

22 秋中俄班数学分析(1)期末试题解答英文版

1. Solution: (1) $\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $|x_n| < \frac{\varepsilon}{2}$ when $n > N_1$.

Let $N_2 = \max \left\{ N_1, \frac{2(|x_1| + |x_2| + \dots + |x_{N_1}|)}{\varepsilon} \right\}$, when $n > N_2$,

$$\begin{aligned} & \left| \frac{x_1 + x_2 + \dots + x_n}{n} - 0 \right| \\ & \leq \frac{|x_1| + |x_2| + \dots + |x_{N_1}| + |x_{N_1+1}| + \dots + |x_n|}{n} \\ & = \frac{|x_1| + |x_2| + \dots + |x_{N_1}|}{n} + \frac{|x_{N_1+1}| + \dots + |x_n|}{n} \\ & < \frac{\varepsilon}{2} + \frac{n - N_1}{n} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence, $\lim_{n \rightarrow \infty} \frac{x_1 + x_2 + \dots + x_n}{n} = 0$.

(2) Suppose $x_n = a + \alpha_n$, $y_n = b + \beta_n$, where $\lim_{n \rightarrow \infty} \alpha_n = 0$,

$\lim_{n \rightarrow \infty} \beta_n = 0$. Then

$$\begin{aligned} & \frac{x_1 y_n + x_2 y_{n-1} + \dots + x_n y_1}{n} \\ & = \frac{(a + \alpha_1)(b + \beta_n) + (a + \alpha_2)(b + \beta_{n-1}) + \dots + (a + \alpha_n)(b + \beta_1)}{n} \\ & = ab + b \cdot \frac{\alpha_1 + \alpha_2 + \dots + \alpha_n}{n} + a \cdot \frac{\beta_1 + \beta_2 + \dots + \beta_n}{n} + \\ & \quad \frac{\alpha_1 \beta_n + \alpha_2 \beta_{n-1} + \dots + \alpha_n \beta_1}{n} \end{aligned}$$

It follows from (1) that

$$\lim_{n \rightarrow \infty} \frac{\alpha_1 + \alpha_2 + \dots + \alpha_n}{n} = \lim_{n \rightarrow \infty} \frac{\beta_1 + \beta_2 + \dots + \beta_n}{n} = 0. \text{ Since}$$

$\lim_{n \rightarrow \infty} \beta_n = 0$, $\exists M > 0$ such that $\forall n \in \mathbb{N}$, it always holds that

$|\beta_n| \leq M$. Thus,

$$\begin{aligned} 0 &\leq \left| \frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n} \right| \leq \frac{|\alpha_1||\beta_n| + |\alpha_2||\beta_{n-1}| + \cdots + |\alpha_n||\beta_1|}{n} \\ &\leq M \cdot \frac{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|}{n} \end{aligned}$$

Since $\lim_{n \rightarrow \infty} |\alpha_n| = 0$, it follows from (1) that

$\lim_{n \rightarrow \infty} \frac{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|}{n} = 0$. Hence, it follows from Squeeze

Theorem that $\lim_{n \rightarrow \infty} \left| \frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n} \right| = 0$, i.e.,

$\lim_{n \rightarrow \infty} \frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n} = 0$. Hence,

$b \cdot \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} + a \cdot \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} + \frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n} \rightarrow 0$

$(n \rightarrow \infty)$, i.e., $\lim_{n \rightarrow \infty} \frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} = ab$.

2. Solution: Since the convergence of numerical series is independent from its first finite terms, we can assume that $\forall n \in \mathbb{N}^*$, it holds that

$b_n \cdot \frac{a_n}{a_{n+1}} - b_{n+1} \geq \delta > 0$. Hence, $\forall n \in \mathbb{N}^*$,

$a_{n+1} \leq \frac{1}{\delta} (a_n b_n - a_{n+1} b_{n+1})$,

$\sum_{k=1}^n a_k \leq a_1 + \frac{1}{\delta} \sum_{k=2}^n (a_{k-1} b_{k-1} - a_k b_k) = a_1 + \frac{1}{\delta} (a_1 b_1 - a_n b_n) < a_1 + \frac{a_1 b_1}{\delta}$

Hence, the numerical series $\sum_{n=1}^{\infty} a_n$ converges.

3. Solution: Suppose that M, m are respectively the upper and lower bound of the infinite subset S of real numbers. Assume that S has no limit point, then $\forall x \in [m, M]$, there exists a neighborhood U_x such that $U_x \cap S$ is finite. Thus, $\{U_x\}_{x \in [m, M]}$ forms an open covering of the closed interval $[m, M]$. By the Finite Covering Theorem, there exists a finite sub-covering $\{U_{x_1}, U_{x_2}, \dots, U_{x_N}\}$ of $[m, M]$, i.e.,

$S \subset [m, M] \subset \bigcup_{i=1}^N U_{x_i}$. However, $\bigcup_{i=1}^N U_{x_i}$ is finite, which contradicts with

the fact that S is infinite. Hence, the assumption is false, i.e., S has limit point.

