Linear transformations of figures

1 Rotation matrix in space. Euler angles

Suppose we are given with coordinate system O, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. Basis is right orthonormal.

Suppose coordinate system was moved into arbitrary point in space and rotated around arbitrary axis with unit direction vector $\boldsymbol{v} \mapsto (x, y, z)$.

Most effective approach is to overlook such arbitrary axis and expand whole rotation into elementary planar rotations corresponding with coordinate axis

1.1 Rotation by coordinate axes

By axis Ox on angle α:

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

By axis Oy on angle β::

$$R_y = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

By axis Oz on angle γ:

$$R_z = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix of whole rotation:

$$R = R_x R_y R_z = \begin{pmatrix} \cos\beta\cos\gamma & -\sin\gamma\cos\beta & \sin\beta \\ \sin\alpha\sin\beta\cos\gamma + \sin\gamma\cos\alpha & -\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & -\sin\alpha\cos\beta \\ \sin\alpha\sin\gamma - \sin\beta\cos\alpha\cos\gamma & \sin\alpha\cos\gamma + \sin\beta\sin\gamma\cos\alpha & \cos\alpha\cos\beta \end{pmatrix}$$

1.2 Euler angles

Leonhard Euler purposed more clear procedure of rotation expansion:

1. Rotation around axis Oz with **precession angle** α, hence $Ox \mapsto ON$
2. Rotation around axis ON with **nutation angle** β, hence $Oz \mapsto OZ$
3. Rotation around axis OZ with **angle of rotation** γ, hence $ON \mapsto OX$

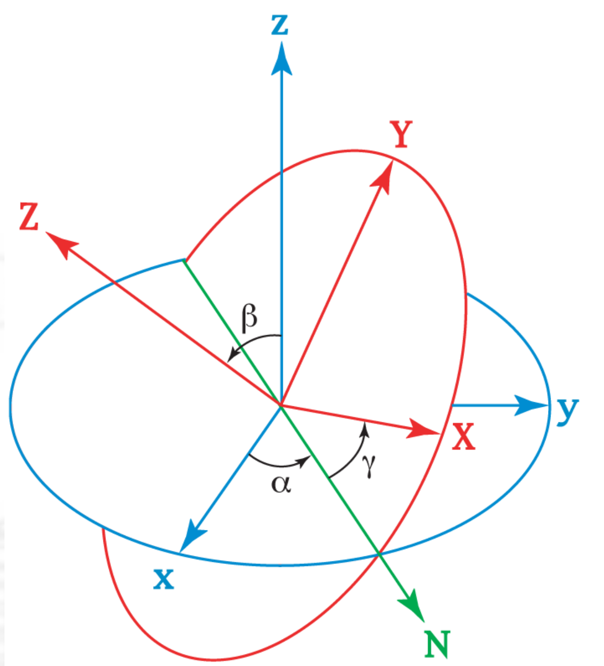

Figure 1: Rotation with Euler angles

Each rotation has expression as a plane rotation matrix for corresponding plane:

$$R_1 = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}$$

$$R_3 = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation matrix expressing whole transformation is now

$$R = R_1 R_2 R_3 = \begin{pmatrix} \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & -\cos\gamma\sin\alpha - \cos\alpha\cos\beta\sin\gamma & \sin\beta\sin\gamma \\ \cos\beta\cos\gamma\sin\alpha + \cos\alpha\sin\gamma & \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\gamma\sin\beta \\ \sin\alpha\sin\beta & \cos\alpha\sin\beta & \cos\beta \end{pmatrix}$$

Steps of this expansion are **non-commutative**

2 Orthogonality of rotation matrix

Definition. We call arbitrary matrix orthogonal if vectors shaped with its columns or rows are pairwise orthogonal unit vectors

