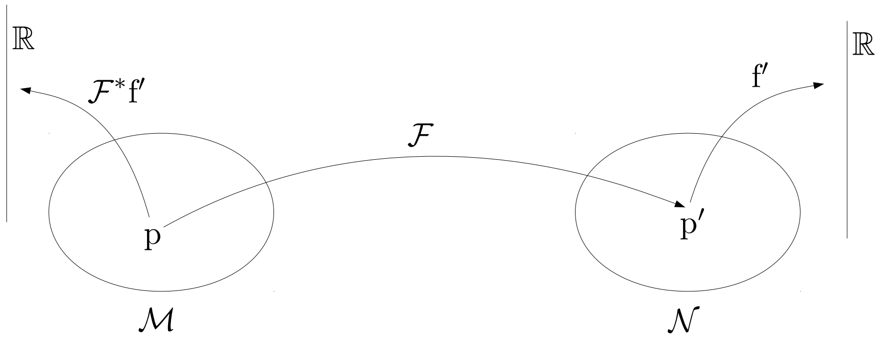

Figure 7.3:

The reason for this is that we want $[\mathcal{F}^* f'](p) = f'(p')$, where $p' = \mathcal{F}(p)$. The situation is shown in Figure 7.3. Now, we will apply equation 7.12 to the cases where $\mathcal{F} = \gamma_t(p)$, i.e to the case of a local flow. We have:

$$\begin{aligned}
\mathcal{L}_V f\Big|_p &= \lim_{t\to 0} \frac{1}{t} \left[\gamma_t^* f_{p'} - f_p\right] \\
&= \lim_{t\to 0} \frac{1}{t} \left[f \circ \gamma_t(p) - f(p)\right] \\
&= \lim_{t\to 0} \frac{1}{t} \left[f(\gamma_t(p)) - f(p)\right] \\
&= \lim_{t\to 0} \frac{1}{t} \left[f(\gamma_t(p)) - f(\gamma_0(p))\right] \\
&= \frac{d}{dt} \left[f \circ \gamma_t(p)\right]\Big|_{t=0} \\
&= \frac{d}{dt} \left[f \circ \gamma_p(t)\right]\Big|_{t=0} \\
&= V_p^\gamma(f) \\
&= V_p(f)
\end{aligned}$$

where to go from the second last line to the last line, we realise that γ is the integral curve of V.

Thus, we see that the Lie derivative of a function (or a 0-form) is just the directional derivative of that function.

Lie derivative of a vector field W

Now, we go back to our original scenario in equation 7.5, reproduced here for convenience:

$$\mathcal{L}_V W\Big|_p = \lim_{t\to 0} \frac{1}{t} \left[\gamma_{-t*} W_{p'} - W_p\right]$$

We shall evaluate equation 7.5 in a local chart (U, ϕ) with coordinates $\phi(p) = (x^1, ..., x^m)$. Our strategy will be first consider $W = \frac{\partial}{\partial x^i}$, i.e we consider a basis field first. Then, we can find the expression for a general vector field.

