1.3. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF DIFFERENTIAL EQUATIONS AND SYSTEMS

The method of solving various classes of equations and other problems using the Laplace transform is called the *operational method*.

1.3.1. Differential equations and systems with constant coefficients

Consider an n-th order linear differential equation with constant coefficients:

$$L(x) \equiv x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_{n-1} x'(t) + a_n x(t) = f(t). \qquad (1.10)$$

Let's set the Cauchy problem: to find a solution to equation (1.10) satisfying the conditions:

$$x(0) = x_0, \ x'(0) = x_1, \ldots, \ x^{(n-1)}(0) = x_{n-1}, \qquad (1.11)$$

where x_i are the specified constants, $i = 0, 1, \ldots, n-1.$

Assuming that the function $f(t)$ is the original, we will look for the solution $x(t)$ of the problem (1.10)–(1.11) on the set of originals.

Let $X(p) \leftrightarrow x(t), \ F(p) \leftrightarrow f(t)$. According to the rule of differentiation of the original and the property of linearity, passing to images in equation (1.10), due to the conditions (1.11), we obtain an equation for an unknown image $X(p)$, which we will call the *operator equation*

$$A(p)X(p) - B(p) = F(p),$$

where

$$A(p) = p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n,$$

$$B(p) = x_0(p^{n-1} + a_1 p^{n-2} + \ldots + a_{n-1}) +$$

$$+x_1(p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-2}) + \ldots + x_{n-2}(p + a_1) + x_{n-1}.$$

Hence

$$X(p) = \frac{B(p) + F(p)}{A(p)}.$$

To find the required solution $x(t)$ of the problem (1.10)– (1.11), it is necessary to restore the original $x(t)$ from its image $X(p)$.

Similarly, the operational method is applied to solving systems of differential equations with constant coefficients.

Example 1

Solve the Cauchy problem:

a) $x' - x = 1,$

$x(0) = -1,$

b) $x'' + x = 2\cos t,$

$x(0) = 0, \quad x'(0) = -1,$

c) $x'' + 2x = t + \dfrac{t^3}{3},$

$x(0) = x'(0) = 0$

d) $x'' - 3x' + 2x = 2e^{3t},$

$x(0)=1, \ x'(0)=3.$

e) $\quad x'' + x' + 6x = 3(\cos 3t - \sin 3t),$

$x(0)=0, \ x'(0)=3.$

Solution:

a) Let $x(t) \leftrightarrow X(p)$.

Then, according to the original differentiation theorem, we get

$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) + 1.$

Let's apply the Laplace transform to both parts of the equation. Let's write out the resulting operator equation

$$pX(p) + 1 - X(p) = \frac{1}{p}.$$

We get

$$X(p) = -\frac{1}{p}.$$

Thus

$x(t) = -1.$

b) Let's move on from the originals to the images

$x(t) \leftrightarrow X(p),$

$x'(t) \leftrightarrow pX(p) - x(0) = pX(p),$

$$x''(t) \leftrightarrow p^2 X(p) - px(0) - x'(0) = p^2 X(p) + 1,$$

$$\cos t \leftrightarrow \frac{p}{p^2 + 1}.$$

Let's write down the equation for the images

$$p^2 X(p) + 1 + X(p) = \frac{2p}{p^2 + 1}.$$

Let's solve the equation for images

$$X(p) = \frac{2p}{\left(p^2 + 1\right)^2} - \frac{1}{p^2 + 1}.$$

According to the image differentiation theorem, we will find the original of the first term

$$\frac{2p}{\left(p^2 + 1\right)^2} = -\left(\frac{1}{p^2 + 1}\right)' \leftrightarrow t \sin t.$$

Therefore, the solution has the form

$$x(t) = t\sin t - \sin t = (t-1)\sin t.$$

c) Let $x(t) \leftrightarrow X(p)$.

