Short course on integral calculus, differential calculus

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CONTENTS

LITERATURE

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Chapter 1

INDEFINITE INTEGRALS

1.1. Primitives

Integrals, together with **Derivatives**, are the basic objects of **Calculus**. Indefinite integrals are defined through **Primitives** (or **Antiderivatives**).

The function $F(x)$ is called a **primitive** (or **antiderivative**) of a function $f(x)$ if

$$F'(x) = f(x) \tag{1}$$

for all x in the domain of $f(x)$.

In other words a **primitive** of $f(x)$ is a function whose derivative equals the given function $f(x)$.

Example 1: The function $F(x)$ is a primitive of $F'(x)$.

Example 2: The function $\ln(1+x^2)$ is a primitive of $\dfrac{2x}{1+x^2}$ since

$$(\ln(1+x^2))' = \frac{1}{1+x^2}\cdot(1+x^2)' = \frac{2x}{1+x^2} \qquad \text{for all } x \in R.$$

Primitives have the following important **property**:

Let $F_1(x)$ and $F_2(x)$ be primitives of f, that is,

$$F_1'(x) = F_2'(x) = f(x)$$

for all x in the domain of $f(x)$.

Then there exists a constant C such that

$$F_1(x) = F_2(x) + C.$$

Indeed, $F_1'(x) = F_2'(x)$ by definition. Therefore, the derivative of the difference between functions $F_1(x)$ and $F_2(x)$ is equal to zero for all x on the given interval:

$$(F_1 - F_2)' = F_1' - F_2' = 0.$$

Hence, the difference $F_1 - F_2$ equals a constant by the corollary to the Mean Value Theorem.

In general, if a function has one primitive, then it has an infinite number of primitives.

However, if we know one primitive $F(x)$ of the function $f(x)$, then we know all primitives of f. The set of all primitives of f can be represented as $F(x)+C$, where C is an arbitrary constant.

Example 3: Both functions, $F_1 = (x+1)^2$ and $F_2 = x^2 + 2x - 4$, are primitives of $f(x) = 2(x+1)$ for all $x \in R$.
One can easily check that the difference between the primitives is a constant:

$$F_1 - F_2 = (x+1)^2 - (x^2 + 2x - 4)$$
$$= (x^2 + 2x + 1) - (x^2 + 2x - 4) = 5.$$

1.2. The Definition and Properties of Indefinite Integrals

The set of all primitives $F(x)$ of $f(x)$ is called the **indefinite integral** of the function $f(x)$.

The **indefinite integral** of $f(x)$ is denoted by the symbol $\int f(x)dx$, which is read as "The integral of $f(x)$ with respect to x".

$$\boxed{\begin{array}{c}\int f(x)dx = F(x) + C \\ \text{if and only if } F'(x) = f(x).\end{array}}$$

- The function $f(x)$ under the integral sign is called the **integrand**.
- The x is the **integration variable**.
- The symbol dx is the differential of x.
- An arbitrary constant C is said to be a **constant of integration**.

All indefinite integrals have the following **properties**:

Differentiation is the inverse operation to indefinite integration:

$$(\int f(x)dx)' = f(x), \tag{1a}$$
$$d\int f(x)dx = f(x)dx. \tag{1b}$$

- These formulas follow from the definition of indefinite integrals and can be easily memorized using the following rule:
  Symbols d and $\int$ cancel each other if they follow one after another.

Integration of the derivative of $f(x)$ yields the function $f(x)$:

$$\int f'(x)dx = \int df(x) = f(x) + C. \tag{2}$$

- This property is evident since the function $f(x)$ is a primitive of $f'(x)$.

Note that integration is the inverse operation to differentiation. However do not forget to add a constant of integration when integration is the last operation!

The following two general formulas allow us to transform a given integral into another integral or integrals.

A constant factor can be taken outside the integral sign.

$$\int cf(x)dx = c\int f(x)dx. \qquad (3)$$

The integral of an algebraic sum of functions equals the algebraic sum of the integrals of each of the functions.

$$\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx. \qquad (4)$$

Both these properties are based on the properties of derivatives. Indeed,

$(\int cf(x)dx)' = cf(x)$ and $(c\int f(x)dx)' = c(\int f(x)dx)' = cf(x)$.

Therefore, both sides in equality (3) represent primitives of the same function.

Property (4) can be obtained in a similar way since the derivative of a sum of functions equals the sum of derivatives of each of the functions.

- Let $\int f(x)dx = F(x) + \mathrm{C}$.

  Then

$$\int f(u)du = F(u) + \mathrm{C} \qquad (5)$$

  for any differentiable function $u = u(x)$.

- This property is based on the invariance of the form of the first differential, according to which the differential formula $dF(x) = F'(x)dx$ holds for any composite function $F(u(x))$:

$$dF(u) = F'(u)du.$$

Advice: Try to memorize and understand all these rules.

Let us consider some elementary examples to illustrate the definition and properties of indefinite integrals before going on.

Examples:

- $\frac{d}{dx}\int \sin^3 3x dx = \sin^3 3x.$ | property (1) |
- $\int \frac{dx}{\cos^2 x} = \int (\tan x)' dx = \tan x + C.$ | property (2) |

- $\int 6x^2dx = 6\int x^2dx = 6\frac{x^3}{3} + C = 2x^3 + C$. | property (3) |

- $\int (2x-3)dx = 2\int xdx - 3\int dx$ | properties (3) and (4) |

$$= 2\frac{x^2}{2} - 3x + C = x^2 - 3x + C.$$

- $\int \frac{\ln^4 x}{x} dx = \int (\ln x)^4 d(\ln x) = \frac{\ln^5 x}{5} + C$. | property (5) |

1.3. A Table of Common Integrals

Let us recall the derivatives of elementary functions. For instance, the power rule states that

$$(x^k)' = kx^{k-1}.$$

This formula can be transformed as follows.
First, we substitute $(n+1)$ for k:

$$(x^{n+1})' = (n+1)x^n.$$

Then we divide both sides of the equality by $(n+1)$) (provided that $n \neq -1$) and read the formula from right to left:

$$x^n = (\frac{x^{n+1}}{n+1})'.$$

Therefore, the function $\frac{x^{n+1}}{n+1}$ is a primitive of x^n, so the power rule for integration is the following:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

The derivatives of all elementary functions can be treated likewise. Then the table of derivatives can be easily transformed into the table of common integrals.
Thus, we have a list of common indefinite integrals.

Table 1

Derivatives	Integrals
$x^n = \left(\frac{x^{n+1}}{(n+1)}\right)'$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$ $\int dx = x + C$
$\frac{1}{x} = (\ln x)'$	$\int \frac{dx}{x} = \ln\lvert x\rvert + C$ $(x \neq 0)$
$a^x = \left(\frac{a^x}{\ln a}\right)'$ $e^x = (e^x)'$	$\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$ $\int e^x dx = e^x + C$
$\sin x = (-\cos x)'$	$\int \sin x dx = -\cos x + C$
$\cos x = (\sin x)'$	$\int \cos x dx = \sin x + C$
$\frac{1}{\cos^2 x} = (\tan x)'$	$\int \frac{dx}{\cos^2 x} = \tan x + C \quad (x \neq \frac{\pi}{2} + \pi n)$
$\frac{1}{\sin^2 x} = (-\cot x)'$	$\int \frac{dx}{\sin^2 x} = -\cot x + C \quad (x \neq \pi n)$
$\frac{1}{\sqrt{1-x^2}} = \begin{cases} (\arcsin x)' \\ (-\arccos x)' \end{cases}$	$\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C \\ -\arccos x + C \end{cases} \quad (\lvert x\rvert \leq 1)$
$\frac{1}{1+x^2} = \begin{cases} (\arctan x)' \\ (-\cot^{-1} x)' \end{cases}$	$\int \frac{dx}{1+x^2} = \begin{cases} \arctan x + C \\ -\cot^{-1} x + C \end{cases}$

Comment

If $x > 0$, then $\ln\lvert x\rvert = \ln x$ and $(\ln\lvert x\rvert)' = \frac{1}{x}$.

If $x < 0$, then $\ln\lvert x\rvert = \ln(-x)$ and $(\ln\lvert x\rvert)' = (\ln(-x))' = \frac{1}{-x}(-1) = \frac{1}{x}$.

Therefore, the function $\frac{1}{x}$ is a primitive of $\ln\lvert x\rvert$ in both cases.

Corollary 1: Each of the functions, $\arcsin x$ and $(-\arccos x)$, is a primitive of $\dfrac{1}{\sqrt{1-x^2}}$. Therefore, the difference between them should be equal to a constant:

$$\arcsin x-(-\arccos x)=\arcsin x+\arccos x=C\,.$$

Setting $x=0$ we find the constant C:

$$C=\arcsin 0+\arccos 0=0+\pi/2=\pi/2\,.$$

Thus,

$$\arcsin x+\arccos x=\pi/2\,.$$

Corollary 2: In a similar way one can get one more formula of elementary mathematics:

$$\arctan x+\cot^{-1}x=\pi/2\,.$$

The result for any particular integral can often be written in many different forms.

In order to solve successfully integration problems, it is necessary to know:

- the properties of integrals;
- the table of common integrals;
- the techniques used for manipulation with integrals.

The best way to acquire enough knowledge of the integral formulas is to use them as many times as possible. Knowledge of the formulas develops the ability to recognize them.

1.4. Techniques of Integration

Evaluating integrals is much more difficult than evaluating derivatives. As for derivatives, there is a systematic procedure based on the chain rule that effectively allows any derivative to be worked out. However, there is not any similar procedure for integrals.

One of the main problems is that it is difficult to know what kinds of functions will be needed to evaluate a particular integral. When we work out a derivative, we always end up with functions that are of the same kind or simpler than the ones we started with. But when we work out integrals, we often end up needing to use functions that are much more complicated than the ones we started with.

Whenever the specific integration formulas do not apply, we have to transform the problem into another problem or problems. One can try to manipulate the integrand algebraically, separate the integrand, if possible, put any constant factors outside of the sign of the integral by making use of the properties of integrals, and so on.

The basic techniques of integration are algebraic manipulation, substitutions, integration by parts, and the method of partial fractions.

1.4.1. Integration by Substitution

The technique of substitutions helps to reduce many integrals to common indefinite integrals, which are given in Table 1.
For convenience sake all substitutions may be subdivided into two classes:

- $u = g(x)$,
- $x = u(t)$.

In both cases we change the variable of integration - in one way or another.
As a rule, the substitution $u = g(x)$ is used when a given integral has the following structure:

$$\int f(g(x))g'(x)dx.$$

Then the substitution $u = g(x)$ implies $du = g'(x)dx$, so that we obtain

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du. \qquad (6)$$

Therefore, the initial integration problem is transformed into another integration problem. However if we can not integrate the function $f(u)$, then another method of integration may be required.

On the other hand, the substitution $x = u(t)$ gives another way of transformation of a given integral.

Now let $\int f(x)dx$ be a given integral.
Then the substitution $x = u(t)$ implies $dx = u'(t)dt$, and we obtain

$$\int f(x)dx = \int f\big(u(t)\big)\, u'(t)\, dt. \qquad (7)$$

As above, we expect that the new integral is easier evaluated. Otherwise, another substitution or integration method may be needed.
As a matter of fact, formulas (6) and (7) give the reverse transformation to each other. They are called the **substitution formulas**.

