Question 1

$(\lambda ab.a)\, u\, (\lambda tcd.\, tcd)\, v$

$a\ [ab := u]$

$u\, (\lambda tcd.\, tcd)\, v$

$tcd\ [tcd := v]$

$\therefore$ β-normal form is uv.

Question 2

$(\lambda pte.pte)(\lambda ab.a)(\lambda x.\, xy)((\lambda v.vv)(\lambda w.ww))$

$pte\ [pte := \lambda ab.a]$

$(\lambda ab.a)(\lambda x.xy)((\lambda v.vv)(\lambda w.ww))$

$a\ [ab := \lambda x.xy]$

$(\lambda x.xy)((\lambda v.vv)(\lambda w.ww))$

$vv\ [v := \lambda w.ww]$

$(\lambda x.\, xy)\underbrace{((\lambda w.ww)(\lambda w.ww))}_{\text{Diverges}}$

$\therefore$ β-normal form is $(\lambda x.xy)$.

Q9.

We want to show that a $2^n \times 2^n$ board with the NW square removed can be covered by V3 blocks.

$$\text{Total Squares} = 2^n \times 2^n$$
$$= \cancel{4n}\ 4^n$$

$$\text{Remaining Squares} = 4^n - 1$$

Since each V3 block covers 3 squares, the number of squares must be divisible by 3.

$P(n) \iff 4^n - 1$ 'mod' $3 = 0$ (divisible by 3)

Base Case: $(n=1)$

~~P(1) true~~ $4^1 - 1 = 3$

3 'mod' $3 = 0$

$P(1) \iff 3$ is divisble by $3 \iff T$

Inductive Hypothesis:

Assume $P(k) \iff T$

$4^k - 1 = 3m$ for $m \in \mathbb{Z}^+$.

Prove $P(k+1) \iff T$

$$4^{k+1} - 1 = 4^k \cdot 4 - 1$$

Using IH: $4^k = 3m + 1$

$$4^{k+1} - 1 = 4(3m+1) - 1$$
$$4^{k+1} - 1 = 12m + 4 - 1$$
$$4^{k+1} - 1 = 12m + 3$$

$$4^{k+1} - 1 = 3(4m+1)$$

$\therefore$ Since $4m+1$ is a +ve integer, we can conclude that $4^{k+1}-1$ is divisible by 3.

$$\therefore P(k) \Rightarrow P(k+1)$$

$\therefore$ By PMI, $P(n)$ holds for all $n \geq 1$:

$$4^n - 1 \text{ 'mod' } 3 = 0 \quad \text{for } n \geq 1.$$

$\therefore$ Entire board can be covered with V3 ~~pieces~~ pieces if a square is removed.