Improper integral

Elena Lebedeva

Saint Petersburg State University

The improper integral is a generalization of the Riemann integral in two directions.

1. To the functions defined on unbounded domains: $[a, \infty)$, $(-\infty, b]$, $(-\infty, \infty)$.
2. To the unbounded functions.

Definition (Local integrability)

We say that a function f is locally integrable on $D \subset \mathbb{R}$, and write $f \in \mathcal{R}_{loc}(D)$, if $f \in \mathcal{R}[c, d]$ for every $[c, d] \subset D$.

Example. $e^x \in \mathcal{R}_{loc}(\mathbb{R})$, $1/x \in \mathcal{R}_{loc}([-1, 1] \setminus \{0\})$.

Definition (Improper integral 1)

Let $f \in \mathcal{R}_{loc}[a, \infty)$. *The quantity*

$$\int_a^{+\infty} f := \lim_{B \to +\infty} \int_a^B f,$$

if the limit exists, is called the **improper integral** *of* f *over* $[a, +\infty)$.
Let $f \in \mathcal{R}_{loc}(-\infty, b]$. *The quantity*

$$\int_{-\infty}^{b} f := \lim_{A \to -\infty} \int_A^b f,$$

if the limit exists, is called the **improper integral** *of* f *over* $(-\infty, b]$.
Let $f \in \mathcal{R}_{loc}(-\infty, +\infty)$. *The quantity*

$$\int_{-\infty}^{+\infty} f := \lim_{A \to -\infty} \int_A^c f + \lim_{B \to +\infty} \int_c^B f,$$

if the limit exists, is called the **improper integral** *of* f *over* $(-\infty, +\infty)$.

Remark. The quantity $\int_{-\infty}^{+\infty} f$ does not depend on the choice of c. Let $d > c$, then by additivity,

$$\int_A^c f + \int_c^B f = \int_A^c f + \int_c^d f + \int_d^B f = \int_A^d f + \int_d^B f.$$

Example.

$$\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2+1} = \lim_{A\to-\infty} \int_A^0 \frac{\mathrm{d}x}{x^2+1} + \lim_{B\to+\infty} \int_0^B \frac{\mathrm{d}x}{x^2+1}$$

$$= \lim_{A\to-\infty} \arctan x\Big|_A^0 - \lim_{B\to+\infty} \arctan x\Big|_0^B = \pi.$$

Definition (Improper integral 2)

Let $f \in \mathcal{R}_{loc}[a, b)$. The quantity

$$\int_a^b f := \lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f,$$

if the limit exists, is called the **improper integral** *of f over $[a, b)$.*
Let $f \in \mathcal{R}_{loc}(a, b]$. The quantity

$$\int_a^b f := \lim_{\varepsilon \to +0} \int_{a+\varepsilon}^{b} f,$$

if the limit exists, is called the **improper integral** *of f over $(a, b]$.*
Let $f \in \mathcal{R}_{loc}[a, c) \cup (c, b]$. The quantity

$$\int_a^b f := \lim_{\varepsilon_1 \to +0} \int_a^{c-\varepsilon_1} f + \lim_{\varepsilon_2 \to +0} \int_{c+\varepsilon}^{b} f,$$

if the limit exists, is called the **improper integral** *of f over $[a, b]$.*

Example. $\int_{-1}^{1} \frac{\arccos x}{\sqrt{1-x^2}}\, \mathrm{d}x.$

The integrand is unbounded at the right half-neighborhood of $x = -1$ only, $\lim\limits_{x \to 1-0} \frac{\arccos x}{\sqrt{1-x^2}} = 1$. So,

$$\int_{-1}^{1} \frac{\arccos x}{\sqrt{1-x^2}} dx = \lim_{\varepsilon \to -1+0} \int_{\varepsilon}^{1} \frac{\arccos x}{\sqrt{1-x^2}} dx = -\lim_{\varepsilon \to -1+0} \int_{\varepsilon}^{1} \arccos x d \arccos x$$

$$= -\frac{1}{2} \lim_{\varepsilon \to -1+0} \arccos^2 x \Big|_{\varepsilon}^{1} = \frac{1}{2} \lim_{\varepsilon \to -1+0} \arccos^2 \varepsilon = \frac{\pi^2}{2}.$$

Definition (Improper integral (many points))

Let $-\infty \le a < c_1 \cdots < c_m < b \le \infty$, $f \in \mathcal{R}_{loc}(a, c_1) \cup \cdots \cup (c_m, b)$. *Then under the improper integral* $\int_a^b f$ *we mean the sum of improper integrals*

$$\int_a^{d_1} f + \int_{d_1}^{c_1} f + \int_{c_1}^{d_2} f + \int_{d_2}^{c_2} f + \cdots + \int_{c_m}^{d_{m+1}} f + \int_{d_{m+1}}^{b} f,$$

where $a < d_1 < c_1 < d_2 < c_2 < \ldots, c_m < d_{m+1} < b$. *If the limits defining the improper integral are finite, then we say that the integral* **converges**, *otherwise, the integral is called* **divergent**.

Remark. If $f \in \mathcal{R}[a, b]$, then since the integral with variable upper limit $\int_a^x f$ is continuous on $[a, b]$, it follows that $\int_a^b f := \lim_{\varepsilon \to +0} \int_a^{b-\varepsilon} f$.
Therefore, for integrable functions the Riemann integral coincides with the improper integral.

Remark. $a, b \in \mathbb{R}$, $f \in \mathcal{R}_{loc}[a, b)$, f is bounded on $[a, b)$ $\Rightarrow$ $f \in \mathcal{R}[a, b]$.

Proof. By the Lebesgue criterion, it is sufficient to check that the set A of the points of discontinuity of f is of measure zero. We choose n_0 : $b - \dfrac{1}{n_0} > a$, and consider $\left[a, b - \dfrac{1}{n_0}\right]$, $\left[b - \dfrac{1}{n}, b - \dfrac{1}{n+1}\right]$, $n > n_0$.

