Surface integrals

Calculate integrals.

1. $\iint \left(x^2+y^2\right) dS$, where:
   (a) S - sphere $x^2+y^2+z^2=R^2$;
   (b) S - cone surface $\sqrt{x^2+y^2} \leqslant z \leqslant 1$.

2. $\iint_S \left(x^2+y^2+z^2\right) dS$, where:
   (a) $S-$ sphere $x^2+y^2+z^2=R^2$;
   (b) S - cube surface $|x| \leqslant a, |y| \leqslant a, |z| \leqslant a$;
   (c) S - octahedron surface $|x|+|y|+|z| \leqslant a$;
   (d) S - full cylinder surface $x^2+y^2 \leqslant r^2, 0 \leqslant z \leqslant H$.

3. (a) $\iint_S xyz dS$; (b) $\iint_S |xy| z dS$; where S is the part of the paraboloid $z=x^2+y^2$ allocated by the condition $z \leqslant 1$.

4. (a) $\iint_S \left(x^2+y^2\right) dS$; (b) $\iint_S \sqrt{x^2+y^2} dS$; where S is the part of the conic surface $z=\sqrt{x^2+y^2}$ allocated by the condition $z \leqslant 1$.

where $f=\sqrt{\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}}$, S - ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

5. $\iint_S z dS$, $S-$surface $x=u\cos v$, $y=u\sin v$, $z=v$, $u \in [0;1]$, $v \in [0;2\pi]$.

6. $\iint_S (2z-x) dydz+(x+2z) dzdx+3z dxdy$, S is the upper side of the triangle $x+4y+z=4$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

7. (a) $\iint_S xz dxdy$;
   (b) $\iint_S yz dydz+zx dzdx+xy dxdy$;
   S inner side of the tetrahedron surface $x+y+z \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

8. (a) $\iint_S y dzdx$
   (b) $\iint_S x^2 dydz$;
   S is the outer side of the sphere $x^2+y^2+z^2=R^2$.

9. $\iint_S (x-1)^3 dydz$, S is the outer side of the hemisphere $x^2+y^2+z^2=2x$, $z \leqslant 0$.

答案 $-\frac{2}{5}\pi$ (×?) $\frac{2}{5}\pi$

10. $\iint_S yz dxdy+zx dydz+xy dzdx$, S is the outer side of the cylinder part $x^2+y^2=r^2$, $x \leqslant 0$, $y \geqslant 0$, $0 \leqslant z \leqslant H$.

11. $\iint_S x^6 dydz+y^4 dzdx+z^2 dxdy$, S is the underside of a part of an elliptical paraboloid $z=x^2+y^2$, $z \leqslant 1$.

MA HW21.

1. $\iint (x^2+y^2)\,dS$, where:

   (a) S - sphere $x^2+y^2+z^2=R^2$;

   (b) S - cone surface $\sqrt{x^2+y^2} \leqslant z \leqslant 1$.

(a) $\begin{cases} x = R\cos\varphi\cos\psi \\ y = R\sin\varphi\cos\psi \\ z = R\sin\psi. \end{cases}$

$E = (-R\sin\varphi\cos\psi)^2 + (R\cos\varphi\cos\psi)^2 + 0 = R^2\cos^2\psi$

$G = (R\cos\varphi\sin\psi)^2 + (-R\sin\varphi\sin\psi)^2 + (R\cos\psi)^2 = R^2$

$F = 0$

$$\iint (x^2+y^2)\,ds = \iint \cdot R^2\cos^2\psi \cdot R^2\cos\psi\, d\varphi\, d\psi = R^4\int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3\psi\, d\psi = \frac{8\pi R^4}{3}$$

(b).
$$\iint_{\Sigma_1} (x^2+y^2)\cdot\sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\left(\frac{y}{\sqrt{x^2+y^2}}\right)}\,dx\,dy = \sqrt{2}\iint (x^2+y^2)\,dx\,dy = 2\sqrt{2}\pi\int_0^1 r^3\,dr = \frac{\sqrt{2}\pi}{2}$$

$$\iint_{\Sigma_2} (x^2+y^2)\,dS = \iint_{\Sigma_2} (x^2+y^2)\,dx\,dy = 2\pi\int_0^1 r^3\,dr = \frac{\pi}{2}.$$

$$\iint_{\Sigma} (x^2+y^2)\,ds = \frac{(\sqrt{2}+1)\pi}{2}.$$

2. $\iint_S (x^2+y^2+z^2)\,dS$, where:

   (a) S – sphere $x^2+y^2+z^2=R^2$;

   (b) S - cube surface $|x| \leqslant a, |y| \leqslant a, |z| \leqslant a$;

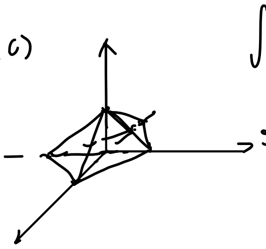
   (c) S - octahedron surface $|x|+|y|+|z| \leqslant a$;

   (d) S - full cylinder surface $x^2+y^2 \leqslant r^2, 0 \leqslant z \leqslant H$.

