Unit 4 — Similarity

Geometry Advanced

2022-2023

1 — Proportions

Advanced Geometry — Similarity Unit

Task 1: Properties of Proportions:

Knowing that $\frac{a}{b} = \frac{c}{d}$ what other equations and proportions can you write?
Find as many as you can. Get creative!
Be sure you can algebraically defend the validity of each of your equations.

Task 2: Geometric Mean

Definition: The **arithmetic mean** of two numbers a and b is the number $\frac{a+b}{2}$.

Definition: The **geometric mean** between a and b is the positive number x such that:

$$\frac{a}{x} = \frac{x}{b}$$

1. Let $a = 4$ and $b = 25$.

   (a) Find the arithmetic mean (i.e., the average) of a and b.

   (b) Find the geometric mean between a and b.

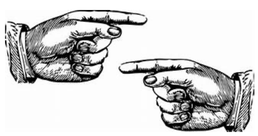
2. Show that the geometric mean of two positive numbers, a and b, is $\sqrt{ab}$.

3. Show that the geometric mean of two positive numbers, a and b, is less than the arithmetic mean

Practice problems

1. In the United Kingdom approximately 3 of every 1500 houses have a trampoline in the backyard. If there are approximately 11,000,000 houses in Great Britain, how many have a trampoline in the backyard? See if you can do this one in your head. Then write down the calculations that you did.

2. The ratio of left-handed students to right handed students at BHS is approximately 2 to 9. If there are 2100 students at BHS, how many are left-handed?



3. A box contains red and yellow marbles in the ratio 2:3 (2 to 3). After adding five red marbles the ratio of red to yellow is 5:6 (5 to 6). How many yellow marbles are in the box?

4. The ratio of red marbles to yellow marbles in a box is 3:4. After adding eight yellow marbles to the box the ratio of red to yellow marbles is 3:5. How many marbles were in the box at the start?

5. During a game players can win or lose marbles. At the start of the game Mo, Larry, and Curly have marbles in the ratio 5:6:7. At the end of the game they have marbles in the ratio 7:9:8. Show that Mo ends the game with more marbles than she had at the start.

6. It takes 6 people who work 8 hours a day for 10 days to build a bridge. How many days would it take to build the bridge if 5 people are working 6 hours per day?

7. State each of the following ratios in lowest terms:

a) 35:75 b) $c^3 : 6c^2$ c) $12 : 18 : 6$

8. True or false?

a) If $\frac{x}{y} = \frac{5}{7}$ then $x = 5$ and $y = 7$

b) If $a = 8$ and $b = 3$ then $\frac{a}{b} = \frac{8}{3}$

c) $\frac{x}{5} = \frac{2}{3}$ if and only if $3x = 10$

d) If $\frac{g}{h} = \frac{4}{9}$ then $\frac{g+4}{h+9} = \frac{4}{9}$

9. Let r be the value of the geometric mean of 3 and 11. Find the value of r^2.

10. Two complementary angles have measures that are in the ratio 5 : 13. Find the measure of the two angles.

11. Two numbers are in the ratio 3 : 5. Their sum is 16. Find the numbers.

12. $\overline{AB}$ contains a point C such that $AC : BC = 3 : 10$. If $AB = 65$, find AC and BC

Properties of Proportions

Can be used in proofs via the reason "property of proportions"

$\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$	$\frac{a}{b} = \frac{c}{d}$ if and only if $\frac{a+b}{b} = \frac{c+d}{d}$
$\frac{a}{b} = \frac{c}{d}$ if and only if $\frac{b}{a} = \frac{d}{c}$	$\frac{a}{b} = \frac{c}{d}$ if and only if $\frac{a-b}{b} = \frac{c-d}{d}$
$\frac{a}{b} = \frac{c}{d}$ if and only if $\frac{a}{c} = \frac{b}{d}$	$\frac{a}{b} = \frac{c}{d}$ if and only if $\frac{a}{b} = \frac{a+c}{b+d}$

These are some of the equivalencies that you might have discovered during Task 1. Use this space to prove any of the properties that you didn't get to in class.

2 – Similar Polygons

Advanced Geometry — Similarity Unit

Similar Polygons: Two *polygons* are similar if and only if…

1. Corresponding angles are congruent

2. The lengths of corresponding sides are proportional (have the same ratio)

Notes about language and notation: We say that…

- Angles are *congruent* (≅)
- Sides are *proportional*
- Triangles are *similar*

- The statement $\Delta ABC \sim \Delta DEF$ is read as "*Triangle ABC is similar to Triangle DEF*"

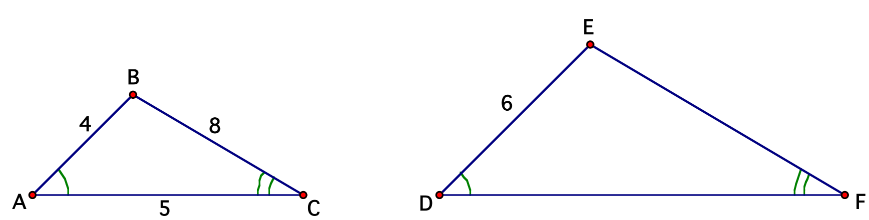
Constant of Proportionality: The ratio of the lengths of any two corresponding sides of two polygons

Ex: If $\Delta ABC \sim \Delta DEF$ and the **C of P** is 2 then $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = 2$

You could also say that the constant of proportionality between these two Δs is ½

Super Interesting Fact: Congruent triangles are just similar triangles with a C of P of 1

EX: In the diagram below It is given that $\Delta ABC \sim \Delta DEF$. Find EF and DF.



