- 99442326-

July 28, 2024

1 Numerical linear algebra Project

Implements numerical methods, specifically LU decomposition and Gaussian elimination with pivoting, to solve systems of linear equations using Python and NumPy.

[19]:
```
import numpy as np
```

1.1 Part 1: LU method to solve the system AX=b where A is an n*n matrix

The code below checks if LU decomposition is possible for a square matrix A. It iterates through each leading principal minor of A and calculates its determinant. If any determinant is close to zero, it returns False; otherwise, it returns True. The function ensures that all leading principal minors are non-zero for LU decomposition to be feasible.

[20]:
```
def is_lu_possible(A):
    n = len(A)
    for i in range(n):
        minor = np.linalg.det(A[:i+1, :i+1])
        if np.isclose(minor, 0.0):
            return False
    return True
```

The lu_decomposition function checks if LU decomposition is possible for a square matrix A. If possible, it initializes lower triangular matrix L as the identity matrix and upper triangular matrix U as a zero matrix. It then iteratively calculates the elements of L and U using nested loops, printing the progress at each step. The function returns the decomposed matrices L and U. If LU decomposition is not possible, it prints a message and returns None, None.

[21]:
```
def lu_decomposition(A):
    if not is_lu_possible(A):
        print("LU decomposition is not possible due to zero leading principal␣
 ↪minors.")
        return None, None

    n = len(A)
    L = np.eye(n)
    U = np.zeros((n, n))
```

```
    for i in range(n):
        for j in range(i, n):
            U[i, j] = A[i, j] - sum(L[i, k] * U[k, j] for k in range(i))
            print(f"Step {i + 1} (U): u({i+1}{j+1}) = {U[i, j]}")
        for j in range(i + 1, n):
            L[j, i] = (A[j, i] - sum(L[j, k] * U[k, i] for k in range(i))) /␣
↪U[i, i]
            print(f"Step {i + 1} (L): l({j+1}{i+1}) = {L[j, i]}")

        print(f"Step {i + 1}:\nL:\n{L}\nU:\n{U}\n")

    return L, U
```

The solve_lu function solves a system of linear equations given the LU decomposition of the coefficient matrix. It performs forward substitution for the lower triangular system Ly = b and back substitution for the upper triangular system Ux = y. The intermediate and final solutions are printed. The function returns the solution vector x.

[22]:
```
def solve_lu(L, U, b):
    n = len(b)
    # Solve Ly = b for y
    y = np.zeros(n)
    for i in range(n):
        y[i] = b[i] - sum(L[i, k] * y[k] for k in range(i))

    print(f"Solved Ly = b, y = {y}\n")

    # Solve Ux = y for x
    x = np.zeros(n)
    for i in range(n - 1, -1, -1):
        x[i] = (y[i] - sum(U[i, k] * x[k] for k in range(i + 1, n))) / U[i, i]

    print(f"Solved Ux = y, x = {x}\n")

    return x
```

Example usage 1

[23]:
```
A = np.array([[1, 2, 3],
              [2, 4, 8],
              [-1, 0, 1]])

b = np.array([1, 0, -1])

if is_lu_possible(A):
    L, U = lu_decomposition(A)
    x = solve_lu(L, U, b)
```

```
    print("Final Solution x:", x)
else:
    print("LU decomposition is not possible due to a zero leading principal␣
 ↪minor.")
```

```
LU decomposition is not possible due to a zero leading principal minor.
```

Example usage 2

[24]:
```
A = np.array([[2, -1, 0],
              [-1, 2, -1],
              [0, -1, 2]])

b = np.array([1, 0, -1])

if is_lu_possible(A):
    L, U = lu_decomposition(A)
    x = solve_lu(L, U, b)
    print("Final Solution x:", x)
else:
    print("LU decomposition is not possible due to a zero leading principal␣
 ↪minor.")
```

```
Step 1 (U): u(11) = 2.0
Step 1 (U): u(12) = -1.0
Step 1 (U): u(13) = 0.0
Step 1 (L): l(21) = -0.5
Step 1 (L): l(31) = 0.0
Step 1:
L:
[[ 1.   0.   0. ]
 [-0.5  1.   0. ]
 [ 0.   0.   1. ]]
U:
[[ 2. -1.  0.]
 [ 0.  0.  0.]
 [ 0.  0.  0.]]

Step 2 (U): u(22) = 1.5
Step 2 (U): u(23) = -1.0
Step 2 (L): l(32) = -0.6666666666666666
Step 2:
L:
[[ 1.          0.          0.        ]
 [-0.5         1.          0.        ]
 [ 0.         -0.66666667  1.        ]]
U:
[[ 2.  -1.   0. ]
```

```
 [ 0.   1.5 -1. ]
 [ 0.   0.   0. ]]

Step 3 (U): u(33) = 1.3333333333333335
Step 3:
L:
[[ 1.          0.          0.        ]
 [-0.5         1.          0.        ]
 [ 0.         -0.66666667  1.        ]]
U:
[[ 2.         -1.          0.        ]
 [ 0.          1.5        -1.        ]
 [ 0.          0.          1.33333333]]

Solved Ly = b, y = [ 1.          0.5        -0.66666667]

Solved Ux = y, x = [ 0.5  0.  -0.5]

Final Solution x: [ 0.5  0.  -0.5]
```

1.2 Part 2: Gaussian elimination with pivoting method to solve the system AX=b

The code below defines a function gaussian_elimination_pivoting to solve a system of linear equations Ax=b using Gaussian elimination with partial pivoting. It converts input matrices to float64, creates an augmented matrix, and performs forward elimination with partial pivoting. The intermediate steps, including pivot rows and elimination, are printed. Then, it conducts backward substitution to find the solution vector x, printing the calculations. The final augmented matrix is printed, and the function returns the solution vector.

