PEKING UNIVERSITY

STELLAR STRUCTURE AND EVOLUTION

Homework 5

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December 4, 2019

1 Tides

(a)

Assuming that orbitals are circular and coplanar, the total angular momentum is combined by four compositions: Earth's spin, Earth's orbital angular momentum, Moon's spin and Moon's orbital angular momentum. Assuming that Earth and Moon are both spherical symmetry, we can get the rotational inertia. After searching the information about mass, orbital period, spin period, radius and distance, we can calculate the angular momentum:

$$\begin{aligned}
I_E &= \frac{2}{5} M_E R_E^2 \\
I_M &= \frac{2}{5} M_M R_M^2 \\
J_{ES} &= I_E \omega_{ES} = I_E \frac{2\pi}{T_{ES}} \\
&= 7.0709 \times 10^{33} kg \cdot m^2/s \\
J_{EO} &= m_E \omega_{MO} r_E^2 = m_E \frac{2\pi}{T_M} \left(\frac{m_M}{m_E} r_M \right)^2 \\
&= 3.4642 \times 10^{32} kg \cdot m^2/s \\
J_{MS} &= I_M \omega_{MS}^2 = I_M \frac{2\pi}{T_{MS}} = 2.3596 \times 10^{29} kg \cdot m^2/s \\
J_{MO} &= m_M \omega_{MO} r_M^2 = m_M \frac{2\pi}{T_M} r_M^2 \\
&= 2.8179 \times 10^{34} kg \cdot m^2/s \\
J_{tot} &= J_{ES} + J_{EO} + J_{MS} + J_{MO} = 3.5597 \times 10^{34} kg \cdot m^2/s
\end{aligned}$$

where J_{ES} is Earth spin angular momentum, J_{EO} is Earth orbital angular momentum, J_{MS} is Moon spin angular momentum, J_{MO} is Moon orbital angular momentum.

From the calculation we can see that orbital angular momentum of Earth and the spin angulat momentum of Moon are much less than other terms.

(b)

Considering the keplarian Third law:

$$\begin{aligned}
r_M^3 \omega_M^2 &= const \\
2\dot{\omega_M} r_M &= -3\omega_M \dot{r_M}
\end{aligned}$$

The orbital angular momentum of Earth and spin angular momentum of Moon are so small that can be ignored, thus the total angular momentum is:

$$\begin{aligned}
J_{tot} &= J_{ES} + J_{EO} + J_{MS} + J_{MO} \\
&= I_E \omega_{ES} + m_M r_M^2 \omega_M \\
&= I_E \omega_{ES} + m_M r_M^2 \omega_M = const
\end{aligned}$$

Then the differential of total angular momentum is:

$$
\begin{aligned}
\dot{J_{tot}} &= I_E\dot{\omega_{ES}} - m_M\frac{3\omega_M\dot{r_M}}{2}r_M + 2m_M\omega_M r_M\dot{r_M} \\
&= I_E\dot{\omega_{ES}} + \frac{1}{2}m_M\omega_M r_M\dot{r_M} = 0
\end{aligned}
$$

Thus the Earth is slowing down its rotation at a rate proportional to the increase in the orbital radius of the Moon.

$$
d\omega_{ES} = -\frac{m_M\omega_M r_M}{2I_E}dr_M
$$

(c)

The changing of daytime on Earth is:

$$
\begin{aligned}
dT &= -\frac{2\pi}{\omega_{ES}^2}d\omega_{Es} \\
&= \frac{2\pi}{\omega_{ES}^2}\frac{m_M\omega_M r_M}{2I_E}dr_M \\
&= 1.6973 \times 10^{-5}s
\end{aligned}
$$

Thus the length of Earths day changing is 1.7×10^{-5} s per year.

