Comparing Methodologies for Pricing Barrier Options

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Written By:
Amod Lamichhane

Andre Sealy

Jeffrey Gebauer

Contents

Chapter 1

Introduction

Barrier options are path-dependent options with price barriers; their price depends on whether the underlying asset's price reaches a certain level during a specific period. Various types of barrier options regularly trade over-the-counter and have done so since 1967 [1]. These exotic options were developed to address the specific hedging concerns and market conditions that European and American options failed to accommodate. Barrier options are very popular for their risk-management solutions, as they allow investors and institutions to take various positions with very specific levels of protection.

As financial markets continued to evolve in the 1990s, barrier options became more standardized and accessible to the financial population. Derivative exchanges and financial institutions offered barrier options on various underlying assets, such as commodities, currencies, and interest rates. The 2008 global financial crisis sparked a renewed interest in derivative products for their risk-management capabilities, and barrier options remained one of the best products for their ability to tailor risk profiles to specific market conditions. In addition, computational tool advancements make pricing these options more manageable, thereby increasing the accessibility to market participants. This report will outline the many analytical and computational tools for pricing barrier options.

Chapter 2

Barrier Options

For some background, we will briefly discuss the attributes of a vanilla option. A call option gives the holder the right, but not the obligation, to buy a particular number of the underlying assets in the future for a pre-agreed price known as the strike price (put options give the holder the right, but not the obligation, to sell). While European options can only be exercised on the expiration date, American options allow the holder to exercise at any time on or before the expiration date. We will focus on European options throughout this research report.

Let S be the price of an underlying asset and K be the strike price, where $S, K \in \mathbb{R}^+$. Then the payoff for a vanilla call option,V_c, is given by the following

$$V_c(S,T) = \begin{cases} S_T - K & \text{if } S_t > K, \forall t \in [0,T) \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Likewise, the payoff for a vanilla put option, V_p, derived by the following:

$$V_p(S,T) = \begin{cases} 0 & \text{if } S_t > K, \forall t \in [0,T) \\ K - S_T & \text{otherwise} \end{cases} \tag{2.2}$$

These formulas drive our intuition of how both vanilla and exotic options can be priced.

As we can see, the payoff of a vanilla option depends only on the terminal value of the underlying asset. However, an exotic option, such as a barrier option, is very different. Its price is determined by whether the underlying asset's price reaches a certain level during a specific period. Barrier options differ from standard vanilla options in several ways.

First, they match the hedging needs more closely than standard options; second, premiums for barrier options are typically lower than vanilla options; and finally, the payoff of a barrier option matches beliefs about the future behavior of the market. These features benefit many different types of investors, regardless of experience or financial needs.

Another significant difference between barrier options and vanilla options is that barrier options are path-dependent. This means that the payoff depends on the process of the underlying asset. Another difference involves the possibility of a rebate. A rebate is a positive discount that a barrier option holder may receive if the barrier is never reached. For the purpose of outlining the analytical framework, we will not discuss rebates.

There are four different types of thresholds, or barriers, to consider which are:

- down-and-out
- up-and-out
- down-and-in
- up-and-in

Combined with calls and puts, we have 8 different types of barrier options in total. The payoff for a barrier option is either "knocked out" or "knocked in" if the price of the underlying crosses the barrier.

For example, let B be the barrier threshold and S_0 be the price of the underlying asset at time $t = 0$. Then, for any K the down-and-out call option with constant barrier $B < S_0$ has a payoff if the underlying prices stays below the barrier value until maturity T:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t > B, \forall t \in [0, T) \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

An up-and-out call option with constant barrier $B < S_0$ has a payoff if the underlying price does not go beyond the barrier value until maturity T:

$$\begin{cases} (S_T - K)^+ & \text{if } S_t < B, \forall t \in [0, T) \\ 0 & \text{otherwise} \end{cases} \tag{2.4}$$

A down-and-in call call option with a constant barrier $B < S_0$ has a payoff if the underlying prices stays below the barrier value until maturity T:

$$\begin{cases} 0 & \text{if } S_t > B, \forall t \in [0, T) \\ (S_T - K)^+ & \text{otherwise} \end{cases} \tag{2.5}$$

An up-and-in call option with a constant barrier $B < S_0$ has a payoff if the underlying prices stays beyond the barrier value until maturity T:

$$\begin{cases} 0 & \text{if } S_t < B, \forall t \in [0, T) \\ (S_T - K)^+ & \text{otherwise} \end{cases} \tag{2.6}$$

There are two main approaches to analytically evaluating the price of a barrier option: the probability method and the partial differential equation (PDE) method. The probability method involves the use of the reflection principal and the Girsanov theorem to estimate the barrier densities. The PDE approach is derived from the institution that all barrier options satisfy the Black-Scholes PDE but with different domains, expiry conditions, and boundary conditions.