[25]:
```python
def gaussian_elimination_pivoting(A, b):
    # Convert A and b to float64 if they are not already
    A = np.array(A, dtype=np.float64)
    b = np.array(b, dtype=np.float64)

    n = len(b)

    # Augmenting the matrix A with the vector b
    augmented_matrix = np.hstack((A, b.reshape(-1, 1)))

    print("Original Augmented Matrix:")
    print(augmented_matrix)

    # Forward elimination
    for i in range(n):
        # Partial pivoting
        max_row_index = np.argmax(np.abs(augmented_matrix[i:, i])) + i
```

```
        augmented_matrix[[i, max_row_index]] = augmented_matrix[[max_row_index,␣
↪i]]

        pivot_row = augmented_matrix[i, :]
        print("\nStep", i+1, "- Pivot Row:")
        print(pivot_row)

        for j in range(i + 1, n):
            factor = augmented_matrix[j, i] / pivot_row[i]
            print("  Subtracting", factor, "* Row", i+1, "from Row", j+1)
            augmented_matrix[j, :] -= factor * pivot_row

        print("\nAugmented Matrix after Step", i+1, ":")
        print(augmented_matrix)

    # Backward substitution
    x = np.zeros(n)
    for i in range(n - 1, -1, -1):
        x[i] = augmented_matrix[i, -1] / augmented_matrix[i, i]
        print("\nCalculating x[{}]: {} / {} = {}".format(i+1,␣
↪augmented_matrix[i, -1], augmented_matrix[i, i], x[i]))
        augmented_matrix[:i, -1] -= augmented_matrix[:i, i] * x[i]

    print("\nFinal Augmented Matrix:")
    print(augmented_matrix)

    return x
```

Example usage for A that is a 6*6 matrix:

A =

```
[[2, -1, 1, 3, 2, -4],

[1,  3, 2, 1,-1,  2],

[4,  2,-3,-1, 2,  1],

[3,  1, 1,-2, 1,  3],

[1, -4,-1, 2,-1,  1],

[-2, 1, 5,-4, 3,  2]]
```

b = [9, 1, -2, 6, -4, 7]