(a). $$\iint_S R^2\cdot R^2\cos\psi\, d\varphi\, d\psi = 2\pi R^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\psi\, d\psi = 4\pi R^4$$

(b). $$\iint_S (x^2+y^2+z^2)\,ds = 6\iint (a^2+x^2+y^2)\,dx\,dy = 6\int_{-a}^{a} dx\int_{-a}^{a}(a^2+x^2+y^2)\,dy = 40a^4$$

(c)



$$\iint_S (x^2+y^2+z^2)\,dS = 8\iint_{S_1}(x^2+y^2+z^2) = 8\sqrt{3}\iint x^2+y^2+(a-x-y)^2\,dx\,dy$$

$$= 8\sqrt{3}\int_0^a dy\int_0^{a-y} 2(x^2+y^2-ax-ay+xy)+a^2\,dx$$

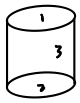
$$= 8\sqrt{3}\int_0^a dy\quad 2\cdot\frac{x^3}{3}+xy^2-\frac{ax^2}{2}-ayx+\frac{yx^2}{2}+a^2x\,\Big|_0^{a-y}$$

$$= 16\sqrt{3}\int_0^a dy\cdot\frac{(a-y)^3}{3}+(a-y)y^2-a\frac{(a-y)^2}{2}-ay(a-y)+y\frac{(a-y)^2}{2}+\frac{1}{2}a^2(a-y)$$

$$= 16\sqrt{3}\int_0^a \cancel{+}\,\frac{5}{3}y^3+\frac{\cancel{1}a}{2}y^2+(\cancel{2}a^2)y+\frac{11a}{6}\;\frac{3}{2} = \frac{a^4}{4}\times 16\sqrt{3} = 4\sqrt{3}a^4$$

$$\left(-\frac{1}{3}-1+\frac{1}{2}\right)y^3+\left(1+1-\frac{1}{2}+1-1\right)ay^2+\left(-1+1-1+\frac{1}{2}-\frac{1}{2}\right)a^2y+\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{2}\right)a^3.$$

(d) $\iint_{\Sigma_1} (R^2+H^2) R\, dR\, d\theta = 2\pi \int_0^r (R^3+RH^2)\, dR = 2\pi\left(\frac{r^4}{4}+\frac{r^2H^2}{2}\right)$ $= -\frac{5}{6}y^3+\frac{3}{2}ay^2-a^2y+\frac{1}{3}a^3$



$\iint_{\Sigma_2} R\cdot R^2\, dR\, d\theta = 2\pi\int_0^r R^3\, dR = \frac{\pi r^4}{2}$ $= -\frac{5}{24}y^4+\frac{a}{2}y^3-\frac{a^2}{2}y^2+\frac{1}{3}a^3y.$

$= \frac{1}{8}\cdot 16\sqrt{3}a^4 = 2\sqrt{3}a^4$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z. \end{cases}$ $E = 1$, $G = 1$, $F = 0$

$\iint_{\Sigma_3} (x^2+y^2+z^2)\, dS = \int_0^{2\pi} d\theta \int_0^H (r^2+z^2)\, r\, dz. = 2\pi H\left(r^3+\frac{H^2}{3}r\right)$

$\iint_S = \iint_{\Sigma_1+\Sigma_2+\Sigma_3} = \pi r^4 + r^2H^2\pi + 2\pi H r^3 + \frac{2\pi H^3}{3} r$

3. (a) $\iint_S xyz dS$; (b) $\iint_S |xy| z dS$; where S is the part of the paraboloid $z = x^2 + y^2$ allocated by the condition $z \leqslant 1$.

