# Comparing Methodologies for Pricing Barrier Options

**FE 620: Pricing and Hedging**
Final Project

**Written By:**
Amod Lamichhane

Andre Sealy

Jeffrey Gebauer

# Contents


**1 Barrier Options** **2**

1.1 Background . . . . . 2

1.2 Options Payoffs . . . . . 2

1.3 Barrier Option Payoffs . . . . . 3

**2 Analytical Solutions for Barrier Options** **5**

2.1 The Black-Scholes Model . . . . . 5

2.2 Analytical Solution to Barrier Options . . . . . 6

2.3 Barrier Option Payoffs . . . . . 6

2.4 Surface of the Barrier Option . . . . . 7

**3 Barrier Option Pricing with Monte Carlo** **8**

**4 Binomial Tree Method** **10**

4.1 Binomial Tree Structure . . . . . 10

4.2 Example Calculation . . . . . 11

4.3 Tree Construction and Results . . . . . 11

**5 The Greeks for Barrier Options** **13**

5.1 Delta ($\Delta$) . . . . . 13

5.1.1 Delta Hedging . . . . . 14

5.2 Gamma ($\Gamma$) . . . . . 14

5.3 Vega ($\nu$) . . . . . 15

5.4 4. Theta ($\Theta$) . . . . . 16

5.5 Rho ($\rho$) . . . . . 17

**A Formulas and Proofs** **20**

A.1 Black-Scholes PDE for Barrier Options . . . . . 20

A.2 Payoffs Formulas for Single Barrier Options . . . . . 21

# Chapter 1

# Barrier Options

## 1.1 Background

Barrier options are path-dependent options with price barriers; their price depends on whether the underlying asset's price reaches a certain level during a specific period. Various types of barrier options regularly trade over-the-counter and have done so since 1967 [1]. These exotic options were developed to address the specific hedging concerns and market conditions that European and American options failed to accommodate. Barrier options are very popular for their risk-management solutions, as they allow investors and institutions to take various positions with very specific levels of protection.

As financial markets continued to evolve in the 1990s, barrier options became more standardized and accessible to the financial population. Derivative exchanges and financial institutions offered barrier options on various underlying assets, such as commodities, currencies, and interest rates. The 2008 global financial crisis sparked a renewed interest in derivative products for their risk-management capabilities, and barrier options remained one of the best products for their ability to tailor risk profiles to specific market conditions. In addition, computational tool advancements make pricing these options more manageable, thereby increasing the accessibility to market participants. This report will outline the many analytical and computational tools for pricing barrier options.

## 1.2 Options Payoffs

For some background, we will briefly discuss the attributes of a vanilla option. A call option gives the holder the right, but not the obligation, to buy a particular number of the underlying assets in the future for a pre-agreed price known as the strike price (put options give the holder the right, but not the obligation, to sell). While European options can only be exercised on the expiration date, American options allow the holder to exercise at any time on or before the expiration date. We will focus on European options throughout this research report.

Let $S$ be the price of an underlying asset and $K$ be the strike price, where $S, K \in \mathbb{R}^+$. Then the payoff for a vanilla call option,$V_c$, is given by the following

$$V_c(S,T) = \begin{cases} S_T - K & \text{if } S_t > K, \forall t \in [0,T) \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

Likewise, the payoff for a vanilla put option, $V_p$, derived by the following:

$$V_p(S,T) = \begin{cases} 0 & \text{if } S_t > K, \forall t \in [0,T) \\ K - S_T & \text{otherwise} \end{cases} \tag{1.2}$$

These formulas drive our intuition of how both vanilla and exotic options can be priced.

## 1.3 Barrier Option Payoffs

As we can see, the payoff of a vanilla option depends only on the terminal value of the underlying asset. However, an exotic option, such as a barrier option, is very different. Its price is determined by whether the underlying asset's price reaches a certain level during a specific period. Barrier options differ from standard vanilla options in several ways.

First, they match the hedging needs more closely than standard options; second, premiums for barrier options are typically lower than vanilla options; and finally, the payoff of a barrier option matches beliefs about the future behavior of the market. These features benefit many different types of investors, regardless of experience or financial needs. Another significant difference between barrier options and vanilla options is that barrier options are path-dependent. This means that the payoff depends on the process of the underlying asset. Another difference involves the possibility of a rebate. A rebate is a positive discount that a barrier option holder may receive if the barrier is never reached. For the purpose of outlining the analytical framework, we will not discuss rebates.

There are four different types of thresholds, or barriers, to consider which are:

- down-and-out
- up-and-out
- down-and-in
- up-and-in

Combined with calls and puts, we have 8 different types of barrier options in total. The payoff for a barrier option is either "knocked out" or "knocked in" if the price of the underlying crosses the barrier.

