§1. First Order Differential Equation. (Existence of solution)

Def. $\dot{x} = X(t,x)$, where $X(t,x)$ is continuous on set $D \subset \mathbb{R}^2$.

First order ordinary differential equation solved w.r.t. derivative.

the solution : $x = \varphi(t)$. defined on $\langle a,b \rangle$.

If. $x = \varphi(t)$ is the solution. with $t \in \langle a,b \rangle$. then.

1) a point $(t, \varphi(t))$. belongs to the set D. for all $t \in \langle a,b \rangle$

2) $\varphi(t)$ is continuously differentiable on $\langle a,b \rangle$

the graph - integral curve.

Def. Cauchy Problem.

Find solution $x = \varphi(t)$. satisfying the condition $x_0 = \varphi(t_0)$ ($t_0 \in \langle a,b \rangle$, $(t_0, x_0) \in D$)

$x(t_0) = x_0$. initial condition. (t_0, x_0). initial data.

$\dot{\varphi}(t_0) = X(t_0, x_0)$, $X(t_0, x_0)$ is the tangent of the slope

$X(t,x)$ on D. defines direction field.

Δ $D = \{(t,x) : |t-t_0| \le a, |x-x_0| \le b\}$ we need a rectangle field of D.

$M = \max |X(t,x)|$.

$h = \min(a, b/M)$

Thm. Peano's Theorem.

For. $X : D \to \mathbb{R}$. $D \subset \mathbb{R}^2$. X is continuous. $\dot{x} = X(t,x)$ on. segment $[t_0 - h, t_0 + h]$ ⇓ peano segment.

there exists a solution. of Cauchy Problem.

(Actually. $\exists$ a neighborhood of t_0 and a function $\varphi : I \to \mathbb{R}$

s.t. for $\forall t \in I$. $X(t, \varphi(t)) = \varphi'(t)$)

Pf: prove the existence of solution on the segment $P = [t_0, t_0 + h]$.

splitting : $d_k = \{ t_0 = t_0^k < t_1^k < \cdots < t_{n_k - 1}^k < t_{n_k}^k = t_0 + h \}$.

set $\lambda_k = \text{rank}\, d_k = \max\limits_{j=1 \cdots n_k} (t_j^k - t_{j-1}^k)$ the rank of splitting.

(when $k \to +\infty$, $\lambda_k \to 0$).

证明这是一个解 For every splitting d_k, build Euler polyline $\varphi_k(t)$.

Let $\boxed{\varphi_k(t_0) = x_0}$. Then (利用递归方法, 切线拟合) when $\lambda_k \to 0$.

$\varphi_k(t) := \varphi_k(t_{j-1}^k) + X(t_{j-1}^k, \varphi_k(t_{j-1}^k))(t - t_{j-1}^k)$. for $t_{j-1}^k \le t \le t_j^k$, $j = 1 \ldots n_k$.

1) we need to show: $|\varphi_k(t) - x_0| \le M(t - t_0)$. holds for all $t \in P$.

(then for any $k \in \mathbb{N}$. defines $\varphi_k(t)$. throughout P). Actually we have $|\varphi_k(t) - x_0| \le b$.

show by induction.

Base : If $t \in [t_0, t_1^k]$. $\varphi_k(t) = x_0 + X(t_0, x_0)(t - t_0)$.

thus $|\varphi_k(t) - x_0| = |X(t_0, x_0)|\,|t - t_0| \le M(t - t_0)$.

by assumption. $|\varphi_k(t_{j-1}^k) - x_0| \le M(t_{j-1}^k - t_0)$. we have $|\varphi_k(t_{j-1}^k) - x_0| < b$. $(t_{j-1}^k, \varphi_k(t_{j-1}^k)) \in D$.

$|\varphi_k(t) - x_0| \le |\varphi_k(t) - \varphi_k(t_{j-1}^k)| + |\varphi_k(t_{j-1}^k) - x_0| \le |X(t_{j-1}^k, \varphi_k(t_{j-1}^k))(t - t_{j-1}^k)| + M(t_{j-1}^k - t_0)$

$\le M(t - t_0)$.

For every splitting d_k of P. define step function $\psi_k(t)$

$$\psi_k(t) = \begin{cases} X(t_{j-1}^k, \varphi_k(t_{j-1}^k)) & \text{if. } t_{j-1}^k \le t \le t_j^k, \ j=1,\dots,n_k. \\ X(t_{n_k}^k, \varphi_k(t_{n_k}^k)) \end{cases}$$

step function

$\psi_k(t)$ derivative / slope of φ_k

2) Now we need to show $\varphi_k(t) = x_0 + \int_{t_0}^{t} \psi_k(\tau)\,d\tau$ (For any $t \in P$. $k \in \mathbb{N}$.).

By induction on j.

Base: $j=1$. $\varphi_k(t) = x_0 + X(t_0, x_0)(t-t_0) = x_0 + \psi_k(t_0)(t-t_0) = x_0 + \int_{t_0}^{t} \psi_k(\tau)\,d\tau$.

hypothesis: $\varphi_k(t_{j-1}^k) = x_0 + \int_{t_0}^{t_{j-1}^k} \psi_k(\tau)\,d\tau$.

$$\begin{aligned} \varphi_k(t) &= \varphi_k(t_{j-1}^k) + \varphi_k(t) - \varphi_k(t_{j-1}^k). \\ &= x_0 + \int_{t_0}^{t_{j-1}^k} \psi_k(\tau)\,d\tau + X(t_{j-1}^k, \varphi_k(t_{j-1}^k))(t - t_{j-1}^k) = \dots \end{aligned}$$

Def. (Uniformly bounded). 一致有界

Functional Sequence $\{\xi_k(t)\}_{k=1}^{+\infty}$ given on the segment $[c,d]$.

there exists a number K. s.t. $|\xi_k(t)| \le K$. for all $t \in [c,d]$. $k \in \mathbb{N}$.

Def. (equicontinuous). (on some segment) 等度连续.

Function Sequence $\{\xi_k(t)\}_{k=1}^{+\infty}$. given on the segment $[c,d]$.

for any $\varepsilon > 0$. $\exists \delta > 0$. s.t. for any $k \in \mathbb{N}$. and any $t_1, t_2 \in [c,d]$. s.t. $|t_1 - t_2| < \delta$.

we have $|\xi_k(t_1) - \xi_k(t_2)| < \varepsilon$.

Arzela - Ascoli lemma.

If the function sequence $\{\xi_k(t)\}_{k=1}^{+\infty}$ is uniformly bounded and equicontinuous on the segment $[c,d]$. then there exists a subsequence $\{\xi_{k_m}(t)\}_{m=1}^{+\infty}$ which converges uniformly on this segment as $m \to +\infty$ to some function $\xi(t)$

(在 $[c,d]$ 上有收敛子列).

We construct Euler polyline sequence $\{\varphi_k(t)\}_{k=1}^{+\infty}$

3) we need to show it's uniformly bounded and equicontinuous on P.

uni. bound: $|\varphi_k(t)| \le |\varphi_k(t) - x_0| + |x_0| \le \underline{M(t-t_0)} + x_0 \le b + |x_0|$ → first conclusion used.

equiconti: $\forall \varepsilon > 0$. Let $\delta = \frac{\varepsilon}{M}$. $\forall t_1, t_2$ s.t. $|t_1 - t_2| < \delta$.

$$|\varphi_k(t_1) - \varphi_k(t_2)| = \underline{\left| \int_{t_2}^{t_1} \psi_k(\tau)\,d\tau \right|} \le |t_1 - t_2| \cdot M = \varepsilon.$$

→ second conclusion used.

From A..A. lemma. we have a subsequence and some function $\underline{\xi(t)}$. 有了 $\varphi(t)$. 证明是解.

By the uniqueness of uniform conv. function. $\lim_{m\to\infty} \varphi_{k_m}(t) = \varphi(t)$.

$\boxed{\text{w.l.g.}}$ we can let. $\{\varphi_{k_m}(t)\}_1 = \{\varphi_k(t)\}_1$ → if not. 调整分割方式. i.e. $\lim_{k\to\infty} \varphi_k(t) = \varphi(t)$

// 到此, 实质是证明了一个类似积分的问题. 即拟合曲线取点不影响结果.

此处是 existence but not uniqueness 的缘由. different splitting.

4) Function sequence $\psi_k(t)$ uniformly on P converges to the function $X(t,\varphi(t))$ at $k\to+\infty$. // If this holds, $\varphi_k(t)=x_0+\int_{t_0}^{t}\psi_k(\tau)d\tau \rightrightarrows \varphi(t)=x_0+\int_{t_0}^{t}X(\tau,\varphi(\tau))d\tau$.

5 Pf: $X(t,x)$ continuous on compact D. by Cantor's thm. it's uni. conv. on D.
$\forall\varepsilon>0.\ \exists\delta>0$: $|t_1-t_2|<\delta$, $|x_1-x_2|<\delta$ it follows that $|X(t_1,x_1)-X(t_2,x_2)|<\frac{\varepsilon}{2}$.
for any $(t_1,x_1),(t_2,x_2)\in D$.

6 Also we have $\lim\limits_{k\to\infty}\varphi_k(t)=\varphi(t)$. $\exists k_1\in\mathbb{N}$. s.t. $|\varphi(t)-\varphi_k(t)|<\delta$ for any $k>k_1$, $t\in P$.
Thus we have $|X(t,\varphi(t))-X(t,\varphi_k(t))|<\frac{\varepsilon}{2}$ (for any $k>k_1$. $t\in P$).

Since $\lambda_k\to 0$. $\exists k_2\in\mathbb{N}$. s.t. $\lambda_k<\min(\delta,\frac{\delta}{M})$. for any $k>k_2$, $t\in P$.

(7) We need to show $|X(t,\varphi_k(t))-\psi_k(t)|<\frac{\varepsilon}{2}$. holds for $k>k_2$, $t\in P$.

Fix $t=t^k_{n_k}$ →正好在"step"端点上. $\psi_k(t)=X(t^k_{n_k},\varphi_k(t^k_{n_k}))$, we have the inequality.

If $t<t^k_{n_k}$, $\exists t^k_{j-1}$ →框定区间. s.t. $|t-t^k_{j-1}|<\delta$. (此处是 $k>k_2$. 不能用 k_1 结论)

$|\varphi_k(t)-\varphi_k(t^k_{j-1})|=|X(t^k_{j-1},\varphi_k(t^k_{j-1}))|(t-t^k_{j-1})\le M\lambda_k=\delta$. (conclusion 2).

⇓ 证明结论 (5) 中. $|x_1-x_2|<\delta$.

thus. $|X(t,\varphi_k(t))-X(t^k_{j-1},\varphi_k(t^k_{j-1}))|<\frac{\varepsilon}{2}$.

Let $k_0=\max\{k_1,k_2\}$, for any $k>k_0$. $t\in P$.

$$|X(t,\varphi(t))-\psi_k(t)|\le|X(t,\varphi(t))-X(t,\varphi_k(t))|+|X(t,\varphi_k(t))-\psi_k(t)|$$
$$<\frac{\varepsilon}{2}\ ((6).\ k>k_1)+\frac{\varepsilon}{2}\ ((7).\ k>k_2)\quad\square$$

$$\varphi_k(t)=x_0+\int_{t_0}^{t}\psi_k(\tau)d\tau \rightrightarrows \varphi(t)=x_0+\int_{t_0}^{t}X(\tau,\varphi(t))d\tau\quad(k\to+\infty).$$

$$(\varphi(t))'=\left(x_0+\int_{t_0}^{t}X(\tau,\varphi(\tau))d\tau\right)'\Rightarrow\varphi'(t)=X(t,\varphi(t))\quad //$$

Technique 1: 解简单一阶常微分方程. (分离变量).

整理: ( ) dt = ( ) dx 两边做积分

Let $X(t,x)$ be continuous in domain G.

Thm 2. Let $(t_0,x_0)\in G$. Then there is $h>0$. s.t. for $|t-t_0|\le h$. the solution of the Cauchy problem $\dot{x}=X(t,x)$ or (t_0,x_0). is define.

Pf: G is open. $\exists a,b\in\mathbb{R}$. $a,b>0$. s.t. $D=\{(t,x):|t-t_0|\le a,\ |x-x_0|\le b\}$. $D\subset G$.
By Peono's thm...

Def. (Uniqueness).

The solution $x=\varphi(t)$ of Cauchy Problem, define for $t\in\langle a,b\rangle$ ($t_0\in\langle a,b\rangle$) is unique on the interval $\langle a,b\rangle$. if any solution of this problem, defined on the same interval, coincide with $\varphi(t)$. ($\forall$ solution $x=\psi(t)$ s.t. $\psi(t)\equiv\varphi(t)$ on $\langle a,b\rangle$)

Thm 3. Let the condition of the Peano's theorem be satisfied, let the solution $x=\varphi(t)$ of Cauchy problem. $\dot{x}=X(t,x)$ s.t. $x(t_0)=x_0$ provided by this thm is unique as a solution defined on the segment $P=[t_0, t_0+h]$.

Then for any sequence of splittings $\{d_k\}_{k=1}^{+\infty}$ of the segment P. satisfying the condition $\lambda_k = \text{rank}\, d_k \underset{k\to+\infty}{\to} 0$. the sequence of Euler polylines $\{\varphi_k(t)\}_{k=1}^{+\infty}$.

uni conv. to $\varphi(t)$ on P. (◎第一段均为前提).

Pf: Assume the converse. $\{d_k\}_{k=1}^{+\infty}$ $\lambda_k = \text{rank}\, d_k \underset{k\to+\infty}{\to} 0$ s.t. $\varphi_k(t)$ not uni. conv. to $\varphi(t)$.

$\exists\, \varepsilon_0>0$. s.t. for any $k_0\in\mathbb{N}$. $\exists\, k>k_0$. $t\in P$. s.t. $|\varphi(t)-\varphi_k(t)| \geq \varepsilon_0$

thus. $\exists\, \{k_j\}_{j=1}^{+\infty}$ (splitting. increasing). and $\{t_j\}_{j=1}^{+\infty}$ (points).

s.t. $|\varphi(t)-\varphi_{k_j}(t_j)| \geq \varepsilon_0$

$\lambda_{k_j}\to 0$. by the proof of Peano's thm 3) Euler polyline $\{\varphi_{k_j}(t)\}_{j=1}^{\infty}$ $\longrightarrow$ choose a subsequence $\{\varphi_{k_{j_m}}(t)\}_{m=1}^{\infty}$.

uni. conv. to some solution $\bar{\varphi}(t)$. $\exists\, m_0\in\mathbb{N}$. for any $m>m_0$, $t\in P$. $|\bar{\varphi}(t)-\varphi_{k_{j_m}}| < \varepsilon_0$.

$|\varphi(t_{j_m})-\bar{\varphi}(t_{j_m})| \geq |\varphi(t_{j_m})-\varphi_{k_{j_m}}(t_{j_m})| - |\varphi_{k_{j_m}}(t_{j_m})-\bar{\varphi}(t_{j_m})| > \varepsilon_0-\varepsilon_0=0$.

at t_{j_m}. the two solution. $\varphi(t)$ and $\bar{\varphi}(t)$ not coincide.

Technique 2

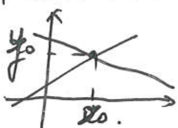
1) Homogeneous Equation.

$M(x,y)\,dx + N(x,y)\,dy = 0$. x, y in same degree

substitution $y=tx$ (t is another variable).

2). $(ax+by+c)\,dx + (a_1x+b_1y+c_1)\,dy = 0$. $\to$ find the intersection point of

$y' = f\left(\frac{ax+by+c}{a_1x+b_1y+c_1}\right)$.



$x_1 = x-x_0$

$y_1 = y-y_0$.

$\to$ to make it homo.

parallel. let. $ax+by=t$.

3) non-homogeneous.

Let $y=z^m$. $y'=m\cdot z^{m-1}\cdot z'$ (解 m. 使齐次)

§2. The problem of uniqueness.

If at (x_0, t_0) the uniqueness condition satisfied. (x_0, t_0) - point of uniqueness for equation $\dot{x} = X(t, x)$

<u>Def</u>. (Uniqueness condition: point & segment.)

At the point $(t_0, x_0) \in G$: if $\exists \Delta > 0$ s.t. $|t - t_0| \leq \Delta$. $\exists x = \varphi(t)$ is unique solution ...

At the segment $[t_0 - \delta, t_0 + \delta]$: for any δ. s.t. $0 < \delta \leq \Delta$. the solution $\varphi(t)$ is unique defined on a segment $[t_0 - \delta, t_0 + \delta]$.

<u>Thm 1</u>. Let $(t_0, x_0) \in G$. and $x = \varphi(t)$ is the solution of the Cauchy Problem defined for $t \in (a, b)$, $t_0 \in (a, b)$

点唯一 ⇒ 函数唯一 If each point $(t, \varphi(t))$, $t \in (a, b)$. is <u>a point of uniqueness</u> the solution $x = \varphi(t)$ is unique on (a, b) (非整段上的唯一. 在小区间上的唯一 $\overset{each}{\Rightarrow}$ 整段).

Pf: Assume. $\exists\, \xi(t)$, on (a, b). s.t. $\xi(t_0) = \varphi(t) = x_0$, $\xi(t^*) \neq \varphi(t^*)$. w.l.g. $t^* \in [t_0, t_0 + \delta]$

define. $u(t) := \varphi(t) - \xi(t)$ on $[t_0, t^*]$. Let $O = \{t : t \in [t_0, t^*], u(t) = 0\}$

since $t_0 \in O$. $O \neq \phi$. O is bounded and closed. (非零点的点构成集合为开集).

$\exists \max O = t_1$. $(t_1 \in O)$. Thus, $t_0 \leq t_1 < t^*$. $u(t_1) = 0$ and $u(t) \neq 0$ holds for $t \in (t_1, t^*]$.

put. $\xi(t_1) = \varphi(t_1) = x_1$ consider Cauchy's Problem on (t_1, x_1).

at point $(t_1, \varphi(t_1))$. the uniqueness point. $\exists \Delta > 0$. $|t - t_0| \leq \Delta$. $\exists x = \bar{\varphi}(t)$. for C.p. and for any δ s.t. $0 < \delta \leq \Delta$. the solution $\bar{\varphi}(t)$ is unique solution for $|t - t_0| \leq \delta$.

Let $\delta < \min(\Delta, t_1 - a, t^* - t_1)$. $(t_1 - \delta, t_1 + \delta) \subset (a, t^*) \subset (a, b)$.

we need $\bar{\varphi}(t) \equiv \varphi(t) \equiv \xi(t)$ in $(t_1 - \delta, t_1 + \delta)$ (due to the uniqueness)

which contradicts with $\varphi(t) \neq \xi(t)$ at. $t_0 \in (t_1, t_1 + \delta)$.

<u>Gronwall's lemma.</u>

Let $u(t)$ be continuous. on the interval $\langle a, b \rangle$. and $u(t) \geq 0$ for $t \in \langle a, b \rangle$

Let. $C \geq 0$. $L > 0$. $\exists\, t_0 \in \langle a, b \rangle$. s.t. for $t \in \langle a, b \rangle$. $u(t) \leq C + L \left| \int_{t_0}^{t} u(\tau)\, d\tau \right|$.

Then $u(t) \leq C \cdot e^{L|t - t_0|}$

Pf: w.l.g. $t > t_0$. let $v(t) = \overset{C = v(t_0).}{C} + L \int_{t_0}^{t} u(\tau)\, d\tau$. $u(t) \leq v(t)$.

$$\frac{d\left(v(t) \cdot e^{-Lt}\right)}{dt} \leq 0. \Rightarrow \text{function } v(t) \cdot e^{-Lt} \text{ decreases.}$$

Therefore $v(t) e^{-Lt} \leq v(t_0) e^{-Lt_0} = C \cdot e^{-Lt_0}$ and $u(t) \leq v(t) \leq Ce^{-L(t - t_0)}$

* when $C = 0$. $u(t) = 0$.

Thm2. (Uniqueness thm)

Assume in the neighborhood of the point $(t_0, x_0) \in G$. partial derivate $\partial X(t,x)/\partial x$. exists and bounded.

Then (t_0, x_0) is the point of uniqueness

Pf: let. $a>0. b>0. M, L>0$.

s.t. $D = \{(t,x): |t-t_0| \le a, |x-x_0| \le b\} \subset G$. and $|X(t,x)| \le M. |\partial X(t,x)/\partial x| \le L$ (for $(t,x) \in D$).

By Peano's thm. on $\{t: |t-t_0| \le h\}$. where $h = \min(a, b/M)$.

$\exists \varphi(t)$. Let show $\Delta = h$. satisfied the def. ($\varphi(t)$ 是从定理中取得. $\xi(t)$ 不是).

Let $0 < \delta \le h$. Denote $x = \xi(t)$. any other solution of $\dot{x} = X(t,x)$. for $|t-t_0| < \delta$.

1). show that $(t, \xi(t)) \in D$ for $|t-t_0| \le \delta$. we have to show $|\xi(t) - x_0| \le b$.

Assume. $\exists t^*$. s.t. $|t^* - t_0| \le \delta$. and $|\xi(t^*) - x_0| > b$. (Contrary).

Notice $t^* \neq t_0$. since. $|\xi(t_0) - x_0| = 0 < b$. w.l.g. $t_0 < t^* \le t_0 + \delta$. put. $t \in [t_0, t^*]$.

$v(t) = |\xi(t) - x_0| - b$ $v(t)$ is continuous. $v(t_0) < 0$. $v(t^*) > 0$.

$\exists \theta \in (t_0, t^*)$. $v(\theta) = 0$.

Therefore. set $O = \{t: t_0 \in [t_0, t^*]. v(t) = 0\} \neq \phi$. closed. bounded. $\exists \min O = t_1. t_1 \in O$.

We now $v(t) \le 0$. for all $t \in [t_0, t_1]$ i.e. any point. $(t, \xi(t)) \in D$. for $t \in [t_0, t_1]$

we have $\dot{\xi}(t) = X(t, \xi(t))$, $t \in [t_0, t_1]$ $\xrightarrow{\text{integrating}}$ $\xi(t_1) - \xi(t_0) = \int_{t_0}^{t_1} X(\tau, \xi(\tau))\, d\tau$.

$\Leftrightarrow |\xi(t_1) - x_0| \le \int_{t_0}^{t_1} |X(\tau, \xi\tau)|\, d\tau \le M \cdot (t_1 - t_0) < M \cdot \delta \le M \cdot h \le b$. $\Leftrightarrow v(t_1) < 0$. (Contra.).