Task 1: Similar Polygons – Congruent Angles and Proportional Sides

By definition, two polygons are similar if and only if corresponding angles are *congruent* **and** corresponding sides are *proportional*.
In this exercise, you will explore whether both conditions are necessary for n-gons in which n >3.

1. As a group, <u>discuss *then* decide</u> whether or not you agree with the following conjectures:

a) **Two quadrilaterals whose corresponding angles are congruent are similar.**
If you believe the conjecture to be true, prove it! If not, draw a counter-example.

Conclusion: ____

b) **Two quadrilaterals whose corresponding side lengths are proportional are similar.**
If you believe the conjecture to be true, prove it! If not, draw a counter-example.

Conclusion: ____

2. Indicate whether the pairs of polygons are always, sometimes, or never similar.

- If **sometimes**, draw an example and counter-example.
- If **never**, draw a counter-example.

(a) Two squares

(b) Two rhombi

(c) Two isosceles triangles

(d) Two right triangles

(e) A trapezoid and a rhombus

(f) Two regular hexagons

(g) Two equilateral triangles

(h) Two regular n-gons

Check your group's responses with another group: _______

Task 2:

Perimeter and Area of Similar Polygons

(a) Explore the relationship between the perimeters of two similar polygons

What did you find? ____

____

(b) Explore the relationship of the areas between two similar polygons

What did you find? ____

____

Check your group's responses with another group: _______

Task 3:

What *does* the definition of similar polygons really say?

You are given that $\Delta ABC \sim \Delta DEF$.

Which of the following statements are true?

Which of the following statements can be justified by the reason "Def. of similar polygons"?

a. $\frac{AB}{AC} = \frac{DE}{DF}$ b. $\frac{BC}{EF} = \frac{CA}{FD}$ c. $\frac{DF}{AC} = \frac{ED}{BA}$ d. $\frac{AB}{DE} = \frac{EF}{BC}$ e. $AB \bullet EF = BC \bullet DE$

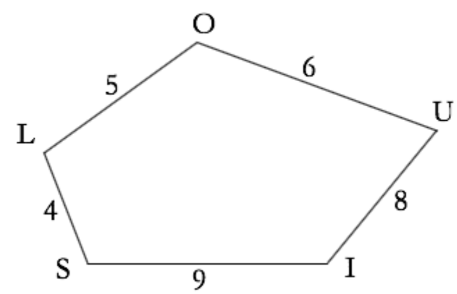
0. Spend 5-10 minutes writing down important information from today in your notebook. A little bit of studying *now* saves you a whole bunch of studying *later*!

1. If $\Delta ABC \sim \Delta DEF$, $AB = 4$, $AC = 10$, $DE = x - 1$, and $DF = x + 2$, find the value of x.

2. If $\Delta ABC \sim \Delta DEF$, $AB = 5$, $AC = 8$, $BC = y + 3$, $DE = 2x + 3$, and $DF = 4x$, and $EF = 5y + 1$, find the values of x and y.

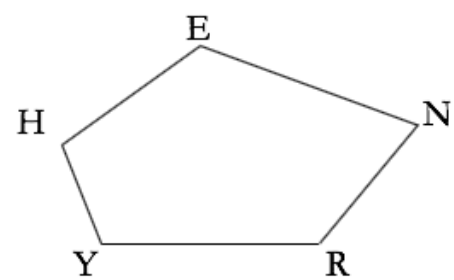
3. Suppose $LOUIS \sim HENRY$, and $NR = 6$.

a. Find the length of each side of HENRY.



b. Find the constant of proportionality of the two pentagons.

c. Find the ratio of the perimeters of the two pentagons.

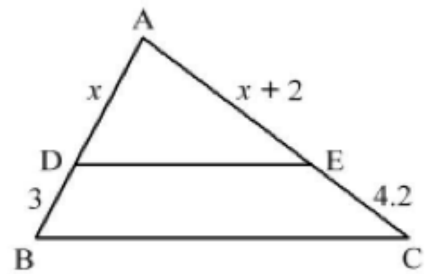


d. If two polygons are similar, what can you conclude about their perimeters? Why?

4. The sides of a pentagon have lengths 5, 8, 9, 11 and 17.
Find the length of the shortest side of a similar polygon whose perimeter is 40.

5. In the diagram to the right it is given that $\Delta ABC \sim \Delta ADE$:

a.) Complete: $\frac{AD}{AB} = \frac{???}{AC}$



b.) Find the value of x.

c.) If the perimeter of ΔADE is 22, find BC.

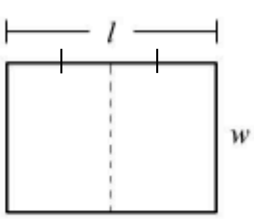
6. Two squares have sides of lengths 9 and 12 respectively.

(a) What is the constant of proportionality between the squares?

(b) What is the ratio of their areas?

(c) How is the ratio of their areas related to the constant of proportionality?

7. The rectangle shown below is folded along the dotted line to create a smaller rectangle that is similar to the original one. Find the ratio $w: l$.