For example, let $B$ be the barrier threshold and $S_0$ be the price of the underlying asset at time $t = 0$. Then, for any $K$ the down-and-out call option with constant barrier $B < S_0$ has a payoff if the underlying prices stays below the barrier value until maturity $T$:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t > B, \forall t \in [0,T) \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

An up-and-out call option with constant barrier $B > S_0$ has a payoff if the underlying price does not go beyond the barrier value until maturity $T$:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t < B, \forall t \in [0,T) \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

A down-and-in call call option with a constant barrier $B < S_0$ has a payoff if the underlying prices stays below the barrier value until maturity $T$:

$$\begin{cases} 0 & \text{if } S_t > B, \forall t \in [0, T) \\ (S_T - K)^+ & \text{otherwise} \end{cases} \tag{1.5}$$

An up-and-in call option with a constant barrier $B < S_0$ has a payoff if the underlying prices stays beyond the barrier value until maturity $T$:

$$\begin{cases} 0 & \text{if } S_t < B, \forall t \in [0, T) \\ (S_T - K)^+ & \text{otherwise} \end{cases} \tag{1.6}$$

There are two main approaches to analytically evaluating the price of a barrier option: the probability method and the partial differential equation (PDE) method. The probability method involves the use of the reflection principal and the Girsanov theorem to estimate the barrier densities. The PDE approach is derived from the intutition that all barrier options satisfy the Black-Scholes PDE but with different domains, expiry conditions, and boundary conditions. Merton was the first to price barrier options using the PDE method, which he used to obtain the theoretical price of a down-and-out call option by using the PDE method to obtain a theoretical price.

# Chapter 2

# Analytical Solutions for Barrier Options

There are closed-form solutions for pricing European-style barrier options. This means we have an explicit mathematical expression that can be used to compute the value of a function without the need for numerical solutions. However, we will continue to compare closed-form solutions to more rigorous methodologies. Unlike their continuous counterparts, no closed-form solutions exist for discrete-time barrier options (even numerical pricing is a challenge). For this reason, we will only focus on continuous-time, single-barrier options.

## 2.1 The Black-Scholes Model

The Black and Scholes model was first published in 1973, named after the two economist who helped to develop it: Fischer Black and Myrion Scholes. (the model is formally known as the Black-Scholes-Merton model) A rigorous derivation of the Wiener process, Ito's lemma, the portfolio process at the risk-free rate gives us the following equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - \frac{\partial f}{\partial} - rf = 0 \tag{2.1}$$

From here, we solve equation (2.1) to arrive at the following equation

$$f(S,t) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) \tag{2.2}$$

where $S$ is the stock price, $K$ is the strike price, $r$ is the risk-free rate, $T$ is the time to expiration, $\sigma$ is the volatility of the stock, $N(\cdot)$ is the cumulative distribution function, and $d_1/d_2$ are derived by the following:

$$d_1 = \frac{\ln\left({}^{S_0}\!/_{K}\right) + \left(r + {}^{\sigma^2}\!/_{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\left({}^{S_0}\!/_{K}\right) + \left(r - {}^{\sigma^2}\!/_{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{2.3}$$

A more rigourous proof for the solution to the Black-Scholes PDE can be found on A.1 of the appendix.

## 2.2 Analytical Solution to Barrier Options

We start by changing the value of $T$ for $\tau = T - t$. From there, in order to have the PDE solutioni for the up-and-out call option, we begin to alter equation (2.2). Let $H$ be the barrier price. Then when $B \geq K$, we have:

$$C_{\text{up-out}}(S,t) = Se^{-q\tau}\left(N(d_1) - \left(\frac{B}{S}\right)^{2\lambda} N(d_1')\right) - Ke^{-r\tau}\left(N(d_2) - \left(\frac{B}{S}\right)^{2\lambda-2} N(d_2')\right) \tag{2.4}$$

where

$$\lambda = \frac{r-q}{\sigma^2} + \frac{1}{2} \tag{2.5}$$

and $d_1'/d_2'$ is derived by the following

$$d_1' = \frac{\ln\left(\frac{B^2}{SK}\right) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{T}}, \quad d_2' = d_1' - \sigma\sqrt{\tau} \tag{2.6}$$

If $S \geq B$ at any time before expiration, the up-and-out call ceases to exist (it is knocked out). If $S < B$ for the entire option's life, the payoff at maturity is just like a standard call, where the payoff is $\max(S_T - K, 0)$

## 2.3 Barrier Option Payoffs

With eight different types of single barrier options comes eight possible payoffs, based on the barrier price. Table (2.3) shows the payoff based on whether the barrier is up or down, whether the stock price in or outside of the barrier, and whether the option type is a call or put. Refer to Appendix A.2

Table 2.1: Theoretical Values of Single Barrier Options

| Down/Up | In/Out | Call/Put | Payoff ($K \leq B$) | Payoff ($K \geq B$) |
|---|---|---|---|---|
| Down | In | Call | $A_1 - A_2 + A_4 + A_5$ | $A_3 + A_5$ |
| Up | In | Call | $A_2 - A_2 + A_4 + A_5$ | $A_1 + A_5$ |
| Down | In | Put | $A_1 + A_5$ | $A_2 - A_3 + A_4 + A_5$ |
| Up | In | Put | $A_3 + A_5$ | $A_1 - A_2 + A_4 + A_5$ |
| Down | Out | Call | $A_2 - A_4 + A_6$ | $A_1 - A_3 + A_6$ |
| Up | Out | Call | $A_1 - A_2 + A_3 - A_4 + A_6$ | $A_6$ |
| Down | Out | Put | $A_6$ | $A_1 - A_2 + A_3 - A_4 + A_6$ |
| Up | Out | Put | $A_1 - A_3 + A_6$ | $A_2 - A_4 + A_6$ |

Throughout this report, we will be deriving our analysis from the up-and-out call and put option, since it is easier to intuitively understand. The payoffs from equations (1.1) and (1.2), still hold for vanilla calls and puts. However, recall for a knocked-out up option, the option losses value once it has reaches the barrier above the underlying stock price.

