Mathematical Logic

Lecture 5

Harbin, 2023

Main Theorem's

Soundness Theorem

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$, for all $\phi \in \mathcal{F}_{\mathcal{S}}$, it holds:
If $\Phi \vdash \phi$, then $\Phi \models \phi$.

Proof. Assume $\Phi \vdash \phi$, i.e., there are $\phi_1, \ldots, \phi_n \in \Phi$, such that the sequent $\phi_1, \ldots, \phi_n \phi$ is derivable in the sequent calculus.

We have already shown: each rule of the sequent calculus is correct (premise-free rules lead to correct sequents, rules with premises preserve correctness). By induction over the length of derivations in the sequent calculus it follows: every sequent that is derivable in the sequent calculus is correct.

So this must hold also for $\phi_1, \ldots, \phi_n \phi$, thus $\{\phi_1, \ldots, \phi_n\} \models \phi$, from where we get $\Phi \models \phi$. ■

Remark. The sequent calculus does not only contain correct rules for $\neg, \vee, \exists, \equiv$, but also for $\wedge, \to, \leftrightarrow, \forall$ by means of the metalinguistic abbreviations that we considered earlier (E.g. $\phi \to \psi := \neg\phi \vee \psi$)

Using such abbreviations we get:

Modus Ponens

$$\begin{array}{c} \Gamma \ \phi \to \psi \\ \Gamma \ \phi \\ \hline \Gamma \ \psi \end{array}$$

proof analogous to:

$$\begin{array}{c} \Gamma \ \phi \vee \psi \\ \Gamma \ \neg\phi \\ \hline \Gamma \ \psi \end{array}$$

We have already defined the notion of derivability for formulas. Some formulas have the property of being derivable without any premises:

$\boxed{\textit{Definition 1.}}$ For all $\phi \in \mathcal{F}_{\mathcal{S}}$:

ϕ is provable iff the (one-element) sequent ϕ is derivable in the sequent calculus (briefly: $\vdash \phi$).

<u>Example.</u> $\phi \vee \neg\phi$ (for arbitrary $\phi \in \mathcal{F}_{\mathcal{S}}$) is provable by "Excluded middle".

Some formulas have the property of not including (explicitly or implicitly) a contradiction:

$\boxed{\textit{Definition 2.}}$ For all $\phi \in \mathcal{F}_{\mathcal{S}}$, $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$:

ϕ is consistent iff there is no $\psi \in \mathcal{F}_{\mathcal{S}}$ with: $\{\phi\} \vdash \psi$, $\{\phi\} \vdash \neg\psi$.

Φ is consistent iff there is no $\psi \in \mathcal{F}_{\mathcal{S}}$ with: $\Phi \vdash \psi$, $\Phi \vdash \neg\psi$.

Example. $P(c)$ is consistent, because: assume $\{P(c)\} \vdash \psi$, $\{P(c)\} \vdash \neg\psi$ $\Rightarrow \{P(c)\} \models \psi$, $\{P(c)\} \models \neg\psi$ by the Soundness Theorem $\Rightarrow$ there are no $\mathfrak{M}, s$ with $\mathfrak{M}, s \models P(c)$.
But this is false: take e.g. $D = \{1\}$, $\mathfrak{I}(c) = 1$, $\mathfrak{I}(P) = D \Rightarrow \mathfrak{M}, s \models P(c)$.

Lemma 1.

Φ is consistent iff there is a $\psi \in \mathcal{F}_{\mathcal{S}}$, such that $\Phi \nvdash \psi$.

Proof. Obvious.■

Lemma 2.

Φ is consistent iff every finite subset $\Phi' \subseteq \Phi$ is consistent.

Proof. Immediate from our definitions.■

Soundness Theorem: Second Version

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$, if Φ is satisfiable, then Φ is consistent.

