AME40453 - Automation and Controls
C7 Pre-Lab Assignment

**For the following questions, please express your answers as algebraic equations written on a separate sheet of paper, and show your work. Then, transcribe the important equations into your lab notebook.**

1. Refer to Figure 1 of the lab handout for the following questions. Assume both tanks have the same cross sectional area $A_T$.
   a. Use conservation laws (rate balance for the volume of water) to derive a system of differential equations for the height of the water in each tank, $h_1$ and $h_2$. Use Poiseuille's Law $p = RQ_{OUT}$ and the hydrostatic pressure $p = \rho gh$ to determine the flow rate out.
   b. If the tanks are identical and both outlet nozzles are the same ($R_1 = R_2 = R$), derive an equation for the time constant $\tau$ in either tank in terms of $R$, $\rho$, $g$, and the cross-sectional area of the tank $A_T = \pi r^2$.
   c. Derive an equation for the steady-state equilibrium height for the top tank $h_{1S}$ in terms of $S$, $\tau$, and tank area $A_T$.
   d. If the tanks are identical and both outlet nozzles are the same ($R_1 = R_2 = R$), derive an equation for the steady-state equilibrium height for the bottom tank $h_{2S}$ in terms of $h_{1S}$. What are the implications for controlling the system?
   e. Derive an equation for flow rate $S_s$ that yields an equilibrium height $h_{1S}$.
   f. Express the governing equations derived in problem 1a in the form $\dot{x} = Ax + Bu$ where $x = \begin{bmatrix} h_1 - h_{1S} \\ h_2 - h_{2S} \end{bmatrix}$ and $u = S - S_s$. In particular, what are $A$ and $B$ in terms of $A_T$ and $\tau$?
   g. Use the lqr() method in Matlab to calculate the optimal gains $k_{p1}$ and $k_{p2}$ (in units of $in^2/s$) for identical tanks, both with a diameter $D = 5''$ and time constant $\tau = 6s$. Assume **R** = [1] and **Q** = **I**. (**R** is the 1x1 identity matrix, and **Q** is the 2x2 identity matrix.)

2. Refer to Figure 2 of the lab handout for the following questions. Assume both tanks have the same cross-sectional area $A_T$. However, we will allow both tanks to have different nozzles, such that the flow resistances $\boldsymbol{R_1 \neq R_2}$.
   a. Similar to Part 1, use conservation laws (rate balance for the volume of water) to derive a system of differential equations for the height of the water in each tank, $h_1$ and $h_2$.
   b. Derive an equation for the time constant $\tau_1$ for Tank 1 in terms of $R_1$, $\rho$, $g$, and the cross-sectional area of the tank $A_T$.

c. Derive an equation for the time constant $\tau_2$ for Tank 2 in terms of $R_2$, $\rho$, $g$, and the cross-sectional area of the tank $A_T$.

d. Derive an equation for the steady-state equilibrium height $h_{1S}$ for Tank 1 in terms of $S_1$, $\tau_1$, and $A_T$.

e. Derive an equation for the steady-state equilibrium height $h_{2S}$ for Tank 2 in terms of $S_2$, $h_{1S}$, $R_2$, $\rho$, and $g$.

f. Assuming $h_{1S}$, $R_1$, $R_2$, $\rho$, and $g$ are constant, derive an equation for the minimum equilibrium height for Tank 2, $\min(h_{2S})$.

g. For a larger range of allowable equilibrium heights $h_{2S}$, do you want $R_2 << R_1$ or $R_2 >> R_1$?

h. Derive an equation for the flow rate $S_{1S}$ necessary to maintain an equilibrium height $h_{S1}$ in terms of *only* the variables $A_T$, $\tau_1$, and $h_{1S}$.

i. Derive an equation for the flow rate $S_{2s}$ necessary to maintain an equilibrium height $h_{2S}$ in terms of *only* the variables $A_T$, $\tau_1$, $\tau_2$, $h_{1S}$, and $h_{2S}$.

j. Express the governing equations derived in problem 2a in the form $\dot{x} = Ax + Bu$ where $x = \begin{bmatrix} h_1 - h_{1S} \\ h_2 - h_{2S} \end{bmatrix}$ and $u = \begin{bmatrix} S_1 - S_{1S} \\ S_2 - S_{2S} \end{bmatrix}$. In particular, what are $A$ and $B$ in terms of $A_T$, $\tau_1$, and $\tau_2$? (Note: $A$ and $B$ will both be 2x2 matrices.)

k. Use the lqr() method in Matlab to calculate the optimal gain matrix (in units of $in^2/s$) for identical tanks, both with a diameter $D$ = 5" and time constants $\tau_1$ = 8s and $\tau_2$ = 2s. Assume $\mathbf{R} = \mathbf{Q} = \mathbf{I}$. (**R** and **Q** are the 2x2 identity matrix.)

l. Assume the flow rates are linearly related to the duty cycle for both pumps, such that $S_1 = a_1(\%PWM_1) + b_1$ and $S_2 = a_2(\%PWM_1) + b_2$. Take Eq. (6) in the handout, and use these calibration equations to replace $S_1$ and $S_2$ with $\%PWM_1$ and $\%PWM_2$. (i.e. Repeat what was done in Eqs. (1) – (4) in Part I of the handout.)