Quadratic Programming (QP) Problem: Portfolio Risk Minimization (Markowitz Model)

Problem Definition

The **Markowitz Portfolio Optimization Model** aims to minimize portfolio risk (variance of returns) while achieving a given level of expected return. The decision variable is the allocation of capital among different assets.
We aim to **minimize the risk (variance)** of a portfolio while achieving a **minimum expected return**. This is a **convex quadratic programming (QP) problem**, where the objective function is quadratic (risk) and constraints are linear.

Stocks Considered

We have **10 well-known stocks** with their respective **expected returns** (based on historical performance and market expectations):

Stock	Ticker	Expected Return (Annualized %)
Apple	AAPL	15.0%
Nvidia	NVDA	20.0%
Google (Alphabet)	GOOG	18.0%
Tesla	TSLA	22.0%
Amazon	AMZN	17.0%
Microsoft	MSFT	16.0%
Meta (Facebook)	META	19.0%
Berkshire Hathaway	BRK.B	13.0%
Johnson & Johnson	JNJ	12.0%
JPMorgan Chase	JPM	14.0%

Problem Inputs

- **Total Budget:** 100% of capital (normalized to 1 for simplicity)
- **Expected Return Constraint:** The portfolio must **at least achieve 16% return**
- **Covariance Matrix:** Represents how stock returns correlate (provided in the code)
- **No Short-Selling Constraint:** No stock can have negative allocation

Mathematical Formulation

Let **x** be the decision variable (vector of stock allocations).

- **Objective:** Minimize portfolio variance:

$$\min \frac{1}{2} x^T \Sigma x$$

- **Constraints:**
  - **Budget Constraint:**

$$\sum x_i = 1$$

  - **Expected Return Constraint:**

$$r^T x \geq 0.16$$

  - **Non-Negativity Constraint:**

$$x_i \geq 0 \quad \forall i$$

Explanation of the Model

- The **objective function** is based on the covariance matrix, ensuring risk minimization.
- The **budget constraint** ensures all capital is invested.
- The **expected return constraint** forces a minimum portfolio return of **16%**.
- The **no short-selling constraint** prevents negative allocations.

Covariance Matrix Explanation

The **covariance matrix (Σ)** is a **key input** in the quadratic programming problem, representing how the returns of different stocks move relative to each other.

What Does It Represent?

- **Diagonal Elements (Σ[i, i])** → Variance of each stock (individual risk).
- **Off-Diagonal Elements (Σ[i, j])** → Covariance between stocks **i** and **j** (how they move together).
  - **Positive covariance** → Stocks tend to move in the **same direction**.
  - **Negative covariance** → Stocks tend to move in **opposite directions**.
  - **Zero covariance** → Stocks are **uncorrelated**.

How Is It Used in the Code?

- `cp.quad_form(x, Sigma)` computes **$x^T \Sigma x$**, which represents **portfolio variance** (total risk).
- **Minimizing this function** ensures that the portfolio is optimized for the **least risk** while maintaining return constraints.

Code Explanation

The code **solves a Quadratic Programming (QP) problem** using the **cvxpy** library in Python.

Libraries Used:

- **cvxpy** → A convex optimization library used to define and solve the optimization problem.
- **numpy** → Used for handling numerical operations (arrays, matrices).

Key Functions & Their Role:

- **cp.Variable(n)** → Defines the decision variable **x**, representing asset allocations.
- **cp.quad_form(x, Sigma)** → Computes the quadratic form $\mathbf{x}^T\boldsymbol{\Sigma}\mathbf{x}$, representing **portfolio variance (risk)**.
- **cp.Minimize()** → Sets up the objective function to **minimize risk**.
- **cp.Problem(objective, constraints)** → Defines the optimization problem.
- **problem.solve()** → Solves the QP problem and finds the optimal asset allocations.

How It Works:

1. Defines **stock data** (returns & covariance matrix).
2. Sets **constraints** (budget, expected return, non-negative weights).
3. Formulates and **solves** the convex optimization problem.
4. Prints **optimal portfolio allocation** and **minimum risk**.

Results:

Optimal Portfolio Allocation:

Apple: 0.2337

Nvidia: 0.1945

Google: -0.0000

Tesla: 0.1325

Amazon: 0.0000

Microsoft: 0.0529

Meta: 0.0000

Berkshire Hathaway: 0.1164

Johnson & Johnson: 0.2252

JPMorgan: 0.0447

Minimum Portfolio Risk (Variance): 0.016499636397344742