Cotorsion pairs in the category of N-complexes

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1) N-complexes:

Let $\mathcal{A}$ be an abelian category and fix a positive integer $N \geq 2$.

An N-complex $X = (X^i, d_X^i)$ is a diagram

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \xrightarrow{d_X^{n+1}} \cdots$$

With $X^i \in \mathcal{A}$ and $d_X^{i+N-1} \cdots d_X^{i+1} d_X^i = 0$ for any $i \in \mathbb{Z}$.

For $0 \leq r < N$ and $i \in \mathbb{Z}$, we often denote

$$d^i_{X,\{r\}} := d_X^{i+r-1} \cdots d_X^i$$

In this notation $d^i_{X,\{1\}} = d_X^i$ and $d^i_{X,\{0\}} = 1_{X_i}$.

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We denote by $\mathbb{C}_N(\mathcal{A})$ the category of unbounded N-complexes over $\mathcal{A}$.

Let $\mathcal{S}_N(\mathcal{A})$ be the collection of short exact sequence in $\mathbb{C}_N(\mathcal{A})$ of which each term is split exact.

- $(\mathbb{C}_N(\mathcal{A}), \mathcal{S}_N(\mathcal{A}))$ is an exact category.
- $(\mathbb{C}_N(\mathcal{A}), \mathcal{S}_N(\mathcal{A}))$ is a Frobenius category.

Indeed,
For any object M of $\mathcal{A}$, $j \in \mathbb{Z}$ and $1 \leq i \leq N$, let

$$D_i^j(M) : \cdots \longrightarrow 0 \longrightarrow X^{j-i+1} \xrightarrow{d_X^{j-i+1}} \cdots \xrightarrow{d_X^{j-2}} X^{j-1} \xrightarrow{d_X^{j-1}} X^j \longrightarrow 0$$

be an N-complex satisfying $X^n = M$ for all $j - i + 1 \leq n \leq j$ and $d_X^n = 1_M$ for all $j - i + 1 \leq n \leq j - 1$.

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- $D_N^i(M)$ is an $\mathcal{S}_N$-projective and $\mathcal{S}_N$-injective object of $(\mathbb{C}_N(\mathcal{A}), \mathcal{S}_N(\mathcal{A}))$.
- Let $X \in \mathbb{C}_N(\mathcal{A})$ be given. Then we have the following exact sequences in $\mathcal{S}_N(\mathcal{A})$

$$0 \longrightarrow \mathrm{Ker}\rho_X \xrightarrow{\varepsilon_X} \bigoplus_{n\in\mathbb{Z}} D_N^n(X^{n-N+1}) \xrightarrow{\rho_X} X \longrightarrow 0$$

and

$$0 \longrightarrow X \xrightarrow{\lambda_X} \bigoplus_{n\in\mathbb{Z}} D_N^n(X^n) \xrightarrow{\eta_X} \mathrm{Cok}\lambda_X \longrightarrow 0$$

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The homotopy category of N-complexes:

A morphism $f : X \longrightarrow Y$ of N-complexes is called null-homotopic if there exists $s^i \in \mathrm{Hom}_{\mathcal{A}}(X^i, Y^{i-N+1})$ such that

$$f^i = \sum_{j=0}^{N-1} d_{Y,\{N-1-j\}}^{i-(N-1-j)} s^{i+j} d_{X,\{j\}}^{i}$$

For morphisms $f, g : X \longrightarrow Y$ in $\mathbb{C}_N(\mathcal{A})$, we denote $f \sim g$ if $f - g$ is null-homotopic.

$\mathbb{K}_N(\mathcal{A})$: The homotopy category of unbounded N-complexes.

- $Obj(\mathbb{K}_N(\mathcal{A})) = Obj(\mathbb{C}_N(\mathcal{A}))$
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Theorem [IKM]

The stable category of the frobenius category $(\mathbb{C}_N(\mathcal{A}), \mathcal{S}_N(\mathcal{A}))$ is the homotopy category $\mathbb{K}_N(\mathcal{A})$ of $\mathcal{A}$.

Therefore,

- $\mathbb{K}_N(\mathcal{A})$ is a triangulated category.

What is the suspension functor of $\mathbb{K}_N(\mathcal{A})$?