## 2.4 Surface of the Barrier Option

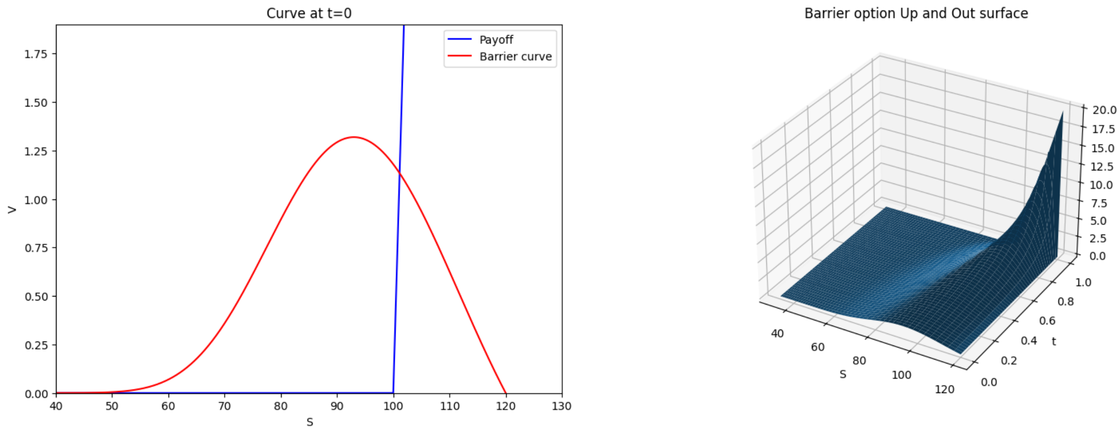


Figure 2.1: At-the-money up-and-out barrier option

Figure (2.1) shows the barrier curve and surface for an at-the-money up-and-out barrier option for $S = 100, K = 100, T = 1, r = 10\%, \sigma = 20\%$, and $B = 120$. The curve assumes the option is close to or at expiration date. As we can see, the option has no value at any price below 100, as the strike price is also 100. However, the call will retain some value, as the option has not reached the barrier of 120. Where the payoff line (blue) and the barrier curve (red) intersects shows the potential payoff at expiration, which is 1.18 according to Black-Scholes.

The surface shows the option value in relation to time and the underlying price. As we can see, the barrier option has the most value when the option is very close to expiration and the stock price is relatively close to the barrier. Otherwise, the option has no value when the underlying is at 40 and the time to expiration is 0. The option also loses it's value when the option is close to the barrier.

# Chapter 3

# Barrier Option Pricing with Monte Carlo

Table 3.1: MC Up-and-out call with $q = 0\%, r = 10\%, T = 1, \sigma = 20\%, K = 100$

| | Price | $MC_{100}$ | $err_{100}$ | $MC_{1000}$ | $err_{1000}$ | $MC_{5000}$ | $err_{5000}$ | $MC_{10000}$ | $err_{10000}$ |
|---|---|---|---|---|---|---|---|---|---|
| $S_0 = 90$ $B = 120$ | 1.2925 (7.3823) | 1.8113 (8.0324) | -0.5188 (-0.6524) | 1.4851 (7.5403) | -0.1926 (-0.1603) | 1.3249 (7.4427) | -0.0324 (-0.0627) | 1.3525 (7.3643) | -0.06 (0.0157) |
| $S_0 = 90$ $B = 130$ | 2.9799 (7.5049) | 3.2746 (6.7301) | -0.2947 (0.7748) | 3.0758 (7.9160) | -0.0960 (-0.4111) | 3.0351 (7.4177) | -0.0552 (0.0872) | 2.9991 (7.4826) | -0.0192 (0.0222) |
| $S_0 = 100$ $B = 120$ | 1.1789 (3.5932) | 1.5790 (4.0435) | -0.4001 (-0.4503) | 1.0808 (3.9701) | 0.0981 (-0.3763) | 1.2127 (3.6234) | -0.0338 (-0.0302) | 1.2060 (3.5990) | -0.0271 (-0.0058) |
| $S_0 = 100$ $B = 130$ | 3.5369 (3.7432) | 4.6782 (4.4019) | -1.1413 (-0.6587) | 3.2785 (3.8816) | 0.2584 (-0.1384) | 3.4410 (3.7508) | 0.0959 (-0.0076) | 3.5303 (3.7498) | 0.0066 (-0.0066) |
| $S_0 = 110$ $B = 120$ | 0.6264 (1.3437) | 0.7961 (1.5428) | -0.1697 (-0.1991) | 0.7819 (1.2988) | -0.1555 (0.0449) | 0.6422 (1.3553) | -0.0159 (-0.0116) | 0.6551 (1.3396) | -0.0362 (0.0041) |
| $S_0 = 110$ $B = 130$ | 2.9014 (1.6735) | 2.4881 (1.5780) | 0.4133 (0.0955) | 3.1815 (1.8227) | -0.2801 (-0.1492) | 2.9130 (1.7561) | -0.0116 (-0.0826) | 2.9700 (1.6167) | -0.0686 (0.0568) |

