Computational Fluid Dynamics: Part 1

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Problem 1

Categorize the following equations by their type (hyperbolic, parabolic, elliptic):

(a) $2u_{xx} - 4u_{xy} + 2u_{yy} + u = \sin(x)$

(b) $uu_x - 2xyu_y = 0$

(c) $\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{r\partial \theta} + G = 0$

Solution

Part a: $2u_{xx} - 4u_{xy} + 2u_{yy} + u = \sin(x)$

This is a second-order PDE with coefficients $a = 2$, $b = -4$, $c = 2$. The discriminant is:

$$\Delta = b^2 - 4ac = (-4)^2 - 4 \cdot 2 \cdot 2 = 16 - 16 = 0$$

Since $\Delta = 0$, the PDE is **parabolic**. The characteristic equation is:

$$a\left(\frac{dy}{dx}\right)^2 - b\frac{dy}{dx} + c = 0$$

Substitute the coefficients:

$$2\left(\frac{dy}{dx}\right)^2 - (-4)\frac{dy}{dx} + 2 = 0$$

$$2\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} + 2 = 0$$

Divide by 2:

$$\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} + 1 = 0$$

$$\left(\frac{dy}{dx} + 1\right)^2 = 0$$

$$\frac{dy}{dx} = -1$$

Integrate:

$$dy = -dx \implies y + x = C$$

Thus, the characteristic curves are:

$$y + x = \text{constant}$$

Part b: $uu_x - 2xyu_y = 0$

This is a first-order PDE. The classification as parabolic, elliptic, or hyperbolic applies to second-order PDEs based on the discriminant Δ, which is not defined here. This equation is **not parabolic, elliptic, or hyperbolic** (it is a first-order quasilinear PDE). We use the method of characteristics.

The general form is:

$$a(x, y, u)u_x + b(x, y, u)u_y = c(x, y, u)$$

Here:

$$\begin{aligned} a &= u \\ b &= -2xy \\ c &= 0 \end{aligned}$$

The characteristic equations are:

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = -2xy, \quad \frac{du}{dt} = 0$$

From $\frac{du}{dt} = 0$:

$$u = \text{constant}$$

Let $u = k$. Then:

$$\frac{dx}{dt} = k \implies x = kt + C_1$$

$$\frac{dy}{dt} = -2xy$$

Substitute $x = kt$ (setting $C_1 = 0$ for simplicity):

$$\frac{dy}{dt} = -2(kt)y$$

$$\frac{dy}{y} = -2kt\,dt$$

Integrate:

$$\ln|y| = -kt^2 + C_2$$

$$y = C_3 e^{-kt^2}, \quad C_3 = e^{C_2}$$

Substitute $t = \frac{x}{k}$:

$$y = C_3 e^{-k\left(\frac{x}{k}\right)^2} = C_3 e^{-\frac{x^2}{k}}$$

Since $u = k$:

$$y = C_3 e^{-\frac{x^2}{u}}$$

Thus:

$$y e^{\frac{x^2}{u}} = C_3 = \text{constant}$$

The characteristic curves are:

$$y e^{\frac{x^2}{u}} = \text{constant}$$

Part c: $\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial z^2} + \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial r^2} + \frac{\partial T}{r\partial \theta} + G = 0$

This PDE involves time t and spatial variables z, r, θ. Consider the spatial part with second-order terms $\frac{\partial^2 T}{\partial z^2}$ and $\frac{\partial^2 T}{\partial r^2}$:
- $a = 1$, $b = 0$, $c = 1$

$$\Delta = b^2 - 4ac = 0 - 4 \cdot 1 \cdot 1 = -4 < 0$$

This suggests elliptic behavior spatially, but the presence of $\frac{\partial T}{\partial t}$ indicates a time-dependent evolution. The PDE is **parabolic** due to the heat equation-like structure. The characteristic behavior for parabolic PDEs is:

$$t = \text{constant}$$

Why $t =$ constant? Unlike hyperbolic PDEs, which have finite-speed characteristic curves, parabolic PDEs (e.g., the heat equation) involve instantaneous diffusion. The solution at time t depends on the initial condition across all spatial points at $t = 0$. The hypersurfaces $t =$ constant represent the levels at which the solution evolves, driven by diffusion in z and r.

Problem 2

Consider the following partial differential equation. Determine the nature of this equation (hyperbolic, parabolic, elliptic) for different values of y and x.

$$(x-1)\frac{\partial^2 u}{\partial x^2} + 2(y+1)\frac{\partial^2 u}{\partial x \partial y} - (x+1)\frac{\partial^2 u}{\partial y^2} = 0$$

Solution

To classify a second-order PDE of the form:

$$au_{xx} + bu_{xy} + cu_{yy} + \text{lower-order terms} = 0,$$

we use the discriminant:

$$\Delta = b^2 - 4ac.$$

The classification is determined by the sign of Δ:

- **Hyperbolic** if $\Delta > 0$,

- **Parabolic** if $\Delta = 0$,
- **Elliptic** if $\Delta < 0$.

