Differential Geometry. Home Assignments

Assignment 4

Problem 1

Consider the surface implicitly defined by the equation

$$xyz = a^3,$$

where x, y, z are positive real numbers. Find the second fundamental form, the Gaussian curvature, and the mean curvature of this surface.

Problem 2

Find the Gaussian and mean curvatures of the surface defined explicitly by

$$z = f(x, y),$$

where f is a smooth function of x and y.

Problem 3

Let an arbitrary surface Φ be parameterised by $\mathbf{r}(u, v)$, with the first and second fundamental forms given by

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

and

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

Consider the paired surface Φ' parameterized by

$$\mathbf{r}' = \mathbf{r}(u, v) + a\mathbf{n}(u, v),$$

where a is a constant scalar and $\mathbf{n}(u, v)$ is the unit normal vector of Φ at the point (u, v), defined as

$$\mathbf{n}(u, v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

Express the coefficients of the first and second fundamental forms of Φ',

$$I' = E' \, du^2 + 2F' \, du \, dv + G' \, dv^2,$$

and

$$II' = L' \, du^2 + 2M' \, du \, dv + N' \, dv^2,$$

in terms of the coefficients E, F, G, L, M, N of the fundamental forms of Φ.

Problem 1

Consider the surface implicitly defined by the equation

$$xyz = a^3,$$

where x, y, z are positive real numbers. Find the second fundamental form, the Gaussian curvature, and the mean curvature of this surface.

Sol: apply the parametrization. $\begin{cases} x = u \\ y = v \\ z = \frac{a^3}{uv} \end{cases}$ $\Rightarrow$ $\vec{r}(u,v) = (u, v, \frac{a^3}{uv})$, $u > 0, v > 0$.

$\vec{r}_u = (1, 0, -\frac{a^3}{u^2v})$. $\vec{r}_{uu} = (0, 0, \frac{2a^3}{u^3v})$

$\vec{r}_v = (0, 1, -\frac{a^3}{uv^2})$. $\vec{r}_{uv} = (0, 0, \frac{a^3}{u^2v^2})$

                        $\vec{r}_{vv} = (0, 0, \frac{2a^3}{uv^3})$.

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -a^3/u^2v \\ 0 & 1 & -a^3/uv^2 \end{vmatrix}}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{(\frac{a^3}{u^2v}, \frac{a^3}{uv^2}, 1)}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{(\frac{a^3}{u}, \frac{a^3}{v}, uv)}{\sqrt{a^6(\frac{1}{u^2}+\frac{1}{v^2}) + u^2v^2}}$$

$E = \vec{r}_u \cdot \vec{r}_u = 1 + \frac{a^6}{u^4v^3}$ $L = \vec{r}_{uu} \cdot \vec{n} = \frac{2a^3}{u^2}(\sqrt{a^6(\frac{1}{u^2}+\frac{1}{v^2}) + u^2v^2})^{-1}$

$F = \vec{r}_u \cdot \vec{r}_v = \frac{a^6}{u^3v^3}$ $M = \vec{r}_{uv} \cdot \vec{n} = \frac{a^3}{uv}(\sqrt{a^6(\frac{1}{u^2}+\frac{1}{v^2}) + u^2v^2})^{-1}$

$G = \vec{r}_v \cdot \vec{r}_v = 1 + \frac{a^6}{u^2v^4}$. $N = \vec{r}_{vv} \cdot \vec{n} = \frac{2a^3}{v^2}(\sqrt{a^6(\frac{1}{u^2}+\frac{1}{v^2}) + u^2v^2})^{-1}$

Gaussian $K = \frac{LN - M^2}{EG - F^2} = \frac{\frac{4a^6/u^2v^2 \cdot (a^6(\frac{1}{u^2}+\frac{1}{v^2}) + u^2v^2)^{-1} - a^6/u^2v^2 (a^6(\frac{1}{u^2}+\frac{1}{v^2}) + u^2v^2)^{-1}}}{(1 + \frac{a^6}{u^4v^3})(1 + \frac{a^6}{u^2v^4}) - \frac{a^{12}}{u^6v^6}}$.

$= \frac{\frac{3a^6}{u^2v^2} \cdot \frac{1}{u^2v^2}}{(1 + \frac{a^6}{u^2v^3}(\frac{1}{u^2} + \frac{1}{v^2}))^2} = \frac{3a^6}{u^4v^4(1 + \frac{a^6}{u^2v^2}(\frac{1}{u^2} + \frac{1}{v^2}))^2}$

mean : $H = \frac{1}{2} \cdot \frac{LG - 2MF + NE}{EG - F^2} = \frac{\frac{a^9}{u^4v^4} + a^3(\frac{1}{u^2}+\frac{1}{v^2})}{1 + \frac{a^6}{u^2v^3}(\frac{1}{u^2} + \frac{1}{v^2}) \cdot \sqrt{a^6(\frac{1}{u^2}+\frac{1}{v^2}) + u^2v^2}}$

$= \frac{a^3(a^6 + u^2v^4 + u^4v^2)}{u^5v^5(1 + \frac{a^6}{u^2v^2}(\frac{1}{u^2} + \frac{1}{v^2}))^{\frac{3}{2}}}$

Thus, in explicit notation.