The technique of substitution is quite general and can be used in a wide variety of problems.
In particular, one can generalize the table of common integrals applying the technique of substitutions. Consider, for instance, the power rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1).$$

Let $u(t)$ be any differentiable function. If we use the substitution $x = u$, then the power rule can be formulated as follows:

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1).$$

This formula is exactly the same as the original power rule. The only difference is the interpretation of the symbol u as a function of a variable t and so $du = u'dt$. Therefore, we obtain the following generalized power rule:

$$\int u^n(t)u'(t)dt = \frac{u^{n+1}(t)}{n+1} + C \qquad (n \neq -1).$$

One can interpret each of the common integrals in a similar way by considering the variable of integration as a function.

1.4.1.1. Examples of Integrating by Substitution

Example 1: Each of the following integrals

1) $\int \frac{(\arctan x)^4}{1+x^2} dx$,

2) $\int \frac{dx}{(\arctan x)(1+x^2)}$,

3) $\int e^{(\arctan x)} \frac{dx}{1+x^2}$

can be written as

$$\int f(\arctan x)\frac{dx}{1+x^2} = \int f(\arctan x)(\arctan x)'dx$$
$$= \int f(\arctan x)d(\arctan x).$$

Therefore, the substitution $u = \arctan x$ is fairly suitable for each of them:

1) $\int \frac{(\arctan x)^4}{1+x^2} dx = \int (\arctan x)^4 d(\arctan x) = \int u^4 du = \frac{u^5}{5} + C$.

Once the solution has been found in terms of u, one needs to replace u in it by the corresponding function of x. So the final solution is the following:

$$\int \frac{(\arctan x)^4}{1+x^2} dx = \frac{(\arctan x)^5}{5} + C.$$

2) $$\int \frac{dx}{(\arctan x)(1+x^2)} = \int \frac{d(\arctan x)}{\arctan x}$$
$$= \int \frac{du}{u} = \ln|u| + C = \ln|\arctan x| + C.$$

3) $$\int e^{(\arctan x)} \frac{dx}{1+x^2} = \int e^{(\arctan x)} d(\arctan x)$$
$$= \int e^u du = e^u + C = e^{\arctan x} + C.$$

One can easily check these solutions by differentiating. Let us check, e.g., the last integral:

$$(e^{\arctan x})' = e^{\arctan x}(\arctan x)' = \frac{e^{\arctan x}}{1+x^2}.$$ That is O.K.

Example 2: Both integrals, $\int \frac{\sin(\ln x)}{x} dx$ and $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$, are easily evaluated by using of the substitution $u = \ln x$. They are just common integrals:

$$\int \frac{\sin(\ln x)}{x} dx = \int \sin u du = -\cos u + C = -\cos(\ln u) + C,$$

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin(\ln x) + C.$$

Example 3: Each of the integrals below is reduced to the table integral

$$\int \frac{dx}{\cos^2 u} = \tan u + C,$$

using the appropriate substitution:

- $\int \frac{dx}{\cos^2(3x-4)} = \frac{1}{3}\tan(3x-4) + C$ ($u = 3x-4$, $du = 3dx$).
- $\int \frac{dx}{\sqrt{x}\cos^2(\sqrt{x})} = 2\tan(\sqrt{x}) + C$ ($u = \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$).
- $\int \frac{x^4 dx}{\cos^2(x^5)} = \frac{1}{5}\tan(x^5) + C$ ($u = x^5$, $du = 5x^4 dx$).
- $\int \frac{dx}{x\cos^2(\ln x)} = \tan(\ln x) + C$ ($u = \ln x$, $du = \frac{dx}{x}$).

- $\int \frac{e^x dx}{\cos^2(e^x)} = \tan e^x + C \qquad\qquad (u = e^x,\ du = e^x dx).$

The formal substitution into the integral really is not necessary.

1.4.1.2. Some Important Integrals

Problem 1: Evaluate the following integral: $\int \frac{dx}{a^2 + x^2}$.

Solution: Let us make the substitution $x = at$. Then

$$\begin{aligned}\int \frac{dx}{a^2 + x^2} &= \int \frac{adt}{a^2 + a^2t^2} = \frac{a}{a^2}\int \frac{dt}{1+t^2} \\ &= \frac{1}{a}\arctan t + C = \frac{1}{a}\arctan\frac{x}{a} + C.\end{aligned} \tag{8}$$

Problem 2: Find the integral $\int \frac{dx}{\sqrt{a^2 - x^2}}$.

Solution: By making use of the same substitution $x = at$ we get:

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2 - x^2}} &= \int \frac{adt}{\sqrt{a^2 - a^2t^2}} = \int \frac{adx}{a\sqrt{1-t^2}} \\ &= \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin\frac{x}{a} + C.\end{aligned} \tag{9}$$

Problem 3: Prove the following formula:

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C\,. \tag{10}$$

Proof: The formula can be verified by differentiation. We have only to check whether the derivative of the function $\ln(x + \sqrt{x^2 \pm a^2})$ equals the integrand.

$$\begin{aligned}(\ln(x + \sqrt{x^2 \pm a^2}))' &= \frac{1}{x + \sqrt{x^2 \pm a^2}}(1 + \frac{1}{2\sqrt{x^2 \pm a^2}}2x) \\ &= \frac{1}{x + \sqrt{x^2 \pm a^2}}\frac{\sqrt{x^2 \pm a^2} + x}{\sqrt{x^2 \pm a^2}} = \frac{1}{\sqrt{x^2 \pm a^2}}.\end{aligned}$$

That is true and hence the formula.

1.4.2. Integration by Parts

The formula for integration by parts states that

$$\int u dv = uv - \int v du \tag{11}$$

for any differentiable functions $u(x)$ and $v(x)$.

This formula allows us to transform one problem of integration into another.

If one of the two integrals, $\int u dv$ or $\int v du$, is easily evaluated, it can be used to find the other one. This is the main idea of the method of integration by parts.

Formula (11) can be derived in the following way:

$$d(uv) = udv + vdu \quad \Rightarrow \quad udv = d(uv) - vdu \quad \Rightarrow$$
$$\int udv = \int d(uv) - \int vdu \quad \Rightarrow \quad \int udv = uv - \int vdu .$$

In practice, the procedure of integrating by parts consists of the following steps:

- First, we introduce intermediary functions $u(x)$ and $v'(x)$ to represent the function $f(x)$ as the product of the factors $u(x)$ and $v'(x)$, so that $f(x)dx = u(x)v'(x)dx = u(x)dv$ and
$$\int f(x)dx = \int udv .$$
For example, one can set $u(x) = f(x)$, which implies $v'(x) = 1$.
- Next we need to differentiate $u(x)$ and integrate $v'(x)$ to get the differential $du = u'(x)dx$ and the function $v(x) = \int v'(x)dx$ respectively. Note that a constant of integration can be taken zero at this step ($C = 0$).
- Then we use formula (11) and try to evaluate the integral $\int vdu$.

FAQ (Frequently Asked Questions): Why do we prefer to deal with the integral $\int vdu$ instead of the initial one?

Answer: It depends on the choice of $u(x)$ whether the integral $\int vdu$ is easier to evaluate in comparison with the initial one. We assume that there exists the right choice.

The main problem one faces when dealing with the method of integration by parts is the choice of the intermediary functions. There is no general rule to follow it. It is a matter of experience and nothing more. At first in order to understand better this technique, it is necessary to make any choice and perform the calculations. If the new integral is simpler than the given one, then the choice is a good one; otherwise, go back and make another choice.

In such a way one can easily appreciate whether the choice of $u(x)$ is the best one. It is possible that you need to evaluate a few integrals before you will start to feel the right choice.

One can apply the following **criteria** to make the right choice.

A: The integral of v' should be easy for evaluation.

B: The derivative of $u(x)$ should be a simple function. Moreover, it is desirable that $u'(x)$ would be more simple function than $u(x)$.

The following examples illustrate the most common cases in which we need to use the technique of integration by parts.

Example 1: Evaluate the integral $\int x^2 \ln x dx$.

Solution: Consider some variants of representation of the above integrand as the product udv.

1) $u = \ln x$, $v' = x^2$ $\Rightarrow$ $du = \frac{dx}{x}$, $v = \int x^2 dx$;

2) $u = x$, $v' = x \ln x$ $\Rightarrow$ $du = dx$, $v = \int x \ln x dx$;

3) $u = x^2$, $v' = \ln x$ $\Rightarrow$ $du = 2x dx$, $v = \int \ln x dx$;

4) $u = x \ln x$, $v' = x$ $\Rightarrow$ $du = d(x \ln x)$, $v = \int x dx$;

5) $u = x^2 \ln x$, $v' = 1$ $\Rightarrow$ $du = d(x^2 \ln x)$, $v = \int dx$.

Let us discuss these choices in detail this time.
Both hypotheses, 2) and 3), do not satisfy criterion A, because it is not clear how to integrate $\ln x$, while hypotheses 4) and 5) contradict to criterion B.
Similar reasons suggest that the first way only is appropriate. Indeed,

- The power function x^2 is easily integrated and its primitive is

$$v = \int x^2 dx = \frac{x^3}{3} \qquad (C = 0).$$

- The derivative of the transcendental function $\ln x$ is the rational function:

$$(\ln x)' = x^{-1}.$$

Therefore, in view of formula (11) we finally get

$$\int x^2 \ln x dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^3 \frac{dx}{x} = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C.$$

Example 2: Evaluate $\int \arctan x dx$.

Solution: Let $u = \arctan x$ and $v' = 1$. Then $du = \frac{dx}{1+x^2}$ and $v = x$.

We integrate by parts:

$$\int \arctan x dx = x \arctan x - \int \frac{x}{1+x^2} dx.$$

To evaluate the new integral we use the substitution $z = 1+x^2$, which implies $dz = d(1+x^2) = 2xdx$, and so $xdx = \frac{1}{2} dz$.

Therefore, $\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{dz}{z} = \frac{1}{2} \ln |z| = \frac{1}{2} \ln(1+x^2)$.

Hence, the final solution is the following:

$$\int \arctan x dx = x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C.$$

In a similar way one can integrate the product of a polynomial $P(x)$ and any inverse trigonometric function, as well as the product of a polynomial $P(x)$ and the logarithmic function.

Each of the following function $P(x) \arcsin x$, $P(x) \arccos x$, $P(x) \arctan x$, $P(x) \cot^{-1} x$ and $P(x) \ln x$ can be integrated by parts. The inverse trigonometric function (or $\ln x$) should be chosen as $u(x)$ and $v'(x) = P(x)$.

It is not always so easy. Sometimes one has to integrate by parts more than once to obtain the result.

Example 3: Evaluate $\int x \ln^2 x dx$.

Solution: Let $u = \ln^2 x$ and $dv = xdx$. Then $du = \frac{2 \ln x dx}{x}$ and $v = \frac{x^2}{2}$.

The formula of integration by parts gives

$$\int x\ln^2 x dx = \frac{x^2}{2}\ln^2 x - \int x\ln x dx .$$

Now we integrate by parts a second time, setting $u = \ln x$ and $dv = xdx$.

After integration and differentiation, we get $du = \frac{dx}{x}$ and $v = \frac{x^2}{2}$.

Therefore,

$$\int x\ln x dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x dx = \frac{x^2}{2}\ln x - \frac{x^2}{4}.$$

The final result is the following:

$$\int x\ln^2 x dx = \frac{x^2}{2}\ln^2 x - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) + C$$

$$= \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C = \frac{x^2}{4}(2\ln^2 x - 2\ln x + 1) + C.$$

Example 4: Evaluate $\int x^2 e^x dx$.

