Higher-Order Hyperbolic Equations with Constant Coefficients

CMOR 505/MATH 423 Partial Differential Equations I

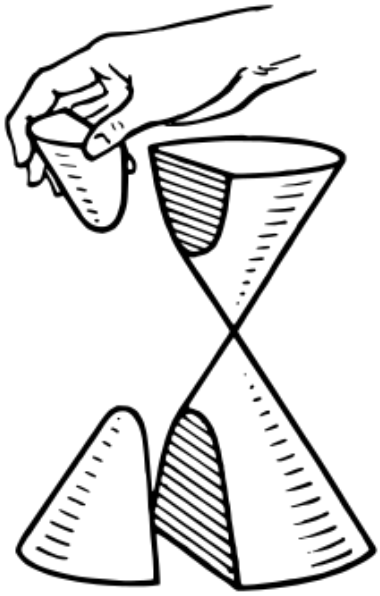
Professor Christos Mantoulidis

Benjamin May
btm2@rice.edu

Bill Nguyen
ttn9@rice.edu

William Pan
pan@rice.edu

Kiara Richardson
kar15@rice.edu



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1 Introduction

A hyperbolic equation is a type of partial differential equation (PDE) that describes wave-like phenomena in various fields of science and engineering, where disturbances propagate with finite speed. Mathematically, hyperbolic equations involve second-order derivatives in both time and space variables.

Hyperbolic equations are characterized by having two families of characteristic curves, along which information travels. These curves intersect transversally, which means they cross without being tangent, allowing for the propagation of disturbances in various directions. This property distinguishes hyperbolic equations from elliptic and parabolic equations, where the propagation of information is fundamentally different.

Higher order hyperbolic equations with constant coefficients are a class of these PDEs. By the notation of [1], these equations typically take the form of

$$P(D,\tau)u = w(x,t) \tag{1}$$

where $P(D,\tau) = P(D_1,\cdots,D_n,\tau)$ is a polynomial of degree m in its $n+1$ arguments.

Having constant coefficients in these equations is significant because it simplifies the analysis and solution techniques. Unlike equations with variable coefficients, where the coefficients depend on the independent variables, equations with constant coefficients allow for straightforward methods for finding solutions such as Fourier transforms, separation of variables, and characteristic equations.

Higher order hyperbolic equations with constant coefficients find applications in various real-life scenarios. In physics, they are fundamental in modeling wave propagation phenomena, including sound waves, electromagnetic waves, and seismic waves. In structural engineering, these equations are used to study the behavior of vibrating structures, such as bridges and buildings, under dynamic loads. Hyperbolic equations are essential for understanding and predicting the behavior of waves and disturbances in various systems. This paper will talk about various methods of solving such equations.

2 Reduction to Standard problem

This section is adapted from John Fritz's textbook with with regards to notation, steps, and structure, page 144-145 [1]. Certain mathematical lines are verbatim from this textbook.

The goal of this section is to solve a general higher order linear partial differential equation that has constant coefficients in terms of standard problems. Throughout this section we will be assuming that the coefficient of τ^m is always nonzero in $P(D,\tau)$. By dividing by the coefficient of τ^m, we can ensure that that $P(0,1) = 1$.

The standard problem is of the form $P(D,\tau)u = 0$ with initial conditions $\tau^k u = 0$ for $k = 0,\ldots,m-2$ and $\tau^{m-1}u = g(x)$ where m is the degree of P.

We will first find a solution (in terms of standard problems) to the problem where $P(0,1) = 1$ and all the initial data has value 0. Thus, for a given $P(D,\tau)$ of degree m with $P(0,1) = 1$, we are trying to find $u(x,t)$ such that $P(D,\tau)u = w(x,t)$ and $\tau^k u = 0$ for $k = 0,\ldots,m-1$. The solution we find will be based on Duhamel's principle. We will first construct a function $U : \mathbb{R}^n \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \to \mathbb{R}$ such that $P(D,\tau)\int_0^t U(x,t,s)ds = w(x,t)$. Write

$$P(D,\tau) = \tau^m + P_1(D)\tau^{m-1} + \cdots + P_m(D) \tag{2}$$

where, for each $i \in \{1,\ldots,m\}$, $P_i(D)$ is a polynomial of D of at most degree i. For a sufficiently regular U, we have

$$
\begin{aligned}
P(D,\tau)\int_0^t U(x,t,s)ds &= \left(\tau^m + P_1(D)\tau^{m-1} + ... + P_{m-1}(D)\tau\right)\int_0^t U(x,t,s)ds \\
&\quad + P_m(D)\int_0^t U(x,t,s)ds \\
&= \Bigg(\left(\tau^{m-1} + P_1(D)\tau^{m-2} + ... + P_{m-1}(D)\right)U(x,t,t) \\
&\quad + \int_0^t (\tau^m + P_1(D)\tau^{m-1} + ... + P_{m-1}(D)\tau)U(x,t,s)ds\Bigg) \\
&\quad + \int_0^t P_m(D)U(x,t,s)ds \\
&= \left(\tau^{m-1} + P_1(D)\tau^{m-2} + ... + P_{m-1}(D)\right)U(x,t,t) \\
&\quad + \int_0^t P(D,\tau)U(x,t,s)ds.
\end{aligned}
$$

Write $U(x,t,s)$ as $U_s(x,t)$. It follows that if for each $s > 0$ we have

$$P(D,\tau)U_s(x,t) = 0 \text{ for } t \geq s \tag{3}$$
$$\tau^k U_s(x,t) = 0 \text{ for } k = 0,\ldots,m-2 \text{ and } t = s \tag{4}$$
$$\tau^{m-1}U_s(x,t) = w(x,s) \text{ at } t = s \tag{5}$$

Then $\int_0^t U(x,t,s)ds$ would satisfy our desired requirements. We implicitly used the fact that $P_{m-1}(D)U(x,t,t)$ would equal 0 for all $t > 0$ by (4).

