2.3. FIRST-ORDER DIFFERENTIAL EQUATIONS, LINEAR WITH RESPECT TO PARTIAL DERIVATIVES

The Cauchy problem for a homogeneous linear equation

To isolate a single particular solution from the general solution, additional conditions must be set. Such conditions, for example, include initial conditions. Initial conditions are often set by fixing one of the independent variables.

We will consider the initial problem, or Cauchy problem, for equation (4) in the following formulation. Among all the solutions of equation (4), find such a solution

$$u = F(x_1, \ldots, x_n), \tag{13}$$

which satisfies the initial conditions:

$$u = \varphi(x_1, \ldots, x_{n-1}) \quad \text{при} \quad x_n = x_n^{(0)}, \tag{14}$$

where φ – is a given continuously differentiable function of variables $x_1, \ldots, x_{n-1}$.

In the case where the desired function depends on two independent variables, Cauchy problem is to find a solution

$$u = F(x, y),$$

which satisfies the initial conditions:

$$u = \varphi(y)$$

at $x = x_0,$

where $\varphi(y)-$ the given function from y.

Geometrically, this means that among all integral surfaces, we are looking for an integral surface $u=F(x,y)$ that passes through a given curve $u=\varphi(y)$ lying in the plane $x=x_0$ parallel to the plane yOu.

Taking into account the general solution of the equation $u=\Phi(\psi_1,...,\psi_{n-1})$, the solution of the Cauchy problem is reduced to determining the type of function Φ such that

$$\Phi(\psi_1,\ldots,\psi_{n-1})\big|_{x_n=x_n^{(0)}}=\varphi(x_1,\ldots,x_{n-1}). \qquad (15)$$

Let's denote

$$\begin{cases} \psi_1(x_1,\ldots,x_{n-1},x_n^{(0)})=\overline{\psi}_1, \\ \psi_2(x_1,\ldots,x_{n-1},x_n^{(0)})=\overline{\psi}_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_{n-1}(x_1,\ldots,x_{n-1},x_n^{(0)})=\overline{\psi}_{n-1}, \end{cases} \qquad (16)$$

then equality (15) can be rewritten as

$$\Phi(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1})=\varphi(x_1,\ldots,x_{n-1}). \qquad (17)$$

The system (16) is solvable with respect to $x_1,...,x_{n-1}$ at least in some neighborhood of the point $\left(x_1^{(0)},...,x_n^{(0)}\right)$ if $A_n\left(x_1^{(0)},...,x_n^{(0)}\right)\neq 0$, which we assume. Resolving the system (16) with respect to $x_1,...,x_{n-1}$, we obtain:

$$\begin{cases} x_1=\omega_1(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \\ x_2=\omega_2(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_{n-1}=\omega_{n-1}(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \end{cases} \qquad (18)$$

If we now take the function

$$\Phi(\psi_1,\dots,\psi_{n-1}) = \varphi(\omega_1(\psi_1,\dots,\psi_{n-1}),\dots,\omega_{n-1}(\psi_1,\dots,\psi_{n-1})),$$

as Φ, then condition (17) will be fulfilled.

Therefore, the function gives the desired solution to the Cauchy problem. Here, the function φ is the function that participates in the initial conditions.

Thus, we come to the following algorithm for solving the Cauchy problem:

1) create the corresponding system of ordinary differential equations and find $n-1$ independent integrals:

$$\begin{cases} \psi_1(x_1,\dots,x_n), \\ \psi_2(x_1,\dots,x_n), \\ \dots\dots\dots\dots\dots\dots, \\ \psi_{n-1}(x_1,\dots,x_n). \end{cases} \tag{19}$$

2) replace the independent variable in integrals (19) with its specified value $x_n^{(0)}$:

$$\begin{cases} \psi_1(x_1,\dots,x_{n-1},x_n^{(0)}) = \overline{\psi}_1, \\ \psi_2(x_1,\dots,x_{n-1},x_n^{(0)}) = \overline{\psi}_2, \\ \dots\dots\dots\dots\dots\dots, \\ \psi_{n-1}(x_1,\dots,x_{n-1},x_n^{(0)}) = \overline{\psi}_{n-1}. \end{cases} \tag{20}$$