Step 1: Identify the Coefficients

From the given PDE:

$$(x-1)u_{xx} + 2(y+1)u_{xy} - (x+1)u_{yy} = 0,$$

we identify:

$$\begin{aligned} a &= x - 1, \\ b &= 2(y+1), \\ c &= -(x+1). \end{aligned}$$

Note that the coefficient of u_{yy} is $-(x+1)$, accounting for the negative sign in the PDE.

Step 2: Compute the Discriminant

The discriminant Δ is:

$$\Delta = b^2 - 4ac.$$

Substitute the coefficients:

$$\Delta = [2(y+1)]^2 - 4(x-1)[-(x+1)].$$

First, compute b^2:

$$[2(y+1)]^2 = 4(y+1)^2.$$

Next, compute $4ac$:

$$4(x-1)[-(x+1)] = -4(x-1)(x+1).$$

Since $(x-1)(x+1) = x^2 - 1$, we have:

$$4ac = -4(x^2 - 1) = -4x^2 + 4.$$

Now, combine the terms:

$$\Delta = 4(y+1)^2 - (-4x^2+4) = 4(y+1)^2 + 4x^2 - 4.$$

Factor out the 4:

$$\Delta = 4\left[(y+1)^2 + x^2 - 1\right].$$

Step 3: Analyze the Sign of Δ

Since the factor 4 is positive, the sign of Δ depends on the term $(y+1)^2 + x^2 - 1$:

- $\Delta > 0$ when $(y+1)^2 + x^2 - 1 > 0$, i.e., $(y+1)^2 + x^2 > 1$,
- $\Delta = 0$ when $(y+1)^2 + x^2 - 1 = 0$, i.e., $(y+1)^2 + x^2 = 1$,
- $\Delta < 0$ when $(y+1)^2 + x^2 - 1 < 0$, i.e., $(y+1)^2 + x^2 < 1$.

Geometrically, $(y+1)^2 + x^2 = 1$ represents a circle in the xy-plane centered at $(0,-1)$ with radius 1.

Step 4: Classify the PDE

The classification of the PDE varies across different regions in the xy-plane:

- **Hyperbolic** when $(y+1)^2 + x^2 > 1$ (outside the circle),
- **Parabolic** when $(y+1)^2 + x^2 = 1$ (on the circle),
- **Elliptic** when $(y+1)^2 + x^2 < 1$ (inside the circle).

Conclusion

The given PDE:

$$(x-1)\frac{\partial^2 u}{\partial x^2} + 2(y+1)\frac{\partial^2 u}{\partial x \partial y} - (x+1)\frac{\partial^2 u}{\partial y^2} = 0$$

is of **mixed type**:

- **Hyperbolic** outside the circle $(y+1)^2 + x^2 = 1$,
- **Parabolic** on the circle $(y+1)^2 + x^2 = 1$,
- **Elliptic** inside the circle $(y+1)^2 + x^2 = 1$.

Problem 3

Through resources, specify the differential equation of concentration transport and explain what physical concept each term represents.

Solution

The differential equation of concentration transport, often referred to as the advection-diffusion equation, describes the transport of a scalar quantity (e.g., concentration c) in a fluid flow. The general form in a continuum mechanics context is:

$$\frac{\partial c}{\partial t} + \mathbf{u} \cdot \nabla c = D\nabla^2 c + S,$$

where:

- $\frac{\partial c}{\partial t}$: The local rate of change of concentration at a fixed point, representing the accumulation or depletion of concentration over time.
- $\mathbf{u} \cdot \nabla c$: The advection term, where $\mathbf{u}$ is the velocity vector and ∇c is the concentration gradient. This term accounts for the transport of concentration due to the bulk motion of the fluid.
- $D\nabla^2 c$: The diffusion term, where D is the diffusion coefficient and $\nabla^2 c$ is the Laplacian of concentration. This term models the spreading of concentration due to molecular diffusion, driven by concentration gradients.
- S: The source/sink term, representing the generation or consumption of the species (e.g., due to chemical reactions, external inputs, or losses).

In the absence of sources or sinks ($S = 0$), the equation simplifies to the advection-diffusion equation, balancing the effects of convection and diffusion.

Problem 4

What is meant by incompressible flow?

Assuming incompressibility for the flow, what changes will the differential transport equations (continuity and momentum) undergo?

Solution

Definition of Incompressible Flow

In continuum mechanics, **incompressible flow** refers to a fluid flow where the density ρ of the fluid remains constant throughout the flow field, i.e., $\rho = \text{constant}$. This implies that the fluid does not undergo significant volume changes under pressure, which is a reasonable assumption for liquids (e.g., water) and for gases under low Mach number conditions (typically $\text{Ma} < 0.3$).