Suppose the asset price follows a Geometric Brownian Motion (GBM):

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{3.1}$$

Where $S_t$ is the asset price at time $t$, $r$ is the risk-free interest rate, $\sigma$ is the volatility of the asset, and $dW_t$ is the increment of a Wiener process. With this process in mind, we can discretize time by dividing the total time into smaller length intervals $\Delta t = T/N$. Afterwards, we simulate the asset price paths by generating a random standard normal variable $Z$ and using the random variable to update the asset price with the discretized GBM.

$$S_{t+\Delta t} = S_t \times \exp\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} \times 2\right) \tag{3.2}$$

Table (3.1) shows the comparison between the Black-Scholes analytical price and the Monte Carlo simulations for the up-and-out call options. We have chosen options with different Barrier values as well as varying stock values.

These results will be compared with those obtained through alternative variations of Monte Carlo methods. We've used simulations for $n = 100, 1000, 5000,$ and 10000 and samples. The errors listed in the tables represent the price deviation from the analytical values. The values in the parenthesis denote the outcome for puts, while the values without parenthesis denote the values for calls. As we can see, the Monte Carlo price gets closer to the analytical solution for $n = 5000$. With the table, we confirm one main aspect of Monte Carlo theory: increasing the number of simulations leads to an improved accuracy for the computation.

# Chapter 4

# Binomial Tree Method

The *binomial tree method* is a useful approach for pricing derivative instruments, including barrier options. It works by breaking down the time from the current moment ($t = 0$) to the option's maturity ($T$) into a finite number of steps ($N$). The more steps you use, the more accurate the price becomes, eventually converging to the theoretical option price.

## 4.1 Binomial Tree Structure

A binomial tree models the possible price movements of the underlying asset over time. At each step, the asset price can either go up or down. The price at any node in the tree is calculated using the formula:

$$S_{i,j} = S_0 \cdot u^j \cdot d^{i-j}$$

where:

- $S_{i,j}$ is the stock price at step $i$, level $j$,
- $S_0$ is the initial stock price,
- $u = e^{\sigma\sqrt{\Delta t}}$ is the up factor,
- $d = \frac{1}{u}$ is the down factor,
- $\Delta t = \frac{T}{N}$ is the time increment per step,
- $\sigma$ is the volatility of the underlying asset.

The risk-neutral probabilities of an upward movement ($p$) and a downward movement ($q$) are:

$$p = \frac{e^{r\Delta t} - d}{u - d}, \quad q = 1 - p$$

where $r$ is the risk-free interest rate.

## 4.2 Example Calculation

Let's consider an up-and-out European style put option with these parameters: $S_0 = 100$, $B = 120$, $K = 100$, $r = 0.05$, $T = 1$ year, $\sigma = 0.2$, $N = 3$.

- The time increment per step is:

$$\Delta t = \frac{T}{N} = \frac{1}{3} \approx 0.3333 \,\text{years}$$

- The up and down factors are:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{0.3333}} \approx 1.1224, \quad d = \frac{1}{u} \approx 0.8909$$

- The risk-neutral probabilities are:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05 \cdot 0.3333} - 0.8909}{1.1224 - 0.8909} \approx 0.5438, q = 1 - p = 0.4562$$

- The payoff for an up-and-out European-style barrier put option at maturity is defined as:

$$\text{Payoff} = \begin{cases} \max(K - S_T, 0), & \text{if } \max(S_t) < B \\ 0, & \text{if } \max(S_t) \geq B \end{cases}$$

where $t \in [0, T]$.