3) solve the system of equations (20) with respect to $x_1,\dots,x_{n-1}$:

$$\begin{cases} x_1 = \omega_1(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}), \\ x_2 = \omega_2(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}), \\ \dots\dots\dots\dots\dots\dots, \\ x_{n-1} = \omega_{n-1}(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}). \end{cases} \tag{21}$$

4) construct a function

$$u = \varphi(\omega_1(\psi_1, \ldots, \psi_{n-1}), \ldots, \omega_{n-1}(\psi_1, \ldots, \psi_{n-1})), \qquad (22)$$

that gives a solution to the Cauchy problem.

Example 1

Find a solution to the Cauchy problem

$$xu'_x + yu'_y + xyu'_z = 0, \quad u(x, y, 0) = x^2 + y^2.$$

<u>Solution:</u>

Let's make a characteristic system

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy}. \qquad (*)$$

We will find the first integral by solving the equation

$$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \frac{x}{y} = C_1.$$

So, $\psi_1(x, y, z) = \dfrac{x}{y}$.

We will find another first integral by considering the second equation of the characteristic system (*)

$$\frac{dy}{y} = \frac{dz}{xy},$$

excluding x from it using the already found first integral ψ_1. Since $x = C_1 y$, we will have

$$\frac{dy}{y} = \frac{dz}{C_1 y^2} \quad \Rightarrow \quad C_1 y dy = dz \quad \Rightarrow \quad C_1 y^2 - 2z = C_2 \quad \Rightarrow \quad xy - 2z = C_2.$$

So, $\psi_2(x,y,z) = xy - 2z$.

The characteristic system corresponding to the equation has the following two first integrals

$$\psi_1(x,y,z) = \frac{x}{y}, \quad \psi_2(x,y,z) = xy - 2z.$$

Considering them at $z = 0$, we make up a system of equations

$$\frac{x}{y} = \overline{\psi}_1, \quad xy = \overline{\psi}_2,$$

from which we find

$$y^2 = \frac{\overline{\psi}_2}{\overline{\psi}_1}, \quad x^2 = \overline{\psi}_1 \overline{\psi}_2.$$

Then for a given function $\varphi(x,y) = x^2 + y^2$ we will have

$$\varphi(\overline{\psi}_1, \overline{\psi}_2) = \left(\overline{\psi}_1 + \frac{1}{\overline{\psi}_1}\right)\overline{\psi}_2.$$

Therefore, the solution to the Cauchy problem has the form

$$u(x,y,z) = \left(\frac{x}{y} + \frac{y}{x}\right)(xy - 2z).$$

Quasi-linear equations

Let the point $x=(x_1,...,x_n)$ belong to the domain $D\subset\mathbb{R}^n$. Consider a quasi-linear equation

$$A_1(x,u)\frac{\partial u}{\partial x_1}+\ldots+A_n(x,u)\frac{\partial u}{\partial x_n}=B(x,u), \qquad (23)$$

assuming that $A_i(x,u)$ $(i=1,...,n)$ and $B(x,u)$ are differentiable functions of the arguments x,u in some domain $G\subset\mathbb{R}^{n+1}$.

Equation (23) corresponds to the following linear equation

$$A_1(x,u)\frac{\partial v}{\partial x_1}+\ldots+A_n(x,u)\frac{\partial v}{\partial x_n}+B(x,u)\frac{\partial v}{\partial u}=0, \qquad (24)$$

with an unknown function $\upsilon=\upsilon(x,u)$.

The method of solving a quasi-linear equation is based on the following theorem:

THEOREM 3.

