I. First order differential equations.

§ 1. Equation solved with respect to the derivative. Peano's theorem.

Equation

$$\dot{x} = X(t,x), \tag{1}$$

where the function $X(t,x)$ is continuous on the set $D \subset R^2$, is called a *first order ordinary differential equation solved with respect to the derivative*.

Definition. *A solution of the equation* (1) is a function $x = \varphi(t)$, defined on the interval $\langle a,b \rangle$ which being substituted into equation (1) turns this equation into an identity.

Remark. If $x = \varphi(t)$ is the solution of (1) with $t \in \langle a,b \rangle$, then

1) a point $(t, \varphi(t))$ belongs to the set D for all $t \in \langle a,b \rangle$,
2) the function $\varphi(t)$ continuously differentiable on the interval $\langle a,b \rangle$. At the same time, if $a \in \langle a,b \rangle$ then under the derivative of the function φ at the point $t = a$ we understand the derivative on the right, and if $b \in \langle a,b \rangle$, and under the derivative of the function φ at the point $t = b$ we understand the derivative on the left.

Graph of the solution $x = \varphi(t)$, $t \in \langle a,b \rangle$ is called *an integral curve.*

Definition. The problem of finding a solution $x = \varphi(t)$ of equation (1), satisfying the condition $x_0 = \varphi(t_0)$ ($t_0 \in \langle a,b \rangle$, $(t_0, x_0) \in D$), is called the *Cauchy problem* for equation (1).

Condition $x(t_0) = x_0$ is called *the initial condition*, and the point (t_0, x_0) is called *the initial data*.

Geometric interpretation of equation (1).

If $x = \varphi(t)$ is the solution of equation (1) passing through the point (t_0, x_0), then $\dot{\varphi}(t_0) = X(t_0, x_0)$. Thus, $X(t_0, x_0)$ is the tangent of the slope of the tangent to the integral curve passing through the point (t_0, x_0). Thus, the right-hand side of the equation (1) on the set D defines the so-called *direction field*.

Mechanical interpretation of the equation (1).

On the set D equation (1) determines the *law of motion of a material point*. At the fixed moment of time $t = t_0$ the right-hand side of the equation specifies the instantaneous velocity of a material point passing through $x_0 = \varphi(t_0)$ in the following way: $\dot{\varphi}(t_0) = X(t_0, \varphi(t_0))$.

With this interpretation, we say that equation (1) defines on the set D *the speed field.*

Now consider equation (1) with continuous function $X(t,x)$ on the set D, which is a rectangle:

$$D = \{(t,x) : |t - t_0| \le a, |x - x_0| \le b\},$$

$a, b \in R$, $a > 0$, $b > 0$.

Set the initial condition

$$x(t_0) = x_0 \quad . \tag{2}$$

According to the Weierstrass theorem on the boundedness of a continuous function on a compact set, there exists a constant $M > 0$ such that

$$|X(t,x)| \le M \text{ for all } (t,x) \in D.$$

Set $h = \min(a, b/M)$.

Theorem 1 (Peano's theorem). Under the above assumptions on the segment $[t_0 - h, t_0 + h]$ there exists a solution of the Cauchy problem (1), (2).

Definition. Segment $[t_0 - h, t_0 + h]$ is called the Peano segment.

Remark. Note that the Peano segment is defined, generally speaking, ambiguously. But for any such segment Theorem 1 is true.

Proof of the Theorem 1. We will prove the existence of the solution on the segment $P = [t_0, t_0 + h]$.

First, we introduce the sequence $\{d_k\}_{k=1}^{+\infty}$ of splittings of the segment P:

$$d_k = \{t_0 = t_0^k < t_1^k < \ldots < t_{n_k - 1}^k < t_{n_k}^k = t_0 + h\}.$$

Let's set $\lambda_k = rank\, d_k = \max\limits_{j=1,\ldots,n_k} \left(t_j^k - t_{j-1}^k\right)$ - *the rank* of the splitting d_k.

The sequence of splittings is chosen in such a way that the condition

$$\lambda_k \underset{k \to +\infty}{\to} 0$$

(3)

holds.

For every splitting d_k we build on a segment P *the Euler polyline* $\varphi_k(t)$ using the following recurrent algorithm.

Let $\varphi_k(t_0) = x_0$. Then we assume

. (4)

Lemma 1. The formula (4) for any $k \in N$ correctly defines the function $\varphi_k(t)$ throughout the segment P, and the inequality

$$|\varphi_k(t) - x_0| \leq M(t - t_0)$$

(5)

holds for all $t \in P$.

Remark. If the function $\varphi_k(t)$ ($k \in N$) is defined and satisfies inequality (5) for some $t \in P$, then the inequality

$$|\varphi_k(t) - x_0| \leq b$$

(6)

holds. Indeed, if $t \in P$, then $t_0 \le t \le t_0 + h$ and $t - t_0 \le h$, and from (5) it follows that $\left|\varphi_k(t) - x_0\right| \le Mh \le M \cdot b/M = b$, that is (6) is true.

Proof of the Lemma 1. We prove the lemma by the method of mathematical induction, carrying out induction on j.

Base of induction. If $t \in \left[t_0, t_1^k\right]$ then $\varphi_k(t) = x_0 + X(t_0, x_0)(t - t_0)$, and hence

$$\left|\varphi_k(t) - x_0\right| = \left|X(t_0, x_0)\right|(t - t_0) \le M(t - t_0),$$

that is inequalities (5) and (6) hold and $\left(t, \varphi_k(t)\right) \in D$ for $t \in \left[t_0, t_1^k\right]$.

Inductive transition. Let us now show that if on the segment $\left[t_0, t_{j-1}^k\right]$ $(j = 2, \ldots, n_k)$ the function $\varphi_k(t)$ is defined and inequality (5) is holds, then on the segment $\left[t_0, t_j^k\right]$ function $\varphi_k(t)$ is also defined and inequality (5) is also holds.

Indeed, by assumption

$$\left|\varphi_k\left(t_{j-1}^k\right) - x_0\right| \le M\left(t_{j-1}^k - t_0\right),$$

and then $\left|\varphi_k\left(t_{j-1}^k\right) - x_0\right| \le b$. Then $\left(t_{j-1}^k, \varphi_k\left(t_{j-1}^k\right)\right) \in D$ and formula (4) defines $\varphi_k(t)$ for $t \in \left[t_{j-1}^k, t_j^k\right]$.

From formula (4) it follows that for $t \in \left[t_{j-1}^k, t_j^k\right]$

$$\left|\varphi_k(t) - \varphi_k\left(t_{j-1}^k\right)\right| \le \left|X\left(t_{j-1}^k, \varphi_k\left(t_{j-1}^k\right)\right)\right|\left(t - t_{j-1}^k\right) \le M\left(t - t_{j-1}^k\right).$$

Hence

$$\left|\varphi_k(t) - x_0\right| \le \left|\varphi_k(t) - \varphi_k\left(t_{j-1}^k\right)\right| + \left|\varphi_k\left(t_{j-1}^k\right) - x_0\right| \le$$
$$\le M\left(t - t_{j-1}^k\right) + M\left(t_{j-1}^k - t_0\right) = M(t - t_0),$$

and inequality (5) is true for $t \in \left[t_0, t_j^k\right]$.

According to the principle of mathematical induction, the function $\varphi_k(t)$ defined for all $t \in P$ and inequalities (5) and (6) hold. Lemma 1 is proved.

Now for every splitting d_k of the segment P we define *the step function* $\psi_k(t)$:

(7)

Lemma 2. For any $t \in P$ and any $k \in N$ the equality

$$\varphi_k(t) = x_0 + \int_{t_0}^{t} \psi_k(\tau) d\tau$$

(8)
holds.

*Proof of the Lemma 2. W*e prove the lemma by the method of mathematical induction, carrying out induction on j.

Base of induction. For $j = 1$ equality (8) is obvious: for $t_0^k \le t < t_1^k$

$$\varphi_k(t) = x_0 + X(t_0, x_0)(t - t_0) = x_0 + \psi_k(t_0)(t - t_0) = x_0 + \int_{t_0}^{t} \psi_k(\tau) d\tau .$$

For $t = t_1^k$ equality (8) is also true (the value of the integrand $\psi_k(t)$ at one point $t = t_1^k$ does not affect the value of the integral).

Inductive transition. Let's assume that for $t \in [t_0, t_{j-1}^k]$ $(j = 2, ..., n_k)$ the equality (8) is true.

Let us show that the equality (8) holds for $t \in [t_0, t_j^k]$.

By the induction hypothesis

$$\varphi_k(t_{j-1}^k) = x_0 + \int_{t_0}^{t_{j-1}^k} \psi_k(\tau) d\tau .$$

According to formulas (4) and (7) for $t \in [t_{j-1}^k, t_j^k)$

$$\varphi_k(t) = \varphi_k(t_{j-1}^k) + X(t_{j-1}^k, \varphi_k(t_{j-1}^k))(t - t_{j-1}^k) =$$

$$= \varphi_k(t_{j-1}^k) + \psi_k(t_{j-1}^k)(t - t_{j-1}^k) = \varphi_k(t_{j-1}^k) + \int_{t_{j-1}^k}^{t} \psi_k(\tau) d\tau,$$

For $t = t_j^k$ the equality

$$\varphi_k(t) = \varphi_k(t_{j-1}^k) + \int_{t_{j-1}^k}^{t} \psi_k(\tau) d\tau$$

is also true.

Hence

$$\varphi_k(t) = x_0 + \int_{t_0}^{t_{j-1}^k} \psi_k(\tau)d\tau + \int_{t_{j-1}^k}^{t} \psi_k(\tau)d\tau = x_0 + \int_{t_0}^{t} \psi_k(\tau)d\tau,$$

therefore equality (8) is true for $t \in \left[t_0, t_j^k\right]$.

According to the principle of mathematical induction equality (8) is true for all $t \in P$. Lemma 2 is proved.

We recall two definitions and the Arzela-Ascoli lemma (from the course of mathematical analysis), which we will use below in the proof of the Peano theorem.

Definition. Function sequence $\left\{\xi_k(t)\right\}_{k=1}^{+\infty}$ given on the segment $[c,d]$ is said to be *uniformly bounded* on this segment if there exists a number K such that $\left|\xi_k(t)\right| \leq K$ for all $t \in [c,d]$ and all $k \in N$.

Definition. Function sequence $\left\{\xi_k(t)\right\}_{k=1}^{+\infty}$ given on the segment $[c,d]$ is called *equicontinuous* on this segment if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $k \in N$ and any $t_1, t_2 \in [c,d]$ from inequality $\left|t_1 - t_2\right| < \delta$ it follows the inequality $\left|\xi_k(t_1) - \xi_k(t_2)\right| < \varepsilon$.

Arzela-Ascoli lemma. If the function sequence $\left\{\xi_k(t)\right\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on the segment $[c,d]$ then there exists a subsequence $\left\{\xi_{k_m}(t)\right\}_{m=1}^{+\infty}$ which converges uniformly on this segment as $m \to +\infty$ to some function $\xi(t)$.

We continue the proof of Peano's theorem.

Lemma 3. Euler polyline sequence $\left\{\varphi_k(t)\right\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on the segment P.

Proof of Lemma 3. Let us show that the functions $\varphi_k(t)$ ($k \in N$) are uniformly bounded on P, using inequality (6):

$$\left|\varphi_k(t)\right| \leq \left|\varphi_k(t) - x_0\right| + \left|x_0\right| \leq b + \left|x_0\right|.$$

Now let's show that $\varphi_k(t)$ ($k \in N$) are equicontinuous.

Fix arbitrary $\varepsilon > 0$ and set $\delta = \varepsilon / M$. Let $t_1, t_2 \in P$, $|t_1 - t_2| < \delta$.

Note that from the definition of $\psi_k(t)$ it follows that $|\psi_k(t)| \le M$. And according to Lemma 2

$$\left|\varphi_k(t_1) - \varphi_k(t_2)\right| = \left|\int_{t_0}^{t_1} \psi_k(\tau) d\tau - \int_{t_0}^{t_2} \psi_k(\tau) d\tau\right| = \left|\int_{t_2}^{t_1} \psi_k(\tau) d\tau\right| \le M|t_1 - t_2| < M\delta = \varepsilon .$$

Lemma 3 is proved.

From the Arzela-Ascoli lemma it follows that from the sequence $\{\varphi_k(t)\}_{k=1}^{+\infty}$ we can choose uniformly converging on the segment P subsequence.

Without loss of generality we will assume that the sequence of splittings $\{d_k\}_{k=1}^{\infty}$ already chosen so that the sequence $\{\varphi_k(t)\}_{k=1}^{+\infty}$ converges uniformly on P at $k \to +\infty$ to the function $\varphi(t)$:

$$\varphi(t) = \lim_{k \to +\infty} \varphi_k(t).$$

Function $\varphi(t)$ is continuous as the uniform limit of continuous functions. Passing to the limit at $k \to +\infty$ in the inequality (6) we get $|\varphi(t) - x_0| \le b$ for $t \in P$. So the function $X(t, \varphi(t))$ exists and continuous on P.

Lemma 4. Function sequence $\psi_k(t)$ uniformly on P converges to the function $X(t, \varphi(t))$ at $k \to +\infty$.

Proof of Lemma 4. Since the function $X(t, x)$ continuous on compact D then by Cantor's theorem it is uniformly continuous on D. This means that for any $\varepsilon > 0$ there exists $\delta > 0$ such that from the condition $|t_1 - t_2| < \delta$, $|x_1 - x_2| < \delta$ it follows that

$$\left|X(t_1, x_1) - X(t_2, x_2)\right| < \varepsilon/2 \tag{9}$$

for any $(t_1, x_1), (t_2, x_2) \in D$.

Function $\varphi(t)$ is the uniform limit of functions $\varphi_k(t)$ at $k \to +\infty$, so there is $k_1 \in N$ such that $\left|\varphi(t) - \varphi_k(t)\right| < \delta$ for any $k > k_1$, $t \in P$.

Therefore, according to (9)

$$\left|X(t, \varphi(t)) - X(t, \varphi_k(t))\right| < \varepsilon/2$$

(10)
for $k > k_1$, $t \in P$.

Since $\lambda_k \underset{k \to +\infty}{\to} 0$ then there exists $k_2 \in N$ such that $\lambda_k < \min(\delta, \delta/M)$ for any $k > k_2$. Let us show that for $k > k_2$ and $t \in P$ inequality

$$\left|X(t, \varphi_k(t)) - \psi_k(t)\right| < \varepsilon/2$$

(11)
holds.

We fix an arbitrary $k > k_2$ and arbitrary $t \in P$.

If $t = t_{n_k}^k$ then according to formula (7)

$$\psi_k(t) = X\left(t_{n_k}^k, \varphi_k\left(t_{n_k}^k\right)\right) \text{ and } \left|X(t, \varphi_k(t)) - \psi_k(t)\right| = 0,$$

therefore inequality (11) holds.

If $t < t_{n_k}^k$ then there exists $j \in \{1, ..., n_k\}$ such that $t_{j-1}^k \le t < t_j^k$ and

$$0 \le t - t_{j-1}^k \le t_j^k - t_{j-1}^k \le \lambda_k < \delta, \tag{12}$$

and according to formula (4)

$$\left|\varphi_k(t) - \varphi_k\left(t_{j-1}^k\right)\right| = \left|X\left(t_{j-1}^k, \varphi_k\left(t_{j-1}^k\right)\right)\right|\left(t - t_{j-1}^k\right) \le M\lambda_k < M \cdot \delta/M = \delta. \tag{13}$$

According to formula (7)

$$\psi_k(t) = X\left(t_{j-1}^k, \varphi_k\left(t_{j-1}^k\right)\right),$$

and from (9), (12), (13) it follows that for $t < t_{n_k}^k$ inequality (11) also holds.

Let $k_0 = \max(k_1, k_2)$. Then for any $k > k_0$ inequalities (10) and (11) are hold. Therefore for any $k > k_0$ and any $t \in P$

$$\left|X\left(t,\varphi(t)\right)-\psi_k(t)\right| \le \left|X\left(t,\varphi(t)\right)-X\left(t,\varphi_k(t)\right)\right|+\left|X\left(t,\varphi_k(t)\right)-\psi_k(t)\right|<$$
$$<\varepsilon/2+\varepsilon/2=\varepsilon.$$

The last inequality proves the Lemma 4.

Uniform convergence of $\psi_k(t)$ to $X\left(t,\varphi(t)\right)$ allows passing in equality (8) to the limit at $k \to +\infty$:

$$\varphi(t)=x_0+\int_{t_0}^{t} X\left(\tau,\varphi(\tau)\right)d\tau . \tag{14}$$

The right-hand side of the equality (14) is continuously differentiable (as an integral with a variable upper limit of a continuous function). Hence, the left-hand side of the equality (the function $\varphi(t)$) is also continuously differentiable.

Differentiating equality (14) we obtain:

$$\dot{\varphi}(t)=X\left(t,\varphi(t)\right),$$

that means that the function $x=\varphi(t)$ is the solution of equation (1) for $t \in P$. Assuming in (14) $t=t_0$ we get $\varphi(t_0)=x_0$.

Using similar reasoning, one can show that on the segment $[t_0-h,t_0]$ there is also a function $x=\varphi(t)$, which is the solution of the Cauchy problem (1), (2).

Let us check that the function $x=\varphi(t)$ defined in this way is the solution of the Cauchy problem (1), (2) on the Peano interval $[t_0-h,t_0+h]$. For this, it suffices to show that $\varphi(t)$ differentiable at the point $t=t_0$.

Let's find one-sided derivatives $\dot{\varphi}_{\pm}(t_0)$ of the function $\varphi(t)$ at the point t_0.

It is clear that $\dot{\varphi}_{+}(t_0)=X(t_0,x_0)$ because $x=\varphi(t)$ is the solution of the Cauchy problem (1), (2) with $t \in [t_0,t_0+h]$. Analogically $\dot{\varphi}_{-}(t_0)=X(t_0,x_0)$. Since one-sided derivatives are equal then $\dot{\varphi}(t_0)$ exists and $\dot{\varphi}(t_0)=X(t_0,x_0)$. The theorem is proved.

Let the right-hand side of the equation (1) (the function $X(t,x)$) be continuous on the set D (not necessarily rectangular) and $(t_0,x_0) \in D$.

Definition. We say that the solution $x=\varphi(t)$ of Cauchy problem (1), (2) defined for $t\in\langle a,b\rangle$ ($t_0\in\langle a,b\rangle$), is *unique* on the interval $\langle a,b\rangle$ if any solution of this problem, defined on the same interval, coincides with $\varphi(t)$. This means that any solution $x=\psi(t)$ of Cauchy problem (1), (2) satisfies the identity $\psi(t)\equiv\varphi(t)$ on $\langle a,b\rangle$.

Theorem 2. Let the conditions of the Peano's theorem be satisfied, and let the solution $x=\varphi(t)$ of the Cauchy problem (1), (2) provided by this theorem is unique as a solution defined on the segment $P=[t_0,t_0+h]$.

Then for any sequence of splittings $\{d_k\}_{k=1}^{+\infty}$ of the segment P satisfying the condition $\lambda_k = rank\, d_k \underset{k\to+\infty}{\to} 0$, the sequence of Euler polylines $\{\varphi_k(t)\}_{k=1}^{+\infty}$ uniformly on P converges to $\varphi(t)$.

Proof. Assume on contrary that there exists a sequence of splittings $\{d_k\}_{k=1}^{+\infty}$ of segment P satisfying the condition $\lambda_k \underset{k\to+\infty}{\to} 0$, such that the Euler polyline sequence $\{\varphi_k(t)\}_{k=1}^{+\infty}$ does not converge uniformly on P to $\varphi(t)$.

Then there exists $\varepsilon>0$ such that for any $k_0\in N$ there are $k>k_0$ and $t\in P$ such that $|\varphi(t)-\varphi_k(t)|\ge\varepsilon$.

And therefore there exists an increasing sequence of natural numbers $\{k_j\}_{j=1}^{+\infty}$ and a sequence of points $\{t_j\}_{j=1}^{+\infty}\subset P$ such that

$$\left|\varphi(t_j)-\varphi_{k_j}(t_j)\right|\ge\varepsilon. \tag{15}$$

We know that $\lambda_{k_j} \underset{j\to+\infty}{\to} 0$. Therefore, as shown in the proof of Peano's theorem, from the sequence of Euler polylines $\{\varphi_{k_j}(t)\}_{j=1}^{\infty}$ we can choose a subsequence $\left\{\varphi_{k_{jm}}(t)\right\}_{m=1}^{\infty}$ uniformly on the segment P converging to some solution $\psi(t)$ of Cauchy problem (1), (2).

Hence there exists $m_0 \in N$ such that

$$\left|\psi(t)-\varphi_{k_{jm}}(t)\right|<\varepsilon$$

(16)

for any $m>m_0$ and any $t \in P$.

Fixing $m>m_0$, and due to inequalities (15) and (16) we have:

$$\left|\varphi\left(t_{j_m}\right)-\psi\left(t_{j_m}\right)\right| \geq \left|\varphi\left(t_{j_m}\right)-\varphi_{k_{jm}}\left(t_{j_m}\right)\right|-\left|\varphi_{k_{jm}}\left(t_{j_m}\right)-\psi\left(t_{j_m}\right)\right|>\varepsilon-\varepsilon=0.$$

This means that at the point $t=t_{j_m}$ the solution $x=\varphi(t)$ does not match the solution $x=\psi(t)$ which contradicts the uniqueness of the solution of the Cauchy problem (1), (2) on the interval P. Theorem 2 is proved.

Now let the function $X(t,x)$ be continuous in the domain G.

Теорема 3. Let $(t_0,x_0)\in G$. Then there is $h>0$ such that for $|t-t_0|\leq h$ the solution of the Cauchy problem (1), (2) is defined.

Proof. The set G is open, hence there are constants $a,b\in R$, $a>0$, $b>0$ such that the rectangle $D=\{(t,x):|t-t_0|\leq a, |x-x_0|\leq b\}$ is contained in G. Now we are in the conditions of Peano's theorem. Theorem 3 is proved.

§ 2. The problem of uniqueness.

Consider the equation

$$\dot{x} = X(t,x), \tag{1}$$

where $X(t,x) \in C(G)$, G - domain in R^2.

Consider also the Cauchy problem of the equation (1)

$$(t_0, x_0). \tag{2}$$

Definition 1. We say that at the point $(t_0, x_0) \in G$ the uniqueness condition is satisfied if there exists $\Delta > 0$ such that for $|t - t_0| \le \Delta$ there exists the solution $x = \varphi(t)$ of the Cauchy problem (1), (2), and for any δ such that $0 < \delta \le \Delta$ the solution $\varphi(t)$ is unique as a solution defined on a segment $[t_0 - \delta, t_0 + \delta]$.

If at the point $(t_0, x_0) \in G$ the uniqueness condition is satisfied, then we say that (t_0, x_0) - point of uniqueness for equation (1).

Example. It is not difficult to check that the solutions of the equation $\dot{x} = 3\sqrt[3]{x^2}$ are the functions $x \equiv 0$ and $x = (t+c)^3$, where $c = const$, $c \in R$.

Every point $(t_0, x_0) \in G = R^2$ is a point of uniqueness if $x_0 \ne 0$. Points $(t_0, 0)$ are not points of uniqueness.

Theorem 1. Let $(t_0, x_0) \in G$ and $x = \varphi(t)$ is the solution of the Cauchy problem (1), (2) defined for $t \in (a,b)$, $t_0 \in (a,b)$.

If each point $(t, \varphi(t))$, $t \in (a,b)$ is a point of uniqueness then the solution $x = \varphi(t)$ is unique on (a,b).

Proof of Theorem 1. We need to prove that any solution $x = \xi(t)$ of Cauchy problem (1), (2), defined for $t \in (a,b)$, satisfies the identity $\xi(t) \equiv \varphi(t)$ on the interval (a,b).

We will prove it by contradiction. Let there be the point $t^* \in (a,b)$ such that $\xi(t^*) \neq \varphi(t^*)$. Notice, that $t^* \neq t_0$ because $\xi(t_0) = \varphi(t_0) = x_0$. We assume for definiteness that $t^* > t_0$.

Now let's define the function $u(t) = \varphi(t) - \xi(t)$ on the segment $[t_0, t^*]$. Let

$$O = \left\{ t : t \in \left[t_0, t^* \right], u(t) = 0 \right\}.$$

O is non-empty set (since $t_0 \in O$). O is bounded and closed (as the set of zeros of a continuous function $u(t)$). Therefore, there is $\max O = t_1$, $t_1 \in O$.

Thus $t_0 \le t_1 < t^*$, $u(t_1) = 0$ and $u(t) \neq 0$ for all $t \in \left(t_1, t^*\right]$.

Let's put $\xi(t_1) = \varphi(t_1) = x_1$, and consider the Cauchy problem

$$(t_1, x_1). \tag{3}$$

According to the conditions of the theorem, at the point $(t_1, \varphi(t_1))$ the uniqueness condition is satisfied. Therefore, there exists $\Delta > 0$ such that for $|t - t_0| \le \Delta$ there exists the solution $x = \bar{\varphi}(t)$ of Cauchy problem (1), (2), and for any δ such that $0 < \delta \le \Delta$ the solution $\bar{\varphi}(t)$ is unique solution determined for $|t - t_0| \le \delta$. Let $0 < \delta < \min\left(\Delta, t_1 - a, t^* - t_1\right)$. Then $a < t_1 - \delta$ and $t_1 + \delta < t^*$. Hence, $(t_1 - \delta, t_1 + \delta) \subset (a, t^*) \subset (a,b)$. Functions $x = \varphi(t)$ and $x = \xi(t)$ are defined on the interval (a,b), therefore they are defined on $(t_1 - \delta, t_1 + \delta)$ and they are both solutions of the Cauchy problem (1), (3). Therefore (due to the uniqueness condition) $\xi(t) \equiv \varphi(t) \equiv \bar{\varphi}(t)$ on the interval $(t_1 - \delta, t_1 + \delta)$, which contradicts the condition $\varphi(t) \neq \xi(t)$ at $t \in (t_1, t_1 + \delta)$. Theorem 1 is proved.

Gronwall's Lemma. Let the function $u(t)$ be continuous on the interval $\langle a,b \rangle$ and $u(t) \ge 0$ for $t \in \langle a,b \rangle$.

Let there be constants $c \ge 0, L > 0$ and exists $t_0 \in \langle a,b \rangle$ such that for $t \in \langle a,b \rangle$ the following inequality holds

$$u(t) \le c + L \left| \int_{t_0}^{t} u(\tau)\, d\tau \right|. \tag{4}$$

Then

$$u(t) \le ce^{L|t-t_0|}. \tag{5}$$

Proof of Gronwall's lemma. Let’s prove the lemma for $t \ge t_0$ (proof for the case $t \le t_0$ is similar).

Inequality (4) and (5) for $t \ge t_0$ could be rewritten in the form

$$u(t) \le c + L\int_{t_0}^{t} u(\tau)\,d\tau, \tag{4'}$$

$$u(t) \le ce^{L(t-t_0)}. \tag{5'}$$

Let's set $v(t) = c + L\int_{t_0}^{t} u(\tau)\,d\tau$. It follows from inequality (4') that $u(t) \le v(t)$. Hence,

$$\frac{d}{dt}\left(v(t)e^{-Lt}\right) = \dot{v}(t)e^{-Lt} - Lv(t)e^{-Lt} = Le^{-Lt}\left(u(t) - v(t)\right) \le 0,$$

that is the function $v(t)e^{-Lt}$ decreases monotonically. Therefore $v(t)e^{-Lt} \le v(t_0)e^{-Lt_0} = ce^{-Lt_0}$ and $u(t) \le v(t) \le ce^{L(t-t_0)}$. The Gronwall's lemma is proved.

Important consequence. If $c = 0$ then from (4) it follows that $u(t) \equiv 0$ on $\langle a,b \rangle$.

Theorem 2 (uniqueness theorem). Let’s assume that in the neighborhood of the point $(t_0, x_0) \in G$ partial derivative $\partial X(t,x)/\partial x$ exists and bounded.

Then (t_0, x_0) is the point of uniqueness.

Proof of Theorem 2. There are constants $a > 0$, $b > 0$ and constants $M > 0$, $L > 0$ such that $D = \{(t,x) : |t - t_0| \le a, |x - x_0| \le b\} \subset G$ and $|X(t,x)| \le M$, $|\partial X(t,x)/\partial x| \le L$ for $(t,x) \in D$.

By Peano's theorem on the interval $\{t: |t-t_0| \le h\}$, where $h=\min(a,b/M)$, there exists the solution $\varphi(t)$ of the Cauchy problem (1), (2). Let's show that $\Delta = h$ satisfies the definition 1.

Let $0<\delta \le h$. Denote by $x=\xi(t)$ any other solution of Cauchy problem (1), (2) for $|t-t_0| \le \delta$.

1. First, let's show that $(t,\xi(t)) \in D$ for $|t-t_0| \le \delta$. So, we have to show that $|\xi(t)-x_0| \le b$.

Let's assume that there is a value t^* such that $|t^*-t_0| \le \delta$ and $|\xi(t^*)-x_0| > b$. Notice, that $t^* \ne t_0$, because $|\xi(t_0)-x_0| = 0 < b$. Let for certainty $t^* > t_0$, that is $t_0 < t^* \le t_0 + \delta$.

Let us put at $t \in [t_0, t^*]$

$$v(t) = |\xi(t)-x_0| - b.$$

Note, that $v(t)$ is continuous function, $v(t_0)<0$, $v(t^*)>0$. According to Cauchy's intermediate value theorem, there is θ such that $t_0 < \theta < t^*$ and $v(\theta)=0$.

Therefore, set $O=\{t: t\in[t_0,t^*],\ v(t)=0\}$ is non-empty, bounded and closed (as the set of zeros of a continuous function) set. Hence there exists $\min O = t_1$, $t_1 \in O$.

Note that $v(t) \le 0$ (since $|\xi(t)-x_0| \le b$) for all $t \in [t_0,t_1]$, and any point $(t,\xi(t))$ belongs to D for $t\in[t_0,t_1]$. It's also easy to see that $\dot{\xi}(t) = X(t,\xi(t))$ for $|t-t_0| \le \delta$. Integrating the last equality from t_0 to t_1, we get:

$$\xi(t_1)-\xi(t_0) = \int_{t_0}^{t_1} X(\tau,\xi(\tau))\,d\tau, \text{ and}$$

$$|\xi(t_1)-x_0| \le \int_{t_0}^{t_1} |X(\tau,\xi(\tau))|\,d\tau \le M(t_1-t_0) < M\delta \le Mh \le b. \tag{6}$$

From inequality (6) it follows that $v(t_1)<0$, which contradicts with the fact that $t_1 \in O$. This contradiction proves that $(t,\xi(t)) \in D$ for $|t-t_0| \le \delta$.

2. Now we fix an arbitrary $t \in [t_0 - \delta, t_0 + \delta]$ and estimate the difference $X(t,\varphi(t)) - X(t,\xi(t))$.

Let us define for $0 \le s \le 1$ the function

$$f(s) = X(t, s\varphi(t) + (1-s)\xi(t)).$$

Let us show that this definition is correct. We should prove that any point $(t, s\varphi(t) + (1-s)\xi(t))$ belongs to D for $s \in [0,1]$.

Note that $|\varphi(t) - x_0| \le b$, as follows from the proof of Peano's theorem, and $|\xi(t) - x_0| \le b$, as proven above, so

$$\begin{aligned} |s\varphi(t) + (1-s)\xi(t) - x_0| &= |s\varphi(t) - sx_0 + (1-s)\xi(t) - (1-s)x_0| \le \\ &\le s|\varphi(t) - x_0| + (1-s)|\xi(t) - x_0| \le sb + (1-s)b = b, \end{aligned}$$

and function $f(s)$ defined correctly for $0 \le s \le 1$.

According to the formula of finite increments (Lagrange's theorem), there is a value $\sigma \in (0,1)$ such that

$$X(t,\varphi(t)) - X(t,\xi(t)) = f(1) - f(0) = f'(\sigma). \tag{7}$$

Also

$$|f'(\sigma)| = \left| \frac{\partial X(t, \sigma\varphi(t) + (1-\sigma)\xi(t))}{\partial x} \right| |\varphi(t) - \xi(t)| \le L|\varphi(t) - \xi(t)|, \tag{8}$$

since $|\partial X(t,x)/\partial x| \le L$ for $(t,x) \in D$.

From (7), (8) it follows that

$$|X(t,\varphi(t)) - X(t,\xi(t))| \le L|\varphi(t) - \xi(t)|. \tag{9}$$

3. Let's now prove that $\xi(t) \equiv \varphi(t)$ on the segment $[t_0 - \delta, t_0 + \delta]$.

$$\dot{\varphi}(t) = X(t,\varphi(t)), \quad \dot{\xi}(t) = X(t,\xi(t))$$

for $|t-t_0|\le\delta$, and hence

$$\dot{\varphi}(t)-\dot{\xi}(t)=X(t,\varphi(t))-X(t,\xi(t)).$$

We integrate the last equality from t_0 to t:

$$\varphi(t)-\varphi(t_0)-\xi(t)+\xi(t_0)=\int_{t_0}^{t}\Big(X(\tau,\varphi(\tau))-X(\tau,\xi(\tau))\Big)d\tau. \qquad (10)$$

Since $\varphi(t_0)=\xi(t_0)=x_0$, then, according to (9) and (10),

$$|\varphi(t)-\xi(t)|\le\left|\int_{t_0}^{t}|X(\tau,\varphi(\tau))-X(\tau,\xi(\tau))|d\tau\right|\le L\left|\int_{t_0}^{t}|\varphi(\tau)-\xi(\tau)|d\tau\right|.$$

Let's put $|\varphi(t)-\xi(t)|=u(t)$, then the last inequality can be rewritten as

$$u(t)\le L\left|\int_{t_0}^{t}u(\tau)d\tau\right|.$$

By corollary of Gronwall's lemma $u(t)\equiv 0$, that is $\xi(t)\equiv\varphi(t)$ on the segment $[t_0-\delta,t_0+\delta]$. This proves the Theorem 2.

Corollary. If $\partial X(t,x)/\partial x$ exists and continuous in the domain G then any point from G is the point of uniqueness. Geometrically, this means that through each point of the domain G goes only one integral curve.

§ 3. Differential equation in symmetric form.

Definition. *Differential equation in symmetric form* is the equation of the following shape

$$M(x,y)dx + N(x,y)dy = 0. \qquad (1)$$

We assume that $M(x,y)$ and $N(x,y)$ are continuous functions in the domain $G \subset R^2$.

Definition. *The solution of the equation* (1) we'll call such function $y = \varphi(x)$, $x \in \langle a,b \rangle$, or such function $x = \psi(y)$, $y \in \langle c,d \rangle$, that, being substituted into the equation (1), turns this equation into an identity.

Let's substitute the function $y = \varphi(x)$ into the equation (1).

$$M(x,\varphi(x))dx + N(x,\varphi(x))\varphi'(x)dx = 0.$$

It follows that

$$M(x,\varphi(x)) + N(x,\varphi(x))\varphi'(x) = 0. \qquad (2)$$

So $y = \varphi(x)$ is the solution of the equation (1) for $x \in \langle a,b \rangle$ if (2) is satisfied.

Similarly, the function $x = \psi(y)$ is the solution of the equation (1) for $y \in \langle c,d \rangle$ if

$$M(\psi(y),y)\psi'(y) + N(\psi(y),y) = 0. \qquad (3)$$

If $N(x_0,y_0) \neq 0$ at the point $(x_0,y_0) \in G$, then $N(x,y) \neq 0$ in some neighborhood of the point (x_0,y_0) due to the continuity of the function $N(x,y)$, and in this neighborhood equation (1) is equivalent to the equation

$$\frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}. \qquad (4)$$

For this equation, according to Peano's theorem, there is the solution $y = \varphi(x)$ such that $y_0 = \varphi(x_0)$.

Similarly, if $M(x_0,y_0)\neq 0$ at the point $(x_0,y_0)\in G$, then $M(x,y)\neq 0$ in some neighborhood (x_0,y_0), and in this neighborhood equation (1) is equivalent to the equation

$$\frac{dx}{dy}=-\frac{N(x,y)}{M(x,y)}, \tag{5}$$

which has the solution $x=\psi(y)$, $x_0=\psi(y_0)$.

