10.10.24

2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

We are considering $u(x, y)$ - unknown function of two independent variables.

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + F\left(x,y,u,u_x,u_y\right) = 0 \quad \textbf{(*)}$$

where $a(x,y),\ b(x,y),\ c(x,y)$ – are defined in the domain D.

$b^2 - ac > 0$ - *hyperbolic type*

$b^2 - ac = 0$ - *parabolic type*

$b^2 - ac < 0$ - *elliptical type*

Depending on what type of equation, you can find such a coordinate transformation:

It is possible to move from coordinates (x,y) to coordinates (ξ,η):

$$(x,y) \to (\xi,\eta).$$

You can find such a transformation and make it.

$$\xi = \varphi(x,y)$$

$$\eta = \psi(x,y)$$

Then the equation (*) can be written much more simply (depending on the type of equation).

And depending on the equation, it will be clear how to solve it.

The coefficients $a(x,y),\ b(x,y),\ c(x,y)$ – may depend on $x,\ y$, so if they are constant, then the equation will always have the same type for all $x,\ y$.

If the coefficients are variable, if they depend on $x,\ y$, then it may happen that the equation in one part of the plane has a hyperbolic type, and in another part of the plane has an elliptical type.

And then in each separate part it is necessary to solve it separately.

It will be necessary to find its canonical form separately for each part.

To bring the equation to a canonical form, it is <u>necessary to determine its type.</u>

Usually, the problem asks you to determine the type of equation.

And then we write <u>the characteristic equation:</u>

$$a(x,y)dy^2 - 2b(x,y)dx\,dy + c(x,y)dx^2 = 0$$

This is an ordinary second-order differential equation.

It can be solved as a quadratic equation with respect to dy.

If we solve such a quadratic equation:

$$dy = \frac{b \pm \sqrt{b^2 - ac}}{a}dx$$

Convert, multiply by a :

$$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$$

And then we see that the solution will depend on what type of equation it is.

If the **equation is hyperbolic**, then there is a positive number under the square root, everything is fine, we have two solutions.

There will be two equations:

$$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0 \quad \Rightarrow \begin{cases} \varphi(x,y) = C_1 \\ \psi(x,y) = C_2 \end{cases}$$

The fact is that if we now use these two independent integrals (these are the functions φ and ψ that we found) in order to replace the variable:

$$\xi = \varphi(x,y)$$

$$\eta = \psi(x,y)$$

Then, when substituting these variables into the equation (*), the equation becomes much simpler.

In the case of a **parabolic type** equation, under the square root of 0, therefore, two equations will not work. Let's look at a specific example.

In the case of an **elliptic type,** there will be a negative number under the square root, respectively, there will be complex numbers. Let's show you a specific example.

You can always find the functions φ, ψ, and make a replacement that will bring the original equation to a much simpler form.

And after that, you can write down the *canonical form* of this equation.

To create a new function from ξ and η, we should theoretically replace it ($u(x,y)$) with a some function $U(\xi,\eta)$:

$$u(x,y) \rightarrow U(\xi,\eta) = U(\varphi(x,y),\psi(x,y)) = u(x,y)$$

In theory, we should replace one function $u(x,y)$ with some other function $U(\xi,\eta)$.

Example 1

To bring the following differential equation to a canonical form:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 0$$

<u>Solution:</u>

The type of equation:

$$a = 1, \;\; b = 1, \;\; c = -3$$

$$b^2 - ac = 1 + 3 = 4 \;\; > 0$$

<u>This equation has a hyperbolic type.</u>

Next, we need to bring it to a canonical form.

We write the <u>characteristic equation:</u>

$$dy^2 - 2dydx - 3dx^2 = 0$$

We solve this equation as a quadratic one.

$$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$$

$$dy - \left(1 \pm \sqrt{4}\right)dx = 0$$

$$\begin{cases} dy - 3dx = 0 \\ dy + dx = 0 \end{cases}$$

It is good that we have an equation with constant coefficients: a, b, c do not depend on x, y.

$$\begin{cases} y-3x=C_1 \\ y+x=C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi=y-3x \\ \eta=y+x \end{cases}$$

When moving to these variables, the equation (*) is greatly simplified.

$$u(x,y)\to U(\xi,\eta)$$

The derivative of the whole function:

$$u_x=U_\xi\xi_x+U_\eta\eta_x$$
$$u_y=U_\xi\xi_y+U_\eta\eta_y$$

Take the second derivatives of $u_{x,}$ u_y.