Example for matrix 3×3:

$$A = \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \end{pmatrix}$$

$$\begin{aligned} |\boldsymbol{a}| = |\boldsymbol{b}| = |\boldsymbol{c}| = 1 \\ \boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{c} = 0 \end{aligned}$$

$$A = \begin{pmatrix} a'^x & b'^x & c'^x \\ a'^y & b'^y & c'^y \\ a'^z & b'^z & c'^z \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}' & \boldsymbol{b}' & \boldsymbol{c}' \end{pmatrix}$$

$$\begin{aligned} |\boldsymbol{a}'| = |\boldsymbol{b}'| = |\boldsymbol{c}'| = 1 \\ \boldsymbol{a}' \cdot \boldsymbol{b}' = \boldsymbol{a}' \cdot \boldsymbol{c}' = \boldsymbol{b}' \cdot \boldsymbol{c}' = 0 \end{aligned}$$

Lemma 2.1. *Inverse matrix of any orthogonal matrix is its transposition.*

Proof. See course of algebra ☐

Corollary 2.2. *Inverse of orthogonal matrix is orthogonal matrix too.*

For rotation matrix on plane:

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

For rows:

$$\begin{aligned} (\cos\varphi)^2 + (\sin\varphi)^2 = \cos^2\varphi + \sin^2\varphi = 1 \\ (-\sin\varphi)^2 + (\cos\varphi)^2 = \sin^2\varphi + \cos^2\varphi = 1 \\ (\cos\varphi) \cdot (\sin\varphi) + (-\sin\varphi) \cdot ((\cos\varphi)) = 0 \end{aligned}$$

For columns:

$$\begin{aligned} (\cos\varphi)^2 + (-\sin\varphi)^2 = \cos^2\varphi + \sin^2\varphi = 1 \\ (\sin\varphi)^2 + (\cos\varphi)^2 = \sin^2\varphi + \cos^2\varphi = 1 \\ (\cos\varphi) \cdot (-\sin\varphi) + (\sin\varphi) \cdot ((\cos\varphi)) = 0 \end{aligned}$$

For any form of space rotation matrix the same check may be done.
Mirror reflection corresponds with

3 Linear transformations of figures

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Suppose arbitrary coordinate system with right orthonormal basis rigidly assigned with arbitrary figure F. Such coordinates in physics are called **material** or **Lagrangian**. Now any deformation of the figure causes transformation of these coordinates and preserves coordinates of each point of figure. Therefore, we need to study transformation of these coordinates instead deformation and displacement of figure.

Coordinates system assigned to arbitrary point in space in this case called **space** or **Euler** coordinates.

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In general case deformation of figure generates **curvilinear material coordinates**.

Here we take a look on some important particular cases

Let figure F be transformed into figure F' with some transformation which preserves linearity of the coordinate axes, hence this transformation of coordinates is expressed with arbitrary transformation matrices S and $T = S^{-1}$ and transition vector $\boldsymbol{r}$.

In components this transformation has expression:

$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14} \\ y' = a_{21}x + a_{22}y + a_{23}z + a_{24} \\ z' = a_{31}x + a_{32}y + a_{33}z + a_{34} \end{cases} \tag{1}$$

Coefficients $a_{ij},\ i, j = 1, 2, 3$ correspond with components of transformation matrix, and coefficients $a_{i4},\ i = 1, 2, 3$ correspond with transition vector.

While rotation and mirror reflection correspond with orthogonal transformation matrix, this condition may be rewritten for them:

$$\begin{cases} \sum\limits_{i=1}^{3} a_{ii}^2 = 1 \\ \sum\limits_{i=1}^{3} a_{ij}^2 = 0, \quad j \neq i \end{cases} \tag{2}$$

Definition. Any linear transformation satisfying (2) is called **orthogonal transformation**.

Lemma 3.1. *Any transformation written in form (1) is orthogonal transformation if conditions (2) are satisfied for its coefficients.*

Proof. Orthogonal transformation is combination of motion and mirror reflection, hence distance between any two points of the figure is invariant against it.

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two distant points of the figure. After apply of transformation they become $A'(x_1', y_1', z_1')$ and $B(x_2', y_2', z_2')$.