Mathematically, incompressibility is expressed through the condition:

$$\nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u}$ is the velocity vector. This condition states that the divergence of the velocity field is zero, meaning there is no net expansion or compression of fluid elements.

Changes to Differential Transport Equations Under Incompressibility

Assuming incompressibility ($\rho = \text{constant}$), the differential transport equations for continuity and momentum in fluid dynamics are modified as follows:

Continuity Equation

The general continuity equation for a compressible fluid is:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

For an incompressible fluid ($\rho = \text{constant}$), the time derivative of density vanishes ($\frac{\partial \rho}{\partial t} = 0$), and ρ can be factored out:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = 0.$$

Since $\rho \neq 0$, this simplifies to:

$$\nabla \cdot \mathbf{u} = 0.$$

This is the continuity equation for incompressible flow, confirming the divergence-free condition stated earlier.

Momentum Equation

The momentum equation (Navier-Stokes equation) for a compressible fluid, assuming a Newtonian fluid, is:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g},$$

where: - p is pressure, - μ is dynamic viscosity, - $\mathbf{g}$ is the gravitational acceleration.

For an incompressible fluid, ρ is constant, and the continuity equation $\nabla \cdot \mathbf{u} = 0$ is used. The viscous term simplifies because the viscosity μ is typically constant for incompressible flows, and the divergence-free condition modifies the stress tensor. The incompressible Navier-Stokes equation becomes:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \rho \mathbf{g}.$$

This equation is simpler than its compressible counterpart because density variations do not affect the flow, and the pressure p acts as a Lagrange multiplier to enforce the incompressibility constraint $\nabla \cdot \mathbf{u} = 0$.

Problem 5

Write the differential energy equation in terms of temperature for fluid flow and state under what conditions each term of this equation can be neglected.

Solution

Differential Energy Equation in Terms of Temperature

The differential energy equation for a fluid flow, expressed in terms of temperature T, accounts for the transport and conversion of thermal energy. For a Newtonian, incompressible fluid with constant properties, the energy equation can be derived from the first law of thermodynamics. Assuming Fourier's law for heat conduction ($\mathbf{q} = -k\nabla T$, where k is thermal conductivity), the energy equation in terms of temperature is:

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = k \nabla^2 T + \Phi + \dot{q},$$

where:

- ρ: Density (constant for incompressible flow),
- c_p: Specific heat at constant pressure,
- $\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T$: The material derivative of temperature, representing the rate of change of temperature following a fluid particle (local change plus advection),
- $k\nabla^2 T$: The heat conduction term, modeling the diffusion of heat within the fluid,
- Φ: The viscous dissipation function, representing the conversion of mechanical energy into heat due to viscous effects. For an incompressible Newtonian fluid in Cartesian coordinates:

$$\begin{aligned}\Phi = \mu \Bigg[& 2\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right) \\ & + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \\ & + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \\ & + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \Bigg]\end{aligned}$$

  where $\mathbf{u} = (u, v, w)$ are the velocity components,
- $\dot{q}$: The volumetric heat source term (e.g., from chemical reactions, radiation, or external heating).

Conditions for Neglecting Terms

Each term in the energy equation can be neglected under specific physical conditions:

- **Local Change Term ($\frac{\partial T}{\partial t}$)**: Neglected in steady-state flows, where the temperature field does not change with time ($\frac{\partial T}{\partial t} = 0$).
- **Advection Term ($\mathbf{u} \cdot \nabla T$)**: Neglected in flows with negligible velocity or when temperature gradients are small (e.g., in a stagnant fluid or uniform temperature field). This term dominates in high-speed flows with significant convection.

- **Conduction Term ($k\nabla^2 T$)**: Neglected when thermal conduction is small compared to convection, which occurs at high Peclet numbers ($\mathrm{Pe} = \frac{\rho c_p U L}{k} \gg 1$), where U and L are characteristic velocity and length scales. Conduction dominates in low Peclet number regimes.

- **Viscous Dissipation Term (Φ)**: Neglected in flows with low viscosity or low velocity gradients, or when the Eckert number ($\mathrm{Ec} = \frac{U^2}{c_p \Delta T}$) is small, indicating that viscous heating is negligible compared to thermal energy changes. This term is significant in high-speed viscous flows (e.g., near walls in boundary layers).

- **Heat Source Term ($\dot{q}$)**: Neglected if there are no internal heat sources or sinks (e.g., no chemical reactions, radiation, or external heating).

In many practical scenarios (e.g., low-speed incompressible flows without heat sources), the energy equation simplifies to a balance between convection and conduction:

$$\rho c_p \left(\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T \right) = k \nabla^2 T.$$