1.3.2. DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Consider an equation of the form

$$a_0(t)x^{(n)}(t)+a_1(t)x^{(n-1)}(t)+\ldots+a_n(t)x(t)=f(t)\,, \qquad (1.12)$$

where $a_i(t),\ i=0,1,\ldots,n$ - polynomials of degree m_i, function $f(t)$ is the original.

Let's denote $m=\max\ \{m_0,m_1,\ldots,m_n\}$.

We will assume that the Cauchy problem for equation (1.12) with the conditions

$$x(0)=x_0\,,\ x'(0)=x_1\,,\ \ldots,\ x^{(n-1)}(0)=x_{n-1}$$

has a solution on the set of originals.

Let $x(t)\leftrightarrow X(p)$.

According to the image differentiation rule, we have

$$t^k x^{(s)}(t)\leftrightarrow(-1)^k\frac{d^k}{dp^k}\left(L\left\{x^{(s)}(t)\right\}\right)=(-1)^k\frac{d^k}{dp^k}\left(p^sX(p)-p^{s-1}x_0-\ldots-x_{s-1}\right).$$

Thus, applying the Laplace transform to both parts of equation (1.12), equation (1.12) is transformed into an m-th order differential equation with respect to the image $X(p)$. After that, the task of integrating equation (1.12) is simplified.

Example 1

Find a solution to the equation

$$ty''(t)-(1+t)y'(t)+y(t)=0$$

<u>Solution:</u>

$y(t)\leftrightarrow Y(p)$

$y'(t)\leftrightarrow pY(p)-y(0)$

Using the image differentiation property ($t\,f(t)\leftrightarrow -F'(p)$), we have

$$t\,y'(t)\leftrightarrow -\left(pY(p)-y(0)\right)'=-\left(Y(p)+pY'(p)\right)=-p\frac{dY(p)}{dp}-Y(p),$$

(because $y(0)=\text{const}$).

$y''(t)\leftrightarrow p^2Y(p)-py(0)-y'(0)$

$$t\,y''(t)\leftrightarrow -\left(p^2Y(p)-py(0)-y'(0)\right)'=-\left(2pY(p)+p^2Y'(p)-y(0)\right)$$

We substitute the images for these terms into the equation and multiply each term by (-1) to get rid of the numerous minus signs.

We have

$$t\,y''(t)-y'(t)-t\,y'(t)+y(t)=0$$

we substitute and get

$$2pY(p)+p^2Y'(p)-y(0)-\left(-pY(p)+y(0)\right)-\left(Y(p)+pY'(p)\right)-Y(p)=0$$

$$2pY(p)+p^2Y'(p)-y(0)+pY(p)-y(0)-Y(p)-pY'(p)-Y(p)=0$$

$$\left(p^2-p\right)Y'(p)+(3p-2)Y(p)=2y(0)=2C_0$$

let's denote an unknown quantity by $C_0=\text{const}$.

In the image space, we did not get an algebraic equation, as before, we get a first-order differential equation (linear inhomogeneous). First, we solve the corresponding homogeneous equation, when in the right part, instead of the term $2C_0$, there is zero.

$$\left(p^2-p\right)Y'(p)+(3p-2)Y(p)=0$$

The resulting equation is an equation with separable variables.

Since $Y'(p)=\dfrac{dY}{dp}$ then

$$\left(p^2-p\right)\frac{dY}{dp}=-(3p-2)Y(p)$$

$$\frac{dY}{Y}=-\frac{(3p-2)}{\left(p^2-p\right)}dp$$

We can integrate and get:

$$\ln Y(p)=-\int\frac{(3p-2)}{p^2-p}dp=-\int\frac{(2p-1)}{p^2-p}dp-\int\frac{(p-1)}{p^2-p}dp=$$

$$==-\int\frac{d\left(p^2-p\right)}{\left(p^2-p\right)}-\int\frac{dp}{p}=-\ln\left(p^2-p\right)-\ln p=-\ln\left(p^3-p^2\right)+\ln C$$

We have received

$$Y_0(p) = \frac{C}{p^3 - p^2}$$

We have set the index to zero, since this is not a solution to our equation.

