THEORY OF SECOND-ORDER CURVES

1 Decomposition of curves

If the left side of the equation of an algebraic curve is decomposed into a product

$$F(x,y) = F_1(x,y) \cdot F_2(x,y),$$

where $F_1(x,y)$ and $F_2(x,y)$ are polynomial functions of x and y, the degree of each of which is greater than or equal to 1, then this curve decomposes into curves defined by equations

$$F_1(x,y) = 0, \qquad F_2(x,y) = 0.$$

Theorem. *If the curve*

$$F(x,y) = 0 \tag{1.1}$$

of order $n > 1$ includes the line

$$Ax + By + C = 0, \tag{1.2}$$

i.e. the coordinates of all points of the line $Ax + By + C = 0$ also satisfy the equation $F(x,y) = 0$, then the left-hand side $F(x,y)$ of the equation (1.1) can be represented as

$$F(x,y) = (Ax + By + C)F_1(x,y),$$

where $F_1(x,y)$ is an polynomial function of x and y whose degree is one less than the degree of $F(x,y)$.

Proof. An polynomial function $F(x,y)$ of degree n in two arguments x and y can be represented as

$$F(x,y) = a_0x^n + a_1(y)x^{n-1} + \ldots + a_n(y), \tag{1.3}$$

where a_0 is a number, and $a_1(y), \ldots, a_n(y)$ are polynomial functions of y, powers not greater than $1, 2, \ldots, n$, respectively. Suppose that $A \neq 0$, then the equation (1.2) can be solved with respect to x

$$x = -\frac{B}{A}y - \frac{C}{A}.$$

By condition

$$F\left(-\frac{B}{A}y - \frac{C}{A}, y\right) = 0,$$

therefore, based on Bezout's theorem[1] function (1.2) divided without remainder by

$$x + \frac{B}{A}y + \frac{C}{A}.$$

Quotient of division

$$\varphi(x, y) = a_0 x^{n-1} + \left[a_1(y) - \left(\frac{B}{A}y + \frac{C}{A}\right) a_0\right] x^{n-2} + \dots$$

is an polynomial function whose degree is 1 less than the degree of $F(x, y)$. So,

$$F(x, y) = \left(x + \frac{B}{A}y + \frac{C}{A}\right) \varphi(x, y),$$

or

$$F(x, y) = (Ax + By + c) F_1(x, y),$$

where

$$F_1(x, y) = \frac{\varphi(x, y)}{A}.$$

2 Center of second order curve

Definition. *The center of a second-order curve is the center of symmetry of this curve, i.e., a point C with the following property: if a point M lies on a curve[a], then on the same curve lies the point M', which is symmetric to the point M with respect to C (Fig. 1).*

[a]In this definition, we also mean complex "points" lying on the given curve.

[1]The theorem states that the remainder of the division of the polynomial $P(x)$ by the binomial $(x - a)$ is equal to $P(a)$.

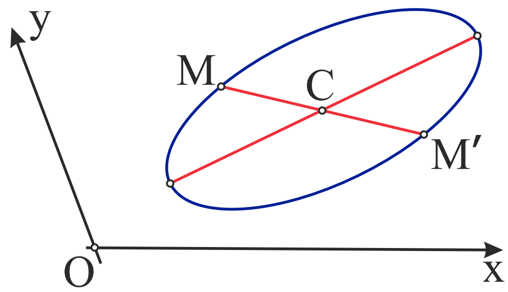

Figure 1

Theorem 1. *Let a second-order curve be given with respect to the general Cartesian coordinate system by the general equation*

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0. \tag{2.1}$$

In order for the origin of coordinates to be its center, it is necessary and sufficient that the equation (2.1) does not contain first-order terms with x and y, i.e., that

$$a_1 = a_2 = 0,$$

otherwise, so that the equation of the curve has the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + a_0 = 0. \tag{2.2}$$

Proof of sufficiency. If $a_1 = a_2 = 0$, then the equation of the curve has the form (2.2), and if the coordinates x and y of point M satisfy it, then the coordinates $-x$, $-y$ of the point M', which is symmetric to M with respect to the origin, also satisfy it.

Proof of necessity. Let the origin be the center of the curve (2.1). Assume contrary to the assertion of the theorem that at least one of the coefficients a_1 or a_2 is different from zero. Take an arbitrary point $M(x, y)$ on the curve (2.1). Its coordinates satisfy the equation (2.1), and since the origin is the center of symmetry of the curve (2.1), the coordinates of the point $M'(-x, -y)$, which is symmetric to the point M with respect to the origin, also satisfy the equation (2.1), i.e.

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 - 2a_1x - 2a_2y + a_0 = 0.$$

From this relation and from the relation (2.1) we find

$$a_1x + a_2y = 0.$$

This equation is satisfied by the coordinates of all points on the line (2.1). Based on the Theorem in Section 1, the function

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0$$

can be represented as a product of two linear functions of x and y, one of which is the form

$$a_1x + a_2y = 0.$$

Thus, this equation splits into two:

$$a_1x + a_2y = 0, \qquad px + qy + r = 0.$$

But since the coordinates of all points of the curve (2.1) satisfy the equation $a_1x + a_2y = 0$, the last two equations are equations of the same straight line, which means that

$$\frac{p}{a_1} = \frac{q}{a_2}, \quad r = 0,$$

and the equation $F(x, y) = 0$ is reduced to the form

$$k(a_1x + a_2y)^2 = 0,$$

i.e., it does not contain first-order terms involving x and y, contrary to the assumption.

Theorem 2. *If, relative to the general Cartesian coordinate system, a second-order curve is given by the general equation*

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0,$$

then the coordinates x_0, y_0 of its center are determined from the system of equations

$$\begin{aligned} a_{11}x + a_{12}y + a_1 &= 0, \\ a_{21}x + a_{22}y + a_2 &= 0, \end{aligned} \tag{2.3}$$

moreover, in the case of inconsistency of this system, the curve has no center (i.e., it is a parabola).

Proof. Let's move the given Cartesian coordinate system so that the point $O'(x_0, y_0)$ becomes the new origin. Denoting the coordinates of an arbitrary point $M(x, y)$ in the new system $O'x'y'$ by x' and y', we have

$$\begin{aligned} x &= x_0 + x', \\ y &= y_0 + y', \end{aligned}$$

and the equation of the function $F(x, y)$ takes the form

$$\begin{gathered} F(x, y) = F'(x', y') = a_{11}(x_0 + x')^2 + 2a_{12}(x_0 + x')(y_0 + y') + \\ a_{22}(y_0 + y')^2 + 2a_1(x_0 + x') + 2a_2(y_0 + y') + a_0 = \\ = a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + 2(a_{11}x_0 + a_{12}y_0 + a_1)x' + 2(a_{21}x_0 + a_{22}y_0 + a_2)y' + a_0' = 0, \end{gathered}$$

where

$$a'_0 = F(x_0, y_0).$$

Based on the previous Theorem, the point $O'(x_0, y_0)$ is the center of the given curve if and only if

$$a_{11}x_0 + a_{12}y_0 + a_1 = 0,$$
$$a_{21}x_0 + a_{22}y_0 + a_2 = 0.$$

If

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0,$$

then the system (2.3) has a unique solution, i.e. the curve (2.1) has a unique center.

If the system (2.3) is indefinite, i.e., has an infinite set of solutions, then the curve (2.1) has an infinite set of centers — the line of centers.

3 Intersection of a second-order curve with a line. Asymptotic directions

Let us assume that, relative to the general Cartesian coordinate system, a second-order curve is given by the general equation

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0 \tag{3.1}$$

We will investigate the intersection of this curve with a line, the equations of which we will take in parametric form:

$$\begin{aligned} x &= x_0 + \alpha t, \\ y &= y_0 + \beta t, \end{aligned} \tag{3.2}$$

Here (x_0, y_0) is some point on the line, and $\{\alpha, \beta\}$ is its direction vector. To find the coordinates of the points of intersection of the line (3.2) with the curve (3.1), we need to find the values of the parameter t, at which the point of the line (3.2) lies on the curve (3.1). Substituting into the equation (3.1) instead of x and y their expressions from the formulas (3.2), we get:

$$\begin{aligned} a_{11}(x_0 + \alpha t)^2 + 2a_{12}(x_0 + \alpha t)(y_0 + \beta t) + a_{22}(y_0 + \beta t)^2 + 2a_1(x_0 + \alpha t) + \\ + 2a_2(y_0 + \beta t) + a_0 = (a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2)t^2 + \\ + 2\left[\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2)\right] t + F(x_0, y_0) = 0. \end{aligned} \tag{3.3}$$

If the coefficient of t^2 in this equation is nonzero, then the equation (3.3) has two roots (real different, imaginary different, or real coinciding), and hence the straight line (3.2) intersects the curve (3.1) at two points (real distinct, complex conjugate, or real coincident, respectively).