Definition. The point $(x_0,y_0)\in G$ such that $M(x_0,y_0)=N(x_0,y_0)=0$ is called *the singular point* of the equation (1).

If $M(x_0,y_0)\neq 0$ or $N(x_0,y_0)\neq 0$ then this point is called *the ordinary point*.

Remark. Let $M(x_0,y_0)\neq 0$ and $N(x_0,y_0)\neq 0$. Then there exists the solution $y=\varphi(x)$, $y_0=\varphi(x_0)$ of the equation (1) and the function $\varphi(x)$ satisfies (2), and therefore satisfies (4):

$$\varphi'(x)=-\frac{M(x,\varphi(x))}{N(x,\varphi(x))},$$

and

$$\varphi'(x_0)=-\frac{M(x_0,y_0)}{N(x_0,y_0)}\neq 0.$$

Hence the function $y=\varphi(x)$ has an inverse function $x=\psi(y)$ in the neighborhood of the point (x_0,y_0), and $\varphi'(x)=1/\psi'(y)$. Hence

$$\psi'(x)=-\frac{N(\psi(y),y)}{M(\psi(y),y)},$$

and

$$\psi'(x_0)=-\frac{N(x_0,y_0)}{M(x_0,y_0)},$$

this means that the function $\psi(y)$ satisfies (5) and satisfies (3), therefore $\psi(y)$ is the solution of the equation (1).

Definition. Function $u(x,y)$ is called *the integral* of the equation (1) in the domain G if the following conditions are satisfied:

1) $u(x,y)$ is continuously differentiable in G,
2) at every ordinary point in the domain G at least one of the partial derivatives ($\partial u/\partial x$ or $\partial u/\partial y$) is not equal to zero,
3) in the domain G the following identity holds:

$$N(x,y)\frac{\partial u(x,y)}{\partial x} - M(x,y)\frac{\partial u(x,y)}{\partial y} \equiv 0. \qquad (6)$$

Theorem 1. Let $y=\varphi(x)$ be the solution of the equation (1) defined for $x\in\langle a,b\rangle$, and the point $(x,\varphi(x))$ be the ordinary point for any $x\in\langle a,b\rangle$.

Let also $u(x,y)$ be the integral of the equation (1) in G. Then $u(x,\varphi(x))=const$, $x\in\langle a,b\rangle$.

Proof of the Theorem 1. Since $y=\varphi(x)$ is the solution of the equation (1), then (2) holds. It follows from the equality (2) that $N(x,\varphi(x))\neq 0$. Indeed, if $N(x,\varphi(x))=0$ at some point $x\in\langle a,b\rangle$, then, according to (2), $M(x,\varphi(x))=0$, and the point $(x,\varphi(x))$ is singular, which contradicts the assumption of theorem.

From the equality (2) we obtain:

$$\varphi'(x) = -\frac{M(x,\varphi(x))}{N(x,\varphi(x))},$$

and

$$\frac{du(x,\varphi(x))}{dx} = \frac{\partial u(x,\varphi(x))}{\partial x} + \frac{\partial u(x,\varphi(x))}{\partial y}\varphi'(x) =$$

$$= \frac{\partial u(x,\varphi(x))}{\partial x} - \frac{\partial u(x,\varphi(x))}{\partial y}\frac{M(x,\varphi(x))}{N(x,\varphi(x))} =$$

$$= \frac{1}{N(x,\varphi(x))}\left(\frac{\partial u(x,\varphi(x))}{\partial x}N(x,\varphi(x)) - \frac{\partial u(x,\varphi(x))}{\partial y}M(x,\varphi(x))\right)$$

Hence $\frac{du(x,\varphi(x))}{dx} = 0$ and $u(x,\varphi(x))=const$. Theorem 1 is proved.

The following theorem could be proved in a similar way.

Theorem 2. Let $x=\psi(y)$ be the solution of the equation (1) defined for $y\in\langle c,d\rangle$, and the point $(\psi(y),y)$ be the ordinary point for any $y\in\langle c,d\rangle$.

Let also $u(x,y)$ be the integral of the equation (1) in G. Then $u(\psi(y),y)=const$, $y\in\langle c,d\rangle$.

Let $u(x,y)$ be the integral of the equation (1) in the domain G and $(x_0,y_0)\in G$. Consider the equation

$$u(x,y)=u(x_0,y_0). \tag{7}$$

Theorem 3. Let's suppose that $N(x_0,y_0)\neq 0$. Then the equation (7) has the solution $y=\varphi(x)$, defined for $x\in(a,b)$, $x_0\in(a,b)$, and $y_0=\varphi(x_0)$. This solution is continuously differentiable on the interval (a,b) and it's also the solution of the equation (1).

Proof of the Theorem 3. First, let's show that $\dfrac{\partial u(x_0,y_0)}{\partial y}\neq 0$.

Indeed, if $\dfrac{\partial u(x_0,y_0)}{\partial y}=0$ then from equality (6) at the point (x_0,y_0) it follows that $\dfrac{\partial u(x_0,y_0)}{\partial x}=0$ and this contradicts with the second condition in the definition of the integral, since the point (x_0,y_0) is ordinary.

From inequality $\dfrac{\partial u(x_0,y_0)}{\partial y}\neq 0$ by the implicit function theorem it follows that equation (7) has the solution $y=\varphi(x)$, defined on some interval (a,b), such that $x_0\in(a,b)$, $y_0=\varphi(x_0)$, and function $\varphi(x)$ continuously differentiable on (a,b).

From the continuity of partial derivatives of the function $u(x,y)$ it follows that $\dfrac{\partial u(x,\varphi(x))}{\partial y}\neq 0$ for $x\in(a,b)$, if the interval (a,b) is small enough.

Differentiating equality $u(x,\varphi(x))=u(x_0,y_0)$, we get:

$$\frac{\partial u(x,\varphi(x))}{\partial x}+\frac{\partial u(x,\varphi(x))}{\partial y}\varphi'(x)=0,$$

and

$$\varphi'(x) = -\frac{\partial u(x,\varphi(x))}{\partial x} \Big/ \frac{\partial u(x,\varphi(x))}{\partial y}. \tag{8}$$

Let's substitute equality (8) into (2):

$$M(x,\varphi(x)) - N(x,\varphi(x))\left(\frac{\partial u(x,\varphi(x))}{\partial x} \Big/ \frac{\partial u(x,\varphi(x))}{\partial y}\right) =$$

$$= \left(\frac{\partial u(x,\varphi(x))}{\partial y}\right)^{-1}\left(M(x,\varphi(x))\frac{\partial u(x,\varphi(x))}{\partial y} - N(x,\varphi(x))\frac{\partial u(x,\varphi(x))}{\partial x}\right) \equiv 0,$$

hence, $y = \varphi(x)$ - solution of equation (1). The Theorem 3 is proved.

The following theorem could be proved in a similar way.

Theorem 4. Let $M(x_0, y_0) \neq 0$. Then the equation (7) has the solution $x = \psi(y)$ defined for $y \in (c,d), y_0 \in (c,d)$, and $x_0 = \psi(y_0)$. This solution is continuously differentiable on the interval (c,d) and it's also the solution of the equation (1).

The corollary of Theorems 3 and 4 is Theorem 5.

Theorem 5. If the point (x_0, y_0) is ordinary then the equation (7) has a solution of the form $y = \varphi(x)$, $x \in (a,b)$, $x_0 \in (a,b)$, $y_0 = \varphi(x_0)$, or the solution of the form $x = \psi(y), y \in (c,d), y_0 \in (c,d), x_0 = \psi(y_0)$. This solution is the solution of the equation (1).

Definition. Let $u(x,y)$ be the integral of equation (1) in the domain G. Equality $u(x,y) = c$, where c is an arbitrary constant, called the general integral of the equation (1) in G.

Example. Function $u(x,y) = x^2 + y^2$ is the integral of the equation $xdx + ydy = 0$. Equality $x^2 + y^2 = c$ is the general integral of the given equation in $G = R^2$.

In the neighborhood of each point $(x_0, y_0) \neq (0,0)$ from the equality $x^2 + y^2 = x_0^2 + y_0^2$ we can express the solution of the equation (1) in the form $y = \varphi(x)$ or in the form $x = \psi(y)$.

§ 4. Equation in total differentials. Integrating multiplier.

Consider the equation

$$M(x,y)dx + N(x,y)dy = 0, \tag{1}$$

where $M(x,y)$ and $N(x,y)$ are continuous functions in the domain $G \subset R^2$.

Definition 1. Equation (1) is called *the equation in total differentials* if there exists continuously differentiable in the domain G function $u(x,y)$ such that for any $(x,y) \in G$

$$du(x,y) = M(x,y)dx + N(x,y)dy. \tag{2}$$

Let us rewrite (2) in the form

$$\frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy = M(x,y)dx + N(x,y)dy.$$

It follows from this equality that equality (2) is equivalent to the equalities

$$\frac{\partial u(x,y)}{\partial x} = M(x,y), \qquad \frac{\partial u(x,y)}{\partial y} = N(x,y) \tag{3}$$

for any $(x,y) \in G$.

Theorem 1. If (1) is the equation in total differentials, then the function $u(x,y)$ is the integral of the equation (1) in the domain G.

Proof of the Theorem 1. Firstly, $u(x,y)$ is the continuously differentiable function. Secondly, at every ordinary point in G at least one of the functions $M(x,y)$, $N(x,y)$ is not equal to zero, and hence at least one of the partial derivatives $\frac{\partial u(x,y)}{\partial x}$, $\frac{\partial u(x,y)}{\partial y}$ is not equal to zero.

Thirdly,

$$N(x,y)\frac{\partial u(x,y)}{\partial x} - M(x,y)\frac{\partial u(x,y)}{\partial y} = N(x,y)M(x,y) - M(x,y)N(x,y) \equiv 0.$$

Thus, all conditions from the definition of the integral are satisfied for the function $u(x,y)$. Theorem is proved.

Theorem 2. Let (1) be the equation in total differentials, and in the domain G there exists continuous partial derivatives $\frac{\partial M(x,y)}{\partial y}$ and $\frac{\partial N(x,y)}{\partial x}$. Then at every point in the domain G

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}. \tag{4}$$

Proof of the Theorem 2. Note, that since (1) is the equation in total differentials, then in the domain G equalities (3) hold. We differentiate the first of equalities (3) with respect to y, and the second with respect to x:

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial^2 u(x,y)}{\partial y \partial x}, \qquad \frac{\partial N(x,y)}{\partial x} = \frac{\partial^2 u(x,y)}{\partial x \partial y}. \tag{5}$$

By assumption both $\frac{\partial M(x,y)}{\partial y}$ and $\frac{\partial N(x,y)}{\partial x}$ are continuous in G. Therefore, both $\frac{\partial^2 u(x,y)}{\partial y \partial x}$ and $\frac{\partial^2 u(x,y)}{\partial x \partial y}$ are also continuous in G. Hence by the theorem on the equality of cross partial derivatives

$$\frac{\partial^2 u(x,y)}{\partial y \partial x} = \frac{\partial^2 u(x,y)}{\partial x \partial y}.$$

From here and from equalities (5) the equality (4) is follows. Theorem is proved.

We just proved that condition (4) is the necessary condition for the fact that (1) is the equation in total differentials. Now let's show that in "rectangular" domains this condition is also sufficient.

Let $G = \{(x,y) : a < x < b, c < y < d\}$. Note, that cases $a = -\infty$, $b = +\infty$, $c = -\infty$, $d = +\infty$ are not excluded.

Theorem 3. Let $\dfrac{\partial M(x,y)}{\partial y}$ and $\dfrac{\partial N(x,y)}{\partial x}$ exist and be continuous in the domain G, and equality (4) is satisfied at each point in G. Then (1) is the equation in total differentials.

Proof of the Theorem 3. Our goal is to construct a continuously differentiable in the domain G function $u(x,y)$, for which conditions (3) are satisfied.

First, we fix the point $(x_0,y_0)\in G$.

Let's find the value of the function u at the point $(x,y)\in G$. Let t lies between x_0 and x (that is $x_0\le t\le x$ or $x\le t\le x_0$). Then $a<t<b$ and the point (t,y) belongs to the domain G.

We know that the function $u(x,y)$ should satisfy (3). Then

$$\frac{\partial u(t,y)}{\partial t}=M(t,y).$$

Let's integrate the last equality over t from x_0 to x (here y considered as a parameter):

$$\int_{x_0}^{x}\frac{\partial u(t,y)}{\partial t}dt=\int_{x_0}^{x}M(t,y)dt.$$

Then

$$u(x,y)-u(x_0,y)=\int_{x_0}^{x}M(t,y)dt,$$

and hence

$$u(x,y)=\int_{x_0}^{x}M(t,y)dt+u(x_0,y). \tag{6}$$

Now let's find $u(x_0,y)$.

Let t lies between y_0 and y. Then $c<t<d$ and the point (x_0,t) belongs to the domain G. It follows from (3) that

$$\frac{\partial u(x_0,t)}{\partial t}=N(x_0,t).$$

Let's integrate the last equality over t from y_0 to y:

$$\int_{y_0}^{y} \frac{\partial u(x_0,t)}{\partial t} dt = \int_{y_0}^{y} N(x_0,t)dt .$$

Then

$$u(x_0,y) - u(x_0,y_0) = \int_{y_0}^{y} N(x_0,t)dt . \tag{7}$$

Obviously function $u(x,y)$ defined up to an additive constant, so we can put $u(x_0,y_0)=0$. From equalities (6) and (7) it follows that

$$u(x,y) = \int_{x_0}^{x} M(t,y)dt + \int_{y_0}^{y} N(x_0,t)dt . \tag{8}$$

Note, that by now we proved only the fact that if the function $u(x,y)$ satisfies conditions (3), then it is given by formula (8). Let us show that the constructed function u is the desired one, that is, the function defined by formula (8) satisfies conditions (3).

From (8) follows:

$$\frac{\partial u(x,y)}{\partial x} = M(x,y),$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{\partial}{\partial y}\int_{x_0}^{x} M(t,y)dt + N(x_0,y) = \int_{x_0}^{x} \frac{\partial M(t,y)}{\partial y} dt + N(x_0,y) =$$

$$= \int_{x_0}^{x} \frac{\partial N(t,y)}{\partial t} dt + N(x_0,y) = N(x,y) - N(x_0,y) + N(x_0,y) = N(x,y).$$

Theorem proved.

Since x and y enter into equation (1) symmetrically, then the following formula is also valid

$$u(x,y) = \int_{x_0}^{x} M(t,y_0)dt + \int_{y_0}^{y} N(x,t)dt . \tag{8'}$$

Integrating factor.

Definition 2. Function $\mu(x,y)$, continuous in the domain G and not vanishing in G, is called the integrating factor of equation (1), if the equation

$$\mu(x,y)M(x,y)dx+\mu(x,y)N(x,y)dy=0 \tag{9}$$

is the equation in total differentials.

We assume that domain G has a rectangular shape and in G there exist continuous derivatives $\frac{\partial M(x,y)}{\partial y}$ and $\frac{\partial N(x,y)}{\partial x}$.

We will look for the integrating factor $\mu(x,y)$ as continuously differentiable in G function.

In order for (9) to be the equation in total differentials, it is necessary and sufficient that

$$\frac{\partial(\mu(x,y)M(x,y))}{\partial y}=\frac{\partial(\mu(x,y)N(x,y))}{\partial x},$$

or

$$N(x,y)\frac{\partial\mu(x,y)}{\partial x}-M(x,y)\frac{\partial\mu(x,y)}{\partial y}=\mu(x,y)\left(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}\right). \tag{10}$$

So the function $\mu(x,y)$ should be the solution of the partial differential equation (10).

We will look for the integrating factor of equation (1), depending only on x. From (10) it follows that $\mu(x)$ satisfies the equation

$$N(x,y)\frac{d\mu(x)}{dx}=\mu(x)\left(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}\right),$$

?

or

$$\frac{1}{\mu(x)}\frac{d\mu(x)}{dx}=\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}\right). \tag{11}$$

The left-hand side of the equality (11) depends only on x, so equality is possible if the right-hand side of (11) also depends only on x. We denote

$$\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}\right)=f(x),$$

and equality (11) we rewrite in the form:

$$\frac{1}{\mu(x)}\frac{d\mu(x)}{dx}=f(x).$$

Integrating the last equation, we get:

$$\mu(x)=\exp\left(\int f(x)dx\right). \tag{12}$$

Thus, we have proved the following statement.

Statement 1. For the equation (1) there exists an integrating factor μ, depending only on x, if the function

$$\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y}-\frac{\partial N(x,y)}{\partial x}\right)$$

depends only on x. $\mu(x)$ in this case is determined by equality (12).

Statement 2 is proved in a similar way.

Statement 2. For the equation (1) there exists an integrating factor μ, depending only on y, if the function

$$\frac{1}{M(x,y)}\left(\frac{\partial N(x,y)}{\partial x}-\frac{\partial M(x,y)}{\partial y}\right)$$

depends only on y.

Example. Linear equation.

Consider the linear equation

$$y'=p(x)y+q(x), \tag{13}$$

where functions $p(x)$ and $q(x)$ are continuous on the interval (a,b).

Let's rewrite (1) in symmetric form:

$$\left(p(x)y+q(x)\right)dx-dy=0. \tag{14}$$

Equations (13) and (14) are equivalent. Equation (2) is given in a rectangular domain $G=\{(x,y): a<x<b, -\infty<y<+\infty\}$.

Here $M(x,y)=p(x)y+q(x)$, $N(x,y)=-1$, $\dfrac{\partial M(x,y)}{\partial y}\neq\dfrac{\partial N(x,y)}{\partial x}$.

We are looking for an integrating factor μ, depending only on x.

Equation (11) in this case will have the following shape:

$$\frac{d\mu(x)}{dx}=-\mu(x)p(x), \tag{15}$$

and hence,

$$\mu(x)=exp\left(-\int_{x_0}^{x}p(s)ds\right),$$

where $x_0\in(a,b)$. Then the equation in total differentials (9) takes the form:

$$\exp\left(-\int_{x_0}^{x}p(s)ds\right)\left(p(x)y+q(x)\right)dx-\exp\left(-\int_{x_0}^{x}p(s)ds\right)dy=0.$$

We use the formula (8'), assuming in this formula $y_0=0$:

$$u(x,y)=\int_{x_0}^{x}q(t)\exp\left(-\int_{x_0}^{t}p(s)ds\right)dt-\int_{0}^{y}\exp\left(-\int_{x_0}^{x}p(s)ds\right)dt.$$

From equality $u(x,y)=c$, which is the general integral of equation (1), we obtain:

$$y=\exp\left(\int_{x_0}^{x}p(s)ds\right)\left(c+\int_{x_0}^{x}q(t)\exp\left(-\int_{x_0}^{t}p(s)ds\right)dt\right). \tag{16}$$

Chapter 2. SYSTEMS OF DIFFERENTIAL EQUATIONS.

A system of equations solved with respect to higher derivatives is called a system of the form

$$\begin{cases} x_1^{(m_1)} = X_1\left(t, x_1, \dot{x}_1, \ddot{x}_1, ..., x_1^{(m_1-1)}, x_2, \dot{x}_2, \ddot{x}_2, ..., x_2^{(m_2-1)}, ..., x_k, \dot{x}_k, \ddot{x}_k, ..., x_k^{(m_k-1)}\right), \\ x_2^{(m_2)} = X_2\left(t, x_1, \dot{x}_1, \ddot{x}_1, ..., x_1^{(m_1-1)}, x_2, \dot{x}_2, \ddot{x}_2, ..., x_2^{(m_2-1)}, ..., x_k, \dot{x}_k, \ddot{x}_k, ..., x_k^{(m_k-1)}\right), \\ \dots \\ x_k^{(m_k)} = X_k\left(t, x_1, \dot{x}_1, \ddot{x}_1, ..., x_1^{(m_1-1)}, x_2, \dot{x}_2, \ddot{x}_2, ..., x_2^{(m_2-1)}, ..., x_k, \dot{x}_k, \ddot{x}_k, ..., x_k^{(m_k-1)}\right), \end{cases} \tag{1}$$

where $x_j^{(s)} = \dfrac{d^s x_j}{dt^s}$, $j = 1,2,...,k$, $s = 1,2,...,m_j$.

Number $n = \sum_{j=1}^{k} m_j$ is called the order of the system (1).

In system (1) the number of equations is always equal to the number of sought functions, the number of arguments of each function X_j equals $(n+1)$.

We assume that the functions X_j are continuous on the set $D \subset R^{n+1}$ for all $j = 1,2,...,k$.

Definition. *The solution of the system (1)* is the set of functions $x_1 = \varphi_1(t)$, $x_2 = \varphi_2(t), ..., x_k = \varphi_k(t)$, defined on the interval $\langle a,b \rangle$, such that being substituted into the system (1) they turn this system into an identity.

Definition. The problem of finding solution $x_1 = \varphi_1(t), x_2 = \varphi_2(t),...,$ $x_k = \varphi_k(t)$ of the system (1) satisfying the conditions

$$\varphi_j(t_0) = x_{j0},\ \dot{\varphi}_j(t_0) = \dot{x}_{j0},\ \ddot{\varphi}_j(t_0) = \ddot{x}_{j0},\ \dots,\ \varphi_j^{(m_j-1)}(t_0) = x_{j0}^{(m_j-1)},$$

where $t_0 \in \langle a,b \rangle$, $\left(t_0, x_{01}, \dot{x}_{01}, ..., x_{01}^{(m_1-1)}, x_{02}, \dot{x}_{02}, ..., x_{02}^{(m_2-1)}, ..., x_{0k}, \dot{x}_{0k}, ..., x_{0k}^{(m_k-1)}\right) \subset D$, $j = 1,2,...,k$, is called the Cauchy problem.

Special cases:

1. If $k=1$, then (1) is called *the equation of order n, solved with respect to the highest derivative*:

$$x^{(n)} = X\left(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}\right). \tag{2}$$

2. If $m_j = 1$ for all $j = 1, 2, \ldots, k$, then (1) is called the *system in normal form*, or *the normal system*:

$$\begin{cases} \dot{x}_1 = X_1\left(t, x_1, x_2, \ldots, x_n\right), \\ \dot{x}_2 = X_2\left(t, x_1, x_2, \ldots, x_n\right), \\ \ldots \\ \dot{x}_n = X_k\left(t, x_1, x_2, \ldots, x_n\right). \end{cases} \tag{3}$$

Let us show that the equation (2) could be reduced to the form (3). Let's denote

$$x_1 = x,\ x_2 = \dot{x},\ x_3 = \ddot{x},\ \ldots,\ x_n = x^{(n-1)}. \tag{4}$$

Then

$$\dot{x}_1 = \dot{x} = x_2,\ \dot{x}_2 = \ddot{x} = x_3,\ \ldots,\ \dot{x}_{n-1} = x^{(n-1)} = x_n,$$

and, as follows from (2),

$$\dot{x}_n = x^{(n)} = X\left(t, x_1, x_2, \ldots, x_n\right). \tag{5}$$

Thus, using notation (4), equation (2) is represented as the normal system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \ldots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = X\left(t, x_1, x_2, \ldots, x_n\right). \end{cases} \tag{6}$$

Note, that the equation (2) and the system (6) are not equivalent in the following sense: the solution of the equation (2) is a function, and the solution of the system (6) is a set of n functions.

But if the function $x=\varphi(t)$ is the solution of the equation (2) determined for $t\in\langle a,b\rangle$, then the function $\varphi(t)$ is n times continuously differentiable on the interval $\langle a,b\rangle$, and the set of n functions $x_1=\varphi(t)$, $x_2=\dot{\varphi}(t)$, $x_3=\ddot{\varphi}(t)$ ..., $x_n=\varphi^{(n-1)}(t)$ is the solution of the system (6) on $\langle a,b\rangle$.

And vice versa, if the set of functions $x_1=\varphi_1(t)$, $x_2=\varphi_2(t)$, ..., $x_n=\varphi_n(t)$, defined on the interval $\langle a,b\rangle$, is the solution of the system (6), then from (4), (5) it follows that each of the functions $\varphi_j(t)$ is continuously differentiable on $\langle a,b\rangle$ (as an element of the solution of the system (6)), $\varphi_j(t)=\varphi_1^{(j-1)}(t)$ for all $j=2,...,n$, and

$$\varphi_1^{(n)}(t)=X\Big(t,\varphi_1(t),\dot{\varphi}_1(t),\ddot{\varphi}_1(t),...,\varphi_1^{(n-1)}(t)\Big),$$

that is, the function $x_1=\varphi_1(t)$ is the solution of the equation (2), defined on $\langle a,b\rangle$.

By applying the procedure described above to each of the equations of system (1), it is possible to reduce system (1) to the normal form. Therefore, further we will consider only normal systems.

§ 1. Vector notation of the normal system.

First, let us recall some concepts and facts from the mathematical analysis course which we will use later.

$\|a\|=\sqrt{a_1^2+a_2^2+...+a_n^2}$ is *the vector norm* of the vector $a=(a_1,a_2,...a_n)^T$.

We say that the sequence of vectors $a^{[k]}=\left(a_1^{[k]},a_2^{[k]},...a_n^{[k]}\right)^T$ *converges* to the vector $a=(a_1,a_2,...a_n)^T$, $k\to+\infty$, if $\left\|a^{[k]}-a\right\|\underset{k\to+\infty}{\to}0$. In this case we will write $a^{[k]}\underset{k\to+\infty}{\to}a$.

Notice, that $a^{[k]}\underset{k\to+\infty}{\to}a$ if and only if $a_j^{[k]}\underset{k\to+\infty}{\to}a_j$ for all $j=1, 2,...,n$.

Note also that for a sequence of vectors the Bolzano-Weierstrass selection principle is valid.

Vector function

$$f(x_1,...,x_m)=\left(f_1(x_1,...,x_m),...,f_n(x_1,...,x_m)\right)^T$$

is continuous on the set $D\subset R^m$ if all functions $f_j(x_1,...,x_m)$ are continuous on the set D. Also f is continuously differentiable with respect to x_k, if all functions $f_j(x_1,...,x_m)$ are continuously differentiable with respect to x_k, $j=1,2,...,n$, $k=1,2,...,m$.

We denote

$$\frac{\partial f(x_1,...,x_m)}{\partial x_k}=\left(\frac{\partial f_1(x_1,...,x_m)}{\partial x_k},...,\frac{\partial f_n(x_1,...,x_m)}{\partial x_k}\right)^T.$$

In particular, for the vector function $u(t)=\left(u_1(t),u_2(t),...u_n(t)\right)^T$ by the above definition $\dot{u}(t)=\left(\dot{u}_1(t),\dot{u}_2(t),...\dot{u}_n(t)\right)^T$.

If the function $u(t)$ is continuous on the segment $[a,b]$, then

$$\int_a^b u(t)dt=\left(\int_a^b u_1(t)dt,\int_a^b u_2(t)dt,...,\int_a^b u_n(t)dt\right)^T.$$

The series

$$\sum_{k=1}^{+\infty}u^{[k]}(t)=\left(\sum_{k=1}^{+\infty}u_1^{[k]}(t),\sum_{k=1}^{+\infty}u_2^{[k]}(t),...,\sum_{k=1}^{+\infty}u_n^{[k]}(t)\right)^T$$

converges on $\langle a,b\rangle$, if the series $\sum_{k=1}^{+\infty}u_j^{[k]}(t)$, $j=1,2,...,n$, converges for all $t\in\langle a,b\rangle$, and *converges uniformly* on $\langle a,b\rangle$ if all components converges uniformly.

For vector series the Weierstrass rule is also valid: if $\left\|u^{[k]}(t)\right\|\le b_k$ for all $t\in\langle a,b\rangle$, $k\in N$ and number series $\sum_{k=1}^{+\infty}b_k$ converges, then the series $\sum_{k=1}^{+\infty}u^{[k]}(t)$ converges uniformly on $\langle a,b\rangle$.

Now consider the system of differential equations in normal form

$$\begin{cases} \dot{x}_1 = X_1(t, x_1, x_2, ..., x_n), \\ \dot{x}_2 = X_2(t, x_1, x_2, ..., x_n), \\ \dots \\ \dot{x}_n = X_k(t, x_1, x_2, ..., x_n), \end{cases} \tag{1}$$

where are all the functions X_j are continuous on the set $D \subset R^{n+1}$, $j = 1, 2, ..., n$.

Let $x = \begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$, $X(t, x) = X(t, x_1, ..., x_n) = \begin{pmatrix} X_1(t, x_1, ..., x_n) \\ \dots \\ X_n(t, x_1, ..., x_n) \end{pmatrix}$.

Then system (1) can be written in vector form

$$\dot{x} = X(t, x). \tag{1'}$$

Solution of system (1') is the vector function $x = \varphi(t) = (\varphi_1(t), ..., \varphi_n(t))^T$, defined on the interval $\langle a, b \rangle$, which being substituted into (1') turns the system into an identity.

The graph of the solution is called the integral curve.

Cauchy problem for system (1') rewrites in the form

$$(t_0, x_0) \in D,$$

where $x_0 = (x_{10}, ..., x_{n0})^T$.

Let

$$D = \{(t, x) : |t - t_0| \le a, \|x - x_0\| \le b\},$$

$a, b \in R$, $a > 0$, $b > 0$.

Let's consider the Cauchy problem

$$(t_0, x_0). \tag{2}$$

According to the Weierstrass's theorem on the boundedness of a continuous function on a compact set, there exists a constant $M > 0$ such that

$$\|X(t,x)\| \le M \text{ for all } (t,x) \in D.$$

Let's set $h = \min(a, b/M)$.

Theorem (Peano's theorem). Under the above assumptions on the segment $[t_0 - h, t_0 + h]$ there is a solution to the Cauchy problem (1), (2).

The proof of Peano's theorem for the system (1) almost word-to-word repeats the proof of the theorem for the first order equation.

§ 2. Lipschitz condition.

Let $x = \begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$, $t \in R$, $X(t,x) = X(t, x_1, \dots, x_n) = \begin{pmatrix} X_1(t, x_1, \dots, x_n) \\ \dots \\ X_n(t, x_1, \dots, x_n) \end{pmatrix}$ be a vector function.

Definition 1. We say that function $X(t,x)$ satisfies *Lipschitz condition* with respect to x on the set $D \subset R^{n+1}$ (denoted as $X(t,x) \in Lip_x(D)$) if there exists a constant $L > 0$ such that for any two points $(t, \bar{x})$, $(t, \bar{\bar{x}}) \subset D$ the inequality

$$\|X(t,\bar{x}) - X(t,\bar{\bar{x}})\| \le L \|\bar{x} - \bar{\bar{x}}\| \tag{1}$$

holds.

Definition 2. We say that function $X(t,x)$ satisfies *locally Lipschitz condition* with respect to x in the domain $G \subset R^{n+1}$ (denoted as $X(t,x) \in Lip_x^{loc}(G)$) if for any point $(t_0, x_0) \in G$ there exists a neighborhood $U(t_0, x_0) \subset G$ such that $X(t,x)$ satisfies Lipschitz condition with respect to x on the set $U(t_0, x_0)$.

Theorem 1. Let $X(t,x)$ be continuous and satisfy locally Lipschitz condition with respect to x in the domain $G \subset R^{n+1}$. Then on any closed bounded set $D \subset G$ function $X(t,x)$ satisfies Lipschitz condition with respect to x.

Proof of Theorem 1. We will prove this theorem by contradiction. Let's assume that there exists closed bounded set $D \subset G$ such that for any constant $L > 0$ there exist points $(t,\overline{x}),(t,\overline{\overline{x}}) \in D$ such that

$$\left\| X(t,\overline{x}) - X(t,\overline{\overline{x}}) \right\| > L \left\| \overline{x} - \overline{\overline{x}} \right\|.$$

Let's take any number sequence $\{L_k\}_{k=1}^{+\infty}$ such that $L_{k+1} > L_k > 0$ for all $k \in N$ and

$$L_k \underset{k\to+\infty}{\to} +\infty. \tag{2}$$

For each L_k there exist points $(t_k,\overline{x}_k),(t_k,\overline{\overline{x}}_k) \in D$ such that

$$\left\| X(t_k,\overline{x}_k) - X(t_k,\overline{\overline{x}}_k) \right\| > L_k \left\| \overline{x}_k - \overline{\overline{x}}_k \right\|. \tag{3}$$

From the sequences of points $\{(t_k,\overline{x}_k)\}_{k=1}^{+\infty}$ and $\{(t_k,\overline{\overline{x}}_k)\}_{k=1}^{+\infty}$ we can choose convergent subsequences (since the set D is closed and bounded). From $\{(t_k,\overline{x}_k)\}_{k=1}^{+\infty}$ we choose a convergent subsequence $\left\{\left(t_{k_j},\overline{x}_{k_j}\right)\right\}_{j=1}^{+\infty}$, then from the sequence $\left\{\left(t_{k_j},\overline{\overline{x}}_{k_j}\right)\right\}_{j=1}^{+\infty}$ we choose a convergent subsequence $\left\{\left(t_{k_{jm}},\overline{\overline{x}}_{k_{jm}}\right)\right\}_{m=1}^{+\infty}$.

Sequences $\left\{\left(t_{k_{jm}},\overline{x}_{k_{jm}}\right)\right\}_{m=1}^{+\infty}$ and $\left\{\left(t_{k_{jm}},\overline{\overline{x}}_{k_{jm}}\right)\right\}_{m=1}^{+\infty}$ converge to some points $(t_0,\overline{x}_0)$ and $(t_0,\overline{\overline{x}}_0)$. Without loss of generality we will assume that

$$(t_k,\overline{x}_k) \underset{k\to+\infty}{\to} (t_0,\overline{x}_0) \text{ and } (t_k,\overline{\overline{x}}_k) \underset{k\to+\infty}{\to} (t_0,\overline{\overline{x}}_0). \tag{4}$$

Now let's consider two cases.

1. If $\overline{x}_0 \neq \overline{\overline{x}}_0$, then from (4), by the continuity of the function $X(t,x)$, it follows:

$$\frac{\left\| X\left(t_k,\bar{x}_k\right) - X\left(t_k,\bar{\bar{x}}_k\right) \right\|}{\left\| \bar{x}_k - \bar{\bar{x}}_k \right\|} \underset{k\to+\infty}{\to} M, \tag{5}$$

where $M \in R$.

According to (5), there is $k_0 \in N$ such that for $k > k_0$

$$\frac{\left\| X\left(t_k,\bar{x}_k\right) - X\left(t_k,\bar{\bar{x}}_k\right) \right\|}{\left\| \bar{x}_k - \bar{\bar{x}}_k \right\|} < M+1, \tag{6}$$

and according to (2) there is $\tilde{k} \in N$ such that

$$L_k > M+1 \tag{7}$$

for all $k > \tilde{k}$.

Let $k > \max\left(k_0,\tilde{k}\right)$. Then from inequality (6) it follows that

$$\left\| X\left(t_k,\bar{x}_k\right) - X\left(t_k,\bar{\bar{x}}_k\right) \right\| < (M+1)\left\| \bar{x}_k - \bar{\bar{x}}_k \right\|,$$

and, since (7) is true, this inequality contradicts (3).

2. Let $\bar{x}_0 = \bar{\bar{x}}_0$. According to the conditions of the theorem and definition 2, there exists neighborhood $U\left(t_0,\bar{x}_0\right) \subset G$ such that $X\left(t,x\right)$ satisfies Lipschitz condition with respect to x on the set $U\left(t_0,\bar{x}_0\right)$. That is, there exists a constant $L > 0$ such that for any two points $\left(t,\bar{x}\right),\left(t,\bar{\bar{x}}\right) \subset U\left(t_0,\bar{x}_0\right)$ inequality (1) holds.

From (4) it follows that there exists the number $k_0 \in N$ such that $\left(t_k,\bar{x}_k\right),\left(t_k,\bar{x}_k\right) \in U\left(t_0,\bar{x}_0\right)$ for all $k > k_0$, and from condition (2) it follows that there is $\tilde{k} \in N$ such that $L_k > L$ for all $k > \tilde{k}$.