Let $\upsilon=V(x,u)$ <u>be the solution of equation</u> (24). Let equation $V(x,u)=0$ define in the domain of D variables $x=(x_1,...,x_n)$ some differentiable function $u=\varphi(x)$, and let $\frac{\partial V}{\partial u}\neq 0$ in D for $u=\varphi$. Then $u=\varphi(x)$ is the solution of equation (23).

We describe *an algorithm for constructing a solution to a quasi-linear equation.*

1) write out the characteristic system for the linear equation (24):

$$\frac{dx_1}{A_1(x,u)} = \ldots = \frac{dx_n}{A_n(x,u)} = \frac{du}{B(x,u)} \tag{25}$$

The characteristics of the linear equation (24) are called **the characteristics of the quasi-linear equation** (23).

2) find the n independent first integrals of the system (25):

$$\psi_1(x,u), \ldots, \psi_n(x,u). \tag{26}$$

(or we can write $\psi_1(x,u)=c_1,\ldots,\psi_n(x,u)=c_n$).

3) using formula (11), construct a general solution to equation (24):

$$v(x,u) = \Phi(\psi_1(x,u), \ldots, \psi_n(x,u)).$$

(or we can write $\upsilon = V(\psi_1(x,u),\ldots,\psi_n(x,u))$).

4) assuming $\upsilon = 0$, write down the equation to determine the set of solutions to equation (23):

$$\Phi(\psi_1(x,u), \ldots, \psi_n(x,u)) = 0. \tag{27}$$

(or we can write $V(\psi_1(x,u),\ldots,\psi_n(x,u))=0$).

The expression (27) is called the **general integral**, or the **general solution**, of equation (23). If u is included only in one of the first integrals (26), for example, in the last one, then the general solution can be written as follows:

$$\psi_n(x,u) = F(\psi_1,\dots,\psi_{n-1}), \tag{28}$$

where $F-$ an arbitrary differentiable function. If it is possible to resolve equality (28) with respect to u, then we obtain a general solution of equation (23) in explicit form.

<u>Comment 1</u>

It is possible that there may be solutions to equation (23) for which equation (24) is not satisfied identically in (x,u), but only when $u=\varphi(x)$ is identical in x. Such solutions are not contained in formula (27) and are called *special*. A special solution is an exceptional case, and therefore we will not consider them further.

<u>Comment 2</u>

The solution of linear equation (2) can also be constructed in the described way.

<u>Comment 3</u>

When constructing the first integrals of the system (25), in some cases it may turn out that the variable u will enter only one of them:

$$\psi_1(x) = c_1,\ \dots,\ \psi_{n-1}(x) = c_{n-1},\ \psi_n(x,\ u) = c_n.$$

Then the general solution will be from the ratio

$$V\big(\psi_1(x),\ \dots,\ \psi_{n-1}(x),\ \psi_n(x,\ u)\big) = 0,$$

which can be rewritten by the implicit function theorem in the form

$$\psi_n(x,u) = F\big(\psi_1(x,u),\dots,\psi_{n-1}(x)\big) \tag{29}$$

By resolving equality (29) with respect to u, we obtain a general solution of equation (23) in explicit form.

The characteristic system (25) will be written in this case as

$$\frac{dx_1}{A_1(x)} = \dots = \frac{dx_n}{A_n(x)} = \frac{du}{0}.$$

A system of n independent first integrals can be chosen as follows

$$\psi_1(x) = c_1, \ \dots, \ \psi_{n-1}(x) = c_{n-1}, \ u = c_n.$$

We see that the variable u will enter only the last first integral. The solution of the equation can be written using the formula (29).

Example 2

Solve the equation

$$x_2 \frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} = x_1 - x_2.$$

<u>Solution:</u>

The characteristic system has the form

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1} = \frac{du}{x_1 - x_2}.$$

The first equality

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1}$$

it will be written in the form of

$$x_1 dx_1 = x_2 dx_2$$

and leads to the first integral

$$x_1^2 - x_2^2 = c_1.$$

To obtain another first integral, we use the property of adding proportions:

$$\frac{d(x_2 - x_1)}{x_1 - x_2} = \frac{du}{x_1 - x_2},$$

where do we get the first integral:

$$u + x_1 - x_2 = c_2.$$

Since u is included in only one of the first integrals obtained, we obtain an explicit solution to the equation

$$u = x_2 - x_1 + \mathrm{F}\left(x_1^2 - x_2^2\right),$$

The function $F(y)-$ is an arbitrary function of class C^1.