(a) $\iint xy\cdot(x^2+y^2)\sqrt{1+4(x^2+y^2)}\, dx\, dy = \int_0^{2\pi} \sin\varphi\cos\varphi\, d\varphi \int_0^1 r^2\cdot r^2\cdot r\sqrt{1+4r^2}\, dr = 0$

(b) $\iint |xy|(x^2+y^2)\sqrt{1+4(x^2+y^2)}\, dx\, dy = 4\int_0^{\frac{\pi}{2}} \sin\varphi\cos\varphi\, d\varphi \int_0^1 r^2\cdot r^2\cdot r\sqrt{1+4r^2}\, dr = 2\int_0^1 r^5\cdot\sqrt{1+4r^2}\, dr.$

$= \frac{1}{6}r^4(1+4r^2)^{\frac{3}{2}} - \frac{1}{30}r^2(1+4r^2)^{\frac{5}{2}} + \frac{1}{420}(1+4r^2)^{\frac{7}{2}}\Big|_0^1.$

$= \frac{5\sqrt{5}}{6} - \frac{5\sqrt{5}}{6} + \frac{25\sqrt{5}}{84} = \frac{25\sqrt{5}}{84} - \frac{1}{420}$

4. (a) $\iint_S (x^2+y^2)\, dS$; (b) $\iint_S \sqrt{x^2+y^2} dS$; where S is the part of the conic surface $z = \sqrt{x^2+y^2}$ allocated by the condition $z \leqslant 1$.

(a) $\iint_S (x^2+y^2)\cdot\sqrt{2}\, dx\, dy = 2\sqrt{2}\pi\int_0^1 r^3\, dr = \frac{\sqrt{2}\pi}{2}$

(b) $\iint_S \sqrt{x^2+y^2}\, dS = 2\sqrt{2}\pi\int_0^1 r^2\, dr = \frac{2\sqrt{2}}{3}\pi$

5. $\iint_S z dS$, S—surface $x = u\cos v$, $y = u\sin v$, $z = v$, $u \in [0; 1]$, $v \in [0; 2\pi]$.

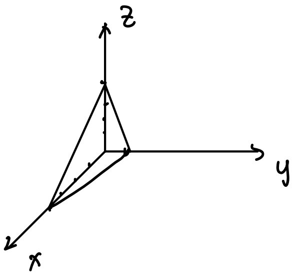
$E = u^2+1$

$G = 1$

$F = (-u\sin v)(\cos v) + (u\cos v)\sin v + 0 = 0.$

$\iint_S z\, dS = \int_0^1\int_0^{2\pi} v\cdot\sqrt{u^2+1}\, du\, dv = \int_0^{2\pi} v\, dv \int_0^1 \sqrt{u^2+1}\, du. = \pi^2(\sqrt{2}+\ln(1+\sqrt{2}))$

6. $\iint_S (2z-x)dydz+(x+2z)dzdx+3zdxdy$, S is the upper side of the triangle $x+4y+z=4$, $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$.



$\vec{n}_0 = (\frac{1}{\cancel{\sqrt6}\,3\sqrt2}, \frac{\cancel{2}\,4}{\cancel{\sqrt6}\,3\sqrt2}, \frac{1}{\cancel{\sqrt6}\,3\sqrt2}).$

$\iint_S \left[\frac{1}{\cancel{2}\,4}(2z-x)+1(x+2z)+\frac{1}{\cancel{2}\,4}\cdot 3z\right]dzdx = \iint_{S_{xz}}\left(\frac{\cancel{19}\,13}{\cancel{2}\,4}z+\frac{3}{\cancel{2}\,4}x\right)dx\,dy$

$=\frac12\int_0^4 dz\int_0^{4-z}(9z+x)dx.$ $\quad \int_0^4 dz\int_0^{4-z}\left(\frac{13}{4}z+\frac34 x\right)dx$

$=\frac12\int_0^4\left[9z(4-z)+\frac{(4-z)^2}{2}\right]dz.$ $\quad =\int_0^4 dz\,\frac{13}{4}zx+\frac38x^2\Big|_0^{4-z}$

$=\frac12\int_0^4\left(-\frac{17}{2}z^2+32z+8\right)dz=\frac12\ -\frac{17}{6}z^3+16z^2+8z\Big|_0^4$

$=\frac{320}{3}$

$=\int_0^4\frac{13}{4}(4-z)z+\frac38(4-z)^2\,dz$

$=\int_0^4-\frac{23}{8}z^2+10z+6\,dz.$

$=-\frac{23}{8\times3}z^3+5z^2+6z\Big|_0^4$

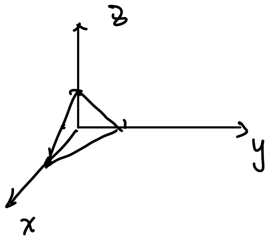
$=-\frac{184}{3}+80+24=\frac{312-184}{3}$

$=\frac{128}{3}$

7. (a) $\iint_S xzdxdy$;

(b) $\iint_S yzdydz+zxdzdx+xydxdy$;

S inner side of the tetrahedron surface $x+y+z\leqslant 1$, $x\geqslant 0$, $y\geqslant 0$, $z\geqslant 0$.