We denote $A_{n_0} := A \cap \left[a, b - \dfrac{1}{n_0}\right]$, $A_n := A \cap \left[b - \dfrac{1}{n}, b - \dfrac{1}{n+1}\right]$.

$f \in \mathcal{R}\left[a, b - \dfrac{1}{n_0}\right]$, $f \in \mathcal{R}\left[b - \dfrac{1}{n}, b - \dfrac{1}{n+1}\right]$

$$\Longrightarrow A_n,\ n \geq n_0 \text{ are of measure } 0.$$

$$\Longleftrightarrow \exists\, (a_k^n, b_k^n),\ k \in \mathbb{N}, \quad A_n \subset \bigcup_k (a_k^n, b_k^n), \quad \sum_k (b_k^n - a_k^n) < \frac{\varepsilon}{2^n}$$

$$\Longrightarrow A \subset \bigcup_{k,n} (a_k^n, b_k^n), \quad \sum_{k,n} (b_k^n - a_k^n) < \sum_n \frac{\varepsilon}{2^n} < \varepsilon \Rightarrow A \text{ is of measure } 0.$$

Therefore, the improper integral over a finite interval $[a, b]$ is a generalization of the Riemann integral to the locally integrable unbounded functions and only to them.

Example.

$$\int_1^{+\infty} \frac{\mathrm{d}x}{x^c} = \lim_{A\to+\infty} \int_1^A \frac{\mathrm{d}x}{x^c} = \lim_{A\to+\infty} \begin{cases} \frac{1}{1-c} x^{1-c}\Big|_1^A, & c \neq 1, \\ \ln x\Big|_1^A, & c = 1, \end{cases}$$

$$= \begin{cases} \frac{1}{c-1}, & c > 1, \\ +\infty, & c \leq 1. \end{cases} \Rightarrow \int_1^{+\infty} \frac{\mathrm{d}x}{x^c} \quad \begin{array}{l} \text{converges for } c > 1, \\ \text{diverges for } c \leq 1. \end{array}$$

Example.

$$\int_0^1 \frac{\mathrm{d}x}{x^c} = \lim_{\varepsilon\to+0} \int_\varepsilon^1 \frac{\mathrm{d}x}{x^c} = \lim_{\varepsilon\to+0} \begin{cases} \frac{1}{1-c} x^{1-c}\Big|_\varepsilon^1, & c \neq 1, \\ \ln x\Big|_\varepsilon^1, & c = 1, \end{cases}$$

$$= \begin{cases} \frac{1}{1-c}, & c < 1, \\ +\infty, & c \geq 1. \end{cases} \Rightarrow \int_0^1 \frac{\mathrm{d}x}{x^c} \quad \begin{array}{l} \text{converges for } c < 1, \\ \text{diverges for } c \geq 1. \end{array}$$

Example.

$$\int_0^\infty \frac{\mathrm{d}x}{x^c} = \int_0^1 \frac{\mathrm{d}x}{x^c} + \int_1^\infty \frac{\mathrm{d}x}{x^c} \text{ diverges for any } c \in \mathbb{R}.$$

Definition (Singular points)

We say that a point p is a **singular point** *of a function f if f is not bounded on any neighbourhood of p or $p = \pm\infty$.*

The case of a “left” singular point can be easily reduced to the case of a “right” singular point via the replacement x by $-x$. Namely, if a is a singular point of f and $f \in \mathcal{R}_{loc}(a, b]$ and $f_1(x) := f(-x)$, then $f_1 \in \mathcal{R}_{loc}[-b, -a)$ and $-a$ is a singular point of f_1. So, it is sufficient to study “right” singular points only.

Since both types of improper integral have the same properties, we will study both of these integrals together and use the following unified notation.

Let $b \in \mathbb{R}$ or $b = +\infty$, $f \in \mathcal{R}_{loc}[a, b)$,

$$\int_a^b f := \begin{cases} \lim\limits_{\varepsilon \to +0} \int_a^{b-\varepsilon} f, & b \in \mathbb{R} \\ \lim\limits_{B \to +\infty} \int_a^{B} f, & b = +\infty. \end{cases}$$

Simple properties.

1. Additivity of improper integral w.r.t an interval.

If $\displaystyle\int_a^b f$ converges, then for any $c \in (a, b)$ $\displaystyle\int_c^b f$ converges as well and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Conversely, if for some $c \in (a, b)$ $\int_c^b f$ converges, then $\int_a^b f$ converges as well.

Proof. For any $A \in (c, b)$ by additivity

$$\int_a^A f = \int_a^c f + \int_c^A f.$$

As $A \to b-$, the limit in both parts exists or does not exist simultaneously. So $\int_c^b f$ and $\int_a^b f$ diverge or converge simultaneously. It remains to pass to the limit $A \to b-$.

Definition

Improper integral $\int_A^b f$ *is called the* **reminder** *of the integral* $\int_a^b f$.

1 implies that the integral and its reminder diverge or converge simultaneously.

2. On the integral and its reminder.
If $\int_a^b f$ converges, then $\int_A^b f \underset{A\to b-}{\longrightarrow} 0$. In other words, the reminder of the convergent integral tends to 0.
Indeed,

$$\int_A^b f = \int_a^b f - \int_a^A f \underset{A\to b-}{\longrightarrow} \int_a^b f - \int_a^b f = 0.$$

3. Linearity.
If $\displaystyle\int_a^b f, \int_a^b g$ converge, $\alpha, \beta \in \mathbb{R}$, then $\displaystyle\int_a^b (\alpha f + \beta g)$ converge and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g. \quad \text{Prove by yourself.}$$

4. Monotonisity. It $\int_a^b f$, $\int_a^b g$ exist in $\overline{\mathbb{R}}$, $f \leq g$ on $[a, b)$, then $\int_a^b f \leqslant \int_a^b g$. Prove by yourself.

Remark. We extrapolate Hölder's and Minkowski's inequality passing to the limit as well.