## 4.3 Tree Construction and Results

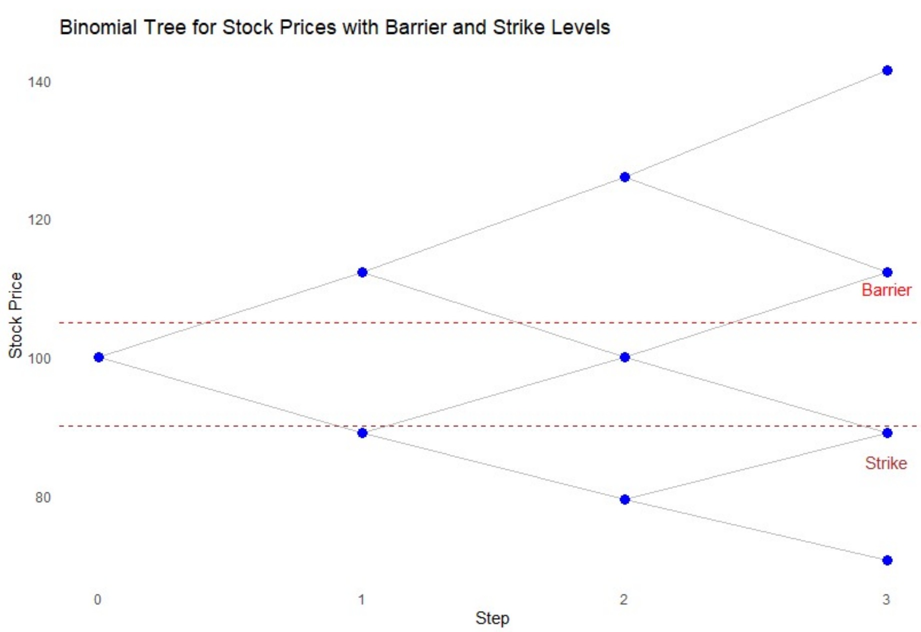


Figure 4.1: Binomial Tree with Barrier Level $B = 90$ and Strike Price $K = 105$.

Figure 4.1 illustrates the binomial tree for the given parameters, showing the possible stock price movements and the resulting option values at each node. The calculated option price, considering the up-and-out barrier condition, is obtained through backward induction along the tree. This ensures that the barrier condition is applied at each step. The final option price using this method is 6.17. For reference, the actual option price using the analytical solution is 5.36.

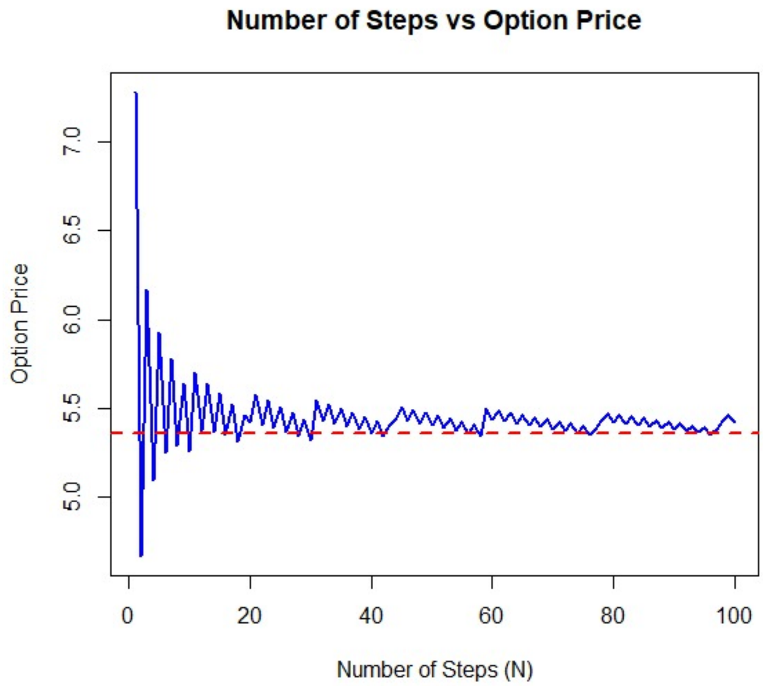


Figure 4.2: Option price converge to analytical solution as number of steps rise.

The binomial tree method is great for pricing various options, including barrier options, because it's intuitive and systematic. As shown in Figure 4.2, the option price gets closer to the analytical solution as the number of steps (N) increases. However, this accuracy comes with higher computational complexity, especially for path-dependent options. In such cases, methods like Monte Carlo simulations or analytical approaches can be more efficient and scalable, providing accuracy without the heavy computational load. Balancing flexibility and efficiency is crucial when choosing the right pricing method for complex financial derivatives.

# Chapter 5

# The Greeks for Barrier Options

The Greeks are key sensitivities in option pricing that measure how the price of an option changes with respect to various factors such as the underlying asset price, time to maturity, volatility, and interest rates. For barrier options, the calculation of the Greeks is more complex due to the added condition of a barrier, which affects the option's price path and behavior.

For the plots in this section, the following parameters will be used:

$$\begin{aligned}
S_0 &= 80.0 \quad \text{(spot stock price)} \\
K &= 50 \quad \text{(strike price)} \\
T &= 1.0 \quad \text{(time to maturity in years)} \\
r &= 0.1 \quad \text{(risk-free rate)} \\
\sigma &= 0.2 \quad \text{(volatility)} \\
B &= 100 \quad \text{(barrier level)}
\end{aligned}$$

## 5.1 Delta ($\Delta$)

Delta for a barrier option measures the sensitivity of the option price to changes in the underlying asset price ($S$). It represents the rate of change of the option's price with respect to small changes in the underlying price.

$$\Delta = \frac{\partial \text{Option Price}}{\partial S}$$

For **knock-in** barrier options, delta behaves similarly to that of a standard European option but is influenced by the presence of the barrier. It reflects how changes in the underlying price affect the probability of the barrier being breached and the option becoming active.