If

$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0, \tag{3.4}$$

then the line with direction vector $\{\alpha, \beta\}$

- crosses the second-order curve at only one point (this will be if and only if the coefficient of t^2 in the equation (3.3) is equal to zero, and the coefficient of t is not equal to zero),
- does not intersect it (this will be if and only if the coefficients of t^2 and t in the equation (3.3) are equal to zero, and the constant term is not equal to zero),
- is part of this line (it will be if and only if the relation (3.3) is an identical with respect to t).

We will say that **a line has an asymptotic direction** with respect to a given second-order curve if the coordinates α, β of its direction vector satisfy the equation (3.4). We will also say that the vector $\{\alpha, \beta\}$ has an asymptotic direction.

Q: how to distinct real coincide and (3.4)

With respect to the asymptotic directions, the second-order curves are divided into three groups.

A. Curves of elliptical type; these curves do not have real asymptotic directions (an ellipse, an imaginary ellipse, pair of imaginary intersecting lines).

B. Curves of hyperbolic type; these curves have two real asymptotic directions (hyperbola, pair intersecting lines).

C. Curves of parabolic type; these curves have one asymptotic direction (parabola, pair parallel lines).

Theorem. *The inequality*

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$$

is a necessary and sufficient condition that the second-order curve given by the general equation (3.1) with respect to the general Cartesian coordinate system has no (real) asymptotic directions, i.e., the curve is an elliptical type.
The inequality

$$\delta < 0$$

is a necessary and sufficient condition for this curve to have two different real asymptotic directions, i.e., to be a curve of hyperbolic type.
The equality

$$\delta = 0$$

is a necessary and sufficient condition that this curve has only one asymptotic direction, i.e., is a curve of parabolic type.

Proof. The coordinates of the vector $\{\alpha, \beta\}$, which has an asymptotic direction, are determined from the equation

$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0.$$

Since the vector $\{\alpha, \beta\}$ is non-zero, it makes sense to consider at least one of the relations

$$k = \frac{\beta}{\alpha} \text{ or } k' = \frac{\alpha}{\beta}.$$

The equation

$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0$$

is therefore equivalent to one of the equations (either $a_{11} \neq 0$ or $a_{22} \neq 0$):

$$a_{22}k^2 + 2a_{12}k + a_{11} = 0$$

or

$$a_{11}{k'}^2 + 2a_{12}k' + a_{22} = 0.$$

In order for the solutions of any of these equations to be complex (conjugate), real different, or coinciding, it is necessary and sufficient to fulfill respectively following conditions

$$\delta > 0, \qquad \delta < 0, \qquad \delta = 0.$$

In the first case ($\delta > 0$), the curve has no real asymptotic directions and is an elliptic type; in the second case ($\delta < 0$) the curve has two different real asymptotic directions and is a curve of hyperbolic type; in the third case ($\delta = 0$), the curve has one (real) asymptotic direction and is a curve of parabolic type. In the latter case, the slope $k = \beta/\alpha$ of the only asymptotic direction is determined by one of the relations

$$k = -\frac{a_{12}}{a_{22}} = -\frac{a_{11}}{a_{12}},$$

if $a_{12} = a_{22} = 0$, then the asymptotic direction is the direction of the Oy axis, since the equation that determines the coordinates of vectors with asymptotic directions becomes $a_{11}\alpha^2 = 0$, whence $\alpha = 0$.

It remains to consider the case $a_{11} = a_{22} = 0$.

The curve equation takes the form

$$2a_{12}xy + 2a_1x + 2a_2y + a_0 = 0,$$

and the equation from which the coordinates of vectors with an asymptotic direction are found:

$$2a_{12}\alpha\beta = 0,$$

therefore, either $\alpha = 0$ or $\beta = 0$, i.e. the curve has two different real asymptotic directions — the directions of the coordinate axes (note that $\delta = -a_{12}^2 < 0$, hence the curve belongs to hyperbolic type).

Conversely, if the coordinate axes have asymptotic directions, then the equation

$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0$$

must be satisfied for both $\alpha = 0$ and $\beta = 0$, i.e.

$$a_{11} = a_{22} = 0,$$

so the equation of the curve has the form

$$2a_{12}xy + 2a_1x + 2a_2y + a_0 = 0.$$

For hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

or for pair intersecting lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

coordinates α, β of vectors with asymptotic direction are determined from the equation

$$\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} = 0,$$

i.e., respectively, these are either the directions of the asymptotes of the hyperbola, or the directions of the lines under consideration.

For a parabola

$$y^2 - 2px = 0$$

the equation defining the coordinates of the vector $\{\alpha, \beta\}$ of the asymptotic direction has the form

$$\beta^2 = 0, \text{ i.e. } \beta = 0,$$

hence, the asymptotic direction of the parabola is the direction of its axis. If, finally, the equation of a second-order curve defines pair parallel (or coinciding) lines, then the asymptotic direction is the direction of these lines.

4 Diameter conjugate to non-asymptotic direction

4.1 General theory

Theorem 1. *The locus of the midpoints of the chords of a second-order curve parallel to the non-asymptotic direction vector is a line; this line is called the **diameter** of the given curve conjugate to the parallel chords under consideration. If a second-order curve is given with respect to a Cartesian coordinate system by the general equation*

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0. \tag{4.1}$$

and its chords are parallel to the non-zero vector $\{\alpha, \beta\}$ (non-asymptotic direction), then the equation for the diameter conjugate to these chords has the form

$$(a_{11}x + a_{12}y + a_1)\alpha + (a_{21}x + a_{22}y + a_2)\beta = 0. \tag{4.2}$$

Proof. Let the line p, collinear to the vector $\{\alpha, \beta\}$ of non-asymptotic direction with respect to the curve (4.1), intersects this curve at the points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$. Denote by M_0 the midpoint of the segment M_1M_2 (see Fig. 2).

The equations of this line can be written as

$$\begin{aligned} x &= x_0 + \alpha t, \\ y &= y_0 + \beta t. \end{aligned} \tag{4.3}$$

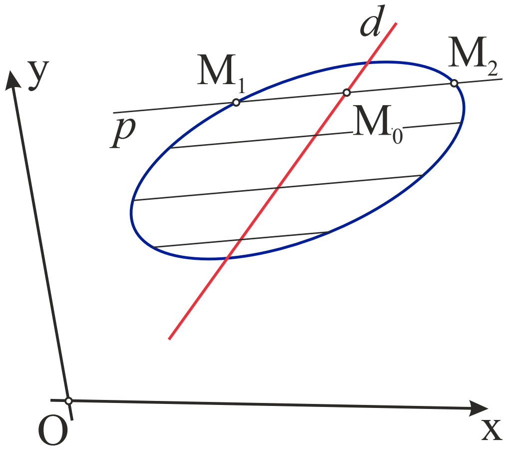

Figure 2

The values of the parameter t corresponding to the coordinates of the points M_1 and M_2 are determined from the equation

$$\begin{aligned}(a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2)t^2 + \\ + 2\left[\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2)\right]t + F(x_0, y_0) = 0,\end{aligned}$$

which we obtain by substituting into the equation (4.1) instead of x and y their expressions from the formulas (4.3).

Let t_1 and t_2 be the roots of this equation. Since t_1 and t_2 are the coordinates of the points M_1 and M_2 on the line p with the origin at the point M_0 and the scale vector $\{\alpha, \beta\}$, the point M_0 is the midpoint of M_1M_2, then the point M_0 on the line p in such coordinate system has the coordinate $t = \dfrac{t_1 + t_2}{2}$. On the other hand, from the relations (4.3) it is clear that $t = 0$ for the point M_0, then $t_1 + t_2 = 0$ and therefore

$$\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2) = 0.$$

This relation is thus a necessary and sufficient condition for a chord having the direction of the vector $\{\alpha, \beta\}$ to be bisected by the point M_0.