2). Now we fix an arbitrary $t \in [t_0 - \delta, t_0 + \delta]$ and estimate and difference

$X(t, \varphi(t)) - X(t, \xi(t))$ (Actually $\le L|\varphi(t) - \xi(t)|$

$\exists s \in [0,1]$. $f(s) = X(t, s\varphi(t) + (1-s)\xi(t))$. $\to$ 1-dim.

Note that. $|\varphi(t) - x_0| \le b$. and $|\xi(t) - x_0| \le b$.

$$|s\varphi(t) + (1-s)\xi(t)| = |s\varphi(t) - s x_0 + (1-s)\xi(t) - (1-s)x_0|$$
$$\le s|\varphi(t) - x_0| + (1-s)|\xi(t) - x_0| \le b.$$

thus we have $(t, s\varphi(t) + (1-s)\xi(t)) \in D$. $f(s)$ is well defined.

$\exists \sigma \in (0,1)$. s.t. $X(t, \varphi(t)) - X(t, \xi(t)) = f(1) - f(0) = f'(\sigma)(1-0)$ [by Lagrange's thm.]

Also. $|f'(\sigma)| = \left|\dfrac{\partial X(t, \sigma\varphi(t) + (1-\sigma)\xi(t))}{\partial x}\right| |\varphi(t) - \xi(t)| \le L|\varphi(t) - \xi(t)|$.
对σ求导

$|X(t, \varphi(t)) - X(t, \xi(t))| \le L|\varphi(t) - \xi(t)|$.

3). Now prove that $\xi(t) \equiv \varphi(t)$ on $[t_0-\delta, t_0+\delta]$

$$\dot\varphi(t) - \dot\xi(t) = X(t, \varphi(t)) - X(t, \xi(t))$$

since $\xi(t_0) = \varphi(t_0) = x_0$.

$$\varphi(t) - \varphi(t_0) - \xi(t) + \xi(t_0) = \int_{t_0}^{t} \left(X(\tau, \varphi(\tau)) - X(\tau, \xi(\tau))\right) d\tau.$$

$$|\varphi(t) - \xi(t)| \le \left| \int_{t_0}^{t} X(\tau, \varphi(\tau)) - X(\tau, \xi(\tau))\, d\tau \right| \le L \left| \int_{t_0}^{t} |\varphi(\tau) - \xi(\tau)|\, d\tau \right|.$$

$u(t) := \varphi(t) - \xi(t) \le L \left| \int_{t_0}^{t} u(\tau)\, d\tau \right|$. (we have the $c=0$ case in Gronwall Lemma)

$u(t) \equiv 0 \iff \xi(t) = \varphi(t)$

Coro. If $\partial X(t,x)/\partial x$ exists and continuous in domain G then any point from G is the point of uniqueness. Geometrically, this means that through each point of the domain G goes only one integral curve.

§ Symmetric form.

Def. (D.E. in symmetric form.) $M(x,y)\,dx + N(x,y)\,dy = 0$. (1)

(Assume M, N are continuous functions in the domain $G \subset \mathbb{R}^2$.)

skill: Let $y = \varphi(x)$. or $x = \psi(y)$. turn the equation into an identity. ($y=\varphi(x)$ is the solution of (1))

$$M(x, \varphi(x))\,dx + N(x, \varphi(x))\,\varphi'(x)\,dx = 0.$$

If $N(x_0, y_0) \neq 0$. at point $(x_0, y_0) \in G$. $\exists\, V(x_0, y_0)$ s.t. $N(x,y) \neq 0$. (continuity of N).

we have $\dfrac{dy}{dx} = -\dfrac{M(x,y)}{N(x,y)}$. by Peano's thm. $\exists\, y = \varphi(x)$. s.t. $y_0 = \varphi(x_0)$.

Def. (singular point).

The point $(x_0, y_0) \in G$. s.t. $M(x_0, y_0) = N(x_0, y_0) = 0$. is singular point of equation (1)

If $M(x_0, y_0) \neq 0$. or $N(x_0, y_0) \neq 0$. ordinary point.

→ 保证了 $\varphi'(x)$. $\psi'(y) \neq 0$. 不会有"⌒"(非单射)

Rem: if $M(x_0, y_0) \cdot N(x_0, y_0) \neq 0$. $y = \varphi(x)$, $x = \psi(y)$ are both solution of (1).

Def: (integral).

Function $u(x,y)$ is called integral of equation (1) in the domain G. if:

1) $u(x,y)$ is continuously differentiable in G.
2). at every ordinary point in the domain G at least one of the partial derivatives ($\partial u/\partial x$ or $\partial u/\partial y$) is not equal to 0.
3) in the domain G. following identity holds:

$$N(x,y)\,\frac{\partial u(x,y)}{\partial x} - M(x,y)\,\frac{\partial u(x,y)}{\partial y} \equiv 0$$

Thm1. Let $y=\varphi(x)$ be the solution of the equation (1) defined for $x\in\langle a,b\rangle$ and the point $(x,\varphi(x))$ be the ordinary point for any $x\in\langle a,b\rangle$. Let $u(x,y)$ be the integral of the (1). in G. Then $u(x,\varphi(x))=\text{const}.\ x\in\langle a,b\rangle$ (the converse also holds).

Pf: First claim $N(x,y)\neq 0$. (otherwise, some point $(x,\varphi(x))$ would be singular).

$$\Rightarrow \varphi'(x)=-\frac{M(x,\varphi(x))}{N(x,\varphi(x))}$$

$$\frac{du(x,\varphi(x))}{dx}=\frac{du(x,\varphi(x))}{\partial x}+\frac{du(x,\varphi(x))}{\partial y}\varphi'(x)=$$

对右侧变量求偏导

$$=\frac{1}{N(x,\varphi(x))}\left[\frac{\partial u(x,\varphi(x))}{\partial x}N(x,\varphi(x))-\frac{\partial u(x,\varphi(x))}{\partial y}M(x,\varphi(x))\right]$$

$=0$ (导函数 $\equiv 0$. 原函数 $=$ const).

Rem. Let $x=\psi(y)$. some condition. $u(\psi(y),y)=\text{const}.\ y\in\langle c,d\rangle$

Thm2. Let $u(x,y)$ be integral of (1). in G. $(x_0,y_0)\in G$. Consider $u(x,y)=u(x_0,y_0)$. Suppose $N(x_0,y_0)\neq 0$. Then equation $u(x,y)=u(x_0,y_0)$ has solution $y=\varphi(x)$. (define. for $x\in(a,b)$. $x_0\in(a,b)$. $y_0=\varphi(x_0)$). This solution is continuously differentiable on the interval (a,b) and it's also the solution of (1)

Pf: First $\dfrac{\partial u(x_0,y_0)}{\partial y}\neq 0$. (def 2. of integral)

by implicit function thm. $u(x,y)=u(x_0,y_0)$ has the solution $y=\varphi(x)$. $\in C'(a,b)$.

From the continuity of partial derivatives of $u(x,y)$. $\Rightarrow \dfrac{\partial u(x,\varphi(x))}{\partial y}\neq 0$. $x\in(a,b)$ small enough.

$$u(x,\varphi(x))=u(x_0,y_0)\Rightarrow \frac{\partial u(x,\varphi(x))}{\partial x}+\frac{\partial u(x,\varphi(x))}{\partial y}\varphi'(x)=0,\ \varphi'(x)=\cdots$$

$$M(x,\varphi(x))-N(x,\varphi(x))\varphi'(x)=\left(\frac{\partial u(x,\varphi(x))}{\partial y}\right)^{-1}\left(M(x,\varphi(x))\frac{\partial u(x,\varphi(x))}{\partial y}-N(x,\varphi(x))\frac{\partial u(x,\varphi(x))}{\partial x}\right)\equiv 0.$$

by def. of integral, $y=\varphi(x)$ is a solution of equation (1).

Coro. If (x_0,y_0) is ordinary. then the $u(x,y)=u(x_0,y_0)$ has a solution of the form $y=\varphi(x)$. $x\in(a,b)$. $x_0\in(a,b)$. $y_0=\varphi(x_0)$ or $x=\psi(y)\ldots$ This is solution of (1).

"Implicit function theorem":

§. Equation in total differential. (全微分/恰当微分方程)

$M(x,y)dx + N(x,y)dy = 0$. M. N are continuous function. domain $G \subset \mathbb{R}^2$ (1)

Def. the equation in total differentials. if there exists continuously differentiable (in the domain G) function $u(x,y)$ s.t. for any $(x,y) \in G$:

$$du(x,y) = M(x,y)\,dx + N(x,y)\,dy$$

($df(x,y) = f_x dx + f_y dy$)

$$\Leftrightarrow \quad \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy = M(x,y)\,dx + N(x,y)\,dy$$

$$\Leftrightarrow \quad \frac{\partial u(x,y)}{\partial x} = M(x,y) \qquad \frac{\partial u(x,y)}{\partial y} = N(x,y) \quad \text{for any } (x,y) \in G$$

Thm 1. If (1) is equation in total differentials. then the function $u(x,y)$ is the integral of equation (1) in the domain G.

Pf: 1). $u(x,y) \in C^1(G)$. 2). at least $M(x,y)$ or $N(x,y) \neq 0$. (def of ordinary point) $\frac{\partial u(x,y)}{\partial x}, \frac{\partial u(x,y)}{\partial y} \neq 0$.

3). Check the equation.

Thm 2. Let (1) be equation in total differentials. in G. $\exists$ continuous partial derivatives $\frac{\partial M(x,y)}{\partial y}$ and $\frac{\partial N(x,y)}{\partial x}$. At every point in G. $\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} = \frac{\partial^2 u(x,y)}{\partial y \partial x}$.

The converse holds when the domain is rectangular $G = \{(x,y): a<x<b,\ c<y<d\}$. ($\pm\infty$ not excluded)

Pf of converse: goal: construct $u(x,y) \in C^1(G)$.

Fix $(x_0,y_0) \in G$. find the value of u at point $(x,y) \in G$. let $x \le t \le x_0$ (or $x_0 \le t \le x$)

Then $a<t<b$. $(t,y) \in G$. $\frac{\partial u(t,y)}{\partial t} = M(t,y)$ (此处是对u的定义，满足M，证其满足N).

$$\xrightarrow{\text{integrate}} \int_{x_0}^{x} \frac{\partial u(t,y)}{\partial t}dt = \int_{x_0}^{x} M(t,y)\,dt \Rightarrow u(x,y) - u(x_0,y) = \int_{x_0}^{x} M(t,y)\,dt.$$

Find $u(x_0,y)$, let $t^* \in [y_0,y] \Rightarrow t^* \in (c,d) \Rightarrow (x_0,t) \in G \Rightarrow \frac{\partial u(x_0,t^*)}{\partial t^*} = N(x_0,t^*)$

$$\xrightarrow{\text{integrate}} u(x_0,y) - u(x_0,y_0) = \int_{y_0}^{y} N(x_0,t^*)\,dt^*.$$

$$\Rightarrow u(x,y) = \int_{x_0}^{x} M(t,y)\,dt + \int_{y_0}^{y} N(x_0,t^*)\,dt^*$$

$$\frac{\partial u(x,y)}{\partial y} = \frac{\partial}{\partial y}\int_{x_0}^{x} M(t,y)\,dt + N(x_0,y) \overset{?}{=} \int_{x_0}^{x} \frac{\partial N(t,y)}{\partial t}dt + N(x_0,y)$$

$$= N(x,y) - N(x_0,y) + N(x_0,y) = N(x,y)$$

(symmetry: $u(x,y) = \int_{x_0}^{x} M(t,y_0)\,dt + \int_{y_0}^{y} N(x,t)\,dt$. also valid).

Integrating factor.

Def. Integrating factor. (multiplier)

Function $\mu(x,y)$. $\in C(G)$. not vanishing in G.

if $\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$. is equation in total differentials.

Assume. G is rectangular. in G. $\exists$ $\frac{\partial M(x,y)}{\partial y} + \frac{\partial N(x,y)}{\partial x}$. $\mu(x,y) \in C'(G)$.

$\mu(x,y)$ will be solution of: $\frac{\partial(\mu(x,y)M(x,y))}{\partial y} = \frac{\partial(\mu(x,y)N(x,y))}{\partial x}$.

Look for integrating factor depending only on x.

$$N(x,y)\frac{d\mu(x)}{dx} = \mu(x)\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right).$$

denote: $f(x) := \frac{1}{\mu(x)}\frac{d\mu(x)}{dx} = \frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right)$.

$f(x) = \frac{1}{\mu(x)}\frac{d\mu(x)}{dx}$. $\xrightarrow{\text{integrating}}$ $\mu(x) = e^{\int f(x)dx}$

statement 1: For equation $M(x,y)dx + N(x,y)dy = 0$, $M(x,y)$. $N(x,y)$. cont. on $G \subset \mathbb{R}^2$ there exists. integrating factor μ, depending only on x. if the function $\frac{1}{N(x,y)}\left(\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right)$ depends only on x.

statement 2: if ... depending only on y. $\frac{1}{M(x,y)}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right)$.

e.g. Linear equation $y' = p(x)y + q(x)$. $p(x)$. $q(x) \in C(a,b)$

$\Rightarrow (p(x)y + q(x))dx - dy = 0$. $(M(x,y) = p(x)y + q(x),\ N(x,y) = -1)$.

find that $\frac{1}{\mu(x)}\frac{d\mu(x)}{dx} = -p(x)$. $\Rightarrow \mu(x) = \exp\left(-\int_{x_0}^{x} p(s)ds\right)$. , $x_0 \in (a,b)$.

§. Systems of differential equations.

Def. A system of equations solved w.r.t. higher derivatives is called a system of the form: (k equations. k variables).

$$\begin{cases} x_1^{(m_1)} = X_1(t, x_1, \dot{x}_1, \ddot{x}_1, \dots x_1^{(m_1-1)}, x_2, \dot{x}_2, \dots x_2^{(m_2-1)}, x_k, \dot{x}_k, \dots x_k^{(m_k-1)}), \\ x_2^{(m_2)} = X_2(t, x_1, \dot{x}_1 \dots), \\ \dots \\ x_k^{(m_k)} = X_k(t, x_1, \dot{x}_1 \dots). \end{cases} \quad (1)$$

where $x_j^{(s)} = \dfrac{d^s x_j}{dt^s}$, $j = 1, 2, \dots k$, $s = 1, 2, \dots, m_j$.

Number. $n = \sum_{j=1}^{k} m_j$. the order of the system.

We Assume functions X_j are Continuous on the set $D \subset \mathbb{R}^{n+1}$ for all $j = 1, 2, \dots k$.

Def. (Solution).

The solution of system. is the set of function $x_1 = \varphi_1(t), \dots x_k = \varphi_k(t)$, defined on $\langle a, b \rangle$. s.t. being substituted into the system (1) they turn this system into an identity.

Def. (Cauchy Problem).

The problem of finding solution $x_1 = \varphi_1(t) \dots x_k = \varphi_k(t)$. of system (1) satisfying the conditions. $\varphi_j(t_0) = x_{j0}$ $\dot{\varphi}_j(t_0) = \dot{x}_{j0}$, $\ddot{\varphi}_j(t_0) = \ddot{x}_{j0}, \dots \varphi_j^{(m_j-1)}(t_0) = x_{j0}^{(m_j-1)}$.
where $t_0 \in \langle a, b \rangle$, $(t_0, x_{01}, \dot{x}_{01}, \dots x_{01}^{(m_1-1)}, \dots x_0, x_{0k}, \dots x_{0k}^{(m_k-1)}) \subset D$, $j = 1, 2, \dots k$.

Special case

1) $k=1$. equation of order n. $x^{(n)} = X(t, x, \dot{x}, \dots x^{(n-1)})$. (2) solution: a function

2) $m_j = 1$. for all $j = 1, \dots k$. the system in normal form (normal system)

$$\begin{cases} \dot{x}_1 = X_1(t, x_1, \dots x_n) \\ \vdots \\ \dot{x}_n = X_k(t, x_1, \dots x_n). \end{cases} \quad (3).$$

* if we denote $x_1 = x$. $x_2 = \dot{x}$ .... $x_n = x^{(n-1)}$. (2) → (3).
the normal system w.r.t. (2)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = X(t, x_1, \dots x_n) \end{cases}$$

solution: n functions.

Consider normal system. (all X_i are continuous. on $D \subset \mathbb{R}^{n+1}$).

Let $x = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$, $X(t,x) = X(t, x_1, \dots, x_n) = \begin{pmatrix} X_1(t, x_1, \dots, x_n) \\ \cdots \\ X_n(t, x_1, \dots, x_n) \end{pmatrix}$

the system can be written in vector form $\dot{x} = X(t,x)$

the solution $x = \varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))^T$.

Cauchy problem. $(t_0, x_0) \in D$. $x_0 = (x_{10}, \dots, x_{n0})^T$.

Lipschitz. Condition. (under normal system).

Def. We say that the function $X(t,x)$ satisfy Lipschitz Condition w.r.t. x. on $D \subset \mathbb{R}^{n+1}$ (denoted as $X(t,x) \in Lip_x(D)$) if there exists a constant $L > 0$. s.t. for any two points $(t, \bar{x})(t, \bar{\bar{x}}) \subset D$ the inequality $\| X(t,\bar{x}), X(t,\bar{\bar{x}}) \| \leq L \| \bar{x} - \bar{\bar{x}} \|$.

(f is lipschitz, if $\exists L$. $\forall x_1, x_2 \in D$, $|f(x_1) - f(x_2)| \leq L|x_1 - x_2|$
f is local lipschitz. $\forall x \in D$. $\exists V_x \subset D$. f is lipschitz).

Def. $X(t,x) \in Lip_x^{loc}(G)$. if for any point $(t_0, x_0) \in G$, there exists a neighborhood $U(t_0, x_0) \subset G$. s.t. $X(t,x)$ satisfies Lipschitz condition w.r.t. x. on $U(t_0, x_0)$.

Thm 1. Let $X(t,x)$ satisfies locally Lipschitz. Condition w.r.t. x in the domain $G \subset \mathbb{R}^{n+1}$: Then on any closed bounded set $D \subset G$. function $X(t,x)$ satisfies. Lipschitz. condition w.r.t. x. (local $\to$ global).

Pf: Proof by contradiction.

$\exists D \subset G$. D is closed and bounded. s.t. $\forall L > 0$. $\exists (t,\bar{x})(t,\bar{\bar{x}}) \in D$. $\| X(t,\bar{x}), X(t,\bar{\bar{x}}) \| > L \| \bar{x} - \bar{\bar{x}} \|$

take $\{L_k\}_{k=1}^{+\infty}$ $L_k \xrightarrow[k\to\infty]{} +\infty$. $L_{k+1} > L_k > 0$.

For each L_k. $\exists (t_k, \bar{x}_k)(t_k, \bar{\bar{x}}_k) \in D$. s.t. $\| X(t_k, \bar{x}_k) - X(t_k, \bar{\bar{x}}_k) \| > L_k \| \bar{x}_k - \bar{\bar{x}}_k \|$.

thus. we have sequence $\{(t_k, \bar{x}_k)\}_{k=1}^{+\infty}$ $\{(t_k, \bar{\bar{x}}_k)\}_{k=1}^{+\infty}$. D is closed and bounded.

we can choose conv. subsequence. $\{(t_{k_j}, \bar{x}_{k_j})\}_{j=1}^{+\infty}$. $\xrightarrow{对应}$ $\{(t_{k_j}, \bar{\bar{x}}_{k_j})\}_{j=1}^{+\infty}$.

(保证对应，且均为原序列子列).

$\{(t_{k_{j_m}}, \bar{x}_{k_{j_m}})\}_{m=1}^{+\infty} \xLeftarrow{对应} \{(t_{k_{j_m}}, \bar{\bar{x}}_{j_m})\}_{m=1}^{+\infty}$

the subsequence conv. to some $(t_0, \bar{x}_0)$. $(t_0, \bar{\bar{x}}_0)$.

1). $\bar{x}_0 \neq \bar{\bar{x}}_0$. by the continuity of $X(t,x)$. $\dfrac{\| X(t_k, \bar{x}_k) - X(t_k, \bar{\bar{x}}_k) \|}{\| \bar{x}_k - \bar{\bar{x}}_k \|} \xrightarrow[k\to+\infty]{} M. \in \mathbb{R}$.

$\exists k_0$. for $k > k_0$. $\dfrac{\| X(t_k, \bar{x}_k) - X(t, \bar{\bar{x}}_k) \|}{\| \bar{x}_k, \bar{\bar{x}}_k \|} < M+1$.

$\exists \tilde{k}$ for $k > \tilde{k}$ $L_k > M+1$

Let. $k > \max(\tilde{k}, k_0)$. both holds.

2) $\bar{x}_0 = \bar{\bar{x}}_0$. $\exists U(t_0, x_0) \subset G$. s.t. $X(t,x)$. is Lipschitz. w.r.t. $U(t_0, x_0)$. $\exists L > 0$, ...

$\exists k_0$. $(t_k, \bar{x}_k)$. $(t_k, \bar{\bar{x}}_k) \in U(t_0, \bar{x}_0)$ for all $k > k_0$

$\exists \tilde{k}$. $L_k > L$.

Let $k > \max(\tilde{k}, k_0)$

Thm2. Let the function $X(t,x)$ be continuously differentiable w.r.t. to x in the domain G. Then $X(t,x) \in Lip_x^{loc}(G)$.

Pf: $X \in C'(G)$. $\frac{\partial X_j(t,x_1,\dots,x_n)}{\partial x_k}$ exists and continuous in G. for all j,k.

$\forall (t_0,x_0) \in G$. G is open. $\exists a,b>0$. $D=\{(t,x) : |t-t_0| \le a, |x-x_0| \le b\} \subset G$.

$\forall (t,\bar{x}), (t,\bar{\bar{x}}) \in D$. estimate the $X_j(t,\bar{x}) - X_j(t,\bar{\bar{x}})$ for $j \in \{1,2,\dots,n\}$.

Let $f(s) = X_j(t, s\bar{x} + (1-s)\bar{\bar{x}})$, $0 \le s \le 1$.

since $|s\bar{x} + (1-s)\bar{\bar{x}} - x_0|| = s||\bar{x} - x_0|| + (1-s)||\bar{\bar{x}} - x_0|| \le b$, the point $(t, s\bar{x}+(1-s)\bar{\bar{x}}) \in D$.

thus. $(t,\bar{x})(t,\bar{\bar{x}}) \in D$. $f(s)$ is correctly defined.

