

For the following questions, please express your answers as algebraic equations written on a separate sheet of paper, and show your work. Then, transcribe the important equations into your lab notebook.

1. Refer to Figure 1 of the lab handout for the following questions. Assume both tanks have the same cross sectional area A_T .
 - a. Use conservation laws (rate balance for the volume of water) to derive a system of differential equations for the height of the water in each tank, h_1 and h_2 . Use Poiseuille's Law $p = RQ_{OUT}$ and the hydrostatic pressure $p = \rho gh$ to determine the flow rate out.
 - b. If the tanks are identical and both outlet nozzles are the same ($R_1 = R_2 = R$), derive an equation for the time constant τ in either tank in terms of R , ρ , g , and the cross-sectional area of the tank $A_T = \pi r^2$.
 - c. Derive an equation for the steady-state equilibrium height for the top tank h_{1S} in terms of S , τ , and tank area A_T .
 - d. If the tanks are identical and both outlet nozzles are the same ($R_1 = R_2 = R$), derive an equation for the steady-state equilibrium height for the bottom tank h_{2S} in terms of h_{1S} . What are the implications for controlling the system?
 - e. Derive an equation for flow rate S_s that yields an equilibrium height h_{1S} .
 - f. Express the governing equations derived in problem 1a in state space form $\dot{x} = Ax + Bu$ where $x = \begin{bmatrix} h_1 - h_{1S} \\ h_2 - h_{2S} \end{bmatrix}$ and $u = S - S_s$. In particular, what are A and B in terms of A_T and τ ?
 - g. Use the lqr() method in Matlab to calculate the optimal gains k_{p1} and k_{p2} (in units of in^2/s) for identical tanks, both with a diameter $D = 5''$. Both tanks have a maximum allowable error of $\Delta h_{max} = 0.5$ in. The pump has a max flow rate $S_{max} = 15 \text{ in}^3/\text{s}$. Each tank has a characteristic time constant $\tau = 6\text{s}$ for draining.
2. Refer to Figure 2 of the lab handout for the following questions. Assume both tanks have the same cross-sectional area A_T . However, we will allow both tanks to have different nozzles, such that the flow resistances $R_1 \neq R_2$.
 - a. Similar to Part 1, use conservation laws (rate balance for the volume of water) to derive a system of differential equations for the height of the water in each tank, h_1 and h_2 .
 - b. Derive an equation for the time constant τ_1 for Tank 1 in terms of R_1 , ρ , g , and the cross-sectional area of the tank A_T .

- c. Derive an equation for the time constant τ_2 for Tank 2 in terms of R_2 , ρ , g , and the cross-sectional area of the tank A_T .
- d. Derive an equation for the steady-state equilibrium height h_{1S} for Tank 1 in terms of S_1 , τ_1 , and A_T .
- e. Derive an equation for the steady-state equilibrium height h_{2S} for Tank 2 in terms of S_2 , h_{1S} , R_2 , ρ , and g .
- f. Assuming h_{1S} , R_1 , R_2 , ρ , and g are constant, derive an equation for the minimum equilibrium height for Tank 2, $\min(h_{2S})$.
- g. For a larger range of allowable equilibrium heights h_{2S} , do you want $R_2 \ll R_1$ or $R_2 \gg R_1$?
- h. Derive an equation for the flow rate S_{1S} necessary to maintain an equilibrium height h_{S1} in terms of *only* the variables A_T , τ_1 , and h_{1S} .
- i. Derive an equation for the flow rate S_{2S} necessary to maintain an equilibrium height h_{2S} in terms of *only* the variables A_T , τ_1 , τ_2 , h_{1S} , and h_{2S} .
- j. Express the governing equations derived in problem 2a in state space form $\dot{x} = Ax + Bu$ where $x = \begin{bmatrix} h_1 - h_{1S} \\ h_2 - h_{2S} \end{bmatrix}$ and $u = \begin{bmatrix} S_1 - S_{1S} \\ S_2 - S_{2S} \end{bmatrix}$. In particular, what are A and B in terms of A_T , τ_1 , and τ_2 ? (Note: A and B will both be 2x2 matrices.)
- k. Use the `lqr()` method in Matlab to calculate the optimal gains k_{p1} and k_{p2} (in units of in^2/s) for identical tanks, both with a diameter $D = 5''$. Tank 1 has a maximum allowable error $\Delta h_{1\max} = 1$ in., and Tank 2 has a maximum error $\Delta h_{2\max} = 0.1$ in. The pumps have max flow rates $S_{1\max} = 15 \text{ in}^3/\text{s}$ and $S_{2\max} = 10 \text{ in}^3/\text{s}$. The tanks have characteristic time constants $\tau_1 = 6\text{s}$ and $\tau_2 = 2\text{s}$ for draining.
- l. Assume the flow rates are linearly related to the duty cycle for both pumps, such that $S_1 = a_1(\% \text{PWM}_1) + b_1$ and $S_2 = a_2(\% \text{PWM}_1) + b_2$. Take Eq. (6) in the handout, and use these calibration equations to replace S_1 and S_2 with $\% \text{PWM}_1$ and $\% \text{PWM}_2$. (i.e. Repeat what was done in Eqs. (1) – (4) in Part I of the handout.)