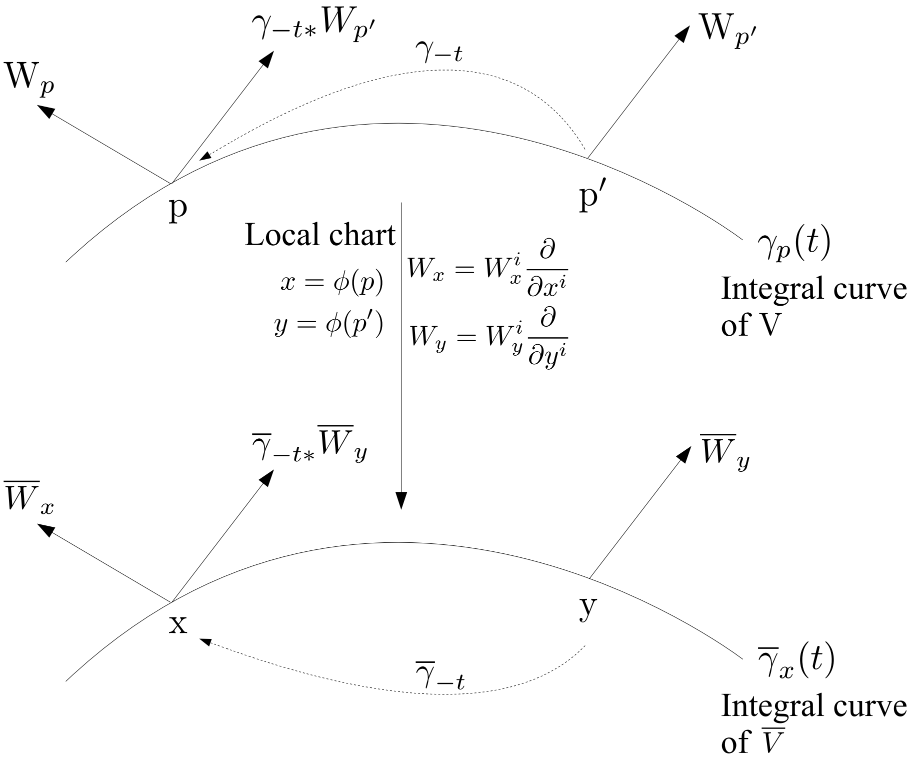

Figure 7.4:

In the local chart (U, ϕ), we shall make the following identifications:

$$\phi(p) = x \equiv (x^1, ..., x^m)$$
$$\phi(p') = y \equiv (y^1, ..., y^m)$$
$$\gamma_t(p) = p' \xRightarrow[(U,\phi)]{\text{Local chart}} \overline{\gamma}_t(x) = y$$
$$W_{p'} \xrightarrow[(U,\phi)]{\text{Local chart}} \overline{W}_y \qquad W_p \xrightarrow[(U,\phi)]{\text{Local chart}} \overline{W}_x$$

Note also that since p and p' lie on the same integral curve $\gamma_p(t)$, we have:

$$\gamma_p(t) = p' \xRightarrow[(U,\phi)]{\text{Local chart}} \overline{\gamma}_x(t) = (x^1(t), ..., x^m(t)) = (y^1, ..., y^m)$$
$$\gamma_p(0) = p' \xRightarrow[(U,\phi)]{\text{Local chart}} \overline{\gamma}_x(0) = (x^1(0), ..., x^m(0)) = (x^1, ..., x^m)$$

where the second line above is a direct consequence of the first. All of this is shown nicely in Figure 7.4. Thus, in a local chart (U, ϕ), the Lie derivative definition looks like:

$$\mathcal{L}_{\overline{V}}\overline{W}\Big|_x = \lim_{t\to 0} \frac{1}{t}\left[\overline{\gamma}_{-t*}\overline{W}_y - \overline{W}_x\right] \tag{7.13}$$

Let's explicitly evaluate equation 7.13 by first evaluating $\overline{\gamma}_{-t*}\overline{W}_y\left(\bar{f}\right)$, where $\bar{f} \in C^{\infty}(\mathbb{R})$ is a function of x. Now,

$$\overline{\gamma}_{-t*}\overline{W}_y\left(\bar{f}\right) = \overline{W}_y\left(\bar{f} \circ \gamma_{-t}(y)\right)$$

Using the fact that according to Figure 7.4, $\gamma_{-t}(y)$ gives us m equations of the form $x^i = x^i(y^1, ..., y^m)$, where $i = 1, .., m$, we have:

$$\begin{aligned}
\overline{\gamma}_{-t*}\overline{W}_y\left(\bar{f}\right) &= \overline{W}_y\left(\bar{f}(x^1(y^1, ..., y^m), ..., x^m(y^1, ..., y^m))\right) \\
&= \overline{W}^k_y \frac{\partial}{\partial y^k}\left(\bar{f}(x^1(y^1, ..., y^m), ..., x^m(y^1, ..., y^m))\right) \\
&= \overline{W}^k_y \frac{\partial \bar{f}}{\partial x^j}\frac{\partial x^j}{\partial y^k} \\
&= \overline{W}^k_y \frac{\partial \bar{f}}{\partial x^j}\frac{\partial \overline{\gamma}^j_{-t}}{\partial y^k} \\
&= \overline{W}^k_y \frac{\partial \overline{\gamma}^j_{-t}}{\partial y^k}\frac{\partial \bar{f}}{\partial x^j}
\end{aligned}$$