Proof. We show: Φ is not consistent $\Rightarrow$ Φ is not satisfiable.
Assume that Φ is not consistent; then $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi \Rightarrow \Phi \models \psi$, $\Phi \models \neg\psi$ by the Soundness theorem, so there are no $\mathfrak{M}, s$ with $\mathfrak{M}, s \models \Phi$, i.e., Φ is not satisfiable. ■

Lemma 3.

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$, $\phi \in \mathcal{F}_{\mathcal{S}}$:
1.) ϕ is provable iff $\varnothing \vdash \phi$.
2.) $\Phi \vdash \phi$ iff $\Phi \cup \{\neg\phi\}$ is not consistent.
3.) ϕ is provable iff $\neg\phi$ is not consistent.

Proof. 1. Follows directly from the definitions.

2. Assume that $\Phi \vdash \phi$. Obviously, this implies that $\Phi \cup \{\neg\phi\} \vdash \phi$. Furthermore, $\Phi \cup \{\neg\phi\} \vdash \neg\phi \Rightarrow \Phi \cup \{\neg\phi\}$ is not consistent. Conversely, if $\Phi \cup \{\neg\phi\}$ is not consistent, then every formula is derivable from $\Phi \cup \{\neg\phi\}$ by Lemma 1, whence $\Phi \cup \{\neg\phi\} \vdash \phi$ i.e., there is a derivation of the following form:
$\Gamma \vdash \phi$ (for $\Gamma \subseteq \phi$), and thus $\Phi \vdash \phi$.
3. Consider $\Phi = \varnothing$ and apply 2. and 1. ■

Lemma 4.

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$, $\phi \in \mathcal{F}_{\mathcal{S}}$:
If Φ is consistent, then $\Phi \cup \{\phi\}$ is consistent or $\Phi \cup \{\neg\phi\}$ is consistent.

Proof. Assume that $\Phi \cup \{\phi\}$, $\Phi \cup \{\neg\phi\}$ are both not consistent. $\Phi \cup \{\neg\phi\}$ is not consistent $\Rightarrow \Phi \vdash \phi$ by the Lemma 3. $\Phi \cup \{\phi\}$ is not consistent $\Rightarrow \Phi \vdash \neg\phi$. In total, Φ is not consistent. ■

Lemma 5.

Let $\mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \ldots$ be a chain of symbol sets. Let $\Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \ldots$ be a chain of sets of formulas, such that:
For all $n \in \{0, 1, 2 \ldots\}$, Φ_n is a set of formulas over the symbol set $\mathcal{S}_n$ and Φ_n is $\mathcal{S}_n$- consistent (i.e., consistent in the sequent calculus for formulas in $\mathcal{F}_{\mathcal{S}_n}$).
Let finally $\mathcal{S} = \cup_{n \in \{0,1,2\ldots\}} \mathcal{S}_n$, $\Phi = \cup_{n \in \{0,1,2\ldots\}} \Phi_n$.
Then Φ is $\mathcal{S}-$consistent.

Proof. Let the symbol sets and formula sets be given as indicated above.
Assume that Φ is not $\mathcal{S}-$consistent $\Rightarrow$ there is a $\Psi \subseteq \Phi$ with Ψ finite, Ψ not $\mathcal{S}-$consistent by the Lemma 2 $\Rightarrow$ there is a $k \in \{0, 1, 2, \ldots\}$ such that $\Psi \subseteq \Phi_k$ (since Ψ is finite).
Ψ is not $\mathcal{S}-$consistent, therefore for some $\psi \in \mathcal{F}_{\mathcal{S}}$, there is an $\mathcal{S}-$derivation of ψ from Ψ, and there is an $\mathcal{S}-$derivation of $\neg\psi$ from Ψ.