Transformations of coordinates are:

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}y_1 + a_{13}z_1 + a_{14} \\ y_1' = a_{21}x_1 + a_{22}y_1 + a_{23}z_1 + a_{24} \\ z_1' = a_{31}x_1 + a_{32}y_1 + a_{33}z_1 + a_{34} \end{cases}$$

and

$$\begin{cases} x_2' = a_{11}x_2 + a_{12}y_2 + a_{13}z_2 + a_{14} \\ y_2' = a_{21}x_2 + a_{22}y_2 + a_{23}z_2 + a_{24} \\ z_2' = a_{31}x_2 + a_{32}y_2 + a_{33}z_2 + a_{34} \end{cases}$$

$$\begin{aligned} (d(A'B'))^2 = (x_1' - x_2')^2 + (y_1' - y_2')^2 + (z_1' - z_2')^2 = \\ = (a_{11}x_1 + a_{12}y_1 + a_{13}z_1 + a_{14} - a_{11}x_2 + a_{12}y_2 + a_{13}z_2 + a_{14})^2 + \\ +(a_{21}x_1 + a_{22}y_1 + a_{23}z_1 + a_{24} - a_{21}x_2 + a_{22}y_2 + a_{23}z_2 + a_{24})^2 + \\ +(a_{31}x_1 + a_{32}y_1 + a_{33}z_1 + a_{34} - a_{31}x_2 + a_{32}y_2 + a_{33}z_2 + a_{34})^2 = \\ = (a_{11}(x_1 - x_2) + a_{12}(y_1 - y_2) + a_{13}(z_1 - z_2))^2 + \\ +(a_{21}(x_1 - x_2) + a_{22}(y_1 - y_2) + a_{23}(z_1 - z_2))^2 + \\ +(a_{31}(x_1 - x_2) + a_{32}(y_1 - y_2) + a_{33}(z_1 - z_2))^2 = ... \end{aligned}$$

After expansion, and rearrange

$$... = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$$

$\square$

Key features of orthogonal transformation:

1. Identity transformation defined by formulas

$$\begin{cases} x' = x \\ y' = y \\ z' = z \end{cases}$$

   is orthogonal.

2. Any sequence of orthogonal transformations is orthogonal transformation.

3. Inverse of orthogonal transformation is orthogonal transformation

Definition. Distance between points is key invariant of orthogonal transformation.

4 Affine transformation

Definition. Any transformation (1) with Coefficients satisfying condition

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

called **affine transformation**.

Therefore, any honest transformation of basis with matrices S and $T = S^{-1}$ defines affine transformation

Key features of orthogonal transformation:

1. Identity transformation defined by formulas
$$\begin{cases} x' = x \\ y' = y \\ z' = z \end{cases}$$
is affine, hence it is transformation with unit matrix Id.

2. Any sequence of orthogonal transformations is affine transformation.

3. Inverse of orthogonal transformation is affine transformation, hence it is transformation with inverse matrix.

Lemma 4.1. *Int is necessary to build mapping between four distant not laying in a single plane points of figure and its mapping to define affine transformation in space explicitly.*

Proof. Affine transformation defined with 16 parameters, 4 parameters for each coordinate.

Mapping of coordinate x for these 4 points is
$$\begin{cases} x_1' = a_{11}x_1 + a_{12}y_1 + a_{13}z_1 + a_{14} \\ x_2' = a_{11}x_2 + a_{12}y_2 + a_{13}z_2 + a_{14} \\ x_3' = a_{11}x_3 + a_{12}y_3 + a_{13}z_3 + a_{14} \\ x_4' = a_{11}x_4 + a_{12}y_4 + a_{13}z_4 + a_{14} \end{cases}$$

Determinant of this system to be solved for a_{11}, a_{12}, a_{13} and a_{14} is
$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix},$$
and must be non-zero to grant solution.

It corresponds with left part of the equation of plane passing through three points and must be zero to satisfy coplanarity.