Let $k > \max\left(k_0,\tilde{k}\right)$. Then

$$\left\| X\left(t_k,\bar{x}_k\right) - X\left(t_k,\bar{\bar{x}}_k\right) \right\| < L\left\| \bar{x}_k - \bar{\bar{x}}_k \right\|,$$

and, since $L_k > L$, this inequality contradicts (3).

The theorem is proved.

Theorem 2. Let the function $X(t,x)$ be continuously differentiable with respect to x in the domain G. Then $X(t,x) \in Lip_x^{loc}(G)$.

Proof of the Theorem 2. According to the assumptions of the theorem, partial derivatives $\dfrac{\partial X_j(t,x_1,...,x_n)}{\partial x_k}$ exist and continuous in G for all $j=1,2,...,n$, $k=1,2,...,n$.

Let's consider an arbitrary point $(t_0,x_0) \in G$. Since G is open, then there exist $a,b \in R$, $a>0$, $b>0$, such that

$$D=\{(t,x):|t-t_0| \le a, \|x-x_0\| \le b\} \subset G.$$

Now consider arbitrary points $(t,\overline{x}),(t,\overline{\overline{x}}) \in D$ and estimate the difference $X_j(t,\overline{x})-X_j(t,\overline{\overline{x}})$ for $j \in \{1,2,...,n\}$.

To do this, we introduce the function $f(s)=X_j(t,s\overline{x}+(1-s)\overline{\overline{x}})$, $0 \le s \le 1$. Let's show that the point $(t,s\overline{x}+(1-s)\overline{\overline{x}})$ belongs to D for any $s \in [0,1]$.

$$\begin{aligned}\|s\overline{x}+(1-s)\overline{\overline{x}}-x_0\| &= \|s\overline{x}-sx_0+(1-s)\overline{\overline{x}}-(1-s)x_0\| \le \\ &\le s\|\overline{x}-x_0\|+(1-s)\|\overline{\overline{x}}-x_0\| \le sb+(1-s)b=b,\end{aligned}$$

since $(t,\overline{x}),(t,\overline{\overline{x}}) \in D$. Hence the function $f(s)$ is defined correctly for $0 \le s \le 1$.

According to the Lagrange’s theorem, there is a value $\sigma \in (0,1)$ such that

$$X_j(t,\overline{x})-X_j(t,\overline{\overline{x}})=f(1)-f(0)=f'(\sigma). \tag{8}$$

Note also, that

$$f'(\sigma)=\sum_{k=1}^{n}\frac{\partial X_j(t,\sigma\overline{x}+(1-\sigma)\overline{\overline{x}})}{\partial x_k}(\overline{x}_k-\overline{\overline{x}}_k),$$

and

$$|f'(\sigma)| \le \sum_{k=1}^{n}\left|\frac{\partial X_j(t,\sigma\overline{x}+(1-\sigma)\overline{\overline{x}})}{\partial x_k}\right||\overline{x}_k-\overline{\overline{x}}_k|. \tag{9}$$

From the continuity of partial derivatives $\partial X_j(t,x)/\partial x_k$ on the compact set D it follows that there exists $K>0$ such that

$$\left|\frac{\partial X_j(t,x)}{\partial x_k}\right| \le K \text{ for all } j=1,2,...,n,\ k=1,2,...,n.$$

Obviously $|\bar{x}_k-\bar{\bar{x}}_k| \le \|\bar{x}-\bar{\bar{x}}\|$ for any $k=1,2,...,n$. Hence, from (8), (9) it follows that

$$\left|X_j(t,\bar{x})-X_j(t,\bar{\bar{x}})\right| \le nK\|\bar{x}-\bar{\bar{x}}\|, \tag{10}$$

and (by definition of the Euclidean norm)

$$\left\|X(t,\bar{x})-X(t,\bar{\bar{x}})\right\| = \sqrt{\sum_{j=1}^{n}\left(X_j(t,\bar{x})-X_j(t,\bar{\bar{x}})\right)^2} \le n\sqrt{n}K\|\bar{x}-\bar{\bar{x}}\|. \tag{11}$$

Denoting $n\sqrt{n}K=L$ we find that for any pair of points $(t,\bar{x}),(t,\bar{\bar{x}})\in D$ inequality (1) is true.

Since the point (t_0,x_0) is an arbitrary point in G, theorem is proved.

§ 3. Picard theorem.

Let's consider the system

$$\dot{x}=X(t,x), \tag{1}$$

$x\in R^n$, $X(t,x)\in C(D)$, $D\subset R^{n+1}$, with the Cauchy problem

$$t=t_0,\ x=x_0, \tag{2}$$

$(t_0,x_0)\in D$.

Consider also the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} X(\tau, x(\tau))d\tau . \qquad (3)$$

Definition. A continuous vector function $x = \varphi(t)$, defined on the interval $\langle a,b \rangle$, $t_0 \in \langle a,b \rangle$, called *the solution of the integral equation* (3) if, being substituted into (3), this function turns equation (3) into an identity.

Statement. Equation (3) is equivalent to the Cauchy problem (1), (2).

Proof of the statement. If function $x = \varphi(t)$ is the solution of (3), then

$$\varphi(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi(\tau))d\tau \qquad (4)$$

for any $t \in \langle a,b \rangle$.

Let us substitute in (4) $t = t_0$. Then we get $\varphi(t_0) = x_0$.

The integrand of the right-hand side of identity (4) is continuous function, and therefore the right-hand side of (4) is continuously differentiable, which means that the function $\varphi(t)$ is also continuously differentiable.

Differentiating (4) by t, we get:

$$\dot{\varphi}(t) = X(t, \varphi(t)), \qquad (5)$$

which means that the function $x = \varphi(t)$ is the solution of the system (1) satisfying the condition $\varphi(t_0) = x_0$.

Vice versa. If function $x = \varphi(t)$ is the solution of the Cauchy problem (1), (2), then (5) holds and $\varphi(t_0) = x_0$. Let's integrate (5) from t_0 to t:

$$\varphi(t) - \varphi(t_0) = \int_{t_0}^{t} X(\tau, \varphi(\tau))d\tau ,$$

that is, the function $\varphi(t)$ satisfies (4). Hence $\varphi(t)$ is the solution of the integral equation (3). The statement is proved.

We will solve equation (3) by the method of successive approximations.

As zero approximation we take the function

$$\varphi_0(t) \equiv x_0.$$

This approximation is defined for $t \in R$.

Let's assume there exists the interval $\langle a_1, b_1 \rangle$ such that $t_0 \in \langle a_1, b_1 \rangle$ and $(t, \varphi_0(t)) \in D$ for any $t \in \langle a_1, b_1 \rangle$. Then for $t \in \langle a_1, b_1 \rangle$ defined the function

$$\varphi_1(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi_0(\tau)) d\tau,$$

which is called Picard first approximation.

If there exists the interval $\langle a_2, b_2 \rangle \subset \langle a_1, b_1 \rangle$, $t_0 \in \langle a_2, b_2 \rangle$ such that $(t, \varphi_1(t)) \in D$ for any $t \in \langle a_2, b_2 \rangle$, then for $t \in \langle a_2, b_2 \rangle$ defined the second approximation

$$\varphi_2(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi_1(\tau)) d\tau.$$

We will continue this process further in a same fashion.

Let's assume that $(k-1)$th approximation $\varphi_{k-1}(t)$ defined on the interval $\langle a_{k-1}, b_{k-1} \rangle$, $k \in N$.

If there exists the interval $\langle a_k, b_k \rangle$ such that $\langle a_k, b_k \rangle \subset \langle a_{k-1}, b_{k-1} \rangle$, $t_0 \in \langle a_k, b_k \rangle$ such that $(t, \varphi_{k-1}(t)) \in D$ for any $t \in \langle a_k, b_k \rangle$, then for $t \in \langle a_k, b_k \rangle$ defined the k-th approximation

$$\varphi_k(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi_{k-1}(\tau)) d\tau. \tag{6}$$

Note that the process of constructing Picard approximations may stop at k-th step, if the interval $\langle a_k, b_k \rangle$ does not exist.

Example. Let $n = 1$, $D = \{(t, x) : t^2 + x^2 \le 1\}$, $(t_0, x_0) = (0, 1)$.

Zero approximation $\varphi_0(t) \equiv x_0$ determined for all t, but $(t, \varphi_0(t)) \notin D$ for $t \neq t_0$. So there is no interval $\langle a_1, b_1 \rangle$ with the above properties, and the Picard process terminates at the first step.

Remark. If $D = R^{n+1}$, then all successive Picard approximations are defined for all $t \in R$.

The proof of the remark follows from the definition of Picard approximations.

Theorem 1 (Picard theorem). Let on a closed bounded set $D \subset R^{n+1}$ vector function $X(t,x)$ is continuous and satisfies the Lipschitz condition according to x. Let's also assume that all Picard approximations $\varphi_k(t)$ defined on the same segment $[a,b]$. Then the sequence of Picard approximations $\varphi_k(t)$ converges uniformly on the segment $[a,b]$ to the function $\varphi(t)$, and $\varphi(t)$ is the solution of the equation (3) (and hence $\varphi(t)$ is the solution of the Cauchy problem (1), (2)).

Proof of Theorem 1. Consider the series

$$\varphi_0(t) + \sum_{k=1}^{+\infty} \left(\varphi_k(t) - \varphi_{k-1}(t) \right), \tag{7}$$

where $t \in [a,b]$.

$\varphi_0(t) + \sum_{k=1}^{m} \left(\varphi_k(t) - \varphi_{k-1}(t) \right) = \varphi_m(t)$ are partial sums of the series (7), therefore the uniform convergence of the series (7) is equivalent to the uniform convergence of the sequence $\{\varphi_k(t)\}_{\kappa=0}^{+\infty}$.

Let us show that the series (7) converges uniformly on the segment $[a,b]$, indicating a convergent number series majorizing (1) on $[a,b]$.

Note, that

$$\left\| \varphi_0(t) \right\| = \left\| x_0 \right\|,$$

$$\left\| \varphi_1(t) - \varphi_0(t) \right\| = \left\| \int_{t_0}^{t} X(\tau, \varphi_0(\tau)) d\tau \right\| \leq \left| \int_{t_0}^{t} \left\| X(\tau, x_0) \right\| d\tau \right|. \tag{8}$$

Function $X(t,x_0)$ is continuous on the segment $[a,b]$, and therefore limited: there is $M>0$ such that $\|X(t,x_0)\|\le M$ for all $t\in[a,b]$. From (8) it follows:

$$\|\varphi_1(t)-\varphi_0(t)\|\le M|t-t_0|. \tag{9}$$

Further,

$$\begin{aligned}\|\varphi_2(t)-\varphi_1(t)\| &= \left\|\int_{t_0}^{t} X(\tau,\varphi_1(\tau))d\tau-\int_{t_0}^{t} X(\tau,\varphi_0(\tau))d\tau\right\|\le \\ &\le\left|\int_{t_0}^{t}\|X(\tau,\varphi_1(\tau))-X(\tau,\varphi_0(\tau))\|d\tau\right|.\end{aligned} \tag{10}$$

According to the assumptions of the theorem $X(t,x)\in Lip_x(D)$, that is, there is the constant $L>0$ such that for any two points $(t,\overline{x}),(t,\overline{\overline{x}})\in D$ the inequality

$$\|X(t,\overline{x})-X(t,\overline{\overline{x}})\|\le L\|\overline{x}-\overline{\overline{x}}\| \tag{11}$$

holds.

By assumption $(t,\varphi_0(t))\in D$ and $(t,\varphi_1(t))\in D$ for all $t\in[a,b]$. Therefore, from (10), (11) it follows that

$$\|\varphi_2(t)-\varphi_1(t)\|\le L\left|\int_{t_0}^{t}\|\varphi_1(\boldsymbol{\tau})-\varphi_0(\boldsymbol{\tau})\|d\tau\right|,$$

and from (9) we conclude:

$$\|\varphi_2(t)-\varphi_1(t)\|\le ML\left|\int_{t_0}^{t}|\tau-t_0|d\tau\right|=\frac{M}{L}\frac{(L|t-t_0|)^2}{2}. \tag{12}$$

Let us prove the inequality

$$\|\varphi_k(t)-\varphi_{k-1}(t)\|\le\frac{M}{L}\frac{(L|t-t_0|)^k}{k!},\ k\in N, \tag{13}$$

using the method of mathematical induction.

Induction base gives inequality (9) or inequality (12).

Induction transition: let's assume that inequality (13) is true. Then

$$\begin{aligned}\left\|\varphi_{k+1}(t)-\varphi_k(t)\right\| &= \left\|\int_{t_0}^{t} X(\tau,\varphi_k(\tau))d\tau - \int_{t_0}^{t} X(\tau,\varphi_{k-1}(\tau))d\tau\right\| \le \\ &\le \left|\int_{t_0}^{t} \left\|X(\tau,\varphi_k(\tau)) - X(\tau,\varphi_{k-1}(\tau))\right\| d\tau\right|.\end{aligned} \tag{14}$$

Since $(t,\varphi_k(t))\in D$ and $(t,\varphi_{k-1}(t))\in D$ for $t\in[a,b]$, then from (11) and (14), using the induction assumption (13), we obtain:

$$\begin{aligned}\left\|\varphi_{k+1}(t)-\varphi_k(t)\right\| &\le L\left|\int_{t_0}^{t}\left\|\varphi_k(\tau)-\varphi_{k-1}(\tau)\right\|d\tau\right| \le \\ &\le L\left|\int_{t_0}^{t}\frac{M}{L}\frac{\left(L|\tau-t_0|\right)^k}{k!}d\tau\right| = \frac{M}{L}\frac{\left(L|t-t_0|\right)^{k+1}}{(k+1)!}.\end{aligned}$$

Since the last inequality proves the induction transition, then (13) is true for all $k\in N$.

Since $t, t_0\in[a,b]$, then $|t-t_0|\le(b-a)$, and therefore from (13) it follows that

$$\left\|\varphi_k(t)-\varphi_{k-1}(t)\right\| \le \frac{M}{L}\frac{\left(L(b-a)\right)^k}{k!} \tag{15}$$

for all $k\in N$.

Inequality (15) means that the series (7) is majorized by convergent number series $\|x_0\|+\frac{M}{L}\sum_{k=1}^{+\infty}\frac{\left(L(b-a)\right)^k}{k!}$, which sum is equal to $\|x_0\|+\frac{M}{L}\left(\exp\left(L(b-a)\right)-1\right)$.

Hence, series (7) converges uniformly on the segment $[a,b]$ to some function $\varphi(t)$. It is obvious that the sequence $\varphi_k(t)$ is also converges uniformly to $\varphi(t)$.

Since $(t,\varphi_k(t))\in D$ for $k\in N$, then $(t,\varphi(t))\in D$, and, according to the Lipschitz condition (11),

$$\|X(t,\varphi_k(t))-X(t,\varphi(t))\|\le L\|\varphi_k(t)-\varphi(t)\|. \qquad (16)$$

The norm of the right-hand side of the inequality (16) tends to zero uniformly on the segment $[a,b]$, and therefore the norm of the left-hand side also uniformly tends to zero. That is, the sequence $X(t,\varphi_k(t))$ converges uniformly on the segment $[a,b]$ to function $X(t,\varphi(t))$. Using this fact, we pass to the limit in equality (6):

$$\varphi(t)=x_0+\int_{t_0}^{t}X(\tau,\varphi(\tau))d\tau.$$

This means that the function $x=\varphi(t)$ is the solution of (3) defined on $[a,b]$, and, according to the statement proven above, $x=\varphi(t)$ is also the solution of the Cauchy problem (1), (2). The theorem is proved.

Corollary 1. Let $D=R^{n+1}$. Let also function $X(t,x)$ be continuous and satisfies the Lipschitz condition with respect to x (globally) on R^{n+1}. Then the Cauchy problem (1), (2) has the solution $x=\varphi(t)$ defined for all $t\in R$.

Proof of Corollary 1. As noted above, all Picard approximations $\varphi_k(t)$ determined for all $t\in R$. Let's take a number sequence $\{h_k\}_{k=1}^{+\infty}$ such that $h_{k+1}>h_k>0$ and $h_k \underset{k\to+\infty}{\to} +\infty$. By Theorem 1 on each of the segments $[t_0-h_k,t_0+h_k]$ sequence of Picard approximations converges uniformly to the solution $x=\varphi(t)$ of Cauchy problem (1), (2). Therefore, the solution is defined for all $t\in R$. The corollary is proved.

Remark. If $D=R^{n+1}$ and function $X(t,x)$ is continuous and satisfies the Lipschitz condition with respect to x locally on R^{n+1}, then the solution $x=\varphi(t)$ of Cauchy problem (1), (2) is not necessarily defined for all $t\in R$.

Indeed, $x = tg\,t$ is the solution of the equation $\dot{x} = x^2 + 1$ with initial conditions $(t_0, x_0) = (0,0)$. This solution is determined only for $t \in (-\pi/2, \pi/2)$, although all Picard's approximations $\varphi_k(t)$ determined for $t \in R$.

Let's assume that the function $X(t,x)$ is continuous on the set

$$D = \{(t,x) : |t - t_0| \le a, \|x - x_0\| \le b\},$$

$a, b \in R$, $a > 0$, $b > 0$.

Then there is $M > 0$ such that $\|X(t,x)\| \le M$ for all $(t,x) \in D$.

Let's put $h = \min(a, b/M)$.

Theorem 2. Let the function $X(t,x)$ be continuous and satisfy the Lipschitz condition on the set D. Then all Picard approximations $\varphi_k(t)$ defined on the interval $[t_0 - h, t_0 + h]$ and sequence $\varphi_k(t)$ converges uniformly on $[t_0 - h, t_0 + h]$ to the solution $x = \varphi(t)$ of the Cauchy problem (1), (2).

Proof of Theorem 2. We'll again use the principle of mathematical induction. Let's prove that every $\varphi_k(t)$ determined for $|t - t_0| \le h$.

Induction base: zero approximation $\varphi_0(t) = x_0$ obviously defined on the segment $[t_0 - h, t_0 + h]$.

Induction transition: let's assume that $\varphi_{k-1}(t)$ determined for $|t - t_0| \le h$ and $(t, \varphi_{k-1}(t)) \in D$ for $t \in [t_0 - h, t_0 + h]$.

Then, according to (6),

$$\varphi_k(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi_{k-1}(\tau))\,d\tau,$$

and $\varphi_k(t)$ determined for $|t - t_0| \le h$.

Since $\|X(t, \varphi_{k-1}(t))\| \le M$ for $t \in [t_0 - h, t_0 + h]$, then

$$\|\varphi_k(t) - x_0\| \le \left| \int_{t_0}^{t} \|X(\tau, \varphi_{k-1}(\tau))\|\,d\tau \right| \le \left| \int_{t_0}^{t} M\,d\tau \right| \le M|t - t_0| \le Mh \le b.$$

It means that $(t, \varphi_k(t)) \in D$. Hence $\varphi_k(t)$ determined for $|t - t_0| \le h$ for all $k \in N$.

Also, as proven in Theorem 1, the sequence $\varphi_k(t)$ converges uniformly on $[t_0 - h, t_0 + h]$ to the solution $x = \varphi(t)$ of the Cauchy problem (1), (2). The theorem is proved.

§ 4. Uniqueness theorem.

We consider the system

$$\dot{x} = X(t,x), \tag{1}$$

where $x \in R^n$, function $X(t,x)$ is continuous and satisfies the Lipschitz condition with respect to x locally in the domain $G \subset R^{n+1}$.

Let $(t_0,x_0) \in G$. Let’s consider also the Cauchy problem:

$$t = t_0,\ x = x_0. \tag{2}$$

Theorem. Let $x = \varphi(t)$ and $x = \xi(t)$ be two solutions of the Cauchy problem (1), (2), defined on the segment $\langle a,b \rangle$. Then $\varphi(t) \equiv \xi(t)$ on $\langle a,b \rangle$.

Proof of the theorem. Let $\theta \in \langle a,b \rangle$. We will show that $\varphi(\theta) = \xi(\theta)$.

Notice, that $\varphi(t_0) \equiv \xi(t_0) = x_0$. Let $\theta \neq t_0$. Without loss of generality we assume that $\theta > t_0$.

Let's denote

$$\Gamma_1 = \{(t,x) : t \in [t_0,\theta],\ x = \varphi(t)\},$$
$$\Gamma_2 = \{(t,x) : t \in [t_0,\theta],\ x = \xi(t)\},$$
$$\Gamma = \Gamma_1 \cup \Gamma_2.$$

The set Γ contained in G, is limited and closed, because $\Gamma_1 \subset G$, $\Gamma_2 \subset G$, and each of the sets Γ_1, Γ_2 is limited and closed.

From Theorem 1 of paragraph 2 it follows that on the set Γ function $X(t,x)$ satisfies the Lipschitz condition with respect to x globally, that is, there is a constant $L > 0$ such that for any two points $(t,\bar{x})$, $(t,\bar{\bar{x}})$ belonging to Γ the following inequality holds

$$\|X(t,\bar{x}) - X(t,\bar{\bar{x}})\| \leq L\|\bar{x} - \bar{\bar{x}}\|.$$

Therefore, for any $t \in [t_0,\theta]$

$$\|X(t,\varphi(t))-X(t,\xi(t))\|\le L\|\varphi(t)-\xi(t)\|. \qquad (3)$$

Since the Cauchy problem (1), (2) is equivalent to the integral equation

$$x(t)=x_0+\int_{t_0}^{t}X(\tau,x(\tau))d\tau,$$

then for $t\in\langle a,b\rangle$

$$\varphi(t)=x_0+\int_{t_0}^{t}X(\tau,\varphi(\tau))d\tau$$

$$\xi(t)=x_0+\int_{t_0}^{t}X(\tau,\xi(\tau))d\tau.$$

Subtracting the second equality from the first equality and considering the norm of the difference, we obtain that for $t\in[t_0,\theta]$

$$\begin{aligned}\|\varphi(t)-\xi(t)\|&=\left\|\int_{t_0}^{t}X(\tau,\varphi(\tau))d\tau-\int_{t_0}^{t}X(\tau,\xi(\tau))d\tau\right\|\le\\ &\le\left|\int_{t_0}^{t}\|X(\tau,\varphi(\tau))-X(\tau,\xi(\tau))\|d\tau\right|\le L\left|\int_{t_0}^{t}\|\varphi(\tau)-\xi(\tau)\|d\tau\right|.\end{aligned} \qquad (4)$$

Let $u(t)=\|\varphi(\tau)-\xi(\tau)\|$. Then inequality (4) takes the form

$$u(t)\le L\left|\int_{t_0}^{t}u(\tau)d\tau\right|,$$

and by corollary of Gronwall's lemma $u(t)\equiv 0$, that is $\varphi(t)\equiv\xi(t)$ on the segment $[t_0,\theta]$ and in particular $\varphi(\theta)=\xi(\theta)$.

Due to the arbitrariness of the choice $\theta\in\langle a,b\rangle$, the theorem is proved.

§5. Continuation of the solutions.

Consider the system

$$\dot{x} = X(t,x), \tag{1}$$

where $x \in R^n$, $X(t,x)$ is continuous and satisfies the Lipschitz condition with respect to x locally in the domain $G \subset R^{n+1}$.

Definition 1. Let $x = \varphi(t)$ be the solution of the system (1) defined on the interval (a,b). This solution *can be continued to the right beyond the point* b, if there is a number $\bar{b} > b$ such that on the interval $(a,\bar{b})$ there exists the solution $x = u(t)$, and $u(t) \equiv \varphi(t)$ on (a,b). We'll also say that the solution $x = u(t)$ is *the continuation of the solution* $x = \varphi(t)$ *to the right until* $\bar{b}$.

The continuation of the solution $x = \varphi(t)$ to the left beyond a defined in a similar way.

Theorem 1. The solution $x = \varphi(t)$ of the system (1), defined on the interval (a,b), can be continued to the right beyond b if and only if there exists the limit $\lim\limits_{t \to b} \varphi(t) = \xi$, and the point (b,ξ) belongs to the domain G.

Proof of Theorem 1. => If the solution $x = \varphi(t)$ of system (1) defined on the interval (a,b) can be continued to the right beyond b then there is a solution $x = u(t)$ defined on the interval $(a,\bar{b})$, where $\bar{b} > b$, such that $u(t) \equiv \varphi(t)$ on (a,b) and therefore there is a limit

$$\lim_{t \to b} \varphi(t) = \lim_{t \to b} u(t) = u(b) = \xi,$$

and, by definition of the solution of the system, point $(b,u(b))$ belongs to the domain G.

<= Let there exists the limit $\lim\limits_{t \to b} \varphi(t) = \xi$, and $(b,\xi) \in G$.

Let's take an arbitrary point $t_0 \in (a,b)$. Let's also consider $x_0 = \varphi(t_0)$ and set the Cauchy problem

$$t = t_0,\ x = x_0. \tag{2}$$

Since $x = \varphi(t)$ solves this problem then function $\varphi(t)$ also satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} X(\tau, x(\tau))\,d\tau, \tag{3}$$

that is

$$\varphi(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi(\tau))\,d\tau. \tag{4}$$

Let us define by continuity the function $\varphi(t)$ putting $\varphi(b) = \xi$. Passing in (4) to the limit at $t \to b$, makes sure that $\varphi(t)$ is a solution of the equation (3) for $t \in (a,b]$, and therefore the solution of the system (1) for $t \in (a,b]$.

Let's set the Cauchy problem

$$t = b,\ x = \xi. \tag{5}$$

Since $\varphi(b) = \xi$, then the function $x = \varphi(t)$ is the solution of the Cauchy problem (1), (5) for $t \in (a,b]$. On the other hand, according to the solution existence theorem, there is $h > 0$ such that for $|t - t_0| \le h$ there exists the solution $x = \psi(t)$ of the Cauchy problem (1), (5). Without loss of generality we will assume that $h < b - a$. Thus, $a < b - h < b < b + h$.

On the segment $[b-h, b]$ two solutions to the Cauchy problem (1), (5) are defined: $\varphi(t)$ and $\psi(t)$. By the uniqueness theorem $\varphi(t) = \psi(t)$ for $t \in [b-h, b]$.

Let's set

$$u(t) = \begin{cases} \varphi(t), & if \ \ t \in (a,b), \\ \psi(t), & if \ \ t \in [b-h, b+h]. \end{cases}$$

Obviously $x = u(t)$ is the continuation of the solution $x = \varphi(t)$ to the right until $b + h$. The theorem is proved.

Let's formulate the similar theorem about continuation of the solution to the left beyond the point a.

Theorem 2. The solution $x = \varphi(t)$ of the system (1), defined on the interval (a,b), can be continued to the left beyond a if and only if there exists the limit $\lim\limits_{t \to a} \varphi(t) = \zeta$, and the point (a,ζ) belongs to the domain G.

Maximum interval of existence.

Theorem 3. Let the solution $x = \varphi(t)$ of the system (1) be defined on the interval (a,b), $b < +\infty$. Then there exists $\beta \geq b$ such that on the interval (a,β) there exists the continuation $x = u(t)$ of the solution $x = \varphi(t)$ to the right until β, and the solution $x = u(t)$ is not continuable to the right beyond β.

Proof of the Theorem 3. If the solution $x = \varphi(t)$ is not continuable to the right beyond b, then we assume $\beta = b$.

Let $x = \varphi(t)$ be continuable to the right beyond b.

Let's denote by $x = u_{\bar{b}}(t)$ the solution of the system (1), which is the continuation of the solution $x = \varphi(t)$ to the right until $\bar{b} > b$. Denote also by B the set of such $\bar{b} > b$ that the solution $x = \varphi(t)$ is continuable to the right until $\bar{b}$. Obviously, if $\bar{b} \in B$ and $b < \bar{\bar{b}} < \bar{b}$, then $\bar{\bar{b}} \in B$.

Let's set $\beta = \sup B$. Note, that it could also happen that $\beta = +\infty$.

By the definition of the supremum, for any t such that $b \leq t < \beta$, there is $\bar{b} \in B$, $t < \bar{b} \leq \beta$, such that there defined the solution $x = u_{\bar{b}}(t)$, $t \in (a,\bar{b})$.

Let us show that (in some sense) the solution $x = u_{\bar{b}}(t)$ does not depend on $\bar{b}$.

Let $b_1, b_2 \in B$, $b < b_1 < b_2 \le \beta$. The solution $x = u_{b_1}(t)$ determined for $t \in (a, b_1)$, the solution $x = u_{b_2}(t)$ determined for $t \in (a, b_2)$, and by the definition of continuation of the solution $u_{b_1}(t) \equiv u_{b_2}(t) \equiv \varphi(t)$ on the interval (a, b).

Let $t_0 \in (a, b)$, $x_0 = \varphi(t_0)$. Then $u_{b_1}(t_0) = u_{b_2}(t_0) = x_0$, and solutions $\varphi(t)$, $u_{b_1}(t)$, $u_{b_2}(t)$ solve the same Cauchy problem. By the uniqueness theorem $u_{b_1}(t) \equiv u_{b_2}(t)$ on the interval (a, b_1), because the $u_{b_1}(t)$ and $u_{b_2}(t)$ both defined on this interval. The last identity proves the independence of the solution $x = u_{\bar{b}}(t)$ from index $\bar{b}$, and hence we can omit this index.

So, no we have the solution $x = u(t)$ such that this solution is determined for $t \in (a, \beta)$ and this is the continuation of the solution $x = \varphi(t)$ to the right beyond β.

It remains to prove that the solution $x = u(t)$ is not continuable to the right beyond β.

If $\beta = +\infty$, then it's obvious.

If $\beta < +\infty$, then let's assume, on contrary to our statement, that $x = u(t)$ is continuable to the right until $\bar{\beta} > \beta$. Then, by definition, there is a solution $x = \bar{u}(t)$ of the system (1), defined on the interval $(a, \bar{\beta})$, such that $u(t) = \bar{u}(t)$ for $t \in (a, \beta)$. And therefore $\varphi(t) \equiv u(t) \equiv \bar{u}(t)$ on the interval (a, b), that is, the solution $x = \varphi(t)$ is continuation to the right until $\bar{\beta}$, and hence $\bar{\beta} \in B$. By the definition of supremum this is impossible. Hence the solution $x = u(t)$ is not continuable to the right beyond β. The theorem is proved.

The following theorem is proved in a similar way.

Theorem 4. Let the solution $x = \varphi(t)$ of the system (1) be defined on the interval (a, b), $a > -\infty$. Then there exists $\alpha \le a$ such that on the interval (α, b) there exists the continuation $x = v(t)$ of the solution $x = \varphi(t)$ to the left until α, and the solution $x = v(t)$ is not continuable to the left beyond α.

A corollary of Theorems 3 and 4 is Theorem 5.

Theorem 5. Let the solution $x=\varphi(t)$ of the system (1) be defined on the interval (a,b). Then there exists $\alpha \le a$ and $\beta \ge b$ such that on the interval (α,β) the continuation $x=\psi(t)$ of the solution $x=\varphi(t)$ is defined, and the solution $x=\psi(t)$ is not continuable to the left beyond α and not continuable to the right beyond β, and $\varphi(t)\equiv\psi(t)$ on the interval (a,b).

Proof of the Theorem 5. If $b<+\infty$, then, according to Theorem 3, there exists $\beta \ge b$, such that on the on the interval (a,β) there exists the continuation $x=u(t)$ of the solution $x=\varphi(t)$ to the right until β, and the solution $x=u(t)$ is not continuable to the right beyond β.

If $b=+\infty$, then we assume $\beta=+\infty$, and $u(t)=\varphi(t)$ for $t\in(a,\beta)$.

If $a>-\infty$, then, according to Theorem 4, there exists $\alpha \le a$ such that on the interval (α,b) there exists the continuation $x=v(t)$ of the solution $x=\varphi(t)$ to the left until α, and the solution $x=v(t)$ is not continuable to the left beyond α.

If $a=-\infty$, then we assume $\alpha=-\infty$, and $v(t)=\varphi(t)$ for $t\in(\alpha,b)$.

Let's set

$$\psi(t)=\begin{cases} u(t), & if \quad t\in(a,\beta), \\ v(t), & if \quad t\in(\alpha,b). \end{cases}$$

This definition is correct since $v(t)\equiv\varphi(t)\equiv u(t)$ on the interval (a,b). Function $x=\psi(t)$ is the solution of the system (1), which, according to Theorems 3 and 4, cannot be continued to the left beyond α, nor to the right beyond β. Moreover, $\varphi(t)\equiv\psi(t)$ on (a,b). The theorem is proved.

Definition 3. If the solution $x=\psi(t)$ of the system (1), defined on the interval (α,β), cannot be continued to the left beyond α, nor to the right for β, then this solution is called maximally continued (or, sometimes, maximally extended), and the interval (α,β) is called the maximum interval of existence of the solution $x=\psi(t)$.

Thus, each point (t_0,x_0) of the domain G corresponds to the interval (α,β) such that on this interval the maximally continued solution of the Cauchy problem (1), (2) is defined.

§6. Behavior of the solutions while approaching to the boundary of the maximum interval of existence.

In this section we again consider the system

$$\dot{x} = X(t,x), \tag{1}$$

where $x \in R^n$, function $X(t,x)$ is continuous and satisfies the Lipschitz condition with respect to x locally in the domain $G \subset R^{n+1}$.

Let us denote by $\overline{G}$ the closure of the set G and through ∂G the boundary of G. So, $\partial G = \overline{G} \setminus G$.

Theorem 1. Let the set G be bounded and the function $X(t,x)$ is also be bounded in G.

If the solution $x = \varphi(t)$ of the system (1), defined on the interval (a,b), $b < +\infty$ is not continuable to the right beyond b, then there is a limit $\lim\limits_{t \to b} \varphi(t) = \xi$, and $(b,\xi) \in \partial G$.

Proof of the Theorem 1. According to the assumptions of the theorem, the function $X(t,x)$ is bounded in G, that is, there is $M > 0$ such that $\|X(t,x)\| \le M$ for $(t,x) \in G$.

Let us fix an arbitrary $\varepsilon > 0$, and put $\delta = \min(b-a,\ \varepsilon/M)$.

Let $t_1, t_2 \in (b-\delta, b)$, $x_1 = \varphi(t_1)$. Then $x = \varphi(t)$ solves the Cauchy problem $t = t_1$, $x = x_1$, and for $t \in (a,b)$

$$\varphi(t) = x_1 + \int_{t_1}^{t} X(\tau, \varphi(\tau))\, d\tau,$$

and in particular,

$$\varphi(t_2) = x_1 + \int_{t_1}^{t_2} X(\tau, \varphi(\tau))\, d\tau.$$

Hence,

$$\left\|\varphi(t_2)-\varphi(t_1)\right\|=\left\|\int_{t_1}^{t_2} X\left(\tau,\varphi(\tau)\right)d\tau\right\|\le\left|\int_{t_1}^{t_2}\left\|X\left(\tau,\varphi(\tau)\right)\right\|d\tau\right|\le\left|\int_{t_1}^{t_2} Md\tau\right|=M\left|t_2-t_1\right|. \quad (2)$$

Since $t_1, t_2 \in (b-\delta,b)$, then $\left|t_2-t_1\right|<\delta$ and from inequality (2) it follows that

$$\left\|\varphi(t_2)-\varphi(t_1)\right\|<M\delta\le\varepsilon.$$

From the last inequality, due to the arbitrariness of ε, it follows the existence of the limit $\lim\limits_{t\to b}\varphi(t)=\xi$.

A point $(t,\varphi(t))$ belongs to the set G for any $t\in(a,b)$, so the limit point (b,ξ) belongs to $\bar{G}$. If the limit point (b,ξ) belongs to G, then, according to the Theorem 1 of the previous paragraph, the solution $x=\varphi(t)$ could be continued to the right beyond b, which contradicts the assumption. That's why $(b,\xi)\in\bar{G}\setminus G=\partial G$. The theorem is proved.

A similar theorem is valid for the left endpoint of the interval.

Theorem 2. Let the set G be bounded and the function $X(t,x)$ is also be bounded in G.

If the solution $x=\varphi(t)$ of the system (1), defined on the interval (a,b), $a>-\infty$, is continuable to the left beyond a, then there is a limit $\lim\limits_{t\to a}\varphi(t)=\zeta$, and $(a,\zeta)\in\partial G$.