Now fix $s > 0$. Let $v : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be a solution to the standard problem for $P(D,\tau)v$ and $f_{m-1} = w(x,s)$. Then, define $U_s(x,t) := v(x,t-s)$. Do this for all s. Note that $U_s(x,t)$ defined in this way satisfies (3), (4), (5) regardless of the choice of s. Thus $u(x,t) = \int_0^t U(x,t,s)ds$ solves $P(D,\tau)u = w(x,t)$ prescribed with zero initial data.

Now denote the solution of the standard problem with respect to $P(D,\tau)$ for f_i as u_i. I claim that

$$
\begin{aligned}
u = u_{m-1} + (\tau + P_1(D))u_{m-2} + (\tau^2 + P_1(D)\tau + P_2(D))u_{m-3} + \ldots \\
+(\tau^{m-1} + P_1(D)\tau^{m-2} + ... + P_{m-1}(D)\tau^0)u_0
\end{aligned}
$$

is a solution to $P(D,\tau)u = 0$ and $\tau^i u = f_i(x)$ when $t = 0$ where f_i are functions for $i = 0, ...m-1$

First notice that it solves $P(D,\tau)u = 0$ since $P(D,\tau)u_i = 0$ for all i.

Let $i \in \{0,\ldots,m-1\}$. At $t = 0$,

$$
\begin{aligned}
\tau^i u &= \tau^i u_{m-1} + (\tau + P_1(D))u_{m-2} + (\tau^{2+i} + P_1(D)\tau^{i+1} + P_2(D))u_{m-3} + \ldots \\
&\quad + (\tau^{m-1+i} + P_1(D)\tau^{m-2+i} + \cdots + P_{m-1}(D)\tau^i)u_0 \\
&= 0 + \ldots + 0 + (\tau^{m-1})u_i + (\tau^{m-0} + P_1(D)\tau^{m-1})u_{i+1} + \ldots \\
&\quad + (\tau^{(m-1+i)} + P_1(D)\tau^{m-2+i} + \cdots + P_i(D)\tau^{m-1})u_0.
\end{aligned}
$$

Since $\tau^j u_k(x,0) = 0$ for all $k = 0,1,\ldots,m-1$, $x \in \mathbb{R}^n$ and $j \leq m-2$, it follows from the above that

$$
\begin{aligned}
\tau^i u &= f_i + P_{D,\tau}\tau u_{i-1} + P_{D,\tau}\tau^2 u_{i-2} + \ldots + P_{D,\tau}\tau^i u_0 \\
&= f_i
\end{aligned}
$$

We have thus proved the claim.

It then follows that a solution to the problem $P(D,\tau)u = w(x,t)$ with initial data $\tau^i u = f_i(x)$ when $t = 0$, is $u = v + g$ where v is a solution to $P(D,\tau)v = 0$ with initial data $\tau^i v = f_i(x)$ when $t = 0$, and g is a solution to $P(D,\tau)g = w(x,t)$ with initial data $\tau^k g = 0$ for $k = 0, ..., m-1$. Since we have shown that we can write v, w in terms of standard problems, we have achieved our goal for this section.

For the rest of the paper, we deal with the following more general problem analogous to that of equations (3 - 5): given a degree-m polynomial differential operator $P(D, \partial/\partial t)$ we seek $u : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}$ such that

$$P\left(D, \frac{\partial}{\partial t}\right) u = 0 \text{ for } t \geq 0 \tag{6}$$

$$\frac{\partial^k}{\partial t^k} u = 0 \text{ for } k = 0, \ldots, m-2 \text{ and } t = 0 \tag{7}$$

$$\frac{\partial^{m-1}}{\partial t^{m-1}} u = g(x) \text{ for } t = 0. \tag{8}$$

The differential equation or systems of equations represented by (6) is **hyperbolic** with respect to the plane $t = 0$ if there exists $u \in C^m(\mathbb{R}^n \times \mathbb{R}_{>0})$ solving equations (6 - 8) for all $g(x) \in C_c^s(\mathbb{R}^n)$ with s sufficiently large.

3 Solutions with the Fourier Transform

For functions $g(x) \in C_c^0(\mathbb{R}^n)$, define their **Fourier transform** $\hat{g} : \mathbb{R}^n \to \mathbb{R}$ by

$$\hat{g}(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int e^{-i\mathbf{x}\cdot\xi} g(x) dx, \tag{9}$$

where $\mathbf{x} \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$.
For $g \in C_0^s(\mathbb{R}^n)$ with sufficiently large s, the **inverse Fourier transform**

$$g(x) = \frac{1}{(\sqrt{2\pi})^n} \int e^{i\mathbf{x}\cdot\xi} \hat{g}(\xi) d\xi \tag{10}$$

holds. We first prove several of properties regarding the Fourier transform. The following proofs are adapted from pages 146 - 147 of [1].

