Equations of mathematical physics

Content

Introduction

The course includes the main sections:

— Fundamentals of operational calculus;

— Classification of partial differential equations;

— Hyperbolic equations;

— Parabolic equations;

— Elliptical equations.

1. FUNDAMENTALS OF OPERATIONAL CALCULUS

1.1. The concepts of the original and the Laplace image. Properties of the Laplace transform

Definition 1. *An original function* is any complex-valued function $f(t)$ of a valid argument t that satisfies the conditions:

1) $f(t)$ is Riemann integrable on any finite interval of the t axis (locally integrable);

2) $f(t)=0$ for all $t<0$;

3) $M>0$ and $\alpha>0$ are constants for which

$$\left|f(t)\right| \le Me^{\alpha t}\,. \tag{1.1}$$

The lower edge α_0 of all numbers α for which the inequality (1.1) is valid is called the *growth index* of the function $f(t)$.

The first condition in definition 1 is sometimes formulated as follows: on any finite interval of the t axis, the function $f(t)$ is continuous, except, perhaps, a finite number of discontinuity points of the first kind.

The simplest original function is the Heaviside function:

$$\theta(t)=\begin{cases}0, & t<0,\\ 1, & t\ge 1.\end{cases}$$

Obviously, for any function $\varphi(t)$ it is true:

$$\varphi(t)\theta(t)=\begin{cases}0, & t<0,\\ \varphi(t), & t\ge 1.\end{cases}$$

If, for $t \geq 0$, the function $\varphi(t)$ satisfies conditions 1 and 3 of definition 1, then the function $\varphi(t)\theta(t)$ is the original. In the future, to shorten the record, we will, as a rule, write $\varphi(t)$ instead of $\varphi(t)\theta(t)$, assuming that the functions we are considering are continued by zero for negative values of the argument t.

Definition 2. *The image of the function* $f(t)$ *according to Laplace* is called the function $F(p)$ of the complex variable $p = s + i\sigma$, defined by the equality

$$F(p) = \int_0^{+\infty} f(t)e^{-pt}dt\,. \qquad (1.2)$$

Theorem 1 (on the analyticity of the image). For any original $f(t)$, its image $F(p)$ is defined and is an analytical function of the variable p in the half—plane $Re\, p > \alpha_0$, where α_0 is the growth index of the function $f(t)$, while the equality is valid:

$$\lim_{\operatorname{Re} p \to +\infty} |F(p)| = 0\,.$$

Theorem 2 (uniqueness). The Laplace image $F(p)$ is unique in the sense that two functions $f_1(t)$ and $f_2(t)$ having the same images coincide at all points of continuity at $t > 0$.

There are several ways to record the correspondence between the original and the image:

$$f(t) \leftrightarrow F(p)\,,\ f(t) \doteq F(p)\,,\ L\{f(t)\} = F(p)\,.$$

Example 1.

Using the definition, find the image of the function $f(t) = sin3t$.

Solution:

For the function $f(t) = sin3t$, we have $\alpha_0 = 0$. Therefore, the image $F(p)$ will be defined and analytically in the half-plane $Re\, p > 0$. Let us apply formula (1.2) to a given function, using the rule of integration in parts and the restriction on the set of values of the variable p, which ensures the convergence of the integral, when performing transformations:

$$F(p) = \int_0^{+\infty} e^{-pt} \sin 3t dt = -\frac{1}{p} e^{-pt} \sin 3t \Big|_0^{+\infty} + \frac{3}{p} \int_0^{+\infty} e^{-pt} \cos 3t dt =$$

$$= \frac{3}{p}\left(-\frac{1}{p} e^{-pt} \cos 3t \Big|_0^{+\infty} - \frac{3}{p} \int_0^{+\infty} e^{-pt} \sin 3t dt \right) = \frac{3}{p^2} - \frac{9}{p^2} F(p)\,.$$

Got equality

$$F(p) = \frac{3}{p^2} - \frac{9}{p^2} F(p)\,.$$

From here we find

$$F(p) = \frac{3}{p^2 + 9}\,.$$

Thus, the following correspondence is valid:

$$\sin 3t \leftrightarrow \frac{3}{p^2+9}, \ \mathrm{Re}\, p > 0.$$

Properties of the Laplace transform

1. **Linearity.** If $f(t) \leftrightarrow F(p)$, $g(t) \leftrightarrow G(p)$, then for any complex λ and μ it is performed

$$\lambda f(t) + \mu g(t) \leftrightarrow \lambda F(p) + \mu G(p), \ \mathrm{Re}\, p > \max(\alpha_0, \beta_0),$$

here and further, α_0, β_0 are the growth indicators of the function $f(t), g(t)$, respectively.

2. **Similarity.** If $f(t) \leftrightarrow F(p)$, then for $\forall \alpha > 0$ it is true

$$f(at) \leftrightarrow \frac{1}{a} F\left(\frac{p}{a}\right), \ \mathrm{Re}\, p > a\alpha_0.$$

3. **Differentiation of the original.** If $f(t), f'(t), ..., f^{(n)}(t)$ are originals and $f(t) \leftrightarrow F(p)$ for $Re\, p > \alpha_0$, then

$$f^{(n)}(t) \leftrightarrow p^n F(p) - p^{n-1} f(+0) - p^{n-2} f'(+0) - \ldots - p f^{(n-2)}(+0) - f^{(n-1)}(+0),$$

where

$$f^{(k)}(+0) = \lim_{t \to +0} f^{(k)}(t), \ k = 0, 1, \ldots, n-1.$$

4. **Image differentiation.** If $f(t) \leftrightarrow F(p)$, then

$$F^{(n)}(p) \leftrightarrow (-t)^n f(t), \ \mathrm{Re}\, p > \alpha_0.$$

5. **Integration of the original.** If $f(t) \leftrightarrow F(p)$, then

$$\int_0^t f(\tau)d\tau \leftrightarrow \frac{F(p)}{p}, \ \operatorname{Re} p > \alpha_0 .$$

6. **Image integration.** If $f(t) \leftrightarrow F(p)$ and $\frac{f(t)}{t}$ are the original, then

$$\int_p^\infty F(\xi)d\xi \leftrightarrow \frac{f(t)}{t}, \ \operatorname{Re} p > \alpha_0 .$$

7. **The delay property.** If $f(t) \leftrightarrow F(p)$ and $f(t)=0$ for $t<\tau$, where $\tau>0$, then

$$f(t-\tau) \leftrightarrow e^{-\tau p} F(p), \ \operatorname{Re} p > \alpha_0 .$$

Remark. The following formulation of the delay property is possible: if $f(t) \leftrightarrow F(p)$, then for any $\tau>0$ there is

$$f(t-\tau)\theta(t-\tau) \leftrightarrow e^{-\tau p} F(p), \ \operatorname{Re} p > \alpha_0 .$$

8. **The displacement property.** If $f(t) \leftrightarrow F(p)$, then for any complex λ

$$e^{\lambda t} f(t) \leftrightarrow F(p-\lambda), \ \operatorname{Re} p > \alpha_0 + \operatorname{Re}\lambda .$$

9. **The image of the convolution.** *The convolution of functions* f *and* g is a function that is denoted by $f \cdot g$ and is defined by equality

$$(f \cdot g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau .$$

The convolution of functions has the property of symmetry, that is,

$$(f \cdot g)(t) = (g \cdot f)(t) .$$

If $f(t) \leftrightarrow F(p)$ and $g(t) \leftrightarrow G(p)$, then

$$(f \cdot g)(t) \leftrightarrow F(p)G(p), \ \operatorname{Re} p > \max(\alpha_0, \beta_0).$$

Here is a table of originals and images of some elementary functions:

Original $f(t)$	Image $F(p)$	Original $f(t)$	Image $F(p)$
1	$\frac{1}{p}$	$sh\, at$	$\frac{a}{p^2 - a^2}$
e^{-at}	$\frac{1}{p+a}$	$ch\, at$	$\frac{p}{p^2 - a^2}$
t	$\frac{1}{p^2}$	$e^{-at} \cos \omega t$	$\frac{p+a}{(p+a)^2 + \omega^2}$
$\sin at$	$\frac{a}{p^2 + a^2}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(p+a)^2 + \omega^2}$
$\cos at$	$\frac{p}{p^2 + a^2}$	$e^{at}\, sh\, \omega t$	$\frac{\omega}{(p-a)^2 - \omega^2}$
$t^n, n \in Z$	$\frac{n!}{p^{n+1}}$	$e^{at}\, ch\, \omega t$	$\frac{p-a}{(p-a)^2 - \omega^2}$
$t^n e^{at}$	$\frac{n!}{(p-a)^{n+1}}$	$t\, sh\, \omega t$	$\frac{2\omega p}{(p^2 - \omega^2)^2}$
$t \sin \omega t$	$\frac{2p\omega}{(p^2 + \omega^2)^2}$	$t\, ch\, \omega t$	$\frac{p^2 + \omega^2}{(p^2 - \omega^2)^2}$
$t \cos \omega t$	$\frac{p^2 - \omega^2}{(p^2 + \omega^2)^2}$	$e^{at}\, t \sin \omega t$	$\frac{2\omega(p-a)}{((p-a)^2 + \omega^2)^2}$
$e^{at}\, t \cos \omega t$	$\frac{(p-a)^2 - \omega^2}{((p-a)^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	$\frac{1}{(p^2 + \omega^2)^2}$
$\frac{1}{2\omega^3}(\omega t\, ch\, \omega t - sh\, \omega t)$	$\frac{1}{(p^2 - \omega^2)^2}$	$\sin(\omega t \pm \varphi)$	$\frac{\omega \cos \varphi \pm p \sin \varphi}{p^2 + \omega^2}$
$\cos(\omega t \pm \varphi)$	$\frac{p \cos \varphi \mp \omega \sin \varphi}{p^2 + \omega^2}$		

Example 2.

Using the properties of the Laplace transform and the table of the main originals and images, find images of the following functions:

1) $f(t) = e^{-4t} \sin 3t \cos 2t$;
2) $f(t) = e^{(t-2)} \sin(t-2)$;
3) $f(t) = t^2 e^{3t}$;
4) $f(t) = \dfrac{\sin^2 t}{t}$.

<u>Solution:</u>

1) Let's transform the expression for the function $f(t)$ as follows:

$$f(t) = e^{-4t} \sin 3t \cos 2t = \frac{1}{2} e^{-4t} (\sin 5t + \sin t) = \frac{1}{2} e^{-4t} \sin 5t + \frac{1}{2} e^{-4t} \sin t .$$

Since $\sin t \leftrightarrow \dfrac{1}{p^2+1}$ and $\sin 5t \leftrightarrow \dfrac{5}{p^2+25}$, then, using the properties of linearity and displacement, for the image of the function $f(t)$ we will have

$$F(p) = \frac{1}{2}\left(\frac{5}{(p+4)^2+25} + \frac{1}{(p+4)^2+1} \right).$$

2) Since $\sin t \leftrightarrow \dfrac{1}{p^2+1}$, $e^t \sin t \leftrightarrow \dfrac{1}{(p-1)^2+1}$, then, using the delay property, we will have

$$f(t) = e^{(t-2)} \sin(t-2) \leftrightarrow F(p) = \frac{e^{-2p}}{(p-1)^2+1} .$$

3) Since $t^2 \leftrightarrow \dfrac{2}{p^3}$, then by the displacement property we have

$$f(t)=t^2e^{3t} \leftrightarrow F(p)=\frac{2}{(p-3)^3}.$$

For comparison, we present a method for constructing an image of function $f(t)=t^2 e^{3t}$ using the image differentiation property:

$$e^{3t} \leftrightarrow \frac{1}{p-3};\ te^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{p-3}\right)=\frac{1}{(p-3)^2};$$

$$t^2e^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{(p-3)^2}\right)=\frac{2}{(p-3)^3}.$$

We got the same result.

4. <u>HOMEWORK (using the image integration property)</u>

For a function defined as follows:

$$f(t)=\begin{cases} 0, & t<t_1, \\ f_1(t), & t_1 \le t < t_2, \\ f_2(t), & t_2 \le t < t_3, \\ \dots & \\ f_{n-1}(t), & t_{n-1} \le t < t_n, \\ f_n(t), & t \ge t_n, \end{cases}$$

using the Heaviside function, you can write an analytical form that is convenient to use when constructing the corresponding image.

It is easy to verify that for a function $g_k(t)$ equal to

$$g_k(t)=\begin{cases} 0, & t<t_k, \\ f_k(t), & t_k \le t < t_{k+1}, \\ 0, & t \ge t_{k+1}, \end{cases}$$

the following representation is valid using the Heaviside function:

$$g_k(t) = f_k(t)\theta(t - t_k) - f_k(t)\theta(t - t_{k+1}) . \qquad (1.3)$$

And for the function

$$g_n(t) = \begin{cases} 0, & t < t_n, \\ f_n(t), & t \geq t_n, \end{cases}$$

there is

$$g_n(t) = f_n(t)\theta(t - t_n) . \qquad (1.4)$$

Assuming that k varies from 1 to $n-1$, the function $f(t)$ can be considered as the sum of the functions $g_k(t)$ and $g_n(t)$:

$$f(t) = \sum_{k=1}^{n-1} g_k(t) + g_n(t) .$$

And then, using expressions (1.3) and (1.4), we get

$$f(t) = f_1(t)\theta(t - t_1) - \sum_{k=2}^{n} \left(f_k(t) - f_{k-1}(t)\right)\theta(t - t_k) . \qquad (1.5)$$

Example 3.

Build an image for the function $f(t)$:

$$f(t) = \begin{cases} 0, & t < a, \\ \varphi(t), & a \leq t < b, \\ 0, & t \geq b. \end{cases}$$

<u>Solution:</u>

Let's write an expression for the function $f(t)$ using the Heaviside function:

$$f(t)=\varphi(t)\theta(t-a)-\varphi(t)\theta(t-b).$$

Since

$$\varphi(t)=\varphi(t-a+a) \text{ и } \varphi(t)=\varphi(t-b+b),$$

then, having found the images for the functions $\varphi(t+a)$ and $\varphi(t+b)$,

$$\varphi(t+a)\leftrightarrow\Phi_1(p),\ \varphi(t+b)\leftrightarrow\Phi_2(p),$$

we construct an image for the function $f(t)$, taking into account the lag property

$$f(t)\leftrightarrow F(p)=\Phi_1(p)e^{-ap}-\Phi_2(p)e^{-bp}.$$

Example 4.

Find the image $F(p)$ of the function $f(t)$:

$$f(t)=\begin{cases}0, & t\in(-\infty,0),\\ 1, & t\in(0,a),\\ \dfrac{2a-t}{a}, & t\in(a,3a),\\ \dfrac{t-4a}{a}, & t\in[3a,\infty).\end{cases}$$

Solution:

Let's find an image of the function $f(t)$, having previously written an expression for it using the Heaviside function $\theta(t)$. To do this, use the formula (1.5). Since for a given function

$t_1=0$, $t_2=a$, $t_3=3a$ and

$$f_1(t) = 1\,,\ f_2(t) = \frac{2a - t}{a}\,,\ f_3(t) = \frac{t - 4a}{a}\,,$$

then we will have

$$f(t) = \theta(t) - \left(\frac{2a - t}{a} - 1\right)\theta(t - a) + \left(\frac{t - 4a}{a} - \frac{2a - t}{a}\right)\theta(t - 3a) =$$

$$= \theta(t) - \frac{t - a}{a}\theta(t - a) + \frac{2(t - 3a)}{a}\theta(t - 3a)\,.$$

Applying the properties of linearity and delay to the constructed expression, we find the desired image $F(p)$:

$$F(p) = \frac{1}{p} - \frac{1}{ap^2}e^{-ap} + \frac{2}{ap^2}e^{-3ap}\,.$$

CONTROL TASKS

Example 1.

1) Check if the following functions are originals and find their growth index:

a) $f(t)=2e^{3t}\sin at,\ a\in R$

b) $f(t)=e^{3+it^2}$

c) $f(t)=\frac{1}{t}$

<u>Solution:</u>

a) The function $f(t)=2e^{3t}\sin at, a\in R$ is continuous at any finite interval $[0,B], B>0$. Therefore, it is integrable on $[0,B]$. Since

$$\left|f(t)\right|=2e^{3t}\left|\sin at\right|\le 2e^{3t},$$

$M=2, \alpha_0=3$, then $f(t)$ is a function of bounded growth with a growth index of $\alpha_0=3$. Therefore, the $f(t)$ -function is the original.

b) The function $f(t)=e^{3+it^2}$ is continuous and, therefore, integrable on any finite interval $[0,B]$.

$$\left|e^{3+it^2}\right|=e^3\left|\cos t^2+i\sin t^2\right|=e^3\le e^3e^{0\cdot t},$$

where $M=e^3, \alpha_0=0$.

The function $f(t)$ is the original with the growth index $\alpha_0=0$.

c) Function $f(t)=\frac{1}{t}$ is not the original. Integral $\int_0^B \frac{1}{t}\,dt = \ln t\Big|_0^B = +\infty$ diverges. Function $f(t)$ is not integrable. The first condition for defining the original function has been violated.

Example 2.

2) The function $F(p)$ is given. Can it be an image of some original in some area? If so, specify this area.

a) $F(p)=1$,

b) $F(p)=\sin p$,

c) $F(p)=\frac{p}{p^2-2p+5}$

<u>Solution:</u>

a) Since

$$\lim_{\mathrm{Re}(p)\to+\infty} F(p)=1,$$

the necessary sign of the existence of an image is not fulfilled for $F(p)$ (Theorem 1). Function $F(p)$ is not an image.

b) Since

$$\lim_{\mathrm{Re}(p)\to+\infty} \sin p$$

does not exist, the necessary sign of the existence of the image is not fulfilled for $F(p)$. Function $F(p)$ is not an image.

c) The necessary indication of the existence of the image

$$\lim_{\mathrm{Re}(p)\to+\infty} \frac{p}{p^2-2p+5}=0$$

has been fulfilled. Function $F(p)$ is analytical in the entire domain except for the zeros of the denominator.

Solving the equation

$$p^2-2p+5=0,$$

we get the simple poles

$$p_{1,2}=1\pm 2i$$

of the function $F(p)$.

Therefore, $F(p)$ will be an image in the region $Re(p)>1$.

In order to verify the correctness of calculations, limiting ratios are used in operational calculus.

Theorem 3 (*on limiting ratios*).

If $f(t), f'(t)$ are originals and $f(t)\leftrightarrow F(p)$, then

$$\lim_{\mathrm{Re}\,p\to+\infty} pF(p)=\lim_{t\to+0} f(t)=f(0), \tag{2}$$

if there is a finite limit of $\lim\limits_{t\to+\infty} f(t)$, then

$$\lim_{p\to 0} pF(p)=\lim_{t\to+\infty} f(t). \tag{3}$$

Example 3.

3) Using the definition, find images of the following functions

a) $f(t)=\theta(t)$

b) $f(t)=e^{4t}$

c) $f(t)=sint$

<u>Solution:</u>

a) Function $f(t)=\theta(t)$ is the original with a growth index of $\alpha_0=0$.

$$F(p)=\int_0^{+\infty}1\cdot e^{-pt}dt=\lim_{B\to+\infty}\int_0^B 1\cdot e^{-pt}dt=\lim_{B\to+\infty}\left(-\frac{1}{p}\cdot e^{-pt}\Big|_0^B\right)=\frac{1}{p}.$$

b) The function $f(t)=e^{4t}$ is the original with the growth index $\alpha_0=4$

$$F(p)=\int_0^{+\infty}e^{4t}\cdot e^{-pt}dt=\lim_{B\to+\infty}\int_0^B e^{-(p-4)t}dt=-\lim_{B\to+\infty}\left(\frac{1}{p-4}e^{-(p-4)t}\Big|_0^B\right)=$$

$$=\lim_{B\to+\infty}\left(\frac{1}{p-4}-\frac{e^{-(p-4)B}}{p-4}\right)=\frac{1}{p-4}.$$

Let's check the calculations using the limit ratios (2) and (3). In this case, $\lim\limits_{t\to+\infty}e^{4t}=+\infty$ (the final limit $f(t)$ does not exist), and condition (2) is fulfilled by

$$\lim_{\mathrm{Re}\,p\to+\infty}pF(p)=\lim_{\mathrm{Re}\,p\to+\infty}\frac{p}{p-4}=1=\lim_{t\to+0}e^{4t}=f(0).$$

c) The function $f(t)=\sin t$ is the original with the growth index $\alpha_0=0$

$$F(p)=\int_0^{+\infty}\sin t\, e^{-pt}\mathrm{d}t=\begin{bmatrix} u=e^{-pt}, & \mathrm{d}v=\sin t\,\mathrm{d}t, \\ \mathrm{d}u=-pe^{-pt}\mathrm{d}t, & v=-\cos t \end{bmatrix}=$$

$$=-e^{-pt}\cos t\Big|_0^{+\infty}-p\int_0^{+\infty}\cos t\, e^{-pt}\mathrm{d}t=\begin{bmatrix} u=e^{-pt}, & \mathrm{d}v=\cos t\,\mathrm{d}t, \\ \mathrm{d}u=-pe^{-pt}\mathrm{d}t, & v=\sin t \end{bmatrix}=$$

$$=1-p\left(pe^{-pt}\sin t\Big|_0^{+\infty}-p\int_0^{+\infty}\sin t\, e^{-pt}\mathrm{d}t\right)=1-p^2\int_0^{+\infty}\sin t\, e^{-pt}\mathrm{d}t.$$

From here

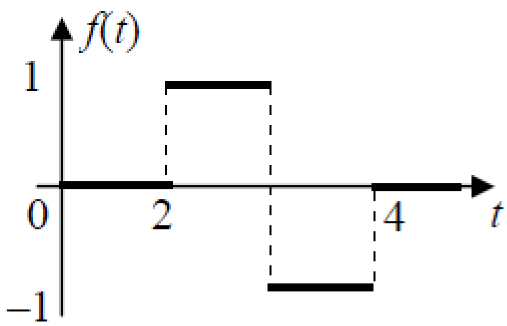
$$\int_0^{+\infty}\sin t\, e^{-pt}\mathrm{d}t=1-p^2\int_0^{+\infty}\sin t\, e^{-pt}\mathrm{d}t.$$

From the obtained equality, we express the desired integral:

$$F(p)=\int_0^{+\infty}\sin t\, e^{-pt}\mathrm{d}t=\frac{1}{p^2+1}.$$

Example 4.

4) Find the image of the function given as follows:



<u>Solution:</u>

The function can be written in analytical form

$$f(t)=\begin{cases}0, & t\le 2,\quad t>4,\\ 1, & 2<t\le 3,\\ -1, & 3<t\le 4.\end{cases}$$

Using the Laplace transform formula:

$$F(p)=\int_0^{+\infty} e^{-pt}f(t)\mathrm{d}t=\int_2^3 e^{-pt}\mathrm{d}t-\int_3^4 e^{-pt}\mathrm{d}t=$$

$$=\frac{1}{p}\left(-e^{-3p}+e^{-2p}+e^{-4p}-e^{-3p}\right)=$$

$$=\frac{e^{-2p}}{p}\left(1-2e^{-p}+e^{-2p}\right)=\frac{\left(e^{-p}\left(1-e^{-p}\right)\right)^2}{p}=\frac{\left(e^{-p}-e^{-2p}\right)^2}{p}.$$

Example 5. (*Using tables and properties of the Laplace transform*)

5) Using the properties of *linearity* and *similarity*, find images of the following functions:

a) $f(t)=\cos t$

b) $f(t)=2-5\cos 2t$

Solution:

a) According to Euler's theorem, $\cos t=\dfrac{e^{it}+e^{-it}}{2}$. Since according to the image table $e^{it}\leftrightarrow\dfrac{1}{p-i}$, $e^{-it}\leftrightarrow\dfrac{1}{p+i}$, then according to the linearity property

$$f(t)=\cos t\leftrightarrow\frac{1}{2}\left(\frac{1}{p-i}+\frac{1}{p+i}\right)=\frac{p}{p^2+1}=F(p).$$

b) By the property of *linearity* and *similarity*

$$f(t)=2-5\cos 2t \leftrightarrow \frac{2}{p}-5\frac{p}{p^2+4}=F(p).$$

Example 6. (*The displacement property*)

6) Find images of the following functions:

a) $f(t)=e^{-3t}\text{ch}2t$,

b) $f(t)=e^{2t}\cos nt$

Solution:

a) According to the image table, we have $ch2t \leftrightarrow \dfrac{p}{p^2-4}$. The presence of a multiplier e^{-3t} implies the use of the displacement theorem (*displacement property*). Therefore:

$$e^{-3t}\,ch2t \leftrightarrow \frac{p+3}{(p+3)^2-4}=F(p)$$

b) Since $\cos nt \leftrightarrow \dfrac{p}{p^2+n^2}$, then

$$f(t)=e^{2t}\cos nt \leftrightarrow \frac{p-2}{(p-2)^2-n^2}=F(p)$$

Example 7. (*Image differentiation.)*

7) Find images of the following functions

a) $f(t)=te^{at}$,

b) $f(t) = te^{t}\cos t$

c) $f(t) = t^{2}\sin t$

<u>Solution:</u>

a) The presence of a multiplier t indicates the need to apply the image differentiation theorem:

Since $e^{at} \leftrightarrow \dfrac{1}{p-a}$, then

$$f(t) = t\,e^{at} \leftrightarrow (-1)^{1}\left(\frac{1}{p-a}\right)' = \frac{1}{(p-a)^{2}} = F(p).$$

b) To find the image, we apply the theorems of image differentiation and displacement

$$\cos t \leftrightarrow \frac{p}{p^{2}+1},$$

$$t\cos t \leftrightarrow -\left(\frac{p}{p^{2}+1}\right)' = -\frac{p^{2}+1-2p^{2}}{(p^{2}+1)^{2}} = \frac{p^{2}-1}{(p^{2}+1)^{2}}.$$

$$f(t) = t\,e^{t}\cos t \leftrightarrow \frac{(p-1)^{2}-1}{((p-1)^{2}+1)^{2}} = \frac{p^{2}-2p}{(p^{2}-2p+2)^{2}} = F(p)$$

c) The presence of a multiplier t^{2} indicates the need to apply the image differentiation theorem

$$\sin t \leftrightarrow \frac{1}{p^2+1},$$

$$t^2 \sin t \leftrightarrow (-1)^2 \left(\frac{1}{p^2+1} \right)''$$

$$\left(\frac{1}{p^2+1} \right)' = \frac{-2p}{(p^2+1)^2},$$

$$\left(\frac{1}{p^2+1} \right)'' = \left(\frac{-2p}{(p^2+1)^2} \right)' = \frac{6p^2-2}{(p^2+1)^3}.$$

$$f(t) = t^2 \sin t \leftrightarrow \frac{6p^2-2}{(p^2+1)^3} = F(p).$$

Example 8. (*Image integration.)*

8) Find images of the following functions:

a) $f(t) = \dfrac{e^t - 1}{t}$,

b) $f(t) = \dfrac{1-\cos t}{t}$

<u>Solution:</u>

a) The function $f(t)$ is continuous for all $t > 0$ and is bounded in the vicinity of zero (according to L'hopital's rule $\lim\limits_{t \to +0} \dfrac{e^t - 1}{t} = 1$).

