# Homework 4

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## 1 The Standard Solar Model and a Polytrope Model

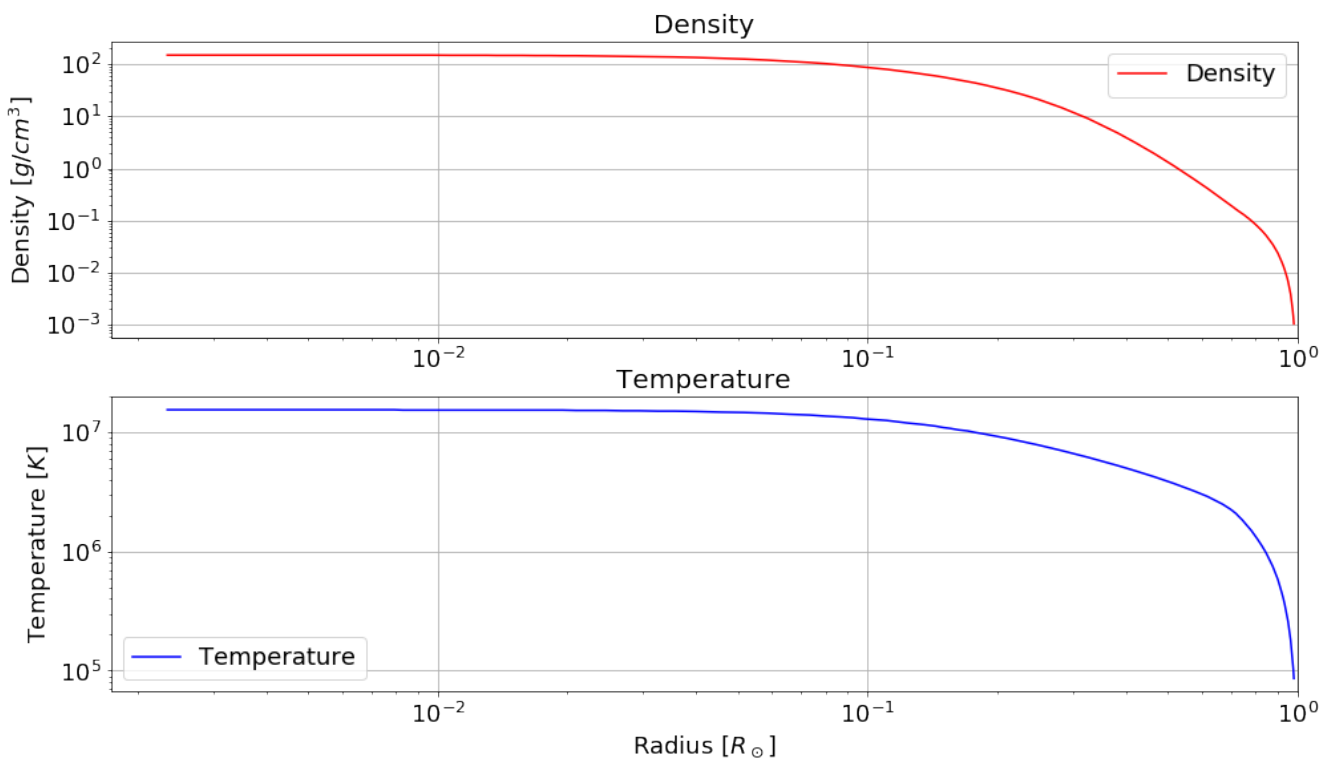


Figure 1: The density and the temperature of the model.

**(b)** Assuming that H and He are purely inoized. So the ion pressure is :

$$P_{ion} = (n_H + n_{He})kT = (\rho X/m_H + \rho Y/m_{He})kT$$

The electron pressure is:

$$P_e = n_e kT = (n_H + 2n_{He})kT = \rho(X/m_H + 2Y/m_{He})kT$$

The radiation pressure is:

