AME40453 - Automation and Controls
C7 Pre-Lab Assignment

For the following questions, please express your answers as algebraic equations written on a separate sheet of paper, and show your work. Then, transcribe the important equations into your lab notebook.

1. Refer to Figure 1 of the lab handout for the following questions. Assume both tanks have the same cross sectional area A_T.
   a. Use conservation laws (rate balance for the volume of water) to derive a system of differential equations for the height of the water in each tank, h_1 and h_2. Use Poiseuille's Law $p = RQ_{OUT}$ and the hydrostatic pressure $p = \rho gh$ to determine the flow rate out.
   b. If the tanks are identical and both outlet nozzles are the same ($R_1 = R_2 = R$), derive an equation for the time constant τ in either tank in terms of R, ρ, g, and the cross-sectional area of the tank $A_T = \pi r^2$.
   c. Derive an equation for the steady-state equilibrium height for the top tank h_{1S} in terms of S, τ, and tank area A_T.
   d. If the tanks are identical and both outlet nozzles are the same ($R_1 = R_2 = R$), derive an equation for the steady-state equilibrium height for the bottom tank h_{2S} in terms of h_{1S}. What are the implications for controlling the system?
   e. Derive an equation for flow rate S_s that yields an equilibrium height h_{1S}.
   f. Express the governing equations derived in problem 1a in the form $\dot{x} = Ax + Bu$ where $x = \begin{bmatrix} h_1 - h_{1S} \\ h_2 - h_{2S} \end{bmatrix}$ and $u = S - S_s$. In particular, what are A and B in terms of A_T and τ?
   g. Use the lqr() method in Matlab to calculate the optimal gains k_{p1} and k_{p2} (in units of in^2/s) for identical tanks, both with a diameter D = 5" and time constant τ = 6s. Assume **R** = [1] and **Q** = **I**. (**R** is the 1x1 identity matrix, and **Q** is the 2x2 identity matrix.)

2. Refer to Figure 2 of the lab handout for the following questions. Assume both tanks have the same cross-sectional area A_T. However, we will allow both tanks to have different nozzles, such that the flow resistances $\boldsymbol{R_1 \neq R_2}$.
   a. Similar to Part 1, use conservation laws (rate balance for the volume of water) to derive a system of differential equations for the height of the water in each tank, h_1 and h_2.
   b. Derive an equation for the time constant τ_1 for Tank 1 in terms of R_1, ρ, g, and the cross-sectional area of the tank A_T.

c. Derive an equation for the time constant τ_2 for Tank 2 in terms of R_2, ρ, g, and the cross-sectional area of the tank A_T.

d. Derive an equation for the steady-state equilibrium height h_{1S} for Tank 1 in terms of S_1, τ_1, and A_T.

e. Derive an equation for the steady-state equilibrium height h_{2S} for Tank 2 in terms of S_2, h_{1S}, R_2, ρ, and g.

f. Assuming h_{1S}, R_1, R_2, ρ, and g are constant, derive an equation for the minimum equilibrium height for Tank 2, $\min(h_{2S})$.

g. For a larger range of allowable equilibrium heights h_{2S}, do you want $R_2 << R_1$ or $R_2 >> R_1$?

h. Derive an equation for the flow rate S_{1S} necessary to maintain an equilibrium height h_{S1} in terms of *only* the variables A_T, τ_1, and h_{1S}.

i. Derive an equation for the flow rate S_{2s} necessary to maintain an equilibrium height h_{2S} in terms of *only* the variables A_T, τ_1, τ_2, h_{1S}, and h_{2S}.

j. Express the governing equations derived in problem 2a in the form $\dot{x} = Ax + Bu$ where $x = \begin{bmatrix} h_1 - h_{1S} \\ h_2 - h_{2S} \end{bmatrix}$ and $u = \begin{bmatrix} S_1 - S_{1S} \\ S_2 - S_{2S} \end{bmatrix}$. In particular, what are A and B in terms of A_T, τ_1, and τ_2? (Note: A and B will both be 2x2 matrices.)

k. Use the lqr() method in Matlab to calculate the optimal gain matrix (in units of in^2/s) for identical tanks, both with a diameter $D = 5$" and time constants $\tau_1 = 8$s and $\tau_2 = 2$s. Assume $\mathbf{R} = \mathbf{Q} = \mathbf{I}$. ($\mathbf{R}$ and $\mathbf{Q}$ are the 2x2 identity matrix.)

l. Assume the flow rates are linearly related to the duty cycle for both pumps, such that $S_1 = a_1(\%\text{PWM}_1) + b_1$ and $S_2 = a_2(\%\text{PWM}_1) + b_2$. Take Eq. (6) in the handout, and use these calibration equations to replace S_1 and S_2 with $\%\text{PWM}_1$ and $\%\text{PWM}_2$. (i.e. Repeat what was done in Eqs. (1) – (4) in Part I of the handout.)