Chapter 16.

Method of weighted residuals for differential equations

Problem: find a solution $y(x)$ of the equation

$$(-P(x)\cdot y')' + Q(x)\cdot y - F(x) = 0$$

$$0 \le x \le L$$

that satisfies the boundary conditions: $y(0)=y(L)=0$

(this can always be obtained by a change of variable y)

If $\bar{y}(x)$ is some approximate solution, then

$$(-P(x)\cdot \bar{y}')' + Q(x)\cdot \bar{y} - F(x) \neq 0,$$

hence, we have a residual (error):

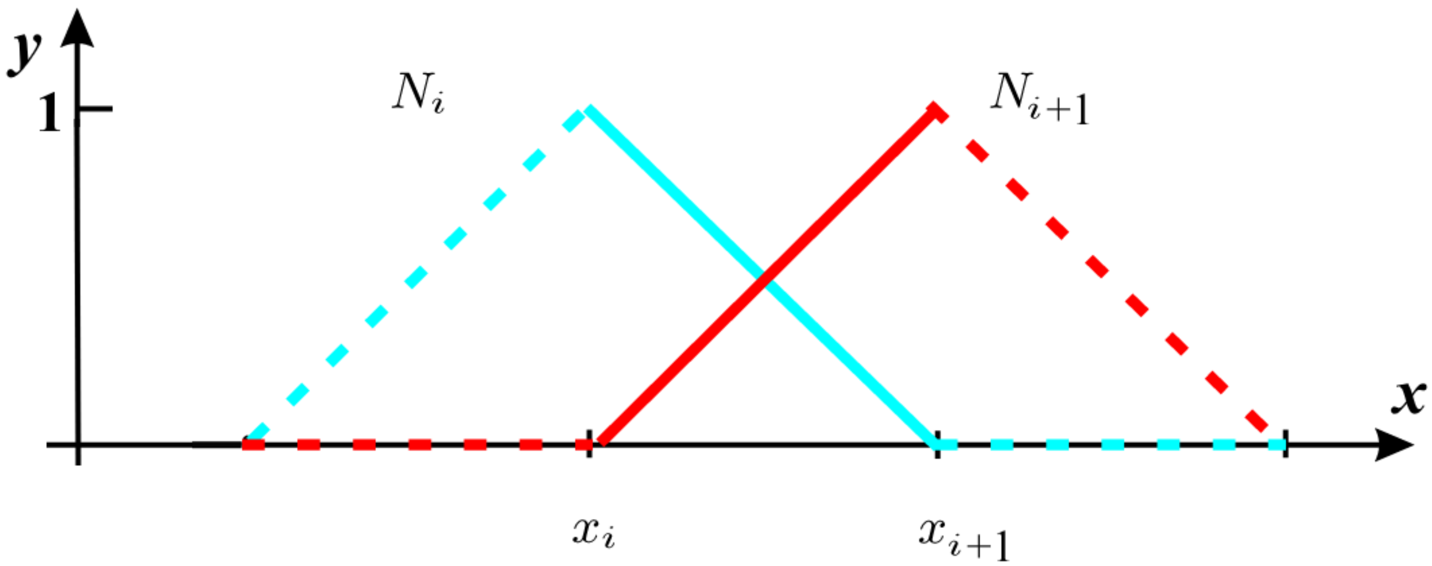
$$(-P(x)\cdot \bar{y}')' + Q(x)\cdot \bar{y} - F(x) = R(x) \quad (1)$$