$$P_{rad} = \frac{1}{3}aT^4$$

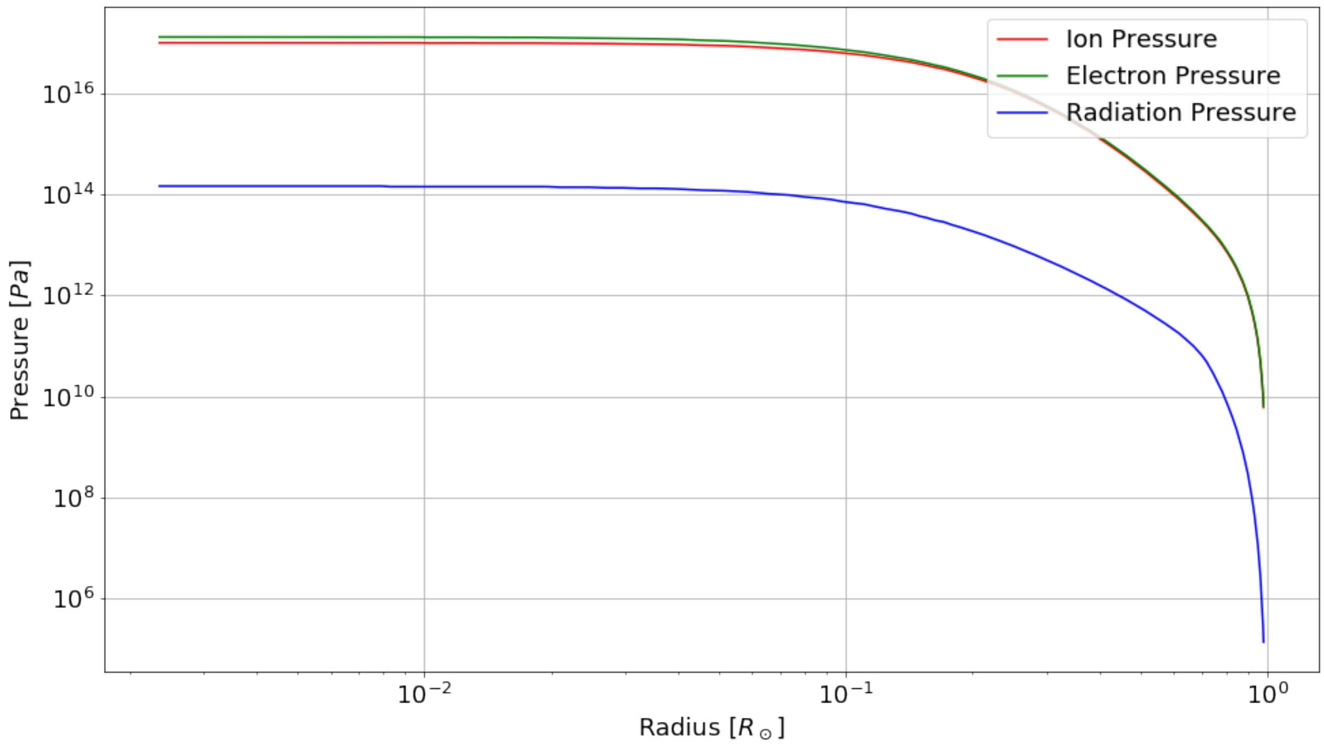


Figure 2: Three types of pressure.

So the ratio is:

$$\beta = \frac{P_{ion} + P_e}{P_{ion} + P_e + P_{rad}}$$

**(c)** The radiation force is much smaller than gravity so the radiation force is not so important in the sun.
**(d)** For polytrope model, $\log P \propto \frac{4}{3} \log \rho$. We can see that Standard model is similar with the polytrope model.
**(e)**
Considering the Eddington's standard model:

$$P = \left( \frac{3R}{a\mu^4} \frac{1-\beta}{\beta^4} \right)^{1/3} \rho^{4/3}$$

Then we can calculate the pressure and density:
**(f)**
Plot density, pressure and radius from two different models in one figure: We can see that the trend of density and pressure in two models is same but polytrope model lost some details in the center of star.