$$u_x=U_\xi\cdot(\text{-}3)+U_\eta\cdot 1$$

$$u_y=U_\xi\cdot 1+U_\eta\cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx}=-3\left(U_{\xi\xi}\cdot\xi_x+U_{\xi\eta}\cdot\eta_x\right)+1\left(U_{\eta\xi}\cdot\xi_x+U_{\eta\eta}\cdot\eta_x\right)=$$

$$=-3\left(U_{\xi\xi}\cdot(-3)+U_{\xi\eta}\cdot 1\right)+1\left(U_{\eta\xi}\cdot(-3)+U_{\eta\eta}\cdot 1\right)=$$

$$=9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}$$

$$u_{xy}=-3\left(U_{\xi\xi}\cdot\xi_y+U_{\xi\eta}\cdot\eta_y\right)+1\left(U_{\eta\xi}\cdot\xi_y+U_{\eta\eta}\cdot\eta_y\right)=$$

$$= -3\left(U_{\xi\xi}\cdot 1 + U_{\xi\eta}\cdot 1\right) + 1\left(U_{\eta\xi}\cdot 1 + U_{\eta\eta}\cdot 1\right) =$$

$$= -3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = 1\left(U_{\xi\xi}\cdot\xi_y + U_{\xi\eta}\cdot\eta_y\right) + 1\left(U_{\eta\xi}\cdot\xi_y + U_{\eta\eta}\cdot\eta_y\right) =$$

$$= 1\cdot U_{\xi\xi} + U_{\xi\eta}\cdot 1 + U_{\eta\xi}\cdot 1 + U_{\eta\eta}\cdot 1 =$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Substituting everything into our equation:

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} + 2\left(-3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - 3\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) +$$

$$+2\left(U_{\xi}\cdot(-3) + U_{\eta}\right) + 6\left(U_{\xi} + U_{\eta}\right) = 0$$

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} - 6U_{\xi\xi} - 4U_{\xi\eta} + 2U_{\eta\eta} - 3U_{\xi\xi} - 6U_{\xi\eta} - 3U_{\eta\eta} -$$

$$-6U_{\xi} + 2U_{\eta} + 6U_{\xi} + 6U_{\eta} = 0$$

$$-16U_{\xi\eta} + 8U_{\eta} = 0$$

$$-U_{\xi\eta} = \frac{1}{2}U_{\eta}$$

This expression is already the canonical form of a hyperbolic equation. This is the answer to this task.

In principle, in general, the **canonical form of a hyperbolic equation** will be as follows:

$$U_{\xi\eta} = \Phi\left(\xi,\eta,U,U_{\xi},U_{\eta}\right).$$

When we moved on to the new variables, there should be a single derivative of this order (mixed) in the left part, with a coefficient equal to one; in the right part - some kind of function Φ, which contains independent variables $\xi,\eta,U,U_{\xi},U_{\eta}$.

Example 2

$$\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial x \partial y} + 13\frac{\partial^2 u}{\partial y^2} = 0$$

Solution:

The type of equation:

$$a = 1,\;\; b = \text{-}3,\;\; c = 13$$

$$b^2 - ac = 9 - 13 = -4 \;\; < 0$$

This equation has an elliptical type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$1 \cdot dy^2 + 6dxdy + 13dx^2 = 0$$

$$dy - \left(-3 \pm \sqrt{-4}\right)dx = 0$$

$$dy-\left(-3\pm 2i\right)dx=0$$

Here is a complex number.

There are two equations here, but they merge into each other during the operation of complex conjugation.

Recall that such a complex conjugation is:

A complex number $z=a+i\,b$

A complex conjugate number $\overline{z}=a-i\,b$

Here, one equation can be obtained from another by complex conjugation. Therefore, it does not make much sense to consider the second equation. We have a real and imaginary part of this equation.

Let's make one equation:

$$dy-\left(-3+2i\right)dx=0$$

We will integrate:

$$y-\left(-3+2i\right)x=A=C_1+iC_2$$

the real part of the equation $y+3x=C_1$

the imaginary part of the equation $-2x=C_2$

$$\begin{cases} y+3x=C_1 \\ -2x=C_2 \end{cases}$$

These are the functions that we can use in this case, as a substitute for variables.

$$\begin{cases} \xi = y+3x \\ \eta = -2x \end{cases}$$

Replace $u(x,y) \to U(\xi,\eta)$.

