Comparing Methodologies for Pricing Barrier Options

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Written By:
Amod Lamichhane

Andre Sealy

Jeffrey Gebauer

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Chapter 1

Barrier Options

1.1 Background

Barrier options are path-dependent options with price barriers; their price depends on whether the underlying asset's price reaches a certain level during a specific period. Various types of barrier options regularly trade over-the-counter and have done so since 1967 [1]. These exotic options were developed to address the specific hedging concerns and market conditions that European and American options failed to accommodate. Barrier options are very popular for their risk-management solutions, as they allow investors and institutions to take various positions with very specific levels of protection.

As financial markets continued to evolve in the 1990s, barrier options became more standardized and accessible to the financial population. Derivative exchanges and financial institutions offered barrier options on various underlying assets, such as commodities, currencies, and interest rates. The 2008 global financial crisis sparked a renewed interest in derivative products for their risk-management capabilities, and barrier options remained one of the best products for their ability to tailor risk profiles to specific market conditions. In addition, computational tool advancements make pricing these options more manageable, thereby increasing the accessibility to market participants. This report will outline the many analytical and computational tools for pricing barrier options.

1.2 Options Payoffs

For some background, we will briefly discuss the attributes of a vanilla option. A call option gives the holder the right, but not the obligation, to buy a particular number of the underlying assets in the future for a pre-agreed price known as the strike price (put options give the holder the right, but not the obligation, to sell). While European options can only be exercised on the expiration date, American options allow the holder to exercise at any time on or before the expiration date. We will focus on European options throughout this research report.

Let S be the price of an underlying asset and K be the strike price, where $S, K \in \mathbb{R}^+$. Then the payoff for a vanilla call option,V_c, is given by the following

$$V_c(S,T) = \begin{cases} S_T - K & \text{if } S_t > K, \forall t \in [0,T) \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

Likewise, the payoff for a vanilla put option, V_p, derived by the following:

$$V_p(S,T) = \begin{cases} 0 & \text{if } S_t > K, \forall t \in [0,T) \\ K - S_T & \text{otherwise} \end{cases} \tag{1.2}$$

These formulas drive our intuition of how both vanilla and exotic options can be priced.

1.3 Barrier Option Payoffs

As we can see, the payoff of a vanilla option depends only on the terminal value of the underlying asset. However, an exotic option, such as a barrier option, is very different. Its price is determined by whether the underlying asset's price reaches a certain level during a specific period. Barrier options differ from standard vanilla options in several ways.

First, they match the hedging needs more closely than standard options; second, premiums for barrier options are typically lower than vanilla options; and finally, the payoff of a barrier option matches beliefs about the future behavior of the market. These features benefit many different types of investors, regardless of experience or financial needs.

Another significant difference between barrier options and vanilla options is that barrier options are path-dependent. This means that the payoff depends on the process of the underlying asset. Another difference involves the possibility of a rebate. A rebate is a positive discount that a barrier option holder may receive if the barrier is never reached. For the purpose of outlining the analytical framework, we will not discuss rebates.

There are four different types of thresholds, or barriers, to consider which are:

- down-and-out
- up-and-out
- down-and-in
- up-and-in

Combined with calls and puts, we have 8 different types of barrier options in total. The payoff for a barrier option is either "knocked out" or "knocked in" if the price of the underlying crosses the barrier.

For example, let B be the barrier threshold and S_0 be the price of the underlying asset at time $t = 0$. Then, for any K the down-and-out call option with constant barrier $B < S_0$ has a payoff if the underlying prices stays below the barrier value until maturity T:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t > B, \forall t \in [0,T) \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

An up-and-out call option with constant barrier $B < S_0$ has a payoff if the underlying price does not go beyond the barrier value until maturity T:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t < B, \forall t \in [0,T) \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

A down-and-in call call option with a constant barrier $B < S_0$ has a payoff if the underlying prices stays below the barrier value until maturity T:

$$\begin{cases} 0 & \text{if } S_t > B, \forall t \in [0, T) \\ (S_T - K)^+ & \text{otherwise} \end{cases} \tag{1.5}$$

An up-and-in call option with a constant barrier $B < S_0$ has a payoff if the underlying prices stays beyond the barrier value until maturity T:

$$\begin{cases} 0 & \text{if } S_t < B, \forall t \in [0, T) \\ (S_T - K)^+ & \text{otherwise} \end{cases} \tag{1.6}$$

There are two main approaches to analytically evaluating the price of a barrier option: the probability method and the partial differential equation (PDE) method. The probability method involves the use of the reflection principal and the Girsanov theorem to estimate the barrier densities. The PDE approach is derived from the institution that all barrier options satisfy the Black-Scholes PDE but with different domains, expiry conditions, and boundary conditions.