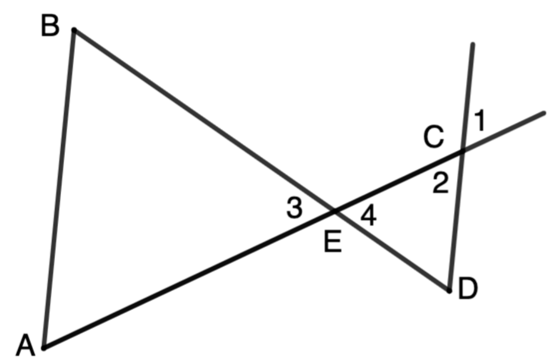
3 – Proving Triangles Similar

Advanced Geometry — Similarity Unit

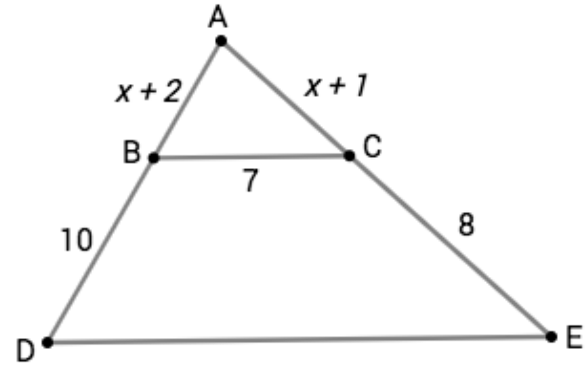
Angle-Angle Similarity ($AA\sim$) theorem:	

1. Given: $\angle 1 \cong \angle A$

   Prove: $\Delta ABE \sim \Delta CDE$



2. In the diagram to the right $\overline{BC} \parallel \overline{DE}$



a) Explain why $\Delta ABC \sim \Delta ADE$.

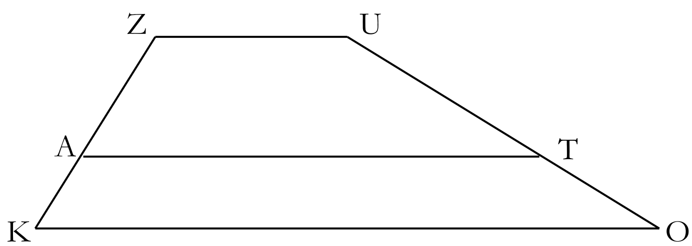
b) Solve for x.

c) Find DE. (Be careful; it's not 14)

3. It is given that $\Delta ABC \sim \Delta DEF$, $AC = 10$, $DF = 5$, and the area of ΔABC is 40. What is the area of ΔDEF?

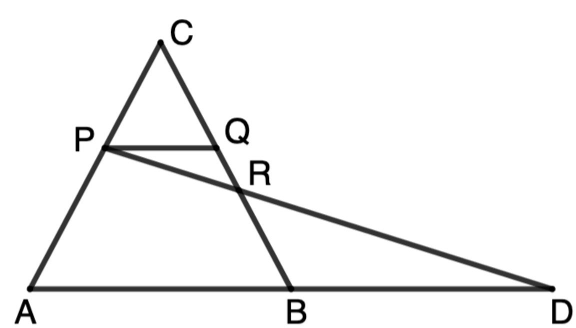
3b. Make a conjecture based on your answer to the previous problem.

4. In trapezoid $KZUO$, $\overline{AT}$ is drawn so that $\overline{KO} \parallel \overline{AT} \parallel \overline{ZU}$. Is $AZUT$ similar to $KZUO$? Why or why not?

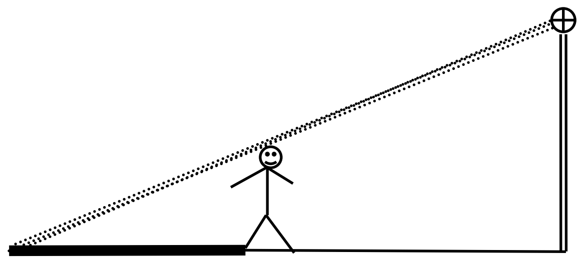


5. Given $\overline{PQ} \parallel \overline{AD}$ and $\overline{AB} \cong \overline{BD}$

Prove: $\frac{CP}{CA} = \frac{PR}{RD}$



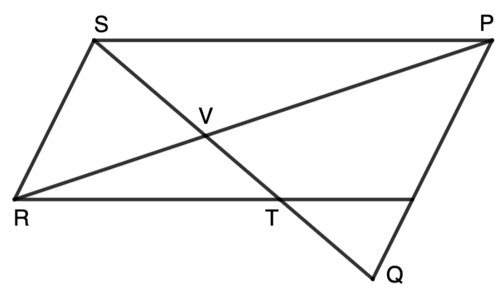
6. It is a dark and stormy night. Spike, who is 5ft 9in tall, is standing near her favorite streetlamp, whose name is Jonze. Jonze is 23 feet tall. The length of Spike's shadow is 4 feet. Determine how far Spike is away from Jonze. *Note: Shown below is Spike and Jonze.*



7. On that same night, Kirby is standing 16 feet away from another street lamp (This streetlamp unfortunately does not have a name). Kirby's shadow is 8 feet long. If the street lamp is 12 feet taller than Kirby, find Kirby's height. *Note: If you're getting 12 feet for an answer, you're making a mistake.*

8. Given: $\overline{SP} \parallel \overline{RT}$
$\overline{SR} \parallel \overline{PQ}$

Prove: $RT \bullet PQ = PS \bullet RS$



9. Spend 5-10 minutes writing down important information from today in your notebook. A little bit of studying *now* saves you a whole bunch of studying *later*!