Theorem 3 (the theorem on the exit of the maximally continued solution from a compact set). Let $x=\varphi(t)$ be the maximally continued solution of the system (1), defined on the interval (α,β), $\beta<+\infty$. Then for any closed bounded set $D\subset G$ there exists $\delta>0$ such that $(t,\varphi(t))\notin D$ for any $t\in(\beta-\delta,\beta)$.

Proof of the Theorem 3. We prove the theorem by contradiction. Let us assume that there is a closed bounded set $D\subset G$, and for this set for any $\delta>0$ there exists $t\in(\beta-\delta,\beta)$ such that $(t,\varphi(t))\in D$.

Let's choose an arbitrary sequence $\{\delta_j\}_{j=1}^{+\infty}$, such that $0<\delta_{j+1}<\delta_j$, $\delta_1<\beta-\alpha$, $\delta_j \underset{j\to+\infty}{\to} 0$. Then for any δ_j there exists $t_j\in(\beta-\delta_j,\beta)$ such that $(t_j,\varphi(t_j))\in D$.

The set D is bounded and closed by assumption, therefore the sequence $\{(t_j,\varphi(t_j))\}_{j=1}^{+\infty}\subset D$ has convergent subsequence $\{(t_k,\varphi(t_k))\}_{k=1}^{+\infty}$, wherein

$$t_k \underset{k\to+\infty}{\to} \beta,\ \ \varphi(t_k) \underset{k\to+\infty}{\to} \xi, \tag{3}$$

and $(\beta,\xi)\in D$.

Since $D\subset G$, then $(\beta,\xi)\in G$. Therefore there exist $a,b\in R$, $a>0$, $b>0$, such that

$$D_0=\{(t,x):|t-\beta|\le 2a, \|x-\xi\|\le 2b\}\subset G.$$

Function $X(t,x)$ is continuous on the compact set D_0, and therefore there exists $M>0$ such that $\|X(t,x)\|\le M$ for $(t,x)\in D_0$.

Let's set $h=\min(a, b/M)$. From assumptions (3) it follows that there exists k_1 such that for any $k>k_1$

$$\beta-h<t_k<\beta, \tag{4}$$

and there exists k_2 such that for any $k>k_2$

$$\|\varphi(t_k)-\xi\|<b, \tag{5}$$

Let's fix $k>\max(k_1,k_2)$.

Let's define the set

$$D_1=\{(t,x):|t-t_k|\le a, \|x-\varphi(t_k)\|\le b\}.$$

Let's show that $D_1\subset D_0$.

Consider $(t,x)\in D_1$. That is

$$|t - t_k| \le a\,,\ \ \|x - \varphi(t_k)\| \le b\,. \tag{6}$$

From inequalities (4) and (6) it follows:

$$|t - \beta| \le |t - t_k| + |t_k - \beta| \le h + a \le 2a\,. \tag{7}$$

Similarly, from (5) and (6) it follows:

$$\|x - \xi\| \le \|x - \varphi(t_k)\| + \|\varphi(t_k) - \xi\| \le b + b = 2b\,. \tag{8}$$

Inequalities (7) and (8) prove that $(t,x) \in D_0$. Hence $D_1 \subset D_0$ and therefore $\|X(t,x)\| \le M$ and for $(t,x) \in D_1$.

Let us set the Cauchy problem

$$t = t_k\,,\ x = \varphi(t_k)\,. \tag{9}$$

Solution $x = \varphi(t)$ defined for $t \in (\alpha, \beta)$ and satisfies (9).
From inequality (4) it follows that

$$\alpha < t_k < \beta < t_k + h\,, \tag{10}$$

On the other hand, due to the choice of h and definition of D_1, from Theorem 2 of the third paragraph it follows that on the interval $[t_k - h, t_k + h]$ there exists the solution $x = \psi(t)$ of the Cauchy problem (1), (9).

Solutions $x = \varphi(t)$ and $x = \psi(t)$ defined for $t \in [t_k, \beta)$ and solve the same Cauchy problem, and by the uniqueness theorem $\varphi(t) \equiv \psi(t)$ on $[t_k, \beta)$.

Let's define the following function:

$$u(t) = \begin{cases} \varphi(t), & if\ \ t \in (\alpha, \beta), \\ \psi(t), & if\ \ t \in [t_k, t_k + h]. \end{cases}$$

Obviously the function $x = u(t)$ is the solution of the system (1). From inequality (10) it follows that $x = u(t)$ is the continuation of the solution $x = \varphi(t)$ to the right beyond β, which contradicts the assumption of the theorem. The resulting contradiction proves the theorem.

A similar theorem is true for the left endpoint of the interval.

Theorem 4. Let $x=\varphi(t)$ be the maximally continued solution of the system (1) defined on the interval (α,β), $\alpha>-\infty$. Then for any closed bounded set $D\subset G$ there exists $\delta>0$ such that $(t,\varphi(t))\notin D$ for $t\in(\alpha,\alpha+\delta)$.

§7. Systems, comparable to the linear systems.

Consider the system

$$\dot{x}=X(t,x), \tag{1}$$

where

$$x=\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}\in R^n,\ X(t,x)=\begin{pmatrix} X_1(t,x_1,\dots,x_n) \\ \dots \\ X_n(t,x_1,\dots,x_n) \end{pmatrix},$$

function $X(t,x)$ is continuous and satisfies the Lipschitz condition with respect to x locally in the domain $G\subset R^{n+1}$,

$$G=\{(t,x):t\in(a,b),\|x\|<+\infty\},$$

(cases $a=-\infty$ or $b=+\infty$ are not excluded).

Definition. System (1) is called *the system comparable to the linear system* if there exist continuous and non-negative on the interval (a,b) functions $M(t)$ and $N(t)$, such that for any $(t,x)\in G$

$$\|X(t,x)\|\le M(t)\|x\|+N(t). \tag{2}$$

Theorem 1. If (1) is the system comparable to the linear system then the maximum interval of existence of any solution of the system (1) is equal to (a,b)

Proof of the Theorem 1. Let's assume, on the contrary, that there exists the solution $x=\varphi(t)$ of the system (1), such that the maximum interval of existence of this solution is (α,β) and (α,β) does not equal (a,b). From the definition of the domain G it follows that $(\alpha,\beta)\subset(a,b)$. Without loss of generality we assume that $\beta<b$.

Let $t_0\in(a,b)$, $x_0=\varphi(t_0)$. Then $x=\varphi(t)$ is the solution of the Cauchy problem $t=t_0$, $x=x_0$, and for any $t\in(\alpha,\beta)$

$$\varphi(t)=x_0+\int_{t_0}^{t}X(\tau,\varphi(\tau))d\tau. \tag{3}$$

Let's estimate $\varphi(t)$ for $t\in[t_0,\beta)$.

Notice, that $\beta<+\infty$ (since $\beta<b\le+\infty$). From (3) we get:

$$\|\varphi(t)\|\le\|x_0\|+\left\|\int_{t_0}^{t}X(\tau,\varphi(\tau))d\tau\right\|\le\|x_0\|+\int_{t_0}^{t}\|X(\tau,\varphi(\tau))\|d\tau,$$

and from inequality (2) it follows that

$$\|\varphi(t)\|\le\|x_0\|+\int_{t_0}^{t}M(\tau)\|\varphi(\tau)\|d\tau+\int_{t_0}^{t}N(\tau)d\tau. \tag{4}$$

Note, that $[t_0,\beta]\subset(a,b)$, since $a\le\alpha<t_0<\beta<b$.

Function $M(t)$ is non-negative and continuous on the interval $[t_0,\beta]$, so there is a constant $L>0$ such that $0\le M(t)\le L$ for $t\in[t_0,\beta]$. From (4) it follows that

$$\|\varphi(t)\|\le\|x_0\|+\int_{t_0}^{\beta}N(\tau)d\tau+L\int_{t_0}^{t}\|\varphi(\tau)\|d\tau \tag{5}$$

for $t\in[t_0,\beta)$.

Let us denote $\|x_0\| + \int_{t_0}^{\beta} N(\tau)d\tau$ by c, and rewrite (5) in the form

$$\|\varphi(t)\| \le c + L\int_{t_0}^{t} \|\varphi(\tau)\| d\tau.$$

Hence the function $\varphi(t)$ satisfies on the interval $[t_0, \beta)$ the conditions of Gronwall's lemma, and then

$$\|\varphi(t)\| \le ce^{L(t-t_0)}.$$

Therefore,

$$\|\varphi(t)\| \le ce^{L(\beta-t_0)} \tag{6}$$

for $t \in [t_0, \beta)$.

The set $D = \left\{(t,x) : t \in [t_0, \beta], \|x\| \le ce^{L(\beta-t_0)}\right\}$ is closed, bounded and contained in G. From (6) it follows that $(t, \varphi(t)) \in D$ for $t \in [t_0, \beta)$, and this contradicts the Theorem 3 of the previous section. The resulting contradiction proves the theorem.

Consider the system

$$\begin{cases} \dot{x}_1 = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + q_1(t), \\ \dot{x}_2 = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + q_2(t), \\ \ldots \\ \dot{x}_n = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + q_n(t), \end{cases} \tag{7}$$

where are all the functions $p_{jk}(t)$, $q_j(t)$ are continuous on the interval (a,b), $j = 1,2,\ldots,n$, $k = 1,2,\ldots,n$.

Definition. System (7) is called the linear system.

Let us show that the linear system satisfies the conditions of Theorem 1 in the domain G.

System (7) is system (1), where

$$X_j(t,x)=\sum_{k=1}^{n} p_{jk}(t)x_k+q_j(t).$$

Functions $X_j(t,x)$ are continuous in the domain G, and satisfy the Lipschitz condition with respect to x locally (since in G there are continuous partial derivatives $\frac{\partial X_j(t,x)}{\partial x_k}=p_{jk}(t)$, $j=1,2,...,n$, $k=1,2,...,n$).

Let's set $p(t)=\max\limits_{j,k=1,...,n}\left(\left|p_{jk}(t)\right|,\left|q_j(t)\right|\right)$. Note that the function $p(t)$ is continuous on the interval (a,b).

Then

$$\left|X_j(t,x)\right|\le\sum_{k=1}^{n}\left|p_{jk}(t)\right|\left|x_k\right|+\left|q_j(t)\right|\le p(t)\left(\sum_{k=1}^{n}\left|x_k\right|+1\right). \tag{8}$$

Since $|x_k|\le\|x\|=\sqrt{x_1^2+...+x_n^2}$, then from (8) it follows that

$$\left|X_j(t,x)\right|\le p(t)\left(n\|x\|+1\right),$$

and

$$\begin{aligned}\|X(t,x)\|&=\sqrt{X_1^2(t,x)+...+X_n^2(t,x)}\le\\&\le\sqrt{np^2(t)\left(n\|x\|+1\right)^2}=n\sqrt{n}p(t)\|x\|+\sqrt{n}p(t).\end{aligned} \tag{9}$$

Assuming in (9) $M(t)=n\sqrt{n}p(t)$, $N(t)=\sqrt{n}p(t)$, we obtain estimation (2) for the right-hand side of the system (7). Thus, the linear system (7) is a system comparable to the linear system, and Theorem 1 is true for it.

Let us formulate the obtained result in the form of a theorem.

Theorem 2. The maximum interval of existence of any solution of the linear system (7) is equal to (a,b).

Chapter 3. LINEAR DIFFERENTIAL EQUATIONS.

Definition. *Linear differential equation of order* n called the equation

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_{n-1}(t)\dot{x} + p_n(t)x = q(t), \tag{1}$$

where $x^{(k)} = \dfrac{d^k x}{dt^k}$, and all functions $p_k(t)$ and function $q(t)$ are continuous on the interval (a,b), $k = 1,2,\ldots,n$.

If $q(t) \equiv 0$ on (a,b) then equation (1) is called *homogeneous*, otherwise equation (1) is *non-homogeneous.*

Let's move from the equation (1) to the system according to the general rule (already explained in the previous chapter).

We denote

$$x_1 = x,\ x_2 = \dot{x},\ x_3 = \ddot{x},\ \ldots,\ x_n = x^{(n-1)}.$$

Then

$$\dot{x}_1 = \dot{x} = x_2,\ \dot{x}_2 = \ddot{x} = x_3,\ \ldots,\ \dot{x}_{n-1} = x^{(n-1)} = x_n,$$

and, as follows from (1),

$$\dot{x}_n = x^{(n)} = -p_1(t)x_n - \ldots - p_{n-1}(t)x_2 - p_n(t)x_1 + q(t).$$

Thus, we get the system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \ldots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -p_n(t)x_1 - p_{n-1}(t)x_2 - \ldots - p_1(t)x_n + q(t). \end{cases} \tag{2}$$

As shown in the last paragraph of the second chapter, all solutions of this system can be continued on the interval (a,b), therefore all solutions of the equation (1) can be continued on (a,b). In what follows, by a solution of the equation (1) we mean a solution defined on the interval (a,b).

Let $t_0 \in (a,b)$, $x_0 = (x_{10},...,x_{n0})^T \in R^n$. The Cauchy problem for system (2) is stated as follows:

$$t = t_0,\ x_1 = x_{10},\ x_2 = x_{20},\ ...,\ x_n = x_{n0}, \tag{3}$$

and the Cauchy problem for equation (1) is

$$t = t_0,\ x = x_{10},\ \dot{x} = x_{20},\ ...,\ x^{(n-1)} = x_{n0}. \tag{4}$$

Solving the Cauchy problem (1), (4) means: finding the solution $x = \varphi(t)$ of the equation (1) such that $\varphi(t_0) = x_{10}$, $\dot{\varphi}(t_0) = x_{20}$, ..., $\varphi^{(n-1)}(t_0) = x_{n0}$.

The right-hand sides of the equations of the system (2) are continuous and continuously differentiable with respect to x_k (for any $k = 1,2,...,n$) in the domain

$$G = \{(t, x_1,...,x_n) : t \in (a,b), |x_k| < +\infty, k = 1,2,...,n\},$$

and, therefore, any Cauchy problem (2), (3) has a unique solution. Hence, the Cauchy problem (4) for the equation (1) has a unique solution defined on the interval (a,b).

Let us denote the left-hand side of the equation (1) by $L(x)$:

$$L(x) = \sum_{k=0}^{n} p_k(t) x^{(n-k)},$$

where $p_0(t) \equiv 1$, $x^{(0)} = x$.

Remark. $L(x)$ is a linear differential operator. That is:

$$L\left(c_1x_1(t)+c_2x_2(t)\right)=c_1L\left(x_1(t)\right)+c_2L\left(x_2(t)\right)$$

for any functions $x_1(t)$, $x_2(t)$, defined on (a,b), and any constants c_1, c_2.

§1. The main property of the solutions of the linear homogeneous equation.

Consider the linear homogeneous equation

$$L(x)=0. \tag{1}$$

Theorem. Let $\varphi_1(t),\varphi_2(t),\dots,\varphi_m(t)$ be the solutions of the equation (1). Then the function

$$\psi(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+\ldots+c_m\varphi_m(t),$$

where $c_1,c_2,\dots,c_m$ are arbitrary constants, is also the solution of the equation (1) on (a,b).

Proof of the Theorem 1. Since $\varphi_j(t)$ is the solution of the equation (1), then $L\left(\varphi_j(t)\right)\equiv 0$ on the interval (a,b) for any $j=1,2,...,m$.

Hence,

$$L\left(\psi(t)\right)=L\left(\sum_{j=1}^{m}c_j\varphi_j(t)\right)=\sum_{j=1}^{m}c_jL\left(\varphi_j(t)\right)=0,$$

and $\psi(t)$ is the solution of the equation (1). The theorem is proved.

§2. Linearly independent functions.

Definition 1. Functions $\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)$, continuous on the interval (a,b), are called linearly dependent on (a,b), if there are constants $c_1,c_2,\dots,c_n$, not all equal to zero, such that

$$c_1\varphi_1(t)+c_2\varphi_2(t)+...+c_n\varphi_n(t)\equiv 0 \tag{1}$$

on (a,b).

Otherwise the functions $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ are called linearly independent on (a,b).

In other words, continuous functions $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ are linearly independent on the interval (a,b), if from identity (1) it follows that $c_k=0$ for all $k=1,2,...,n$.

Let's assume that the functions $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ are continuously differentiable $(n-1)$ times on interval (a,b). Let's consider the determinant

$$W(t)=W(\varphi_1(t),\varphi_2(t),...,\varphi_n(t))=\begin{vmatrix} \varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t) \\ \dot{\varphi}_1(t) & \dot{\varphi}_2(t) & \cdots & \dot{\varphi}_n(t) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_1^{(n-1)}(t) & \varphi_2^{(n-1)}(t) & \cdots & \varphi_n^{(n-1)}(t) \end{vmatrix}. \tag{2}$$

Definition 2. Determinant (2) is called the Wronski determinant for a system of functions $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ on the interval (a,b) or simply Wronskian.

Theorem 1. Let the functions $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ be linearly dependent and continuously differentiable $(n-1)$ times on the interval (a,b). Then $W(t)\equiv 0$ on (a,b).

Proof of the Theorem 1. According to the assumptions of the theorem, there are constants $c_1,c_2,\ldots,c_n$, not all equal to zero, such that on (a,b) the identity (1) holds. Let us differentiate this identity $(n-1)$ times and build the system of identities:

$$\begin{cases} c_1\varphi_1(t)+c_2\varphi_2(t)+\ldots+c_n\varphi_n(t)\equiv 0, \\ c_1\dot{\varphi}_1(t)+c_2\dot{\varphi}_2(t)+\ldots+c_n\dot{\varphi}_n(t)\equiv 0, \\ \ldots \\ c_1\varphi_1^{(n-1)}(t)+c_2\varphi_2^{(n-1)}(t)+\ldots+c_n\varphi_n^{(n-1)}(t)\equiv 0. \end{cases} \tag{3}$$

For arbitrary $t\in(a,b)$ let's consider the linear algebraic system

$$\begin{cases} z_1\varphi_1(t)+z_2\varphi_2(t)+\ldots+z_n\varphi_n(t)=0, \\ z_1\dot{\varphi}_1(t)+z_2\dot{\varphi}_2(t)+\ldots+z_n\dot{\varphi}_n(t)=0, \\ \ldots \\ z_1\varphi_1^{(n-1)}(t)+z_2\varphi_2^{(n-1)}(t)+\ldots+z_n\varphi_n^{(n-1)}(t)=0. \end{cases} \tag{4}$$

The determinant of this system is the Wronskian $W(t)$ of functions $\varphi_1(t)$, $\varphi_2(t),\ldots,\varphi_n(t)$ at the point $t\in(a,b)$. According to (3), system (4) has a non-zero solution $z_1=c_1, z_2=c_2,\ldots,z_n=c_n$, so the determinant $W(t)$ of the system (4) is equal to zero. From arbitrariness of t it follows that $W(t)\equiv 0$ on (a,b). The theorem is proved.

Remark. The converse of the Theorem 1 is, generally speaking, not true. The following example proves this.

Example. Let $n=2$,

$$\varphi_1(t)=\begin{cases} 0, & if \quad -1<t\le 0, \\ t^2, & if \quad 0<t<1, \end{cases}$$

$$\varphi_2(t)=\begin{cases} t^2, & if \quad -1<t\le 0, \\ 0, & if \quad 0<t<1. \end{cases}$$

Functions $\varphi_1(t)$ and $\varphi_2(t)$ are continuously differentiable on the interval $(-1,1)$,

$$W(\varphi_1(t),\varphi_2(t))=\begin{vmatrix} 0 & t^2 \\ 0 & 2t \end{vmatrix}, \quad \text{if } -1<t\le 0,$$

and

$$W\left(\varphi_1(t),\varphi_2(t)\right)=\begin{vmatrix} t^2 & 0 \\ 2t & 0 \end{vmatrix}, \qquad \text{if } 0<t<1.$$

Hence, $W(t)\equiv 0$ on $(-1,1)$.

Let us show that the functions $\varphi_1(t)$ and $\varphi_2(t)$ are linearly independent on $(-1,1)$. Consider the identity

$$c_1\varphi_1(t)+c_2\varphi_2(t)\equiv 0. \tag{5}$$

Let's put at first $t=1/2$ in (5). Then we get $c_1=0$. Let us now put $t=-1/2$ in (5), and we get $c_2=0$. Thus, from identity (5) it follows that $c_1=c_2=0$, and functions $\varphi_1(t)$ and $\varphi_2(t)$ are linearly independent on $(-1,1)$.

Let us now consider the linear homogeneous equation

$$L(x)=0, \tag{6}$$

where, as before, $L(x)=\sum_{k=0}^{n} p_k(t)x^{(n-k)}$, $p_0(t)\equiv 1$, $x^{(0)}=x$, and functions $p_k(t)$ are continuous on the interval (a,b) for $k=1,2,...,n$.

If $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ are solutions of the equation (6), then the stronger statement than the converse of the Theorem 1 is true.

Theorem 2. Let $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ be the solutions of the equation (6).

If there is a point $t_0\in(a,b)$ such that $W(t_0)=0$, then the functions $\varphi_1(t)$, $\varphi_2(t),\ldots,\varphi_n(t)$ are linearly dependent on (a,b).

Proof of the Theorem 2. Consider a linear homogeneous algebraic system

$$
\begin{cases}
z_1\varphi_1(t_0)+z_2\varphi_2(t_0)+...+z_n\varphi_n(t_0)=0,\\
z_1\dot{\varphi}_1(t_0)+z_2\dot{\varphi}_2(t_0)+...+z_n\dot{\varphi}_n(t_0)=0,\\
...\\
z_1\varphi_1^{(n-1)}(t_0)+z_2\varphi_2^{(n-1)}(t_0)+...+z_n\varphi_n^{(n-1)}(t_0)=0.
\end{cases}
\tag{7}
$$

The determinant of this system is the Wronskian $W(t_0)$, and $W(t_0)=0$. Therefore, system (7) has the non-zero solution $z_1=c_1, z_2=c_2,\dots,z_n=c_n$.

Let's consider

$$\psi(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+...+c_n\varphi_n(t). \tag{8}$$

According to the main property of solutions of the linear homogeneous equation, $x=\psi(t)$ is the solution of the equation (6).

Since $c_1, c_2,\dots,c_n$ is the solution of the system (7), then

$$\psi(t_0)=0,\ \dot{\psi}(t_0)=0,\ ...,\ \psi^{(n-1)}(t_0)=0,$$

that is $x=\psi(t)$ is the solution of the Cauchy problem

$$t=t_0, x=0, \dot{x}=0,..., x^{(n-1)}=0$$

for the equation (6).

The same Cauchy problem is solved by a trivial solution $x(t)\equiv 0$.

From the uniqueness theorem it follows that $\psi(t)\equiv 0$ on the interval (a,b). And, since $c_1, c_2,\dots,c_n$ is non-zero solution of system (7), it follows from (8) the linear dependence of the functions $\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)$. The theorem is proved.

Let $\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)$ be the solutions of the equation (6). Next two statements follow from Theorems 1 and 2.

Corollary 1. If there exists the point $t_0\in(a,b)$ such that $W(t_0)=0$, then $W(t)\equiv 0$, and solutions $\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)$ are linearly dependent on (a,b).

Corollary 2. If there is the point $t_1 \in (a,b)$ such that $W(t_1) \neq 0$, then $W(t) \neq 0$ for any $t \in (a,b)$, and solutions $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$ are linearly independent on (a,b).

§3. Fundamental system of solutions.

Consider the linear homogeneous equation

$$L(x) = 0, \tag{1}$$

where $L(x) = \sum_{k=0}^{n} p_k(t) x^{(n-k)}$, $p_0(t) \equiv 1$, $x^{(0)} = x$, and functions $p_k(t)$ are continuous on the interval (a,b) for all $k = 1,2,\dots,n$.

Definition 1. The set $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$, consisting of n linearly independent solutions of the equation (1) is called *the fundamental system of solutions* of the equation (1).

Theorem 1. Homogeneous linear equation (1) has the fundamental system of solutions.

here we fixed some a_{jk}.

Proof of the Theorem 1. Let $A = \{a_{jk}\}_{j,k=1}^{n}$ be an arbitrary square matrix of order n, such that $\det A \neq 0$.

the function $x, \dot{x}, \dots$ is given in equation (1)

Let's consider an arbitrary point $t_0 \in (a,b)$, and set n Cauchy problems of the equation (1):

? $\begin{cases} x(t_0) = a_{1k} \\ \dot{x}(t_0) = a_{2k} \\ \dots \end{cases}$

how can we prove those equation will holds simultaneously?

$$t = t_0,\ x = a_{1k},\ \dot{x} = a_{2k},\ \dots,\ x^{(n-1)} = a_{nk},$$

where $k = 1,2,\dots,n$.

Let $\varphi_k(t)$ be the solution of k-th Cauchy problem. Let's compose a Wronskian for the system of functions $\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)$ on (a,b).

According to our choice of Cauchy problems

$$W(t_0)=\begin{vmatrix} \varphi_1(t_0) & \varphi_2(t_0) & \dots & \varphi_n(t_0) \\ \dot{\varphi}_1(t_0) & \dot{\varphi}_2(t_0) & \dots & \dot{\varphi}_n(t_0) \\ \dots & \dots & \dots & \dots \\ \varphi_1^{(n-1)}(t_0) & \varphi_2^{(n-1)}(t_0) & \dots & \varphi_n^{(n-1)}(t_0) \end{vmatrix} = \det A \neq 0,$$

and, by the Corollary 2 from the end of the previous paragraph, solutions $\varphi_1(t)$, $\varphi_2(t),\dots,\varphi_n(t)$ are linearly independent on (a,b). That is, they form the fundamental system of solutions. The theorem is proved.

Definition 2. Let $\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)$ be the fundamental system of solutions of the equation (1), and $c_1, c_2,\dots,c_n$ be arbitrary constants. Let's consider the formula:

$$x(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+\dots+c_n\varphi_n(t). \tag{2}$$

The right-hand side of the formula (2) is called *the general solution* of the equation (1).

Theorem 2. Let $\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)$ be the fundamental system of solutions of the equation (1). Then

1) for any set of constants $c_1, c_2,\dots,c_n$ formula (2) gives a solution of the equation (1);

2) if $x=\xi(t)$ is the solution of the equation (1), then there is a set of constants $\bar{c}_1$, $\bar{c}_2,\dots,\bar{c}_n$ such that $\xi(t)=\bar{c}_1\varphi_1(t)+\bar{c}_2\varphi_2(t)+\dots+\bar{c}_n\varphi_n(t)$.

Proof of the Theorem 2. The first statement of the theorem directly follows from the main property of solutions of a linear homogeneous equation (from the Theorem 1 of the first paragraph).

Let's prove the second statement. Let's take an arbitrary point $t_0 \in (a,b)$ and consider the linear non-homogeneous algebraic system

$$\begin{cases} z_1\varphi_1(t_0)+z_2\varphi_2(t_0)+\ldots+z_n\varphi_n(t_0)=\xi(t_0), \\ z_1\dot{\varphi}_1(t_0)+z_2\dot{\varphi}_2(t_0)+\ldots+z_n\dot{\varphi}_n(t_0)=\dot{\xi}(t_0), \\ \ldots \\ z_1\varphi_1^{(n-1)}(t_0)+z_2\varphi_2^{(n-1)}(t_0)+\ldots+z_n\varphi_n^{(n-1)}(t_0)=\xi^{(n-1)}(t_0). \end{cases} \tag{3}$$

The determinant of this system is the Wronskian $W(t_0)$, and $W(t_0)\neq 0$, since the set $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ is the fundamental system of solutions of the equation (1). Hence, system (3) has the unique solution $z_1=\overline{c}_1, z_2=\overline{c}_2,\ldots,z_n=\overline{c}_n$.

Let's consider the function

$$\eta(t)=\overline{c}_1\varphi_1(t)+\overline{c}_2\varphi_2(t)+\ldots+\overline{c}_n\varphi_n(t). \tag{4}$$

The function $x=\eta(t)$ is the solution of the equation (1), and, according to (3)

$$\eta(t_0)=\xi(t_0),\ \dot{\eta}(t_0)=\dot{\xi}(t_0),\ \ldots,\ \eta^{(n-1)}(t_0)=\xi^{(n-1)}(t_0),$$

that is $x=\xi(t)$ and $x=\eta(t)$ solve the same Cauchy problem

$$t=t_0,\ x=\xi(t_0),\ \dot{x}=\dot{\xi}(t_0),\ \ldots,\ x^{(n-1)}=\xi^{(n-1)}(t_0)$$

of the equation (1). From the uniqueness theorem it follows that $\xi(t)\equiv\eta(t)$ on the interval (a,b). The theorem is proved.

§4. Linear non-homogeneous equation.

Consider the linear non-homogeneous equation

$$L(x)=q(t), \tag{1}$$

where $L(x)=\sum_{k=0}^{n}p_k(t)x^{(n-k)}$, $p_0(t)\equiv 1$, $x^{(0)}=x$, functions $p_k(t)$, $k=1,2,\ldots,n$, and $q(t)$ are continuous on the interval (a,b).

The homogeneous equation corresponding to equation (1) is:

$$L(x)=0. \tag{2}$$

Theorem 1. Let $x=\psi(t)$ be the solution of the equation (1) and $x=\varphi(t)$ be the solution of the equation (2). Then $x=\varphi(t)+\psi(t)$ is the solution of the equation (1).

Proof of the Theorem 1. Function $\psi(t)$ is the solution of the equation (1), and $\varphi(t)$ is the solution of the equation (2). This means that $L(\psi(t))\equiv q(t)$ and $L(\varphi(t))\equiv 0$ on the interval (a,b). Hence,

$$L(\varphi(t)+\psi(t))=L(\varphi(t))+L(\psi(t))=0+q(t)=q(t),$$

and function $\varphi(t)+\psi(t)$ is the solution of the equation (1). The theorem is proved.

Definition 2. Let the function $\psi(t)$ be the solution of the equation (1), the set of functions $\varphi_1(t),\varphi_2(t),\dots,\varphi_n(t)$ be the fundamental system of solutions of the equation (2), and $c_1,c_2,\dots,c_n$ be arbitrary constants.

Let's consider the formula:

$$x(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+\dots+c_n\varphi_n(t)+\psi(t). \tag{3}$$

The right-hand side of the formula (3) is called *the general solution* of the equation (1).

Theorem 2. Let $\psi(t)$ be the solution of the equation (1), and $\varphi_1(t),\varphi_2(t),\dots,$ $\varphi_n(t)$ be the fundamental system of solutions of the equation (2). Then

1) for any set of constants $c_1,c_2,\dots,c_n$ formula (3) gives a solution of the equation (1);

2) if $x=\xi(t)$ is the solution of the equation (1), then there is the set of constants $\overline{c}_1,\overline{c}_2,\dots,\overline{c}_n$ such that $\xi(t)=\overline{c}_1\varphi_1(t)+\overline{c}_2\varphi_2(t)+\dots+\overline{c}_n\varphi_n(t)+\psi(t)$.

Proof of the Theorem 2. The first statement of the theorem follows from the main property of solutions of the linear homogeneous equation (Theorem 1 of the first paragraph) and Theorem 1.

Let's prove the second statement. Let's consider an arbitrary point $t_0 \in (a,b)$ and form a linear non-homogeneous algebraic system

$$\begin{cases} z_1\varphi_1(t_0) + z_2\varphi_2(t_0) + \ldots + z_n\varphi_n(t_0) + \psi(t_0) = \xi(t_0), \\ z_1\dot{\varphi}_1(t_0) + z_2\dot{\varphi}_2(t_0) + \ldots + z_n\dot{\varphi}_n(t_0) + \dot{\psi}(t_0) = \dot{\xi}(t_0), \\ \ldots \\ z_1\varphi_1^{(n-1)}(t_0) + z_2\varphi_2^{(n-1)}(t_0) + \ldots + z_n\varphi_n^{(n-1)}(t_0) + \psi^{(n-1)}(t_0) = \xi^{(n-1)}(t_0). \end{cases} \tag{4}$$

The determinant of this system is the Wronskian $W(t_0)$, and $W(t_0) \neq 0$, since the set $\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)$ is the fundamental system of solutions of the equation (2). Consequently, system (4) has the unique solution $z_1 = \overline{c}_1, z_2 = \overline{c}_2, \ldots, z_n = \overline{c}_n$.

Let's set

$$\eta(t) = \overline{c}_1\varphi_1(t) + \overline{c}_2\varphi_2(t) + \ldots + \overline{c}_n\varphi_n(t) + \psi(t). \tag{5}$$

The function $x = \eta(t)$ is the solution of the equation (1), and, according to (4)

$$\eta(t_0) = \xi(t_0),\ \dot{\eta}(t_0) = \dot{\xi}(t_0),\ \ldots,\ \eta^{(n-1)}(t_0) = \xi^{(n-1)}(t_0),$$

that is, $x = \xi(t)$ and $x = \eta(t)$ solve the same Cauchy problem

$$t = t_0,\ x = \xi(t_0),\ \dot{x} = \dot{\xi}(t_0),\ \ldots,\ x^{(n-1)} = \xi^{(n-1)}(t_0)$$

of the equation (1). From the uniqueness theorem it follows that $\xi(t) \equiv \eta(t)$ on the interval (a,b). The theorem is proved.

§5. Lagrange's method.

Consider the non-homogeneous linear equation

$$L(x) = q(t), \tag{1}$$

and the corresponding homogeneous linear equation

$$L(x)=0, \tag{2}$$

$L(x)=\sum_{k=0}^{n} p_k(t)x^{(n-k)}$, $p_0(t)\equiv 1$, $x^{(0)}=x$, where functions $p_k(t)$, $k=1,2,...,n$, and $q(t)$ are continuous on the interval (a,b).

Let $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ be the fundamental system of solutions of the equation (2). Particular solution $x=\psi(t)$ of the equation (1) will be sought in the form

$$\psi(t)=\sum_{j=1}^{n} u_j(t)\varphi_j(t), \tag{3}$$

where $u_j(t)$ are unknown functions, $j=1,2,...,n$.

We have

$$\dot{\psi}(t)=\sum_{j=1}^{n} u_j(t)\dot{\varphi}_j(t)+\sum_{j=1}^{n} \dot{u}_j(t)\varphi_j(t).$$

Let’s set

$$\sum_{j=1}^{n} \dot{u}_j(t)\varphi_j(t)=0, \tag{4}$$

and hence,

$$\dot{\psi}(t)=\sum_{j=1}^{n} u_j(t)\dot{\varphi}_j(t),$$

and

$$\ddot{\psi}(t)=\sum_{j=1}^{n} u_j(t)\ddot{\varphi}_j(t)+\sum_{j=1}^{n} \dot{u}_j(t)\dot{\varphi}_j(t).$$

Let’s set

$$\sum_{j=1}^{n} \dot{u}_j(t)\dot{\varphi}_j(t) = 0,$$

and similarly, we will further assume that for all $s = 1,...,(n-2)$

$$\sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(s)}(t) = 0. \tag{5}$$

Then

$$\psi^{(s)}(t) = \sum_{j=1}^{n} u_j(t)\varphi_j^{(s)}(t) \tag{6}$$

for all $s = 1,...,(n-1)$, and

$$\psi^{(n)}(t) = \sum_{j=1}^{n} u_j(t)\varphi_j^{(n)}(t) + \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t). \tag{7}$$

Now let's put the function $\psi(t)$ together with its derivatives into the equation (1), using formulas (3), (6), (7) and the fact that $L(\varphi_j(t)) \equiv 0$ on the interval (a,b).

$$L(\psi(t)) = L\left(\sum_{j=1}^{n} u_j(t)\varphi_j(t)\right) = \sum_{k=0}^{n} p_k(t)\left(\sum_{j=1}^{n} u_j(t)\varphi_j^{(n-k)}(t)\right) + \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t) =$$

$$= \sum_{j=1}^{n} u_j(t)\left(\sum_{k=0}^{n} p_k(t)\varphi_j^{(n-k)}(t)\right) + \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t) =$$

$$= \sum_{j=1}^{n} u_j(t)L(\varphi_j(t)) + \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t) = \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t),$$

and $x = \psi(t)$ is the solution of the equation (1), if

$$\sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t) = q(t). \tag{8}$$

So the function $\psi(t)$, defined by the formula (3), is the solution of the equation (1) if conditions (4), (5), (8) holds.