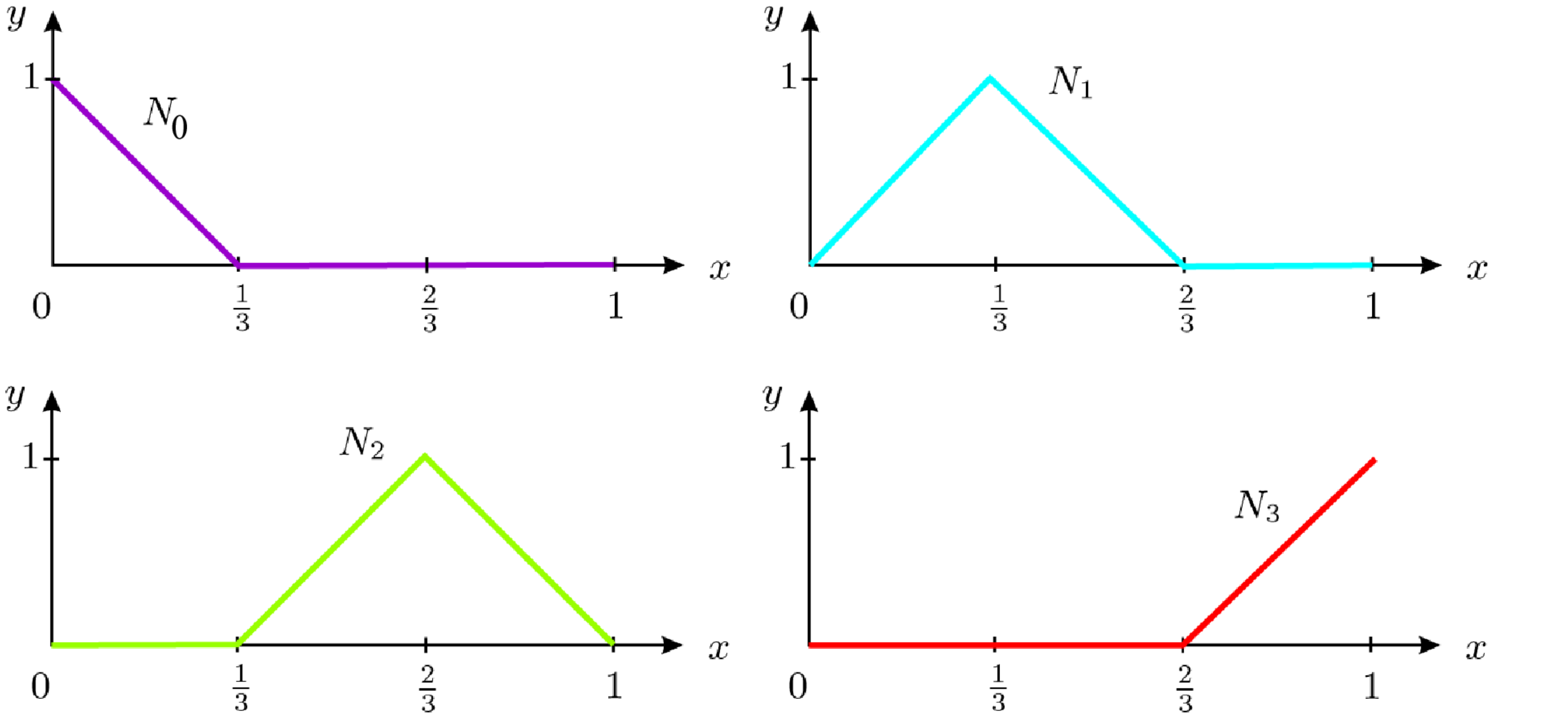
We will search $\bar{y}(x)$ in the form

$$\bar{y}(x) = y_0 N_0(x) + \ldots + y_i N_i(x) + y_{i+1} N_{i+1}(x) + \ldots + y_n N_n(x)$$

where $N_i(x)$ are piecewise linear functions, and we will find parameters y_i that minimize error $|R(x)|$.

$$N_i(x) = \begin{cases} 0, & x \leqslant x_{i-1} \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}, & x_{i-1} \leqslant x \leqslant x_i \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & x_i \leqslant x \leqslant x_{i+1} \\ 0, & x \geqslant x_{i+1} \end{cases}$$





y
1
N_0
0
$\frac{1}{3}$
$\frac{2}{3}$
1
x
N_1
N_2
N_3

The function $\bar{y}(x)$ cannot be inserted into (1), because $\bar{y}''$ does not exist. Therefore, we cannot substitute $\bar{y}(x)$ into left-hand side of (1) and search parameters y_1, $y_2, \ldots, y_n$ which deliver a minimum of $|R(x)|$.

In order to avoid $\bar{y}''$, we will minimize $|R(x)|$ with the method of weighted residuals: multiply (1) by weight functions $w_i(x)$, integrate the product, and require the integral to be equal to zero:

$$\int_0^L R(x) \cdot w_i(x)\, dx = 0 \qquad (2)$$

$$i = 1, 2, 3, \ldots m$$

$w_i(x)$ must be linearly independent functions, then expression (2) can be treated as orthogonality of $R(x)$ to basis functions $w_i(x)$ in Hilbet space.