Theorem 3.1. *Let $g \in C_c^s(\mathbb{R}^n)$ for some $s \in \{0, 1, 2, \ldots\}$, and denote its Fourier transform as $\hat{g}$. Differentiation for g is transformed into multiplication for $\hat{g}$, i.e. for any multi-index $\alpha \in Z^n$ with $|\alpha| \leq s$,*

$$(i\xi)^\alpha \hat{g} = \widehat{D^\alpha g}. \tag{11}$$

Proof. We have from (9) that for arbitrary $k \in \{1, \ldots, n\}$,

$$\widehat{D_k g} = \frac{1}{(\sqrt{2\pi})^n} \int e^{-i\mathbf{x}\cdot\xi} D_k g(x) dx.$$

Invoking integration by parts on the above integral results in boundary terms that resolve to 0, since g is compactly supported. Thus applying integration by parts to the integral above yields

$$\begin{aligned}
\widehat{D_k g} &= -\frac{1}{(\sqrt{2\pi})^n} \int D_k(e^{-i\mathbf{x}\cdot\xi}) g(x) dx \\
&= -\frac{1}{(\sqrt{2\pi})^n} \int -i\xi_k \cdot e^{-i\mathbf{x}\cdot\xi} g(x) dx \\
&= i\xi_k \hat{g}.
\end{aligned}$$

For arbitrary derivatives D^α with $|\alpha| \leq s$, repeated application of the above result yields equation (11), as desired. □

Theorem 3.2. *For functions $g \in C_c^{n+1}(\mathbb{R}^n)$, the integral in the inverse Fourier transform in (10) converges absolutely.*

Proof. Take $\xi \in \mathbb{R}^n$ and write $\xi = (\xi_1, \ldots, \xi_n)$. We have that

$$|\xi| = \sqrt{\sum_{k=1}^{n} \xi_k^2} \leq \sqrt{n} \max_{k \in \{1,\ldots,n\}} |\xi_k|.$$

Thus by the binomial theorem,

$$\begin{aligned}
(1 + |\xi|)^{n+1} &= \sum_{k=0}^{n+1} \binom{n+1}{k} |\xi|^k \\
(1 + |\xi|)^{n+1} &\leq \sum_{k=0}^{n+1} 2^{n+1} \cdot \left[n^{k/2} \left(\max_{i \in \{1,\ldots,n\}} |\xi_i| \right)^k \right] \\
(1 + |\xi|)^{n+1} &\leq 2^{n+1} n^{(n+1)/2} \sum_{|\alpha| \leq n+1} |\xi^\alpha| \\
(1 + |\xi|)^{n+1} \cdot |\hat{g}(\xi)| &\leq 2^{n+1} n^{(n+1)/2} \sum_{|\alpha| \leq n+1} |(i\xi)^\alpha \hat{g}(\xi)|.
\end{aligned}$$

Since $g \in C^{n+1}(\mathbb{R}^n)$, we can invoke Theorem 3.1 to write the right-hand side as

$$(1 + |\xi|)^{n+1} \cdot |\hat{g}(\xi)| \leq 2^{n+1} n^{(n+1)/2} \sum_{|\alpha| \leq n+1} \int |D^\alpha g(x)| dx.$$

Since g is compactly supported, the right-hand side resolves to a finite constant independent of ξ. Denote it as M. Then

$$|\hat{g}(\xi)| \leq \frac{M}{(1 + |\xi|)^{n+1}}$$

so as $|\xi| \to \infty$, $\hat{g}(\xi) \to 0$. Since $\hat{g}$ decreases with the inverse Fourier transform in (10) converges absolutely. □

A formal solution of equations (6 - 8) can be obtained through a Fourier transformation with respect to the spatial variables $x_1, \ldots, x_n$ by reformulating the equations as an initial-value problem described by an ordinary differential equation, which is much easier to deal with. We derive this solution in the following theorem, adapted from pages 147 - 148 of [1].

Theorem 3.3. *If there exists a solution $u : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}$ with compact support in x and sufficiently often differentiable to equations (6 - 8) for a given polynomial operator P of degree m, then u is given by*

$$u(x,t) = \frac{1}{(\sqrt{2\pi})^n} \int e^{ix \cdot \xi} Z(\xi, t) \hat{g}(\xi) d\xi \tag{12}$$

where Z as a function of t denotes the solution of the ordinary differential equation problem

$$P\left(i\xi, \frac{\partial}{\partial t}\right) Z(\xi, t) = 0 \text{ for } t \geq 0 \tag{13}$$

$$\frac{\partial^k}{\partial t^k} Z(\xi, 0) = \begin{cases} 0 & \text{for } k = 0, \ldots, m-2 \\ \hat{g}(\xi) & \text{for } k = m-1. \end{cases} \tag{14}$$

in which $\hat{g}$ is the Fourier transform of $g : \mathbb{R}^n \to \mathbb{R}_{>0}$ defined by

$$g(x) = \left(\frac{\partial^{m-1}}{\partial t^{m-1}} u(x,t) \right) \Big|_{t=0}$$

Proof. Give the assumption that u is sufficiently often differentiable, the integral in the inverse Fourier transform (10) is well-defined. By differentiation under the integral sign, equation (6) for the given polynomial operator P and solution u can be reformulated as

$$P\left(D, \frac{\partial}{\partial t}\right) u(x,t) = \frac{1}{(\sqrt{2\pi})^n} \int e^{ix\cdot\xi} P\left(D, \frac{\partial}{\partial t}\right) \hat{u}(\xi,t) d\xi$$
$$0 = \frac{1}{(\sqrt{2\pi})^n} \int e^{ix\cdot\xi} P\left(i\xi, \frac{\partial}{\partial t}\right) \hat{u}(\xi,t) d\xi \quad \text{by (11)}$$

and equations (7) and (8) can be reformulated as

$$\frac{\partial^k}{\partial t^k} u(x,0) = \begin{cases} 0 & \text{for } k = 0, \ldots, m-2 \\ \frac{1}{(\sqrt{2\pi})^n} \int e^{ix\cdot\xi} \hat{g}(\xi) d\xi & \text{for } k = m-1. \end{cases}$$