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Define functors $\Sigma, \Sigma^{-1} : \mathbb{C}_N(\mathcal{A}) \longrightarrow \mathbb{C}_N(\mathcal{A})$ by

$$\Sigma^{-1}X = \mathrm{Ker}\rho_X \qquad \text{and} \qquad \Sigma X = \mathrm{Cok}\lambda_X$$

$$(\Sigma X)^m = \bigoplus_{i=m+1}^{m+N-1} X^i, \qquad d^m_{\Sigma X} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -d^{m+1}_{\{N-1\}} & -d^{m+2}_{\{N-2\}} & \cdots & \cdots & \cdot & -d^{m+N-1} \end{bmatrix}$$

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2)Derived category of N-complexes

Let X be an N-complex in $\mathcal{A}$

For $0 \leq r \leq N$ and $i \in \mathbb{Z}$ define

$$\mathrm{Z}^i_r(X) := \mathrm{Ker} d^i_{X,\{r\}} \quad , \quad \mathrm{B}^i_r(X) := \mathrm{Im} d^{i-r}_{X,\{r\}}$$

$$\mathrm{H}^i_r(X) := \mathrm{Z}^i_r(X)/\mathrm{B}^i_{N-r}(X).$$

For example, $\mathrm{Z}^n_N(X) = \mathrm{B}^n_0(X) = X^n$ and $\mathrm{Z}^n_0(X) = \mathrm{B}^n_N(X) = 0$

we can understand a homology as follows

$$\mathrm{H}^n_r(X) = \mathrm{Cok}(\mathrm{Z}^{n-N+r}_N(X) \xrightarrow{d^{n-N+r}_N} \cdots \xrightarrow{d^{n-2}_{r+2}} \mathrm{Z}^{n-1}_{r+1}(X) \xrightarrow{d^{n-1}_{r+1}} \mathrm{Z}^n_r(X))$$

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$$\mathrm{H}^n_r(X) = \mathrm{Cok}(\mathrm{Z}^{n-N+r}_N(X) \xrightarrow{d^{n-N+r}_N} \cdots \xrightarrow{d^{n-2}_{r+2}} \mathrm{Z}^{n-1}_{r+1}(X) \xrightarrow{d^{n-1}_{r+1}} \mathrm{Z}^n_r(X))$$

2)Derived category of N-complexes

Let X be an N-complex in $\mathcal{A}$
For $0 \leq r \leq N$ and $i \in \mathbb{Z}$ define

$$\mathrm{Z}^i_r(X) := \mathrm{Ker} d^i_{X,\{r\}} \quad , \quad \mathrm{B}^i_r(X) := \mathrm{Im} d^{i-r}_{X,\{r\}}$$

$$\mathrm{H}^i_r(X) := \mathrm{Z}^i_r(X)/\mathrm{B}^i_{N-r}(X).$$

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Definition:

Let $X \in \mathbb{K}_N(\mathcal{A})$. We say X is N-exact if $\mathrm{H}_r^i(X) = 0$ for each $i \in \mathbb{Z}$ and all $r = 1, 2, ..., N-1$. We denote the full subcategory of $\mathbb{K}_N(\mathcal{A})$ consisting of N-exact complexes by $\mathbb{K}_N^{\mathrm{ac}}(\mathcal{A})$.

Definition:

A morphism $f : X \longrightarrow Y$ of $\mathbb{K}_N(\mathcal{A})$ is called an N-quasi-isomorphism if $\mathrm{H}_r^i(f) : \mathrm{H}_r^i(X) \longrightarrow \mathrm{H}_r^i(Y)$ is an isomorphism for any $0 < r < N$ and $i \in \mathbb{Z}$.

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3) Cotorsion Pairs

Definition

A pair $(\mathcal{F}, \mathcal{C})$ of classes of object of $\mathcal{A}$ is said to be a cotorsion pair if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$, where the left and right orthogonals are defined as follows

$$^{\perp}\mathcal{C} := \{A \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(A, Y) = 0, \text{ for all } Y \in \mathcal{C}\}$$

and

$$\mathcal{F}^{\perp} := \{A \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(W, A) = 0, \text{ for all } W \in \mathcal{F}\}.$$

3) Cotorsion Pairs

A cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called complete if for every $A \in \mathcal{A}$ there exist exact sequences

$$0 \to Y \to W \to A \to 0 \quad \text{and} \quad 0 \to A \to Y' \to W' \to 0,$$

where $W, W' \in \mathcal{F}$ and $Y, Y' \in \mathcal{C}$.

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where $W, W' \in \mathcal{F}$ and $Y, Y' \in \mathcal{C}$.

If there is a set $\mathcal{S}$ of objects of $\mathcal{A}$ such that $\mathcal{S}^{\perp} = \mathcal{C}$ for some cotorsion pair $(\mathcal{F}, \mathcal{C})$, then the pair is said to be cogenerated by a set.

4) Cotorsion pair in $\mathbb{C}_N(\mathcal{A})$

Definition:

Let $\mathcal{A}$ be an abelian category. Given two classes of objects $\mathcal{X}$ and $\mathcal{U}$ in $\mathcal{A}$ with $\mathcal{U} \subseteq \mathcal{X}$. We denote by $\widetilde{\mathcal{U}}_{\mathcal{X}_N}$ the class of all N-exact complexes U with each degree $U^i \in \mathcal{U}$ and each cycle $\mathrm{Z}^i_r(U) \in \mathcal{X}$ for all $1 \leq r \leq N-1$ and $i \in \mathbb{Z}$.