[26]:
```
A = np.array([[2, -1, 1, 3, 2, -4],
              [1, 3, 2, 1, -1, 2],
              [4, 2, -3, -1, 2, 1],
```

```
                [3, 1, 1, -2, 1, 3],
                [1, -4, -1, 2, -1, 1],
                [-2, 1, 5, -4, 3, 2]])

b = np.array([9, 1, -2, 6, -4, 7])

solution = gaussian_elimination_pivoting(A, b)
print("\nSolution:", solution)
```

```
Original Augmented Matrix:
[[ 2. -1.  1.  3.  2. -4.  9.]
 [ 1.  3.  2.  1. -1.  2.  1.]
 [ 4.  2. -3. -1.  2.  1. -2.]
 [ 3.  1.  1. -2.  1.  3.  6.]
 [ 1. -4. -1.  2. -1.  1. -4.]
 [-2.  1.  5. -4.  3.  2.  7.]]

Step 1 - Pivot Row:
[ 4.  2. -3. -1.  2.  1. -2.]
  Subtracting 0.25 * Row 1 from Row 2
  Subtracting 0.5 * Row 1 from Row 3
  Subtracting 0.75 * Row 1 from Row 4
  Subtracting 0.25 * Row 1 from Row 5
  Subtracting -0.5 * Row 1 from Row 6

Augmented Matrix after Step 1 :
[[ 4.    2.   -3.   -1.    2.    1.   -2.  ]
 [ 0.    2.5   2.75  1.25 -1.5   1.75  1.5 ]
 [ 0.   -2.    2.5   3.5   1.   -4.5  10.  ]
 [ 0.   -0.5   3.25 -1.25 -0.5   2.25  7.5 ]
 [ 0.   -4.5  -0.25  2.25 -1.5   0.75 -3.5 ]
 [ 0.    2.    3.5  -4.5   4.    2.5   6.  ]]

Step 2 - Pivot Row:
[ 0.   -4.5  -0.25  2.25 -1.5   0.75 -3.5 ]
  Subtracting 0.4444444444444444 * Row 2 from Row 3
  Subtracting 0.1111111111111111 * Row 2 from Row 4
  Subtracting -0.5555555555555556 * Row 2 from Row 5
  Subtracting -0.4444444444444444 * Row 2 from Row 6

Augmented Matrix after Step 2 :
[[ 4.          2.         -3.         -1.          2.          1.
  -2.        ]
 [ 0.         -4.5        -0.25        2.25       -1.5         0.75
  -3.5       ]
 [ 0.          0.          2.61111111  2.5         1.66666667 -4.83333333
  11.55555556]
```

```
 [ 0.          0.          3.27777778 -1.5        -0.33333333  2.16666667
   7.88888889]
 [ 0.          0.          2.61111111  2.5        -2.33333333  2.16666667
  -0.44444444]
 [ 0.          0.          3.38888889 -3.5         3.33333333  2.83333333
   4.44444444]]

Step 3 - Pivot Row:
[ 0.          0.          3.38888889 -3.5         3.33333333  2.83333333
  4.44444444]
  Subtracting 0.9672131147540983 * Row 3 from Row 4
  Subtracting 0.7704918032786886 * Row 3 from Row 5
  Subtracting 0.7704918032786886 * Row 3 from Row 6

Augmented Matrix after Step 3 :
[[ 4.          2.         -3.         -1.          2.          1.
  -2.        ]
 [ 0.         -4.5        -0.25        2.25       -1.5         0.75
  -3.5       ]
 [ 0.          0.          3.38888889 -3.5         3.33333333  2.83333333
   4.44444444]
 [ 0.          0.          0.          1.8852459  -3.55737705 -0.57377049
   3.59016393]
 [ 0.          0.          0.          5.19672131 -4.90163934 -0.01639344
  -3.86885246]
 [ 0.          0.          0.          5.19672131 -0.90163934 -7.01639344
   8.13114754]]

Step 4 - Pivot Row:
[ 0.          0.          0.          5.19672131 -4.90163934 -0.01639344
 -3.86885246]
  Subtracting 0.36277602523659297 * Row 4 from Row 5
  Subtracting 1.0 * Row 4 from Row 6

Augmented Matrix after Step 4 :
[[ 4.          2.         -3.         -1.          2.          1.
  -2.        ]
 [ 0.         -4.5        -0.25        2.25       -1.5         0.75
  -3.5       ]
 [ 0.          0.          3.38888889 -3.5         3.33333333  2.83333333
   4.44444444]
 [ 0.          0.          0.          5.19672131 -4.90163934 -0.01639344
  -3.86885246]
 [ 0.          0.          0.          0.         -1.77917981 -0.56782334
   4.99369085]
 [ 0.          0.          0.          0.          4.         -7.
  12.        ]]
```

```
Step 5 - Pivot Row:
[ 0.  0.  0.  0.  4. -7. 12.]
  Subtracting -0.44479495268138813 * Row 5 from Row 6

Augmented Matrix after Step 5 :
[[ 4.          2.         -3.         -1.          2.          1.
  -2.        ]
 [ 0.         -4.5        -0.25        2.25       -1.5         0.75
  -3.5       ]
 [ 0.          0.          3.38888889 -3.5         3.33333333  2.83333333
   4.44444444]
 [ 0.          0.          0.          5.19672131 -4.90163934 -0.01639344
  -3.86885246]
 [ 0.          0.          0.          0.          4.         -7.
  12.        ]
 [ 0.          0.          0.          0.          0.         -3.68138801
  10.33123028]]

Step 6 - Pivot Row:
[ 0.          0.          0.          0.          0.         -3.68138801
 10.33123028]

Augmented Matrix after Step 6 :
[[ 4.          2.         -3.         -1.          2.          1.
  -2.        ]
 [ 0.         -4.5        -0.25        2.25       -1.5         0.75
  -3.5       ]
 [ 0.          0.          3.38888889 -3.5         3.33333333  2.83333333
   4.44444444]
 [ 0.          0.          0.          5.19672131 -4.90163934 -0.01639344
  -3.86885246]
 [ 0.          0.          0.          0.          4.         -7.
  12.        ]
 [ 0.          0.          0.          0.          0.         -3.68138801
  10.33123028]]

Calculating x[6]: 10.331230283911673 / -3.6813880126182967 = -2.8063410454155955

Calculating x[5]: -7.6443873179091675 / 4.0 = -1.9110968294772919

Calculating x[4]: -13.282365459985673 / 5.19672131147541 = -2.5559125964010283

Calculating x[3]: 9.820372750642672 / 3.388888888888889 = 2.8978149100257067

Calculating x[2]: 2.2133676092544987 / -4.5 = -0.49185946872322195

Calculating x[1]: 11.749785775492715 / 4.0 = 2.9374464438731787
```

```
Final Augmented Matrix:
[[  4.           2.          -3.          -1.           2.
    1.          11.74978578]
 [  0.          -4.5         -0.25         2.25        -1.5
    0.75         2.21336761]
 [  0.           0.           3.38888889  -3.5          3.33333333
    2.83333333   9.82037275]
 [  0.           0.           0.           5.19672131  -4.90163934
   -0.01639344 -13.28236546]
 [  0.           0.           0.           0.           4.
   -7.          -7.64438732]
 [  0.           0.           0.           0.           0.
   -3.68138801  10.33123028]]

Solution: [ 2.93744644 -0.49185947  2.89781491 -2.5559126  -1.91109683
-2.80634105]
```

Example usage for A that is a 12*12 matrix:

A =

```
[[2, -1, 1, 3, 1, 5, 4, 1, 3, 6, 2, 8],

[4, 1, -1, 5, 3, 7, 3, 2, 5, 1, 7, 2],

[6, 3, 2, 8, 5, 9, 6, 4, 1, 3, 5, 1],

[8, 5, 3, 2, 7, 1, 8, 7, 4, 2, 1, 9],

[3, 4, 2, 1, 6, 7, 2, 9, 1, 5, 3, 6],

[1, 2, 6, 4, 8, 3, 1, 5, 9, 7, 2, 4],

[2, 1, 3, 5, 9, 6, 4, 8, 1, 2, 7, 3],

[5, 7, 1, 6, 4, 2, 9, 3, 6, 8, 4, 1],

[9, 8, 4, 7, 2, 3, 7, 1, 8, 4, 9, 5],

[3, 6, 5, 1, 8, 4, 5, 6, 7, 9, 1, 2],

[1, 2, 4, 6, 3, 8, 2, 1, 3, 7, 5, 9],

[7, 5, 9, 2, 1, 4, 6, 3, 2, 8, 4, 7]]
```

b = [7, 15, 29, 24, 21, 15, 8, 12, 20, 18, 23, 16]

