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SURVEYS

Structured Preferences: A Literature Survey

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Abstract—A survey of papers on practically significant restrictions on the preference profile of a collective is carried out, including single-peaked preferences, group-separable preferences, preferences with the single-crossing property, and Euclidean preferences and their extensions. Both ordinal and dichotomous preferences are considered. For structured preferences, we present characterization in terms of forbidden subprofiles and the probability of the appearance of a profile with a given property. For group-separable preferences, we describe an algorithm for constructing a hierarchical tree. Structured preferences leading to a unique stable matching in the marriage problem are considered separately.

Keywords: preference domain, matching, single-peakedness

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1. INTRODUCTION

There are many approaches to modeling preferences and setting the problem of their aggregation [1]. The underlying preference model is a strict linear order. The tuple of linear orders of all agents is called the preference profile. The standard statement of the collective choice problem is the definition of public choice (preference) as a function of the preference profile.

One main theoretical problem in this area is the Condorcet paradox: the preference of the team constructed based on the simple majority rule can have cycles. This paradox arises when there are three or more alternatives. K.J. Arrow [2] stated a number of axioms that the procedure for aggregating the team preferences should have. The system of axioms turned out to be inconsistent given three or more alternatives. A. Gibbard and M.A. Satterthwaite [3, 4] showed that, under similar assumptions, any aggregation rule can be manipulated, i.e., allows for a benefit to the participant from misrepresenting one's own preference.

One solution of Arrow's paradox is to narrow the scope of the aggregation rules. On the domain of preference profiles that do not give rise to cycles when using the simple majority rule, the simple majority rule satisfies Arrow's axioms, and any rule that differs from the simple majority rule on this domain will violate conditions close to the Arrow's axioms [5, 6]. The simple majority rule has several axiomatic justifications [7–11]; it is not difficult from the computational point of view and intuitively understandable to voters.

The problem of finding all preference profiles for which the simple majority rule gives an acyclic option comparison relation has been solved only for the case of three and four alternatives [12]. A concise necessary and sufficient condition on the preference profile has been established for the case of three alternatives—one of the alternatives must be number one option for at least half of the agents or one of the options must be the last to choose for at least half of the agents [13].

The need to narrow down the scope of preference profiles under consideration has led to the theory of preference domains. A preference domain is a subset of linear orders on a fixed set of alternatives. The domain generates preference profiles consisting only of the linear orders that make up the domain.

Restricting the preferences of members of a society to preferences from a certain domain is a simplification, but since preferences in a society are not formed independently, a considerable part of society members have a limited set of linear orders, and a restriction on a preference domain seems to be a reasonable premise for modeling collective choice. Domains as restrictions on rationality are considered in [14].

A *Condorcet domain* is a subset of preferences such that any preference profile with an odd number of agents leads to an acyclic simple majority decision. All Condorcet domains were found [15, 16] for three, four, and five alternatives. They differ significantly in structure and size. For six alternatives, the maximum-size Condorcet domain contains 45 linear orders, and for seven alternatives, it is a figure of 100 linear orders. For a larger number of alternatives, the problem of finding the maximum-size Condorcet domain remains unsolved [17]. Recent results on maximum Condorcet domains are presented in [18–20]. Some generalization of the concept of a Condorcet domain is discussed in [21, 22].

Some Condorcet domains are natural well-interpreted restrictions on the preference setting domain. K. Inada [23, 24] and A.K. Sen [25] described the domains of single-peaked preferences, single-dipped preferences, and group-separable preferences. J. Mirrlees and K. Roberts [26, 27] proposed the concept of single-crossing preferences. Various constraints on the preference profile have been called *structured preferences* [28]. All the types of structured preferences mentioned above have many useful properties and applications, which are discussed in this survey.

In addition to guaranteeing the existence of a solution of the aggregation problem in Arrow's statement, structured preferences reduce the computational complexity of many problems in the collective choice theory, create a convenient graphical representation of preferences, and have an axiomatic justification.

This survey presents the results on characterizing structured preference profiles in terms of forbidden subprofiles and on finding the number of profiles with given properties. The characterization of structured preferences is important for identifying key internal structures and for constructing algorithms for recognizing structured preferences. The number of structured preference profiles permits one to find the probability of occurrence of such profiles in a discrete equiprobable distribution model. Although such a distribution does not reflect real data, finding probabilities is of interest for comparing the probabilities of different preference types with each other.

Within the framework of the equiprobable distribution model, general formulas have been obtained (for an arbitrary number of agents and alternatives) for group-separable preferences [29] and for some subclasses of single-peaked preferences and single-crossing preferences [30]. For a small number of alternatives, the problem of counting the number of single-peaked preference profiles was solved in [31–33].

In addition to Condorcet domains, the literature contains works on the study of dictatorial domains, i.e., domains for which the only procedure that satisfies Arrow's axioms [2] is the dictatorial procedure. Examples of dictatorial domains are cyclic group domains [34], circular domains [35], top-circular domains [36], and domains that are single-peaked on a circle [37]. The dictatorial domain of the minimum size contains only six types of linear orders for any number of alternatives [38]. Dictatorial domains have far fewer applications than Condorcet domains and will not be discussed in this survey.

In addition to structured domains in the ordinal preference aggregation problem, the survey considers structured preference profiles in the marriage problem (generalized matchings) and structured dichotomous preferences.

The survey is organized as follows. Section 2 introduces the basic concepts and the theory of maximal Condorcet domains. Section 3 is devoted to structured ordinal preferences. Section 4 contains an analysis of structured preference profiles in the marriage problem. Section 5 explores dichotomous preferences. Section 6 contains the conclusions.

2. BASIC CONCEPTS

The following notation is adopted in the present paper: the set of alternatives is denoted by $A = \{1, \ldots, m\}$ and the set of agents, by $\mathscr{N} = \{1, \ldots, n\}$. Each agent $i \in \mathscr{N}$ has a linear preference order P_i over the set A (hereinafter, unless otherwise stated, by preference we mean a linear order). The maximum element in this order is best, and the minimum is worst. For aP_ic and bP_ic we introduce the notation $\{a, b\}P_ic$. Let $\mathscr{L}(A)$ be the set of all possible linear orders on the set X. A tuple of n linear orders is a preference profile, $\mathscr{P} = (P_1, \ldots, P_n) \in \mathscr{L}(A)^n$.

A preference domain is a subset of linear orders. A preference profile is generated by a domain if the preferences of all agents belong to the domain. Here the structured preference domain refers to the maximum structured preference domain; i.e., if any additional linear order is added to such a domain, the domain loses a key property, for example, single-peakedness.

In the present paper, we use the assumption that each agent has an independent and equiprobable occurrence of each linear order of preference as the only probability distribution model. The number of elementary events (preference profiles) is equal to $(m!)^n$. Having counted the number of preference profiles with a certain property $\#structured(m, n)$, we can calculate the probability of a structured preference profile as $\frac{\#structured(m,n)}{(m!)^n}$.