Since

$$e^t - 1 \leftrightarrow \frac{1}{p-1} - \frac{1}{p},$$

then by the image integration theorem we obtain

$$f(t)=\frac{e^t-1}{t}\leftrightarrow\int_p^{\infty}\left(\frac{1}{z-1}-\frac{1}{z}\right)dz=\left(ln|z-1|-ln|z|\right)\Big|_p^{\infty}=$$

$$=ln\left|\frac{z-1}{z}\right|_p^{\infty}=ln\frac{p}{p-1}=F(p).$$

b) Since $\lim\limits_{t\to+0}\frac{1-\cos t}{t}=\lim\limits_{t\to+0}\frac{2\sin^2\frac{t}{2}}{t}=\lim\limits_{t\to+0}\sin\frac{t}{2}=0$, then $f(t)$ is continuous and bounded at $t>0$. Let's apply the image integration theorem. Since $1-\cos t\leftrightarrow\frac{1}{p}-\frac{p}{p^2+1}$, then

$$f(t)=\frac{1-\cos t}{t}\leftrightarrow\int_p^{\infty}\left(\frac{1}{z}-\frac{z}{z^2+1}\right)dz=\left(ln\,z-\frac{1}{2}ln\left(z^2+1\right)\right)\Big|_p^{\infty}=$$

$$=ln\frac{z}{\sqrt{z^2+1}}\Big|_p^{\infty}=ln\frac{\sqrt{p^2+1}}{p}=F(p).$$

Example 9. *(Differentiation of the original.)*

9) Find images of the following functions:

a) $f(t)=\sin^2 t$,

b) $f(t)=te^t$.

<u>Solution:</u>

a) Let $f(t) \leftrightarrow F(p)$. Since $f(0)=0$, then $f'(t) \leftrightarrow pF(p)-f(0)=pF(p)$. Calculate the derivative of the function $f(t)$ and find the image for $f'(t)$

$$f'(t)=(\sin^2 t)'=2\sin t\cos t=\sin 2t \leftrightarrow \frac{2}{p^2+4}$$

Thus, according to the original differentiation theorem, to

determine the image $F(p)$ we have the equation $pF(p)=\frac{2}{p^2+4}$,

solving which we get $F(p)=\frac{2}{p(p^2+4)}$.

b) Let $f(t) \leftrightarrow F(p)$. Since $f(0)=0$, then

$f'(t) \leftrightarrow pF(p)-f(0)=pF(p)$.

Let's find the image for the derivative:

$$f'(t)=\left(te^t\right)'=e^t+te^t \leftrightarrow =\frac{1}{p-1}+F(p)$$

Thus, to determine $F(p)$, we have the equation

$$\frac{1}{p-1}+F(p)=pF(p)$$

Therefore

$$F(p)=\frac{1}{(p-1)^2}$$

Example 10. *(Integrating the original).*

10) Find images of the following functions:

a) $f(t)=\int_0^t \sin\tau d\tau$,

b) $f(t)=\int_0^t \tau^2 e^{-\tau} d\tau$

Solution:

a) Since $\sin t \leftrightarrow \frac{1}{p^2+1}$, then by the original integration theorem

$$\int_0^t \sin\tau\, d\tau \leftrightarrow \frac{1}{p}\cdot\frac{1}{p^2+1}=\frac{1}{p(p^2+1)}$$

b) By the delay theorem $t^2 e^{-t} \leftrightarrow \frac{2!}{(p+1)^3}$. Then we get

$$\int_0^t \tau^2 e^{-\tau} d\tau \leftrightarrow \frac{1}{p}\cdot\frac{2}{(p+1)^3}=\frac{2}{p(p+1)^3}$$

Theorem *(on convolution, Borel's theorem).*

The convolution of originals

$$(f_1\cdot f_2)(t)=\int_0^t f_1(\tau)f_2(t-\tau)d\tau=\int_0^t f_1(t-\tau)f_2(\tau)d\tau$$

corresponds to the product of images

$$(f_1 \cdot f_2)(t) \leftrightarrow F_1(p)F_2(p).$$

Notation for examples (The convolution of functions):

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau = \int_0^t f_1(t-\tau) f_2(\tau)\,\mathrm{d}\tau$$

Example 11.

11) Find the convolution and the image of the convolution (by the properties of the Laplace transform and by the convolution theorem).

a) $t * e^t$

b) $\sin t * t$.

<u>Solution:</u>

a) Let's find the convolution using the formula

$$t * e^t = \int_0^t (t-\tau)e^\tau\,\mathrm{d}\tau = t\left(e^t - 1\right) - \int_0^t \tau e^\tau\,\mathrm{d}\tau = \begin{vmatrix} u = \tau & \mathrm{d}v = e^\tau \mathrm{d}\tau \\ \mathrm{d}u = \mathrm{d}\tau & v = e^\tau \end{vmatrix} =$$

$$= t\left(e^t - 1\right) - \tau e^\tau\Big|_0^t + \int_0^t e^\tau\,\mathrm{d}\tau = t\left(e^t - 1\right) - \left(\tau \cdot e^\tau - e^\tau\right)\Big|_0^t =$$

$$= te^t - t - te^t + e^t - 1 = e^t - t - 1.$$

Let's find the convolution image using the properties of linearity,

displacement

$$t * e^t = e^t - t - 1 \leftrightarrow \frac{1}{p-1} - \frac{1}{p^2} - \frac{1}{p} = \frac{1}{p^2(p-1)}$$

Let's find the convolution image according to Borel's theorem

$$t \leftrightarrow \frac{1}{p^2}, \quad e^t \leftrightarrow \frac{1}{p-1}, \quad t * e^t \leftrightarrow \frac{1}{p^2(p-1)}$$

b) <u>HOMEWORK №1.</u>

Example 12.

12) Find images of the following functions:

a) $f(t) = \int_0^t \cos(t-\tau) e^{2\tau} d\tau$,

b) $f(t) = \int_0^t e^{2(\tau - t)} \tau^2 d\tau$

<u>Solution:</u>

a) The function $f(t)$ is a convolution of $f(t) = f_1(t) * f_2(t)$, where $f_1(t) = \cos t,\ f_2(t) = e^{2t}$.

Since $\cos t \leftrightarrow \frac{p}{p^2+1},\ e^{2t} \leftrightarrow \frac{1}{p-2}$, then

$$\int_0^t \cos(t-\tau) e^{2\tau} d\tau = \cos t * e^{2t} \leftrightarrow \frac{p}{p^2+1} \cdot \frac{1}{p-2} = \frac{p}{(p^2+1)(p-2)}.$$

b) <u>HOMEWORK №2.</u>

Delay theorem *(delay property, Remark).*

If $f(t)\theta(t) \leftrightarrow F(p)$ and $\tau > 0$, then

$$f(t-\tau)\theta(t-\tau) \leftrightarrow e^{-p\tau} F(p)$$

Example 13.

13) Find images of the following functions:

a) $f(t) = e^{t-3}\,\theta(t-3)$

b) $f(t) = (t-1)^2\,\theta(t-1)$

Solution:

a) For the function $e^t\,\theta(t) \leftrightarrow \dfrac{1}{p-1}$. By the delay theorem

$$e^{t-3}\,\theta(t-3) \leftrightarrow \frac{e^{-3p}}{p-1}.$$

It should be noted that $e^{t-3}\,\theta(t) = e^{-3}\,e^t\,\theta(t) \leftrightarrow \dfrac{e^{-3}}{p-1}$

b) For the function $t^2\,\theta(t) \leftrightarrow \dfrac{2}{p^3}$. By the delay theorem

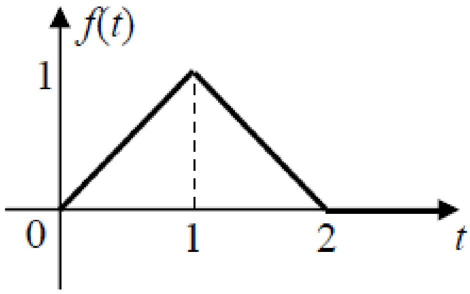
$$(t-1)^2\,\theta(t-1) \leftrightarrow \frac{2e^{-p}}{p^3}.$$

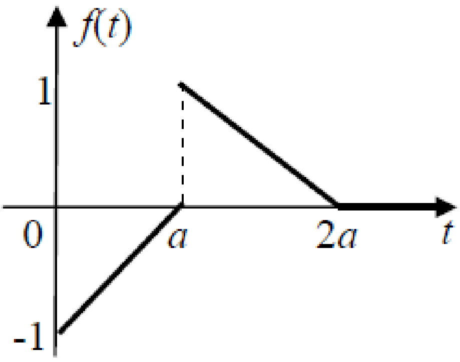
It should be noted that

$$(t-1)^2\,\theta(t) = (t^2-2t+1)\theta(t) \leftrightarrow \frac{2}{p^3} - \frac{2}{p^2} + \frac{1}{p}.$$

Example 14.

14) Find images of the following functions defined graphically:

a) 

b) 

<u>Solution:</u>

a) The function can be written in analytical form

$$f(t) = \begin{cases} 0, & t \le 0, \quad t \ge 2, \\ t, & 0 < t \le 1, \\ 2-t, & 1 < t \le 2. \end{cases}$$

Since

$$f_1(t)=\begin{cases} t, & t\in[0,1] \\ 0, & t\notin[0,1] \end{cases}$$

$$f_1(t)=t\theta(t)-t\theta(t-1),$$

$$f_2(t)=\begin{cases} 2-t, & t\in[1,2] \\ 0, & t\notin[1,2] \end{cases}$$

$$f_2(t)=(2-t)\theta(t-1)-(2-t)\theta(t-2),$$

the composite function $f(t)=f_1(t)+f_2(t)$ is represented by one analytical expression in the form:

$$f(t)=t\theta(t)-t\theta(t-1)+(2-t)\theta(t-1)-(2-t)\theta(t-2)=$$

$$=t\theta(t)-2(t-1)\theta(t-1)+(t-2)\theta(t-2).$$

Applying the delay theorem, we find the image of the function

$$f(t)\leftrightarrow\frac{1}{p^2}-\frac{2e^{-p}}{p^2}+\frac{e^{-2p}}{p^2}=\frac{1}{p^2}(1-e^{-p})^2.$$

b) <u>HOMEWORK №3.</u>

The property of partial degeneracy of the original (the advance theorem).

If $f(t) \leftrightarrow F(p)$ and $\tau > 0$, then

$$f(t+\tau) \leftrightarrow e^{p\tau}\left(F(p) - \int_0^\tau f(t)e^{-pt}\,dt\right).$$

Here $f(t+\tau) = f(t+\tau)\theta(t)$.

Figure 1 shows graphs of the original functions $f(t)$, $f(t-\tau)\theta(t-\tau)$, $f(t+\tau)$, where $\tau > 0$. To calculate the images of functions $f(t-\tau)\theta(t-\tau)$, $f(t+\tau)$ from the known image $F(p) \leftrightarrow f(t)$, the delay and advance theorems are used, respectively.

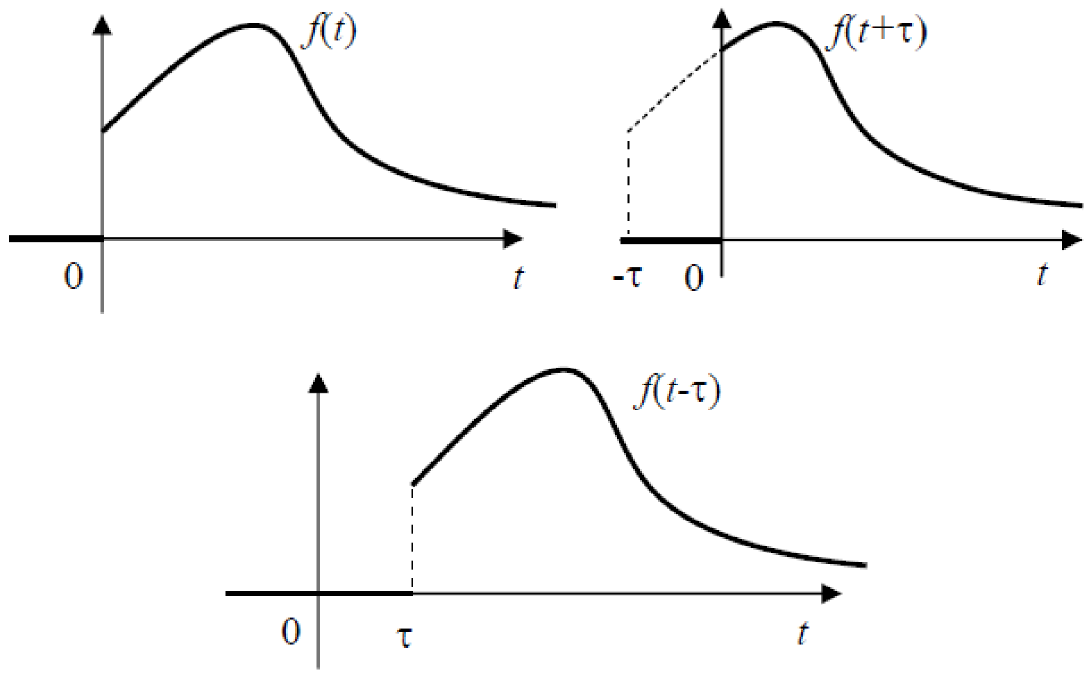


Fig. 1. Graphs of the original functions

Example 15.

15) Find images of the following functions:

a) $f(t) = \sin(t+\tau), \quad \tau > 0,$

b) $f(t) = \cos(t+\tau), \quad \tau > 0.$

<u>Solution:</u>

a) For the function $sin\, t \leftrightarrow \dfrac{1}{p^2+1}$. By the advance theorem

$$sin(t+\tau) \leftrightarrow e^{p\tau}\left(\frac{1}{p^2+1} - \int_0^{\tau} \sin t e^{-pt} \mathrm{d}t\right).$$

Since

$$\int_0^{\tau} \sin t e^{-pt} \mathrm{d}t = \left[\begin{array}{c}\text{integration}\\ \text{by parts,}\\ \text{twice}\end{array}\right] = \frac{-p\sin t - \cos t}{p^2+1} e^{-pt}\Bigg|_0^{\tau} =$$

$$= \frac{-p\sin\tau - \cos\tau}{p^2+1} e^{-p\tau} - \frac{1}{p^2+1},$$

then according to the advance theorem

$$\sin(t+\tau) \mapsto \frac{p\sin\tau + \cos\tau}{p^2+1}.$$

b) For the function $cos\, t \leftrightarrow \dfrac{p}{p^2+1}$. According to the advance theorem

$$cos(t+\tau) \leftrightarrow e^{p\tau}\left(\frac{p}{p^2+1} - \int_0^{\tau} \cos t e^{-pt} \mathrm{d}t\right).$$

Since

$$\int_0^{\tau}\cos t e^{-pt}\,dt = \left[\begin{array}{c}\text{integration}\\ \text{by parts,}\\ \text{twice}\end{array}\right] = \frac{-p\cos t+\sin t}{p^2+1}e^{-pt}\Big|_0^{\tau} =$$

$$= \frac{-p\cos\tau+\sin\tau}{p^2+1}e^{-p\tau}+\frac{p}{p^2+1},$$

then we will get

$$sin(t+\tau)\leftrightarrow \frac{p\cos\tau-\sin\tau}{p^2+1}.$$

The image of the periodic function.

Let the original function $f(t)$ have a period T.

Then if $f_0(t)\leftrightarrow F_0(p)$, where

$$f_0(t)=\begin{cases} f(t) & \text{when } 0<t<T, \\ 0 & \text{when } \; t<0 \text{ and } t>T, \end{cases}$$

then

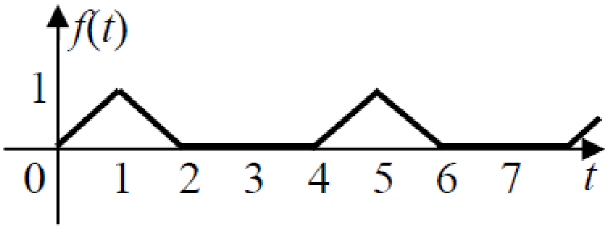
$$f(t)\leftrightarrow \frac{F_0(p)}{1-e^{-pT}}.$$

Example 16.

Find images of the following periodic functions:

a) $f(t)=|\cos t|$,

b) $f(t) = |\sin t|$.



c)

<u>Solution:</u>

a) The function $f(t) = |\cos t|$ is periodic with a period $T = \pi$. Consider the function

$$f_0(t) = \begin{cases} |\cos t|, & 0 \le t \le \pi, \\ 0, & t < 0,\ t > \pi. \end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t) \leftrightarrow \int_0^{\pi} |\cos t| e^{-pt} \mathrm{d}t = \int_0^{\pi/2} \cos t e^{-pt} \mathrm{d}t - \int_{\pi/2}^{\pi} \cos t e^{-pt} \mathrm{d}t =$$

$$= \left[\int \cos t e^{-pt} \mathrm{d}t = \frac{e^{-pt}}{p^2+1}(-p\cos t + \sin t) \right] =$$

$$= \frac{1}{p^2+1}\left(2e^{-\frac{\pi}{2}p} + p\left(1 - e^{-\pi p}\right) \right).$$

According to the formula for the image of the periodic function

$$|\cos t| \leftrightarrow \frac{2e^{-\frac{\pi}{2}p} + p\left(1 - e^{-\pi p}\right)}{\left(p^2+1\right)\left(1 - e^{-\pi p}\right)}.$$

b) The function $f(t)=|sin t|$ is periodic with a period $T=\pi$.

Consider the function

$$f_0(t)=\begin{cases}|\sin t|, & 0\le t\le \pi,\\ 0, & t<0,\ t>\pi.\end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t)\leftrightarrow \int_0^{\pi}|\sin t|e^{-pt}\mathrm{d}t=\int_0^{\pi}\sin t e^{-pt}\mathrm{d}t=$$

$$=\left[\int \sin t e^{-pt}\mathrm{d}t=-\frac{e^{-pt}}{p^2+1}(p\sin t+\cos t)\right]=\frac{1}{p^2+1}\left(1+e^{-\pi p}\right)$$

According to the formula for the image of the periodic function

$$|sin t|\leftrightarrow \frac{1+e^{-\pi p}}{\left(p^2+1\right)\left(1-e^{-\pi p}\right)}.$$

c) The function is periodic with period $T=4$. Consider the function

$$f_0(t)=\begin{cases}t, & 0\le t\le 1,\\ 2-t, & 1<t\le 2,\\ 0, & 2<t\le 4.\end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t)\leftrightarrow \int_0^{1}te^{-pt}\mathrm{d}t+\int_1^{2}(2-t)e^{-pt}\mathrm{d}t=$$

$$=-e^{-pt}\left(\frac{t}{p}+\frac{1}{p^2}\right)\Bigg|_0^1+e^{-pt}\left(\frac{t-2}{p}+\frac{1}{p^2}\right)\Bigg|_1^2=$$

$$=-e^{-p}\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{1}{p^2}+\frac{e^{-2p}}{p^2}+e^{-p}\left(\frac{1}{p}-\frac{1}{p^2}\right)=\frac{\left(1-e^{-p}\right)^2}{p^2}.$$

According to the formula for the image of the periodic function

$$f(t)\leftrightarrow\frac{\left(1-e^{-p}\right)^2}{p^2\left(1-e^{-4p}\right)}$$

The theorem of differentiation by parameter.

If, for any fixed x, the function $f(x,t)$ is the original, and $F(p,x)$ is its image, and if in the integral $F(p,x)=\int\limits_0^{+\infty}f(x,t)e^{-pt}dt$ differentiation by parameter x under the sign of the integral is possible, then $\frac{\partial f(x,t)}{\partial x}\leftrightarrow\frac{\partial F(p,x)}{\partial x}$.

This property is used in solving partial differential equations.

1.2. RESTORING THE ORIGINAL IMAGE

1.2.1 The elementary method

In many cases, a given image can be converted to a form where the original is easily restored directly using the properties of the Laplace transform and the table of originals and images.

In this case, the method of decomposing a rational fraction into the sum of the simplest ones is widely used to transform the image.

Let $F(p)$ be a rational function, to find the original, we represent the function $F(p)$ as the sum of the simplest fractions of the form

$$\frac{A}{p-a},\ \frac{Ap+B}{(p-a)^2+b^2},\ \frac{A}{(p-a)^k},\ \frac{Ap+B}{\left((p-a)^2+b^2\right)^k},\ k=2,3,\ldots$$

(A,B,a,b are some constants), for each of which we can construct the corresponding original.

Indeed, using the displacement property and the table of originals and images, we find

$$\frac{A}{p-a}\leftrightarrow Ae^{at},\ \frac{A}{(p-a)^k}\leftrightarrow\frac{A}{(k-1)!}t^{k-1}e^{at},\ k=2,3,\ldots;$$

$$\frac{Ap+B}{(p-a)^2+b^2}=\frac{A(p-a)+B+Aa}{(p-a)^2+b^2}\leftrightarrow Ae^{at}\cos bt+\frac{B+Aa}{b}e^{at}\sin bt\,.$$

Let $S_k(p)\leftrightarrow s_k(t)$. Let's build the original for the image

$$S_k(p)=\frac{Ap+B}{\left((p-a)^2+b^2\right)^k},\ k=2,3,\ldots$$

Consider the expression for image $S_2(p)$. Since

$$S_2(p)=\frac{Ap+B}{(p-a)^2+b^2}\frac{1}{(p-a)^2+b^2},$$

then, applying the convolution image property, we construct the corresponding original:

$$s_2(t)=\frac{1}{b}\int_0^t s_1(t-\tau)e^{a\tau}\sin b\tau d\tau .$$

Here the original $s_1(t)$ is defined by the expression

$$s_1(t)=Ae^{at}\cos bt+\frac{B+Aa}{b}e^{at}\sin bt .$$

Further, since

$$S_3(p)=S_2(p)\frac{1}{(p-a)^2+b^2},$$

then

$$s_3(t)=\frac{1}{b}\int_0^t s_2(t-\tau)e^{a\tau}\sin b\tau d\tau .$$

Similar reasoning leads to the following relation:

$$s_k(t)=\frac{1}{b}\int_0^t s_{k-1}(t-\tau)e^{a\tau}\sin b\tau d\tau ,\ k\geq 2 .$$

Example 1.

Find the original corresponding to the image

$$F(p)=\frac{1}{p^3-p}.$$

Solution:

Decomposing a given image into the sum of the simplest fractions

$$\frac{1}{p^3-p}=\frac{1}{p(p-1)(p+1)}=-\frac{1}{p}+\frac{1}{2(p-1)}+\frac{1}{2(p+1)},$$

we'll find the original

$$f(t)=-1+\frac{1}{2}e^{t}+\frac{1}{2}e^{-t}=-1+\operatorname{ch}t\,.$$

Example 2.

Find the original corresponding to the image

$$F(p)=\frac{1}{(p^2+4)^2}\,.$$

Solution:

Applying the convolution image property, we will have

$$F(p)=\frac{1}{(p^2+4)^2}=\frac{1}{p^2+4}\frac{1}{p^2+4}\leftrightarrow\frac{1}{4}\int_0^t\sin 2(t-\tau)\sin 2\tau d\tau\,.$$

Having calculated the integral, we get the desired expression for the original

$$f(t)=\frac{1}{16}\sin 2t-\frac{1}{8}t\cos 2t\,.$$

Example 3.