Let's collect these conditions into one system.

$$\begin{cases} \dot{u}_1(t)\varphi_1(t)+\dot{u}_2(t)\varphi_2(t)+...+\dot{u}_n(t)\varphi_n(t)=0, \\ \dot{u}_1(t)\dot{\varphi}_1(t)+\dot{u}_2(t)\dot{\varphi}_2(t)+...+\dot{u}_n(t)\dot{\varphi}_n(t)=0, \\ \cdots \\ \dot{u}_1(t)\varphi_1^{(n-2)}(t)+\dot{u}_2(t)\varphi_2^{(n-2)}(t)+...+\dot{u}_n(t)\varphi_n^{(n-2)}(t)=0, \\ \dot{u}_1(t)\varphi_1^{(n-1)}(t)+\dot{u}_2(t)\varphi_2^{(n-1)}(t)+...+\dot{u}_n(t)\varphi_n^{(n-1)}(t)=q(t). \end{cases} \tag{9}$$

Definition. System (9) is called the system in variations.

For any $t\in(a,b)$ system (9) is a linear non-homogeneous algebraic system with respect to $\dot{u}_1(t),\dot{u}_2(t),\ldots,\dot{u}_n(t)$. The determinant of this system is the Wronskian $W(t)$, and $W(t)\neq 0$. Therefore, for any $t\in(a,b)$ system (9) has a solution $\dot{u}_j(t)=f_j(t)$, $j=1,2,...,n$.

All coefficients of the linear system (9) continuously depend on t on the interval (a,b), and the solutions of this system continuously depend on the coefficients. Hence, all the functions $f_j(t)$ are continuous for (a,b), $j=1,2,...,n$, and

$$u_j(t)=\int f_j(t)dt,\ j=1,2,...,n.$$

According to formula (3),

$$\psi(t)=\sum_{j=1}^{n}\varphi_j(t)\int f_j(t)dt,$$

and now formula (3) of the previous paragraph for the general solution of equation (1) takes the form

$$x(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+...+c_n\varphi_n(t)+\sum_{j=1}^{n}\varphi_j(t)\int f_j(t)dt.$$

§6. Linear homogeneous equation with constant coefficients.

Let

$$L(x)=\sum_{k=0}^{n} a_k x^{(n-k)}, \tag{1}$$

where $a_0=1$, a_k are real numbers, $k=1,2,...,n$.

Consider the linear homogeneous equation

$$L(x)=0. \tag{2}$$

We seek for a solution of the equation (1) in the form $x=e^{\lambda t}$. Let's substitute the function $e^{\lambda t}$ in $L(x)$:

$$L\left(e^{\lambda t}\right)=P(\lambda)e^{\lambda t}, \tag{3}$$

where

$$P(\lambda)=\sum_{k=0}^{n} a_k \lambda^k. \tag{4}$$

Definition. Polynomial $P(\lambda)$ is called *the characteristic polynomial*. Equation

$$P(\lambda)=0 \tag{5}$$

is called *the characteristic equation*, its roots are called *characteristic numbers* of the equation (2).

We'll show, that the function $x=e^{\lambda t}$ is a solution of the equation (2) if and only if λ is the root of the characteristic equation (5).

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be roots of the equation (5). Let us construct the fundamental system of solutions of the equation (2). We'll split our argumentation for several cases.

1. If all characteristic numbers $\lambda_1, \lambda_2, ..., \lambda_n$ are real and simple (that is, not multiple), then the equation (2) has the fundamental system of solutions

$$e^{\lambda_1 t},\ e^{\lambda_2 t}, ...,\ e^{\lambda_n t}.$$

We will prove the linear independence of these functions later, after considering of all the cases.

2. Let all characteristic numbers $\lambda_1, \lambda_2, ..., \lambda_n$ be simple, but among them there are complex ones.

Lemma. If $u(t)$ and $v(t)$ are real functions, and complex-valued function $w(t) = u(t) + iv(t)$ (where $i = \sqrt{-1}$) is the solution of the equation (2), then $u(t)$ and $v(t)$ are also solutions of (2).

Proof of the lemma. We know, that $L(w(t)) = 0$. We also know, that

$$L(w(t)) = L(u(t) + iv(t)) = L(u(t)) + iL(v(t)).$$

Hence, $L(u(t)) + iL(v(t)) = 0$, and therefore $L(u(t)) = 0$ and $L(v(t)) = 0$. From the last two equalities it follows that $u(t)$ and $v(t)$ are solutions of the equation (2). The lemma is proved.

Let $\lambda_j = \alpha + i\beta$, $j \in \{1, 2, ..., n\}$, be a root of the characteristic equation (5), and $\beta \neq 0$. Then $\bar{\lambda}_j = \alpha - i\beta$ is also the root of the characteristic equation, and the functions $e^{\lambda_j t}$ and $e^{\bar{\lambda}_j t}$ are solutions of the equation (2).

According to Euler's formula

$$e^{\lambda_j t} = e^{(\alpha + i\beta)t} = e^{\alpha t}\cos(\beta t) + ie^{\alpha t}\sin(\beta t),$$

and, due to the above lemma, the functions

$$e^{\alpha t}\cos(\beta t) \text{ and } e^{\alpha t}\sin(\beta t)$$

are solutions of the equation (2). This pair of functions in the fundamental system of solutions corresponds to the pair of roots λ_j and $\overline{\lambda}_j$ of characteristic equation (5).

3. Let there be multiples among the roots of the characteristic equation, and the root λ_j, $j \in \{1,2,...,n\}$, has the multiplicity $d \geq 2$. It means that

$$P(\lambda_j) = P'(\lambda_j) = P''(\lambda_j) = ... = P^{(d-1)}(\lambda_j) = 0. \qquad (6)$$

Let us differentiate equality (3) m times with respect to λ.

On the one side,

$$\begin{aligned} \frac{\partial^m}{\partial \lambda^m} L\left(e^{\lambda t}\right) = \frac{\partial^m}{\partial \lambda^m} \sum_{k=0}^{n} a_k \frac{\partial^{n-k}}{\partial t^{n-k}} e^{\lambda t} = \sum_{k=0}^{n} a_k \frac{\partial^{n-k}}{\partial t^{n-k}} \left(\frac{\partial^m}{\partial \lambda^m} e^{\lambda t} \right) = \\ = \sum_{k=0}^{n} a_k \frac{\partial^{n-k}}{\partial t^{n-k}} \left(t^m e^{\lambda t}\right) = L\left(t^m e^{\lambda t}\right), \end{aligned} \qquad (7)$$

and on the other side, according to Leibniz's formula,

$$\frac{\partial^m}{\partial \lambda^m}\left(P(\lambda) e^{\lambda t}\right) = \sum_{s=0}^{m} C_m^s P^{(s)}(\lambda) \frac{\partial^{m-s}}{\partial \lambda^{m-s}} e^{\lambda t} = \sum_{s=0}^{m} C_m^s P^{(s)}(\lambda) t^{m-s} e^{\lambda t}. \qquad (8)$$

From equalities (3), (7) and (8) it follows that

$$L\left(t^m e^{\lambda t}\right) = \sum_{s=0}^{m} C_m^s P^{(s)}(\lambda) t^{m-s} e^{\lambda t}. \qquad (9)$$

Let us now set in (9) $\lambda = \lambda_j$, and $m = 1,2,...,(d-1)$. Then from (6) it follows that the right-hand side of equality (9) is equal to zero, and $L\left(t^m e^{\lambda_j t}\right) = 0$ for all $m = 1,2,...,(d-1)$. Hence the equation (2) has exactly d solutions

$$e^{\lambda_j t},\ t e^{\lambda_j t},\ t^2 e^{\lambda_j t}, ...,\ t^{d-1} e^{\lambda_j t}, \qquad (10)$$

which in the fundamental system correspond to the root λ_j of multiplicity d.

If $\lambda_j = \alpha + i\beta$, where $\beta \neq 0$ is the root of equation (5) of multiplicity d, then $\overline{\lambda}_j = \alpha - i\beta$ is the root of equation (5) of multiplicity d. Separating the real and the imaginary parts in the solutions (10), we obtain $2d$ solutions

$$
\begin{aligned}
&e^{\alpha t}\cos(\beta t),\ te^{\alpha t}\cos(\beta t),\ t^2e^{\alpha t}\cos(\beta t),...,\ t^{d-1}e^{\alpha t}\cos(\beta t),\\
&e^{\alpha t}\sin(\beta t),\ te^{\alpha t}\sin(\beta t),\ t^2e^{\alpha t}\sin(\beta t),...,\ t^{d-1}e^{\alpha t}\sin(\beta t),
\end{aligned}
$$

corresponding in the fundamental system of solutions to the pair of roots λ_j and $\overline{\lambda}_j$ of multiplicity d of the characteristic equation (5).

So, we built n various solutions corresponding to n roots of the equation (5). Let us prove that actually the system of linearly independent solutions has been constructed.

Theorem. The constructed system of solutions is fundamental.

Proof of the Theorem. Let $\lambda_1, \lambda_2, ..., \lambda_s$ be different roots of the characteristic equation (5) and d_j be multiplicity of the root λ_j. Let's assume that the solutions

$$
\begin{aligned}
&e^{\lambda_1 t},\ te^{\lambda_1 t},\ t^2e^{\lambda_1 t},...,\ t^{d_1-1}e^{\lambda_1 t},\\
&...\\
&e^{\lambda_s t},\ te^{\lambda_s t},\ t^2e^{\lambda_s t},...,\ t^{d_s-1}e^{\lambda_s t}
\end{aligned}
$$

of the equation (2) are linearly dependent. That is, there are constants c_{jm}, $j = 1,2,...,s$, $m = 1,2,...,d_j$, not all equal to zero, such that

$$
\begin{aligned}
&c_{11}e^{\lambda_1 t} + c_{12}te^{\lambda_1 t} + c_{13}t^2e^{\lambda_1 t} + ... + c_{1d_1}t^{d_1-1}e^{\lambda_1 t} + ... +\\
&+c_{s1}e^{\lambda_s t} + c_{s2}te^{\lambda_s t} + c_{s3}t^2e^{\lambda_s t} + ... + c_{sd_s}t^{d_s-1}e^{\lambda_s t} \equiv 0
\end{aligned}
$$

for $t \in R$, or

$$
R_{10}(t)e^{\lambda_1 t} + R_{20}(t)e^{\lambda_2 t} + ... + R_{s0}(t)e^{\lambda_s t} \equiv 0, \tag{11}
$$

where $R_{j0}(t)$ is the polynomial of $(d_j - 1)$ degree, and not all of these polynomials are identically equal to zero.

Let for certainty $R_{s0}(t) \neq 0$. Let us rewrite identity (11) in the form

$$R_{10}(t) + R_{20}(t) e^{(\lambda_2 - \lambda_1)t} + \ldots + R_{s0}(t) e^{(\lambda_s - \lambda_1)t} \equiv 0, \tag{12}$$

and differentiate (12) d_1 times. We get the identity

$$R_{21}(t) e^{(\lambda_2 - \lambda_1)t} + \ldots + R_{s1}(t) e^{(\lambda_s - \lambda_1)t} \equiv 0, \tag{13}$$

where $R_{j1}(t)$ are polynomials of the same degree as $R_{j0}(t)$, $j = 2,3,\ldots,s$. Note, that $R_{j1}(t) \equiv 0$ if, and only if $R_{j0}(t) \equiv 0$.

From identity (13) it follows that

$$R_{21}(t) + R_{31}(t) e^{(\lambda_3 - \lambda_2)t} + \ldots + R_{s1}(t) e^{(\lambda_s - \lambda_2)t} \equiv 0. \tag{14}$$

Differentiating (14) d_2 times, we get the identity

$$R_{32}(t) e^{(\lambda_3 - \lambda_2)t} + \ldots + R_{s2}(t) e^{(\lambda_s - \lambda_2)t} \equiv 0,$$

where $R_{j2}(t)$ are polynomials of the same degree as $R_{j0}(t)$, $j = 3,\ldots,s$. Note, that $R_{j2}(t) \equiv 0$ if, and only if $R_{j0}(t) \equiv 0$.

After completing the described procedure $(s-1)$ times, we get the identity

$$R_{s(s-1)}(t) e^{(\lambda_s - \lambda_{s-1})t} \equiv 0,$$

and then

$$R_{s(s-1)}(t) \equiv 0$$

which is true only if $R_{s0}(t) \equiv 0$. But according to our assumption $R_{s0}(t) \neq 0$. The resulting contradiction proves the theorem.

511 ~ 831

§7. Linear non-homogeneous equation with constant coefficients.

Let again $L(x)=\sum_{k=0}^{n} a_k x^{(n-k)}$, where $x^{(0)}=x$, $a_0=1$, and all a_k are real numbers, $k=1,2,...,n$.

We consider the linear non-homogeneous equation

$$L(x)=q(t), \tag{1}$$

where function $q(t)$ is continuous, $t\in R$.

The homogeneous equation corresponding to equation (1) is

$$L(x)=0. \tag{2}$$

Charactereristic equation for equation (2) is

$$P(\lambda)=0, \tag{3}$$

where $P(\lambda)=\sum_{k=0}^{n} a_k \lambda^k$.

Let $\varphi_1(t),\varphi_2(t),\ldots,\varphi_n(t)$ be the fundamental system of solutions of the equation (2) (such a system was built in the previous paragraph).

In order to obtain a general solution of the non-homogeneous equation (1), it remains to find only a particular solution $x=\psi(t)$ of the equation (1). This solution can be found by the Lagrange's method, but if the equation (2) is an equation with constant coefficients, and the function $q(t)$ has a special form, then the solution $x=\psi(t)$ can be obtained by the method of indetermined coefficients, which is outlined below.

1. Let the non-homogeneity of the equation (1) have the form

$$q(t)=R_m(t)e^{\lambda_0 t}, \tag{4}$$

where $R_m(t)$ is m-degree polynomial: $R_m(t)=\sum_{j=0}^{m} r_j t^{m-j}$.

Theorem 1. If λ_0 is not the root of the characteristic equation (3) (that is, $P(\lambda_0) \neq 0$), then the equation (1) with nonlinearity (4) has a solution of the form

$$\psi(t) = Q_m(t)e^{\lambda_0 t}, \tag{5}$$

where $Q_m(t)$ is m-degree polynomial: $Q_m(t) = \sum_{j=0}^{m} q_j t^{m-j}$.

计算复杂、理论较简单，未讲评

Proof of the Theorem 1. Let's find the coefficients of the polynomial $Q_m(t)$ by substituting function (5) into the equation (1).

$$L(\psi(t)) = L\left(Q_m(t)e^{\lambda_0 t}\right) = L\left(\sum_{j=0}^{m} q_j t^{m-j} e^{\lambda_0 t}\right) = \sum_{j=0}^{m} q_j L\left(t^{m-j} e^{\lambda_0 t}\right).$$

According to the formula (9) of the previous paragraph

$$L\left(t^{m-j} e^{\lambda_0 t}\right) = \sum_{s=0}^{m-j} C_{m-j}^{s} P^{(s)}(\lambda_0) t^{m-j-s} e^{\lambda_0 t}, \tag{6}$$

hence,

$$L(\psi(t)) = e^{\lambda_0 t} \sum_{j=0}^{m} q_j \left(\sum_{s=0}^{m-j} C_{m-j}^{s} P^{(s)}(\lambda_0) t^{m-j-s}\right). \tag{7}$$

Let's substitute (7) into the equation

$$L(\psi(t)) = R_m(t)e^{\lambda_0 t}.$$

Then

$$\begin{aligned} & q_0 \sum_{s=0}^{m} C_m^s P^{(s)}(\lambda_0) t^{m-s} + q_1 \sum_{s=0}^{m-1} C_{m-1}^{s} P^{(s)}(\lambda_0) t^{m-1-s} + \\ & + q_2 \sum_{s=0}^{m-2} C_{m-2}^{s} P^{(s)}(\lambda_0) t^{m-2-s} + \ldots + q_{m-1} \sum_{s=0}^{1} C_1^s P^{(s)}(\lambda_0) t^{1-s} + q_m C_0^0 P(\lambda_0) = \\ & = r_0 t^m + r_1 t^{m-1} + \ldots + r_{m-1} t + r_m. \end{aligned} \tag{8}$$

Equating in (8) the coefficients at the same degrees of t, we obtain the system for determining the coefficients of the polynomial $Q_m(t)$:

$$\begin{cases} q_0 P(\lambda_0) = r_0, \\ q_0 C_m^1 P'(\lambda_0) + q_1 P(\lambda_0) = r_1, \\ q_0 C_m^2 P''(\lambda_0) + q_1 C_{m-1}^1 P'(\lambda_0) + q_2 P(\lambda_0) = r_2, \\ \ldots \\ q_0 P^{(m)}(\lambda_0) + q_1 P^{(m-1)}(\lambda_0) + \ldots + q_m P(\lambda_0) = r_m. \end{cases} \tag{9}$$

From the system (9) we can recursively find q_j, $j = 0,1,\ldots,m$, since $P(\lambda_0) \neq 0$. The theorem is proved.

Theorem 2. If λ_0 is the root of the characteristic equation (3) of the multiplicity $d \geq 1$:

$$P(\lambda_0) = P'(\lambda_0) = P''(\lambda_0) = \ldots = P^{(d-1)}(\lambda_0) = 0, \qquad P^{(d)}(\lambda_0) \neq 0, \tag{10}$$

then the equation (1) with nonlinearity (4) has a solution of the form

$$\psi(t) = t^d Q_m(t) e^{\lambda_0 t}, \tag{11}$$

where $Q_m(t)$ is m-degree polynomial: $Q_m(t) = \sum\limits_{j=0}^{m} q_j t^{m-j}$.

Proof of the Theorem 2. As in the proof of the Theorem 1, the coefficients of the polynomial $Q_m(t)$ we find by substituting (11) into the equation (1).

$$L(\psi(t)) = L\left(t^d Q_m(t) e^{\lambda_0 t}\right) = L\left(\sum_{j=0}^{m} q_j t^{m+d-j} e^{\lambda_0 t}\right) = \sum_{j=0}^{m} q_j L\left(t^{m+d-j} e^{\lambda_0 t}\right),$$

and according to formula (6),

$$L\left(t^{m+d-j} e^{\lambda_0 t}\right) = \sum_{s=0}^{m+d-j} C_{m+d-j}^{s} P^{(s)}(\lambda_0) t^{m+d-j-s} e^{\lambda_0 t}.$$

From equalities (10) it follows that

$$L\left(t^{m+d-j}e^{\lambda_0 t}\right)=$$
$$=\sum_{s=d}^{m+d-j} C_{m+d-j}^{s} P^{(s)}(\lambda_0) t^{m+d-j-s} e^{\lambda_0 t} = \sum_{s=0}^{m-j} C_{m+d-j}^{s+d} P^{(s+d)}(\lambda_0) t^{m-j-s} e^{\lambda_0 t},$$

and

$$L(\psi(t)) = e^{\lambda_0 t} \sum_{j=0}^{m} q_j \left(\sum_{s=0}^{m-j} C_{m+d-j}^{s+d} P^{(s+d)}(\lambda_0) t^{m-j-s} \right). \tag{12}$$

Substituting the resulting equality (12) into the equation

$$L(\psi(t)) = R_m(t) e^{\lambda_0 t}$$

and equating the coefficients at the same degrees of t, we obtain the system for determining the coefficients of the polynomial $Q_m(t)$:

$$\begin{cases} q_0 P^{(d)}(\lambda_0) = r_0, \\ q_0 C_{d+m}^{d+1} P^{(d+1)}(\lambda_0) + q_1 C_{d+m-1}^{d} P^{(d)}(\lambda_0) = r_1, \\ q_0 C_{d+m}^{d+2} P^{(d+2)}(\lambda_0) + q_1 C_{d+m-1}^{d+1} P^{(d+1)}(\lambda_0) + q_2 C_{d+m-2}^{d} P^{(d)}(\lambda_0) = r_2, \\ \ldots \\ q_0 P^{(d+m)}(\lambda_0) + q_1 P^{(d+m-1)}(\lambda_0) + \ldots + q_m P^{(d)}(\lambda_0) = r_m. \end{cases} \tag{13}$$

Since $P^{(d)}(\lambda_0) \neq 0$, then from the system (13) we can recursively find all q_j, $j = 0,1,\ldots,m$. The theorem is proved.

2. Let the non-homogeneity of the equation (1) have the form

$$q(t) = e^{\alpha_0 t}\left(\tilde{R}_{m_1}(t)\cos(\beta_0 t) + \hat{R}_{m_2}(t)\sin(\beta_0 t)\right), \tag{14}$$

where $\tilde{R}_{m_1}(t)$ and $\hat{R}_{m_2}(t)$ are m_1- and m_2-degree polynomials respectively.

Theorem 3. If $\lambda_0 = \alpha_0 + i\beta_0$ is not the root of the characteristic equation (3), then the equation (1) with nonlinearity (14) has a solution of the form

$$\psi(t) = e^{\alpha_0 t}\left(\tilde{Q}_m(t)\cos(\beta_0 t) + \hat{Q}_m(t)\sin(\beta_0 t)\right), \tag{15}$$

where $m = \max(m_1, m_2)$, $\tilde{Q}_m(t)$ and $\hat{Q}_m(t)$ are m-degree polynomials.

Theorem 4. If $\lambda_0 = \alpha_0 + i\beta_0$ is the root of the characteristic equation (3) of the multiplicity d, then the equation (1) with nonlinearity (14) has a solution of the form

$$\psi(t) = t^d e^{\alpha_0 t}\left(\tilde{Q}_m(t)\cos(\beta_0 t) + \hat{Q}_m(t)\sin(\beta_0 t)\right), \tag{16}$$

where $m = \max(m_1, m_2)$, $\tilde{Q}_m(t)$ and $\hat{Q}_m(t)$ are m-degree polynomials.

Before proving Theorems 3 and 4, we prove a simple statement.

Statement. If $\psi_1(t)$ is the solution of the equation $L(x) = q_1(t)$, and $\psi_2(t)$ is the solution of the equation $L(x) = q_2(t)$, then the function $\psi_1(t) + \psi_2(t)$ is the solution of the equation $L(x) = q_1(t) + q_2(t)$.

Proof of the statement. By assumption $L(\psi_1(t)) = q_1(t)$ and $L(\psi_2(t)) = q_2(t)$, hence,

$$L(\psi_1(t) + \psi_2(t)) = L(\psi_1(t)) + L(\psi_2(t)) = q_1(t) + q_2(t).$$

Thus, the function $\psi_1(t) + \psi_2(t)$ is the solution of the equation $L(x) = q_1(t) + q_2(t)$. The statement is proved.

Proof of the Theorems 3 and 4. According to Euler's formulas

$$e^{\alpha_0 t}\cos(\beta_0 t) = \frac{1}{2}\left(e^{\lambda_0 t} + e^{\bar{\lambda}_0 t}\right),$$

$$e^{\alpha_0 t}\sin\left(\beta_0 t\right)=\frac{1}{2i}\left(e^{\lambda_0 t}-e^{\bar{\lambda}_0 t}\right),$$

where $\lambda_0=\alpha_0+i\beta_0$, $\bar{\lambda}_0=\alpha_0-i\beta_0$.

Then

$$q\left(t\right)=\frac{1}{2}\tilde{R}_{m_1}\left(t\right)\left(e^{\lambda_0 t}+e^{\bar{\lambda}_0 t}\right)+\frac{1}{2i}\hat{R}_{m_2}\left(t\right)\left(e^{\lambda_0 t}-e^{\bar{\lambda}_0 t}\right)=$$
$$=e^{\lambda_0 t}\frac{1}{2i}\left(i\tilde{R}_{m_1}\left(t\right)+\hat{R}_{m_2}\left(t\right)\right)+e^{\bar{\lambda}_0 t}\frac{1}{2i}\left(i\tilde{R}_{m_1}\left(t\right)-\hat{R}_{m_2}\left(t\right)\right).$$

Let's denote

$$q_1\left(t\right)=e^{\lambda_0 t}\frac{1}{2i}\left(i\tilde{R}_{m_1}\left(t\right)+\hat{R}_{m_2}\left(t\right)\right),$$

$$q_2\left(t\right)=e^{\bar{\lambda}_0 t}\frac{1}{2i}\left(i\tilde{R}_{m_1}\left(t\right)-\hat{R}_{m_2}\left(t\right)\right).$$

Then $q\left(t\right)=q_1\left(t\right)+q_2\left(t\right)$, and $q_1\left(t\right)$, $q_2\left(t\right)$ are functions of the form (4). The degree of polynomials

$$\frac{1}{2i}\left(i\tilde{R}_{m_1}\left(t\right)+\hat{R}_{m_2}\left(t\right)\right)\quad\text{and}\quad\frac{1}{2i}\left(i\tilde{R}_{m_1}\left(t\right)-\hat{R}_{m_2}\left(t\right)\right)$$

is equal to $m=\max\left(m_1,m_2\right)$.

According to the Theorems 1 and 2, the equations

$$L\left(x\right)=q_1\left(t\right)\quad\text{and}\quad L\left(x\right)=q_2\left(t\right)$$

have correspondingly complex-valued solutions $\psi_1\left(t\right)$ and $\psi_2\left(t\right)$ of the form (5) if λ_0 is not the root of the characteristic equation (3), and type (11) if λ_0 is the root of the characteristic equation (3) of the multiplicity d.

Hence, according to the statement proven above, equation (1) has the solution $\psi_0\left(t\right)=\psi_1\left(t\right)+\psi_2\left(t\right)$ of the form

$$\psi_0(t)=\tilde{U}_m(t)e^{\lambda_0 t}+\hat{U}_m(t)e^{\bar{\lambda}_0 t} \tag{17}$$

if λ_0 is not the root of the characteristic equation (3), and of the form

$$\psi_0(t)=t^d\left(\tilde{U}_m(t)e^{\lambda_0 t}+\hat{U}_m(t)e^{\bar{\lambda}_0 t}\right) \tag{18}$$

if λ_0 is the root of the characteristic equation (3) of the multiplicity d, where $\tilde{U}_m(t)$ and $\hat{U}_m(t)$ are m-degree polynomials with complex coefficients.

To complete the proof we need to prove a short lemma.

Lemma. If the complex-valued function $w(t)=u(t)+iv(t)$ is the solution of the equation (1) with $q(t)=f(t)+ih(t)$, where $u(t)$, $v(t)$, $f(t)$, $h(t)$ are real functions, then $u(t)$ is the solution of the equation $L(x)=f(t)$, and $v(t)$ is the solution of the equation $L(x)=h(t)$.

Proof of the lemma. On the one side, $L(w(t))=f(t)+ih(t)$, and on the other side,

$$L(w(t))=L(u(t)+iv(t))=L(u(t))+iL(v(t)).$$

From equality $L(u(t))+iL(v(t))=f(t)+ih(t)$ it follows that $L(u(t))=f(t)$ and $L(v(t))=h(t)$. Hence, $u(t)$ and $v(t)$ are solutions of the equations $L(x)=f(t)$ and $L(x)=h(t)$ respectively. The lemma is proved.

Separating the real and imaginary parts in the solution of the form (17), according to the lemma, we obtain the statement of the Theorem 3. Similarly, separating the real and imaginary parts in the solution of the form (18), we obtain the statement of the Theorem 4.

The theorems are proved.

Chapter 4. LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS.

Let's consider the linear system

$$\begin{cases} \dot{x}_1 = p_{11}(t)x_1 + p_{12}(t)x_2 + \ldots + p_{1n}(t)x_n + q_1(t), \\ \dot{x}_2 = p_{21}(t)x_1 + p_{22}(t)x_2 + \ldots + p_{2n}(t)x_n + q_2(t), \\ \cdots \\ \dot{x}_n = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \ldots + p_{nn}(t)x_n + q_n(t), \end{cases} \tag{1}$$

where all the functions $p_{jk}(t)$, $q_j(t)$ are continuous on the interval (a,b).

System (1) is a system in normal form. The right-hand sides of all equations of the system (1)

$$X_j(t,x) = \sum_{k=1}^{n} p_{jk}(t)x_k + q_j(t)$$

are continuous in the domain

$$G = \{(t,x) : t \in (a,b), \|x\| < +\infty\}$$

and satisfy the Lipschitz condition with respect to x locally, since in G there are continuous partial derivatives $\dfrac{\partial X_j(t,x)}{\partial x_k} = p_{jk}(t)$.

Therefore, the Cauchy problem for system (1)

$$t = t_0,\ x_1 = x_{10},\ x_2 = x_{20},\ \ldots,\ x_n = x_{n0}, \tag{2}$$

where $t_0 \in (a,b)$, $x_0 = (x_{10},\ldots,x_{n0})^T \in R^n$, has the unique solution.

In addition, in the seventh paragraph of the second chapter it is proven that all solutions of a linear system are defined on the interval (a,b). Hence, further we assume that all solutions (1) are defined on (a,b).

Definition. System (1) is called homogeneous if $q_j(t) \equiv 0$, $j = 1,2,\ldots,n$. Otherwise, the system is called non-homogeneous.

§ 1. Vector notation of a linear system.

First, let us recall information from matrix theory that we will use later.

Matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}$$

we will write it in the form $A_{[n\times m]} = \{a_{jk}\}$ or $A = \{a_{jk}\}_{[n\times m]}$, or in the form $A = (a_1, a_2, ..., a_m)$, where a_j is j-th column of the matrix A.

If $A_{[n\times m]} = \{a_{jk}\}$, $B_{[n\times m]} = \{b_{jk}\}$, then $(A+B)_{[n\times m]} = \{a_{jk} + b_{jk}\}$.

If $B_{[s\times n]} = \{b_{jl}\} = (b_1, b_2, ..., b_n)$, $A_{[n\times m]} = \{a_{lk}\}$, then $(BA)_{[s\times m]} = \left\{ \sum_{l=1}^{n} b_{jl} a_{lk} \right\}$, or $(BA)_{[s\times m]} = (Ba_1, Ba_2, ..., Ba_m)$.

If matrix A is n-dimensional square matrix, then the norm of the matrix A is the operator norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\|,$$

where (as before) $\|x\| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ is Euclidean norm of the vector $x = (x_1, ..., x_n)^T$.

It is not difficult to show that the following inequalities are true

$$\|A+B\| \le \|A\| + \|B\|,$$

$$\|A \cdot B\| \le \|A\| \cdot \|B\|.$$

If $x = (x_1, ..., x_n)^T$, then

$$\|Ax\| \le \|A\| \cdot \|x\|$$

(this inequality is called the property of consistency between the operator norm of a matrix and the Euclidean norm of a vector).

Matrix $U(t)_{[s\times n]} = \{u_{jl}(t)\}$ is continuous on the interval $\langle a,b\rangle$ if all functions $u_{jl}(t)$ are continuous on $\langle a,b\rangle$, and is continuously differentiable if all $u_{jl}(t)$ are continuously differentiable on $\langle a,b\rangle$. Wherein

$$\dot{U}(t)_{[s\times n]} = \{\dot{u}_{jl}(t)\}.$$

If $U(t)_{[s\times n]} = \{u_{jl}(t)\}$ and $V(t)_{[n\times m]} = \{v(t)_{lk}\}$ are continuously differentiable matrices on $\langle a,b\rangle$, then their product is also continuously differentiable on $\langle a,b\rangle$, and it is easy to show that

$$\frac{d}{dt}\big(U(t)V(t)\big) = \dot{U}(t)V(t) + U(t)\dot{V}(t).$$

If matrix $U(t)_{[s\times n]} = \{u_{jl}(t)\}$ is continuous on the segment $[a,b]$, then by definition

$$\int_a^b U(t)\,dt = \left\{\int_a^b u_{jl}(t)\,dt\right\}.$$

It is not difficult to show that for a n-dimensional square matrix $U(t)$

$$\left\|\int_a^b U(t)\,dt\right\| \le \int_a^b \|U(t)\|\,dt.$$

Let $x=\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$, $P(t)_{[n\times n]}=\{p_{jk}(t)\}$, $q(t)=\begin{pmatrix} q_1(t) \\ \dots \\ q_n(t) \end{pmatrix}$.

We write the linear system (1) in the vector form

$$\dot{x}=P(t)x+q(t). \qquad (1)$$

We assume that the matrix $P(t)$ and the vector $q(t)$ are continuous on the interval (a,b).

Recall that the solution of the system (1) is a vector function $x=\varphi(t)$, defined on (a,b), which, being substituted into (1), turns the system into an identity.

Cauchy problem for system (1) is

$$t=t_0,\ x=x_0,$$

where $t_0\in(a,b)$, $x_0\in R^n$.

§ 2. Matrix equation.
The main property of a linear homogeneous system.

Let us first consider the linear homogeneous system

$$\dot{x}=P(t)x, \qquad (1)$$

where the matrix $P(t)_{[n\times n]}=\{p_{jk}(t)\}$ is continuous on the interval (a,b).

Let the vector functions $\varphi_1(t),\varphi_2(t),\dots,\varphi_m(t)$ be the solutions of the system (1). Let's build the matrix

$$\Phi_m(t)=(\varphi_1(t),\varphi_2(t),\dots,\varphi_m(t)). \qquad (2)$$

Consider the matrix equation

$$\dot{X}=P(t)X, \qquad (3)$$

where X is any matrix with n lines.

Solution of the equation (3) called the matrix $X = X(t)$, continuously differentiable on (a,b) and satisfying the equation (3).

Statement. $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are solutions of the system (1) if and only if the matrix $\Phi_m(t)$ is the solution of the equation (3).

Proof of the statement. If $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are solutions of the system (1), then $\dot{\varphi}_j(t) = P(t)\varphi_j(t)$ for all $j = 1,2,...,m$. According to (2),

$$\dot{\Phi}_m(t) = (\dot{\varphi}_1(t), \dot{\varphi}_2(t), ..., \dot{\varphi}_m(t)) =$$
$$= (P(t)\varphi_1(t), P(t)\varphi_2(t), ..., P(t)\varphi_m(t)) = P(t)\Phi_m(t),$$

and hence the matrix $\Phi_m(t)$ is the solution of the equation (3).

Vice versa, if $\Phi_m(t)$ is the solution of the equation (3), then

$$(\dot{\varphi}_1(t), \dot{\varphi}_2(t), ..., \dot{\varphi}_m(t)) = \dot{\Phi}_m(t) =$$
$$= P(t)\Phi_m(t) = (P(t)\varphi_1(t), P(t)\varphi_2(t), ..., P(t)\varphi_m(t)),$$

and $\dot{\varphi}_j(t) = P(t)\varphi_j(t)$ for any $j = 1,2,...,m$. Hence, $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are solutions of the system (1). The statement is proved.

Remark. For matrix equation (3) we can set the Cauchy problem

$$t = t_0, \ X = A_0,$$

where $t_0 \in (a,b)$ and A_0 is the matrix with n lines. Of course for solution of this problem the existence and uniqueness theorems are true.

The main property of a linear homogeneous system.

Theorem. Let $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ be solutions of the system (1). Then the vector function

$$\psi(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + ... + c_m\varphi_m(t),$$

where $c_1, c_2, \ldots, c_m$ are arbitrary constants, is also the solution of the system (1) on (a,b).

Proof of the Theorem 1. Since the functions $\varphi_j(t)$, $j = 1,2,...,m$ are solutions of the system (1), then the matrix $\Phi_m(t)$ is the solution of the equation (3).

Notice, that $\psi(t) = \Phi_m(t)c$, where $c = (c_1, c_2, ..., c_m)^T$ is m-dimensional constant vector, and

$$\dot{\psi}(t) = \dot{\Phi}_m(t)c = P(t)\Phi_m(t)c = P(t)\psi(t).$$

Hence, $\psi(t)$ is the solution of the equation (1). The theorem is proved.

§3. Linearly independent solutions of a linear homogeneous system.

We consider a linear homogeneous system

$$\dot{x} = P(t)x, \tag{1}$$

where matrix $P(t)_{[n\times n]} = \{p_{jk}(t)\}$ is continuous on the interval (a,b).

Let the vector functions $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ be solutions of the system (1), and

$$\Phi_m(t) = (\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)) \tag{2}$$

is a matrix made up of solutions.