Substituting the above result into equation 7.13, we have:

$$\begin{aligned}
\mathcal{L}_{\overline{V}}\overline{W}\Big|_x \bar{f} &= \lim_{t\to 0}\frac{1}{t}\left[\overline{\gamma}_{-t*}\overline{W}_y - \overline{W}_x\right]\left(\bar{f}\right) \\
&= \lim_{t\to 0}\frac{1}{t}\left[\overline{W}^k_y \frac{\partial \overline{\gamma}^j_{-t}}{\partial y^k}\frac{\partial \bar{f}}{\partial x^j} - \overline{W}^k_x \frac{\partial \bar{f}}{\partial x^k}\right]
\end{aligned}$$

Now, we restrict ourselves to the basis field $\overline{W} = \dfrac{\partial}{\partial x^i}$, which gives us:

$$\overline{W}^k_x = \overline{W}^k_x = \delta^k_i$$

Then, we have:

$$\begin{aligned}
\mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^i}\right)\Big|_x \bar{f} &= \lim_{t\to 0}\frac{1}{t}\left[\delta^k_i \frac{\partial \overline{\gamma}^j_{-t}}{\partial y^k}\frac{\partial \bar{f}}{\partial x^j} - \delta^k_i \frac{\partial \bar{f}}{\partial x^k}\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\frac{\partial \overline{\gamma}^j_{-t}}{\partial y^i}\frac{\partial \bar{f}}{\partial x^j} - \delta^k_i \frac{\partial \bar{f}}{\partial x^k}\right]
\end{aligned}$$

Using some mathematical trickery, i.e $\delta^k_i = \dfrac{\partial y^k}{\partial y^i}$, we have:

$$\begin{aligned}
\mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^i}\right)\Big|_x \bar{f} &= \lim_{t\to 0}\frac{1}{t}\left[\frac{\partial \overline{\gamma}^j_{-t}}{\partial y^i}\frac{\partial \bar{f}}{\partial x^j} - \frac{\partial y^k}{\partial y^i}\frac{\partial \bar{f}}{\partial x^k}\right] \\
&= \lim_{t\to 0}\frac{1}{t}\left[\frac{\partial \overline{\gamma}^j_{-t}}{\partial y^i}\frac{\partial \bar{f}}{\partial x^j} - \frac{\partial \overline{\gamma}^j_{0}}{\partial y^i}\frac{\partial \bar{f}}{\partial x^j}\right]
\end{aligned}$$

where we note that γ_0 is the identity map, i.e $\gamma_0(y) = y$. Making the y dependence of γ^j_{-t} explicit, we have[7.2]:

$$\begin{aligned}\mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^i}\right)\Big|_x \bar{f} &= \lim_{t\to 0}\frac{1}{t}\left[\frac{\partial\overline{\gamma}^j_{-t}(y^1,...,y^m)}{\partial y^i}\frac{\partial \bar{f}}{\partial x^j} - \frac{\partial\overline{\gamma}^j_0(y^1,...,y^m)}{\partial y^i}\frac{\partial \bar{f}}{\partial x^j}\right] \\ &= \lim_{t\to 0}\frac{1}{t}\left[\frac{\partial\overline{\gamma}^j_{-t}(y^1,...,y^m)}{\partial y^i} - \frac{\partial\overline{\gamma}^j_0(y^1,...,y^m)}{\partial y^i}\right]\frac{\partial \bar{f}}{\partial x^j}\end{aligned}$$

