HW3. Version 5

Find the largest and smallest values of the function $z = 5xy - 4$ if the variables x *and* y are positive and satisfy the coupling equation $\dfrac{x^2}{8} + \dfrac{y^2}{2} - 1 = 0$.

Solution: denote $u(x, y, \lambda) = 5xy - 4 + \lambda\left(\dfrac{x^2}{8} + \dfrac{y^2}{2} - 1\right) = 0$

$$\begin{cases} \dfrac{\partial u}{\partial x} = 0 \\[2mm] \dfrac{\partial u}{\partial y} = 0 \\[2mm] \dfrac{x^2}{8} + \dfrac{y^2}{2} - 1 = 0 \end{cases} \Rightarrow \begin{cases} 5y + \dfrac{\lambda x}{4} = 0 \\[2mm] 5x + \lambda y = 0 \\[2mm] \dfrac{x^2}{8} + \dfrac{y^2}{2} - 1 = 0 \end{cases}$$

first two equation has nontrival solution iff

$$\begin{vmatrix} \dfrac{\lambda}{4} & 5 \\[2mm] 5 & \lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda = \pm 10.$$

(trivial solution do not satisfy the 3-rd equation.)

① $\lambda_1 = 10$. $x = -2y$. we have stationary points $(-2, 1)$ $(2, -1)$

② $\lambda_2 = -10$ $x = 2y$ we have stationary points $(-2, -1)$ $(2, 1)$

$$d^2u = \dfrac{\lambda}{4} dx^2 + 10\, dx\, dy + \lambda\, dy^2$$

$$d\left(\dfrac{x^2}{8} + \dfrac{y^2}{2} - 1\right) = \dfrac{x\, dx}{4} + y\, dy = 0. \qquad \Rightarrow dy = -\dfrac{x}{4y} dx.$$

① $\lambda_1 = 10$ $d^2u = \dfrac{5}{2} dx^2 + 10\, dx \cdot \dfrac{-xy}{-4y} dx + 10\left(\dfrac{1}{2} dx\right)^2 = 10\, dx^2 > 0.$

② $\lambda_2 = -10$ $d^2u = -\dfrac{5}{2} dx + 10\, dx \cdot \left(-\dfrac{1}{2} dx\right) + 10\left(-\dfrac{1}{2} dx\right)^2 = -5\, dx^2 < 0.$

$(-2, 1)$ $(2, -1)$ is strictly conditional minimum

$(-2, -1)$ $(-2, -1)$ is strictly conditional maximum

and since x, y are positive.

$z_{max} = 6$. where $\begin{cases} x = 2 \\ y = 1 \end{cases}$ z_{min} not exist

HW4. Version 5.

Calculate: $\displaystyle\lim_{x \to 0} \dfrac{e^x \sin x - x(1+x)}{x^3}$

Solution: $\displaystyle\lim_{x \to 0} \dfrac{e^x \sin x - x(1+x)}{x^3}$

$= \displaystyle\lim_{x \to 0} \dfrac{\left(1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + o(x^3)\right)\left(x - \dfrac{x^3}{6} + o(x^3)\right) - x(1+x)}{x^3}$

$= \displaystyle\lim_{x \to 0} \dfrac{x - \dfrac{x^3}{6} + x^2 + \dfrac{x^3}{2} - x(1+x) + o(x^3)}{x^3} = \dfrac{1}{3}$