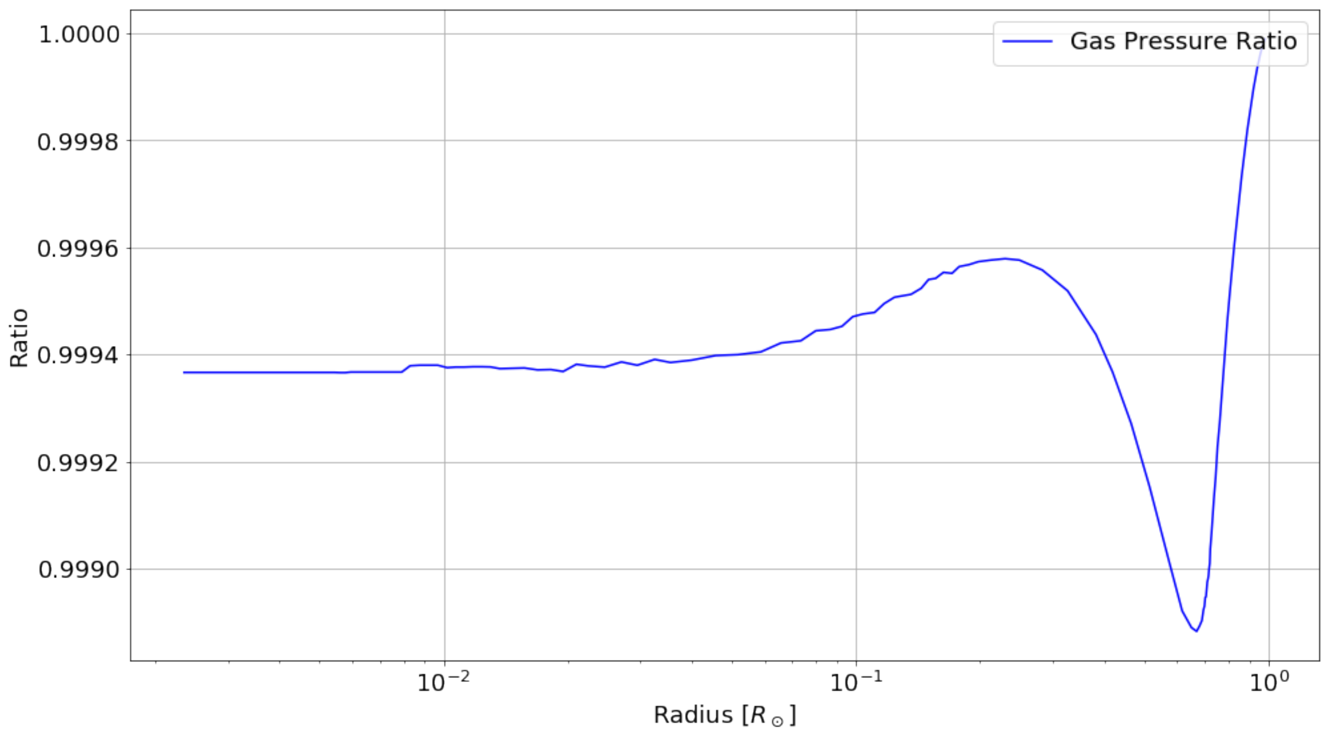


Figure 3: $\beta$

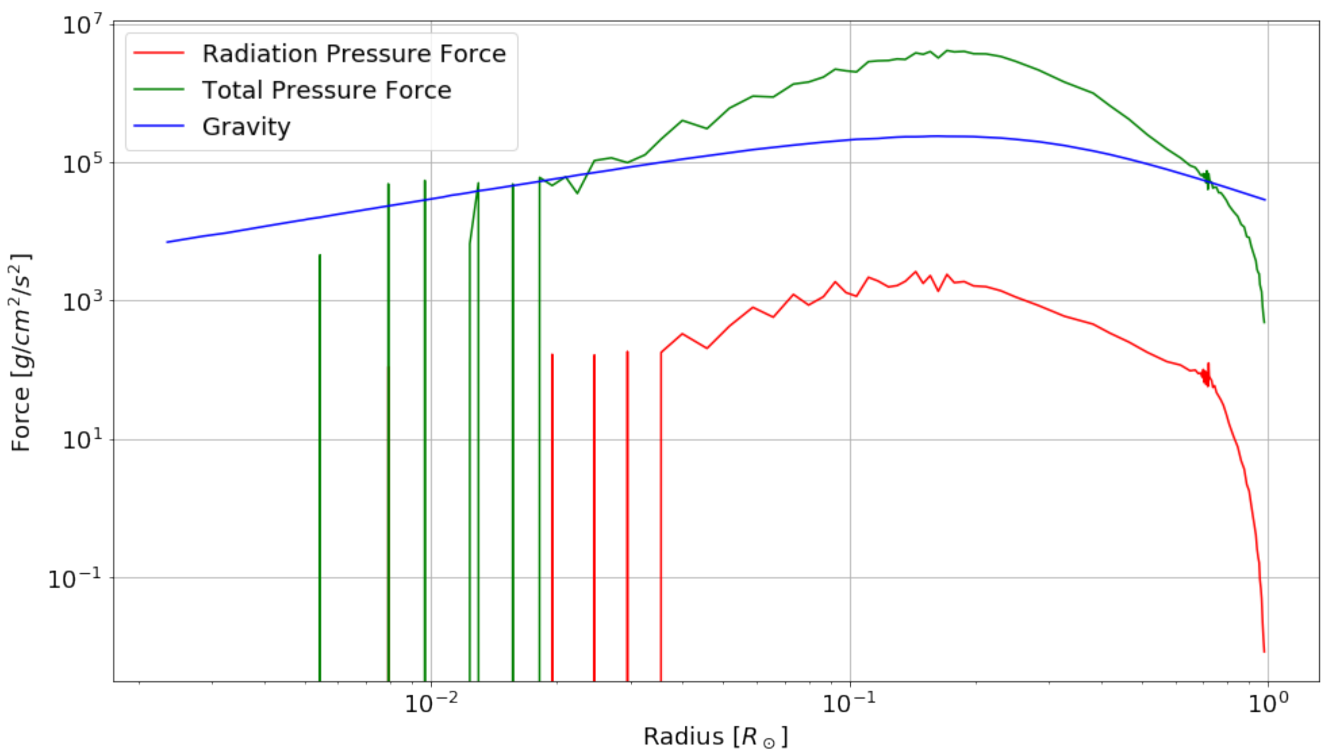


Figure 4: Radiation force compared with gravity, we can see that radiation force is much smaller than fravity.

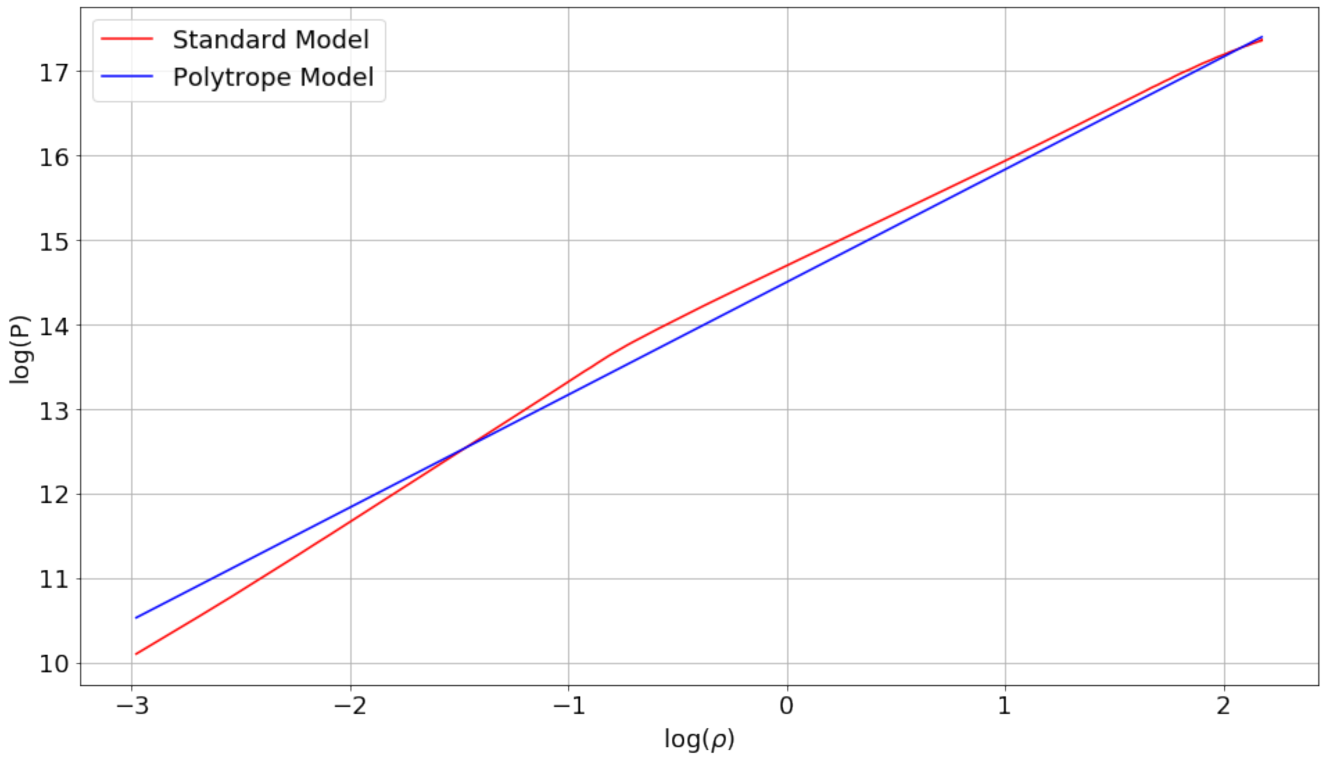


Figure 5: Standard Solar Model and polytrope. They are similar.

