Real Analysis 2024. Homework 6.

1. Consider a space $(\mathbb{R}^2, \mathcal{B}_2, \mu_2)$ and a function $f(x,y) = \frac{1}{|x|+|y|}$ prove that $f \in L^1([-1,1]^2)$.

Proof. Consider

$$E_k = \left\{ (x,y) \in [-1,1]^2 : \frac{1}{k+1} < |x|\,, |y| \le \frac{1}{k} \right\}.$$

Then

$$\int_{E_k} \frac{dxdy}{|x|+|y|} \le 2(k+1)\mu_2(E_k) \le \frac{8(k+1)}{k^2(k+1)^2} \le \frac{8}{k^3}.$$

Hence,

$$\int_{[-1,1]^2} \frac{dxdy}{|x|+|y|} \le \sum_{k=1}^{\infty} \frac{1}{k^3} < +\infty.$$

□

2. Consider $\mathbb{N}$ with counting measure. Describe space $L^p(\mathbb{N})$.

$$L^p(\mathbb{N}) = \left\{ \{c_k\}_{k=1}^{\infty} : \left\|\{c_k\}\right\|_p = \left(\sum_{k=1}^{\infty} |c_k|^p < \infty \right)^{1/p} \right\}.$$

3. Prove Hölder's inequality for simple functions using Hölder's inequality for finite sums. Apply theorem on monotone sequence to obtain Hölder's inequality for nonnegative measurable functions.

Proof. Suppose $\frac{1}{p} + \frac{1}{q} = 1$. Let Let

$$f = \sum_{k=1}^{n} a_k \chi_{A_k}, \quad g = \sum_{j=1}^{m} b_j \chi_{B_j},$$

where

$$X = \bigcup_{k=1}^{n} A_k = \bigcup_{j=1}^{m} B_j;$$

$$A_{k_1} \cap A_{k_2} = \varnothing,\ k_1 \ne k_2;\ \ B_{j_1} \cap B_{j_2} = \varnothing,\ j_1 \ne j_2.$$

Consequently,

$$fg = \sum_{\substack{1\le k\le n\\ 1\le j\le m}} a_k b_j \chi_{A_k\cap B_j}$$

and

$$\int_X |fg|\, d\mu = \sum_{\substack{1\le k\le n\\ 1\le j\le m}} |a_k b_j|\, \mu(A_k \cap B_j) = \sum_{\substack{1\le k\le n\\ 1\le j\le m}} |a_k|\, \mu(A_k\cap B_j)^{1/p}\, |b_j|\, \mu(A_k\cap B_j)^{1/q} \le$$

$$\left(\sum_{\substack{1\le k\le n\\ 1\le j\le m}} |a_k|^p\, \mu(A_k\cap B_j)\right)^{1/p} \left(\sum_{\substack{1\le k\le n\\ 1\le j\le m}} |b_j|^q\, \mu(A_k\cap B_j)\right)^{1/q} =$$

$$\left(\sum_{1\le k\le n} |a_k|^p\, \mu(A_k)\right)^{1/p} \left(\sum_{1\le j\le m} |b_j|^q\, \mu(B_j)\right)^{1/q} = \left(\int_X |f|^p\, d\mu\right)^{1/p} \left(\int_X |g|^q\, d\mu\right)^{1/q}.$$

If $f \in L^p(X,\mu)$ and $g \in L^q(X,\mu)$ consider increasing sequences of simple functions φ_n, ψ_n such that

$$\varphi_n(x) \le \varphi_{n+1}(x),\ f(x) = \lim_{n\to+\infty} \varphi_n(x), \quad x\in X;$$

$$\psi_n(x) \le \psi_{n+1}(x),\ g(x) = \lim_{n\to+\infty} \psi_n(x), \quad x\in X.$$

Then

$$\varphi_n(x)\psi_n(x) \le \varphi_{n+1}(x)\psi_{n+1}(x),\ fg(x) = \lim_{n\to+\infty} \varphi_n(x)\psi_n(x), \quad x\in X;$$

and by the monotone convergence theorem we have

$$\int_X |fg|\, d\mu = \lim_{n\to+\infty} \int_X \varphi_n\psi_n d\mu \le$$

$$\lim_{n\to+\infty} \left(\int_X |\varphi_n|^p\, d\mu\right)^{1/p} \left(\int_X |\psi_n|^q\, d\mu\right)^{1/q} =$$

$$\left(\int_X |f|^p\, d\mu\right)^{1/p} \left(\int_X |g|^q\, d\mu\right)^{1/q}.$$

□

4. Prove the General Lebesgue Dominated Convergence Theorem. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g_n and dominates $\{f_n\}$ on E in the sense that

$$|f_n| \leq g_n \text{ a.e. on } E \text{ for all } n.$$

Prove that if

$$\lim_{n\to\infty} \int_E g_n = \int_E g < \infty,$$

then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. The following assertions hold simultaneously a.e.

(a) $|f_n(x)| \leqslant g_n(x)$ for every $n \in \mathbb{N}$;

(b) $f_n(x) \to f(x)$.

Considering the limit in the first one we see that $|f(x)| \leqslant g(x)$ for a.e. $x \in E$. Consequently, $f_n, f \in L(E, \mu)$ and by the Chebyshev inequality functions $f_n, f, g_n g$ are finite a.e. Noticing that $g_n + f_n \geqslant 0$ a.e. and applying Fatou's theorem we obtain

$$\begin{aligned}\int_E g + \int_E f = \int_E (g+f) \leqslant \underline{\lim} \int_E (g_n + f_n) = \\ = \underline{\lim} \left(\int_E g_n + \int_E f_n \right) = \int_E g + \underline{\lim} \int_E f_n.\end{aligned}$$

Hence

$$\int_E f \leqslant \underline{\lim} \int_E f_n \tag{1}$$

Analogously,

$$\begin{aligned}\int_E g - \int_E f = \int_E (g-f) \leqslant \underline{\lim} \int_E (g_n - f_n) = \\ = \int_E g + \underline{\lim} \left(- \int_E f_n \right) = \int_E \Phi - \overline{\lim} \int_E f_n.\end{aligned}$$

Consequently,

$$\overline{\lim} \int_E f_n \leqslant \int_E f. \tag{2}$$

Considering estimates (1) and (2) we see that

$$\int_E f \leqslant \underline{\lim} \int_E f_n \leqslant \overline{\lim} \int_E f_n \leqslant \int_E f.$$

This implies the existence of the limit $\lim \int_E f_n$ and that this limit is equal to the integral $\int_E f$. □

5. Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

Proof. Notice that

$$\left| \int_E |f| - \int_E |f_n| \right| \leq \int_E ||f| - |f_n|| \leq \int_E |f - f_n|$$

which proves that if $\int_E |f - f_n| \to 0$ then $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$.

Assume that $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. Let $g_n = |f_n| + |f|$. Then

$$|f_n - f| \leq |f_n| + |f|,$$

while

$$\int_E (|f_n| + |f|) \to 2 \int_E |f|.$$

Consequently, by Lebesgue dominated convergence theorem

$$\int_E |f_n - f| \to 0$$

since $|f_n - f| \to 0$ a.e. on E. □

6. For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? Is

it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

Solution. Both are not true. Consider a function

$$f(x) = \begin{cases} 1, & x \in [n, n+1/2),\ n \in \mathbb{N}; \\ -1, & x \in [n+1/2, n+1),\ n \in \mathbb{N}. \end{cases}$$

Then $a_n = 0$. The series converges absolutely and while

$$\int\limits_0^{+\infty} |f|\, dx = +\infty.$$