On the other hand, each line parallel to the vector $\{\alpha, \beta\}$ intersects the curve (4.1) at two points (actually different, imaginary different, or actually coinciding), and therefore, the locus of the midpoints of the chords of the curve (4.1) parallel to the vector $\{\alpha, \beta\}$ is the entire line whose equation

$$\alpha(a_{11}x + a_{12}y + a_1) + \beta(a_{21}x + a_{22}y + a_2) = 0. \tag{4.4}$$

In this equation, the coefficients of x and y do not vanish simultaneously, since if we had

$$a_{11}\alpha + a_{12}\beta = 0,$$
$$a_{21}\alpha + a_{22}\beta = 0,$$

then the relation

$$\alpha(a_{11}\alpha + a_{12}\beta) + \beta(a_{21}\alpha + a_{22}\beta) = 0$$

or

$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0,$$

i.e. the vector $\{\alpha, \beta\}$ would have an asymptotic direction.

From the equation (4.4) we find the coordinates α' and β' of the direction vector of the diameter conjugate to the chords parallel to the non-zero vector $\{\alpha, \beta\}$:

$$\alpha' = -(a_{12}\alpha + a_{22}\beta), \qquad \beta' = a_{11}\alpha + a_{21}\beta. \tag{4.5}$$

Multiplying the first of these relations by $-\beta'$, the second by α' and adding, we get

$$a_{11}\alpha\alpha' + a_{12}(\alpha\beta' + \alpha'\beta) + a_{22}\beta\beta' = 0. \tag{4.6}$$

This is the *necessary* condition relating the coordinates of the non-zero vector $\{\alpha, \beta\}$ parallel to the chords of the second-order curve given by the general equation (4.1) with respect to the affine coordinate system, and coordinates of a non-zero vector $\{\alpha', \beta'\}$ parallel to the diameter conjugate to these chords.

The condition (4.6) is *sufficient*, since it implies that

$$\alpha' : \beta' = -(a_{21}\alpha + a_{22}\beta) : (a_{11}\alpha + a_{12}\beta),$$

i.e. $\{\alpha', \beta'\}$ is a non-zero vector parallel to the diameter (4.4). The relation (4.6) is fulfilled for the asymptotic direction of the second-order curve if we put $\alpha = \alpha'$ and $\beta = \beta'$ in it (because then we get $a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0$), so the asymptotic direction of a second-order curve is often called *self-conjugate*.

Theorem 2. *If a second-order curve has a single center, and if we consider a family of parallel chords of this curve that do not have an asymptotic direction, then the diameter conjugate to it also does not have an asymptotic direction; if we take a family of chords of a second-order curve parallel to this diameter, then the diameter conjugate to it will be parallel to the chords of the initial family.*

Proof. Let a second-order curve be given by the general equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0.$$

Let us assume that this curve has a single center, i.e.

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

Let take any non-zero vector $\{\alpha, \beta\}$ that does not have an asymptotic direction and consider the diameter equation

$$\alpha(a_{11}x + a_{12}y + a_1) + \beta(a_{21}x + a_{22}y + a_2) = 0,$$

conjugate to chords parallel to the vector $\{\alpha, \beta\}$. The direction vector $\{\alpha', \beta'\}$ of this diameter has the coordinates

$$\alpha' = -(a_{21}\alpha + a_{22}\beta), \qquad \beta' = a_{11}\alpha + a_{12}\beta.$$

The vectors $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ are not collinear because

$$\begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix} = \alpha\beta' - \alpha'\beta = \alpha(a_{11}\alpha + a_{12}\beta) + \beta(a_{21}\alpha + a_{22}\beta) =$$
$$= a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 \neq 0.$$

The vector $\{\alpha', \beta'\}$ is not asymptotic direction. Indeed, since $\delta \neq 0$, the values $a_{11}\alpha' + a_{12}\beta'$ and $a_{21}\alpha' + a_{22}\beta'$ do not turn to zero at the same time. Therefore, the equation

$$(a_{11}\alpha' + a_{12}\beta')\alpha + (a_{21}\alpha' + a_{22}\beta')\beta = 0, \tag{4.7}$$

where α and β are treated as unknown cannot have two linearly independent solutions. But one of its solutions is a pair of coordinates of the vector $\{\alpha, \beta\}$. To the direction of $\{\alpha, \beta\}$ the diameter with the direction vector $\{\alpha', \beta'\}$ is conjugate, and it is non-collinear to the vector $\{\alpha, \beta\}$. This means that the coordinates of the vector $\{\alpha', \beta'\}$ do not satisfy the equation (4.7), i.e.

$$(a_{11}\alpha' + a_{12}\beta')\alpha' + (a_{21}\alpha' + a_{22}\beta')\beta' \neq 0$$

or

$$a_{11}{\alpha'}^2 + 2a_{12}\alpha'\beta' + a_{22}{\beta'}^2 \neq 0,$$

i.e., $\{\alpha', \beta'\}$ is a vector with non-asymptotic direction (4.1).

Now from the relation (4.6)

$$a_{11}\alpha\alpha' + a_{12}(\alpha\beta' + \alpha'\beta) + a_{22}\beta\beta' = 0$$

by virtue of its symmetry with respect to the pairs of numbers α, β and α', β', it follows that if a second-order curve has a single center, then the diameter conjugate to the chords parallel to the vector $\{\alpha', \beta'\}$ is collinear with the vector $\{\alpha, \beta\}$.

Definition. *Two diameters of a curve with a single center, of which each bisects chords parallel to the other, are called* ***conjugate****.*

Condition

$$a_{11}\alpha\alpha' + a_{12}(\alpha\beta' + \alpha'\beta) + a_{22}\beta\beta' = 0$$

can now be interpreted as *necessary and sufficient condition* for the conjugacy of two diameters of a second-order curve having a single center and given by the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0.$$

If $\alpha \neq 0$ and $\alpha' \neq 0$, then this necessary and sufficient conjugacy condition can be written as

$$a_{11} + a_{12}(k + k') + a_{22}kk' = 0,$$

where $k = \beta/\alpha$ and $k' = \alpha/\beta$ are the slopes of conjugate diameters.

Theorem 3. *If a second-order curve has a single center, then any non-asymptotic line passing through its center is the diameter of this line.*

Proof. If the curve given by the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0,$$

has a single center, then the lines

$$a_{11}x + a_{12}y + a_1 = 0,$$
$$a_{21}x + a_{22}y + a_2 = 0$$

intersect at its center ($\delta \neq 0$), but in this case the equation of any line p passing through the center of the curve can be written as

$$\alpha(a_{11}x + a_{12}y + a_1) + \beta(a_{21}x + a_{22}y + a_2) = 0,$$

where at least one of the numbers α or β is not equal to 0.

Hence, the line p is the diameter of the curve conjugate to the chords parallel to the vector $\{\alpha, \beta\}$ if this vector has a non-asymptotic direction. But the non-asymptotic nature of the direction of the line p implies the non-asymptotic nature of its conjugate direction $\{\alpha, \beta\}$, hence the line p is a diameter.

Theorem 4. *If a second-order curve is a curve of parabolic type, then the diameter conjugate to the chords of the curve parallel to the non-asymptotic direction has an asymptotic direction.*

Proof. The vector $\{\alpha', \beta'\}$, collinear to the diameter conjugate to chords with non-asymptotic direction $\{\alpha, \beta\}$, has coordinates

$$\alpha' = -(a_{21}\alpha + a_{22}\beta), \qquad \beta' = a_{11}\alpha + a_{12}\beta$$

(see the proof of Theorem 2). Hence, due to $\delta = 0$, we find

$$a_{11}\alpha' + a_{12}\beta' = -\delta\beta = 0,$$
$$a_{21}\alpha' + a_{22}\beta' = \delta\alpha = 0,$$

hence,

$$\alpha'(a_{11}\alpha' + a_{12}\beta') + \beta'(a_{21}\alpha' + a_{22}\beta') = 0,$$

or

$$a_{11}\alpha'^2 + 2a_{12}\alpha'\beta' + a_{22}\beta'^2 = 0.$$

Note that the vectors $\{-a_{12}, a_{11}\}$ and $\{-a_{22}, a_{21}\}$, at least one of which is non-zero, have an asymptotic direction for parabolic curves.