Example 3

Solve the equation

$$\left(x_2 + 2u^2\right)\frac{\partial u}{\partial x_1} - 2x_1^2 u\frac{\partial u}{\partial x_2} = x_1^2.$$

Solution:

The characteristic system has the form

$$\frac{dx_1}{x_2 + 2u^2} = \frac{dx_2}{-2x_1^2 u} = \frac{du}{x_1^2}.$$

From the second ratio we have

$$dx_2 = -2udu,$$

where do we get the first integral:

$$x_2 + u^2 = c_1. \qquad (*)$$

Consider the ratio

$$\frac{dx_1}{x_2 + 2u^2} = \frac{du}{x_1^2}$$

Substituting $x_2 = c_1 - u^2$ into the ratio, we get

$$\frac{dx_1}{c_1 + u^2} = \frac{du}{x_1^2}$$

from which

$$x_1^2\, dx_1 = \left(c_1 + u^2\right) du,$$

and therefore

$$\frac{x_1^3}{3} = c_1 u + \frac{u^3}{3} + c_2.$$

Then the general solution of the equation will be written in an implicit form

$$V\left(x_2 + u^2,\ x_1^3 - 3\left(x_2 + u^2\right)u - u^3\right) = 0,$$

where the function V is an arbitrary function of class C^1 and such that

$$\frac{\partial V\left(x_2 + u^2,\ x_1^3 - 3\left(x_2 + u^2\right)u - u^3\right)}{\partial u} \neq 0.$$

Example 4

Find a solution to the equation

$$\sin y \cdot u_x + e^x \cdot u_y = 2x \sin y \cdot u^2.$$

Solution:

Obviously, the given equation is quasi-linear. To construct a general solution, we find two independent first integrals of the system of equations:

$$\frac{dx}{\sin y} = \frac{dy}{e^x} = \frac{du}{2x \sin y \cdot u^2}.$$

Considering two equations

$$\frac{dx}{\sin y} = \frac{dy}{e^x} \quad \text{and} \quad \frac{dx}{\sin y} = \frac{du}{2x \sin y \cdot u^2},$$

we obtain

$$\psi_1(x,y,u) = e^x + \cos y, \qquad \psi_2(x,y,u) = x^2 + \frac{1}{u}.$$

Then the general integral of the given equation has the form:

$$\Phi\left(e^x + \cos y, x^2 + \frac{1}{u}\right) = 0.$$

Solving this equation with respect to the second argument, we obtain

$$x^2 + \frac{1}{u} = F(e^x + \cos y) \quad \Rightarrow \quad u = \left(F(e^x + \cos y) - x^2\right)^{-1}.$$

where F is an arbitrary continuously differentiable function.

Example 5

Find a solution to the equation

$$(2y - u)\, u'_x + y\, u'_y = u.$$

Solution:

Let's make up a characteristic system

$$\frac{dx}{2y-u} = \frac{dy}{y} = \frac{du}{u}.$$

Solving the equation

$$\frac{dy}{y} = \frac{du}{u} \quad \Rightarrow \quad \frac{u}{y} = C_1,$$

we find the first integral

$$\psi_1(x,y,u) = \frac{u}{y}.$$

Using the rule of equal fractions, we will make an integrable combination

$$\frac{dx}{2y-u} = \frac{2dy - du}{2y-u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

Where do we get another first integral

$$\psi_2(x, y, u) = x - 2y + u.$$

Therefore, the general integral of the given equation has the form

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0,$$

where Φ is an arbitrary continuously differentiable function.