2.1. Maximal Condorcet Domains

Condorcet's paradox is a situation in which the collective preference constructed according to the simple majority rule contains a cycle. The simplest example is a preference profile with three agents and three alternatives (preferences are written in a column without the binary relation sign P)

$$\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{array}.$$

Further in the paper, we define various configurations—certain fragments of domains whose presence or absence determines the properties of the domains.

Configuration 1 (Condorcet's cycle). There exist three agents $i, j, k \in \mathscr{N}$ and three alternatives $a, b, c \in A$ such that aP_ibP_ic, cP_jaP_jb, and bP_kcP_ka.

If the preference domain contains Configurations 1, then among the preference profiles generated by this domain there is a profile with Condorcet's paradox. The definition of a Condorcet domain through the acyclic relation of collective preferences built according to the simple majority rule is nonoperational; therefore, for practical application, the equivalent statements presented in Theorem 1 are used.

Theorem 1 [25]. *The following definitions of a Condorcet domain are equivalent*:

(i) *A preference domain is a Condorcet domain if and only if it does not contain Configuration 1.*

(ii) *A preference domain is a Condorcet domain if and only if for each triple of alternatives there exists an alternative from this triple for which it is true that it either never comes first in the restriction of the domain to this triple*, *or never comes second in the restriction of the domain to this triple*, *or never takes the third place in the restriction of the domain to this triple.*

Condition (ii) in Theorem 1 reflects three ways to avoid Configuration 1, where each alternative is in the first, second, and third place. More equivalent definitions of the Condorcet domain can be found in the survey [39].

One large Condorcet domain is the Fishburn domain [40], which satisfies the following interleaved scheme. When ordering the alternatives $a_1, \ldots, a_m$, for each triple a_i, a_j, a_k of alternatives it is true that if the median of the numbers $i \leq j \leq k$ is even (odd), then the alternative a_j is never at the first place in the restriction of the domain to this triple, and if the median of the numbers $i \leq j \leq k$ is odd (even), then the alternative a_j never comes last in the restriction of the domain to this triple. The Fishburn domain is associated with many combinatorial structures such as wire diagrams [17], rhombus tilings [41], and finite Coxeter groups [42].

The number of linear orders in the Fishburn domain is [17]

$$|F_m| = (m+3)\,2^{m-3} - \begin{cases} m - \dfrac{3}{2}\dbinom{m-2}{\frac{m}{2}-1} & \text{for even } \ m \\[2ex] \dfrac{m-1}{2}\dbinom{m-1}{\frac{m-1}{2}} & \text{for odd } \ m. \end{cases}$$

The Fishburn domain has the maximum size among all Condorcet domains with at most seven alternatives and is a basis for constructing large Condorcet domains for any number of alternatives [20, 40], but this domain does not have a clear interpretation and has no practical application. It is well known that for large m there exists a Condorcet domain containing at least 2.189^m linear orders [20]. Section 3 explores smaller domains with a well-interpreted structure.

3. STRUCTURED ORDINAL PREFERENCES

3.1. Single-Peaked Preferences

Single-peaked preferences presuppose some ordering of alternatives along the so-called axis (by *axis* we mean a linear order of alternatives; the concept of distance between alternatives is not introduced). Each agent has an ideal alternative on this axis. If two alternatives are located on the same side of the ideal alternative, including a possible match with the ideal one, then the alternative that is closer to the ideal point is preferred to the alternative that is farther away. Alternatives on different sides of the ideal one can be ordered in any way.

Axes with a direct order of alternatives and with a reverse order of alternatives correspond to identical single-peaked domains. In this sense, these axes are equivalent. For example, the axes 1234 and 4321 are equivalent. Thus, there are $m!/2$ distinct single-peaked preference axes and domains. Single-peaked preference domains each contain 2^{m-1} linear orders [43]. For example, the axis 1234 defines the following domain of eight linear orders:

$$\begin{array}{cccccccc} 1 & 2 & 2 & 2 & 3 & 3 & 3 & 4 \\ 2 & 1 & 3 & 3 & 2 & 2 & 4 & 3 \\ 3 & 3 & 1 & 4 & 1 & 4 & 2 & 2 \\ 4 & 4 & 4 & 1 & 4 & 1 & 1 & 1 \end{array}.$$

Single-peaked domains may have a nonempty intersection. For a single-peak preference profile, one cannot unambiguously determine by which single-peak domain it is generated. For example, the profile $(123, 123, 213)$ can be generated both by the single-peaked domain $\{123, 213, 231, 321\}$ with axis 123 and by the single-peaked domain with axis 312. The algorithms [44, 45] for recognizing single-peaked preference profiles answer the question of the possibility of constructing a suitable axis and construct one of the possible axes.

Single-peaked preferences have a lot of applications in economics, psychology, and political sciences. The reason for this is the theoretical properties of single-peaked preferences. First, single-peaked preferences are highly interpretable and easy to visualize. In political science, a scale of right-left candidates or parties is common; in marketing, agents are distinguished by loyalty to a particular brand in such a way that they line up. Second, the median voter theorem [46], which says that the political position of two candidates under free competition tends to the position of the median voter, provides an additional justification for the simple majority rule. Third, many computational problems in the collective choice theory are greatly simplified under single-peaked preferences [47, 48], including, for example, manipulation problems (for more details on manipulation problems, see [49]). Fourth, single-peaked preferences have an axiomatic justification [50]. The single-peaked preference domain for each alternative has at least one linear order with this alternative in the first place. All linear orders in this domain are related to each other; i.e., one linear order can be obtained from another by successively inverting pairs of adjacent linear order alternatives, with all intermediate linear orders also belonging to the same domain. In addition, there are a couple of linear orders in this domain that have the reverse order of alternatives with respect to each other. Thus, the single-peaked preference domain consists of fairly close preferences, but at the same time it is quite diverse, since it contains a pair of opposite preferences.

Single-dipped preferences are the complete opposite of single-peaked preferences. All alternatives are also on the axis, but each agent has its own worst alternative on this axis, and the farther from this alternative, the better. In economics, these preferences occur when modeling preferences for a public bad (for example, the preferences of local residents about the location of a polluting enterprise on the coast). The analysis done for single-peaked preferences can almost always be applied to single-dipped preferences, so single-dipped preferences will not be explored further in this paper.

The worst alternative for each of the agents with single-peaked preferences is one of the end points of the axis. Thus, a preference profile cannot have three agents with different worst-case alternatives.

Configuration 2. There exist three agents $i, j, k \in \mathscr{N}$ and three alternatives $a, b, c \in A$ such that $\{a, b\}P_i c$, $\{a, c\}P_j b$, and $\{b, c\}P_k a$.