Now, making the substitution $s = -t$, we have:

$$\begin{aligned}\mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^i}\right)\Big|_x \bar{f} &= -\lim_{s\to 0}\frac{1}{s}\left[\frac{\partial\overline{\gamma}^j_{s}(y^1,...,y^m)}{\partial y^i} - \frac{\partial\overline{\gamma}^j_0(y^1,...,y^m)}{\partial y^i}\right]\frac{\partial \bar{f}}{\partial x^j} \\ &= -\lim_{s\to 0}\left[\frac{\partial}{\partial y^i}\left(\frac{\overline{\gamma}^j_{s}(y^1,...,y^m) - \overline{\gamma}^j_0(y^1,...,y^m)}{s}\right)\right]\frac{\partial \bar{f}}{\partial x^j}\end{aligned}$$

Now, the limit in the last line above is a tricky limit to take. We first realise that $s \to 0 \implies t \to 0$, and $t \to 0 \implies y \to x$. Thus, this allows us to do:

$$\mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^i}\right)\Big|_x \bar{f} = -\lim_{s\to 0}\left[\frac{\partial}{\partial x^i}\left(\frac{\overline{\gamma}^j_{s}(x^1,...,x^m) - \overline{\gamma}^j_0(x^1,...,x^m)}{s}\right)\right]\frac{\partial \bar{f}}{\partial x^j}$$

which then gives us:

$$\begin{aligned}\mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^i}\right)\Big|_x \bar{f} &= -\frac{\partial}{\partial x^i}\left(\frac{d\overline{\gamma}^j_{s}(x^1,\dots,x^m)}{ds}\Big|_{s=0}\right)\frac{\partial \bar{f}}{\partial x^j} \\ &= -\frac{\partial}{\partial x^i}\left(\frac{d\overline{\gamma}^j_{x}(s)}{ds}\Big|_{s=0}\right)\frac{\partial \bar{f}}{\partial x^j}\end{aligned}$$

Realising that $\dfrac{d\overline{\gamma}^j_x(s)}{ds}$ is j-th component of the tangent vector to the integral curve at the point x, which is just another way to refer to the j-th component of the vector field $\overline{V}_x$, we have:

$$\mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^i}\right)\Big|_x = -\frac{\partial}{\partial x^i}\left(\overline{V}^j_x\right)\frac{\partial \bar{f}}{\partial x^j} \tag{7.14}$$

Now that we have equation 7.14, we can easily extend our results to include any field $\overline{W} = \overline{W}^i\dfrac{\partial}{\partial x^i}$ using equation 7.10 and equation 7.11. I.e, we can write:

$$\begin{aligned}\overline{W} &= \overline{W}^i\frac{\partial}{\partial x^i} \\ &= \overline{W}^1\frac{\partial}{\partial x^1} + \overline{W}^2\frac{\partial}{\partial x^2} + \dots + \overline{W}^m\frac{\partial}{\partial x^m}\end{aligned}$$

[7.2] The derivation from here onwards departs from Kuldip's treatment, not sure how legit it is. The departure is because Kuldip's treatment seems sketchy...

which gives us

$$
\begin{aligned}
\mathcal{L}_{\overline{V}}\overline{W} &= \mathcal{L}_{\overline{V}}\left(\overline{W}^i \frac{\partial}{\partial x^i}\right) \\
&= \mathcal{L}_{\overline{V}}\left(\overline{W}^1 \frac{\partial}{\partial x^1}\right) + \mathcal{L}_{\overline{V}}\left(\overline{W}^2 \frac{\partial}{\partial x^2}\right) + \ldots + \mathcal{L}_{\overline{V}}\left(\overline{W}^m \frac{\partial}{\partial x^m}\right)
\end{aligned}
$$

and note that for each individual term in the summation, say $\mathcal{L}_{\overline{V}}\left(\overline{W}^1 \frac{\partial}{\partial x^1}\right)$, we can treat $\overline{W}^1$ as a 0-form and $\frac{\partial}{\partial x^1}$ as a vector and then apply equation 7.11. E.g:

$$
\begin{aligned}
\mathcal{L}_{\overline{V}}\left(\overline{W}^1 \frac{\partial}{\partial x^1}\right) &= \mathcal{L}_{\overline{V}}\left(\overline{W}^1\right) + \frac{\partial}{\partial x^1}\overline{W}^1 \mathcal{L}_{\overline{V}}\left(\frac{\partial}{\partial x^1}\right) \\
&= V^j \frac{\partial \overline{W}^1}{\partial x^j}\frac{\partial}{\partial x^1} + \overline{W}^1\left(-\frac{\partial}{\partial x^1}\left(\overline{V}^j\right)\frac{\partial}{\partial x^j}\right) \\
&= V^j \frac{\partial \overline{W}^1}{\partial x^j}\frac{\partial}{\partial x^1} - \overline{W}^1 \frac{\partial \overline{V}^j}{\partial x^1}\frac{\partial}{\partial x^j}
\end{aligned}
$$