4 – Triangle Proportionality Theorem (and others)

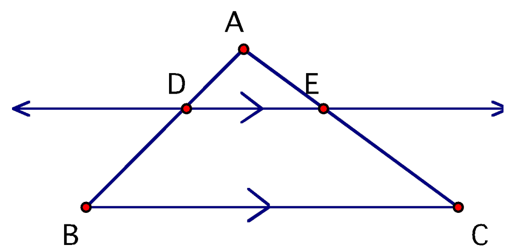
Advanced Geometry — Similarity Unit

Triangle Proportionality Theorem (Δ Prop Thm): A line that intersects two sides of a triangle is parallel to the third side of the triangle if and only if it divides the two sides proportionally.

Ex: In the diagram to the right,

$$\overleftrightarrow{DE} \parallel \overline{BC} \leftrightarrow \frac{AD}{DB} = \frac{AE}{EC}$$

(and equivalent proportions)

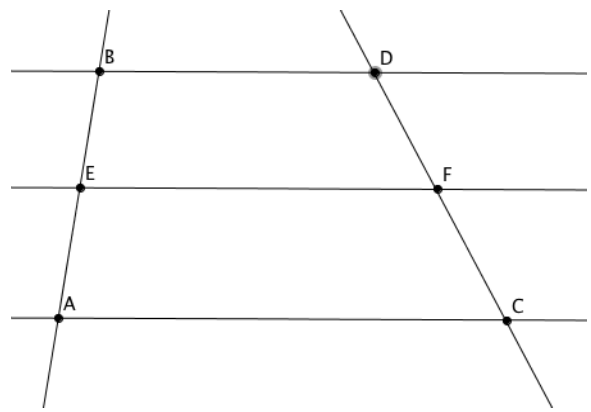


Corollary:

If three parallel lines are cut by two transversals then the parallel lines divide the transversals proportionally.

Ex: In the diagram to the right,

$$\overleftrightarrow{BD} \parallel \overleftrightarrow{EF} \parallel \overleftrightarrow{AC} \rightarrow \frac{BE}{EA} = \frac{DF}{FC}$$

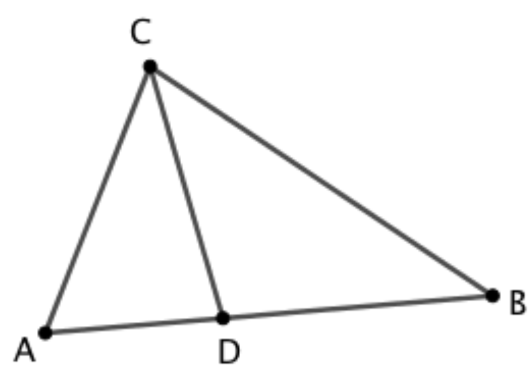


Angle Bisector Proportionality Theorem (∠ Bisector Prop Thm):

The angle bisector of a triangle divides the opposite side of the triangle in the same ratio as that of the two adjacent sides of the angle that is bisected.

Ex: In the diagram below,

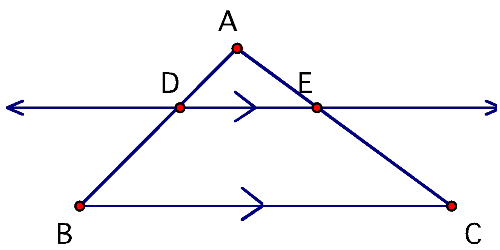
$$\overline{CD} \text{ bisects } \angle ACB \text{ if and only if } \frac{AD}{BD} = \frac{AC}{BC}$$



Let's prove these theorems and corollaries!

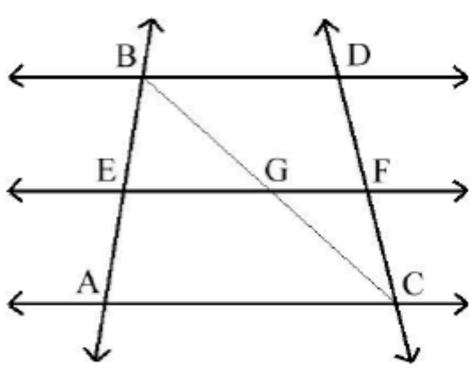
1. Given: $\overleftrightarrow{DE} \parallel \overline{BC}$
Prove: $\frac{AD}{DB} = \frac{AE}{EC}$

(this proof can just be reversed for the converse)

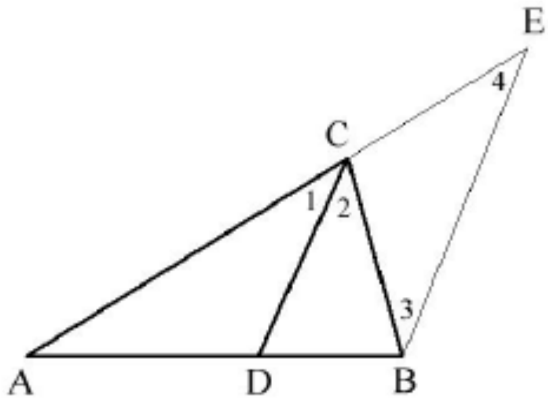


2. Given: $\overleftrightarrow{BD} \parallel \overleftrightarrow{EF} \parallel \overleftrightarrow{AC}$
Prove: $\frac{BE}{EA} = \frac{DF}{FC}$

(Hint: Draw BC as shown)



3. In this problem, we will prove the Angle Bisector Theorem ($\overline{CD}$ bisects $\angle ACB$ if and only if $\frac{AD}{BD} = \frac{AC}{BC}$)



a. Explain the theorem in your own words.