$\text{II} = L dx^2 + 2M dx dy + N dy^2 = \frac{\frac{2a^3}{x^2}(dx)^2 + \frac{2a^3}{xy}(dx dy) + \frac{2a^3}{y^2}(dy)^2}{\sqrt{a^6(\frac{1}{x^2} + \frac{1}{y^2}) + x^2y^2}}$

Gaussian curvature: $K = \frac{3a^6}{x^4y^2(1 + \frac{a^6}{x^2y^2}(\frac{1}{x^2} + \frac{1}{y^2}))^2}$

mean curvature $H = \frac{a^3(a^6 + x^4y^2 + x^2y^4)}{x^5y^5(1 + \frac{a^6}{x^2y^2}(\frac{1}{x^2} + \frac{1}{y^2}))^{\frac{3}{2}}}$.

Problem 2
Find the Gaussian and mean curvatures of the surface defined explicitly by

$$z = f(x,y),$$

where f is a smooth function of x and y.

$$\vec{r} = \vec{r}(x, y, f(x,y)).$$

$$\vec{r}_x = (1, 0, f_x) \qquad E = (\vec{r}_x)^2 = 1 + f_x^2 \qquad \vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} = \frac{(-f_x, -f_y, 1)}{\sqrt{f_x^2 + f_y^2 + 1}}$$

$$\vec{r}_y = (0, 1, f_y) \qquad F = \vec{r}_x \cdot \vec{r}_y = f_x f_y$$

$$G = (\vec{r}_y)^2 = 1 + f_y^2$$

$$\vec{r}_{xx} = (0, 0, f_{xx}) \qquad L = \vec{r}_{xx} \cdot \vec{n} = f_{xx} \cdot \left(\sqrt{f_x^2 + f_y^2 + 1}\right)^{-1}$$

$$\vec{r}_{xy} = (0, 0, f_{xy}) \qquad M = \vec{r}_{xy} \cdot \vec{n} = f_{xy} \left(\sqrt{f_x^2 + f_y^2 + 1}\right)^{-1}$$

$$\vec{r}_{yy} = (0, 0, f_{yy}) \qquad N = \vec{r}_{yy} \cdot \vec{n} = f_{yy} \cdot \left(\sqrt{f_x^2 + f_y^2 + 1}\right)^{-1}$$

Gaussian $\quad K = \dfrac{LN - M^2}{EG - F^2} = \dfrac{f_{xx} f_{yy} - (f_{xy})^2}{(f_x^2 + f_y^2 + 1) \cdot (1 + f_x^2 + f_y^2)} = \dfrac{f_{xx} f_{yy} - (f_{xy})^2}{(1 + f_x^2 + f_y^2)^2}$

Mean $\quad H = \dfrac{1}{2} \cdot \dfrac{LG - 2MF + NE}{EG - F^2} = \dfrac{1}{2} \cdot \dfrac{f_{xx}(1 + f_y^2) + f_{yy}(1 + f_x^2) - 2 f_x f_y f_{xy}}{(f_x^2 + f_y^2 + 1)^{3/2}}$

$$= \dfrac{f_{xx} f_y^2 + f_x^2 f_{yy} + f_{xx} + f_{yy} - 2 f_x f_y f_{xy}}{2(f_x^2 + f_y^2 + 1)^{3/2}}$$

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$$\mathbf{n}(u,v) = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

Express the coefficients of the first and second fundamental forms of Φ',

$$I' = E'\,du^2 + 2F'\,du\,dv + G'\,dv^2,$$

and

$$II' = L'\,du^2 + 2M'\,du\,dv + N'\,dv^2,$$

in terms of the coefficients E, F, G, L, M, N of the fundamental forms of Φ.

Sol: $\quad \vec{n}_u = \dfrac{-LG + MF}{EG - F^2} \vec{r}_u + \dfrac{LF - ME}{EG - F^2} \vec{r}_v \qquad \vec{r}_u' = \vec{r}_u + a\vec{n}_u$

$$\vec{n}_v = \dfrac{-MG + NF}{EG - F^2} \vec{r}_u + \dfrac{MF - NE}{EG - F^2} \vec{r}_v \qquad \vec{r}_v' = \vec{r}_v + a\vec{n}_v$$

$$E' = (\vec{r_u'})^2 = \vec{r_u}^2 + 2a\,\vec{r_u}\,\vec{n_u} + a^2\,\vec{n_u}^2 =$$