Find the original corresponding to the image

$$F(p)=\frac{e^{-\frac{p}{2}}}{p(p+1)(p^2+4)}.$$

Solution:

Let's imagine the fraction included in the expression as the simplest fractions:

$$\frac{1}{p(p+1)(p^2+4)}=\frac{A}{p}+\frac{B}{p+1}+\frac{Cp+D}{p^2+4}.$$

Applying the method of undefined coefficients to the decomposition,

we obtain

$$A=\frac{1}{4},\ B=D=-\frac{1}{5},\ C=-\frac{1}{20}.$$

The image has the form

$$F(p)=\frac{1}{4}\frac{e^{-\frac{p}{2}}}{p}-\frac{1}{5}\frac{e^{-\frac{p}{2}}}{p+1}-\frac{1}{20}\frac{pe^{-\frac{p}{2}}}{p^2+4}-\frac{1}{5}\frac{e^{-\frac{p}{2}}}{p^2+4}. \qquad (*)$$

Using the ratios

$$\frac{1}{p}\leftrightarrow\theta(t),\ \frac{1}{p+1}\leftrightarrow e^{-t}\theta(t),\ \frac{p}{p^2+4}\leftrightarrow\cos 2t\theta(t),\ \frac{1}{p^2+4}\leftrightarrow\frac{1}{2}\sin 2t\theta(t)$$

and given the delay property, we get the desired original for the image (*)

$$f(t)=\left(\frac{1}{4}-\frac{1}{5}e^{-\left(t-\frac{1}{2}\right)}-\frac{1}{20}\cos(2t-1)-\frac{1}{10}\sin(2t-1)\right)\theta\left(t-\frac{1}{2}\right).$$

Example 4.

Find the original corresponding to the image

$$F(p)=\frac{e^{-\frac{p}{3}}}{p(p^2+1)}.$$

Solution:

Applying the convolution property and the correspondence table of originals and images, we get

$$\frac{1}{p(p^2+1)}\leftrightarrow\int_0^t \sin\tau d\tau=-\cos\tau\Big|_0^t=(1-\cos t)\theta(t).$$

When constructing the original for a given image, we apply the delay property and get

$$f(t)=\left(1-\cos\left(t-\frac{1}{3}\right)\right)\theta\left(t-\frac{1}{3}\right).$$

1.2.2 The conversion formula. Decomposition theorems

Theorem 1 (Laplace transform conversion formula, Riemann-Mellin formula).

Let $f(t)$ be the original and $F(p)$ be its image. If the function $f(t)$ is continuous at point t and has finite one-sided derivatives at this point, then

$$f(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{pt}F(p)dp. \tag{1.6}$$

The improper integral (1.6) is taken along any straight line $\operatorname{Re} p = b > \alpha_0$, where α_0 is the growth index of the function $f(t)$ and is understood in the sense of the main value, that is

$$f(t) = \int_{b-i\infty}^{b+i\infty} e^{pt} F(p)dp = \lim_{R\to+\infty} \int_{b-iR}^{b+iR} e^{pt} F(p)dp \, .$$

The Riemann-Mellin formula (1.6) is the inverse of the formula $F(p) = \int_0^{+\infty} f(t) e^{-pt}\, dt$ and is called *the inverse Laplace transform*.

The direct application of the conversion formula to restore the original $f(t)$ from the $F(p)$ image is difficult. Decomposition theorems are usually used to find the original.

Theorem 2 (the first decomposition theorem).

Let the function $F(p)$ be regular at point $p=\infty, F(\infty)=0$ and its Laurent series in the vicinity of point $p=\infty$ has the form

$$F(p) = \sum_{k=0}^{\infty} \frac{c_k}{p^{k+1}} = \frac{c_0}{p} + \frac{c_1}{p^2} + \frac{c_2}{p^3} + \ldots,$$

then the function

$$f(t) = \sum_{k=0}^{\infty} c_k \cdot \frac{t^k}{k!} = c_0 + c_1 t + c_2 \cdot \frac{t^2}{2!} + \ldots, \quad t \geq 0$$

is the original with the image $F(p)$.

Definition. A function $F(p)$ is called *meromorphic* in the complex plane if it is regular in any bounded region of the complex plane, with the possible exception of a finite number of singular points of the pole type.

Theorem 3 (the second decomposition theorem).

Let the meromorphic function $F(p)$ be regular in the half-plane $\operatorname{Re} p = \alpha$ and satisfy the conditions:

1) there is a system of circles

$$C_n : |p| = R_n, \; R_1 < R_2 < \ldots < R_n \to \infty \; (n \to \infty)$$

such that $\max\limits_{p \in C_n} |F(p)| \to 0 \; (n \to \infty)$;

2) for $\forall a > \alpha$, the integral $\int\limits_{-\infty}^{\infty} |F(a + i\sigma)| d\sigma$ converges.

Then $F(p)$ is an image, the original for which is the function

$$f(t) = \sum_{p_k} \underset{p = p_k}{res} \left[F(p) e^{pt} \right],$$

where the sum is taken over all poles p_k of the function $F(p)$.

<u>Consequence. </u>If $F(p) = \dfrac{A_n(p)}{B_m(p)}$, where $A_n(p), B_m(p)$ are polynomials of degree n and m, respectively, having no common zeros, and if $n < m$, then

$$f(t)=\sum_{k=1}^{l}\frac{1}{(m_k-1)!}\frac{d^{m_k-1}}{dp^{m_k-1}}\left\{F(p)e^{pt}(p-p_k)^{m_k}\right\}\Big|_{p=p_k}, \qquad (1.7)$$

where $p_1,...,p_l$ are different zeros of the polynomial $B_m(p)$, and m_k is the multiplicity of zero $p_k : \sum_{k=1}^{l} m_k = m$.

In particular, if all poles of the function $F(p)$ are simple, then formula (1.7) takes the form:

$$f(t)=\sum_{k=1}^{m}\frac{A_n(p_k)}{B'_m(p_k)}e^{p_k t}. \qquad (1.8)$$

Example 5.

Find the original corresponding to the image

$$F(p)=\frac{p^2+2}{p^3-p^2-6p}.$$

Solution:

Since $p^3-p^2-6p=p(p-3)(p+2)$, the function $F(p)$ has three simple poles: $p_1=0$, $p_2=3$, $p_3=-2$. Let's construct the corresponding original using the formula (1.8):

$$f(t)=\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=0}+\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=3}+\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=-2}=-\frac{1}{3}+\frac{11}{5}e^{3t}+\frac{3}{5}e^{-2t}.$$

1.2. RESTORING THE ORIGINAL IMAGE

To find the original function for a given image, it requires knowledge of the tables of correspondence between the originals and images, the application of the properties of the Laplace transform, the decomposition of the image into the simplest fractions, the use of decomposition theorems.

Using the properties of the Laplace transform

First of all, it is necessary to bring the function to a simpler, "tabular" form.

If the denominator of a fraction contains a square trinomial, then the full square is allocated in it.

It is convenient to use the original integration theorem to find the original fraction $\frac{F(p)}{p^n}$ if the original $f(t)$ of the $F(p)$ image is known.

The presence of the $e^{-p\tau}, \tau > 0$ multiplier in image $F(p)$ indicates the need to apply the delay theorem.

If the image is represented as $F(p) = F_1(p)F_2(p)$ or $F(p) = pF_1(p)F_2(p)$ and the originals $f_1(t) \leftrightarrow F_1(p)$, $f_2(t) \leftrightarrow F_2(p)$ are known, then *Borel's theorem* and *Duhamel's integral* are used to find the original $f(t) \leftrightarrow F(p)$, respectively.

Duhamel's integral

If

$$f_1 * f_2 = \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau \leftrightarrow F_1(p) F_2(p)$$

then

$$f_1(t)\, f_2(0) + \int_0^t f_1(\tau)\, f_2'(t-\tau)\, d\tau \leftrightarrow pF_1(p) F_2(p).$$

Comment

1) Due to the symmetry of the convolution $f_1 * f_2 = f_2 * f_1$

$$f_1(t)\, f_2(0) + \int_0^t f_1(\tau)\, f_2'(t-\tau)\, d\tau = f_1(t)\, f_2(0) + \int_0^t f_1(t-\tau)\, f_2'(\tau)\, d\tau$$

2) Obviously

$$f_1(0)\, f_2(t) + \int_0^t f_1'(\tau) f_2(t-\tau)\, d\tau = f_1(0)\, f_2(t) + \int_0^t f_1'(t-\tau) f_2(\tau)\, d\tau$$

Example 6.

Find the original corresponding to the image (using the Duhamel integral):

a) $F(p) = \dfrac{2p^2}{\left(p^2+1\right)^2}$,

b) $F(p) = \dfrac{p^3}{\left(p^2+1\right)\left(p^2+4\right)}$,

c) $F(p)=\dfrac{2p}{(p-1)\left(p^2-2p-3\right)}$.

Solution:

a) Let's write an image in the form

$$\frac{2p^2}{\left(p^2+1\right)^2}=2p\cdot\frac{1}{p^2+1}\cdot\frac{p}{p^2+1}.$$

Since

$$\frac{1}{p^2+1}\leftrightarrow \sin t=f_1(t),\ \frac{p}{p^2+1}\leftrightarrow \cos t=f_2(t),$$

$$f_1(0)=\sin 0=0,\ f_1'(t)=\cos t,$$

then based on the Duhamel formula we have

$$2p\cdot\frac{1}{p^2+1}\cdot\frac{p}{p^2+1}\leftrightarrow 0+2\int_0^t \cos\tau\cos(t-\tau)\,d\tau=$$

$$=2\int_0^t\frac{\cos t+\cos(t-2\tau)}{2}d\tau=\left(\tau\cos t-\frac{1}{2}\sin(t-2\tau)\right)\Bigg|_0^t=t\cos t+\sin t.$$

b) Let's write an image in the form

$$\frac{p^3}{\left(p^2+1\right)\left(p^2+4\right)}=p\cdot\frac{p}{p^2+1}\cdot\frac{p}{p^2+4}.$$

$$\frac{p}{p^2+1}\leftrightarrow \cos t=f_1(t),\ \frac{p}{p^2+4}\leftrightarrow \cos 2t=f_2(t),$$

$$f_1(0)=\cos 0=1,\quad f_1'(t)=-\sin t.$$

Then we get

$$f(t)=f_1(0)f_2(t)+\int_0^t f_1'(\tau)f_2(t-\tau)d\tau=$$

$$=\cos 0\cdot\cos 2t+\int_0^t \cos'\tau\cdot\cos 2(t-\tau)d\tau=$$

$$=\cos 2t-\int_0^t \sin\tau\cos 2(t-\tau)d\tau=$$

$$=\cos 2t-\frac{1}{2}\int_0^t(\sin(3\tau-2t)+\sin(2t-\tau))d\tau=$$

$$=\cos 2t+\frac{1}{6}\cos(3\tau-2t)\Big|_0^t-\frac{1}{2}\cos(2t-\tau)\Big|_0^t=\frac{4}{3}\cos 2t-\frac{1}{3}\cos t.$$

c) <u>HOMEWORK №1</u>

Example 7.

Using the properties of the Laplace transform to find the original corresponding to the image:

a) $F(p)=\dfrac{p}{p^2-2p+26}$,

b) $F(p)=\dfrac{1}{p\left(p^2+4\right)}$,

c) $F(p)=\dfrac{e^{-p}}{p+1}$,

d) $F(p)=\dfrac{p}{(p^2+4)^2}$

<u>Solution:</u>

a) Let's transform the image by highlighting the full square in the denominator. To find the original, we will use the displacement theorem, the linearity property and the image table.

$$\frac{p}{p^2-2p+26}=\frac{(p-1)+1}{(p-1)^2+25}=\frac{p-1}{(p-1)^2+25}+\frac{1}{(p-1)^2+25}\leftrightarrow$$

$$\leftrightarrow e^t\left(\cos 5t+\frac{1}{5}\sin 5t\right)$$

b) From the table of images, we have $\frac{2}{p^2+4}\leftrightarrow \sin 2t$.

Using the linearity and integration properties of the original, we find

$$F(p)=\frac{1}{p\left(p^2+4\right)}=\frac{1}{2}\cdot\frac{1}{p}\cdot\frac{2}{p^2+4}\leftrightarrow$$

$$\frac{1}{2}\int_0^t \sin 2\tau d\tau=-\frac{1}{4}\cos 2\tau\Big|_0^t=\frac{1}{4}(1-\cos 2t).$$

You can also find the original by representing the original function as the sum of the simplest fractions,

$$F(p)=\frac{1}{4}\left(\frac{1}{p}-\frac{p}{p^2+4}\right)\leftrightarrow$$

$$\leftrightarrow\frac{1}{4}(1-\cos 2t)$$

c) From the table of images we have $\frac{1}{p+1}\leftrightarrow e^{-t}$. The presence of a multiplier e^{-p} indicates the need to apply the delay theorem. Therefore

$$\frac{e^{-p}}{p+1}\leftrightarrow e^{-(t-1)}\,\theta(t-1)$$

d) Let's write an image in the form

$$F(p)=\frac{p}{(p^2+4)^2}=\frac{1}{p^2+4}\cdot\frac{p}{p^2+4},$$

$$\frac{1}{p^2+4}\leftrightarrow \sin 2t,\ \frac{p}{p^2+4}\leftrightarrow \cos 2t.$$

Let's apply the image multiplication theorem (Borel's theorem)

$$F(p)\leftrightarrow \sin 2t*\cos 2t=\int_0^t \sin 2\tau\cos 2(t-\tau)d\tau=$$

$$=\frac{1}{2}\int_0^t(\sin(4\tau-2t)+\sin 2t)d\tau=\frac{1}{2}\left(-\frac{\cos(4\tau-2t)}{4}+\tau\cdot\sin 2t\right)\Bigg|_0^t=$$

$$=\frac{1}{2}\left(-\frac{\cos 2t}{4}+t\sin 2t+\frac{\cos(-2t)}{4}\right)=\frac{t}{2}\sin 2t.$$

1.2.1 The elementary method

Example 8.

Find the original corresponding to the image:

a) $F(p)=\dfrac{-5}{p(p-1)\left(p^2+4p+5\right)}$,

b) $F(p)=\dfrac{1}{p(p-1)\left(p^2+4\right)}$.

Solution:

a) Let's imagine $F(p)$ as the sum of elementary fractions:

$$F(p)=\frac{-5}{p(p-1)\left(p^2+4p+5\right)}=\frac{A}{p}+\frac{B}{p-1}+\frac{Cp+D}{p^2+4p+5}.$$

To find A,B,C,D we have the equation

$$A(p-1)\left(p^2+4p+5\right)+Bp\left(p^2+4p+5\right)+(Cp+D)p(p-1)=-5.$$

Substituting different values of p, we obtain a system for determining the coefficients

$$p=0:\quad -5A=-5, \qquad p=1:\quad 10B=-5,$$

$$p=-1:\quad -4A-2B-2(-C+D)=-5,$$

$$p=-2:\quad -3A-2B+6(-2C+D)=-5.$$

We find the coefficients:

$$A=1,\quad B=-\frac{1}{2},\quad C=-\frac{1}{2},\quad D=-\frac{3}{2}.$$

$$F(p)=\frac{1}{p}-\frac{1}{2}\cdot\frac{1}{p-1}-\frac{1}{2}\cdot\frac{p+3}{p^2+4p+5}=$$

$$=\frac{1}{p}-\frac{1}{2}\frac{1}{p-1}-\frac{1}{2}\left(\frac{p+2}{(p+2)^2+1}+\frac{1}{(p+2)^2+1}\right)\leftrightarrow$$

$$1-\frac{1}{2}e^{t}-\frac{1}{2}e^{-2t}(\cos t+\sin t).$$

b) HOMEWORK №2

1.2.2 The conversion formula. Decomposition theorems

Theorem 1 (Riemann-Mellin).

Let the function $f(t)$ be the original with the growth index α_0, and $F(p)$ be its image. Then at any point t of the continuity of the original $f(t)$, the Riemann-Mellin formula is valid

$$f(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}F(p)e^{pt}dp \qquad (1.6)$$

where integration is performed along any straight line $\operatorname{Re} p = b$, $b > \alpha_0$, and the integral is understood in the sense of the principal value.

Equality takes place at every point at which $f(t)$ is continuous. At the point t_0, which is the point of discontinuity of the 1st kind of the function $f(t)$, the right side of the Riemann-Mellin formula is equal to

$$\frac{1}{2}\left(f(t_0 - 0) + f(t_0 + 0)\right).$$

The Riemann-Mellin formula (1.6) is the inverse of the formula $F(p) = \int_0^{+\infty} f(t) e^{-pt}\, dt$ and is called *the inverse Laplace transform*.

The direct application of the conversion formula to restore the original $f(t)$ from the $F(p)$ image is difficult. Decomposition theorems are usually used to find the original.

Theorem 2 (the first decomposition theorem).

If the function $F(p)$ in the vicinity of point $p = \infty$ can be represented as a Laurent series (point $p = \infty$ is the zero of the function $F(p)$ and $F(p)$ is analytic in the vicinity of this point)

$$F(p) = \sum_{k=0}^{\infty} \frac{c_k}{p^{k+1}} = \frac{c_0}{p} + \frac{c_1}{p^2} + \frac{c_2}{p^3} + \ldots,$$

then the function

$$f(t)=\sum_{k=0}^{\infty} c_k \cdot \frac{t^k}{k!} = c_0 + c_1 t + c_2 \cdot \frac{t^2}{2!} + \dots, \quad t \ge 0$$

is the original with the image $F(p)$.

Example 9.

Find the original corresponding to the image using the first decomposition theorem:

a) $F(p)=\dfrac{p}{p^2+1}$,

b) $F(p)=\dfrac{1}{p(p^4+1)}$,

c) $F(p)=\dfrac{1}{p}e^{\frac{1}{p^2}}$.

<u>Solution:</u>

a) Decompose the function $F(p)$ into a Laurent series

$$\frac{p}{p^2+1}=\frac{1}{p\left(1+\frac{1}{p^2}\right)}=\frac{1}{p}\left(1-\frac{1}{p^2}+\frac{1}{p^4}-\dots\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{2n+1}}, \quad |p|>1.$$

Since $\dfrac{1}{p^{2n+1}} \leftrightarrow \dfrac{t^{2n}}{(2n)!}$, then according to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{2n+1}} \leftrightarrow \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!} = \cos t = f(t).$$

b) Decompose the function $F(p)$ into a Laurent series

$$\frac{1}{p\left(p^4+1\right)}=\frac{1}{p^5\left(1+\frac{1}{p^4}\right)}=\frac{1}{p^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n+5}},\quad |p|>1\,.$$

Since $\frac{1}{p^{4n+5}}\leftrightarrow\frac{t^{4(n+1)}}{(4(n+1))!}$, then according to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n+5}}\leftrightarrow\sum_{n=0}^{\infty}\frac{(-1)^n t^{4(n+1)}}{(4(n+1))!}=f(t)$$

c) Using the power series expansion of the function

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

we obtain

$$\frac{1}{p}e^{\frac{1}{p^2}}=\frac{1}{p}\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n}}=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}}\,.$$

According to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}}\leftrightarrow\sum_{n=0}^{\infty}\frac{t^{2n}}{n!(2n)!}=f(t)\,.$$

$$F(p)=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}}\leftrightarrow\sum_{n=0}^{\infty}\frac{t^{2n}}{n!(2n)!}=f(t)$$

Theorem 3 (the second decomposition theorem).

Let the function $F(p)$ of the complex variable p be analytic in the entire plane, with the exception of a finite number of isolated singularity points $p_1, p_2, ..., p_n$, located in the half-plane $\operatorname{Re} p < \alpha_0$.

If $\lim\limits_{p\to\infty} F(p) = 0$, and $F(p)$ is absolutely integrable along any vertical line $\operatorname{Re} p = b,\ b > \alpha_0$, then $F(p)$ is an image, and the original $f(t)$ corresponding to the image $F(p)$ is determined by the formula

$$F(p) \leftrightarrow \sum_{k=1}^{n} \operatorname*{Res}_{p=p_k}\left[F(p)e^{pt}\right] = f(t)$$

If p_k is a pole of order m_k, then

$$\operatorname*{Res}_{p=p_k}\left[F(p)e^{pt}\right] = \lim_{p\to p_k}\left\{\frac{d^{m_k-1}}{dp^{m_k-1}}\left((p-p_k)^{m_k} F(p)e^{pt}\right)\right\} =$$
$$= \sum_{j=0}^{m_k-1} \frac{t^{m_k-1-j}}{j!(m_k-1-j)!} \lim_{p\to p_k}\left\{\frac{d^j}{dp^j}\left((p-p_k)^{m_k} F(p)\right)\right\}.$$

If $F(p) = \dfrac{P(p)}{Q(p)}$ is a rational regular irreducible fraction, p_k are poles of the order m_k, $(k = 1,2,...,n)$ of the function $F(p)$, then the original $f(t)$ corresponding to the image $F(p)$ is determined by the formula

$$F(p) \leftrightarrow$$

$$\sum_{k=1}^{n} \frac{1}{(m_k-1)!} \lim_{p\to p_k}\left\{\frac{d^{m_k-1}}{dp^{m_k-1}}\left((p-p_k)^{m_k} F(p)e^{pt}\right)\right\} = f(t). \tag{1.7}$$

In particular, if $p_1, p_2, ..., p_n$ are the simple poles of $F(p)$, then the function

$$f(t)=\sum_{k=1}^{n}\frac{P(p_k)}{Q'(p_k)}e^{p_k t} \qquad (1.8)$$

Example 10.

Find the original corresponding to the image

$$F(p)=\frac{p^2+2}{p^3-p^2-6p}.$$

<u>Solution:</u>

Since $p^3-p^2-6p=p(p-3)(p+2)$, the function $F(p)$ has three simple poles: $p_1=0$, $p_2=3$, $p_3=-2$. Let's construct the corresponding original using the formula (1.8):

$$f(t)=\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=0}+\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=3}+\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=-2}=-\frac{1}{3}+\frac{11}{5}e^{3t}+\frac{3}{5}e^{-2t}.$$

Example 11.

Using the second decomposition theorem, find the original corresponding to the image

a) $F(p)=\dfrac{p^2+p-1}{(p-2)(p-5)(p+4)}$

b) $F(p)=\dfrac{p-1}{(p+1)\left(p^2+4\right)}$

<u>Solution:</u>

a) Function $F(p)=\dfrac{p^2+p-1}{(p-2)(p-5)(p+4)}$ has simple poles (zeros of the denominator) $p_1=2, p_2=5, p_3=-4$. Let's denote

$$P(p)=p^2+p-1,\quad Q(p)=p^3-3p^2-18p+40,$$

$$Q'(p)=3p^2-6p-18.$$

Then for $p_1=2$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_1=2}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_1=2}=-\frac{5}{18},$$

for $p_2=5$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_2=5}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_2=5}=\frac{29}{27},$$

for $p_3=-4$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_3=-4}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_3=-4}=\frac{11}{54}.$$

Therefore, according to the formula (1.8)

$$F(p)=\frac{p^2+p-1}{(p-2)(p-5)(p+4)}\leftrightarrow$$

$$-\frac{5}{18}e^{2t}+\frac{29}{27}e^{5t}+\frac{11}{54}e^{-4t}=\frac{1}{54}\left(11e^{-4t}+58e^{5t}-15e^{2t}\right)=f(t).$$

b) <u>HOMEWORK №3</u>

1.3. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF DIFFERENTIAL EQUATIONS AND SYSTEMS

The method of solving various classes of equations and other problems using the Laplace transform is called the *operational method*.

1.3.1.Differential equations and systems with constant coefficients

Consider an n -th order linear differential equation with constant coefficients:

$$L(x) \equiv x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_{n-1}x'(t) + a_n x(t) = f(t). \qquad (1.10)$$

Let's set the Cauchy problem: to find a solution to equation (1.10) satisfying the conditions:

$$x(0) = x_0 ,\ x'(0) = x_1 ,.., \ x^{(n-1)}(0) = x_{n-1} , \qquad (1.11)$$

where x_i are the specified constants, $i = 0,1,...,n-1$.

Assuming that the function $f(t)$ is the original, we will look for the solution $x(t)$ of the problem (1.10)–(1.11) on the set of originals.

Let $X(p) \leftrightarrow x(t), F(p) \leftrightarrow f(t)$. According to the rule of differentiation of the original and the property of linearity, passing to images in equation (1.10), due to the conditions (1.11), we obtain an equation for an unknown image $X(p)$, which we will call the *operator equation*

$$A(p)X(p) - B(p) = F(p) ,$$

where

$$A(p) = p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n ,$$

$$B(p) = x_0 (p^{n-1} + a_1 p^{n-2} + \ldots + a_{n-1}) +$$
$$+ x_1 (p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-2}) + \ldots + x_{n-2} (p + a_1) + x_{n-1} .$$

Hence

$$X(p) = \frac{B(p) + F(p)}{A(p)} .$$

To find the required solution $x(t)$ of the problem (1.10)– (1.11), it is necessary to restore the original $x(t)$ from its image $X(p)$.

Similarly, the operational method is applied to solving systems of differential equations with constant coefficients.