By Lagrange's thm, $\exists \sigma \in (0,1)$ s.t. $X_j(t,\bar{x}) - X_j(t,\bar{\bar{x}}) = f(1) - f(0) = f'(\sigma)$

$$f'(\sigma) = \sum_{k=1}^{n} \frac{\partial X_j(t, \sigma\bar{x} + (1-\sigma\bar{\bar{x}})}{\partial x_k} \cdot (\bar{x}_k - \bar{\bar{x}}_k).$$

partial derivates $\frac{\partial X_j(t,x)}{\partial x_k}$ is continuous on compact set D. $\exists K$. $\left|\frac{\partial X_j(t,x)}{\partial x_k}\right| \le K$.

$$||X(t,\bar{x}) - X(t,\bar{\bar{x}})|| = \sqrt{\sum_{j=1}^{n} (X_j(t,\bar{x}) - X_j(t,\bar{\bar{x}}))^2} \le \sqrt{n \cdot (n^\circ K ||\bar{x} - \bar{\bar{x}}||)^2} = n^{\frac{3}{2}} K ||\bar{x} - \bar{x}||$$

Denote $n^{\frac{3}{2}}K = L$. Since (t_0,x_0) is arbitrary....

Picard Theorem.

Let's consider the system $\dot{x} = X(t,x)$. (1) $x \in \mathbb{R}^n$, $X(t,x) \in C(D)$. $D \subset \mathbb{R}^{n+1}$.
with Cauchy Problem $(t_0, x_0) \in D$.
The integral equation $x(t) = x_0 + \int_{t_0}^{t} X(\tau, x(\tau))\, d\tau$. (2).

Def. A cont. vector function $x = \varphi(t)$, defined on $\langle a,b \rangle$. $t_0 \in \langle a,b \rangle$, called the solution of the integral equation $x(t) = x_0 + \int_{t_0}^{t} X(\tau, x(\tau))\, d\tau$, if $x = \varphi(t)$ substited into the equation, turns it into an identity.

(Actually. the equation (2). equivalent to Cauchy problem (1).). [代入 $\varphi(t)=x$. 积分/求导]

Def. Picard approximation.

zero approximation: take function $\varphi_0(t) \equiv x_0$, $t \in \mathbb{R}$.)

first approximation $\exists \langle a_1, b_1 \rangle$. s.t. $t_0 \in \langle a_1, b_1 \rangle$ and $(t, \varphi_0(t)) \in D$. for any $t \in \langle a_1, b_1 \rangle$.

Then for $t \in \langle a_1, b_1 \rangle$ defined the function $\varphi_1(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi_0(\tau))\, d\tau$.

continue: $\langle a_2, b_2 \rangle \subset \langle a_1, b_1 \rangle$. $t_0 \in \langle a_2, b_2 \rangle$. s.t. $(t, \varphi_1(t)) \in D$. for any $t \in \langle a_2, b_2 \rangle$.

second approximation $\varphi_2(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi_1(\tau))\, d\tau$.

⋮

k-th approximation. $\varphi_k(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi_{k-1}(\tau))\, d\tau$.

if the $\langle a_k, b_k \rangle$ not exist. the Picard approximation with stop.

If $D = \mathbb{R}^{n+1}$, all successive Picard approximation are defined for all $t \in \mathbb{R}$.

Thm. (Picard Theorem).

On a closed bounded set $D \subset \mathbb{R}^{n+1}$ vector function $X(t,x)$ is continuous and satisfy the Lip. Condition according to x. Assume all Picard Approximation $\varphi_k(t)$ defined on the same segment $[a,b]$. Then the sequence of Picard approximation $\varphi_k(t)$ conv. uni. on the segment $[a,b]$ to the function $\varphi(t)$, and $\varphi(t)$ is the solution. ($\varphi_k \rightrightarrows \varphi$).

Pf: Consider series. $\varphi_0(t) + \sum_{k\geq 1}^{+\infty} (\varphi_k(t) - \varphi_{k-1}(t))$. $t \in [a,b]$

the uni. conv. of the series $\Leftrightarrow$ the uni. conv. of $\{\varphi_k(t)\}_{k=0}^{+\infty}$ (partial sum).

note: $\|\varphi_0(t)\| = \|x_0\|$

$$\|\varphi_1(t) - \varphi_0(t)\| = \left\| \int_{t_0}^{t} X(\tau, \varphi_0(\tau))d\tau \right\| \leq \left| \int_{t_0}^{t} \|X(\tau, x_0)\| d\tau \right|$$

Function $X(t,x_0)$ is cont. on $[a,b]$: $\exists M > 0$ s.t. $\|X(t,x_0)\| \leq M$. for all $t \in [a,b]$.

it follows: $\|\varphi_1(t) - \varphi_0(t)\| \leq M|t-t_0|$

Further: $$\|\varphi_2(t) - \varphi_1(t)\| = \left\| \int_{t_0}^{t} X(\tau, \varphi_1(\tau))d\tau - \int_{t_0}^{t} X(\tau, \varphi_0(\tau))d\tau \right\|$$
$$\leq \left| \int_{t_0}^{t} \| X(\tau, \varphi_1(\tau))d\tau - X(\tau, \varphi_0(\tau))d\tau \| \right|.$$

Since. $X(t,x) \in Lip_x(D)$. $(t, \varphi_0(t)) \in D$. $(t, \varphi_1(t)) \in D$.

$$\|\varphi_2(t) - \varphi_1(t)\| \leq \left| \int_{t_0}^{t} L(\varphi_1(\tau) - \varphi_0(\tau))d\tau \right|.$$

Conclude: $\|\varphi_2(t) - \varphi_1(t)\| \leq ML \left| \int_{t_0}^{t} |\tau - t_0| d\tau \right| = \frac{M}{L} \frac{(L|t-t_0|)^2}{2}$

Pf $\|\varphi_k(t) - \varphi_{k-1}(t)\| \leq \frac{M}{L} \frac{(L|t-t_0|)^k}{k!}$, $k \in \mathbb{N}$. by induction.

$$\{\varphi_k(t)\} \leq \|x_0\| + \frac{M}{L} \sum_{k\geq 1}^{+\infty} \frac{(L(b-a))^k}{k!} \to \|x_0\| + \frac{M}{L} \cdot e^{L(b-a)-1}.$$

Now show the $\varphi_k(t) \rightrightarrows \varphi(t)$.

since $(t, \varphi_k(t)) \in D$. for $k \in \mathbb{N}$. then $(t, \varphi(t)) \in D$.

$$\|X(t, \varphi_k(t)) - X(t, \varphi(t))\| \leq L \|\varphi_k(t) - \varphi(t)\|. \to 0.$$

$$X(t, \varphi_k(t)) \rightrightarrows X(t, \varphi(t)) \quad \text{on } [a,b].$$

Thm 2. Let the function $X(t,x)$ be continuous and satisfy the Lipschitz. Condition on D. Then all Picard approximations $\varphi_k(t)$ defined on the interval $[t_0-h, t_0+h]$ and sequence $\varphi_k(t)$. conv. uni. on $[t_0-h, t_0+h]$ to $\varphi(t)$ $x=\varphi(t)$ is the solution.

Pf by introduction, mainly to show for every k. $(t, \varphi_k(t)) \in D$. $\varphi_k(t)$ determined for $|t-t_0| \leq h$

Uniqueness Thm.

Consider $\dot{x} = X(t,x)$. $x \in \mathbb{R}^n$. $X(t,x) \in C(G)$ and $X(t,x) \in Lip_x^{loc}(G)$. (domain $G \subset \mathbb{R}^{n+1}$).

Let $(t_0, x_0) \in G$. Cauchy problem: $t = t_0$, $x = x_0$.

Thm. Let $x = \varphi(t)$. $x = \xi(t)$ be two solutions of Cauchy problem. defined on $\langle a,b \rangle$.

Then $\varphi(t) \equiv \xi(t)$ on $\langle a,b \rangle$

Pf: $\forall \theta \in (a,b)$. show $\varphi(\theta) = \xi(\theta)$.

since $\xi(t_0) = \varphi(t_0) = x_0$ already proved. Let $\theta \neq t_0$. w.l.g. $\theta > t_0$

denote

$$T_1 = \{(t,x) : t \in [t_0, \theta],\ x = \varphi(t)\}$$

$$T_2 = \{(t,x) : t \in [t_0, \theta],\ x = \xi(t)\}$$

$$T = T_1 \cup T_2$$

the set T is contained in G, limited and closed, because $T_1 \subset G$. $T_2 \subset G$.

and each T_1. T_2 is limited and closed.

$\exists L > 0$. for any $t \in [t_0, \theta]$

$\| X(t, \varphi(t)) - X(t, \xi(t)) \| \leq L \| \varphi(t) - \xi(t) \|$.

On the other hand. Cauchy Problem here is equivalent to

$$x(t) = x_0 + \int_{t_0}^{t} X(\tau, x(\tau))\, d\tau.$$

for $t \in \langle a,b \rangle$. $\xi(t)$. $\varphi(t)$...

$$\|\varphi(t) - \xi(t)\| = \left\| \int_{t_0}^{t} X(\tau, \varphi(\tau))\, d\tau - \int_{t_0}^{t} X(\tau, \xi(\tau))\, d\tau \right\|$$

$$\leq \left| \int_{t_0}^{t} \| X(\tau, \varphi(\tau)) - X(\tau, \xi(\tau)) \|\, d\tau \right|$$

$$\leq L \left| \int_{t_0}^{t_0} \| \varphi(\tau) - \xi(\tau) \|\, d\tau \right|.$$ (the Gronwall Lemma where $c = 0$).

* Banach Thm.

Def. Contraction. (压缩映射).

(X,d) metric space. $E \subset X$. contraction (mapping) / contractor is a function $f: E \subset X$ with the property that there exists a number $q \in (0,1)$.

s.t. $d(f(x), f(y)) \leq q\, d(x,y)$. $x, y \in E$.

Lipschitz constant: the smallest value of q (the coefficient of contraction of f).

Fact: A contraction is uniformly continuous on E.

Def. fixed point.

A point x is a fixed point of function f if $x = f(x)$.

Banach fixed-point. thm.

Assume that a metric space (X,d) is complete and a set $E \subset X$ is closed. Then the contraction $f: E \to E$ has unique fixed point.

Pf: unique: $d(x_1,x_2) = d(f(x_1),f(x_2)) \le q \cdot d(x_1,x_2) \Rightarrow d(x_1,x_2)=0$.

existence: $x_0 \in E$. $x_n = f(x_{n-1})$. prove $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence.

w.l.g. $n<m$. $k = m-n$

$$d(x_k,x_0) \le d(x_k,x_{k-1}) + \cdots + d(x_1,x_0) \le (q^{k-1}+q^{k-2}+\cdots+1)\, d(x_1,x_0) \le \frac{d(x_1,x_0)}{1-q}$$

then $d(x_m,x_n) = d(f(x_{m-1}),f(x_{n-1})) \le q \cdot d(f(x_{m-1}),f(x_{n-1})) = \cdots = q^n d(x_{m-n},x_0)$

$$\le \frac{d(x_1,x_0)}{1-q} \cdot q^n.$$

极限在E内 有极限

$\{x_n\}$ has limit $a \in E$. since X is complete and E is closed.

$$a = \lim x_{n+1} = \lim f(x_n) = f(a)$$

(use the continuity of f)

Def. (Lipschitz function).

Assume that $f:[a,b]\to[a,b]$. $|f(x_1)-f(x_2)| \le K|x_1-x_2|$. f is Lip. function; $K<1$.

f is contraction. a sequence $\{x_n\}$. $x_n = f(x_{n-1})$. conv. to unique solution of $x=\varphi(x)$

Tech.

The solution of the equation $\varphi(x)=0$ where $\varphi(a)<0<\varphi(b)$ and $0<K_1\le\varphi'(x)\le K_2$.

on $[a,b]$ can be considered as the solution of equation $x = f(x) = x - \lambda\varphi(x)$. (1)

to let $|f'(x)|<1$

* Choose λ is sufficiently small we get into the equation (1).

use the contraction conclu.

$$-1 < 1-\lambda K_2 \le f'(x) \le 1-\lambda K_1 < 1.$$

Tech 1. $y' + a(x)y = b(x)$.

1). Solve $y'+a(x)y=0$. $\Rightarrow$ get $y = \varphi(x)\cdot C(x)$

constant, w.r.t. x.

2) Put $y=\varphi(x)C(x)$ to the original equation. find $C(x)$.

Tech 2. $y' + a(x)y = b(x)y^n$. Bernulli Equation.

1) $y\equiv 0$ 2) $/y^n$. $z = \frac{1}{y^{n-1}}$

Continuation of the solution. (Extension)

Consider the system. $\dot{x} = X(t,x)$, where $x \in \mathbb{R}^n$, $X(t,x)$ is continuous and satisfies the Lip. condition w.r.t x locally in $G \subset \mathbb{R}^{n+1}$.

Def.1. (Continued solution).

Let $x = \varphi(t)$ be solution of $\dot{x} = X(t,x)$ defined on (a,b). The solution can be continued to the right beyond the point b, if $\exists\ \bar{b} > b$. s.t. on $(a,\bar{b})$ $\exists$ solution $x = u(t)$ and $u(t) \equiv \varphi(t)$ on (a,b), We say $x = u(t)$ is the continuation (extension) of solution $x = \varphi(t)$ to the right until $\bar{b}$.

(Left beyond / until def. similarly).

Thm 1. (Existence of the continuation)

The solution $x = \varphi(t)$ of $\dot{x} = X(t,x)$, defined on (a,b), can be continue to the right beyond b iff the limit $\lim\limits_{t\to b} \varphi(t) = \xi$. and the point (b, ξ) belongs to G.

Pf $\Rightarrow$. $\exists\ \bar{b} > b$. $x = u(t)$.

therefore. $\exists \lim\limits_{t\to b} \varphi(t) = \lim\limits_{t\to b} u(t) = u(b) := \xi$.

by def. of solution of the system, $(b, u(b)) \in G$.

$\Leftarrow$ $\exists \lim\limits_{t\to b} \varphi(t) = \xi$. and $(b,\xi) \in G$.

1). $\forall t_0 \in (a,b)$, take $x_0 = \varphi(t_0)$ set Cauchy Problem $t = t_0$, $x = x_0$ (1).

Since $x = \varphi(t)$ solves this problem, $\varphi(t)$ satisfies: $\varphi(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi(\tau))d\tau$.

Put $\varphi(b) = \xi$ be continuity of φ. Passing the limit at $t \to b$, makes sure that $\varphi(t)$ is a solution of $x(t) = x_0 + \int_{t_0}^{t} X(\tau, x(\tau))d\tau$. for $t \in (a,b]$

2). set Cauchy Problem $t = b$, $x = \xi$ (2). Since $\varphi(b) = \xi$. $x = \varphi(t)$ is the solution of (1)(2) for $t \in (a,b]$

According to Peano. thm. $\exists h > 0$. for $|t - t_0| \le h$. $\exists\ x = \psi(t)$ of Cauchy problem (1)(2).

w.l.g. assume $h < b-a$, Thus $a < b-h < b < b+h$.

On the segment $[b-h, b]$, two solution to Cauchy Problem (1)(2). are defined. $\varphi(t)$. $\psi(t)$.

By the uniqueness. $\varphi(t) = \psi(t)$ for $t \in [b-h, b]$

set $u(t) = \begin{cases} \varphi(t), & t \in (a,b). \\ \psi(t), & t \in (b-h, b+h) \end{cases}$

$x = u(t)$ is the continuation of the solution $x = \varphi(t)$ to the right until $b+h$. $\square$

Maximum Interval of Existence.

Thm 3. Let the solution $x=\varphi(t)$ of $\dot{x}=X(t,x)$ be defined on the interval (a,b) $b<+\infty$. Then $\exists\,\beta\geqslant b$ s.t. on (a,β). $\exists\, x=u(t)$ as the continuation of $x=\varphi(t)$ to the right until β. and the solution $x=u(t)$ is not continuable to the right beyond β. $(\beta\in[b,+\infty])$

Pf: 1) $x=\varphi(t)$ is not continuable to the right beyond b, $\beta=b$.

2) Let $x=\varphi(t)$ be continuable to the right beyond b.
denote $x=u_{\bar b}(t)$. the cont. of $x=\varphi(t)$ to the right until $\bar b>b$
denote B the set of $\bar b>b$. s.t. $x=u_{\bar b}$ the cont. of $x=\varphi(t)$ to the right until $\bar b>b$.
set $\beta=\sup B$. ($\beta=+\infty$ not excluded).
By def of sup. $\forall t$. s.t. $b\le t<\beta$. $\exists\,\bar b\in B$. $t<\bar b\le\beta$. s.t. $x=u_{\bar b}(t)$. $t\in(a,\bar b)$

Let $b_1,b_2\in B$. $b<b_1<b_2\le\beta$. $u_{b_1}(t)\equiv u_{b_2}(t)\equiv\varphi(t)$ on (a,b).
Let $t_0\in(a,b)$, $x_0=\varphi(t_0)$. Then $u_{b_1}(t_0)=u_{b_2}(t_0)=x_0$. the solution solve the some C.P.
By uniqueness $u_{b_1}(t)\equiv u_{b_2}(t)$ on (a,b_1). (proved that. the solution $x_{\bar b}(t)$ doesn't depend on $\bar b$. thus we omit the index).

It remains to prove that $x=u(t)$ is not continuable to the right beyond β.
1) if $\beta=+\infty$. obvious.
2) if $\beta<+\infty$. Assume the converse. $x=u(t)$ cont. to $\bar\beta>\beta$.
then $\exists\; x=\bar u(t)$ of $\dot{x}=X(t,x)$ defined on $(a,\bar\beta)$. s.t. $u(t)=\bar u(t)$ for $t\in(a,\beta)$.
Therefore. $\varphi(t)\equiv u(t)\equiv\bar u(t)$ on (a,b). then the $x=\varphi(t)$ is cont. to the right until $\bar\beta$
$\bar\beta\in B$. which contradicts with the def. of sup.

Thm 5. Let $x=\varphi(t)$ be the solution of $\dot{x}=X(t,x)$, on the (a,b). Then there exist $\alpha\le a$ and $\beta\geqslant b$ s.t. on the interval (α,β). the continuation $x=\psi(t)$ of the solution $x=\varphi(t)$ is defined. and the solution $x=\psi(t)$ is not cont. to left beyond α and right beyond β. and $\varphi(t)\equiv\psi(t)$ on (a,b)

Pf: 1) if $b<+\infty$. by thm 3. $\exists\,\beta\geqslant b$. s.t. on (a,β). $\exists$ contiuation $x=u(t)$ of the solution $x=\varphi(t)$ to the right until β, and the solution $x=u(t)$ is not continuable to the right beyond β.
2) $b=+\infty$. we assume $\beta=+\infty$.
a and α completely same. $\exists\; x=v(t)$
Let set $\psi(t)=\begin{cases}u(t) & \text{if } t\in(a,\beta).\\ v(t) & \text{if } t\in(\alpha,b)\end{cases}$ since $v(t)\equiv\varphi(t)\equiv u(t)$ on (a,b). the def. is correct.
* (α,β) the maximal interval of extension.
Function $x=\psi(t)$. is the solution of $\dot{x}=X(t,x)$ cannot be continued to the left (right) beyond α (β). And $\varphi(t)\equiv\psi(t)$ on (a,b).

Def. Maximally continued. (maximally extended).
the solution $x=\varphi(t)$. of $\dot{x}=X(t,x)$. defined on (α,β) can't be continued to the left beyond α (right beyond β). the solution is ~. the interval (α,β). is called the ~~ma~~ maximum interval of existence of the solution $x=\varphi(t)$.

§. Behavior of the Solution while Approaching to the Boundary of the Maximal Interval of Existence.

In the section. consider $\dot{x}=X(t,x)$. $x\in\mathbb{R}^n$. X cont. and satisfies the Lip. Condition w.r.t. x locally in the domain $G\subset\mathbb{R}^{n+1}$.
Denote $\bar{G}$, closure of G. and $\partial G=\bar{G}\setminus G$.

Thm 1. Let G be bounded and $X(t,x)$ bounded in G. If $x=\varphi(t)$ is the solution of $\dot{x}=X(t,x)$ defined on (a,b). $b<+\infty$. is not continuable to the right beyond b, then there is a limit $\lim\limits_{t\to b}\varphi(t)=\xi$. and $(b,\xi)\in\partial G$.

Pf: 1) Existence of limit.
$\exists M>0$. s.t. $\|X(t,x)\|\leq M$. for $(t,x)\in G$.
fixed arbitrary $\varepsilon>0$. put $\delta=\min(b-a,\frac{\varepsilon}{M})$.
Let $t_1,t_2\in(b-\delta,b)$, $x_1=\varphi(t_1)$. Then $x=\varphi(t)$ solves Cauchy Problem $t=t_1$, $x=x_1$.
and for $t\in(a,b)$. $\varphi(t)=x_1+\int_{t_1}^{t}X(\tau,\varphi(\tau))d\tau$.
in particular $\varphi(t_2)=x_1+\int_{t_1}^{t_2}X(\tau,\varphi(\tau))d\tau$.
Hence, $\|\varphi(t_2)-\varphi(t_1)\|=\|\int_{t_1}^{t_2}X(\tau,\varphi(\tau))d\tau\|\leq\left|\int_{t_1}^{t_2}\|X(\tau,\varphi(\tau))\|d\tau\right|\leq\left|\int_{t_1}^{t_2}Md\tau\right|=M|t_2-t_1|$
Since $t_1,t_2\in(b-\delta,b)$ then $|t_2-t_1|<\delta$. and $\|\varphi(t_2)-\varphi(t_1)\|<M\delta\leq\varepsilon$
Since ε is arbitrary. $\exists\ \xi=\lim\limits_{t\to b}\varphi(t)$.
2) A point $(t,\varphi(t))$ belongs to the set G for any $t\in(a,b)$. so the limit point (b,ξ). belongs to $\bar{G}$.
If $(b,\xi)\in G$. $x=\varphi(t)$ could be continued to the right beyond b.
Thus $(b,\xi)\in\bar{G}\setminus G=\partial G$. □.

Thm2. $(a,\xi)\in\partial G$. Left.