Remark. Convergence of $\int_a^b f$, $\int_a^b g$ does not imply convergence of $\int_a^b fg$. For example, $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}}$ converge, but $\int_0^1 \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{x}} \mathrm{d}x$ diverge.

Theorem (Integration by parts in the improper integral)

If f, g is differentiable on $[a, b)$ $b \in \overline{\mathbb{R}}$, $f', g' \in \mathcal{R}_{loc}[a, b)$, then

$$\int_a^b f'g = fg\Big|_a^b - \int_a^b fg'.$$

Proof. Let $B \in [a, b)$, then $\int_a^B f'g = fg\Big|_a^B - \int_a^B fg'$. It remains to pass to the limit $B \to b$.

Example. $J = \displaystyle\int_0^{\pi/2} \log \sin x \, \mathrm{d}x.$ Let us check the convergence.

$$J = x \log \sin x \Big|_{+0}^{\pi/2} - \int_0^{\pi/2} x \cot x \, \mathrm{d}x =$$

$$= - \lim_{x \to +0} (x \ln \sin x) - \int_0^{\pi/2} x \cot x \, \mathrm{d}x = - \int_0^{\pi/2} x \cot x \, \mathrm{d}x.$$

The last integral is finite, therefore J is finite.

$$J = [x = 2t] = 2 \int_0^{\pi/4} \log \sin 2t \, \mathrm{d}t = 2 \int_0^{\pi/4} (\log 2 + \log \sin t + \log \cos t) \, \mathrm{d}t$$

$$= \frac{\pi}{2} \log 2 + 2 \int_0^{\pi/4} \log \sin t dt + 2 \int_0^{\pi/4} \log \cos t \, \mathrm{d}t.$$

The change of variable $t = \pi/2 - u$ in the last integral yields

$\displaystyle\int_{\pi/4}^{\pi/2} \log \sin u \, \mathrm{d}u, \quad J = \frac{\pi}{2} \log 2 + 2J, \quad J = -\frac{\pi}{2} \log 2.$

Theorem (Change of variables in the improper integral)

If $f \in C[a,b)$, $g : [c,d) \to [a,b)$, g is differentiable on $[c,d)$, $g' \in \mathcal{R}_{loc}[c,d)$, $\exists \lim_{t \to d-0} g(t) = g(d-0) \in \overline{\mathbb{R}}$, then

$$\int_c^d (f \circ g) g' = \int_{g(c)}^{g(d-0)} f.$$

Proof. Consider $t \in [c,d)$, $x \in [a,b)$. We denote $\Phi(t) := \int_c^t (f \circ g) g'$, $F(x) := \int_{g(c)}^{x} f$. By the theorem on change of variables in the Riemann integral, $\Phi(t) = F(g(t))$. $\Rightarrow$ then pass the limit.

1. Suppose $\exists \int_{g(c)}^{g(d-0)} f := K$, that is $\lim\limits_{x \to g(d-0)} F(x) = K$. Let us prove that $\int_c^d (f \circ g) g' = K$, that is $\lim\limits_{t \to d-0} \Phi(t) = K$. Let $t_n \in [c,d)$, $\lim\limits_{n \to \infty} t_n = d$ be given. Since $\exists \lim_{t \to d-0} g(t)$, it follows that

(连续函数性质 $\lim\limits_{t \to d-} g(t) = g(d-)$
海涅定理 $\lim\limits_{n \to \infty} g(t_n) = g(d-)$)

F连续. 与极限运算交换次序

$\lim\limits_{n \to \infty} g(t_n) = g(d-0) \Longrightarrow \lim\limits_{n \to \infty} \Phi(t_n) = \lim\limits_{n \to \infty} F(g(t_n)) = K.$

再用一次海涅定理

$\lim \phi(t) = \int^d (f \circ g) g'$

$t \to d-$ $\int_c$

2. Suppose $\exists \displaystyle\int_c^d (f \circ g) g' =: L$, that is $\displaystyle\lim_{t \to d-0} \Phi(t) = L$. Let us prove that $\displaystyle\int_{g(c)}^{g(d-0)} f = L$, that is $\displaystyle\lim_{x \to g(d-0)} F(x) = L$.

(a) If $g(d-0) \in [a, b)$, then $\displaystyle\int_{g(c)}^{g(d-0)} f$ is a Riemann integral and by **1.** it is equal to L.

(b) $g(d-0) = b$. Let $x_n \in [a, b)$, $\displaystyle\lim_{n \to \infty} x_n = b$. Without loss of generality (WLOG) $x_n \in [g(c), b)$. $g(d-0) = b \Rightarrow \exists\, \tau_n \in [c, d)\ x_n \in [g(c), g(\tau_n)]$. By the Bolzano intermediate value theorem, $\exists\, t_n \in [c, d)\ x_n = g(t_n)$. Let us prove that $\displaystyle\lim_{n \to \infty} t_n = d - 0$. Assume the converse, then $\exists\, t_{n_k}$: $\displaystyle\lim_{k \to \infty} t_{n_k} = \tau < d$. Then by the continuity of g, we get $\displaystyle\lim_{k \to \infty} g(t_{n_k}) = g(\tau) < b$, but $\displaystyle\lim_{k \to \infty} g(t_{n_k}) = \lim_{k \to \infty} x_{n_k} = b$. Finally,

$$F(x_n) = F(g(t_n)) = \Phi(t_n) \to L \text{ as } n \to \infty. \quad \square$$

Example. $I = \int_a^b \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}, \quad a, b \in \mathbb{R}, \quad b > a.$

$$I = \big[x = a\cos^2 t + b\sin^2 t, \quad t \in (0, \pi/2), \ \alpha = 0, \ \beta = \pi/2,$$

$$x - a = (b-a)\sin^2 t, \ b - x = (b-a)\cos^2 t, \ \mathrm{d}x = 2(b-a)\sin t\cos t\,\mathrm{d}t\big]$$

$$\int_a^b \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}} = 2\int_0^{\pi/2} \mathrm{d}t = \pi.$$

Example.