Configuration 2 refines condition (ii) in Theorem 1: in each triple of alternatives, there is an alternative from this triple for which it is true that it never ranks third in the restriction of the domain to this triple. A preference profile that does not contain Configuration 2 is said to be Arrow single-peaked (for more details on these preferences, see [2, 20, 31, 51, 52]).

Configuration 3. There exist two agents $i, j \in \mathscr{N}$ and four alternatives $a, b, c, d \in A$ such that $\{a, d\}P_i b P_i c$ and $\{c, d\}P_j b P_j a$.

Configuration 3 specifies a limit on the agent pairs. When fixing one of the agents, the set of possible preferences of the second agent corresponds to some permutation pattern, which is specified through forbidden combinations (for more details on permutation patterns, see [53]). The number of permutations in the pattern gives the number of profiles with two agents.

Theorem 2 [54]. *A preference profile is single-peaked if and only if it does not contain Configurations 2 and 3.*

Theorem 3 [31]. *The number of single-peaked preference profiles is asymptotically equal to*

$$\#SP(m, n) \approx \frac{m!}{2} 2^{(m-1)n}.$$

The exact number of single-peaked profiles is known for two agents and an arbitrary number of alternatives [33] and for three, four [31], and five alternatives [32] and an arbitrary number of agents.

Single-peaked preferences have many extensions. One direction is to consider various classes of graphs instead of an axis of alternatives. Examples of such an extension are preferences singled-peaked on trees [55], preferences single-peaked on a circle [37], and preferences single-peaked on an arbitrary graph [56, 57].

In the model of single-peaked preferences on a tree, the alternatives are the vertices of a connected acyclic graph (tree), and each path represents an axis for the corresponding subprofile of single-peaked preferences. Due to the variety of trees, these preferences are very diverse. There are algorithms for finding a tree representing a given profile [58, 59] as well as algorithms for finding a tree with some desired properties, for example, with a small number of leaves [60]. The special shape of the trees makes it easier to find the committee in the proportional representation problem [61].

Single-peaked preferences on a circle are defined by analogy with classical single-peaked preferences. All alternatives are located on a circle. Each agent has its own best and worst alternatives. Along the way from the worst alternative to the best one, alternative are ordered in ascending order of preference. Alternatives on different sides of the best (worst) alternative can be ordered in any way.

In total, there are $\frac{(m-1)!}{2}$ domains of single-peaked preferences on a circle, each of which has $m2^{m-2}$ linear orders. The following three configurations characterize single-peaked preferences on a circle.

Configuration 4. There exist two agents $i, j \in \mathscr{N}$ and five alternatives $a, b, c, d, e \in A$ such that $\{a, b\}P_i c P_i \{d, e\}$ and $\{a, e\}P_j c P_j \{b, d\}$.

Configuration 4 defines a permutation pattern known as square permutations [62].

Configuration 5. There exist three agents $i, j, k \in \mathcal{N}$ and four alternatives $a, b, c, d \in A$ such that $\{a, b\}P_i\{c, d\}$, $\{a, c\}P_j\{b, d\}$, and $\{a, d\}P_k\{b, c\}$.

Configuration 6. There exist three agents $i, j, k \in \mathcal{N}$ and four alternatives $a, b, c, d \in A$ such that $\{a, b\}P_i\{c, d\}$, $\{b, c\}P_j\{a, d\}$, and $\{c, a\}P_k\{b, d\}$.

Theorem 4 [37]. *A preference profile is single-peaked on a circle if and only if it does not contain Configurations 4, 5, and 6.*

Knowing the number of domains and the number of linear orders in each domain, we obtain the approximate number of single-peaked preference profiles on a circle.

Theorem 5. *The number of preference profiles single-peaked on a circle is asymptotically equal to*

$$\#SPC(m, n) \approx \frac{(m-1)!}{2}\left(m2^{m-2}\right)^n.$$

Preferences single-peaked on a circle have mainly computational applications in the area of the proportional representation problem [37].

An even wider class of preferences is modeled by preferences close to single-peaked ones. This is a class of preferences that are reduced to single-peaked preferences after removing either k alternatives or k agents, or splitting the set of agents or alternatives into k parts, or transforming preferences through k pairwise inversions of standing alternatives in some preferences, etc. The parameter k in this definition serves as a measure characterizing the distance from a given profile to the nearest single-peaked profile. This distance can be defined in different ways (removal of agents, removal of alternatives, etc.) for different purposes. The main problem of this approach is the difficulty in determining whether a profile belongs to one or another class of preferences. A considerable number of such problems are NP-hard [63], but there are approximation algorithms [64]. At the same time, some classes of preferences close to single-peaked ones are easy to identify and permit one to reduce the complexity of the voting outcome manipulation and control problems [65–72], proportional representation problems [73], and the Kemeny ranking problem [74].

Another extension of single-peaked preferences is single-peaked preferences for incomplete data [75]. If the input data is a partial rather than linear order, then the problem of possible reconstruction of preferences to a single-peaked preference profile is NP-hard. If the original data is a weak order, then there is a polynomial complexity algorithm to determine whether the preference profile can be restored to a single-peaked one.

3.2. Group-Separable Preferences

Inada [23, 24] introduced the group-separability property in the following way. A preference profile is *group-separable* if for any subset $X \subseteq A$, $|X| \geq 2$, there exists a partition into nonempty subsets X' and $X \backslash X'$ such that for each agent $i \in \mathcal{N}$ one has either aP_ib for any $a \in X'$ and $b \in X \backslash X'$ or bP_ia for any $a \in X'$ and $b \in X \backslash X'$. Examples of maximal group-separable preference domains are given by

$$D_1 = \left\{ \begin{array}{cccccccc} 1 & 2 & 1 & 2 & 3 & 3 & 4 & 4 \\ 2 & 1 & 2 & 1 & 4 & 4 & 3 & 3 \\ 3 & 3 & 4 & 4 & 1 & 2 & 1 & 2 \\ 4 & 4 & 3 & 3 & 2 & 1 & 2 & 1 \end{array} \right\},$$

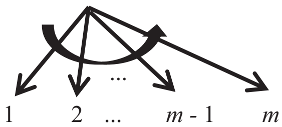

Fig. 1. Hierarchical tree for the linear order $12\ldots m$.

$$D_2 = \left\{ \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 & 3 & 4 & 4 & 3 \\ 3 & 4 & 4 & 3 & 4 & 3 & 2 & 2 \\ 4 & 3 & 2 & 2 & 1 & 1 & 1 & 1 \end{array} \right\}.$$

All maximal single-peaked preference domains have the same structure; i.e., with a fixed number of alternatives, one domain can be obtained from another by renaming the alternatives. Unlike single-peaked domains, maximal group-separable domains differ structurally from one another and cannot be reduced to a single domain by renaming alternatives (see the examples above).