"Domain" → open and connected. "Solution" – connect. (每个部分都是解).

Thm 3. (Exit of Maximally Continued Solution from a compact set).

Let $x=\varphi(t)$ be the maximally continued solution of $\dot{x}=X(t,x)$. defined on the interval (α,β). $\beta<+\infty$. Then for any closed bounded set $D\subset G$. $\exists\delta>0$. s.t. $(t,\varphi(t))\notin D$ for any $t\in(\beta-\delta,\beta)$.

Pf: Assume the converse. $\exists D\subset G$. for this set for any $\delta>0$. $\exists t\in(\beta-\delta,\beta)$ s.t. $(t,\varphi(t))\in D$.

$\forall\{\delta_j\}_{j=1}^{+\infty}$ s.t. $0<\delta_{j+1}<\delta_j$. $\delta_1<\beta-\alpha$.

Then for any δ_j $\exists$ $t_j\in(\beta-\delta_j,\beta)$. s.t. $(t_j,\varphi(t_j))\in D$.

The set D is bounded and closed by assumption. $\{(t_j,\varphi(t_j))\}_{j=1}^{+\infty}\subset D$ has conv. subsequence $\{(t_k,\varphi(t_k))\}_{k=1}^{+\infty}$. $t_k\underset{k\to+\infty}{\to}\beta$. $\varphi(t_k)\underset{k\to+\infty}{\rightrightarrows}\xi$. and $(\beta,\xi)\in D\subset G$.

There fore there exist $a,b\in\mathbb{R}$. $a>0$. $b>0$

s.t. $D_0=\{(t,x):|t-\beta|\le 2a, \|x-\xi\|\le 2b\}\subset G$. $\exists M$. s.t. $\|X(t,x)\|\le M$ for $(t,x)\in D_0$.

set $h=\min(a,\frac{b}{M})$. $\exists k_1$. s.t. for any $k>k_1$. $\beta-h<t_k<\beta$.

$\exists k_2$ s.t. for any $k>k_2$. $\|\varphi(t_k)-\xi\|<b$.

fix $k>\max(k_1,k_2)$ define $D_1=\{(t,x):|t-t_k|\le a, \|x-\varphi(t_k)\|\le b\}$

$D_1\subset D$. since. $\forall(t,x)\in D_1$. $|t-\beta|\le|t-t_k|+|t_k-\beta|\le h+a\le 2a$. $\|x-\xi\|\le 2b$.

Let's set Cauchy Problem $t=t_k$, $x=\varphi(t_k)$. solution $x=\varphi(t)$ defined for (α,β).

since $\beta-h<t_k<\beta$. we have $\alpha<t_k<\beta<t_k+h$.

On the other hand. on $[t_k-h,t_k+h]$ $\exists x=\psi(t)$ of Cauchy Problem.

$x=\varphi(t)$ and $x=\psi(t)$ defined for $t\in[t_k,\beta)$ and solve the same Cauchy Problem, by uniqueness thm. $\varphi(t)\equiv\psi(t)$ on $[t_k,\beta)$.

Let $u(t)=\begin{cases}\varphi(t) & t\in(\alpha,\beta)\\ \psi(t). & t\in[t_k,t_k+h]\end{cases}$ $x=u(t)$ is the solution of $\dot{x}=X(t,x)$.

which means $x=u(t)$ is the continuation to the right beyond β. □.

Thm 4. Left. $t\in(\alpha,\alpha+\delta)$.

§. Systems (comparable to the L.S.)

Consider the system $\dot{x} = X(t,x)$, function $X(t,x)$ is cont. and Local Lip. condition w.r.t x.

where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \quad X(t,x) = \begin{pmatrix} X_1(t, x_1, \dots, x_n) \\ \vdots \\ X_n(t, x_1, \dots, x_n) \end{pmatrix}$ (1)

in the domain $G \subset \mathbb{R}^{n+1}$, $G = \{(t,x) : t \in (a,b),\ \|x\| < +\infty\}$. ($a = -\infty$, $b = +\infty$, included).

Def. System (1) is called the system comparable to the linear system if there exist continuous and non-negative on the interval (a,b) functions $M(t)$ and $N(t)$ s.t. for any $(t,x) \in G$, $\|X(t,x)\| \le M(t)\|x\| + N(t)$.

Thm1. If (1) is the system comparable to the linear system then the maximum interval of existence of any solution of the system (1) is equal to (a,b).

Pf: Assume the converse. $\exists$ solution $\varphi(t)$ s.t. the maximum interval is $(\alpha,\beta) \subsetneq (a,b)$. 定域为$(a,b)$
不意味不可延伸.
只是说明不可更小.

Let $t_0 \in (\alpha,\beta)$. $x_0 = \varphi(t_0)$.

for any $t \in (\alpha,\beta)$. $\varphi(t) = x_0 + \int_{t_0}^{t} X(\tau, \varphi(\tau))\, d\tau$.

Estimate $\varphi(t)$ for $t \in [t_0, \beta)$.

$$\|\varphi(t)\| \le \|x_0\| + \left\| \int_{t_0}^{t} X(\tau, \varphi(\tau))\, d\tau \right\| \le \|x_0\| + \int_{t_0}^{t} \|X(\tau, \varphi(\tau))\|\, d\tau.$$

$$\|\varphi(t)\| \le \|x_0\| + \int_{t_0}^{t} M(\tau)\|\varphi(\tau)\|\, d\tau + \int_{t_0}^{t} N(\tau)\, d\tau$$

since $[t_0, \beta] \subset (a,b)$. $M(t)$ is cont. on (a,b). $\exists L > 0$. $|M(t)| \le L$.

$\|\varphi(t)\| \le \|x_0\| + \int_{t_0}^{\beta} N(\tau)\, d\tau + L\int_{t_0}^{t}\|\varphi(\tau)\|\, d\tau$. for $t \in [t_0, \beta)$. since $N(t) \ge 0$.

From Gronwall's Lemma $\|\varphi(t)\| \le \left(\|x_0\| + \int_{t_0}^{\beta} N(\tau)\, d\tau\right) e^{L(t-t_0)} \le C \cdot e^{L(\beta - t_0)}$

The set $D = \{(t,x) : t \in [t_0, \beta],\ \|x\| \le C \cdot e^{L(\beta - t_0)}\}$ is closed, bounded and $D \subset G$.

We also ~~we~~ have $(t, \varphi(t)) \in D$, $t \in [t_0, \beta)$ contradicts with Thm3. □

Consider the system.

$$\begin{cases} \dot{x}_1 = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + q_1(t) \\ \dot{x}_2 = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + q_2(t) \\ \cdots \\ \dot{x}_n = p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + q_n(t). \end{cases} \quad \text{or} \quad \dot{x} = p(t)x + q(t). \qquad (T).$$

$\dot{x}, x, q(t) \in \mathbb{R}^n$. $P(t) \in M_{n\times n}$.

Def. System above is called the linear system. (satisfy the condition of Def. t.s.c.t.t.l.s).

$X_j(t,x) = \sum_{k=1}^{n} p_{jk}(t)x_k + q_j(t) \qquad \frac{\partial X_j(t,x)}{\partial x_k} = p_{jk}(t). \Rightarrow X_j(t,x)$ satisfy the local Lip.

set $p(t) = \max(|p_{jk}(t)|, |q_j(t)|)$. $p(t)$ is cont. on (a,b).

$$|X_j(t,x)| \le p(t)\left(\sum_{k=1}^{n}|x_k| + 1\right) \overset{|x_k| \le \|x\| = (\sum x_k^2)^{\frac{1}{2}}}{\le} p(t)(n\|x\| + 1).$$

$$\|X(t,x)\| = \left(\sum_{j=1}^{n} X_j^2(t,x)\right)^{\frac{1}{2}} \le \sqrt{np^2(t)(n\|x\|+1)^2} = \underbrace{n\sqrt{n}\,p(t)}_{M(t)}\|x\| + \underbrace{\sqrt{n}\,p(t)}_{N(t)}$$

: nonnegative, continuous

★ So for linear system, we also have thm 1.

Chapter 3. Linear Differential Equation.

Def. Linear differential equation of order n:

$$x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_{n-1}(t)\dot{x} + p_n(t)x = q(t) \qquad (1)$$

where $x^{(k)} = \frac{d^k x}{dt^k}$. $p_k(t)$ and $q(t)$ are cont. on (a,b)

If $q(t) \equiv 0$ on (a,b) then equation (1). is called homogeneous.

"equation" → "system. $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, …, $x_n = x^{(n-1)}$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \cdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -p_n(t)x_1 - p_{n-1}(t)x_2 - \cdots - p_1(t)x_n + q(t). \end{cases} \qquad (2)$$

(All solutions of this system can be continued on interval (a,b)).

The Cauchy Problem: find the solution $x = \varphi(t)$ of the (1). s.t. $\varphi(t_0) = x_{10}$, $\dot{\varphi}(t_0) = x_{20}$, … $\varphi^{(n-1)}(t_0) = x_{n0}$.

1) In the domain $G = \{(t, x_1, \ldots x_n) : t \in (a,b), |x_k| < +\infty, k = 1,2,\ldots,n\}$. the RHS of (2) are cont. and cont. diff. the solution is unique.

2) denote LHS of (1) by $L(x)$. $L(x) = \sum_{k=0}^{n} p_k(t)x^{(n-k)}$. $p_0(t) \equiv 1$, $x^{(0)} = x$.

Rem: $L(x)$ is linear differential operator. i.e. $L(c_1x_1(t) + c_2x_2(t)) = c_1L(x_1(t)) + c_2L(x_2(t))$.

Main property of the solutions of l. homo. equation $L(x) = 0$ (1).

Thm 1. Let $\varphi_1(t), \varphi_2(t), \ldots \varphi_m(t)$ be the solution equation (1). Then the function $\psi(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + \cdots + c_m\varphi_m(t)$ is the solution of (1). (where $c_1, c_2, \ldots c_m$ are arbitrary constants)

Pf: $\varphi_j(t)$ is solution $\Rightarrow L(\varphi_j(t) \equiv 0)$ on the interval (a,b). for any $j = 1,2,\ldots,m$.

Hence, $L(\psi(t)) = L\left[\sum_{j=1}^{m} c_j\varphi_j(t)\right] = \sum_{j=1}^{m} c_jL(\varphi_j(t)) = 0$. 默认考虑线性相关性的前提是连续

Def. Function $\varphi_1(t), \ldots \varphi_n(t)$, cont. on the interval (a,b). are called l.d. on (a,b). if there are constant $c_1, c_2, \ldots, c_n$, not all $\neq 0$. s.t. $c_1\varphi_1(t) + c_2\varphi_2(t) + \cdots + c_n\varphi_n(t) \equiv 0$

Assume $\varphi_k(t)$ are continuously diff. $(n-1)$ times. on (a,b)

$$W(t) = W(\varphi_1(t), \varphi_2(t), \ldots \varphi_n(t)) = \begin{vmatrix} \varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t) \\ \dot{\varphi}_1(t) & \dot{\varphi}_2(t) & \cdots & \dot{\varphi}_n(t) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_1^{(n-1)}(t) & \varphi_2^{(n-1)}(t) & \cdots & \varphi_n^{(n-1)}(t) \end{vmatrix}$$

Wronski. determinant.

只要是满足条件的φ..所有.
↓ 与微分方程的解无关.

Thm. Let the functions $\varphi_1(t), \varphi_2(t), \ldots \varphi_n(t)$ be linearly dependent and cont. diff. $(n-1)$ times on the interval (a,b). Then $W(t) \equiv 0$ on (a,b).

Pf: construct: $\begin{cases} c_1\varphi_1(t) + \cdots + c_n\varphi_n(t) = 0 \\ c_1\dot{\varphi}_1(t) + \cdots + c_n\dot{\varphi}_n(t) = 0 \\ \cdots \\ c_1\varphi_1^{(n-1)}(t) + \cdots + c_n\varphi_n^{(n-1)}(t) = 0. \end{cases}$ Fix t. $\begin{bmatrix} \varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t) \\ \vdots & & & \vdots \\ \varphi_1^{(n-1)}(t) & \cdots & \cdots & \varphi_n^{(n-1)}(t) \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \mathbf{0}$ (1).

the system (1) has non-zero solution $z_k = c_k$. the det $|\quad| = 0$. t is arbitrary.

The converse is not true. ($W(t) \equiv 0 \Rightarrow$ l.i / l.d. both possible.

counter-e.g. $\varphi_1 = \begin{cases} 0, & -1 \le t < 0 \\ t^2, & 0 \le t \le 1 \end{cases}$ $\varphi_2 = \begin{cases} t^2, & -1 \le t \le 0 \\ 0, & 0 \le t \le 1 \end{cases}$ $W(t) \begin{cases} = \begin{vmatrix} 0 & t^2 \\ 0 & 2t \end{vmatrix}, & -1 \le t \le 0. \\ = \begin{vmatrix} t^2 & 0 \\ 2t & 0 \end{vmatrix}, & 0 \le t \le 1 \end{cases}$

Now consider linear homogenous equation $L(x)=0$ (1).

$L(x)=\sum_{k=0}^{n} p_k(t)\,x^{(n-k)}$, $p_0(t)\equiv 1$. $x^{(0)}=x$, $p_k(t)$ are cont. on the interval (a,b) for $k=1,2,\dots,n$.

(the stronger statement than the converse of Thm 1. is true).

Thm.2. Let $\varphi_1(t), \varphi_2(t), \dots \varphi_n(t)$ be the solutions of the $L(x)=0$. If there is a point $t_0\in(a,b)$ s.t. $W(t_0)=0$, then the functions $\varphi_1(t),\dots\varphi_n(t)$ are linearly dependent on (a,b)

Pf: Consider a linear homogeneous algebraic system.

$$\begin{cases} z_1\varphi_1(t_0)+z_2\varphi_2(t_0)+\cdots+z_n\varphi_n(t_0)=0 \\ z_1\dot\varphi_1(t_0)+z_2\dot\varphi_2(t_0)+\cdots+z_n\dot\varphi_n(t_0)=0 \\ \cdots \\ z_1\varphi_1^{(n-1)}(t_0)+z_2\varphi_2^{(n-1)}(t_0)+\cdots+z_n\varphi_n^{(n-1)}(t_0)=0 \end{cases} \quad (1)$$

The determinant Wronskian $W(t_0)$, and $W(t_0)=0$.
non-zero solution: $z_1=C_1, z_2=C_2, \dots z_n=C_n$.

consider $\psi(t)=C_1\varphi_1(t)+C_2\varphi_2(t)+\cdots+C_n\varphi_n(t)$

since $C_1, C_2, \dots C_n$ are solutions of (1), $\psi(t_0)=\dot\psi(t_0)=\cdots=\psi^{(n-1)}(t_0)=0$.

that is $x=\psi(t)$ is the solution of the Cauchy problem. (本身是解，现在有了初值条件，建立柯西问题)

$t=t_0,\ x=0,\ \dot x=0,\ \dots,\ x^{(n-1)}=0$.

the Cauchy Problem has trivial solution $\psi(t)\equiv 0$.

by the uniqueness, $\psi(t)\equiv 0$. have not all zero $C_1,\dots C_n$.

Coro 1. If there exists the point $t_0\in(a,b)$ s.t. $W(t_0)=0$, then $W(t)\equiv 0$, and solutions $\varphi_1(t), \varphi_2(t), \dots \varphi_n(t)$ are linearly dependent on (a,b)

Coro 2. If there exists $t_1\in(a,b)$ s.t. $W(t_1)\neq 0$, then $W(t)\neq 0$ for any $t\in(a,b)$, and solution $\varphi_1(t),\dots\varphi_n(t)$ are linear independent on (a,b).

(在已知 $\varphi_k(t)$ 是解的情况下，只须考虑任意一点的 Wronskian 即可分析线性相关性.

§. Fundamental System of Solutions.

Consider . l.h.e. $L(X)=0$. (1)

where $L(X)=\sum_{k=0}^{n} p_k(t)\, x^{(n-k)}$, $p_0(t)\equiv 1$, $x^{(0)}=x$, and functions $p_k(t)$ are continuous on the interval (a,b) for all $k=1,2,\dots n$.

(基础解系)

Def. The set $\varphi_1(t), \varphi_2(t), \dots \varphi_n(t)$. consisting of n l.i. solution of the equation (1) is called the fundamental system of solutions of equation (1)

Thm 1. Homogeneous linear equation (1) has the fundamental system of solutions.

Pf: Let $A=\{a_{jk}\}_{j,k=1}^{n}$ be arbitrary square matrix of order n. s.t. $\det A\neq 0$.

Let's consider $\forall t_0\in(a,b)$. set n Cauchy Problem of (1).

$t=t_0$. $x=a_{1k}$, $\dot{x}=a_{2k}$, ... $x^{(n-1)}=a_{nk}$. $1\leq k\leq n$, $k=1,2,\dots,n$.

Let $\varphi_k(t)$ be the solution of k-th Cauchy Problem $W(t_0)=\begin{vmatrix} \varphi_1(t_0) & \cdots & \varphi_n(t_0) \\ \cdots & & \cdots \\ \varphi_1^{(n-1)}(t_0) & \cdots & \varphi_n^{(n-1)}(t_0) \end{vmatrix}=\det A\neq 0$.

By coro 2. $\varphi_k(t)$ are linearly independent on (a,b)

Thus, they form the fundamental system of solutions.

Def: Let $\varphi_1(t) \dots \varphi_n(t)$ be the fundamental system of solutions of equation (1) and $c_1, c_2, \dots, c_n$ be arbitrary constants. Let's consider:

$x(t)=c_1\varphi_1(t)+c_2\varphi_2(t)+\dots+c_n\varphi_n(t)$ (2)

The right hand side of (2). is called general solution of the equation.

Thm 2. Let $\varphi_1(t), \varphi_2(t) \dots \varphi_n(t)$ be the fundamental system of the solution of equation (1).

1) for any set of constant $c_1, c_2, \dots c_n$; (2) gives a solution of equation (1)

2) if $x=\xi(t)$ is the solution of (1). then there exists a set constants $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n$

s.t. $\xi(t)=\bar{c}_1\varphi_1(t)+\bar{c}_2\varphi_2(t)+\dots+\bar{c}_n\varphi_n(t)$

Pf 1). main property of the solutions of a l.h.e.

2) $\forall t_0\in(a,b)$. $\begin{cases} z_1\varphi_1(t_0)+\dots+z_n\varphi_n(t_0)=\xi(t_0) \\ \vdots \\ z_1\varphi_1^{(n-1)}(t_0)+\dots+z_n\varphi_n^{(n-1)}(t_0)=\xi^{(n-1)}(t_0) \end{cases}$ (3) the Wronskian $W(t_0)\neq 0$. (since $\varphi_1(t), \dots \varphi_n(t)$ is l.i.).

Thus, system has the unique solution $z_1=\bar{c}_1$, $z_2=\bar{c}_2$, ... $z_n=\bar{c}_n$

Let's consider $\eta(t)=\sum_{k=1}^{n}\bar{c}_k\varphi_k(t)$. The function $x=\eta(t)$ is the solution of (1) and $\eta^{(k)}(t_0)=\xi^{(k)}(t_0)$.

ξ, η solve the same Cauchy Problem, by the uniqueness $\xi(t)\equiv\eta(t)$ on the interval (a,b).

§. Linear non-homogenous equation.

Consider $L(x)=q(t)$. (1). $L(x)=\sum_{k=1}^{n} p_k(t)x^{(n-k)}$, $p_0(t)\equiv 1$. $x^{(0)}=x$. $p_k(t)\in C(a,b)$

$L(x)=0$ (2) corresponding homogenous solution.

Thm 1. if $x=\psi(t)$ is solution of (1). $x=\varphi(t)$ is solution of (2). $\psi(t)+\varphi(t)$ is solution of (1).

Pf: use the linearity.

Def 2. general solution: $x(t)=c_1\varphi_1(t)+\cdots+c_n\varphi_n(t)+\psi(t)$. (3).

$\psi(t)$ is the solution of (1). $\varphi_k(t)$ is the fundamental solution of (2).

Thm 2. Let $\psi(t)$ solution of (1). $\varphi_k(t)$ be the fundamental system of (2). Then

1) for any set of constant $c_1,\dots c_n$ formula (3) gives a solution of equation (1)

2) if $x=\xi(t)$ is the solution of (1). then there is the set of constants $\bar{c}_1,\bar{c}_2,\dots\bar{c}_n$ s.t. $\xi(t)=\bar{c}_1\varphi_1(t)+\bar{c}_2\varphi_2(t)+\cdots+\bar{c}_n\varphi_n(t)+\psi(t)$

Pf: 1) linearity

2) $\forall t_0\in(a,b)$.
$$\begin{cases}\sum z_k\varphi_k(t_0)=\xi(t_0)\\ \vdots\\ \sum z_k\varphi_k^{(n-1)}(t_0)=\xi^{(n-1)}(t_0)\end{cases}$$
$W(t_0)\neq 0$ (fundamental system $\to$ l.i.).

similarly as we proved in l.h. $\exists\, \bar{c}_k=z_k$.

set $\eta(t)=\bar{c}_1\varphi_1(t)+\bar{c}_2\varphi_2(t)+\cdots+\bar{c}_n\varphi_n(t)+\psi(t)$

get $\eta(t)\equiv\xi(t)$ similarly.

§.5. Lagrange's variation method. (求非齐次线性微分方程组的特解，也证明了特解存在性)

Consider $L(x)=q(t)$ (1) $L(x)=\sum_{k=1}^{n}p_k(t)x^{(n-k)}$, $p_0(t)\equiv 1$. $x^{(0)}=x$. where $p_k(t), q(t)$ cont.

$L(x)=0$ (2)

Let $\varphi_1(t),\varphi_2(t),\dots\varphi_n(t)$ be f.s. of solutions of the equation (2).

Particular solution $x=\psi(t)$ of the equation (1). will be sought in the form.