$\vec{n}_0=(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}).$

(a). $\iint_{S_1}xz\,dx\,dy=\iint_S xz\,dz\,dx=\int_0^1x\,dx\int_0^{1-x}z\,dz$

$\int_0^1\frac{(1-x)^2}{2}\cdot x\,dx=\int_0^1\frac{x^3}{2}-x^2+\frac12x\,dx=\frac{x^4}{8}-\frac{x^3}{3}+\frac{x}{\cancel{2}\,4}\Big|_0^1$

$=\frac{7}{24}\ \frac{1}{24}$ ×

$-\int_0^1x\,dx\int_0^{1-x}dy\int_0^{1-x-y}dz.$

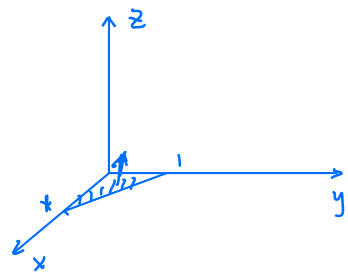
$-\int_0^1\frac{(1-x)^2}{2}x\cdot dx=-\int_0^1\frac{x^3}{2}-x^2+\frac{x}{2}dx$

$=-\frac{x^4}{8}-\frac{x^3}{3}+\frac{x^2}{4}\Big|_0^1$

$=-\frac{3-8+6}{24}=-\frac{1}{24}.$

$\iint_{S_2}xz\,dx\,dy\overset{z=0}{=}0$

$\iint_S=-\iint_{S_1+S_2+S_3+S_4}=-\cancel{\frac{7}{24}}.\ \frac{1}{24}.$



$\iint_S=\iint_{\sigma_{xy}}xy\,dxdy.$

$=\int_0^1x\,dx\int_0^{1-x}y\,dy$

$=\int_0^1\frac{(1-x)^2}{2}x\,dx=\frac{1}{24}$

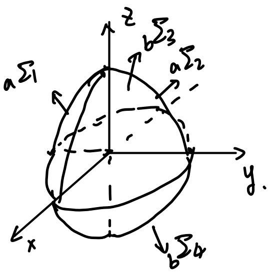
(b) $\iint_{S_1}=3\iint xz\,dz\,dx=\cancel{\frac{7}{8}}.\ \frac18$

$\iint_{S_2}=\cancel{\frac{7}{24}}.\ \frac{1}{24}.$ $\quad \iint_S=\iint_{S_1^-+S_2+S_3+S_4}=0.$ ✓

8. (a) $\iint_S y\,dz\,dx$

(b) $\iint_S x^2 dy\,dz$;

S is the outer side of the sphere $x^2+y^2+z^2=R^2$.



(a) $y=\sqrt{R^2-x^2-z^2}$

$$\iint_S y\,dz\,dx = 2\iint_{S_{xz}} \sqrt{R^2-x^2-z^2}\,dz\,dx = 2\cdot 2\pi\int_0^R \sqrt{R^2-r^2}\cdot r\,dr = 2\pi\int_0^R \sqrt{R^2-r^2}\,d(r^2)$$

$$= -2\pi\cdot\frac{2}{3}\cdot(R^2-r^2)^{\frac{3}{2}}\Big|_0^R = \frac{4\pi R^3}{3}$$

(b). $$\iint_S x^2 dy\,dz = 2\iint_{\Sigma_3} x^2 dy\,dz = \cancel{4}\iint_{S_{xy}}(R^2-y^2-z^2)\,dy\,dz = 2\cancel{4}\pi\int_{-R}^{R}(R^2-r^2)\,r\,dr = 2\cancel{4}\pi\cdot\frac{r^2}{2}R^2-\frac{r^4}{4}\Big|_{-R}^{R}$$

$$= \pi R^4$$

0 ×

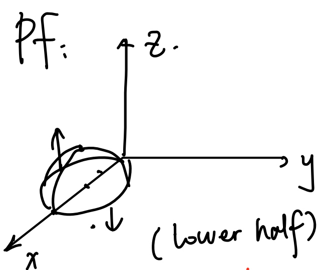
9. $\iint_S (x-1)^3 dy\,dz$, S is the outer side of the hemisphere $x^2+y^2+z^2=2x$, $z\leqslant 0$.

