Chapter 15. *ill*-conditioned systems and matrices

If small changes in coefficients of the system of linear algebraic equations produce large changes in the solution, then the system is said to be ill-conditioned (Chapter 03-2). This can usually be expected when determinant $|A|$ of matrix A is small.

The quantity

$$C = \|A\| \cdot \|A^{-1}\| \quad ,$$

where $\|A\|$ is any matrix norm, gives a measure of the condition of the matrix. It is called the condition number. Large condition number indicates that the matrix is ill-conditioned. Otherwise, it is well-conditioned.

$$A=\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \end{bmatrix}$$

By definition, ***ill*-conditioned systems admit large changes in the solution small under small changes in coefficients**

$$\left.\begin{aligned} 2x + y &= 2 \\ 2x + 1.01y &= 2.01 \end{aligned}\right\}$$

has the solution

$$x = 0.5 \quad \text{and} \quad y = 1.$$

But the system

$$\left.\begin{aligned} 2x + y &= 2 \\ 2.01x + y &= 2.05 \end{aligned}\right\}$$

has the solution $x = 5$ and $y = -8$.

Also, *ill*-conditioned systems are highly sensitive to round-off errors

Example

$$A=\begin{pmatrix} 0.913 & 0.659 \\ 0.780 & 0.563 \end{pmatrix} \quad b=\begin{pmatrix} 0.254 \\ 0.217 \end{pmatrix}$$

$$\begin{cases} 0.913\,x_1 + 0.659\,x_2 = 0.254 \\ 0.780\,x_1 + 0.563\,x_2 = 0.217 \end{cases}$$

$-0.780/0.913 = -0.8543264\ldots$ **≈ -0.85433**

$-0.659 * 0.85433 + 0.563 = -0.0000035\ldots$

$-0.254 * 0.85433 + 0.217 = 0.0000002\ldots$

$$\begin{cases} 0.913\,x_1 + 0.659\,x_2 = 0.254 \\ -0.0000035\,x_2 = 0.0000002 \end{cases} \rightarrow \quad x_2 = -0.57143 \quad x_1 = 0.69066$$

det(A)= $\quad$ c= $\qquad$ True solution: $x_2 = -1 \quad x_1 = 1$

Example of using the pivoting to avoid sensitivity to round-off errors

Apply Gaussian elimination to the system

$$\begin{aligned} E_1: &\quad 0.003000x_1 + 59.14x_2 = 59.17 \\ E_2: &\quad 5.291x_1 - 6.130x_2 = 46.78, \end{aligned}$$

using four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

Solution The first pivot element, $a_{11}^{(1)} = 0.003000$, is small, and its associated multiplier,

$$m_{21} = \frac{5.291}{0.003000} = 1763.\overline{6},$$

rounds to the large number 1764. Performing $(E_2 - m_{21}E_1) \rightarrow (E_2)$ and the appropriate rounding gives the system

$$\begin{aligned} 0.003000x_1 + 59.14x_2 &\approx 59.17 \\ -104300x_2 &\approx -104400, \end{aligned}$$

$$x_2 \approx 1.001,$$

which is a close approximation to the actual value, $x_2 = 1.000$. However, because of the small pivot $a_{11} = 0.003000$,

$$x_1 \approx \frac{59.17 - (59.14)(1.001)}{0.003000} = -10.00$$

This contradicts the exact value $x_1 = 10.00$.

Now let us use pivoting, i.e., interchange of rows:

$$5.291x_1 - 6.130x_2 = 46.78,$$
$$0.003000x_1 + 59.14x_2 = 59.17.$$

The multiplier for this system is

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = 0.0005670,$$

and the operation $(E_2 - m_{21}E_1) \rightarrow (E_2)$ reduces the system to

$$\begin{aligned} 5.291x_1 - 6.130x_2 &\approx 46.78, \\ 59.14x_2 &\approx 59.14. \end{aligned}$$

The four-digit answers resulting from the backward substitution are the correct values $x_1 = 10.00$ and $x_2 = 1.000$. ■

Determinant of a matrix:

- The determinant of a 2×2 matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

- The determinant of a 3×3 matrix:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

If A is an $n \times n$ matrix, then the **determinant** of A is obtained by multiplying each element of the first row by its cofactor and then adding the results. In symbols,

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11}\boldsymbol{C}_{11} + a_{12}\boldsymbol{C}_{12} + \cdots + a_{1n}\boldsymbol{C}_{1n}$$

What is a cofactor?

