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Homework 1 ECE 4710J

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1. Fundamental Linear Algebra

Ben, Tom, and Amy are shopping for fruit at a grocery store. A fruit bowl contains some fruit and the price of fruit bowl is the total price of all of its individual fruit.
The store has apples for \$2, bananas for \$1, and oranges for \$4. The price of each of these can be written in a vector:

$$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

The store sells the following fruit bowls:

1. 2 of each fruit
2. 5 apples and 8 bananas
3. 2 bananas and 3 oranges
4. 10 oranges

(a) Define a matrix B such that

$$B\vec{v}$$

evaluates to a length 4 column vector containing the price of each fruit bowl. The first entry of the result should be the cost of fruit bowl 1, the second entry the cost of fruit bowl 2, etc.

Since $B\vec{v} = \begin{bmatrix} 14 \\ 18 \\ 14 \\ 40 \end{bmatrix}$ and B is a 4×3 matrix, we have

$$\Rightarrow B = \begin{bmatrix} 2 & 2 & 2 \\ 5 & 8 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 10 \end{bmatrix}$$

(b) Ben, Tom, and Amy make the following purchases:

- Ben buys 2 fruit bowl 1s and 1 fruit bowl 2.
- Tom buys 1 of each fruit bowl.
- Amy buys 10 fruit bowl 4s

Define a matrix A such that the matrix expression

$$AB\vec{v}$$

evaluates to a length 3 column vector containing how much each of them spent. The first entry of the result should be the total amount spent by Ben, the second entry the amount sent by Tom, etc.

Since $AB\vec{v} = \begin{bmatrix} 46 \\ 86 \\ 400 \end{bmatrix}$ and A is a 3×4 matrix, we have

$$\Rightarrow A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

(c) Let's suppose the store changes their fruit prices, but you don't know what they changed their prices to. Ben, Tom, and Amy buy the same quantity of fruit baskets and the number of fruit in each basket is the same, but now they each spent these amounts:

$$\vec{x} = \begin{bmatrix} 80 \\ 80 \\ 100 \end{bmatrix}$$

In terms of A, B, and $\vec{x}$, determine $\vec{v_2}$ (the new prices of each fruit).

Since $AB = \begin{bmatrix} 8 & 12 & 4 \\ 7 & 12 & 15 \\ 0 & 0 & 100 \end{bmatrix}$ and $AB\bar{v}_2 = \bar{x} = \begin{bmatrix} 80 \\ 80 \\ 100 \end{bmatrix}$, we have

Let $\bar{v}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, then $\begin{cases} 8a + 12b + 4c = 80 \\ 7a + 12b + 15c = 80 \\ 100c = 100 \end{cases} \Rightarrow \begin{cases} a = \frac{11}{2} \\ b = \frac{53}{24} \\ c = 1 \end{cases}$

$$\Rightarrow \bar{v}_2 = \begin{bmatrix} \frac{11}{2} \\ \frac{53}{24} \\ 1 \end{bmatrix}$$

2. Calculus

Let $\sigma(x) = \frac{1}{1+e^{-x}}$.

(a) Show that $\sigma(-x) = 1 - \sigma(x)$

(b) show that the derivative can be written as:
$$\frac{d}{dx}\sigma(x) = \sigma(x)(1-\sigma(x))$$

(c) Make a plot for $\sigma(x)$ and $\frac{d}{dx}\sigma(x)$ in the same coordinate system for $x \in [-5, 5]$

(a) $\sigma(-x) = \dfrac{1}{1+e^{x}}$

$1 - \sigma(x) = 1 - \dfrac{e^x}{e^x+1} = \dfrac{1}{e^x+1}$

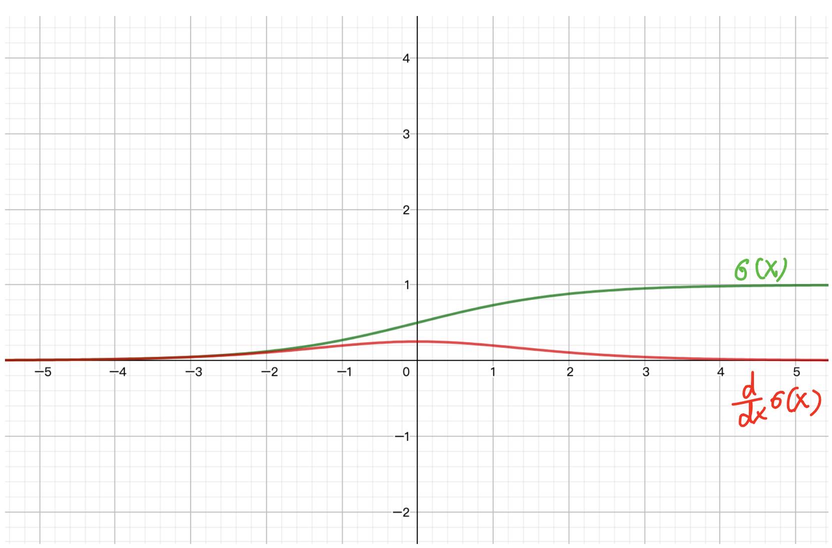
$\Rightarrow$ Thus we have $\sigma(-x) = 1 - \sigma(x)$

(b) $\dfrac{d}{dx}\sigma(x) = -\dfrac{1}{(1+e^{-x})^2} \cdot (-e^{-x}) = \dfrac{1}{e^{-x}+2+e^{x}}$