But in these two sequent calculus derivations only finitely many symbols in $\mathcal{S}$ can occur. Thus there is an $m \in \mathbb{N}$, such that $\mathcal{S}_m$ contains all the symbols in these two derivations. Without loss of generality, we can assume that $m \geq k \Rightarrow \Psi$ is not $\mathcal{S}_m$-consistent $\Rightarrow \Phi_k$ is not $\mathcal{S}_m$-consistent, and since $m \geq k$, it also follows that $\Phi_k \subseteq \Phi_m \Rightarrow \Phi_m$ is not $\mathcal{S}_m$-consistent, which leads to contradiction $\Rightarrow \Phi$ is $\mathcal{S}$-consistent. ■

The Completeness Theorem

Let $\mathcal{S}$ be a symbol set that we keep fixed.

$\boxed{\textit{Definition 3.}}$ Let $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$:

- Φ is maximally consistent iff Φ is consistent and for all $\phi \in \mathcal{F}_{\mathcal{S}}$: $\Phi \vdash \phi$ or $\Phi \vdash \neg\phi$.
- Φ contains instances iff for every formula of the form $\exists x \phi$ there is a $t \in \mathcal{T}_{\mathcal{S}}$ such that $\Phi \vdash (\exists x \phi \rightarrow \phi\frac{t}{x})$

Now let Φ be maximally consistent with instances (i.e., it is maximally consistent and contains instances).
To show: Φ is satisfiable.

Let us consider an example first: $\Phi = \{P(c_1)\}$, so Φ is consistent (since it is satisfiable). But Φ is not maximally consistent (e.g., $\Phi \nvdash P(c_2)$, $\Phi \nvdash \neg P(c_2)$, because $\Phi \cup \{\neg P(c_2)\}$, $\Phi \cup \{P(c_2)\}$ consistent since they are satisfiable.
Furthermore, Φ does not contain instances for all formulas (only for some).

E.g., $\Phi \nvdash (\exists x \neg P(x) \to \neg P(t))$ for arbitrary $t \in \mathcal{T}_S$
Since: choose any model of $\Phi \cup \{\neg(\exists x \neg P(x) \to \neg P(t))\}$ such a model exists and thus this formula set is consistent. But e.g.
$\Phi \vdash (\exists x P(x) \to P(c_1))$, since $\Phi \vdash P(c_1)$ and thus by the $\vee$-introduction rule in the consequent: $\Phi \vdash \neg\exists x P(x) \vee P(c_1)$

Another example:

Let $\mathfrak{M}, s$ be such that for every $d \in D$ there is a $t \in \mathcal{T}_S$ suchn that: $Val_{\mathfrak{M},s}(t) = d$ (which implies that D is countable).
Consider $\Phi = \{\phi \in \mathcal{F}_S | \mathfrak{M}, s \models \phi\} \Rightarrow \Phi$ is consistent and $\Phi \vdash \phi$ or $\Phi \vdash \neg\phi$ for arbitrary $\phi \in \mathcal{F}_S$ (because $\mathfrak{M}, s \models \phi$ or $\mathfrak{M}, s \models \neg\phi \Rightarrow \phi \in \Phi$ or $\neg\phi \in \Phi$). $\Rightarrow \Phi$ is maximally consistent.

Furthermore, for all formulas $\exists x \phi$ thee is a t, such that $\Phi \vdash (\exists x \phi \to \phi\frac{t}{x})$
Case 1: $\mathfrak{M}, s \not\models \exists x \phi \Rightarrow \mathfrak{M}, s \models \exists x \phi \to \phi\frac{t}{x}$ for all $t \in \mathcal{T}_S$

Case 2: $\mathfrak{M}, s \models \exists x\phi \Rightarrow$ there is a $d \in D$, such that $\mathfrak{M}, s\frac{d}{x} \models \phi$, and $d = Val_{\mathfrak{M},s}(t)$ for some $t \in \mathcal{T}_S \Rightarrow$ there is a $t \in \mathcal{T}_S$, such that: $\mathfrak{M}, s\frac{Val_{\mathfrak{M},s}(t)}{x} \models \phi \Rightarrow$ there is a $t \in \mathcal{T}_S$, such that $\mathfrak{M}, s \models \phi\frac{t}{x}$ by the Substitution Lemma $\Rightarrow$ there is a $t \in \mathcal{T}_S$, such that $\mathfrak{M}, s \models \exists x\phi \to \phi\frac{t}{x}$.
For such a t it follows:
$\exists x\phi \to \phi\frac{t}{x} \in \Phi$.