$$= E + 2a\left(\frac{-LG + MF}{EG - F^2}\,E + \frac{LF - ME}{EG - F^2}\,F\right) + a^2\left(\left(\frac{-LG + MF}{EG - F^2}\right)^2 E + \left(\frac{LF - ME}{EG - F^2}\right)^2 G + \frac{-LG + MF}{EG - F^2}\frac{LF - ME}{EG - F^2}\cdot F\right.$$

$$F' = (\vec{r_u'} \cdot \vec{r_v'}) = \vec{r_u}\cdot\vec{r_v} + a\vec{r_u}\cdot\vec{n_v} + a\vec{r_v}\cdot\vec{n_u} + a^2\vec{n_v}\cdot\vec{n_u}$$

$$= F + a\left[\frac{-LG + MF}{EG - F^2}\,F + \frac{LF - ME}{EG - F^2}\,G + \frac{-MG + NF}{EG - F^2}\,E + \frac{MF - NE}{EG - F^2}F\right]$$

$$+ a^2\left[\frac{-LG + MF}{EG - F^2}\frac{-MG + NF}{EG - F^2}E + \frac{-LG + MF}{EG - F^2}\cdot\frac{MF - NE}{EG - F^2}\cdot F + \frac{-MG + NF}{EG - F^2}\cdot\frac{LF - ME}{EG - F^2}\cdot F + \frac{LF - ME}{EG - F^2}\cdot\frac{MF - NE}{EG - F^2}\cdot G\right]$$

$$G' = \vec{r_v'}^2 + 2a\,\vec{r_v}\cdot\vec{n_v} + a^2\vec{n_v}^2 = F + 2a\left(\frac{-MG + NF}{EG - F^2}\,F + \frac{MF - NE}{EG - F^2}\,G\right) +$$

$$a^2\left[\left(\frac{-MG + NF}{EG - F^2}\right)^2 E + \left(\frac{MF - NE}{EG - F^2}\right)^2 G + 2\frac{-MG + NF}{EG - F^2}\frac{MF - NE}{EG - F^2}\,F\right]$$

since $\quad\vec{r_u}\cdot\vec{n} = 0 \qquad\qquad \vec{r_u}\times\vec{r_v} = \vec{r_u}\times\vec{r_v}$

$\qquad\qquad\vec{r_v}\cdot\vec{n} = 0. \qquad\qquad$ thus $\vec{n'} = \vec{n}$.

$$L' = \vec{r_{uu}'}\cdot n = L + a\,\vec{n_{uu}}\cdot\vec{n}$$

$$M' = \vec{r_{uv}'}\cdot n = M + a\,\vec{n_{uv}}\cdot\vec{n}$$

$$N' = \vec{r_{vv}'}\cdot n = N + a\,\vec{n_{vv}}\cdot\vec{n}$$

consider $\quad\dfrac{d\,(n\cdot n)}{du^2} = \vec{n_{uu}}\cdot\vec{n} + \vec{n_u}\cdot\vec{n_u} = 0.\quad$ since $\vec{n}$ has unit length.

$\Rightarrow \vec{n_{uu}}\cdot\vec{n} = -\|\vec{n_u}\|^2 \quad$ similarly. $\quad \vec{n_{uv}}\cdot\vec{n} = -\vec{n_v}\cdot\vec{n_u} \qquad\qquad \vec{n_{vv}}\cdot\vec{n} = -\|\vec{n_v}\|^2$

$$\|\vec{n_u}\|^2 = \frac{(-LG + MF)^2 E + 2(-LG + MF)(LF - ME))F + (LF - ME)^2 G.}{(EG - F^2)^2}$$

$$\vec{n_v}\cdot\vec{n_u} = \frac{-LG + MF}{EG - F^2}\frac{-MG + NF}{EG - F^2}E + \frac{-LG + MF}{EG - F^2}\cdot\frac{MF - NE}{EG - F^2}\cdot F + \frac{-MG + NF}{EG - F^2}\cdot\frac{LF - ME}{EG - F^2}\cdot F + \frac{LF - ME}{EG - F^2}\cdot\frac{MF - NE}{EG - F^2}\cdot G$$

$$\|\vec{n_v}\|^2 = \left(\frac{-MG + NF}{EG - F^2}\right)^2 E + \left(\frac{MF - NE}{EG - F^2}\right)^2 G + 2\frac{-MG + NF}{EG - F^2}\frac{MF - NE}{EG - F^2}F$$

$$L' = L - \frac{a}{(EG - F^2)^2}\left[(-LG + MF)^2 E + 2(-LG + MF)(LF - ME))F + (LF - ME)^2 G\right]$$

$$M' = M - \frac{a}{(EG - F^2)^2}\left((-LG + MF)(-MG + NF)E + \left[(-LG + MF)(MF - NE) + (-MG + NF)(LF - ME)\right]\cdot F + (LF - ME)(MF - NE)G\right.$$

$$N' = N - \frac{a}{(EG - F^2)^2}\left[(-MG + NF)^2 E + (MF - NE)^2 G + 2(-MG + NF)(MF - NE)F\right]$$