Problem 1

(5 points)
Find partial derivatives and $\frac{\partial^2 z}{\partial x \partial y}$ for the implicit function $z$ defined by the equation

$$xyz = x + y + z.$$

Solution: $yz + xy\dfrac{\partial z}{\partial x} = 1 + \dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial x} = \dfrac{1 - yz}{xy - 1}$.

$$z + y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x} + xy\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y},$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{z + y\frac{\partial z}{\partial y} + x\frac{\partial z}{\partial x}}{xy - 1}.$$

$$xz + xy\frac{\partial z}{\partial y} = 1 + \frac{\partial z}{\partial y}, \quad \frac{\partial z}{\partial y} = \frac{1 - xz}{xy - 1}.$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{z + y\frac{1 - xz}{xy - 1} + x\frac{1 - yz}{xy - 1}}{xy - 1} = \frac{-xyz + z - y + xyz - x + xyz}{(xy - 1)^2}$$

$$= \frac{xyz - x - y + z}{(xy - 1)^2}$$

Problem 2

(5 points)
Check the differentiability of the function $f(x, y) = \sqrt[3]{xy}$ at the point $(0, 0)$.

Solution A: $f'_x(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{0 - 0}{x} = 0$.

Similarly, $f'_y(0,0) = 0$.

$$\lim_{\rho\to 0^+}\frac{f(\Delta x,\Delta y)-f(0,0)-f'_x(0,0)\Delta x-f'_y(0,0)\Delta y}{\rho}$$

$$=\lim_{\rho\to 0^+}\frac{\sqrt[3]{\Delta x\,\Delta y}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\xlongequal{\Delta y=\Delta x}\lim_{\Delta x\to 0}\frac{(\Delta x)^{\frac{2}{3}}}{\sqrt{2}\,|\Delta x|}=\frac{1}{\sqrt{2}}\lim_{\Delta x\to 0}\frac{1}{|\Delta x|^{\frac{1}{3}}}=+\infty$$

Hence, $f(x,y)$ is not differentiable at point $(0,0)$.

Solution B:

$$\lim_{\rho\to 0^+}\frac{f\left(0+\rho\frac{\sqrt{2}}{2},0+\rho\frac{\sqrt{2}}{2}\right)-f(0,0)}{\rho}=\lim_{\rho\to 0^+}\frac{\sqrt[3]{\frac{1}{2}\rho^2}}{\rho}=\sqrt[3]{\frac{1}{2}}\lim_{\rho\to 0^+}\frac{1}{\rho^{\frac{1}{3}}}=+\infty$$

, which means that along the direction $\left\{\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\}$, function $f(x,y)$ doesn't have directional derivative at point $(0,0)$. Hence, $f(x,y)$ is not differentiable at point $(0,0)$.

Problem 3

(5 points)
Find extremal points of the function

$$f(x,y,z)=x^2+y^2+z^2+12xy+2z.$$

Solution: $\mathrm{d}f=2x\,\mathrm{d}x+2y\,\mathrm{d}y+2z\,\mathrm{d}z+12(y\,\mathrm{d}x+x\,\mathrm{d}y)+2\,\mathrm{d}z$,

$\mathrm{d}^2f=2\,\mathrm{d}x^2+2\,\mathrm{d}y^2+2\,\mathrm{d}z^2+12(\mathrm{d}x\,\mathrm{d}y+\mathrm{d}x\,\mathrm{d}y)=2\,\mathrm{d}x^2+2\,\mathrm{d}y^2+2\,\mathrm{d}z^2+24\,\mathrm{d}x\,\mathrm{d}y$

Set $\boldsymbol{A}=\begin{bmatrix}2&12&0\\12&2&0\\0&0&2\end{bmatrix}$, then

$$|\boldsymbol{A}-\lambda\boldsymbol{E}|=\begin{vmatrix}2-\lambda & 12 & 0\\ 12 & 2-\lambda & 0\\ 0 & 0 & 2-\lambda\end{vmatrix}=[(\lambda-2)^2-12^2](2-\lambda)$$
$$=(\lambda-14)(\lambda+10)(2-\lambda)=0$$

$\lambda_1=14$, $\lambda_2=-10$, $\lambda_3=2$. Hence $\boldsymbol{A}$ is neither positive definite nor negative definite, i.e., $f(x,y,z)$ has no extremal point.