$$\int_0^1 \frac{e^x}{\sqrt{1-x^2}}\,\mathrm{d}x = \left[f(x) = \frac{e^x}{\sqrt{1-x^2}},\ g(t) = \sin t,\ g : \left[0, \frac{\pi}{2}\right) \to [0,1) \right]$$

$$= \int_0^{\pi/2} \frac{e^{\sin t}}{\sqrt{1-\sin^2 t}} \cos t\,\mathrm{d}t = \int_0^{\pi/2} e^{\sin t}\,\mathrm{d}t.$$

Lemma

Suppose $f \in \mathcal{R}_{loc}[a,b)$, $f \geq 0$, *then* $\int_a^b f$ *is convergent iff* $F(x) := \int_a^x f$ *is bounded from above.*

Proof. $x_1, x_2 \in [a,b)$, $x_1 < x_2 \Rightarrow F(x_2) - F(x_1) = \int_{x_1}^{x_2} f \geq 0$.

F is nondecreasing $\Rightarrow \int_a^b f = \lim\limits_{x \to b-0} F(x)$ is finite iff F is bounded from above. $\square$

Theorem (A comparison test for convergence of improper integrals)

$f, g \in \mathcal{R}_{loc}[a, b)$, $f, g \geq 0$, $f(x) = O(g(x))$, $x \to b - 0$.

1. *If* $\int_a^b g$ *converges, then* $\int_a^b f$ *converges.*
2. *If* $\int_a^b f$ *diverges, then* $\int_a^b g$ *diverges.*

Proof. 1. $f(x) = O(g(x))$, $x \to b - 0 \Leftrightarrow$ $\exists\, K\ \exists\, c\ \forall\, x \in [c, b)\ f(x) \leq Kg(x) \Rightarrow \int_c^{b-\varepsilon} f \leq K \int_c^{b-\varepsilon} g$. If $\int_a^b g$ converges, then $\int_c^b g = \int_a^b g - \int_a^c g$ converges. Passing to the limit $\varepsilon \to 0+$, we get that $\int_c^b f$ converges. Then $\int_a^b f = \int_c^b f + \int_a^c f$ converges.

2. Assume the converse and apply **1.** $\square$

Corollary (Limit comparison test for convergence of improper integral)

$f, g \in \mathcal{R}_{loc}[a, b),\ f, g \geq 0,\ \lim\limits_{x\to b-0} \dfrac{f(x)}{g(x)} = k \in [0, +\infty].$

$+\infty$

$\cos x$

1. *If* $k \in (0, +\infty)$, *then* $\int_a^b f$ *and* $\int_a^b g$ *converge or diverge simultaneously.*

   $f(x) = \frac{\cos x}{x^p}$

   $g(x) = \frac{1}{x^p}$

2. *If* $k = 0$, *then the convergence of* $\int_a^b g$ *implies the converges of* $\int_a^b f$.
3. *If* $k = +\infty$, *then the convergence of* $\int_a^b f$ *implies the converges of* $\int_a^b g$.

Example. $I = \displaystyle\int_2^{+\infty} \frac{\mathrm{d}x}{x^\alpha \log^\beta x}$, $\alpha, \beta \in \mathbb{R}$. We know that $\log x = o(x^q)$ as $x \to +\infty$, $q > 0$. So, for any $p \in \mathbb{R}$ and $q > 0$ we get $\log^p x = o(x^q)$ as $x \to +\infty$. For $p \le 0$ it is obvious, for $p > 0$

$\log^\beta x = o(x^{\frac{1-\alpha}{2}})$

$$\frac{x^{\frac{1-\alpha}{2}}}{\log^{-\beta} x} = +\infty \qquad \frac{\log^p x}{x^q} = \left(\frac{\log x}{x^{q/p}}\right)^p \xrightarrow[x \to +\infty]{} 0.$$

If $\alpha > 1$, then $\displaystyle\lim_{x\to+\infty} \frac{\log^{-\beta} x}{x^{\frac{\alpha-1}{2}}} = 0$ implies $\displaystyle\frac{1}{x^\alpha \log^\beta x} = O\left(\frac{1}{x^{\frac{\alpha+1}{2}}}\right)$ as $x \to +\infty$. By $\dfrac{\alpha+1}{2} > 1$, the integral $\int_2^{+\infty} \frac{dx}{x^{\frac{\alpha+1}{2}}}$ converges, so I converges by comparison test for any β.

If $\alpha < 1$, then we proceed the same way. $\displaystyle\frac{1}{x^{\frac{1+\alpha}{2}}} = O\left(\frac{1}{x^\alpha \log^\beta x}\right)$ as $x \to +\infty$. By $\dfrac{1+\alpha}{2} < 1$, the integral $\displaystyle\int_2^{+\infty} \frac{\mathrm{d}x}{x^{\frac{1+\alpha}{2}}}$ diverges, so I diverges by comparison test for any β.

If $\alpha = 1$, then prove by yourself that I converges for $\beta > 1$ and diverges for $\beta \le 1$.

Summary: $\displaystyle\int_2^{+\infty} \frac{\mathrm{d}x}{x^\alpha \log^\beta x}$ converges iff $\alpha > 1$ and β is arbitrary or $\alpha = 1$ and $\beta > 1$.

Example. The Gamma function $\quad \Gamma(x) := \displaystyle\int_0^{+\infty} t^{x-1} e^{-t}\, \mathrm{d}t.$

Let us prove that $\Gamma(x)$ converges iff $x > 0$.

$$\Gamma(x) = \int_0^1 t^{x-1} e^{-t}\, \mathrm{d}t + \int_1^{+\infty} t^{x-1} e^{-t}\, \mathrm{d}t =: J_1 + J_2.$$

Since $\displaystyle\lim_{t\to 0} \frac{t^{x-1}e^{-t}}{t^{x-1}} = 1$ and $\displaystyle\int_0^1 t^{x-1}\, \mathrm{d}t < +\infty \Leftrightarrow x > 0$, it follows by limit comparison test that $J_1 < +\infty \Leftrightarrow x > 0$.