$$\iiint 3\cos^2\varphi\,\cos^3\psi\, r^4$$

$$\iiint_V 3(x-1)^2\,dx\,dy\,dz.$$

$$\frac{3}{5}\int_0^{\frac{\pi}{2}}\cos^3\psi\int_0^{2\pi}\cos^2\varphi\int_0^1 r^4\,dr.$$

Pf:



(lower half)

$(x-1)^2+y^2+z^2=1$ $(z\geqslant 0)$.

$$\iint_{S_1}(1-y^2-z^2)^{\frac{3}{2}}\,dy\,dz$$

$$-\int_{2\pi}^{\pi} d\varphi\int_0^1 (1-r^2)^{\cancel{\frac{3}{2}}}\,r\,dr. = -\frac{1}{2}\int_0^{\pi} d\varphi\cdot\frac{2}{5}\cdot(1-r^2)^{\frac{5}{2}}\Big|_0^1 = \frac{2}{5}\pi$$

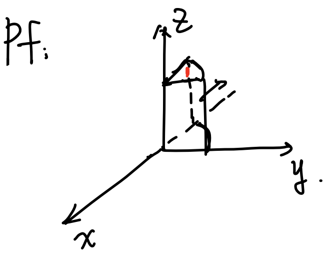
不能投影到zOy! 不是1-to-1

$$= \frac{\cancel{2}\pi}{5}$$

bottom. $\iint_{S_2} = 0$. $\iint_S = \iint_{S_1+S_2^-} = \frac{\pi}{5}$

10. $\iint_S yz\,dx\,dy + zx\,dy\,dz + xy\,dz\,dx$, S is the outer side of the cylinder part $x^2+y^2=r^2$, $x\leqslant 0$, $y\geqslant 0$, $0\leqslant z\leqslant H$.

Pf:



$\vec{n} = (-x, y, 0).$

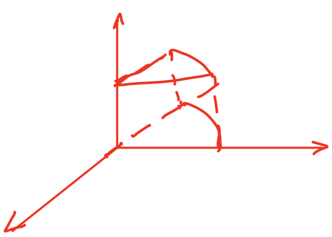
$$\iint_S 0 + xy\cdot\frac{y}{x}\,dy\,dz + zx\,dy\,dz$$

$$= \iint_{S_{zy}} -y^2 - z\sqrt{r^2-y^2}\,dy\,dz.$$

$$= -H\cdot\int_0^r y^2\,dy - \int_0^H z\,dz\int_r^0\sqrt{r^2-y^2}\,dy$$

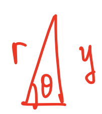
$$\frac{H^2}{2}r^2\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta. = \frac{\pi r^2 H^2}{8}$$

$$= \frac{Hr^3}{3} + \frac{\pi r H^2}{8}$$



$$= -\frac{Hr^3}{3} + \frac{\pi r^2 H^2}{8}$$

$\sqrt{r^2-y^2} = r\cos\theta.$

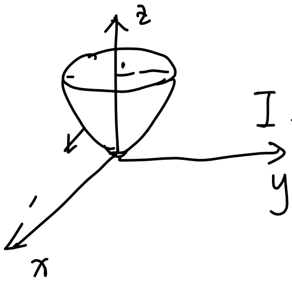


$y = r\sin\theta$

$dy = r\cos\theta\,d\theta.$

11. $\iint_S x^6 dydz + y^4 dzdx + z^2 dxdy$, S is the underside of a part of an elliptical paraboloid $z = x^2 + y^2$, $z \leqslant 1$.

$$\frac{d}{dxdy} = \frac{2x}{-1}$$



$$\vec{n} = (2x, 2y, -1).$$

$$I = -\iint_S x^6 \cdot \overset{-2x}{\left(-\frac{1}{2x}\right)} dxdy + y^4 \cdot \overset{-2y}{\left(-\frac{1}{2y}\right)} dxdy + (x^2+y^2)\, dxdy$$

$$= \frac{1}{2}\iint_{S_{xy}} x^5 + y^3 - 2(x^2+y^2)^2\, dxdy.$$

$$= -2\pi \int_0^1 r^2 \cdot r\, dr = -\frac{\pi}{2}. \quad X \quad -\frac{\pi}{3}$$

$$\iint 2x^7 + 2y^5 - (x^2+y^2)^2\, dxdy = -2\pi \cdot \int_0^1 r^4 \cdot r\, dr = -\frac{\pi}{3}$$