As known, $R(x) \longrightarrow 0$ when $m \longrightarrow \infty$, because only identical zero can be orthogonal to all basis functions in a Hilbert space, see course of Functional Analysis.

Therefore, the larger m is used (that is the larger number of conditions (2) are imposed), the smaller value $|R(x)|$ can be expected .

Let us choose the number of conditions m <u>equal</u> to the number n-1 of unknown parameters y_i in $\bar{y}(x)$.

In the formula

$$\int_0^L R(x)\cdot w_i(x)\, dx = 0 \qquad (2)$$

we will use basis functions $N_i(x)$ as weight functions $w_i(x)$:

$$\int_0^L R(x)\cdot N_i(x)\, dx = 0$$

$i=1, 2, \ldots, n-1$ **- inner points**

Now we recall that by $R(x)$ we denoted the left-hand side of expression (1):

$$\int_0^L [(-P(x)\cdot \bar{y}')' + Q(x)\cdot\bar{y} - F(x)]\cdot N_i(x)dx = 0$$

Integration by parts **eliminates** $\bar{y}''$ **:**

$$\int_0^L [Q(x)\cdot\bar{y} - F(x)]\cdot N_i(x)dx +$$

$$+\int_0^L P(x)\cdot \bar{y}'\cdot N'_i(x)dx - \underbrace{P(x)\cdot \bar{y}'\cdot N_i(x)\Big|_0^L}_{= 0 \text{ as } N_i(0) = N_i(L) = 0} = 0$$

Therefore

$$\int_0^L [Q(x)\cdot\bar{y} - F(x)]\cdot N_i(x)dx + $$

$$+\int_0^L P(x)\cdot\bar{y}'\cdot N'_i(x)dx = 0 \quad (2')$$

Now we can substitute

$$\bar{y}(x) = y_0 N_0(x) + \ldots + y_i N_i(x) + \ldots + y_n N_n(x)$$

and obtain a system of algebraic equations with respect to y_i

At first, this will be demonstrated in the particular case

$$P(x) \equiv 1\ ,\quad Q(x) \equiv 0\ :$$

$$y''(x) + F(x) = 0\ ; \qquad y(0) = y(1) = 0,\ \ L = 1$$

$$\int_0^1 [\ \bar{y}'(x) \cdot N'_i(x) - F(x) \cdot N_i(x)]\, dx = 0$$

$$0,\quad x_1,\quad x_2, \ldots,\ x_{n-1}\ ,\quad 1$$

$$0,\quad y_1,\quad y_2, \ldots,\ y_{n-1}\ ,\quad 0$$

$$\bar{y}(x) = \cancel{y_0 N_0(x)} + y_1 N_1(x) + y_2 N_2(x) \ldots + \cancel{y_n N_n(x)}$$

$$\int_0^1 \sum_{j=1}^{n-1} y_j N'_j(x) \cdot N'_i(x)\, dx - \int_0^1 F(x) \cdot N_i(x)\, dx = 0$$

We change the order of summation and integration:

$$\sum_{j=1}^{n-1} y_j \int_0^1 N'_j(x) \cdot N'_i(x)\, dx - \int_0^1 F(x) \cdot N_i(x)\, dx = 0 \qquad (3)$$

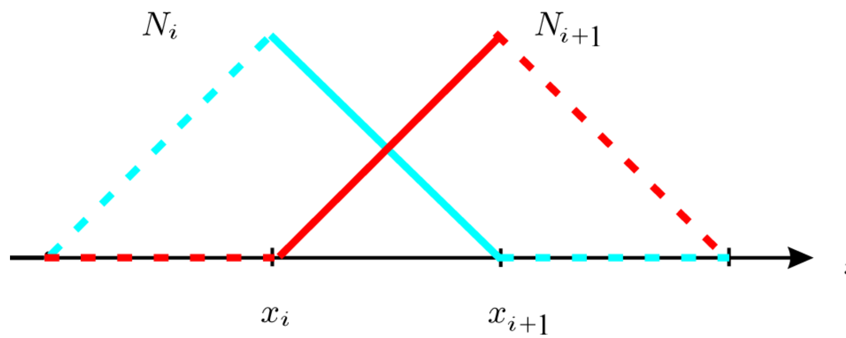
$$\boxed{\sum_{i=1}^{n-1} y_j K_{ij} = b_i}$$

where

$$K_{ij} = \int_0^1 N'_i(x) \cdot N'_j(x)\, dx \qquad b_i = \int_0^1 F(x) \cdot N_i(x)\, dx$$

