Combinatorics

Lecture 1

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Combinatorics – a section of mathematics in which one studies questions about how many different configurations (combinations), subject to certain conditions, you can compose from given objects.

The main problem of combinatorics is counting the number elements in a finite set.

Three principles of combinatorics

When counting the number of different combinations in combinatorics the following two basic rules are used.

Multiplication principle: If object A can be selected in m various ways and after each of these choices object B can be selected by n different ways, then the selection of two objects A and B in the specified order can be done in mn ways.

Addition principle: If object A can be selected in m various ways, and object B can be selected in other n various ways (provided that the simultaneous choice of A and B is impossible), then A or B can be done in $m + n$ ways.

Pigeonhole principle: If n objects are distributed over m places, and if $n > m$, then some place receives at least two objects.

Examples.

Example 1. If your closet contains 3 hats, 2 coats and 2 scarves. Assuming you are comfortable with wearing any combination of hat, coat and scarf, (and you need a hat, coat and scarf today), how many different outfit could you select from your closet?
Answer: $3 \cdot 2 \cdot 2 = 12$

Example 2. How many License plates, consisting of 2 letters from the English alphabet followed by 4 digits are possible?
Answer: $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6760000$

Example 3. A group of 5 boys and 3 girls is to be photographed. How many ways can they be arranged in one row?
Answer: There are 8 people so there are
$8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$

Examples

Example 4. How many different 4 letter words (including nonsense words) can you make from the letters of the word "MATHEMATICS" if

(a) letters cannot be repeated. Answer: "MATHEMATICS" has 8 distinct letters. Hence the answer is $8 \cdot 7 \cdot 6 \cdot 5 = 1680$

(b) letters can be repeated. There are still only 8 distinct letters so the answer is $8^4 = 4096$

(c) letters cannot be repeated and the word must start with a vowel. Answer: ???

Examples

Example 5. Let A and B be two disjoint sets. What is $|A \cup B|$?
Answer: By the addition principle, $|A| + |B|$

Example 6. Let A and B denote two finite sets; show that
$|A \setminus B| = |A| - |A \cap B|$
Answer: The key observation is that $A \setminus B$ and $A \cap B$ are disjoint. Further, the union of $A \setminus B$ and $A \cap B$ is A. Therefore, by the addition principle, $|A| = |A \setminus B| + |A \cap B|$

Example 7. How many 4 digit numbers begin with a 4 or a 5?
Answer: Using the multiplication principle, the number of 4 digit numbers which begin with 4 is 1000. Likewise, the number of 4 digit numbers which begin with 5 is 1000. The set of 4 digit numbers which begin with 4 and the set of 4 digit numbers which begin with 5 are disjoint. Thus, the number of 4 digit numbers which begin with a 4 or a 5 is $1000 + 1000 = 2000$, using the addition principle.

Examples

Example 8. Show that if 51 positive integers between 1 and 100 are chosen, then one of them must divide the other.

Answer: Let $n_1, n_2, \ldots, n_{51}$ denote the chosen numbers. Every number can be expressed as a product of prime numbers. Therefore, each $n_i = 2^{k_i} \cdot b_i$ where b_i is some odd number, such that $1 \leq b_i \leq 99$. There are exactly 50 odd numbers between 1 and 99. Therefore, $b_i = b_j$, for some pair (n_i, n_j) (pigeonhole principle). In other words, we must have $n_i = 2^{k_i} \cdot b_i$ and $n_j = 2^{k_j} \cdot b_j$ for some i and j. Depending on whether $k_i \geq k_j$ or vice versa, one of n_i and n_j must divide the other.

Mappings and partial permutations

Let sets X and Y be given, and the set X contains n elements ($|X| = n$), and the set Y contains m elements ($|Y| = m$). Under these conditions, the problem can be formulated as follows: how many mappings are there that satisfy some conditions? Note also that each such mapping f can be associated with the "word" $f(x_1), \ldots, f(x_n)$ in the alphabet of m "letters". We obtain an equivalent formulation of our problem: count the number of words in the alphabet that satisfy the given conditions.