Theorem 5. *If a curve is a parabola, then the diameter is any line that has an asymptotic direction with respect to this curve.*

Proof. If a curve is a parabola, then by virtue of Theorem 4 all its diameters have an asymptotic direction. Let us prove that the opposite is also true: any line p that has an asymptotic direction of a parabola is its diameter. Take an arbitrary point (x_0, y_0) on the line p and choose a non-zero vector $\{\alpha, \beta\}$ that has non-asymptotic direction and such that its coordinates satisfy the relation

$$\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2) = 0.$$

For this it is enough to put

$$\alpha = -(a_{21}x_0 + a_{22}y_0 + a_2),$$
$$\beta = a_{11}x_0 + a_{12}y_0 + a_1.$$

The vector $\{\alpha, \beta\}$ is non-zero, because for the parabola given by the general equation, the system

$$a_{11}x + a_{12}y + a_1 = 0,$$
$$a_{21}x + a_{22}y + a_2 = 0$$

incompatible. This vector does not have an asymptotic direction, since, assuming the opposite, from the last relations (due to $\delta = 0$) we find

$$a_{11}\alpha + a_{12}\beta = a_{12}a_1 - a_{11}a_2 = 0,$$
$$a_{21}\alpha + a_{22}\beta = a_{22}a_1 - a_{12}a_2 = 0$$

and the system would be compatible.

Hence, for the indicated choice of α and β, the equation

$$\alpha(a_{11}x + a_{12}y + a_1) + \beta(a_{21}x + a_{22}y + a_2) = 0$$

will be the equation of the line p.

Note. If a curve has a line of centers, then each point of this line must belong to each of the curve's diameters. Thus, the line of centers turns out to be the only diameter of the curve; since in the case under consideration the the second-order curve is a pair of parallel (or coinciding) lines, and the line of centers is a line lying in the middle between them, then this last line will be the only diameter of the curve (split into pair parallel or coinciding lines).

4.2 Diameters of second-order curves given by canonical equations

If the ellipse is given by the canonical equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then the equation of the diameter conjugate to chords which are parallel to the non-zero vector $\{\alpha, \beta\}$ has the form

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 0,$$

and the equation of the diameter conjugate to it is

$$\beta x - \alpha y = 0.$$

Figure 3 shows an ellipse, its two conjugate diameters, and chords parallel to each of them.

If the hyperbola is given by the canonical equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and the non-zero vector $\{\alpha, \beta\}$ has no asymptotic direction (i.e., it is not collinear to any of the vectors $\{a, \pm b\}$), then the equation of the diameter conjugate to the chords parallel to this vector, has the form

$$\frac{\alpha x}{a^2} - \frac{\beta y}{b^2} = 0,$$

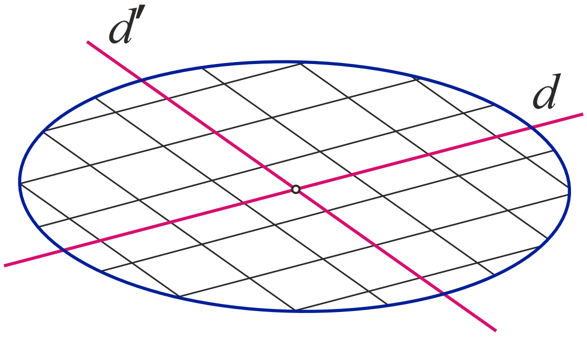

Figure 3

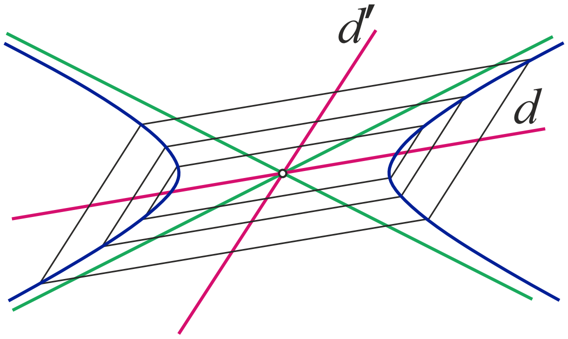

Figure 4

and the equation of the diameter conjugate to it,

$$\beta x - \alpha y = 0.$$

Figure 4 shows a hyperbola, its conjugate diameters d and d', and the chords parallel to each of them.

For parabola

$$y^2 - 2px = 0,$$

the equation of diameter conjugate to chords parallel to the non-zero vector $\{\alpha, \beta\}$ has the form

$$-\alpha p + \beta y = 0 \text{ or } y = \frac{p}{k},$$

where k is the slope of the chords.

All diameters of a parabola are parallel to its axis. Figure 5 shows one of the families of parallel chords of a parabola and its conjugate diameter.

If a given curve splits into a pair of parallel lines, then it has one single diameter (the "centerline" with respect to the two given lines); this diameter is the locus of the midpoints of the chords of any direction, it is the line of the centers of our curve, its direction is special, it is conjugate to any direction (Fig. 6).

5 Tangent to second-order curve

Let, the second-order curve be given by the general equation

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0. \tag{5.1}$$

We will call a point $M_0(x_0, y_0)$ lying on this curve non-singular if among the numbers[2]

$$a_{11}x_0 + a_{12}y_0 + a_1,$$
$$a_{21}x_0 + a_{22}y_0 + a_2$$

[2] These numbers are the values of partial derivatives $F'_x/2$ and $F'_y/2$ at the point $M_0(x_0, y_0)$.

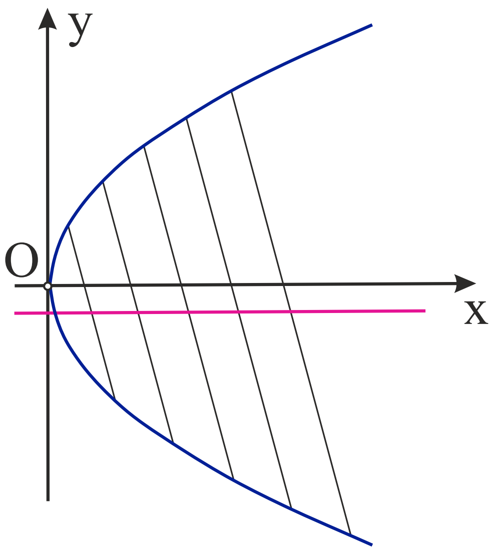

Figure 5

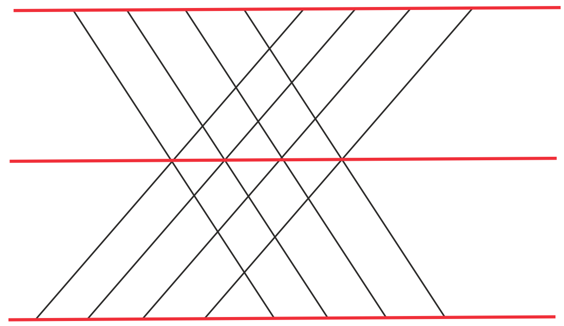

Figure 6

there is at least one that is not equal to zero.

It is clear that the point M_0 lying on the curve (5.1) is singular if and only if it is the center of the curve (5.1).

Thus, among the curves of the elliptical type, only the curve that splits into pair imaginary intersecting lines has a singular point (this is the point of their intersection); among curves of hyperbolic type, a pair of intersecting lines has a singular point (this is also the point of their intersection).

Definition. *A tangent to a second-order curve at a non-singular point lying on this curve is a line passing through this point, intersecting the given curve at a double point, or merging with the line included in the given curve.*

Theorem 1. *Let $M_0(x_0, y_0)$ be a non-singular point of the second-order curve defined by the equation (5.1)*

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0.$$

Then the equation of the tangent to this curve at the point M_0 has the form

$$(a_{11}x_0 + a_{12}y_0 + a_1)x + (a_{21}x_0 + a_{22}y_0 + a_2)y + a_1x_0 + a_2y_0 + a_0 = 0. \quad (5.2)$$

Proof. Consider the equations of the line

$$\begin{aligned} x &= x_0 + \alpha t, \\ y &= y_0 + \beta t, \end{aligned} \tag{5.3}$$

the curve (5.1) passing through the given non-singular point $M_0(x_0, y_0)$. Let's find the intersection points of the line (5.3) with the curve (5.1). Substituting $x_0 + \alpha t$ and $y_0 + \beta t$ into the equation (5.1) instead of x and y, we get

$$\begin{aligned} a_{11}(x_0+\alpha t)^2 + 2a_{12}(x_0+\alpha t)(y_0+\beta t) + a_{22}(y_0+\beta t)^2 + \\ + 2a_1(x_0+\alpha t) + 2a_2(y_0+\beta t) + a_0 = 0 \end{aligned}$$

or

$$\begin{aligned} &(a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2)t^2 + \\ &\qquad 2[\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2)]t + \\ &\qquad\qquad + a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 + 2a_1x_0 + 2a_2y_0 + a_0 = 0. \end{aligned}$$

But by assumption, the point $M_0(x_0, y_0)$ lies on the given curve, so

$$a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 + 2a_1x_0 + 2a_2y_0 + a_0 = 0$$

and the last equation becomes

$$\begin{aligned} &(a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2)t^2 + \\ &\qquad 2[\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2)]t = 0. \end{aligned} \tag{5.4}$$

One of the roots of this equation is $t = 0$; hence, from the relations (5.3) we find $x = x_0$, $y = y_0$, i.e., the coordinates of the point M_0.