2 The Pre-main Sequence

(a)

From energy conservation:

$$\dot{E_{tot}} = L_{nuc} - L$$

Total energy of star is: $E_{tot} = E_{grav} + E_{int} = \Omega + U$, where Ω is gravitational potential energy and U is internal energy. From virial theorem: $2U + \Omega = 0$, $E_{tot} = 0.5\Omega$ Thus:

$$L = L_{nuc} - \dot{E_{tot}} = L_{nuc} - 0.5\dot{\Omega} = L_{nuc} + L_{grav}$$

(b)

If the star has no nuclear sources, then $L = -0.5\dot{\Omega}$, where $\Omega = -\frac{GM^2}{R}$. Thus:

$$\begin{aligned} L &= -0.5\frac{d\Omega}{dt} = \frac{2GM^2}{R^2}\frac{dR}{dt} \\ \frac{2L}{GM^2}t &= \frac{1}{R_0} - \frac{1}{R} \\ R(t) &= \frac{1}{\frac{1}{R_0} - \frac{2L}{GM^2}t} = \frac{R_0}{1 + t/\tau} \end{aligned}$$

where $\tau = -0.5GM^2/R_0L = 0.5\Omega(R_0)/L$

(c)

Considering the polytrope model with n = 1.5, we can know the relation between central pressure and density:

$$\begin{aligned} P_c &= K\rho_c^{(n+1)/n} \\ K &= N_n GM^{(n-1)/n}R^{(3-n)/n} \end{aligned}$$

where $N_n = 0.42422$. Assume that gas in the center is ideal, then the temperature is:

$$\begin{aligned} T &= \frac{K\rho_c^{2/3}m_p}{k_B} \\ &= \frac{Km_p}{k_B}\rho_c^{2/3} \\ &= \frac{Gm_p}{k_B}\frac{N_n z_n^2}{(4\pi\Theta_n)^{2/3}}\frac{M}{R} \end{aligned}$$

The gravitational potential energy for star is:

$$\begin{aligned} \Omega &= -\frac{3}{5-n}\frac{GM^2}{R} \\ &= -\frac{6}{7}\frac{GM^2}{R} \end{aligned}$$

The central temperatue is:

$$\begin{aligned} T &= \frac{Gm_p}{k_B} \frac{N_n z_n^2}{(4\pi\Theta_n)^{2/3}} \frac{M}{R} \\ &= \frac{Gm_p}{k_B} \frac{N_n z_n^2}{(4\pi\Theta_n)^{2/3}} \frac{M}{R_0}(1+t/\tau) \end{aligned}$$

(d)

The initial condition is:

$$R_0 = 4R_\odot$$

Then the radius is:

$$\begin{aligned} R(t) &= \frac{R_0}{1+t/\tau} \\ &= \frac{2.7828 \times 10^9}{1 + \frac{t}{8.4108\times 10^5\ yr}}\ m \end{aligned}$$

The central temperature is:

$$\begin{aligned} T_c &= \frac{Gm_p}{k_B} \frac{N_n z_n^2}{(4\pi\Theta_n)^{2/3}} \frac{M}{R_0}(1+t/\tau) \\ &= 3.1112 \times 10^6\ (1 + \frac{t}{8.4108 \times 10^5})\ K \end{aligned}$$

(d)

From figure 2 we can see it takes about 1.86×10^6 years for the star to contract to the main sequence. The radius is about $1.245 \times \mathrm{R}_\odot$.

The thermal timescale is:

$$\begin{aligned} \tau_{KH} &\approx 11.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^2 \frac{R_\odot}{R} \frac{L_\odot}{L}\ yr \\ &= 9.375 \times 10^5\ yr \end{aligned}$$

The thermal timescale is less than this time.