$$u_x = U_\xi \cdot \xi_x + U_\eta \cdot \eta_x = U_\xi \cdot 3 + U_\eta \cdot (-2)$$

$$u_y = U_\xi \cdot \xi_y + U_\eta \cdot \eta_y = U_\xi \cdot 1 + U_\eta \cdot 0$$

$$u_x = 3U_\xi - 2U_\eta$$

$$u_y = U_\xi$$

$$u_{xx} = 3\left(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x\right) - 2\left(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x\right) =$$

$$= 3\left(U_{\xi\xi} \cdot 3 + U_{\xi\eta} \cdot (-2)\right) - 2\left(U_{\eta\xi} \cdot 3 + U_{\eta\eta} \cdot (-2)\right) =$$

$$= 9U_{\xi\xi} - 12U_{\xi\eta} + 4U_{\eta\eta}$$

$$u_{xy} = 3\left(U_{\xi\xi}\xi_y + U_{\xi\eta}\eta_y\right) - 2\left(U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y\right) =$$

$$= 3\left(U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0\right) - 2\left(U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 0\right) =$$

$$= 3U_{\xi\xi} - 2U_{\eta\xi}$$

$$u_{yy} = U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0 = U_{\xi\xi}$$

We substitute all derivatives:

$$9U_{\xi\xi} - 12U_{\xi\eta} + 4U_{\eta\eta} - 18U_{\xi\xi} + 12U_{\xi\eta} + 13U_{\xi\xi} = 0$$

$$4U_{\xi\xi} + 4U_{\eta\eta} = 0$$

$$U_{\xi\xi} + U_{\eta\eta} = 0$$

This is the canonical form of the elliptical type.

General view (canonical view) elliptic equations:

$$U_{\xi\xi} + U_{\eta\eta} = \Phi\left(\xi, \eta, U, U_{\xi}, U_{\eta}\right)$$

with a coefficient equal to one (on the left side). If you get something else, you should look for an error.

Example 3

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 1\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + cu = 0$$

<u>Solution:</u>

The type of equation:

$$a = 1,\ \ b = \text{-}1,\ \ c = 1$$

$$b^2 - ac = (-1)^2 - 1 = 0$$

<u>This equation has a parabolic type.</u>

Next, we need to bring it to a canonical form.

We write the <u>characteristic equation:</u>

$$dy^2 + 2dxdy + dx^2 = 0$$

$$(dy + dx)^2 = 0$$

$$dy + dx = 0$$

$$y + x = C_1$$

That is, $\xi = y + x$.

Where do we get η from?

In the case of parabolic type equations, the function η can generally be taken any (linearly independent, which is written for ξ):

$$\eta = x.$$

Usually take x, the constant will not be linearly independent.

That is,

$$\begin{cases} \xi = y + x \\ \eta = x \end{cases}$$

If you doubt that these functions are linearly independent (or if ξ is a composite function), then we remember about replacing variables, and turn to topics about integrals if we do some kind of transformation (replacing variables from $(x,y) \to (\xi,\eta)$).

This transformation will be non-degenerate, so the functions will be linearly independent.

The Jacobian of the transition (determinant) will not be equal to 0.

We can make such a determinant, calculate, the transformation is non-degenerate, therefore we can choose the function as $\eta = x$.

$$(x,y) \to (\xi,\eta)$$

The transformation is non-degenerate and the Jacobian of the transition is non-zero.

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

The transformation is non-degenerate.

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi \cdot 1 + U_\eta \cdot 1;$$
$$u_y = U_\xi;$$
$$u_{xx} = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 1 + U_{\eta\xi} \cdot 1 + U_{\eta\eta} =$$
$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$
$$u_{xy} = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0 + U_{\eta\xi} \cdot 1 =$$
$$= U_{\xi\xi} + U_{\eta\xi}$$
$$u_{yy} = U_{\xi\xi}$$

All derivatives have been found.

Let's substitute it into the original equation:

$$U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta} - 2U_{\xi\xi} - 2U_{\eta\xi} + U_{\xi\xi} + \alpha U_\xi + \alpha U_\eta + \beta U_\xi + cU = 0$$

$$U_{\eta\eta} = -(\alpha + \beta)U_\xi - \alpha U_\eta - cU$$

A parabolic type equation has been reduced to a canonical form.

In general, the canonical form of a parabolic equation is:

$$U_{\eta\eta} = \Phi(\xi, \eta, U, U_\xi, U_\eta).$$

Example 4 (EQUATION WITH VARIABLE COEFFICIENTS)

$$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} - 2x \frac{\partial u}{\partial x} = 0$$

<u>Solution:</u>

The type of equation:

$$a = y^2, \;\; b = 0, \; c = -x^2$$

$$b^2 - ac = x^2 y^2 \;\; > 0, \qquad x \neq 0, \, y \neq 0$$

<u>The hyperbolic type is everywhere on the entire plane (except the axes).</u>

Next, we need to bring it to a canonical form.

