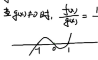
3、函数 $f(x)=(x^2-x-2)|x^3-x|$ 的不可导的点的个数为（ ）

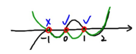
A、0； B、1； C、2； D、3 .

$g(x)=(x^2-x-2)(x^3-x)=(x-2)(x+1)\cdot x(x-1)(x+1)=(x-2)(x+1)^2x(x-1)$

当 $g(x)\neq 0$ 时，$\frac{f(x)}{g(x)}=\frac{|x^3-x|}{x^3-x}$

$x(x-1)(x+1)$





$g(x)=(x-2)(x+1)^2x(x-1)$

$$\lim_{x\to -1}\frac{g(x)-g(-1)}{x+1}=\lim_{x\to -1}(x-2)(x+1)x(x-1)=0$$

$$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}=\lim_{x\to 0}(x-2)(x+1)^2(x-1)\neq 0$$

$$\lim_{x\to 1}\frac{g(x)-g(1)}{x-1}=\lim_{x\to 1}(x-2)(x+1)^2x\neq 0$$

$f'(x)(1+x^2)-2xf(x)=0$

10. 设 $f(x)$ 在 $(-\infty,+\infty)$ 上有界且可导，证明方程 $f'(x)(1+x^2)=2xf(x)$ 至少有一个实根.

证：设 $F(x)=\frac{f(x)}{1+x^2}$. 因为 $f(x)$ 在 $(-\infty,+\infty)$ 上有界且

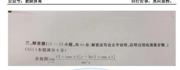
$\lim_{x\to\infty}\frac{1}{1+x^2}=0$，所以 $\lim_{x\to\infty}F(x)=\lim_{x\to\infty}\frac{f(x)}{1+x^2}=0$.

(i) 若 $f(x)\equiv 0\ (\forall x\in(-\infty,+\infty))$，则 $F(x)\equiv 0$，$F'(x)\equiv 0$.

(ii) 若 $\exists x_0\in(-\infty,+\infty)$ 使得 $f(x_0)\neq 0$，则 $F(x_0)\neq 0$. 因为 $\lim_{x\to-\infty}F(x)=\lim_{x\to+\infty}F(x)=0$ 且 $F(x)$ 在 $(-\infty,+\infty)$ 上连续，所以 $\exists x_1\in(-\infty,x_0)$, $x_2\in(x_0,+\infty)$ 使得 $F(x_1)=F(x_2)=\frac{1}{2}F(x_0)$. 由罗尔中值定理可知，$\exists\xi\in(x_1,x_2)\subset(-\infty,+\infty)$ 使得 $F'(\xi)=0$.

综上，无论何种情形下，总是 $\exists\xi\in(-\infty,+\infty)$ 使得 $F'(\xi)=0$，即 $\frac{f'(\xi)(1+\xi^2)-f(\xi)\cdot 2\xi}{(1+\xi^2)^2}=0$，

$f'(\xi)(1+\xi^2)=2\xi f(\xi)$，得证.



三、解答题（15～23 小题，共 94 分. 解答应写出文字说明、证明过程或演算步骤.）

（15）（本题满分 9 分）

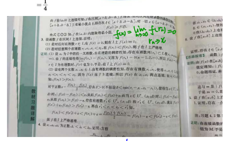
求极限 $\lim_{x\to 0}\frac{(1-\cos x)[x-\ln(1+\tan x)]}{\sin^4 x}$.

解析：

$$\lim_{x\to 0}\frac{(1-\cos x)[x-\ln(1+\tan x)]}{\sin^4 x}$$
$$=\lim_{x\to 0}\frac{\frac{1}{2}x^2[x-\ln(1+\tan x)]}{x^4}$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{x-\ln(1+\tan x)}{x^2}$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{x-\tan x+\tan x-\ln(1+\tan x)}{x^2}$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{x-\tan x}{x^2}+\frac{1}{2}\lim_{x\to 0}\frac{\tan x-\ln(1+\tan x)}{x^2}$$
$$=\frac{1}{2}\lim_{x\to 0}\frac{1-\sec^2 x}{2x}+\frac{1}{2}\lim_{x\to 0}\frac{\frac{1}{2}\tan^2 x}{x^2}$$
$$=-\frac{1}{4}\lim_{x\to 0}\frac{\tan^2 x}{x}+\frac{1}{4}$$
$$=\left(-\frac{1}{4}\right)\cdot 0+\frac{1}{4}$$
$$=\frac{1}{4}$$