Merton was the first to price barrier options using the PDE method, which he used to obtain the theoretical price of a down-and-out call option by using the PDE method to obtain a theoretical price. We will demonstrate this in the next section.

Chapter 3

Analytical Solutions for Barrier Options

There are closed-form solutions for pricing European-style barrier options. This means we have an explicit mathematical expression that can be used to compute the value of a function without the need for numerical solutions. However, we will continue to compare closed-form solutions to more rigorous methodologies. Unlike their continuous counterparts, no closed-form solutions exist for discrete-time barrier options (even numerical pricing is a challenge). For this reason, we will only focus on continuous-time, single-barrier options.

3.1 The Black-Scholes Model

The Black and Scholes model was first published in 1973, named after the two economist who helped to develop it: Fischer Black and Myrion Scholes. (the model is formally known as the Black-Scholes-Merton model) A rigorous derivation of the Wiener process, Ito's lemma, the portfolio process at the risk-free rate gives us the following equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - \frac{\partial f}{\partial} - rf = 0 \tag{3.1}$$

From here, we solve equation (3.1) to arrive at the following equation

$$f(S,t) = Se^{-qT}N(d_1) - Ke^{-rT}N(d_2) \tag{3.2}$$

where S is the stock price, K is the strike price, r is the risk-free rate, T is the time to expiration, σ is the volatility of the stock, $N(\cdot)$ is the cumulative distribution function, and d_1/d_2 are derived by the following:

$$d_1 = \frac{\ln\left({}^{S_0}/_{K}\right) + \left(r + {}^{\sigma^2}/_{2}\right)T}{\sigma\sqrt{T}}, \quad d_2 = \frac{\ln\left({}^{S_0}/_{K}\right) + \left(r - {}^{\sigma^2}/_{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{3.3}$$

A more rigourous proof for the solution to the Black-Scholes PDE can be found on A.1 of the appendix.

3.2 Black-Scholes Solution to Barrier Options

An essential concept in pricing barrier options is the in-out parity:

$$\text{Down-and-in call} + \text{ Down-and-out call} = \text{ Standard European Call}$$

which we can use to derive the following relationship for a down-and-in call optionV_{DIC} as:

$$V_{\text{DIC}}(S,t) = V_{\text{Call}}(S,t) - V_{\text{DOC}}(S,t).$$

We start by changing the value of T for $\tau = T - t$. From there, in order to have the PDE solutioni for the down-and-out call option, we begin to alter equation (3.2). Let H be the barrier price. Then when $B \geq K$, we have:

$$V_{\text{DOC}}(S,t) = Se^{-q\tau}\left(N(d_1) - \left(\frac{B}{S}\right)^{2\lambda} N(d_1')\right) - Ke^{-r\tau}\left(N(d_2) - \left(\frac{B}{S}\right)^{2\lambda-2} N(d_2')\right) \tag{3.4}$$

where

$$\lambda = \frac{r-q}{\sigma^2} + \frac{1}{2} \tag{3.5}$$

and d_1'/d_2' is derived by the following

$$d_1' = \frac{\ln\left(\frac{B^2}{SK}\right) + (r - q + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{T}}, \quad d_2' = d_1' - \sigma\sqrt{\tau} \tag{3.6}$$

Subtracting equation (3.4) from equation (3.2), we get the following:

$$V_{\text{DIC}}(S,t) = Se^{-q\tau}\left(\frac{H}{S}\right)^{2\lambda} N(d_1') - Ke^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N(d_2') \tag{3.7}$$

3.3 Barrier Option Payoffs

With eight different types of single barrier options comes eight possible payoffs, based on the barrier price. Table (3.3) shows the payoff based on whether the barrier is up or down, whether the stock price in or outside of the barrier, and whether the option type is a call or put. Refer to Appendix A.2

Table 3.1: Theoretical Values of Single Barrier Options

Down/Up	In/Out	Call/Put	Payoff ($K \leq B$)	Payoff ($K \geq B$)
Down	In	Call	$A_1 - A_2 + A_4 + A_5$	$A_3 + A_5$
Up	In	Call	$A_2 - A_2 + A_4 + A_5$	$A_1 + A_5$
Down	In	Put	$A_1 + A_5$	$A_2 - A_3 + A_4 + A_5$
Up	In	Put	$A_3 + A_5$	$A_1 - A_2 + A_4 + A_5$
Down	Out	Call	$A_2 - A_4 + A_6$	$A_1 - A_3 + A_6$
Up	Out	Call	$A_1 - A_2 + A_3 - A_4 + A_6$	A_6
Down	Out	Put	A_6	$A_1 - A_2 + A_3 - A_4 + A_6$
Up	Out	Put	$A_1 - A_3 + A_6$	$A_2 - A_4 + A_6$

Throughout this report, we will be deriving our analysis from the down-and-out call, since it is easier to intutively understand.