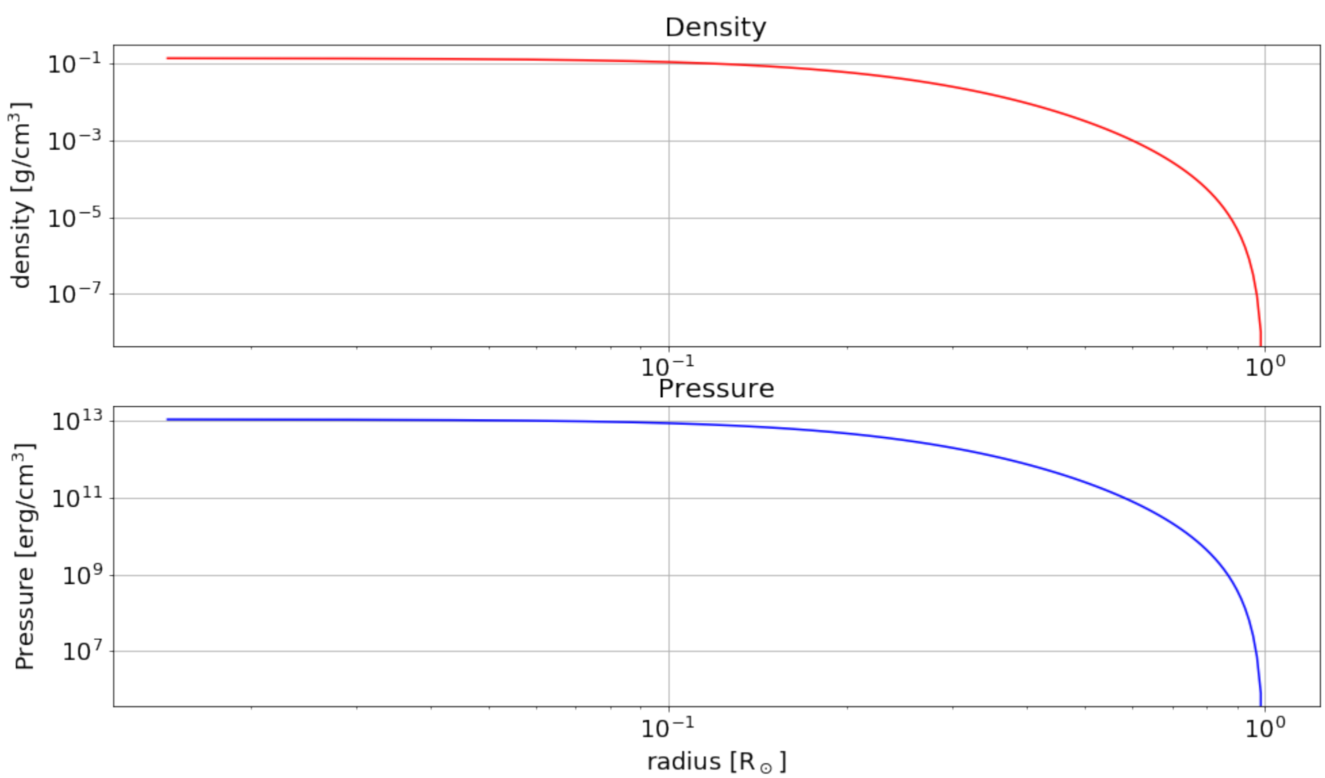


Figure 6: The density and pressure from polytrope model.

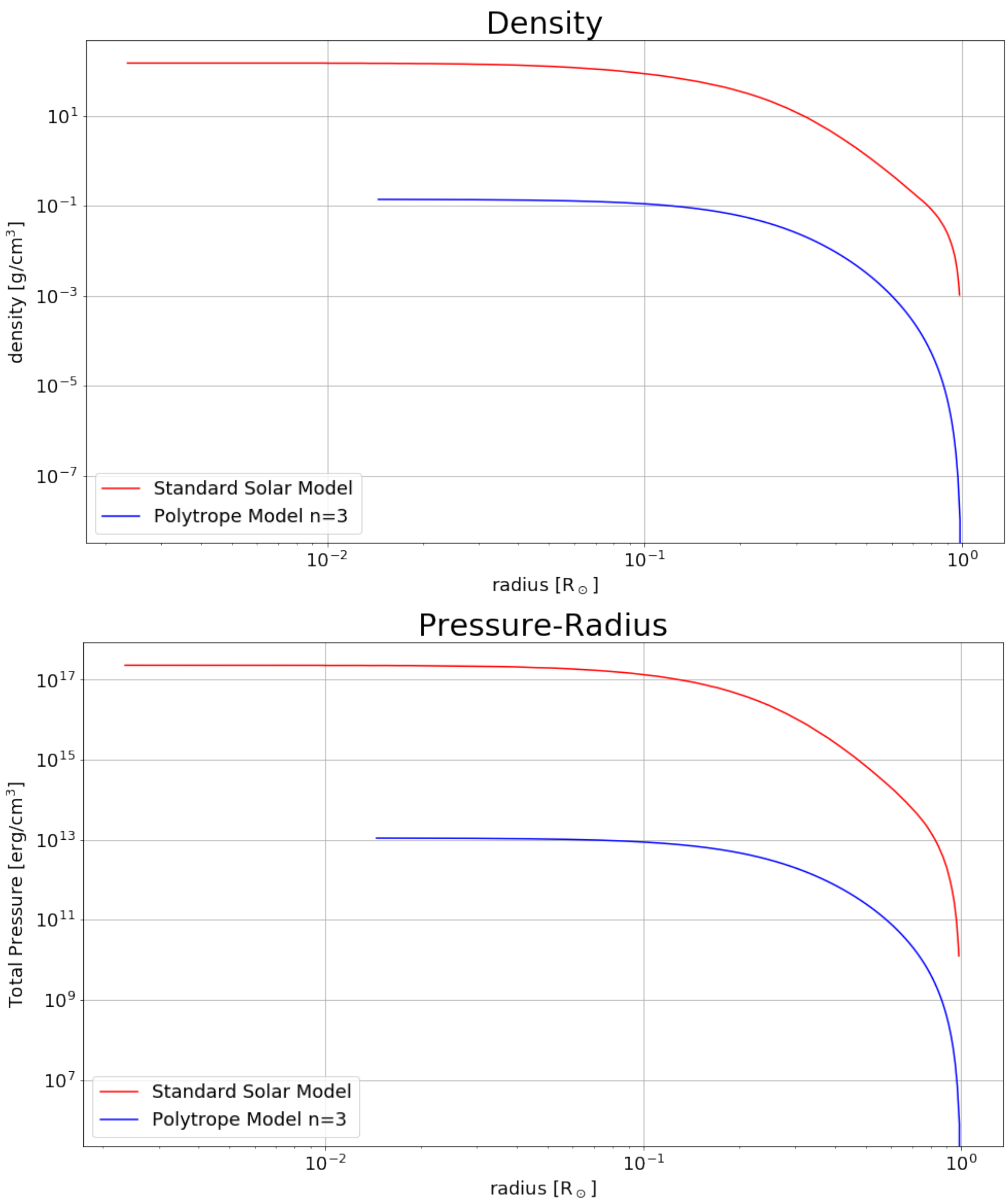


Figure 7: Density, pressure vs radius from two different models.