C_{ij}

row column

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

Find: $C_{11} = ?$

$C_{21} = ?$

$$A = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & 8 \\ -3 & 2 & 1 \end{vmatrix} \qquad M_{11} = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & 8 \\ -3 & 2 & 1 \end{vmatrix} \qquad M_{21} = \begin{vmatrix} 1 & -1 & -1 \\ 2 & 3 & 8 \\ -3 & 2 & 1 \end{vmatrix}$$

What is Minor?

A minor M_{ij} of a matrix element is the <u>determinant</u> of the matrix obtained by eliminating the row and column of the matrix to which that element belongs.

Finding the inverse of matrix $\mathbf{A}$:

$$\mathbf{A}^{-1} = \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}$$

the transpose of the matrix of cofactors

$$= \frac{1}{\det(A)} \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^{T}$$

Inverse of a 2×2 Matrix

If $ad - bc \neq 0$, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Calculation of the inverse of 3x3 matrix using cofactors

$$A=\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

$C_{11}=6$ $C_{12}=-24$ $C_{13}=10$ $C_{21}=5$ $C_{22}=17$ $C_{23}=-7$ $C_{31}=1$ $C_{32}=-3$ $C_{33}=1$

$$\begin{pmatrix} 6 & -24 & 10 \\ -5 & 17 & -7 \\ 1 & -3 & 1 \end{pmatrix}$$

matrix of cofactors

$$\begin{pmatrix} 6 & -5 & 1 \\ -24 & 17 & -3 \\ 10 & -7 & 1 \end{pmatrix}$$

transverse

determinant = -2

A^{-1} =

Calculation of the inverse of matrix with the Gauss Method

X will be the inverse of A if

$$AX = I, \tag{7.34}$$

where I is the unit matrix of the same order as A. It is required to determine the elements of X such that Eq. (7.34) is satisfied. For example, for third-order matrices, Eq. (7.34) may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This equation is equivalent to three equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can therefore apply the Gaussian elimination method to each of these systems and the result in each case will be the *corresponding* column of A^{-1}.

Example

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 + x_2 + x_3 = 1$$

$$3x_1 + 2x_2 + 3x_3 = 0$$

$$x_1 + 4x_2 + 9x_3 = 0$$

$$x_2 + 3x_3 = -3$$

$$7x_2 + 17x_3 = -1$$

$$-4x_3 = 20$$

$x_3 = -5$ $\quad$ $x_2 = 12$ $\quad$ $x_1 = -3$

$$A^{-1} = \begin{pmatrix} -3 & & \\ 12 & & \\ -5 & & \end{pmatrix}$$

$$2x_1 + x_2 + x_3 = 0$$

$$3x_1 + 2x_2 + 3x_3 = 1$$

$$x_1 + 4x_2 + 9x_3 = 0$$

$$x_2 + 3x_3 = 1$$

$$7x_2 + 17x_3 = 0$$

$$x_3 = 7/2 \qquad x_2 = -17/2 \qquad x_1 = 5/2$$

$$A^{-1} = \begin{pmatrix} -3 & 5/2 & \\ 12 & -17/2 & \\ -5 & 7/2 & \end{pmatrix}$$

$$x_2 + 3x_3 = 0$$

$$7x_2 + 17x_3 = 1$$

$x_3 = -1/2$ $\quad$ $x_2 = 3/2$ $\quad$ $x_1 = -1/2$

$$A^{-1} = \begin{pmatrix} -3 & 5/2 & -1/2 \\ 12 & -17/2 & 3/2 \\ -5 & 7/2 & -1/2 \end{pmatrix}$$

For ill-conditioned systems, the accuracy of an approximate solution can be improved by an iterative procedure. Consider the system

$$2x+y = 2$$
$$2x+1.01y=2.01$$

Let an approximate solution of the given system be given by

$$x^{(1)} = 1 \quad \text{and} \quad y^{(1)} = 1.$$

Substituting these values in the given system, we obtain

$$\left.\begin{aligned} 2x^{(1)} + y^{(1)} &= 3 \\ \text{and}\ \ 2x^{(1)} + 1.01y^{(1)} &= 3.01 \end{aligned}\right\} \tag{i}$$

Subtracting each equation of (i) from the corresponding equation of the given system, we get

$$2(x - x^{(1)}) + (y - y^{(1)}) = -1$$

and

$$2(x - x^{(1)}) + 1.01(y - y^{(1)}) = -1.$$

Solving the above system of equations, we obtain

$$x - x^{(1)} = -\frac{1}{2} \quad \text{and} \quad y - y^{(1)} = 0.$$

Hence

$$x = \frac{1}{2} \quad \text{and} \quad y = 1,$$

which is the exact solution of the given system.