For simplicity we suppose $x_{i+1} - x_i = h$, then

$N'_i = \dfrac{1}{h}$ at interval (x_{i-1}, x_i)



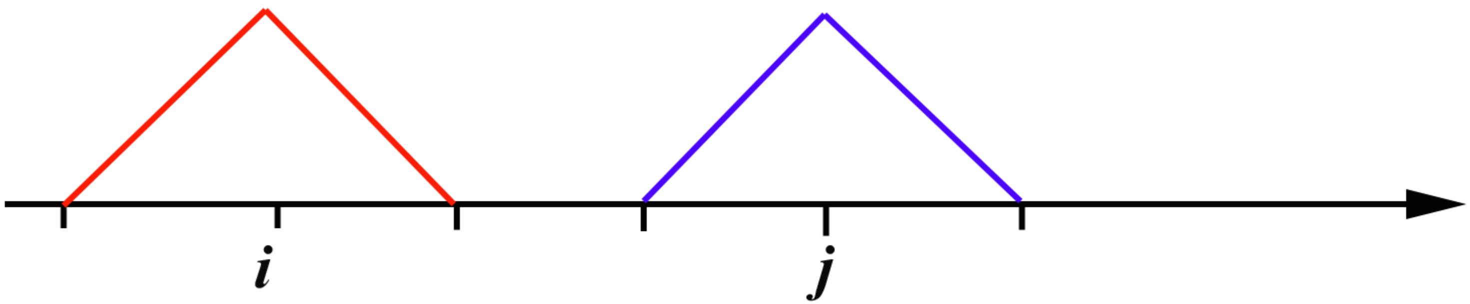
$N'_i = -\dfrac{1}{h}$ at interval $(x_i; x_{i+1})$.

$N'_i = 0$ outside of (x_{i-1}, x_{i+1})

$$K_{ij} = \int_0^1 N'_i(x) \cdot N'_j(x)\, dx$$

Four combinations of i and j in matrix K_{ij} are possible:

1) if difference between j and i is larger or equal to 2, then $K_{ij}=0$



2) if i=j (diagonal of the matrix), then

$$K_{ij} = \int_{x_{i-1}}^{x_{i+1}} (N_i')^2\, dx = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h}\right)^2 dx =$$

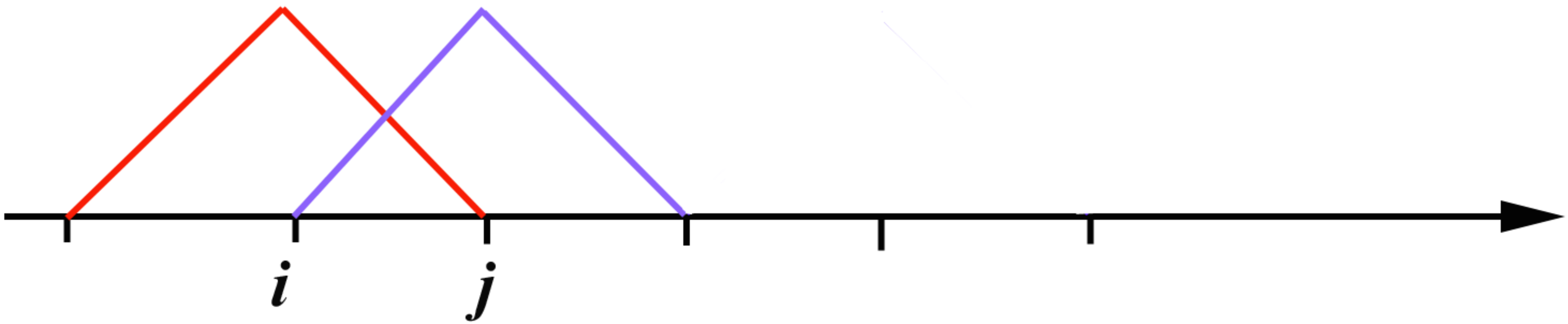
$$= \frac{1}{h^2}\left(x\Big|_{x_{i-1}}^{x_i} + x\Big|_{x_i}^{x_{i+1}} \right) = \frac{1}{h^2}\left(x_i - x_{i-1} + x_{i+1} - x_i\right) =$$

$$= \frac{1}{h^2}\left(x_{i+1} - x_{i-1}\right) = \frac{1}{h^2} \cdot 2h = \frac{2}{h}$$