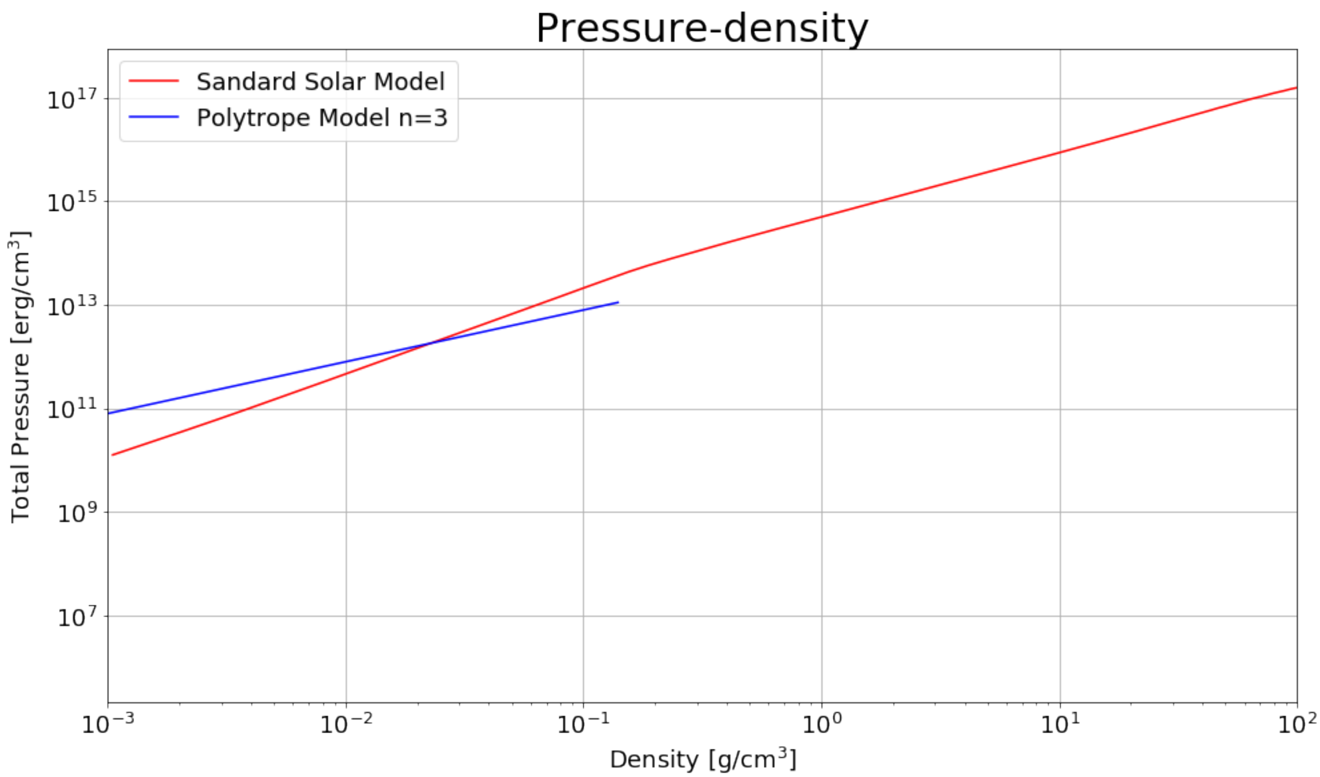


Figure 8: Density vs pressure from two different models.

## 2 Convection

**(a)** The energy flux is :

$$F_{conv} = \rho c_P T \left(\frac{l_c}{H_P}\right)^2 \sqrt{\frac{1}{2} g H_P} (\nabla - \nabla_{ad})^{3/2}$$

where $F_{conv} = L_\odot / 4\pi (R_\odot/2)^2 = L_\odot/\pi R_\odot^2$, $g = Gm(r)/r^2 = \frac{2GM_\odot}{R_\odot^2}$, thus $H_P = \frac{P}{\rho g} = \frac{\rho T}{\mu m_H g}$.

$$\nabla - \nabla_{ad} = \left(\frac{LR}{M}\right)^{2/3} \frac{R}{GM} = 1.37 \times 10^{-8}$$

Take $T = 10^7 K$, $\rho = 1 g/cm^3$, $X = 0.733, Y = 0.253, Z = 0.014$:

$$\frac{\delta T}{T} = \frac{l_c}{H_P} (\nabla - \nabla_{ad}) \approx 3.838 \times 10^{-8}$$

**(b)** The average velocity of the convective element is:

$$v_c = \sqrt{\frac{1}{2} l_c g \frac{\delta T}{T}} = 1.91 \times 10^3 cm/s$$

The sound speed :

$$c_s = \left(\frac{kT}{\mu m_H}\right)^{1/2} = 3.7 \times 10^7 cm/s$$

Thus $v_c/c_s \approx 5 \times 10^{-5}$. $v_c$ is much less than sound speed.
**(c)**

$$\beta = \frac{\rho v_c^2}{P} \approx \left(\frac{v_c}{c_s}\right)^2 \approx 2.5 \times 10^{-9}$$

The convection doesn't alter significantly the hydrostatic structure of the region.
**(d)**
The crossing time is:

$$t_c = \frac{l_c}{v_c} \approx 5.8 \times 10^{-2} yr$$

The nuclear timescale is:

$$t = \frac{\epsilon q M_\odot c^2}{L_\odot} \approx 10^{10} yr$$

The thermal timescale is:

$$t_{KH} = \frac{GM_\odot^2}{R_\odot L_\odot} \approx 10^7 yr$$

thus $t/t_c \approx 10^{11}$, $t_{KH}/t_c \approx 10^8$, which means convective elements can cross the region in a typical timescale for many times which means that the convective region is mixed well and have a uniform chemical composition.