We can see that these are satisfied when $\hat{u}$ is a solution of the initial-value problem given by equations (13) and (14) for each $\xi \in \mathbb{R}^n$. Since equation (13) describes an ordinary differential equation, there exists a function $Z : \mathbb{R}^n \times \mathbb{R}_{>0} \to \mathbb{R}$ such that $Z(\xi, \cdot)$ solves equations (13) and (14) for each $\xi \in \mathbb{R}^n$. Substituting Z into the Fourier transform for u completes the proof. □

Note that for u defined in (12) to be well-defined, the integral must converge absolutely. To verify this, it is sufficient to verify that, for all $\alpha \in Z^n$ and k such that $|\alpha| + k \leq m$,

$$(1+\xi)^{n+1} \left| \frac{\partial^k}{\partial t^k} \xi^\alpha Z(\xi,t) \hat{g}(\xi) \right|$$

is bounded uniformly in ξ and t for all $\xi \in \mathbb{R}^n$ and for t restricted to any finite interval $[0,T]$. For sufficiently regular u, the proof of Theorem 3.2 posits that we can control $\hat{g}(\xi)$. Thus to verify uniform boundedness of the expression above it suffices to show that there exists a constant N such that for all $\xi \in \mathbb{R}^n$, $t \in [0,T]$, $k = 0, 1, \ldots, m$,

$$\left| \frac{\partial^k}{\partial t^k} Z(\xi,t) \right| \leq (1+|\xi|)^k N. \tag{15}$$

There is a specific condition that is easier to check under which the above holds. Gårding's textbook titled *Linear Hyperbolic Partial Differential Equations with Constant Coefficients* [2] explains this condition like so:

Let $v = (v_1, v_2, \ldots, v_n)$ be a vector of coefficients corresponding to the highest-order spatial derivatives in the polynomial P. Then, ***Gårding's hyperbolicity condition*** *requires that the quadratic form $Q(v)$, defined by*

$$Q(v) = \sum_{i,j=1}^{n} a_{ij} v_i v_j$$

is definite, i.e. either positive definite or negative definite, where a_{ij} are coefficients in the expansion of P. Equivalently, the eigenvalues of the matrix a_{ij} should have the same signs.

Equivalently, Fritz John [1] formulates that equation (6) is hyperbolic if there exists a real number c such that for all $\xi \in \mathbb{R}^n$ and all complex λ with $\mathrm{Im}(\lambda) \leq -c$, $P(i\xi, i\lambda) \neq 0$. That is, all m roots λ of

$$P(i\xi, i\lambda) = 0$$

lie in the half plane $\text{Im}(\lambda) > -c$. We prove the following theorem adapted from pages 149 - 150 of Fritz John's textbook [1], which will also give us more insights into the form of the solution given by Theorem 3.3.

Theorem 3.4. *If P satisfies the Gårding hyperbolicity condition, then there exists a unique solution u satisfying equations (6 - 8).*

Proof. Represent the solution Z of equations (13) and (14) as the Cauchy integral

$$Z(\xi, t) = \frac{1}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda, \tag{16}$$

where Γ is a curve that winds around each root λ of $P(i\xi, i\lambda) = 0$ once in the counterclockwise direction. We first verify that equation (16) indeed satisfies equations (13) and (14); differentiating under the integral achieves

$$\begin{aligned} P\left(i\xi, \frac{\partial}{\partial t}\right) Z(x,t) &= \frac{1}{2\pi} \int_\Gamma P\left(i\xi, \frac{\partial}{\partial t}\right) \cdot \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda \\ &= \frac{1}{2\pi} \int_\Gamma P\left(i\xi, i\lambda\right) \cdot \frac{e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda \\ &= \frac{1}{2\pi} \int_\Gamma e^{i\lambda t} d\lambda \\ &= 0. \quad \text{by Cauchy's integral theorem} \end{aligned}$$

Thus equation (13) holds. Then, expanding $P(i\xi, i\lambda)$ in the manner of equation (2), we get

$$P(i\xi, i\lambda) = (i\lambda)^m + P_1(i\xi)(i\lambda)^{m-1} + \cdots + P_m(i\xi). \tag{17}$$

From this, we can write, at $t = 0$,

$$\frac{\partial^k}{\partial t^k} Z(x, 0) = \frac{1}{2\pi} \int_\Gamma \frac{i^k \lambda^k}{(i\lambda)^m + P_1(i\xi)(i\lambda)^{m-1} + \cdots + P_m(i\xi)}$$

which has value 0 for $k = 0, \ldots, m-2$ by Cauchy's integral theorem and value 1 for $k = m-1$. Thus equation (14) also holds.
Note from equation (17) we can estimate

$$|P_k(i\xi) \leq M(1 + |\xi|)^k$$

for each $k = 1, \ldots, m$, all $\xi \in \mathbb{R}^n$, and some constant M. Then for a root λ of $P(i\xi, i\lambda) = 0$, moving the $(i\lambda)^m$ term to the right-hand side yields

$$\begin{aligned} |\lambda|^m &\leq M \sum_{k=1}^m (1 + |\xi|)^k |\lambda|^{m-k} \\ \left(\frac{|\lambda|}{1 + |\xi|}\right)^m &\leq M \sum_{k=1}^m \left(\frac{|\lambda|}{1 + |\xi|}\right)^{m-k}. \end{aligned}$$