Thus, summing over all these individual terms, we have:

$$
\mathcal{L}_{\overline{V}}\left(\overline{W}^i \frac{\partial}{\partial x^i}\right) = V^j \frac{\partial \overline{W}^i}{\partial x^j}\frac{\partial}{\partial x^i} - \overline{W}^i \frac{\partial \overline{V}^j}{\partial x^i}\frac{\partial}{\partial x^j}
$$

Doing a renaming of dummy variables for the first term, i.e the good old $i \leftrightarrow j$, we have:

$$
\begin{aligned}
\mathcal{L}_{\overline{V}}\left(\overline{W}^i \frac{\partial}{\partial x^i}\right) &= V^i \frac{\partial \overline{W}^j}{\partial x^i}\frac{\partial}{\partial x^j} - \overline{W}^i \frac{\partial \overline{V}^j}{\partial x^i}\frac{\partial}{\partial x^j} \\
&= \left(V^i \frac{\partial \overline{W}^j}{\partial x^i} - \overline{W}^i \frac{\partial \overline{V}^j}{\partial x^i}\right)\frac{\partial}{\partial x^j}
\end{aligned}
$$

With reference to equation 4.9, notice that the last line is exactly how the the commutator of $\overline{V}$ and $\overline{W}$ looks like. This gives us our very interesting result:

$$
\mathcal{L}_{\overline{V}}\overline{W} = \left[\overline{V}, \overline{W}\right] \tag{7.15}
$$

and this is the reason why the commutator is also called the Lie bracket.

Lie derivative of a one-form

Here, we want to evaluate equation 7.6 in a local chart (U, ϕ), which we will write as:

$$
\mathcal{L}_{\overline{V}}\overline{\theta} = \lim_{t\to 0}\frac{1}{t}\left[\overline{\gamma}_t^*\overline{\theta}_y - \overline{\theta}_x\right] \tag{7.16}
$$

To get from equation 7.6 to equation 7.16, we have made the folllowing substitutions:

$$\phi(p) = x, \qquad \phi(p') = y$$
$$p' = \gamma_t(p) \xRightarrow[(U,\phi)]{\text{Local Chart}} y = \gamma_t(x)$$
$$\theta_{p'} \xrightarrow[(U,\phi)]{\text{Local Chart}} \overline{\theta}_y$$
$$\theta_p \xrightarrow[(U,\phi)]{\text{Local Chart}} \overline{\theta}_x$$

Note that the second equation above gives us m coordinate transformation equations of the form: $y^i = y^i(x^1, ..., x^m)$ for $i = 1, 2, ..., m$. Now, we first evaluate the $\overline{\gamma}_t^* \overline{\theta}_y$ term. We first have:

$$\left\langle \overline{\gamma}_t^* \overline{\theta}_y,\ \frac{\partial}{\partial x^i} \right\rangle = \left\langle \overline{\theta}_y,\ \overline{\gamma}_{t*} \frac{\partial}{\partial x^i} \right\rangle$$

Next, we note that, for $\bar{f}' \in C^\infty(\mathcal{M})$ where $\bar{f}'$ is a function of y, we have, using theorem 4.1,

$$\begin{aligned}
\left[\overline{\gamma}_{t*} \frac{\partial}{\partial x^i} \bar{f}'\right](y) &= \frac{\partial}{\partial x^i}\left(\bar{f}' \circ \overline{\gamma}_t(x^1, ..., x^m)\right) \\
&= \frac{\partial \bar{f}'\left(\overline{\gamma}_t(x^1, ..., x^m)\right)}{\partial x^i} \\
&= \frac{\partial \bar{f}'\left(y^1(x^1, ..., x^m), ..., y^m(x^1, ..., x^m)\right)}{\partial x^i} \\
&= \frac{\partial \bar{f}'}{\partial y^j} \frac{\partial y^j}{\partial x^i}
\end{aligned}$$