Problem 4

(5 points)
Calculate the integral

$$\iint_G \frac{dxdydz}{(x+y)(x+y+z)},$$

where

$$G=\{(x,y,z)\in\mathbb{R}^3 : 1<x<2,\ 1<x+y<3,\ 1<x+y+z<4\}.$$

Solution: Set $\begin{cases} u=x \\ v=x+y \\ w=x+y+z \end{cases}$, then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}1&0&0\\1&1&0\\1&1&1\end{vmatrix}=1$,

$\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=1$, $G=\{(u,v,w)\,|\,1\leqslant u\leqslant 2, 1\leqslant v\leqslant 3, 1\leqslant w\leqslant 4\}$.

$$\iiint_G \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{(x+y)(x+y+z)}=\iiint_G \frac{\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w}{vw}=\int_1^2 \mathrm{d}u\int_1^3\frac{\mathrm{d}v}{v}\int_1^4\frac{\mathrm{d}w}{w}$$
$$=(2-1)(\ln 3-\ln 1)(\ln 4-\ln 1)=\ln 3\cdot\ln 4$$

Problem 5

(5 points)

Find the conditional extremum

$$f(x, y) = x + 2y, \quad x^2 + y^2 = 4.$$

Solution: Set $F(x,y) = x + 2y + \lambda(x^2 + y^2 - 4)$, from the system of

equations $\begin{cases} F_x' = 1 + 2\lambda x = 0 \\ F_y' = 2 + 2\lambda y = 0 \\ x^2 + y^2 - 4 = 0 \end{cases}$, we obtain solutions $\begin{cases} x = \dfrac{2}{\sqrt{5}} \\ y = \dfrac{4}{\sqrt{5}} \\ \lambda = -\dfrac{\sqrt{5}}{4} \end{cases}$,

$\begin{cases} x = -\dfrac{2}{\sqrt{5}} \\ y = -\dfrac{4}{\sqrt{5}} \\ \lambda = \dfrac{\sqrt{5}}{4} \end{cases}$.

$\mathrm{d}F = \mathrm{d}x + 2\,\mathrm{d}y + \lambda(2x\,\mathrm{d}x + 2y\,\mathrm{d}y)$, $\mathrm{d}^2 f = \mathrm{d}^2 F = 2\lambda(\mathrm{d}x^2 + \mathrm{d}y^2)$,

$2x\,\mathrm{d}x + 2y\,\mathrm{d}y = 0$.

At point $\left(\dfrac{2}{\sqrt{5}}, \dfrac{4}{\sqrt{5}}\right)$,

$\mathrm{d}^2 f\Big|_{\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)} = 2\left(-\dfrac{\sqrt{5}}{4}\right)\left(\mathrm{d}x^2 + \dfrac{1}{4}\mathrm{d}x^2\right) = -\dfrac{5\sqrt{5}}{8}\mathrm{d}x^2 < 0$, hence

$\left(\dfrac{2}{\sqrt{5}}, \dfrac{4}{\sqrt{5}}\right)$ is the conditional maximum.

At point $\left(-\dfrac{2}{\sqrt{5}}, -\dfrac{4}{\sqrt{5}}\right)$,

$$\mathrm{d}^2 f\Big|_{\left(-\frac{2}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right)} = 2\cdot\frac{\sqrt{5}}{4}\left(\mathrm{d}x^2+\frac{1}{4}\mathrm{d}x^2\right) = \frac{5\sqrt{5}}{8}\mathrm{d}x^2 > 0$$, hence $\left(-\frac{2}{\sqrt{5}},-\frac{4}{\sqrt{5}}\right)$ is the conditional minimum.

Problem 6

(5 points)
Calculate the Curve Intergal of a first kind $\int_L (x^2+y^2+z^2)dl$, where $L = \{(x,y,z) : x = cost,\ y = sint,\ z = t,\ 0 \leqslant t \leqslant 2\pi\}$

Solution:

$$\int_L (x^2+y^2+z^2)\mathrm{d}s$$
$$=\int_0^{2\pi} (\cos^2 t+\sin^2 t+t^2)\sqrt{(-\sin t)^2+\cos^2 t+1^2}\,\mathrm{d}t$$
$$=\int_0^{2\pi}(1+t^2)\sqrt{2}\,\mathrm{d}t = \sqrt{2}\left(t+\frac{1}{3}t^3\right)\Big|_0^{2\pi} = \sqrt{2}\left(2\pi+\frac{8}{3}\pi^3\right)$$

Problem 7

(5 points)
Calculate the Curve Intergal of a second kind $\int_L(-2xy^2dx+y^3dy)$, where L - part of the curve $x^3+2x^2+y^2=3$ from point $A(-1,\sqrt{2})$ to point $B(1,0)$

Solution:

$$\int_L (-2xy^2)\mathrm{d}x + y^3\mathrm{d}y$$
$$=\int_L (-2x)(3-x^3-2x^2)\mathrm{d}x+y^3\mathrm{d}y = \int_L (2x^4+4x^3-6x)\mathrm{d}x + \int_L y^3\mathrm{d}y$$
$$=\int_{-1}^{1}(2x^4+4x^3-6x)\mathrm{d}x + \int_{\sqrt{2}}^{0} y^3\mathrm{d}y = 2\cdot 2\cdot\frac{1}{5}+\left(\frac{1}{4}y^4\right)\Big|_{\sqrt{2}}^{0} = \frac{4}{5}-1 = -\frac{1}{5}$$