Solution: We have to make the right choice between differentiation and integration of x^2. Note that every differentiation of a polynomial decreases its degree, and hence, the polynomial vanishes after a few steps, while integration of a polynomial increases its degree.

Therefore, the right choice is the following:

$$u = x^2 \quad \text{and} \quad dv = e^x dx \quad \Rightarrow \quad du = 2xdx \text{ and } v = e^x.$$

The formula of integration by parts yields:

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx . \tag{12}$$

We need to integrate by parts once more.

Let $u = x$ and $dv = e^x dx$ which imply $du = dx$ and $v = e^x$.

Therefore,

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x .$$

From here and equality (12) we obtain

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C .$$

The examples above illustrate that the single integration by parts can not be enough to obtain the answer, and so some extra work may be needed, e.g., another integration by parts or using some other techniques.

The last example can be generalized:

Each of the following integrals

$$\int P(x)e^{ax}dx,$$

$$\int P(x)\sin axdx \text{ and}$$

$$\int P(x)\cos axdx$$

can be evaluated using the integration by parts.

In order to get the solution, it is necessary to use integration by parts n times if the degree of the polynomial equals n.

The summary table below includes some suggested substitutions and formulas.

Table 2

Integrals	Substitutions	Basic Formulas
$\int P(x)\begin{pmatrix}\arcsin x\\ \arccos x\\ \arctan x\\ \cot^{-1} x\\ \ln x\end{pmatrix}dx$	$u=\begin{cases}\arcsin x\\ \arccos x\\ \arctan x\\ \cot^{-1} x\\ \ln x\end{cases}$ $dv = P(x)dx$	$du=\begin{cases}\dfrac{dx}{\sqrt{1-x^2}}\\ -\dfrac{dx}{\sqrt{1-x^2}}\\ \dfrac{dx}{1+x^2}\\ -\dfrac{dx}{1+x^2}\\ \dfrac{dx}{x}\end{cases}$ $v=\int P(x)dx$
$\int P(x)\begin{pmatrix}e^{ax}\\ \sin ax\\ \cos ax\end{pmatrix}dx$	$u = P(x)$ $dv=\begin{pmatrix}e^{ax}\\ \sin ax\\ \cos ax\end{pmatrix}dx$	$du = P'(x)dx$ $v=\dfrac{1}{a}\begin{cases}e^{ax}\\ -\cos ax\\ \sin ax\end{cases}$

By making use of integration by parts we sometimes come to an equation for the integral but not an explicit formula. However, by solving this equation we obtain the desired result. Let us consider a typical problem of such a kind.

Problem 4: Find the integral $I = \int e^{ax}\cos(bx)dx$.

Solution: Let $u = e^{ax}$ and $dv = \cos bx dx$, so that $du = ae^{ax}dx$ and $v = \sin bx/b$. The formula of integration by parts gives

$$\int e^{ax}\cos bx dx = \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx dx.$$

The new integral is similar to the initial one. Let us integrate by parts the second time. Note that we have to use again e^{ax} as u. Otherwise, we would come back to the original integral and nothing more.

Thus, now let $u = e^{ax}$ and $dv = \sin bx dx$. Then $du = ae^{ax}dx$ and $v = -\cos bx/b$.

In this case we have

$$\int e^{ax}\sin bx dx = -\frac{1}{b}e^{ax}\cos bx - (-\frac{a}{b}\int e^{ax}\cos bx dx).$$

Combining both formulas yields

$$\int e^{ax}\cos bx dx = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}\int e^{ax}\cos bx dx).$$

This equality can be considered as a linear equation with respect to the given integral I: $\quad I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I$.

By combining of similar terms and making use of simple algebraic manipulations, we get

$$(b^2 + a^2)I = e^{ax}(b\sin bx + a\cos bx) \qquad \Rightarrow$$

$$I = \frac{a\cos bx + b\sin bx}{a^2 + b^2}e^{ax}.$$

Hence, the final solution is

$$\int e^{ax}\cos bx dx = \frac{a\cos bx + b\sin bx}{a^2 + b^2}e^{ax} + C. \tag{13}$$

In a similar way one can obtain another formula of this kind:

$$\int e^{ax}\sin bx dx = \frac{a\sin bx - b\cos bx}{a^2 + b^2}e^{ax} + C. \tag{14}$$

1.5. Integration of Rational Functions

1.5.1. Main Definitions

Let us start from the definition chain:

Rational Functions → Proper Fractions → Partial Fractions

A **rational function** is a function that can be expressed as the ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}.$$

A rational function $\frac{P(x)}{Q(x)}$ is said to be a **proper fraction** if the degree of the polynomial $P(x)$ is less than that of $Q(x)$.

For example, the following functions

$$\frac{x^3}{2x+7}, \qquad \frac{3x-2}{5x^3+x-1}, \qquad \frac{1}{(x+5)^4}.$$

are the rational functions. Furthermore, the last two functions are the proper fractions.

Fractions of the following form

$$1) \quad \frac{1}{(x-a)^n} \qquad (n \geq 1), \tag{15}$$

$$2) \quad \frac{Ax+B}{(x^2+px+q)^n} \qquad (n \geq 1) \tag{16}$$

are called the **partial fractions**, where the quadratic polynomial x^2+px+q is assumed to be irreducible, that is, the discriminant $D = p^2 - 4q$ is negative.

The problem of integration of rational functions can be subdivided into several separate problems such as:

1) Integration of partial fractions.
2) Decomposition of a proper fraction into a sum of partial fractions.
3) Reduction of any rational function to a proper fraction.

Consider the procedure of integration of a ration function $f(x) = \frac{P(x)}{Q(x)}$.

It comprises the following steps.

- Assume that $f(x)$ is a proper fraction. Otherwise it is necessary first to perform the **polynomial long division** in order to represent the function

$f(x)$ as a sum of some polynomial and the remainder term (which is a proper fraction). Any polynomial is easily integrated, so in both cases we can deal only with proper fractions.

Therefore, the problem of integration of rational functions can always be reduced to the one of integration of proper fractions, keeping in mind that any rational function either is a proper fraction or can be expressed through a proper fraction.

- In order to decompose the given function (or the remainder term) into the sum of partial fractions, the denominator $Q(x)$ has to be factored into irreducible polynomials, that is, linear and irreducible quadratic polynomials. The corresponding method is called Decomposition of Rational Functions into a Sum of Partial Fractions (in short form: **Partial Fraction Decomposition**).
- To integrate each of the obtained partial fractions.

1.5.2. Integration of Partial Fractions

We attach importance to the partial fractions because any proper fraction can be decomposed into a sum of partial fractions.

Partial fractions of the first type (expression (15)) are easily integrated in view of common integrals:

$$\int \frac{dx}{x-a} = \ln|x-a| + C, \qquad (16)$$

$$\int \frac{dx}{(x-a)^n} = \frac{1}{(-n+1)(x-a)^{n-1}} + C \qquad (n \neq 1). \qquad (17)$$

In order to integrate partial fractions of the second type (expression (16)), one has to complete the square for the polynomial $x^2 + px + q$, e.g., making use of the substitution $t = x + p/2$, that is, $x = t - p/2$.

Hence,

$$x^2 + px + q = (t - \frac{p}{2})^2 + p(t - \frac{p}{2}) + q$$

$$= t^2 - pt + \frac{p^2}{4} + pt - \frac{p^2}{2} + q = t^2 + (q - \frac{p^2}{4}) = t^2 + a^2,$$

where the positive constant $q - \frac{p^2}{4}$ is denoted as a^2.

Thereto, $dx = dt$ and $Ax + B = At + (B - A\frac{p}{2}) = At + B_1$, where $B_1 = B - p/2$.

Then we apply the properties of integrals to obtain

$$\int \frac{Ax+B}{(x^2+px+q)^n}dx = \int \frac{At+B_1}{(t^2+a^2)^n}dt$$
$$= A\int \frac{tdt}{(t^2+a^2)^n} + B_1\int \frac{dt}{(t^2+a^2)^n}. \tag{18}$$

The first integral on the right-hand side is easily evaluated:

$$\int \frac{tdt}{(t^2+a^2)^n} = \frac{1}{2}\int \frac{d(t^2)}{(t^2+a^2)^n}$$
$$= \frac{1}{2}\int \frac{d(t^2+a^2)}{(t^2+a^2)^n} = \begin{cases} \frac{1}{2}\ln(t^2+a^2)+C, & \text{if} \quad n=1; \\ \frac{1}{2(-n+1)(t^2+a^2)^{n-1}}+C, & \text{if} \quad n>1. \end{cases}$$

Now let us apply the technique of integration by parts to find integrals

$$I_n = \int \frac{dt}{(t^2+a^2)^n} \qquad (n \geq 1). \tag{19}$$

Let $u = \frac{1}{(t^2+a^2)^n}$ and $dv = dt$.

Then $du = \frac{-2ntdt}{(t^2+a^2)^{n+1}}$ and $v = t$.

Therefore,

$$\int \frac{dt}{(t^2+a^2)^n} = \frac{t}{(t^2+a^2)^n} - (-2n)\int \frac{t^2}{(t^2+a^2)^{n+1}}dt$$
$$= \frac{t}{(t^2+a^2)^n} + 2n\int \frac{(t^2+a^2)-a^2}{(t^2+a^2)^{n+1}}dt$$
$$= \frac{t}{(t^2+a^2)^n} + 2n\int \frac{dt}{(t^2+a^2)^n} - 2na^2\int \frac{dt}{(t^2+a^2)^{n+1}}.$$

Then we combine the similar terms and express the integral I_{n+1} through the integral I_n:

$$\int \frac{dt}{(t^2+a^2)^{n+1}} = \frac{1}{2na^2}\left((2n-1)\int \frac{dt}{(t^2+a^2)^n} + \frac{t}{(t^2+a^2)^n}\right). \tag{20}$$

This recurrence formula allows us:

- to find integral I_2 if integral I_1 is known (setting $n=1$),
- to find integral I_3 if integral I_2 is known (setting $n=2$), and so on.

Note that integral I_1 can be found by using of simple methods. (See equality (8).)

$$I_1 = \int \frac{dt}{t^2 + a^2} = \frac{1}{a}\arctan\frac{t}{a} + C.$$

Then setting $n=1$ we have by recurrence formula (20)

$$I_2 = \int \frac{dt}{(t^2 + a^2)^2} = \frac{1}{2a^2}\left(\frac{1}{a}\arctan\frac{t}{a} + \frac{t}{t^2+a^2}\right) + C, \quad etc.$$

Thus, the problem of integration of partial fractions is completely solved.

1.5.3. Partial Fraction Decomposition

1.5.3.1. The Main Idea of the Method

In simple cases the decomposition of proper fractions into a sum of partial fractions can be easily obtained by means of algebraic manipulations.
Here are typical **examples**:

- $$\frac{1}{(x-a)(x-b)} = \frac{b-a}{(b-a)(x-a)(x-b)}$$
$$= \frac{(x-a)-(x-b)}{(b-a)(x-a)(x-b)} = \frac{1}{b-a}\left(\frac{1}{x-b} - \frac{1}{x-a}\right).$$

- $$\frac{1}{(x^2-49)} = \frac{1}{(x-7)(x+7)} = \frac{1}{14}\left(\frac{1}{x-7} - \frac{1}{x+7}\right).$$

- $$\frac{1}{x(x^2+4)} = \frac{1}{4}\frac{4}{x(x^2+4)} = \frac{1}{4}\frac{(x^2+4)-x^2}{x(x^2+4)}$$
$$= \frac{1}{4}\left(\frac{(x^2+4)}{x(x^2+4)} - \frac{x^2}{x(x^2+4)}\right) = \frac{1}{4}\left(\frac{1}{x} - \frac{x}{x^2+4}\right).$$

In more complicated cases one has to use the Method of Partial Fractions Decomposition.
The main idea of this method can be illustrated by the following simple example.