[27]:
```
A = np.array([[2, -1, 1, 3, 1, 5, 4, 1, 3, 6, 2, 8],
              [4, 1, -1, 5, 3, 7, 3, 2, 5, 1, 7, 2],
```

```
                [6, 3, 2, 8, 5, 9, 6, 4, 1, 3, 5, 1],
                [8, 5, 3, 2, 7, 1, 8, 7, 4, 2, 1, 9],
                [3, 4, 2, 1, 6, 7, 2, 9, 1, 5, 3, 6],
                [1, 2, 6, 4, 8, 3, 1, 5, 9, 7, 2, 4],
                [2, 1, 3, 5, 9, 6, 4, 8, 1, 2, 7, 3],
                [5, 7, 1, 6, 4, 2, 9, 3, 6, 8, 4, 1],
                [9, 8, 4, 7, 2, 3, 7, 1, 8, 4, 9, 5],
                [3, 6, 5, 1, 8, 4, 5, 6, 7, 9, 1, 2],
                [1, 2, 4, 6, 3, 8, 2, 1, 3, 7, 5, 9],
                [7, 5, 9, 2, 1, 4, 6, 3, 2, 8, 4, 7]])

b = np.array([7, 15, 29, 24, 21, 15, 8, 12, 20, 18, 23, 16])

solution = gaussian_elimination_pivoting(A, b)
print("\nSolution:", solution)
```

```
Original Augmented Matrix:
[[ 2. -1.  1.  3.  1.  5.  4.  1.  3.  6.  2.  8.  7.]
 [ 4.  1. -1.  5.  3.  7.  3.  2.  5.  1.  7.  2. 15.]
 [ 6.  3.  2.  8.  5.  9.  6.  4.  1.  3.  5.  1. 29.]
 [ 8.  5.  3.  2.  7.  1.  8.  7.  4.  2.  1.  9. 24.]
 [ 3.  4.  2.  1.  6.  7.  2.  9.  1.  5.  3.  6. 21.]
 [ 1.  2.  6.  4.  8.  3.  1.  5.  9.  7.  2.  4. 15.]
 [ 2.  1.  3.  5.  9.  6.  4.  8.  1.  2.  7.  3.  8.]
 [ 5.  7.  1.  6.  4.  2.  9.  3.  6.  8.  4.  1. 12.]
 [ 9.  8.  4.  7.  2.  3.  7.  1.  8.  4.  9.  5. 20.]
 [ 3.  6.  5.  1.  8.  4.  5.  6.  7.  9.  1.  2. 18.]
 [ 1.  2.  4.  6.  3.  8.  2.  1.  3.  7.  5.  9. 23.]
 [ 7.  5.  9.  2.  1.  4.  6.  3.  2.  8.  4.  7. 16.]]

Step 1 - Pivot Row:
[ 9.  8.  4.  7.  2.  3.  7.  1.  8.  4.  9.  5. 20.]
  Subtracting 0.4444444444444444 * Row 1 from Row 2
  Subtracting 0.6666666666666666 * Row 1 from Row 3
  Subtracting 0.8888888888888888 * Row 1 from Row 4
  Subtracting 0.3333333333333333 * Row 1 from Row 5
  Subtracting 0.1111111111111111 * Row 1 from Row 6
  Subtracting 0.2222222222222222 * Row 1 from Row 7
  Subtracting 0.5555555555555556 * Row 1 from Row 8
  Subtracting 0.2222222222222222 * Row 1 from Row 9
  Subtracting 0.3333333333333333 * Row 1 from Row 10
  Subtracting 0.1111111111111111 * Row 1 from Row 11
  Subtracting 0.7777777777777778 * Row 1 from Row 12

Augmented Matrix after Step 1 :
[[ 9.          8.          4.          7.          2.          3.
   7.          1.          8.          4.          9.          5.
```

```
  20.        ]
 [ 0.         -2.55555556 -2.77777778  1.88888889  2.11111111  5.66666667
  -0.11111111  1.55555556  1.44444444 -0.77777778  3.         -0.22222222
   6.11111111]
 [ 0.         -2.33333333 -0.66666667  3.33333333  3.66666667  7.
   1.33333333  3.33333333 -4.33333333  0.33333333 -1.         -2.33333333
  15.66666667]
 [ 0.         -2.11111111 -0.55555556 -4.22222222  5.22222222 -1.66666667
   1.77777778  6.11111111 -3.11111111 -1.55555556 -7.          4.55555556
   6.22222222]
 [ 0.          1.33333333  0.66666667 -1.33333333  5.33333333  6.
  -0.33333333  8.66666667 -1.66666667  3.66666667  0.          4.33333333
  14.33333333]
 [ 0.          1.11111111  5.55555556  3.22222222  7.77777778  2.66666667
   0.22222222  4.88888889  8.11111111  6.55555556  1.          3.44444444
  12.77777778]
 [ 0.         -0.77777778  2.11111111  3.44444444  8.55555556  5.33333333
   2.44444444  7.77777778 -0.77777778  1.11111111  5.          1.88888889
   3.55555556]
 [ 0.          2.55555556 -1.22222222  2.11111111  2.88888889  0.33333333
   5.11111111  2.44444444  1.55555556  5.77777778 -1.         -1.77777778
   0.88888889]
 [ 0.         -2.77777778  0.11111111  1.44444444  0.55555556  4.33333333
   2.44444444  0.77777778  1.22222222  5.11111111  0.          6.88888889
   2.55555556]
 [ 0.          3.33333333  3.66666667 -1.33333333  7.33333333  3.
   2.66666667  5.66666667  4.33333333  7.66666667 -2.          0.33333333
  11.33333333]
 [ 0.          1.11111111  3.55555556  5.22222222  2.77777778  7.66666667
   1.22222222  0.88888889  2.11111111  6.55555556  4.          8.44444444
  20.77777778]
 [ 0.         -1.22222222  5.88888889 -3.44444444 -0.55555556  1.66666667
   0.55555556  2.22222222 -4.22222222  4.88888889 -3.          3.11111111
   0.44444444]]

Step 2 - Pivot Row:
[ 0.          3.33333333  3.66666667 -1.33333333  7.33333333  3.
  2.66666667  5.66666667  4.33333333  7.66666667 -2.          0.33333333
 11.33333333]
  Subtracting -0.6999999999999998 * Row 2 from Row 3
  Subtracting -0.6333333333333332 * Row 2 from Row 4
  Subtracting 0.4 * Row 2 from Row 5
  Subtracting 0.3333333333333333 * Row 2 from Row 6
  Subtracting -0.23333333333333328 * Row 2 from Row 7
  Subtracting 0.7666666666666666 * Row 2 from Row 8
  Subtracting -0.8333333333333333 * Row 2 from Row 9
  Subtracting -0.7666666666666666 * Row 2 from Row 10
  Subtracting 0.3333333333333333 * Row 2 from Row 11
```

```
  Subtracting -0.3666666666666667 * Row 2 from Row 12

Augmented Matrix after Step 2 :
[[ 9.          8.          4.          7.          2.          3.
   7.          1.          8.          4.          9.          5.
  20.        ]
 [ 0.          3.33333333  3.66666667 -1.33333333  7.33333333  3.
   2.66666667  5.66666667  4.33333333  7.66666667 -2.          0.33333333
  11.33333333]