Limit comparison test $\displaystyle\lim_{t\to+\infty} \frac{t^{x-1}e^{-t}}{e^{-t/2}} = \lim_{t\to+\infty} \frac{t^{x-1}}{e^{t/2}} = 0$ and $\displaystyle\int_1^{+\infty} e^{-t/2}\, \mathrm{d}t = -2e^{-t/2}\Big|_1^{+\infty} = 2e^{-1/2}$ implies $J_2 < +\infty$.

Example. $I = \int_0^1 \frac{\mathrm{d}x}{x^p + x^q}$.

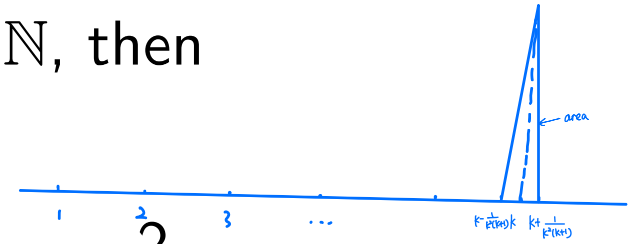
If $p = q$, then I converges iff $p = q < 1$.

Let $p < q$.
The equality $\frac{1}{x^p + x^q} = \frac{1}{x^p\left(1 + x^{q-p}\right)}$ and $x^{q-p} \to 0$ as $x \to +0$ implies that the functions $f(x) = \frac{1}{x^p + x^q}$ and $g(x) = \frac{1}{x^p}, p > 0$ are equivalent as $x \to +0$. If $p \leq 0$, then the integral I is proper. So, by comparison test I converges, if $p < 1$, and it diverges if $p \geq 1$.

For arbitrary p and q the integral I converges, if $\min\{p, q\} < 1$, and it diverges, if $\min\{p, q\} \geq 1$.

Example. We give an example of a function f such that $\displaystyle\int_1^{+\infty} f$ converges and $\displaystyle\lim_{x\to+\infty} f(x) \neq 0$.

Let $E = \displaystyle\bigcup_{k=1}^{\infty} \left(k - \frac{1}{k^2(k+1)}, k + \frac{1}{k^2(k+1)} \right)$, $f(k) = k$ for all $k \in \mathbb{N}$, $f(x) = 0$ for all $x \in [0, +\infty) \setminus E$, f is linear on $\left[k - \dfrac{1}{k^2(k+1)}, k \right]$ and on $\left[k, k + \dfrac{1}{k^2(k+1)} \right]$. Then $f \in C[0, +\infty)$. If $N \in \mathbb{N}$, then



$$\int_0^{N+1/2} f = \sum_{k=1}^{N} \int_{k-\frac{1}{k^2(k+1)}}^{k+\frac{1}{k^2(k+1)}} f = \sum_{k=1}^{N} \frac{1}{2} \cdot \frac{2}{k^2(k+1)} \cdot k$$

$$= \sum_{k=1}^{N} \frac{1}{k(k+1)} = \sum_{k=1}^{N} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{N+1} \xrightarrow[N\to\infty]{} 1.$$

So, $\displaystyle\int_0^{+\infty} f = 1$. At the same time, f is even not bounded as $x \to \infty$.

Theorem (Cauchy criterion for convergence of an improper integral)

Suppose $f \in \mathcal{R}_{loc}[a, b)$, *then* $\int_a^b f$ *converges iff*

$$\forall \varepsilon \; \exists b_0 \in [a, b) \; \forall b_1, b_2 \in [b_0, b) \left| \int_{b_1}^{b_2} f \right| < \varepsilon$$

Proof.

$$F(x) := \int_a^x f, \quad \left| \int_{b_1}^{b_2} f \right| = |F(b_2) - F(b_1)| \quad \square$$

Definition (Absolutely convergent integral)

If the integral $\int_a^b |f|$ *converges, then we say that* $\int_a^b f$ *is* ***absolutely convergent***.

Theorem

If $\int_a^b f$ *is absolutely convergent, then it is convergent.*

The first proof. Cauchy criterion $\left|\int_{b_1}^{b_2} f\right| \le \left|\int_{b_1}^{b_2} |f|\right|$

The second proof. Consider functions

$$f_+ := \max\{f, 0\}, \quad f_- := \max\{-f, 0\}.$$

They are called a **positive** and a **negative parts** of the function f respectively. $f = f_+ - f_-$, $|f| = f_+ + f_-$, $f_+, f_- \le |f|$.

If $\int_a^b |f|$ converges, then by the comparison test $\int_a^b f_+$, $\int_a^b f_-$ converge,

therefore, $\int_a^b f = \int_a^b f_+ - \int_a^b f_-$ converges. $\square$

Example. Prove that if $\int_a^{+\infty} f(x)\,\mathrm{d}x$ converges and f is monotone, then

⇒ speed . to 0 . "faster than"

$f(x) = o\left(\dfrac{1}{x}\right)$ as $x \to +\infty$.

It follows from convergence of the integral that $|f(x)| \to 0$ as $x \to +\infty$. (Assume the converse.) By Cauchy's criterion,

$$\forall\ \varepsilon > 0\ \ \exists\ A > a\ \ \forall x_1 > A\ \ \forall x_2 > A\ \ \left|\int_{x_1}^{x_2} f(x)\,\mathrm{d}x\right| < \varepsilon.$$

Fix $x_0 > A$, and $x > x_0$, consider $x_0 < t < x$.

1) If $f > 0$, then $f(t) > f(x)$, so $\left|\int_{x_0}^{x} f\right| = \int_{x_0}^{x} f > f(x)(x - x_0)$.

2) If $f < 0$, then $-f > 0$, then $-f(t) > -f(x)$, so
$\left|\int_{x_0}^{x} f\right| = \int_{x_0}^{x} (-f) > (-f)(x)(x - x_0)$.