Example 1:

1) The sum of partial fractions, $\frac{2}{x-1}$ and $\frac{5}{x+4}$, can be combined into a more complicated fraction:

$$\frac{2}{x-1}+\frac{5}{x+4}=\frac{2(x+4)+5(x-1)}{(x-1)(x+4)}=\frac{7x+3}{(x-1)(x+4)}.$$

When we read this formula from left to right, we say about reduction of fractions to the common denominator.
We can also read the same formula from right to left:

$$\frac{7x+3}{(x-1)(x+4)}=\frac{2}{x-1}+\frac{5}{x+4}.$$

In this case we say about decomposition of the compound fraction into the sum of partial fractions.

2) Let us assume that we need to decompose the fraction $\frac{7x+3}{(x-1)(x+4)}$ into partial fractions. It looks in a general form as follows:

$$\frac{7x+3}{(x-1)(x+4)}=\frac{A}{x-1}+\frac{B}{x+4},$$

where A and B are undetermined constants.
If we multiply across by $(x-1)(x+4)$, then we get

$$7x+3=A(x+4)+B(x-1).$$

This equality is the equation for constants A and B but at the same time it is the identity with respect to x. So one can substitute any value for x to find the constants.
Setting $x=1$ we get the equality $10=5A$ which implies $\underline{A=2}$.
Setting $x=-4$, we obtain $(-25)=-5B \Rightarrow \underline{B=5}$.

Therefore, $$\frac{7x+3}{(x-1)(x+4)}=\frac{2}{x-1}+\frac{5}{x+4}.$$

as it was desired.

The Method of Partial Fractions Decomposition proceeds in the opposite direction in comparison with the reduction to a common denominator, that is, it transforms a compound fraction into a sum of partial fractions.
Partial Fractions Decomposition is the reverse procedure to reduction to the common denominator.

1.5.3.2. Partial Fraction Decomposition: The Main Rules

There are a few rules to decompose any proper fraction into a sum of partial fractions.

Rule 1: Let $\dfrac{P(x)}{Q(x)}$ be a proper fraction and let $Q(x) = (x-a)\,Q_1(x)$.

Then there exists a unique proper fraction $\dfrac{P_1(x)}{Q_1(x)}$ and a unique constant A, such that the given proper fraction can be represented in the form

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{P_1(x)}{Q_1(x)}.$$

Note that the degree of a polynomial $Q_1(x)$ is less than the degree of the given polynomial $Q(x)$: $\text{degree}(Q_1) = \text{degree}(Q) - 1$.

Then one can apply this rule to the proper fraction $P_1(x)/Q_1(x)$, if the denominator $Q_1(x)$ includes a linear factor, that is, $Q_1(x) = (x-b)\,Q_2(x)$. Therefore,

$$\frac{P(x)}{(x-a)(x-b)Q_2(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{P_2(x)}{Q_2(x)}.$$

Each such transformation decreases the degree of the denominator of the proper fraction.

Corollary: If the denominator $Q(x)$ of the proper fraction $\dfrac{P(x)}{Q(x)}$ consists of n different linear factors, that is, $Q(x) = (x-a_1)(x-a_2)...(x-a_n)$, then

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-a_1)(x-a_2)...(x-a_n)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}.$$

One can say that each linear factor $(x-a_k)$ in the denominator of the proper fraction yields the partial fraction $\dfrac{A_k}{x-a_k}$, where A_k is a constant.

The structure of decomposition of any proper fraction depends only on the factors, which the denominator consists of. For instance, both fractions below have the same structure of decomposition into partial fractions:

$$\frac{1}{x(x-3)(x+2)} = \frac{A_1}{x} + \frac{A_2}{x-3} + \frac{A_3}{x+2}, \tag{21}$$

$$\frac{5x-1}{x(x-3)(x+2)}=\frac{A_1}{x}+\frac{A_2}{x-3}+\frac{A_3}{x+2}. \qquad (22)$$

The numerator determines numerical values of the constants A_1, A_2, A_3.
Let us find, e.g., numerical values of the constants in decomposition (21).
First, we multiply both sides by $x(x-3)(x+2)$:

$$1=A_1(x-3)(x+2)+A_2x(x+2)+A_3x(x-3).$$

One can see that all fractions have disappeared.
Then we take for x such values that make some of the terms vanish:

$$x=0 \quad\Rightarrow\quad 1=A_1(-3)2=-6A_1 \quad\Rightarrow\quad A_1=-1/6,$$
$$x=3 \quad\Rightarrow\quad 1=15A_2 \quad\Rightarrow\quad A_2=1/15,$$
$$x=-2 \quad\Rightarrow\quad 1=10A_3 \quad\Rightarrow\quad A_3=1/10.$$

Finally, it remains to put the constants back into the original partial fractions:

$$\frac{1}{x(x-3)(x+2)}=-\frac{1}{6x}+\frac{1}{15(x-3)}+\frac{1}{10(x+2)}.$$

If the denominator of a proper fraction includes nth power of the factor $(x-a)$, then one can use the following rule of decomposition into a sum of partial fractions:

Rule 2: Let $\dfrac{P(x)}{Q(x)}$ be a proper fraction and let $Q(x)=(x-a)^n\,Q_1(x)$.

Then there exists a unique proper fraction $P_1(x)/Q_1(x)$ and unique constants $A_1, A_2, ..., A_n$ such that the given rational function can be represented in the form

$$\frac{P(x)}{(x-a)^nQ_1(x)}=\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+...+\frac{A_n}{(x-a)^n}+\frac{P_1(x)}{Q_1(x)}.$$

Then one can apply the above rules to the proper fraction $\dfrac{P_1(x)}{Q_1(x)}$, if the denominator $Q_1(x)$ includes a linear factor (repeated or not).

Example 2: The decomposition of any proper fraction $\dfrac{P(x)}{Q(x)}$ with denominator $Q(x)=(x-a)(x-b)^3$ has the following form:

$$\frac{P(x)}{(x-a)(x-b)^3}=\frac{A}{x-a}+\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\frac{B_3}{(x-b)^3}.$$

Example 3: Decompose the fraction $\dfrac{1}{(x+1)(x-4)^2}$ into partial fractions.

Solution: By applying the above rules we have:

$$\frac{1}{(x+1)(x-4)^2}=\frac{A_1}{x+1}+\frac{A_2}{x-4}+\frac{A_3}{(x-4)^2}.$$

Then we multiply both sides by $(x+1)(x-4)^2$:

$$1=A_1(x-4)^2+A_2(x+1)(x-4)+A_3(x+1).$$

To solve this equation with respect to A_1, A_2 and A_3, we take for x a few values:

$$x=-1 \;\Rightarrow\; 1=25A_1 \qquad\Rightarrow\qquad \underline{A_1=1/25};$$

$$x=4 \;\Rightarrow\; 1=5A_3 \qquad\Rightarrow\qquad \underline{A_3=1/5};$$

$$x=0 \;\Rightarrow\; 1=16A_1-4A_2+A_3=16/25-4A_2+1/5 \;\Rightarrow\; \underline{A_2=-1/25}.$$

Thus,

$$\frac{1}{(x+1)(x-4)^2}=\frac{1}{25}\left(\frac{1}{x+1}-\frac{1}{x-4}+\frac{5}{(x-4)^2}\right).$$

Consider now the case when the denominator of a proper fraction includes the irreducible factor (x^2+px+q).

Rule 3: Let $\dfrac{P(x)}{Q(x)}$ be a proper fraction and let

$$Q(x)=(x^2+px+q)\,Q_1(x).$$

Then there exists a unique proper fraction $\dfrac{P_1(x)}{Q_1(x)}$ and unique constants A and B such that the given rational function can be represented in the form

$$\frac{P(x)}{(x^2+px+q)Q_1(x)}=\frac{Ax+b}{x^2+px+q}+\frac{P_1(x)}{Q_1(x)}.$$

Note that $\mathrm{degree}(Q_1)=\mathrm{degree}(Q)-2$.

Then one can apply the above rules to the proper fraction $\dfrac{P_1(x)}{Q_1(x)}$, if its denominator includes either linear or irreducible factors.

Example 4: The proper fraction $\dfrac{1}{(x-3)(x^2-x+2)}$ is decomposed into partial fractions as follows:

$$\frac{1}{(x-3)(x^2-x+2)}=\frac{A_1}{(x-3)}+\frac{A_2x+B_2}{(x^2-x+2)}.$$

As above we get the equality

$$1=A_1(x^2-x+2)+(A_2x+B_2)(x-3)$$

and solve it with respect to A_1, A_2 and B_2:

$$x=3 \quad\Rightarrow\quad 1=8A_1 \quad\Rightarrow\quad A_1=1/8;$$

$$x=0 \quad\Rightarrow\quad 1=2A_1-3B_2 \quad\Rightarrow\quad \frac{3}{4}=-3B_2 \quad\Rightarrow B_2=-1/4;$$

$$x=1 \;\Rightarrow\; 1=2A_1+(A_2+B_2)(-2) \;\Rightarrow\; 1=\frac{1}{4}-2A_2+\frac{1}{2} \;\Rightarrow\; A_2=-\frac{1}{8}.$$

Thus, $$\frac{1}{(x-3)(x^2-x+2)}=\frac{1}{8}\Big(\frac{1}{(x-3)}-\frac{x+2}{(x^2-x+2)}\Big).$$

At last, we need only to consider the case when the denominator of a proper fraction includes n times repeated irreducible factor (x^2+px+q).

Rule 4: Let $\dfrac{P(x)}{Q(x)}$ be a proper fraction and let

$$Q(x)=(x^2+px+q)^n\,Q_1(x).$$

Then there exists a unique proper fraction $\dfrac{P_1(x)}{Q_1(x)}$ and unique constants $A_1, A_2, ..., A_n$ and $B_1, B_2, ..., B_n$ such that the given rational function can be represented in the form

$$\frac{P(x)}{(x^2+px+q)^n Q_1(x)}=\frac{A_1x+B_1}{x^2+px+q}+\frac{A_2x+B_2}{(x^2+px+q)^2}+...$$

$$+\frac{A_nx+B_n}{(x^2+px+q)^n}+\frac{P_1(x)}{Q_1(x)}.$$

Here $\text{degree}(Q_1)=\text{degree}(Q)-2n$.

Example 5: The partial decomposition technique gives

$$\frac{1}{(x-3)(x^2-x+2)^2}=\frac{A_1x+B_1}{x^2-x+2}+\frac{A_2x+B_2}{(x^2-x+2)^2}+\frac{A_3}{x-3}.$$

1.5.3.3. Factoring

One of the steps of decomposition of a proper fraction into a sum of partial fractions consists of factoring of the denominator $Q(x)$.

It is appropriate to mention here t**he fundamental theorem of algebra:**

Every polynomial can be factored into linear factors (polynomials of degree 1) and irreducible polynomials of degree 2.

