Equations of Mathematical Physics

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CONTENTS

LITERATURE

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Chapter 1

LINEAR DIFFERENTIAL OPERATORS

1.1. Definition and examples

We introduce notations that are convenient when using functions of several variables and differential operators.

A multi-index α is a set of $\alpha = (\alpha_1, ..., \alpha_n)$, where $\alpha_j \in \mathbb{Z}_+$ (that is, α_j are non-negative integers).

If α is a multi-index, then we will put $|\alpha| = \alpha_1 + ... + \alpha_n$, $\alpha! = \alpha_1!..\alpha_n!$, and if vector $x = (x_1, ..., x_n) \in \mathbb{C}^n$ is given, then $x^\alpha = x_1^{\alpha_1}, ..., x_n^{\alpha_n}$.

If Ω is an open subset in $\mathbb{R}^n$, then by $C^\infty(\Omega)$ we denote **the set of complex-valued infinitely differentiable functions** in Ω, and by $C_0^\infty(\Omega)$ we denote **the set of finite infinitely differentiable functions** in Ω, that is, such $\varphi \in C^\infty(\Omega)$ that there exists a compact $K \subset \Omega$, outside of which the function φ turns to 0 (compact K depends on the function φ).

Let's say

$$\partial_j = \frac{\partial}{\partial x_j} : C^\infty(\Omega) \to C^\infty(\Omega); \quad D_j = -i\partial_j,$$

where $i = \sqrt{-1}$; $\partial = (\partial_1, \dots, \partial_n)$; $D = (D_1, \dots, D_n)$; $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$; $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$.

Thus,

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

is a mixed derivative operator, $\partial^\alpha : C^\infty(\Omega) \to C^\infty(\Omega)$, $D^\alpha = i^{-|\alpha|}\partial^\alpha$.

If $f \in C^\infty(\Omega)$, we instead $\partial^\alpha f$ will also sometimes write $f^{(\alpha)}$.

Linear differential operator is an operator

$$A: C^\infty(\Omega) \to C^\infty(\Omega)$$

$$A = \sum_{|\alpha| \leqslant m} a_\alpha(x) D^\alpha, \tag{1.2}$$

where $a_\alpha(x) \in C^\infty(\Omega)$.

Instead D^α we can write ∂^α, but written through D^α more convenient, as will be seen from further.

Here $m \in \mathbb{Z}_+$ and we will say that A - operator order $\le m$.

We will say that A operator order m, if it is written in the form (1.2) and there is a multi-index α, which $|\alpha| = m$ and $a_\alpha(x) \neq 0$.

Examples.

1. Laplace operator

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} = -(D_1^2 + \ldots + D_n^2).$$

2. Thermal conductivity operator

$$\frac{\partial}{\partial t} - \Delta$$

(here the number of variables is equal to $n+1$ and they are denoted by $t, x_1, \ldots, x_n$).

3. Wave operator or d'Alambertian

$$\Box = \frac{\partial^2}{\partial t^2} - \Delta.$$

4. Operator of Sturm-Liouville, determined by formula

$$Lu = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u,$$

where $p,q \in C^{\infty}((a,b))$, $n=1$, $\Omega=(a,b)$.

Examples operators 1)-3) – are operators with constant coefficients, the operator of Sturm-Liouville has variable coefficients.

1.2. Full and main symbols

The symbol or *the full symbol* of the operator A of the order m is called a function of the form

$$a(x, \xi) = \sum_{|\alpha| \leqslant m} a_{\alpha}(x)\xi^{\alpha}, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n, \tag{1.3}$$

and the *main symbol* is a function of the form

$$a_m(x, \xi) = \sum_{|\alpha| = m} a_{\alpha}(x)\xi^{\alpha}, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n. \tag{1.4}$$

The symbol belongs to $C^{\infty}(\Omega)[\xi]$, that is, it is a polynomial of $\xi_1,...,\xi_n$ with coefficients from the ring $C^{\infty}(\Omega)$, and the full symbol is a homogeneous polynomial of $\xi_1,...,\xi_n$ degree m with coefficients from $C^{\infty}(\Omega)$.