For the line (5.3) to be tangent to the curve (5.1), it is necessary and sufficient that the equation

$$\begin{aligned} &(a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2)t + \\ &\qquad 2[\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2)] = 0 \end{aligned} \tag{5.5}$$

has a second root equal to zero $t = 0$, and for this it is necessary and sufficient that the condition

$$\alpha(a_{11}x_0 + a_{12}y_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_2) = 0$$

be satisfied.

Thus, the coordinates of the direction vector of the tangent

$$\begin{aligned} \alpha &= -(a_{21}x_0 + a_{22}y_0 + a_2), \\ \beta &= a_{11}x_0 + a_{12}y_0 + a_1 \end{aligned} \tag{5.6}$$

(this vector is non-zero, since the point $M_0(x_0, y_0)$ is non-singular by assumption). If the vector $\{\alpha, \beta\}$, whose coordinates are determined by the relations (5.6), has a non-asymptotic direction, i.e.

$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 \neq 0,$$

then the equation (5.5) has only the root $t = 0$, and if the vector $\{\alpha, \beta\}$ has an asymptotic direction, i.e.

$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0,$$

then the equation (5.5) becomes an identity, the line (5.3) is a part of the given curve (5.1) and, therefore, according to the accepted definition, is tangent to the curve (5.1) at the point $M_0(x_0, y_0)$.

Thus, the equations of the tangent to the curve (5.1) at its non-singular point $M_0(x_0, y_0)$ have the form

$$\begin{aligned} x &= x_0 - (a_{21}x_0 + a_{22}y_0 + a_2)t, \\ y &= y_0 + (a_{11}x_0 + a_{12}y_0 + a_1)t, \end{aligned}$$

or

$$(a_{11}x_0 + a_{12}y_0 + a_1)(x - x_0) + (a_{21}x_0 + a_{22}y_0 + a_2)(y - y_0) = 0,$$

or

$$\begin{aligned} &(a_{11}x_0 + a_{12}y_0 + a_1)x + (a_{21}x_0 + a_{22}y_0 + a_2)y- \\ &\qquad\qquad - a_{11}x_0^2 - 2a_{12}x_0y_0 - a_{22}y_0^2 - a_1x_0 - a_2y_0 = 0, \end{aligned}$$

and since

$$a_{11}x_0^2 + 2a_{12}x_0y_0 + a_{22}y_0^2 + 2a_1x_0 + 2a_2y_0 + a_0 = 0$$

we finally get

$$(a_{11}x_0 + a_{12}y_0 + a_1)x + (a_{21}x_0 + a_{22}y_0 + a_2)y + a_1x_0 + a_2y_0 + a_0 = 0.$$

Theorem 2. *Let the second-order curve be given by the general equation (5.1)*

$$F(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0.$$

Let the diameter

$$(a_{11}x + a_{12}y + a_1)\alpha + (a_{21}x + a_{22}y + a_2)\beta = 0 \tag{5.7}$$

of this curve conjugate to chords having a non-asymptotic direction $\{\alpha, \beta\}$ *intersects the curve under consideration at a non-singular point* $M_0(x_0, y_0)$. *Then the tangent to this line at the point* M_0 *is parallel to the chords conjugate to the diameter (5.7).*

Proof. Since the diameter (5.7) passes through the point $M_0(x_0, y_0)$, then

$$(a_{11}x_0 + a_{12}y_0 + a_1)\alpha + (a_{21}x_0 + a_{22}y_0 + a_2)\beta = 0$$

and since $M_0(x_0, y_0)$ is a non-singular point of the curve under consideration, we can assume that

$$\alpha = -(a_{21}x_0 + a_{22}y_0 + a_2),$$
$$\beta = a_{11}x_0 + a_{12}y_0 + a_1,$$

and these are the coordinates of the direction vector of the tangent to the considered curve of the second-order at its non-singular point (see formulas (5.6) above).

6 Curve equation in coordinate system with axes of conjugate directions

Let a second-order curve be given by its general equation

$$F(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0. \tag{6.1}$$

Firstly, let consider the diameters conjugate to the directions of the coordinate axes. The equation for the diameter conjugate to the $\{\alpha, \beta\}$ direction is

$$(a_{11}\alpha + a_{12}\beta)x + (a_{21}\alpha + a_{22}\beta)y + (a_1\alpha + a_2\beta) = 0. \tag{6.2}$$

If $\alpha = 1$, $\beta = 0$ (i.e. the vector $\{\alpha, \beta\}$ is the direction vector of the x-axis), then the equation (6.2) becomes

$$a_{11}x + a_{12}y + a_1 = 0. \tag{6.3}$$

If $\alpha = 0$, $\beta = 1$, then the conjugate diameter equation is

$$a_{21}x + a_{22}y + a_2 = 0. \tag{6.4}$$

So, the diameter conjugate to the direction of the x-axis has the equation (6.3), and the diameter conjugate to the direction of the y-axis has the equation (6.4).

Let us now assume that the y-axis has an arbitrary, non-asymptotic direction for the given curve, and the x-axis is a diameter conjugate to the direction of the ordinate axis. Then the equation (6.4) is the equation of the x-axis, i.e. it expresses the same line as the equation

$$y = 0.$$

Therefore, the coefficients

$$a_{21}, \qquad a_{22}, \qquad a_2$$

of equation (6.4) must be proportional to the coefficients

$$0, \qquad 1, \qquad 0$$

of x-axis equation, which means that

$$a_{21} = 0, \qquad a_{22} \neq 0, \qquad a_2 = 0.$$

Therefore, in our coordinate system, the curve (6.1) has the equation

$$\boxed{a_{11}x^2 + a_{22}y^2 + 2a_1x + a_0 = 0.} \tag{6.5}$$

Let's consider two cases separately:

1. Curve (6.1) is of central type.

2. Curve (6.1) is of parabolic type (a parabola or a pair of parallel lines).

In the *first case*, from the fact that the x-axis is a diameter conjugate to the direction of the y-axis, it follows that the y-axis also has a direction conjugate to the x-axis. *If in this case the origin of coordinates lies in the center of the curve, then both coordinate axes are conjugate diameters.*

But then the Oy, being the diameter conjugate to the Ox, has the equation (6.3), which should be equivalent to the equation

$$x = 0.$$

So the coefficients

$$a_{11}, \qquad a_{12}, \qquad a_1$$

of equation (6.3) must be proportional to the coefficients

$$1, \qquad 0, \qquad 0$$

of equation of the y-axis, i.e.

$$a_{11} \neq 0, \qquad a_{12} = 0, \qquad a_1 = 0;$$

hence the equation (6.5) takes the form

$$\boxed{a_{11}x^2 + a_{22}y^2 + a_0 = 0.} \tag{6.6}$$

So, *if the coordinate axes form a pair of conjugate diameters of a given (arbitrary) second-order central curve, then the equation of this curve in this coordinate system has the form (6.6) (Fig. 7).*

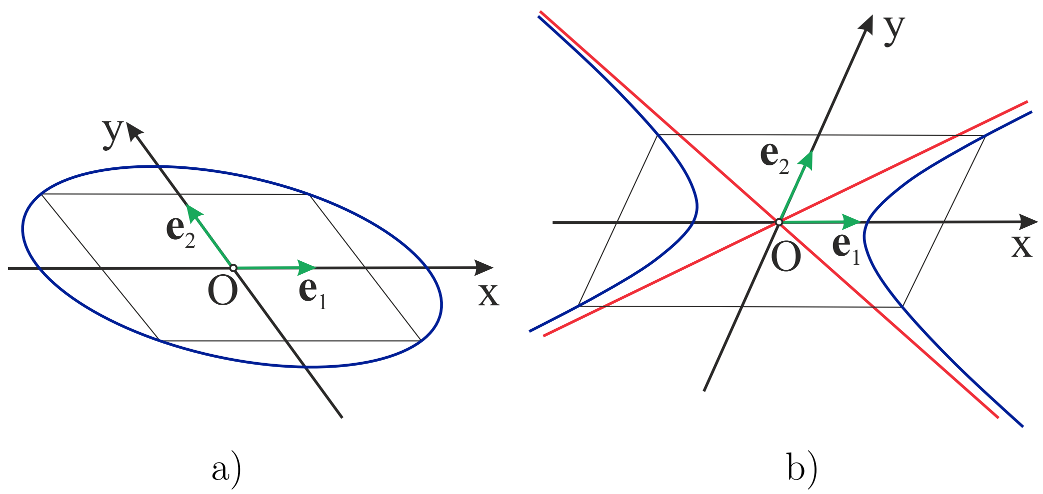

Figure 7

The case of a decomposing central curve is characterized by the fact that in the equation (6.6) we have $a_0 = 0$ (the center, i.e. the origin, is a point of the curve) (Fig. 8).