Example 1

Solve the Cauchy problem:

a) $x' - x = 1,$

$x(0) = -1,$

b) $x'' + x = 2\cos t,$

$x(0) = 0, \quad x'(0) = -1,$

c) $x'' + 2x = t + \dfrac{t^3}{3},$

$x(0) = x'(0) = 0$

d) $x'' - 3x' + 2x = 2e^{3t},$

$x(0)=1,\ x'(0)=3.$

e) $x''+x'+6x=3(\cos 3t-\sin 3t),$

$x(0)=0, x'(0)=3.$

<u>Solution:</u>

a) Let $x(t)\leftrightarrow X(p)$.

Then, according to the original differentiation theorem, we get

$x'(t)\leftrightarrow\ pX(p)-x(0)=pX(p)+1.$

Let's apply the Laplace transform to both parts of the equation. Let's write out the resulting operator equation

$$pX(p)+1-X(p)=\frac{1}{p}.$$

We get

$$X(p)=-\frac{1}{p}.$$

Thus

$$x(t)=-1.$$

b) Let's move on from the originals to the images

$x(t)\leftrightarrow X(p),$

$x'(t)\leftrightarrow\ pX(p)-x(0)=pX(p),$

$x''(t) \leftrightarrow \quad p^2X(p) - px(0) - x'(0) = p^2X(p) + 1,$

$\cos t \leftrightarrow \quad \dfrac{p}{p^2+1}.$

Let's write down the equation for the images

$$p^2X(p) + 1 + X(p) = \frac{2p}{p^2+1}.$$

Let's solve the equation for images

$$X(p) = \frac{2p}{\left(p^2+1\right)^2} - \frac{1}{p^2+1}.$$

According to the image differentiation theorem, we will find the original of the first term

$$\frac{2p}{\left(p^2+1\right)^2} = -\left(\frac{1}{p^2+1}\right)' \leftrightarrow t\sin t.$$

Therefore, the solution has the form

$$x(t) = t\sin t - \sin t = (t-1)\sin t.$$

c) Let $x(t) \leftrightarrow X(p)$.

Let's move on to the images in the equation

$$p^2X(p) - px(0) - x'(0) + 2X(p) = \frac{1}{p^2} + \frac{1}{3}\frac{3!}{p^4}.$$

Since $x(0) = x'(0) = 0$, then

$$\left(p^2+2\right)X=\frac{p^2+2}{p^4}$$

$$X(p)=\frac{1}{p^4}.$$

Having found the original for this image, we get a solution to the Cauchy problem

$$X(p)=\frac{1}{p^4}=\frac{3!}{3!p^4}\leftrightarrow\frac{t^3}{3!}=\frac{t^3}{6}=x(t).$$

d) Let's move on from the originals to the images

$$x(t)\leftrightarrow X(p)$$

$$x'(t)\leftrightarrow pX(p)-1,$$

$$x''(t)\leftrightarrow p^2X(p)-p-3,$$

$$e^{3t}\leftrightarrow\frac{1}{p-3}.$$

Let's write down the equation for the images

$$p^2X(p)-p-3-3pX(p)+3+2X(p)=\frac{2}{p-3}.$$

Let's solve the equation for images

$$\left(p^2-3p+2\right)X(p)=\frac{2}{p-3}+p,$$

$$X(p)=\frac{p^2-3p+2}{(p-3)\left(p^2-3p+2\right)}=\frac{1}{p-3}.$$

Let's find the original for the function $X(p)$

$$X(p)=\frac{1}{p-3}\leftrightarrow e^{3t}=x(t)$$

e) <u>HOMEWORK</u>

Example 2

Find a solution of the differential equation

$$x'(t)+x(t)=e^{-t},$$

satisfying the condition $x(0)=1$ (Cauchy problem).

<u>Solution:</u>

Let $x(t)\leftrightarrow X(p)$. Since

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-1,$$

$$e^{-t}\leftrightarrow\frac{1}{p+1},$$

applying the Laplace transform to a given equation using the linearity property, we obtain an algebraic equation for $X(p)$:

$$pX(p)-1+X(p)=\frac{1}{p+1}.$$

From where we find for $X(p)$:

$$X(p)=\frac{1}{(p+1)^2}+\frac{1}{p+1}.$$

Since

$$\frac{1}{p+1}\leftrightarrow e^{-t},\quad \frac{1}{(p+1)^2}\leftrightarrow te^{-t},$$

we have

$$X(p)\leftrightarrow x(t)=te^{-t}+e^{-t}.$$

Check.

We show that the found function is indeed a solution to the Cauchy problem. We substitute the expression for the function $x(t)$ and its derivative

$$x'(t)=-te^{-t}+e^{-t}-e^{-t}=-te^{-t}$$

into the given equation

$$-te^{-t}+te^{-t}+e^{-t}=e^{-t}.$$

After addition of similar terms in the left part of the equation, we get the correct identity: $e^{-t}\equiv e^{-t}$.

Thus, the constructed function is a solution to the equation.

Let's check if it satisfies the initial condition $x(0)=1$:

$$x(0)=0\cdot e^{-0}+e^{-0}=1.$$

Therefore, the found function is a solution to the Cauchy problem.

Example 3

Find a solution of the differential equation

$$x''(t)+3x'(t)=e^{t},$$

satisfying the condition $x(0)=0,\ x'(0)=-1$.

<u>Solution:</u>

We apply the Laplace transform to the equation. Using the property of linearity and considering that

$$x(t)\leftrightarrow X(p),$$
$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-0=pX(p),$$
$$x''(t)\leftrightarrow p^2X(p)-px(0)-x'(0)=p^2X(p)-p\cdot 0-(-1)=p^2X(p)+1,$$
$$e^{t}\leftrightarrow \frac{1}{p-1},$$

we obtain an algebraic equation for $X(p)$:

$$p^2X(p)+1+3pX(p)=\frac{1}{p-1},\ \Leftrightarrow\ (p^2+3p)X(p)=\frac{1}{p-1}-1.$$

We will find a fundamental solution:

$$H(p)=\frac{1}{(p^2+3p)}=\frac{1}{3}\left(\frac{1}{p}-\frac{1}{p+3}\right)\leftrightarrow h(t)=\frac{1}{3}\left(1-e^{-3t}\right).$$

Then, since

$$X(p)=\left(\frac{1}{p-1}-1\right)H(p)=\frac{1}{p-1}H(p)-H(p),$$

using the convolution image property, we will write the solution of the given equation in the form

$$x(t)=\frac{1}{3}\int_0^t e^{t-\tau}(1-e^{-3\tau})d\tau-\frac{1}{3}(1-e^{-3t}).$$

Having calculated the integrals and addition of similar terms, we get the final answer:

$$x(t)=-\frac{2}{3}+\frac{1}{4}e^{t}+\frac{5}{12}e^{-3t}.$$

<u>Check.</u>

We have

$$x(t)=-\frac{2}{3}+\frac{1}{4}e^{t}+\frac{5}{12}e^{-3t},\ x'(t)=\frac{1}{4}e^{t}-\frac{5}{4}e^{-3t},\ x''(t)=\frac{1}{4}e^{t}+\frac{15}{4}e^{-3t}.$$

We substitute everything into a given equation

$$\frac{1}{4}e^{t}+\frac{15}{4}e^{-3t}+3\left(\frac{1}{4}e^{t}+\frac{5}{4}e^{-3t}\right)\equiv e^{t}.$$

As a result, we get the identity $e^{t}\equiv e^{t}$. Therefore, the found function is a solution to the equation. Let's check the fulfillment of the initial conditions:

$$x(0)=-\frac{2}{3}+\frac{1}{4}e^{0}+\frac{5}{12}e^{-0}=0;\ x'(0)=\frac{1}{4}e^{0}-\frac{5}{4}e^{-0}=-1.$$

Therefore, the found function is a solution of the Cauchy problem.

Example 4

Find a solution of the differential equation

$$x'''(t)+2x''(t)+5x'(t)=0,$$

satisfying the conditions: $x(0)=-1,\ x'(0)=2,\ x''(0)=0$.

<u>Solution:</u>

Let $x(t)\leftrightarrow X(p)$.

Since, take into account the given conditions, we have

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-(-1)=pX(p)+1,$$

$$x''(t)\leftrightarrow p^2X(p)-px(0)-x'(0)=p^2X(p)-p(-1)-2=p^2X(p)+p-2,$$

$$x'''(t)\leftrightarrow p^3X(p)-p^2x(0)-px'(0)-x''(0)=$$

$$=p^3X(p)-p^2(-1)-p2-0=p^3X(p)+p^2-2p,$$

then, after applying the Laplace transform for a given equation, we obtain the following operator equation:

$$p^3X(p)+p^2-2p+2p^2X(p)+2p-4+5pX(p)+5=0,$$

or after the transformations:

$$X(p)(p^3+2p^2+5p)=-p^2-1.$$

Solving this equation for $X(p)$, we obtain

$$X(p)=\frac{-p^2-1}{p(p^2+2p+5)}.$$

The resulting expression is decomposed into the simplest fractions:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{A}{p}+\frac{Bp+C}{p^2+2p+5}.$$

Using the method of undefined coefficients, we find A, B, C.

To do this, we bring the fractions to a common denominator and equate the coefficients with the same degrees of p:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{Ap^2+2Ap+5A+Bp^2+Cp}{p(p^2+2p+5)}.$$

We obtain a system of algebraic equations for A,B,C:

$$A+B=-1,\ 2A+C=0,\ 5A=-1,$$

the solution of which will be: $A=-\frac{1}{5}, B=-\frac{4}{5}, C=\frac{2}{5}$.

Then

$$X(p)=-\frac{1}{5p}+\frac{1}{5}\frac{-4p+2}{p^2+2p+5}.$$

To find the original of the second fraction, select the full square in its denominator: $p^2+2p+5=(p+1)^2+4$, then select the summand $p+1$ in the numerator:

$$-4p+2=-4(p+1)+6,$$

and decompose the fraction into the sum of two fractions:

$$\frac{1}{5}\frac{-4p+2}{p^2+2p+5}=-\frac{4}{5}\frac{p+1}{(p+1)^2+4}+\frac{3}{5}\frac{2}{(p+1)^2+4}.$$

Next, using the displacement property and the table of correspondence between images and originals, we obtain a solution to the original equation:

$$x(t)=-\frac{1}{5}-\frac{4}{5}e^{-t}\cos 2t+\frac{3}{5}e^{-t}\sin 2t.$$

Example 5

Find a solution to the system:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 2x_2 + \sin t, \\ \dfrac{dx_2}{dt} = -x_1 + x_2 + 1, \end{cases}$$

satisfying the initial conditions $x_1(0) = 1,\ x_2(0) = 0$.

Solution:

Let's construct a solution using the Laplace transform, first reducing the system to one equivalent second-order equation.

Let's express the unknown function $x_2(t)$ from the first equation of the system

$$x_2 = \frac{1}{2}\left(\frac{dx_1}{dt} - x_1 - \sin t\right),$$

$$\frac{dx_2}{dt} = \frac{1}{2}\left(\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt} - \cos t\right)$$

and substitute it into the second equation

$$\frac{1}{2}\left(\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt} - \cos t\right) = -x_1 + \frac{1}{2}\left(\frac{dx_1}{dt} - x_1 - \sin t\right) + 1.$$

Let's transform the resulting equation by entering the notation $f(t)$ for the right side:

$$\frac{d^2x_1}{dt^2} - 2\frac{dx_1}{dt} + 3x_1 = \cos t - \sin t \equiv f(t). \qquad (*)$$

Let's find the initial conditions

$$x_1(t)\big|_{t=0}=1\,;\; x_1'(t)\big|_{t=0}=(x_1+2x_2+\sin t)\big|_{t=0}=1\,. \qquad (**)$$

Let's apply the Laplace transform to the equation (*) with initial conditions (**).

Let $X_1(p)\leftrightarrow x_1(t)$, $F(p)\leftrightarrow f(t)$, then

$$p^2X_1(p)-p-1-2pX_1(p)+2+3X_1(p)=F(p)\,,$$
$$X_1(p)(p^2-2p+3)=F(p)+p-1\,,$$
$$X_1(p)=\frac{F(p)}{p^2-2p+3}+\frac{p-1}{p^2-2p+3}\,.$$

We will find a fundamental solution:

$$h(t)\leftrightarrow H(p)=\frac{1}{p^2-2p+3}=\frac{1}{(p-1)^2+2}\leftrightarrow\frac{1}{\sqrt{2}}e^t\sin\sqrt{2}t\,.$$

Let's find the original $x_1(t)$, given that $h'(t)\leftrightarrow pH(p)-h(0)=pH(p)$,

$$x_1(t)=\int_0^t h(t-\tau)f(\tau)d\tau+h'(t)-h(t)\,.$$

The expression for the function $x_2(t)$ can be constructed using the second equation of a given system, substituting the found expression for the function $x_1(t)$ into it:

$$x_2=\frac{1}{2}\left(\frac{dx_1}{dt}-x_1-\sin t\right).$$

As a result

$$x_1(t)=\frac{2}{3}-\frac{1}{2}t+\frac{1}{3}e^t\cos\sqrt{2}t+\frac{7\sqrt{12}}{12}e^t\sin\sqrt{2}t\,,$$
$$x_2(t)=-\frac{1}{3}-\frac{1}{4}\cos t-\frac{1}{4}\sin t+\frac{7}{12}e^t\cos\sqrt{2}t-\frac{\sqrt{2}}{6}e^t\sin\sqrt{2}t\,.$$

Example 4

Find a solution of the differential equation

$$x'''(t)+2x''(t)+5x'(t)=0,$$

satisfying the conditions: $x(0)=-1,\ x'(0)=2,\ x''(0)=0$.

<u>Solution:</u>

Let $x(t)\leftrightarrow X(p)$.

Since, take into account the given conditions, we have

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-(-1)=pX(p)+1,$$

$$x''(t)\leftrightarrow p^2X(p)-px(0)-x'(0)=p^2X(p)-p(-1)-2=p^2X(p)+p-2,$$

$$x'''(t)\leftrightarrow p^3X(p)-p^2x(0)-px'(0)-x''(0)=$$

$$=p^3X(p)-p^2(-1)-p2-0=p^3X(p)+p^2-2p,$$

then, after applying the Laplace transform for a given equation, we obtain the following operator equation:

$$p^3X(p)+p^2-2p+2p^2X(p)+2p-4+5pX(p)+5=0,$$

or after the transformations:

$$X(p)(p^3+2p^2+5p)=-p^2-1.$$

Solving this equation for $X(p)$, we obtain

$$X(p)=\frac{-p^2-1}{p(p^2+2p+5)}.$$

The resulting expression is decomposed into the simplest fractions:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{A}{p}+\frac{Bp+C}{p^2+2p+5}.$$

Using the method of undefined coefficients, we find A, B, C.

To do this, we bring the fractions to a common denominator and equate the coefficients with the same degrees of p:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{Ap^2+2Ap+5A+Bp^2+Cp}{p(p^2+2p+5)}.$$

We obtain a system of algebraic equations for A,B,C:

$$A+B=-1,\ 2A+C=0,\ 5A=-1,$$

the solution of which will be: $A=-\frac{1}{5}, B=-\frac{4}{5}, C=\frac{2}{5}$.

Then

$$X(p)=-\frac{1}{5p}+\frac{1}{5}\frac{-4p+2}{p^2+2p+5}.$$

To find the original of the second fraction, select the full square in its denominator: $p^2+2p+5=(p+1)^2+4$, then select the summand $p+1$ in the numerator:

$$-4p+2=-4(p+1)+6,$$

and decompose the fraction into the sum of two fractions:

$$\frac{1}{5}\frac{-4p+2}{p^2+2p+5}=-\frac{4}{5}\frac{p+1}{(p+1)^2+4}+\frac{3}{5}\frac{2}{(p+1)^2+4}.$$

Next, using the displacement property and the table of correspondence between images and originals, we obtain a solution to the original equation:

$$x(t) = -\frac{1}{5} - \frac{4}{5}e^{-t}\cos 2t + \frac{3}{5}e^{-t}\sin 2t.$$

Example 5

Solve the Cauchy problem:

$$x''' - x'' - 6x' = 0,$$
$$x(0) = 15, x'(0) = 2, x''(0) = 56.$$

Solution:

Let's move on from the originals to the images:

$x(t) \leftrightarrow X(p)$,

$x'(t) \leftrightarrow pX(p) - 15$,

$x'''(t) \leftrightarrow p^3 X(p) - 15p^2 - 2p - 56$.

Let's solve the equation for images

$$\left(p^3 - p^2 - 6p\right)X(p) = 15p^2 - 13p - 36,$$

$$X(p) = \frac{15p^2 - 13p - 36}{p(p-3)(p+2)}.$$

The function $X(p)$ is a proper rational irreducible fraction for which the points $p_1 = 0,\ p_2 = 3,\ p_3 = -2$ are simple poles. Since

$$P(p) = 15p^2 - 13p - 36,$$

$$Q(p) = p^3 - p^2 - 6p,$$

$$Q'(p) = 3p^2 - 2p - 6,$$

that's for

$p_1 = 0$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_1=0} = \frac{-36}{-6} = 6,$$

for

$p_2 = 3$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_2=3} = \frac{60}{15} = 4,$$

For

$p_3 = -2$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_3=-2} = \frac{50}{10} = 5,$$

and by the second decomposition theorem we get

$$x(t) = 6 + 5e^{-2t} + 4e^{3t}.$$

<u>Remark</u>

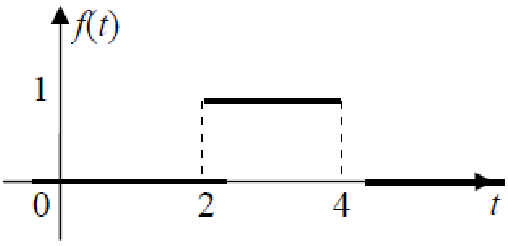
In many practical problems, the right side of the differential equation is given graphically. In this case, the solution algorithm does not change, and the delay theorem and methods from properties are used to find the image of the original given by the graph.

Example 6

Solve the Cauchy problem for a differential equation with the right-hand side given graphically:

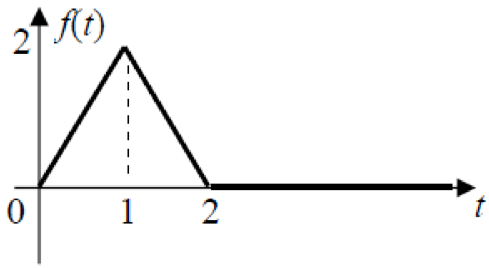
a)

$$x'' + x = f(t),$$
$$x(0) = x'(0) = 0,$$



b)

$$x'' + 4x = f(t),$$
$$x(0) = x'(0) = 0,$$



<u>Solution:</u>

a) Let's move on from the originals to the images:

$$x(t) \leftrightarrow X(p)$$

$$x'(t) \leftrightarrow pX(p)$$

$$x''(t) \leftrightarrow p^2 X(p)$$

$$f(t) = \theta(t-2) - \theta(t-4) \leftrightarrow \frac{1}{p}\left(e^{-2p} - e^{-4p}\right)$$

Let's solve the equation for images

$$\left(p^2+1\right)X(p) = \frac{1}{p}\left(e^{-2p} - e^{-4p}\right),$$

$$X(p)=\frac{1}{p\left(p^2+4\right)}\left(e^{-2p}-e^{-4p}\right).$$

Since

$$\frac{1}{p\left(p^2+1\right)}=\frac{1}{p}-\frac{p}{p^2+1}\leftrightarrow 1-cos\,t\,,$$

then

$$x(t)=\left(1-cos\left(t-2\right)\right)\theta\left(t-2\right)-\left(1-cos\left(t-4\right)\right)\theta\left(t-4\right).$$

The solution of the Cauchy problem can be presented in an analytical form:

$$x(t)=\begin{cases} 0, & t<2, \\ 1-\cos(t-2), & 2\le t<4, \\ \cos(t-4)-\cos(t-2), & t\ge 4. \end{cases}$$

b) Let's move on from the originals to the images:

$$x(t)\leftrightarrow X(p)$$

$$x'(t)\leftrightarrow pX(p)$$

$$x''(t)\leftrightarrow p^2X(p)$$

$$f(t)=2t\theta(t)-2t\theta(t-1)+(4-2t)\theta(t-1)-(4-2t)\theta(t-2)=$$

$$=2t\theta(t)-4(t-1)\theta(t-1)+2(t-2)\theta(t-2)\leftrightarrow$$

$$\leftrightarrow\frac{2}{p^2}-\frac{4}{p^2}e^{-p}+\frac{2}{p^2}e^{-2p}=\frac{2}{p^2}\left(1-2e^{-p}+e^{-2p}\right)$$

Let's solve the equation for images

$$\left(p^2+4\right)X(p)=\frac{2}{p^2}\left(1-2e^{-p}+e^{-2p}\right),$$

$$X(p)=\frac{2}{p^2\left(p^2+4\right)}\left(1-2e^{-p}+e^{-2p}\right).$$

Since

$$\frac{2}{p^2\left(p^2+4\right)}=\frac{1}{2}\left(\frac{1}{p^2}-\frac{1}{p^2+4}\right)\leftrightarrow\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right),$$

Then

$$x(t)=\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)\theta(t)-\left(t-1-\frac{1}{2}\sin 2(t-1)\right)\theta(t-1)+$$

$$+\frac{1}{2}\left(t-2-\frac{1}{2}\sin 2(t-2)\right)\theta(t-2).$$

Remark

If $t=t_0\neq 0$ is taken for the initial time in the Cauchy problem, then a new variable $\tau=t-t_0$ is introduced. Then $\tau=0$ for $t=t_0$.

Example 7

Solve the Cauchy problem

a)

$$x''+x'=t,$$
$$x(1)=1,\quad x'(1)=0,$$

b)

$$x'' + x = -2\sin t,$$
$$x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1.$$

<u>Solution:</u>

a) Let $t = \tau + 1,\ x(t) = x(\tau + 1) = z(\tau)$.

Then the equation and the initial conditions will take the form

$$z'' + z = \tau + 1, \quad z(0) = 1, \quad z'(0) = 0.$$

Let's move on from the originals to the images

$$z(\tau) \leftrightarrow Z(p)$$

$$z'(\tau) \leftrightarrow pZ(p) - 1$$

$$z''(\tau) \leftrightarrow p^2 Z(p) - p$$

$$\tau + 1 \leftrightarrow \frac{1}{p^2} + \frac{1}{p}.$$

Let's write down the equation for the images

$$p^2 Z(p) - p + pZ(p) - 1 = \frac{1}{p^2} + \frac{1}{p}.$$

Solving the operator equation and moving on to the originals, we get

$$Z(p) = \frac{1}{p^3} + \frac{1}{p} \leftrightarrow 1 + \frac{\tau^2}{2} = z(\tau).$$

Returning to the original variable t, we obtain a solution to the Cauchy problem

$$x(t) = 1 + \frac{(t-1)^2}{2}.$$

b) Let

$$t=\tau+\frac{\pi}{2},\quad x(t)=x\left(\tau+\frac{\pi}{2}\right)=z(\tau).$$

Then the equation and the initial conditions will take the form

$$z''+z=-2\sin\left(\tau+\frac{\pi}{2}\right),$$

$$z(0)=0,\quad z'(0)=1.$$

Let's move on from the originals to the images

$$z(\tau)\leftrightarrow Z(p)$$

$$z'(\tau)\leftrightarrow pZ(p)$$

$$z''(\tau)\leftrightarrow p^2Z(p)-1$$

$$-2\sin\left(\tau+\frac{\pi}{2}\right)=-2\cos\tau\leftrightarrow\frac{-2p}{p^2+1}.$$

Let's write down the equation for the images

$$p^2Z(p)-1+Z(p)=\frac{-2p}{p^2+1}.$$

Let's solve the equation for images

$$Z(p)=\frac{1}{p^2+1}-\frac{2p}{\left(p^2+1\right)^2}.$$

Turning to the originals, we get

$$\frac{1}{p^2+1}\leftrightarrow\sin\tau$$

$$\frac{p}{\left(p^2+1\right)^2} \leftrightarrow \sin\tau * \cos\tau = \frac{1}{2}\tau\sin\tau$$

$$z(\tau) = (1-\tau)\sin\tau \,.$$

Returning to the original variable t, we obtain a solution to the original Cauchy problem

$$x(t) = \left(1 - t + \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right) = \left(t - 1 - \frac{\pi}{2}\right)\cos t \,.$$

Example 8

Solve systems of differential equations with given initial conditions:

a)

$$\begin{cases} x' + y = 2e^t, \\ y' + x = 2e^t, \end{cases}$$
$$x(0) = y(0) = 1,$$

b)

$$\begin{cases} x'' + x' + y'' - y = e^t, \\ x' + 2x - y' + y = e^{-t}, \end{cases}$$

$$x(0) = y(0) = y'(0) = 0, \; x'(0) = 1.$$

<u>Solutions:</u>

a) Let

$$x(t) \leftrightarrow X(p)$$

$$y(t) \leftrightarrow Y(p)$$

Considering that

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - 1$$

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p) - 1$$

$$e^t \leftrightarrow \frac{1}{p-1}$$

we obtain an operator system of linear equations

$$\begin{cases} pX(p) - 1 + Y(p) = \dfrac{2}{p-1}, \\ pY(p) - 1 + X(p) = \dfrac{2}{p-1}. \end{cases} \Rightarrow \begin{cases} pX(p) + Y(p) = \dfrac{p+1}{p-1}, \\ pY(p) + X(p) = \dfrac{p+1}{p-1}. \end{cases}$$

Solving the system, we get $X(p) = Y(p) = \dfrac{1}{p-1}$.