Thus we must either have $|\lambda|/(1+|\xi|) < 1$, or $[|\lambda|/(1+|\xi|)]^m < Mm[|\lambda|/(1+|\xi|)]^{m-1}$, which is equivalent to $|\lambda|/(1 + |\xi|) < Mm$. Thus we reach the inequality

$$\frac{|\lambda|}{1 + |\xi|} < 1 + Mm. \tag{18}$$

Denote the m roots of $P(i\xi, i\lambda) = 0$ as $\lambda_k(\xi)$ for each $k = 1, \ldots, m$. Then for all $\lambda \in \mathbb{R}_{>0}$ we can write

$$P(i\xi, i\lambda) = i^m \prod_{k=1}^{m} (\lambda - \lambda_k(\xi)). \tag{19}$$

Consider the case where Γ is the boundary of the union of open disks of center λ_k and radius 1. Then Γ satisfies our conditions given in equation (16) and has a total length of less than or equal to $2m\pi$. Note that by construction, for any $\lambda \in \Gamma$, we have that $|\lambda - \lambda_k(\xi)| \geq 1$ for every $k = 1, \ldots, m$. Thus

$$|P(i\xi, i\lambda)| \geq 1. \tag{20}$$

Also since for each $\lambda \in \Gamma$ there exists k such that $|\lambda - \lambda_k(\xi)| = 1$, we can write

$$|\lambda| \leq 1 + (1 + Mm)(1 + |\xi|) \qquad \leq (2 + Mm)(1 + |\xi|).$$

From the Gårding hyperbolicity condition, there exists c such that $\text{Im}\lambda \geq -c - 1$, so for $t \geq 0$ and $\lambda \in \Gamma$,

$$|e^{i\lambda t}| \leq e^{(1+c)t}. \tag{21}$$

Thus differentiating under the integral sign in equation (16) and invoking equations (20 - 21) yields

$$\begin{aligned} \left| \frac{\partial^k}{\partial t^k} Z(\xi, t) \right| &= \left| \frac{1}{2\pi} \int \frac{(i\lambda)^k e^{i\lambda t}}{P(i\xi, i\lambda)} d\lambda \right| \\ &\leq m(2 + Mm)^k (1 + |\xi|)^k e^{(1+c)T} \end{aligned}$$

for $t \in [0, T]$, $\xi \in \mathbb{R}^n$, and $k = 0, 1, \ldots, m$. This follows the form of the inequality in (15), which completes the proof. □

Thus Gårding's hyperbolicity condition determines whether a hyperbolic partial differential equation is well-posed, i.e. it has a unique solution that depends continuously on the initial or boundary data. In geometric terms, Gårding's condition ensures that the primary component of the hyperbolic equation exhibits characteristic behavior similar to a hyperbolic equation. This implies that the speeds associated with the highest-order spatial derivatives remain uniform across the domain. This consistency is essential for establishing the well-defined nature of the hyperbolic problem, which in turn assures the existence and uniqueness of solutions.

4 Plane Waves

This section is based off of John Fritz's textbook pages 158-161 [1]. A function $F : \mathbb{R}^n \to \mathbb{R}$ is a **plane wave function** if it can be written in the form $F(x) = G(x \cdot \xi)$ for some $G : \mathbb{R}^n \to \mathbb{R}$ and $\xi \in \mathbb{R}^n$.

Throughout this section let us assume that $P(D, \tau)$ is homogeneous and has degree m where m is even. Additionally let us assume that the coefficient of τ^m is 1. Notice that, by the chain rule,

$$\frac{\partial^{k_1}}{\partial x_1^{k_1}} \cdots \frac{\partial^{k_n}}{\partial x_n^{k_n}} \tau^{k_\tau} G(x \cdot \xi + \lambda t) = \xi_1^{k_1} \cdots \xi_n^{k_n} \lambda^{k_\tau} G^{(k_1 + \cdots + k_n + k_\tau)}$$

Thus, as P is homogeneous,

$$P(D, \tau) G(x \cdot \xi + \lambda t) = P(\xi, \lambda) G^{(m)}(x \cdot \xi + \lambda t)$$

We can see that $u = G(x \cdot \xi + \lambda t)$ solves $P(D, \tau)u = 0$ when $P(\xi, \lambda) = 0$.

We will now use this to try to find a solution to $P(D,\tau)u = 0$ with initial conditions at $\tau^k = 0$ at $t = 0$ for $k = 0, 1, \ldots, m-2$ and $\tau^{m-1} = g$, where $g = g(x \cdot \xi)$ is a plane wave function. Also assume $\xi \neq 0$, and P is strictly hyperbolic, i.e. every root of P is real and distinct when $\xi \neq 0$. Let $G \in C^m(\mathbb{R})$ such that $G^{(m-1)}(s) = g(s)$. Denote $\frac{\partial}{\partial\lambda}P(\xi,\lambda)$ as $P_\lambda(\xi,\lambda)$.

We will show that

$$u(x,t) = \sum_{k=1}^{m} \frac{G(x\cdot\xi + \lambda_k t)}{P_\lambda(\xi,\lambda_k)}$$

is a solution.

This paper will not be going into the specifics of residues, singularities and poles, or even define them, but we will be stealing some information relating to them. Firstly that for any $j \in \mathbb{N}$, $\frac{\lambda^j}{P(\xi,\lambda)}$ as a function of λ over $\mathbb{C}$ will have isolated singularities precisely at $\lambda_1, ..., \lambda_m$, and that $\lambda_1, ..., \lambda_m$ are simple zeros of $\frac{1}{P(\xi,\lambda)}$. Residue theory tells us that

$$\frac{\lambda_k^j}{P_\lambda(\xi,\lambda_k)} = \text{Residue of } \frac{\lambda^j}{P(\xi,\lambda)} \text{ at } \lambda_k \tag{22}$$

All of this information used about residues is based off Topics 7-8 in the lecture notes of Orloff's course [4]. We will also be using Cauchy's residue theorem which gives us that, for a function f that is holomorphic on $\mathbb{C}$ except at some isolated singularities, $\int_C f(z)dz = (2\pi i\Sigma$ residues of f at the singularities of f inside C), where C is a simple closed curve over the complex plane that is oriented counterclockwise that does not go through any singularities of f.