We use the method of variation of parameters (variation of constants), the Lagrange method to

$$\left(p^2 - p\right)Y'(p) + (3p-2)Y(p) = 2y(0) = 2C_0$$

The solution of an inhomogeneous equation is sought in the same form as the solution of a homogeneous one, only instead of an arbitrary constant C, a new unknown function $C(p)$ is put.

$$Y_0(p) = \frac{C}{p^3 - p^2} \rightarrow Y(p) = \frac{C(p)}{p^3 - p^2}$$

This change of variables is substituted into the inhomogeneous equation. Only first you need to calculate $Y'(p)$:

$$Y'(p) = \frac{C'(p)}{p^3 - p^2} - \frac{C(p)\left(3p^2 - 2p\right)}{\left(p^3 - p^2\right)^2}$$

$$\left(p^2 - p\right)\left[\frac{C'(p)}{p^3 - p^2} - \frac{C(p)\left(3p^2 - 2p\right)}{\left(p^3 - p^2\right)^2}\right] + (3p-2)\left[C(p)\frac{1}{p^3 - p^2}\right] = 2C_0$$

$$\frac{C'(p)}{p} - \frac{C(p)\left(3p^2 - 2p\right)}{p\left(p^3 - p^2\right)} + \frac{(3p-2)C(p)}{p^3 - p^2} = 2C_0$$

$$C'(p) = 2C_0 p$$

$$C(p) = C_0 p^2 + C_1$$

Substituting the found function in $Y(p) = \frac{C(p)}{p^3 - p^2}$ and get

$$Y(p) = \frac{C_0 p^2 + C_1}{p^3 - p^2} = \frac{C_0}{p-1} + \frac{C_1}{(p-1)p^2}$$

From the image of the solution, you need to calculate the inverse Laplace transform (find the original)

$$\frac{1}{p-1} \leftrightarrow e^t$$

For the second term, let's use the property ($\int_0^t f(u)\,du \leftrightarrow \frac{F(p)}{p}$):

$$\frac{1}{p(p-1)} \leftrightarrow \int_0^t e^u du = e^t - 1$$

$$\frac{1}{p^2(p-1)} \leftrightarrow \int_0^t (e^u - 1)\,du = e^t - 1 - t$$

Our solution is

$$y(t) = C_0 e^t + C_1 (e^t - 1 - t)$$

Let's denote

$$C_0 + C_1 = \tilde{C}_0$$

$$-C_1 = \tilde{C}_1$$

$$y(t) = e^t\left(C_0 + C_1\right) + C_1\left(-1 - t\right)$$

$$y(t) = \tilde{C}_0\, e^t + \tilde{C}_1\left(1 + t\right)$$

Example 2

Find a solution to the equation

$$tx''(t) - (1+t)x'(t) + 2(1-t)x(t) = 0 \;.$$

<u>Solution:</u>

Let $x(t) \leftrightarrow X(p)$.

Then, using the property of differentiating the original and differentiating the image, we write:

$$x'(t) \leftrightarrow pX(p) - x(0) \;,$$

$$x''(t) \leftrightarrow p^2 X(p) - px(0) - x'(0) \,,$$

$$tx(t) \leftrightarrow -\frac{dX(p)}{dp} \;,$$

$$tx'(t) \leftrightarrow -\frac{d}{dp}\{pX(p) - x(0)\} = -p\frac{dX}{dp} - X(p) \,,$$

$$tx''(t) \leftrightarrow -\frac{d}{dp}\left\{p^2 X(p) - px(0) - x'(0)\right\} = -p^2\frac{dX}{dp} - 2pX(p) + x(0) \,.$$