Problem 8

Calculate the Surface Intergal of a first kind $\iint_S (x^2+y^2+z)dS$, where

$S$ - upper hemisphere $x^2+y^2+z^2=a^2$, $0 \leqslant z$

Solution: (Solution A) $S:\ z=\sqrt{a^2-x^2-y^2}$,

$(x,y)\in D_{xy}=\{(x,y)|x^2+y^2\leqslant a^2\}$. $\dfrac{\partial z}{\partial x}=\dfrac{-x}{\sqrt{a^2-x^2-y^2}}$,

$\dfrac{\partial z}{\partial y}=\dfrac{-y}{\sqrt{a^2-x^2-y^2}}$,

$$\mathrm{d}S=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y=\frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$\iint_S (x^2+y^2+z)\,\mathrm{d}S$$

$$=\iint_S (x^2+y^2)\,\mathrm{d}S+\iint_S z\,\mathrm{d}S$$

$$=\iint_{D_{xy}}(x^2+y^2)\frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y+\iint_{D_{xy}}\sqrt{a^2-x^2-y^2}\frac{a}{\sqrt{a^2-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$=a\iint_{D_{xy}}\frac{r^2}{\sqrt{a^2-r^2}}\,r\,\mathrm{d}r\,\mathrm{d}\theta+a\iint_{D_{xy}}\mathrm{d}x\,\mathrm{d}y=a\int_0^{2\pi}\mathrm{d}\theta\int_0^a\frac{r^3}{\sqrt{a^2-r^2}}\,\mathrm{d}r+a\cdot\pi a^2$$

$$=2\pi a\int_0^a\frac{r^3}{\sqrt{a^2-r^2}}\,\mathrm{d}r+\pi a^3=2\pi a\int_0^{\frac{\pi}{2}}\frac{a^3\sin^3\theta}{a\cos\theta}\,a\cos\theta\,\mathrm{d}\theta+\pi a^3$$

$$=2\pi a^4\int_0^{\frac{\pi}{2}}\sin^3\theta\,\mathrm{d}\theta+\pi a^3=2\pi a^4\cdot\frac{2}{3}+\pi a^3=\frac{4\pi}{3}a^4+\pi a^3$$

(Solution B) $S:\ \rho=a,\ 0\leqslant\varphi\leqslant\dfrac{\pi}{2},\ 0\leqslant\theta\leqslant 2\pi$.

$\mathrm{d}S=a^2\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta$.

$$\iint_S (x^2+y^2+z)\,\mathrm{d}S$$

$$=\iint_S (x^2+y^2)\,\mathrm{d}S+\iint_S z\,\mathrm{d}S$$

$$=\iint_S a^2\sin^2\varphi\cdot a^2\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta+\iint_S a\cos\varphi\cdot a^2\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta$$

$$=a^4\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\sin^3\varphi\,\mathrm{d}\varphi+a^3\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\,\mathrm{d}\varphi$$

$$=a^4\cdot 2\pi\cdot\frac{2}{3}+a^3\cdot 2\pi\int_0^{\frac{\pi}{2}}\sin\varphi\,\mathrm{d}(\sin\varphi)$$

$$=\frac{4\pi}{3}a^4+a^3\cdot 2\pi\cdot\left(\frac{1}{2}\sin^2\varphi\Big|_0^{\frac{\pi}{2}}\right)=\frac{4\pi}{3}a^4+\pi a^3$$

(Solution C) $\vec{\boldsymbol{n}}^0=\left\{\frac{x}{a},\frac{y}{a},\frac{z}{a}\right\}$,

$$\iint_S (x^2+y^2+z)\,\mathrm{d}S$$

$$=a\iint_S\left(x\cdot\frac{x}{a}+y\cdot\frac{y}{a}+\frac{z}{a}\right)\mathrm{d}S=a\iint_S x\,\mathrm{d}y\,\mathrm{d}z+y\,\mathrm{d}z\,\mathrm{d}x+\mathrm{d}x\,\mathrm{d}y$$

$$=a\iint_{D_{xy}}\left[x\left(-\frac{\partial z}{\partial x}\right)+y\left(-\frac{\partial z}{\partial y}\right)+1\right]\mathrm{d}x\,\mathrm{d}y$$

$$=a\left[\iint_{D_{xy}}\left(x\cdot\frac{x}{\sqrt{a^2-x^2-y^2}}+y\cdot\frac{y}{\sqrt{a^2-x^2-y^2}}\right)\mathrm{d}x\,\mathrm{d}y+\iint_{D_{xy}}\mathrm{d}x\,\mathrm{d}y\right]$$

$$=a\iint_{D_{xy}}\frac{x^2+y^2}{\sqrt{a^2-x^2-y^2}}\mathrm{d}x\,\mathrm{d}y+a\cdot\pi a^2=\frac{4\pi}{3}a^4+\pi a^3$$