Noting that $\dfrac{\partial y^j}{\partial x^i} = \dfrac{\partial \overline{\gamma}_t^j(x^1, ..., x^m)}{\partial x^i}$, our result above becomes:

$$\left[\overline{\gamma}_{t*} \frac{\partial}{\partial x^i} \bar{f}'\right](y) = \frac{\partial \overline{\gamma}_t^j(x^1, ..., x^m)}{\partial x^i} \frac{\partial \bar{f}'}{\partial y^j}$$

which gives us:

$$\overline{\gamma}_{t*} \frac{\partial}{\partial x^i} = \frac{\partial \overline{\gamma}_t^j(x^1, ..., x^m)}{\partial x^i} \frac{\partial}{\partial y^j}$$

Continuing with our evaluation of $\overline{\gamma}_t^* \overline{\theta}_y$, we have:

$$
\begin{aligned}
\left\langle \overline{\gamma}_t^* \overline{\theta}_y, \frac{\partial}{\partial x^i} \right\rangle &= \left\langle \overline{\theta}_y, \overline{\gamma}_{t*} \frac{\partial}{\partial x^i} \right\rangle \\
&= \left\langle \overline{\theta}_y, \frac{\partial \overline{\gamma}_t^j (x^1, ..., x^m)}{\partial x^i} \frac{\partial}{\partial y^j} \right\rangle \\
&= \frac{\partial \overline{\gamma}_t^j (x^1, ..., x^m)}{\partial x^i} \left\langle \overline{\theta}_y, \frac{\partial}{\partial y^j} \right\rangle \\
&= \frac{\partial \overline{\gamma}_t^j (x^1, ..., x^m)}{\partial x^i} \left(\overline{\theta}_y \right)_j \\
&= \frac{\partial \overline{\gamma}_x^j (t)}{\partial x^i} \left(\overline{\theta}_y \right)_j
\end{aligned}
$$

where in the last line, we note that for a local flow $\overline{\gamma}_t$, we have $\overline{\gamma}_t(x) = \overline{\gamma}_x(t)$. Thus, this gives us:

$$
\overline{\gamma}_t^* \overline{\theta}_y = \frac{\partial \overline{\gamma}_x^j (t)}{\partial x^i} \left(\overline{\theta}_y \right)_j dx^i
$$

Substituting the above result into equation 7.16, we have:

$$
\begin{aligned}
\mathcal{L}_{\overline{V}} \overline{\theta} &= \lim_{t \to 0} \frac{1}{t} \left[\overline{\gamma}_t^* \overline{\theta}_y - \overline{\theta}_x \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[\frac{\partial \overline{\gamma}_x^j (t)}{\partial x^i} \left(\overline{\theta}_y \right)_j dx^i - \left(\overline{\theta}_x \right)_i dx^i \right]
\end{aligned}
$$

Now, we first consider a basis one-form, i.e we consider $\overline{\theta} = \delta_i^k dx^i = dx^k$. Then, we have:

$$
\begin{aligned}
\mathcal{L}_{\overline{V}} dx^k &= \lim_{t \to 0} \frac{1}{t} \left[\frac{\partial \overline{\gamma}_x^j (t)}{\partial x^i} \delta_j^k dx^i - \delta_i^k dx^i \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[\frac{\partial \overline{\gamma}_x^k (t)}{\partial x^i} dx^i - \frac{\partial x^k}{\partial x^i} dx^i \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[\frac{\partial \overline{\gamma}_x^k (t)}{\partial x^i} dx^i - \frac{\partial \overline{\gamma}_x^k (0)}{\partial x^i} dx^i \right] \\
&= \lim_{t \to 0} \frac{1}{t} \left[\frac{\partial \overline{\gamma}_x^k (t)}{\partial x^i} - \frac{\partial \overline{\gamma}_x^k (0)}{\partial x^i} \right] dx^i \\
&= \lim_{t \to 0} \frac{\partial}{\partial x^i} \left[\frac{\gamma_x^k (t) - \gamma_x^k (0)}{t} \right] dx^i \\
&= \frac{\partial}{\partial x^i} \left(\frac{d \overline{\gamma}_x^k}{dt} \right) \bigg|_{t=0} dx^i
\end{aligned}
$$