$\psi(t):=\sum_{j=1}^{n}u_j(t)\varphi_j(t)$, where $u_j(t)$ are unknown functions. $j=1,2,3,\dots,n$. find $u_j(t)$

① take the derivative. $\dot\psi(t)=\sum\dot u_j(t)\varphi_j(t)+\sum u_j(t)\dot\varphi_j(t)$

② Let's set $\sum_{j=1}^{n}\dot u_j(t)\varphi_j(t)=0$. $\Rightarrow$ $\dot\psi(t)=\sum u_j(t)\dot\varphi_j(t)$ (先假设，后说明可行).

repeat ①. $\ddot\psi(t)=\sum u_j(t)\ddot\varphi_j(t)+\sum\dot u_j(t)\dot\varphi_j(t)$

repeat ② $\sum\dot u_j(t)\dot\varphi_j(t)=0$. $\Rightarrow$ $\ddot\psi(t)=\sum u_j(t)\ddot\varphi_j(t)$

repeat ①② $\psi^{(s)}(t)=\sum^{n}u_j(t)\varphi_j^{(s)}(t)$ for all $s=1,2,\dots,n-1$.

and $\psi^{(n)}(t)=\sum_{j=1}^{n}u_j(t)\varphi_j^{(n)}(t)+\sum_{j=1}^{n}\dot u_j(t)\varphi_j^{(n-1)}(t)$

$$L(\psi(t)) = L\Big(\sum_{j=1}^{n} u_j(t)\varphi_j(t)\Big) = \sum_{k=0}^{n} p_k(t)\,\psi^{(n-k)}(t) = \sum_{k=0}^{n} p_k(t)\Big[\sum_{j=1}^{n} u_j(t)\varphi_j^{(n-k)}(t)\Big] + \underbrace{\sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t)}_{\text{对于 } \psi^{(n)},\ p_0(t)=1.}$$

$$= \sum_{j=1}^{n} u_j(t)\Big[\sum_{k=0}^{n} p_k(t)\varphi_j^{(n-k)}(t)\Big] + \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t)$$

$$= \sum_{j=1}^{n} u_j(t)\, L(\varphi_j(t)) + \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t) = \sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t)$$

if $x=\psi(t)$ is the solution of the equation (1), $\sum_{j=1}^{n} \dot{u}_j(t)\varphi_j^{(n-1)}(t) = q(t)$ (3)

Collect the condition

$$\begin{cases} \dot{u}_1(t)\varphi_1(t) + \cdots + \dot{u}_n(t)\varphi_n(t) = 0 & \text{(def of } \psi(t)) \\ \dot{u}_1(t)\dot{\varphi}_1(t) + \cdots + \dot{u}_n(t)\dot{\varphi}_n(t) = 0 & \text{(obtained by step ②).} \\ \cdots & \cdots \\ \dot{u}_1(t)\varphi_1^{(n-2)}(t) + \cdots + \dot{u}_n(t)\varphi_n^{(n-2)}(t) = 0 & \\ \dot{u}_1(t)\varphi_1^{(n-1)}(t) + \cdots + \dot{u}_n(t)\varphi_n^{(n-1)}(t) = q(t) & (3). \end{cases} \qquad (4)$$

Def. System (4) is called the system in variations ($\dot{u}_j$ unknown).

For any fixed $t \in (a,b)$. system (4) is lin-h.s. w.r.t. $\dot{u}_1(t), \dots \dot{u}_n(t)$.

the determinant of (4) is Wronskian $W(t) \neq 0$. the system (4) has non-trivial solutions, denote as $\dot{u}_j(t) = f_j(t)$

关于 $f_j(t) \in C(a,b)$ 所有φ连续. f_j 可表示为行列式的比 $f_j = \frac{|A_j|}{|A|}$, f_j 表示为连续函数的$+-\times\div$. 且$|A| \neq 0$.

$f_j(t) \in C(a,b)$. $u_j(t) = \int f_j(t)\,dt$, $j=1,2,\dots n$.

此处只是 for each t. 还不是函数. coefficient conti. depend on t, solution will be cont. depend on coefficient. also on t.

By the def. of $\psi(t)$. $\psi(t) = \sum_{j=1}^{n} \varphi_j(t)\int f_j(t)\,dt$

the general solution of (1). takes form $x(t) = C_1\varphi_1(t) + \cdots + C_n\varphi_n(t) + \sum_{j=1}^{n}\varphi_j(t)\int f_j(t)\,dt$

可以看做系统的"常数变易"

Tech. Ricatti Equation:

$\dot{y} + a(x)y + b(x)y^2 = c(x)$.

if we know the solution of this equation $y=\psi$. then substitution $y = z+\psi$.

§. Linear homogenous equation with constant coefficients.

Let $L(x) = \sum_{k=0}^{n} a_k x^{(n-k)}$, where $a_0 = 1$, a_k are real numbers $k = 1, 2, \dots, n$.

Consider the linear homogenous equation $L(x) = 0$.

Seek for a solution of the equation (1) in the form $x = e^{\lambda t}$.

Let's substitute the function $e^{\lambda t}$ in $L(x)$:

$L(e^{\lambda t}) = P(\lambda) e^{\lambda t}$, where $P(\lambda) = \sum_{k=0}^{n} a_k \lambda^{n-k}$. ($\lambda$ can be complex, $(e^{\lambda t})' = \lambda e^{\lambda t}$. $\lambda \in \mathbb{C}$)

Def. Polynomial $P(\lambda)$ is characteristic polynomial.

Equation $P(\lambda) = 0$ is characteristic equation. the roots are characteristic numbers. (of equation $L(x) = 0$)

Fact: $x = e^{\lambda t}$ is a solution of $L(x) = 0 \Leftrightarrow \lambda$ is the root of the characteristic equation.

1. If. all characteristic number $\lambda_1, \dots, \lambda_n$. real and simple (not multiple).
   $L(x)$ has f.s.o.s. $e^{\lambda_1 t}, e^{\lambda_2 t}, \dots e^{\lambda_n t}$ (linear, independent).
2. all $\lambda_1, \dots \lambda_n$ are simple, some $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$.

Lemma. If $u(t)$. $v(t)$ real valued functions, and complex-valued function $w(t) = u(t) + i v(t)$. if $w(t)$ is the solution of $L(x) = 0$. then $u(t)$ and $v(t)$ are also solution of (2).

Pf: $L(w(t)) = L(u(t) + i v(t)) = L(u(t)) + i L(v(t)) = 0$.

$L(u(t)), L(v(t)) \in \mathbb{R}$, $L(u(t)) = L(v(t)) = 0$.

If $\lambda_j = \alpha + i\beta$ be a root. of $P(\lambda) = 0$, $e^{\lambda_j t}$ and $e^{\bar{\lambda}_j t}$ are solutions.

$e^{\lambda_j t} = e^{(\alpha + i\beta)t} = e^{\alpha t} \cos(\beta t) + i \cdot e^{\alpha t} \sin(\beta t)$

thus, the functions $e^{\alpha t} \cos \beta t$. $e^{\alpha t} \sin(\beta t)$ are solutions. (corre. to λ_j, $\bar{\lambda}_j$).

Thus we can build f.s.o.s. by real function (Although it has complex root).

3. let λ_j be multiple root. has the multiplicity $d \geq 2$.

It means. $P(\lambda_j) = P'(\lambda_j) = P''(\lambda_j) = \dots = P^{(d-1)}(\lambda_j) = 0$.

diff. equation $L(e^{\lambda t}) = P(\lambda) \cdot e^{\lambda t}$ m times w.r.t. λ.

$$\frac{\partial^m}{\partial \lambda^m} L(e^{\lambda t}) = \frac{\partial^m}{\partial \lambda^m} \left(\sum a_k \frac{\partial^{n-k}}{\partial t^{n-k}} e^{\lambda t}\right) = \sum_{k=0}^{n} a_k \frac{\partial^{n-k}}{\partial t^{n-k}} \cdot \left(\frac{\partial^m}{\partial \lambda^m} e^{\lambda t}\right)$$

$$= \sum a_k \cdot \frac{\partial^{n-k}}{\partial t^{n-k}} \cdot t^m \cdot e^{\lambda t} = L(t^m \cdot e^{\lambda t}).$$

$$\frac{\partial^m}{\partial \lambda^m} (P(\lambda) \cdot e^{\lambda t}) = \sum_{s=0}^{m} C_m^s P^{(s)}(\lambda) \cdot \frac{\partial^{m-s}}{\partial \lambda^{m-s}} e^{\lambda t} = \sum_{s=0}^{m} C_m^s P^{(s)}(\lambda) t^{m-s} e^{\lambda t}$$

Thus we have $L(t^m e^{\lambda t}) = \sum_{s=0}^{m} C_m^s P^{(s)}(\lambda) t^{m-s} e^{\lambda t}$.

put $\lambda = \lambda_j$, $m = 1, 2, \dots d-1$. $L(t^m \cdot e^{\lambda_j t}) = 0$.

the equation $L(x) = 0$ has d solution: $e^{\lambda_j t}, te^{\lambda_j t}, t^2 e^{\lambda_j t}, \dots t^{d-1} e^{\lambda_j t}$.

4. If. $\lambda_j = \alpha + i\beta$. $(\beta \neq 0)$ is multiple root of multiplicity d.

we have $2d$ solution. $e^{\alpha t}\cos(\beta t)$ $te^{\alpha t}\cos\beta t$ $\dots$ $t^{d-1} e^{\alpha t}\cos\beta t$

$e^{\alpha t}\sin(\beta t)$ $te^{\alpha t}\sin\beta t$ $\dots$ $t^{d-1} e^{\alpha t}\sin\beta t$.

It remains to show the system of solution we built is fundamental (Check the linear independence).

Pf: Let. $\lambda_1, \lambda_2, \dots \lambda_s$. be different roots. d_j. be multiplicity of the root λ_j.

Assume $e^{\lambda_1 t}, te^{\lambda_1 t} \dots t^{d_1-1} e^{\lambda_1 t}$ are linear independent.

$\vdots$

$e^{\lambda_s t}, te^{\lambda_s t} \dots t^{d_s-1} e^{\lambda_s t}$.

$\exists$ not all zero $C_{11} e^{\lambda_1 t} + C_{12} te^{t\lambda_1} + \dots + C_{1d_1} t^{d_1-1} e^{\lambda_1 t} + \dots + C_{s1} e^{\lambda_s t} + \dots C_{sd_s} t^{d_s-1} e^{\lambda_s t} = 0$.

rewrite as polynomial w.r.t. t $(t \in \mathbb{R})$. $R_{10}(t) e^{\lambda_1 t} + R_{20}(t) e^{\lambda_2 t} + \dots + R_{s0}(t) e^{\lambda_s t} = 0$

Let. $R_{s0}(t) \neq 0$. for certainty. $R_{10}(t) + R_{20}(t) e^{(\lambda_2 - \lambda_1)t} + \dots + R_{s0}(t) e^{(\lambda_s - \lambda_1)t} = 0$.

diff. d_1 times. $R_{21}(t) e^{(\lambda_2 - \lambda_1)t} + \dots + R_{s1}(t) \cdot e^{(\lambda_s - \lambda_1)t}$

($R_{j1}(t)$ are polynomials of same degree as $R_{j0}(t)$; $j = 2, 3, \dots s$; $R_{j1}(t) \equiv 0$ iff $R_{j0}(t) \equiv 0$)

Repeat the procedure $(s-1)$ times. $R_{s(s-1)}(t) \cdot e^{(\lambda_s - \lambda_{s-1})t} \equiv 0$. ($R_{j1} = (\lambda_j - \lambda_1) R_{j1} + R_{j1}'$. 若$R_{j0} \neq 0$. 前后项次数不同. $R_{j1} \neq 0$).

since $\lambda_s \neq \lambda_{s-1}$. $R_{s(s-1)}(t) \equiv 0$. $\Rightarrow$ $R_{s0}(t) \equiv 0$. $\square$

§. Linear non-homogeneous equation with constant coefficients

$L(x) = \sum a_k x^{(n-k)}$, where $x^{(0)} = x$. $a_0 = 1$. $\quad L(x) = q(t) \quad (1)$

$L(x) = 0 \quad (2)$

Characteristic equation for (2) is $P(\lambda) = \sum_{k=0}^{n} a_k \lambda^k = 0 \quad (3)$

Let $\varphi_1, \dots \varphi_n$ be fundmental system of solutions of the (2).

Find particular solution $x = \psi(t)$ of (1) - method of indetermined coefficients.

1. Let non-homo of (1). have form $q(t) = R_m(t) e^{\lambda_0 t}$ ($R_m(t)$ is $R_m(t) = \sum_{j=0}^{m} r_j t^{m-j}$.)

Thm 1. If λ_0 is not the root of the characteristic equation (3) $(P(\lambda_0) \neq 0)$. then (1) with nonlinearity $q(t)$ has a solution of the form $\psi(t) = Q_m(t) e^{\lambda_0 t}$ ($Q_m(t) = \sum_{j=0}^{m} q_j t^{m-j}$).

Thm2. If λ_0 is the root of characteristic equation (3). of multiplicity $d \geq 1$.

$p(\lambda_0) = p'(\lambda_0) = p''(\lambda_0) = \cdots = p^{(d-1)}(\lambda_0) = 0$. $\quad p^{(d)}(\lambda_0) \neq 0$.

then the equation (1) with nonlinearity $q(t)$ has a solution $\varphi(t) = t^d Q_m(t) e^{\lambda_0 t}$.

Rem: $L(X) = q_1 + q_2$.

$\Rightarrow$ solve $\begin{cases} L(\varphi_1) = q_1 \\ L(\varphi_2) = q_2 \end{cases} \Rightarrow L(\varphi_1 + \varphi_2) = q_1 + q_2$ (linearity)

Let the non-homogeneity of $L(X) = q(t)$ have the form.

$q(t) = e^{\alpha_0 t}(\tilde{R}_{m_1}(t)\cos(\beta_0 t) + \hat{R}_{m_2}(t)\sin(\beta_0 t))$.

($\tilde{R}_{m_1}(t)$. $\hat{R}_{m_2}$. m_1-/m_2-degree of polynomials).

Thm3. If $\lambda_0 = \alpha_0 + i\beta_0$ is not the root of characteristic equation (3). then (1). with $q(t)$. has the solution of the form

$$\varphi(t) = e^{\alpha_0 t}(\tilde{Q}_m(t)\cos(\beta_0 t) + \hat{Q}_m(t)\sin(\beta_0 t)).$$

$m = \max(m_1, m_2)$. $\tilde{Q}_m$. $\hat{Q}_m$ are m-degree polynomial.

if $\lambda_0 = \alpha_0 + i\beta_0$ is the root of (3)

$$\varphi(t) = t^d e^{\alpha_0 t}(\tilde{Q}_m(t)\cos(\beta_0 t) + \hat{Q}_m(t)\sin(\beta_0 t)).$$

Tech. $a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_n y = f(x)$.

$\Rightarrow x = e^t$.

Chapter 4. Linear Systems of differential Equation.

Consider
$$\begin{cases} \dot{x}_1 = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + q_1(t) \\ \dot{x}_2 = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + q_2(t) \\ \cdots \\ \dot{x}_n = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + q_n(t) \end{cases}$$
normal form (1).

where $p_{kj}(t)$. $q_j(t) \in C(a.b)$

RHS $X_j(t,x) = \sum_{k=1}^{n} p_{jk}(t)x_k + q_j(t)$ are cont. in the domain $G = \{(t,x) : t \in (a,b), \|x\| < +\infty\}$

$X_j \in Lip^{loc}\, G.$, since $\frac{\partial X_j(t,x)}{\partial x_k} = p_{jk}(t) \in C(G)$.

Construct Cauchy Problem $t = t_0$, $x_1 = x_{10}, \ldots x_n = x_{n0}$ ($t_0 \in (a.b)$. $x_0 = (x_{10}, \ldots x_{n0})^T \in \mathbb{R}^n$) has the unique solution.

Def. System (1) is homogeneous if $q_j(t) \equiv 0$, $j = 1,2,\ldots n$.
Otherwise, the system is called non-homogeneous.

△ Notation of matrix. and vector

$U(t)_{[s\times n]} = \{u_{jl}(t)\}$, derivate, continuity, differentiability.
$U \Leftrightarrow$ all $u_{jl}(t)$.

$$\frac{d}{dt}(U(t)\,V(t)) = \dot{U}(t)V(t) + U(t)\dot{V}(t)$$

$$\int_a^b U(t)\,dt = \left\{ \int_a^b u_{jl}(t)\,dt \right\}. \quad \left\| \int_a^b U(t)\,dt \right\| \le \int_a^b \|U(t)\|\,dt.$$

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $P(t)_{[n\times n]} = \{p_{jk}(t)\}$. $q(t) = \begin{pmatrix} q_1(t) \\ \vdots \\ q_n(t) \end{pmatrix}$ $\quad (1) \Leftrightarrow \dot{x} = P(t)x + q(t)$

§. Matrix. equation.

1. Linear homogeneous system. $\dot{x} = P(t)x$ (1). $P(t)_{[n\times n]} = \{p_{jk}(t)\} \in C(a,b)$.

Let vector function $\varphi_1, \ldots \varphi_m$ be solution of (1).

build matrix: $\Phi_m(t) = (\varphi_1(t), \ldots \varphi_m(t))$. matrix equation $\dot{X} = P(t)X$ (2) (X has n row).

Solution of (2) called the matrix $X = X(t)$. cont. differentiable (a,b). and satisfying the equation.

$\varphi_1, \ldots \varphi_m$ are solutions of the system (1) iff the matrix $\Phi_m(t)$ is the solution of (2).

Pf $\Rightarrow$ if $\varphi_1 \ldots \varphi_m$ are solution of system (1). $\dot{\varphi}_j = P(t)\varphi_j(t)$ for all $j = 1,2,\ldots m$.

According to (2). $\dot{\Phi}_m(t) = (\dot{\varphi}_1, \dot{\varphi}_2, \ldots \dot{\varphi}_m) = (P(t)\varphi_1(t), P(t)\varphi_2(t), \ldots P(t)\varphi_m(t)) = P(t)\Phi_m(t)$

$\Leftarrow$ $(\dot{\varphi}_1 \ldots \dot{\varphi}_m) = \dot{\Phi}_m = P(t)\Phi_m(t) = (P(t)\varphi_1(t), \ldots, P(t)\varphi_m(t))$.

$\dot{\varphi}_j(t) = P(t)\varphi_j(t)$ for any $j = 1,2,\ldots m$.

Cauchy Problem. $t = t_0$. $X = A_0$. ($t_0 \in (a,b)$).

main property

thm.1. Let $\varphi_1 \dots \varphi_m$ be solution of (1). Then vector function
$\psi(t) = c_1\varphi_1 + \dots + c_m\varphi_m$. where $c_1, c_2, \dots, c_m$ are arbitrary constant. is also solution of (1) on (a,b)

Pf: write. $\psi(t) = \Phi_m(t)\,c$. $c = (c_1, \dots c_m)^T$.
$\dot\psi(t) = \dot\Phi_m(t)\cdot c = P(t)\Phi_m(t)\,c = P(t)\psi(t)$.

§. Linear Independent solutions. of linear homogeneous system.

Def.1. Solution $\varphi_1 \dots \varphi_m$ of (1). call l.d. on (a,b). if $\exists\, c_1 \dots c_m$ not all zero.
s.t. $c_1\varphi_1(t) + \dots + c_m\varphi_m(t) \equiv 0$

Def.2. l.d. if. $\exists$ m-dimensional constant vector $c = (c_1 \dots c_m)^T$ s.t. $c \neq 0$. $\Phi_m(t)\,c \equiv 0$ on (a,b)

Thm 1. Let solution $\varphi_1 \dots \varphi_m$ be linear dependent on (a,b). for any $t \in (a,b)$
rank $\Phi_m(t) < m$.

Pf: $\exists\, \bar c \neq 0$, $\dim \bar c = m$. s.t. $\Phi_m(t)\bar c \equiv 0$.
consider l.h. algebraic system. $\Phi_m(t)\, z = 0$. ($z = (z_1 \dots z_m)^T$ is unknown vector)
the system has a non-zero solution $z = \bar c$ for each $t \in (a,b)$.
then $r(\Phi_m(t)) < m$ ($|\Phi_m(t)| \neq 0$).

Thm 2. Let solution $\varphi_1 \dots \varphi_m$ of (1). $\exists\, t_0 \in (a,b)$ s.t. rank $\Phi_m(t_0) < m$.
then $\varphi_1 \dots \varphi_m$ are linearly dependent on (a,b).

Pf: Consider $\Phi_m(t_0)\, z = 0$. has a non-zero solution $z = \bar c$.
$\psi(t) = \Phi_m(t)\bar c \to x = \psi(t)$ is also the solution of system (1).
$\psi(t_0) = \Phi_m(t_0)\bar c = 0$. Cauchy Problem. $t = t_0$. $x = 0$. $\to$ solve by $x(t) \equiv 0$.
the uniqueness follows $\psi(t) \equiv 0$. on (a,b). $\varphi_1 \dots \varphi_m$ are l.d. $\square$.

Coro. if. $\exists\, t_1 \in (a,b)$. rank $\Phi_m(t_1) = m$. $\Rightarrow$ rank $\Phi_m(t) = m$ for all $t \in (a,b)$.
$\varphi_1 \dots \varphi_m$ are linearly independent on (a,b).

Thm 3. System (1) can't have more than n l.i. solution.

Pf: Let $\varphi_1 \dots \varphi_m$ are solutions. $m > n$.
build matrix Φ_m. size: $n \times m$. rank $\Phi_m \leq n < m$. $\varphi_1 \dots \varphi_m$ are l.d. on (a,b).

Def. Determinant of $\Phi(t) = (\varphi_1 \dots \varphi_n)$. $\varphi_1 \dots \varphi_n$ are solution of (1) on (a,b).
is called the Wronski determinant $W(t) = \det \Phi(t)$.

Coro. $\exists\, t_0 \in (a,b)$. $W(t_0) = 0$, then $W(t) \equiv 0$ on (a,b). solutions $\varphi_1 \dots \varphi_n$ are l.d.

Coro. $\exists\, t_1 \in (a,b)$ $W(t_1) \neq 0$. then $W(t) \neq 0$ on (a,b). solutions $\varphi_1 \dots \varphi_n$ are l.i.

§. Fundamental System of solution.

consider l.h. $\dot{x} = P(t)x$. $P(t)_{n\times n} = \{p_{jk}(t)\} \in C(a,b)$ (1)

Def. The set of l.i. solution $\varphi_1, \dots \varphi_n$ of (1) is called fundamental system of solution of system (1). The matrix $\Phi(t) = (\varphi_1, \dots \varphi_n)$. fundamental matrix of (1)

Thm 1. Linear system (1) has a fundamental system of solution.

