Probability Club 1!!!

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1. Generalised Mean: The generalised mean $G_n(x_i)$, for a set x_i of N positive real numbers and any $n \in \mathbb{R}$, is given by:

$$n \neq 0: \; G_n(x_i) = \left(\sum_{i=1}^{N} \frac{x_i^n}{N} \right)^{1/n}, \; G_0(x_i) = \prod_{i=1}^{N} x_i^{1/N}$$

Hence G_2 is the rms mean, G_1 is the arithmetic mean, G_0 is the geometric mean, G_{-1} is the harmonic mean etc.
It turns out that for given x_i, $p \leq q \implies G_p \leq G_q$, which gives a convenient way of remembering the HM-GM-AM-RMS inequality. The proof of this is rather complicated however.

(a) Show that G_n is continuous over n. This is obviously true for $n \neq 0$, so what I'm effectively asking is to show that:

$$\lim_{n \to 0} \left(\sum_{i=1}^{N} \frac{x_i^n}{N} \right)^{1/n} = \prod_{i=1}^{N} x_i^{1/N}$$

which gives intuition for the choice of G_0.

(b) Work out expressions for $G_\infty = \lim_{n \to \infty} G_n$ and $G_{-\infty} = \lim_{n \to -\infty} G_n$.

(c) Given the generalised weighted mean defined analogously:

$$n \neq 0: \; G_n^w(x_i, w_i) = \left(\frac{\sum_{i=1}^{N} w_i x_i^n}{\sum_{i=1}^{N} w_i} \right)^{1/n}$$

determine G_0^w, G_∞^w, $G_{-\infty}^w$. Hint: you can treat the w_i's as integers to simplify things massively, I am not looking for a formal proof.

2. Use of probability theory in evaluating wacky integrals, from MIT integration bee: Evaluate

$$\int_0^1 \max(\{x\}, \{\sqrt{2}x\}, \{\sqrt{5}x\})\,\mathrm{d}x$$

where $\{y\}$ is the fractional part of y (in $[0,1)$ obviously).
Hint: since the max will be between 0 and 1, rewrite the integral as:

$$\int_0^1 aP(\max = a)\mathrm{d}a$$

Now, unless you are well acquainted with deep results in ergodic theory and order statistics, or this is somehow just obvious to you, I will tell you that due to $(1, \sqrt{2}, \sqrt{5})$ being linearly independent over $\mathbb{Q}$, and given that the max function only depends on the relative order rather than the proper correlation of the variables, $\{x\}$, $\{\sqrt{2}x\}$, and $\{\sqrt{5}x\}$ function effectively as independent random variables, call them X, Y, Z, each with uniform distribution in $[0, 1)$. Use this to calculate $P(\max = a)$.
Hint 2: Calculate this by considering the cdf $F(\max \leq a)$.

Hints and Solutions - ONLY WHEN NECESSARY

1. (a) Take the ln of the LHS, noting since ln is a continuous function you can bring the $n \to 0$ limit to the outside.

(b) The most effective way to do this, for the ∞ case, is to bring $\max_i(x_i)$ outside the bracket, and then note the leftover bracket goes to 1 in the limit. Hence you will get $G_\infty = \max_i(x_i)$ and similarly $G_{-\infty} = \min_i(x_i)$. This is a bit cheap however, as it implies you already suspect the answer :/

(c) Treating the w_i's as integers, it is easy to see that we can replace N with $\sum_i w_i$ with w_i repeats of x_i, hence:

$$G_0^w = \prod_{i=1}^{N} x_i^{(w_i / \sum_i w_i)}, \; G_\infty^w = \max_i(x_i), \; G_{-\infty}^w = \min_i(x_i)$$

2. Start by working out the cdf $F(\max \leq a)$. Imagine the 3d sample space $[0,1)^3$. The graph $\max(X,Y,Z) \leq a$ is a cube side length a, hence since X, Y, Z are uniform we get $F(\max \leq a) = a^3$, then $P(\max = a) = 3a^2$, and:

$$I = \int_0^1 3a^3 \, \mathrm{d}a = \frac{3}{4}$$

Note that as alluded to in the question, this method can only be applied for functions like $\max(X,Y)$ or $|X - Y|$ that are strictly based on order. XY or even $X + Y$ do not work due to them depending explicitly on the correlation between X and Y, which you can try for yourself.