Let's move on to the images in the equation

$$p^2 X(p) - px(0) - x'(0) + 2X(p) = \frac{1}{p^2} + \frac{1}{3}\frac{3!}{p^4}.$$

Since $x(0) = x'(0) = 0$, then

$$\left(p^2+2\right)X = \frac{p^2+2}{p^4}$$

$$X(p) = \frac{1}{p^4}.$$

Having found the original for this image, we get a solution to the Cauchy problem

$$X(p) = \frac{1}{p^4} = \frac{3!}{3!\,p^4} \underset{\leftrightarrow}{} \frac{t^3}{3!} = \frac{t^3}{6} = x(t).$$

d) Let's move on from the originals to the images

$$x(t) \leftrightarrow X(p)$$

$$x'(t) \leftrightarrow pX(p)-1,$$

$$x''(t) \leftrightarrow p^2X(p)-p-3,$$

$$e^{3t} \leftrightarrow \frac{1}{p-3}.$$

Let's write down the equation for the images

$$p^2X(p)-p-3-3pX(p)+3+2X(p) = \frac{2}{p-3}.$$

Let's solve the equation for images

$$(p^2 - 3p + 2)X(p) = \frac{2}{p-3} + p,$$

$$X(p) = \frac{p^2 - 3p + 2}{(p-3)(p^2 - 3p + 2)} = \frac{1}{p-3}.$$

Let's find the original for the function $X(p)$

$$X(p) = \frac{1}{p-3} \leftrightarrow e^{3t} = x(t)$$

e) <u>HOMEWORK</u>

Example 2

Find a solution of the differential equation

$$x'(t) + x(t) = e^{-t},$$

satisfying the condition $x(0) = 1$ (Cauchy problem).

<u>Solution:</u>

Let $x(t) \leftrightarrow X(p)$. Since

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - 1,$$

$$e^{-t} \leftrightarrow \frac{1}{p+1},$$

applying the Laplace transform to a given equation using the linearity property, we obtain an algebraic equation for $X(p)$:

$$pX(p) - 1 + X(p) = \frac{1}{p+1}.$$

From where we find for $X(p)$:

$$X(p) = \frac{1}{(p+1)^2} + \frac{1}{p+1}.$$

Since

$$\frac{1}{p+1} \leftrightarrow e^{-t}, \quad \frac{1}{(p+1)^2} \leftrightarrow te^{-t},$$

we have

$$X(p) \leftrightarrow x(t) = te^{-t} + e^{-t}.$$

Check.

We show that the found function is indeed a solution to the Cauchy problem. We substitute the expression for the function $x(t)$ and its derivative

$$x'(t) = -te^{-t} + e^{-t} - e^{-t} = -te^{-t}$$

into the given equation

$$-te^{-t} + te^{-t} + e^{-t} = e^{-t}.$$

After addition of similar terms in the left part of the equation, we get the correct identity: $e^{-t} \equiv e^{-t}$.

Thus, the constructed function is a solution to the equation.

Let's check if it satisfies the initial condition $x(0) = 1$:

$$x(0) = 0 \cdot e^{-0} + e^{-0} = 1.$$

Therefore, the found function is a solution to the Cauchy problem.

Example 3

Find a solution of the differential equation

$$x''(t) + 3x'(t) = e^t,$$

satisfying the condition $x(0) = 0, \ x'(0) = -1$.

<u>Solution:</u>

We apply the Laplace transform to the equation. Using the property of linearity and considering that

$$x(t) \leftrightarrow X(p),$$

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - 0 = pX(p),$$

$$x''(t) \leftrightarrow p^2 X(p) - px(0) - x'(0) = p^2 X(p) - p \cdot 0 - (-1) = p^2 X(p) + 1,$$

$$e^t \leftrightarrow \frac{1}{p-1},$$

we obtain an algebraic equation for $X(p)$:

$$p^2 X(p) + 1 + 3pX(p) = \frac{1}{p-1}, \ \Leftrightarrow \ (p^2 + 3p)X(p) = \frac{1}{p-1} - 1.$$