Merton was the first to price barrier options using the PDE method, which he used to obtain the theoretical price of a down-and-out call option by using the PDE method to obtain a theoretical price. We will demonstrate this in the next section.

Chapter 2

Analytical Solutions for Barrier Options

There are closed-form solutions for pricing European-style barrier options. This means we have an explicit mathematical expression that can be used to compute the value of a function without the need for numerical solutions. However, we will continue to compare closed-form solutions to more rigorous methodologies. Unlike their continuous counterparts, no closed-form solutions exist for discrete-time barrier options (even numerical pricing is a challenge). For this reason, we will only focus on continuous-time, single-barrier options.

2.1 The Black-Scholes Model

The Black and Scholes model was first published in 1973, named after the two economist who helped to develop it: Fischer Black and Myrion Scholes. (the model is formally known as the Black-Scholes-Merton model) A rigorous derivation of the Wiener process, Ito's lemma, the portfolio process at the risk-free rate gives us the following equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - \frac{\partial f}{\partial} - rf = 0 \tag{2.1}$$

From here, we solve equation (2.1) to arrive at the following equation

$$f(S,t) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) \tag{2.2}$$

where S is the stock price, K is the strike price, r is the risk-free rate, T is the time to expiration, σ is the volatility of the stock, $N(\cdot)$ is the cumulative distribution function, and d_1/d_2 are derived by the following:

$$d_1 = \frac{\ln\left({}^{S_0}/_{K}\right) + \left(r + {}^{\sigma^2}/_{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\left({}^{S_0}/_{K}\right) + \left(r - {}^{\sigma^2}/_{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{2.3}$$

A more rigourous proof for the solution to the Black-Scholes PDE can be found on A.1 of the appendix.

2.2 Black-Scholes Solution to Barrier Options

An essential concept in pricing barrier options is the in-out parity:

$$C_{\text{vanilla}} = C_{\text{up-in}} + C_{\text{up-out}}$$

which we can use to derive the following relationship for a up-and-in call optionV_{DIC} as:

$$C_{\text{up-out}} = C_{\text{vanilla}} - C_{\text{up-in}}$$

We start by changing the value of T for $\tau = T - t$. From there, in order to have the PDE solutioni for the up-and-out call option, we begin to alter equation (2.2). Let H be the barrier price. Then when $B \geq K$, we have:

$$C_{\text{up-out}}(S,t) = Se^{-q\tau}\left(N(d_1) - \left(\frac{B}{S}\right)^{2\lambda} N(d_1')\right) - Ke^{-r\tau}\left(N(d_2) - \left(\frac{B}{S}\right)^{2\lambda-2} N(d_2')\right) \tag{2.4}$$

where

$$\lambda = \frac{r-q}{\sigma^2} + \frac{1}{2} \tag{2.5}$$

and d_1'/d_2' is derived by the following

$$d_1' = \frac{\ln\left(\frac{B^2}{SK}\right) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{T}}, \quad d_2' = d_1' - \sigma\sqrt{\tau} \tag{2.6}$$

If $S \geq B$ at any time before expiration, the up-and-out call ceases to exist (it is knocked out). If $S < B$ for the entire option's life, the payoff at maturity is just like a standard call, where the payoff is $\max(S_T - K, 0)$

2.3 Barrier Option Payoffs

With eight different types of single barrier options comes eight possible payoffs, based on the barrier price. Table (2.3) shows the payoff based on whether the barrier is up or down, whether the stock price in or outside of the barrier, and whether the option type is a call or put. Refer to Appendix A.2

