Complex Analysis 2024. Homework 14.

1. Provide an example of a function with a nonisolated singular point.

$$f(z) = \frac{1}{\sin \frac{1}{z}}$$

has singular points at 0 and $\frac{1}{\pi k}$, $k \in \mathbb{Z}$.

2. Let $D \subset \mathbb{C}$ be simply connected domain, $f \in H(D)$ and $f(D) \subset \mathbb{C}\backslash\{0\}$. Prove that there exist functions g and h holomorphic in domain D such that

$$f = g^2, \quad f = e^h, \text{ in } D.$$

Proof. Consider the composition $\sqrt{f(z)}$. By monodromy theorem in domain D it consists of two single-valued analytic functions. If g is one of these branches then $g \in H(D)$ and $g^2 = f$ in D.

Analogously $f = e^h$ for any function h obtained by the composition of $Lnf(z)$. □

Proof. Let $h_1(z)$ is antiderivative of function $f'(z)/f(z)$ in D (that exists since D is simply connected). Then

$$e^{h_1(z)} = Cf(z) \quad z \in D,$$

for some $C \in \mathbb{C}\backslash\{0\}$ since

$$\left(e^{h_1(z)}/f(z)\right)' = (h_1' e^{h_1} f - f' e^{h_1})/f^2 = 0.$$

Let $a = \ln(C)$ be some value of $Ln(C)$ and $h = h_1(z)/a$. Then $h \in \mathcal{O}(D)$ and $e^h = f$. Finally, function $g(z) = e^{h(z)/2}$ is holomorphic in D and $g^2 = f$. □

3. Let $D \subset \mathbb{C}$ be simply connected domain, $f \in H(D)$, $f \not\equiv 0$. Prove that $f = g^2$ for some $g \in H(D)$ if and only if orders of all zeros of f are even.

Proof. Suppose that $f = g^2$ for some $g \in H(D)$. Notice that $f(a) = 0$ iff $g(a) = 0$.

Let $f(a) = 0$ then $g(z) = (z-a)^m h(z)$, $h \in H(D)$, $h(a) \neq 0$. Consequently, $f(z) = (z-a)^{2m} h^2(z)$ and the order of zero a for f is even.

Suppose that all zeroes of function f have even order. $\square$

4. Describe all values z^z, $z \neq 0$.

   Answer:
   $$z^z = e^{z \ln z} = e^{z(\ln|z| + iargz) + 2\pi kzi)}, \quad k \in \mathbb{Z}.$$

5. Consider a holomorphic branch f of a function Lnz such that $f(1) = 0$. Let $x \in (0, +\infty)$. Calculate
   $$\lim_{t \to 0+} (f(x+it) - f(x-it)).$$

   Answer $2\pi i$.

6. Let f_0 be holomorphic branch of $z^{1/3}$ such that $f(1) = e^{2\pi i/3}$. Consider a path $\gamma(t) = (1+t)e^{7\pi it}$, $0 \leq t \leq 1$. Calculate value of analytic continuation of f_0 at point -2.

   Answer: $-\sqrt{2}$.

7. Suppose that analytic functions $\mathcal{F}$ and $\mathcal{G}$ have single-valued branches in domain D. Prove that $\mathcal{F} + \mathcal{G}$ and $\mathcal{F}\mathcal{G}$ have single-valued branches in domain D. Use this property to prove that $\sqrt{1-z^2}$ has single-valued branch in $\mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$. Suppose that $f(z)$ is a branch such that $f(0) = 1$. Calculate $f(i)$.

   Proof. Suppose that $f, g \in H(D)$ are holomorphic branches of $\mathcal{F}$ and $\mathcal{G}$. Let $a \in D$ then f and g can be obtained as result of analytic continuation along all paths of some elements (U_1, f) and (U_2, g) centered at a. So $f + g$ is result of analytic continuation of element $(U, f+g)$, where $U = U_1 \cap U_2$. Analogously for fg.

   Function $\sqrt{1-z^2}$ has holomorphic branches defined by branches of $\sqrt{1-z}$ in $\mathbb{C} \setminus [1, +\infty)$ and $\sqrt{1+z}$ in $\mathbb{C} \setminus (-\infty, -1]$. Consider branch

$f_1(z)$ of $\sqrt{1-z}$ in $\mathbb{C}\backslash[1,+\infty)$ such that $f_1(0)=1$ and branch $f_2(z)$ of $\sqrt{1+z}$ in $\mathbb{C}\backslash(-\infty,0]$ such that $f_2(0)=1$. Hence,

$$f_1(i)=\sqrt{2}e^{i\pi/4}; \quad f_2(i)=\sqrt{2}e^{i\pi/4}$$

and $f(i)=\sqrt{2}$. □

8. Let $a,b\in\mathbb{C}$. Prove that function $z^a(1-z)^b$ has single-valued branch in domain $\mathbb{C}\backslash[0,1]$ if and only if $a+b\in\mathbb{Z}$.

   Proof. This function has single value if and only if every analytic continuation along closed path around $[0,1]$ conicides with itself. Let $z_0\in\mathbb{C}\backslash[0,1]$, $\gamma:[0,1]\to D$ be a closed simple path around $[0,1]$ such that $\gamma(0)=\gamma(1)=z_0$. Consider a holomorphic branch f of $z^a(1-z)^b$ in neighbourhood U of z_0 then $f=gh$, where g and h are holomorphci branches of z^a and $(1-z)^b$ in U. Analytic continuations of branches g and h along γ are equal to $e^{2\pi a}g$ and $e^{2\pi b}h$ respectively. Consequently, analytic continuation of $f=gh$ along path γ is equal to $e^{2\pi(a+b)}f$ and

   $$e^{2\pi(a+b)}f=f \;\Leftrightarrow\; a+b\in\mathbb{Z}.$$

   □

9. Prove that $Ln(z)Ln(1+z^2)$ has single-valued branch in $D=\mathbb{C}\backslash(-i\infty,i]$.

   Proof. Lnz has holomorphic branch in D since D is simply connected and $0\notin D$.

   Let $g(z)=1+z^2$. Then $g(D)=\mathbb{C}\backslash(-\infty,0]$. Consequently, Lnz has holomorphic branch in $g(D)$ which composition will be holomorphic branch of $Ln(1+z^2)$. Multiplying we obtain result. □