Let j $\in \{0, ..., m-2\}$. In this case, at t = 0, we can write

$$\tau^j u = G^{(j)}(x*\xi)\Sigma_{k=1}^m \frac{\lambda_k^j}{P_\lambda(\xi,\lambda_k)}$$

Cauchy's residue theorem tells us that for sufficiently large r with $B_r(0) \subset \mathbb{C}$ containing $\lambda_1, ..., \lambda_k$, we have

$$\tau^j u = G^{(j)}(x*\xi) * \frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{\lambda^j}{P(\xi,\lambda)}d\lambda$$

by using (22). Looking at the limit as r goes to infinity makes it apparent that $\tau^j u = 0$. Finally, at $t = 0$,

$$\tau^{m-1}u = g\Sigma_{k=1}^m \frac{\lambda_k^{m-1}}{P_\lambda(\xi,\lambda_k)}$$

which by the residue theorem gives us that, for r sufficiently large,

$$\tau^{m-1}u = g\frac{1}{2\pi i}\int_{\partial B_r(0)} \frac{\lambda^{m-1}}{P(\xi,\lambda)}d\lambda$$

Since the coefficient of λ^m in $P(\xi,\lambda)$ is 1, by taking the limit as $r \to \infty$ we can see that $\tau^{m-1}u = g$ at $t = 0$ as desired. Thus u has our desired properties.

5 Numerical Methods for the Wave Equation

Modern PDE solvers either tend to implement numerical methods such as the finite differences method (FDM), finite elements method (FEM), multi-grid method, spectral method, or a neural network-based formulation of the above methods. In either case, it is computationally efficient and mathematically more

digestible to write these methods in terms of convolution over the discrete grid over which the PDE is being solved so that our operations can be implemented in the Fourier (frequency) domain.

When dealing with hyperbolic equations, Fourier transforms enable the transformation of the problem from the spatial domain (physical space) to the frequency domain (fourier space), facilitating the analysis and solution process [3]. Notedly, in numerical methods, this change is done to reduce computational cost from $O(N^2)$ operations in the physical space to $O(N \log_2 N)$ operations in the frequency space [3].

The N referenced here represents the size of the spatial domain or the number of grid points used in numerical methods. So, for example, if you're discretizing a one-dimensional spatial domain into N grid points, then N represents the number of grid points along that dimension. The computational complexity is often expressed in terms of N.

Fourier transforms decompose a function into its constituent frequencies, allowing us to represent complex functions as a sum of simpler sinusoidal components. This transformation is useful when dealing with hyperbolic equations because it converts differential equations involving spatial derivatives into algebraic equations involving the transformed variables, typically frequency or wave number denoted by some k. This enables us to transform the problem into a more manageable form and obtain insights into the behavior of waves and disturbances described by the equations. In this section we will go over two notable method types, the finite difference method and the spectral method, and show their Fourier formulations.

5.1 The Finite Difference Method

The finite differences method (FDM) is a common method of solving PDEs which approximates derivatives with differences, yielding a system of discrete linear equations that can then be solved algebraically.

For a candidate solution $u(x,t)$ to some PDE, the x derivative in FDM is approximated using the derivative definition at some point $(x,t) \in (\mathbb{R} \times [0,\infty))$ as follows:

$$\frac{du}{dx} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$

And the approximation of the second derivative (derived using Taylor expansion of the first derivative):

$$\frac{d^2u}{dx^2} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}$$

And when generalized to higher spatial dimensions, the first and second order derivatives become the gradient and Laplacian operators respectively, and these values at a certain (x,t) point would be a sum of the differential-shifted function values (in the form of $u(x+\Delta x,t)$) for all neighbors that are Δx distance away from the (x,t) point in one of the available dimensions. For example, for a given Δx and Δt where t is plotted as another spatial dimension, one can write the following Laplacian:

$$\Delta u(x,t) = \frac{u(x+\Delta x,t) + u(x-\Delta x,t) - 4u(x,t) + u(x,t+\Delta t) + u(x,t-\Delta t)}{(\Delta x)^2(\Delta t)^2}$$

Writing this derivative approximation in this per-point way as a matrix (in $\mathbb{R} \times \mathbb{R}_+$) or a tensor (in $\mathbb{R}^n$) creates what is known as the differentiation stencil, which is convolved over the output space of u. The **convolution** of two functions $f, g : \Omega \to \mathbb{R}$ is given by

$$(f \circledast g)(x) = \int_\Omega f(y)g(x-y)dy$$

In essence, convolution is an accumulation functional that acts as a localized inner product between two functions centered on a point (x), which is exactly what our differentiation stencils are trying to accomplish.

Then we can write this convolution (assume $\mathbb{R} \times \mathbb{R}_+$ case) as an infinite-sized diagonal-constant Toeplitz matrix defined as follows: $TA = A \circledast S$ for any matrix A of appropriate size, and

$$T = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & 0 & -1 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & -1 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & -1 & 1 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for a forward first-difference matrix.
For center-difference matrices or higher-dimension matrices, the "edge" values between -1 and 1 are scaled differently, and can be arranged differently to find higher-order features such as crosses/circles (2D) or spheres (n-D), etc. Performing this computation for every point in the discretized Ω yields a system of linear equations that can be solved algebraically or using other numerical optimization techniques.
Then,

$$\frac{\partial u}{\partial x} = U \circledast S = TU$$

Where we define the stencil S centered on point x as $[u(x - \Delta x), -u(x), u(x + \Delta x)]$, and that centered on (x, t) as:

$$\begin{bmatrix} 0 & u(x, t + \Delta t) & 0 \\ u(x - \Delta x, t) & u(x, t) & u(x + \Delta x, t) \\ 0 & u(x, t - \Delta t) & 0 \end{bmatrix}$$