Table 2.1: Theoretical Values of Single Barrier Options

Down/Up	In/Out	Call/Put	Payoff ($K \leq B$)	Payoff ($K \geq B$)
Down	In	Call	$A_1 - A_2 + A_4 + A_5$	$A_3 + A_5$
Up	In	Call	$A_2 - A_2 + A_4 + A_5$	$A_1 + A_5$
Down	In	Put	$A_1 + A_5$	$A_2 - A_3 + A_4 + A_5$
Up	In	Put	$A_3 + A_5$	$A_1 - A_2 + A_4 + A_5$
Down	Out	Call	$A_2 - A_4 + A_6$	$A_1 - A_3 + A_6$
Up	Out	Call	$A_1 - A_2 + A_3 - A_4 + A_6$	A_6
Down	Out	Put	A_6	$A_1 - A_2 + A_3 - A_4 + A_6$
Up	Out	Put	$A_1 - A_3 + A_6$	$A_2 - A_4 + A_6$

Throughout this report, we will be deriving our analysis from the up-and-out call and put option, since it is easier to intuitively understand. The payoffs from equations (1.1) and (1.2), still hold for vanilla calls and puts. However, recall for a knocked-out up option, the option losses value once it has reaches the barrier above the underlying stock price.

2.4 Surface of the Barrier Option

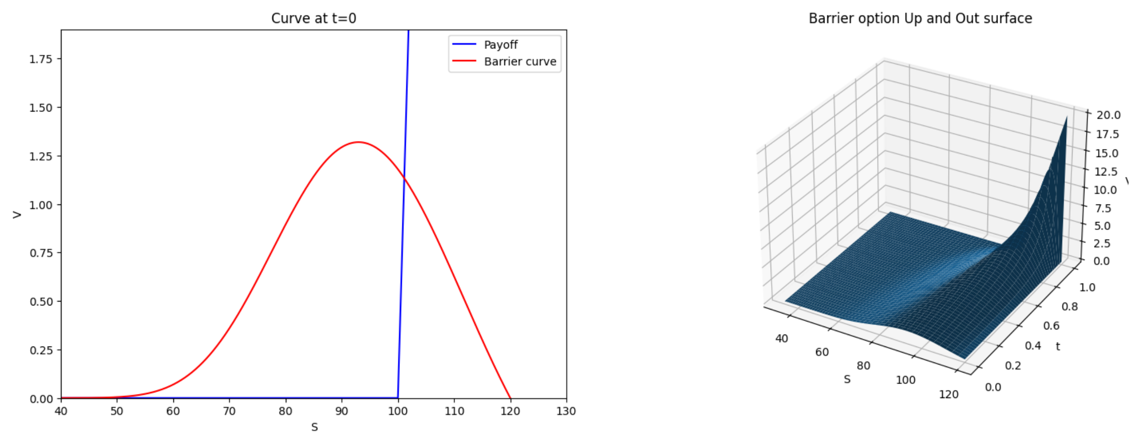

Figure 2.1: At-the-money up-and-out barrier option

Figure (2.1) shows the barrier curve and surface for an at-the-money up-and-out barrier option for $S = 100, K = 100, T = 1, r = 10\%, \sigma = 20\%$, and $B = 120$. The curve assumes the option is close to or at expiration date. As we can see, the option has no value at any price below 100, as the strike price is also 100. However, the call will retain some value, as the option has not reached the barrier of 120. Where the payoff line (blue) and the barrier curve (red) intersects shows the potential payoff at expiration, which is 1.18 according to Black-Scholes.

The surface shows the option value in relation to time and the underlying price. As we can see,

Chapter 3

Barrier Option Pricing with Monte Carlo

Suppose the asset price follows a Geometric Brownian Motion (GBM):

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{3.1}$$

Where S_t is the asset price at time t, r is the risk-free interest rate, σ is the volatility of the asset, and dW_t is the increment of a Wiener process. With this process in mind, we can discretize time by dividing the total time into smaller length intervals $\Delta t = T/N$. Afterwards, we simulate the asset price paths by generating a random standard normal variable Z and using the random variable to update the asset price with the discretized GBM.