$\therefore$ Pf: Let A be any $n\times n$ matrix. $\det A \neq 0$.

take $\forall t_0 \in (a,b)$. set Cauchy Problem $t=t_0$, $X=A$. for $\dot{X} = P(t)X$ (2)

Cauchy Problem has a solution $X = \Phi(t)$. Columns $\varphi_1, \varphi_2, \dots \varphi_n$ of $\Phi(t)$ are solutions of (1). Where $\det \Phi(t_0) = W(t_0) = \det A \neq 0$.

thus $W(t) \neq 0$. for any $t \in (a,b)$, $\varphi_1 \dots \varphi_n$ are l.i. form a fundamental system of solution

Def. (general solution)

Let $\Phi(t)$. fundamental solution of (1). c be arbitrary n-dim vector.

Consider $x(t) = \Phi(t)c$. the RHS is called general solution of (1).

Thm 2. Let $\Phi(t)$ be fundamental matrix of (1). Then.

1) for any constant vector c, $x(t) = \Phi(t)\cdot c$ gives the solution.

2). if $x = \xi(t)$ is solution (1). $\exists \bar{c} \in \mathbb{R}^n$, $\xi(t) = \Phi(t)\bar{c}$ for any $t \in (a,b)$

Pf. 1). use the linearity

2). $\forall t_0 \in (a,b)$. $\Phi(t_0)z = \xi(t_0)$. z is unknown.

the determinant of this system is $W(t_0)$. and $W(t_0) \neq 0$. since $\Phi(t)$. fundmental matrix.

Hence $\Phi(t_0)z = \xi(t_0)$ has unique solution $z = \bar{c}$.

set $\eta(t) = \Phi(t)\bar{c}$. $\eta(t)$ and $\xi(t)$ solve the same Cauchy Problem. $\eta \equiv \xi$.

General expression for fundamental matrix.

Thm 3. Let $\Phi(t)$ be fundamental. matrix of system (1)

1). if B is an arbitrary square matrix of order n. s.t. $\det B \neq 0$. then $\Psi(t) = \Phi(t)B$ is also fundamental matrix of the system (1)

2) if Ψ is fundamental matrix of (1). $\exists B$ of $n\times n$. $\det B \neq 0$. s.t. $\Psi(t) = \Phi(t)B$ for any $t \in (a,b)$

Pf: 1), $\dot{\Psi}(t) = \dot{\Phi}(t)B = P(t)\Phi(t)B = P(t)\Psi(t)$.

Moreover, $\det \Psi(t) = \det \Phi(t) \cdot \det B \neq 0$.

2). Let $\Psi(t)$ be f.m. of (1). consider $\forall t_0 \in (a,b)$. set $B = \Phi^{-1}(t_0)\Psi(t_0)$ $\det B \neq 0$.

$X(t) = \Phi(t)B$ is f.m. of (1). $X(t_0) = \Phi(t_0)B = \Phi^{-1}(t_0)\Phi(t_0)\Psi(t_0) = \Psi(t_0)$

(X, Ψ. solves. same C.P. $X \equiv \Psi$).

Def3. Let $\Phi(t)$ be fundamental matrix of (1). B be arbitrary $n\times n$. matrix. s.t. $\det B \neq 0$. Consider the formula: $X(t) = \Phi(t)B$. RHS is called the general expression for the fundamental matrices of (1).

Fundamental matrix $X(t,t_0)$. turns into. identity matrix for $t=t_0$. – fundamental Cauchy matrix

Rem: $\forall \Phi(t)$. f.m. of (1). $X(t,t_0) = \Phi(t)\Phi^{-1}(t_0)$ forms fundment Cauchy matrix (可以当作定义).

§. Liouville formula.

$$W(t) = c\cdot \exp\left(\int \operatorname{Tr} P(t)\,dt\right) = W(t_0)\exp\left(\int_{t_0}^{t} \operatorname{Tr} P(\tau)\,d\tau\right).$$

Wronskian 由初值.系数对角线决定. $t_0 \in (a,b)$.

Remark: From the Liouville formula. for l.s. $\Rightarrow$ l.e.

$$x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_n(t)x = 0 \Rightarrow \begin{cases} x_1 = x \\ x_2 = \dot{x} \\ \vdots \\ x_n = x^{(n-1)} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \cdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -p_n(t)x_1 - p_{n-1}(t)x_2 - \cdots - p_1(t)x_n \end{cases}$$

$$W(t) = c\exp\left(-\int p_1(t)\,dt\right) = W(t_0)\exp\left(-\int_{t_0}^{t} p_1(\tau)\,d\tau\right).$$

Tech. find particular solution ① – linear homo – denote as φ_1. use $W(t)$. find $\varphi_2 \ldots \varphi_n$.

①② – linear nonhomo – denote ①–② $= \varphi_1$. find $\varphi_2 \ldots \varphi_n$.

Proof of Liouville formula: $\dot{x}_j = \sum_{k=1}^{n} p_{jk}(t)x_k \quad j = 1,2,\ldots n.$ (linear. homo.) (1)

Let $\varphi_1(t), \ldots \varphi_n(t)$ be solutions of system (1). $\varphi_m(t) = \begin{pmatrix} \varphi_{1m}(t) \\ \cdots \\ \varphi_{nm}(t) \end{pmatrix}$ $m = 1,2,\ldots,n$.

$$W(t) = \begin{vmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \cdots & \varphi_{1n}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) & \cdots & \varphi_{2n}(t) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix} \qquad W(t) = \sum_{\Delta} (-1)^{\alpha} \varphi_{1j_1}(t) \cdots \varphi_{nj_n}(t)$$

$$\dot{W}(t) = \sum_{s=1}^{n}\sum_{\Delta}(-1)^{\alpha}\varphi_{1j_1}(t)\cdots\dot{\varphi}_{sj_s}(t)\cdots\varphi_{nj_n}(t) = \sum_{s=1}^{n}\begin{vmatrix} \varphi_{11}(t) & \cdots & \varphi_{1n}(t) \\ \cdots & & \cdots \\ \dot{\varphi}_{s1}(t) & \cdots & \dot{\varphi}_{sn}(t) \\ \cdots & & \cdots \\ \varphi_{n1}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix}$$

since $\varphi_m(t)$ are solution of (1).

$$\dot{\varphi}_{sm}(t) = \sum_{k=1}^{n} p_{sk}(t)\varphi_{km}(t) \quad s = 1,2,\ldots,n.$$

$$\dot{W}(t) = \sum_{s=1}^{n}\begin{vmatrix} \varphi_{11}(t) & \varphi_{12}(t) & \cdots & \varphi_{1n}(t) \\ \cdots & \cdots & & \cdots \\ \sum_{k=1}^{n}p_{sk}(t)\varphi_{k1}(t) & \sum_{k=1}^{n}p_{sk}(t)\varphi_{k2}(t) & \cdots & \sum_{k=1}^{n}p_{sk}(t)\varphi_{kn}(t) \\ \cdots & \cdots & & \cdots \\ \varphi_{n1}(t) & \varphi_{n2}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix} = \sum_{s=1}^{n}\sum_{k=1}^{n} p_{sk}(t)\cdot\begin{vmatrix} \varphi_{11}(t) & \cdots & \varphi_{1n}(t) \\ \cdots & & \cdots \\ \varphi_{k1}(t) & \cdots & \varphi_{kn}(t) \\ \cdots & & \cdots \\ \varphi_{n1}(t) & \cdots & \varphi_{nn}(t) \end{vmatrix} \to s\text{th row}$$

when $s \neq k$. $D = 0$. $\dot{W}(t) = \sum_{s=1}^{n} p_{ss}(t)W(t)$

$$\Rightarrow W(t) = c\,e^{\int \operatorname{Tr} P(t)\,dt}.$$

In particular, for equation $x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_{n-1}(t)x^{(1)} + p_n(t)x = 0$.

$$\begin{cases} \dot{x}_1 = x_2 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = -p_n(t)x_1 - \cdots - p_1(t)x_n \end{cases} \Rightarrow W(t) = C e^{-\int p_1(t)dt}$$

§. Linear homogeneous system with constant coefficients.

Consider a linear homogeneous system $\dot{x} = Ax$ (1). A is $n\times n$ constant matrix.

Looking for a solution of system (1) in the form $x = \gamma e^{\lambda t}$, where γ is constant n-dimensional vector, λ is scalar.

$$\dot{x} = Ax \overset{x=\gamma e^{\lambda t}}{\Longleftrightarrow} \lambda \gamma e^{\lambda t} = A\gamma e^{\lambda t} \Longleftrightarrow (A - \lambda E)\gamma = 0. \qquad (2).$$

The l.s. has non-trivial solution iff $\det(A-\lambda E) = 0$.

Def. characterise.

Equation $\det(A-\lambda E)=0$ is called the characteristic equation of the system (1).

The roots of equation $\det(A-\lambda E) = 0$ are eigenvalues of matrix A. (characteristic number)

$x = \gamma e^{\lambda t}$ is a solution of (1) $\Leftrightarrow$ λ is eigenvalue of the matrix A.
γ is eigenvector corr. to this number

Let. $\lambda_1, \dots \lambda_n$; $\gamma_1 \dots \gamma_n$. be eigenvalue and corr. eigenvector. To construct f.s. of (1).

1. All characteristic numbers $\lambda_1, \lambda_2, \dots \lambda_n$ are real and simple (not multiple).
f.s.o.s: $\gamma_1 e^{\lambda_1 t}, \gamma_2 e^{\lambda_2 t}, \dots, \gamma_n e^{\lambda_n t}$.

2. All characteristic numbers are simple but some are complex.

Lemma. If $u(t)$. $v(t)$ are real vector functions, and complex-value function $w(t) = u(t) + i v(t)$ is solution of (1). $u(t)$. $v(t)$ are also solutions of (1).

Pf: $\dot{w}(t) = \dot{u}(t) + i\dot{v}(t)$.
$\dot{w}(t) = Aw(t) = A(u(t) + iv(t)) = Au(t) + iAv(t)$.
$\Rightarrow$ $\dot{u}(t) = Au(t)$ and $\dot{v}(t) = Av(t)$.

Let. $\lambda_j = \alpha + i\beta$. ($\beta \neq 0$). be eigenvalue. $\bar{\lambda}_j$ also eigenvalue. (γ_j. $\bar{\gamma}_j$ similarly).

function. $Re(\gamma_j e^{\lambda_j t})$, $Im(\gamma_j e^{\lambda_j t})$. are solutions of system (1). (corr. to λ_j, $\bar{\lambda}_j$).

3. Some multiple roots.
If real root λ_j. $j \in \{1, 2, \dots n\}$ has a multiplicity $d \geq 2$.
solution should be in the form $x = \gamma_j(t) e^{\lambda_j t}$. ($\gamma_j(t)$. poly. of deg $d-1$).
(thus we have d l.i. solution of this type).
if $\lambda_j = \alpha \pm i\beta$. of multiplicity d. $\Rightarrow$ $2d$ solution.

$\Rightarrow$ T.b.c. show this n solutions are l.i.

§. Matrix method of integrating of a linear homogenous system with con. coefficient.

Matrix theory introduction:

convergence: $A_k = \{a_{jm}^{[k]}\}_{[n\times n]} \xrightarrow[k\to\infty]{} A = \{a_{jm}\}_{[n\times n]}$ if $\|A_k - A\| \xrightarrow[k\to+\infty]{} 0$

$\iff a_{jm}^{[k]} \xrightarrow[k\to+\infty]{} a_{jm}$ for all $j = 1,2,\dots n$. $m = 1,2,\dots n$.

Series convergence: $\sum_{k=1}^{+\infty} A_k \to A$. iff its partial sum conv. to A. note $\sum_{k=1}^{+\infty} A_k = A$.

1. $\sum_{k=1}^{+\infty} A_k = A$ iff $\sum_{k=1}^{S} a_{jm}^{[k]} \xrightarrow[S\to+\infty]{} a_{jm}$ for all j, m.

2. if numerical series $\sum b_k$ conv. $\|A_k\| \le b_k$ for all $k \in \mathbb{N}$.
   then $\sum_{k=1}^{+\infty} A_k$ also conv.

3. $A = \{a_{jm}\}_{[n\times n]}$ constant matrix. $\sum_{k=0}^{+\infty} \frac{1}{k!} A^k$ $(A^0 = E)$, $\|A^k\| \le \|A\|^k$.
   and $\sum_{k=0}^{+\infty} \frac{1}{k!} \|A\|^k = e^{\|A\|}$ conv., so $\sum \frac{1}{k!} A^k$ conv.

Def. Sum of $\sum_{k=0}^{+\infty} \frac{1}{k!} A^k$ - the matrix exponent of A. The notation $\sum_{k=0}^{+\infty} \frac{1}{k!} A^k = e^A$

(Rem. $A_{[n\times n]}$. $B_{[n\times n]}$. $e^{A+B} \ne e^A e^B$. in general case.

Lemma.1. Let $A_{[n\times n]}$ and $B_{[n\times n]}$ be square matrices. If $AB = BA$.
then $e^{A+B} = e^A e^B$

Pf: $e^{A+B} = \sum_{k=0}^{+\infty} \frac{1}{k!} (A+B)^k$.

$$e^A e^B = \left(\sum \frac{1}{k!} A^k\right)\left(\sum \frac{1}{m!} B^m\right) = \sum_{k=0}^{+\infty}\left(\sum_{s=0}^{k} \frac{1}{s!(k-s)!} A^{k-s} B^s\right)$$

1) $k=0$. $LHS = E = RHS$. $k=1$. $LHS = A+B = RHS$.

2) $k=2$. $LHS = \frac{1}{2}(A+B)^2$
$RHS = \frac{1}{2}(A^2 + 2AB + B^2)$.

3) $k=n$ $\sum_{s=0}^{k} \frac{1}{s!(k-s)!} A^{k-s} B^s = \frac{1}{k!}(A+B)^k$

Lemma 2. Let $A_{[n\times n]}$ and $B_{[n\times n]}$ be square matrix. there is a matrix $S_{[n\times n]}$ s.t. $\det S \ne 0$.
$A = SBS^{-1}$. Then $e^A = Se^B S^{-1}$.

Pf: $\left\|\sum_{k=0}^{m} \frac{1}{k!} A^k - Se^B S^{-1}\right\| = \left\|\sum \frac{1}{k!} (SBS^{-1})^k - Se^B S^{-1}\right\| = \left\|\sum \frac{1}{k!} SB^k S^{-1} - Se^B S^{-1}\right\|$

$$= \left\|S\left(\sum_{k=0}^{m} \frac{1}{k!} B^k - e^B\right) S^{-1}\right\| \le \|S\| \cdot \left\|\sum_{k=0}^{m} \frac{1}{k!} B^k - e^B\right\| \cdot \|S^{-1}\| \to 0.$$

Consider l.h.s. $\dot{x} = Ax$ (1). $A_{[n\times n]}$. constant matrix.

Thm. e^{At} is fundamental matrix of (1)

Pf: Let $A^k = \{a_{jm}^{[k]}\}_{[n\times n]}$. Then $e^{At} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} A^k = \left\{ \sum_{k=0}^{+\infty} \frac{t^k}{k!} a_{jm}^{[k]} \right\}_{[n\times n]}$

by def. $\frac{de^{At}}{dt} = \left\{ \frac{d}{dt} \sum \frac{t^k}{k!} a_{jm}^{[k]} \right\}_{[n\times n]} = \left\{ \sum_{k=1}^{+\infty} \frac{t^{k-1}}{(k-1)!} a_{jm}^{[k]} \right\}_{[n\times n]} = \sum_{k=1}^{+\infty} \frac{t^{k-1}}{(k-1)!} A^k$.

$$= A \sum_{k=1}^{+\infty} \frac{t^{k-1}}{(k-1)!} A^{k-1} = Ae^{At},$$

i.e. e^{At} satisfy $\dot{X} = AX$, the column of e^{At} are solutions of (1).

$W(t) = \det e^{At}$. $W(0) = \det e^{A\cdot 0} = 1. \neq 0$. e^{At} is fundamental matrix of (1).

Tech. $F(x, y^{(k)}, y^{(n)}) = 0$. let. $y^{(k)} = z$.

$F(y, y', \ldots y^{(n)}) = 0$ Let $y' = p(y)$

$F(x, y, y', \ldots y^{(n)}) = 0$. and $F(x, y, \ldots y^{(n)}) = F(x, ky, \ldots ky^{(n)})$. homo. w.r.t. $y, \ldots y^{(n)}$

Let. $y' = yz$.

造结构. $yy'' = (y')^2 \Rightarrow \frac{y''}{y'} = \frac{y'}{y} \Rightarrow (\ln y')' = (\ln y)'$

Let B be Jordan form of the matrix A, and S be reducing matrix, $\det S \neq 0$. and $A = SBS^{-1}$. Then $e^{At} = Se^{Bt}S^{-1}$ (by lemma 2).

Find e^{Bt}. B is block-diagonal matrix with blocks $B_1, B_2, \ldots B_d$;

$B = \operatorname{diag}(B_1, B_2, \ldots B_d)$

Let V_s be the size of the block B_s, $s = 1, 2, \ldots d$.

Then $B_s = \begin{bmatrix} \lambda_p & & & \\ 1 & \lambda_p & & \\ & 1 & \ddots & \\ & & & \lambda_p \end{bmatrix}_{V_s \times V_s}$ $\quad B_s = \lambda_p E_s + H_s$.

$H_s = \{h_{jm}\}$. $\begin{cases} h_{jm} = 1 \text{ if } m = j-1, (j = 2, 3, \ldots V_s). \\ h_{jm} = 0 \text{ if } m \neq j-1. \end{cases}$

it follows $e^{B_s t} = e^{(\lambda_p E_s + H_s)t} = e^{\lambda_p E_s t} e^{H_s t}$ (commute)

$$e^{\lambda_p E_s t} = \sum_{k=0}^{+\infty} \frac{(\lambda_p t)^k}{k!} E_s^k = \sum_{k=0}^{+\infty} \frac{(\lambda_p t)^k}{k!} E_s = E_s e^{\lambda_p t}$$

Consider $H_s^2 = \{h_{jm}^{[2]}\}$. $h_{jm} = \begin{cases} 1, & m = j-2, \ j = 3, \ldots V_s. \\ 0, & m \neq j-2. \end{cases}$

similarly, define $H_s^k = \{h_{jm}^{[k]}\}$, $k \leq V_s - 1$. where $h_{jm} = 1$. only if $m = j-k$

Also, $H_s^k = 0$ if $k \geq V_s$, where $0 = 0_{[V_s \times V_s]}$.

$$e^{H_s t} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} H_s^k = E_s + tH_s + \frac{t^2}{2} H_s^2 + \cdots + \frac{t^{V_s - 1}}{(V_s - 1)!} H_s^{V_s - 1}$$

$$= \begin{bmatrix} 1 & & & & & \\ t & 1 & & & & \\ \frac{t^2}{2} & t & 1 & & & \\ \cdots & \frac{t^2}{2} & t & 1 & & \\ \cdots & \cdots & \frac{t^2}{2} & t & \ddots & \\ \frac{t^{V_s-1}}{(V_s-1)!} & \cdots & & \ddots & t & 1 \end{bmatrix}_{[V_s \times V_s]}$$

$$e^{B_s t} = E_s \cdot e^{\lambda_p t} \cdot e^{H_s t} = \begin{bmatrix} e^{\lambda_p t} & & & \\ te^{\lambda_p t} & e^{\lambda_p t} & & \\ \frac{t^2}{2} e^{\lambda_p t} & te^{\lambda_p t} & e^{\lambda_p t} & \\ \frac{t^{V_s-1}}{(V_s-1)!} e^{\lambda_p t} & \cdots & \ddots & e^{\lambda_p t} \end{bmatrix}$$

$$e^{Bt} = \operatorname{diag}(e^{B_1 t} \cdots e^{B_d t}).$$

from $e^{At} = Se^{Bt}S^{-1} \Rightarrow e^{At}S = Se^{Bt}$. S is constant matrix, $\det S \neq 0$.

by thm 3. e^{At} is f.m. $\Rightarrow e^{At}S$ is f.m. $\Rightarrow Se^{Bt}$ is f.m.

the column of matrix Se^{Bt} has the form $x = \gamma_p(t) e^{\lambda_p t}$. $\dim \gamma_p \leq d-1$. d is mul-of λ_p.

To find particular solution of $x=\psi(t)$ of l.n.h.s. $\dot{x} = Ax + q(t)$ (1)

A is constant matrix of $n\times n$. $q(t)$ is cont. vector function (of some special form)

The linear homogeneous system corr. to (1). has form $\dot{x} = Ax$. (2)

Consider characteristic equation of (2). $\det(A-\lambda E)=0$. (3)

1. let $q(t) = R_m(t)\, e^{\lambda_0 t}$. (where $R_m(t)$ is vector polynomial of deg. m)

Thm 1. if λ_0 is not the root of (3), $\psi(t) = Q_m(t)\, e^{\lambda_0 t}$. vector polynomial Q_m. $\deg = m$.

Thm 2. if λ_0 is the root of (3). with multiplicity d $(d \geq 1)$

$$\psi(t) = Q_{m+d}(t)\cdot e^{\lambda_0 t}.$$

→ 注意此处与 linear equation 的区别 equation 时直接乘 t^d. 而此处则是多项式 $Q_d(t)$

(Q_{m+d} is vector polynomial. $\deg Q = m+d$).

2. Let $q(t) = e^{\alpha_0 t}\left(R_{m_1} \cos(\beta_0 t) + R_{m_2} \cos(\beta_0 t)\right)$.

Thm 3. if $\lambda_0 = \alpha_0 + i\beta_0$ is not root.

$$\psi(t) = e^{\alpha_0 t}\left(\tilde{Q}_m(t)\cos\beta_0 t + \hat{Q}_m(t)\sin\beta_0 t\right)$$

$\deg \tilde{Q}_m,\ \hat{Q}_m = m$, $m = \max(m_1, m_2)$

Thm 4. if λ_0 is root. with multiplicity $d \geq 1$.

$$\psi(t) = e^{\alpha_0 t}\left(\tilde{Q}_{m+d}(t)\cos\beta_0 t + \hat{Q}_{m+d}(t)\sin\beta_0 t\right)$$

Chapter 5. Differential Properties of Solutions as Functions of Initial Condition and Parameters.

Consider system of equation. RHS depends on the parameter $\dot{x} = X(t, x, \mu)$ 不同μ定义不同系统 用μ表示一系列相似的系统.
where $x \in \mathbb{R}^n$, vector function $X(t, x, \mu) \in C$ at $(t, x) \in G \subset \mathbb{R}^{n+1}$; $\mu \in F \subset \mathbb{R}$.
Denote $x = \varphi(t, \theta, \xi, \mu)$. the solution of (1). with initial condition $t = \theta$. $x = \xi$.