Figure 5.1 shows the delta behavior of an up-and-out put option as a function of the underlying stock price ($S$). When the stock price is well below the barrier level ($B$), delta behaves similarly to a vanilla European put option. As the price approaches the barrier, delta tends toward zero because the likelihood of the option being knocked out increases, reducing its sensitivity.

To further analyze the delta across different barrier options, refer to the table below:

Table 5.1: Delta behavior for different types of barrier options.

| **Barrier Type** | **Price Far from Barrier** | **Price Near the Barrier** | **Intuition** |
|---|---|---|---|
| **Up-and-Out (Knock-Out)** | Similar to vanilla options | $\Delta \to 0$ as $S \to B$ | Probability of knock-out increases as $S$ nears $B$. |
| **Down-and-Out (Knock-Out)** | Similar to vanilla options | $\Delta \to 0$ as $S \to B$ | Probability of knock-out increases as $S$ nears $B$. |
| **Up-and-In (Knock-In)** | $\Delta \approx 0$ far from barrier | $\Delta \uparrow$ as $S \to B$ | Option becomes active as the price hits the barrier. |
| **Down-and-In (Knock-In)** | $\Delta \approx 0$ far from barrier | $\Delta \uparrow$ as $S \to B$ | Option becomes active as the price hits the barrier. |

**Key Intuition:**

- **Knock-Out Options:** Lose their sensitivity as $\Delta$ tends toward 0 as the likelihood of being knocked out near the barrier increases.
- **Knock-In Options:** Show increasing sensitivity as the underlying price nears the barrier because the probability of activation rises.

The visual analysis in Figure 5.1 and the insights from Table 5.1 provide a comprehensive understanding of how delta behaves across different types of barrier options.

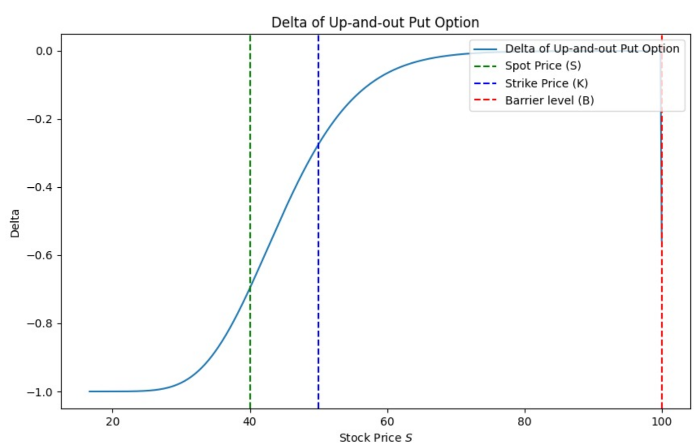


Figure 5.1: Delta of up-and-out Put Option vs. the Stock Price.

### 5.1.1 Delta Hedging

## 5.2 Gamma ($\Gamma$)

Gamma ($\Gamma$) for a barrier option measures the rate of change of delta ($\Delta$) with respect to changes in the underlying asset price ($S$). It represents the curvature or sensitivity of delta in response to movements in the underlying price. It provides insights into how rapidly an option's delta will change as the underlying price fluctuates.

$$\Gamma = \frac{\partial^2 \text{Option Price}}{\partial S^2}$$

For **knock-in** and **knock-out** barrier options, gamma exhibits unique behavior due to the presence of the barrier. The interaction between the price of the stock, the barrier level and the time to maturity influences how gamma behaves in different scenarios.

Table 5.2: Gamma behavior for Knock-In and Knock-Out barrier options.

| **Barrier Type** | **Price Far from Barrier** | **Price Near the Barrier** | **Intuition** |
|---|---|---|---|
| **Knock-Out** | Similar to vanilla options | $\Gamma \uparrow$ as $S \to B$ | Gamma spikes as the price nears the barrier due to increased risk of knock-out. |
| **Knock-In** | $\Gamma \approx 0$ far from barrier | $\Gamma \uparrow$ as $S \to B$ | Gamma rises as the stock price approaches the barrier due to increased probability of activation. |

Figure 5.2 illustrates the gamma behavior of an up-and-out put option as a function of the underlying stock price ($S$).

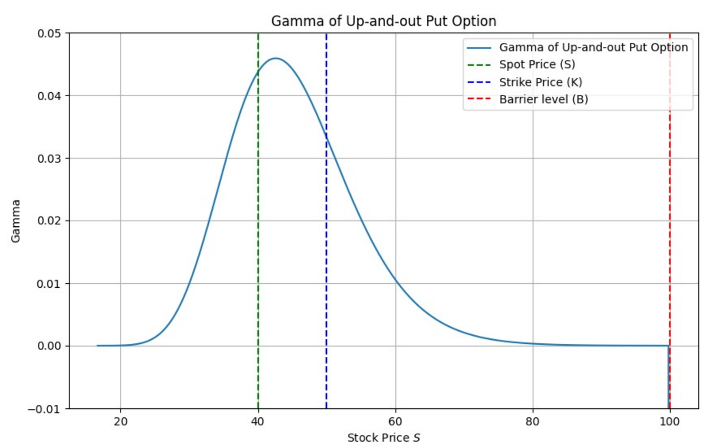


Figure 5.2: Gamma behavior of an up-and-out Put Option vs. the Stock Price.