Or, denote $S'$: $z=0$, $(x,y)\in D_{xy}$, downward.

$$\iint_S (x^2+y^2+z)\,\mathrm{d}S$$

$$=a\iint_S\left(x\cdot\frac{x}{a}+y\cdot\frac{y}{a}+\frac{z}{a}\right)\mathrm{d}S=a\iint_S x\,\mathrm{d}y\,\mathrm{d}z+y\,\mathrm{d}z\,\mathrm{d}x+\mathrm{d}x\,\mathrm{d}y$$

$$=a\left(\oiint_{S+S'} x\,\mathrm{d}y\,\mathrm{d}z+y\,\mathrm{d}z\,\mathrm{d}x+\mathrm{d}x\,\mathrm{d}y-\iint_{S'} x\,\mathrm{d}y\,\mathrm{d}z+y\,\mathrm{d}z\,\mathrm{d}x+\mathrm{d}x\,\mathrm{d}y\right)$$

$$=a\iiint_\Omega (1+1+0)\,\mathrm{d}\Omega-a(-1)\iint_{D_{xy}}\mathrm{d}x\,\mathrm{d}y$$

$$=a\cdot 2\cdot\frac{1}{2}\cdot\frac{4\pi}{3}a^3+a\cdot\pi a^2=\frac{4\pi}{3}a^4+\pi a^3$$

Problem 9

(5 points)
Calculate the Intergal $\int_L (x^2+y)dx+(y^2+z)dy+(z^2+x)dz$, where L - ellipse $x^2+y^2=4$, $x+z=2$ positive oriented on the upper side of the plane

Solution: Set $\Sigma:\ z=2-x,\ (x,y)\in D_{xy}=\{(x,y)|x^2+y^2\leqslant 4\}$,

upward.

$$\oint_L (x^2+y)\,\mathrm{d}x+(y^2+z)\,\mathrm{d}y+(z^2+x)\,\mathrm{d}z$$

$$=\iint_\Sigma\begin{vmatrix}\mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2+y & y^2+z & z^2+x\end{vmatrix}=\iint_\Sigma(-1)\,\mathrm{d}y\,\mathrm{d}z+(-1)\,\mathrm{d}z\,\mathrm{d}x+(-1)\,\mathrm{d}x\,\mathrm{d}y$$

$$=\iint_{D_{xy}}\left[(-1)\left(-\frac{\partial z}{\partial x}\right)+(-1)\left(-\frac{\partial z}{\partial y}\right)+(-1)\right]\mathrm{d}x\,\mathrm{d}y$$

$$=\iint_{D_{xy}}[(-1)\cdot 1+(-1)\cdot 0+(-1)]\,\mathrm{d}x\,\mathrm{d}y=\iint_{D_{xy}}(-2)\,\mathrm{d}x\,\mathrm{d}y=(-2)\cdot 4\pi=-8\pi$$

## Problem 10

(5 points)
Formulate the Next Theorems and Definitions:

(a) Gamma Function Definition:
We recall that the gamma function is defined for $x > 0$ by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{1}$$

(b) Weierstrass approximation Theorem:
Let $f \in C(\mathbb{R}^m)$. Then for any $R > 0$ and $\varepsilon > 0$ there exists a polynomial $P$ in $m$ variables such that

$$|f(x) - P(x)| < \varepsilon \quad \text{for all } x \text{ in } \bar{B}(R).$$

(c) The first definition of a smooth manifold
A set $M, M \subset \mathbb{R}^m$, is called a *$k$-dimensional manifold of class $C^r$* if every point $p$ from $M$ has a neighborhood $U$ such that the intersection $U \cap M$ is a simple $k$-dimensional manifold of class $C^r$. Its parametrization is called a local parametrization of the manifold $M$ in the vicinity of the point $p$.

(d) Definition of potential vector field
A vector field $V = (V_1, \ldots, V_m)$ defined in a domain $\mathcal{O}$ is called potential if there exists a smooth function $F$ on $\mathcal{O}$ (a potential of $V$ ) such that $V(x) = \operatorname{grad} F(x)$ for all points $x \in \mathcal{O}$.

(e) The Gauss-Ostrogradski formula
Let $f$ be a function smooth on a standard compact set $K \subset \mathbb{R}^m$.

Then for every unit vector $e \in \mathbb{R}^m$, one has

$$\int_K \frac{\partial f}{\partial e}(x) dx = \int_{\partial K} f(x) \langle v(x), e \rangle d\sigma(x).$$