Some **Examples** of **Factoring:**

- The polynomial $x^3 - 5x^2 - x - 15$ can be factored into a linear factor and an irreducible factor of degree 2:
$$x^3 - 5x^2 - x - 15 = (x-3)(x^2 - 2x + 5).$$
- The polynomial $x^2 + 6x + 9$ has a twice repeated linear factor (of degree 1):
$$x^2 + 6x + 9 = (x+3)^2.$$
- The polynomial $x^4 + 2x^2 + 1$ has a twice repeated irreducible factor of degree 2:
$$x^4 + 2x^2 + 1 = (x^2 + 1)^2.$$
- Both factors of the polynomial $x^4 + 1$ are irreducible ones of degree 2:
$$x^4 + 1 = (x^4 + 2x^2 + 1) - 2x^2 = (x^2 + 1)^2 - 2x^2$$
$$= (x^2 + 1 - \sqrt{2}x)(x^2 + 1 + \sqrt{2}x).$$

FAQ: How can we know whether a quadratic polynomial is irreducible or it can be factored further into two linear factors?
Answer: A reducible quadratic polynomial has two zeros or one repeated zero; an irreducible quadratic polynomial has no zeros. So if the quadratic formula results in a negative expression under the radical (the discriminant), the associated polynomial is irreducible.

- The quadratic polynomial $x^2 - 5x + 4$ has two zeros: $x_1 = 1$ and $x_2 = 4$. Therefore, it can be factored into two linear factors as follows:
$$x^2 - 5x + 4 = (x-1)(x-4).$$
- The quadratic polynomial $(x^2 - 4x + 4)$ has one repeated zero:
$x_1 = x_2 = 2$. Therefore, $x^2 - 4x + 4 = (x-2)^2$.
- Using the quadratic formula for the polynomial $(x^2 - 2x + 4)$ yields:
$$x_{1,2} = 1 \pm \sqrt{1-4} = 1 \pm \sqrt{-3}.$$
Since the discriminant is negative, the polynomial is irreducible.

1.5.4. Polynomial Long Division

Let $\dfrac{P(x)}{Q(x)}$ be a rational function, and let the degree of the polynomial $P(x)$ be greater than or equal to the degree of the polynomial $Q(x)$. Then there exist the uniquely determined polynomials $S(x)$ and $R(x)$, such that the rational function $\dfrac{P(x)}{Q(x)}$ can be represented in the form

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $\dfrac{R(x)}{Q(x)}$ is a proper fraction.

The polynomial $S(x)$ is called the quotient; the term $Q(x)$ is the divisor and the expression $R(x)$ is called the remainder. In the special case when the remainder equals zero, it is said that $Q(x)$ divides evenly into $P(x)$.

Let us consider the division algorithm in detail for particular examples.

Example 6: Perform polynomial long division if

$$f(x) = \frac{5x^3 - x^2 + 4x + 7}{x^2 + 3x - 1}.$$

First, we write the expression in a form of long division:

$$\begin{array}{l|l} 5x^3 - x^2 + 4x + 7 & x^2 + 3x - 1 \\ \cline{2-2} & \end{array}$$

Next we divide the leading term $5x^3$ in the numerator of the given polynomial by the leading term x^2 of the divisor, and write the answer $5x$ under the line:

$$\begin{array}{l|l} 5x^3 - x^2 + 4x + 7 & x^2 + 3x - 1 \\ \cline{2-2} & 5x \end{array}$$

Now we multiply the term $5x$ to the divisor $x^2 + 3x - 1$, and write the answer

$$5x(x^2 + 3x - 1) = 5x^3 + 15x^2 - 5x$$

under the numerator polynomial, lining up the terms of equal degree:

$$\begin{array}{l|l} 5x^3 \quad - x^2 + 4x + 7 & x^2 + 3x - 1 \\ \cline{2-2} 5x^3 + 15x^2 - 5x & 5x \end{array}$$

Then subtract the last line from the line above it:

$$\begin{array}{rl} - \begin{array}{l} 5x^3 \quad - x^2 + 4x + 7 \\ 5x^3 + 15x^2 - 5x \\ \hline \quad -16x^2 + 9x + 7 \end{array} & \begin{array}{|l} x^2 + 3x - 1 \\ \hline 5x \end{array} \end{array}$$

Now we have to repeat the procedure: to divide the leading term $(-16x^2)$ of the polynomial in the last line by the leading term x^2 of the divisor to obtain (-16), and add this term to the $5x$ under the line on the right-hand side:

$$\begin{array}{rl} - \begin{array}{l} 5x^3 \quad - x^2 + 4x + 7 \\ 5x^3 + 15x^2 - 5x \\ \hline \quad -16x^2 + 9x + 7 \end{array} & \begin{array}{|l} x^2 + 3x - 1 \\ \hline 5x - 16 \end{array} \end{array}$$

Then multiply the term (-16) by the divisor $x^2 + 3x - 1$, and write the answer

$$-16(x^2 + 3x - 1) = -16x^2 - 48x + 16$$

under the last line polynomial, lining up terms of equal degree:

$$\begin{array}{rl} - \begin{array}{l} 5x^3 \quad - x^2 + 4x + 7 \\ 5x^3 + 15x^2 - 5x \\ \hline \quad -16x^2 \quad + 9x + 7 \\ \quad -16x^2 - 48x + 16 \end{array} & \begin{array}{|l} x^2 + 3x - 1 \\ \hline 5x - 16 \end{array} \end{array}$$

Subtract the last line from the line above it:

$$\begin{array}{rl} - \begin{array}{l} 5x^3 \quad - x^2 + 4x + 7 \\ 5x^3 + 15x^2 - 5x \\ \hline - \begin{array}{l} -16x^2 \quad + 9x + 7 \\ -16x^2 - 48x + 16 \\ \hline \qquad 57x - 9 \end{array} \end{array} & \begin{array}{|l} x^2 + 3x - 1 \\ \hline 5x - 16 \end{array} \end{array}$$

At the next step we would divide the term $57x$ by the leading term x^2 of the divisor, not yielding a polynomial expression.
Therefore, the division procedure is terminated. The remainder is in the last line: $57x - 9$, and the quotient is $5x - 16$. One can see that the remainder $(57x - 9)$ has degree 1, which is less than the degree of the divisor.

Thus, we finally get:

$$\frac{5x^3 - x^2 + 4x + 7}{x^2 + 3x - 1} = (5x - 16) + \frac{57x - 9}{x^2 + 3x - 1}.$$

The easiest way to check the answer algebraically is to multiply both sides by the divisor:

$$5x^3 - x^2 + 4x + 7 = (5x - 16)(x^2 + 3x - 1) + (57x - 9).$$

Then we multiply out and simplify the right side:

$$\begin{aligned} 5x^3 - x^2 + 4x + 7 &= (5x - 16)(x^2 + 3x - 1) + (57x - 9) \\ &= 5x^3 + 15x^2 - 5x - 16x^2 - 48x + 16 + 57x - 9 \\ &= 5x^3 - x^2 + 4x + 7. \end{aligned}$$

Thus, we have the identity and so the answer is correct.

Example 7: Perform polynomial long division if $f(x) = \dfrac{x^3 - 4x^2 - x - 6}{x^2 - x + 2}$.

In a similar way as above we get:

$$\begin{array}{rl|l} - & x^3 - 4x^2 - x - 6 & x^2 - x + 2 \\ \cline{3-3} & x^3 - x^2 + 2x & x - 3 \\ \cline{1-2} - & -3x^2 + 3x - 6 & \\ & -3x^2 + 3x - 6 & \\ \cline{1-2} & 0 & \end{array}$$

In this case, the remainder equals zero, so $(x - 3)$ divides evenly into $(x^2 - x + 2)$.

Therefore, $$\frac{x^3 - 4x^2 - x - 6}{x^2 - x + 2} = x - 3.$$

Multiplying both sides by the divisor yields:

$$x^3 - 4x^2 - x - 6 = (x^2 - x + 2)(x - 3).$$

By polynomial long division, the polynomial $x^3 - 4x^2 - x - 6$ is factored, that is, it is written as the product of polynomials with lower degrees.

Summary example: Evaluate the integral $\displaystyle\int \frac{x^4 + 1}{x^3 - 9x} dx$ using the technique of integrating rational functions.

Solution: Since the degree of the numerator is greater than that of the denominator, we have to perform the polynomial long division to get

$$\frac{x^4+1}{x^3-9x} = x + \frac{9x^2+1}{x^3-9x}.$$

Next we factor the denominator:

$$x^3-9x = x(x^2-9) = x(x-3)(x+3).$$

Then we use the method of partial fractions to split the fraction $\frac{9x^2+1}{x^3-9x}$ into easily integrable ones:

$$\frac{9x^2+1}{x^3-9x} = \frac{9x^2+1}{x(x-3)(x+3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+3}.$$

Now we simplify this equality to get

$$9x^2+1 = A(x-3)(x+3) + Bx(x+3) + Cx(x-3).$$

To solve this equation with respect to the constants we take for x a few values:

$$\begin{array}{lllll} x=0 & \Rightarrow & 1=-9A & \Rightarrow & A=-1/9; \\ x=3 & \Rightarrow & 82=18B & \Rightarrow & B=41/9, \\ x=-3 & \Rightarrow & 82=18c & \Rightarrow & C=41/9. \end{array}$$

Therefore,

$$\frac{9x^2+1}{x^3-9x} = \frac{1}{9}\left(-\frac{1}{x} + \frac{41}{x-3} + \frac{41}{x+3}\right),$$

which implies

$$\frac{x^4+1}{x^3-9x} = x + \frac{9x^2+1}{x^3-9x} = x + \frac{1}{9}\left(-\frac{1}{x} + \frac{41}{x-3} + \frac{41}{x+3}\right).$$

Finally, we get:

$$\begin{aligned} \int \frac{x^4+1}{x^3-9x}dx &= \int \left(x - \frac{1}{9x} + \frac{41}{9(x-3)} + \frac{41}{9(x+3)}\right)dx \\ &= \int xdx - \frac{1}{9}\int \frac{dx}{x} + \frac{41}{9}\left(\int \frac{dx}{x-3} + \int \frac{dx}{x+3}\right) \\ &= \frac{x^2}{2} - \frac{1}{9}\ln|x| + \frac{41}{9}(\ln|x-3| + \ln|x+3|) + C \\ &= \frac{x^2}{2} - \frac{1}{9}\ln|x| + \frac{41}{9}\ln|x^2-9| + C. \end{aligned}$$

1.6. Integration of Trigonometric Functions

1.6.1. Integrals of the Form $\int \sin^m x \cos^n x \, dx$

We consider here two cases: either both exponents, m and n, are even numbers or at least one of them is odd.

Case 1: Let m and n be even numbers, that is, $m = 2k$ and $n = 2l$.

Then the powers of sines and cosines can be reduced step by step, using the following trigonometric identities:

$$2\sin^2 x = 1 - \cos 2x, \qquad (23a)$$

$$2\cos^2 x = 1 + \cos 2x, \qquad (23b)$$

$$2\sin x \cos x = \sin 2x. \qquad (24)$$

Indeed,

$$\int \sin^{2k} x \cos^{2l} x dx = \int (\sin^2 x)^k (\cos^2 x)^l dx$$

$$= \frac{1}{4} \int (1 - \cos 2x)^k (1 + \cos 2x)^l dx.$$

By removing parentheses, we obtain the sum of simpler integrals, some of which have to be further simplified in a similar way as above.