## 5.3 Vega ($\nu$)

**Vega** ($\nu$) measures the sensitivity of the option price to changes in volatility. For barrier options, the calculation of vega incorporates adjustments to account for the probability of breaching the barrier level under varying volatility conditions.

$$\nu = \frac{\partial \text{Option Price}}{\partial \sigma}$$

- **Knock-In Options:** Increased volatility raises the likelihood of the underlying asset price crossing the barrier level, making the option more valuable. As a result, $\nu$ (vega) will generally be positive.
- **Knock-Out Options:** Increased volatility can lead to a higher probability of the option being knocked out, reducing its value. Therefore, $\nu$ will tend to be negative for these types of options.

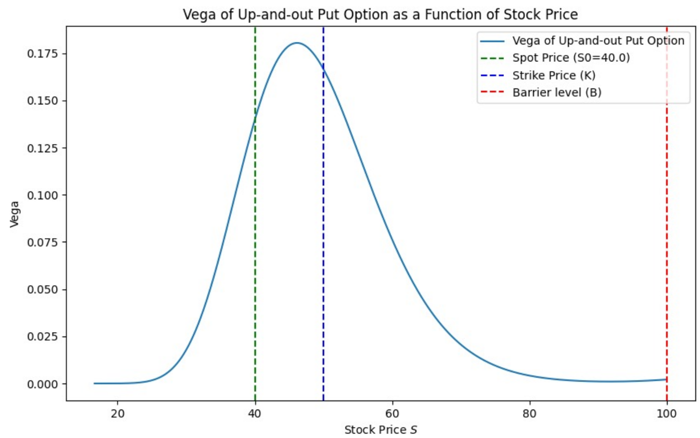


Figure 5.3: Vega vs Stock Price of an Up-and-Out Put Option

The visualization shown in Figure 5.3 provides insights into how Vega behaves with changes in stock price across up-and-out options. The sensitivity patterns are influenced by the interplay between volatility and the probability of breaching the barrier level.

## 5.4 4. Theta ($\Theta$)

Theta ($\Theta$) measures the rate of change of an option's price with the passage of time, assuming all other variables remain constant. It is a critical "Greek" that reflects time decay, which is the gradual erosion of the option's value as expiration approaches. In the case of barrier options, theta is influenced not only by the time remaining but also by the interaction with the barrier level and the path dependency of the option.

Barrier options are unique because their payoff depends on the underlying asset price crossing (or not crossing) a specified barrier level during the option's lifetime. As such, the sensitivity of theta changes depending on proximity to the barrier, time remaining until maturity, and whether the option is near the knock-in or knock-out condition.

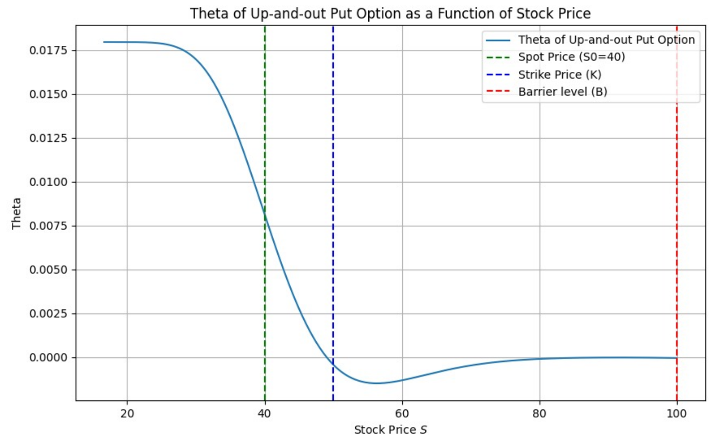


Figure 5.4: Theta vs Stock Price of an Up-and-Out Put Option

- **Knock-In Options:** As time to maturity decreases, the value of knock-in options tends to decline, especially when the underlying asset price is far from the barrier. This is because the likelihood of triggering the option by crossing the barrier becomes lower as time runs out.
- **Knock-Out Options:** For knock-out options, time decay ($\Theta$) has a more pronounced effect, especially if the underlying asset price is close to the barrier. As time to maturity approaches, the risk of the underlying price reaching the barrier and knocking out the option increases, thereby accelerating time decay.
- **Proximity to the Barrier:** Theta becomes more sensitive when the underlying price is near the barrier level. This reflects the increased path dependency and likelihood of barrier activation.

The visualization in Figure 5.4 illustrates the relationship between stock price and theta for an up-and-out put option. This analysis highlights how theta's sensitivity varies depending on proximity to the barrier and the remaining time in the option's life. It is critical to consider these patterns when employing barrier options in trading strategies, especially as expiration approaches.