Hence, it is not zero for non-coplanar points. □

While, affine transformation establishes explicit correspondence, between original and image points, it never mereges several points into single point, and never splits single point into group of points.

Lemma 4.2. *Affine transformation maps plane into plane and line into line and preserves parallelism.*

Proof. Suppose σ is plane with equation

$$ax + by + cz + d = 0.$$

Substituting (2) into it and regrouping, we yield

$$a'x' + b'y' + c'z' + d' = 0,$$

and σ is part of this plane, say σ'.

Inverse transform all whole plane σ' yields original equation, hence transformation maps $\sigma \mapsto \sigma'$ and vice-versa □

While affine transformation for points is bijective, it maps distant planes into distant planes. As particular case, while parallel planes do not contain common points, images will not contain common points and will be parallel.

If we define line as crossing of two planes, distant lines must be mapped into distant lines.

Constructing parallel lines as a section of parallel planes with third common plane, we grant mapping of parallel lines into parallel lines.

5 Key invariant of affine transformation. Ratio of three points

Definition. Let A, B and C be three points on a line.

We call number

$$(ABC) = \frac{AB}{BC}$$

the **ratio of three points on line**

Lemma 5.1. *Ratio of three points is invariant against any affine transformation.*

Proof. While orthogonal transformation is a particular case of affine transformation, we can assume without any leak of generalization that line in question is axis Ox and its image is axis $O'x'$.

If it is not true, we apply auxiliary orthogonal transformations to grant it.

$$(ABC) = \frac{|x_A - x_B|}{|x_B - x_C|}$$

Image of each point on this line has coordinate $x' = a_{11}x + a_{14}$.

$$(A'B'C') = \frac{|x'_{A'} - x'_{B'}|}{|x'_{B'} - x'_{C'}|} = \frac{|a_{11}x_A + a_{14} - a_{11}x_B + a_{14}|}{a_{11}x_B + a_{14} - a_{11}x_C + a_{14}} = \frac{|a_{11}||x_A - x_B|}{|a_{11}||x_B - x_C|} = \frac{|x_A - x_B|}{|x_B - x_C|} = (ABC)$$

□

6 Projective transformation

Definition. Projective transformation is a transformation defined with formulas

$$\begin{cases} x' = \dfrac{a_{11}x + a_{12}y + a_{13}z + a_{14}}{a_{41}x + a_{42}y + a_{43}z + a_{44}} \\ y' = \dfrac{a_{21}x + a_{22}y + a_{23}z + a_{24}}{a_{41}x + a_{42}y + a_{43}z + a_{44}} \\ z' = \dfrac{a_{31}x + a_{32}y + a_{33}z + a_{34}}{a_{41}x + a_{42}y + a_{43}z + a_{44}}, \end{cases} \tag{3}$$

while coefficients satisfy condition:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \neq 0$$

Discussed above affine transformation is particular case of projective transformation with $a_{41} = a_{42} = a_{43} = 0$, and $a_{44} = 1$.

Condition of non-zero denominator, yields that transformed figure must not intersect planes

$$\sigma_\infty : \quad a_{41}x + a_{42}y + a_{43}z + a_{44} = 0$$

All three key features of affine transformation are preserved:

1. Identity transformation defined by formulas

$$\begin{cases} x' = x \\ y' = y \\ z' = z \end{cases}$$

   is projective.

2. Any sequence of projective transformations is projective transformation.

3. Inverse of projective transformation is projective transformation

Projective transformation is also explicit and transforms lines into lines

Definition. Suppose A, B, C and D are points laying on a line, and $\boldsymbol{e}$ is non-zero vector, not perpendicular with line. Cross ratio of this point is a number.

$$(ABCD) = \frac{\boldsymbol{e} \cdot \overrightarrow{AC}}{\boldsymbol{e} \cdot \overrightarrow{BC}} : \frac{\boldsymbol{e} \cdot \overrightarrow{AB}}{\boldsymbol{e} \cdot \overrightarrow{BD}}$$

Any basis vector may be treated as $\boldsymbol{e}$ if it is not perpendicular with line.