 [ 0.          0.          1.9         2.4         8.8         9.1
   3.2         7.3        -1.3         5.7        -2.4        -2.1
  23.6       ]
 [ 0.          0.          1.76666667 -5.06666667  9.86666667  0.23333333
   3.46666667  9.7        -0.36666667  3.3        -8.26666667  4.76666667
  13.4       ]
 [ 0.          0.         -0.8        -0.8         2.4         4.8
  -1.4         6.4        -3.4         0.6         0.8         4.2
   9.8       ]
 [ 0.          0.          4.33333333  3.66666667  5.33333333  1.66666667
  -0.66666667  3.          6.66666667  4.          1.66666667  3.33333333
   9.        ]
 [ 0.          0.          2.96666667  3.13333333 10.26666667  6.03333333
   3.06666667  9.1         0.23333333  2.9         4.53333333  1.96666667
   6.2       ]
 [ 0.          0.         -4.03333333  3.13333333 -2.73333333 -1.96666667
   3.06666667 -1.9        -1.76666667 -0.1         0.53333333 -2.03333333
  -7.8       ]
 [ 0.          0.          3.16666667  0.33333333  6.66666667  6.83333333
   4.66666667  5.5         4.83333333 11.5        -1.66666667  7.16666667
  12.        ]
 [ 0.          0.          0.03333333  0.86666667  7.73333333  7.96666667
   1.93333333  5.9         4.76666667  5.1         1.46666667  0.03333333
  14.8       ]
 [ 0.          0.          2.33333333  5.66666667  0.33333333  6.66666667
   0.33333333 -1.          0.66666667  4.          4.66666667  8.33333333
  17.        ]
 [ 0.          0.          7.23333333 -3.93333333  2.13333333  2.76666667
   1.53333333  4.3        -2.63333333  7.7        -3.73333333  3.23333333
   4.6       ]]

Step 3 - Pivot Row:
[ 0.          0.          7.23333333 -3.93333333  2.13333333  2.76666667
  1.53333333  4.3        -2.63333333  7.7        -3.73333333  3.23333333
  4.6       ]
  Subtracting 0.2442396313364055 * Row 3 from Row 4
  Subtracting -0.11059907834101382 * Row 3 from Row 5
  Subtracting 0.5990783410138247 * Row 3 from Row 6
  Subtracting 0.4101382488479262 * Row 3 from Row 7
```

```
  Subtracting -0.5576036866359446 * Row 3 from Row 8
  Subtracting 0.43778801843317966 * Row 3 from Row 9
  Subtracting 0.00460829493087562 * Row 3 from Row 10
  Subtracting 0.32258064516129026 * Row 3 from Row 11
  Subtracting 0.2626728110599078 * Row 3 from Row 12

Augmented Matrix after Step 3 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -4.10599078e+00
   9.34562212e+00 -4.42396313e-01  3.09216590e+00  8.64976959e+00
   2.76497696e-01  1.41935484e+00 -7.35483871e+00  3.97695853e+00
   1.22764977e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00 -1.23502304e+00
   2.63594470e+00  5.10599078e+00 -1.23041475e+00  6.87557604e+00
  -3.69124424e+00  1.45161290e+00  3.87096774e-01  4.55760369e+00
   1.03087558e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.02304147e+00
   4.05529954e+00  9.21658986e-03 -1.58525346e+00  4.23963134e-01
   8.24423963e+00 -6.12903226e-01  3.90322581e+00  1.39631336e+00
   6.24423963e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  4.74654378e+00
   9.39170507e+00  4.89861751e+00  2.43778802e+00  7.33640553e+00
   1.31336406e+00 -2.58064516e-01  6.06451613e+00  6.40552995e-01
   4.31336406e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  9.40092166e-01
  -1.54377880e+00 -4.23963134e-01  3.92165899e+00  4.97695853e-01
  -3.23502304e+00  4.19354839e+00 -1.54838710e+00 -2.30414747e-01
  -5.23502304e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  2.05529954e+00
   5.73271889e+00  5.62211982e+00  3.99539171e+00  3.61751152e+00
   5.98617512e+00  8.12903226e+00 -3.22580645e-02  5.75115207e+00
   9.98617512e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  8.84792627e-01
   7.72350230e+00  7.95391705e+00  1.92626728e+00  5.88018433e+00
   4.77880184e+00  5.06451613e+00  1.48387097e+00  1.84331797e-02
   1.47788018e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
```

```
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  3.43317972e+00
   8.23963134e+00  8.37327189e+00  2.79723502e+00  6.17050691e+00
  -6.08294931e-01  3.67741935e+00 -1.41935484e+00 -2.94930876e+00
   2.23917051e+01]]

Step 4 - Pivot Row:
[ 0.          0.          0.          6.93548387 -0.35483871  5.77419355
 -0.16129032 -2.38709677  1.51612903  1.51612903  5.87096774  7.29032258
 15.51612903]
  Subtracting -0.17807308970099664 * Row 4 from Row 5
  Subtracting 0.8684385382059799 * Row 4 from Row 6
  Subtracting 0.6843853820598007 * Row 4 from Row 7
  Subtracting 0.13554817275747508 * Row 4 from Row 8
  Subtracting 0.29634551495016614 * Row 4 from Row 9
  Subtracting 0.1275747508305649 * Row 4 from Row 10
  Subtracting -0.5920265780730896 * Row 4 from Row 11
  Subtracting 0.4950166112956812 * Row 4 from Row 12

Augmented Matrix after Step 4 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   2.57275748e+00  6.13421927e+00 -1.25913621e+00  6.45049834e+00
  -3.42126246e+00  1.72159468e+00  1.43255814e+00  5.85581395e+00
   1.30717608e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   4.36345515e+00 -5.00531561e+00 -1.44518272e+00  2.49700997e+00
   6.92757475e+00 -1.92956811e+00 -1.19534884e+00 -4.93488372e+00
  -7.23056478e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
```

```