Chapter 4

Barrier Option Pricing with Monte Carlo

	Price	MC_{100}	err_{100}	MC_{1000}	err_{1000}	MC_{5000}	err_{5000}	MC_{10000}	err_{10000}
$S_0 = 100$ $B = 95$	5.05								
$S_0 = 100$ $B = 90$	8.5462								
$S_0 = 110$ $B = 110$	17.6635								
$S_0 = 120$ $B = 100$	23.4640								
$S_0 = 120$ $B = 110$	13.2648								

Chapter 5

Binomial Tree Method

The *binomial tree method* is a widely used approach to price derivative instruments, including barrier options. The method approximates the price evolution of the underlying asset by discretizing time into a finite number of steps, N, between the current time, $t = 0$, and the maturity, T. The accuracy of the method improves as the number of steps increases, with the binomial tree price converging to the theoretical option price.

5.1 Binomial Tree Structure

A binomial tree models the possible price movements of the underlying asset over time. At each step, the asset price can move either up or down. The price at any node in the tree is calculated using the formula:

$$S_{i,j} = S_0 \cdot u^j \cdot d^{i-j} \tag{5.1}$$

where:

- $S_{i,j}$ is the stock price at step i, level j,
- S_0 is the initial stock price,
- $u = e^{\sigma\sqrt{\Delta t}}$ is the up factor,
- $d = \dfrac{1}{u}$ is the down factor,
- $\Delta t = \dfrac{T}{N}$ is the time increment per step, and
- σ is the volatility of the underlying asset.

The risk-neutral probability of an upward movement, p, and a downward movement, q, are defined as:

$$p = \frac{e^{r\Delta t} - d}{u - d}, \quad q = 1 - p \tag{5.2}$$

where r is the risk-free interest rate.

5.2 Example Calculation

Let us consider an up-and-in European call option with the following parameters:

- Spot price (S_0): 100,
- Barrier level (B): 105,
- Strike price (K): 90,
- Risk-free rate (r): 0.05,
- Time to maturity (T): 1 year,
- Volatility (σ): 0.2,
- Number of steps (N): 3.

Using these parameters:

- The time increment per step is:

$$\Delta t = \frac{T}{N} = \frac{1}{3} \approx 0.3333\,\text{years.}$$

- The up and down factors are:

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.2\sqrt{0.3333}} \approx 1.1224, \quad d = \frac{1}{u} \approx 0.8909$$

- The risk-neutral probabilities are:

$$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05\cdot 0.3333} - 0.8909}{1.1224 - 0.8909} \approx 0.5438, \quad q = 1 - p = 0.4562.$$

5.3 Payoff Calculation

To calculate the option price, we build the binomial tree and track the stock price at each node. At maturity ($t = T$), the payoff for an up-and-in European call option is defined as:

$$\text{Payoff} = \begin{cases} \max(S_T - K, 0), & \text{if } \max(S_t) \geq B \text{ for } t \in [0, T], \\ 0, & \text{otherwise.} \end{cases} \tag{5.3}$$

For the given parameters:

- If the stock price reaches or exceeds $B = 90$ during its lifetime, the payoff is calculated as $\max(S_T - K, 0)$.
- Otherwise, the option expires worthless, with a payoff of 0.

5.4 Tree Construction and Results

Figure **??** illustrates the binomial tree for the given parameters, showing the possible stock price movements and the resulting option values at each node. The calculated option price, considering the up-and-in barrier condition, is obtained through backward induction along the tree. This ensures that the barrier condition is applied at each step. The final option price is 16.42.

The binomial tree method provides a flexible and intuitive framework for pricing options, including barrier options. However, as shown in this example, its computational requirements grow significantly with the number of steps N, making it less efficient for path-dependent options. Alternative methods, such as Monte Carlo simulations or analytical solutions, may be more suitable for pricing complex derivatives.

Chapter 6

The Greeks for Barrier Options

The Greeks are key sensitivities in option pricing that measure how the price of an option changes with respect to various factors such as the underlying asset price, time to maturity, volatility, and interest rates. For barrier options, the calculation of the Greeks is more complex due to the added condition of a barrier, which affects the option's price path and behavior. This chapter provides an explanation of how to calculate the Greeks for barrier options.

1. Delta (Δ)

Delta for a barrier option measures the sensitivity of the option price to changes in the underlying asset price. It represents the rate of change of the option price with respect to small changes in the asset price.

For a **knock-in** barrier option, delta is calculated similarly to standard options, but with the added complexity of the barrier. The delta reflects how changes in the underlying price affect the probability of hitting the barrier level and, thus, the likelihood of the option becoming active.