1. Theorem on integral continuity.

Lemma 1. Consider two system. $\dot{x} = X(t, x)$ (1) $\dot{y} = Y(t, y)$ (2); $x = \varphi(t)$, $y = \psi(t)$ be solution of (1), (2). defined on $[c, d]$
[$x, y \in \mathbb{R}^n$. $X(t,x)$. $Y(t,y) \in C(D)$. $D \subset \mathbb{R}^{n+1}$. $X(t,x) \in Lip_x D$]
$\theta \in [c, d]$. for any $t \in [c,d]$. $\|\varphi(t) - \psi(t)\| \le \left[\|\varphi(\theta) - \psi(\theta)\| + \int_c^d \|X(t, \psi(t)) - Y(t, \psi(t))\| d\tau\right] e^{L|t-\theta|}$.

Pf: $x = \varphi(t)$. $y = \psi(t)$ satisfy the integral equation:

$$\varphi(t) = \varphi(\theta) + \int_\theta^t X(\tau, \varphi(\tau))\, d\tau. \qquad \psi(t) = \psi(\theta) + \int_\theta^t Y(\tau, \psi(\tau))\, d\tau.$$

$$\Rightarrow \|\varphi(t) - \psi(t)\| \le \|\varphi(\theta) - \psi(\theta)\| + \left|\int_\theta^t \|X(\tau, \varphi(\tau)) - Y(\tau, \psi(\tau))\| d\tau\right| \qquad (3).$$

evaluate the integral: $\left|\int_\theta^t \|X(\tau, \varphi(\tau)) - Y(\tau, \psi(\tau))\| d\tau\right|$

$$\le \left|\int_\theta^t \|X(\tau, \varphi(\tau)) - X(\tau, \psi(\tau))\| d\tau\right| + \left|\int_\theta^t \|X(\tau, \psi(\tau)) - Y(\tau, \psi(\tau))\| d\tau\right|.$$

$$\le L\left|\int_\theta^t \|\varphi(\tau) - \psi(\tau)\| d\tau\right| + \left|\int_\theta^t \|X(\tau, \psi(\tau)) - Y(\tau, \psi(\tau))\| d\tau\right| \quad (X \text{ is Lip}).$$

denote. $C = \|\varphi(\theta) - \psi(\theta)\| + \left|\int_\theta^t \|X(\tau, \psi(\tau)) - Y(\tau, \psi(\tau))\| d\tau\right|$.

(3) implies. $\|\varphi(t) - \psi(t)\| \le C + L\left|\int_\theta^t \|\varphi(t) - \psi(t)\| d\tau\right|$

by Gronwall's lemma. $\le C \cdot e^{L|t-\theta|}$

□.

Next, we assume. $X(t, x, \mu) \in Lip_x$, for any $\mu \in F \subset \mathbb{R}^m$

△ X satisfy local. Lip. w.r.t. x for any μ∈F.

Thm 1. Let the solution $x = \varphi(t, \theta_0, \xi_0, \mu_0)$ of the system $\dot{x} = X(t, x, \mu)$ is determined for $t \in [a, b]$, where $\theta_0 \in [a,b]$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ s.t. if

$|\theta - \theta_0| < \delta$, $\theta \in [a, b]$. $\|\xi - \xi_0\| < \delta$, $\|\mu - \mu_0\| < \delta$. (1)

then on the segment $[a, b]$, there exists the solution $x = \psi(t, \theta, \xi, \mu)$. of the $\dot{x} = X(t, x, \mu)$ and $\|\varphi(t, \theta_0, \xi_0, \mu_0) - \psi(t, \theta, \xi, \mu)\| < \varepsilon$. for any $t \in [a, b]$.

Pf: denote $\varphi(t) = \varphi(t, \theta_0, \xi_0, \mu_0)$. $\psi(t) = \psi(t, \theta, \xi, \mu)$

closed and bounded set $\mathcal{T} = \{(t, x) : t \in [a,b], x = \varphi(t)\}$

Let $\varepsilon > 0$. let the set $D = \{(t, x) : t \in [a, b], \|x - \varphi(t)\| \le \varepsilon\}$.

By Weierstrass's thm. $\exists M > 0$. $\forall (t, x) \in D$. $\|X(t, x, \mu_0)\| \le M$.

$X(t, x, \mu_0)$ satisfy. Lip. w.r.t. x. (local $\xrightarrow{\text{compact}}$ global) $\exists L > 0$ s.t. $\|X(t, \bar{x}, \mu_0) - X(t, \bar{\bar{x}}, \mu_0)\| \le L\|\bar{x} - \bar{\bar{x}}\|$. for $\forall (t, \bar{x}), (t, \bar{\bar{x}}) \in D$

Choose $\Delta > 0$. s.t. $(1 + M + b - a)\, e^{L(b-a)} \Delta < \varepsilon$

for such Δ. $\exists \delta_1 > 0$. s.t. $\|X(t,x,\mu_0) - X(t,x,\mu)\| < \Delta$, for any μ s.t. $\|\mu - \mu_0\| < \delta_1$.

Set $\delta = \min(\Delta, \delta_1)$. show that under this δ. the thm. satisfy. by contradiction

a) $\exists\, \theta, \xi, \mu$. satisfy (1). s.t. $\psi(t) = \psi(t,\theta,\xi,\mu)$ defined not for all $t \in [a,b]$.

there is a solution. $x = \psi(t)$. on (α, β). either $\alpha \geq a$ or $\beta \leq b$.

While approaching the endpoint of maximum interval of existence, solution $x = \psi(t)$ leave the compact set D. Therefore. $\exists\, t_0 \in [a,b]$. satisfy $\|\varphi(t_0) - \psi(t_0)\| \geq \varepsilon$.

b) for any θ, ξ, μ satisfy (1). $\psi(t) = \psi(t,\theta,\xi,\mu)$ defined on $[a,b]$. as the solution.

but $\exists\, t^* \in [a,b]$ s.t. $\|\varphi(t^*) - \psi(t^*)\| \geq \varepsilon$.

Estimate. the norm. of difference $\varphi(\theta) - \psi(\theta)$.

$$\varphi(\theta) = \varphi(\theta_0) + \int_{\theta_0}^{\theta} X(\tau, \varphi(\tau), \mu_0)\, d\tau.$$

$$\Leftrightarrow \|\varphi(\theta) - \varphi(\theta_0)\| \leq \left|\int_{\theta_0}^{\theta} M\, d\tau\right| = M|\theta - \theta_0|.$$

Also. $\|\varphi(\theta) - \psi(\theta)\| \leq \|\varphi(\theta) - \varphi(\theta_0)\| + \|\varphi(\theta_0) - \psi(\theta)\| \leq M|\theta - \theta_0| + \|\xi - \xi_0\|$

$$\leq M\delta + \delta \leq (M+1)\Delta < \varepsilon.$$

Thus. $t^* \neq \theta$. w.l.g. assume $t^* > \theta$. there is $T \in (\theta, t^*]$ s.t. $\|\varphi(t) - \psi(t)\| < \varepsilon$. for any $t \in (\theta, T)$. and $\|\varphi(T) - \psi(T)\| = \varepsilon$. (by the continuity).

Assume $[c,d] = [\theta, T]$. $X(t,x) = X(t,x,\mu_0)$ and $Y(t,y) = X(t,x,\mu)$, by the Lemma 1.

$$\|\varphi(T) - \psi(T)\| \leq \left[\|\varphi(\theta) - \psi(\theta)\| + \int_{\theta}^{T} \|X(t,\psi(t),\mu_0) - X(t,\psi(t),\mu)\|\, d\tau\right] e^{L(T-\theta)}.$$

$$\leq \left[\|\varphi(\theta) - \psi(\theta)\| + \int_{\theta}^{T} \Delta\, d\tau\right] e^{L(T-\theta)}$$

$$\leq \left(M|\theta - \theta_0| + \|\xi - \xi_0\| + \Delta|T - \theta|\right) \cdot e^{L(T-\theta)}.$$

$$\leq (1+M)\Delta + \Delta(b-a).\ e^{L(b-a)} < \varepsilon. \text{ Contradicts.}$$

Thm 2. (without parameter).

Let the solution $x = (t, \theta_0, \xi_0)$ of $\dot{x} = X(t,x)$. be defined for $t \in [a,b]$, where $\theta_0 \in [a,b]$.

$\exists\, \delta > 0$. $M > 0$, $L > 0$. s.t. if $|\theta - \theta_0| < \delta$. $\theta \in [a,b]$. $\|\xi - \xi_0\| < \delta$.

then on $[a,b]$ defined the solution $x = \psi(t, \theta, \xi)$

and $\|\varphi(t,\theta_0,\xi_0) - \psi(t,\theta,\xi)\| \leq M|\theta - \theta_0| + \|\xi - \xi_0\|)\, e^{L|t-\theta|}$ for any $t \in [a,b]$

Thm 3. Let the solution $x = \varphi(t, \theta_0, \xi_0)$ of $\dot{x} = X(t, x)$. defined for $t \in [a,b]$, $\theta_0 \in [a,b]$.
Then there exist $\delta > 0$. s.t. if $|\theta - \theta_0| < \delta$, $\theta \in (a,b)$. $\|\xi - \xi_0\| < \delta$.
then the solution $x = \psi(t, \theta, \xi)$. is defined on the segment $[a,b]$ is continuously differentiable w.r.t. θ and ξ.

Chapter 6. Lyapunov Stability.

Consider $\dot{y} = Y(t, y)$ (1) $y \in \mathbb{R}^n$. $Y(t, y) \in C(G)$. $Y(t, y) \in \mathrm{Lip}_y^{loc}\ G \subset \mathbb{R}^{n+1}$.
Let $y = \varphi(t)$ be solution of (1). defined for $t \in [t_0, +\infty)$

反之：先选解，再定δ.

Def 1. Solution $y = \varphi(t)$ is Lyapunov stable if $\forall \varepsilon > 0$. $\exists \delta > 0$. s.t. any solution $y = \psi(t)$ of (1) with initial conditional satisfy $\|\psi(t_0) - \varphi(t_0)\| < \delta$. defined for $t \in [t_0, +\infty)$ satisfy $\|\psi(t) - \varphi(t)\| < \varepsilon$. for any $t \in [t_0, +\infty)$. Otherwise $y = \varphi(t)$ is Lyapunov unstable.

Def 2. Solution $y = \varphi(t)$ of (1). is asymptotically Lyapunov stable if it's Lyapunov stable and $\exists \Delta > 0$. s.t. for any $y = \psi(t)$. with initial condition satisfy:
$\|\varphi(t_0) - \psi(t_0)\| < \Delta$, we have $\|\psi(t) - \varphi(t)\| \xrightarrow[t \to +\infty]{} 0$

Study the stability of the solution:
make the substitution $y = \varphi(t) + x$. Then $\begin{cases} \dot{\varphi}(t) + \dot{x} = Y(t, \varphi(t) + x). \\ \dot{\varphi}(t) = Y(t, \varphi(t)) \end{cases}$
denote $X(t, x) = Y(t, \varphi(t) + x) - Y(t, \varphi(t))$, (1) $\Rightarrow \dot{x} = X(t, x)$ (2)
Solution $y = \varphi(t)$ corr. to $x \equiv 0$ of (2). $\Rightarrow$ only need to study stability of $x \equiv 0$ of (1).

§ Stability of Solutions of Linear System.

Consider l.n-h.s. $\dot{y} = P(t)y + q(t)$ (1) matrix. $P(t)$. vector $q(t)$ is cont. on $[t_0, +\infty)$
l.h.s. $\dot{y} = P(t)y$ (2)

Thm 1. Type of stability of any solution $y = \varphi(t)$ of (1) coincides with the type of stability of solution $y \equiv 0$ of (2).

Pf: $y = \varphi(t) + x$. $Y(t, y) = P(t)y + q(t)$
$X(t, x) = Y(t, \varphi(t) + x) - Y(t, \varphi(t)) = P(t)(\varphi(t) + x) + q(t) - (P(t)\varphi(t) + q(t)) = P(t)x$.
$y = \varphi(t)$ of (1) $\Rightarrow$ $y \equiv 0$ of (2).

Rem: all solutions of (1) have same type of stability. $\Rightarrow$ Linear system (1) is stable / un. / asy. if $y \equiv 0$ of (2) is stable / un. / asy.

Thm 2. (stability criterion for l.h.s.) TFAE:

1. Solution $y \equiv 0$ of (2) is Lyapunov stable
2. Any solution of (2) is bounded on $+\infty$. (each $[t_0, +\infty)$)
3. Any fundamental matrix of (2) is bounded on $+\infty$.
4. There exists a bounded on $+\infty$ fundamental matrix of (2)

Pf: $1 \Rightarrow 2 \overset{obv.}{\Rightarrow} 3 \overset{obv.}{\Rightarrow} 4 \Rightarrow 1$ 因为ε取定，δ取定，不能使所有 $\|\psi(t_0)\| < \delta$ (给定的δ)

"$1 \Rightarrow 2$" $\forall \varepsilon > 0$. $\exists \delta > 0$, for any $\|\psi(t_0)\| < \delta$, $t \in [t_0, +\infty)$ s.t. $\|\psi(t)\| < \varepsilon$

thus $\psi(t)$ is bounded.

consider $y = \psi(t)$ with initial condition $\|\psi(t_0)\| \geq \delta$. fix $\alpha > 0$ s.t. $\|\alpha \psi(t_0)\| < \delta$.

the $\xi(t) = \alpha \psi(t)$ also solution of (2). $\|\xi(t)\| < \varepsilon$. $\Rightarrow \|\psi(t)\| < \frac{\varepsilon}{\alpha}$. bounded.

"$4 \Rightarrow 1$" Let $\Phi(t)$ be bounded fundament matrix of (2). $\exists M > 0$. $\|\Phi(t)\| \leq M$ for any $t \in [t_0, +\infty)$.

fix $\varepsilon > 0$. Solution $y = \psi(t)$ of (2) $\psi(t) = \Phi(t)\Phi^{-1}(t_0)\psi(t_0)$ $C = \|\Phi^{-1}(t_0)\| > 0$.

If $\|\psi(t_0)\| < \delta$. then $\|\psi(t)\| < MC\delta$. set $\delta = \frac{\varepsilon}{MC}$. $y \equiv 0$ is Lyapunov stable.

Thm 4. (asymptotic stability criterion for l.h.s) TFAE:

1. Solution $y \equiv 0$ of (2) is asymptotic stable.
2. $\|\psi(t)\| \xrightarrow[t \to +\infty]{} 0$ for any solution $y = \psi(t)$ of (2)
3. $\|\Phi(t)\| \xrightarrow[t \to +\infty]{} 0$ for any fundamental matrix $\Phi(t)$ of (2).
4. There is fundamental matrix $\Phi(t)$ of (2). s.t. $\|\Phi(t)\| \xrightarrow[t \to +\infty]{} 0$

Pf: "$1 \Rightarrow 2$" $\exists \Delta$. for any solution with initial condition $\|\psi(t_0)\| < \Delta$. $\|\psi(t)\| \to 0$.

for any initial condition $\|\psi(t_0)\| \geq \Delta$, fix α...

"$4 \Rightarrow 1$" $\|\Phi(t)\| \to 0$. $\Phi(t)$ is bounded, by thm 2. $y \equiv 0$ is stable.

similar $\psi(t) = \Phi(t)\Phi^{-1}(t_0)\psi(t_0)$ as pf. of 4.

§. Stability of solutions of a l.h.s. with constant coefficient.

Consider l.h.s. $\dot{y} = Ay$ where A is constant square matrix of $n\times n$. $t\in\mathbb{R}$. $y\in\mathbb{R}^n$. (1)

Study the stability of solution $y\equiv 0$, $t\in[0,+\infty)$

$\Phi(t) = e^{At}$ is fundamental matrix of (1)

Let J be Jordan canonical form of A. $\det S\neq 0$, $A = SJS^{-1}$

Lemma 1.

1. Matrix e^{At} is bounded for $t\in[0,+\infty)$ iff the matrix e^{Jt} is bounded for $t\in[0,+\infty)$
2. $\|e^{At}\| \xrightarrow[t\to+\infty]{} 0$ iff $\|e^{Jt}\| \xrightarrow[t\to\infty]{} 0$

Pf: $e^{At} = S e^{Jt} S^{-1} \Rightarrow \|e^{At}\| \leq \|S\|\cdot\|e^{Jt}\|\cdot\|S^{-1}\|$

$e^{Jt} = S^{-1} e^{At} S$

Lemma 2. If $Re\,\lambda_j < 0$. for all $j = 1, 2, \dots n$. then $\|e^{At}\| \xrightarrow[t\to+\infty]{} 0$

Pf: each element of e^{Jt} either $= 0$ or $= \frac{t^k}{k!} e^{\lambda_j t}$

$\left|\frac{t^k}{k!} e^{\lambda_j t}\right| = \frac{t^k}{k!} e^{Re\,\lambda_j t} \xrightarrow[t\to+\infty]{} 0$ each element $\to 0$. $\|e^{Jt}\| \to 0$

Thm 1. (estimate the norm of e^{At}).

Let $\lambda_1, \lambda_2, \dots \lambda_n$ be eigenvalues of the matrix A. Then for any $\delta > \max\limits_{j=1,2,\dots n} (Re\,\lambda_j)$ there exists a constant $K \geq 1$. s.t. $\|e^{At}\| \leq K e^{\delta t}$

Pf: fix δ. $\mu_j = \lambda_j - \delta$. are eigenvalues of $B = A - \delta E$. and $Re\,\mu_j < 0$ for any j.

By lemma 2. $\|e^{Bt}\| \to 0$. $e^{At} = e^{Bt + \delta E t} = e^{Bt} e^{\delta E t} = e^{Bt} E e^{\delta t} = e^{Bt} e^{\delta t}$

denote $K := \sup\limits_{t\in[0,+\infty)} \|e^{Bt}\|$ $(K\geq 1)$. $\|e^{At}\| \leq K e^{\delta t}$

Thm 2. Let $\lambda_1, \lambda_2, \dots \lambda_n$ be eigenvalue. of matrix A. (non-critical case. $\lambda_j \neq 0$).

1. If $Re\,\lambda_j < 0$ for all j. then solution $y\equiv 0$ of (1) is asymptotically stable.
2. If $\exists\,\lambda_j$, $j\in\{1,2,\dots,n\}$, s.t. $Re\,\lambda_j > 0$. then solution $y\equiv 0$ of (1) is unstable.

Pf: 1. lemma 2 $\|e^{At}\| \to 0 \overset{\text{Thm 4.}}{\Longrightarrow} y\equiv 0$ is asymptotically stable.

2. sufficient to check e^{Jt} is unbounded

each element : 0 or $\frac{t^k}{k!} e^{\lambda_j t}$ $\exists\, Re\,\lambda_j > 0$. $|e^{\lambda_j t}| = e^{Re\,\lambda_j t} \xrightarrow[t\to+\infty]{} +\infty$

Thm 3. Let $\lambda_1 \dots \lambda_n$ be eigenvalue of A. $\mathrm{Re}\,\lambda_j \le 0$ for any j, $\exists \lambda_j$, $\mathrm{Re}\,\lambda_j < 0$.

1. If each eigenvalue λ_j s.t. $\mathrm{Re}\,\lambda_j = 0$. corr. to 1-dim Jordan block, then the solution $y \equiv 0$ of (1). is stable, but not asymptotically.
2. If $\exists \lambda_j$ s.t. $\mathrm{Re}\,\lambda_j = 0$. this $\lambda_j = 0$ corr. to Jordan block of dim > 1. solution $y \equiv 0$ of (1) is unstable.

Pf: 1. Each element of e^{Jt}. - 0 or $\frac{t^k}{k!} e^{\lambda_j t}$

if $\mathrm{Re}\,\lambda_j = 0$, λ_j corr. 1-dim Jordan block. $|e^{\lambda_j t}| = e^0 = 1$.

thus, it's stable (bounded). not asymptotically.

2. $\exists \lambda_j = 0$. matrix e^{Jt} has element $te^{\lambda_j t}$, $|t^{\lambda_j t}| = t \to +\infty$. unbounded.

(corr. to Jordan block dim > 1).

§. Stability at the first approximation

Consider system $\dot{y} = Y(t,y)$ (1) where $y \in \mathbb{R}^n$. $Y(t,y)$ cont. on t. cont. diff. w.r.t. y in G.

Let $y = \varphi(t)$ be solution of (1), defined for $t \in [t_0, +\infty)$

Assume $\exists \rho_0 > 0$. s.t. $\{(t,y) : t \in [t_0, +\infty), \|y - \varphi(t)\| < \rho_0\} \subset G$.

Substitute $y = \varphi(t) + x$. $\Rightarrow \dot{x} = Y(t, \varphi(t) + x) - Y(t, \varphi(t))$

$\Rightarrow \dot{x} = P(t)x + g(t,x)$ (2) $P(t) = \dfrac{\partial Y(t, \varphi(t))}{\partial y}$, $g(t,0) = 0$. $\dfrac{\|g(t,x)\|}{\|x\|} \xrightarrow[\|x\| \to 0]{} 0$

Def. system $\dot{x} = P(t)x$ is first approximation system for (2)

Let $\Phi(t,\tau)$ be fundamental Cauchy matrix of system $\dot{x} = P(t)x$

$\to$ 初值.

Thm 1. (on the stability of zero solution of first approximation system).

Let there exist constants $K \ge 1$. $\sigma > 0$. $c \in (0, \frac{\sigma}{K})$. s.t. $\|\Phi(t,\tau)\| \le Ke^{-\sigma(t-\tau)}$ for any $t, \tau \in \mathbb{R}$. $t_0 \le \tau < t < +\infty$ and $\|g(t,x)\| < c\|x\|$ for any $t \ge t_0$ $\|x\| < \rho_0$

Then solution $x \equiv 0$ of (2) is asymptotically stable.

Pf. Firstly show all solutions of system (2) with initial condition closed to zero are defined for $t \ge t_0$.

Prove by contradiction: Let $x = \xi(t)$ be solution of (2). with i.c. $(t_0, \xi(t_0))$, $\|\xi(t_0)\| < \rho < \rho_0$.