Case 2: Let n be an odd number: $n = 2k + 1$.
Then for any number m we get:

$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x \cos^{2k} x \cos x dx$$

$$= \int \sin^m x (1 - \sin^2 x)^k \cos x dx.$$

This form suggests the substitution $t = \sin x$, which implies $dt = \cos x dx$, and so

$$\int \sin^m x \cos^{2k+1} x dx = \int t^m (1 - t^2)^k dt.$$

If m is an odd number, then by making use of the substitution $t = \cos x$, we obtain

$$\int \sin^{2k+1} x \cos^n x dx = -\int (1 - t^2)^k t^n dt.$$

Thus, the problem of integration is reduced to a simple procedure of term-by-term integration of a linear combination of power functions.

Examples:

- $\int \cos^2 xdx = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{2}(x+\frac{1}{2}\sin 2x)+C$.
- $\int \sin^2 3x\cos^2 3xdx = \frac{1}{4}\int \sin^2 6xdx$ | by formula (24) |

$$= \frac{1}{8}\int (1-\cos 12x)dx = \frac{1}{8}(x-\frac{1}{12}\sin 12x)+C.$$

- $\int \cos^5 xdx = \int (1-\sin^2 x)^2 \cos xdx$ | by substitution $t=\sin x$ |

$$= \int (1-2t^2+t^4)dt = t-\frac{2}{3}t^3+\frac{1}{5}t^5+C$$

$$= \sin x-\frac{2}{3}\sin^3 x+\frac{1}{5}\sin^5 x+C.$$

- $\int \frac{\sin^5 x}{\cos x}dx = \int \frac{(1-\cos^2 x)^2}{\cos x}\sin xdx$ | by substitution $t=\cos x$ |

$$= -\int \frac{(1-t^2)^2}{t}dt = \int (\frac{1}{t}-2t+t^3)dt$$

$$= \ln|t|-t^2+\frac{t^4}{4}+C$$

$$= \ln|\cos x|-\cos^2 x+\frac{\cos^4 x}{4}+C.$$

1.6.2. Integration of Powers of Trigonometric Functions

1.6.2.1. Integrals of the Form $\int \frac{\mathbf{dx}}{\mathbf{sin^n x}}$ and $\int \frac{\mathbf{dx}}{\mathbf{cos^n x}}$

The power n is assumed to be a natural number. So there are two possible cases.

Case 1: Let n be an odd number, that is, $n=2k-1$.

In this case, both problems of integration, $\int \frac{dx}{\sin^n x}$ and $\int \frac{dx}{\cos^n x}$, can be solved by using of the substitutions $\cos x = t$ or $\sin x = t$, correspondingly:

$$\int \frac{dx}{\sin^{2k-1} x} = \int \frac{\sin xdx}{\sin^{2k} x} = \int \frac{\sin xdx}{(1-\cos^2 x)^k} = -\int \frac{d(\cos x)}{(1-\cos^2 x)^k} = -\int \frac{dt}{(1-t^2)^k},$$

$$\int \frac{dx}{\cos^{2k-1} x} = \int \frac{\cos xdx}{(1-\sin^2 x)^k} = \int \frac{dt}{(1-t^2)^k}.$$

Hence, the given integrals are transformed to integrals of proper fractions.

Case 2: Let n be an even number, that is, $n = 2k$.

The integral $\int \frac{dx}{\sin^n x}$ can be transformed by using of the trigonometric identities:

$$\frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = 1 + \cot^2 x \qquad \Rightarrow$$

$$\frac{1}{\sin^{2k} x} = (\frac{1}{\sin^2 x})^{k-1} \frac{1}{\sin^2 x} = (1 + \cot^2 x)^{k-1} \frac{1}{\sin^2 x}$$

$$\Rightarrow$$

$$\int \frac{dx}{\sin^{2k} x} = \int (1 + \cot^2 x)^{k-1} \frac{dx}{\sin^2 x} = -\int (1 + t^2)^{k-1} dt,$$

where $t = \cot x$.

As above, the integral $\int \frac{dx}{\cos^n x}$ can be evaluated by the substitution $t = \tan x$:

$$\int \frac{dx}{\cos^{2k} x} = \int (1 + \tan^2 x)^{k-1} \frac{dx}{\cos^2 x} = \int (1 + t^2)^{k-1} dt.$$

Thus, we have the integral of a polynomial.

Examples:

- $$\int \frac{dx}{\cos x} = \int \frac{\cos x dx}{\cos^2 x} = \int \frac{d(\sin x)}{1 - \sin^2 x} = \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + C.$$

- $$\int \frac{dx}{\cos^4 x} = \int (1 + \tan^2 x) \frac{dx}{\cos^2 x}$$

$$= \int (1 + t^2) dt = t + \frac{t^3}{3} + C = \tan x + \frac{\tan^3 x}{3} + C.$$

1.6.2.2. Integrals of the Form $\int \tan^n x\,dx$ and $\int \cot^n x\,dx$

As usual, the power n is assumed to be natural unless otherwise is stipulated.
Note that the given integrals are easily evaluated for $n = 1$ and $n = 2$. For instance,

$$\int \tan^2 xdx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \frac{dx}{\cos^2 x} - \int dx$$
$$= \tan x - \int dx = \tan x - x + C.$$

Hence, the problem of integration consists in lowering of the power n of tangents and cotangents, that can be easily carried out by using of trigonometric identities:

$$\int \tan^n xdx = \int \tan^{n-2} x\left(\frac{1}{\cos^2 x} - 1\right)dx = \int \tan^{n-2} x \frac{dx}{\cos^2 x} - \int \tan^{n-2} xdx.$$

Taking into account that

$$\int \tan^{n-2} x \frac{dx}{\cos^2 x} = \int \tan^{n-2} xd(\tan x) = \frac{\tan^{n-1} x}{n-1} + C,$$

we obtain the following reduction formula:

$$\int \tan^n xdx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} xdx. \tag{25}$$

Therefore, the problem of integration of $\tan^n x$ is reduced to that of integration of $\tan^{n-2} x$. In this way one can lower any natural power n to 1 or zero.
Similarly, one can also get the reduction formula for the cotangent function:

$$\cot^n x = \cot^{n-2} x \cot^2 x = \cot^{n-2} x\left(\frac{1}{\sin^2 x} - 1\right) \qquad \Rightarrow$$
$$\int \cot^n xdx = \int \cot^{n-2} x \frac{dx}{\sin^2 x} - \int \cot^{n-2} xdx \qquad \Rightarrow$$
$$\int \cot^n xdx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} xdx. \tag{26}$$

In particular, $\int \cot^3 xdx = -\frac{\cot^2 x}{2} - \int \cot xdx = -\frac{\cot^2 x}{2} - \ln|\sin x| + C$.

1.6.3. Integration of Products of Sines and Cosines

Each of the following integrals

$$\int \sin ax \cos bx dx, \qquad \int \sin ax \sin bx dx \quad \text{and} \quad \int \cos ax \cos bx dx$$

can be easily evaluated with the help of trigonometric identities:

$$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha-\beta)+\sin(\alpha+\beta)), \tag{27}$$

$$\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta)-\cos(\alpha+\beta)), \tag{28}$$

$$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha-\beta)+\cos(\alpha+\beta)), \tag{29}$$

Examples:

- $\int \sin 2x \cos x dx = \frac{1}{2}\int (\sin 3x + \sin x)dx = -\frac{1}{6}\cos 3x - \frac{1}{2}\cos x + C$.
- $\int \sin 5x \sin 3x dx = \frac{1}{2}\int (\cos 2x - \cos 8x)dx = \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$.

Sometimes it is necessary to apply identities (27) – (29) more than once to obtain the final result.

Example: In order to evaluate $\int \sin 2x \cos 3x \cos 4x dx$ it is necessary to transform the product of trigonometric function into their linear combination.
By identity (29) we have

$$\cos 3x \cos 4x = \frac{1}{2}(\cos x + \cos 7x).$$

Then we use identity (27):

$$\begin{aligned}\sin 2x \cos 3x \cos 4x &= \frac{1}{2}\sin 2x(\cos x + \cos 7x)\\ &= \frac{1}{2}(\sin 2x \cos x + \sin 2x \cos 7x)\\ &= \frac{1}{4}(\sin x + \sin 3x + \sin(-5x) + \sin 9x).\end{aligned}$$

Hence,

$$\int \sin 2x \cos 3x \cos 4x dx = -\frac{1}{4}(\cos x + \frac{1}{3}\cos 3x - \frac{1}{5}\cos 5x + \frac{1}{9}\cos 9x) + C.$$

1.6.4. Rational Expressions of Trigonometric Functions

1.6.4.1. General Substitution $\mathbf{t = \tan\frac{x}{2}}$

Let $P(x, y)$ and $Q(x, y)$ be polynomials with respect to variables x and y.

The quotient $R(x, y) = \dfrac{P(x, y)}{Q(x, y)}$ of two polynomials is a rational expression of x and y.

Likewise, the quotient

$$R(\sin x, \cos x) = \frac{P(\sin x, \cos x)}{Q(\sin x, \cos x)}$$

is called a rational expression of sine and cosine.

Note that all the other trigonometric functions are rational functions of sine and cosine.

Example 1: Such expressions as

$$\frac{2 - 3\sin x}{7 - 4\cos^2 x + 2\sin x}, \qquad \frac{1}{1 + \sqrt{3}\cos^5 x}, \qquad \frac{\cos x}{2 + 5\cos^3 x \sin x}$$

are rational ones of sine and cosine, but the expression $\dfrac{1}{1 + \sqrt{\cos x}}$ is not that.

Theorem: Let $R(\sin x, \cos x)$ be a rational expression of sine and cosine. Then there exists a rational function $f(t)$ such that

$$\int R(\sin x, \cos x)dx = \int f(t)dt\,.$$

Note: Any integral of a rational function can be evaluated. Therefore, the theorem states that any integral of a rational expression $R(\sin x, \cos x)$ can be transformed into the integral of a rational function and hence, can also be evaluated.

Proof: Let $t = \tan\dfrac{x}{2}$. Then $\sin x$ and $\cos x$ can be expressed through rational functions with respect to t by using of double-angle formulas:

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})} = \frac{2\tan\frac{x}{2}}{1 + \tan^2(\frac{x}{2})} = \frac{2t}{1 + t^2}, \qquad (30)$$

$$\cos x = \frac{\cos^2(\frac{x}{2}) - \sin^2(\frac{x}{2})}{\cos^2(\frac{x}{2}) + \sin^2(\frac{x}{2})} = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} = \frac{1-t^2}{1+t^2}. \qquad (31)$$

Moreover, from $x = 2\arctan t$ it follows that $dx = \dfrac{2dt}{1+t^2}$.

Therefore,

$$\int R(\sin x, \cos x)dx = \int R(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})\frac{2dt}{1+t^2} = \int f(t)dt,$$

where $f(t) = R(\dfrac{2t}{1+t^2}, \dfrac{1-t^2}{1+t^2})\dfrac{2}{1+t^2}$ is some rational function.

This completes the proof.