Lemma 1.

If $|X| = n$ and $|Y| = m$, then the number of all functions $f : X \to Y$ is equal to m^n.

Proof. Let $X = \{1, \ldots, n\}$, $Y = \{1, \ldots, m\}$, then each function can be identified with the sequence $f(1), \ldots, f(n)$. So, each element $f(i)$ can be selected in m ways, which gives exactly m^n possibilities (multiplication principle!).$\blacksquare$

Definitions.

- A function $f : X \to Y$ is surjective if and only if for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$ (i.e., f(X) = Y).
- A function $f : X \to Y$ is injective if and only if whenever $f(x) = f(y)$, $x = y$.
- A function $f : X \to Y$ is bijective if it is a one-to-one correspondence between those sets, in other words both injective and surjective.
- If $x \in \mathbb{R}$, then we denote $[x]_n = x(x-1)\dots(x-n+1)$.

Definition. Let $f : X \to Y$. A function $g : Y \to X$ is the inverse of f if $f \circ g = 1_Y$ and $g \circ f = 1_X$ (such function g is denoted as f^{-1}).

Proposition 2.

A function $f : X \to Y$ is bijective if and only if it is invertible.

Proof. $\Rightarrow$: Let $f : X \to Y$ be bijective. We will define a function $f^{-1} : Y \to X$ as follows. Let $y \in Y$; since f is surjective, there exists $x \in X$ such that $f(x) = y$. Let $f^{-1}(y) = x$. Since f is injective, this a is unique, so f^{-1} is well-defined.

Now we much check that f^{-1} is the inverse of f . First we will show that $f^{-1} \circ f = 1_X$. Let $x \in X$. Let $y = f(x)$. Then, by definition, $f^{-1}(y) = x$. Then $f^{-1} \circ f(x) = f^{-1}(y) = x$.

Now we will show that $f \circ f^{-1} = 1_Y$. Let $y \in Y$. Let $x = f^{-1}(y)$. Then, by definition, $f(x) = y$. Then $f \circ f^{-1}(y) = f(x) = y$.

$\Leftarrow$: First, we will show that f is surjective. Suppose $y \in Y$. Let $x = f^{-1}(y)$. Then $f(x) = f \circ f^{-1}(y) = 1_Y(y) = y$. So f is surjective.

Now, we will show that f is injective. Let $x_1, x_2 \in X$ be such that $f(x_1) = f(x_2)$. We will show that $x_1 = x_2$. Let $y = f(x_1)$ and $x = f^{-1}(y)$. Then $x_2 = f^{-1} \circ f(x_2) = f^{-1}(y) = x$.

But at the same time, $x_1 = f^{-1} \circ f(x_1) = f^{-1}(y) = x$. ∎

Lemma 3.

The total number of injective mappings from a finite set X with n elements to set Y with m elements is $[m]_n$.
(Equivalent statement. Number of words of length n without repetition of letters in the alphabet with m letters is $[m]_n$).

Proof. We want to find the number of sequences $y_1, \ldots, y_n$ with distinct elements. Element y_1 can be chosen in m ways, y_2 can be chosen in $m - 1$ ways and so on. Thus, in total we get $[m]_n$ such sequences. ■

Combinations, partial permutations and permutations

Let $A = \{a_1, \dots, a_n\}$ be a set of n objects.

Definitions.

- An arbitrary ordered set (tuple) of k elements of a given set, among which there may be duplicates, is called a partial permutation of length k from n-set with repetition (denoted as $\tilde{A}_n^k$).
- Accordingly, if the elements cannot be repeated, then the set is called an partial permutation of length k from n-set (denoted as A_n^k). Extremely important particular case $k = n$, such partial permutationss are called permutations.
- A combination of n elements by k is a set of k elements of this set. Sets that differ only in the order of elements are considered the same (denoted as C_n^k or $\binom{n}{k}$).
- Combinations with repetitions are defined similarly (denoted as $\tilde{C}_n^k$).