Examples.

1. The Laplace operator has matching full and main symbols

$\xi^2 = \xi_1^2 + \ldots + \xi_n^2$.

2. The thermal conductivity operator $\frac{\partial}{\partial t} - \Delta$ has a full symbol $a(t, x, \tau, \xi) = i\tau + \xi^2$, and its main symbol is $a_2(t, x, \tau, \xi) = \xi^2$.

3. The wave operator has matching full and main symbols

$$a(t, x, \tau, \xi) = a_2(t, x, \tau, \xi) = -\tau^2 + \xi^2.$$

4. The Sturm-Liouville operator has the full symbol $a(x, \xi) = -p(x)\xi^2 + ip'(x)\xi + q(x)$, and the main symbol $a_2(x, \xi) = -p(x)\xi^2.$.

The symbol of the operator A is restored according to the operator by the formula

$$a(x, \xi) = e^{-ix\cdot\xi} A(e^{ix\cdot\xi}), \tag{1.5}$$

in which the operator A is applied according to x.

The notation $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$ is used here.

The formula (1.5) is obtained from the ratio

$$D^\alpha e^{ix\cdot\xi} = \xi^\alpha e^{ix\cdot\xi},$$

which is easily verified by induction according to $|\alpha|$. It also follows from (1.5) that the coefficients $a_\alpha(x)$ of the operator A are uniquely determined.

It is useful to apply the operator A to a more general exponent than $e^{ix\xi}$ to the exponent $e^{i\lambda\varphi(x)}$, where $\varphi \in C^\infty(\Omega)$, λ is a parameter.

Lemma 1.1. If $f \in C^\infty(\Omega)$, then $e^{-i\lambda\varphi(x)} A(f(x)e^{i\lambda\varphi(x)})$ is a polynomial of λ degree $\leq m$ with coefficients from $C^\infty(\Omega)$, and

$$e^{-i\lambda\varphi(x)} A\big(f(x)e^{i\lambda\varphi(x)}\big) = \lambda^m f(x) a_m(x, \varphi_x) + \lambda^{m-1}(\ldots) + \ldots, \tag{1.6}$$

that is, the highest coefficient of this polynomial (for λ^m) is $f(x)a_m(x,\varphi_x)$, where $\varphi_x = (\frac{\partial\varphi}{\partial x_1},...,\frac{\partial\varphi}{\partial x_n}) = grad\,\varphi$ is the gradient vector φ.

Proof: We have

$$D_j\left[f(x)\,e^{i\lambda\varphi(x)}\right] = \lambda(\partial_j\varphi)(x)\,e^{i\lambda\varphi(x)} + (D_jf)\,e^{i\lambda\varphi(x)},$$

whence the statement of the lemma is obtained for operators of order less than 1 (≤ 1). In the future, when finding arbitrary $D^\alpha e^{i\lambda\varphi(x)}$, it will also be necessary to differentiate products of the form $f(x)e^{i\lambda\varphi(x)}$, where $f \in C^\infty(\Omega)$. In this case, a new multiplier λ appears only when the exponent is differentiated. Therefore, it is clear that

$$D^\alpha e^{i\lambda\varphi(x)} = \lambda^{|\alpha|}\varphi_x^\alpha e^{i\lambda\varphi(x)} + \lambda^{|\alpha|-1}(\dots) + \dots,$$

whence the statement of the Lemma follows.

Corollary 1.2. Let A,B be two linear differential operators in Ω, k,l be their orders, and a_k, b_l are the main symbols. Next, let $C = A \circ B$ be the composition of these operators, and c_{k+l} be its main symbol. Then

$$c_{k+l}(x,\xi) = a_k(x,\xi)\,b_l(x,\xi). \tag{1.7}$$

Comment. Obviously, C is a differential operator of the order $\leq k+l$.