For **knock-out** options, delta will approach zero as the underlying asset approaches the barrier level, since the probability of the option being knocked out increases, making the option's price less sensitive to further changes in the underlying asset price.

2. Gamma (Γ)

Gamma for barrier options measures the rate of change of delta with respect to changes in the underlying asset price. It is the second derivative of the option price with respect to the asset price.

Gamma for barrier options can be derived from the standard Black-Scholes model, but with an adjustment for the probability of barrier activation. The higher the volatility, the larger the gamma for barrier options, especially near the barrier level.

For **knock-in** options, gamma is typically larger as the option approaches the barrier level, as it is more sensitive to changes in the underlying asset price.

3. Vega (ν)

Vega for barrier options reflects the sensitivity of the option price to changes in volatility. The formula for vega is similar to the standard Black-Scholes model but adjusted for the likelihood of hitting the barrier level.

For **knock-in** options, increased volatility tends to increase the likelihood of hitting the barrier, thus increasing the option price, and therefore vega will be positive. For **knock-out** options, increased volatility increases the chance of the option being knocked out, reducing the option price. Hence, vega for knock-out options will generally be negative.

4. Theta (Θ)

Theta for barrier options measures the rate of change of the option's price as time to maturity decreases. Barrier options are path-dependent, and their theta is influenced by the time remaining until the option either knocks in, knocks out, or expires. The closer the option is to the barrier, the more sensitive the theta becomes.

For **knock-in** options, as time to maturity decreases, the value of the option tends to decrease, particularly when the underlying asset price is far from the barrier. For **knock-out** options, time decay may have a more pronounced effect, especially if the underlying price is near the barrier.

5. Rho (ρ)

Rho for barrier options measures the sensitivity of the option price to changes in interest rates. It represents how much the option price changes when the risk-free interest rate changes by 1%.

For **knock-in** options, an increase in interest rates generally increases the option price, since higher rates make holding the option more attractive relative to other assets. For **knock-out** options, the effect of interest rate changes may be less pronounced, especially if the option is likely to be knocked out before maturity.

Bibliography

[1] Cox, John C. ”Options Markets.” (1985).

Appendix A

Formulas and Proofs

A.1 Black-Scholes PDE for Barrier Options

A.2 Payoffs Formulas for Single Barrier Options

The formula for Single Barrier options have been proved by Reiner and Rubinstein. The payoff of a Single Barrier call or put depends on the following formulas:

$$\begin{aligned}
A_1 &= Se^{-q\tau}aN(ax_1) - Ke^{-r\tau}N(ax_1 - a\sigma\sqrt{\tau}) \\
A_2 &= Se^{-q\tau}aN(ax_1) - Ke^{-r\tau}N(ax_2 - a\sigma\sqrt{\tau}) \\
A_3 &= Se^{-q\tau}a\left(\tfrac{B}{S}\right)^{2\lambda+2}N(bx_3) - K\left(\tfrac{B}{S}\right)^{2\lambda}e^{-r\tau}N(bx_3 - b\sigma\sqrt{\tau}) \\
A_4 &= Se^{-q\tau}a\left(\tfrac{B}{S}\right)^{2\lambda+2}N(bx_4) - K\left(\tfrac{B}{S}\right)^{2\lambda}e^{-r\tau}N(bx_4 - b\sigma\sqrt{\tau}) \\
A_5 &= Ke^{-rT}\left[N(bx_2 - b\sigma\sqrt{\tau}) - \left(\tfrac{B}{S}\right)^{2\lambda}N\left(bx_4 - b\sigma\sqrt{\tau}\right)\right] \\
A_6 &= Ke^{-rT}\left[N(bx_5 - b\sigma\sqrt{\tau}) - \left(\tfrac{B}{S}\right)^{2\lambda}N\left(bx_5 - b\sigma\sqrt{\tau}\right)\right]
\end{aligned}$$

with

$$\begin{cases} a = 1, -1 & \text{call or put} \\ b = 1, -1 & \text{out or in} \end{cases}$$

where x_1, x_2, x_3, x_4, x_5 is the following:

$$x_1 = \frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}, \quad x_2 = \frac{\ln\left(\frac{S}{B}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}$$

$$x_3 = \frac{\ln\left(\frac{B^2}{SK}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}, \quad x_4 = \frac{\ln\left(\frac{B}{S}\right)}{\sigma\sqrt{\tau}} + (1+\mu)\sigma\sqrt{\tau}$$

$$x_5 = \frac{\ln\left(\frac{B}{S}\right)}{\sigma\sqrt{\tau}} + \lambda\sigma\sqrt{\tau}$$

where μ and λ is the following

$$\mu = \frac{r - q - \frac{\sigma^2}{2}}{\sigma^2}, \quad \lambda = \sqrt{\mu^2 + \frac{2q}{\sigma^2}}$$