We will find a fundamental solution:

$$H(p) = \frac{1}{(p^2 + 3p)} = \frac{1}{3}\left(\frac{1}{p} - \frac{1}{p+3}\right) \leftrightarrow h(t) = \frac{1}{3}\left(1 - e^{-3t}\right).$$

Then, since

$$X(p) = \left(\frac{1}{p-1} - 1\right)H(p) = \frac{1}{p-1}H(p) - H(p),$$

using the convolution image property, we will write the solution of the given equation in the form

$$x(t) = \frac{1}{3}\int_0^t e^{t-\tau}(1 - e^{-3\tau})d\tau - \frac{1}{3}(1 - e^{-3t}).$$

Having calculated the integrals and addition of similar terms, we get the final answer:

$$x(t) = -\frac{2}{3} + \frac{1}{4}e^t + \frac{5}{12}e^{-3t}.$$

Check.

We have

$$x(t) = -\frac{2}{3} + \frac{1}{4}e^t + \frac{5}{12}e^{-3t}, \quad x'(t) = \frac{1}{4}e^t - \frac{5}{4}e^{-3t}, \quad x''(t) = \frac{1}{4}e^t + \frac{15}{4}e^{-3t}.$$

We substitute everything into a given equation

$$\frac{1}{4}e^t + \frac{15}{4}e^{-3t} + 3\left(\frac{1}{4}e^t + \frac{5}{4}e^{-3t}\right) \equiv e^t.$$

As a result, we get the identity $e^t \equiv e^t$. Therefore, the found function is a solution to the equation. Let's check the fulfillment of the initial conditions:

$$x(0) = -\frac{2}{3} + \frac{1}{4}e^0 + \frac{5}{12}e^{-0} = 0; \quad x'(0) = \frac{1}{4}e^0 - \frac{5}{4}e^{-0} = -1.$$

Therefore, the found function is a solution of the Cauchy problem.

Example 4

Find a solution of the differential equation

$$x'''(t) + 2x''(t) + 5x'(t) = 0,$$

satisfying the conditions: $x(0) = -1, \; x'(0) = 2, \; x''(0) = 0$.

Solution:

Let $x(t) \leftrightarrow X(p)$.

Since, take into account the given conditions, we have

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - (-1) = pX(p) + 1,$$

$$x''(t) \leftrightarrow p^2X(p) - px(0) - x'(0) = p^2X(p) - p(-1) - 2 = p^2X(p) + p - 2,$$

$$x'''(t) \leftrightarrow p^3X(p) - p^2x(0) - px'(0) - x''(0) =$$

$$= p^3X(p) - p^2(-1) - p2 - 0 = p^3X(p) + p^2 - 2p,$$

then, after applying the Laplace transform for a given equation, we obtain the following operator equation:

$$p^3X(p) + p^2 - 2p + 2p^2X(p) + 2p - 4 + 5pX(p) + 5 = 0,$$

or after the transformations:

$$X(p)(p^3 + 2p^2 + 5p) = -p^2 - 1.$$

Solving this equation for $X(p)$, we obtain

$$X(p) = \frac{-p^2 - 1}{p(p^2 + 2p + 5)}.$$

The resulting expression is decomposed into the simplest fractions:

$$\frac{-p^2 - 1}{p(p^2 + 2p + 5)} = \frac{A}{p} + \frac{Bp + C}{p^2 + 2p + 5}.$$

Using the method of undefined coefficients, we find A, B, C.