As particular case we take $\boldsymbol{e} = \boldsymbol{e}_1$, therefore

$$(ABCD) = \frac{x_C - x_A}{x_C - x_B} : \frac{x_D - x_A}{x_D - x_B}$$

Application of auxiliary orthogonal transformations grants that both original and image points lay on such line without any leak of generalization

Lemma 6.1. *Cross ratio is invariant against any projective transformation.*

Proof. While orthogonal transformation is a particular case of affine transformation, we can assume without any leak of generalization that line in question is axis Ox and its image is axis $O'x'$.

If it is not true, we apply auxiliary orthogonal transformations to grant it.

x coordinate now has transformation

$$x' = \frac{a_{11}x + a_{14}}{a_{44}x + a_{44}}$$

Substituting this identity into formula, we yield desired result. □

Theorem 6.2. *Any planar figure F' on a plane α which is an image of application of any non-affine projective transformation to figure F laying on the same plane is an image of central projection with arbitrary center C of a figure F''equal with F.*

And any image of such projection is image of projective transformation.

Suppose without leak of generalization that plane α coincide with xOy.

Let $A(x, y, 0)$ be point of F', and $A''(xx'', y'', z'')$ is corresponding point of F'', $S(x_0, y_0, z_0)$ is a center of projection, and $A'(x', y', 0)$.

Central projection means that S, A' and A'' lay on a single line:

$$\frac{x' - x_0}{x'' - x_0} = \frac{y' - y_0}{y'' - y_0} = \frac{-z_0}{z'' - z_0}.$$

Hence,

$$x' = \frac{-z_o x'' + x_0 z''}{z'' - z_0}, x' = \frac{-z_o y'' + y_0 z''}{z'' - z_0}$$

Transformation of F'' into F is orthogonal, therefore there is projective transformation F into F'.

7 Homogeneous coordinates

Definition. Homogeneous coordinates of a point on plane is triplet of points x_1, x_2, x_3, at least one is non-zero, having correspondence with Cartesian coordinates

$$x = \frac{x_1}{x_3} \quad y = \frac{y_1}{x_3}$$

Remark. For any non-zero ρ homogeneous coordinates ρx_1, ρ_2, ρx_3 define the same point as x_1, x_2, x_3.

Any line on plane has in Cartesian coordinates representation (basis is right orthogonal):

$$a_1 x + a_2 y + a_3 = 0 \quad (a_1^2 + a_2^2 \neq 0)$$

In homogeneous coordinates we write:

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

For any point on plane $A(x, y)$ we can assign homogeneous coordinates $(x, y, 1)$.
But for triplet $(x_1, x_2, 0)$ there is no corresponding point on plane.
Hence, we **compliment** line with infinity-distant point corresponding with such triplet

Definition. On complimented plane equation

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0$$

govern arbitrary line.

Equation

$$a_3 x_3 = 0$$

govern infinity distant line

While system

$$\begin{cases} a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 \\ b_1 x_1 + b_2 x_2 + b_3 x_3 = 0 \end{cases}$$

must have non-trivial solution, any pair of lines on this complimented plane are crossing.

Parallel lines are crossing in infinity-distant point.

Projective transformation on plane written in homogeneous coordinates has form:

$$\begin{aligned} x_1' &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\ x_2' &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\ x_3' &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3 \end{aligned}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

Natural form for non-complimented plane may be yielded with division of first and second line with third for any non-infinite point.

In space we define homogeneous coordinates as quadruplet x_1, x_2, x_3, x_4:

$$x = \frac{x_1}{x_4} \quad y = \frac{x_2}{x_4} \quad z = \frac{z_1}{x_4}$$

In this case we compliment space with infinity-distant points, lines and planes corresponding with $x_4 = 0$.

Any equation

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0$$

governs a plane.

Any pair of equations

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = 0 b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0$$

governs a line.

Projective transformation now has form:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \\ x_3' &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \\ x_4' &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \end{aligned}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \neq 0$$