Proof of corollary 1.2:

$$\begin{aligned} C\left(e^{i\lambda\varphi(x)}\right) &= A\left(\lambda^l b_l(x,\varphi_x)e^{i\lambda\varphi(x)} + \lambda^{l-1}(\dots) + \dots\right) = \\ &= \lambda^{k+l}a_k(x,\varphi_x)\,b_l(x,\varphi_x)\,e^{i\lambda\varphi(x)} + \lambda^{k+l-1}(\dots) + \dots \end{aligned}$$

Hence, according to Lemma 1.1. it follows

$$c_{k+l}(x, \varphi_x) = a_k(x, \varphi_x)\, b_l(x, \varphi_x) \tag{1.8}$$

for any function $\varphi \in C^\infty(\Omega)$.

But then, choosing $\varphi(x) = x \cdot \xi$, we get $\varphi_x = \xi$ and (1.8) goes to

$$c_{k+l}(x, \xi) = a_k(x, \xi)\, b_l(x, \xi),$$

which was to be proved.

1.3. Change of variables

Let's give a diffeomorphism $\varkappa \colon \Omega \to \Omega_1$, where Ω, Ω_1 are regions in $\mathbb{R}^n$. Such a diffeomorphism can be defined by a set of functions $y_1(x_1,...,x_n),...,y_n(x_1,...,x_n),\ y_j \in C^\infty(\Omega)$, equal to the coordinates of point x at point $\varkappa(x)$.

If $f \in C^\infty(\Omega_1)$, then we will put $\varkappa^* f = f \circ \varkappa$,

that is, $(\varkappa^* f)(x) = f(\varkappa(x))$ or $(\varkappa^* f)(x) = f(y_1(x), \dots, y_n(x))$.

Essentially, $\varkappa^* f$ is obtained from f by replacing variables or moving to coordinates $x_1,...,x_n$ from coordinates $y_1,...,y_n$. The diffeomorphism $\varkappa$ has an inverse map $\varkappa^{-1}$, which is also a diffeomorphism. It is clear that $\varkappa^*$ is a linear isomorphism of $\varkappa^* : C^\infty(\Omega_1) \to C^\infty(\Omega)$, and $(\varkappa^*)^{-1} = (\varkappa^{-1})^*$.

Let's give the operator $A \colon C^\infty(\Omega) \to C^\infty(\Omega)$. Let's define the operator $A_1 : C^\infty(\Omega_1) \to C^\infty(\Omega_1)$ using a commutative diagram

$$\begin{array}{ccc}
C^\infty(\Omega) & \xrightarrow{A} & C^\infty(\Omega) \\
\uparrow{\scriptstyle \varkappa^*} & & \uparrow{\scriptstyle \varkappa^*} \\
C^\infty(\Omega_1) & \xrightarrow{A_1} & C^\infty(\Omega_1)
\end{array}$$

that is, $A_1 = (\varkappa^*)^{-1} A \varkappa^* = (\varkappa^{-1})^* A \varkappa^*$.

In other words,

$$A_1 v(y) = \left\{ A[v(y(x))] \right\} \Big|_{x=x(y)}, \qquad (1.9)$$

where $y(x) = \varkappa(x)$, $x(y) = \varkappa^{-1}(y)$.

Thus, the A_1 operator is essentially just a written A operator in y coordinates.

If A is a linear differential operator, then from (1.9) and the formula for differentiating a complex function it is easily verified that A_1 is also a linear differential operator.

Let's find out how the main symbols of the operators A and A_1 are connected.

Reminder: *cotangent vector* or *covector* at point $x \in \Omega$ is a linear functional on the tangent space Ω to at point x (we will denote the tangent space to Ω at point x by $T_x\Omega$, and the cotangent space - the set of all cotangent vectors at point x - by $T_x^*\Omega$).

By $T^*\Omega$ we denote the union of $\bigcup_x T_x^*\Omega \cong \Omega \times \mathbb{R}^n$.