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
  -1.49568106e+00 -1.20664452e+00  3.94352159e+00  8.21262458e-01
  -3.44053156e+00  3.98803987e+00 -2.34418605e+00 -1.21860465e+00
  -7.33820598e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   5.83787375e+00  3.91096346e+00  4.04318937e+00  4.32491694e+00
   5.53687708e+00  7.67973422e+00 -1.77209302e+00  3.59069767e+00
   5.38803987e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   7.76877076e+00  7.21727575e+00  1.94684385e+00  6.18471761e+00
   4.58538206e+00  4.87109635e+00  7.34883721e-01 -9.11627907e-01
   1.27993355e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.13554817e+00  2.97607973e+00  2.99667774e+00  7.23654485e+00
   1.17408638e+00  2.31694352e+00 -3.87906977e+00  8.29302326e+00
   2.14624585e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   8.41528239e+00  5.51495017e+00  2.87707641e+00  7.35215947e+00
  -1.35880399e+00  2.92691030e+00 -4.32558140e+00 -6.55813953e+00
   1.47109635e+01]]

Step 5 - Pivot Row:
[ 0.          0.          0.          0.          9.6345515   0.94684385
  2.54817276  8.97009967  0.27574751 -1.29568106  2.04651163 -4.34883721
 -6.30564784]
  Subtracting 0.45289655172413795 * Row 5 from Row 6
  Subtracting 0.2670344827586207 * Row 5 from Row 7
  Subtracting -0.1552413793103448 * Row 5 from Row 8
  Subtracting 0.6059310344827586 * Row 5 from Row 9
  Subtracting 0.8063448275862068 * Row 5 from Row 10
  Subtracting 0.9482068965517241 * Row 5 from Row 11
  Subtracting 0.8734482758620687 * Row 5 from Row 12

Augmented Matrix after Step 5 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
```

```
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00 -5.43413793e+00 -2.59924138e+00 -1.56551724e+00
   6.80268966e+00 -1.34275862e+00 -2.12220690e+00 -2.96531034e+00
  -4.37475862e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  5.88137931e+00 -1.93958621e+00  4.05517241e+00
  -3.49489655e+00  2.06758621e+00  8.86068966e-01  7.01710345e+00
   1.47555862e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00 -1.05965517e+00  4.33910345e+00  2.21379310e+00
  -3.39772414e+00  3.78689655e+00 -2.02648276e+00 -1.89372414e+00
  -8.31710345e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  3.33724138e+00  2.49917241e+00 -1.11034483e+00
   5.36979310e+00  8.46482759e+00 -3.01213793e+00  6.22579310e+00
   9.20882759e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  2.07827586e+00  5.80482759e-01 -1.26896552e+00
   9.12620690e-01  3.54551724e+00 -5.81958621e+00  1.24166207e+01
   2.74415172e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  4.68793103e+00  6.51379310e-01 -4.82758621e-01
  -1.59965517e+00  4.05862069e+00 -6.11310345e+00 -2.75965517e+00
   2.02186207e+01]]

Step 6 - Pivot Row:
[ 0.          0.          0.          0.          0.          6.4537931
 -0.10786207 -1.04827586  4.36303448  5.91586207 -0.91531034  2.59503448
 17.88386207]
  Subtracting 0.9113058345800384 * Row 6 from Row 7
  Subtracting -0.16419106646719386 * Row 6 from Row 8
  Subtracting 0.5170976704424023 * Row 6 from Row 9
  Subtracting -0.8420068390681767 * Row 6 from Row 10
  Subtracting 0.32202393673861923 * Row 6 from Row 11
  Subtracting 0.726383842701432 * Row 6 from Row 12
```

```
Augmented Matrix after Step 6 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00 -1.84129087e+00  5.01047232e+00
  -7.47095533e+00 -3.32357341e+00  1.72019662e+00  4.65223338e+00
  -1.54208164e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
  -5.38073306e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  2.55494764e+00 -5.68283821e-01
   3.11367814e+00  5.40574909e+00 -2.53883308e+00  4.88390682e+00
  -3.88758282e-02]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00 -2.69006198e+00 -2.44817269e+00
   1.04763945e+01  3.63843770e+00 -2.89290447e+00 -7.80273563e-01
   1.06835756e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  6.15216927e-01 -9.31395597e-01
  -4.92380851e-01  1.64046805e+00 -5.52483437e+00  1.15809575e+01
   2.16824856e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  7.29728574e-01  2.78692028e-01
  -4.76889293e+00 -2.38565933e-01 -5.44823680e+00 -4.64464629e+00
```

```
   7.22807224e+00]]

Step 7 - Pivot Row:
[ 0.          0.          0.          0.          0.          0.
  4.32139346  2.04167557 -2.68135285  4.75822825 -2.17676854 -1.46764266
 -5.38073306]
  Subtracting -0.4260873005667713 * Row 7 from Row 8
  Subtracting 0.5912323564031298 * Row 7 from Row 9
  Subtracting -0.6224987388599293 * Row 7 from Row 10
  Subtracting 0.14236540420775687 * Row 7 from Row 11
  Subtracting 0.16886418263286493 * Row 7 from Row 12

Augmented Matrix after Step 7 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
  -5.38073306e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  5.88040436e+00
  -8.61344573e+00 -1.29615278e+00  7.92703192e-01  4.02688948e+00
  -3.83474367e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00 -1.77538848e+00
   4.69898070e+00  2.59253059e+00 -1.25185709e+00  5.75162465e+00
```

```
   3.14238766e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00 -1.17723222e+00
   8.80725576e+00  6.60042879e+00 -4.24794014e+00 -1.69387927e+00