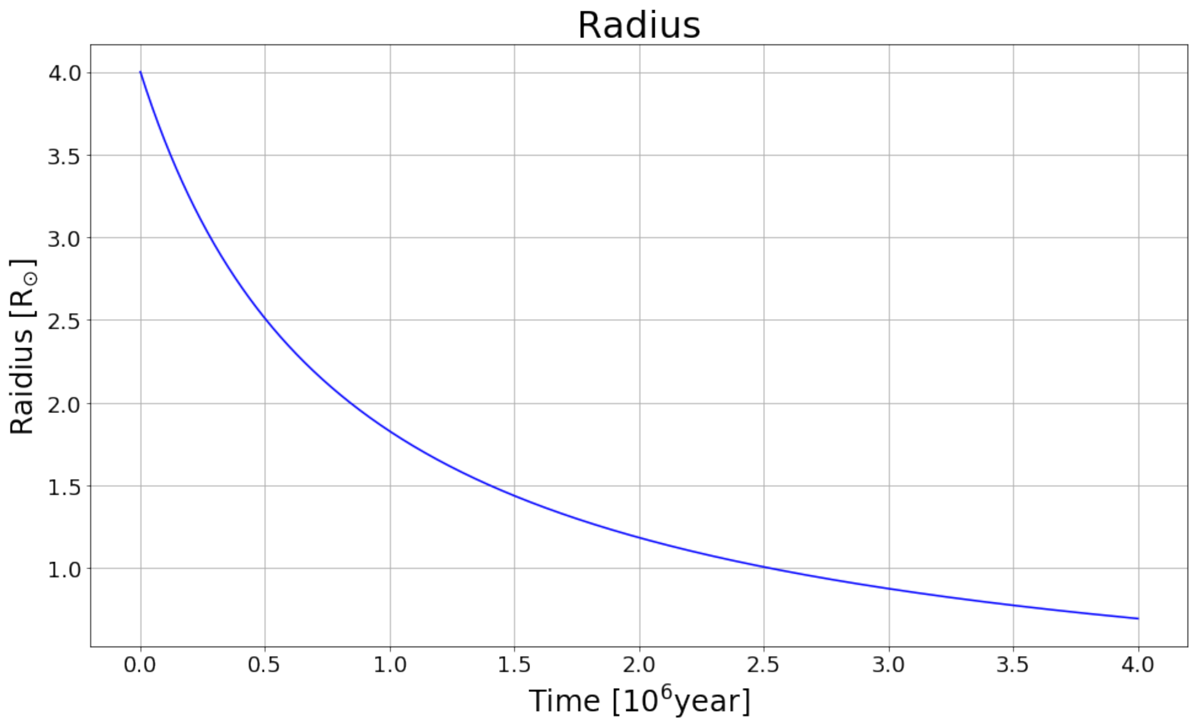

Figure 1: The radius as a function of time.

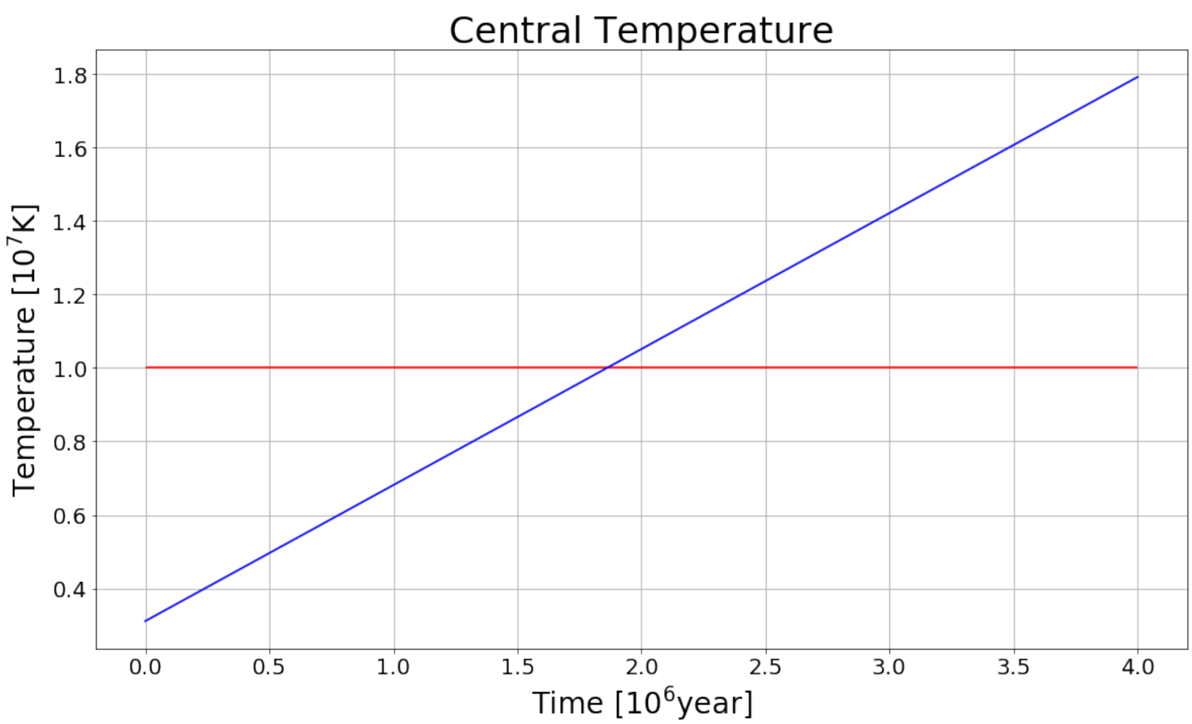

Figure 2: Central temperature as a function of time.