Definition 1. Solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ of the system (1) called linearly dependent on the interval (a,b), if there are constants $c_1, c_2, \ldots, c_m$, not all equal to zero, such that on (a,b)

$$c_1\varphi_1(t) + c_2\varphi_2(t) + ... + c_m\varphi_m(t) \equiv 0. \tag{3}$$

Definition 1 can be reformulated using matrix (2).

Solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ of the system (1) called linearly dependent on the interval (a,b) if there exists m-dimensional constant vector $c = (c_1, c_2, ..., c_m)^T$ such that $c \neq 0$, and $\Phi_m(t)c \equiv 0$ on (a,b).

Definition 2. Solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ of the system (1) are linearly independent on (a,b) if they are not linearly dependent on this interval.

In other words, solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are linearly independent on (a,b) if from the identity $\Phi_m(t)c \equiv 0$ it follows that the vector c is equal to zero.

Theorem 1. Let the solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ of the system (1) be linearly dependent on the interval (a,b). Then for any $t \in (a,b)$

$$rank\, \Phi_m(t) < m.$$

Proof of Theorem 1. Since $\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t)$ are linearly dependent, then there is m-dimensional constant vector $\overline{c} \neq 0$, such that $\Phi_m(t)\overline{c} \equiv 0$ on (a,b).

Consider a linear homogeneous algebraic system

$$\Phi_m(t)z = 0, \tag{4}$$

where $z = (z_1, z_2, ..., z_m)^T$ is unknown vector.

Since system (4) has a non-zero solution $z = \overline{c}$ for each $t \in (a,b)$, then $rank\, \Phi_m(t) < m$. The theorem is proved.

Remark. While proving Theorem 1, we did not use the fact that $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are solutions of the system (1). Theorem 1 is true for any set of vector functions defined on the interval (a,b).

Theorem 2. Let $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ be solutions of the system (1).

If there exists the point $t_0 \in (a,b)$ such that

$$rank\, \Phi_m(t_0) < m,$$

then $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are linearly dependent on (a,b).

Proof of the Theorem 2. Let's consider the linear homogeneous algebraic system

$$\Phi_m(t_0)z = 0, \tag{5}$$

where vector $z = (z_1, z_2, ..., z_m)^T$ is the sought vector.

Since $rank\, \Phi_m(t_0) < m$, then the system (5) has a non-zero solution $z = \overline{c}$.

Let's set $\psi(t) = \Phi_m(t)\overline{c}$. According to the main property of solutions of a linear homogeneous equation, $x = \psi(t)$ is also the solution of the system (1).

Since $z = \overline{c}$ is the solution of the system (5), then $\psi(t_0) = \Phi_m(t_0)\overline{c} = 0$. Hence $x = \psi(t)$ is the solution of the Cauchy problem $t = t_0$, $x = 0$ for system (1). The same Cauchy problem is solved by a trivial solution $x(t) \equiv 0$.

From the uniqueness theorem it follows that $\psi(t) \equiv 0$ on the interval (a,b). Hence the functions $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are linearly dependent (since $\overline{c} \neq 0$). The theorem is proved.

Let $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ be solutions of the system (1), $t \in (a,b)$. Then two statements follow from Theorems 1 and 2.

Corollary 1. If there is a point $t_0 \in (a,b)$ such that $rank\, \Phi_m(t_0) < m$, then $rank\, \Phi_m(t) < m$ for all $t \in (a,b)$, and solutions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t)$ are linearly dependent on (a,b).

Corollary 2. If there is a point $t_1 \in (a,b)$ such that $rank\, \Phi_m(t_1) = m$, then $rank\, \Phi_m(t) = m$ for all $t \in (a,b)$, and solutions $\varphi_1(t), \varphi_2(t), \ldots, \varphi_m(t)$ are linearly independent on (a,b).

Theorem 3. System (1) cannot have more than n linearly independent solutions.

Proof of Theorem 3. Let $m > n$ and $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ be solutions of the system (1), $t \in (a,b)$.

Let's build matrix (2). This matrix is $n \times m$ matrix, and hence $rank\, \Phi_m(t) \le n < m$. From Theorem 1 it follows that $\varphi_1(t), \varphi_2(t), ..., \varphi_m(t)$ are linearly dependent on (a,b). The theorem is proved.

Let $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ be a set of n solutions of the system (1). Let's create the matrix

$$\Phi(t) = (\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)).$$

Definition. Determinant of $\Phi(t)$ is called the Wronski determinant for functions $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ on the interval (a,b) or Wronskian.

$$W(t) = \det \Phi(t).$$

Corollary 3. If there exists the point $t_0 \in (a,b)$ such that $W(t_0) = 0$, then $W(t) \equiv 0$ on (a,b), and solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ are linearly dependent.

Corollary 4. If there exists the point $t_1 \in (a,b)$ such that $W(t_1) \ne 0$, then $W(t) \ne 0$ for all $t \in (a,b)$, and solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ are linearly independent on (a,b).

Remark. The definition of the Wronskian for a linear system (1) is a generalization of the corresponding definition for a linear homogeneous equation of n-th order.

It is not difficult to show that while moving from an equation to a system according to the standard rule described at the beginning of the third chapter, the definition of the Wronskian for the system follows from the definition of the Wronskian for the equation.

§4. Fundamental system of solutions. General solution.

We consider a linear homogeneous system

$$\dot{x} = P(t)x, \tag{1}$$

where the matrix $P(t)_{[n\times n]} = \{p_{jk}(t)\}$ is continuous on the interval (a,b).

Definition 1. The set of linearly independent solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ of the system (1) is called the *fundamental system of solutions* of the system (1).

The matrix

$$\Phi(t) = (\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)), \tag{2}$$

composed of these solutions is called the *fundamental matrix* of system (1).

Theorem 1. Linear system (1) has a fundamental system of solutions.

Proof of the Theorem 1. Let A be any square matrix of order n, such that $\det A \neq 0$.

Let's take an arbitrary point $t_0 \in (a,b)$, and set the Cauchy problem

$$t = t_0, X = A$$

for the matrix equation

$$\dot{X} = P(t)X. \tag{3}$$

As was shown in the second paragraph, this Cauchy problem has a solution $X = \Phi(t)$. Columns $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ of $\Phi(t)$ are solutions of system (1). Wherein

$$\det \Phi(t_0) = W(t_0) = \det A \neq 0,$$

and by 4 corollary of the previous paragraph, solutions $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$ are linearly independent on (a,b), and hence they form a fundamental system of solutions. The theorem is proved.

Definition 2. Let $\Phi(t)$ be fundamental matrix of system (1). Let also c be an arbitrary constant n-dimensional vector. Consider the formula:

$$x(t) = \Phi(t)c. \tag{4}$$

The right-hand side of the formula (2) is called *the general solution* of the system (1).

Theorem 2. Let $\Phi(t)$ be the fundamental matrix of the system (1). Then

1) for any constant vector c formula (4) gives the solution of the system (1),

2) if $x=\xi(t)$ is solution (1), then there is a vector $\overline{c}$ such that $\xi(t)=\Phi(t)\overline{c}$ for any $t\in(a,b)$.

Proof of the Theorem 2. The first statement of the theorem follows from the main property of solutions of a linear homogeneous system.

Let's prove the second statement. Let's consider an arbitrary point $t_0\in(a,b)$ and form a linear non-homogeneous algebraic system

$$\Phi(t_0)z=\xi(t_0), \tag{5}$$

where n-dimensional vector z is unknown vector.

The determinant of this system is the Wronskian $W(t_0)$, and $W(t_0)\neq 0$, since $\Phi(t)$ is fundamental matrix. Hence, system (5) has the unique solution $z=\overline{c}$.

Let's set

$$\eta(t)=\Phi(t)\overline{c}.$$

$x=\eta(t)$ is the solution of the system (1), and, according to (5), $\eta(t_0)=\xi(t_0)$. Hence $x=\xi(t)$ and $x=\eta(t)$ solve the same Cauchy problem of the system (1). From the uniqueness theorem it follows that $\xi(t)\equiv\eta(t)$ on the interval (a,b). The theorem is proved.

General expression for the fundamental matrix of the system (1).

Theorem 3. Let $\Phi(t)$ be the fundamental matrix of the system (1).

1) If B is an arbitrary square matrix of order n, such that $\det B\neq 0$, then the matrix $\Psi(t)=\Phi(t)B$ is also the fundamental matrix of the system (1).

2) If $\Psi(t)$ is the fundamental matrix of the system (1), then there is a square matrix B of order n, $\det B\neq 0$, such that $\Psi(t)=\Phi(t)B$ for any $t\in(a,b)$.

Proof of the Theorem 3. Let's prove the first statement. Since $\Phi(t)$ is a solution of the matrix equation (3) and $\Psi(t)=\Phi(t)B$, then

$$\dot{\Psi}(t)=\dot{\Phi}(t)B=P(t)\Phi(t)B=P(t)\Psi(t).$$

Hence $\Psi(t)$ is also a solution of the equation (3). Moreover, $\det\Psi(t)=\det\Phi(t)\cdot\det B\neq 0$ for any $t\in(a,b)$. Hence, $\Psi(t)$ is fundamental matrix of the system (1).

Let's prove the second statement. Let $\Psi(t)$ be fundamental matrix of the system (1). Let's consider an arbitrary point $t_0\in(a,b)$ and set

$$B=\Phi^{-1}(t_0)\Psi(t_0).$$

$\det B=\det\Phi^{-1}(t_0)\cdot\det\Psi(t_0)\neq 0$, and, according to what was proven above, the matrix $X(t)=\Phi(t)B$ is the fundamental matrix of the system (1). Moreover,

$$X(t_0)=\Phi(t_0)B=\Phi^{-1}(t_0)\Phi(t_0)\Psi(t_0)=\Psi(t_0),$$

Hence, the matrices $X(t)$ and $\Psi(t)$ solve the same Cauchy problem for matrix equation (3). From the uniqueness theorem it follows that $X(t)\equiv\Psi(t)$ on the interval (a,b). The theorem is proved.

Definition 3. Let $\Phi(t)$ be fundamental matrix of solutions of the system (1). Let B be arbitrary square matrix of order n, such that $\det B\neq 0$.

Let's consider the formula:

$$X(t)=\Phi(t)B. \tag{6}$$

The right side of the formula (6) is called the general expression for the fundamental matrices of system (1).

Fundamental matrix $X(t,t_0)$, which turns into the identity matrix for $t=t_0$ and is called the *fundamental Cauchy matrix*.

Remark. If $\Phi(t)$ is any fundamental matrix of the system (1), then $X(t,t_0)=\Phi(t)\Phi^{-1}(t_0)$ is fundamental Cauchy matrix.

Indeed, $X(t_0,t_0)=\Phi(t_0)\Phi^{-1}(t_0)=E$.

§5. Liouville formulas.

Let us write the linear homogeneous system in the form

$$\dot{x}_j=\sum_{k=1}^{n}p_{jk}(t)x_k\,,\quad j=1,2,...,n. \tag{1}$$

Let $\varphi_1(t),\varphi_2(t),...,\varphi_n(t)$ be solutions of the system (1),

$$\varphi_m(t)=\begin{pmatrix}\varphi_{1m}(t)\\ \dots\\ \varphi_{nm}(t)\end{pmatrix},$$

$m=1,2,...,n$.

Consider the Wronskian of this system of solutions:

$$W(t)=\begin{vmatrix}\varphi_{11}(t) & \varphi_{12}(t) & \dots & \varphi_{1n}(t)\\ \varphi_{21}(t) & \varphi_{22}(t) & \dots & \varphi_{2n}(t)\\ \dots & \dots & \dots & \dots\\ \varphi_{n1}(t) & \varphi_{n2}(t) & \dots & \varphi_{nn}(t)\end{vmatrix}.$$

By definition,

$$W(t)=\sum_{\Delta}(-1)^{\alpha}\varphi_{1j_1}(t)...\varphi_{nj_n}(t),$$

where the sum sign means summation according to the determinant rule, that is, the summation is carried out over all permutations $\Delta=(j_1,j_2,...,j_n)$ of natural numbers $(1,2,...,n)$. Wherein $\alpha=0$, if permutation $(j_1,j_2,...,j_n)$ is even, and $\alpha=1$, if the permutation is odd.

$$\dot{W}(t)=\sum_{\Delta}(-1)^{\alpha}\sum_{s=1}^{n}\varphi_{1j_1}(t)\ldots\varphi_{(s-1)j_{s-1}}(t)\dot{\varphi}_{sj_s}(t)\varphi_{(s+1)j_{s+1}}(t)\ldots\varphi_{nj_n}(t)=$$

$$=\sum_{s=1}^{n}\sum_{\Delta}(-1)^{\alpha}\varphi_{1j_1}(t)\ldots\varphi_{(s-1)j_{s-1}}(t)\dot{\varphi}_{sj_s}(t)\varphi_{(s+1)j_{s+1}}(t)\ldots\varphi_{nj_n}(t).$$

The internal sum in the last equality is a determinant, which differs from the Wronskian only by that in s-th row there are derivatives of the corresponding components of the solutions of system (1).

$$\dot{W}(t)=\sum_{s=1}^{n}\begin{vmatrix}\varphi_{11}(t) & \varphi_{12}(t) & \ldots & \varphi_{1n}(t)\\ \ldots & \ldots & \ldots & \ldots\\ \dot{\varphi}_{s1}(t) & \dot{\varphi}_{s2}(t) & \ldots & \dot{\varphi}_{sn}(t)\\ \ldots & \ldots & \ldots & \ldots\\ \varphi_{n1}(t) & \varphi_{n2}(t) & \ldots & \varphi_{nn}(t)\end{vmatrix}.$$

Since $\varphi_m(t)$ are solutions of the system (1), then

$$\dot{\varphi}_{sm}(t)=\sum_{k=1}^{n}p_{sk}(t)\varphi_{km}(t).$$

Hence,

$$\dot{W}(t)=\sum_{s=1}^{n}\begin{vmatrix}\varphi_{11}(t) & \varphi_{12}(t) & \ldots & \varphi_{1n}(t)\\ \ldots & \ldots & \ldots & \ldots\\ \sum_{k=1}^{n}p_{sk}(t)\varphi_{k1}(t) & \sum_{k=1}^{n}p_{sk}(t)\varphi_{k2}(t) & \ldots & \sum_{k=1}^{n}p_{sk}(t)\varphi_{kn}(t)\\ \ldots & \ldots & \ldots & \ldots\\ \varphi_{n1}(t) & \varphi_{n2}(t) & \ldots & \varphi_{nn}(t)\end{vmatrix},$$

or

$$\dot{W}(t)=\sum_{s=1}^{n}\sum_{k=1}^{n}p_{sk}(t)\begin{vmatrix}\varphi_{11}(t) & \varphi_{12}(t) & \ldots & \varphi_{1n}(t)\\ \ldots & \ldots & \ldots & \ldots\\ \varphi_{k1}(t) & \varphi_{k2}(t) & \ldots & \varphi_{kn}(t)\\ \ldots & \ldots & \ldots & \ldots\\ \varphi_{n1}(t) & \varphi_{n2}(t) & \ldots & \varphi_{nn}(t)\end{vmatrix}.$$

Determinants under the sum sign in the last formula where $s\neq k$ differ from Wronskian by that instead s -th line they have k -th line. And these determinants are equal to zero because they have two identical lines.

For $s=k$ the determinants under the sum sign coincide with the Wronskian. Hence,

$$\dot{W}(t)=\sum_{s=1}^{n}p_{ss}(t)W(t),$$

or

$$\dot{W}(t)=W(t)TrP(t), \tag{2}$$

where $P(t)=\{p_{jk}(t)\}_{[n\times n]}$.

Integrating equation (2), we obtain

$$W(t)=c\exp\left(\int Tr\,P(t)dt\right), \tag{3}$$

or

$$W(t)=W(t_0)\exp\left(\int_{t_0}^{t}TrP(\tau)d\tau\right), \tag{4}$$

where $t_0\in(a,b)$.

Definition. Formulas (3) and (4) are called Liouville formulas.

Remark. From the Liouville formulas for a linear homogeneous system we can easily obtain similar formulas for a linear homogeneous equation of n-th order

$$x^{(n)}+p_1(t)x^{(n-1)}+\ldots+p_{n-1}(t)\dot{x}+p_n(t)x=0. \tag{5}$$

According to the standard procedure described at the beginning of the third chapter, using the notation

$$x_1=x, x_2=\dot{x}, x_3=\ddot{x},\ldots,x_n=x^{(n-1)},$$

equation (5) can be reduced to a linear homogeneous system

$$
\begin{cases}
\dot{x}_1 = x_2, \\
\dots \\
\dot{x}_{n-1} = x_n, \\
\dot{x}_n = -p_n(t)x_1 - p_{n-1}(t)x_2 - \dots - p_1(t)x_n.
\end{cases}
\tag{6}
$$

From Liouville formulas (3), (4) for the Wronskian of system (6), we obtain the Liouville formulas for the Wronskian of equation (5):

$$
W(t) = c\exp\left(-\int p_1(t)\,dt\right),
$$

$$
W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} p_1(\tau)\,d\tau\right),
$$

where $t_0 \in (a,b)$.

§7. Linear homogeneous system with constant coefficients.

We consider a linear homogeneous system

$$\dot{x} = Ax, \tag{1}$$

where A is constant square matrix of n*-th* order.

We are looking for a solution of system (1) in the form $x = \gamma e^{\lambda t}$, where γ is constant n-dimensional vector, λ is scalar.

Let's substitute the vector function $\gamma e^{\lambda t}$ in (1). We get

$$\lambda\gamma e^{\lambda t} = A\gamma e^{\lambda t}.$$

Hence,

$$(A - \lambda E)\gamma = 0. \tag{2}$$

Linear system (2) has a non-zero solution if and only if

$$\det(A - \lambda E) = 0. \tag{3}$$

Definition. Equation (3) is called the *characteristic equation* of the system (1). Obviously the roots of equation (3) are eigenvalues of the matrix A. They are called *characteristic numbers*.

From equality (2) it follows that the vector function $x = \gamma e^{\lambda t}$ is a solution of the system (1) if and only if λ is eigenvalue of the matrix A, and γ is the eigenvector corresponding to this number.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of the matrix A, and $\gamma_1, \gamma_2, ..., \gamma_n$ be the eigenvectors corresponding to these numbers.

Let us construct a fundamental system of solutions of the system (1). We will prove the linear independence of the constructed system in the next section.

1. If all characteristic numbers $\lambda_1, \lambda_2, ..., \lambda_n$ are real and simple (that is, not multiple), then system (1) has the fundamental system of solutions

$$\gamma_1 e^{\lambda_1 t},\ \gamma_2 e^{\lambda_2 t}, ...,\ \gamma_n e^{\lambda_n t}.$$

2. Let all characteristic numbers $\lambda_1, \lambda_2,..., \lambda_n$ be simple, but among them there are complex ones.

Lemma. If $u(t), v(t)$ are real vector functions, and complex-valued function $w(t)=u(t)+iv(t)$ is the solution of the system (1), then $u(t)$ and $v(t)$ are also solutions of the system (1).

Proof of the lemma. On the one side, $\dot{w}(t)=\dot{u}(t)+i\dot{v}(t)$, and on the other side,

$$\dot{w}(t)=Aw(t)=A\big(u(t)+iv(t)\big)=Au(t)+iAv(t).$$

Hence $\dot{u}(t)=Au(t)$ and $\dot{v}(t)=Av(t)$. Then $u(t)$ and $v(t)$ are solutions of the system (1). The lemma is proved.

Let $\lambda_j=\alpha+i\beta$, $\beta\neq 0$, $j\in\{1,2,...,n\}$ be eigenvalue of the matrix A, and γ_j be corresponding eigenvector. Then $\bar{\lambda}_j=\alpha-i\beta$ is also eigenvalue of the matrix A, and $\bar{\gamma}_j$ is corresponding eigenvector.

Functions $\gamma_j e^{\lambda_j t}$ and $\bar{\gamma}_j e^{\bar{\lambda}_j t}$ are solutions of the system (1), and, according to the lemma, real functions $\mathrm{Re}\left(\gamma_j e^{\lambda_j t}\right)$ and $\mathrm{Im}\left(\gamma_j e^{\lambda_j t}\right)$ are also solutions of the system (1). This pair of functions in the fundamental system of solutions corresponds to a pair of characteristic numbers λ_j and $\bar{\lambda}_j$.

3. Let there be multiple roots among the roots of the characteristic equation (3).

If the real root λ_j, $j\in\{1,2,...,n\}$, has a multiplicity $d\geq 2$, then the solutions of system (1) should be searched in the form

$$x=\gamma_j(t)e^{\lambda_j t}, \tag{4}$$

where $\gamma_j(t)$ is vector polynomial of degree $(d-1)$. There is exactly d linearly independent solutions of this type.

If $\lambda_j = \alpha + i\beta$, where $\beta \neq 0$ is characteristic number of multiplicity d, then $\overline{\lambda}_j = \alpha - i\beta$ is also characteristic number of multiplicity d. Separating the real and imaginary parts in solutions of type (4), we obtain $2d$ linearly independent solutions.

Using the method described above, we can build n various solutions that correspond to n roots of the equation (5). In the next section we will prove that such a system is indeed fundamental. Also we will clarify the form of solutions (4) in the case of multiple eigenvalues of the matrix A.

§8. Matrix method of integrating of a linear homogeneous system with constant coefficients.

First, let us recall some information from matrix theory.

Sequence of matrices $A_k = \left\{ a_{jm}^{[k]} \right\}_{[n\times n]}$ converges to the matrix $A = \left\{ a_{jm} \right\}_{[n\times n]}$ (we just write $A_k \underset{k\to+\infty}{\to} A$), if $\|A_k - A\| \underset{k\to+\infty}{\to} 0$. Notice, that $A_k \underset{k\to+\infty}{\to} A$ if and only if $a_{jm}^{[k]} \underset{k\to+\infty}{\to} a_{jm}$ for all $j = 1,2,...,n$, $m = 1,2,...,n$.

Series of matrices $\sum\limits_{k=1}^{+\infty} A_k$ converges to the matrix A if sequence of its partial sums converges to A. The notation is $\sum\limits_{k=1}^{+\infty} A_k = A$

It's obvious that $\sum\limits_{k=1}^{+\infty} A_k = A$ if and only if $\sum\limits_{k=1}^{s} a_{jm}^{[k]} \underset{s\to+\infty}{\to} a_{jm}$ for all $j = 1,2,...,n$, $m = 1,2,...,n$.

If number series $\sum\limits_{k=1}^{+\infty} b_k$ converges, and $\|A_k\| \le b_k$ for all $k \in N$, then the series $\sum\limits_{k=1}^{+\infty} A_k$ also converges.

Let $A = \left\{ a_{jm} \right\}_{[n\times n]}$ be constant matrix. Consider the series

$$\sum_{k=0}^{+\infty} \frac{1}{k!} A^k, \tag{1}$$

where $A^0 = E_{[n\times n]}$.

$\left\| A^k \right\| \le \left\| A \right\|^k$, and number series $\sum_{k=0}^{+\infty} \frac{1}{k!} \left\| A \right\|^k$ converges ($\sum_{k=0}^{+\infty} \frac{1}{k!} \left\| A \right\|^k = e^{\left\| A \right\|}$), so series (1) also converges.

Definition. Sum of series $\sum_{k=0}^{+\infty} \frac{1}{k!} A^k$ called the matrix exponent of A. The notation is $\sum_{k=0}^{+\infty} \frac{1}{k!} A^k = e^A$.

Let $A_{[n\times n]}$ and $B_{[n\times n]}$ be arbitrary square matrices. Then, generally speaking, e^{A+B} not equal $e^A e^B$. But if A and B commute, then the following statement is true.

Lemma 1. Let $A_{[n\times n]}$ and $B_{[n\times n]}$ be square matrices. If $AB = BA$, then

$$e^{A+B} = e^A e^B.$$

Proof of the Lemma 1. Let us compare the series

$$e^{A+B} = \sum_{k=0}^{+\infty} \frac{1}{k!} \left(A + B \right)^k$$

and

$$e^A e^B = \left(\sum_{k=0}^{+\infty} \frac{1}{k!} A^k \right) \left(\sum_{m=0}^{+\infty} \frac{1}{m!} B^k \right) = \sum_{k=0}^{+\infty} \left(\sum_{s=0}^{k} \frac{1}{s!\left(k-s\right)!} A^{k-s} B^s \right).$$

Obviously, the first two terms of these series (for $k = 0$ and $k = 1$) match up.

For $k = 2$ the first series term is equal to $\frac{1}{2}\left(A + B \right)^2 = \frac{1}{2}\left(A^2 + AB + BA + B^2 \right)$ and the term of the second series is equal to $\frac{1}{2}\left(A^2 + 2AB + B^2 \right)$. If $AB = BA$, then these terms are equal to each other.

The further proof is no different from the proof of the identity $e^{a+b} = e^a e^b$ in the scalar case:

$$\sum_{s=0}^{k}\frac{1}{s!(k-s)!}A^{k-s}B^{s}=\frac{1}{k!}(A+B)^{k}$$

if $AB=BA$. The lemma is proved.

Lemma 2. Let $A_{[n\times n]}$ and $B_{[n\times n]}$ be square matrices, and there is a matrix $S_{[n\times n]}$ such that $\det S\neq 0$, and $A=SBS^{-1}$. Then

$$e^{A}=S\,e^{B}S^{-1}.$$

Proof of the Lemma 2. Let's estimate the norm $\left\|\sum_{k=0}^{m}\frac{1}{k!}A^{k}-S\,e^{B}S^{-1}\right\|$.

$$A^{k}=\left(SBS^{-1}\right)^{k}=SBS^{-1}SBS^{-1}...SBS^{-1}=SB^{k}S^{-1},$$

and

$$\left\|\sum_{k=0}^{m}\frac{1}{k!}A^{k}-Se^{B}S^{-1}\right\|=\left\|\sum_{k=0}^{m}\frac{1}{k!}\left(SBS^{-1}\right)^{k}-Se^{B}S\right\|=\left\|\sum_{k=0}^{m}\frac{1}{k!}SB^{k}S^{-1}-Se^{B}S^{-1}\right\|=$$
$$=\left\|S\left(\sum_{k=0}^{m}\frac{1}{k!}B^{k}-e^{B}\right)S^{-1}\right\|\leq\|S\|\cdot\left\|\sum_{k=0}^{m}\frac{1}{k!}B^{k}-e^{B}\right\|\cdot\left\|S^{-1}\right\|\underset{m\to+\infty}{\to}0.$$

Hence, $e^{A}=S\,e^{B}S^{-1}$. The lemma is proven.

Let's consider a linear homogeneous system

$$\dot{x}=Ax, \qquad (2)$$

where A is constant square matrix of n-*th* order.

Theorem. e^{At} is fundamental matrix of (2).

Proof of the theorem. Let $A^{k}=\left\{a_{jm}^{[k]}\right\}_{[n\times n]}$. Then

$$e^{At} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k = \left\{ \sum_{k=0}^{+\infty} \frac{t^k}{k!} a_{jm}^{[k]} \right\}_{[n\times n]} .$$

The elements of this matrix are convergent for any t series. By definition

$$\frac{d}{dt} e^{At} = \left\{ \frac{d}{dt} \sum_{k=0}^{+\infty} \frac{t^k}{k!} a_{jm}^{[k]} \right\}_{[n\times n]} = \left\{ \sum_{k=1}^{+\infty} \frac{t^{k-1}}{(k-1)!} a_{jm}^{[k]} \right\}_{[n\times n]} = \sum_{k=1}^{+\infty} \frac{t^{k-1}}{(k-1)!} A^k ,$$

$$\sum_{k=1}^{m} \frac{t^{k-1}}{(k-1)!} A^k = A \sum_{k=1}^{m} \frac{t^{k-1}}{(k-1)!} A^{k-1} \underset{m\to+\infty}{\to} Ae^{At} .$$

Hence e^{At} satisfies the matrix equation

$$\dot{X} = AX .$$

Then the columns of the matrix e^{At} are solutions of the system (2). $W(t) = \det e^{At}$ and $W(0) = \det e^{A\cdot 0} = \det E = 1$. Hence, e^{At} is the fundamental matrix of (2). The theorem is proved.

Let B be Jordan form of the matrix A, and S be reducing matrix, $\det S \neq 0$, and $A = SBS^{-1}$. Then $e^{At} = Se^{Bt}S^{-1}$ by Lemma 2.

Let's find the type of the matrix e^{Bt}.

B is block-diagonal matrix with blocks $B_1, B_2, ..., B_d$:

$$B = diag(B_1, B_2, ..., B_d). \tag{3}$$

Let v_s be the size of the block B_s, $s = 1,2,...,d$. Then

$$B_s = \begin{pmatrix} \lambda_p & 0 & ... & 0 & 0 \\ 1 & \lambda_p & 0 & ... & 0 \\ ... & 1 & ... & ... & ... \\ ... & ... & ... & \lambda_p & 0 \\ 0 & 0 & ... & 1 & \lambda_p \end{pmatrix}_{[v_s \times v_s]} .$$

Block B_s can be represented in the form

$$B_s = \lambda_p E_s + H_s, \tag{4}$$

where λ_p is eigenvalue of the matrix A, E_s is identity matrix of dimension v_s, and $H_s = \{h_{jm}\}$ is square matrix of dimension v_s, which has ones under the main diagonal and all other elements are equal to zero. In other words, $h_{jm} = 1$ if $m = j-1$, $j = 2,3,...,v_s$, and $h_{jm} = 0$, if $m \neq j-1$.

From the equality (4) it follows, that

$$e^{B_s t} = e^{(\lambda_p E_s + H_s)t},$$

and according to Lemma 1, since the identity matrix commutes with any matrix, then

$$e^{B_s t} = e^{\lambda_p E_s t} e^{H_s t}. \tag{5}$$

From the definition of matrix exponent, we obtain

$$e^{\lambda_p E_s t} = \sum_{k=0}^{+\infty} \frac{(\lambda_p t)^k}{k!} E_s^k = \sum_{k=0}^{+\infty} \frac{(\lambda_p t)^k}{k!} E_s = E_s \sum_{k=0}^{+\infty} \frac{(\lambda_p t)^k}{k!} = E_s e^{\lambda_p t}. \tag{6}$$

Let's consider $H_s^2 = \{h_{jm}^{[2]}\}$. Obviously, $h_{jm}^{[2]} = 1$, if $m = j-2$, $j = 3,4,...,v_s$, and $h_{jm}^{[2]} = 0$, if $m \neq j-2$.

Similarly, if $k \leq v_s - 1$, then $H_s^k = \{h_{jm}^{[k]}\}$, where $h_{jm}^{[k]} = 1$, if $m = j-k$, $j = (k+1),...,v_s$, and $h_{jm}^{[k]} = 0$, if $m \neq j-k$.

Also, $H_s^k = 0$ if $k \geq v_s$, where $0 = 0_{[v_s \times v_s]}$ is zero matrix.

Therefore,

$$e^{H_s t} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} H_s^k = E_s + tH_s + \frac{t^2}{2} H_s^2 + ... + \frac{t^{v_s-1}}{(v_s-1)!} H_s^{v_s-1}, \tag{7}$$

and then

$$e^{H_s t} = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ t & 1 & 0 & \dots & 0 \\ \frac{t^2}{2} & t & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & 0 \\ \frac{t^{v_s-1}}{(v_s-1)!} & \dots & \dots & t & 1 \end{pmatrix}_{[v_s \times v_s]}$$

From equalities (5)-(7) we obtain:

$$e^{B_s t} = e^{\lambda_p t} \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ t & 1 & 0 & \dots & 0 \\ \frac{t^2}{2} & t & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & 0 \\ \frac{t^{v_s-1}}{(v_s-1)!} & \dots & \dots & t & 1 \end{pmatrix}_{[v_s \times v_s]}$$

or

$$e^{B_s t} = \begin{pmatrix} e^{\lambda_p t} & 0 & \dots & 0 & 0 \\ te^{\lambda_p t} & e^{\lambda_p t} & 0 & \dots & 0 \\ \frac{t^2}{2}e^{\lambda_p t} & te^{\lambda_p t} & \dots & \dots & \dots \\ \dots & \dots & \dots & e^{\lambda_p t} & 0 \\ \frac{t^{v_s-1}}{(v_s-1)!}e^{\lambda_p t} & \dots & \dots & te^{\lambda_p t} & e^{\lambda_p t} \end{pmatrix}_{[v_s \times v_s]} . \quad (8)$$

From the definition of the matrix exponent and formula (3) it follows that e^{Bt} is block-diagonal matrix:

$$e^{Bt} = diag\left(e^{B_1 t}, e^{B_2 t}, ..., e^{B_d t}\right), \tag{9}$$

and each block of this matrix has the form (8).

Multiplying equality $e^{At} = Se^{Bt}S^{-1}$ by S from the right, we get:

$$e^{At}S = Se^{Bt}.$$

S is constant matrix, $\det S \neq 0$, and according to Theorem 3 of the fourth paragraph, $e^{At}S$ is fundamental matrix of system (2). Hence, Se^{Bt} is also fundamental matrix of (2). From (8) and (9) it follows that the columns of this matrix has the form

$$x = \gamma_p(t)e^{\lambda_p t},$$

where $\gamma_p(t)$ is vector polynomial of degree no higher than $(d-1)$, where d is multiplicity of eigenvalue λ_p.

This justifies method of constructing a fundamental system of solutions of the system (2) (from the 7 paragraph) and proves the linear independence of the solutions that form this fundamental system.

Comment. To find a particular solution $x = \psi(t)$ of a linear non-homogeneous system

$$\dot{x} = Ax + q(t), \tag{10}$$

where $A_{[n\times n]}$ is a constant matrix, and $q(t)$ is continuous vector function of a special form, there is a method of undetermined coefficients, similar to the corresponding method for linear non-homogeneous equations.

We present the theorems of this method without proof (the proof here is similar to the proof of the corresponding theorems for linear equations).

The linear homogeneous system corresponding to (10) has the form (2). Consider the characteristic equation of the system (2)

$$\det(A - \lambda E) = 0. \tag{11}$$

1. Let the non-homogeneity of system (10) has the form

$$q(t)=R_m(t)e^{\lambda_0 t}, \tag{12}$$

where $R_m(t)$ is vector polynomial of degree m.

Theorem 1. If λ_0 is not the root of the characteristic equation (11), then system (10) with nonlinearity (12) has a solution of the form

$$\psi(t)=Q_m(t)e^{\lambda_0 t},$$

where $Q_m(t)$ is vector polynomial of degree m.

Theorem 2. If λ_0 is the root of the characteristic equation (11) of the multiplicity $d\geq 1$, then the system (10) with nonlinearity (12) has a solution of the form

$$\psi(t)=Q_{m+d}(t)e^{\lambda_0 t},$$

where $Q_{m+d}(t)$ is vector polynomial of degree $(m+d)$.

2. Let the non-homogeneity of equation (1) has the form

$$q(t)=e^{\alpha_0 t}\left(\tilde{R}_{m_1}(t)\cos(\beta_0 t)+\hat{R}_{m_2}(t)\sin(\beta_0 t)\right), \tag{13}$$

where $\tilde{R}_{m_1}(t)$ and $\hat{R}_{m_2}(t)$ are vector polynomials of degree m_1 and m_2 respectively.

Theorem 3. If $\lambda_0=\alpha_0+i\beta_0$ is not the root of the characteristic equation (11), then equation (10) with nonlinearity (13) has a solution of the form

$$\psi(t)=e^{\alpha_0 t}\left(\tilde{Q}_m(t)\cos(\beta_0 t)+\hat{Q}_m(t)\sin(\beta_0 t)\right),$$

where $m=\max(m_1,m_2)$, $\tilde{Q}_m(t)$ and $\hat{Q}_m(t)$ are vector polynomials of degree m.

Theorem 4. If $\lambda_0 = \alpha_0 + i\beta_0$ is the root of the characteristic equation (11) of the multiplicity $d \geq 1$, then equation (10) with nonlinearity (13) has a solution of the form

$$\psi(t) = e^{\alpha_0 t}\left(\tilde{Q}_{m+d}(t)\cos(\beta_0 t) + \hat{Q}_{m+d}(t)\sin(\beta_0 t)\right),$$

where $m = \max(m_1, m_2)$, $\tilde{Q}_{m+d}(t)$ and $\hat{Q}_{m+d}(t)$ are vector polynomials of degree $(m+d)$.