# 3 Nuclear Reactions II

**(a)**
For reactions in PPI cycle:

$$\begin{aligned}
{}^1H + {}^1H &\rightarrow {}^2H + e^+ + \nu_e && (1)\\
{}^2H + {}^1H &\rightarrow {}^3He + \gamma && (2)\\
{}^3He + {}^3He &\rightarrow {}^4He + 2\,{}^1H && (3)\\
e^+ + e^- &\rightarrow 2\gamma && (4)
\end{aligned}$$

The released energy in each reaction is:

$$\begin{aligned}
E_1 &= (2m_{\,^1H} - m_{\,^2H} - m_e)c^2 = 420.22201 keV && (1)\\
E_2 &= (m_{\,^2H} + m_{\,^1H} - m_{\,^3He})c^2 = 5493.47799 keV && (2)\\
E_3 &= (2m_{\,^3He} - m_{\,^4He} - 2m_{\,^1H})c^2 = 12859.57366 keV && (3)\\
E_4 &= (m_{\,e^+} + m_{\,e^-})c^2 = 1021.99789 keV && (4)
\end{aligned}$$

**(b)**
The whole PPI reaction is made up of: $2 \times (1) + 2 \times (2) + (3) + 2 \times (4)$, thus the total energy released is: $E = 2E_1 + 2E_2 + E_3 + 2E_4 = 26.73097 MeV$.
**(c)**
The PPII reaction is made up of :

$$\begin{aligned}
{}^1H + {}^1H &\rightarrow {}^2H + e^+ + \nu_1 && (1)\\
{}^2H + {}^1H &\rightarrow {}^3He + \gamma && (2)\\
{}^3He + {}^4He &\rightarrow {}^7Be + \gamma && (3)\\
{}^7Be + e^- &\rightarrow {}^7Li + \nu_2 && (4)\\
{}^7Li + {}^1H &\rightarrow {}^4He + {}^4He && (5)\\
e^+ + e^- &\rightarrow 2\gamma && (6)
\end{aligned}$$

The whole PPII reaction is $(1) + (2) + (3) + (4) + (5) + (6)$:

$$4\,{}^1H + 2e^- \rightarrow {}^4He + 4\gamma + \nu_1 + \nu_2$$

thus the energy released is

$$\begin{aligned}
E_2 &= (4 \times m_{{}^1H} + 2 \times m_{e^-} - m_{{}^4He})c^2 - E_{\nu_1} - E_{\nu_2}\\
&= 26.73097 - 0.265 - 0.814\\
&= 25.65197 MeV
\end{aligned}$$

The PPIII reaction is made up of :

$$\begin{aligned}
{}^1H + {}^1H &\rightarrow {}^2H + e^+ + \nu_1 && (1)\\
{}^2H + {}^1H &\rightarrow {}^3He + \gamma && (2)\\
{}^3He + {}^4He &\rightarrow {}^7Be + \gamma && (3)\\
{}^7Be + {}^1H &\rightarrow {}^8B + \gamma && (4)\\
{}^8B &\rightarrow {}^8Be + e^+ + \nu_3 && (5)\\
{}^8Be &\rightarrow {}^4He + {}^4He && (6)\\
e^+ + e^- &\rightarrow 2\gamma && (7)
\end{aligned}$$

The whole PPIII reaction is $(1) + (2) + (3) + (4) + (5) + (6) + 2 \times (7)$:

$$4\,{}^1H + 2e^- \rightarrow {}^4He + 7\gamma + \nu_1 + \nu_3$$

thus the energy released is

$$\begin{aligned}
E_3 &= (4 \times m_{{}^1H} + 2 \times m_{e^-} - m_{{}^4He})c^2 - E_{\nu_1} - E_{\nu_3}\\
&= 26.73097 - 0.265 - 6.71\\
&= 19.75597 MeV
\end{aligned}$$

Considering the neutrino derived in PPI reaction, the energy released is:

$$\begin{aligned} E_1 &= (4 \times m_{\,^1H} + 2 \times m_{e^-} - m_{\,^4He})c^2 - 2 \times E_{\nu_1} \\ &= 26.73097 - 0.265 \times 2 \\ &= 26.20097 MeV \end{aligned}$$

Most energy is from PPI reaction, then the energy is from PPII reaction, PPIII reaction releases the least energy to the star.
For each reaction, the fuel is $4\,^1H + 2e^-$, so the energy released per unit mass is:

$$\epsilon_1 = E_1/(4m_{\,^1H} + 2e^-) = 6.27257 \times 10^{18} erg/g \tag{1}$$
$$\epsilon_2 = E_2/(4m_{\,^1H} + 2e^-) = 6.14113 \times 10^{18} erg/g \tag{2}$$
$$\epsilon_3 = E_3/(4m_{\,^1H} + 2e^-) = 4.72962 \times 10^{18} erg/g \tag{3}$$

**d**
The $3\alpha$ reaction is $3\,^4He \rightarrow \,^{12}C + \gamma$, thus the energy released is:

$$E = (3m_{\,^4He} - m_{\,^{12}C})c^2 = 7.27475 MeV$$

The energy generated per unit mass is:

$$\epsilon = E/(3m_{\,^4He}) = 5.84702 \times 10^{17} erg/g$$

**(e)**
If the stellar core is burning H by the PPI chain, the lifetime of it is:

$$\begin{aligned} t &= \frac{m\epsilon_1}{L_\odot} \\ &= \frac{0.2 \times 1.989 \times 10^{33} g\ 6.27257 \times 10^{12} erg/g}{3.828 \times 10^{33} erg/s} \\ &= 2.071 \times 10^{10} yr \end{aligned}$$

If it is burning He:

$$\begin{aligned} t &= \frac{m\epsilon}{L_\odot} \\ &= \frac{0.2 \times 1.989 \times 10^{33} g\ 5.84702 \times 10^{17} erg/g}{3.828 \times 10^{33} erg/s} \\ &= 1.927 \times 10^{10} yr \end{aligned}$$

# 4 Rotational Velocities

**(a)**

Considering a small particle on the surface of stellar, if it doesn't leave stellar surface, its velocity will not larger than first cosmic velocity of the stellar, which means:

$$m\omega^2 R \ll \frac{GMm}{R^2}$$
$$\delta = \frac{\omega^2 R^3}{GM} \ll 1$$

**(b)**

For the data in RotationalVelocitiesMainSequence.txt, considering the $M_\odot = 1.989 \times 10^{33}$g, $R_\odot = 6.963 \times 10^{10}$cm, calculated $\delta$ for each type of star, their $\delta$ is:

| SpT | O5 | B2.5 | B5 | A0 | A7 | F0 | F5 | G0 | G2 |
|---|---|---|---|---|---|---|---|---|---|
| $\delta$ | 37.87 | 96.81 | 135.90 | 137.09 | 115.63 | 43.79 | 3.185 | 0.711 | 0.028 |

From this table we can see that $\delta$ for A0 star is the largest.