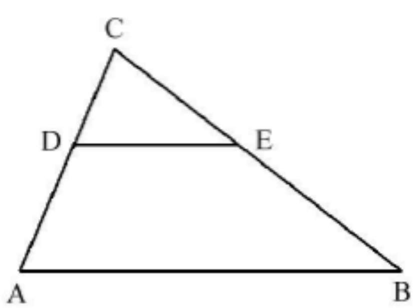
b. To begin the proof, we draw $\overline{BE}$ parallel to $\overline{CD}$.
We also extend $\overline{AC}$ to point E.
Explain why that makes $\frac{AD}{BD} = \frac{AC}{CE}$.

c. Show that $BC = CE$.

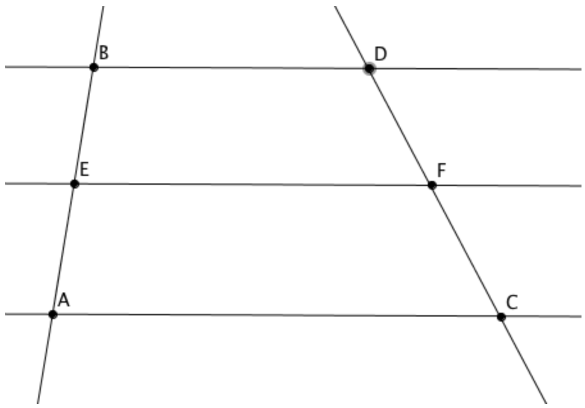
d. Show that $\frac{AD}{BD} = \frac{AC}{BC}$.

---

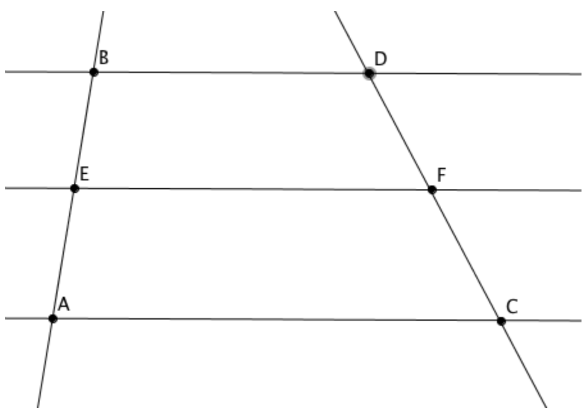
4. In the diagram below, $\overline{DE} \parallel \overline{AB}$, $CE = x - 1$, $BE = x + 1$, $CD = 4$, and $AC = 9$. Find x.



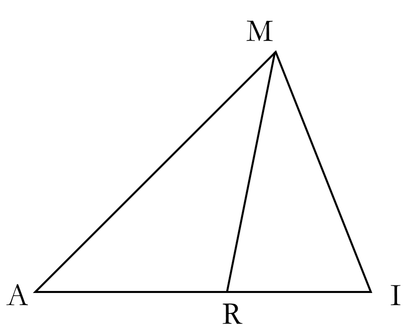
5. Let $AE = x$, $BE = x - 5$, $CF = x - 3$, and $DF = x - 6$. Find the value of x.



6. This time, let $AE = 24$, $BE = 16$, and $CD = 65$. Find CF.



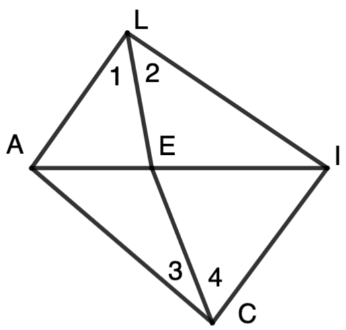
7. $\overline{MR}$ is an angle bisector of ΔAMI. $RI = 6$, $AM = 12$, and $AR = 16$. Find MI.



8. Spend 5-10 minutes writing down important information from today in your notebook. A little bit of studying *now* saves you a whole bunch of studying *later*!

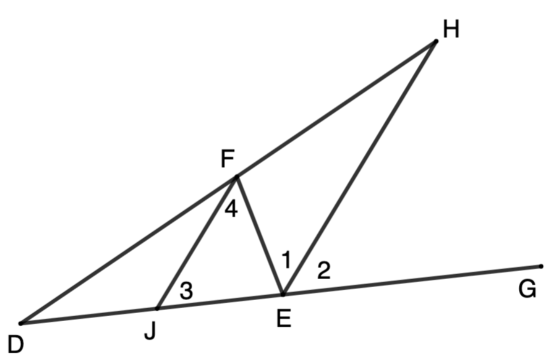
9. Given: $\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$

Prove: $\frac{AL}{LI} = \frac{CA}{CI}$



10. Given: $\overline{EH}$ bisects $\angle FEG$
$\overline{FJ} \parallel \overline{HE}$

Prove: $\frac{DH}{FH} = \frac{DE}{FE}$



5 – More Ways of Proving Triangles Similar

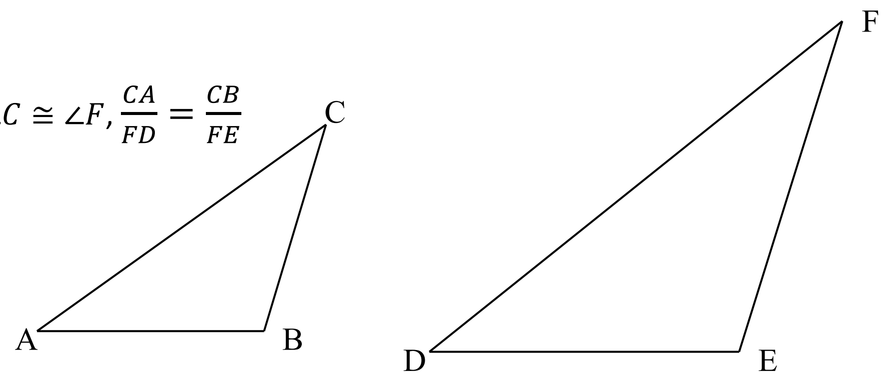
Advanced Geometry — Similarity Unit

> **Side-Angle-Side Similarity Theorem (SAS~):** If one angle of one triangle is congruent to one angle of another triangle and the lengths of the sides including these angles are proportional, then the triangles are similar.