Table 3.1: MC Up-and-out call with $q = 0\%, r = 10\%, T = 1, \sigma = 20\%, K = 100$

	Price	MC_{100}	err_{100}	MC_{1000}	err_{1000}	MC_{5000}	err_{5000}	MC_{10000}	err_{10000}
$S_0 = 100$ $B = 0.01$	11.1605	10.4632	0.6973	11.4493	-0.2888	10.8657	0.2948	11.1015	0.0590
$S_0 = 90$ $B = 120$	1.2925 (7.3823)	1.8113 (8.0324)	-0.5188 (-0.6524)	1.4851 (7.5403)	-0.1926 (-0.1603)	1.3249 (7.4427)	-0.0324 (-0.0627)	1.3525 (7.3643)	-0.06 (0.0157)
$S_0 = 90$ $B = 130$	2.9799 (7.5049)	3.2746 (6.7301)	-0.2947 (0.7748)	3.0758 (7.9160)	-0.0960 (-0.4111)	3.0351 (7.4177)	-0.0552 (0.0872)	2.9991 (7.4826)	-0.0192 (0.0222)
$S_0 = 100$ $B = 120$	1.1789 (3.5932)	1.5790 (4.0435)	-0.4001 (-0.4503)	1.0808 (3.9701)	0.0981 (-0.3763)	1.2127 (3.6234)	-0.0338 (-0.0302)	1.2060 (3.5990)	-0.0271 (-0.0058)
$S_0 = 100$ $B = 130$	3.5369 (3.7432)	4.6782 (4.4019)	-1.1413 (-0.6587)	3.2785 (3.8816)	0.2584 (-0.1384)	3.4410 (3.7508)	0.0959 (-0.0076)	3.5303 (3.7498)	0.0066 (-0.0066)
$S_0 = 110$ $B = 120$	0.6264 (1.3437)	0.7961 (1.5428)	-0.1697 (-0.1991)	0.7819 (1.2988)	-0.1555 (0.0449)	0.6422 (1.3553)	-0.0159 (-0.0116)	0.6551 (1.3396)	-0.0362 (0.0041)
$S_0 = 110$ $B = 130$	2.9014 (1.6735)	2.4881 (1.5780)	0.4133 (0.0955)	3.1815 (1.8227)	-0.2801 (-0.1492)	2.9130 (1.7561)	-0.0116 (-0.0826)	2.9700 (1.6167)	-0.0686 (0.0568)

$$S_{t+\Delta t} = S_t \times \exp\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t} \times 2\right) \tag{3.2}$$

Table (3.1) shows the comparison between the Black-Scholes analytical price and the Monte Carlo simulations for the up-and-out call options. We have chosen options with different Barrier values as well as varying stock values. These results will be compared with those obtained through alternative variations of Monte Carlo methods. We've used simulations for $n = 100, 1000, 5000,$ and 10000 and samples. The errors listed in the tables represent the price deviation from the analytical values. The values in the parenthesis denote the outcome for puts, while the values without parenthesis denote the values for calls.

As we can see, the Monte Carlo price gets closer to the analytical solution for $n = 5000$. With the table, we confirm one main aspect of Monte Carlo theory: increasing the number of simulations leads to an improved accuracy for the computation.

Chapter 4

The Greeks for Barrier Options

The Greeks are key sensitivities in option pricing that measure how the price of an option changes with respect to various factors such as the underlying asset price, time to maturity, volatility, and interest rates. For barrier options, the calculation of the Greeks is more complex due to the added condition of a barrier, which affects the option's price path and behavior. This chapter provides an explanation of how to calculate the Greeks for barrier options.

1. Delta (Δ)

Delta for a barrier option measures the sensitivity of the option price to changes in the underlying asset price. It represents the rate of change of the option price with respect to small changes in the asset price.

For a **knock-in** barrier option, delta is calculated similarly to standard options, but with the added complexity of the barrier. The delta reflects how changes in the underlying price affect the probability of hitting the barrier level and, thus, the likelihood of the option becoming active.

For **knock-out** options, delta will approach zero as the underlying asset approaches the barrier level, since the probability of the option being knocked out increases, making the option's price less sensitive to further changes in the underlying asset price.

2. Gamma (Γ)

Gamma for barrier options measures the rate of change of delta with respect to changes in the underlying asset price. It is the second derivative of the option price with respect to the asset price.

Gamma for barrier options can be derived from the standard Black-Scholes model, but with an adjustment for the probability of barrier activation. The higher the volatility, the larger the gamma for barrier options, especially near the barrier level.