Using the image table, we will find $x(t) = y(t) = e^t$.

b) We have

$$x(t) \leftrightarrow X(p)$$

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p)$$

$$x''(t) \leftrightarrow p^2 X(p) - x'(0) = pX(p) - 1$$

$$y(t) \leftrightarrow Y(p)$$

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p)$$

$$y''(t) \leftrightarrow p^2 Y(p) - y'(0) = p^2 Y(p)$$

$$e^{t} \leftrightarrow \frac{1}{p-1}$$

$$e^{-t} \leftrightarrow \frac{1}{p+1}$$

Let's write down a system of operator equations

$$\begin{cases} p^2X-1+pX+p^2Y-Y=\dfrac{1}{p-1}, \\ pX+2X-pY+Y=\dfrac{1}{p+1}. \end{cases} \Rightarrow \begin{cases} \left(p^2+p\right)X+\left(p^2-1\right)Y=\dfrac{p}{p-1}, \\ (p+2)X+(1-p)Y=\dfrac{1}{p+1}. \end{cases}$$

Let's solve a system of linear equations with respect to X and Y using Kramer's formulas:

$$\Delta=\begin{vmatrix} p^2+p & p^2-1 \\ p+2 & 1-p \end{vmatrix}=p(p+1)(1-p)-(p+2)\left(p^2-1\right)=$$
$$=2(1+p)^2(1-p),$$

$$\Delta_x=\begin{vmatrix} \dfrac{p}{p-1} & p^2-1 \\ \dfrac{1}{p+1} & 1-p \end{vmatrix}=1-2p, \quad \Delta_y=\begin{vmatrix} p^2+p & \dfrac{p}{p-1} \\ p+2 & \dfrac{1}{p+1} \end{vmatrix}=\frac{3p}{1-p}.$$

Thus

$$X(p)=\frac{\Delta_x}{\Delta}=\frac{1-2p}{2(p+1)^2(1-p)}=\frac{1}{8}\frac{1}{p-1}+\frac{3}{4}\frac{1}{(p+1)^2}-\frac{1}{8}\frac{1}{p+1},$$

$$Y(p)=\frac{\Delta_y}{\Delta}=\frac{3p}{2(p+1)^2(1-p)^2}=\frac{3p}{2\left(p^2-1\right)^2}.$$

Let's move on to the originals. Since

$e^{-t} \leftrightarrow \dfrac{1}{p+1}$ and $\operatorname{sh} t \leftrightarrow \dfrac{1}{p^2-1}$

then, according to the image differentiation theorem, we find

$$\left(\frac{1}{p+1}\right)' = -\frac{1}{\left(p+1\right)^2} \leftrightarrow -te^{-t}$$

$$\left(\frac{1}{p^2-1}\right)' = -\frac{2p}{\left(p^2-1\right)^2} \leftrightarrow -t\,\mathrm{sh}\,t$$

Therefore, the solution of the system will be

$$x(t) = \frac{1}{4}\mathrm{sh}\,t + \frac{3}{4}te^{-t}, \quad y(t) = \frac{3}{4}t\,\mathrm{sh}\,t\,.$$

Example 9 (HOMEWORK)

Solve system of differential equations with given initial conditions:

$$\begin{cases} x'(t) = -x(t) + y(t) + e^t, \\ y'(t) = x(t) - y(t) + e^t \end{cases}$$

$$x(0) = y(0) = 1\,.$$

* This homework and homework from 09.09.24 should be done by September 15, 2024.

1.3.2. DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Consider an equation of the form

$$a_0(t)x^{(n)}(t)+a_1(t)x^{(n-1)}(t)+\ldots+a_n(t)x(t)=f(t), \qquad (1.12)$$

where $a_i(t),\ i=0,1,\ldots,n$ - polynomials of degree m_i, function $f(t)$ is the original.

Let's denote $m=\max\{m_0,m_1,\ldots,m_n\}$.

We will assume that the Cauchy problem for equation (1.12) with the conditions

$$x(0)=x_0,\ x'(0)=x_1,\ \ldots,\ x^{(n-1)}(0)=x_{n-1}$$

has a solution on the set of originals.

Let $x(t)\leftrightarrow X(p)$.

According to the image differentiation rule, we have

$$t^k x^{(s)}(t)\leftrightarrow(-1)^k\frac{d^k}{dp^k}\left(L\left\{x^{(s)}(t)\right\}\right)=(-1)^k\frac{d^k}{dp^k}\left(p^sX(p)-p^{s-1}x_0-\ldots-x_{s-1}\right).$$

Thus, applying the Laplace transform to both parts of equation (1.12), equation (1.12) is transformed into an m-th order differential equation with respect to the image $X(p)$. After that, the task of integrating equation (1.12) is simplified.

Example 1

Find a solution to the equation

$$ty''(t)-(1+t)y'(t)+y(t)=0$$

<u>Solution:</u>

$y(t)\leftrightarrow Y(p)$

$y'(t)\leftrightarrow pY(p)-y(0)$

Using the image differentiation property ($t\,f(t)\leftrightarrow -F'(p)$), we have

$$t\,y'(t)\leftrightarrow -\left(pY(p)-y(0)\right)'=-\left(Y(p)+pY'(p)\right)=-p\frac{dY(p)}{dp}-Y(p),$$

(because $y(0)=\text{const}$).

$y''(t)\leftrightarrow p^2Y(p)-py(0)-y'(0)$

$$t\,y''(t)\leftrightarrow -\left(p^2Y(p)-py(0)-y'(0)\right)'=-\left(2pY(p)+p^2Y'(p)-y(0)\right)$$

We substitute the images for these terms into the equation and multiply each term by (-1) to get rid of the numerous minus signs.

We have

$$t\,y''(t)-y'(t)-t\,y'(t)+y(t)=0$$

we substitute and get

$$2pY(p)+p^2Y'(p)-y(0)-\left(-pY(p)+y(0)\right)-\left(Y(p)+pY'(p)\right)-Y(p)=0$$

$$2pY(p)+p^2Y'(p)-y(0)+pY(p)-y(0)-Y(p)-pY'(p)-Y(p)=0$$

$$\left(p^2-p\right)Y'(p)+(3p-2)Y(p)=2y(0)=2C_0$$

let's denote an unknown quantity by $C_0=\text{const}$.

In the image space, we did not get an algebraic equation, as before, we get a first-order differential equation (linear inhomogeneous). First, we solve the corresponding homogeneous equation, when in the right part, instead of the term $2C_0$, there is zero.

$$\left(p^2-p\right)Y'(p)+(3p-2)Y(p)=0$$

The resulting equation is an equation with separable variables.

Since $Y'(p)=\frac{dY}{dp}$ then

$$\left(p^2-p\right)\frac{dY}{dp}=-(3p-2)Y(p)$$

$$\frac{dY}{Y}=-\frac{(3p-2)}{\left(p^2-p\right)}dp$$

We can integrate and get:

$$\ln Y(p)=-\int\frac{(3p-2)}{p^2-p}dp=-\int\frac{(2p-1)}{p^2-p}dp-\int\frac{(p-1)}{p^2-p}dp=$$

$$==-\int\frac{d\left(p^2-p\right)}{\left(p^2-p\right)}-\int\frac{dp}{p}=-\ln\left(p^2-p\right)-\ln p=-\ln\left(p^3-p^2\right)+\ln C$$

We have received

$$Y_0(p) = \frac{C}{p^3 - p^2}$$

We have set the index to zero, since this is not a solution to our equation.

We use the method of variation of parameters (variation of constants), the Lagrange method to

$$(p^2 - p)Y'(p) + (3p-2)Y(p) = 2y(0) = 2C_0$$

The solution of an inhomogeneous equation is sought in the same form as the solution of a homogeneous one, only instead of an arbitrary constant C, a new unknown function $C(p)$ is put.

$$Y_0(p) = \frac{C}{p^3 - p^2} \to Y(p) = \frac{C(p)}{p^3 - p^2}$$

This change of variables is substituted into the inhomogeneous equation. Only first you need to calculate $Y'(p)$:

$$Y'(p) = \frac{C'(p)}{p^3 - p^2} - \frac{C(p)(3p^2 - 2p)}{(p^3 - p^2)^2}$$

$$(p^2 - p)\left[\frac{C'(p)}{p^3 - p^2} - \frac{C(p)(3p^2 - 2p)}{(p^3 - p^2)^2}\right] + (3p-2)\left[C(p)\frac{1}{p^3 - p^2}\right] = 2C_0$$

$$\frac{C'(p)}{p} - \frac{C(p)(3p^2 - 2p)}{p(p^3 - p^2)} + \frac{(3p-2)C(p)}{p^3 - p^2} = 2C_0$$

$$C'(p) = 2C_0 p$$

$$C(p)=C_0 p^2+C_1$$

Substituting the found function in $Y(p)=\dfrac{C(p)}{p^3-p^2}$ and get

$$Y(p)=\frac{C_0 p^2+C_1}{p^3-p^2}=\frac{C_0}{p-1}+\frac{C_1}{(p-1)p^2}$$

From the image of the solution, you need to calculate the inverse Laplace transform (find the original)

$$\frac{1}{p-1}\leftrightarrow e^t$$

For the second term, let's use the property ($\int_0^t f(u)du \leftrightarrow \dfrac{F(p)}{p}$):

$$\frac{1}{p(p-1)}\leftrightarrow \int_0^t e^u du=e^t-1$$

$$\frac{1}{p^2(p-1)}\leftrightarrow \int_0^t \left(e^u-1\right)du=e^t-1-t$$

Our solution is

$$y(t)=C_0 e^t+C_1\left(e^t-1-t\right)$$

Let's denote

$$C_0+C_1=\tilde{C}_0$$

$$-C_1=\tilde{C}_1$$

$$y(t) = e^t\left(C_0 + C_1\right) + C_1\left(-1 - t\right)$$

$$y(t) = \tilde{C}_0\, e^t + \tilde{C}_1\left(1 + t\right)$$

Example 2

Find a solution to the equation

$$tx''(t) - (1+t)x'(t) + 2(1-t)x(t) = 0 .$$

<u>Solution:</u>

Let $x(t) \leftrightarrow X(p)$.

Then, using the property of differentiating the original and differentiating the image, we write:

$$x'(t) \leftrightarrow pX(p) - x(0) ,$$

$$x''(t) \leftrightarrow p^2 X(p) - px(0) - x'(0) ,$$

$$tx(t) \leftrightarrow -\frac{dX(p)}{dp} ,$$

$$tx'(t) \leftrightarrow -\frac{d}{dp}\{pX(p) - x(0)\} = -p\frac{dX}{dp} - X(p) ,$$

$$tx''(t) \leftrightarrow -\frac{d}{dp}\left\{p^2 X(p) - px(0) - x'(0)\right\} = -p^2\frac{dX}{dp} - 2pX(p) + x(0) .$$

Applying the Laplace transform to a given equation, we obtain the following operator equation:

$$-p^2\frac{dX}{dp}-2pX(p)+x(0)-$$

$$-pX(p)+x(0)+p\frac{dX(p)}{dp}+X(p)+2X(p)+2\frac{dX(p)}{dp}=0\ ,$$

which can be easily reduced to the form

$$(p^2-p-2)\frac{dX}{dp}+3(p-1)X(p)=2x(0)\,.$$

Having solved the obtained ordinary differential equation, for example, by the method of variation of parameters, we construct its general solution:

$$X(p)=\frac{x(0)}{p-2}+\frac{c}{(p-2)(p+1)^2}\,.$$

Here c is an arbitrary constant. Further, since

$$\frac{1}{p-2}\leftrightarrow e^{2t}\ ,$$

$$\frac{1}{(p+1)^2}\leftrightarrow te^{-t}\ ,$$

$$\frac{1}{(p-2)(p+1)^2}\leftrightarrow\int_0^t\tau e^{-\tau}e^{2(t-\tau)}d\tau=\frac{1}{9}\left(e^{2t}-(3t+1)e^{-t}\right),$$

then the general solution of the given equation will have the form

$$x(t)=x(0)e^{2t}+c\left(\frac{1}{9}e^{2t}-\frac{1}{9}(3t+1)e^{-t}\right)=\left(x(0)+c\right)e^{2t}-c\left(3t+1\right)e^{-t}\,.$$

1.4. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF DIFFERENTIAL EQUATIONS WITH A DELAYED ARGUMENT

Consider a linear differential equation with a delayed argument with constant coefficients:

$$x^{(n)}(t)=\sum_{k=0}^{n-1}a_k x^{(k)}(t-\tau_k)+f(t)\,,\ 0<t<+\infty\,, \qquad (1.13)$$

where $a_k=\text{const}$, $\tau_k=\text{const}\geq 0$.

Let's assume that

$$x(t)=x'(t)=\ldots=x^{(n-1)}(t)\equiv 0\,,$$

$$\text{for } \forall t<0\,.$$

Let it be required to find a solution to equation (1.13) satisfying the initial conditions:

$$x(0)=x'(0)=\ldots=x^{(n-1)}(0)=0\,. \qquad (1.14)$$

Applying the Laplace transform to both parts of equation (1.13) and taking into account the delay property of the original, we obtain the operator equation for the image $X(p)\leftrightarrow x(t)$:

$$p^n X(p)=\sum_{k=0}^{n-1}a_k p^k X(p)e^{-\tau_k p}+F(p)\,, \qquad (1.15)$$

where $F(p)\leftrightarrow f(t)$.

From (1.15) for $X(p)$ we will have

$$X(p)=\frac{F(p)}{p^n-\sum_{k=0}^{n-1}a_k p^k e^{-\tau_k p}}. \qquad (1.16)$$

The original for the image (1.16) defines the solution of equation (1.13) satisfying the conditions (1.14).

Let's formulate a problem for an equation with a delayed argument describing a *process with an aftereffect*. It is required to find a continuously differentiable solution $x(t)$ for $t>t_0$ of the equation

$$x'(t)=f\left(t,x(t),x(t-\tau)\right),\ \tau=\text{const}>0\,, \qquad (1.17)$$

if it is known that

$$x(t)=\varphi(t),\ t\in[t_0-\tau,t_0]\,. \qquad (1.18)$$

The initial function $\varphi(t)$ is a given continuously differentiable function.

The segment $\left[t_0-\tau,t_0\right]$ on which the function $\varphi(t)$ is defined is called the *initial set.*

If equation (1.17) is linear, then its solution satisfying condition (1.18) can be found using the Laplace transform. Let $t_0=0$, then when constructing the corresponding operator equation, it should be taken into account that for the image of the function $x(t-\tau)$ we have

$$x(t-\tau)\leftrightarrow\int_0^\infty e^{-pt}x(t-\tau)dt=\int_{-\tau}^\infty e^{-p(\eta+\tau)}x(\eta)d\eta=$$

$$=\int_{-\tau}^0 e^{-p(\eta+\tau)}x(\eta)d\eta+\int_0^\infty e^{-p(\eta+\tau)}x(\eta)d\eta=e^{-p\tau}\int_{-\tau}^0 e^{-p\eta}\varphi(\eta)d\eta+e^{-p\tau}X(p).$$

When restoring originals from known images, you can use the following decomposition:

$$\frac{1}{1-\dfrac{\gamma e^{-np}}{(p+a)^m}}=1+\frac{\gamma e^{-np}}{(p+a)^m}+\left(\frac{\gamma e^{-np}}{(p+a)^m}\right)^2+\ldots=\sum_{k=0}^{\infty}\left(\frac{\gamma e^{-np}}{(p+a)^m}\right)^k, \quad (1.19)$$

which is true for any $n, m\in N$, on condition $\operatorname{Re} p>0$.

Example 1

Find a solution of the equation

$$x'(t)=x(t-1)+1,\ x(0)=0.$$

<u>Solution:</u>

Assuming that $x(t)\equiv 0$ for $t\in[-1,0]$, applying the Laplace transform to a given equation, we obtain the following operator equation:

$$pX(p)=X(p)e^{-p}+\frac{1}{p}.$$

Where from

$$X(p)=\frac{1}{p}\frac{1}{pe^{-p}}=\frac{1}{p^2}\frac{1}{1-\dfrac{e^{-p}}{p}}.$$

Further, applying the formula (1.19), we have

$$X(p)=\frac{1}{p^2}\sum_{k=0}^{\infty}\left(\frac{e^{-p}}{p}\right)^k=\sum_{k=0}^{\infty}\frac{e^{-pk}}{p^{k+2}}.$$

Given the delay property, we construct an expression for the corresponding original $x(t)$ in the form

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\theta(t-k).$$

Example 2

Find a solution of the equation

$$x'(t)=x(t-1),$$

if $x(t)\equiv 2$ for $\forall t\in[-1,0]$.

Solution:

Let $x(t)\leftrightarrow X(p)$.

It follows from the condition that $x(0)=2$, so we have

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-2.$$

We apply the Laplace transform to both parts of the given equation. For the right side of the equation, we have

$$x(t-1)\leftrightarrow\int_0^{\infty}e^{-pt}x(t-1)dt=\int_{-1}^{\infty}e^{-p(z+1)}x(z)dz=$$

$$=\int_{-1}^{0}e^{-p(z+1)}x(z)dz+\int_{0}^{\infty}e^{-p(z+1)}x(z)dz=$$

$$=2\int_{-1}^{0}e^{-p(z+1)}dz+e^{-p}X(p)=\frac{2}{p}\left(1-e^{-p}\right)+e^{-p}X(p).$$

Therefore, the corresponding operator equation has the form

$$pX(p)-2=\frac{2}{p}\left(1-e^{-p}\right)+e^{-p}X(p).$$

From here we get

$$X(p)=2\frac{p+1-e^{-p}}{p^2-pe^{-p}}=\frac{2}{p}+\frac{2}{p(p-e^{-p})}.$$

Using the result of the previous example, we will construct the original in the form

$$x(t)=2\left(\theta(t)+\sum_{k=0}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\theta(t-k)\right).$$

Example 3

Find a solution of the equation

$$x'(t)+2x(t)-x(t-1)=f(t),$$

if $x(0)=0$ and $x(t)\equiv 0$ for $\forall t<0$.

<u>Solution:</u>

Let $x(t)\leftrightarrow X(p)$, $f(t)\leftrightarrow F(p)$.

Since under the given conditions $x(t-1)\leftrightarrow e^{-p}X(p)$, the operator equation corresponding to the given one has the form

$$pX(p)+2X(p)-e^{-p}X(p)=F(p).$$

The solution of this equation is written as a product:

$$X(p)=\frac{1}{p+2-e^{-p}}F(p).$$

Let's build the original for the function

$$Y(p)=\frac{1}{p+2-e^{-p}},$$

by performing the following transformations:

$$Y(p)=\frac{1}{p+2-e^{-p}}=\frac{1}{p+2}\left(\frac{1}{1-\dfrac{e^{-p}}{p+2}}\right)=\sum_{k=0}^{\infty}\frac{e^{kp}}{(p+2)^{k+1}}.$$

The last equality is written, taking into account the formula (1.19). Turning to the originals for the summands of the sum of the series, using $t^k \leftrightarrow \dfrac{k!}{p^{k+1}}$, $t<0$ and the delay property, we find

$$Y(p)\leftrightarrow y(t)=\sum_{k=0}^{\infty}\frac{(t-k)^k}{k!}e^{-2(t-k)}\theta(t-k).$$

The solution to this problem will be the function $x(t)$, which is a convolution of the functions $f(t)$ and $y(t)$:

$$x(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^t f(t-\tau)(\tau-k)^k e^{-2(\tau-k)}\theta(\tau-k)d\tau.$$

1.5. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF INTEGRAL EQUATIONS AND SYSTEMS

1.5.1.The Volterra equation of the second kind

Consider the Volterra linear integral equation of the second kind with a kernel $K(t)$ of the form

$$y(t) = f(t) + \int_0^t K(t-\tau)y(\tau)d\tau, \qquad (1.20)$$

where $K(t), f(t)$ - given functions, $y(t)$ - the desired function.

Let $y(t) \leftrightarrow Y(p)$, $f(t) \leftrightarrow F(p)$, $K(t) \leftrightarrow K^*(p)$. Passing to the images in equation (1.20) and using the convolution image property, we obtain the corresponding operator equation

$$Y(p) = F(p) + K^*(p)Y(p).$$

From here

$$Y(p) = \frac{F(p)}{1 - K^*(p)}.$$

The original for image $Y(p)$ has the desired solution to equation (1.20).

1.5.2. The Volterra equation of the first kind

Consider a linear integral Volterra equation of the first kind with a kernel $K(t)$ of the form

$$\int_0^t K(t-\tau)y(\tau)d\tau = f(t), \qquad (1.21)$$

where $K(t), f(t)$ - given functions, $y(t)$ - the desired function.

Let $y(t) \leftrightarrow Y(p)$, $f(t) \leftrightarrow F(p)$, $K(t) \leftrightarrow K^*(p)$. Then, applying the Laplace transform to equation (1.21), we obtain the operator equation.

$$K^*(p)Y(p) = F(p) \;\Rightarrow\; Y(p) = \frac{F(p)}{K^*(p)}.$$

The original for $Y(p)$ gives the desired solution to equation (1.21).

1.5.3.Systems of Volterra integral equations

Consider a system of Volterra integral equations of the form

$$y_i(t) = f_i(t) + \sum_{k=1}^{s} \int_0^t K_{ik}(t-\tau) y_k(\tau) d\tau \,,\; i = 1,2,\dots s \,, \qquad (1.22)$$

where $K_{ik}(t)$, $f_i(t)$ - given functions, $i,k = 1,2,\dots,s$.

Let

$$F_i(p) \leftrightarrow f_i(t) \,,\; K_{ik}^*(p) \leftrightarrow K_{ik}(t) \,,\; Y_i(p) \leftrightarrow y_i(t) \,.$$

Applying the Laplace transform to both parts of the equations (1.22), we obtain a system of operator equations

$$Y_i(p) = F_i(p) + \sum_{k=1}^{s} K_{ik}^*(p) Y_k(p) \,,\; i = 1,2,\dots s \,, \qquad (1.23)$$

linear with respect to the images $Y_i(p)$. Solving the system (1.23), we find $Y_i(p)$, the originals for which will be the solution of the original system of integral equations (1.22).

What will be useful to us:

$\int\limits_0^\infty y(t)e^{-pt}\,dt = Y(p)$ - Laplace transform

$\int\limits_0^\infty y'(t)e^{-pt}\,dt = pY(p) - y(0)$ - Laplace transform of the derivative

$\int\limits_0^\infty y^{(n)}(t)e^{-pt}\,dt = p\int\limits_0^\infty y^{(n-1)}(t)e^{-pt}dt - y^{(n-1)}(0)$ - Laplace transform from high order derivatives

$\int\limits_0^\infty \left(\int\limits_0^t y(\tau)\,d\tau\right)e^{-pt}\,dt = \dfrac{Y(p)}{p}$ - Laplace transform from the integral

$\int\limits_0^\infty \left(\int\limits_0^x g(x-t)\,y(t)\,dt\right)e^{-px}\,dx = G(p)Y(p)$ - Laplace transform from a convolution type integral

Let we have $K(x-t)$ - the kernel of the integral operator (difference kernel).

The Volterra integral equation of the second kind looks like this:

$$y(x) = f(x) + \int\limits_0^x K(x-t)\,f(t)\,dt$$

Let's move everything to the left side:

$$y(x) - f(x) - \int\limits_0^x K(x-t)\,f(t)\,dt = 0$$

$$\int_0^{\infty}\left[y(x)-f(x)-\int_0^{x}K(x-t)f(t)dt\right]e^{-px}dx=0$$

We have obtained three Laplace transformations:

$$\int_0^{\infty}y(x)e^{-px}dx=\int_0^{\infty}f(x)e^{-px}dx+\int_0^{\infty}\left(\int_0^{x}K(x-t)y(t)dt\right)e^{-px}dx$$

The image of the first integral $Y(p)$, the second is $F(p)$, and the third is a convolution type integral.

The integral equation turns into an algebraic equation for images:

$$Y(p)=F(p)+K^{*}(p)Y(p)$$

$$Y(p)=\frac{F(p)}{1-K^{*}(p)}\ ,$$

where $F(p),K^{*}(p)-$ we know.

Example 1

Solve the integral equation

$$y(x) = \sin x + \int_0^x (x-t)y(t)dt .$$

<u>Solution:</u>

Let $y(x) \leftrightarrow Y(p)$.