(for first-order differences and only immediate neighbors, this stencil differentiates in the shape of a cross)
And $U_{x,t} = u(x, t)$

We can also work some algebra (Thm. 3.1) to show that convolving two functions in a spatial/temporal dimension is equivalent to taking the inverse Fourier transform of the multiplication of the two functions' Fourier transforms. Let $\mathbb{F}$ be the 1D Discrete Fourier Transform (DFT) matrix (generalizable to n-D as a tensor multiply, but n-D DFT matrices are separable, meaning one can write an n-D DFT as a composition of n 1D DFTs on each axis) with the dimensions equal to that of the number of input grid points: Let $\omega = e^{i2\pi/N}$ where N is the length of the domain of the function we are Fourier transforming (in 1D) measure of area over the domain Ω, in terms of discretized grid points. Let f be such a function. Then,

$$\mathbb{F} = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & ... \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & ... \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & ... \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & ... \\ \omega^0 & \omega^4 & \omega^8 & \omega^1 2 & ... \\ ... & & & & \end{bmatrix}$$

To do that, since our convolution operation is already defined in matrix-multiplication form, we simply multiply the appropriate DFT matrices to the equation as follows:

$$\frac{du}{dx} \approx \mathbb{F}^{-1}(\mathbb{F}T)(\mathbb{F}U)$$

where in 1D, $\mathbb{F}T$ in Fourier space is approximating a multiplication by i onto $\mathbb{F}U$, whereas in higher dimensions it would approximate some other kind of rotation. As such, we will keep the above formulation for generality.

Formulating this extension for systems where $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}_+$:

$$\nabla_x U \approx \sum_i^n \prod_j^n (\mathbb{F}_j^{-1})(\mathbb{F}_j T_{ij})(\mathbb{F}_j U)$$

Where i represents the dimension along which we are differentiating, and j represents the dimension along which we are Fourier transforming.
Using the separability property of the DFT matrix mentioned above, we can apply the DFT matrix to each dimension (that being $n \times n$ T_j and U_j) n times then summing up the derivative along each dimension.

FDM, when formulated this way, (according to the Sampling Theorem, which we will not be covering in this paper) creates a basis that is local in space, and thus is local and periodic in frequency. Since hyperbolic equations' solution spaces are often smooth, we can use the same formulation, but using a basis whose atoms are instead local and non-periodic in frequency, resulting in smooth global basis functions in space. A common basis for this is the Fourier basis, which would become the base for spectral methods. In that case, our formulation of $\mathbb{F}_j T_{ij}$ would simply become a diagonal matrix whose nonzero entries are the eigenvalues of the Fourier eigenfunctions.

5.2 The Spectral Method

The spectral method is a numerical technique used for solving differential equations. It accomplishes this task by representing the solution to the differential equation as a weighted combination of basis functions, also known as trial functions. These basis functions are carefully chosen to possess desirable mathematical properties, such as orthogonality, smoothness, and global support, making them well-suited for accurately approximating the behavior of the solution across the entire domain of interest.

Unlike finite difference or finite element methods, which typically use local approximations over small regions of the domain, the spectral method considers the entire domain as a unified entity. This method is particularly great for problems characterized by smooth solutions and regular domains, like the wave equation or heat equation.

Spectral methods are categorized within the weighted residual methods (WRMs) framework. WRMs, a cornerstone in various numerical methodologies like finite element, spectral, finite volume, and boundary element methods, encapsulate a distinct set of approximation [3]. These methods aim to minimize residuals or errors in a defined manner, giving rise to specific approaches such as Galerkin, Petrov-Galerkin, collocation, and tau formulations.

When understanding WRMs, we consider the general problem

$$\partial_t u(x,t) - \mathcal{L}u(x,t) = \mathcal{N}(u)(x,t), \quad t > 0, \quad x \in \Omega, \tag{23}$$

where $\mathcal{L}$ is a leading spatial derivative operator, and $\mathcal{N}$ is a lower-order linear or nonlinear operator involving only spatial derivatives [1]. In this definition Ω denotes a bounded domain of $\mathbb{R}^d$, where $d = 1, 2, 3$. Equation 23 requires an initial condition along with appropriate boundary conditions for completeness.

In this section, we consider the one-dimensional wave equation for a domain extending from 0 to 2π and a wave speed $c = 1$. The wave equation is a hyperbolic PDE that describes how waves propagate through a medium. Its conditions are specified as follows:

$$u_{tt} = c^2 u_{xx}, \quad 0 \le x \le 2\pi$$
$$u(x,0) = \sin(x), \quad u(x,t) = \sin(x - ct)$$

The initial step in the WRM involves approximating the solution of our chosen wave equation, denoted by $u(x)$, of equation 23 by a finite sum

$$u(x) \approx u_N(x) = \sum_{k=0}^{N} a_k \phi_k(x) \tag{24}$$

where ϕ_k are the basis functions and a_k are the to be determined expansion coefficients. Because we will be solving the wave equation, the basis function we will use shall be the Fourier basis functions $\phi_k(x) = e^{ikx}$, therefore implying that we will be using the Fourier Spectral Method.

Moving forward, differentiation can be done either in the physical space or in the frequency space when using the Spectral Method. Differentiation in the physical space leads to an enormous matrix multiplication procedure (see above section about convolution) that requires $O(N^2)$ operations. This method is quite long and detailed for a section in this paper, so we refer you to the textbook to see further computation in that space [1] . In the frequency space, this can be performed with $O(N \log_2 N)$ operations using the Fast Fourier Transform in any coding language. Therefore, we transfer to the frequency, or Fourier, space using a finite Fourier series.