   7.33407600e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00 -1.22205957e+00
  -1.10648968e-01  9.63060960e-01 -5.21493783e+00  1.17898990e+01
   2.24485158e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00 -6.60738484e-02
  -4.31610847e+00 -1.04206026e+00 -5.08065856e+00 -4.39681401e+00
   8.13668533e+00]]

Step 8 - Pivot Row:
[ 0.          0.          0.          0.          0.          0.
  0.          5.88040436 -8.61344573 -1.29615278  0.79270319  4.02688948
 -3.83474367]
  Subtracting -0.301916054250357 * Row 8 from Row 9
  Subtracting -0.20019579379750813 * Row 8 from Row 10
  Subtracting -0.2078189681128606 * Row 8 from Row 11
  Subtracting -0.011236276351843252 * Row 8 from Row 12

Augmented Matrix after Step 8 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
```

```
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
  -5.38073306e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  5.88040436e+00
  -8.61344573e+00 -1.29615278e+00  7.92703192e-01  4.02688948e+00
  -3.83474367e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  2.22044605e-16
   2.09844315e+00  2.20120126e+00 -1.01252727e+00  6.96740723e+00
   1.98461698e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   7.08288015e+00  6.34094445e+00 -4.08924429e+00 -8.87712930e-01
   6.56637645e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
  -1.90068637e+00  6.93695827e-01 -5.05019907e+00  1.26267630e+01
   2.16515833e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
  -4.41289152e+00 -1.05662419e+00 -5.07175153e+00 -4.35156677e+00
   8.09359709e+00]]

Step 9 - Pivot Row:
[ 0.          0.          0.          0.          0.          0.
  0.          0.          7.08288015  6.34094445 -4.08924429 -0.88771293
  6.56637645]
  Subtracting 0.29626975295748886 * Row 9 from Row 10
  Subtracting -0.2683493621807861 * Row 9 from Row 11
  Subtracting -0.6230363113270692 * Row 9 from Row 12

Augmented Matrix after Step 9 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
```

```
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
  -5.38073306e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  5.88040436e+00
  -8.61344573e+00 -1.29615278e+00  7.92703192e-01  4.02688948e+00
  -3.83474367e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   7.08288015e+00  6.34094445e+00 -4.08924429e+00 -8.87712930e-01
   6.56637645e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  2.22044605e-16
   0.00000000e+00  3.22571209e-01  1.98992126e-01  7.23040972e+00
   3.91982533e-02]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
  -2.22044605e-16  2.39528423e+00 -6.14754517e+00  1.23885458e+01
   2.34136663e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  2.89401445e+00 -7.61949921e+00 -4.90464416e+00
   1.21846881e+01]]

Step 10 - Pivot Row:
[ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
  0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
  0.00000000e+00  2.89401445e+00 -7.61949921e+00 -4.90464416e+00
  1.21846881e+01]
  Subtracting 0.8276683701994918 * Row 10 from Row 11
  Subtracting 0.11146150585660264 * Row 10 from Row 12

Augmented Matrix after Step 10 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
```

```
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
  -5.38073306e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  5.88040436e+00
  -8.61344573e+00 -1.29615278e+00  7.92703192e-01  4.02688948e+00
  -3.83474367e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   7.08288015e+00  6.34094445e+00 -4.08924429e+00 -8.87712930e-01
   6.56637645e+00]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  2.89401445e+00 -7.61949921e+00 -4.90464416e+00
   1.21846881e+01]
 [ 0.00000000e+00  0.00000000e+00  1.83779296e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
  -2.22044605e-16  0.00000000e+00  1.58873322e-01  1.64479647e+01
   1.33287854e+01]
 [ 0.00000000e+00  0.00000000e+00  2.47494260e-17  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  2.22044605e-16
   0.00000000e+00  0.00000000e+00  1.04827298e+00  7.77708874e+00
  -1.31892543e+00]]

Step 11 - Pivot Row:
[ 0.00000000e+00  0.00000000e+00  2.47494260e-17  0.00000000e+00
  0.00000000e+00  0.00000000e+00  0.00000000e+00  2.22044605e-16
```

```
  0.00000000e+00  0.00000000e+00  1.04827298e+00  7.77708874e+00
 -1.31892543e+00]
  Subtracting 0.15155720368380796 * Row 11 from Row 12

Augmented Matrix after Step 11 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
  -5.38073306e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  5.88040436e+00
  -8.61344573e+00 -1.29615278e+00  7.92703192e-01  4.02688948e+00
  -3.83474367e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   7.08288015e+00  6.34094445e+00 -4.08924429e+00 -8.87712930e-01
   6.56637645e+00]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  2.89401445e+00 -7.61949921e+00 -4.90464416e+00
   1.21846881e+01]
 [ 0.00000000e+00  0.00000000e+00  2.47494260e-17  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  2.22044605e-16
   0.00000000e+00  0.00000000e+00  1.04827298e+00  7.77708874e+00
```

```
  -1.31892543e+00]
 [ 0.00000000e+00  0.00000000e+00  1.80028342e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00 -3.36524594e-17
