Complex Analysis 2024. Homework 1.

1. Calculate the following complex numbers, indicate the real and imaginary parts, calculate their absolute value.

$$z_1 = (1-2i)(3-4i); \quad z_2 = (-2i)^7; \quad z_3 = i^{2024}; \quad z_4 = \frac{1+2i}{2-i}.$$

Solution.

$$z_1 = -7-10i, \quad \operatorname{Re} z_1 = -5;\ \operatorname{Im} z_1 = -10;\ |z_1| = 5\sqrt{5};$$

$$z_2 = 2^7 i, \quad \operatorname{Re} z_2 = 0;\ \operatorname{Im} z_1 = 2^7;\ |z_1| = 2^7;$$

$$z_3 = 1, \quad \operatorname{Re} z_3 = 1;\ \operatorname{Im} z_3 = 0;\ |z_4| = 1;$$

$$z_4 = i, \quad \operatorname{Re} z_4 = 0;\ \operatorname{Im} z_4 = 1;\ |z_4| = 1;$$

2. Describe and draw sets defined by the equations

(a) $\operatorname{Re} z = 2 \operatorname{Im} z$;
Solution. This equation describes a line

$$x = 2y.$$

(b) $|z-1-i| = 3$;
Solution. This equation describes a circle of radius 3 with center at $(1,1)$.

(c) $|z-2| = \operatorname{Re} z$.
Solution. This equation describes a parabola

$$y^2 - 4x + 4 = 0; \quad x = \frac{y^2}{4} + 1.$$

3. Prove the parallelogram identity

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Solution. First

$$|z_1 + z_2|^2 = (z_1+z_2)\overline{z_1 + z_2} = (z_1+z_2)(\overline{z}_1+\overline{z}_2) = |z_1|^2+|z_2|^2+z_1\overline{z}_2+z_2\overline{z}_1 = |z_1|^2+|z_2|^2+2\,\mathrm{Re}($$

Then

$$\begin{aligned} &|z_1 + z_2|^2 + |z_1 - z_2|^2 = \\ &\qquad |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z}_2) + |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\overline{z}_2) = \\ &\qquad\qquad 2(|z_1|^2 + |z_2|^2) \quad (2) \end{aligned}$$

4. Prove that $z^{-1} = \bar{z}/\,|z|^2$.

Solution.

$$z(\bar{z}/\,|z|^2) = |z|^2\,/\,|z|^2 = 1.$$