Applying the Laplace transform to a given equation, we obtain the following operator equation:

$$-p^2\frac{dX}{dp}-2pX(p)+x(0)-$$

$$-pX(p)+x(0)+p\frac{dX(p)}{dp}+X(p)+2X(p)+2\frac{dX(p)}{dp}=0\ ,$$

which can be easily reduced to the form

$$(p^2-p-2)\frac{dX}{dp}+3(p-1)X(p)=2x(0)\,.$$

Having solved the obtained ordinary differential equation, for example, by the method of variation of parameters, we construct its general solution:

$$X(p)=\frac{x(0)}{p-2}+\frac{c}{(p-2)(p+1)^2}\,.$$

Here c is an arbitrary constant. Further, since

$$\frac{1}{p-2}\leftrightarrow e^{2t}\ ,$$

$$\frac{1}{(p+1)^2}\leftrightarrow te^{-t}\ ,$$

$$\frac{1}{(p-2)(p+1)^2}\leftrightarrow\int_0^t \tau e^{-\tau}e^{2(t-\tau)}d\tau=\frac{1}{9}\left(e^{2t}-(3t+1)e^{-t}\right),$$

then the general solution of the given equation will have the form

$$x(t)=x(0)e^{2t}+c\left(\frac{1}{9}e^{2t}-\frac{1}{9}(3t+1)e^{-t}\right)=\left(x(0)+c\right)e^{2t}-c\left(3t+1\right)e^{-t}\,.$$

1.4. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF DIFFERENTIAL EQUATIONS WITH A DELAYED ARGUMENT

Consider a linear differential equation with a delayed argument with constant coefficients:

$$x^{(n)}(t)=\sum_{k=0}^{n-1} a_k x^{(k)}(t-\tau_k)+f(t),\ 0<t<+\infty, \qquad (1.13)$$

where $a_k=\text{const}$, $\tau_k=\text{const}\geq 0$.

Let's assume that

$$x(t)=x'(t)=\ldots=x^{(n-1)}(t)\equiv 0,$$

$$\text{for } \forall t<0.$$

Let it be required to find a solution to equation (1.13) satisfying the initial conditions:

$$x(0)=x'(0)=\ldots=x^{(n-1)}(0)=0. \qquad (1.14)$$

Applying the Laplace transform to both parts of equation (1.13) and taking into account the delay property of the original, we obtain the operator equation for the image $X(p)\leftrightarrow x(t)$:

$$p^n X(p)=\sum_{k=0}^{n-1} a_k p^k X(p) e^{-\tau_k p}+F(p), \qquad (1.15)$$

where $F(p)\leftrightarrow f(t)$.

From (1.15) for $X(p)$ we will have

$$X(p)=\frac{F(p)}{p^n-\sum_{k=0}^{n-1}a_k p^k e^{-\tau_k p}}. \qquad (1.16)$$

The original for the image (1.16) defines the solution of equation (1.13) satisfying the conditions (1.14).

Let's formulate a problem for an equation with a delayed argument describing a *process with an aftereffect*. It is required to find a continuously differentiable solution $x(t)$ for $t>t_0$ of the equation

$$x'(t)=f\left(t,x(t),x(t-\tau)\right),\ \tau=\text{const}>0, \qquad (1.17)$$

if it is known that

$$x(t)=\varphi(t),\ t\in[t_0-\tau,t_0]. \qquad (1.18)$$

The initial function $\varphi(t)$ is a given continuously differentiable function.

The segment $[t_0-\tau,t_0]$ on which the function $\varphi(t)$ is defined is called the *initial set.*

If equation (1.17) is linear, then its solution satisfying condition (1.18) can be found using the Laplace transform. Let $t_0=0$, then when constructing the corresponding operator equation, it should be taken into account that for the image of the function $x(t-\tau)$ we have

$$x(t-\tau)\leftrightarrow\int_0^{\infty}e^{-pt}x(t-\tau)dt=\int_{-\tau}^{\infty}e^{-p(\eta+\tau)}x(\eta)d\eta=$$

$$=\int_{-\tau}^{0}e^{-p(\eta+\tau)}x(\eta)d\eta+\int_0^{\infty}e^{-p(\eta+\tau)}x(\eta)d\eta=e^{-p\tau}\int_{-\tau}^{0}e^{-p\eta}\varphi(\eta)d\eta+e^{-p\tau}X(p).$$

When restoring originals from known images, you can use the following decomposition:

$$\frac{1}{1-\frac{\gamma e^{-np}}{(p+a)^m}}=1+\frac{\gamma e^{-np}}{(p+a)^m}+\left(\frac{\gamma e^{-np}}{(p+a)^m}\right)^2+\ldots=\sum_{k=0}^{\infty}\left(\frac{\gamma e^{-np}}{(p+a)^m}\right)^k, \quad (1.19)$$

which is true for any $n, m \in N$, on condition $\operatorname{Re} p > 0$.