Example 2: By applying the substitution $t = \tan\dfrac{x}{2}$ we get

$$\int \frac{dx}{\sin x} = \int \frac{1}{\frac{2t}{1+t^2}} \frac{2dt}{(1+t^2)} = \int \frac{dt}{t} = \ln|t| + C = \ln|\tan\frac{x}{2}| + C.$$

Note that

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(x+\frac{\pi}{2})} = \int \frac{d(x+\frac{\pi}{2})}{\sin(x+\frac{\pi}{2})} = \ln|\tan(\frac{x}{2}+\frac{\pi}{4})| + C.$$

Example 3: Find $\displaystyle\int \frac{dx}{2+\cos x - \sin x}$

Solution: Let $t = \tan\dfrac{x}{2}$. Using simple algebraic manipulations we obtain:

$$\int \frac{dx}{2+\cos x-\sin x} = \int \frac{1}{2+\frac{1-t^2}{1+t^2}-\frac{2t}{1+t^2}} \cdot \frac{2dt}{1+t^2}$$

$$= 2\int \frac{dt}{2(1+t^2)+1-t^2-2t} = 2\int \frac{dt}{t^2-2t+3} = 2\int \frac{dt}{(t+1)(t-3)}.$$

The technique of partial fraction decomposition gives

$$2\int \frac{dt}{(t+1)(t-3)} = \frac{1}{2}\left(\int \frac{dt}{t-3} - \int \frac{dt}{t+1}\right)$$
$$= \frac{1}{2}(\ln|t-3| - \ln|t+1|) + C = \frac{1}{2}\ln\left|\frac{t-3}{t+1}\right| + C.$$

It remains to substitute $\tan\frac{x}{2}$ for t:

$$\int \frac{dx}{2+\cos x - \sin x} = \frac{1}{2}\ln\left|\frac{\tan x/2 - 3}{\tan x/2 + 1}\right| + C.$$

1.6.4.2. Other Substitutions

General substitution $t = \tan\frac{x}{2}$ enables us to evaluate integrals of the form $\int R(\sin x, \cos x)dx$ but very often in a complicated way.

However, there are a few specific cases when a rational expression $R(\sin x, \cos x)$ has even-odd symmetry. In these cases, integrals $\int R(\sin x, \cos x)dx$ can be transformed into integrals of rational functions by another trigonometric substitutions, which turn out often to be more preferable for integration of rational functions.

Let us consider these cases.

Case 1: If

$$R(-\sin x, \cos x) = -R(\sin x, \cos x),$$

then one can apply the substitution $t = \cos x$.

Case 2: If

$$R(\sin x, -\cos x) = -R(\sin x, \cos x),$$

then the suitable substitution is $t = \sin x$.

Case 3: If

$$R(-\sin x, -\cos x) = R(\sin x, \cos x),$$

then both substitutions, $t = \tan x$ and $t = \cot x$, are suitable.

As an example, let us give reasoning for Case 1.

Proof: The expression $R(\sin x, \cos x)$ is an odd rational function with respect to $\sin x$. Hence, we have

$$R(\sin x, \cos x) = \sin x \frac{R(\sin x, \cos x)}{\sin x} = \sin x \cdot R_1(\sin x, \cos x).$$

Here $R_1(\sin x, \cos x)$ is some even rational function with respect to $\sin x$ containing only even powers of sine.
Hence,

$$R_1(\sin x, \cos x) = R_2(\sin^2 x, \cos x) = R_2(1-\cos^2 x, \cos x).$$

However, the last rational expression is some rational function f with respect to $\cos x$:

$$R_2(1-\cos^2 x, \cos x) = f(\cos x).$$

Therefore, by making use of the substitution $\cos x = t$, we obtain

$$\int R(\sin x, \cos x)dx = \int f(\cos x)\sin xdx = \int f(t)dt$$

Hence, the desired result.
Other cases can be treated similarly.

Example 1: Find $\int \frac{\sin^3 x}{4-\cos^2 x}dx$.

Solution: This is Case 1, that is $R(-\sin x, \cos x) = -R(\sin x, \cos x)$.
Indeed,

$$\frac{(-\sin x)^3}{4-\cos^2 x} = -\frac{\sin^3 x}{4-\cos^2 x}.$$

Then for the substitution $t = \cos x$ we have $dt = -\sin xdx$ and

$$\sin^3 xdx = \sin^2 x\sin xdx = (1-\cos^2 x)\sin xdx = -(1-t^2)dt.$$

Therefore,

$$\int \frac{\sin^3 x}{4-\cos^2 x}dx = -\int \frac{1-t^2}{4-t^2}dt = \int (-1+\frac{3}{4-t^2})dt = -t+3\int \frac{dt}{4-t^2}$$

$$= -t + \frac{3}{4}\ln\left|\frac{t+2}{t-2}\right| + C = -\cos x + \frac{3}{4}\ln\left|\frac{\cos x+2}{\cos x-2}\right| + C.$$

Example 2: Find $\int \frac{\sin x\cos x}{3\sin x+1}dx$.

Solution: Here we have Case 2 due to the identity

$$\frac{\sin x(-\cos x)}{3\sin x+1} = -\frac{\sin x\cos x}{3\sin x+1}.$$

So we make the substitution

$$t = \sin x,$$

which gives $\cos xdx = dt$.

Therefore,

$$\int \frac{\sin x \cos x}{3\sin x+1}dx = \int \frac{t}{3t+1}dt = \frac{1}{3}\int (1-\frac{1}{3t+1})dt$$
$$= \frac{1}{3}(t-\frac{1}{3}\ln|3t+1|)+C$$
$$= \frac{1}{3}(\sin x-\frac{1}{3}\ln|3\sin x+1|)+C.$$

Example 3: Find $\int \frac{dx}{2\sin x\cos x-4\sin^2 x+5}$.

Solution: This is Case 3 since:

$$\frac{1}{2(-\sin x)(-\cos x)-4(-\sin x)^2+5} = \frac{1}{2\sin x\cos x-4\sin^2 x+5},$$

that is, $R(-\sin x,-\cos x) = R(\sin x,\cos x)$.

First, we transform the integrand:

$$\frac{1}{2\sin x\cos x-4\sin^2 x+5} = \frac{1}{2\sin x\cos x-4\sin^2 x+5(\sin^2 x+\cos^2 x)}$$
$$= \frac{1}{\sin^2 x+2\sin x\cos x+5\cos^2 x} = \frac{1}{\cos^2 x(\tan^2 x+2\tan x+5)}.$$

Therefore, the rational expression of sine and cosine has been transformed into the rational function of t by the above formula. So we have reduced the initial problem to integration of the rational function:

$$\int \frac{dx}{2\sin x\cos x-4\sin^2 x+5} = \int \frac{dt}{t^2+2t+5}$$
$$= \int \frac{d(t+1)}{(t+1)^2+4} = \frac{1}{2}\arctan\frac{t+1}{2}+C = \frac{1}{2}\arctan\frac{\tan x+1}{2}+C$$

For convenience sake, let us summarize the main results. The table gives substitutions and basic formulas for all the cases.

Table 3

Properties	Substitutions	Basic Formulas
$R(-\sin x, \cos x) =$ $= -R(\sin x, \cos x)$	$\cos x = t$	$-\sin x dx = dt$ $\sin^2 x = 1 - t^2$
$R(\sin x, -\cos x) =$ $= -R(\sin x, \cos x)$	$\sin x = t$	$\cos x dx = dt$ $\cos^2 x = 1 - t^2$
$R(-\sin x, -\cos x) =$ $= R(\sin x, \cos x)$	$\tan x = t$	$\dfrac{dx}{\cos^2 x} = dt$ $\sin^2 x = \dfrac{t^2}{1+t^2}$ $\cos^2 x = \dfrac{1}{1+t^2}$
	$\cot x = t$	$\dfrac{dx}{\sin^2 x} = -dt$ $\sin^2 x = \dfrac{1}{1+t^2}$ $\cos^2 x = \dfrac{t^2}{1+t^2}$
Any rational expression $R(\sin x, \cos x)$	$\tan \dfrac{x}{2} = t$	$dx = \dfrac{2dt}{1+t^2}$ $\sin x = \dfrac{2t}{1+t^2}$ $\cos x = \dfrac{1-t^2}{1+t^2}$

1.7. Integrals Involving Rational Exponents

1. Integrals with rational exponents $x^{\frac{1}{n}}$ can be transformed to integrals of rational functions by making use the substitution $x = u^n$, which implies $\sqrt[n]{x} = u$ and $dx = nu^{n-1}du$.

   Example 1: Let $\int \frac{dx}{\sqrt{x}+3}$ be a given integral.

   The substitution $x = u^2$ yields $\sqrt{x} = u$ and $dx = 2udu$, so that

$$\int \frac{dx}{\sqrt{x}+3} = 2\int \frac{udu}{u+3} = 2\int \frac{(u+3-3)du}{u+3}$$

$$= 2\int du - 6\int \frac{du}{u+3} = 2u - 6\ln|u+3| + C$$

$$= 2\sqrt{x} - 6\ln|\sqrt{x}+3| + C.$$

2. Integrals with a few rational exponents can be evaluated by the substitution $x = u^n$, where n is the least common multiple of the denominators of the exponents.

   Example 2: Consider the integral $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

   The substitution $x = u^6$ allows us to get rid of both square and cube radical signs without getting new fractional exponents. Then $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$ and $dx = 6u^5du$, so that

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = 6\int \frac{u^5du}{u^3+u^2} = 6\int \frac{u^3du}{u+1}.$$

   This integral of the rational function can be easily evaluated by employing a polynomial long division:

$$\int \frac{u^3du}{u+1} = \int (u+1)^2du - 3\int (u+1)du + 3\int du - \int \frac{du}{u+1}$$

$$= \frac{(u+1)^3}{3} - \frac{3(u+1)^2}{2} + 3u - \ln|u+1| + C$$

$$= \frac{(\sqrt[6]{x}+1)^3}{3} - \frac{3(\sqrt[6]{x}+1)^2}{2} + 3\sqrt[6]{x} - \ln|\sqrt[6]{x}+1| + C.$$

Therefore,

$$\int \frac{dx}{\sqrt[3]{u}+\sqrt{x}} = \frac{(\sqrt[6]{x}+1)^3}{3} - \frac{3(\sqrt[6]{x}+1)^2}{2} + 3\sqrt[6]{x} - \ln|\sqrt[6]{x}+1| + C.$$

3. Integrals involving expressions of the form $\sqrt[n]{\dfrac{ax+b}{cx+d}}$ can be evaluated by the substitution $\dfrac{ax+b}{cx+d} = u^n$, which eliminates the radical sign and yields x as a rational function of u: $x = \dfrac{u^n d - b}{a - u^n c}$.

1.8. Integrals Involving Radicals $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$

Consider integrals that involve the following radicals:

$$\sqrt{a^2 - x^2}, \qquad \sqrt{a^2 + x^2} \quad \text{or} \quad \sqrt{x^2 - a^2}.$$

In order to eliminate the radical sign, one needs to use appropriate substitutions, e.g., trigonometric substitutions.

Problem 1: Eliminate the radical sing for $\sqrt{a^2 - x^2}$.

Solution: The trigonometric identity $1 - \sin^2 x = \cos^2 x$ suggests the substitution $x = a\sin u$. Indeed,

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin^2 u} \\ &= \sqrt{a^2(1-\sin^2 u)} = \sqrt{a^2\cos^2 u} = a\cos u.\end{aligned}$$

Note: The same idea works for the cosine-substitution: $x = a\cos u$. In this case $\sqrt{a^2 - x^2} = a\sin u$.