Choose N to be any power of two. Given any function $u(x)$ with a periodic domain, its finite Fourier series can be expressed as:

$$u(x) = \sum_{k=-N/2}^{N/2} \hat{u}_k e^{ikx}, \quad \hat{u}_{N/2} = \hat{u}_{-N/2} \tag{25}$$

where $\hat{u}_k$ denotes the k^{th} Fourier coefficient. The n^{th} derivative is computed as follows:

$$u^n(x_j) = \sum_{k=-N/2}^{(N/2)-1} (ik)^n \cdot \hat{u}_k e^{ikx_j} \tag{26}$$

where k is the wave number vector given by $k = (\frac{-N}{2}, ..., \frac{N}{2} - 1)$. So, when we move into the frequency space using the Fast Fourier Transform in our code, we initiate the wave number vector, compute the multiplicative factor $(ik)^n$, and then use it to multiply the frequency vector $\hat{u} = (\hat{u}_1, ..., \hat{u}_N)$.

Now that we understand the basics, here are the steps we took to solving the 1D wave equation using the Fourier Spectral Method:

1. Discretize the spatial domain and evaluate the initial condition at the discrete points.

```
x = linspace(0, L, N+1);
u0val = u0(x(1:end-1));
```

2. Use the Fast Fourier Transform to transform the initial condition into the frequency space. Note: use **fft** for a 1D problem and **fft2** for a 2D problem.

```
u0_hat = fft(u0val);
```

3. In the frequency space, compute the multiplicative factors of the derivatives using the wave number vector $i * k$.

```
k = (-N/2:(N/2)-1) * (2*pi/L);
k = fftshift(k);
```

[1] The computation of this can be seen in chapter 2 of textbook [3].

4. Still in the frequency space, the Fourier Spectral method computes the temporal evolution by multiplying the Fourier coefficients $\hat{u}(k,t)$ with the exponential of the wave number vector scaled by the wave speed and time step, $e^{-ick\Delta t}$, effectively applying the wave equation's differential operator.

```
u_hat = exp(-1i*c*k*dt) .* u0_hat;
```

5. Next, use a time-stepping scheme to obtain the solution at the next time level. For our code we adopt the Exponential Time Differencing scheme. Finally, use the Inverse Fast Fourier Transform to move out of the Frequency space. The solution at this point becomes the initial condition and this process is repeated until the final time is reached.

```
u_hat = exp(-1i*c*k*dt) .* u0_hat;
u0val = real(ifft(u_hat));
```

After following these steps and setting up some initial parameters, we analyze the results to evaluate the effectiveness of the Fourier Spectral Method in solving the 1D wave equation.

Error Analysis: We calculate the L^2 norm of the difference between the numerical solution and the exact analytical solution at the final time, providing a quantitative measure of the method's accuracy. This error metric helps us assess convergence properties as we vary the number of spatial discretization points N.

```
errs_spectral(j) = sqrt(h * sum((us - u_tval).^2));
```

Graphical Results: By plotting both the numerical and exact solutions, we visually compare the wave profiles. This visualization is crucial for understanding how well the numerical method captures the dynamics of the wave equation.

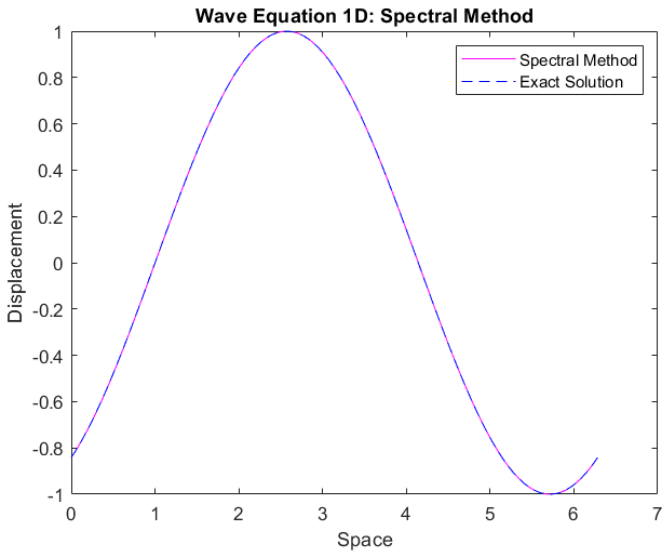

Figure 1: Fourier Spectral Method

Convergence Study: Finally, we systematically vary the number of discretization points and observe the impact on the accuracy and computational cost. This study helps to find an optimal balance between computational efficiency and solution accuracy.

Spatial Discretization (N)	Spectral Method Error
20	0.00089906
40	0.00089267
60	0.00089053
80	0.00088945
100	0.00088881

Figure 2: Table of Convergence

The results from the Table of Convergence clearly demonstrate the high accuracy and effectiveness of the Fourier Spectral Method in solving the 1D wave equation. The errors are exceptionally small across all spatial discretizations, indicating that even at lower resolutions (e.g., $N = 20$), the spectral method provides a very accurate approximation of the exact solution. The slight decrease in error as the number of discretization points increases suggests that the method converges towards the exact solution as the spatial resolution improves. Additionally, the close alignment of the spectral and exact solution in the graphical representation further confirms the method's capability to accurately capture the wave behavior over the domain.

6 Conclusion

Hyperbolic equations, especially its poster child the wave equation, exhibit characteristics that lend themselves naturally to Fourier solutions, such as the non-attenuating and propagating wavelike nature of the derivatives. As such, this paper went over analytic solutions with similar intuition to Fourier, such as the reduction of a hyperbolic equation into simpler PDEs, then looking at analytic Fourier solutions and superposition of plane waves. Then, with that intuition in mind, we see how we can leverage the power of numerical optimization to solve these PDEs while showing that their formulation still remains faithful to this intuition of "decompose into constituent waves, solve for each and add together."

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