$|f(x)| \cdot x < \left|\int_{x_0}^{x} f\right| + x_0 |f(x)| < 2\varepsilon$

$\wedge$ ε

$\exists \tilde{x}.\ \forall x \in \tilde{x}$

$f(x) < \frac{\varepsilon}{x_0}$, $x_0|f(x)| < \varepsilon$

Thus, $\quad |f(x)|\,(x - x_0) < \left|\int_{x_0}^{x} f(t)dt\right| < \varepsilon.$

By the last inequality and $\lim_{x\to+\infty} x_0|f(x)| = 0$, we obtain

$\lim_{x\to+\infty} xf(x) = 0$, that is $f(x) = o\left(\dfrac{1}{x}\right)$ as $x \to +\infty$.

Definition (Conditionally convergent integral)

If an improper integral converges but not absolutely, we say that it **converges conditionally**.

Theorem (Abel's and Dirichlet's tests for convergence of an integral)

Let $f \in \mathcal{R}_{loc}[a, b)$, g be monotonic on $[a, b)$. If one of the following pairs of conditions hold

1. *Abel's test*

   1.1 $\int_a^b f$ *converges,* **1.2** g *is bounded on* $[a, b)$,

   or

2. *Dirichlet's test*

   2.1 $F(x) := \int_a^x f$ *is bounded on* $[a, b)$, **2.2** $\lim_{x \to b-0} g(x) = 0$,

then $\int_a^b fg$ *converges.*

Proof. By Cauchy criterion for convergence, it is sufficient to check that

$$\forall\,\varepsilon\ \exists\, b_0 \in [a,b)\ \forall\, b_1, b_2 \in [b_0,b)\ \left|\int_{b_1}^{b_2} fg\right| < \varepsilon.$$

By the second mean value theorem, (WLOG $b_1 < b_2$)

$$\left|\int_{b_1}^{b_2} fg\right| = \left|g(b_1)\int_{b_1}^{c} f + g(b_2)\int_{c}^{b_2} f\right| =: A, \quad c \in (b_1, b_2)\,.$$

1. g is bounded on $[a,b) \Rightarrow \exists\, M\ \forall\, x \in [a,b)\ |g(x)| < M$.
$\displaystyle\int_a^b f$ converges $\Rightarrow \forall\,\varepsilon\ \exists\, b_0 \in [a,b)\ \forall\, b_1, b_2, c \in [b_0,b)$

$$\left|\int_{b_1}^{c} f\right| < \frac{\varepsilon}{2M},\ \left|\int_{c}^{b_2} f\right| < \frac{\varepsilon}{2M}.$$

Thus,

$$A \le |g(b_1)|\left|\int_{b_1}^{c} f\right| + |g(b_2)|\left|\int_{c}^{b_2} f\right| \le M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} = \varepsilon.$$

By Cauchy criterion for convergence, it is sufficient to check that

$$\forall \varepsilon \ \exists\, b_0 \in [a, b) \ \forall\, b_1, b_2 \in [b_0, b) \left| \int_{b_1}^{b_2} fg \right| < \varepsilon.$$

By the second mean value theorem, (WLOG $b_1 < b_2$)

$$\left| \int_{b_1}^{b_2} fg \right| = \left| g(b_1) \int_{b_1}^{c} f + g(b_2) \int_{c}^{b_2} f \right| =: A, \quad c \in (b_1, b_2)\,.$$

2. $F(x) := \displaystyle\int_a^x f$ is bounded on $[a, b) \Rightarrow \exists\, K \ \forall\, x \in [a, b) \ |F(x)| < K$

$$\Rightarrow \left| \int_{b_1}^{c} f \right| = |F(c) - F(b_1)| \le |F(b_1)| + |F(c)| \le 2K\,, \ \left| \int_{c}^{b_2} f \right| \le 2K.$$

$\lim\limits_{x \to b} g(x) = 0 \Rightarrow \forall \varepsilon \ \exists\, b_0 \in [a, b) \ \forall\, b_1, b_2 \in [b_0, b) \ |g(b_1)|\,, |g(b_2)| < \dfrac{\varepsilon}{4K}.$

Thus,

$$A \le |g(b_1)| \left| \int_{b_1}^{c} f \right| + |g(b_2)| \left| \int_{c}^{b_2} f \right| \le \frac{\varepsilon}{4K} 2K + \frac{\varepsilon}{4K} 2K = \varepsilon. \ \square$$

Example. $\int_a^{+\infty} g(x)\sin x\,\mathrm{d}x$, $\int_a^{+\infty} g(x)\cos x\,\mathrm{d}x$, where g is monotonic and nonnegative on $[a,+\infty)$, $a \in \mathbb{R}$.

- The Dirichlet integral $\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$.
- The Fresnel integrals $\int_0^{+\infty} \sin(x^2)\,\mathrm{d}x$, $\int_0^{+\infty} \cos(x^2)\,\mathrm{d}x$.

$$2\int_0^{+\infty} \sin(x^2)\,\mathrm{d}x = \left[x=\sqrt{t},\, x'(t)=\frac{1}{2\sqrt{t}}\right] = \int_0^{+\infty} \frac{\sin t}{\sqrt{t}}\,\mathrm{d}t = \underbrace{\int_0^1}_{abs.conv.} + \int_1^{+\infty}$$

We consider $\int_1^{+\infty} g(x)\sin x\,\mathrm{d}x$.

1. $\int_a^{+\infty} g$ converges.

By $|g(x)\sin x| \le g(x)$ and the comparison test, $\int_1^{+\infty} |g(x)\sin x|\,\mathrm{d}x$ converges $\Rightarrow \int_a^{+\infty} g(x)\sin x\mathrm{d}x$ absolutely converges.

Example. $\int_a^{+\infty} g(x)\sin x\,\mathrm{d}x$, $\int_a^{+\infty} g(x)\cos x\,\mathrm{d}x$, where g is monotonic and nonnegative on $[a, +\infty)$.

2. $\int_a^{+\infty} g$ diverges. **2.1.** $\lim\limits_{x\to\infty} g(x) = 0$.