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Proposition:

Let $\mathcal{A}$ be an abelian category with injective cogenerator J. Let $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{X}, \mathcal{Y})$ be two cotorsion pairs with $\mathcal{U} \subseteq \mathcal{X}$ in $\mathcal{A}$. Then $(\widetilde{\mathcal{U}}_{\mathcal{X}_N}, (\widetilde{\mathcal{U}}_{\mathcal{X}_N})^{\perp})$ is a cotorsion pair in $\mathbb{C}_N(\mathcal{A})$ and $(\widetilde{\mathcal{U}}_{\mathcal{X}_N})^{\perp}$ is the class of all N-complexes V for which each $V^i \in \mathcal{V}$ and for each map $U \to V$ is null-homotopic whenever $U \in \widetilde{\mathcal{U}}_{\mathcal{X}_N}$.

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Proposition:

Let $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{X}, \mathcal{Y})$ be two cotorsion pairs with $\mathcal{U} \subseteq \mathcal{X}$ in $\mathcal{A}$. Then $({}^{\perp}(\widetilde{\mathcal{Y}}_{\mathcal{V}_N}), \widetilde{\mathcal{Y}}_{\mathcal{V}_N})$ is a cotorsion pair in $\mathbb{C}_N(\mathcal{A})$ and ${}^{\perp}(\widetilde{\mathcal{Y}}_{\mathcal{V}_N})$ is the class of all N-complexes X for which each $X^i \in \mathcal{X}$ and for each map $X \to Y$ is null-homotopic whenever $Y \in \widetilde{\mathcal{Y}}_{\mathcal{V}_N}$.

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in $\mathcal{A}$ and consider the following subclasses of $\mathbb{C}_N(\mathcal{A})$

(1) $C_N(\mathcal{F}) = \{X \in \mathbb{C}_N(\mathcal{A}) \,|\, X^i \in \mathcal{F}\}$

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(5) $\mathrm{dg}\widetilde{\mathcal{F}}_N = \{X \in \mathbb{C}_N(\mathcal{F}) \,|\, \mathrm{Hom}_{\mathbb{K}_N(\mathcal{A})}(X, C) = 0 \;, \forall C \in \widetilde{\mathcal{C}}_N\}$

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- If we consider $(\mathcal{U},\mathcal{V}) = (\mathcal{F},\mathcal{C})$ then clearly $\widetilde{\mathcal{F}}_{\mathcal{F}_N} = \widetilde{\mathcal{F}}_N$ and $(\widetilde{\mathcal{F}}_{\mathcal{F}_N})^{\perp} = dg\widetilde{\mathcal{C}}_N$, so $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ is a cotorsion pairs.
- Dually $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$ is a cotorsion pair.
- If we set $(\mathcal{U},\mathcal{V}) = (\mathcal{A},\mathcal{I})$ (the usual injective cotorsion pair in $\mathcal{A}$), then $\mathcal{F} \subseteq \mathcal{A}$, therefore $\widetilde{\mathcal{F}}_{\mathcal{A}_N} = \mathrm{ex}\widetilde{\mathcal{F}}_N$, so $(\mathrm{ex}\widetilde{\mathcal{F}}_N, (\mathrm{ex}\widetilde{\mathcal{F}}_N)^{\perp})$ is a cotorsion pair.
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Proposition:

Let $(\mathcal{F},\mathcal{C})$ be a cotorsion pair in $\mathcal{A}$. Then $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ and $({}^{\perp}\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$ are cotorsion pairs in $\mathbb{C}_N(\mathcal{A})$.

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- If we consider $(\mathcal{U},\mathcal{V}) = (\mathcal{F},\mathcal{C})$ then clearly $\widetilde{\mathcal{F}}_{\mathcal{F}_N} = \widetilde{\mathcal{F}}_N$ and $(\widetilde{\mathcal{F}}_{\mathcal{F}_N})^{\perp} = dg\widetilde{\mathcal{C}}_N$, so $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ is a cotorsion pairs.
- Dually $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$ is a cotorsion pair.
- If we set $(\mathcal{U},\mathcal{V}) = (\mathcal{A},\mathcal{I})$ (the usual injective cotorsion pair in $\mathcal{A}$), then $\mathcal{F} \subseteq \mathcal{A}$, therefore $\widetilde{\mathcal{F}}_{\mathcal{A}_N} = \mathrm{ex}\widetilde{\mathcal{F}}_N$, so $(\mathrm{ex}\widetilde{\mathcal{F}}_N, (\mathrm{ex}\widetilde{\mathcal{F}}_N)^{\perp})$ is a cotorsion pair.
- Dually $({}^{\perp}(\mathrm{ex}\widetilde{\mathcal{C}}_N), \mathrm{ex}\widetilde{\mathcal{C}}_N)$ is a cotorsion pair.