<ins>Proof</ins>:

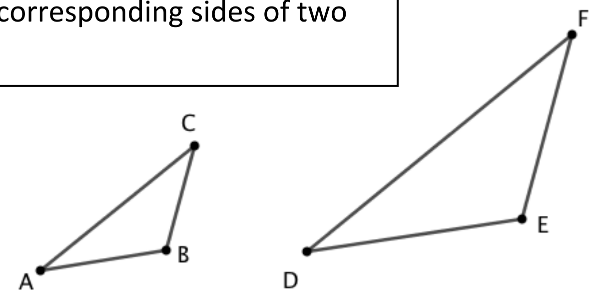
Given: ΔABC and ΔDEF, $\angle C \cong \angle F$, $\frac{CA}{FD} = \frac{CB}{FE}$

Prove: $\Delta ABC \sim \Delta DEF$



Claims	Reasons
1. $\angle C \cong \angle F, \frac{CA}{FD} = \frac{CB}{FE}$	1. Given
2. Construct point X on $\overrightarrow{FD}$ & Y on $\overrightarrow{FE}$ such that $FX = CA$ and $FY = CB$	2. ______________________________
3. Construct $\overline{XY}$	3. ______________________________
4. ______________________________	4. Substitution; 1, 2
5. ______________________________	5. Property of Proportions; 4
6. $\overline{XY} \parallel \overline{DE}$	6. ______________________________
7. $\angle D \cong \angle FXY$	7. ______________________________
8. $\overline{CA} \cong \overline{FX}$ and $\overline{CB} \cong \overline{FY}$	8. Def $\cong$; 2
9. ______________________________	9. SAS$\cong$; 1, 8
10. $\angle A \cong \angle FXY$	10. ______________________________
11. ______________________________	11. Transitive; 7, 10
12. $\Delta ABC \sim \Delta DEF$	12. ______________________________

Side-Side-Side Similarity Theorem (SSS~): If the lengths of the corresponding sides of two triangles are proportional, then the triangles are similar.



Proof:

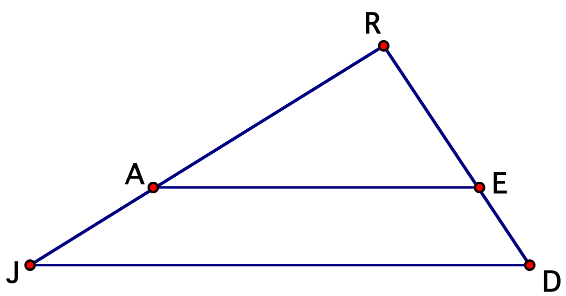
Given: $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ Prove: $\Delta ABC \sim \Delta DEF$

Claims	Reasons
1. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$	1. Given
2. Construct point X on $\overline{FD}$ such that $AC = XF$	2. ________________
3. $\overline{AC} \cong \overline{XF}$	3. ________________
4. Construct point Y on $\overline{FE}$ and then construct $\overline{XY}$ such that $\overline{XY} \parallel \overline{DE}$.	4. ________________
5. $\angle FXY \cong \angle D$	5. ________________
6. ________________	6. Reflexive
7. $\Delta XYF \sim \Delta DEF$	7. ________________
8. $\frac{XY}{DE} = \frac{XF}{DF} = \frac{YF}{EF}$	8. ________________
9. $\frac{XY}{DE} = \frac{\quad}{DF}$	9. Substitution; 8, 2
10. $\frac{XY}{DE} = \frac{AB}{DE}$	10. ________________
11. ________________	11. Reflexive
12. $XY = AB$	12. ________________
13. ________________	13. Def of ≅; 12
14. $\frac{BC}{EF} = \frac{YF}{EF}$	14. ________________
15. ________________	15. Reflexive
16. $BC = YF$	16. ________________
17. ________________	17. Def of ≅
18. ________________	18. SSS≅; 3, 13, 17
19. $\angle C \cong \angle F$	19. ________________
20. $\Delta ABC \sim \Delta DEF$	20. ________________

1. Complete two of the following three proofs (using separate paper for space if needed)

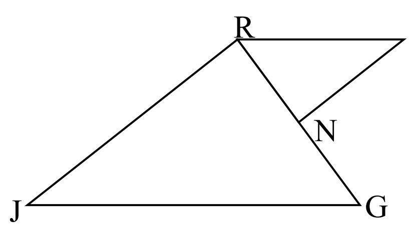
(a) Given: $\frac{JR}{AR} = \frac{DR}{ER}$

Prove: $\frac{JD}{AE} = \frac{JR}{AR}$



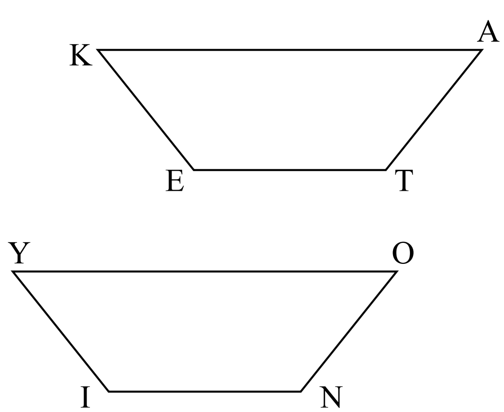
(b) Given: N is the midpoint of RG
$RJ = 2ON$
$JG = 2RO$