For a matrix 3 x 3:

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_2 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \tag{7.55}$$

Let $x_1^{(1)}$, $x_2^{(1)}$ and $x_3^{(1)}$ be an approximate solution. Substituting these values in the left side of Eq. (7.55), we get new values of b_1, b_2 and b_3. Let these new values be $b_1^{(1)}$, $b_2^{(1)}$ and $b_3^{(1)}$. The new system of equations is given by

$$\left.\begin{aligned} a_{11}x_1^{(1)} + a_{12}x_2^{(1)} + a_{13}x_3^{(1)} &= b_1^{(1)} \\ a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + a_{23}x_3^{(1)} &= b_2^{(1)} \\ a_{31}x_1^{(1)} + a_{32}x_2^{(1)} + a_{33}x_3^{(1)} &= b_3^{(1)} \end{aligned}\right\} \tag{7.56}$$

Subtracting each equation given in Eq. (7.56) from the corresponding equation given in Eq. (7.55), we obtain

$$\left.\begin{aligned} a_{11}e_1 + a_{12}e_2 + a_{13}e_3 = d_1 \\ a_{21}e_1 + a_{22}e_2 + a_{23}e_3 = d_2 \\ a_{31}e_1 + a_{32}e_2 + a_{33}e_3 = d_3 \end{aligned}\right\} \tag{7.57}$$

where $e_i = x_i - x_i^{(1)}$ and $d_i = b_i - b_i^{(1)}$. We now solve the system (7.57) for e_1, e_2 and e_3. Since $e_i = x_i - x_i^{(1)}$, we obtain

$$x_i = x_i^{(1)} + e_i, \tag{7.58}$$

which is a better approximation for x_i.

The procedure can be repeated.

When dealing with ill-conditioned systems of equations, it's essential to employ techniques for refining the solution.

1. **Iterative Methods**: Instead of solving the system directly, consider using iterative methods such as the Gauss-Seidel method or the Jacobi method. These methods can often converge to a solution even for ill-conditioned systems.
2. **Preconditioning**: Preconditioning involves transforming the original system into an equivalent system that has better conditioning properties. This can be done through various techniques such as scaling, reordering equations, or using specialized preconditioning matrices.
3. **Regularization**: Regularization techniques add a regularization term to the system of equations to stabilize the solution. This can help mitigate the effects of ill-conditioning and improve the numerical stability of the solution.
4. **Refinement**: After obtaining an initial solution, perform iterative refinement techniques such as iterative improvement or backward error analysis. These techniques involve computing residuals and using them to improve the solution iteratively.
5. **Perturbation Analysis**: Analyze the sensitivity of the solution to perturbations in the input data. This can help identify which parts of the input are most influential in determining the solution and guide the refinement process.
6. **Higher Precision Arithmetic**: Sometimes, using higher precision arithmetic (e.g., using double precision instead of single precision) can help mitigate the effects of numerical errors, especially when dealing with ill-conditioned systems.
7. **Solver Selection**: Choose appropriate numerical solvers that are specifically designed to handle ill-conditioned systems. Some solvers are better suited for such systems than others.
8. **Model Improvement**: If possible, consider improving the mathematical model itself to reduce ill-conditioning. This might involve simplifying the equations, removing redundant variables, or incorporating additional constraints.
9. **Data Cleaning**: Ensure that the input data is clean and free from errors or inconsistencies that could exacerbate ill-conditioning.
10. **Dimensionality Reduction**: If the problem permits, consider reducing the dimensionality of the system by using techniques such as model reduction or sparse matrix representations. This can help improve computational efficiency and numerical stability.

By employing these techniques, you can effectively deal with ill-conditioned systems of equations and obtain more accurate and reliable solutions.