So Φ is actually maximally consistent with instances. Now we that we have seen an example of a maximally consistent set of formulas that contains instances, let us consider such sets in general. We will show that every such set is satisfiable.

Completeness Theorem I

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$ it holds:
If Φ is consistent, the Φ is satisfiable.

Proof. Without proof. ■

Completeness Theorem II

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$, for all $\phi \in \mathcal{F}_{\mathcal{S}}$, it holds:
If $\Phi \models \phi$, then $\Phi \vdash \phi$.

Proof. Without proof. ■

Thus, if ϕ follows logically from Φ, then ϕ is derivable from Φ on the basis of the sequent calculus; thus the sequent calculus is complete.

Remark.

- ϕ is provable iff ϕ is logically true.
- Φ is consistent iff Φ is satisfiable.
- Since logical consequence and satis

ability are independent of the particular choice of $\mathcal{S}$, the same must hold for derivability and consistency.

Applications

Loewenheim-Skolem Theorem

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$, if Φ is satisfiable, then there are $\mathfrak{M}, s$, such that $\mathfrak{M}, s \models \Phi$, and the domain D of $\mathfrak{M}$ is countable.

Proof. Without proof. ■

Compactness Theorem

For all $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$, $\phi \in \mathcal{F}_{\mathcal{S}}$:

1. $\Phi \models \phi$ if and only if there is a $\Psi \subseteq \Phi$ with Ψ finite and $\Psi \models \phi$.
2. Φ is satisfiable if and only if for all $\Psi \subseteq \Phi$ with Ψ finite: Ψ is satisfiable.

Proof. We already know that the proof-theoretic analogues to these claims hold (by Lemma 1 from the previous lecture and Lemma 2). But this means we are done by the soundness and the completeness theorem.■

The theorem of Loewenheim-Skolem and the compactness theorem are important tools in model theory and have several surprising implications and applications.

Examples.

I. Consider the first-order theory of set theory: let $\mathcal{S}_{Set} = \{\in\}$.

List of set-theoretic definitions and axioms:

(i) definition of $\varnothing$:

$$\forall y(\varnothing = y \leftrightarrow \forall z \, \neg z \in y)$$

(ii) definition of $\subseteq$:

$$\forall x \forall y(x \subseteq y \leftrightarrow \forall z(z \in x \to z \in y))$$

(iii) definition of $\{,\}$:

$$\forall x \forall y \forall z(\{x, y\} = z \leftrightarrow \forall w(w \in z \leftrightarrow w = x \vee w = y))$$

(iv) definition of $\cup$:

$$\forall x \forall y \forall z(x \cup y = z \leftrightarrow \forall w(w \in z \leftrightarrow (w \in x \vee w \in y)))$$

(v) definition of $\cap$:

$$\forall x \forall y \forall z (x \cap y = z \leftrightarrow \forall w (w \in z \leftrightarrow (w \in x \wedge w \in y)))$$

I'. Axioms:

(i) Axiom of Extensionality ("Two sets that have the same members are equal"):

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \to x = y)$$

(ii) Axiom Schema of Separation:

$$\forall x_1 \ldots \forall x_n \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \phi[z, x_1, \ldots, x_n])$$

Explanation: For every set x and for every property E that is expressed by a formula ϕ with free variables $z, x_1, \ldots, x_n$ there is a set $\{z \in x \mid z \text{ has the property } E\}$

(iii) Axiom of Pairs ("For every two sets x, y there is the pair set $\{x, y\}$"):

$$\forall x \forall y \exists z \forall w (w \in z \leftrightarrow w = x \vee w = y)$$