$\ln(1+x)=x-\frac{1}{2}x^2+o(x^2)$

$x-\ln(1+x)\sim\frac{1}{2}x^2,\ x\to 0$



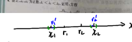
3. 设函数 f 在区间 I 上连续，证明：

(1) 若对任何有理数 $r\in I$，有 $f(r)=0$，则在 I 上 $f(x)\equiv 0$；

(2) 若对任意两个有理数 r_1,r_2，$r_1<r_2$，有 $f(r_1)<f(r_2)$，则 f 在 I 上严格增.

$f(x)=\lim_{n\to\infty}f(r_n)=0$

$r_n\to x$



$|f(x_1)-f(r_1')|<\varepsilon$

$-\varepsilon<f(x_1)-f(r_1')<\varepsilon$

$f(x_1)<f(r_1')+\varepsilon<f(r_1)+\varepsilon=f(r_1)+\frac{f(r_2)-f(r_1)}{2}$

$=\frac{f(r_1)+f(r_2)}{2}$

7 设 f 在 $[a,b]$ 上连续，且对任何 $x\in[a,b]$，存在 $y\in[a,b]$，使得

$$|f(y)|\leqslant\frac{1}{2}|f(x)|.$$

证明：存在 $\xi\in[a,b]$，使得 $f(\xi)=0$.

提示：函数 $|f|$ 在 $[a,b]$ 上有最小值 $m=f(\xi)$，若 $m=0$，则已得证；若 $m>0$，

问题：$\lim$ $f(a)=0$?

$x_0,x_1,x_2,\cdots\in[a,b]$

致密性定理：有界数列必有收敛的子列

$\lim_{k\to\infty}x_{n_k}=x$, $f(x)=0$?

$|f(x_{n_k})|\leqslant\frac{1}{2^{n_k}}|f(x_0)|$

$0\leqslant\lim_{k\to\infty}|f(x_{n_k})|\leqslant\lim_{k\to\infty}\frac{|f(x_0)|}{2^{n_k}}=0$

$\Rightarrow\lim_{k\to\infty}f(x_{n_k})=0$，即 $f(x)=0$.

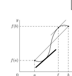
Theorem 1 (Lagrange's finite-increment theorem) *If a function $f:[a,b]\to\mathbb{R}$ is continuous on a closed interval $[a,b]$ and differentiable on the open interval $]a,b[$, there exists a point $\xi\in]a,b[$ such that*

$$\boxed{f(b)-f(a)=f'(\xi)(b-a).}$$

$v(\xi)=\frac{s(t_2)-s(t_1)}{t_2-t_1}$ 平均速率

$\Leftrightarrow f'(\xi)=\frac{f(b)-f(a)}{b-a}$

切线的斜率 割线的斜率



Proof: (method A) $f(b)-f(a)=f'(\xi)(b-a)$

$\Leftrightarrow f'(\xi)-\frac{f(b)-f(a)}{b-a}=0.$

Let $F(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$.

$F(a)=f(a)-\frac{f(b)-f(a)}{b-a}a=\frac{f(a)b-f(a)a-f(b)a+f(a)a}{b-a}$

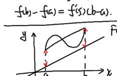
$=\frac{f(a)b-f(b)a}{b-a}$

$F(b)=f(b)-\frac{f(b)-f(a)}{b-a}b=\frac{f(b)b-f(b)a-f(b)b+f(a)b}{b-a}$

$=\frac{f(a)b-f(b)a}{b-a}$

$F(a)=F(b)$. By Rolle's Theorem, there exists $\xi\in(a,b)$ such that $F'(\xi)=0$, i.e., $f'(\xi)-\frac{f(b)-f(a)}{b-a}=0$,

$f(b)-f(a)=f'(\xi)(b-a)$.