For **knock-in** options, gamma is typically larger as the option approaches the barrier level, as it is more sensitive to changes in the underlying asset price.

3. Vega (ν)

Vega for barrier options reflects the sensitivity of the option price to changes in volatility. The formula for vega is similar to the standard Black-Scholes model but adjusted for the likelihood of hitting the barrier level.

For **knock-in** options, increased volatility tends to increase the likelihood of hitting the barrier, thus increasing the option price, and therefore vega will be positive. For **knock-out** options, increased volatility increases the chance of the option being knocked out, reducing the option price. Hence, vega for knock-out options will generally be negative.

4. Theta (Θ)

Theta for barrier options measures the rate of change of the option's price as time to maturity decreases. Barrier options are path-dependent, and their theta is influenced by the time remaining until the option either knocks in, knocks out, or expires. The closer the option is to the barrier, the more sensitive the theta becomes.

For **knock-in** options, as time to maturity decreases, the value of the option tends to decrease, particularly when the underlying asset price is far from the barrier. For **knock-out** options, time decay may have a more pronounced effect, especially if the underlying price is near the barrier.

5. Rho (ρ)

Rho for barrier options measures the sensitivity of the option price to changes in interest rates. It represents how much the option price changes when the risk-free interest rate changes by 1%.

For **knock-in** options, an increase in interest rates generally increases the option price, since higher rates make holding the option more attractive relative to other assets. For **knock-out** options, the effect of interest rate changes may be less pronounced, especially if the option is likely to be knocked out before maturity.

Bibliography

[1] Cox, John C. "Options Markets." (1985).

Appendix A

Formulas and Proofs

A.1 Black-Scholes PDE for Barrier Options

A.2 Payoffs Formulas for Single Barrier Options

The formula for Single Barrier options have been proved by Reiner and Rubinstein. The payoff of a Single Barrier call or put depends on the following formulas:

$$
\begin{aligned}
A_1 &= Se^{-q\tau}aN(ax_1) - Ke^{-r\tau}N(ax_1 - a\sigma\sqrt{\tau}) \\
A_2 &= Se^{-q\tau}aN(ax_1) - Ke^{-r\tau}N(ax_2 - a\sigma\sqrt{\tau}) \\
A_3 &= Se^{-q\tau}a\left(\tfrac{B}{S}\right)^{2\lambda+2}N(bx_3) - K\left(\tfrac{B}{S}\right)^{2\lambda}e^{-r\tau}N(bx_3 - b\sigma\sqrt{\tau}) \\
A_4 &= Se^{-q\tau}a\left(\tfrac{B}{S}\right)^{2\lambda+2}N(bx_4) - K\left(\tfrac{B}{S}\right)^{2\lambda}e^{-r\tau}N(bx_4 - b\sigma\sqrt{\tau}) \\
A_5 &= Ke^{-rT}\left[N(bx_2 - b\sigma\sqrt{\tau}) - \left(\tfrac{B}{S}\right)^{2\lambda}N\left(bx_4 - b\sigma\sqrt{\tau}\right)\right] \\
A_6 &= Ke^{-rT}\left[N(bx_5 - b\sigma\sqrt{\tau}) - \left(\tfrac{B}{S}\right)^{2\lambda}N\left(bx_5 - b\sigma\sqrt{\tau}\right)\right]
\end{aligned}
$$

with

$$
\begin{cases}
a = 1, -1 & \text{call or put} \\
b = 1, -1 & \text{out or in}
\end{cases}
$$

where x_1, x_2, x_3, x_4, x_5 is the following:

$$
x_1 = \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}, \quad x_2 = \frac{\ln\left(\frac{S}{B}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}
$$

$$
x_3 = \frac{\ln\left(\frac{B^2}{SK}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}, \quad x_4 = \frac{\ln\left(\frac{B}{S}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}
$$

$$
x_5 = \frac{\ln\left(\frac{B}{S}\right)}{\sigma\sqrt{\tau}} + \lambda\sigma\sqrt{\tau}
$$

where μ and λ is the following

$$
\mu = \frac{r - q - \frac{\sigma^2}{2}}{\sigma^2}, \quad \lambda = \sqrt{\mu^2 + \frac{2q}{\sigma^2}}
$$