Example. Let $A = \{1, 2, 3, 4\}$, and let $k = 2$.
Partial permutations with repetitions:
$(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2),$
$(3,3), (3,4), (4,1), (4,2), (4,3), (4,4)$. So, $\tilde{A}_4^2 = 16$.

Partial permutations: $(1,2), (1,3), (1,4), (2,1), (2,3),$
$(2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)$. So, $A_4^2 = 12$.

Combinations: $(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)$, and $C_4^2 = 6$.

Combinations with repetitions:
$(1,2), (1,3), (1,4), (2,3), (2,4), (3,4), (1,1), (2,2), (3,3), (4,4)$ and $\tilde{C}_4^2 = 10$.

Proposition 4.

a.) $\tilde{A}_n^k = n^k$.
b.) $A_n^k = \frac{n!}{(n-k)!}$.
c.) $C_n^k = \frac{n!}{k!(n-k)!}$.
d.) $\tilde{C}_n^k = C_{n+k-1}^k$.

Proof. We already know the first two equalities.
From the initial set of objects $\{a_1, \ldots, a_n\}$, we form k combinations: the first combination is $\{a_1, \ldots, a_k\}$, etc., the last combination is $\{a_{n-k+1}, \ldots, a_n\}$ The number of the last combination according to the definition is C_n^k. Now let's order each of the obtained combinations in $k!$ ways and get the partial permutations. Therefore, the number of partial permutations is on the one hand A_n^k and on the other side – $k!C_n^k$, whence $C_n^k = \frac{A_n^k}{k!} = \frac{n!}{k!(n-k)!}$.

Let us show the fourth equality. Fix some combination with repetitions $(a_1, \dots, a_n)$. We now construct a sequence of 0 and 1 according to the following rule: first, write 1 as many times as the element a_1 occurs in combination. After that, we put 0 and write 1 as many times as a_2 occurs in combination and so on. At the end of the sequence, there will be 1 as many times as a_n occurs in the combination and we will not put 0 at the end. It is easy to understand that according to the described algorithm, two identical sequences of 0 and 1 are obtained only for the same combinations. Note that in each such sequence exactly k ones and $n-1$ zeroes. Hence, the mapping described above is a bijection between k-combinations with repetitions and sequences of 0 and 1 of length $n+k-1$ and containing exactly k ones. There are exactly C_{n+k-1}^{k} such sequences.■

Reinterpretation: Balls in urns

There is another way to look at the main result of the previous section. Suppose that we have n urns $U_1, \ldots, U_n$. We have k indistinguishable balls. How many ways can we put the balls in the urns?

If x_i is the number of balls we put into the ith urn, then $x_1, \ldots, x_n$ are non-negative integers which add up to k. So the number of ways of putting the balls into the urns is $C^k_{n+k-1} = \binom{n+k-1}{k}$.

The conditions can be varied in many ways. Suppose, for example, that we have to distribute k balls among n urns as above, but with the requirement that no urn should be empty. This asks that $x_i \geq 1$ for all i. If we define new variables $y_1, \ldots, y_n$ by $y_i = x_i - 1$, then the sum of the y's is $k - n$; so the number of choices of the y's is

$$\binom{n + (k-n) - 1}{k-n} = \binom{k-1}{k-n}$$

The simple way to think about this is: suppose each urn is to be non-empty. Then we first take n balls and put one in each urn. Then we distribute the remaining $k-n$ balls into the urns in any way. This gives the same result as above.

Example. How many ways can I distribute 100 sweets to a class of 30 boys and 20 girls, if it is required that each boy has at least one sweet and each girl has at least two sweets?
Solution: To solve this, I first give one sweet to each boy and two to each girl, using up $30+2\cdot 20=70$ sweets. Then I distribute the remaining 30 sweets among the 50 children, which can be done in

$$\binom{30+50-1}{30}=\binom{79}{30}$$

Proposition 5.

i) $C_n^k = C_n^{n-k}$
ii) $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$
iii) $C_n^0 + C_n^1 + \ldots + C_n^n = 2^n$
iv) $(C_n^0)^2 + (C_n^1)^2 + \ldots + (C_n^n)^2 = C_{2n}^n$

Proof. i) and ii) obviously follow from the formula above.