Problem 2: Eliminate the radical sign for $\sqrt{a^2 + x^2}$.
Solution: The trigonometric identity

$$1 + \tan^2 u = \frac{1}{\cos^2 u},$$

hints at the substitution $x = a\tan u$. Then

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2 u} = \sqrt{a^2(1+\tan^2 u)} = \sqrt{\frac{a^2}{\cos^2 u}} = \frac{a}{\cos u}.$$

Note: One can also use the substitution $x = \cot u$, which gives $\sqrt{a^2 + x^2} = \dfrac{a}{\sin u}$.

Problem 3: Eliminate the radical sign for $\sqrt{x^2 - a^2}$.
Solution: Since the difference

$$\frac{1}{\sin^2 u} - 1 = \frac{1 - \sin^2 u}{\sin^2 u} = \frac{\cos^2 u}{\sin^2 u} = \cot^2 u$$

is the perfect square, the substitution $x = \dfrac{a}{\sin u}$ is suitable for eliminating of the radical:

$$\sqrt{x^2 - a^2} = \sqrt{\frac{a^2}{\sin^2 u} - a^2} = \sqrt{a^2 (\frac{1}{\sin^2 u} - 1)} = \sqrt{a^2 \cot^2 u} = a \cot u .$$

Note: The identity $\dfrac{1}{\cos^2 u} - 1 = \tan u$ suggests the substitution $x = \dfrac{a}{\cos u}$, which is also suitable for eliminating of the radical $\sqrt{x^2 - a^2}$.
In this case $\sqrt{x^2 - a^2} = a \tan u$.

The following examples illustrate applications of the above trigonometric substitutions for elimination of radical signs.

Example 1: Find $\displaystyle\int \frac{\sqrt{3 - x^2}}{x^2} dx$.
Solution: Let $x = \sqrt{3} \sin u$. Then

$$\int \frac{\sqrt{3 - x^2}}{x^2} dx = \int \frac{\sqrt{3} \cos u}{3 \sin^2 u} \sqrt{3} \cos u du$$

$$= \int \cot^2 u du = \int (\frac{1}{\sin^2 u} - 1) du = -\cot u - u + C$$

The solution is found in terms of u, and we have to express it in terms of x:

$$x = \sqrt{3} \sin u \qquad \Rightarrow \qquad u = \arcsin \frac{x}{\sqrt{3}},$$

$$\cot u=\frac{\cos u}{\sin u}=\frac{\sqrt{1-\sin^2 u}}{\sin u}=\frac{\sqrt{1-\sin^2\arcsin\frac{x}{\sqrt{3}}}}{\sin\arcsin\frac{x}{\sqrt{3}}}=\frac{\sqrt{1-\frac{x^2}{3}}}{\frac{x}{\sqrt{3}}}=\frac{\sqrt{3-x^2}}{x}.$$

Therefore, the final solution is

$$\int\frac{\sqrt{3-x^2}}{x^2}dx=-\frac{\sqrt{3-x^2}}{x}-\arcsin\frac{x}{\sqrt{3}}+C.$$

Example 2: Find $\int\frac{\sqrt{9+x^2}}{x^4}dx$.

Solution: Let $x=3\tan u$. Then $dx=\frac{3du}{\cos^2 u}$ and

$$\sqrt{9+x^2}=\sqrt{9+9\tan^2 u}=\sqrt{9(1+\tan^2 u)}=\sqrt{\frac{9}{\cos^2 u}}=\frac{3}{\cos u}.$$

Therefore,

$$\int\frac{\sqrt{9+x^2}}{x^4}dx=\int\frac{9}{81\tan^4 u\cos^3 u}du$$

$$=\frac{1}{9}\int\frac{\cos u du}{\sin^4 u}=\frac{1}{9}\int\frac{dt}{t^4} \qquad |\,\text{substitution} \quad t=\sin u\,|$$

$$=-\frac{1}{27t^3}+C=-\frac{1}{27\sin^3 u}+C.$$

It remains to express the answer in terms of x:

$$x=3\tan u \qquad \Rightarrow \qquad u=\arctan\frac{x}{3},$$

$$\sin u=\frac{\sin u}{\sqrt{\cos^2 u+\sin^2 u}}=\frac{\tan u}{\sqrt{1+\tan^2 u}}$$

$$=\frac{\tan(\arctan\frac{x}{3})}{\sqrt{1+\tan^2(\arctan\frac{x}{3})}}=\frac{\frac{x}{3}}{\sqrt{1+(\frac{x}{3})^2}}=\frac{x}{\sqrt{9+x^2}}.$$

Finally we have

$$\int \frac{\sqrt{9+x^2}}{x^4}dx = -\frac{(9+x^2)\sqrt{9+x^2}}{27x^3} + C.$$

Example 3: Find $\int \frac{dx}{x^2\sqrt{x^2-5}}$

Solution: Let $x = \frac{\sqrt{5}}{\sin u}$. Then $dx = -\frac{\sqrt{5}\cos u}{\sin^2 u}du$,

$$\sqrt{x^2-5} = \sqrt{\frac{5}{\sin^2 u} - 5} = \sqrt{\frac{5(1-\sin^2 u)}{\sin^2 u}} = \sqrt{5}\cot u$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{x^2-5}} = -\int \frac{\sqrt{5}\cos u \sin^2 u du}{5\sqrt{5}\cot u \sin^2 u} = -\frac{1}{5}\int \sin u du = \frac{\cos u}{5} + C.$$

Now we have to return to the initial variable x:

$$x = \frac{\sqrt{5}}{\sin u} \Rightarrow \qquad \sin u = \frac{\sqrt{5}}{x} \qquad \Rightarrow$$

$$\cos u = \sqrt{1-\sin^2 u} = \sqrt{1-(\frac{\sqrt{5}}{x})^2} = \sqrt{1-\frac{5}{x^2}} = \frac{\sqrt{x^2-5}}{x}.$$

Therefore,

$$\int \frac{dx}{x^2\sqrt{x^2-5}} = \frac{\sqrt{x^2-5}}{5x} + C.$$

Problem 4: Eliminate the radical sign for $\sqrt{\pm x^2 + px + q}$.

Solution: In order to evaluate an integral of expression involving the radical of this type, one has to complete the square of the quadratic trinomial. Then the previous methods can be used to solve the integrals.

Example 4: Consider the radical $\sqrt{x^2-6x+10}$.

Let us transform the quadratic polynomial under the radical sign to get a perfect square:

$$x^2 - 6x + 25 = (x^2 - 6x + 9) + 16 = (x-3)^2 + 4^2.$$

Then we can use the tangent-substitution $x - 3 = 4\tan u$ to solve the problem.

1.9. Integrals of the Form $\int x^{\mathbf{m}}(a+bx^{\mathbf{n}})^{\mathbf{p}}\,dx$

Chebyshev proved the following **theorem**:
Let m, n and p be rational numbers.
Then the following integral

$$\int x^m (a+bx^n)^p\,dx$$

is evaluated in terms of elementary functions if and only if there is an integer among the numbers p, $\frac{m+1}{n}$ and $\frac{m+1}{n}+p$.

Proof: Consider three cases.

1) Let the number p be an integer, and let s be the least common multiple of the denominators of the exponents m and n. Then by substitution $x=t^s$, the given integral can be transformed to the integral of a rational function. Therefore, it can be evaluated in terms of elementary functions.

2) Let the number $\frac{m+1}{n}$ be an integer. By making the substitution $x^n = z$, that is, $x = z^{1/n}$ we get

$$\int x^m (a+bx^n)^p\,dx = \int z^{\frac{m}{n}}(a+bz)^p \frac{1}{n} z^{\frac{1}{n}-1}dz = \frac{1}{n}\int z^{\frac{m+1}{n}-1}(a+bz)^p\,dz\,.$$

If s is the denominator of the rational number p, then by substitution $a+bz=t^s$ we obtain the integral of a rational function.

3) Let the number $\frac{m+1}{n}+p$ be an integer. The last integral can be written as:

$$\int z^{\frac{m+1}{n}-1}(a+bz)^p\,dz = \int z^{\frac{m+1}{n}+p-1}\left(\frac{a+bz}{z}\right)^p dz\,.$$

Therefore, by substitution $\frac{a+bz}{z}=t^s$ it is transformed to the integral of a rational function (s is the denominator of the rational number p).

Thus, all the cases are investigated.

In conclusion, we give the table of substitutions for all these cases.

Table 4

Integer	Substitutions
p	$x = u^s$ s is the least common multiple of the denominators of the rational exponents m and n
$\dfrac{m+1}{n}$	$a + bx^n = t^s$ s is the denominator of the rational exponent p
$\dfrac{m+1}{n} + p$	$\dfrac{a}{x^n} + b = t^s$ s is the denominator of the rational exponent p

Example: Consider $\int \sqrt[4]{1+x^2}\,dx$.

Here all numbers, $p = 1/4$, $\dfrac{m+1}{n} = \dfrac{0+1}{2} = \dfrac{1}{2}$ and $\dfrac{m+1}{n} + p = \dfrac{3}{4}$, are not integer. Hence, the given integral cannot be expressed through a finite number of elementary functions.

1.10. Some Irreducible Integrals

Integrals of rational functions are evaluated straightforward, and the answer is expressed in terms of rational functions, logarithms, and inverse trigonometric functions.

But it is still possible to find even fairly simple looking integrals that just cannot be done in terms of elementary functions such as exponentials, logarithms, trigonometric functions and so on.

Liouville showed that the integrals given below cannot be expressed in terms of a finite number of elementary functions:

$$\int e^{-x^2}dx, \quad \int \frac{e^x}{x}dx, \quad \int \frac{\sin x}{x}dx, \quad \int \frac{\cos x}{x}dx, \quad \int \frac{dx}{\ln x}.$$

Each of the following integrals is also irreducible:

$$\int x^x dx, \quad \int \frac{\arctan x}{x}dx, \quad \int \frac{\ln x}{x+1}dx.$$

1.11. Extended List of Common Indefinite Integrals

The table below gives the list of the most important indefinite integrals.

Table 5

$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$	$\int \frac{dx}{x-a} = \ln\lvert x-a\rvert + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C$	
$\int \cos(ax+b)dx = \frac{1}{a}\sin(ax+b) + C$	
$\int \frac{dx}{\cos^2 x} = \tan x + C$	$\int \frac{dx}{\sin^2 x} = -\cot x + C$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \begin{cases} \arcsin\frac{x}{a} + C \\ -\arccos\frac{x}{a} + C \end{cases}$	$\int \frac{dx}{a^2+x^2} = \begin{cases} \frac{1}{a}\arctan\frac{x}{a} + C \\ -\frac{1}{a}\cot^{-1}\frac{x}{a} + C \end{cases}$
$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}) + C$	$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} + C$
$\int \tan x dx = -\ln\lvert\cos x\rvert + C$	$\int \cot x dx = \ln\lvert\sin x\rvert + C$
$\int \frac{dx}{\sin x} = \ln\lvert\tan\frac{x}{2}\rvert + C$	$\int \frac{dx}{\cos x} = \ln\lvert\tan(\frac{x}{2} + \frac{\pi}{4})\rvert + C$
$\int e^{ax}\sin bx dx = \frac{a\sin bx - b\cos bx}{a^2+b^2}e^{ax} + C$	
$\int e^{ax}\cos bx dx = \frac{a\cos bx + b\sin bx}{a^2+b^2}e^{ax} + C$	
$\int \frac{dx}{(x^2+a^2)^{n+1}} = \frac{1}{2a^2}\left(\int \frac{dx}{(x^2+a^2)^n} + \frac{x}{(x^2+a^2)^n}\right)$	
$\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$	