  -2.22044605e-16  0.00000000e+00  0.00000000e+00  1.52692908e+01
   1.35286780e+01]]

Step 12 - Pivot Row:
[ 0.00000000e+00  0.00000000e+00  1.80028342e-16  0.00000000e+00
  0.00000000e+00  0.00000000e+00  0.00000000e+00 -3.36524594e-17
 -2.22044605e-16  0.00000000e+00  0.00000000e+00  1.52692908e+01
  1.35286780e+01]

Augmented Matrix after Step 12 :
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
   2.00000000e+01]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   1.13333333e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   4.60000000e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   1.55161290e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -6.30564784e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   1.78838621e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
  -5.38073306e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  5.88040436e+00
  -8.61344573e+00 -1.29615278e+00  7.92703192e-01  4.02688948e+00
  -3.83474367e+00]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   7.08288015e+00  6.34094445e+00 -4.08924429e+00 -8.87712930e-01
```

```
   6.56637645e+00]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  2.89401445e+00 -7.61949921e+00 -4.90464416e+00
   1.21846881e+01]
 [ 0.00000000e+00  0.00000000e+00  2.47494260e-17  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  2.22044605e-16
   0.00000000e+00  0.00000000e+00  1.04827298e+00  7.77708874e+00
  -1.31892543e+00]
 [ 0.00000000e+00  0.00000000e+00  1.80028342e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00 -3.36524594e-17
  -2.22044605e-16  0.00000000e+00  0.00000000e+00  1.52692908e+01
   1.35286780e+01]]

Calculating x[12]: 13.528678019906518 / 15.269290844513339 = 0.886005653940879

Calculating x[11]: -8.209470023375943 / 1.0482729817374499 = -7.831423843214233

Calculating x[10]: -43.14129728378364 / 2.894014453570328 = -14.907077340460582

Calculating x[9]: 69.85323923819925 / 7.082880153674582 = 9.862264745784179

Calculating x[8]: 64.43163658572198 / 5.880404356126172 = 10.957007832054519

Calculating x[7]: 53.877242653037875 / 4.3213934601410555 = 12.46756240781631

Calculating x[6]: 66.4060197273686 / 6.453793103448276 = 10.28945593125502

Calculating x[5]: -148.25722957475458 / 9.634551495016611 = -15.388077966207286

Calculating x[4]: 25.976225029429997 / 6.935483870967742 = 3.7454091902899065

Calculating x[3]: 66.11334375312703 / 7.233333333333334 = 9.14009360642309

Calculating x[2]: 25.04745832682871 / 3.3333333333333335 = 7.514237498048613

Calculating x[1]: -154.43131757543716 / 9.0 = -17.159035286159686

Final Augmented Matrix:
[[ 9.00000000e+00  8.00000000e+00  4.00000000e+00  7.00000000e+00
   2.00000000e+00  3.00000000e+00  7.00000000e+00  1.00000000e+00
   8.00000000e+00  4.00000000e+00  9.00000000e+00  5.00000000e+00
  -1.54431318e+02]
 [ 0.00000000e+00  3.33333333e+00  3.66666667e+00 -1.33333333e+00
   7.33333333e+00  3.00000000e+00  2.66666667e+00  5.66666667e+00
   4.33333333e+00  7.66666667e+00 -2.00000000e+00  3.33333333e-01
   2.50474583e+01]
 [ 0.00000000e+00  0.00000000e+00  7.23333333e+00 -3.93333333e+00
```

```
   2.13333333e+00  2.76666667e+00  1.53333333e+00  4.30000000e+00
  -2.63333333e+00  7.70000000e+00 -3.73333333e+00  3.23333333e+00
   6.61133438e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  6.93548387e+00
  -3.54838710e-01  5.77419355e+00 -1.61290323e-01 -2.38709677e+00
   1.51612903e+00  1.51612903e+00  5.87096774e+00  7.29032258e+00
   2.59762250e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   9.63455150e+00  9.46843854e-01  2.54817276e+00  8.97009967e+00
   2.75747508e-01 -1.29568106e+00  2.04651163e+00 -4.34883721e+00
  -1.48257230e+02]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  6.45379310e+00 -1.07862069e-01 -1.04827586e+00
   4.36303448e+00  5.91586207e+00 -9.15310345e-01  2.59503448e+00
   6.64060197e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  4.32139346e+00  2.04167557e+00
  -2.68135285e+00  4.75822825e+00 -2.17676854e+00 -1.46764266e+00
   5.38772427e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  5.88040436e+00
  -8.61344573e+00 -1.29615278e+00  7.92703192e-01  4.02688948e+00
   6.44316366e+01]
 [ 0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   7.08288015e+00  6.34094445e+00 -4.08924429e+00 -8.87712930e-01
   6.98532392e+01]
 [ 0.00000000e+00  0.00000000e+00 -2.22044605e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  0.00000000e+00
   0.00000000e+00  2.89401445e+00 -7.61949921e+00 -4.90464416e+00
  -4.31412973e+01]
 [ 0.00000000e+00  0.00000000e+00  2.47494260e-17  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00  2.22044605e-16
   0.00000000e+00  0.00000000e+00  1.04827298e+00  7.77708874e+00
  -8.20947002e+00]
 [ 0.00000000e+00  0.00000000e+00  1.80028342e-16  0.00000000e+00
   0.00000000e+00  0.00000000e+00  0.00000000e+00 -3.36524594e-17
  -2.22044605e-16  0.00000000e+00  0.00000000e+00  1.52692908e+01
   1.35286780e+01]]

Solution: [-17.15903529   7.5142375    9.14009361   3.74540919 -15.38807797
  10.28945593  12.46756241  10.95700783   9.86226475 -14.90707734
  -7.83142384   0.88600565]
```