Choosing the basis of vectors $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ in each tangent space $T_x\Omega$, we construct a dual basis in $T_x^*\Omega$: it consists of such functionals $dx_1, \ldots, dx_n$ that $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol ($\delta_{ij} = 0, i \neq j$ and $\delta_{ij} = 1, i = j$).

An **example** of a tangent vector is the velocity vector of a curve $x(t)$: $\dot{x}(0) \in T_{x(0)}\Omega$.

Its coordinates in the basis $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ are equal to $\frac{dx_1}{dt}\Big|_{t=0}, \dots, \frac{dx_n}{dt}\Big|_{t=0}$.

An **example** of a cotangent vector is the differential or gradient of the function $f \in C^\infty(\Omega)$: this is a functional on $T_x\Omega$, whose value on the tangent vector $\dot{x}(0)$ is equal to $\frac{df(x(t))}{dt}\Big|_{t=0}$; its coordinates in the basis $dx_1, \dots, dx_n$ are equal to $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$.

If the diffeomorphism $\varkappa: \Omega \to \Omega_1$ is given, then there is a natural mapping (the differential of the mapping $\varkappa$):

$$\varkappa_*: T_x\Omega \to T_{\varkappa(x)}\Omega_1$$

and a dual mapping on linear functionals

$${}^t\varkappa_*: T^*_{\varkappa(x)}\Omega_1 \to T^*_x\Omega.$$

Choosing the bases in $T_x\Omega,\ T_{\varkappa(x)}\Omega_1,\ T^*_{\varkappa(x)}\Omega_1,\ T^*_x\Omega$, respectively, $\left\{\frac{\partial}{\partial x_j}\right\}_{j=1}^n, \left\{\frac{\partial}{\partial y_j}\right\}_{j=1}^n, \{dy_j\}_{j=1}^n, \{dx_j\}_{j=1}^n$, we get that $\varkappa_*$ has a matrix called the Jacobian matrix: $(\varkappa_*)_{kl} = \frac{\partial y_k}{\partial x_l}$, and ${}^t\varkappa_*$ has a matrix transposed to the Jacobian matrix $\varkappa_*$.

Note that the maps $\varkappa_*$ and ${}^t\varkappa_*$ are isomorphisms at each point of $x \in \Omega$.

Next, it is convenient to set the cotangent vector at point x by specifying a pair (x,ξ), where $\xi \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_n)$ is the set of coordinates of this vector in the basis $\{dx_j\}_{j=1}^n$.

With isomorphism, $({}^t\varkappa_*)^{-1}: T^*\Omega \to T^*\Omega_1$ the convector (x,ξ) corresponds to the convector (y,η), where $y = \varkappa(x)$, $\eta = ({}^t\varkappa_*)^{-1}\xi$.

Theorem 1.3.

The main symbol a_m of operator A takes the same value on vector (x, ξ) as the main symbol a'_m of operator A_1 on the corresponding vector (y, η), that is,

$$a_m(x, \xi) = a'_m(\varkappa(x), ({}^t\varkappa_*)^{-1}\xi). \tag{1.10}$$

In other words, the main symbol is a well-defined function on $T^*\Omega$ (independent of the choice of curved coordinates in Ω).

Proof:

The cotangent vector at point x can be written as the gradient $\varphi_x(x)$ of the function $\varphi \in C^\infty(\Omega)$ at point x.

From Lemma 1.1. it is clear that $a_m(x,\varphi_x(x))$ does not depend on the choice of coordinates. But on the other side, it is clear that this value does not depend on the choice of the function φ with a given differential at point x. Therefore, the main symbol is correctly assigned to $T^*\Omega$, which was required.

➔1.4. Reduction of operators of the 2nd order with constant coefficients to the canonical form.

➔ **1.5. Characteristics. Ellipticity and hyperbolicity**

➔ **1.6. Characteristics and reduction to the canonical form of operators and equations of the 2nd order at n=2**

➔ **1.7. General solution of a homogeneous hyperbolic equation with constant coefficients at n=2**