$|F(x)| = \left|\int_a^x \sin t\,\mathrm{d}t\right| = |\cos a - \cos x| \le 2$. By the Dirichlet test, $\int_a^{+\infty} g(x)\sin x\,\mathrm{d}x$ converges.

$|\sin x| \ge \sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$. By the Dirichlet test, $\int_a^{+\infty} g(x)\frac{\cos 2x}{2}\,\mathrm{d}x$ converges, $\int_a^{+\infty} \frac{g}{2}$ diverges $\Rightarrow \int_a^{+\infty} g(x)\left(\frac{1}{2} - \frac{\cos 2x}{2}\right)\mathrm{d}x$ diverges.

By the comparison test, $\int_a^{+\infty} |g(x)\sin x|\,\mathrm{d}x$ diverges.

Thus, $\int_a^{+\infty} g(x)\sin x\,\mathrm{d}x$ conditionally converges.

Example. $\int_a^{+\infty} g(x)\sin x\,\mathrm{d}x$, $\int_a^{+\infty} g(x)\cos x\,\mathrm{d}x$, where g is monotonic and nonnegative on $[a,+\infty)$.

2. $\int_a^{+\infty} g$ diverges. **2.2.** $\lim\limits_{x\to\infty} g(x) = c > 0$.

$\lim\limits_{x\to\infty} g(x) = c > 0 \;\Rightarrow\; \exists\, A \;\; \forall\, x > A \;\; g(x) \geq \frac{c}{2}$.

$\sin x \geq \frac{1}{2} \;\Leftrightarrow x \;\in \left[\frac{\pi}{6} + 2\pi k, \frac{5\pi}{6} + 2\pi k\right], k \in \mathbb{Z}$.

$$\Rightarrow \int_{\frac{\pi}{6}+2\pi k}^{\frac{5\pi}{6}+2\pi k} g(x)\sin x\,\mathrm{d}x \geq \frac{c}{2}\cdot\frac{1}{2}\cdot\frac{2\pi}{3} = \frac{c\pi}{6}.$$

Therefore, $\exists\, \varepsilon_0 = \frac{c\pi}{6} \;\; \forall\, b_0 \in [1,+\infty) \;\; \exists\, b_1 = \frac{\pi}{6} + 2\pi\left\lceil\frac{b_0}{2\pi}\right\rceil$,

$b_2 = \frac{5\pi}{6} + 2\pi\left\lceil\frac{b_0}{2\pi}\right\rceil \in (b_0,+\infty) \;\; \int_{b_1}^{b_2} g(x)\sin x\,\mathrm{d}x \geq \varepsilon_0$

Thus, by Cauchy criterion, $\int_a^{+\infty} g(x)\sin x\,\mathrm{d}x$ diverges.

Example. $I = \displaystyle\int_0^{+\infty} \frac{\sin\left(x + \frac{1}{x}\right)}{x^\alpha}\,\mathrm{d}x.$

Let $I = I_1 + I_2 + I_3 + I_4$, where

$$I_1 = \int_0^1 \frac{\sin x \cos \frac{1}{x}}{x^\alpha}\,dx, \quad I_2 = \int_1^{+\infty} \frac{\sin x \cos \frac{1}{x}}{x^\alpha}\,dx,$$

$$I_3 = \int_0^1 \frac{\cos x \sin \frac{1}{x}}{x^\alpha}\,dx, \quad I_4 = \int_1^{+\infty} \frac{\cos x \sin \frac{1}{x}}{x^\alpha}\,dx.$$

Changing the variable $\frac{1}{x} = t$ in I_1 and I_3, we get

$$I_1 = \int_1^{+\infty} \frac{\cos t \sin \frac{1}{t}}{t^{2-\alpha}}\,\mathrm{d}t, \quad I_3 = \int_1^{+\infty} \frac{\sin t \cos \frac{1}{t}}{t^{2-\alpha}}\,\mathrm{d}t,$$

→ 对应的应是 $\int_{+\infty}^{1}$. 后面提出一个负号 $\int_1^{+\infty}$

so the integrals $\boxed{I_1, I_4}$ and $\boxed{I_2, I_3}$ are of the same type. Thus, it is sufficient to investigate I_2 and I_4. Since $\left(\dfrac{\cos(1/x)}{x^\alpha}\right)' < 0$ for $x > x_1 > 1$, it follows that $\dfrac{\cos(1/x)}{x^\alpha}$ decays to 0 as $x \to +\infty$ and $\alpha > 0$. The function

$F(x) = \int_1^x \sin x\,dx = -\cos x + \cos 1$ bounded

$x \mapsto \sin x, 1 \le x < +\infty$ has a bounded primitive $\forall\, x \in [1, +\infty)$. Thus, by Dirichlet's test, I_2 converges for $\alpha > 0$.

→ use Cauchy's criterion.

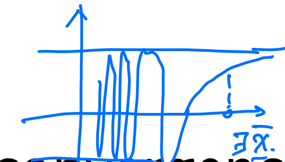
Let us show that I_2 diverges for $\alpha \le 0$. Fix $0 < \varepsilon < 1$. Set $\beta = -\alpha$, find $n \in \mathbb{N}$ such that $\cos \frac{1}{x} > \frac{1}{2}$ for $x \ge 2n\pi$. By the first mean value theorem, we get

$$= \xi_n^{\beta} \cos \frac{1}{\xi_n^{\beta}} \overbrace{\boxed{\int_{2n\pi}^{(2n+1)\pi} \sin x \, dx}}^{=2}$$

$$\left| \int_{2n\pi}^{(2n+1)\pi} x^{\beta} \underline{\sin \overset{\ge 0}{x}} \cos \frac{1}{x} dx \right| = 2\underset{\text{for each } n.}{\xi_n^{\beta}} \cos \frac{1}{\xi_n} > \xi_n^{\beta} \ge 1, \;\; 2n\pi \le \xi_n \le (2n+1)\pi.$$

