

Equations of Mathematical Physics Homework 4

1 September 9th

1.1 Find solutions of differential equation

$$x'' + x' + 6x = 3(\cos 3t - \sin 3t) \quad x(0) = 0, x'(0) = 3$$

Solution:

Denote that $x(t) \longleftrightarrow X(p)$ $f(t) = 3(\cos 3t - \sin 3t) \longleftrightarrow F(p)$, then we have:

$$\begin{aligned} F(p) &= 3\left(\frac{p}{p^2 + 9} - \frac{3}{p^2 + 9}\right) = \frac{3(p - 3)}{p^2 + 9} \\ A(p) &= p^2 + p + 6 \quad B(p) = 3 \\ X(p) &= \frac{F(p) + B(p)}{A(p)} = \frac{3(p - 3) + 3(p^2 + 9)}{(p^2 + 9)(p^2 + p + 6)} = \frac{3}{p^2 + 9} \end{aligned}$$

Thus, by the table of original and image of elementary functions, $x(t) = \sin 3t$.

2 September 11th

2.1 Find solutions of differential system

$$\begin{cases} x'(t) = -x(t) + y(t) + e^t \\ y'(t) = x(t) - y(t) + e^t \end{cases} \quad x(0) = y(0) = 1$$

Solution:

Denote that $x(t) \longleftrightarrow X(p)$ $y(t) \longleftrightarrow Y(p)$, then we have:

$$\begin{aligned} x'(t) &\longleftrightarrow pX(p) - X(0) = pX(p) - 1 \\ y'(t) &\longleftrightarrow pY(p) - Y(0) = pY(p) - 1 \end{aligned}$$

Apply Laplace transform to both sides, we have:

$$\begin{cases} pX(p) - 1 + X(p) - Y(p) = \frac{1}{p - 1} \\ pY(p) - 1 - X(p) + Y(p) = \frac{1}{p - 1} \end{cases}$$

Rewrite these equations, we get:

$$\begin{cases} (p+1)X(p) - Y(p) = \frac{p}{p-1} \\ -X(p) + (p+1)Y(p) = \frac{p}{p-1} \end{cases}$$

Solve this linear system,

$$\Delta = \begin{vmatrix} p+1 & -1 \\ -1 & p+1 \end{vmatrix} = p^2 + 2p$$

$$\Delta_x = \begin{vmatrix} \frac{p}{p-1} & -1 \\ \frac{p}{p-1} & p+1 \end{vmatrix} = \frac{p^2 + 2p}{p-1} \quad \Delta_y = \begin{vmatrix} p+1 & \frac{p}{p-1} \\ -1 & \frac{p}{p-1} \end{vmatrix} = \frac{p^2 + 2p}{p-1}$$

$$X(p) = \frac{1}{p-1} \quad Y(p) = \frac{1}{p-1}$$

Then we restore originals from the images, we get the solutions:

$$x(t) = e^t$$

$$y(t) = e^t$$