For iii) note that the number of all subsets of a set of n elements is 2^n. On the other hand, for each $k \leq n$ the number of $k-$element subsets is C_n^k.

Let us show the fourth equality. Consider the set $X = \{x_1, \ldots, x_n, \ldots, x_{2n}\}$. Total $n-$combinations is C_{2n}^n. On the other hand, consider $i \in \{0, 1, \ldots, n\}$, and all $n-$combinations, each of which contains exactly i objects from $a_1, \ldots, a_n$. The number of such combinations (for each i) is equal to $C_n^i C_n^{n-i}$ (multiplication rule!). Thus we get

$$C_{2n}^n = \sum_{i=0}^{n} C_n^i C_n^{n-i} = \sum_{i=0}^{n} (C_n^i)^2 \blacksquare$$

Theorem 6 (Binomial formula).

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$$

Proof. $(x+y)^n = \underbrace{(x+y)\dots(x+y)}_{n}$ and from each bracket we must take either x or y. Let x be taken from k brackets, then y is taken from $n-k$ brackets. When we change these variables, we get a term of the form $x^k y^{n-k}$. How many of these terms will there be? Exactly as many times as we take concrete k brackets (from which we take x) from n possible. This can be done in C_n^k ways. Summing over k, we obtain the required equality. ■

Let's get back to permutations. We will consider permutations as bijections $\{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$. The set of all permutations will be denoted as S_n, also note that $|S_n| = n!$. In fact, permutations form a group (under composition).

We're going to introduce a more efficient way of writing permutations. This involves thinking about a special kind of permutation called a cycle. Let $m > 0$, let $a_0, a_1, \ldots, a_{m-1}$ be distinct positive integers. Then $\tau := (a_0, a_1, \ldots, a_{m-1})$ is defined to be the permutation (in S_n) such that

- $\tau(a_i) = a_{i+1}$ for $i < m - 1$
- $\tau(a_{m-1}) = a_0$
- $\tau(x) = x$ for any number x which isn't equal to one of the a_i.

A permutation of the form $(a_0, a_1, \ldots, a_{m-1})$ is called an m-cycle. A permutation which is an m-cycle for some is called a cycle.

Examples. (i) In S_3, the 2-cycle $(1,2)$ is the permutation that sends 1 to 2, 2 to 1, and 3 to 3.
(ii) In S_4, the 3-cycle is the permutation that sends 1 to 1, 2 to 4, 4 to 3, and 3 to 2.
(iii) In S_5, $(3,2,5) = (5,3,2) = (2,5,3)$.

In general, every $m-$cycle can be written m different ways since you can put any one of the m things in the cycle first.

Definition. Two cycles $(a_0, a_1, \ldots, a_{m-1})$ and $(b_0, b_1, \ldots, b_{k-1})$ are disjoint if no a_i equals any b_j .

Example. a) $(1,2,7)$ is disjoint from $(5,4)$.
b.) $(1,2,3)$ and $(3,5)$ are not disjoint.

One reason disjoint cycles are important is that disjoint cycles commute, that is,
if σ and τ are disjoint cycles then, $\sigma \circ \tau = \tau \circ \sigma$

For every permutation π there is an inverse permutation π^{-1} such that $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = Id$. How do we find the inverse of a cycle? Let $\sigma = (a_0, a_1, \ldots, a_m)$, then σ^{-1} sends a_i to a_{i+1} for all i (and every number not equal to an a_i to itself).
In other words, σ^{-1} is the cycle $(a_{m-1}, a_{m-2}, \ldots, a_0)$.

Lemma 7.