4. Solution:

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\tan(\tan x) - \sin(\sin x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{\tan(\tan x) - \sin(\tan x) + \sin(\tan x) - \sin(\sin x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{\tan(\tan x) - \sin(\tan x)}{x^3} + \lim_{x \rightarrow 0} \frac{\sin(\tan x) - \sin(\sin x)}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{\frac{1}{2} \tan^3 x}{x^3} + \lim_{x \rightarrow 0} \frac{2 \cos\left(\frac{\tan x + \sin x}{2}\right) \sin\left(\frac{\tan x - \sin x}{2}\right)}{x^3} \\
&= \frac{1}{2} + 2 \lim_{x \rightarrow 0} \frac{\frac{\tan x - \sin x}{2}}{x^3} = \frac{1}{2} + \lim_{x \rightarrow 0} \frac{\frac{1}{2} x^3}{x^3} = \frac{1}{2} + \frac{1}{2} = 1
\end{aligned}$$

5. Solution: “Sufficiency” Define $f(a) = f(a + 0)$,

$f(b) = f(b - 0)$, then $f(x)$ is continuous on the closed interval $[a, b]$.

Hence, $f(x)$ is uniformly continuous on $[a, b]$, and consequently also uniformly continuous in (a, b) .

“Necessity” Since $f(x)$ is uniformly continuous in (a, b) , for

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that for any $x_1, x_2 \in (a, b)$ and $|x_1 - x_2| < \delta$, it holds that $|f(x_1) - f(x_2)| < \varepsilon$. In particular, if $x_1, x_2 \in (a, a + \delta)$, it also holds that $|f(x_1) - f(x_2)| < \varepsilon$. By Cauchy Convergence Principle, $f(a + 0)$ exists. Similarly, $f(b - 0)$ also exists.

6. Solution: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{1+t^2}}{\frac{2t}{1+t^2}} = \frac{1}{2t}$,

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = \frac{d\left(\frac{dy}{dt}\right)}{dx} = \frac{-\frac{1}{2t^2}}{\frac{2t}{1+t^2}} = -\frac{1+t^2}{4t^3}, \quad \left.\frac{d^2y}{dx^2}\right|_{t=1} = -\frac{1}{2}.$$

7. Solution: let $F(x) = f(x)(b-x)^a$, then $F(x)$ is continuous on $[a, b]$, differentiable in (a, b) , and $F(a) = f(a)(b-a)^a = 0$ ($f(a) = 0$), $F(b) = f(b)(b-b)^a = 0$, by Rolle’s Theorem, $\exists \xi \in (a, b)$ such that $F'(\xi) = 0$, i.e.,

$$f'(\xi)(b-\xi)^a + f(\xi)a(b-\xi)^{a-1}(-1) = 0,$$
equivalently

$$f(\xi) = \frac{b-\xi}{a} f'(\xi).$$

8. Solution: Without loss of generality, assume that $f(x)$ reaches its minimum value -1 at $x = x_0$, where $x_0 \in (0, 1)$, then $f'(x_0) = 0$.

By Taylor's Theorem,

$$f(0) = f(x_0) + \frac{f'(x_0)}{1!}(0 - x_0) + \frac{f''(\xi_1)}{2!}(0 - x_0)^2, \quad \xi_1 \in (0, x_0)$$

$$f(1) = f(x_0) + \frac{f'(x_0)}{1!}(1 - x_0) + \frac{f''(\xi_2)}{2!}(1 - x_0)^2, \quad \xi_2 \in (x_0, 1)$$

$$\text{i.e., } 0 = -1 + \frac{f''(\xi_1)}{2!}x_0^2, \quad 0 = -1 + \frac{f''(\xi_2)}{2!}(1 - x_0)^2. \text{ Hence,}$$

$$f''(\xi_1) = \frac{2}{x_0^2}, \quad f''(\xi_2) = \frac{2}{(1 - x_0)^2}. \text{ By the continuity of } f''(x),$$

$\exists \xi_3 \in [\xi_1, \xi_2]$ such that

$$f''(\xi_3) = \frac{1}{2}[f''(\xi_1) + f''(\xi_2)] = \frac{1}{x_0^2} + \frac{1}{(1 - x_0)^2}.$$

Next, study the minimum value of the function $y = \frac{1}{x^2} + \frac{1}{(1 - x)^2}$ in

$(0, 1)$.

$$y'(x) = \frac{-2}{x^3} + \frac{2}{(1 - x)^3} = 2 \left[\frac{x^3 - (1 - x)^3}{x^3(1 - x)^3} \right] = 2 \frac{(2x - 1)(x^2 - x + 1)}{x^3(1 - x)^3}$$

If $x \in \left(0, \frac{1}{2}\right)$, $y'(x) < 0$, $y(x)$ is monotonically decreasing; if

$x \in \left(\frac{1}{2}, 1\right)$, $y'(x) > 0$, $y(x)$ is monotonically increasing. Thus,

$y\left(\frac{1}{2}\right) = 8$ is the minimum value. Hence,

$$M = \max_{0 \leq x \leq 1} f''(x) \geq f''(\xi_3) \geq 8.$$