$\sigma(x)(1-\sigma(x)) = \dfrac{1}{1+e^{-x}} \cdot \dfrac{1}{e^x+1} = \dfrac{1}{e^x+e^{-x}+2}$

$\Rightarrow$ Thus the derivative $\frac{d}{dx}\sigma(x) = \sigma(x)(1-\sigma(x))$

(c)



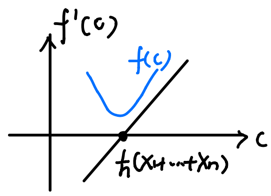
3. Minimization

Consider the function $f(c) = \frac{1}{n}\Sigma_{i=1}^{n}(x_i - c)^2$. In this scenario, suppose that our data points $x_1, x_2, ..., x_n$ are fixed, and that c is the only variable.
Using calculus, determine the value of c that minimizes $f(c)$. You must justify that this is indeed a minimum, and not a maximum.

Since $x_1, x_2 \cdots x_n$ are fixed, they are constants.

$$f(c) = \frac{1}{n}\left[(x_1-c)^2 + \cdots (x_n-c)^2\right] = \frac{1}{n}(x_1^2 + \cdots x_n^2) - \frac{2c}{n}(x_1+x_2+\cdots+x_n) + c^2$$

$$f'(c) = 2c - \frac{2}{n}(x_1+x_2+\cdots+x_n)$$



① when $c < \frac{1}{n}(x_1+x_2+\cdots+x_n)$, $f'(c) < 0 \Rightarrow f(c)$ is monotonically decreasing

② when $c > \frac{1}{n}(x_1+x_2+\cdots+x_n)$, $f'(c) > 0 \Rightarrow f(c)$ is monotonically increasing

$\Rightarrow$ Thus when $c = \frac{1}{n}(x_1+x_2+\cdots+x_n)$, $f(c)$ is minimum.

4. Probability

Only 1% of 40-year-old women who participate in a routine mammography test have breast cancer. 80% of women who have breast cancer will test positive, but 9.6% of women who don't have breast cancer will also get positive tests.

Suppose we know that a woman of this age tested positive in a routine screening. What is the probability that she actually has breast cancer? (Note: You must show all of your work, and also simplify your final answer to 3 decimal places.)

Let "has breast cancer" be A_1 , "healthy" be A_2 , "test positive" be B .

then $P[A_1] = 1\%$, $P[A_2] = 99\%$

$P[B|A_1] = 80\%$, $P[B|A_2] = 9.6\%$

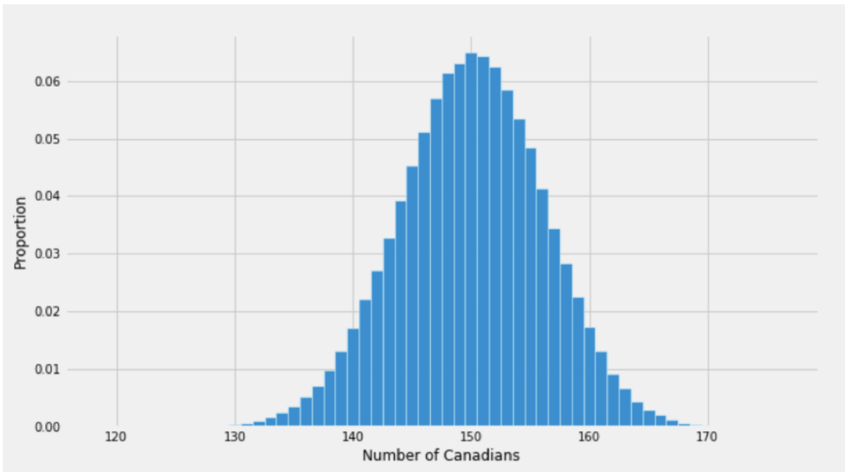
$$\Rightarrow P[A_1|B] = \frac{P[B|A_1]\cdot P[A_1]}{P[B|A_1]\cdot P[A_1] + P[B|A_2]\cdot P[A_2]} = \frac{0.8 \times 0.01}{0.8 \times 0.01 + 0.096 \times 0.99} \approx 7.764\%$$

Thus, the probability is $7.764\% \approx 0.078$

5. Statistics

Suppose we collected a sample of 200 students at University A, and 150 of them happened to be Canadian (so, if we were to select a student uniformly at random from our sample, there is a 0.75 chance that they are Canadian).

For inferential purposes, we choose to bootstrap this sample 500,000 times. That is, we simulate the act of re-sampling (with replacement) 200 students from our observed sample, and each time we record the number of Canadians in our re-sample. We provide a histogram of the sampling distribution below.



正态分布（二项分布）：

E[X] = miu = np
Var[X] = np(1-p)

What is the standard deviation of the sampling distribution shown above? Select the closest option below, and **explain your answer**.

a 1.5

b 6.1

c 12.4

d 10.1

Hint: While it is possible to calculate the answer, the histogram has all of the information you need.

I choose b.

Since ① the histogram is symmetrical

② the expectation $E[X]=150$, which is equal to $np = 200 \cdot 0.75 = 150$

The distribution conforms to the normal distribution $X \sim N(\mu, \sigma)$

$$Var[X] = np(1-p) = 37.5$$

$$\Rightarrow \sigma = \sqrt{Var[X]} \approx 6.1$$

Thus, the standard deviation is 6.1, choose b.