3) if j=i+1 , see the element immediately on the right of diagonal in line i

$$K_{ij} = \int_{0}^{1} N'_i(x) \cdot N'_j(x)\, dx = \int_{x_i}^{x_{i+1}} N'_i(x) \cdot N'_j(x)\, dx = -1/h$$

as $N'_i(x) = -1/h \qquad N'_j(x) = 1/h$



4) if j=i-1, then again K_{ij} = -1/h .

As a consequence, matrix K is three-diagonal :

$$K = \frac{1}{h}\begin{vmatrix} 2 & -1 & 0 & & & & \\ -1 & 2 & -1 & & & 0 & \\ 0 & -1 & 2 & & & & \\ & & & \cdots & & & \\ & & & & 2 & -1 & 0 \\ & 0 & & & -1 & 2 & -1 \\ & & & & 0 & -1 & 2 \end{vmatrix}$$

Solving the obtained system of algebraic equations with respect to y_i, we arrive at an approximate solution of the problem for differential equation.

\- -

If one considers the differential equation with the extra term *y(x)* on the left:

$$y''(x) + y(x) + F(x) = 0\ ; \qquad y(0)=y(1)=0, \qquad (Q\equiv -1)$$

then the extra term

$$-\int_0^1 \bar{y}(x)\cdot N_i(x)\,dx$$

appears in the left-hand side of system (3), that is

$$- \sum_{j=1}^{n-1} y_j \int_0^{1} N_j(x)\cdot N_i(x)\, dx$$

An analysis of 4 combinations for i and j reveals extra terms in matrix *K*, which becomes (in case n=5):

$$\begin{pmatrix} 2(1/h-h/3) & -(1/h+h/6) & 0 & 0 \\ -(1/h+h/6) & 2(1/h-h/3) & -(1/h+h/6) & 0 \\ 0 & -(1/h+h/6) & 2(1/h-h/3) & -(1/h+h/6) \\ 0 & 0 & -(1/h+h/6) & 2(1/h-h/3) \end{pmatrix}$$

For example, for i=j we obtain :

$$- \int_{0}^{1} N_j(x)\cdot N_i(x)\, dx = \quad - \int_{x_{i-1}}^{x_{i+1}} N_i(x)\cdot N_i(x)\, dx =$$

$$-2 \int_{x_{i-1}}^{x_i} [(x-x_{i-1})/h]^2 \; dx = \; -2\, (x-x_{i-1})^3/3h^2 \Big|_{x_{i-1}}^{x_i} =$$