# 5 Code

## 5.1 The Standard Solar Model and a Polytrope Model

```
import numpy as np
import matplotlib.pyplot as plt
from astropy.io import fits
model = np.loadtxt('SolarModel.txt')
m = model[:,0]
r = model[:,1]
T = model[:,2]
rho = model[:,3]
F = model[:,4]
X = model[:,5]
Y = model[:,6]
plt.figure(figsize=(16,9))
ax1 = plt.subplot(2,1,1)
ax1.plot(r,rho,'r',label='Density')
#ax1.set_xlabel(r'Radius [$R_\odot$]',fontsize=18)
ax1.set_ylabel(r'Density [$g/cm^3$]',fontsize=18)
ax1.set_title('Density',fontsize=20)
ax1.grid()
ax1.legend(fontsize=18)
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.loglog()
plt.xlim(0,1)
ax2 = plt.subplot(2,1,2)
ax2.plot(r,T,'b',label='Temperature')
ax2.set_xlabel(r'Radius [$R_\odot$]',fontsize=18)
ax2.set_ylabel(r'Temperature [$K$]',fontsize=18)
ax2.set_title('Temperature',fontsize=20)
```

```
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.legend(fontsize=18)
plt.grid()
plt.loglog()
plt.xlim(0,1)
me = 9.11*10**(-28)
mu = 1.67*10**(-24)
#mu_e = me/0.5/mu
mu_e = 1
h = 1.38*10**(-16)
c = 3*10**10
e = 4.83*10**(-10)
Z = 1
A = 1
E = 170
h = 6.626*10**(-27)
u = 0.5
mu = 1.67*10**(-24)
me = 9.11*10**(-28)
k = 1.38*10**(-16)
R = k/mu
print(R/10**7)
K_NR = h**2/20/me/mu**(5/3)*(3/np.pi)**(2/3)
print(K_NR/10**12)
K_ER = h*c/8/mu**(4/3)*(3/np.pi)**(1/3)
print(K_ER/10**15)
#K_NR = 1.0036*10**13 #
#K_ER = 1.2435*10**15 #
#K_NR = h**2/20/me/mu**(5/3)*(3/np.pi)**(2/3)
#K_ER = h*c/8/mu**(4/3)*(3/np.pi)**(1/3)
R = 8.314*10**7
a = 7.56*10**(-15)
rho_g = np.log10(mu_e*(K_ER/K_NR)**3)
C = 1*(4.83*10**(-10))**2/k/170*(4*np.pi/3/mu)**(1/3)
plt.figure(figsize=(21,12))
rho = np.linspace(-10,10,1000)
rho1 = np.linspace(-10,rho_g,1000)
rho2 = np.linspace(rho_g,10,1000)
T1 = 2/3*rho1+np.log10(K_NR*0.5/R/mu_e**(5/3))
T2 = 1/3*rho2+np.log10(K_ER*0.5/R/mu_e**(4/3))
T3 = 1/3*rho+1/3*np.log10(3*R/a)
T4 = 1/3*rho+np.log10(C)
plt.plot(rho1,T1,'r',linewidth=2)
plt.plot(rho2,T2,'r',linewidth=2)
plt.plot(rho,T3,'b',linewidth=2)
plt.plot(rho,T4,'yellow',linewidth=2)
```

```
plt.vlines(rho_g,5.2,8.9,'green','--',linewidth=2)
plt.ylim(2,12)
plt.xlim(-10,10)
plt.xlabel(r'Log $\rho$ [$g \cdot cm^{-3}$]',fontsize=18)
plt.ylabel(r'Log $T$ [$K$]',fontsize=18)
plt.title(r'log $T$ - log $\rho$',fontsize=24)
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.text(-7.5,9,'Radiation',fontsize=30)
plt.text(-9,3,'Ideal gas',fontsize=28)
plt.text(2.5,6,'Non-relativistic',fontsize=18)
plt.text(2.5,5.5,'Degenerate Electrons',fontsize=18)
plt.text(7.5,7.5,'Relativistic',fontsize=18)
plt.text(7.5,7,'Degenerate Electrons',fontsize=18)
plt.text(5,3,'Crystallization of the ions',fontsize=18)
x1,y1 = np.array(4),np.array(6)
x2,y2 = 8,8
x3,y3 = np.log10(1.622*10**2),np.log10(1.571*10**7)
plt.scatter(x1,y1,s=np.array(500),c='yellow',marker='*',label='Low mass White Dwarf')
plt.scatter(x2,y2,s=np.array(500),c='red',marker='*',label='High mass White Dwarf')
plt.scatter(x3,y3,s=np.array(500),c='blue',marker='*',label='Sun')
T = model[:,2]
rho = model[:,3]
plt.plot(np.log10(rho),np.log10(T),c='black',label='Solar Model',linewidth=5)
plt.legend(loc='upper left',fontsize=18)
plt.grid()
#plt.savefig('Contours.pdf')
k = 1.38*10**(-16)
a = 7.56*10**(-15)
m_H = 1.00784*1.661*10**(-24)
m_He = 4.002602*1.661*10**(-24)

n_H = rho*X/m_H
n_He = rho*Y/m_He
P_ion = (n_H+n_He)*k*T
P_e = (n_H+2*n_He)*k*T
P_rad = 1/3*a*T**4
plt.figure(figsize=(16,9))
plt.plot(r,P_ion,'r',label='Ion Pressure')
plt.plot(r,P_e,'green',label='Electron Pressure')
plt.plot(r,P_rad,'b',label='Radiation Pressure')
plt.legend(loc='upper right',fontsize=18)
plt.xlabel(r'Radius [$R_\odot$]',fontsize=18)
plt.ylabel(r'Pressure [$Pa$]',fontsize=18)
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.grid()
plt.loglog()