Since the integral included in the given equation is a convolution of two functions t and $y(t)$, then its image will be the product of images of these functions, that is $\frac{1}{p^2}Y(p)$. Applying the Laplace transform to the equation, we obtain the following operator equation:

$$Y(p) = \frac{1}{p^2+1} + \frac{1}{p^2}Y(p) .$$

His solution has the form

$$Y(p) = \frac{p^2}{(p^2-1)(p^2+1)} = \frac{p}{(p^2-1)}\frac{p}{(p^2+1)} .$$

Since

$$\frac{p}{\left(p^2-1\right)} \leftrightarrow \operatorname{ch} x , \ \frac{p}{\left(p^2+1\right)} \leftrightarrow \cos x ,$$

then the original corresponding to the image $Y(p)$ is a convolution of two functions — $\operatorname{ch} x$ and $\cos x$:

$$y(x) = \int_0^x \operatorname{ch}(x-t)\cos t dt$$

Having calculated the integral, we get the desired solution:

$$y(x)=\frac{1}{2}\sin x+\frac{1}{4}e^{x}-\frac{1}{4}e^{-x}.$$

Example 2

Solve the integral equation

$$y(x)=\cos x+\int_0^x (x-t)y(t)dt;$$

<u>Solution:</u>

$$\cos x \leftrightarrow \frac{p}{p^2+1}$$

$$Y(p)=\frac{p}{p^2+1}+\frac{1}{p^2}Y(p)$$

$$Y(p)=\frac{p^3}{(p^2-1)(p^2+1)}=p\cdot\frac{p}{p^2-1}\cdot\frac{p}{p^2+1}$$

$$\frac{p}{p^2-1}\leftrightarrow ch(t)=f_1(t)$$

$$\frac{p}{p^2+1}\leftrightarrow \cos(t)=f_2(t)$$

We found the original corresponding to the image using the Duhamel integral.

Duhamel integral:

$$f(t)=f_1(0)f_2(t)+\int_0^t f_1'(\tau)f_2(t-\tau)d\tau$$

We have

$$y(t)=1\cdot cos(t)+\int_0^t sh(\tau)cos(t-\tau)d\tau$$

Using twice integration by parts in the integral, we have

$$\int sh(\tau)cos(\tau-t)d\tau=\frac{sh(\tau)sin(\tau-t)+ch(\tau)cos(\tau-t)}{2}+C=$$

$$=\frac{e^{-\tau}\left[\left(e^{2\tau}-1\right)sin(\tau-t)+\left(e^{2\tau}+1\right)cos(\tau-t)\right]}{2}+C$$

And

$$\int_0^t sh(\tau)cos(\tau-t)d\tau=\frac{e^{-t}\left(e^{2t}+1\right)}{4}-\frac{cos(t)}{2}$$

$$y(t)=1\cdot cos(t)+\frac{e^{-t}\left(e^{2t}+1\right)}{4}-\frac{cos(t)}{2}=\frac{e^{-t}\left(e^{2t}+1\right)}{4}+\frac{cos(t)}{2}$$

Our solution is

$$y(x)=\frac{e^{-x}\left(e^{2x}+1\right)}{4}+\frac{cos(x)}{2}$$

Example 3

Solve a system of integral equations:

$$\begin{cases} y(x) = e^x + \int\limits_0^x y(t)dt - \int\limits_0^x e^{(x-t)} z(t)dt, \\ z(x) = -x - \int\limits_0^x (x-t) y(t)dt - \int\limits_0^x z(t)dt. \end{cases}$$

<u>Solution:</u>

Let $y(x) \leftrightarrow Y(p)$, $z(x) \leftrightarrow Z(p)$.

We apply the Laplace transform to each equation of the system. Using the properties on the integration of the original and on convolution to construct images of the original equations, we obtain

$$\begin{cases} Y(p) = \dfrac{1}{p-1} + \dfrac{Y(p)}{p} - \dfrac{Z(p)}{p-1}, \\ Z(p) = -\dfrac{1}{p^2} - \dfrac{Y(p)}{p^2} - \dfrac{Z(p)}{p}. \end{cases}$$

Solving a system of algebraic equations, we find the images

$$Y(p) = \frac{1}{p-2},$$

$$Z(p) = -\frac{1}{p(p-2)} = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{p-2}\right),$$

which correspond to the originals:

$$y(x) = e^{2x},$$

$$z(x) = \frac{1}{2} - \frac{1}{2}e^{2x}.$$

Example 4

Solve the integral-differential equation

$$y''(x)+2y'(x)-2\int_0^x \sin(x-t)y'(t)dt=\cos x\,,\ \ y(0)=y'(0)=0\,.$$

<u>Solution:</u>

Let $y(x)\leftrightarrow Y(p)$.

We apply the Laplace transform to a given equation:

$$p^2Y(p)+2pY(p)-2\frac{1}{p^2+1}pY(p)=\frac{p}{p^2+1}\,.$$

Solving the equation with respect to $Y(p)$, we obtain

$$Y(p)=\frac{1}{p(p+1)^2}=\frac{1}{p}-\frac{1}{(p+1)^2}-\frac{1}{p+1}\leftrightarrow y(x)=1-e^{-x}x-e^{-x}\,.$$

In that way

$$y(x)=1-e^{-x}x-e^{-x}\,.$$

Example 5

Solve the integral-differential equation

$$y''(x)-2y'(x)+y(x)+2\int_0^x \cos(x-t)y''(t)dt+$$

$$+2\int_0^x \sin(x-t)y'(t)dt=\cos x\,,\ \ y(0)=y'(0)=0\,.$$

<u>Solution:</u>

Let $y(x) \leftrightarrow Y(p)$.

We apply the Laplace transform to a given equation:

$$p^2Y(p) - 2pY(p) + Y(p) + 2\frac{p}{p^2+1}p^2Y(p) + \frac{2}{p^2+1}pY(p) = \frac{p}{p^2+1}.$$

Solving the equation with respect to $Y(p)$, we obtain

$$Y(p) = \frac{p}{p^2+1}\frac{1}{p^2+1} \leftrightarrow y(x) = \int_0^x \cos(x-t)\sin t dt = \frac{x}{2}\sin x.$$

When switching to the original, the convolution image property was used.

1.5. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF INTEGRAL EQUATIONS AND SYSTEMS

Example 1

Solve the integral equation

$$\int_0^t e^{2(t-\tau)} y(\tau)\,d\tau = t^2 e^t$$

<u>Solution:</u>

In this equation $K(t-\tau) = e^{2(t-\tau)}$, therefore $K(t) = e^{2t}$.

Let's find an image of this function $\frac{1}{p-2} \leftrightarrow e^{2t}$.

Let's find the image of the right side of the equation, that is, the function $t^2 e^t$:

$$\frac{2}{(p-1)^3} \leftrightarrow t^2 e^t$$

Let's write down the equation

$$Y(p)\frac{1}{p-2} = \frac{2}{(p-1)^3}$$

From here

$$Y(p) = \frac{2p-4}{(p-1)^3}$$

Using the method of undetermined coefficients, we will find the decomposition of a fraction into the simplest fractions:

$$Y(p) = \frac{2p-4}{(p-1)^3} = \frac{A}{(p-1)^3} + \frac{B}{(p-1)^2} + \frac{C}{p-1}$$

Let's bring the right part to the common denominator and equate the numerators of the resulting and the original fraction:

$$A+B(p-1)+C(p-1)^2 \equiv 2p-4$$

Let's find the coefficients A, B, C.

$$p=1 \qquad A=-2$$

$$p=2 \qquad A+B+C=0$$

$$p=0 \qquad A-B+C=-4$$

$$2B=4 \quad \Rightarrow \quad B=2;$$

$$2A+2C=-4 \quad \Rightarrow \quad A+C=-2 \quad \Rightarrow \quad C=0$$

In that way

$$Y(p)=\frac{2p-4}{(p-1)^3}=-\frac{2}{(p-1)^3}+2\frac{1}{(p-1)^2}$$

We will find the original corresponding to the image:

$$y(t)=-t^2e^t+2te^t=te^t(2-t)$$

So, the solution of this integral equation is the function

$$y(t)=te^t(2-t)$$

Example 2

Solve the integral equation

$$y(t)=1+t+\int_0^t cos(t-\tau)y(\tau)d\tau$$

<u>Solution:</u>

In this case $\qquad f(t)=1+t \quad \Rightarrow \quad F(p)=\frac{1}{p}+\frac{1}{p^2}$;

$$K(t)=cos\,t \quad \Rightarrow \quad K^*(p)=\frac{p}{p^2+1}$$

The integral $\int_0^t \cos(t-\tau)\,y(\tau)\,d\tau$ is a convolution of the function $\cos t$ and $y(t)$.

Image of the equation:

$$Y(p)=\frac{1}{p}+\frac{1}{p^2}+Y(p)\frac{p}{p^2+1}$$

We'll find $Y(p)$.

$$Y(p)\left(1-\frac{p}{p^2+1}\right)=\frac{1}{p}+\frac{1}{p^2} \quad \Rightarrow$$

$$Y(p)\frac{p^2-p+1}{p^2+1}=\frac{p+1}{p^2} \quad \Rightarrow$$

$$Y(p)=\frac{(p^2+1)(p+1)}{p^2(p^2-p+1)}$$

Let's imagine the image of the solution as the sum of the simplest fractions:

$$Y(p)=\frac{(p^2+1)(p+1)}{p^2(p^2-p+1)}=\frac{A}{p^2}+\frac{B}{p}+\frac{Cp+D}{p^2-p+1}$$

Let's find the decomposition coefficients.

$$A(p^2-p+1)+Bp(p^2-p+1)+Cp^3+Dp^2 \equiv p^3+p^2+p+1$$

We equate the coefficients with the same degrees p in the right and left parts of the identity:

p^3	$B+C=1$
p^2	$A-B+D=1$
p	$-A+B=1$
p^0	$A=1$

We will get $A=1, B=2, C=-1, D=2.$

Therefore, the decomposition has the form:

$$Y(p)=\frac{\left(p^2+1\right)(p+1)}{p^2\left(p^2-p+1\right)}=\frac{1}{p^2}+\frac{2}{p}+\frac{2-p}{p^2-p+1}\ .$$

Let's convert all fractions into table fractions:

$$Y(p)=\frac{1}{p^2}+\frac{2}{p}+\frac{2-p}{p^2-p+1}=\frac{1}{p^2}+2\frac{1}{p}+\frac{\frac{3}{2}+\frac{1}{2}-p}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$=\frac{1}{p^2}+2\frac{1}{p}+\sqrt{3}\frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}$$

We will find the corresponding original:

$$y(t)=t+2+e^{\frac{t}{2}}\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}t-\cos\frac{\sqrt{3}}{2}t\right)$$

This function is the solution of a given integral equation.

Example 3

Solve the integral equation

$$y(t)=cos\,t-\int_0^t e^{t-\tau}y(\tau)\,d\tau$$

Solution:

Here $f(t)=\cos t;\quad K(t)=e^t$.

Since $\int_0^t e^{t-\tau}y(\tau)\,d\tau$ is a convolution of functions e^t and $y(t)$, that is,

$$e^t * y(t) = \int_0^t e^{t-\tau} y(\tau)\,d\tau \leftrightarrow K^*(p)Y(p)\text{, where}$$

$$K(t) \leftrightarrow K^*(p) = \frac{1}{p-1},\ \ f(t) \leftrightarrow F(p) = \frac{p}{p^2+1};$$

$$y(t) \leftrightarrow Y(p).$$

The image of the integral equation takes the form

$$Y(p) = \frac{p}{p^2+1} - \frac{1}{p-1} Y(p)$$

$$Y(p) + \frac{1}{p-1} Y(p) = \frac{p}{p^2+1} \quad \Rightarrow$$

$$Y(p) \frac{p}{p-1} = \frac{p}{p^2+1} \quad \Rightarrow$$

$$Y(p) = \frac{p-1}{p^2+1}$$

Based on the image of the solution, we will find its original:

$$Y(p) = \frac{p-1}{p^2+1} = \frac{p}{p^2+1} - \frac{1}{p^2+1} \quad \leftrightarrow \quad \cos t - \sin t = y(t)$$

So, the solution of the integral equation is the function:

$$y(t) = \cos t - \sin t$$

Example 4 (HOMEWORK 5, the deadline is September 24th)

№1	№2
$y(x) = x + \int_0^x \sin(x-t) y(t)\,dt$;	$y''(x) - 2y'(x) + y(x) + 2\int_0^x \cos(x-t) y''(t)\,dt +$ $+2\int_0^x \sin(x-t) y'(t)\,dt = \sin x,\ \ y(0) = y'(0) = 0$;

2. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

2.1. PARTIAL DIFFERENTIAL EQUATIONS

Denote by D the region of the n -dimensional space R^n of points $x=(x_1,x_2,...,x_n)$, $x_1,x_2,...,x_n$, $n \geq 2$ — Cartesian coordinates of point x.

An equation of the form

$$F\left(x,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},...,\frac{\partial^k u}{\partial x_1^{i_1}...\partial x_n^{i_n}},\frac{\partial u}{\partial x_n^m}\right)=0,\ x\in D, \qquad (2.1)$$

$$\sum_{j=1}^{n} i_j = k,\ k=0,1,...,m,\ m\geq 1$$

is called a *partial differential equation of the order* m with respect to an unknown function $u=u(x)$, where $F=F\left(x,u,\frac{\partial u}{\partial x_1},...\right)-$ is a given real function of points $x\in D$, an unknown function u and its partial derivatives. The left side of equality (2.1) is called a *partial differential operator of order* m.

The real function $u=u(x_1,x_2,...,x_n)$, defined in the domain D of the assignment of equation (2.1), continuous together with its partial derivatives included in this equation and converting it into an identity, is called the *classical (regular) solution* of equation (2.1).

Equation (2.1) is called linear if F depends linearly on all variables of the form

$$\frac{\partial^k u}{\partial x_1^{i_1}...\partial x_n^{i_n}},\ 0\leq k\leq m.$$

The linear equation can be written as

$$\sum_{k=0}^{m}\sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}(x)\frac{\partial^k u}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}} = f(x)\,,\ \sum_{j=1}^{n} i_j = k\ ,\ x\in D$$

or in the form of

$$Lu = f(x)\,,\ x\in D\,,$$

where $L-$ linear differential operator of order m:

$$L \equiv \sum_{k=0}^{m}\sum_{i_1,\ldots,i_n} a_{i_1\ldots i_n}(x)\frac{\partial^k}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}\ ,\ \sum_{j=1}^{n} i_j = k\ .$$

A linear equation is called homogeneous if $f(x)\equiv 0$, inhomogeneous if $f(x)\neq 0$.

Equation (2.1) of order m is called quasilinear if F linearly depends only on partial derivatives of order m:

$$\frac{\partial^m u}{\partial x_1^{i_1}\ldots\partial x_n^{i_n}}\ ,\ \sum_{j=1}^{n} i_j = m\ .$$

Control tasks

Example 1

Find out if the following equalities are partial differential equations:

1. $\cos(u_x+u_y)-\cos u_x\cos u_y+\sin u_x\sin u_y=0$;

Solution:

1. By converting the cosine of the sum into the product of cosines and sines, we obtain the identity

$$\cos(u_x+u_y)=\cos u_x\cdot\cos u_y-\sin u_x\cdot\sin u_y$$

$$\cos u_x \cdot \cos u_y - \sin u_x \cdot \sin u_y - \cos u_x \cdot \cos u_y + \sin u_x \cdot \sin u_y = 0$$

$$0 = 0$$

This is not a differential equation.

2. $u_{xx}^2 + u_{yy}^2 - \left(u_{xx} - u_{yy}\right)^2 = 0$;

<u>Solution:</u>

We open the brackets, give similar terms, and get

$$u_{xx}^2 + u_{yy}^2 - \left(u_{xx}^2 - 2u_{xx}u_{yy} + u_{yy}^2\right) = 0$$

$$u_{xx}^2 + u_{yy}^2 - u_{xx}^2 + 2u_{xx}u_{yy} - u_{yy}^2 = 0$$

$$2u_{xx}u_{yy} = 0$$

Equation (2) is a differential equation.

3. $\sin^2\left(u_{xx} + u_{xy}\right) + \cos^2\left(u_{xx} + u_{xy}\right) - u = 1$;

<u>Solution:</u>

Using the basic trigonometric identity, we obtain

$$\sin^2\left(u_{xx} + u_{xy}\right) + \cos^2\left(u_{xx} + u_{xy}\right) = 1$$

$$1 - u = 1$$

$$-u = 0$$

Equation (3) is not a differential equation.

Example 2

Determine the order of the equations:

1. $\ln\left|u_{xx}u_{yy}\right| - \ln\left|u_{xx}\right| - \ln\left|u_{yy}\right| + u_x + u_y = 0$

Solution:

1. Converting the sum of the logarithms into the logarithm of the product and giving similar terms, we get

$$\ln\left|u_{xx}\right| + \ln\left|u_{yy}\right| = \ln\left|u_{xx}\,u_{yy}\right|$$

$$\ln\left|u_{xx}u_{yy}\right| - \ln\left|u_{xx}u_{yy}\right| + u_x + u_y = 0$$

$$u_x + u_y = 0$$

The order of the differential equation (1) is the first.

2. $u_x u_{xy}^2 + \left(u_{xx}^2 - 2u_{xy}^2 + u_y\right)^2 - 2xy = 0$

Solution:

We open the brackets, give similar terms, and get

$$\left(u_{xx}^2 - 2u_{xy}^2 + u_y\right)^2 = u_{xx}^4 - 4u_{xx}^2u_{xy}^2 + 4u_{xy}^4 + 2u_{xx}^2u_y - 4u_{xy}^2u_y + u_y^2$$

$$u_x u_{xy}^2 + u_{xx}^4 - 4u_{xx}^2u_{xy}^2 + 4u_{xy}^4 + 2u_{xx}^2u_y - 4u_{xy}^2u_y + u_y^2 - 2xy = 0$$

The order of the differential equation (2) is second.

Example 3

Find out which of the following equations are linear and which are nonlinear (quasilinear):

1. $2\sin(x+y)u_{xx} - x\cos y u_{xy} + xyu_x - 3u + 1 = 0$;

Solution:

In this equation, the coefficients before the second and first derivatives are functions of x and y, so the equation is linear. The function $f(x,y)=1$, that is, the equation is inhomogeneous.

Equation (1) is linear and inhomogeneous.

2. $x^2yu_{xxy} + 2e^x y^2 u_{xy} - (x^2y^2+1)u_{xx} - 2u = 0$;

Solution:

In this equation, the coefficients before the second and third derivatives are functions of x and y, so the equation is linear, of the third order.

The function $f(x,y)=0$, that is, the equation is homogeneous.

Equation (2) is linear and homogeneous.

3. $3u_{xy} - 6u_{xx} + 7u_y - u_x + 8x = 0$;

Solution:

In this equation, the coefficients before the second and first derivatives are constant values, so the equation is linear with constant coefficients. The function $f(x,y)=8x$, that is, the equation is inhomogeneous.

Equation (3) is linear with constant coefficients and inhomogeneous.

4. $u_x u_{xy}^2 + 2xuu_{yy} - 3xyu_y - u = 0$;

<u>Solution:</u>

The first term can be represented as follows: $u_x u_{xy}^2 = u_x u_{xy} u_{xy}$. It was found that the coefficient before the highest second derivative also depends on the second derivative u_{xy}, that is, the equation is nonlinear.

Equation (4) is nonlinear.

2.2. EXAMPLES OF THE SIMPLEST PARTIAL DIFFERENTIAL EQUATIONS

Let's look at some examples of partial differential equations.

Example 1

Find the function $u = u(x, y)$ satisfying the differential equation:

$$\frac{\partial u}{\partial x} = 1$$

<u>Solution:</u>

Integrating, we get

$$u = x + \varphi(y),$$

where $\varphi(y)$ - an arbitrary function. This is the general solution of this differential equation.

Example 2

Solve the equation

$$\frac{\partial^2 u}{\partial y^2} = 6y,$$

where $u = u(x, y)$.

<u>Solution:</u>

Integrating twice by y, we get

$$\frac{\partial u}{\partial y} = 3y^2 + \varphi(x),$$

$$u = y^3 + y\varphi(x) + \psi(x),$$

where $\varphi(x)$ and $\psi(x)$ an arbitrary functions.

Example 3

Solve the equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0.$$

Solution:

Integrating the equation with respect to x, we have

$$\frac{\partial u}{\partial y} = f(y).$$

Integrating obtained result by y, we find

$$u = \varphi(x) + \psi(y),$$

where $\psi(y) = \int f(y)dy$, $\varphi(x)$ and $\psi(y)$ - arbitrary functions.

Example 4

Solve the equation

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 2x \frac{\partial u}{\partial y} = 0, \; x \neq 0.$$

Solution:

This equation can be reduced to the form

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial y} \right) = 0 .$$

Integrating the equation with respect to the variable x, we obtain

$$x^2 \frac{\partial u}{\partial y} = f(y) ,$$

where $f(y)$ - arbitrary function.

Integrating the result obtained with respect to the variable y, we find

$$u = \varphi(x) + \psi(y) ,$$

where $\psi(y) = \frac{1}{x^2} \int f(y) dy$, $\varphi(x)$ and $\psi(y)$ - are arbitrary functions.

2.3. FIRST-ORDER DIFFERENTIAL EQUATIONS, LINEAR WITH RESPECT TO PARTIAL DERIVATIVES

Partial differential equation of the first order

Some problems of classical mechanics, continuum mechanics, acoustics, optics, hydrodynamics, and radiation transfer are reduced to partial differential equations of the first order. Analytical methods developed in the theory of ordinary differential equations are applicable to the solution of some of them.

1. Basic concepts. Classification of equations

A partial differential equation of the first order is an equation of the form [1]

$$F\left(x_1, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right) = 0, \qquad (1)$$

where $x_1, \dots, x_n$ – are independent variables, $u = u(x_1, \dots, x_n)$ – is an unknown function, $F(x_1, x_2, \dots, x_n, u, p_1, p_2, \dots, p_n)$ – is a given continuously differentiable function [2] in some region $G \subset \mathbb{R}^{2n+1}$, and at each point in the region G

$$\sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i}\right)^2 \neq 0.$$

Equation (1) can be abbreviated as [3]

$$F(x, u, \operatorname{grad} u) = 0, \qquad (1')$$

where $x=\left(x_1,...,x_n\right)$ and $\operatorname{grad} u=\left(\dfrac{\partial u}{\partial x_1},...,\dfrac{\partial u}{\partial x_n}\right)$.

---

[1] Next, another entry of the first-order partial derivative will be used too. For example, for the function $u=u\left(x,y,z\right)$:

$$u'_x=\frac{\partial u}{ux},\ u'_y=\frac{\partial u}{\partial y},\ u'_z=\frac{\partial u}{\partial z}.$$

[2] Here, for the function F, the arguments $p_i,\ i=1,...,n,$ denote partial derivatives u'_{x_i}.

[3] Often, to write the gradient of the function $u\left(x\right)$, the operator ∇ («nabla») is used, that is, instead of $\operatorname{grad} u$, they write ∇u. Then equation $(1')$ can be written in the form $F\left(x,u,\nabla u\right)=0$.

Definition 1. The function $u=\varphi\left(x_1,...,x_n\right)$ given in the domain $D\subset\mathbb{R}^n$ is called the **solution of equation** (1) if:

1) $\varphi\left(x_1,...,x_n\right)-$ is a continuously differentiable function in D,

2) for all points $x=\left(x_1,...,x_n\right)\in D$ point $\left(x,\varphi,\varphi'_{x_1},...,\varphi'_{x_n}\right)\in G$,

3) $F\left(x_1,...,x_n,\varphi,\varphi'_{x_1},...,\varphi'_{x_n}\right)\equiv 0$ for any $\left(x_1,...,x_n\right)\in D$.

The solution of equation (1) in the $\left(n+1\right)$ - dimensional space of variables $x_1,...,x_n,u$ defines some smooth surface of dimension n, which is called the **integral surface of equation** (1).

Depending on how the unknown function u and its partial derivatives enter equation (1), **linear** and **nonlinear** equations are distinguished.

A linear partial differential equation of the first order is an equation of the form

$$A_1(x)\frac{\partial u}{\partial x_1}+\ldots+A_n(x)\frac{\partial u}{\partial x_n}=B(x)u+f(x), \qquad (2)$$

where $A_i(x)$ $(i=1,...,n)$, $B(x)$ and $f(x)$ - the given functions of point $x=(x_1,...,x_n)\in D$, moreover, $\sum_{i=1}^{n}A_i^2(x)\neq 0$ for any $x\in D$.

The functions A_i and B are called the coefficients of the equation. The linearity of the equation is determined by the fact that the unknown function $u(x)$ and all its partial derivatives enter the equation linearly.

If $f(x)\equiv 0$, then equation (2) is called **homogeneous**, otherwise- **inhomogeneous**.

If equation (1) cannot be written as (2), then it is called **nonlinear**. If in it the function F is linear with respect to all derivatives of the unknown function $\frac{\partial u}{\partial x_i}$, then equation (1) is called **quasi-linear**. The quasi-linear equation can be written as follows

$$A_1(x,u)\frac{\partial u}{\partial x_1}+\ldots+A_n(x,u)\frac{\partial u}{\partial x_n}=B(x,u). \qquad (3)$$

2. Homogeneous linear equation

Let the point $x=(x_1,...,x_n)$ belong to the domain $D\subset\mathbb{R}^n$, $n\geq 2$. In the domain D, consider a homogeneous linear partial differential equation of the first order of the form

$$A_1(x_1,\dots,x_n)\frac{\partial u}{\partial x_1}+\dots+A_n(x_1,\dots,x_n)\frac{\partial u}{\partial x_n}=0. \qquad (4)$$

Let the coefficients $A_i(x)$ $(i=1,\dots,n)-$ be continuously differentiable functions in D, for which

$$\sum_{i=1}^{n} A_i^2(x)\neq 0, \quad \forall\, x\in D.$$

Equation (4) can be given the following geometric interpretation. If we consider the coefficients $A_i(x)$ to be components of the vector $\mathbf{A}(x)$ in n-dimensional space, then equation (4) means that the derivative of the function $u(x)$ is equal to zero along the direction of vector $\mathbf{A}$.