Chapter 5. DIFFERENTIAL PROPERTIES OF SOLUTIONS AS FUNCTIONS OF INITIAL CONDITION AND PARAMETERS.

In this chapter we consider a system of equations, the right-hand side of which depends on the parameter

$$\dot{x} = X(t, x, \mu), \tag{1}$$

where $x \in R^n$, vector function $X(t,x,\mu)$ is continuous for $(t,x) \in G \subset R^{n+1}$, and $\mu \in F \subset R^m$.

Let us denote by $x = \varphi(t,\theta,\xi,\mu)$ the solution of the system (1) with initial condition $t = \theta$, $x = \xi$. We will study the differential properties of this solution on all arguments.

§ 1. Theorem on integral continuity.

Lemma. Let's consider two systems

$$\dot{x} = X(t,x), \tag{1}$$

$$\dot{y} = Y(t,y), \tag{2}$$

where $x \in R^n$, $y \in R^n$, functions $X(t,x)$ and $Y(t,y)$ are continuous on $D \subset R^{n+1}$, and $X(t,x)$ satisfies the Lipschitz condition with respect to x on the set D:

$$\left\| X(t,\bar{x}) - X(t,\bar{\bar{x}}) \right\| \le L \left\| \bar{x} - \bar{\bar{x}} \right\| \tag{3}$$

for any two points $(t,\bar{x})$, $(t,\bar{\bar{x}})$ from the set D.

Let $x = \varphi(t)$ be solution of system (1), and $y = \psi(t)$ be solution of the system (2), defined on the segment $[c,d]$, $\theta \in [c,d]$.

Then for any $t \in [c,d]$

$$\left\| \varphi(t) - \psi(t) \right\| \le \left(\left\| \varphi(\theta) - \psi(\theta) \right\| + \int_c^d \left\| X(t,\psi(t)) - Y(t,\psi(t)) \right\| d\tau \right) e^{L|t-\theta|}. \tag{4}$$

Proof of the lemma. Functions $x=\varphi(t)$ and $y=\psi(t)$ satisfy the integral equations:

$$\varphi(t)=\varphi(\theta)+\int_{\theta}^{t}X(\tau,\varphi(\tau))d\tau,$$

$$\psi(t)=\psi(\theta)+\int_{\theta}^{t}X(\tau,\psi(\tau))d\tau.$$

We subtract one from another and move to the norms:

$$\|\varphi(t)-\psi(t)\|\le\|\varphi(\theta)-\psi(\theta)\|+\left|\int_{\theta}^{t}\|X(t,\varphi(t))-Y(t,\psi(t))\|d\tau\right|. \quad (5)$$

Let us evaluate the integral in formula (5).

$$\begin{aligned}&\left|\int_{\theta}^{t}\|X(t,\varphi(t))-Y(t,\psi(t))\|d\tau\right|=\\ &=\left|\int_{\theta}^{t}\|X(t,\varphi(t))-X(t,\psi(t))+X(t,\psi(t))-Y(t,\psi(t))\|d\tau\right|\le \\ &\le\left|\int_{\theta}^{t}\|X(t,\varphi(t))-X(t,\psi(t))\|d\tau\right|+\left|\int_{\theta}^{t}\|X(t,\psi(t))-Y(t,\psi(t))\|d\tau\right|.\end{aligned} \quad (6)$$

Hence, according to Lipschitz condition,

$$\left|\int_{\theta}^{t}\|X(t,\varphi(t))-Y(t,\psi(t))\|d\tau\right|\le$$

$$\le L\left|\int_{\theta}^{t}\|\varphi(t)-\psi(t)\|d\tau\right|+\int_{c}^{d}\|X(t,\psi(t))-Y(t,\psi(t))\|d\tau.$$

And from inequality (5) it follows that

$$\|\varphi(t)-\psi(t)\|\le C+L\left|\int_{\theta}^{t}\|\varphi(t)-\psi(t)\|d\tau\right|, \quad (7)$$

where C such constant, that

$$C = \left\|\varphi(\theta) - \psi(\theta)\right\| + \int_c^d \left\|X(t,\psi(t)) - Y(t,\psi(t))\right\| d\tau .$$

From inequality (7), according to Gronwall's lemma, we obtain:

$$\left\|\varphi(t) - \psi(t)\right\| \le Ce^{L|t-\theta|} .$$

The last inequality is inequality (4). The lemma is proved.

Next we consider the system

$$\dot{x} = X(t,x,\mu), \tag{8}$$

where $x \in R^n$, vector function $X(t,x,\mu)$ is continuous for $(t,x) \in G$, $\mu \in F$, G is domain in R^{n+1}, and $F \subset R^m$.

We assume that $X(t,x,\mu)$ satisfies the local Lipschitz condition with respect to x for any $\mu \in F$.

Theorem 1. Let the solution $x = \varphi(t,\theta_0,\xi_0,\mu_0)$ of the system (8) be defined for $t \in [a,b]$, and $\theta_0 \in [a,b]$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if

$$|\theta - \theta_0| < \delta ,\theta \in [a,b], \|\xi - \xi_0\| < \delta , \|\mu - \mu_0\| < \delta , \tag{9}$$

then on the segment $[a,b]$ there exists the solution $x = \psi(t,\theta,\xi,\mu)$ of the system (8), and

$$\left\|\varphi(t,\theta_0,\xi_0,\mu_0) - \psi(t,\theta,\xi,\mu)\right\| < \varepsilon \tag{10}$$

for any $t \in [a,b]$.

Proof of the Theorem 1. Let's denote

$$\varphi(t) = \varphi(t,\theta_0,\xi_0,\mu_0),\ \psi(t) = \psi(t,\theta,\xi,\mu).$$

The set $\Gamma = \{(t,x) : t \in [a,b], x = \varphi(t)\}$ is closed, bounded, and $\Gamma \subset G$. Let's choose ε small enough that the set

$$D = \{(t,x) : t \in [a,b], \|x - \varphi(t)\| \le \varepsilon\}$$

contained in the domain G.

D is a closed, bounded set, and by Weierstrass's theorem there exists $M > 0$ such that for any $(t,x) \in D$

$$\|X(t,x,\mu_0)\| \le M .$$

Since $X(t,x,\mu_0)$ satisfies the Lipschitz condition with respect to x on the set D, then there is a constant $L > 0$ such that

$$\|X(t,\bar{x},\mu_0) - X(t,\bar{\bar{x}},\mu_0)\| \le L\|\bar{x} - \bar{\bar{x}}\|$$

for any points $(t,\bar{x})$, $(t,\bar{\bar{x}})$ from the set D.

Let's choose $\Delta > 0$, such that

$$(1 + M + b - a)e^{L(b-a)}\Delta < \varepsilon . \tag{11}$$

According to Cantor's theorem for such Δ there exists $\delta_1 > 0$ such that

$$\|X(t,x,\mu_0) - X(t,x,\mu)\| < \Delta, \tag{12}$$

for $(t,x) \in D$ and $\|\mu - \mu_0\| < \delta_1$.

Let's set $\delta = \min(\Delta, \delta_1)$. We'll show that this δ is what we are looking for. We will prove it by contradiction. There are two possible cases:

a) there exist θ, ξ, μ, satisfying condition (9), such that the solution $\psi(t) = \psi(t,\theta,\xi,\mu)$ defined not for any $t \in [a,b]$,

b) for any θ, ξ, μ, satisfying condition (9), solutions $\psi(t)=\psi(t,\theta,\xi,\mu)$ of the system (8) are defined on the interval $[a,b]$, but there are θ, ξ, μ, satisfying (9), and there exists $t^* \in [a,b]$, such that inequality (10) does not holds for $t=t^*$:

$$\left\|\varphi(t^*)-\psi(t^*)\right\| \geq \varepsilon . \tag{13}$$

In the case a) there is a solution $x=\psi(t)$ of system (8), defined on the interval (α,β), and the segment $[a,b]$ does not contained in (α,β). Hence, either $\alpha \geq a$, or $\beta \leq b$. While approaching the endpoint of the maximum interval of existence, the solution $x=\psi(t)$ leaves the compact set D (§6, Chapter 2), therefore, in this case, as in case b), there is $t^* \in [a,b]$, satisfying inequality (13).

Let's estimate the norm of the difference $\varphi(\theta)-\psi(\theta)$.

Solution $x=\varphi(t)$ satisfies the integral equation

$$\varphi(\theta)=\varphi(\theta_0)+\int_{\theta_0}^{\theta} X(\tau,\varphi(\tau),\mu_0)\,d\tau .$$

Since $(t,\varphi(t)) \in D$ for any $t \in [a,b]$, then $\left\|X(t,\varphi(t),\mu_0)\right\| \leq M$. Hence,

$$\left\|\varphi(\theta)-\varphi(\theta_0)\right\| \leq \left|\int_{\theta_0}^{\theta} \left\|X(\tau,\varphi(\tau),\mu_0)\right\| d\tau\right| \leq \left|\int_{\theta_0}^{\theta} M d\tau\right| = M\left|\theta-\theta_0\right|. \tag{14}$$

Also

$$\left\|\varphi(\theta)-\psi(\theta)\right\| \leq \left\|\varphi(\theta)-\varphi(\theta_0)\right\| + \left\|\varphi(\theta_0)-\psi(\theta)\right\|,$$

$\varphi(\theta_0)=\xi_0$, $\psi(\theta)=\xi$, and from the last inequality and (14) it follows that

$$\left\|\varphi(\theta)-\psi(\theta)\right\| \leq M\left|\theta-\theta_0\right| + \left\|\xi-\xi_0\right\|. \tag{15}$$

Since $\delta \leq \Delta$ then from inequalities (9), (11) and (15) it follows that

$$\|\varphi(\theta)-\psi(\theta)\| \le M\delta+\delta \le (M+1)\Delta < \varepsilon. \tag{16}$$

From (13) and (16) it follows that $t^* \ne \theta$. Without loss of generality we will assume that $t^* > \theta$. Then there is $T \in (\theta, t^*]$ such that $\|\varphi(t)-\psi(t)\| < \varepsilon$ for $t \in (\theta, T)$, and

$$\|\varphi(T)-\psi(T)\| = \varepsilon. \tag{17}$$

Let us apply the lemma proved above. We will assume that (in the notation of the lemma)

$$[c,d]=[\theta,T],\ X(t,x)=X(t,x,\mu_0) \text{ and } Y(t,y)=X(t,x,\mu).$$

Then inequality (4) for $t=T$ takes the form

$$\begin{aligned}&\|\varphi(T)-\psi(T)\| \le \\ &\le \left(\|\varphi(\theta)-\psi(\theta)\| + \int_\theta^T \left\|X(t,\psi(t),\mu_0)-X(t,\psi(t),\mu)\right\| d\tau \right) e^{L(T-\theta)}.\end{aligned} \tag{18}$$

Since $\delta \le \delta_1$, $(t,\psi(t)) \in D$ for $t \in [\theta,T]$, and from inequalities (9) and (12) it follows that for any $t \in [\theta,T]$

$$\left\|X(t,\psi(t),\mu_0)-X(t,\psi(t),\mu)\right\| < \Delta,$$

therefore, from inequalities (15) and (18) we obtain:

$$\|\varphi(T)-\psi(T)\| \le \left(\|\varphi(\theta)-\psi(\theta)\| + \int_\theta^T \Delta\, d\tau \right) e^{L(T-\theta)},$$

and

$$\|\varphi(T)-\psi(T)\| \le \left(M|\theta-\theta_0| + \|\xi-\xi_0\| + \Delta|T-\theta| \right) e^{L(T-\theta)}.$$

According to conditions (9), (11) and (16), we obtain

$$\|\varphi(T)-\psi(T)\| \le (1+M+b-a)e^{L(b-a)}\Delta < \varepsilon .$$

The last inequality contradicts (17). The resulting contradiction proves the theorem.

Consider a system without a parameter

$$\dot{x} = X(t,x), \tag{19}$$

where $x \in R^n$, vector function $X(t,x)$ is continuous and satisfies the local Lipschitz condition with respect to x in the domain $G \subset R^{n+1}$.

Theorem 2. Let the solution $x = \varphi(t,\theta_0,\xi_0)$ of the system (19) be defined for $t \in [a,b]$, where $\theta_0 \in [a,b]$.

Then there are constants $\delta > 0$, $M > 0$ and $L > 0$, such that if

$$|\theta-\theta_0| < \delta,\ \theta \in [a,b],\ \|\xi-\xi_0\| < \delta, \tag{20}$$

then on the segment $[a,b]$ there defined the solution $x = \psi(t,\theta,\xi)$ of the system (19), and

$$\|\varphi(t,\theta_0,\xi_0)-\psi(t,\theta,\xi)\| \le \left(M|\theta-\theta_0| + \|\xi-\xi_0\|\right)e^{L|t-\theta|} \tag{21}$$

for any $t \in [a,b]$.

Proof of the Theorem 2. The proof of Theorem 2 overall repeats the proof of the previous theorem. Let us introduce the following notation:

$$\varphi(t) = \varphi(t,\theta_0,\xi_0), \quad \psi(t) = \psi(t,\theta,\xi).$$

The set $\Gamma = \{(t,x) : t \in [a,b], x = \varphi(t)\}$ is closed, bounded, and $\Gamma \subset G$.

Therefore there exists $\varepsilon > 0$, such that the set $D = \{(t,x) : t \in [a,b], \|x-\varphi(t)\| \le \varepsilon\}$ also contained in the domain G.

Since D is closed, bounded set, then, according to Weierstrass's theorem, there exists $M > 0$, such that for any $(t,x) \in D$

$$\|X(t,x)\| \le M .$$

Since $X(t,x)$ satisfies the Lipschitz condition with respect to x on the set D, then there exists $L>0$, such that inequality (3) is satisfied for any points $(t,\overline{x})$, $(t,\overline{\overline{x}})$ from the set D.

According to the proof of the Theorem 1, there is $\delta>0$ such that, if conditions (20) are satisfied, then the solution $x=\psi(t,\theta,\xi)$ of the system (19) is defined on the interval $[a,b]$, and

$$\|\varphi(t,\theta_0,\xi_0)-\psi(t,\theta,\xi)\|<\varepsilon$$

for any $t\in[a,b]$.

Let's apply the lemma proved above, assuming $[c,d]=[a,b]$ and $X(t,x)=Y(t,y)$. Inequality (4) in this case has the form

$$\|\varphi(t)-\psi(t)\| \le \left(\|\varphi(\theta)-\psi(\theta)\|\right)e^{L|t-\theta|}$$

for any $t\in[a,b]$.

For $\|\varphi(\theta)-\psi(\theta)\|$ estimation (15) is satisfied, therefore,

$$\|\varphi(t)-\psi(t)\| \le \left(M|\theta-\theta_0|+\|\xi-\xi_0\|\right)e^{L|t-\theta|}.$$

The theorem is proved.

Corollary. Let the solution $x=\varphi(t,\theta_0,\xi_0)$ of the system (19) be defined for $t\in[a,b]$, $\theta_0\in[a,b]$.

Then there exist constants $\delta>0$ and $K>0$ such that if conditions (20) are hold then on the segment $[a,b]$ there exists the solution $x=\psi(t,\theta,\xi)$ of the system (19), and

$$\|\varphi(t,\theta_0,\xi_0)-\psi(t,\theta,\xi)\| \le K\left(|\theta-\theta_0|+\|\xi-\xi_0\|\right) \tag{22}$$

for any $t\in[a,b]$.

Proof of the corollary. From the Theorem 2 it follows that if conditions (20) are hold, then the solution $x=\psi(t,\theta,\xi)$ of the system (19) is defined on $[a,b]$, and satisfies inequality (21).

Notice, that $|t-\theta|\le b-a$ for any $t\in[a,b]$, $\theta\in[a,b]$.

Let $K=\max(1,M)e^{L(b-a)}$. With this K from (21) the estimation (22) follows. The corollary is proved.

Consider the system

$$\dot{x}=X(t,x), \tag{23}$$

where $x\in R^n$, and vector function $X(t,x)$ is continuously differentiable with respect to x in the domain $G\subset R^{n+1}$.

Theorem 3. Let the solution $x=\varphi(t,\theta_0,\xi_0)$ of the system (23) be defined for $t\in[a,b]$, where $\theta_0\in[a,b]$.

Then there exists $\delta>0$ such that if

$$|\theta-\theta_0|<\delta,\ \theta\in(a,b),\ \|\xi-\xi_0\|<\delta,$$

then the solution $x=\psi(t,\theta,\xi)$ of the system (23) is defined on the segment $[a,b]$ and is continuously differentiable with respect to θ and ξ.

Chapter 6. LYAPUNOV STABILITY.

§ 1. Definition of Lyapunov stability.

Consider the system

$$\dot{y} = Y(t, y), \tag{1}$$

where $y \in R^n$, $Y(t,y)$ is continuous and satisfies the Lipschitz condition with respect to y locally in the domain $G \subset R^{n+1}$.

Let $y = \varphi(t)$ be solution of the system (1), defined for $t \in [t_0, +\infty)$.

Definition 1. Solution $y = \varphi(t)$ is called *Lyapunov stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $y = \psi(t)$ of the system (1) with initial condition satisfying the inequality

$$\|\psi(t_0) - \varphi(t_0)\| < \delta,$$

defined for $t \in [t_0, +\infty)$, and satisfies the inequality

$$\|\psi(t) - \varphi(t)\| < \varepsilon$$

for any $t \in [t_0, +\infty)$.

Otherwise the solution $y = \varphi(t)$ is called *Lyapunov unstable*.

Definition 2. Solution $y = \varphi(t)$ of the system (1) is called *asymptotically Lyapunov stable* if it's Lyapunov stable and there exists $\Delta > 0$ such that for any solution $y = \psi(t)$ of the system (1) with initial condition satisfying the inequality

$$\|\psi(t_0) - \varphi(t_0)\| < \Delta,$$

condition

$$\|\psi(t) - \varphi(t)\| \underset{t \to +\infty}{\to} 0$$

holds.

Remark. On the finite segment $[a,b]$ ε-closeness of solutions with δ-close initial condition is guaranteed by the theorem on integral continuity. But for stability it's required ε-closeness of solutions with δ-close initial data on the half-line $[t_0,+\infty)$.

Let's make the substitution in the system (1)

$$y=\varphi(t)+x.$$

Then

$$\dot{\varphi}(t)+\dot{x}=Y(t,\varphi(t)+x)$$

and

$$\dot{\varphi}(t)=Y(t,\varphi(t)).$$

Hence, the system (1) is reduced to the system

$$\dot{x}=X(t,x), \tag{2}$$

where

$$X(t,x)=Y(t,\varphi(t)+x)-Y(t,\varphi(t)).$$

Solution $y=\varphi(t)$ of the system (1) corresponds to the solution $x\equiv 0$ of the system (2), and studying of the stability of the solution $y=\varphi(t)$ of the system (1) is reduced to studying of the stability of the solution $x\equiv 0$ of the system (2).

§ 2. Stability of solutions of a linear system.

We consider a linear non-homogeneous system

$$\dot{y}=P(t)y+q(t), \tag{1}$$

where the matrix $P(t)$ and vector $q(t)$ are continuous for $t\in[t_0,+\infty)$.

Consider also homogeneous system

$$\dot{y} = P(t)y. \tag{2}$$

Theorem 1. Type of the stability of any solution $y = \varphi(t)$ of the system (1) coincides with the type of the stability of the solution $y \equiv 0$ of the system (2).

Proof of the theorem. Let's make the substitution $y = \varphi(t) + x$, suggested at the end of the first paragraph. In our case

$$Y(t, y) = P(t)y + q(t),$$

$$X(t, x) = Y(t, \varphi(t) + x) - Y(t, \varphi(t)) =$$
$$= P(t)(\varphi(t) + x) + q(t) - (P(t)\varphi(t) + q(t)) = P(t)x,$$

and studying of the stability of the solution $y = \varphi(t)$ of the system (1) is reduced to the studying of the stability of the solution $y \equiv 0$ of the system (2). The theorem is proved.

Thus, all solutions of the system (1) have the same type of the stability.

Definition. Linear system (1) (and linear system (2)) is called stable, asymptotically stable or unstable if the solution $y \equiv 0$ of homogeneous system is stable, asymptotically stable or unstable (2).

Theorem 2 (stability criterion for a linear homogeneous system).

The following conditions are equivalent.

1. Solution $y \equiv 0$ of the system (2) is Lyapunov stable.
2. Any solution of the system (2) is bounded on $+\infty$ (on each $[t_0, +\infty)$).
3. Any fundamental matrix of system (2) is bounded on $+\infty$.
4. There exists a bounded on $+\infty$ fundamental matrix of system (2).

(Everywhere below by «bounded» we mean « bounded on $+\infty$ »).

Proof of Theorem 2. Obviously, the third condition follows from the second condition, since the columns of each fundamental matrix of the system (2) are solutions of the system. Obviously, the third condition implies the fourth.

It remains to prove that the first condition implies the second, and the fourth condition implies the first.

Let us prove that the first condition implies the second.

Since the solution $y \equiv 0$ of the system (2) is Lyapunov stable, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $y = \psi(t)$ of the system (2) with initial conditions satisfying the inequality $\|\psi(t_0)\| < \delta$, satisfies the inequality $\|\psi(t)\| < \varepsilon$ for any $t \in [t_0, +\infty)$.

Let's consider $\varepsilon = 1$. For this ε there exists $\delta > 0$ such that any solution $y = \psi(t)$ of the system (2) with initial conditions satisfying the inequality $\|\psi(t_0)\| < \delta$, satisfies the inequality $\|\psi(t)\| < 1$ for all $t \in [t_0, +\infty)$. Hence, $y = \psi(t)$ is bounded.

Let's now consider the solution $y = \psi(t)$ with initial condition $\|\psi(t_0)\| \geq \delta$. We fix the number $\alpha > 0$ such that $\|\alpha\psi(t_0)\| < \delta$.

Function $\xi(t) = \alpha\psi(t)$ is also the solution of the system (2), and $\|\xi(t_0)\| = \|\alpha\psi(t_0)\| < \delta$. Hence, $\|\xi(t)\| < 1$ for any $t \in [t_0, +\infty)$.

Thus, $\|\psi(t)\| < 1/\alpha$ for any $t \in [t_0, +\infty)$. Therefore the solution $y = \psi(t)$ is bounded. So, we proved that the first condition implies the second.

Let us now prove that the fourth condition implies the first.

Let $\Phi(t)$ be bounded fundamental matrix of the system (2). Then there is a value $M > 0$, such that $\|\Phi(t)\| \leq M$ for any $t \in [t_0, +\infty)$.

Let's prove that the solution $y \equiv 0$ of the system (2) is stable.

We fix $\varepsilon > 0$. Solution $y = \psi(t)$ of the system (2) can be represented it in the form

$$\psi(t) = \Phi(t)\Phi^{-1}(t_0)\psi(t_0).$$

If $\|\psi(t_0)\| < \delta$, then

$$\|\psi(t)\| \leq \|\Phi(t)\| \cdot \|\Phi^{-1}(t_0)\| \cdot \|\psi(t_0)\| < MC\delta,$$

where $C = \|\Phi^{-1}(t_0)\| = const$.

Let's set $\delta = \varepsilon/(MC)$. Then from the last inequality it follows that $\|\psi(t)\| < \varepsilon$. The theorem is proved.

Theorem 2 can be reformulated as follows.

Theorem 3 (unstability criterion for a linear homogeneous system).

The following conditions are equivalent.

1. Solution $y \equiv 0$ of the system (2) is unstable.
2. There exists an unbounded on $+\infty$ solution of the system (2).
3. There exists an unbounded on $+\infty$ fundamental matrix of the system (2).
4. Any fundamental matrix of the system (2) is unbounded on $+\infty$.

Theorem 4 (criterion for the asymptotic stability of a linear homogeneous system).

The following conditions are equivalent.

1. Solution $y \equiv 0$ of the system (2) is asymptotically stable.
2. $\|\psi(t)\| \underset{t\to+\infty}{\to} 0$ for any solution $y = \psi(t)$ of the system (2).
3. $\|\Phi(t)\| \underset{t\to+\infty}{\to} 0$ for any fundamental matrix $\Phi(t)$ of the system (2).
4. There is a fundamental matrix $\Phi(t)$ of the system (2), such that $\|\Phi(t)\| \underset{t\to+\infty}{\to} 0$.

Proof of the Theorem 4. Obviously, the second condition implies the third, and the third condition implies the fourth.

It remains to prove that the first condition implies the second, and the fourth condition implies the first.

Let us prove that the first condition implies the second.

Solution $y \equiv 0$ of the system (2) is asymptotically stable. Therefore, this solution is Lyapunov stable, and there exists $\Delta > 0$ such that for any solution $y = \psi(t)$ of the system (2) with initial condition satisfying the inequality $\|\psi(t_0)\| < \Delta$, the condition $\|\psi(t)\| \underset{t\to+\infty}{\to} 0$ holds.

Let's now consider the solution $y = \psi(t)$ with initial condition $\|\psi(t_0)\| \geq \Delta$. We fix the number $\alpha > 0$ such that $\|\alpha\psi(t_0)\| < \Delta$.

Function $\xi(t) = \alpha\psi(t)$ is also the solution of the system (2), and $\|\xi(t_0)\| = \|\alpha\psi(t_0)\| < \Delta$. Hence, $\|\xi(t)\| \underset{t\to+\infty}{\to} 0$.

Then, $\left\|\psi(t)\right\| = \dfrac{1}{\alpha}\left\|\xi(t)\right\| \underset{t\to+\infty}{\to} 0$. So, we proved that the first condition implies the second.

Let us now prove that the fourth condition implies the first.

Let $\Phi(t)$ be the fundamental matrix of the system (2), such that $\left\|\Phi(t)\right\| \underset{t\to+\infty}{\to} 0$. Hence, $\Phi(t)$ is bounded, and, according to Theorem 2, the solution $y \equiv 0$ of the system (2) is stable.

It remains to prove that there exists $\Delta > 0$ such that for any solution $y = \psi(t)$ of the system (2) with initial condition satisfying the inequality $\left\|\psi(t_0)\right\| < \Delta$, the condition $\left\|\psi(t)\right\| \underset{t\to+\infty}{\to} 0$ holds.

Solution $y = \psi(t)$ of the system (2) can be represented it in the form

$$\psi(t) = \Phi(t)\Phi^{-1}(t_0)\psi(t_0).$$

If $\left\|\psi(t_0)\right\| < \Delta$, then

$$\left\|\psi(t)\right\| \le \left\|\Phi(t)\right\| \cdot \left\|\Phi^{-1}(t_0)\right\| \cdot \left\|\psi(t_0)\right\| < \left\|\Phi(t)\right\| C\Delta,$$

where $C = \left\|\Phi^{-1}(t_0)\right\| = const$.

And from the condition $\left\|\Phi(t)\right\| \underset{t\to+\infty}{\to} 0$ it follows that $\left\|\psi(t)\right\| \underset{t\to+\infty}{\to} 0$. The theorem is proved.

§ 3. Stability of solutions of a linear homogeneous system with constant coefficients.

Consider a linear homogeneous system

$$\dot{y} = Ay, \tag{1}$$

where A is constant square matrix of dimension n, $t \in R$, $y \in R^n$.

We study the stability of the solution $y \equiv 0$, $t \in [0, +\infty)$.

$\Phi(t) = e^{At}$ is fundamental matrix of the system (1).

First, we prove some properties of the matrix e^{At}, $t \geq 0$.

Let J be Jordan canonical form of the matrix A, and S be such a matrix, that $\det S \neq 0$, $A = SJS^{-1}$.

Lemma 1.

1. Matrix e^{At} is bounded for $t \in [0,+\infty)$ if and only if the matrix e^{Jt} is bounded for $t \in [0,+\infty)$.
2. $\left\| e^{At} \right\| \underset{t\to+\infty}{\to} 0$ if and only if $\left\| e^{Jt} \right\| \underset{t\to+\infty}{\to} 0$.

Proof of the lemma. $e^{At} = Se^{Jt}S^{-1}$ (according to Lemma 2 of Paragraph 8 of Chapter 4). Then,

$$\left\| e^{At} \right\| \leq \left\| S \right\| \cdot \left\| e^{Jt} \right\| \cdot \left\| S^{-1} \right\|, \tag{2}$$

and from the boundedness of the matrix e^{Jt} it follows the boundedness of the matrix e^{At}.

If $\left\| e^{Jt} \right\| \underset{t\to+\infty}{\to} 0$, then from the estimate (2) it follows that $\left\| e^{At} \right\| \underset{t\to+\infty}{\to} 0$.

$e^{Jt} = S^{-1}e^{At}S$, and similarly it is proved that from the boundedness of the matrix e^{At} it follows the boundedness of the matrix e^{Jt}. And if $\left\| e^{At} \right\| \underset{t\to+\infty}{\to} 0$, then $\left\| e^{Jt} \right\| \underset{t\to+\infty}{\to} 0$. The lemma is proved.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of the matrix A.

Lemma 2. If $\operatorname{Re}\lambda_j < 0$ for any $j = 1,2,...,n$, then $\left\| e^{At} \right\| \underset{t\to+\infty}{\to} 0$.

Proof of Lemma 2. As follows from the Paragraph 8 of the Chapter 4, that each element of the matrix e^{Jt} either equal to zero or has the form $\frac{t^k}{k!}e^{\lambda_j t}$.

Note, that if $\operatorname{Re}\lambda_j < 0$, then

$$\left| \frac{t^k}{k!}e^{\lambda_j t} \right| = \frac{t^k}{k!}e^{\operatorname{Re}\lambda_j t} \underset{t\to+\infty}{\to} 0.$$

Each element of the matrix e^{Jt} tends to zero at $t \to +\infty$. Hence, $\left\| e^{Jt} \right\| \underset{t\to+\infty}{\to} 0$, and therefore $\left\| e^{At} \right\| \underset{t\to+\infty}{\to} 0$. The lemma is proved.

Theorem 1 (on estimating of the norm of the fundamental matrix e^{At}).

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of the matrix A.

Then for any $\delta > \max\limits_{j=1,2,...,n} \left(\operatorname{Re} \lambda_j \right)$ there exists a constant $K \geq 1$ such that

$$\left\| e^{At} \right\| \leq K e^{\delta t}.$$

Proof of the theorem. We fix $\delta > \max\limits_{j=1,2,...,n} \left(\operatorname{Re} \lambda_j \right)$.

Obviously $\mu_j = \lambda_j - \delta$ are eigenvalues of the matrix $B = A - \delta E$, and $\operatorname{Re} \mu_j < 0$ for any $j = 1,2,...,n$. From Lemma 2 it follows that $\left\| e^{Bt} \right\| \underset{t\to+\infty}{\to} 0$. Then

$$e^{At} = e^{Bt+\delta Et} = e^{Bt} e^{\delta Et} = e^{Bt} E e^{\delta t} = e^{Bt} e^{\delta t}. \qquad (3)$$

Let $K = \sup\limits_{t\in[0,+\infty)} \left\| e^{Bt} \right\|$. ($K \geq 1$, since $\left\| e^{Bt} \right\| = 1$ for $t = 0$).

From equality (3) it follows that $\left\| e^{At} \right\| \leq K e^{\delta t}$. The theorem is proved.

Theorem 2 (type of stability of a homogeneous system with constant coefficients, non-critical case).

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of the matrix A.

1. If $\operatorname{Re} \lambda_j < 0$ for all $j = 1,2,...,n$, then the solution $y \equiv 0$ of the system (1) is asymptotically stable.
2. If there exists λ_j, $j \in \{1,2,...,n\}$, such that $\operatorname{Re} \lambda_j > 0$, then the solution $y \equiv 0$ of the system (1) is unstable.

Proof of the theorem. The proof of the first statement follows from the Lemma 2 and the Theorem 4 of the previous paragraph.

For proving the second statement, we will show, that e^{At} is unbounded and apply Theorem 3 of the previous paragraph. From the Lemma 1 it follows that it is sufficient to prove the unboundedness of the matrix e^{Jt}.

Each element of the matrix e^{Jt} either equal to zero or has the form $\frac{t^k}{k!}e^{\lambda_j t}$. If there exists λ_j such that $\mathrm{Re}\lambda_j > 0$, then the matrix e^{Jt} contain an element $e^{\lambda_j t}$, and $\left|e^{\lambda_j t}\right| = e^{\mathrm{Re}\lambda_j t} \underset{t\to+\infty}{\to} +\infty$. Hence, e^{Jt} is unbounded matrix. The theorem is proved.

Theorem 3 (type of stability of a homogeneous system with constant coefficients, critical case).

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of the matrix A, $\mathrm{Re}\lambda_j \le 0$ for any $j = 1,2,...,n$, and there are eigenvalues λ_j, such that $\mathrm{Re}\lambda_j = 0$.

1. If each eigenvalue λ_j, such that $\mathrm{Re}\lambda_j = 0$, corresponds to a one-dimensional Jordan block (in a Jordan form of the matrix), then the solution $y \equiv 0$ of the system (1) is stable, but not asymptotically.
2. If there exists λ_j such that $\mathrm{Re}\lambda_j = 0$, and this λ_j corresponds to a Jordan block of dimension greater than one, then the solution $y \equiv 0$ of the system (1) is unstable.

Proof of the theorem. Each element of the matrix e^{Jt} either equal to zero or has the form $\frac{t^k}{k!}e^{\lambda_j t}$.

If the $\mathrm{Re}\lambda_j < 0$, then

$$\left|\frac{t^k}{k!}e^{\lambda_j t}\right| = \frac{t^k}{k!}e^{\mathrm{Re}\lambda_j t} \underset{t\to+\infty}{\to} 0.$$

If $\mathrm{Re}\lambda_j = 0$, and λ_j corresponds to a one-dimensional Jordan block, then each such block represents one element equal to $e^{\lambda_j t}$, and $\left|e^{\lambda_j t}\right| = e^0 = 1$.

Thus, all elements of the matrix e^{Jt} are bounded, and then e^{Jt} is bounded matrix. Hence, according to Lemma 1 and Theorem 2 of the previous paragraph, the solution $y \equiv 0$ of system (1) is stable, but not asymptotically.

If $\mathrm{Re}\lambda_j = 0$, and λ_j corresponds to a Jordan block of dimension greater than one, then the matrix e^{Jt} has an element equal to $te^{\lambda_j t}$, and $\left|te^{\lambda_j t}\right| = t \underset{t\to+\infty}{\to} +\infty$. Hence, e^{Jt} is unbounded matrix. And according to Theorem 3 of the previous paragraph, the solution $y \equiv 0$ of the system (1) is unstable. The theorem is proved.

§ 4. Stability at the first approximation.

Consider the system

$$\dot{y} = Y(t, y), \tag{1}$$

where $y \in R^n$, $Y(t, y)$ is continuous on t and continuously differentiable with respect to y in the domain $G \subset R^{n+1}$.

Let $y = \varphi(t)$ be solution of the system (1), defined for $t \in [t_0, +\infty)$.

We assume, that there exists $\rho_0 > 0$ such that the set

$$\{(t, y) : t \in [t_0, +\infty), \|y - \varphi(t)\| < \rho_0\}$$

contained in the domain G.

Substitution $y = \varphi(t) + x$ reduce the system (1) to the system

$$\dot{x} = Y(t, \varphi(t) + x) - Y(t, \varphi(t)),$$

or (in other words) to the system

Taylor ?

$$\dot{x} = P(t)x + g(t, x), \tag{2}$$

where $P(t) = \dfrac{\partial Y(t, \varphi(t))}{\partial y}$, $g(t, 0) = 0$, $\dfrac{\|g(t, x)\|}{\|x\|} \underset{\|x\| \to 0}{\to} 0$.

Definition. System

$$\dot{x} = P(t)x, \tag{3}$$

is called the first approximation system for the system (2).

Let $\Phi(t,\tau)$ be the fundamental Cauchy matrix of the system (3).

Theorem 1 (on the stability of the zero solution of the first approximation system).