The set of group-separable preference profiles is closed with respect to the operation of removing alternatives from the profile. Thus, the restriction of a group-separable preference profile to any subset of alternatives is group-separable.

A set $X \subseteq A$ of alternatives is called a *set of clones* for a profile $\mathscr{P}$ if the alternatives in X are sequential in the preferences of each agent. *Nontrivial* sets of clones will be sets of clones that consist of more than one alternative and do not coincide with the set of all alternatives. For example, the sets $\{1, 2\}$ and $\{3, 4\}$ are nontrivial sets of clones for D_1. For D_2, these sets are $\{3, 4\}$ and $\{2, 3, 4\}$. Clone sets combine alternatives that are in some sense close to each other for each agent. Clone sets are a well-established tool of analysis in the collective choice theory [76, 77].

An acyclic directed graph in which only one vertex has no incoming arcs and the remaining vertices have a single incoming arc is an outgoing tree. A vertex with no incoming arcs is a root. Vertices that do not have outgoing arcs are terminal. A vertex-induced subtree is a part of an outgoing tree that contains some tree vertex, all descendants of this vertex, and the arcs connecting them. The terminal vertices and the outgoing tree itself are vertex-induced subtrees.

An outgoing tree T in which the alternatives are leaves of this tree and the outgoing arcs of each vertex are ordered will be called a *hierarchical tree* representing a profile $\mathscr{P}$ of group-separable preferences if for each set X of clones of the profile $\mathscr{P}$ it is true that there exists a vertex and a set of vertex-induced subtrees whose roots are successive outgoing arcs of this vertex such that the set of leaves of the subtrees is the set X. For each vertex, the order of outgoing arcs is defined, and accordingly, so is the order of child vertices and vertex-induced subtrees for which these vertices are root ones.

The planar representation of a hierarchical tree displays not only the arcs and vertices but also the order of the arcs. For example, Fig. 1 represents a hierarchical tree for the preference profile consisting of a single linear order $12\ldots m$. The semicircular arrow shows the order of the arcs. In this preference profile, all sets of the form $\{i, \ldots, j\}$, where $i < j$, are clone sets. Each set of clones forms a sequence of terminal vertices. Thus, for each set of clones, there is a set of vertex-induced subtrees whose roots are successive outgoing arcs such that the set of leaves of the subtrees is the given set of clones. The inversion of the order of outgoing arcs does not violate the possibility of finding the desired sequence of outgoing arcs for each set of clones. The ability of a hierarchical

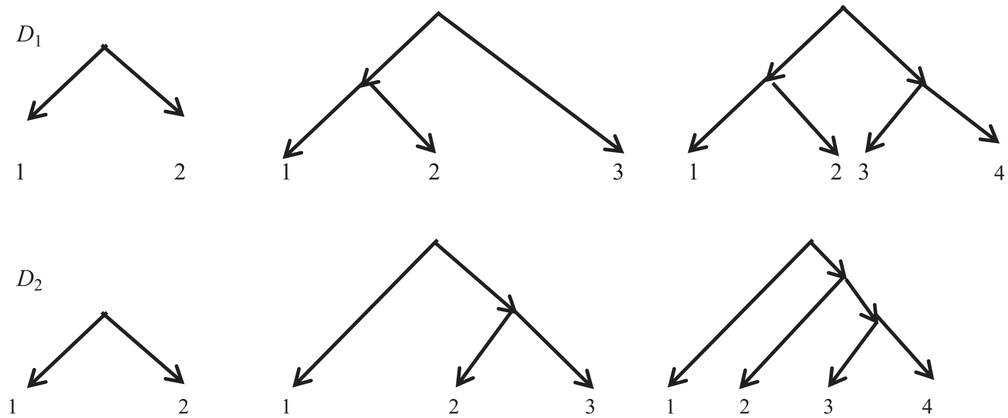

Fig. 2. Example of the implementation of the algorithm for constructing a hierarchical tree of alternatives for the domains $D1$ and $D2$.

tree and its planar representation to represent a preference profile is invariant under the outgoing arc order inversion operation. Arbitrary permutation of arcs can lead to the impossibility of finding a set of consecutive arcs for some set of clones. The considered type of trees is a special case of PQ-trees, see [78].

The class of binary hierarchical trees with all possible orders of outgoing arcs uniquely corresponds to a binary tree with unordered arcs. Therefore, each maximal group-separable domain corresponds to a binary tree with unordered arcs. One can construct this tree for a given domain as follows. An arbitrary linear order is taken from the domain. Hierarchical trees are built successively for two, three, etc. best alternatives to the chosen linear order. The algorithm stops after constructing a hierarchical tree for m alternatives. For two alternatives, the hierarchical tree consists of a root and two leaves. At each step, this tree is completed with a new nonterminal vertex and a new terminal vertex as follows. The tree constructed at the previous step contains a vertex and the corresponding vertex-induced subtree such that the set of leaves (alternatives) of this subtree is adjacent to the new alternative in all restrictions of the linear orders of the domain to the set of alternatives of the tree under construction (the set of alternatives of the previous step combined with the new alternative). If the vertex found is not a root, then the new vertex divides the incoming arc of the vertex found into two arcs; otherwise a new root and an arc connecting the new root with the old one are added to the tree. An additional arc connects the new nonterminal vertex to the new terminal vertex.

An illustration of the algorithm for constructing hierarchical trees for the domains D_1 and D_2 defined at the beginning of this section is shown in Fig. 2. In this example, the linear order 1234 is taken as a basis.

Based on the hierarchical tree, one can reconstruct the maximal group-separable domain. For the chosen planar representation of the tree, the linear order belonging to the domain can be constructed recursively by performing a depth-first search starting from the root (according to the order of outgoing arcs, we iterate outgoing arcs; for the vertex to which the arc leads, we run the same algorithm; after finding the terminal vertex, we add it to the linear order and return to the vertex that still has unconsidered arcs). Next, it is necessary to enumerate all inversions of the intervals of the constructed linear order corresponding to all vertex-induced subtrees, and all linear orders constructed in this way will belong to the group-separable domain.

Each maximal group-separable domain corresponds to a sequential partition of alternatives into two subsets.

For a group-separable preference profile and for a nonmaximal domain, the iterative algorithm for constructing a hierarchical tree is modified. To describe the algorithm, we introduce a new concept. A *minimum cover of a clone set* X for a profile $\mathscr{P}$ is the clone set $C(X)$ that contains the set X, does not coincide with the set X, and does not contain another nontrivial clone set containing the set X as a subset. The minimum cover may not be unique, but the number of minimum covers cannot exceed two [77]. For example, for a preference profile consisting of a single linear order 123, we have $C(\{1\}) = \{1, 2\}$, and for the set $\{2\}$ there are two minimal covers, $\{1, 2\}$ and $\{2, 3\}$.

