Beam Deflection Problems

1 Beam Deflection Problems

1.1 Cantilever Beam Under Uniform Load

Differential Equation:

$$EI\frac{d^4w}{dx^4} = -q \tag{1}$$

Boundary Conditions:

- $w(0) = 0$ (No deflection at fixed end)

- $w'(0) = 0$ (No slope at fixed end)

- $w''(L) = 0$ (No moment at free end)

- $w'''(L) = 0$ (No shear at free end)

Analytical Solution:

$$w(x) = \frac{q}{24EI}\left(x^4 - 4Lx^3 + 6L^2x^2\right) \tag{2}$$

Maximum Deflection:

$$w_{\max} = \frac{qL^4}{8EI} \quad \text{(at free end)} \tag{3}$$

1.2 Fully Restrained Beam Under Uniform Load

Figure 2: Fully restrained beam with uniform load q

Differential Equation:

$$EI\frac{d^4w}{dx^4} = -q \tag{4}$$

Boundary Conditions:

- $w(0) = w(L) = 0$

- $w'(0) = w'(L) = 0$

Analytical Solution:

$$w(x) = \frac{q}{24EI}\left(x^4 - 2Lx^3 + L^3x\right) \tag{5}$$

Maximum Deflection:

$$w_{\max} = \frac{qL^4}{384EI} \quad \text{(at mid-span)} \tag{6}$$

1.3   Fully Restrained Beam Under Mid-Span Point Load

Differential Equation:

$$EI\frac{d^4w}{dx^4} = -P\delta\left(x - \frac{L}{2}\right) \tag{7}$$

Boundary Conditions:

- $w(0) = w(L) = 0$

- $w'(0) = w'(L) = 0$

Analytical Solution (Piecewise):

$$w(x) = \begin{cases} \dfrac{P}{48EI}\left(3Lx^2 - 4x^3\right) & 0 \leq x \leq \dfrac{L}{2} \\ \dfrac{P}{48EI}\left(3Lx^2 - 4x^3 + L^3 - 6L^2\left(x - \dfrac{L}{2}\right)\right) & \dfrac{L}{2} \leq x \leq L \end{cases} \tag{8}$$

Maximum Deflection:

$$w_{\max} = \frac{PL^3}{192EI} \quad \text{(at mid-span)} \tag{9}$$

Figure 3: Fully restrained beam with point load P at mid-span

Notation

- E: Young's modulus (Pa)

- I: Moment of inertia (m^4)

- L: Beam length (m)

- q: Uniform load intensity (N/m)

- P: Point load magnitude (N)

Image Sources: Replace `cantilever_uniform_load.png`, `fixed_fixed_uniform.png`, and `fixed_fixed_point.png` with actual image files

Reference: Beam diagrams generated using SkyCiv Beam Calculator