Assume the solution is defined for $t \in [t_0, \beta)$. $\beta < +\infty$, $[t_0, \beta)$ is the maximal interval of existence.

$\dot{\xi}(t) = P(t)\xi(t) + g(t, \xi(t)) \xrightarrow[g(t,\xi(t)) = q(t)]{\text{denote}}$ $\xi(t)$ is solution of $\dot{x} = P(t)x + q(t)$.

$\xi(t) = \Phi(t, t_0)\xi(t_0) + \int_{t_0}^{t} \Phi(t,\tau) q(\tau)\, d\tau$.

$\|\xi(t)\| \le Ke^{-\sigma(t-t_0)} \cdot \|\xi(t_0)\| + \left| \int_{t_0}^{t} Ke^{-\sigma(t-\tau)} c \|\xi(\tau)\|\, d\tau \right|$

By Gronwalls Lemma. $\|\xi(t)\| \cdot e^{\sigma(t-t_0)} \le K \cdot \|\xi(t_0)\| e^{cK(t-t_0)}$.

By the assumption, $c \in (0, \frac{\sigma}{K})$, $cK - \sigma < 0 \Rightarrow e^{(cK-\sigma)(t-t_0)} < 1 \Rightarrow \|\xi(t)\| \le K\|\xi(t_0)\|$ for all $t \in [t_0, \beta)$.

set $\Delta = \frac{\rho}{2K}$, and $\|\xi(t_0)\| < \Delta < \rho$. $\Rightarrow$ $\|\xi(t)\| < \frac{\rho}{2}$. (solution is bounded).

for any $t \in [t_0, \beta)$. and if $\beta < +\infty$, then the graph of $x = \xi(t)$ contained in the compact set. $D = \{(t, x) ; t \in [t_0, \beta], \|x\| < \frac{\rho}{2}\}$. $\Rightarrow$ contradicts with exit of maximal extended solution.

There fore, $\beta = +\infty$.

now shows asymptotic stability of $x \equiv 0$.

fix $\varepsilon > 0$. set $\delta = \min\left(\frac{\varepsilon}{2K}, \Delta\right)$. $\|\xi(t)\| < K\delta < \varepsilon$. (if $\|\xi(t_0)\| < \delta$).

$\|\xi(t)\| \xrightarrow[t \to +\infty]{} 0$.

Now consider $\dot{x} = Ax + g(t, x)$. A. constant square $n \times n$. $g(t, x) \in C(G)$. (1).

$\{(t, x) : t \in [t_0, +\infty), \|x\| < \rho_0\} \subset G$. for some $\rho_0 > 0$, and $g(t, 0) = 0$.

it follows that (1) has solution $x \equiv 0$. ($t \in [t_0, +\infty)$)

We assume $\frac{\|g(t,x)\|}{\|x\|} \xrightarrow[\|x\| \to 0]{} 0$ uniformly on $t \in [t_0, +\infty)$

The first approximation system of (1): $\dot{x} = Ax$. (2).

Thm 2.

Let $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ be eigenvalues of matrix A, $Re\,\lambda_j < 0$. for any $j = 1, 2, \dots n$. and $\frac{\|g(t,x)\|}{\|x\|} \xrightarrow[\|x\| \to 0]{} 0$ uniformly on $t \in [t_0, +\infty)$. Then the solution $x \equiv 0$ of (1) is asymptotically stable.

Pf: Show for system (1), condition of Thm 1. satisfied.

$\Phi(t) = e^{At}$ is fundamental matrix of (1)

$\Phi(t, \tau) = \Phi(t)\Phi^{-1}(\tau) = e^{At}e^{-A\tau} = e^{A(t-\tau)}$

choose δ. s.t. $\max(Re\,\lambda_j) < \delta < 0$. let $\sigma = -\delta$ $\|e^{A(t-\tau)}\| \le Ke^{-\sigma(t-\tau)}$

consider $\forall c \in (0, \frac{\sigma}{K})$. $\frac{\|g(t,x)\|}{\|x\|} \to 0$. $\exists \rho < \rho_0$, for $\|x\| < \rho$. it follows. $\frac{\|g(t,x)\|}{\|x\|} < c$.

for system (1) all condition of Thm 1. is satisfied 有界性.

Thm 3. (unstability).

Let among the eigenvalue $\lambda_1, \lambda_2 \dots \lambda_n$ of matrix A. there exist λ_j. s.t. $Re\,\lambda_j > 0$.

and $\frac{\|g(t,x)\|}{\|x\|} \xrightarrow[\|x\| \to 0]{} 0$ uniformly on t. Then the solution $x \equiv 0$ of (1) is unstable.

Rem: If $Re\,\lambda_j \le 0$ for any $j = 1, 2, \dots n$. $\exists$ ~~Re~~ $\lambda_i = 0$. the stability of solution $x \equiv 0$ (1) depends on function $g(t, x)$. [Thm. for l.h.s. not holds.

Tech: 1) if the n.h.s. has non-zero equilibrium solution. use. affine (linear) change $x' = x - x_0$.

2) use Taylor extension. $\dot{x} = Ax + g(t, x)$. let $g(t, x) = o(\rho)$.

3) calculate eigenvalues.

Chapter 7. Autonomous Systems. (自治系统)

Let's consider $\dot{x} = F(x)$. (RHS doesn't on t). (1)

$x \in M \subset \mathbb{R}^n, t \in \mathbb{R}$. $F(X) \in C(M)$, $F(X) \in Lip_x^{loc}(M)$ → 注意 F 比 $X(t,x)$ 的定义域少一维.

Def. A system of differential equations is autonomous if RHS of system doesn't explicitly depend on t. Domain M is phase space (相空间) of autonomous system (1)

Def. Projection of the integral curve $x = \varphi(t)$ to the phase space M. is trajectory (轨道) of the solution.

integral curve: solution graph $\Gamma_{\varphi(t)} = \{(t,x) : x = \varphi(t), t \in \langle a,b \rangle\}$.

denote: $L\varphi(t) = \{\varphi(t) : t \in \langle a,b \rangle\}$

Thm 1. Characteristic Property

Let $x = \varphi(t)$ be solution of (1). and (a,b) is maximal interval of existence.

Then $\forall c$ function $x = \varphi(t+c)$ is the solution of (1). $(a-c, b-c)$ is m.i.o.e.

Pf: fix $c > 0$. $\xi(t) = \varphi(t+c)$

$\dot{\xi}(t) = \dot{\varphi}(t+c) = F(\varphi(t+c)) = F(\xi(t))$ define $t+c \in (a,b)$.

if $(a-c, b-c)$ not maximal. for $x = \varphi(t) = \xi(t+c)$. can be continued right/left beyond b/a.

Thm 2. Trajactories corr. to different solutions neither intersect nor coincide.

Pf: let $x = \varphi(t)$. $t \in (a_1, b_1)$. $x = \psi(t)$, $t \in (a_2, b_2)$. are solution.

Assume $\exists t_1 \in (a_1, b_1)$. $t_2 \in (a_2, b_2)$. $\varphi(t_1) = \psi(t_2)$

$\varphi(t_1) = \psi(t_1 + t_2 - t_1) = \psi(t_1 + c)$. the solution solve the same Cauchy Problem $(t_1, \varphi_1(t))$

By uniqueness. $(a_1, b_1) = (a_2 - c, b_2 - c)$. $\varphi(t) \equiv \psi(t+c)$. on (a_1, b_1). $L\varphi(t) = L\varphi(t+c) = L\psi(t)$

Types of Trajectory.

1. equilibrium $\varphi(t) \equiv x_0$.
2. cycle $x = \varphi(t)$ periodical
3. general trajectory

Let $\varphi(t) \equiv x_0$, $x_0 \in M$. be solution of (1). the trajectory: $L\varphi(t) = \{x_0\}$.

Def. Trajectory $L\varphi(t) = \{x_0\}$ is equilibrium / singular point of system (1)

Lemma 1. The point x_0 is equilibrium of (1) iff $F(x_0) = 0$.

Pf: "$\Rightarrow$". $\varphi(t) \equiv x_0 \Rightarrow \dot{\varphi}(t) \equiv 0 \Rightarrow F(\varphi(t)) = F(x_0) = 0$.

"$\Leftarrow$" $F(x_0) = 0 \Rightarrow \varphi(t) \equiv x_0$ is solution of (1). $\Rightarrow x_0$ is equilibrium.

Lemma 2. (Condition for periodicity of solution)

Let $x = \varphi(t)$ be solution of (1). on m.i.o.e. (a,b). $\varphi(t) \not\equiv$ const.

there exist $t_1; t_2 \in (a,b)$. $t_1 \neq t_2$, $\varphi(t_1) = \varphi(t_2)$

Then $(a,b) = \mathbb{R}$. and $\varphi(t + (t_2 - t_1)) = \varphi(t)$ for any $t \in \mathbb{R}$. Hence $x = \varphi(t)$ is periodic solution.

Def. If $x = \varphi(t)$ is periodic solution. $\varphi(t) \not\equiv$ const. then trajectory $L\varphi(t)$ is a closed curve – cycle.

Pf: set $c = t_2 - t_1$. $\xi(t) = \varphi(t)$ is solution on $(a-c, b-c)$. and $\xi(t_1) = \varphi(t_1 + c) = \varphi(t_2) = \varphi(t_1)$

ξ, φ solve same C.P. $(t_1, \varphi(t_1))$; $\varphi(t) \equiv \xi(t)$ for any $t \in (a,b)$.

and $(a,b) = (a-c, b-c)$ (maximal interval coincide). $c \neq 0$. $(a,b) = \mathbb{R}$.

and $\varphi(t) = \varphi(t+c)$

Thm 3. (Structure of a general trajectory).

Let $L_{\varphi(t)}$ be general trajectory corr. to the solution $x=\varphi(t)$, $t\in(a,b)$.

Then the map $\varphi:(a,b)\mapsto L_{\varphi(t)}$ is <u>regular</u> (diffeomorphism: 1. one-to-one 2. $f, f^{-1}\in C'$

(general trajectory: $\varphi(t)\not\equiv$ const and non-periodical, define for $t\in\mathbb{R}$. or $(a,b)\neq\mathbb{R}$).

Pf: Map $\varphi:(a,b)\to L_{\varphi(t)}\in C'(a,b)$.; one-to-one.

Check $\dot\varphi(t)\neq 0$ for any $t\in(a,b)$. Assume $\exists\, t_0\in(a,b)$. s.t. $\dot\varphi(t_0)=0$.

$x_0=\varphi(t_0)$ is equilibrium. contradicts with thm 2.

Hence. $\dot\varphi(t)\neq 0$. for any $t\in(a,b)$. thus. φ is one to one.

Thm 4. (Structure of a cycle).

Let $x=\varphi(t)$ be periodical solution of (1). $\varphi(t)\not\equiv$ const.

Then there is the smallest positive period of the function $\varphi(t)$ and the cycle $L_{\varphi(t)}$ is regular image of a circle.

Pf: let $T=\{J\in(0,+\infty):\varphi(t+J)=\varphi(t),\ t\in\mathbb{R}\}$

since $T\neq\phi$. then there exists $\omega=\inf T$ and $\omega\geqslant 0$.

1) prove that $\omega\neq 0$: if $\omega=0$, $\{J_k\}_{k=1}^{+\infty}\subset T$. $J_k\xrightarrow[k\to+\infty]{}0$

fix $t\in\mathbb{R}$. $t=m_kJ_k+r_k$. $m_k\in\mathbb{Z}$. $r_k\in[0,J_k)$. $r_k\xrightarrow[k\to+\infty]{}0$

By def of set T, $\varphi(t)=\varphi(m_kJ_k+r_k)=\varphi(r_k)$. $\xrightarrow[\text{pass the limit}]{}$ $\varphi(t)=\varphi(0)\Rightarrow\varphi(t)\equiv\varphi(0)$.

2) prove that $\omega\in T$. if $\omega\notin T$. $\exists\{J_k\}_{k=1}^{+\infty}\subset T$. $J_k\xrightarrow[k\to+\infty]{}\omega$

fix $t\in\mathbb{R}$. $\varphi(t+J_k)=\varphi(t)$ for any $J_k\xrightarrow[\text{pass the limit}]{}$ $\varphi(t+\omega)=\varphi(t)$. $\Rightarrow\omega\in T$. there is a smellest period.

3). Let S' be unit circle: $S'=\{(\cos\theta,\sin\theta):\theta\in\mathbb{R}\}$.

Points <u>on</u> a circle are equivalent classes $[(\cos\theta,\sin\theta)]=\{(\cos\theta_1,\sin\theta_1):\theta_1\equiv\theta\pmod{2\pi}\}$.

$L_{\varphi(t)}$ is cycle corr. to a periodic solution $x=\varphi(t)$ with smallest positive period ω.

Points on trajectory (cycle) $L_{\varphi(t)}$ are equivalent classes $[\varphi(t)]=\{\varphi(t_1):t_1\equiv t\pmod{\omega}\}$.

Denote map $h:[(\cos\theta,\sin\theta)]\longrightarrow[\varphi(\theta\cdot\frac{\omega}{2\pi})]$.

Check the correctness: if $\theta_1\equiv\theta \bmod 2\pi$. $t=\theta\cdot\frac{\omega}{2\pi}$. $t_1=\theta_1\frac{\omega}{2\pi}$. $\Rightarrow t_1\equiv t\pmod{\omega}$.

$h\in C'$ since $\varphi(t)\in C'$ (需要拓扑/微分几何证明)

Check h is one-to-one: if $\varphi(\theta_1\frac{2\omega}{2\pi})=\varphi(\theta_2\frac{\omega}{2\pi})$. then $\theta_1\frac{\omega}{2\pi}\equiv\theta_2\frac{\omega}{2\pi}\pmod{\omega}$. Hence $\theta_1\equiv\theta_2\pmod{2\pi}$

Let $\exists\, t_0\in(a,b)$. s.t. $\dot\varphi(t_0)=0$. Then $x_0=\varphi(t_0)$ is equilibrum. and $x_0\in L_{\varphi(t)}$. contradicts with thm 2.

Therefore. h is regular

§. Poincaré classification of singular points of 2nd-order l.h.s.

Consider l.h. autonomous system of <u>second order</u> $\dot{y} = Ay$. (1).

where $y = (y_1, y_2)^T \in \mathbb{R}^2$, A is constant $[2\times2]$ matrix. $\det A \neq 0$.

Let λ_1, λ_2 be eigenvalues of A. J is Jordan form of A. $A = S^{-1}JS$. $\det S \neq 0$.

$(1) \xRightarrow{x=Sy} \dot{x} = Jx$. (2). — has unique equilibrium $x=0$.

Study the trajectory:

1. $\lambda_1, \lambda_2 \in \mathbb{R}$. $\lambda_1 \neq \lambda_2$. $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. and system (2). $\rightarrow \begin{cases} \dot{x}_1 = \lambda_1 x_1 \\ \dot{x}_2 = \lambda_2 x_2 \end{cases}$

solution: $\begin{cases} x_1 = c_1 e^{\lambda_1 t} \\ x_2 = c_2 e^{\lambda_2 t} \end{cases}$, $t \in \mathbb{R}$

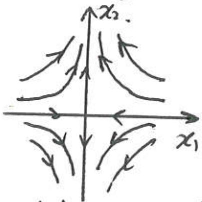
① $c_1 c_2 = 0$. trajectory. L_0. $L_{1,0}$, $L_{-1,0}$, $L_{0,1}$, $L_{0,-1}$.

② $c_1 c_2 \neq 0$. $L\varphi(t) = \left\{ (x_1, x_2) : x_2 = c_2 \left(\frac{x_1}{c_1}\right)^{\frac{\lambda_2}{\lambda_1}} \right\}$

a) $\lambda_1 \lambda_2 < 0$.

w.l.g. $\lambda_1 < 0$, $\lambda_2 > 0$.

箭头是t增加方向.



hyperbolas lying depending on sign of c_1, c_2.

(此时 eigenvector $(1,0)^T$ $(0,1)^T$)

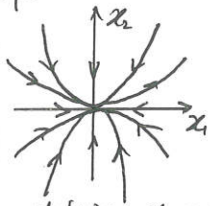
解题时要先计算 eigenvector.

(可理解为建立新坐标轴)

equilibrium $x=0$ is saddle. $\lambda_1 < 0$, $\lambda_2 > 0$

b) $\lambda_1 \lambda_2 > 0$.

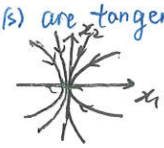
w.l.g. $\lambda_1 < 0$, $\lambda_2 < 0$. $|\lambda_1| < |\lambda_2|$



parabolas.

$|\lambda_1| < |\lambda_2|$ trajectory(s) are tangent to horizontal axis. 与水平轴相切.

(if. $|\lambda_2| < |\lambda_1|$



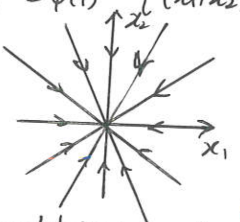
equilibrium $x=0$ is node

2. λ_1, λ_2 be real. $\lambda_1 = \lambda_2 = \lambda$.

a) $J = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$. system (2) $\rightarrow \begin{cases} \dot{x}_1 = \lambda x_1 \\ \dot{x}_2 = \lambda x_2 \end{cases} \Rightarrow$ solution $\begin{cases} x_1 = c_1 e^{\lambda t} \\ x_2 = c_2 e^{\lambda t} \end{cases}$

trajectory $L\varphi(t) = \left\{ (x_1, x_2) : x_2 = \left(\frac{c_2}{c_1}\right) x_1 \right\}$ $(c_1, c_2 \neq 0)$.

w.l.g. $\lambda < 0$



rays. (depend on signs of c_1, c_2.

仅存形如 $\begin{cases} \dot{x} = ax \\ \dot{y} = ay \end{cases}$

临界结点

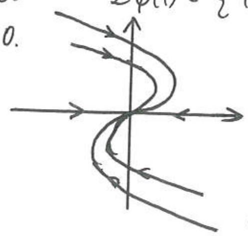
equilibrium $x=0$ is dicritical node $\lambda < 0$.

b) $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. system (2) $\rightarrow \begin{cases} \dot{x}_1 = \lambda x_1 + x_2 \\ \dot{x}_2 = \lambda x_2 \end{cases} \Rightarrow$ solution $\begin{cases} x_1 = (c_1 + c_2 t) e^{\lambda t} \\ x_2 = c_2 e^{\lambda t} \end{cases}$

①. $c_2 = 0$. $L_0 = \{0\}$. $L_{1,0} = \{(x_1, 0) : x_1 > 0\}$. $L_{-1,0} = \{(x_1, 0) : x_1 < 0\}$.

② $c_2 \neq 0$. $L\varphi(t) = \left\{ (x_1, x_2) : x_1 = \left[c_1 + \frac{c_2}{\lambda} \ln \frac{x_2}{c_2}\right] \frac{x_2}{c_2} \right\}$.

w.l.g $\lambda < 0$.



curves. (lie in upper/lower half-plane).

equilibrium $x=0$ degenerate node.

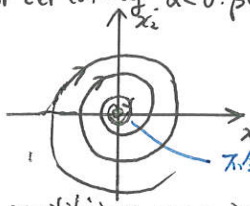
3. Let λ_1, λ_2 be complex. (conjugate). $\lambda_1 = \alpha + \beta i$. $\lambda_2 = \alpha - \beta i$. $\beta \neq 0$.

$J = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$. system: $\begin{cases} \dot{x}_1 = \alpha x_1 - \beta x_2 \\ \dot{x}_2 = \beta x_1 + \alpha x_2 \end{cases} \xrightarrow[\begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}\ \theta \in \mathbb{R}]{\text{polar change}} \begin{cases} \dot{r}\cos\theta - r\dot{\theta}\sin\theta = \alpha r\cos\theta - \beta r\sin\theta. \ ① \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta = \beta r\cos\theta + \alpha r\sin\theta \ ② \end{cases}$

$\begin{matrix} ① \times \cos\theta + ② \times \sin\theta \Rightarrow \\ ① \times \sin\theta + ② \times \cos\theta \Rightarrow \end{matrix} \begin{cases} \dot{r} = \alpha r \\ \dot{\theta} = \beta. \end{cases} \Longrightarrow$ solution $\begin{cases} r = r_0 e^{\alpha t} \\ \theta = \theta_0 + \beta t. \end{cases}$

a) if $\alpha \neq 0$. spiral. (tend to equilibrium positive direction $\alpha > 0$ / negative direction $\alpha < 0$.
if $\alpha > 0$. spiral sink. $\alpha < 0$. spiral soure. ("向外"和"向里"线的区别).

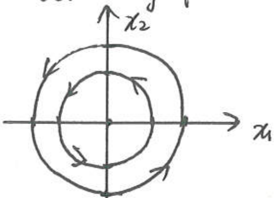
for certainty $\alpha < 0$. $\beta < 0$. $\beta < 0$. clockwise. $\beta > 0$ anti-clockwise



不会从equilibrium 开始. 任意取一个 (1,0). 可计算 $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$ 方向导数. 可知旋转方向.

equilibrium. $x = 0$ is focus.

b) if $\alpha = 0$. circles.
for certainty $\beta > 0$ clockwise ($t \uparrow \theta \uparrow$).



equilibrium $x = 0$ is center.

Remark: The phase portrait of (2) $\xrightarrow[\text{transformation}]{\text{affine coordinate}}$ ~ of (1)

§. Equilibrium of a second order system. (非线性. 可有多个 equilibrium)

Consider autonomous system of second $\dot{x} = Ax + g(x)$. (1)

$x = (x_1, x_2)^T \in \mathbb{R}^2$. A is constant matrix 2×2. $\det A \neq 0$. $g(x) \in C^1(M)$.

M is domain in $\mathbb{R}^2$. $0 \in M$. $g(0) = 0$ and. $\dfrac{\|g(x)\|}{\|x\|} \xrightarrow[\|x\| \to 0]{} 0$ ∴ (g(x)是较高阶的无穷小.

Assume equilibrium $x = 0$ is isolated (in some neighborhood. no other equilibrium). 对A的作用不大)

Consider the linearization of (1). $\dot{x} = Ax$. in the neighborhood of a point $x = 0$

Thm (Poincare) Let λ_1, λ_2 be eigenvalues of matrix A.

If $\operatorname{Re} \lambda_j \neq 0$. $j = 1, 2$. then the equilibrium $x = 0$ of the system (1) has same Poincaré type as the equilibrium $x = 0$ of (2)