3 Supernovae and Neutron Stars

(a)

Considering the neutrino is a kind of relativistic particle, the relation between its energy and momentum is $P = E/c$. Assume mass of neutron star is $1.4M_\odot$, from momentum conservation we can know the kick velocity of star is:

$$\begin{aligned} P_{NS} &= P_\nu \\ m_{NS}v_{NS} &= 1\%E_{tot}/c \\ v_{NS} &= 119.82\ km/s \end{aligned}$$

(b) The rotation energy of neutron star is:

$$\begin{aligned} E_{rot} &= \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}I\frac{4\pi^2}{P^2} \end{aligned}$$

where I is rotational inertia, ω is angular velocity. Then the power of energy loss of a neutron star is:

$$dE/dt = \frac{1}{2}I\frac{-8\pi}{P^3}\dot{P}$$

Thus the pular spin-down time is:

$$\begin{aligned} t &= E_{rot}/|dE/dt| \\ &= \frac{P}{2\dot{P}} \\ &= 16666.67\ yr \end{aligned}$$

(c)

The cross section for neutrino with energy 1MeV is:

$$\sigma_\nu = 10^{-45}A^2\left(\frac{1MeV}{511keV}\right)^2 = 7.5\times10^{-43}\ cm^2$$

The number of particles in a man's body is:

$$N = \frac{70kg}{14 * m_p} = 2.99\times10^{27}$$

Assume that the lethal dose for men is 10J/kg:

$$\begin{aligned} \frac{10^{53}erg}{4\pi d^2}N\sigma_\nu &= 10J/kg\times 70kg \\ d &= 5.05\times10^{13}cm \end{aligned}$$

4 Half-life of Ni

(a)

The number density of ^{56}Ni is:

$$\begin{aligned}\frac{dN_{Ni}}{dt} &= -N_{Ni}\lambda_{Ni} \\ &= -N_{Ni}\frac{\ln 2}{\tau_{1/2Ni}}\end{aligned}$$

where N_{Ni} is the number density of ^{56}Ni, $\tau_{1/2Ni}$ is 6.1 d.

The number density of ^{56}Co is:

$$\begin{aligned}\frac{dN_{Co}}{dt} &= -N_{Co}\lambda_{Co} + N_{Ni}\lambda_{Ni} \\ &= -N_{Co}\frac{\ln 2}{\tau_{1/2Co}} + N_{Ni}\frac{\ln 2}{\tau_{1/2Ni}}\end{aligned}$$

where N_{Co} is the number density of ^{56}Co, $\tau_{1/2Co}$ is 77.1d, N_{Ni} is the number density of ^{56}Ni, $\tau_{1/2Ni}$ is 6.1d.

The number density of ^{56}Fe is:

$$\begin{aligned}\frac{dN_{Fe}}{dt} &= N_{Co}\lambda_{Co} \\ &= N_{Co}\frac{\ln 2}{\tau_{1/2Co}}\end{aligned}$$

(b)

Assume that the supernova produced ^{56}Ni number density is N_0, then the number density of ^{56}Ni is:

$$N_{Ni} = N_0 e^{-\lambda_{Ni} t}$$

The number density of ^{56}Co is:

$$N_{Co} = \frac{N_0\lambda_{Ni}}{\lambda_{Co} - \lambda_{Ni}}\left(e^{-\lambda_{Ni}t} - e^{-\lambda_{Co}}\right)$$

The number density of ^{56}Fe is:

(c) From above we can know the luminocity is:

$$\begin{aligned}L(t) &= -Q_{Ni}\frac{dN_{Ni}}{dt} - Q_{Co}\frac{dN_{Co}}{dt} \\ &= \lambda_{Ni}N_0 e^{-\lambda_{Ni}t}\left(Q_{Co}\frac{\lambda_{Co}}{\lambda_{Co} - \lambda_{Ni}} + Q_{Ni} - Q_{Co}\frac{\lambda_{Co}}{\lambda_{Co} - \lambda_{Ni}}e^{-(\lambda_{Co}-\lambda_{Ni})t}\right)\end{aligned}$$