Let $a_0, \ldots, a_m$ be distinct numbers. Then

$$(a_0, a_1)(a_1, a_2, \ldots, a_m) = (a_0, a_1, \ldots, a_m)$$

Proof. Let $\gamma = (a_0, a_1, \ldots, a_m)$, $\delta = (a_0, a_1)$, $\epsilon = (a_1, a_2, \ldots, a_m)$ so that we have to show $\gamma(x) = \delta(\epsilon(x))$ for all x. This is verified by an elementary direct calculation. ■

Theorem 8.

Every $\sigma \in S_n$ equals a product of disjoint cycles.

Proof. By induction on n. It is certainly true for $n = 1$ and $n = 2$. Now let $\sigma \in S_n$ and suppose that every permutation in S_{n-1} is a product of disjoint cycles. If $\sigma(n) = n$, then we can consider σ as a permutation of $\{1, 2, \ldots, n-1\}$, so it equals a product of disjoint cycles by the inductive hypothesis. If $\sigma(n) = k$, then consider the permutation $\tau = (n, k) \circ \sigma$.
$\tau(n) = n$, so we can consider τ as a permutation in S_{n-1} and therefore by induction we can write τ as a product of disjoint cycles

$$\tau = c_1 \ldots c_r$$

where the cycles $c_1, \ldots, c_r$ only contain the numbers $1, 2, \ldots, n-1$.

Nextly, $(n,k)(n,k)\sigma = (n,k)c_1 \dots c_r$, so

$$\sigma = (n,k)c_1 \dots c_r$$

None of the cycles c_i contain n . If none of them contain k then this is an expression for σ as a product of disjoint cycles, so we are done. If one of them contains k, then because disjoint cycles commute we can assume that it is c_1. Recall that we can write c_1 starting with any one of its elements. We choose to write it starting with k, so that for some numbers $a_1, \dots, a_m$

$$c_1 = (k, a_1, \dots, a_m)$$

By Lemma 7,

$$(n,k)c_1 = (n, k, a_1, \dots, a_m)$$

and therefore $\sigma = (n, k, a_1, \dots, a_m)c_2 c_3 \dots c_r$ This is a product of disjoint cycles since neither k nor n belongs to any of $c_2, \dots, c_r$. , so we are done.■

Example.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

$\sigma = (1, 7, 4)(2, 6, 5)(3)$ Since any 1-cycle is equal to the identity

function, and because 1-cycles look confusingly like what we write when we evaluate a function, we usually omit 1-cycles like (3) from disjoint cycle decompositions, so we'd write the permutation σ of the previous example as $(1, 7, 4)(2, 6, 5)$.

Definition. A transposition is a 2-cycle. (i.e., of the form (i, j) .)

Lemma 9.

Every cycle equals a product of transpositions.

Proof. Obvious. ■

Theorem 10.

Every permutation in S_n is equal to a product of transpositions.

Proof. Immediately from Theorem 8 and Lemma 9. ■

Example.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 7 & 1 & 3 & 2 & 6 & 4 & 5 \end{pmatrix}$$

$$\sigma = (1,8,4)(2,9,5,3,7,6) = (1,8)(8,4)(2,9)(9,5)(5,3)(3,7)(7,6)$$

Exercises

Exercise 1. A student has to answer 10 questions, choosing at least 4 from each of Parts A and B. If there are 6 questions in Part A and 7 in Part B, in how many ways can the student choose 10 questions?

Exercise 2. Suppose m men and n women are to be seated in a row so that no two women sit together. If $m > n$, show that the number of ways in which they can be seated is

$$\frac{m!(m+1)!}{(m-n+1)!}$$

Exercise 3. Find the number of permutations of n different things taken r at a time such that two specific things occur together.

Exercises

Exercise 4. Let $|A| = n$.
(a) How many reflexive binary relations on A?
(b) How many irreflexive binary relations on A?
(c) How many symmetric binary relations?
(d) How many antisymmetric binary relations ?
(e) How many asymmetric binary relations ?

Exercise 5. (i) Count the number of permutations of S_9 whose disjoint cycle decompositions have exactly one cycle.
(ii) Count the number of permutations of S_7 whose disjoint cycle decompositions consist of two cycles, one of them of length 3.