(iv) Axiom of Unions ("For every set x there is a set y, which contains precisely the members of the members of x"):

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$$

(v) Powerset Axiom ("For every set x there is the power set y of x"):

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subseteq x)$$

(vi) Axiom of Infinity :

$$\exists x(\varnothing \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x))$$

(vii) Axiom of Choice:

$\forall x(\neg\varnothing \in x \wedge \forall u \forall v(u \in x \wedge v \in x \wedge \neg u \equiv v \rightarrow u \cap v \equiv \varnothing) \rightarrow \exists y \forall w(w \in x \rightarrow \exists!$

Explanation : for every set x that has non-empty and pairwise disjoint sets as its members there is a (choice) set y that contains for each set in x precisely one member.

Remark. Practically all theorems of standard mathematics can be derived from this set of de
nitions and axioms. At the same time, Loewenheim-Skolem tells that if this set of definitions and axioms is consistent, then it has a model with a countable domain!

II. Let $\mathcal{S}_{arithm} = \{\tilde{0}, \tilde{1}, \tilde{+}, \tilde{\cdot}\}$ be a standard model of arithmetic: $(\mathbb{N}, \mathfrak{I})$ with $\mathfrak{I}$ as expected (so $\mathfrak{I}(\tilde{0}) = 0$, $\mathfrak{I}(\tilde{1}) = 1$, $\mathfrak{I}(\tilde{+}) = +$ and $\mathfrak{I}(\tilde{\cdot}) = \cdot$ on $\mathbb{N}$).

Let Φ_{arithm} be the set of $\mathcal{S}_{arithm}$−sentences that are satisfied by this model, i.e.:

$$\Phi_{arithm} = \{\phi \in \mathcal{F}_{\mathcal{S}_{arithm}} \mid \phi \text{ sentence}, \ (\mathbb{N}, \mathfrak{I}) \models \phi\}$$

Now consider

$$\Psi = \Phi_{arithm} \cup \{\neg x \equiv 0, \neg x \equiv 1, \neg x \equiv 1 + 1, \neg x \equiv (1 + 1) + 1, \ldots\}$$

It holds that every finite subset of Ψ is satisfiable: just take the standard model of artithmetic and choose s in the way that $s(x)$ is a sufficiently large natural number (for a given finite subset of Ψ, $s(x)$ has to be large enough to be greater than any number denoted by any of the right-hand sides of the negated equations in the subset).

By the compactness theorem, this implies: Ψ is satisfiable, i.e., there is a model $\mathfrak{M}'$ and a variable assignment s', such that $\mathfrak{M}', s' \models \Psi$.

It follows:

$s'(x) \neq Val_{\mathfrak{M}',s'}(0)$, since $\neg x \equiv 0 \in \Psi$
$s'(x) \neq Val_{\mathfrak{M}',s'}(1)$, since $\neg x \equiv 1 \in \Psi$
$s'(x) \neq Val_{\mathfrak{M}',s'}(1 + 1)$, since $\neg x \equiv 1 + 1 \in \Psi$

If we finally identify the objects $Val_{\mathfrak{M}',s'}(1 + \ldots + 1)$ with our standard natural numbers, we get:
there exists a model of the set of true arithmetical sentences, such that the domain of this model contains a "new number" $s'(x)$ that is different from any of the "old" natural numbers $0, 1, 2, 3, \ldots$!

Exercises.

Exercise. Let $\mathcal{S}$ be an arbitrary symbol set.

Let $\Phi = \{v_0 \equiv t \mid t \in \mathcal{T}_{\mathcal{S}}\} \cup \{\exists v_1 \exists v_2 \neg v_1 \equiv v_2\}$

Show that:

(a) Φ is consistent.

(b) There is no formula set $\Psi \subseteq \mathcal{F}_{\mathcal{S}}$ with $\Phi \subseteq \Psi$, such that Ψ is consistent and contains instances.