Let now the curve (6.1) (whose equation has already been reduced to the form (6.5)) is a *parabolic curve.* Then $a_{11}a_{22} = a_{12}^2$; but in the equation (6.5) the coefficient $a_{12} = 0$, so $a_{11}a_{22} = 0$, and since $a_{22} \neq 0$, then $a_{11} = 0$.

So, *if in the case of a parabolic curve the y-axis is directed along an arbitrary non-asymptotic direction, and the x-axis is the diameter conjugate to this direction (and, therefore, having an asymptotic direction), then the equation of the curve in this coordinate system has the form*

$$\boxed{a_{22}y^2 + 2a_1x + a_0 = 0.} \tag{6.7}$$

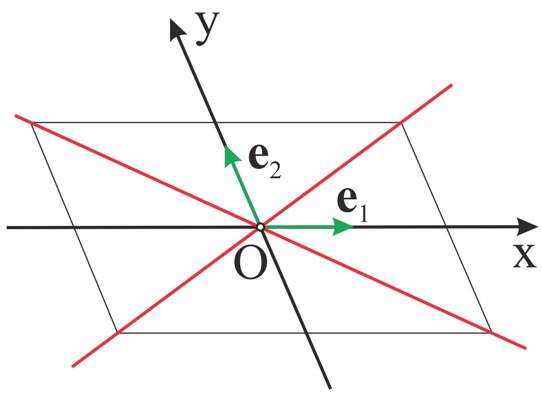

Figure 8

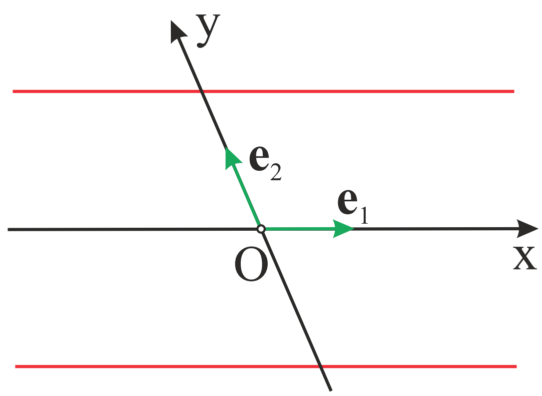

Figure 9

If our curve decomposes into a pair of parallel lines (Fig. 9), then $\Delta = 0$, i.e.

$$\begin{vmatrix} 0 & 0 & a_1 \\ 0 & a_{22} & 0 \\ a_1 & 0 & a_0 \end{vmatrix} = -a_1^2 a_{22} = 0.$$

Since $a_{22} \neq 0$, then $a_1 = 0$, and the equation (6.7) must have the form

$$a_{22}y^2 + a_0 = 0, \qquad y = \pm\sqrt{-\frac{a_0}{a_{22}}}. \tag{6.8}$$

But if our curve is a non-decomposing parabolic curve, then $a_1 \neq 0$ is necessarily.

Let us now move the origin to the intersection point O' of the curve with the x-axis, i.e. coordinate transformation

$$x = x' + x_0, \qquad y = y',$$

where x_0 is defined by the requirement that the point $O' = (x_0, 0)$ satisfies the equation (6.7), i.e. that

$$a_{22} \cdot 0 + 2a_1x_0 + a_0 = 0.$$

In the transformed coordinate system, the equation (6.7) becomes

$$a_{22}y'^2 + 2a_1x' + (2a_1x_0 + a_0) = 0,$$

hence

$$a_{22}y'^2 + 2a_1x' = 0. \tag{6.9}$$

Find the points of intersection of the y'-axis with the parabola. For this we put in the equation (6.9) $x' = 0$; we get $a_{22}y'^2 = 0$, i.e. axis $O'y'$ intersects the parabola at two coincident points (coinciding with the point O') and, therefore, is tangent to the parabola (Fig. 10). If, in this case, the abscissa axis is perpendicular to the ordinate axis, then, dividing the chords perpendicular to it in half, it will turn out to be the axis of the parabola (Fig. 11).

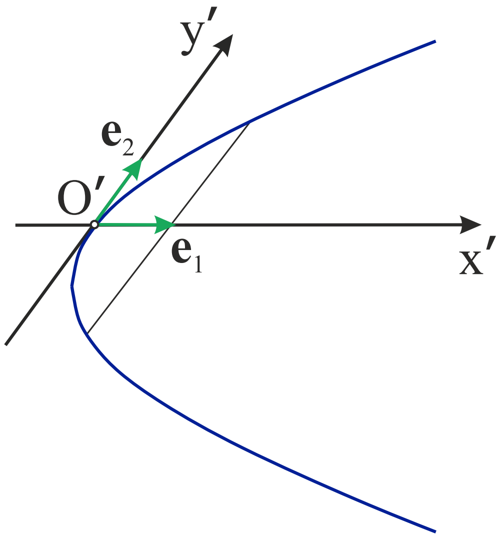

Figure 10

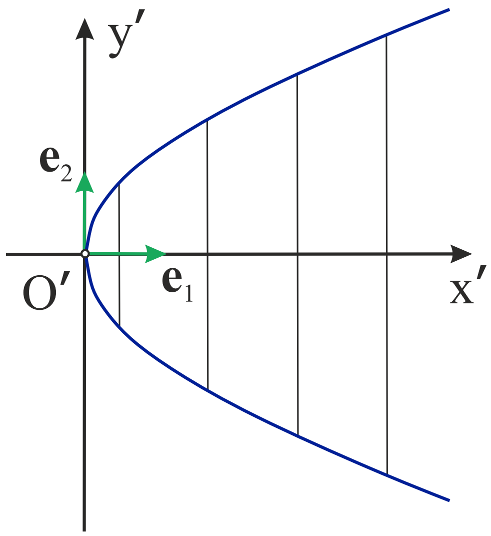

Figure 11

7 Principal directions and principal diameters

Definition. *A non-asymptotic direction of a second-order curve is called* ***principal*** *if it is perpendicular to the diameter conjugate to the chords having this direction. This diameter is called the* ***principal diameter*** *of the second order curve. It is the axis of symmetry of the curve.*

Theorem 1. *Coordinates* α, β *of a (non-zero) vector having a principal direction with respect to a second-order curve given by the general equation*

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0, \tag{7.1}$$

are determined from the system

$$\begin{aligned} a_{11}\alpha + a_{12}\beta &= \lambda\alpha, \\ a_{21}\alpha + a_{22}\beta &= \lambda\beta, \end{aligned} \tag{7.2}$$

where λ *is a non-zero root of the quadratic equation*

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0, \tag{7.3}$$

or

$$\lambda^2 - S\lambda + \delta = 0, \tag{7.4}$$

where
$$S = a_{11} + a_{22}, \tag{7.5}$$
$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \tag{7.6}$$

Proof. Let a non-zero vector $\{\alpha, \beta\}$ have no asymptotic direction with respect to the curve (7.1). Then the equation of the diameter conjugate to the chords of the curve (7.1) parallel to the vector $\{\alpha, \beta\}$ has the form (Section 4)

$$(a_{11}x + a_{12}y + a_1)\alpha + (a_{21}x + a_{22}y + a_2)\beta = 0,$$

or
$$(a_{11}\alpha + a_{12}\beta)x + (a_{21}\alpha + a_{22}\beta)y + a_1\alpha + a_2\beta = 0. \tag{7.7}$$

Since the coordinate system is rectangular, this diameter is perpendicular to the chords parallel to the non-zero vector $\{\alpha, \beta\}$ if and only if the non-zero vectors

$$\{a_{11}\alpha + a_{12}\beta, a_{21}\alpha + a_{22}\beta\} \text{ and } \{\alpha, \beta\}$$

are collinear, i.e., if and only if there exists a number λ such that

$$\begin{aligned} a_{11}\alpha + a_{12}\beta &= \lambda\alpha, \\ a_{21}\alpha + a_{22}\beta &= \lambda\beta. \end{aligned} \tag{7.8}$$

In these relations $\lambda \neq 0$, since otherwise we would have

$$a_{11}\alpha + a_{12}\beta = 0, \qquad a_{21}\alpha + a_{22}\beta = 0,$$

where
$$(a_{11}\alpha + a_{12}\beta)\alpha + (a_{21}\alpha + a_{22}\beta)\beta = 0$$

or
$$a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0,$$

i.e. the vector $\{\alpha, \beta\}$ would have an asymptotic direction.