(d) The mass of ^{56}Ni is $1M_\odot$, so the total number of ^{56}Ni is:

$$N_0 = \frac{1M_\odot}{56m_p} = 2.123 \times 10^{55}$$

Thus the luminosity of supernova as a function of time is:

$$L = 7.80088 \times 10^{43} \left(\frac{1}{2}\right)^{t/6.1d} + 1.75406 \times 10^{43} \left(\frac{1}{2}\right)^{t/77.1d} \quad erg/s$$

the light curve of supernova is:

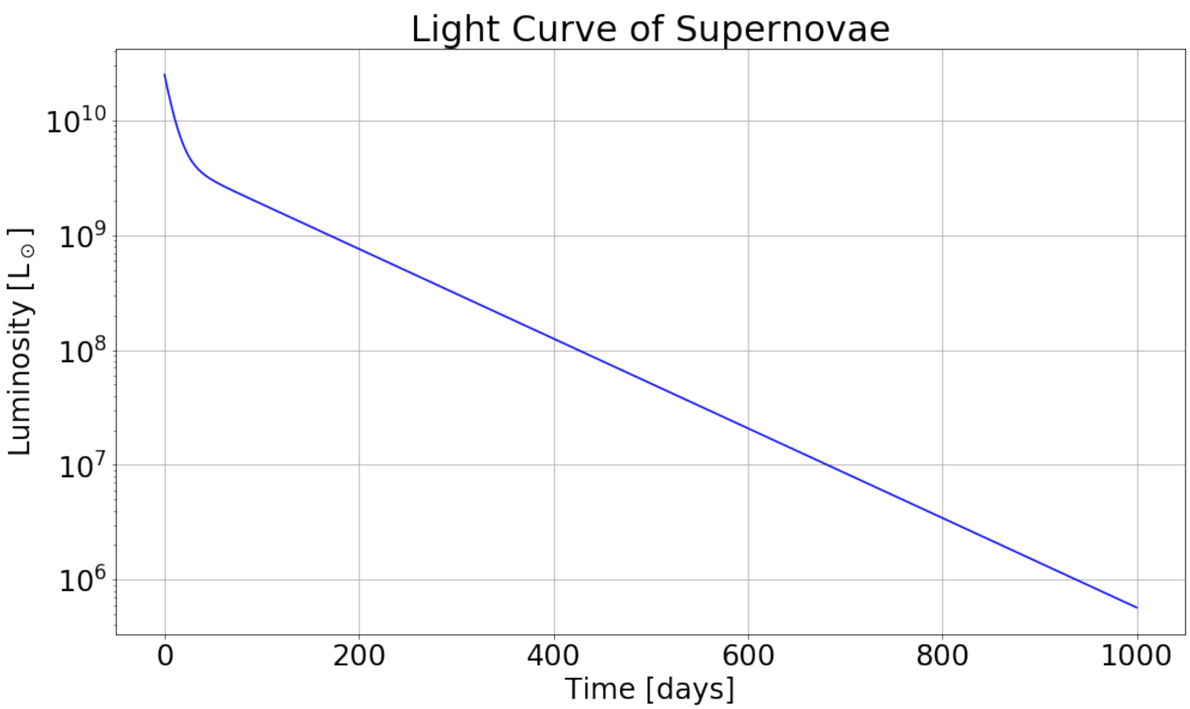

Figure 3: The light curve of supernova.

A Tides

```
import numpy as np
import matplotlib.pyplot as plt
from astropy import constants as c, units as u
M_moon = 7.342*10**22
M_earth = 5.97237*10**24
T_moon_rotation = 27.321661*24*3600
T_earth_spin = 0.99726968*24*3600
R_moon = 1737.4*10**3
R_earth = 6371*10**3
D_moon = M_earth/(M_earth+M_moon)*384402*10**3
D_earth = M_moon/(M_earth+M_moon)*384402*10**3
I_earth = 2/5*M_earth*R_earth**2
I_moon = 2/5*M_moon*R_moon**2
omega_earth = 2*np.pi/T_earth_spin
omega_moon = 2*np.pi/T_moon_rotation
J_earth_spin = I_earth*omega_earth
J_earth_orbital = M_earth*omega_moon*D_earth**2
J_moon_spin = I_moon*omega_moon
J_moon_orbital = M_moon*omega_moon*D_moon**2
print(J_earth_spin,J_earth_orbital,J_moon_spin,J_moon_orbital)
print(J_earth_spin+J_earth_orbital+J_moon_spin+J_moon_orbital)
dD_Moon = 0.038*M_earth/(M_earth+M_moon)
domega_ES = -0.5*M_moon*omega_moon*D_moon/I_earth*dD_Moon
dT = -2*np.pi/omega_earth**2*domega_ES
print(dT)
```