The hierarchical tree construction algorithm starts by selecting one (arbitrary) linear order from the preference profile or from a nonmaximal domain and constructing a hierarchical tree to narrow the preference profile to a set of the top two alternatives of the chosen preference. For the two alternatives, the hierarchical tree consists of a root and two leaves. The order of arcs for a given tree is chosen arbitrarily. Further, at each step, the restriction of the preference profile to the new subset of alternatives is constructed by adding the next preferred alternative in the chosen linear order. Thus, a group-separable preference profile is constructed on some set of alternatives in which one of the alternatives is new (alternative x). This alternative is the worst in one of the linear orders. There is a unique minimal cover for the set consisting of the alternative x.

According to the definition of group-separable preferences, the minimal cover $C(\{x\})$ constructed can be divided into two sets of clones. By the construction of the minimal cover, this partition is represented by the sets $\{x\}$ and $C(\{x\})\backslash x$.

The number of minimal covers of the set $C(\{x\})\backslash x$ can be one or two. In the first case, it is true that $C(\{x\}) = C(C(\{x\})\backslash x)$. In this case, the tree is completed with a new nonterminal vertex, a new terminal vertex (alternative x), and new arcs in the manner described in the algorithm for constructing a hierarchical tree for a maximal group-separable domain. The order of outgoing arcs of the new nonterminal vertex is chosen arbitrarily. If the tree was binary, then it will remain binary. In the second case, the tree is completed with a new terminal vertex and a new arc in such a way that the new terminal vertex is a child of the same vertex as the root vertex of the vertex-induced subtree whose leaves are the set $C(\{x\})\backslash x$. In addition, the new terminal vertex will be simultaneously adjacent to the vertex that is the root of the vertex-induced subtree whose leaves are the set $C(\{x\})\backslash x$ and that is extreme (first or last) in the corresponding order of child vertices. In this case, the resulting hierarchical tree will not be binary.

The algorithm stops after performing the step for the worst alternative in the chosen linear order.

Different group-separable preference profiles, as well as different nonmaximal group-separable preference domains, can have the same hierarchical trees. P. Faliszewski et al. [79] found the number of group-separable preference profiles with a fixed number of agents and alternatives leading to a given hierarchical tree and described an algorithm for the equiprobable generation of such profiles.

An example of a hierarchical tree of alternatives is shown in Fig. 3. The set of alternatives corresponds to the set of leaves of the tree. The vertices of the tree correspond to the categories of alternatives, which are further divided into subcategories. According to Fig. 2, the whole set of drinks is divided into cold and hot ones. Each agent prefers all cold drinks over hot ones, or conversely, all hot drinks over cold ones. For each subsequent partition of alternatives, one subcategory is better than another of the same level.

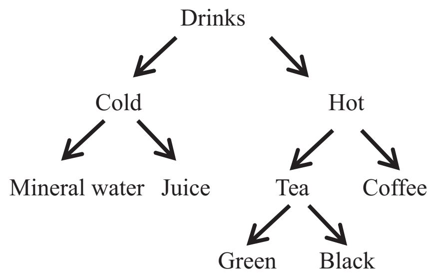

Fig. 3. Hierarchical tree of alternatives borrowed from [79].

Each of the $(2m-3)!$ maximal group-separable domains contains 2^{m-1} preferences. Each maximal group-separable domain contains the same number of linear orders as the maximal single-peaked preference domain, but since the number of group-separable domains is greater than the number of single-peaked ones, the number of group-separable preference profiles exceeds the number of single-peaked preference profiles.

Theorem 6 [29]. *The number of group-separable preference profiles is asymptotically equal to*

$$\#GS(m,n) \approx (2m-3)!\, 2^{(m-1)n}.$$

Configuration 7. There exist three agents $i, j, k \in \mathscr{N}$ and three alternatives $a, b, c \in A$ such that one has $(aP_ibP_ic$ or $cP_ibP_ia)$, $(bP_jcP_ja$ or $aP_jcP_jb)$, and $(bP_kaP_kc$ or $cP_kaP_kb)$.

Configuration 7 provides a variety of median alternatives. The prohibition on Configuration 7 limits the number of alternatives in any triple to two, which corresponds to one of the ways to implement constraint (ii) in Theorem 1.

Configuration 8. There exist two agents $i, j \in \mathscr{N}$ and four alternatives $a, b, c, d \in A$ such that one has $aP_ibP_icP_id$ and $(bP_jdP_jaP_jc$ or $cP_jaP_jdP_jb)$.

Configuration 8 defines a permutation pattern called separable permutations (for more on separable permutations, see [80]).

Theorem 7 [54]. *A preference profile is group-separable if and only if it does not contain Configurations 7 and 8.*

Configuration 9. There exist two agents $i, j \in \mathscr{N}$ and four alternatives $a, b, c, d \in A$ such that one has $aP_ibP_icP_id$ and $(bP_jaP_jdP_jc$ or $cP_jdP_jaP_jb)$.

Configuration 9 is used in the definition of *enriched group-separable preferences*, whose characterization is given in Theorem 9.

Theorem 8 [81]. *A preference profile is enriched group-separable if and only if it does not contain Configurations 7, 8, and 9.*

Enriched group-separable preferences are of interest primarily from the point of view of their combinatorial properties [81].

Group-separable preferences simplify the classical problems of the collective choice theory only in the case of hierarchical trees of bounded height [79]. Group-separable preferences create conditions for the existence of a solution of the problem of random distribution of indivisible objects [82]. When the domain is limited to tier preferences, the solution of the problem becomes unique [83].

A preference domain is a *tier* one if there exists a weak order on the set of alternatives such that each equivalence class consists of one or two alternatives, and the preferences included in the domain are a linearization of this weak order. The maximum-size tier domain contains $2^{\lfloor \frac{m}{2} \rfloor}$ preferences. The tier preference domain is not a maximal Condorcet domain, but in the problem of the distribution of indivisible objects, a collective preference is not constructed and the relation generated by the simple majority rule does not play any role.

3.3. Single-Crossing Preferences

A *single-crossing preference domain* is characterized by a linear order (axis) on the set of agents. For each pair of alternatives, the subset of agents that have the same preferences on this pair is an interval on the initial linear order of the agents. Thus, when moving along the agent axis, preferences change at most once for each pair of alternatives.

A single-crossing preference domain naturally arises when modeling preferences with respect to tax rates [25, 26]. Single-crossing preference domains simplify the solution of some proportional representation problems [84].

Examples of single-crossing preference domains include

$$D_3 = \left\{ \begin{matrix} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 1 & 4 & 4 & 2 & 3 \\ 3 & 3 & 4 & 1 & 3 & 3 & 2 \\ 4 & 4 & 3 & 3 & 1 & 1 & 1 \end{matrix} \right\},$$

$$D_4 = \left\{ \begin{matrix} 1 & 2 & 2 & 2 & 2 & 4 & 4 \\ 2 & 1 & 3 & 3 & 4 & 2 & 3 \\ 3 & 3 & 1 & 4 & 3 & 3 & 2 \\ 4 & 4 & 4 & 1 & 1 & 1 & 1 \end{matrix} \right\}.$$

Just as group-separable preference domains, single-crossing preference domains are diverse and structurally different from each other.