Proposition:

Let $(\mathcal{F},\mathcal{C})$ be a cotorsion pair in $\mathcal{A}$. Then $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ and $({}^{\perp}\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$ are cotorsion pairs in $\mathbb{C}_N(\mathcal{A})$.

Main Theorem:

Let $\mathcal{G}$ be a Grothendieck category endowed with a faithful functor $U : \mathcal{G} \to \mathbf{Set}$,
We will also assume that there exists an infinite regular cardinal λ such that for each $G \in \mathcal{G}$ and any set $S \subseteq G$ with $|S| < \lambda$, there is a subobject $X \subseteq G$ such that $S \subseteq X \subseteq G$ and $|X| < \lambda$.

Theorem:

Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion pair in $\mathcal{G}$ cogenerated by a set such that $\mathcal{F}$ contains a generator G of $\mathcal{G}$. Then the induced pairs

(1) $(\widetilde{\mathcal{F}}_N, dg\widetilde{\mathcal{C}}_N)$ and $(dg\widetilde{\mathcal{F}}_N, \widetilde{\mathcal{C}}_N)$

(2) $(\mathbb{C}_N(\mathcal{F}), \mathbb{C}_N(\mathcal{F})^{\perp})$ and $({}^{\perp}\mathbb{C}_N(\mathcal{C}), \mathbb{C}_N(\mathcal{C}))$

(3) $(ex\widetilde{\mathcal{F}}_N, (ex\widetilde{\mathcal{F}}_N)^{\perp})$ and $({}^{\perp}(ex\widetilde{\mathcal{C}}_N), ex\widetilde{\mathcal{C}}_N)$

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5) Applications

Proposition:

If $(\mathcal{F}, \mathcal{C})$ is a complete cotorsion pair in $\mathbb{C}_N(\mathcal{A})$ and if $\mathcal{F}$ is closed under taking suspensions, then the embeddings $\mathbb{K}_N(\mathcal{F}) \to \mathbb{K}_N(\mathcal{A})$ and $\mathbb{K}_N(\mathcal{C}) \to \mathbb{K}_N(\mathcal{A})$ have right and left adjoints respectively.

- Consider the complete cotorsion pair $(\mathrm{Prj}\text{-}R, \mathrm{Mod}\text{-}R)$.
- Then we have $(\mathbb{C}_N(\mathrm{Prj}\text{-}R), \mathbb{C}_N(\mathrm{Prj}\text{-}R)^{\perp})$ in $\mathbb{C}_N(R)$.
- $\mathbb{K}_N(\mathrm{Prj}\text{-}R) \to \mathbb{K}_N(\mathrm{Mod}\text{-}R)$ has right adjoint functor $j : \mathbb{K}_N(\mathrm{Mod}\text{-}R) \to \mathbb{K}_N(\mathrm{Prj}\text{-}R)$.
- Then the natural inclusion $j_! : \mathbb{K}_N(\mathrm{Prj}\text{-}R) \to \mathbb{K}_N(\mathrm{Flat}\text{-}R)$ has a right adjoint $j^* = j|_{\mathbb{K}(\mathrm{Flat}\text{-}R)}$. ($N$-complex version of Neeman result [Neeman1, Proposition 8.1])

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By general theory on Bousfield localization we can say that the right adjoint functor $j^* : \mathbb{K}_N(\mathrm{Flat}\text{-}R) \to \mathbb{K}_N(\mathrm{Prj}\text{-}R)$ is a Verdier quotient.

In fact this adjoint functor identifies $\mathbb{K}_N(\mathrm{Prj}\text{-}R)$ with the Verdier quotient map

$$\mathbb{K}_N(\mathrm{Prj}\text{-}R) \to \mathbb{K}_N(\mathrm{Flat}\text{-}R)/\mathbb{K}_N(\mathrm{Prj}\text{-}R)^{\perp}$$

where

$$\mathbb{K}_N(\mathrm{Prj}\text{-}R)^{\perp} = \{Y \in \mathbb{K}_N(\mathrm{Flat}\text{-}R) \,|\, \mathrm{Hom}(j_! X, Y) = 0 \,:\, \forall X \in \mathbb{K}_N(\mathrm{Prj}\text{-}R)\}$$

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But $(\widetilde{\mathrm{Flat}\text{-}R_N}, \widetilde{\mathrm{Flat}\text{-}R_N}^{\perp})$ is complete cotorsion pair

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Therefore $\mathbb{K}_N(\mathrm{Flat}\text{-}R) \xrightarrow{j^*} \mathbb{K}_N(\mathrm{Prj}\text{-}R)$ has a right adjoint functor.(N-complex version of Neeman result [Neeman2, Theorem 0.1])

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Thank you all for your attention