Prove: $\Delta RJG \sim \Delta NOR$



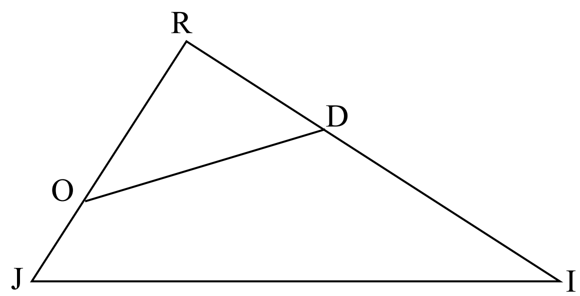
(c) Given: $KETA \sim YINO$.

Prove: $\frac{EA}{IO} = \frac{KA}{YO}$.



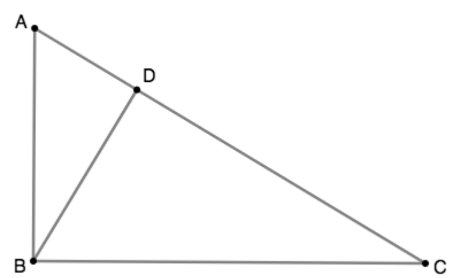
2. Spend 5-10 minutes writing down important information from today in your notebook. A little bit of studying *now* saves you a whole bunch of studying *later*!

3. In the diagram below, $JO = 1$, $RO = 3$, $RD = 2$, $DI = 4$, and $OD = 4$. Find JI.



4. In the diagram to the right $AD = 2$, $DC = 4$, $\overline{AB} \perp \overline{BC}$, and $\overline{BD} \perp \overline{AC}$.

   a. Show that $\Delta ADB \sim \Delta ABC$.

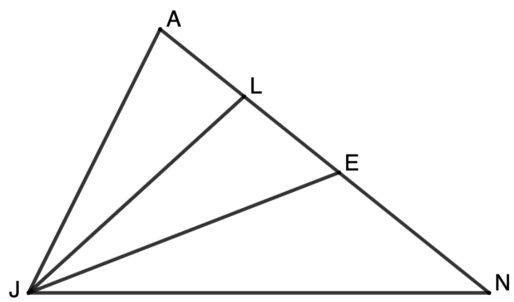


   b. Show that $\Delta BDC \sim \Delta ABC$.

   c. Show that $\Delta ADB \sim \Delta BDC$.

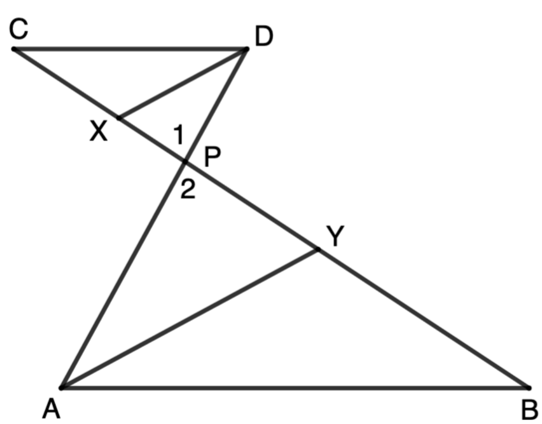
   d. Find BD, AB, and BC.

5. In the diagram below, $\angle AJL \cong \angle LJE \cong \angle EJN$.
Explain why it is *impossible* to have $AL = 1$, $LE = 2$, and $EN = 4$. (Hint: Use **indirect** reasoning.)



6. Given: $\frac{CX}{BY} = \frac{CD}{BA} = \frac{DX}{AY}$

Prove: $AP \bullet CP = BP \bullet DP$.



6 – Areas of Similar Polygons

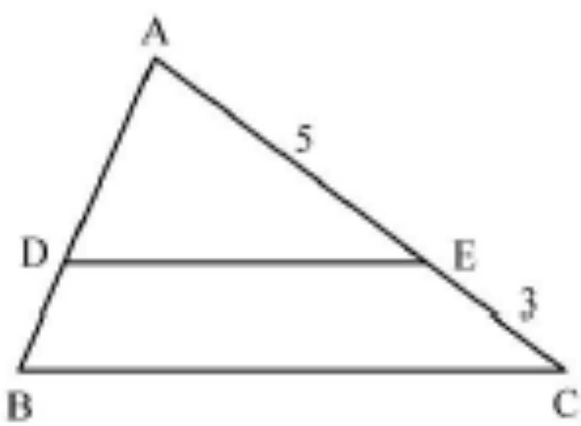
Advanced Geometry — Similarity Unit

> **Area of Similar Polygons Theorem (AST):** If two polygons are similar, then the ratio of their areas is equal to the square of the constant of proportionality.

1. The constant of proportionality of two similar triangles is 4:7. If the area of the larger Δ is 441, find the area of the smaller Δ.

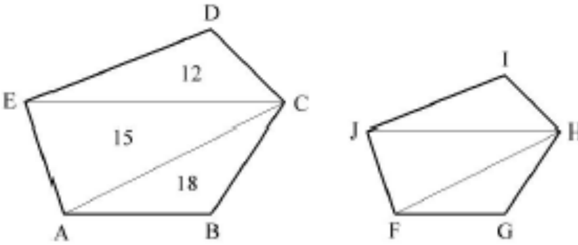
2. Two similar triangles have areas 12 and 27. One side of the smaller triangle has length 6. Find the length of the corresponding side of the larger triangle. Hint: it helps to reduce the area ratio.