Obviously, equation (4) has a solution of the form $u\equiv C$, where C is a constant. But equation (4) also has infinitely many solutions other than the constant.

For example, the solution to the equation $\frac{\partial u}{\partial x_1}=0$ is any continuous function Φ that does not depend on x_1, that is, $u(x_1,\dots,x_n)=\Phi(x_2,\dots,x_n)$ (this solution is obtained by integrating the equation with respect to the variable x_1). In general, the search for solutions to equation (4) is reduced to constructing solutions to a system of ordinary differential equations.

Let us compare equation (4) with a system of ordinary differential equations called **equations of characteristics:**

$$\frac{dx_1}{A_1(x_1,\ldots,x_n)} = \frac{dx_2}{A_2(x_1,\ldots,x_n)} = \ldots = \frac{dx_n}{A_n(x_1,\ldots,x_n)}. \qquad (5)$$

This system is called a system of differential equations in a *symmetric form* corresponding to a homogeneous linear partial differential equation (4) (or a **characteristic system**).

In the case of two independent variables, it consists of a single equation. Under the assumptions made regarding the coefficients $A_1(x_1,\ldots,x_n),\ldots,A_n(x_1,\ldots,x_n)$, system (5) has exactly $n-1$ independent first integrals.

Definition 2. The first integral of the system (5) is called the function $\psi(x_1,\ldots,x_n)$, which differs from the constant, which is identically equal to some constant at all points $(x_1,\ldots,x_n)$ of the integral curve of the system (5).

Often, the first integral is not called the function ψ, but the ratio $\psi = C$, where C is a constant.

The integral curves of the system of equations (5) are called the **characteristics of the partial differential equation** (4).

THEOREM 1. Every solution $\varphi(x_1,...,x_n)$ of equation (4) is the first integral of system (5), and, conversely, every first integral $\psi(x_1,...,x_n)$ of system (5) is the solution of equation (4).

For example, the equation

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0 \qquad (*)$$

corresponds to the system of differential equations

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{-z},$$

which has the following integrals

$$\psi_1 = xz, \qquad \psi_2 = x\sqrt{y}.$$

Then the functions $u_1 = xz$ and $u_2 = x\sqrt{y}$ are solutions to this equation. You can verify by substituting the expressions for u_1 and u_2 into the equation (*).

Let's assume that $A_n(x) \neq 0$ in D and $n-1$ independent first integrals of the system (5) are found:

$$\psi_1(x_1,\ldots,x_n),\ldots,\psi_{n-1}(x_1,\ldots,x_n). \qquad (6)$$

The condition for the independence of integrals (6) is the difference from the zero of the Jacobian[4]

$$J = \frac{D(\psi_1, \dots, \psi_{n-1})}{D(x_1, \dots, x_{n-1})} \neq 0, \quad \forall x \in D. \qquad (7)$$

[4] The Jacobian of functions $\psi_i(x_1,\dots,x_m,x_{m+1},\dots,x_n)$, $i=1,2,\dots,m,$ is a determinant of order m, the i -th row of which contains partial derivatives of the first order of the function ψ_i with respect to variables $x_1,\dots,x_m$. It is briefly indicated by the symbol $\frac{D(\psi_1,\dots,\psi_m)}{D(x_1,\dots,x_m)}$.

Let's introduce new independent variables

$$\begin{aligned} &\xi_1 = \psi_1(x_1, \dots, x_n), \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ &\xi_{n-1} = \psi_{n-1}(x_1, \dots, x_n), \\ &\xi_n = \psi_n(x_1, \dots, x_n), \end{aligned} \qquad (8)$$

where the function $\psi_n(x_1,\dots,x_n)$ can be any continuously differentiable function in D, but in which the transformation (8) is non-degenerate, and a new notation with $\upsilon=\upsilon(\xi_1,\dots,\xi_n)$ is used for the dependent variable, and $u(x)=\upsilon(\psi_1(x),\dots,\psi_n(x))$.

$$v = v(\xi_1, \dots, \xi_n),$$

$$u(x) = v(\psi_1(x), \dots, \psi_n(x)). \qquad (9)$$

We show that when replacing (8)-(9), equation (4) is reduced to the simplest form when it is easy to construct its solution.

Indeed, we will express the derivatives included in equation (4) in terms of new variables using the rule of differentiation of a complex function:

$$\frac{\partial u}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial v}{\partial \xi_i}\frac{\partial \xi_i}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial v}{\partial \xi_i}\frac{\partial \psi_i}{\partial x_k}, \quad k = 1, \ldots, n.$$

Substituting these expressions into equation (4) and grouping the terms, we obtain the equation

$$\sum_{i=1}^{n-1} \left(A_1(x)\frac{\partial \psi_i}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_i}{\partial x_n} \right) \frac{\partial v}{\partial \xi_i} +$$
$$+ \left(A_1(x)\frac{\partial \psi_n}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_n}{\partial x_n} \right) \frac{\partial v}{\partial \xi_n} = 0$$

Since the functions ψ_i for $i = 1,...,n-1$ are the first integrals of the system (5), then, according to Theorem 1, they are solutions of equation (4). Therefore, the last equation takes the form

$$\left(A_1(x)\frac{\partial \psi_n}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_n}{\partial x_n} \right) \frac{\partial v}{\partial \xi_n} = 0.$$

And since the transformation (8) is non-degenerate, the function ψ_n cannot be a solution to equation (4), and therefore we will have

$$\frac{\partial v}{\partial \xi_n} = 0. \tag{10}$$

Thus, using the non-degenerate transformation (8), equation (4) is reduced to the form (10), which is called canonical.

Integrating equation (10) by ξ_n, we obtain its solution

$$v(\xi_1, \ldots, \xi_n) = \Phi(\xi_1, \ldots, \xi_{n-1}),$$

where Φ is an arbitrary function that does not depend on ξ_n and has continuous derivatives with respect to the variables $\xi_1,\ldots,\xi_{n-1}$. Returning to the old variables, we obtain the solution of equation (4).

THEOREM 2. Any solution $u=\varphi(x_1,\ldots,x_n)$ of equation (4) is represented as

$$u = \Phi(\psi_1(x_1,\ldots,x_n),\ldots,\psi_{n-1}(x_1,\ldots,x_n)), \qquad (11)$$

where $\Phi(\psi_1,\ldots,\psi_{n-1})$ is some differentiable function of its arguments $\psi_1,\ldots,\psi_{n-1}$, and $\psi_i(x_1,\ldots,x_n)$ $(i=1,\ldots,n-1)$ are the first integrals of the system (5) satisfying the independence condition (7).

Formula (11) represents **the general solution** of equation (4).

Thus, the problem of constructing a general solution to equation (4) is equivalent to the problem of finding n− 1 independent first integrals of the corresponding system of ordinary differential equations (5).

Example 1

Find the general solution $u=u(x,y)$ of the equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0.$$

<u>Solution:</u>

Let's write down the equation of characteristics

$$\frac{dx}{y} = \frac{dy}{-x}.$$

This equation has a solution

$$x^2 + y^2 = C.$$

Therefore, the first integral is the function:

$$\psi = x^2 + y^2.$$

Then the general solution has the form

$$u = \Phi(x^2 + y^2),$$

and it represents a family of surfaces of rotation with the axis of rotation Ou. In particular, for $\Phi(\psi) = \psi$ we obtain a paraboloid of rotation:

$$u = x^2 + y^2,$$

when $\Phi(\psi) = \sqrt{\psi}$, we get a cone

$$u = \sqrt{x^2 + y^2}.$$

It is often possible to construct the first integrals of the characteristic system (5) for the case $n > 2$ by finding integrable combinations. *An integrable combination* is called a differential equation, which is a consequence of the system of equations (5) and is integrated in quadratures. The first integral of the system (5) is obtained from each integrable combination.

Example 2

Find the general solution $u=u(x,y,z)$ of the equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$$

<u>Solution:</u>

Let's write down the equations of characteristics in a symmetric form

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

Solving: $\frac{dy}{y}=\frac{dx}{x}$ and $\frac{dz}{z}=\frac{dx}{x}$.

we find the first two integrals:

$$\frac{y}{x}=C_1 \text{ and } \frac{z}{x}=C_2$$

Then the general solution of the given equation has the form

$$u(x,y,z)=\Phi\left(\frac{y}{x},\frac{z}{x}\right)$$

To make integrable combinations of system (5), you can use the following rule of **equal fractions.**

If there are equal fractions

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n},$$

and arbitrary numbers $\lambda_1,\ldots,\lambda_n$ such that $\lambda_1 b_1+\ldots+\lambda_n b_n \neq 0$, then

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n}=\frac{\lambda_1 a_1+\ldots+\lambda_n a_n}{\lambda_1 b_1+\ldots+\lambda_n b_n}.$$

Example 3

Find a solution of the equation

$$(x+z)u'_x+(y+z)u'_y+(x+y)u'_z=0.$$

Solution:

To find the independent first integrals, we make up the equations of characteristics in a symmetric form

$$\frac{dx}{x+z}=\frac{dy}{y+z}=\frac{dz}{x+y}.$$

By the property of equal fractions, we have

$$\frac{dx-dz}{z-y}=\frac{dy-dz}{z-x}\ \Rightarrow\ (x-z)d(x-z)=(y-z)d(y-z).$$

Integrating the last equality, we get the first integral

$$\psi_1(x,y,z)=(x-z)^2-(y-z)^2=(x-y)(x+y-2z).$$

According to the property of equal fractions, we will make another equality

$$\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{x-y}\ \Leftrightarrow\ \frac{d(x+y+z)}{x+y+z}=\frac{2d(x-y)}{x-y},$$

the integration of which gives another first integral

$$\psi_2(x,y,z) = \frac{x+y+z}{(x-y)^2}$$

Then the general solution of the given equation has the form:

$$u(x,y,z) = \Phi\left((x-y)(x+y-2z), \frac{x+y+z}{(x-y)^2}\right).$$

where $\Phi(a,b)-$ is an arbitrary continuously differentiable function.

Example 4

Find a solution of the equation

$$xu'_x + yu'_y + xyu'_z = 0, \quad x \neq 0, \ y \neq 0.$$

<u>Solution:</u>

Let's make up a characteristic system

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy}. \qquad (*)$$

We will find the first integral by solving the equation

$$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \frac{x}{y} = C_1.$$

So, $\psi_1(x,y,z) = \dfrac{x}{y}$.

We will find another first integral by considering the second equation of the characteristic system (*)

$$\frac{dy}{y} = \frac{dz}{xy},$$

excluding x from it using the already found first integral ψ_1. Since $x = C_1 y$, we will have

$$\frac{dy}{y} = \frac{dz}{C_1 y^2} \quad \Rightarrow \quad C_1 y dy = dz \quad \Rightarrow \quad C_1 y^2 - 2z = C_2 \quad \Rightarrow \quad xy - 2z = C_2.$$

So, $\psi_2(x,y,z) = xy - 2z$ and the general solution of the given equation will be written as

$$u(x,y,z) = \Phi\left(\frac{x}{y}, xy - 2z\right).$$

2.3. FIRST-ORDER DIFFERENTIAL EQUATIONS, LINEAR WITH RESPECT TO PARTIAL DERIVATIVES

The Cauchy problem for a homogeneous linear equation

To isolate a single particular solution from the general solution, additional conditions must be set. Such conditions, for example, include initial conditions. Initial conditions are often set by fixing one of the independent variables.

We will consider the initial problem, or Cauchy problem, for equation (4) in the following formulation. Among all the solutions of equation (4), find such a solution

$$u = F(x_1, \ldots, x_n), \tag{13}$$

which satisfies the initial conditions:

$$u = \varphi(x_1, \ldots, x_{n-1}) \quad \text{при} \quad x_n = x_n^{(0)}, \tag{14}$$

where φ – is a given continuously differentiable function of variables $x_1,\ldots,x_{n-1}$.

In the case where the desired function depends on two independent variables, Cauchy problem is to find a solution

$$u = F(x, y),$$

which satisfies the initial conditions:

$$u = \varphi(y)$$

at $x = x_0,$

where $\varphi(y)-$ the given function from y.

Geometrically, this means that among all integral surfaces, we are looking for an integral surface $u=F(x,y)$ that passes through a given curve $u=\varphi(y)$ lying in the plane $x=x_0$ parallel to the plane yOu.

Taking into account the general solution of the equation $u=\Phi(\psi_1,...,\psi_{n-1})$, the solution of the Cauchy problem is reduced to determining the type of function Φ such that

$$\Phi(\psi_1,\dots,\psi_{n-1})\big|_{x_n=x_n^{(0)}}=\varphi(x_1,\dots,x_{n-1}). \qquad (15)$$

Let's denote

$$\begin{cases} \psi_1(x_1,\dots,x_{n-1},x_n^{(0)})=\overline{\psi}_1, \\ \psi_2(x_1,\dots,x_{n-1},x_n^{(0)})=\overline{\psi}_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \psi_{n-1}(x_1,\dots,x_{n-1},x_n^{(0)})=\overline{\psi}_{n-1}, \end{cases} \qquad (16)$$

then equality (15) can be rewritten as

$$\Phi(\overline{\psi}_1,\dots,\overline{\psi}_{n-1})=\varphi(x_1,\dots,x_{n-1}). \qquad (17)$$

The system (16) is solvable with respect to $x_1,...,x_{n-1}$ at least in some neighborhood of the point $\left(x_1^{(0)},...,x_n^{(0)}\right)$ if $A_n\left(x_1^{(0)},...,x_n^{(0)}\right)\neq 0$, which we assume. Resolving the system (16) with respect to $x_1,...,x_{n-1}$, we obtain:

$$\begin{cases} x_1=\omega_1(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}), \\ x_2=\omega_2(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}), \\ \dots\dots\dots\dots\dots\dots\dots\dots, \\ x_{n-1}=\omega_{n-1}(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}), \end{cases} \qquad (18)$$

If we now take the function

$$\Phi(\psi_1,\ldots,\psi_{n-1}) = \varphi(\omega_1(\psi_1,\ldots,\psi_{n-1}),\ldots,\omega_{n-1}(\psi_1,\ldots,\psi_{n-1})),$$

as Φ, then condition (17) will be fulfilled.

Therefore, the function gives the desired solution to the Cauchy problem. Here, the function φ is the function that participates in the initial conditions.

Thus, we come to the following algorithm for solving the Cauchy problem:

1) create the corresponding system of ordinary differential equations and find $n-1$ independent integrals:

$$\begin{cases} \psi_1(x_1,\ldots,x_n), \\ \psi_2(x_1,\ldots,x_n), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \psi_{n-1}(x_1,\ldots,x_n). \end{cases} \tag{19}$$

2) replace the independent variable in integrals (19) with its specified value $x_n^{(0)}$:

$$\begin{cases} \psi_1(x_1,\ldots,x_{n-1},x_n^{(0)}) = \overline{\psi}_1, \\ \psi_2(x_1,\ldots,x_{n-1},x_n^{(0)}) = \overline{\psi}_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \psi_{n-1}(x_1,\ldots,x_{n-1},x_n^{(0)}) = \overline{\psi}_{n-1}. \end{cases} \tag{20}$$

3) solve the system of equations (20) with respect to $x_1,\ldots,x_{n-1}$:

$$\begin{cases} x_1 = \omega_1(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \\ x_2 = \omega_2(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_{n-1} = \omega_{n-1}(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}). \end{cases} \tag{21}$$

4) construct a function

$$u = \varphi(\omega_1(\psi_1, \dots, \psi_{n-1}), \dots, \omega_{n-1}(\psi_1, \dots, \psi_{n-1})), \qquad (22)$$

that gives a solution to the Cauchy problem.

Example 1

Find a solution to the Cauchy problem

$$xu'_x + yu'_y + xyu'_z = 0, \quad u(x, y, 0) = x^2 + y^2.$$

<u>Solution:</u>

Let's make a characteristic system

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy}. \qquad (*)$$

We will find the first integral by solving the equation

$$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \frac{x}{y} = C_1.$$

So, $\psi_1(x, y, z) = \frac{x}{y}$.

We will find another first integral by considering the second equation of the characteristic system (*)

$$\frac{dy}{y} = \frac{dz}{xy},$$

excluding x from it using the already found first integral ψ_1. Since $x = C_1 y$, we will have

$$\frac{dy}{y} = \frac{dz}{C_1 y^2} \quad \Rightarrow \quad C_1 y dy = dz \quad \Rightarrow \quad C_1 y^2 - 2z = C_2 \quad \Rightarrow \quad xy - 2z = C_2.$$

So, $\psi_2(x,y,z)=xy-2z$.

The characteristic system corresponding to the equation has the following two first integrals

$$\psi_1(x,y,z)=\frac{x}{y},\quad \psi_2(x,y,z)=xy-2z.$$

Considering them at $z=0$, we make up a system of equations

$$\frac{x}{y}=\overline{\psi}_1,\quad xy=\overline{\psi}_2,$$

from which we find

$$y^2=\frac{\overline{\psi}_2}{\overline{\psi}_1},\quad x^2=\overline{\psi}_1\overline{\psi}_2.$$

Then for a given function $\varphi(x,y)=x^2+y^2$ we will have

$$\varphi(\overline{\psi}_1,\overline{\psi}_2)=\left(\overline{\psi}_1+\frac{1}{\overline{\psi}_1}\right)\overline{\psi}_2.$$

Therefore, the solution to the Cauchy problem has the form

$$u(x,y,z)=\left(\frac{x}{y}+\frac{y}{x}\right)(xy-2z).$$

Quasi-linear equations

Let the point $x=(x_1,...,x_n)$ belong to the domain $D\subset\mathbb{R}^n$. Consider a quasi-linear equation

$$A_1(x,u)\frac{\partial u}{\partial x_1}+\ldots+A_n(x,u)\frac{\partial u}{\partial x_n}=B(x,u), \qquad (23)$$

assuming that $A_i(x,u)$ $(i=1,...,n)$ and $B(x,u)$ are differentiable functions of the arguments x,u in some domain $G\subset\mathbb{R}^{n+1}$.

Equation (23) corresponds to the following linear equation

$$A_1(x,u)\frac{\partial v}{\partial x_1}+\ldots+A_n(x,u)\frac{\partial v}{\partial x_n}+B(x,u)\frac{\partial v}{\partial u}=0, \qquad (24)$$

with an unknown function $\upsilon=\upsilon(x,u)$.

The method of solving a quasi-linear equation is based on the following theorem:

THEOREM 3.

Let $\upsilon=V(x,u)$ <u>be the solution of equation</u> (24). Let equation $V(x,u)=0$ define in the domain of D variables $x=(x_1,...,x_n)$ some differentiable function $u=\varphi(x)$, and let $\frac{\partial V}{\partial u}\neq 0$ in D for $u=\varphi$. Then $u=\varphi(x)$ is the solution of equation (23).

We describe *an algorithm for constructing a solution to a quasi-linear equation.*

1) write out the characteristic system for the linear equation (24):

$$\frac{dx_1}{A_1(x,u)} = \ldots = \frac{dx_n}{A_n(x,u)} = \frac{du}{B(x,u)} \qquad (25)$$

The characteristics of the linear equation (24) are called **the characteristics of the quasi-linear equation** (23).

2) find the n independent first integrals of the system (25):

$$\psi_1(x,u), \ldots, \psi_n(x,u). \qquad (26)$$

(or we can write $\psi_1(x,u) = c_1, \ldots, \psi_n(x,u) = c_n$).

3) using formula (11), construct a general solution to equation (24):

$$v(x,u) = \Phi(\psi_1(x,u), \ldots, \psi_n(x,u)).$$

(or we can write $\upsilon = V(\psi_1(x,u), \ldots, \psi_n(x,u))$).

4) assuming $\upsilon = 0$, write down the equation to determine the set of solutions to equation (23):

$$\Phi(\psi_1(x,u), \ldots, \psi_n(x,u)) = 0. \qquad (27)$$

(or we can write $V(\psi_1(x,u), \ldots, \psi_n(x,u)) = 0$).

The expression (27) is called the **general integral**, or the **general solution**, of equation (23). If u is included only in one of the first integrals (26), for example, in the last one, then the general solution can be written as follows:

$$\psi_n(x,u) = F(\psi_1,\ldots,\psi_{n-1}), \tag{28}$$

where $F-$ an arbitrary differentiable function. If it is possible to resolve equality (28) with respect to u, then we obtain a general solution of equation (23) in explicit form.

Comment 1

It is possible that there may be solutions to equation (23) for which equation (24) is not satisfied identically in (x,u), but only when $u=\varphi(x)$ is identical in x. Such solutions are not contained in formula (27) and are called *special*. A special solution is an exceptional case, and therefore we will not consider them further.

Comment 2

The solution of linear equation (2) can also be constructed in the described way.

Comment 3

When constructing the first integrals of the system (25), in some cases it may turn out that the variable u will enter only one of them:

$$\psi_1(x) = c_1,\ \ldots,\ \psi_{n-1}(x) = c_{n-1},\ \psi_n(x,\ u) = c_n.$$

Then the general solution will be from the ratio

$$V\big(\psi_1(x),\ \ldots,\ \psi_{n-1}(x),\ \psi_n(x,\ u)\big) = 0,$$

which can be rewritten by the implicit function theorem in the form

$$\psi_n(x,u) = F\big(\psi_1(x,u),\ldots,\psi_{n-1}(x)\big) \tag{29}$$

By resolving equality (29) with respect to u, we obtain a general solution of equation (23) in explicit form.

The characteristic system (25) will be written in this case as

$$\frac{dx_1}{A_1(x)} = \ldots = \frac{dx_n}{A_n(x)} = \frac{du}{0}.$$

A system of n independent first integrals can be chosen as follows

$$\psi_1(x) = c_1,\ \ldots,\ \psi_{n-1}(x) = c_{n-1},\ u = c_n.$$

We see that the variable u will enter only the last first integral. The solution of the equation can be written using the formula (29).

Example 2

Solve the equation

$$x_2 \frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} = x_1 - x_2.$$

Solution:

The characteristic system has the form

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1} = \frac{du}{x_1 - x_2}.$$

The first equality

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1}$$

it will be written in the form of

$$x_1 dx_1 = x_2 dx_2$$

and leads to the first integral

$$x_1^2 - x_2^2 = c_1.$$

To obtain another first integral, we use the property of adding proportions:

$$\frac{d\left(x_2 - x_1\right)}{x_1 - x_2} = \frac{du}{x_1 - x_2},$$

where do we get the first integral:

$$u + x_1 - x_2 = c_2.$$

Since u is included in only one of the first integrals obtained, we obtain an explicit solution to the equation

$$u = x_2 - x_1 + \mathrm{F}\left(x_1^2 - x_2^2\right),$$

The function $F(y) -$ is an arbitrary function of class C^1.

Example 3

Solve the equation

$$\left(x_2 + 2u^2\right)\frac{\partial u}{\partial x_1} - 2x_1^2 u \frac{\partial u}{\partial x_2} = x_1^2.$$

Solution:

The characteristic system has the form

$$\frac{dx_1}{x_2 + 2u^2} = \frac{dx_2}{-2x_1^2 u} = \frac{du}{x_1^2}.$$

From the second ratio we have

$$dx_2 = -2u du,$$

where do we get the first integral:

$$x_2 + u^2 = c_1. \qquad (*)$$

Consider the ratio

$$\frac{dx_1}{x_2 + 2u^2} = \frac{du}{x_1^2}$$

Substituting $x_2 = c_1 - u^2$ into the ratio, we get

$$\frac{dx_1}{c_1 + u^2} = \frac{du}{x_1^2}$$

from which

$$x_1^2\,dx_1 = \left(c_1 + u^2\right)du,$$

and therefore

$$\frac{x_1^3}{3} = c_1 u + \frac{u^3}{3} + c_2.$$

Then the general solution of the equation will be written in an implicit form

$$V\left(x_2 + u^2,\ x_1^3 - 3\left(x_2 + u^2\right)u - u^3\right) = 0,$$

where the function V is an arbitrary function of class C^1 and such that

$$\frac{\partial V\left(x_2 + u^2,\ x_1^3 - 3\left(x_2 + u^2\right)u - u^3\right)}{\partial u} \neq 0.$$

Example 4

Find a solution to the equation

$$\sin y \cdot u_x + e^x \cdot u_y = 2x \sin y \cdot u^2.$$

<u>Solution:</u>

Obviously, the given equation is quasi-linear. To construct a general solution, we find two independent first integrals of the system of equations:

$$\frac{dx}{\sin y} = \frac{dy}{e^x} = \frac{du}{2x \sin y \cdot u^2}.$$

Considering two equations

$$\frac{dx}{\sin y} = \frac{dy}{e^x} \quad \text{and} \quad \frac{dx}{\sin y} = \frac{du}{2x \sin y \cdot u^2},$$

we obtain

$$\psi_1(x,y,u) = e^x + \cos y, \qquad \psi_2(x,y,u) = x^2 + \frac{1}{u}.$$

Then the general integral of the given equation has the form:

$$\Phi\left(e^x + \cos y, x^2 + \frac{1}{u}\right) = 0.$$

Solving this equation with respect to the second argument, we obtain

$$x^2 + \frac{1}{u} = F(e^x + \cos y) \quad \Rightarrow \quad u = \left(F(e^x + \cos y) - x^2\right)^{-1}.$$

where F is an arbitrary continuously differentiable function.