## 5.5 Rho ($\rho$)

Rho for barrier options measures the sensitivity of the option price to changes in the risk-free interest rate ($r$). It represents how much the value of the option changes when the risk-free rate changes by 1%. Rho is important in understanding the impact of monetary policy, such as changes in interest rates, on the value of barrier options.

$$\rho = \frac{\partial \text{Option Price}}{\partial r}$$

The visual behavior in Figure 5.5 illustrates how changes in the risk-free interest rate ($r$) impact the value of an up-and-out barrier option under varying interest rate conditions.

When interest rates rise:

- **Knock-In Options:** Tend to show a positive relationship with Rho ($\rho$). This reflects that higher interest rates increase the present value of the potential payoff, making the knock-in option more valuable.
- **Knock-Out Options:** Tend to show muted sensitivity to changes in the interest rate. This is because the likelihood of being knocked out (and thus losing value) dominates any benefit gained from higher interest rates.

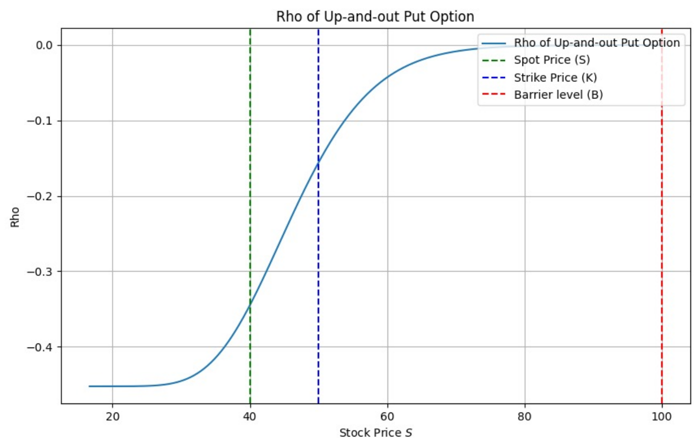


Figure 5.5: Rho vs Stock Price for an Up-and-Out Put Option

The visual behavior in Figure 5.5 depicts how the effect of interest rate changes impacts the value of an up-and-out barrier option under various interest rate conditions. When rates rise, knock-in options tend to show a positive relationship with Rho, reflecting the increased value from higher risk-free rates. Conversely, knock-out options tend to show muted sensitivity, as their value diminishes if the likelihood of being knocked out rises.

# Bibliography

[1] Cox, John C. "Options Markets." (1985).

# Appendix A

# Formulas and Proofs

## A.1 Black-Scholes PDE for Barrier Options

## A.2 Payoffs Formulas for Single Barrier Options

The formula for Single Barrier options have been proved by Reiner and Rubinstein. The payoff of a Single Barrier call or put depends on the following formulas:

$$
\begin{aligned}
A_1 &= Se^{-q\tau} a N(ax_1) - Ke^{-r\tau} N(ax_1 - a\sigma\sqrt{\tau}) \\
A_2 &= Se^{-q\tau} a N(ax_1) - Ke^{-r\tau} N(ax_2 - a\sigma\sqrt{\tau}) \\
A_3 &= Se^{-q\tau} a \left(\tfrac{B}{S}\right)^{2\lambda+2} N(bx_3) - K\left(\tfrac{B}{S}\right)^{2\lambda} e^{-r\tau} N(bx_3 - b\sigma\sqrt{\tau}) \\
A_4 &= Se^{-q\tau} a \left(\tfrac{B}{S}\right)^{2\lambda+2} N(bx_4) - K\left(\tfrac{B}{S}\right)^{2\lambda} e^{-r\tau} N(bx_4 - b\sigma\sqrt{\tau}) \\
A_5 &= Ke^{-rT}\left[N(bx_2 - b\sigma\sqrt{\tau}) - \left(\tfrac{B}{S}\right)^{2\lambda} N\left(bx_4 - b\sigma\sqrt{\tau}\right)\right] \\
A_6 &= Ke^{-rT}\left[N(bx_5 - b\sigma\sqrt{\tau}) - \left(\tfrac{B}{S}\right)^{2\lambda} N\left(bx_5 - b\sigma\sqrt{\tau}\right)\right]
\end{aligned}
$$

with

$$
\begin{cases}
a = 1, -1 & \text{call or put} \\
b = 1, -1 & \text{out or in}
\end{cases}
$$

where $x_1, x_2, x_3, x_4, x_5$ is the following:

$$
x_1 = \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}, \quad x_2 = \frac{\ln\left(\frac{S}{B}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}
$$

$$
x_3 = \frac{\ln\left(\frac{B^2}{SK}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}, \quad x_4 = \frac{\ln\left(\frac{B}{S}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}
$$

$$
x_5 = \frac{\ln\left(\frac{B}{S}\right)}{\sigma\sqrt{\tau}} + \lambda\sigma\sqrt{\tau}
$$

where $\mu$ and $\lambda$ is the following

$$
\mu = \frac{r - q - \frac{\sigma^2}{2}}{\sigma^2}, \quad \lambda = \sqrt{\mu^2 + \frac{2q}{\sigma^2}}
$$