Configuration 10. There exist three agents $i, j, k \in \mathscr{N}$ and not necessarily distinct alternatives $a, b, c, d, e, f \in A$ such that bP_ia, cP_id, eP_if, aP_jb, dP_jc, eP_jf, aP_kb, cP_kd, and fP_ke.

Configuration 11. There exist four agents $i, j, k, l \in \mathscr{N}$ and not necessarily distinct alternatives $a, b, c, d \in A$ such that aP_ib, cP_id, aP_jb, dP_jc, bP_ka, cP_kd, bP_la, and dP_lc.

Theorem 9 [85]. *A preference profile has the single-crossing property if and only if it does not contain Configurations 10 and 11.*

Forbidden subprofiles contain three or four agents, so all preference profiles with two agents are single-crossing preference profiles.

Each single-crossing preference domain corresponds to a maximal chain in the weak Bruhat order constructed over a set of linear orders with a nesting relation of sets of pairs of alternatives that have the reverse order with respect to the minimal element. Thus, the weak Bruhat order has two linear orders inverse to each other as the minimum and maximum elements. Each single-crossing preference domain contains linear orders that form a chain from the minimum to the maximum element in the weak Bruhat order. Each single-crossing preference domain contains $\frac{m(m-1)}{2}+1$ preferences [85]. According to [86], the number of maximal chains in the weak Bruhat order is

$$\frac{\binom{m}{2}!}{1^{m-1}3^{m-2}5^{m-3}\cdots(2m-3)^{1}}.$$

Far from all single-crossing preference domains are maximal Condorcet domains. For each number of alternatives, only two domains are both single-crossing preference domains and maximal Condorcet domains [87].

For three alternatives, the number of single-crossing preference profiles is greater than the number of single-peaked preference profiles and the number of group-separable preference profiles, but the opposite is true for more alternatives. Theorem 10 reveals why there are a large number of single-crossing preference profiles for three alternatives.

Theorem 10. *A preference profile with three alternatives is a single-crossing preference profile if and only if it is single-peaked or single-dipped one.*

Single-crossing preferences have many extensions such as single-crossing preferences on a tree [88], on graphs [89], preferences close to single-crossing preferences [63, 90–92] (these are preferences that reduce to single-crossing preferences after removing either k alternatives or k agents, or splitting the set of agents or alternatives into k parts, or transforming preferences via k pairwise replacements of successive alternatives in some preferences, and so on), and single-crossing preferences under incomplete information [93].

Single-crossing preferences fix some ordering of agents that can be compared with the distribution of agents (voters) by income, education, etc.; thus, one can find an explanation for the existing preferences. The paper [94] is devoted to studying the correlation of single-crossing preferences and exogenous attributes.

Configuration 12 (minimal diversity). For each alternative, there exists an order of preferences with this alternative as the maximal one.

Theorem 11 [95]. *A single-crossing preference profile containing Configuration 12 is single-peaked.*

Theorem 12 [95]. *A single-crossing preference profile is single-peaked if and only if it can be augmented with preferences such that the augmented profile is a single-crossing preference profile and contains Configuration 12.*

Theorem 12 characterizes domains that are the intersection of a single-peaked preference domain and a single-crossing preference domain. The interest in this domain arose because it is close to the domain of Euclidean preferences, which is the subject of the next subsection.

3.4. One-Dimensional Euclidean Preferences

Unless otherwise stated, Euclidean preferences mean one-dimensional Euclidean preferences in this section. A preference profile is *Euclidean* if there is an axis on which alternatives are located and ideal points of agents are such that the preferences of agents are formed based on the geometric proximity of the alternatives. Each agent prefers alternatives closer to his/her ideal point over more distant ones. It is impossible to find the coordinates of alternatives and agents for every single-peaked preference profile in such a way that it becomes Euclidean [96].

It is Euclidean preferences that are used in many models of spatial economics, including H. Hotelling's model of a linear city [97], and models of political competition originating from the model by A. Downs [98].

Theorem 13 [99]. *For up to five alternatives, a preference profile is Euclidean if and only if it is both a single-peaked and a single-crossing one.*

Starting from six alternatives, all Euclidean preference profiles are both single-peaked and single-crossing preference profiles, but there are examples where the converse is not true. Three agents are sufficient for an example.

Theorem 14 [99]. *For two agents, the preference profile is Euclidean if and only if it is single-peaked.*

Theorem 15 [100]. *There is no finite number of forbidden subprofiles characterizing Euclidean preference profiles.*

Theorem 15 does not prohibit characterization in terms of infinitely many forbidden subprofiles. Such a characterization could be quite compact and applicable in applied research, such as, for example, the characterization of matrices with the property of consecutive ones [101]. Despite the absence of characterization in terms of a finite number of forbidden subprofiles, there is a polynomial complexity algorithm for recognizing Euclidean preference profiles [102, 103].

A natural generalization of Euclidean preferences are multidimensional Euclidean preferences, in which the alternatives and ideal points of agents are represented by points in a multidimensional Euclidean space, but the problem of recognizing the profile of k-dimensional Euclidean preferences becomes NP-hard for any $k > 1$ [104].

Another generalization is Euclidean preferences on a circle. These preferences have found application in the theory of industrial markets [105] and political competition [106].

4. STRUCTURED PREFERENCES IN THE MARRIAGE PROBLEM

Let us consider two finite disjoint sets: the set $W = \{w_1, \ldots, w_n\}$ of women and the set $M = \{m_1, \ldots, m_n\}$ of men. Each agent $i \in W \cup M$ has a preference over the set of agents of the opposite sex (preferences are linear orders). The problem is to create the kind of male–female pairings that the agents would like to keep.

The marriage problem is a tuple $(W, M, \mathscr{P})$ that consists of the set of women, the set of men, and a preference profile of $2n$ linear orders. A *matching* is a mapping $\mu : W \cup M \to W \cup M$ such that $\forall w \in W$ we have $\mu(w) \in M$ and $\mu(\mu(w)) = w$ and that $\forall m \in M$ we have $\mu(m) \in W$ and $\mu(\mu(m)) = m$. A matching μ is said to be *stable* if there exists no blocking pair, i.e., a pair $(X, x) \in W \times M$ such that $XP_x\mu(x)$ and $xP_X\mu(X)$. A stable matching is a solution of the marriage problem. A proof of the existence of a solution of the marriage problem and an algorithm for constructing a stable matching were obtained by D. Gale and L.S. Shapley in [107].