Example 1

Find a solution of the equation

$$x'(t)=x(t-1)+1, \; x(0)=0.$$

<u>Solution:</u>

Assuming that $x(t) \equiv 0$ for $t \in [-1,0]$, applying the Laplace transform to a given equation, we obtain the following operator equation:

$$pX(p)=X(p)e^{-p}+\frac{1}{p}.$$

Where from

$$X(p)=\frac{1}{p}\frac{1}{pe^{-p}}=\frac{1}{p^2}\frac{1}{1-\frac{e^{-p}}{p}}.$$

Further, applying the formula (1.19), we have

$$X(p)=\frac{1}{p^2}\sum_{k=0}^{\infty}\left(\frac{e^{-p}}{p}\right)^k=\sum_{k=0}^{\infty}\frac{e^{-pk}}{p^{k+2}}.$$

Given the delay property, we construct an expression for the corresponding original $x(t)$ in the form

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\theta(t-k).$$

Example 2

Find a solution of the equation

$$x'(t)=x(t-1),$$

if $x(t)\equiv 2$ for $\forall t\in[-1,0]$.

Solution:

Let $x(t)\leftrightarrow X(p)$.

It follows from the condition that $x(0)=2$, so we have

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-2.$$

We apply the Laplace transform to both parts of the given equation. For the right side of the equation, we have

$$x(t-1)\leftrightarrow\int_{0}^{\infty}e^{-pt}x(t-1)dt=\int_{-1}^{\infty}e^{-p(z+1)}x(z)dz=$$

$$=\int_{-1}^{0}e^{-p(z+1)}x(z)dz+\int_{0}^{\infty}e^{-p(z+1)}x(z)dz=$$

$$=2\int_{-1}^{0}e^{-p(z+1)}dz+e^{-p}X(p)=\frac{2}{p}\left(1-e^{-p}\right)+e^{-p}X(p).$$

Therefore, the corresponding operator equation has the form

$$pX(p)-2=\frac{2}{p}\left(1-e^{-p}\right)+e^{-p}X(p).$$

From here we get

$$X(p)=2\frac{p+1-e^{-p}}{p^2-pe^{-p}}=\frac{2}{p}+\frac{2}{p(p-e^{-p})}.$$

Using the result of the previous example, we will construct the original in the form

$$x(t)=2\left(\theta(t)+\sum_{k=0}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\theta(t-k)\right).$$

Example 3

Find a solution of the equation

$$x'(t)+2x(t)-x(t-1)=f(t),$$

if $x(0)=0$ and $x(t)\equiv 0$ for $\forall t<0$.

Solution:

Let $x(t)\leftrightarrow X(p)$, $f(t)\leftrightarrow F(p)$.

Since under the given conditions $x(t-1)\leftrightarrow e^{-p}X(p)$, the operator equation corresponding to the given one has the form

$$pX(p)+2X(p)-e^{-p}X(p)=F(p).$$

The solution of this equation is written as a product:

$$X(p)=\frac{1}{p+2-e^{-p}}F(p).$$

Let's build the original for the function

$$Y(p)=\frac{1}{p+2-e^{-p}},$$

by performing the following transformations:

$$Y(p)=\frac{1}{p+2-e^{-p}}=\frac{1}{p+2}\left(\frac{1}{1-\frac{e^{-p}}{p+2}}\right)=\sum_{k=0}^{\infty}\frac{e^{kp}}{(p+2)^{k+1}}.$$

The last equality is written, taking into account the formula (1.19). Turning to the originals for the summands of the sum of the series, using $t^k \leftrightarrow \frac{k!}{p^{k+1}}$, $t<0$ and the delay property, we find

$$Y(p)\leftrightarrow y(t)=\sum_{k=0}^{\infty}\frac{(t-k)^k}{k!}e^{-2(t-k)}\theta(t-k).$$

The solution to this problem will be the function $x(t)$, which is a convolution of the functions $f(t)$ and $y(t)$:

$$x(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^t f(t-\tau)(\tau-k)^k e^{-2(\tau-k)}\theta(\tau-k)d\tau.$$