Since $\forall \;\; x_0 > 1 \;\; \exists n \in \mathbb{N}$ such that $2n\pi > x_0$ it follows by Cauchy's criterion that I_2 diverges for $\alpha \le 0$. So, I_2 converges iff $\alpha > 0$. Analogously, I_3 converges iff $2 - \alpha > 0$, that is $\alpha < 2$. Thus, I_2 and I_3 converge simultaneously iff $0 < \alpha < 2$. (check I_2's related function I_3)

Let us apply Dirichlet's test for I_4. The inequalities $\left(\dfrac{\sin(1/x)}{x^{\alpha}} \right)' < 0$,
$0 < \dfrac{\sin(1/x)}{x^{\alpha}} < \dfrac{1}{x^{\alpha+1}}$ valid for $x > 1, \alpha + 1 > 0$ and boundedness of $x \mapsto \displaystyle\int_1^x \cos t \, \mathrm{d}t, 1 \le x < +\infty$ imply that I_4 converges for $\alpha + 1 > 0$, that is $\alpha > -1$. Therefore, I_1 converges for $\alpha < 3$, and both integrals converge simultaneously for $-1 < \alpha < 3$. By $(-1, 3) \cap (0, 2) = (0, 2)$, I converges for $0 < \alpha < 2$.



$\exists \bar{x}.$ for $x > \bar{x}$. $\cos x > \frac{1}{2}$

Let us study an absolute convergence of I_2. It follows from the inequalities

(去绝对值) $\frac{\sin^2 x}{2x^\alpha} \leqslant \frac{1}{2}\frac{|\sin x|}{x^\alpha} \leqslant$

comparison test. conv. for $\alpha > 1$.

$\frac{1}{4x^\alpha}$ div. on $\alpha \leq 1$.

$$\frac{1-\cos 2x}{4x^\alpha} = \frac{\sin^2 x}{2x^\alpha} < \frac{\left|\sin x \cos \frac{1}{x}\right|}{x^\alpha} \leqslant \frac{1}{x^\alpha},$$

holding true, say, for $x > 2$, that I_2 absolutely converges for $\alpha > 1$, and absolutely diverges for $\alpha \leq 1$.

Analogously, I_3 absolutely converges for $2 - \alpha > 1$, that is $\alpha < 1$. The sets $\{\alpha \in \mathbb{R} : \alpha > 1\}$ and $\{\alpha \in \mathbb{R} : \alpha < 1\}$ are disjoint, that is why there is no $\alpha \in \mathbb{R}$ such that I_2 and I_3 simultaneously absolutely converge. Thus, I is absolutely divergent.

Definition (The principal value of the improper integral)

Let $f \in \mathcal{R}_{loc}[a,c) \cup (c,b]$. *The quantity*

$$p.v. \int_a^b f := \lim_{\varepsilon \to +0} \left(\int_a^{c-\varepsilon} f + \int_{c+\varepsilon}^b f \right),$$

if the limit exists, is called the **principal value of the improper integral** *of* f *over* $[a,b]$.

Let $f \in \mathcal{R}_{loc}(-\infty,+\infty)$. *The quantity*

$$p.v. \int_{-\infty}^{+\infty} f := \lim_{A \to +\infty} \int_{-A}^{A} f,$$

if the limit exists, is called the **principal value of the improper integral** *of* f *over* $\mathbb{R}$.

Compare with ordinary improper integrals

$$\int_a^b f := \lim_{\varepsilon_1 \to +0} \int_a^{c-\varepsilon_1} f + \lim_{\varepsilon_2 \to +0} \int_{c+\varepsilon}^b f, \quad \int_{-\infty}^{+\infty} f := \lim_{A \to -\infty} \int_A^c f + \lim_{B \to +\infty} \int_c^B f.$$

Example.

1. $\displaystyle\int_{-\infty}^{+\infty} \sin x \, \mathrm{d}x$ diverges,

$$p.v. \int_{-\infty}^{+\infty} \sin x \, \mathrm{d}x = \lim_{A \to +\infty} \int_{-A}^{A} \underbrace{\sin x}_{odd} \, \mathrm{d}x = 0.$$

2. $\displaystyle\int_{-1}^{1} \frac{\mathrm{d}x}{x}$ diverges,

$$p.v. \int_{-1}^{1} \frac{\mathrm{d}x}{x} = \lim_{\varepsilon \to +0} \left(\int_{-1}^{-\varepsilon} \frac{\mathrm{d}x}{x} + \int_{+\varepsilon}^{1} \frac{\mathrm{d}x}{x} \right)$$

$$= \lim_{\varepsilon \to +0} \left(\ln|x| \Big|_{-1}^{-\varepsilon} + \ln|x| \Big|_{\varepsilon}^{1} \right) = 0.$$

Example. v.p. $\displaystyle\int_0^{+\infty} \frac{dx}{x^2-3x+2}$.

also correct (in French.)

$$\text{v. p.} \int_0^{+\infty} \frac{dx}{x^2-3x+2} = \text{v. p.} \int_0^{3} \frac{dx}{x^2-3x+2} + \int_3^{+\infty} \frac{dx}{x^2-3x+2}$$

$$= \lim_{\substack{\mu\to+0\\ \varepsilon\to+0}} \left(\log\left|\frac{x-2}{x-1}\right|\Bigg|_0^{1-\varepsilon} + \log\left|\frac{x-2}{x-1}\right|\Bigg|_{1+\varepsilon}^{2-\mu} + \log\left|\frac{x-2}{x-1}\right|\Bigg|_{2+\mu}^{3} \right)$$

$$+ \lim_{x\to+\infty} \log\frac{t-2}{t-1}\Bigg|_3^x$$

$$= -\log 2 + \lim_{\substack{\varepsilon\to+0\\ \mu\to+0}} \left(\log\frac{1+\varepsilon}{1-\varepsilon} + \log\frac{1+\mu}{1-\mu} \right) + \lim_{x\to+\infty} \log\frac{x-2}{x-1} = \log\frac{1}{2}.$$