To do this, we bring the fractions to a common denominator and equate the coefficients with the same degrees of p:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{Ap^2+2Ap+5A+Bp^2+Cp}{p(p^2+2p+5)}.$$

We obtain a system of algebraic equations for A,B,C:

$$A+B=-1,\ 2A+C=0,\ 5A=-1,$$

the solution of which will be: $A=-\frac{1}{5},\ B=-\frac{4}{5},C=\frac{2}{5}.$

Then

$$X(p)=-\frac{1}{5p}+\frac{1}{5}\frac{-4p+2}{p^2+2p+5}.$$

To find the original of the second fraction, select the full square in its denominator: $p^2+2p+5=(p+1)^2+4$, then select the summand $p+1$ in the numerator:

$$-4p+2=-4(p+1)+6,$$

and decompose the fraction into the sum of two fractions:

$$\frac{1}{5}\frac{-4p+2}{p^2+2p+5}=-\frac{4}{5}\frac{p+1}{(p+1)^2+4}+\frac{3}{5}\frac{2}{(p+1)^2+4}.$$

Next, using the displacement property and the table of correspondence between images and originals, we obtain a solution to the original equation:

$$x(t)=-\frac{1}{5}-\frac{4}{5}e^{-t}\cos 2t+\frac{3}{5}e^{-t}\sin 2t.$$

Example 5

Find a solution to the system:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 2x_2 + \sin t, \\ \dfrac{dx_2}{dt} = -x_1 + x_2 + 1, \end{cases}$$

satisfying the initial conditions $x_1(0) = 1$, $x_2(0) = 0$.

Solution:

Let's construct a solution using the Laplace transform, first reducing the system to one equivalent second-order equation.

Let's express the unknown function $x_2(t)$ from the first equation of the system

$$x_2 = \frac{1}{2}\left(\frac{dx_1}{dt} - x_1 - \sin t\right),$$

$$\frac{dx_2}{dt} = \frac{1}{2}\left(\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt} - \cos t\right)$$

and substitute it into the second equation

$$\frac{1}{2}\left(\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt} - \cos t\right) = -x_1 + \frac{1}{2}\left(\frac{dx_1}{dt} - x_1 - \sin t\right) + 1.$$

Let's transform the resulting equation by entering the notation $f(t)$ for the right side:

$$\frac{d^2x_1}{dt^2} - 2\frac{dx_1}{dt} + 3x_1 = \cos t - \sin t \equiv f(t). \qquad (*)$$

Let's find the initial conditions

$$x_1(t)\big|_{t=0}=1\,;\ x_1'(t)\big|_{t=0}=(x_1+2x_2+\sin t)\big|_{t=0}=1.\qquad (**)$$

Let's apply the Laplace transform to the equation (*) with initial conditions (**).

Let $X_1(p)\leftrightarrow x_1(t)$, $F(p)\leftrightarrow f(t)$, then

$$p^2X_1(p)-p-1-2pX_1(p)+2+3X_1(p)=F(p),$$

$$X_1(p)(p^2-2p+3)=F(p)+p-1,$$

$$X_1(p)=\frac{F(p)}{p^2-2p+3}+\frac{p-1}{p^2-2p+3}.$$

We will find a fundamental solution:

$$h(t)\leftrightarrow H(p)=\frac{1}{p^2-2p+3}=\frac{1}{(p-1)^2+2}\leftrightarrow \frac{1}{\sqrt{2}}e^t\sin\sqrt{2}t.$$

Let's find the original $x_1(t)$, given that $h'(t)\leftrightarrow pH(p)-h(0)=pH(p)$,

$$x_1(t)=\int_0^t h(t-\tau)f(\tau)d\tau+h'(t)-h(t).$$

The expression for the function $x_2(t)$ can be constructed using the second equation of a given system, substituting the found expression for the function $x_1(t)$ into it:

$$x_2=\frac{1}{2}\left(\frac{dx_1}{dt}-x_1-\sin t\right).$$

As a result

$$x_1(t)=\frac{2}{3}-\frac{1}{2}t+\frac{1}{3}e^t\cos\sqrt{2}t+\frac{7\sqrt{12}}{12}e^t\sin\sqrt{2}t,$$

$$x_2(t)=-\frac{1}{3}-\frac{1}{4}\cos t-\frac{1}{4}\sin t+\frac{7}{12}e^t\cos\sqrt{2}t-\frac{\sqrt{2}}{6}e^t\sin\sqrt{2}t.$$