Let there exist constants $K \ge 1$, $\sigma > 0$ and $c \in (0, \sigma/K)$ such that

$$\|\Phi(t,\tau)\| \le Ke^{-\sigma(t-\tau)} \tag{4}$$

for any $t, \tau \in R$, $t_0 \le \tau < t < +\infty$, and

$$\|g(t,x)\| < c\|x\| \tag{5}$$

for any $t \ge t_0$.

Then the solution $x \equiv 0$ of the system (2) is asymptotically stable.

Proof of the theorem. Let us first prove that all solutions of system (2) with initial condition close to zero are defined for $t \ge t_0$.

We prove it by contradiction.

Let $x = \xi(t)$ be solution of the system (2) with initial condition $(t_0, \xi(t_0))$, where $\|\xi(t_0)\| < \rho < \rho_0$.

Let's assume that this solution is defined for $t \in [t_0, \beta)$, $\beta < +\infty$, and $[t_0, \beta)$ is the maximum interval of existence. Since $x = \xi(t)$ is solution, then

$$\dot{\xi}(t) = P(t)\xi(t) + g(t, \xi(t)).$$

Let's denote $g(t, \xi(t)) = q(t)$. Then the function $\xi(t)$ is the solution of the linear system

$$\dot{x} = P(t)x + q(t), \tag{6}$$

and this solution can be written as

$$\xi(t) = \Phi(t, t_0)\xi(t_0) + \int_{t_0}^{t} \Phi(t,\tau)q(\tau)d\tau. \tag{7}$$

Hence,

$$\|\xi(t)\| \le \|\Phi(t,t_0)\| \cdot \|\xi(t_0)\| + \left|\int_{t_0}^{t} \|\Phi(t,\tau)\| \cdot \|q(\tau)\| d\tau\right|,$$

and taking into account inequalities (4), (5), we get

$$\|\xi(t)\| \le Ke^{-\sigma(t-t_0)} \cdot \|\xi(t_0)\| + \left|\int_{t_0}^{t} Ke^{-\sigma(t-\tau)} c\|\xi(\tau)\| d\tau\right|. \tag{8}$$

Let us multiply both sides of inequality (8) by $e^{\sigma(t-t_0)}$:

$$e^{\sigma(t-t_0)}\|\xi(t)\| \le K \cdot \|\xi(t_0)\| + cK\left|\int_{t_0}^{t} e^{\sigma(\tau-t_0)}\|\xi(\tau)\| d\tau\right|,$$

and apply Gronwall's lemma to the function $u(t) = e^{\sigma(t-t_0)}\|\xi(t)\|$. We get

$$e^{\sigma(t-t_0)}\|\xi(t)\| \le K \cdot \|\xi(t_0)\| e^{cK(t-t_0)},$$

or

$$\|\xi(t)\| \le K \cdot \|\xi(t_0)\| e^{(cK-\sigma)(t-t_0)} \tag{9}$$

for any $t \in [t_0, \beta)$.

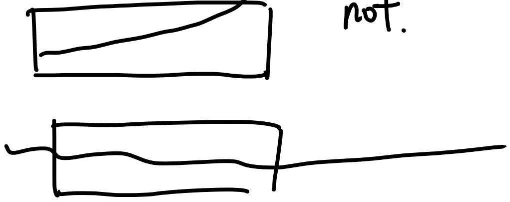
According to the assumptions of the theorem, $c \in (0, \sigma/K)$. Hence, $cK - \sigma < 0$, and

$$\|\xi(t)\| \le K \cdot \|\xi(t_0)\| \tag{10}$$

for any $t \in [t_0, \beta)$.

Let's set $\Delta = \dfrac{\rho}{2K}$, and $\|\xi(t_0)\| < \Delta < \rho$. Then from inequality (10) it follows that

$$\|\xi(t)\| < \frac{\rho}{2} \tag{11}$$



not.

for any $t \in [t_0, \beta)$. And if $\beta < +\infty$, then the graph of $x = \xi(t)$ contained in the compact set

$$D = \{(t, x) : t \in [t_0, \beta], \|x\| \le \rho/2\},$$

which contradicts the theorem on the exit of the maximally extended solution from a compact set (§6, Chapter 2).

Therefore, our assumption is incorrect, and $\beta = +\infty$.

We have proved that all solutions of system (2) with initial data $(t_0, \xi(t_0))$, where $\|\xi(t_0)\| < \Delta < \rho$, defined for $t \in [t_0, +\infty)$.

Let us now prove the asymptotic stability of the zero solution of the system (2).

We fix $\varepsilon > 0$, and set $\delta = \min\left(\dfrac{\varepsilon}{2K}, \Delta\right)$. From inequality (10) it follows that $\|\xi(t)\| < K\delta < \varepsilon$, if $\|\xi(t_0)\| < \delta$.

Moreover, from inequality (9) it follows that $\|\xi(t)\| \underset{t \to +\infty}{\to} 0$.

Therefore, the solution $x \equiv 0$ of the system (2) is asymptotically stable. The theorem is proved.

Let us now consider the system

$$\dot{x} = Ax + g(t, x), \tag{12}$$

where A is constant square matrix of dimension n, function $g(t, x)$ is continuous in the domain $G \subset R^{n+1}$, $\{(t, x) : t \in [t_0, +\infty), \|x\| < \rho_0\} \subset G$ for some $\rho_0 > 0$, and $g(t, 0) = 0$.

From these conditions it follows that the system (12) has the solution $x \equiv 0$, defined for $t \in [t_0, +\infty)$.

We assume that $\dfrac{\|g(t, x)\|}{\|x\|}$ tends to zero at $\|x\| \to 0$ uniformly on t for $t \in [t_0, +\infty)$.

The first approximation system for system (12) is a linear homogeneous system with constant coefficients

$$\dot{x} = Ax. \tag{13}$$

Theorem 2 (on the asymptotic stability of the zero solution of the first approximation system with a constant matrix in the linear part).

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of the matrix A, $\operatorname{Re}\lambda_j < 0$ for any $j = 1, 2, ..., n$, and $\dfrac{\|g(t,x)\|}{\|x\|}$ tends to zero at $\|x\| \to 0$ uniformly on $t \in [t_0, +\infty)$.

Then the solution $x \equiv 0$ of the system (12) is asymptotically stable.

Proof of the theorem. Let us show that for system (12) all the conditions of Theorem 1 are satisfied.

$\Phi(t) = e^{At}$ is fundamental matrix of the system (13), and

$$\Phi(t,\tau) = \Phi(t)\Phi^{-1}(\tau) = e^{At}e^{-A\tau} = e^{A(t-\tau)}.$$

Let's choose an arbitrary δ, satisfying the inequality

$$\max_{j=1,2,...,n} \left(\operatorname{Re}\lambda_j\right) < \delta < 0.$$

By Theorem 1 of the third paragraph (on estimating the norm of the fundamental matrix) for the chosen δ there is a constant $K \geq 1$, and the inequality

$$\left\| e^{A(t-\tau)} \right\| \leq K e^{\delta(t-\tau)}$$

holds for any $t, \tau \in R$ such that $t_0 \leq \tau < t < +\infty$.

Let's set $\sigma = -\delta$, and then the condition (4) is satisfied.

Let's consider an arbitrary $c \in (0, \sigma/K)$.

$\dfrac{\|g(t,x)\|}{\|x\|}$ tends to zero at $\|x\| \to 0$ uniformly on t. Hence, there exists ρ, $0 < \rho < \rho_0$, such that from the inequality $\|x\| < \rho$ it follows inequality $\dfrac{\|g(t,x)\|}{\|x\|} < c$, or $\|g(t,x)\| < c\|x\|$. So, the condition (5) is satisfied.

Thus, for system (12) all the conditions of Theorem 1 are satisfied, and according to this theorem, the solution $x \equiv 0$ of the system (12) is asymptotically stable. The theorem is proved.

We also present the unstability theorem without proof.

Theorem 2 (on the unstability of the zero solution of the first approximation system with a constant matrix in the linear part).

Let among the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of the matrix A there exists λ_j, such that $\mathrm{Re}\,\lambda_j > 0$, and $\frac{\|g(t,x)\|}{\|x\|}$ tends to zero at $\|x\| \to 0$ uniformly on t.

Then the solution $x \equiv 0$ of the system (12) is unstable.

Remark. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be eigenvalues of the matrix A. If $\mathrm{Re}\,\lambda_j \le 0$ for any $j = 1, 2, ..., n$, and there are eigenvalues λ_j, such that $\mathrm{Re}\,\lambda_j = 0$, then the stability type of the solution $x \equiv 0$ system (12) depends on the function $g(t,x)$

CHAPTER 7. AUTONOMOUS SYSTEMS.

§ 1. Basic definitions. Characteristic property of autonomous systems.

Let's consider the system

$$\dot{x} = F(x), \tag{1}$$

the right-hand side of which does not explicitly depend on t. Here $x \in M \subset R^n$, $t \in R$, vector function $F(x)$ is continuous and satisfies the Lipschitz condition locally in the domain M.

Definition. A system of differential equations is called autonomous if the right-hand side of the system does not explicitly depend on t.

Domain M is called the phase space of the autonomous system (1).

Let $x = \varphi(t)$ be a solution of the system (1), defined for $t \in \langle a,b \rangle$.

Recall that the solution graph, that is, the set $\Gamma_{\varphi(t)} = \{(t,x) : x = \varphi(t), t \in \langle a,b \rangle\}$ is called the integral curve.

Definition. Projection of the integral curve $x = \varphi(t)$ to the phase space M, that is, the set $L_{\varphi(t)} = \{\varphi(t) : t \in \langle a,b \rangle\}$, is called the trajectory of this solution.

Geometric interpretation of an autonomous system (1).

System (1) on phase space M induces a vector field.

Every point $x \in M$ corresponds to the vector $F(x)$, which is tangent to the trajectory passing through this point. Variable t becomes a parameter for parametrization of this curve.

Theorem 1 (characteristic property of autonomous systems).

Let $x = \varphi(t)$ be solution of the system (1), and (a,b) be the maximum interval of the existence of the solution $\varphi(t)$.

Then for any constant c function $x = \varphi(t+c)$ is also the solution of the system (1), and $(a-c, b-c)$ is the maximum interval of existence of this solution.

Proof of the theorem. We fix an arbitrary constant c. Let's set $\xi(t) = \varphi(t+c)$. Then

$$\dot{\xi}(t)=\dot{\varphi}(t+c)=F(\varphi(t+c))=F(\xi(t)),$$

and hence $x=\xi(t)$ is the solution of the system (1), defined for $t+c\in(a,b)$.

If $(a-c,b-c)$ is not the maximum interval of existence of the solution $x=\xi(t)$, then this solution can be continued to the right beyond $(b-c)$ or left beyond $(a-c)$. Then the solution $x = \varphi(t) = \xi(t - c)$ can be continued to the right beyond b or left beyond a, which contradicts the assumption. Hence, $(a-c,b-c)$ is maximum interval of existence of the solution $x=\varphi(t+c)$. The theorem is proved.

Remark. It is obvious that the solutions $x=\varphi(t)$, $t\in(a,b)$, and $x=\varphi(t+c)$, $t\in(a-c,b-c)$, corresponds to the same trajectory: $L_{\varphi(t)}=L_{\varphi(t+c)}$.

Theorem 2. Trajectories corresponding to different solutions either do not intersect or coincide.

Proof of the theorem. Let the solution $x=\varphi(t)$ be defined on the interval (a_1,b_1), and (a_1,b_1) is maximum interval of existence of $\varphi(t)$, and the solution $x=\psi(t)$ be defined on the interval (a_2,b_2), and (a_2,b_2) is maximum interval of existence of $\psi(t)$.

Let there be points $t_1\in(a_1,b_1)$ and $t_2\in(a_2,b_2)$ such that $\varphi(t_1)=\psi(t_2)$. Then

$$\varphi(t_1)=\psi(t_1+(t_2-t_1))=\psi(t_1+c),$$

where $c=t_2-t_1$, and solutions $x=\varphi(t)$, $x=\psi(t+c)$ solve the same Cauchy problem $(t_1,\varphi(t_1))$. By the theorem of the uniqueness of the solutions $(a_1,b_1)=(a_2-c,b_2-c)$, and $\varphi(t)\equiv\psi(t+c)$ on the interval (a_1,b_1). Hence, $L_{\varphi(t)}=L_{\psi(t+c)}=L_{\psi(t)}$. The theorem is proved.

§ 2. Types of trajectories.

Let's consider an autonomous system

$$\dot{x}=F(x), \tag{1}$$

where $x\in M\subset R^n$, $t\in R$, function $F(x)$ is continuous and satisfies the Lipschitz condition locally in the domain M.

Let $\varphi(t) \equiv x_0$, $x_0 \in M$ be solution of the system (1). Then the trajectory corresponding to this solution is the point: $L_{\varphi(t)} = \{x_0\}$.

Definition. Trajectory $L_{\varphi(t)} = \{x_0\}$ (or point x_0) is called an *equilibrium* or *singular point* of the system (1).

Lemma 1. The point x_0 is equilibrium of the system (1) if and only if $F(x_0) = 0$.

Proof of Lemma 1. If x_0 is equilibrium, then $\varphi(t) \equiv x_0$ is solution of system (1), and $\dot{\varphi}(t) = 0$. Hence, $F(\varphi(t)) = F(x_0) = 0$.

If $F(x_0) = 0$, then $\varphi(t) \equiv x_0$ is solution of system (1), and x_0 is equilibrium. The lemma is proved.

Lemma 2 (condition for the periodicity of the solution). Let $x = \varphi(t)$ be solution of the system (1), defined on the interval (a,b), $\varphi(t) \neq const$, and (a,b) be maximum interval of the existence of the solution $\varphi(t)$.

Let there exist $t_1 \in (a,b)$ and $t_2 \in (a,b)$, such that $t_1 \neq t_2$, $\varphi(t_1) = \varphi(t_2)$.

Then $(a,b) = R$ and $\varphi(t + (t_2 - t_1)) = \varphi(t)$ for any $t \in R$, and hence $x = \varphi(t)$ is periodic solution with period $(t_2 - t_1)$.

Definition. If $x = \varphi(t)$ is periodic solution, $\varphi(t) \neq const$, then the trajectory $L_{\varphi(t)}$ is a closed curve, which is called a cycle.

Proof of the lemma. Let's set $c = t_2 - t_1$. Hence, $\xi(t) = \varphi(t + c)$ is the solution of the system (1), defined on the interval $(a - c, b - c)$, and

$$\xi(t_1) = \varphi(t_1 + c) = \varphi(t_2) = \varphi(t_1).$$

Thus, $x = \varphi(t)$ and $x = \xi(t)$ solve the same Cauchy problem $(t_1, \varphi(t_1))$. By the theorem of uniqueness of solutions $(a,b) = (a - c, b - c)$, and $\varphi(t) \equiv \xi(t)$ for any $t \in (a,b)$.

Since $(a,b)=(a-c,b-c)$, $c\neq 0$, then the interval (a,b) is invariant under the shift. It follows that $(a,b)=R$ and $\varphi(t)=\varphi(t+c)$ for any $t\in R$. The lemma is proved.

Thus, any trajectory of system (1) is a trajectory of one of three types:

1) $L_{\varphi(t)}$ is equilibrium. In this case $\varphi(t)\equiv x_0$, and this solution is defined for any $t\in R$.

2) $L_{\varphi(t)}$ is cycle. In this case $x=\varphi(t)$ is periodic solution, and this solution is defined for any $t\in R$.

3) $L_{\varphi(t)}$ is *general trajectory*. In this case $\varphi(t)\neq const$ and $x=\varphi(t)$ is non-periodic solution, and this solution can be defined for any $t\in R$ or on the interval $(a,b)\neq R$.

Theorem 1 (about the structure of a general trajectory). Let $L_{\varphi(t)}$ be general trajectory corresponding to the solution $x=\varphi(t)$, $t\in(a,b)$.

Then the map $\varphi:(a,b)\to L_{\varphi(t)}$ is regular.

Proof of the theorem. Map $\varphi:(a,b)\to L_{\varphi(t)}$ is continuously differentiable and (by Lemma 2) one-to-one.

Let's prove that $\dot{\varphi}(t)\neq 0$ for any $t\in(a,b)$. Let there exists a point $t_0\in(a,b)$ such that $\dot{\varphi}(t_0)=0$. Then $x_0=\varphi(t_0)$ is equilibrium, and $x_0\in L_{\varphi(t)}$, which contradicts Theorem 2 of the previous paragraph (trajectories corresponding to different solutions do not intersect or coincide). Hence, $\dot{\varphi}(t)\neq 0$ for any $t\in(a,b)$, and map $\varphi:(a,b)\to L_{\varphi(t)}$ is regular. The theorem is proved.

Theorem 2 (about the structure of a cycle). Let $x=\varphi(t)$ be periodic solution of the system (1), $\varphi(t)\neq const$.

Then there is the smallest positive period of the function $\varphi(t)$, and the cycle $L_{\varphi(t)}$ is regular image of a circle.

Proof of the theorem. Let

$$T=\{\tau\in(0,+\infty):\varphi(t+\tau)=\varphi(t),t\in R\}.$$

Since $T\neq\varnothing$, then there exists $\omega=\inf T$, and $\omega\geq 0$.

Let us first prove that $\omega\neq 0$.

Assume, on contrary, that that $\omega=0$. Then in the set T there are points arbitrarily close to zero. Thus, there exists a sequence $\{\tau_k\}_{k=1}^{+\infty}\subset T$ such that $\tau_k \underset{k\to+\infty}{\to} 0$. We fix an arbitrary $t\in R$. For each τ_k number t representable in the form $t=m_k\tau_k+r_k$, where m_k is integer, $0\leq r_k<\tau_k$, and $r_k \underset{k\to+\infty}{\to} 0$. Then, by the definition of the set T,

$$\varphi(t)=\varphi(m_k\tau_k+r_k)=\varphi(r_k) \text{ for any } k\in N.$$

Passing in the last equality to the limit at $k\to+\infty$, we get: $\varphi(t)=\varphi(0)$ for any $t\in R$, which contradicts with the assumption $\varphi(t)\neq const$. Hence, $\omega>0$.

Let's prove that $\omega\in T$.

Assume, on contrary, that $\omega\notin T$. Since $\omega=\inf T$, then in the set T there are points arbitrarily close to ω. Thus, there is a sequence $\{\tau_k\}_{k=1}^{+\infty}\subset T$ such that $\tau_k \underset{k\to+\infty}{\to} \omega$. We fix an arbitrary $t\in R$. By of a set T,

$$\varphi(t+\tau_k)=\varphi(t) \text{ for any } k\in N.$$

Passing in the last equality to the limit at $k\to+\infty$, we get: $\varphi(t+\omega)=\varphi(t)$ for any $t\in R$, which contradicts the condition $\omega\notin T$. Thus, $\omega\in T$, and ω is the smallest positive period of the function $\varphi(t)$.

Let S^1 be unit circle: $S^1=\{(\cos\theta,\sin\theta):\theta\in R\}$. Points on a circle are equivalence classes

$$[(\cos\theta,\sin\theta)]=\{(\cos\theta_1,\sin\theta_1):\theta_1\equiv\theta(\bmod 2\pi)\}.$$

$L_{\varphi(t)}$ is cycle corresponding to a periodic solution $x=\varphi(t)$ with the smallest positive period ω.

At the same time, points on the trajectory $L_{\varphi(t)}$ are equivalence classes

$$\left[\varphi(t)\right]=\{\varphi(t_1):\, t_1 \equiv t(\operatorname{mod}\omega)\}.$$

Let the mapping h map the point $\left[(\cos\theta, \sin\theta)\right]$ lying on a circle, to the point $\left[\varphi\left(\theta\frac{\omega}{2\pi}\right)\right]$, lying on the trajectory $L_{\varphi(t)}$.

Note that the mapping h is defined correctly: if $\theta_1 \equiv \theta(\operatorname{mod} 2\pi)$, $t=\theta\frac{\omega}{2\pi}$, $t_1=\theta_1\frac{\omega}{2\pi}$, then $t_1 \equiv t(\operatorname{mod}\omega)$.

Mapping h is continuously differentiable, since the function $\varphi(t)$ is continuously differentiable.

Let's prove, that h is one-to-one mapping. Indeed, if $\varphi\left(\theta_1\frac{\omega}{2\pi}\right)=\varphi\left(\theta_2\frac{\omega}{2\pi}\right)$, then $\theta_1\frac{\omega}{2\pi}\equiv\theta_2\frac{\omega}{2\pi}(\operatorname{mod}\omega)$. Hence, $\theta_1 \equiv \theta_2(\operatorname{mod} 2\pi)$.

Let's prove that the derivative h does not vanish. It's enough to show, that $\dot{\varphi}(t)\neq 0$ for any $t\in R$. Let there be a point $t_0\in(a,b)$ such that $\dot{\varphi}(t_0)=0$. Then $x_0=\varphi(t_0)$ is equilibrium, and $x_0\in L_{\varphi(t)}$, which contradicts Theorem 2 of the previous paragraph. Therefore, the mapping h is regular. The theorem is proved.

§ 3. Poincaré classification of singular points of a second-order linear homogeneous system.

We consider a linear homogeneous autonomous system of the second order

$$\dot{y}=Ay, \tag{1}$$

where $y=(y_1,y_2)^T\in R^2$, A is constant $[2\times 2]$ matrix, $\det A\neq 0$.

System (1) has the unique equilibrium $y=0$.

Let λ_1,λ_2 be eigenvalues of the matrix A, J the Jordan form of A, $A=S^{-1}JS$, $\det S\neq 0$.

Then the system (1) could be rewritten in the form $\dot{y} = S^{-1} J S y$, or $S\dot{y} = J S y$, and substitution $x = S y$ leads to the system

$$\dot{x} = Jx. \tag{2}$$

Let's study the behavior of the trajectories of the system (2) on the phase plane. Note that the system (2) also has the unique equilibrium $x = 0$.

1. Let λ_1, λ_2 be real, $\lambda_1 \neq \lambda_2$. Then $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, and the system (2) can be rewritten in the form

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1, \\ \dot{x}_2 = \lambda_2 x_2. \end{cases} \tag{3}$$

Solution of the system (3) is

$$\begin{cases} x_1 = c_1 e^{\lambda_1 t}, \\ x_2 = c_2 e^{\lambda_2 t}, \ t \in R. \end{cases} \tag{4}$$

The solution $x_1 = 0$, $x_2 = 0$, corresponds to the trajectory $L_0 = \{0\}$. Solutions $x_1 = c_1 e^{\lambda_1 t}$, $x_2 = 0$, $c_1 > 0$ corresponds to the trajectory $L_{1,0} = \{(x_1, 0) : x_1 > 0\}$, and if $c_1 < 0$, then to the trajectory $L_{-1,0} = \{(x_1, 0) : x_1 < 0\}$. Similarly, solutions $x_1 = 0$, $x_2 = c_2 e^{\lambda_2 t}$, $c_2 > 0$ corresponds to the trajectory $L_{0,1} = \{(0, x_2) : x_2 > 0\}$, and if $c_2 < 0$, then to the trajectory $L_{0,-1} = \{(0, x_2) : x_2 < 0\}$.

Solutions of (4) with $c_1 c_2 \neq 0$ correspond to the trajectory

$$L_{\varphi(t)} = \left\{(x_1, x_2) : x_2 = c_2 \left(x_1 / c_1\right)^{\lambda_2/\lambda_1}\right\}. \tag{5}$$

a) If $\lambda_1 \lambda_2 < 0$, then trajectories (5) are hyperbolas lying depending on the signs c_1 and c_2 in one of the coordinate quarters.

If $c_1 > 0$, $c_2 > 0$, then trajectory (5) is determined at $x_1 > 0$ and lies in the first coordinate quadrant. If $c_1 > 0, c_2 < 0$, then this trajectory is determined at $x_1 > 0$ and lies in the fourth quarter. Similarly, when $c_1 < 0$ trajectory (5) is defined for $x_1 < 0$. This trajectory lies in the second coordinate quarter if $c_2 > 0$, and in the third quarter, if $c_2 < 0$.

With this disposition of trajectories on phase plane singular point $x=0$ of system (2) is called the *saddle*. Trajectories $L_{1,0}$, $L_{-1,0}$, $L_{0,1}$ and $L_{0,-1}$ are called saddle separatrices.

Let for certainty $\lambda_1 < 0, \lambda_2 > 0$.

Then with increasing t phase point $(x_1, 0)$ moves along trajectories $L_{1,0}$ and $L_{-1,0}$ to the equilibrium $x=0$, and the phase point $(0, x_2)$ moves along trajectories $L_{0,1}$ and $L_{0,-1}$ out from the equilibrium. In Figure 1 those movements are shown by arrows. Phase point movement along trajectories of type (5) in Figure 1 is also shown with arrows.

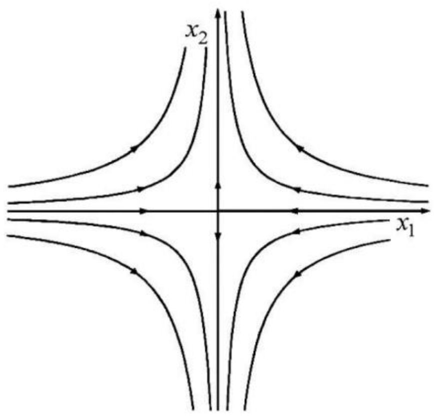


Figure 1 (equilibrium $x=0$ is saddle, $\lambda_1 < 0$, $\lambda_2 > 0$).

b) If $\lambda_1 \lambda_2 > 0$, then trajectories (5) are parabolas lying (depending on the signs c_1 and c_2) in one of the coordinate quarters.

With this disposition of trajectories on the phase plane, the equilibrium $x=0$ of system (2) is called the *node*.

Let for certainty $\lambda_1 < 0$ and $\lambda_2 < 0$, and $|\lambda_1| < |\lambda_2|$.

Then with increasing t phase point (x_1, x_2) moves along all trajectories to the equilibrium $x=0$, in Figure 2 this movement is shown by arrows. Since $|\lambda_1| < |\lambda_2|$, then trajectories (5) are tangent to the horizontal axis at the origin.

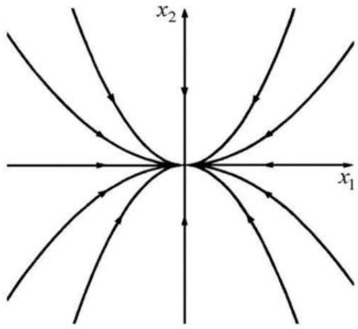


Figure 2 (equilibrium $x=0$ is node, $\lambda_1<0, \lambda_2<0$, $|\lambda_1| < |\lambda_2|$.).

2. Let λ_1, λ_2 be real, $\lambda_1=\lambda_2=\lambda$. Then the Jordan form of the matrix A looks like $J=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$ or $J=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$.

a) If $J=\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$, then the system (2) can be written in the form

$$\begin{cases}\dot{x}_1=\lambda x_1,\\ \dot{x}_2=\lambda x_2.\end{cases} \tag{6}$$

Solution of the system (6) is

$$\begin{cases}x_1=c_1e^{\lambda t},\\ x_2=c_2e^{\lambda t},\ t\in R.\end{cases} \tag{7}$$

The solution $x_1=0$, $x_2=0$, corresponds to the trajectory $L_0=\{0\}$. Solutions $x_1=c_1e^{\lambda_1 t}$, $x_2=0$, $c_1>0$ corresponds to the trajectory $L_{1,0}=\{(x_1,0):x_1>0\}$, and if $c_1<0$, then to the trajectory $L_{-1,0}=\{(x_1,0):x_1<0\}$. Similarly, solutions $x_1=0$, $x_2=c_2e^{\lambda_2 t}$, $c_2>0$ corresponds to the trajectory $L_{0,1}=\{(0,x_2):x_2>0\}$, and if $c_2<0$, then to the trajectory $L_{0,-1}=\{(0,x_2):x_2<0\}$.

Solutions of (6) with $c_1c_2\neq 0$ correspond to the trajectory

$$L_{\varphi(t)}=\{(x_1,x_2):x_2=(c_2/c_1)x_1\}, \tag{8}$$

which are rays lying (depending on the signs c_1 and c_2) in one of the coordinate quarters.

With this disposition of trajectories on the phase plane, the singular point $x=0$ of the system (2) is called a dicritical node.

Let for certainty $\lambda<0$.

Then with increasing t phase point (x_1,x_2) moves along all trajectories to the equilibrium $x=0$, in Figure 3 this movement is shown by arrows.

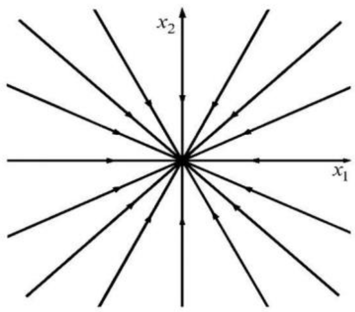


Figure 3 (equilibrium $x=0$ is dicritical node, $\lambda<0$).

b). If $J=\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$, then the system (2) can be written in the form

$$\begin{cases}\dot{x}_1=\lambda x_1+x_2,\\ \dot{x}_2=\lambda x_2.\end{cases} \tag{9}$$

Solution of system (9) is

$$\begin{cases}x_1=(c_1+c_2t)e^{\lambda t},\\ x_2=c_2e^{\lambda t},\ t\in R.\end{cases} \tag{10}$$

The solution $x_1=0$, $x_2=0$, corresponds to the trajectory $L_0=\{0\}$. Solutions $x_1=c_1e^{\lambda_1 t}$, $x_2=0$, $c_1>0$ corresponds to the trajectory $L_{1,0}=\{(x_1,0):x_1>0\}$, and if $c_1<0$, then to the trajectory $L_{-1,0}=\{(x_1,0):x_1<0\}$.

Solutions (9) with $c_2\neq 0$ correspond to the trajectory

$$L_{\varphi(t)}=\left\{(x_1,x_2):x_1=\left(c_1+\frac{c_2}{\lambda}\ln\frac{x_2}{c_2}\right)\frac{x_2}{c_2}\right\}, \tag{11}$$

which are curves that lie (depending on the sign c_2) in the upper or lower half-plane.

With this disposition of trajectories on the phase plane, the equilibrium $x=0$ is called a degenerate node.

Let for certainty $\lambda<0$.

Then with increasing t phase point (x_1, x_2) moves along all trajectories to the equilibrium $x=0$, in Figure 4 this movement is shown by arrows.

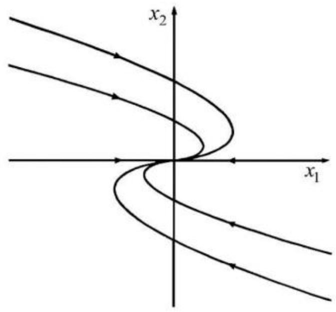


Figure 4 (equilibrium $x=0$ is degenerate node, $\lambda<0$).

3. Let λ_1, λ_2 be complex-conjugate: $\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$, $\beta\neq 0$. Then $J=\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$, and system (2) can be written in the form

$$\begin{cases}\dot{x}_1=\alpha x_1-\beta x_2,\\ \dot{x}_2=\beta x_1+\alpha x_2.\end{cases} \tag{12}$$

Let's rewrite system (12) in polar coordinates:

$$\begin{cases}x_1=r\cos\theta,\\ x_2=r\sin\theta,\ \theta\in R.\end{cases} \tag{13}$$

We obtain

$$\begin{cases}\dot{r}\cos\theta-r\dot{\theta}\sin\theta=\alpha r\cos\theta-\beta r\sin\theta,\\ \dot{r}\sin\theta+r\dot{\theta}\cos\theta=\beta r\cos\theta+\alpha r\sin\theta,\end{cases}$$

or

$$\begin{cases} \dot{r} = \alpha r, \\ \dot{\theta} = \beta. \end{cases} \tag{14}$$

Solution of system (14) is

$$\begin{cases} \dot{r} = r_0 e^{\alpha t}, \\ \dot{\theta} = \theta_0 + \beta t. \end{cases} \tag{15}$$

a). If $\alpha \neq 0$, then the trajectories corresponding to solution (15) are spirals tending to equilibrium at the positive direction if $\alpha > 0$ (*spiral sink*), and in the negative direction if $\alpha < 0$ (*spiral source*).

With this disposition of trajectories on the phase plane, the singular point $x = 0$ system (2) is called *focus*.

Let for certainty $\alpha < 0$, $\beta < 0$.
Then with increasing t phase point moves along spirals to the equilibrium. On Figure 5 this movement is shown by arrows.

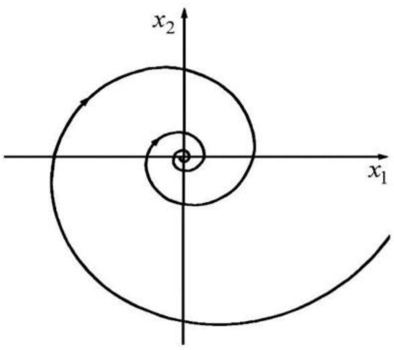


Figure 5 (equilibrium $x = 0$ is focus, $\alpha < 0$, $\beta < 0$).

b). If $\alpha = 0$, then the trajectories (5) are circles.

With this disposition of trajectories on the phase plane, the equilibrium $x = 0$ is called the *center*.

Let for certainty $\beta > 0$.
Then with increasing t the phase point (x_1, x_2) moves along circles in a positive direction. In Figure 6 this movement is shown by arrows.

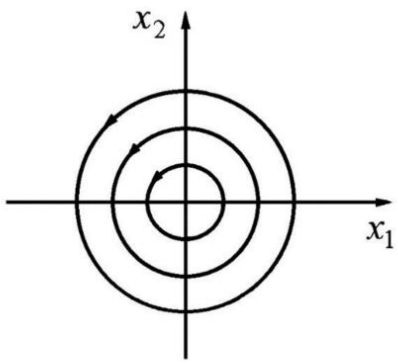


Figure 6 (equilibrium $x=0$ is center, $\alpha=0$, $\beta>0$).

相轨迹图.

Remark. The phase portrait of system (1) is obtained from the phase portrait of system (2) using an affine coordinate transformation. Equilibrium $y=0$ of system (1) has the same Poincaré type as the equilibrium $x=0$ of system (2). In Figure 7 the phase portrait of system (1) is shown in the case when the singular point of the system is a saddle. The directions of the saddle 分界线 separatrices are determined by the eigenvectors of matrix A.

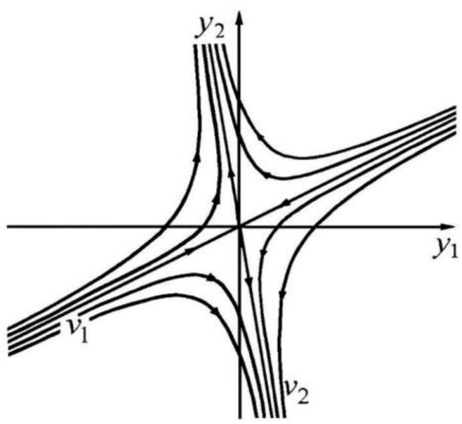


Figure 7 (equilibrium $y = 0$ is saddle, $\lambda_1 < 0, \lambda_2 > 0$).

§ 4. Equilibrium of a second order system.

We consider an autonomous system of the second order

$$\dot{x} = Ax + g(x), \tag{1}$$

where $x=(x_1,x_2)^T \in R^2$, A is constant $[2\times 2]$ matrix, $\det A \neq 0$, $g(x)\in C^2(M)$, M is domain in R^2, $0\in M$, $g(0)=0$, and

$$\frac{\|g(x)\|}{\|x\|} \underset{\|x\| \to 0}{\to} 0.$$

We assume that the equilibrium $x = 0$ is isolated (that is, in some neighborhood of the point $x = 0$ there are no other equilibrium of the system (1)).

Let's consider the linearization of the system (1)

$$\dot{x} = Ax \tag{2}$$

in the neighborhood of a point $x = 0$.

Theorem (Poincare). Let λ_1, λ_2 be eigenvalues of the matrix A.

If $\operatorname{Re}\lambda_j \neq 0$, $j = 1,2$, then the equilibrium $x = 0$ of the system (1) has the same Poincaré type as the equilibrium $x = 0$ of the system (2).