3. Suppose $\overline{DE} \parallel \overline{BC}$ and the area of ΔADE is 10. Find the area of trapezoid $BDEC$.



4. In this problem you will show that *all* similar polygons follow the same rule for area, not just triangles.
Consider the similar pentagons shown below.

a) Show that $\Delta ABC \sim \Delta FGH$, $\Delta ACE \sim \Delta FHJ$, and $\Delta DEC \sim \Delta IJH$
These don't have to be full proofs, just outlines of your arguments.

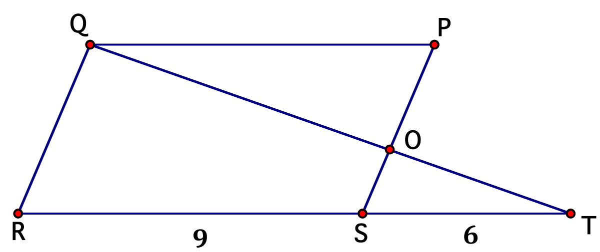


b) Suppose $\frac{AB}{FG} = \frac{3}{2}$. Also suppose that ΔABC, ΔACE and ΔDEC have areas 18, 15, and 12, respectively.
Find the areas of ΔFGH, ΔFHJ, and ΔIJH.

c) Using your results from (b), find the ratio of the areas of the two pentagons.
How does this ratio relate to the constant of proportionality of the two pentagons?

5. In the diagram below, $PQRS$ is a parallelogram.
Find the ratio of the areas for each pair of triangles.

a) ΔTOS and ΔQOP.



b) ΔTOS and ΔTQR.

6. Spend 5-10 minutes writing down important information from today in your notebook. A little bit of studying *now* saves you a whole bunch of studying *later*!

7. One side of a pentagon is 1 cm shorter than the corresponding side of a similar pentagon.
   The areas of the pentagons are 2cm^2 and 18 cm^2, respectively.
   Find the lengths of the two corresponding sides.

8. The length of a side of an n-sided regular polygon is 36.
   The area of a second polygon similar to the first is 81/16 times the area of the first.
   The difference between the perimeters is 855.
   Find the value of n.

9. In ΔLMN, altitude LK is 12 cm long. Through point J of LK, a line is drawn parallel to MN, dividing the triangle into two regions with equal areas. Find LJ.

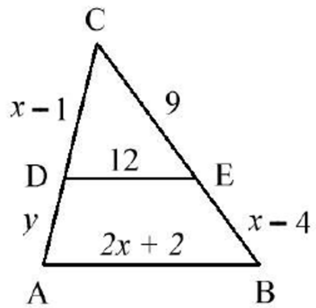
8 – Unit Review Problems

Advanced Geometry — Similarity Unit

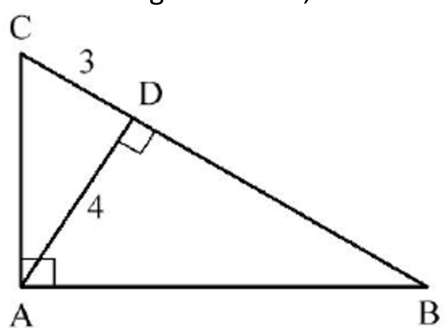
1. a) $\overline{PQ}$ has length 48 and point R is between P and Q so that $PR : QR = 3 : 13$. Find PR.

   b) The angles of a pentagon are in the ratio $1 : 4 : 7 : 7 : 8$. Find the measure of the largest angle.

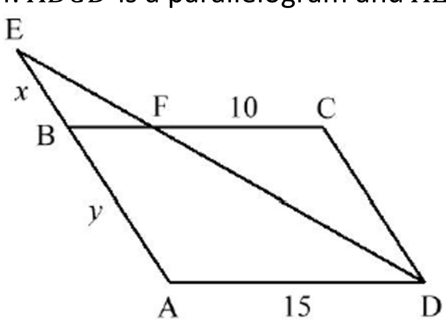
2. In the diagram below, $\overline{DE} \parallel \overline{AB}$. Find x and y.



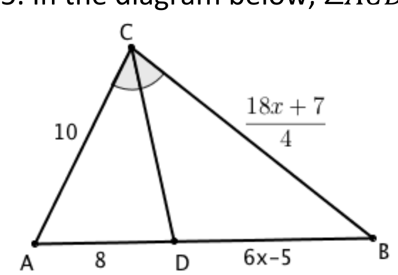
3. In the diagram below, $\overline{AC} \perp \overline{AB}$ and $\overline{AD} \perp \overline{BC}$. Find AB.



4. $ABCD$ is a parallelogram and $AE = 12$. Find x and y.

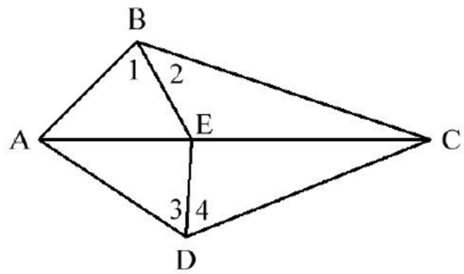


5. In the diagram below, $\angle ACD \cong \angle BCD$. Solve for x.



6. Given: $\angle 1 \cong \angle 2$
$\angle 3 \cong \angle 4$

Prove: $AB \bullet CD = BC \bullet AD$



7. Given: $\frac{CE}{AC} = \frac{CD}{BC}$

Prove: $EF \bullet AF = DF \bullet BF$