A pair that prefers each other to all other agents of the opposite sex is called a *fixed pair*. In any stable matching, agents from each fixed pair will be matched against each other. J. Eeckhout [108] formulated the *sequential preference condition*: there is a permutation σ of women and a permutation τ of men such that

$$\forall i, j \in \{1, \ldots, n\} \text{ such that } j > i \text{ we have } m_{\tau(i)} P_{w_{\sigma(i)}} m_{\tau(j)};$$
$$\forall i, j \in \{1, \ldots, n\} \text{ such that } j > i \text{ we have } w_{\sigma(i)} P_{m_{\tau(i)}} w_{\sigma(j)}.$$

The sequential preference condition implies the existence of a fixed pair $(w_{\sigma(1)}, m_{\tau(1)})$.

Theorem 16 [108]. *The sequential preference condition is sufficient for the uniqueness of a stable matching.*

The permutations from the sequential preference condition define the matching $\mu\left(m_{\tau(i)}\right) = w_{\sigma(i)}$. The sequential preference condition allows an agent to prefer someone to his/her matching pair, but only if the preferred ones are "higher" in the linear order corresponding to the permutation. Imagine that men and women are ordered by height, and each agent is paired with an agent of the opposite sex of the same rank in height. If for each agent, among the agents that are better than his/her matching pair, there are only agents that are taller, then the rank-based matching is the only stable matching. Let us prove the following theorem.

Theorem 17. *The probability of a profile appearing that satisfies the sequential preference condition satisfies the following recurrence relation*:

$$s(n) = \sum_{i=0}^{n-1} (-1)^{n-i+1} \binom{n}{i}^2 \frac{(n-i)!}{n^{2(n-i)}} s(i),$$

where

$$s(0) = 1.$$

Proof. All preference profiles that satisfy the sequential preference condition can be divided into sets of equal cardinality with preference profiles that differ in the final matching. All preference profiles with a given matching can be divided into sets with different numbers of fixed pairs. There are $\binom{n}{i}$ ways to choose i pairs. After eliminating these i pairs, we obtain a preference profile with $n-i$ pairs, and this profile will satisfy the sequential preference condition. For a given smaller preference profile, there are $\binom{n}{i}^{2(n-i)} i!^{2(n-i)}$ ways to add i pairs to the profile and $(n-1)!^{2i}$ ways to define the preferences of i pairs of agents. Applying the inclusion–exclusion principle, we obtain the number of preference profiles that satisfy the condition of sequential preferences,

$$SPC(n) = n! \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} \binom{n}{i}^{2(n-i)} i!^{2(n-i)} (n-1)!^{2i} SPC(n-i),$$

where

$$SPC(0) = 1.$$

Calculating the probability

$$s(n) = \frac{SPC(n)}{n!^{2n}}$$

and introducing the new summation index $i' = n - i$, we obtain the final result. ∎

The present author proposed [109] a more general α-*condition* for the uniqueness of a stable matching. There exists a stable matching μ, a permutation σ of women, and a permutation τ of men such that

$$\forall i, j \in \{1, \ldots, n\} \text{ such that } j > i, \text{ we have } \mu(w_{\sigma(i)}) P_{w_{\sigma(i)}} m_{\tau(j)},$$

and a permutation σ' of women and a permutation τ' of men such that

$$\forall i, j \in \{1, \ldots, n\} \text{ such that } j > i, \text{ we have } \mu(m_{\tau'(i)}) P_{m_{\tau'(i)}} w_{\sigma'(j)}.$$

Example [109]. The preference profile

$$\begin{aligned} w_1 &: m_1 P_{w_1} m_3 P_{w_1} m_2, \quad m_1 : w_3 P_{m_1} w_1 P_{m_1} w_2; \\ w_2 &: m_1 P_{w_2} m_2 P_{w_2} m_3, \quad m_2 : w_2 P_{m_2} w_1 P_{m_2} w_3; \\ w_3 &: m_2 P_{w_3} m_3 P_{w_3} m_1, \quad m_3 : w_2 P_{m_3} w_3 P_{m_3} w_1; \end{aligned}$$

violates the sequential preference condition, since it does not have a fixed pair, but it has a unique stable matching $w_1 \leftrightarrow m_1$, $w_2 \leftrightarrow m_2$, $w_3 \leftrightarrow m_3$. This profile satisfies the α-condition for $\sigma = \tau = 123$ and $\sigma' = \tau' = 231$.

Theorem 18 [109]. *The α-condition is sufficient for the uniqueness of a stable matching.*

The α-condition permits one to have different criteria for constructing orderings. Suppose that, in addition to the preferences profile of agents, there are two orderings expressing "female look" and "male look," respectively. For example, women order everyone by height and men, by age. In the α-condition, the first ordering determines the permutations σ and τ, and the second ordering determines the permutations σ' and τ'. Imagine that men and women are ordered according to the given criteria, and each agent is paired with an agent of the opposite sex of the same rank according to the criterion of his/her gender. If for each agent, among the agents that are better than his/her pair according to the matching, there are only agents that are higher according to the criterion for the corresponding gender, then the rank-based matching is the only stable matching. The α-condition can be applied when developing decentralized algorithms in matching markets [110, 111].

5. DICHOTOMOUS PREFERENCES

Dichotomous preferences are given by splitting the set of alternatives into two subsets—supported alternatives and unacceptable alternatives.

Dichotomous preferences are used in approval voting [112], in the proportional representation problem [113], in the matching problem [114], in the resource allocation problem [115, 116], in hedonic games [117], and other applications.

Despite the simple structure, many computational problems associated with dichotomous preferences turn out to be NP-hard; this has led to the creation of the theory of dichotomous preference domains—restrictions on the set of possible preferences. E. Elkind and M. Lackner [118] showed that many natural restrictions on the preference profile lead to a simplification of computational proportional representation problems.

As before, we have the set $A = \{1, \ldots, m\}$ of alternatives and the set $\mathscr{N} = \{1, \ldots, n\}$ of agents. Each agent $i \in \mathscr{N}$ approves the set $A_i \subseteq A$. Let us compose an $m \times n$ matrix X, where $x_{ij} = 1$

if agent i approves of alternative j and $x_{ij} = 0$ if agent j does not approve of alternative i. The matrix $X_{m\times n}$ is a preference profile. The set of all possible $m \times n$ binary matrices will be denoted by $\mathscr{M}_{m\times n}$. There are 2^{mn} distinct matrices and hence distinct preference profiles. The preference domain is the subset of possible columns in the binary matrix.

We say that matrices X and Y are *equivalent* with the notation $X \equiv Y$ if the matrix X is equal to the matrix Y after interchanging the rows and columns,

$$X \equiv Y \quad \text{if and only if} \quad x_{ij} = y_{\sigma(i)\tau(j)}, \quad \sigma(i) \in S_m, \quad \tau(j) \in S_n,$$

where S_k is the set of all permutations on the set $\{1, \dots, k\}$.