We write the <u>characteristic equation:</u>

$$y^2 dy^2 - x^2 dx^2 = 0$$

$$(ydy)^2 = (xdx)^2$$

$$ydy = \pm xdx$$

$$2ydy = \pm 2xdx$$

We will integrate:

$$\begin{cases} y^2 = x^2 + C_1 \\ y^2 = -x^2 + C_2 \end{cases}$$

$$\begin{cases} y^2 - x^2 = C_1 \\ y^2 + x^2 = C_2 \end{cases}$$

Substitution:

$$\begin{cases} \xi = y^2 - x^2 \\ \eta = y^2 + x^2 \end{cases}$$

We substitute all this, and we get the canonical form:

$$\xi_x = -2x$$
$$\xi_y = 2y$$
$$\eta_x = 2x$$
$$\eta_y = 2y$$

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi(-2x) + U_\eta(2x) = 2x(-U_\xi + U_\eta)$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y = U_\xi(2y) + U_\eta(2y) = 2y(U_\xi + U_\eta)$$
$$u_{xx} = \left[2x \cdot (-U_\xi + U_\eta)\right]'_x = 2(-U_\xi + U_\eta) + 2x\left[-U_\xi + U_\eta\right]'_x =$$
$$= 2(-U_\xi + U_\eta) + 2x(-U_{\xi\xi}\xi_x - U_{\xi\eta}\eta_x + U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x) =$$
$$= 2(-U_\xi + U_\eta) + 2x(-U_{\xi\xi}(-2x) - U_{\xi\eta}(2x) + U_{\eta\xi}(-2x) + U_{\eta\eta}(2x)) =$$
$$= 2(-U_\xi + U_\eta) + 4x^2(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta})$$

We used the formula: $(u\upsilon)' = u'\upsilon + \upsilon' u$

$\left[-U_\xi + U_\eta\right]'_x$ - and here we take it as a derivative of a complicated function

u_{xy} we don't need it, it's not in the equation.

$$u_{yy} = 2\left(U_{\xi} + U_{\eta}\right) + 2y\left(U_{\xi\xi}\xi_{y} + U_{\xi\eta}\eta_{y} + U_{\eta\xi}\xi_{y} + U_{\eta\eta}\eta_{y}\right) =$$
$$= 2\left(U_{\xi} + U_{\eta}\right) + 2y\left(U_{\xi\xi}\left(2y\right) + U_{\xi\eta}\left(2y\right) + U_{\eta\xi}\left(2y\right) + U_{\eta\eta}\left(2y\right)\right) =$$
$$= 2\left(U_{\xi} + U_{\eta}\right) + 4y^{2}\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right)$$

We substitute it into the equation:

$$2y^{2}\left(-U_{\xi} + U_{\eta}\right) + 4x^{2}y^{2}\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) -$$
$$-2x^{2}\left(U_{\xi} + U_{\eta}\right) - 4x^{2}y^{2}\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) -$$
$$-4x^{2}\left(-U_{\xi} + U_{\eta}\right) = 0$$

$$-16x^{2}y^{2}U_{\xi\eta} + \left(-2y^{2} + 2x^{2}\right)U_{\xi} + \left(2y^{2} - 6x^{2}\right)U_{\eta} = 0$$

Let's express x^{2} and y^{2}:

$$\begin{cases} \xi = y^{2} - x^{2} & (1) \\ \eta = y^{2} + x^{2} & (2) \end{cases}$$

let's add two equations:

$$y^{2} = \frac{\xi + \eta}{2}$$

Subtract from (2) equation (1) equation:

$$x^{2} = \frac{\eta - \xi}{2}$$

$$-16\frac{\eta^2-\xi^2}{4}U_{\xi\eta}+\left(-\xi-\eta+\eta-\xi\right)U_{\xi}+\left(\xi+\eta-3\eta+3\xi\right)U_{\eta}=0$$

$$-4\left(\eta^2-\xi^2\right)U_{\xi\eta}-2\xi U_{\xi}+\left(4\xi-2\eta\right)U_{\eta}=0$$

$$U_{\xi\eta}=\frac{1}{4\left(\eta^2-\xi^2\right)}\left(-2\xi U_{\xi}+\left(4\xi-2\eta\right)U_{\eta}\right)$$