Further, rewriting the system (7.8) as

$$\begin{aligned} (a_{11} - \lambda)\alpha + a_{12}\beta &= 0, \\ a_{21}\alpha + (a_{22} - \lambda)\beta &= 0 \end{aligned}$$

and noticing that it has a non-zero solution α, β, we conclude that

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0. \tag{7.9}$$

Conversely, the equation (7.9) always has real roots λ_1 and λ_2; in fact, rewriting it as

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}^2 = 0, \tag{7.10}$$

find the discriminant:

$$D = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2) = (a_{11} - a_{22})^2 + 4a_{12}^2 > 0.$$

Case 1. $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2$, (real or imaginary ellipse, hyperbola, pair intersecting lines, real or imaginary).

In this case, the system (7.8) with $\lambda = \lambda_1$ takes the form

$$\begin{aligned} a_{11}\alpha + a_{12}\beta &= \lambda_1\alpha, \\ a_{21}\alpha + a_{22}\beta &= \lambda_1\beta \end{aligned} \tag{7.11}$$

and due to the conditions

$$\begin{vmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{vmatrix} = 0$$

has a non-zero solution α_1, β_1.

From the relations (7.11) for this solution, which take the form

$$\begin{aligned} a_{11}\alpha_1 + a_{12}\beta_1 &= \lambda_1\alpha_1, \\ a_{21}\alpha_1 + a_{22}\beta_1 &= \lambda_1\beta_1, \end{aligned} \tag{7.12}$$

we find

$$(a_{11}\alpha_1 + a_{12}\beta_1)\alpha_1 + (a_{21}\alpha_1 + a_{22}\beta_1)\beta_1 = \lambda_1(\alpha^2 + \beta^2) \neq 0,$$

or

$$a_{11}\alpha_1^2 + 2a_{12}\alpha_1\beta_1 + a_{22}\beta_1^2 \neq 0.$$

This means that this solution does not have an asymptotic direction of the curve (7.1), and due to the relations (7.11), the non-zero vector $\{\alpha_1, \beta_1\}$ has principal direction (7.1).

Similarly, for $\lambda = \lambda_2$ from the system (7.8) we find a non-zero vector $\{\alpha_2, \beta_2\}$ such that

$$\begin{aligned} a_{11}\alpha_2 + a_{12}\beta_2 &= \lambda_2\alpha_2, \\ a_{21}\alpha_2 + a_{22}\beta_2 &= \lambda_2\beta_2, \end{aligned} \tag{7.13}$$

also having a principal direction relative to the curve (7.1).

Let us prove that the vectors $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ are mutually perpendicular. From the relations (7.12) and (7.13) we find

$$\begin{aligned} (a_{11}\alpha_1 + a_{12}\beta_1)\alpha_2 + (a_{21}\alpha_1 + a_{22}\beta_1)\beta_2 &= \lambda_1(\alpha_1\alpha_1 + \beta_1\beta_2), \\ (a_{11}\alpha_2 + a_{12}\beta_2)\alpha_1 + (a_{21}\alpha_2 + a_{22}\beta_2)\beta_1 &= \lambda_2(\alpha_1\alpha_1 + \beta_1\beta_2). \end{aligned}$$

The left-hand sides of these equalities are the same, so

$$\lambda_1(\alpha_1\alpha_1 + \beta_1\beta_2) = \lambda_2(\alpha_1\alpha_1 + \beta_1\beta_2),$$

or

$$(\lambda_1 - \lambda_2)(\alpha_1\alpha_1 + \beta_1\beta_2) = 0,$$

whence ($\lambda_1 \neq \lambda_2$)

$$\alpha_1\alpha_1 + \beta_1\beta_2 = 0,$$

i.e., the vectors $\{\alpha_1, \beta_1\}$ and $\{\alpha_2, \beta_2\}$ are mutually perpendicular. This also implies that in the case of $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, and $\lambda_1 \neq \lambda_2$, the system (7.8) for $\lambda = \lambda_1$ has a non-zero solution, but cannot have two linearly independent solutions (same with $\lambda = \lambda_2$). In other words, *the curve has two and only two mutually perpendicular and principal diameters.*

Case 2. $\lambda_1 = \lambda_2 \neq 0$ (real, zero, or imaginary circle).

In this case, there are no asymptotic directions, and any direction is principal; the equation (7.7) of the diameter conjugate to the chords parallel to the non-zero vector $\{\alpha, \beta\}$ takes the form

$$a_{11}\alpha x + a_{11}\beta y + a_1\alpha + a_2\beta = 0 \tag{7.14}$$

and is the equation of a line perpendicular to the vector

$$\{\alpha, \beta\}\,(a_{11} \neq 0).$$

The principal diameter in this case is any straight line (7.14) passing through the center of the line.

Case 3. $\lambda_1 = 0$, $\lambda_2 \neq 0$ (parabola, pair parallel or coinciding lines).

In this case $\delta = 0$. The system (7.8) with $\lambda = \lambda_2$ has a non-zero solution α_2, β_2; the vector $\{\alpha_2, \beta_2\}$ has no asymptotic direction (7.1) and has a principal direction.

For $\lambda = \lambda_1 = 0$, the system (7.8) becomes

$$\begin{aligned} a_{11}\alpha + a_{12}\beta &= 0,\\ a_{21}\alpha + a_{22}\beta &= 0, \end{aligned}$$

this system has a non-zero solution α_1, β_1, but the vector $\{\alpha_1, \beta_1\}$ has an asymptotic direction, since from the relations

$$\begin{aligned} a_{11}\alpha_1 + a_{12}\beta_1 &= 0,\\ a_{21}\alpha_1 + a_{22}\beta_1 &= 0, \end{aligned}$$

follows that

$$a_{11}\alpha_1^2 + 2a_{12}\alpha_1\beta_1 + a_{22}\beta_1^2 = 0.$$

The vectors $\{\alpha_1, \beta_1\}$, $\{\alpha_2, \beta_2\}$ are also orthogonal here (due to the fact that $\lambda_1 \neq \lambda_2$, the proof is given above). It follows from this that the system

$$\begin{aligned} a_{11}\alpha + a_{12}\beta = 0, \\ a_{21}\alpha + a_{22}\beta = 0, \end{aligned}$$

in this case has one non-zero solution, but does not have two linearly independent solutions. It follows from the last relations that the vectors

$$\{-a_{12}, a_{11}\} \text{ and } \{-a_{22}, a_{21}\}$$

have an asymptotic direction (if the equation (7.1) is the equation of a parabola, or pair parallel or coinciding lines); moreover, at least one of these vectors is non-zero.

Since the vector with the principal direction is perpendicular to the asymptotic direction, the principal direction of the line (7.1) in the case of $\lambda_1 = 0$, $\lambda_2 \neq 0$ is determined by one of the vectors or $\{a_{11}, a_{12}\}$, or $\{a_{21}, a_{22}\}$ (one of which is non-zero).

Thus, in the case of a curve of parabolic type, there is only one principal diameter; its equation

$$a_{11}(a_{11}x + a_{12}y + a_1) + a_{12}(a_{21}x + a_{22}y + a_2) = 0,$$

or

$$a_{21}(a_{11}x + a_{12}y + a_1) + a_{22}(a_{21}x + a_{22}y + a_2) = 0.$$

In this case, it is necessary to take that of the equations in which the coefficients of $a_{11}x + a_{12}y + a_1$ and $a_{21}x + a_{22}y + a_2$ are not simultaneously equal to zero, and any of them, if both vectors $\{a_{11}, a_{12}\}$ and $\{a_{21}, a_{22}\}$ are non-zero.

In the case of a parabola, this is its axis of symmetry; for pair parallel or coinciding lines, the principal diameter coincides with the location of the line centers.