A matrix $Q_{k\times l}$ is a *pattern* in a matrix $A_{m\times n}$ if there exists a submatrix $B_{k\times l}$ of the matrix $A_{m\times n}$ such that $Q_{k\times l} \equiv B_{k\times l}$. Patterns play the role of configurations in ordinal preferences.

For each linear order, one can define a set of dichotomous preferences that can be achieved with some level of approval by the agent.

For single-peaked preferences, the approval sets (some number of best alternatives) necessarily form an interval on the alternative axis (the notion of a single-peaked preference axis was introduced in Sec. 3.1). Thus, we obtain a preference domain with consecutive 1s in the columns (candidate intervals).

Pattern 1.

$$\left[\begin{array}{cccccccc|c}
1 & 0 & 0 & & & & & & 1 \\
1 & 1 & 0 & \dots & & 0 & & & 0 \\
0 & 1 & 1 & & & & & & \\
 & \vdots & & \ddots & & \vdots & & & \dots \\
 & & & & 1 & 1 & 0 & & \\
 & 0 & & \cdots & 0 & 1 & 1 & & 0 \\
 & & & & 0 & 0 & 1 & & 1
\end{array}\right] \in \mathscr{M}_{k+2\times k+2}, \quad k \geq 1.$$

Pattern 2.

$$\left[\begin{array}{ccccccc|cc}
1 & 0 & 0 & & & & & 1 & 0 \\
1 & 1 & 0 & \dots & & 0 & & 1 & 1 \\
0 & 1 & 1 & & & & & & \\
 & \vdots & & \ddots & & \vdots & & \multicolumn{2}{c}{\dots\dots} \\
 & & & & 1 & 1 & 0 & & \\
 & 0 & & \cdots & 0 & 1 & 1 & 1 & 1 \\
 & & & & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & & & \dots & & 0 & 0 & 1 & 1
\end{array}\right] \in \mathscr{M}_{k+2\times k+3}, \quad k \geq 1.$$

Pattern 3.

$$\left[\begin{array}{ccccccc|c}
1 & 0 & 0 & & & & & 0 \\
1 & 1 & 0 & \dots & & 0 & & 1 \\
0 & 1 & 1 & & & & & \\
 & \vdots & & \ddots & & \vdots & & \dots \\
 & & & & 1 & 1 & 0 & \\
 & 0 & & \dots & 0 & 1 & 1 & 1 \\
 & & & & 0 & 0 & 1 & 0 \\
\hline
0 & & & \dots & & 0 & 0 & 1
\end{array}\right] \in \mathscr{M}_{k+3\times k+2}, \quad k \geq 1.$$

Pattern 4.

$$\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}.$$

Pattern 5.

$$\begin{bmatrix}
1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}.$$

Theorem 19 [101]. *A dichotomous preference profile satisfies the property of consecutive* 1*s in columns if and only if the profile does not contain Patterns* 1–5*.*

The characteristic in Theorem 21 contains infinitely many patterns, but since the patterns are similar to each other, the problem of finding a pattern of the minimum size is solved in polynomial time [119]. There is a simple algorithm based on PQ trees for testing the property of consecutive 1s in a matrix [78].

Transposed Patterns 1–5 will characterize the domain with consecutive 1s in rows (agent intervals) discussed in [118]. In the literature, one can find characterizations of binary matrices with the circular property of consecutive 1s [120] and with the property of consecutive 1s on a tree [121].

Z. Terzopoulou et al. [122] investigated preference domains with limited agent intervals. These intervals are formed only at the edges of the axis (preference domain with *voter external intervals, hereinafter referred to as VEI*) or only at one edge of the axis (preference domain with *single-sided voter external intervals, hereinafter referred to as SVEI*). In the case of voting for candidates from political parties, voters can be ordered according to the degree of loyalty to one of the two parties. The most loyal voters will approve of all the candidates of their party, and the less loyal voters may approve a certain number of candidates from both parties. An example of a preference profile with VEI is

$$\begin{pmatrix} 1 & 1 & & 1 & 0 \\ 1 & 1 & & 0 & 0 \\ 1 & 0 & & 0 & 0 \\ 0 & 0 & & 0 & 1 \\ 0 & 0 & & 1 & 1 \\ 0 & 1 & & 1 & 1 \end{pmatrix}.$$

Each candidate receives approval from the interval of agents located at the edge of the axis.

Pattern 6.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Pattern 7.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Pattern 8.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Pattern 9.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Pattern 10.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Theorem 20 [122]. *A profile of dichotomous preferences satisfies the VEI property if and only if the profile does not contain Patterns 6–10.*

Pattern 11.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Theorem 21 [122]. *A dichotomous preference profile satisfies the SVEI property if and only if the profile does not contain Pattern 11.*

Matrices that do not contain Pattern 11 can be uniquely reconstructed from their sums of rows and columns [123].

Theorem 22 [123]. *The number of profiles of dichotomous preferences that satisfy the SVEI property is*

$$\#SVEI(m,n) = \sum_{k=1}^{\min(m,n)} k!^2 S\,(n+1,k+1)\,S\,(m+1,k+1),$$

where $S\,(a,b)$ is the Stirling number of the second kind (see [124]).

The paper [122] also considers sufficient and necessary conditions for finding structured dichotomous preferences in the case of incomplete information, when there is no information about some cells in the preference profile. The possible use of interval preferences on the preference axis is extended to interval preferences on trees in [125].

Another example of structured dichotomous preferences is partition. The dichotomous preference profile satisfies the *partition* property if each agent's approval set matches one of the sets that belong to some partition of the set of alternatives.

Pattern 12.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Theorem 23 [122]. *A dichotomous preference profile satisfies the partition property if and only if the profile does not contain Pattern 12.*

Theorem 24 [126]. *The number of profiles of dichotomous preferences that satisfy the partition property is*

$$\#PART(m,n) = \sum_{k=1}^{\min(m,n)} k!S\,(n,k)\,S\,(m,k),$$

where $S\,(a,b)$ is the Stirling number of the second kind (see [124]).

The partition property is very restrictive, and the number of profiles satisfying this property is less than the number of profiles possessing the SVEI property.

6. CONCLUSIONS

On the one hand, structured preference profiles are a very limited combination of preferences and cannot reflect the diversity of opinions in real aggregation problems. On the other hand, structured preference models usually provide solutions with good properties. The task of extending structured preference domains to increase the number of profiles covered while maintaining the desired properties remains open. Many extension attempts face both computational and interpretational challenges.

Promising but little touched areas of preference modeling are studies of incomplete preferences, namely preferences represented by weak or partial orders, preferences having a linear order over a subset of alternatives, and so on. The analysis of dichotomous preferences presented in this survey is the first step in this direction.

A natural direction of research is the analysis of real data on preference profiles. Preference profile libraries [127, 128] are the starting point for much research in this area.

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