Theory of Multi-Objective Optimization in Convex Optimization

1. Introduction to Multi-Objective Optimization

Multi-objective optimization (MOO), also known as multi-criteria or vector optimization, deals with problems where multiple conflicting objectives must be optimized simultaneously. Unlike single-objective optimization, which seeks a unique optimal solution, MOO typically results in a set of **Pareto-optimal** solutions.

In the context of **convex optimization**, multi-objective problems (MOPs) have convex objective functions and feasible regions defined by convex constraints, which ensures desirable properties such as well-behaved Pareto frontiers and efficient solution methods.

2. Mathematical Formulation

A general multi-objective optimization problem can be written as:

$$\min_{x \in X} \quad F(x) = (f_1(x), f_2(x), ..., f_k(x))$$

where:

- x is the decision variable in a convex feasible set $X \subseteq \mathbb{R}^n$,
- $F(x)$ is a vector-valued function consisting of k objective functions $f_i(x)$,
- Each $f_i(x)$ is convex and differentiable in X,
- The feasible set is defined by convex constraints:

$$g_j(x) \leq 0, \quad j = 1, ..., m$$

where $g_j(x)$ are convex functions.

3. Pareto Optimality and Efficient Solutions

Since multiple objectives often conflict (e.g., cost vs. quality in engineering), a single optimal solution usually does not exist. Instead, we define:

- **Pareto Dominance:** A solution x^* **dominates** another solution x if:

$$f_i(x^) \leq f_i(x), \quad \forall i = 1, ..., k, \quad \text{and} \quad \exists i \text{ such that } f_i(x^) < f_i(x).$$

- **Pareto Optimal Solution:** A solution x^* is Pareto optimal if no other feasible solution dominates it.
- **Pareto Front:** The set of all Pareto-optimal solutions forms the **Pareto front**, which represents the trade-off surface among objectives.

4. Scalarization Techniques for Solving MOO

To solve multi-objective convex optimization problems, scalarization methods convert the problem into a single-objective form. The key methods include:

4.1. Weighted Sum Method

This method converts the problem into a single-objective optimization using a weighted sum of objectives:

$$\min_{x \in X} \quad \sum_{i=1}^{k} \lambda_i f_i(x), \quad \lambda_i \geq 0, \quad \sum_{i=1}^{k} \lambda_i = 1.$$

If each $f_i(x)$ is convex, the resulting function is convex, and standard convex optimization techniques can be applied.

✅ **Pros:** Simple and widely used.
❌ **Cons:** Cannot find solutions in non-convex regions of the Pareto front.

4.2. ε-Constraint Method

This method optimizes one objective while converting the others into constraints with upper bounds:

$$\min_{x \in X} f_1(x), \quad \text{s.t. } f_i(x) \leq \epsilon_i, \quad i = 2, ..., k.$$

✅ **Pros:** Can generate any Pareto-optimal solution.
❌ **Cons:** Requires solving multiple constrained problems.

4.3. Lexicographic Method

This method orders objectives by priority and optimizes them sequentially:

1. Solve $\min f_1(x)$.
2. Restrict the solution space to this optimal $f_1(x)$ and optimize $f_2(x)$, and so on.

✅ **Pros:** Works well when one objective is dominant.
❌ **Cons:** Ignores trade-offs between lower-priority objectives.

4.4. Normal-Boundary Intersection (NBI)

This method constructs evenly distributed Pareto-optimal solutions by systematically exploring the trade-off surface.

✅ **Pros:** Uniformly distributed solutions.
❌ **Cons:** Computationally expensive.

5. Optimality Conditions in Multi-Objective Convex Optimization

- **Karush-Kuhn-Tucker (KKT) Conditions:** If x^* is a Pareto-optimal point, there exist multipliers \lambda_i such that:

$$\sum_{i=1}^{k} \lambda_i \nabla f_i(x) + \sum_{j=1}^{m} \mu_j \nabla g_j(x) = 0, \quad \lambda_i \geq 0.$$

- **Convexity and Pareto Efficiency:** If all $f_i(x)$ are convex, then any local minimizer obtained from the weighted sum or ε-constraint method is a global Pareto-optimal solution.

6. Conclusion

Multi-objective convex optimization provides a structured way to balance conflicting objectives. Convexity ensures well-behaved Pareto fronts and guarantees optimal solutions via scalarization methods. By choosing an appropriate method based on problem characteristics, one can efficiently find trade-off solutions for real-world problems like energy planning, economics, and engineering design.