Note that in the case of $\delta \neq 0$ the direction of any principal diameter is principal, and in the case of $\delta = 0$ the principal direction is the direction perpendicular to the single principal diameter.

8 Location of a second-order curve in relation to a rectangular coordinate system

To determine the location of the second-order curve given by the general equation

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0 \qquad (8.1)$$

relative to a rectangular coordinate system, it is sufficient to know the parameters characterizing the given curve and the coordinate system in which the curve equation is canonical.

If the equation (8.1) is an equation of an ellipse, then we need to find its semiaxes, the center and the direction of the focal axis (or the axis perpendicular to it).

If the equation (8.1) is the equation of a hyperbola, then we need to find its semiaxis, center and direction of the real (or imaginary) axis.

If the equation (8.1) is the equation of a parabola, then we need to find its parameter, the vertex and the direction of one of the two axis, for example, the one on which the focus lies.

If the curve is reduced to one point, then you need to find its coordinates.

Finally, if the curve splits into pair real lines, then it is necessary to find (in the given coordinate system) the equation of each of them[3].

If in the equation (8.1) $a_{12} = 0$, then the location of the curve is determined using translation of axes (see examples below). Let $a_{12} \neq 0$. As was proved earlier, the equation (8.1) can be reduced to one of the following forms by rotating the Oxy coordinate axes by the angle α and then translating them:

$$\begin{aligned} &(I) && \lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0, \\ &(II) && \lambda_1 x^2 \pm 2\sqrt{-\frac{\Delta}{S}}\, y = 0, \\ &(III) && \lambda_1 x^2 + \frac{K}{S} = 0. \end{aligned} \tag{8.2}$$

Rewriting formulas (1.3) from the previous topic as

$$\begin{aligned} a'_{11} &= \lambda_1 = a_{11}\cos^2\alpha + 2a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha, \\ a'_{12} &= 0 = -a_{11}\cos\alpha\sin\alpha + a_{12}(\cos^2\alpha - \sin^2\alpha) + a_{22}\cos\alpha\sin\alpha \end{aligned}$$

we have

$$\begin{aligned} \lambda_1\cos\alpha &= a_{11}\cos^3\alpha + 2a_{12}\cos^2\alpha\sin\alpha + a_{22}\sin^2\alpha\cos\alpha, \\ 0 &= -\sin\alpha a_{12}(\cos^2\alpha - \sin^2\alpha) + (a_{11} - a_{22})\cos\alpha\sin^2\alpha, \end{aligned}$$

and adding these equations, we get

$$\lambda_1\cos\alpha = a_{11}\cos\alpha + a_{12}\sin\alpha,$$

whence the slope of the new axis $O'x$ for each of the reduced equations (8.2) of second-order curves

$$\tan\alpha = \frac{\lambda_1 - a_{11}}{a_{12}}, \tag{8.3}$$

[3]We do not consider the question of the location of imaginary curves.

where λ_1 is the root of the characteristic equation, which is the coefficient of x^2 in each of the reduced equations (8.2).

1. If the equation (8.1) is an ellipse equation, then the reduced equation has the form

$$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0.$$

Assuming that λ_1 denotes the root of the characteristic equation, which is smaller in absolute value ($|\lambda_1| < |\lambda_2|$), and rewriting the last equation in the form

$$\frac{x^2}{-\dfrac{\Delta}{\delta\lambda_1}} + \frac{y^2}{-\dfrac{\Delta}{\delta\lambda_2}} = 1.$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where

$$a = \sqrt{-\frac{\Delta}{\delta\lambda_1}}, \qquad b = \sqrt{-\frac{\Delta}{\delta\lambda_2}},$$

we conclude that $a > b$, so the formula (8.3) determines the slope of the major axis of the ellipse.

The coordinates of the center of the ellipse are found from the system

$$\begin{aligned} a_{11}x + a_{12}y + a_1 &= 0, \\ a_{21}x + a_{22}y + a_2 &= 0. \end{aligned} \tag{8.4}$$

2. If the equation (8.1) is an equation of a hyperbola, then its reduced equation again has the form (I). Denoting by λ_1 root of the characteristic equation, which has the same sign as Δ, we rewrite the equation (I) in the form

$$\frac{x^2}{-\dfrac{\Delta}{\delta\lambda_1}} - \frac{y^2}{-\dfrac{\Delta}{\delta\lambda_2}} = 1$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where

$$a = \sqrt{-\frac{\Delta}{\delta\lambda_1}}$$

is the length of the real semi-axis, and

$$b = \sqrt{\frac{\Delta}{\delta\lambda_2}}$$

is the length of the imaginary semiaxis.

Using the formula (8.3), we can determine the slope of the real axis $O'x$. We find the center of coordinates as well from the system (8.4).

3. If the equation (8.1) is the equation of a parabola, then its the reduced equation is

$$\lambda_1 x^2 \pm 2\sqrt{-\frac{\Delta}{S}}y = 0, \ (S = \lambda_1),$$

whence the parabola parameter

$$p = \sqrt{-\frac{\Delta}{S^3}}.$$

The vertex of the parabola is found as follows: take a point (x, y) on the parabola. The coordinates of the vector $\mathbf{n}$ normal to the tangent to the parabola at this point are:

$$a_{11}x + a_{12}y + a_1, \qquad a_{21}x + a_{22}y + a_2.$$

For the point (x, y) to be the vertex of the parabola, it is necessary and sufficient that the vector $\mathbf{n}$ has the direction of the diameters of the parabola (the asymptotic direction), i.e., that the condition

$$\begin{aligned} a_{11}x + a_{12}y + a_1 &= -a_{12}t, \\ a_{21}x + a_{22}y + a_2 &= a_{11}t \end{aligned} \tag{8.5}$$

(see Theorem 4, Section 4). Multiplying these equalities by $-a_{12}$ and a_{11}, respectively, and adding them term by term, we get

$$a_2a_{11} - a_1a_{12} = (a_{11}^2 + a_{12}^2)t,$$

where

$$t = \frac{a_2a_{11} - a_1a_{12}}{a_{11}^2 + a_{12}^2}. \tag{8.6}$$

Rewriting the parabola equation as

$$(a_{11}x + a_{12}y + a_1)x + (a_{21}x + a_{22}y + a_2)y + a_1x + a_2y + a_0 = 0,$$

due to relations (8.5) we have

$$(-a_{12}x + a_{11}y)t + a_1x + a_2y + a_0 = 0. \tag{8.7}$$

Thus, to find the coordinates of the vertex of the parabola, it is necessary to solve the linear system (8.5), (8.7), where t is determined by the formula (8.6).

Using the formula (8.3) ($\lambda_1 \neq 0$), we find the slope of the tangent to the parabola at its vertex (and if in the formula (8.3) we take $\lambda_1 = 0$, then it will determine the slope of the parabola diameters).

To find a vector collinear to the diameters of a parabola and going towards its concavity, we note that the equation (8.1) due to $\delta = 0$ can always be rewritten in the form

$$(\alpha x + \beta y)^2 + px + qy + r = 0. \tag{8.8}$$

Point (x_0, y_0) of intersection of lines

$$\alpha x + \beta y = 0, \qquad px + qy + r = 0$$

always lies on the given parabola (these lines always intersect if the equation (8.1) is the equation of a parabola). The equation

$$px + qy + r = 0$$

is tangent to the parabola at this point $M_0(x_0, y_0)$, and

$$\alpha x + \beta y = 0$$

equation of the diameter passing through the point of contact. The parabola, whose equation is written as (8.8), lies in the negative half-plane from the line $px + qy + r = 0$, and the main vector $\{p, q\}$ of this tangent is directed into the positive half-plane from the line $px + qy + r = 0$. Therefore, if the non-zero vector $\{-a_{12}, a_{11}\}$ ($a_{12} \neq 0$), collinear with the diameters of the parabola (8.1), is directed to the negative half-plane from the line $px + qy + r = 0$, i.e. $-a_{12}p + a_{11}q < 0$, then the vector $\{-a_{12}, a_{11}\}$, collinear to the diameters of the parabola, is directed towards its concavity, and if $-a_{12}p + a_{11}q > 0$, then towards convexity.

In the case of $-a_{12}p + a_{11}q > 0$, the vector $\{a_{12}, -a_{11}\}$ is collinear to the parabola axis and directed towards its concavity.

Remark. The last question of determining the direction vector of parabola diameters (8.1) is solved assuming that the equation (8.1) of the parabola is given with respect to the general Cartesian coordinate system.