B The Pre-main Sequence

```
#for polytrope
n = 1.5
W_n = 0.770140
N_n = 0.42422
Theta_n = 2.71406
z_n = 3.65375
A = c.G*c.m_p/c.k_B*N_n*z_n**2/(4*np.pi*Theta_n)**(2/3)
A.to('K m/kg')
M = c.M_sun
R_0 = 4*c.R_sun
Omega_0 = -6/7*c.G*M**2/R_0
L = 4*c.L_sun
tau = 0.5*Omega_0/L
a = c.G*c.m_p/c.k_B*N_n*z_n**2/(4*np.pi*Theta_n)**(2/3)*M/R_0
T_0 = a.to('K')
```

```
T_0
t = np.linspace(0,0.4*10**7,1000)*u.yr
R = R_0/(1-t/tau)
r = R.to('R_sun')
T = a*(1-t/tau)
plt.figure(figsize=(16,9))
plt.plot(t,r,'b')
plt.grid()
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.xlabel('Time [year]',fontsize=24)
plt.ylabel(r'Raidius [R$_{\odot}$]',fontsize=24)
plt.title('Radius',fontsize=30)
T = T_0*(1-t/tau)
plt.figure(figsize=(16,9))
plt.plot(t,T,'b')
plt.hlines(10**7,0,0.4*10**7,'r')
plt.grid()
plt.xticks(fontsize=18)
plt.yticks(fontsize=18)
plt.xlabel('Time [year]',fontsize=24)
plt.ylabel(r'Temperature [K]',fontsize=24)
plt.title('Central Temperature',fontsize=30)
t[465]
f = abs(T-10**7*u.K)
np.where(f == f.min())
R[465]
1.5/16*10**7
```

C Supernovae and Neutron Stars

```
m_NS = 1.4*c.M_sun
E = 10**53*u.erg
v = 0.01*E/c.c/m_NS
v.to('km/s')
P = 0.1*u.s
Pdot = 3*10**(-6)*u.s/u.yr

E = 4*np.pi/P**2
Edot = 8*np.pi/P**3*Pdot

t = E/Edot
print(t)
```

D Half-life of Ni

```
import sympy as sy
from sympy import *
t = sy.symbols('t')
f = sy.Function('f')
N_Co = Function('N_{Co}')
L_Co = symbols('\lambda_{Co}')
L_Ni = symbols('\lambda_{Ni}')
N_0 = symbols('N_0')
sy.dsolve(sy.diff(f(t),t,1)+f(t)*L_Co-N_0*E**(-L_Ni*t)*L_Ni)
N_Fe = sy.Function('g')
sy.dsolve(sy.diff(N_Fe(t),t)-N_0*L_Ni/(L_Co-L_Ni)*(E**(-L_Ni*t)-1))
Q_Co = 4.564*u.MeV
Q_Ni = 2.136*u.MeV
t = np.linspace(0,10,1000)*u.day.to('s')*u.s
import astropy.constants as c
N = 1*c.M_sun/56/c.m_p
t1 = 6.1*u.day.to('s')*u.s
t2 = 77.1*u.day.to('s')*u.s
L = N*np.log(2)/t1*0.5**(t/t1)*(Q_Co*77.1/(77.1-6.1)-Q_Ni)
Lum = L.to('erg/s')
plt.figure(figsize=(16,9))
plt.plot(t*u.s.to('day'),Lum,'b')
plt.grid()
plt.xlabel('Days',fontsize=24)
plt.ylabel('Luminosity',fontsize=24)
plt.xticks(fontsize=24)
plt.yticks(fontsize=24)
plt.title('Light Curve of Supernovae',fontsize=30)
```