$F(x)=f(x)-\frac{f(b)-f(a)}{b-a}x$

割线方程：$y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a)$

$y=f(a)+\frac{f(b)-f(a)}{b-a}(x-a)$

$F(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a)$,

(method B) $F(a)=f(a)-f(a)=0$,

$F(b)=f(b)-f(a)-f(b)+f(a)=0$,

- - -

Corollary 1 (Criterion for monotonicity of a function) *If the derivative of a function is nonnegative (resp. positive) at every point of an open interval, then the function is nondecreasing (resp. increasing) on that interval.*

Proof: $\forall x_1,x_2\in I$ such that $x_1<x_2$, then by Lagrange's Theorem, $f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)$ where $\xi\in(x_1,x_2)$. Since $f'(x)>0\ (x\in I)$, $f(x_2)-f(x_1)>0$, i.e., $f(x_2)>f(x_1)$. $f(x)$ is increasing on I.

Corollary 2 (Criterion for a function to be constant) *A function that is continuous on a closed interval* $[a, b]$ *is constant on it if and only if its derivative equals zero at every point of the interval* $[a, b]$ *(or only the open interval* $]a, b[$*).*

Proof: "$\Rightarrow$" Obviously.

"$\Leftarrow$" $\forall x_1, x_2 \in [a,b]$, By Lagrange's Theorem, there exists ξ between x_1, x_2 such that

$f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2)$. Since $f'(x) \equiv 0$ in (a,b),

it follows that $f(x_1) - f(x_2) = 0 \cdot (x_1 - x_2) = 0$, i.e, $f(x_1) = f(x_2)$.

Hence, $f(x) \equiv C$, $\forall x \in [a,b]$.

Proposition 2 (Cauchy's finite-increment theorem) *Let* $x = x(t)$ *and* $y = y(t)$ *be functions that are continuous on a closed interval* $[\alpha, \beta]$ *and differentiable on the open interval* $]\alpha, \beta[$.

Then there exists a point $\tau \in]\alpha, \beta[$ *such that*

$$x'(\tau)\big(y(\beta) - y(\alpha)\big) = y'(\tau)\big(x(\beta) - x(\alpha)\big).$$

If in addition $x'(t) \neq 0$ *for each* $t \in]\alpha, \beta[$, *then* $x(\alpha) \neq x(\beta)$ *and we have the equality*

$$\frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} = \frac{y'(\tau)}{x'(\tau)}. \tag{5.48}$$

Proof: $x'(\tau)(y(\beta) - y(\alpha)) = y'(\tau)(x(\beta) - x(\alpha))$

$\Leftrightarrow x'(\tau)(y(\beta) - y(\alpha)) - y'(\tau)(x(\beta) - x(\alpha)) = 0.$

Let $F(t) = x(t)[y(\beta) - y(\alpha)] - y(t)[x(\beta) - x(\alpha)]$

$F(\alpha) = x(\alpha)[y(\beta) - y(\alpha)] - y(\alpha)[x(\beta) - x(\alpha)]$

$= x(\alpha)y(\beta) - \cancel{x(\alpha)y(\alpha)} - x(\beta)y(\alpha) + \cancel{x(\alpha)y(\alpha)}$

$= x(\alpha)y(\beta) - x(\beta)y(\alpha).$

$F(\beta) = x(\beta)[y(\beta) - y(\alpha)] - y(\beta)[x(\beta) - x(\alpha)]$

$= \cancel{x(\beta)y(\beta)} - x(\beta)y(\alpha) - \cancel{x(\beta)y(\beta)} + x(\alpha)y(\beta)$

$= x(\alpha)y(\beta) - x(\beta)y(\alpha)$

$F(\alpha) = F(\beta)$. By Rolle's Theorem, there exists $\tau \in (\alpha, \beta)$ such that $F'(\tau) = 0$, ...

$\underbrace{\cancel{f(b) - f(a)}}_{0} = f'(\xi)(b-a) \Rightarrow f'(\xi) = 0$

$\boxed{\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(\xi)}{g'(\xi)}}$ $\qquad \dfrac{f(b) - f(a)}{b - a} = \dfrac{f'(\xi)}{1}$

$g(x) = x$ $\qquad f(b) - f(a) = f'(\xi)(b-a)$

$$\frac{y(\beta) - y(\alpha)}{x(\beta) - x(\alpha)} = \frac{y'(\tau)}{x'(\tau)}.$$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t \in I$, $y = y(x)$