Example 5

Find a solution to the equation

$$(2y - u)\, u'_x + y\, u'_y = u.$$

Solution:

Let's make up a characteristic system

$$\frac{dx}{2y-u} = \frac{dy}{y} = \frac{du}{u}.$$

Solving the equation

$$\frac{dy}{y} = \frac{du}{u} \quad \Rightarrow \quad \frac{u}{y} = C_1,$$

we find the first integral

$$\psi_1(x,y,u) = \frac{u}{y}.$$

Using the rule of equal fractions, we will make an integrable combination

$$\frac{dx}{2y-u} = \frac{2dy - du}{2y-u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

Where do we get another first integral

$$\psi_2(x, y, u) = x - 2y + u.$$

Therefore, the general integral of the given equation has the form

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0,$$

where Φ is an arbitrary continuously differentiable function.

Example 1.

To find a general solution of a quasi-linear inhomogeneous partial differential equation of the first order

$$x^2u\frac{\partial u}{\partial x}+y^2u\frac{\partial u}{\partial y}=x+y$$

<u>Solution:</u>

The characteristic system, which corresponds to this quasi-linear equation, in a symmetric form, has the form:

$$\frac{dx}{x^2u}=\frac{dy}{y^2u}=\frac{dz}{x+y}$$

The first integrated combination

$$\frac{dx}{x^2u}=\frac{dy}{y^2u}$$

after the reduction by u:

$$\frac{dx}{x^2}=\frac{dy}{y^2}$$

and integration, it gives

$$\frac{1}{x}-\frac{1}{y}=C_1.$$

To obtain another first integral, we make an integrable combination

$$\frac{dx-dy}{x^2u-y^2u}=\frac{du}{x+y}$$

Where from

$$\frac{d\left(x-y\right)}{u\left(x^2-y^2\right)}=\frac{du}{x+y}$$

or

$$\frac{d\left(x-y\right)}{x-y}=udu$$

After integration, we get

$$\ln\left|x-y\right|-\frac{u^2}{2}=C_2\ .$$

Therefore, the general solution of this quasi-linear differential equation will be:

$$\Phi\left(\frac{1}{x}-\frac{1}{y},\ln\left|x-y\right|-\frac{u^2}{2}\right)=0\ ,$$

where Φ is an arbitrary function. Since u is included in only one of the first integrals, the general solution can be written as

$$\ln\left|x-y\right|-\frac{u^2}{2}=F\left(\frac{1}{x}-\frac{1}{y}\right),$$

where F is an arbitrary function.

Example 2.

Find a general solution to a quasi-linear partial differential equation of the first order

$$xy\frac{\partial u}{\partial x}+\left(x-2u\right)\frac{\partial u}{\partial y}=yu\ .$$

Solution:

The characteristic system corresponding to this quasi-linear equation takes the form in a symmetric form:

$$\frac{dx}{xy} = \frac{dy}{x-2u} = \frac{du}{yu}$$

From the first integrable combination

$$\frac{dx}{xy} = \frac{du}{yu}$$

From the first integrable combination we obtain

$$\frac{u}{x} = C_1 .$$

To find the second independent integral of the characteristic system, we rewrite it as:

$$\begin{cases} x' = xy, \\ y' = x-2u, \\ u' = yu. \end{cases}$$

Let's differentiate the second equation of the system

$$y'' = x' - 2u',$$

substitute instead of x' and y' their expressions from the first and third equations of the system:

$$y'' = x' - 2u' = xy - 2yu = y(x-2u) = yy',$$

as a result, we obtain a second-order equation

$$y'' = yy'.$$

By replacing $y' = p$, where $p = p(y)$, $y'' = pp'$, we lower the order of the equation:

$$pp' = yp.$$

From where we have two equations: $p = 0$ and $p' = y$. The first one gives a trivial solution

$$y' = 0, \ \ y = const,$$

which we are not interested in. Solving the second equation, we get

$$p = \frac{y^2}{2} + \tilde{C}_2$$

or

$$y' = \frac{y^2}{2} + \tilde{C}_2$$

$$x - 2u = \frac{y^2}{2} + \tilde{C}_2$$

$$2x - 4u = y^2 + 2\tilde{C}_2$$

$$2x - 4u - y^2 = 2\tilde{C}_2 = C_2.$$

Thus, the general solution of the original quasi-linear differential equation is

$$\Phi\left(\frac{u}{x}, 2x - 4u - y^2\right) = 0,$$

where Φ is an arbitrary function.

Example 3.

Find a general solution to the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2.$$

Solution:

Let's make up the equations of characteristics:

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-x^2 - y^2}.$$

The first equation of this system can be solved separately from the second, since it does not contain u (the variable u, which stands in the left and right sides of this equation, is reduced). From equality

$$\frac{dx}{x} = \frac{dy}{y}$$

we get by integrating

$$\ln|x| = \ln|y| + \ln C,$$

where do we find the first integral of the system:

$$\frac{y}{x} = C_1.$$

The second equation of this system contains all three variables

To exclude the variable y, let's use the first integral found. Since the desired integral curve lies on one of the surfaces defined by the first integral found, at each point of this curve $y = C_1 x$ (the value of the constant C_1 is the same at all points of the desired integral curve, but may differ if you switch to another integral curve). Replace y with C_1x in the second equation, after the transformations we get

$$-(1+C_1^2)x\,dx = u\,du.$$

Integrating, we find the dependence

$$(1+C_1^2)x^2 + u^2 = C_2.$$

This ratio containing C_2 is not the first integral, since it also contains an arbitrary constant C_1. Given that for the found curve $C_1 = \frac{y}{x}$, we rewrite the ratio as

$$x^2 + y^2 + u^2 = C_2.$$

In this form of notation, the relation is performed for any of the integral curves, that is, it is the first integral.

The general solution of the first-order equation has the form (in an implicit form)

$$\Phi\left(\frac{y}{x}, x^2 + y^2 + u^2\right) = 0,$$

where Φ is an arbitrary differentiable function. It is possible to get an explicit solution from the last expression:

$$u = \pm\sqrt{f\left(\frac{y}{x}\right) - x^2 - y^2},$$

where f is an arbitrary differentiable function.

Comment

When finding the first integrals of a system written in symmetric form, derivative proportions are often used to obtain integrable combinations, for example

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}.$$

In the above example, we will rewrite the system in the form

$$\frac{x\,dx}{x^2} = \frac{y\,dy}{y^2} = \frac{u\,du}{-x^2-y^2},$$

then we will use the derived proportion by adding the numerators

and denominators of the first and second ratios:

$$\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{x\,dx}{x^2} = \frac{y\,dy}{y^2} = \frac{u\,du}{-x^2-y^2}.$$

Comparing the first relation with the last one, we get

$$x\,dx + y\,dy = -u\,du,$$

hence the first integral

$$x^2 + y^2 + u^2 = C_2.$$

FINDING A PARTICULAR SOLUTION (CAUCHY PROBLEM)

In order to single out one definite solution from the infinite set of solutions given by formula

$$\Phi\left(\psi_1(x,y,u),\psi_2(x,y,u)\right)=0,$$

it is necessary to find the function Φ included in the solution.

This can be done under additional conditions. We formulate the problem of finding a partial solution to equation

$$P(x,y,u)\frac{\partial u}{\partial x}+Q(x,y,u)\frac{\partial u}{\partial y}=R(x,y,u) \qquad (1)$$

— the Cauchy problem.

(*A general view of a first-order quasilinear equation with two independent variables, where $u=u(x,y)$ is the desired function; $P(x,y,u)$, $Q(x,y,u)$, $R(x,y,u)$ are continuous changes in the variables of the function in the considered area that do not vanish at the same time.*) We just wrote it down in this form.

Along some curve L_1 of the plane (x,y), the values of the desired function are set

$$\begin{cases} y = f(x), \\ u = g(x), \end{cases}$$

where $f(x)$, $g(x)$ are differentiable functions. Find such a solution $u=u(x,y)$ of equation (1) in the vicinity of the line L_1 so that $u=u(x,f(x))=g(x)$.

Geometric illustration of the Cauchy problem: through a space curve L with a continuously varying tangent (smooth curve), draw the integral surface of equation (1) (Fig. 1).

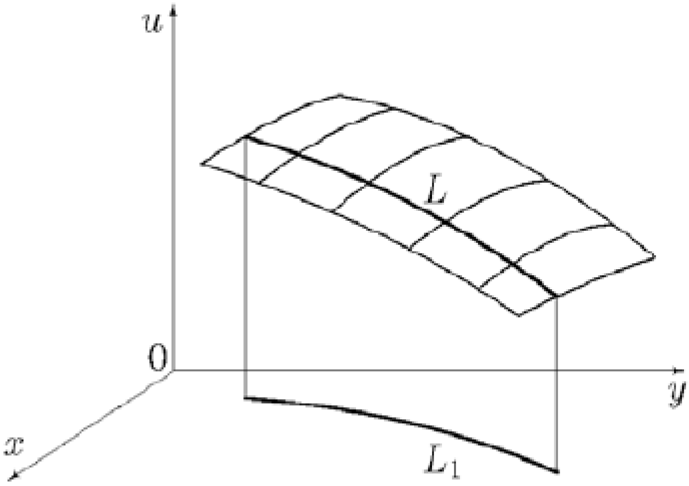


Fig.1.

The L line can be defined in a more general way:

$$\begin{cases} x = \varphi(\sigma), \\ y = \varkappa(\sigma), \\ u = \chi(\sigma). \end{cases} \tag{2}$$

The first two of the equations (2) define the curve L_1 in parametric form, all three equations define the curve L in space (x,y,u), for which L_1 is a projection onto the plane (x,y). In this case, the condition

$$u(\varphi(\sigma), \varkappa(\sigma)) = \chi(\sigma)$$

must be fulfilled.

The geometric solution to the Cauchy problem is obvious:

a characteristic must be drawn through each point of a given line L.

The set of characteristics passing through all points of the line L form the desired integral surface (see Fig. 1).

A system of differential equations

$$\frac{dx}{P(x,y,u)} = \frac{dy}{Q(x,y,u)} = \frac{du}{R(x,y,u)}$$

can be integrated without knowledge of the integral surface. The general integral of the system

$$\begin{cases} \psi_1(x,y,u) = C_1, \\ \psi_2(x,y,u) = C_2 \end{cases}$$

defines a family of characteristics (depending on two parameters C_1 and C_2), which has the following property: one characteristic passes through each point of the domain where the conditions for the existence and uniqueness of the solution are met. Constants C_1 and C_2 are independent, each of them can be assigned any values; a two-parameter family of characteristics is obtained as a result of the intersection of each surface of one family with each surface of the other (Fig.2).

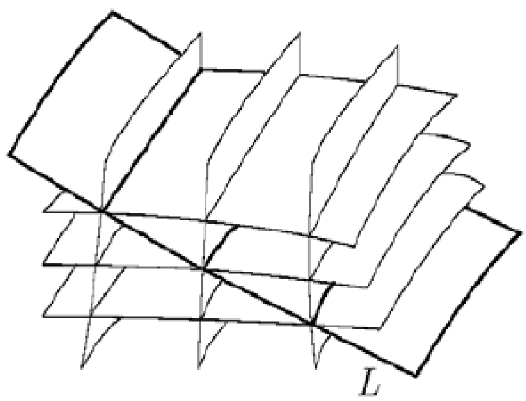


Fig.2.

Of all the surfaces defined by the first integrals, we need to leave only such pairs of surfaces whose intersection lines pass through the points of the line L; for this, we must learn how to select pairs C_1 and C_2 accordingly. In other words, between arbitrary constants C_1 and C_2 in the general integral of the system, it is necessary to establish some dependence $\Phi(C_1, C_2) = 0$.

A surface from the family $\psi_1(x,y,u)=C_1$ can be drawn through each point of the line L. Substituting the coordinates of this point, given as functions of the parameter σ, into the equation of the surface, we establish a relationship between the value of the constant C_1 defining the surface and the value of the parameter σ corresponding to the point of intersection of the line L with this surface:

$$C_1 = \psi_1(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_1(\sigma). \qquad (3)$$

A similar ratio

$$C_2 = \psi_2(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_2(\sigma) \qquad (4)$$

gives the dependence between the parameter σ of the point of the curve L and the constant C_2 of the surface of another family intersecting the line L at this point. Any pair of values C_1 and C_2 calculated by formulas (3) and (4) for one value σ will determine a pair of surfaces whose intersection line (characteristic) passes through the point of the line L corresponding to this value σ. Therefore, formulas (3) and (4) taken together define in parametric form the desired dependence between C_1 and C_2:

$$\begin{cases} C_1 = C_1(\sigma), \\ C_2 = C_2(\sigma). \end{cases} \quad (5)$$

Thus, the system of equations

$$\begin{cases} \psi_1(x, y, u) = C_1(\sigma), \\ \psi_2(x, y, u) = C_2(\sigma) \end{cases} \quad (6)$$

defines a family of characteristics passing through the line L. This family of characteristics, depending on one parameter σ, forms the desired integral surface (solution of the Cauchy problem).

Excluding the parameter σ from equations (5), can obtain a dependence of the form:

$$\Phi(C_1, C_2) = 0. \quad (7)$$

Accordingly, the solution of the Cauchy problem will be presented as

$$\Phi(\psi_1(x, y, u), \psi_2(x, y, u)) = 0. \quad (8)$$

To exclude σ, for example, equation (3) with respect to σ should be resolved and the expression $\sigma(C_1)$ should be substituted into the left part of the relation (4). This is possible when σ enters the left parts of equations (3) and (4).

If the entire curve L lies, for example, on the surface $\psi_1(x,y,u) = C_1^0$, then the ratio

$$\psi_1(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_1^0$$

cannot be resolved relative to the σ parameter. But then this surface itself $\psi_1(x,y,u)=C_1^0$ is the integral surface in the Cauchy problem.

Finally, if a given curve L lies simultaneously on two surfaces of different families, then it itself is a characteristic, and the parameter σ is not included in any of the relations (3), (4). The Cauchy problem becomes indefinite, since each characteristic belongs to an infinite set of integral surfaces. Indeed, if only the constants C_1^0 and C_2^0 satisfy equation

$$\Phi(C_1, C_2) = 0,$$

where Φ is an arbitrary differentiable function, then equation

$$\Phi(\psi_1(x,y,u), \psi_2(x,y,u)) = 0$$

defines an integral surface passing through the line L. Thus, countless integral surfaces can be drawn through a given line L.

Example 4.

Find the integral surface of the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2,$$

passing through the curve

$$\begin{cases} x = a, \\ u = \sqrt{y^2 + a^2}, \end{cases}$$

where a is a constant.

<u>Solution:</u>

In Example 3, the first integrals of the equations of characteristics are found,

$$\frac{y}{x} = C_1, \qquad x^2 + y^2 + u^2 = C_2.$$

The first equation defines a set of planes passing through the Oz -axis, the second one defines spheres of different radii centered at the origin. The characteristics of the original partial differential equation can be represented as meridians on spheres.

In the equations of a given curve, we take y as an independent variable.

Substituting the coordinates of points into the first integrals of the system, we obtain

$$\begin{cases} C_1 = \dfrac{y}{a}, \\ C_2 = a^2 + y^2 + (y^2 + a^2). \end{cases}$$

From here

$$C_2 = 2a^2(1 + C_1^2).$$

Replacing C_1 and C_2 in this dependence with the functions on the left sides of the first integrals, we find the desired solution

$$x^2 + y^2 + u^2 = 2a^2 \frac{x^2 + y^2}{x^2}.$$

Example 5.

Find the integral surface of the equation

$$(x^2 - y^2 - u^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 2xu,$$

passing through the curve $L: \; x = 0, y = 2a\cos t, u = 2a\sin t$.

Solution:

Integrating a system of equations

$$\frac{dx}{x^2 - y^2 - u^2} = \frac{dy}{2xy} = \frac{du}{2xu}.$$

From the second equation

$$\frac{dy}{y} = \frac{du}{u}$$

we get the first integral

$$\frac{u}{y} = C_1.$$

To obtain another first integral, we present the system as

$$\frac{x\,dx}{x^2 - y^2 - u^2} = \frac{y\,dy}{2y^2} = \frac{u\,du}{2u^2},$$

and then we apply the derivative proportion, adding up all the numerators and all the denominators of the relations:

$$\frac{x\,dx + y\,dy + u\,du}{x^2 + y^2 + u^2} = \frac{dy}{2y}.$$

Thus, an integrable combination is obtained

$$\frac{d(x^2 + y^2 + u^2)}{x^2 + y^2 + u^2} = \frac{dy}{y},$$

where from

$$\frac{x^2+y^2+u^2}{y} = C_2.$$

The first found integrals define planes passing through the Ox axis and spheres centered on the Oy axis passing through the origin. The characteristics will be circles passing through the origin, for which the Ox axis is tangent.

The desired surface will be obtained by rotating a circle of radius a around the Ox axis, touching the axis at the origin. Let's find the equation of this surface. Substituting the coordinates of the points of a given line, expressed in terms of t, into the first integrals, we obtain

$$C_1 = \operatorname{tg} t, \qquad C_2 = \frac{2a}{\cos t}.$$

Excluding t, we find the relationship between C_1 and C_2:

$$4a^2(C_1^2+1) = C_2^2,$$

and then the solution of the Cauchy problem

$$4a^2(u^2+y^2) = (x^2+y^2+u^2)^2.$$

In spherical coordinates $(x = r\sin\theta,\ y = r\cos\theta\cos\varphi,\ u = r\cos\theta\sin\varphi)$ the equation of the surface takes the form $r = 2a\cos\theta$.

September 30th, 2024

Example 1

Find the general integral of the equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u\,.$$

Solution:

Consider a system of equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{du}{u}\,.$$

Solving the equation

$$\frac{dx}{x}=\frac{dy}{y}\,,$$

we get

$$\frac{y}{x}=C_1\,;$$

the solution of the equation

$$\frac{dx}{x}=\frac{du}{u}$$

is

$$\frac{u}{x}=C_2\,.$$

Now we can find the general integral of the given equation:

$$\Phi\left(\frac{y}{x},\frac{u}{x}\right)=0$$

or

$$\frac{u}{x}=\psi\left(\frac{y}{x}\right),$$

so

$$u = x\psi\left(\frac{y}{x}\right),$$

where ψ is an arbitrary function.

Example 2

Find the general integral of the equation

$$\left(x^2 + y^2\right)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 0 .$$

Solution:

Let's write down a system of equations

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{du}{0} .$$

Using the property of proportion, we present the equation

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy}$$

as

$$\frac{dx + dy}{x^2 + y^2 + 2xy} = \frac{dx - dy}{x^2 + y^2 - 2xy},$$

$$\frac{d(x+y)}{(x+y)^2} = \frac{d(x-y)}{(x-y)^2} .$$

Integrating, we get

$$-\frac{1}{x+y} = -\frac{1}{x-y} + C_1 ,$$

$$\frac{1}{x-y} - \frac{1}{x+y} = C_1 ,$$

$$\frac{2y}{x^2 - y^2} = C_1 .$$

The last equality can be rewritten as

$$\frac{y}{x^2-y^2}=C_1.$$

The second equation of the system:

$$du=0.$$

$$u=C_2.$$

The general integral of a given equation has the form

$$\Phi\left(\frac{y}{x^2-y^2},u\right)=0,$$

or

$$u=\psi\left(\frac{y}{x^2-y^2}\right),$$

where ψ is an arbitrary function.

Example 3

Find solutions to the equation

$$(2y-u)\,u'_x+y\,u'_y=u.$$

Solution:

Let's make up a characteristic system

$$\frac{dx}{2y-u}=\frac{dy}{y}=\frac{du}{u}.$$

Solving the equation

$$\frac{dy}{y}=\frac{du}{u} \quad \Rightarrow \quad \frac{u}{y}=C_1,$$

we find the first integral

$$\psi_1(x,y,u)=\frac{u}{y}.$$

Using the rule of equal fractions, we will make an integrable combination

$$\frac{dx}{2y-u} = \frac{2dy-du}{2y-u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

From where we get another first integral

$$\psi_2(x,y,u) = x - 2y + u.$$

Therefore, the general integral of the given equation has the form

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0,$$

where Φ is an arbitrary continuously differentiable function.

Example 4

Find a general solution to the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2.$$

<u>Solution:</u>

Let's make up the equations of characteristics:

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-x^2-y^2}.$$

The first equation of this system can be solved separately from the second, since it does not contain u (the variable u, which stands in the left and right sides of this equation, is reduced).

From equality,

$$\frac{dx}{x} = \frac{dy}{y}$$

we obtain by integrating

$$\ln|x| = \ln|y| + \ln C,$$

from where we find the first integral of the system

$$\frac{y}{x} = C_1.$$

The second equation of this system

$$\frac{dx}{xu} = \frac{du}{-x^2 - y^2}$$

contains all three variables. To exclude the variable y, let's use the first integral found.

Since the desired integral curve lies on one of the surfaces defined by the first integral found, at each point of this curve $y = C_1 x$ (the value of the constant C_1 is the same at all points of the desired integral curve, but may differ if you switch to another integral curve).

Replace y with $C_1 x$ in the second equation, and after the transformations we get

$$-(1 + C_1^2)x\,dx = u\,du.$$

Integrating, we find the dependence

$$(1 + C_1^2)x^2 + u^2 = C_2.$$

This ratio containing C_2 is not the first integral, since it also contains an arbitrary constant C_1. Given that for the found curve

$$C_1 = \frac{y}{x},$$

we rewrite the ratio as

$$x^2 + y^2 + u^2 = C_2.$$

In this form of notation, the relation is performed for any of the integral curves, that is, it is the first integral.

The general solution of the first-order equation has the form (in an implicit form)

$$\Phi\left(\frac{y}{x}, x^2 + y^2 + u^2\right) = 0,$$

where Φ is an arbitrary differentiable function. It is possible to get an explicit solution from the last expression:

$$u = \pm\sqrt{f\left(\frac{y}{x}\right) - x^2 - y^2},$$

where f is an arbitrary differentiable function.

Example 5

Find a general solution to the partial differential equation

$$e^x \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = ye^x$$

Solution:

Creating a system

$$\frac{dx}{e^x} = \frac{dy}{y^2} = \frac{du}{ye^x}$$

From the first equation, we find one first integral

$$\frac{1}{y} - e^{-x} = C_1$$

and from the second, taking into account the equality

$$e^x = \frac{y}{1 - yC_1}$$

another first integral

$$u - \frac{\ln|y| - x}{e^{-x} - y^{-1}} = C_2$$

follows.

Thus, the general integral of this equation will be

$$\Phi\left(\frac{1}{y} - e^{-x}, \frac{\ln|y| - x}{e^{-x} - y^{-1}} - u\right) = 0$$

The general solution has the form

$$u = \frac{\ln|y| - x}{e^{-x} - y^{-1}} + \varphi\left(\frac{1}{y} - e^{-x}\right).$$

Example 6

Find a solution to the equation

$$u\frac{\partial u}{\partial x} + (u^2 - x^2)\frac{\partial u}{\partial y} + x = 0$$

under additional conditions:

a) $y = 2x^2,\ u = x;$

b) $y = 1 + x^2, \quad u = x.$

Solution:

Let's write down the equations of characteristics

$$\frac{dx}{u} = \frac{dy}{u^2 - x^2} = \frac{du}{-x}.$$

Let's find the first integral of the system:

$$\frac{dx}{u} = \frac{du}{-x};$$

$$x\,dx + u\,du = 0;$$

$$x^2 + u^2 = C_1.$$

To determine the next integral, we take

$$\frac{x\,dx + u\,du}{u^2 - x^2} = \frac{dy}{u^2 - x^2}.$$

Comparing the first and second relations, we get

$$x\,dx + u\,du = dy,$$

or

$$d(xu) = dy.$$

Another first integral has been found

$$xu - y = C_2$$

The general solution

$$\Phi(x^2 + u^2, xu - y) = 0.$$

a) Solving the Cauchy problem, it is convenient to take x as the parameter σ on this curve. Substituting into the first integrals x,

$$y = 2x^2,\ u = x,$$

we get

$$C_1 = 2x^2,\ C_2 = -1.$$

Therefore,

$$C_1 = -2C_2.$$

Then the particular solution has the form

$$x^2 + u^2 = -2xu + 2y,$$

or

$$(x+u)^2 = 2y.$$

b) Let's solve the problem under another initial conditions.

Substituting $x,\ y = 1 + x^2,\ u = x$ into the first integrals, we get $C_1 = 2x^2$, $C_2 = -1$. The independence of C_2 from x means that the curve given by the initial conditions lies on the surface $xu - y = -1$.

At the same time, C_1 depends on x, which means that different surfaces of the first family $x^2 + u^2 = C_1$ correspond to different points of a given line. Therefore, the line specified by the initial conditions is not a characteristic, and $xu - y = -1$ is the only integral surface satisfying the initial conditions.

Homework №6. (The deadline is the 4th of October).

Solve the Cauchy problem for the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xy,$$

satisfying the conditions $y = x,\ u = x^2.$