1. The Primal Problem in Convex Optimization

In a standard **convex optimization problem**, we seek to minimize an objective function f(x) subject to a set of constraints:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} f(x) \\
&\text{s.t.} g_i(x) \leq 0, \quad i = 1, ..., m \\
&\quad h_j(x) = 0, \quad j = 1, ..., p
\end{aligned}
$$

where:

- $f(x)$ is the **objective function** (assumed convex).
- $g_i(x)$ are **inequality constraints** (assumed convex).
- $h_j(x)$ are **equality constraints** (assumed affine or differentiable).

2. Duality and the Lagrangian Function

To analyze the primal problem, we introduce the **Lagrangian function**:

$$
\mathcal{L}(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \nu_j h_j(x),
$$

where:

- λ_i (dual variables) are **Lagrange multipliers** for the inequality constraints.
- ν_j are **Lagrange multipliers** for the equality constraints.

We define the **Lagrange dual function** by minimizing the Lagrangian over x:

$$
d(\lambda, \nu) = \inf_{x} \mathcal{L}(x, \lambda, \nu).
$$

The **dual problem** is then:

$$
\max_{\lambda \geq 0, \nu} d(\lambda, \nu).
$$

This formulation gives us a **lower bound** on the optimal primal solution. If **strong duality** holds (which is always true for convex problems under Slater's condition), then the **optimal dual value equals the optimal primal value**.

3. The Karush-Kuhn-Tucker (KKT) Conditions

For a **convex optimization problem**, the **KKT conditions** characterize the optimal solution. They are **necessary** for any local optimum (and **sufficient** for convex problems).

The KKT conditions consist of the following:

(1) Stationarity

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \nu_j \nabla h_j(x^*) = 0.$$

This condition ensures that at the optimal point x^*, the gradient of the objective function is balanced by the gradients of the constraints.

(2) Primal Feasibility

$$g_i(x^*) \leq 0, \quad h_j(x^*) = 0.$$

This ensures that x^* satisfies the original constraints.

(3) Dual Feasibility

$$\lambda_i \geq 0, \quad \forall i.$$

This ensures that the **dual variables** (Lagrange multipliers for inequality constraints) are non-negative.

(4) Complementary Slackness

$$\lambda_i g_i(x^*) = 0, \quad \forall i.$$

This condition states that either:

- The constraint $g_i(x)$ is **inactive** at x^* (i.e., $g_i(x^*) < 0 \Rightarrow \lambda_i = 0$).
- The constraint is **active** (i.e., $g_i(x^*) = 0 \Rightarrow \lambda_i \geq 0$).

Together, these conditions provide a powerful tool to **verify** whether a given point x* is an optimal solution.

4. Why KKT Conditions Help in Optimization

By solving the KKT conditions, we can:

- **Determine optimal solutions**: Any x* satisfying KKT conditions (for convex problems) is optimal.
- **Check feasibility**: Primal feasibility ensures constraints are satisfied.
- **Analyze duality**: The complementary slackness condition links the primal and dual solutions.
- **Simplify computation**: In many cases, KKT conditions allow us to convert constrained problems into simpler algebraic equations.

5. Interior Point Methods for Convex Optimization

Interior point methods are a class of algorithms used for solving constrained optimization problems efficiently.

Barrier Function Approach

Instead of dealing with constraints directly, we introduce a **barrier function**:

$$\min f(x) - \frac{1}{t} \sum_i \log(-g_i(x)).$$

As $t \to \infty$, the solution of this modified problem converges to the solution of the original constrained problem.

Interior point methods iteratively update the solution by:

1. **Solving a sequence of unconstrained problems** (where constraints appear as penalties).
2. **Using Newton's method** to optimize the search direction.
3. **Dynamically updating the barrier parameter** to move closer to the constraint boundary.

These methods are widely used in large-scale convex optimization due to their efficiency.

6. Equality Constrained Optimization and Lagrange Multipliers

For an optimization problem with only **equality constraints**:

$$\min_x f(x)$$
$$\text{s.t.} h_j(x) = 0, \quad j = 1, ..., p.$$

We define the **Lagrangian function**:

$$\mathcal{L}(x, \nu) = f(x) + \sum_{j=1}^{p} \nu_j h_j(x).$$

The **optimality conditions** (first-order conditions) are:

$$\nabla f(x^{)} + \sum_{j=1}^{p} \nu_j \nabla h_j(x^{)} = 0, \quad h_j(x^*) = 0.$$

These conditions allow us to solve equality-constrained problems efficiently by finding x^* and the Lagrange multipliers ν_j.

https://youtu.be/uh1Dk68cfWs?si=gZGvyPykRhmxh5k0