Noting that $\frac{\partial}{\partial x^i}\left(\frac{d\overline{\gamma}_x^k}{dt}\right)\Big|_{t=0}$ is nothing more than the k-th component of the tangent vector to the integral curve $\gamma_x(t)$, which is the k-th component of the vector field $\overline{V}$ at x, we have:

$$\begin{aligned}\mathcal{L}_{\overline{V}} dx^k &= \frac{\partial}{\partial x^i}\left(V_x^k\right) dx^i \\ &= \frac{\partial V^k}{\partial x^i} dx^i \end{aligned} \tag{7.17}$$

The equation above can then be used to evaluate the Lie derivative of any arbitrary one-form $\theta_i dx^i$, using both equation 7.10 and equation 7.11. This is left as an exercise to...future KH or whoever reads this notes[7.3].

Now, we shall state one very important result in the evaluation of Lie derivatives:

Theorem 7.6. *In a local chart, for a vector field $\overline{V}$ and a one-form $\overline{\omega}$ we have:*

$$\mathcal{L}_{\overline{V}}\overline{\omega} = d\left(\overline{\omega}(\overline{V})\right) + (d\overline{\omega})(\overline{V}) \tag{7.18}$$

where d stands for the exterior derivative operator[7.4].

Proof. To fill in next time □

Remark. According to Kuldip's lecture notes, theorem 7.6 holds for 2-forms too. I wonder if it holds for n forms; we can try to prove this next time.

7.2.3 Properties of the Lie derivative (Part 2)

Now that we can explicitly evaluate the Lie derivative for a scalar field, vector field and one-form, we can show that for a tensor T in general, the following two properties are true:

$$\mathcal{L}_{aV+bW}T = a\mathcal{L}_V T + b\mathcal{L}_W T \tag{7.19a}$$

$$\mathcal{L}_{[V,W]}T = \mathcal{L}_V\left(\mathcal{L}_W(T)\right) - \mathcal{L}_W\left(\mathcal{L}_V(T)\right) \tag{7.19b}$$

Rough sketch of a proof for now. Since an arbitrary tensor is just of the form $T = T^{def\dots}_{abc\dots}\partial_d \otimes \partial_e \otimes \partial_f \otimes dx^a \otimes dx^b \otimes dx^c$, we can first prove these two equations individually for a scalar field, vector field and for a one-form (To be done in the future[7.5].), then apply equation 7.10 and equation 7.11. □

Now, the above two properties naturally lead us to the notion of Lie algebras. But first, let us define what an invariant tensor field is first.

[7.3] This exercise is rather simple, just follow the same steps as when finding the Lie derivative of an arbitary vector field given the Lie derivative of the basis vector.

[7.4] To be introduced in a later chapter.

[7.5] Alternatively, see Edward Teo's PC4274 tutorial 3

Definition 7.5 (Invariant tensor field). A tensor field T is said to be invariant under a vector field V if

$$\mathcal{L}_V(T) = 0$$

We sometimes also say that T is Lie-dragged by V.

Remark. In other words, this means that as we move along the local flow generated by the vector field V, the tensor T remains unchanged.

Now, equation 7.19 implies that when T is invariant under V and W, then it is invariant under $aV + bW$ and $[V, W]$. We say that the set of all fields which T is invariant under forms a Lie algebra. This shows the intimate relationship beween invariances (or symmetries) and Lie algebras. This idea will be explored further in a later chapter on Lie algebras.

7.2.4 Lie derivative and the coordinate basis

See Edward teo's notes chapter 3 for now, Will add later when I have time.

Things yet to be written:

1. Chapter 1, basic topo
2. Chapter 6, contraction of tensors and how that relates to the tensor product
3. Chapter 7, all the red stuff
4. All the exterior algebra stuff