```

```
beta = (P_ion+P_e)/(P_ion+P_e+P_rad)
plt.figure(figsize=(16,9))
plt.plot(r, beta, 'b', label='Gas Pressure Ratio')
plt.legend(loc='upper right',fontsize=18)
plt.xlabel(r'Radius [$R_\odot$]',fontsize=18)
plt.ylabel(r'Ratio',fontsize=18)
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.grid()
plt.semilogx()
G = 6.67*10**(-8)
R_sun = 6.963*10**10
P_tot = P_ion+P_e+P_rad
F_p = []
F_ptot = []
for i in range(len(r)-1):
    F_p.append((-P_rad[i+1]+P_rad[i])/R_sun/(r[i+1]-r[i])*4*np.pi*r[i]**2)
    F_ptot.append((-P_tot[i+1]+P_tot[i])/R_sun/(r[i+1]-r[i])*4*np.pi*r[i]**2)
F_g = G*m/(r*R_sun)**2*2*10**33
plt.figure(figsize=(16,9))
plt.plot(r[:-1], F_p, 'r', label='Radiation Pressure Force')
plt.plot(r[:-1], F_ptot,'g',label='Total Pressure Force')
plt.plot(r, F_g, 'b', label = 'Gravity')
plt.legend(loc='upper left',fontsize=18)
plt.xlabel(r'Radius [$R_\odot$]',size=18)
plt.ylabel(r'Force [$g/cm^2/s^2$]',size=18)
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.loglog()
plt.grid()
P_tot = P_ion+P_e+P_rad
P_pol = 4/3*np.log10(rho)+14.5
plt.figure(figsize=(16,9))
plt.plot(np.log10(rho), np.log10(P_tot),'r', label='Standard Model')
plt.plot(np.log10(rho), P_pol, 'b', label='Polytrope Model')
plt.xlabel(r'log($\rho$)',fontsize=18)
plt.ylabel('log(P)',fontsize=18)
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.legend(loc='upper left',fontsize=18)
plt.grid()
import astropy.constants as c
import astropy.units as u
Lane = np.loadtxt('polytrope_n3.txt')
Xi = Lane[:,0]
Theta = Lane[:,1]
n = 3
z_n = 6.89685
```

```
mu = 2
beta_p = 0.9995853

R = c.R
R_sun = c.R_sun

#K = (3*c.R**4/a/mu**4*(1-beta_p)/beta_p**4)**(1/3)
#alpha = c.R_sun/z_n
#rho_c = (alpha**2*4*np.pi*c.G/(n+1)/K)**(n/(n-1))

K = (3*R**4/a/mu**4*(1-beta_p)/beta_p**4)**(1/3)
alpha = R_sun/z_n
rho_c = (alpha**2*4*np.pi*c.G/(n+1)/K)**(n/(n-1))

r_p = Xi*alpha
rho_p = rho_c*Theta**n

plt.figure(figsize=(16,9))
ax1 = plt.subplot(2,1,1)
ax1.semilogy(r_p.to(u.Rsun),rho_p.cgs,'r')
plt.title('Density',fontsize=20)
plt.tick_params(labelsize=18)
#plt.xlabel(r'$r(\mathrm{R_\odot})$',fontsize=18)
plt.ylabel(r'density $[\mathrm{g/cm^3}]$',fontsize=18)
plt.semilogx()
plt.grid()

P_p = K*rho_p

ax2 = plt.subplot(2,1,2)
plt.semilogy(r_p.to(u.Rsun),P_p.cgs,'b')
plt.tick_params(labelsize=18)
plt.title('Pressure',fontsize=20)
plt.xlabel(r'radius $[\mathrm{R_\odot}]$',fontsize=18)
plt.ylabel(r'Pressure $[\mathrm{erg/cm^3}]$',fontsize=18)
plt.semilogx()
plt.grid()
plt.show()
plt.figure(figsize=(16,9))
plt.semilogy(r,rho, 'r',label='Standard Solar Model')
plt.semilogy(r_p.to(u.Rsun),rho_p.cgs, 'b',label='Polytrope Model n=3')
plt.title('Density',fontsize=30)
plt.tick_params(labelsize=18)
plt.xlabel(r'radius $[\mathrm{R_\odot}]$',fontsize=18)
plt.ylabel(r'density $[\mathrm{g/cm^3}]$',fontsize=18)
plt.legend(fontsize=18)
plt.semilogx()
plt.grid()
```

```
plt.show()
plt.figure(figsize=(16,9))
plt.semilogy(r,P_tot, 'r',label='Standard Solar Model')
plt.semilogy(r_p.to(u.Rsun),P_p.cgs, 'b',label='Polytrope Model n=3')
#plt.title('Density-Radius',fontsize=30)
plt.tick_params(labelsize=18)
plt.title('Pressure-Radius',fontsize=30)
plt.xlabel(r'radius $[\mathrm{R_\odot}]$',fontsize=18)
plt.ylabel(r'Total Pressure $[\mathrm{erg/cm^3}]$',fontsize=18)
plt.legend(fontsize=18)
plt.semilogx()
plt.grid()
plt.show()
plt.figure(figsize=(16,9))
plt.loglog(rho,P_tot, 'r',label='Sandard Solar Model')
plt.loglog(rho_p.cgs,P_p.cgs, 'b',label='Polytrope Model n=3')
plt.legend(fontsize=18)
plt.tick_params(labelsize=18)
plt.title('Pressure-density',fontsize=30)
plt.xlabel(r'$\rho(\mathrm{g/cm^3})$',fontsize=18)
plt.ylabel(r'$P_\mathrm{t}(\mathrm{erg/cm^3})$',fontsize=18)
plt.xlim((10**-3,10**2))
plt.semilogx()
plt.grid()
plt.show()
```