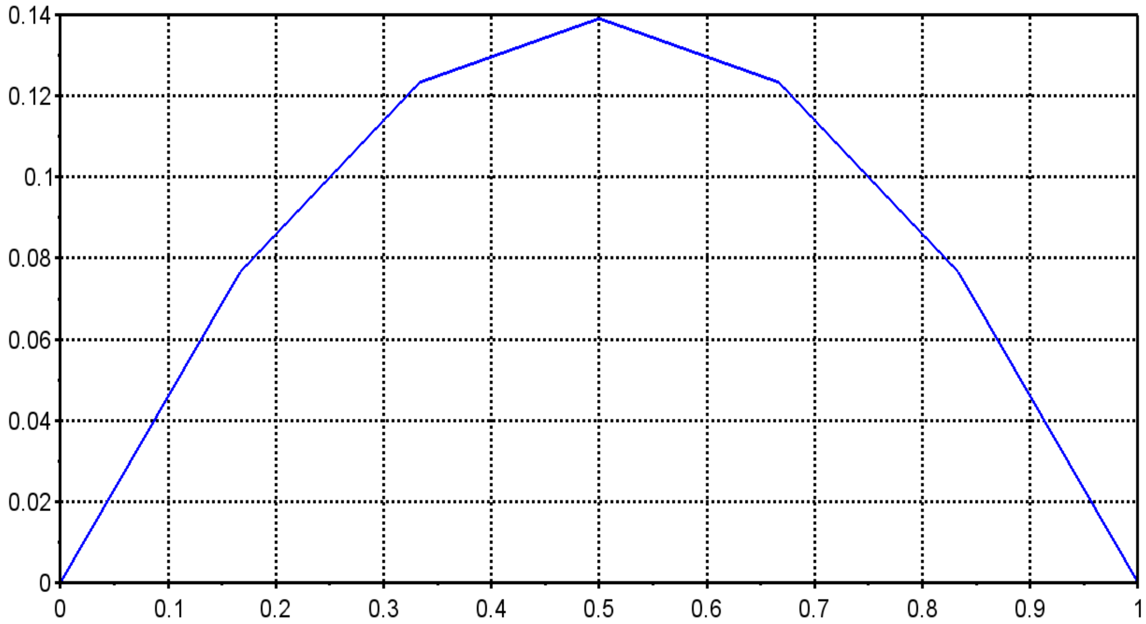
$$= -2\, h^3/3h^2 = -2\, h/3$$

Example: F=1

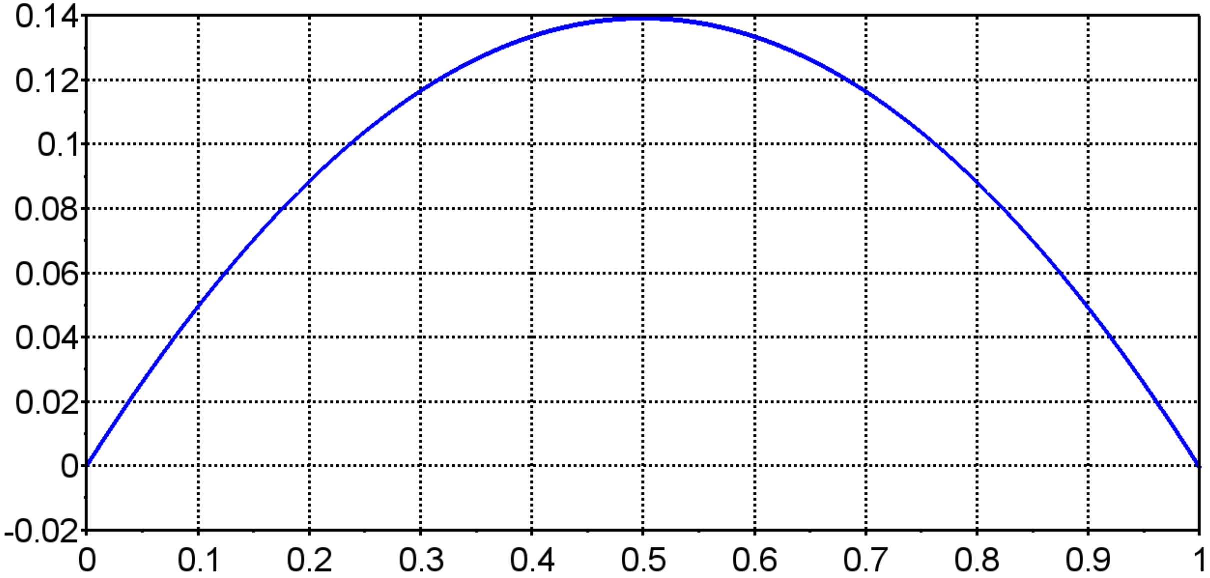
```
clear
 //   6 elements ,  7 points, 5 inner points
n=6
h=1/n
for i=1: 5
b(i)=h
end
K=[ 2*(1/h-h/3)    -1/h-h/6            0              0           0 ; ...
      -1/h-h/6        2*(1/h-h/3)     -1/h-h/6       0           0 ; ...
       0             -1/h-h/6        2*(1/h-h/3)   -1/h-h/6     0 ; ...
       0              0             -1/h-h/6     2*(1/h-h/3)  -1/h-h/6; ...
     0              0              0             -1/h-h/6      2*(1/h-h/3)];
K1=inv(K)
y=K1*b
```

```
for i=1:7
    yy(i)=0
    xx(i)=h*(i-1)
end
for i=1:5
    yy(i+1)=y(i)
end
plot(xx,yy)
xgrid
```

Solution:



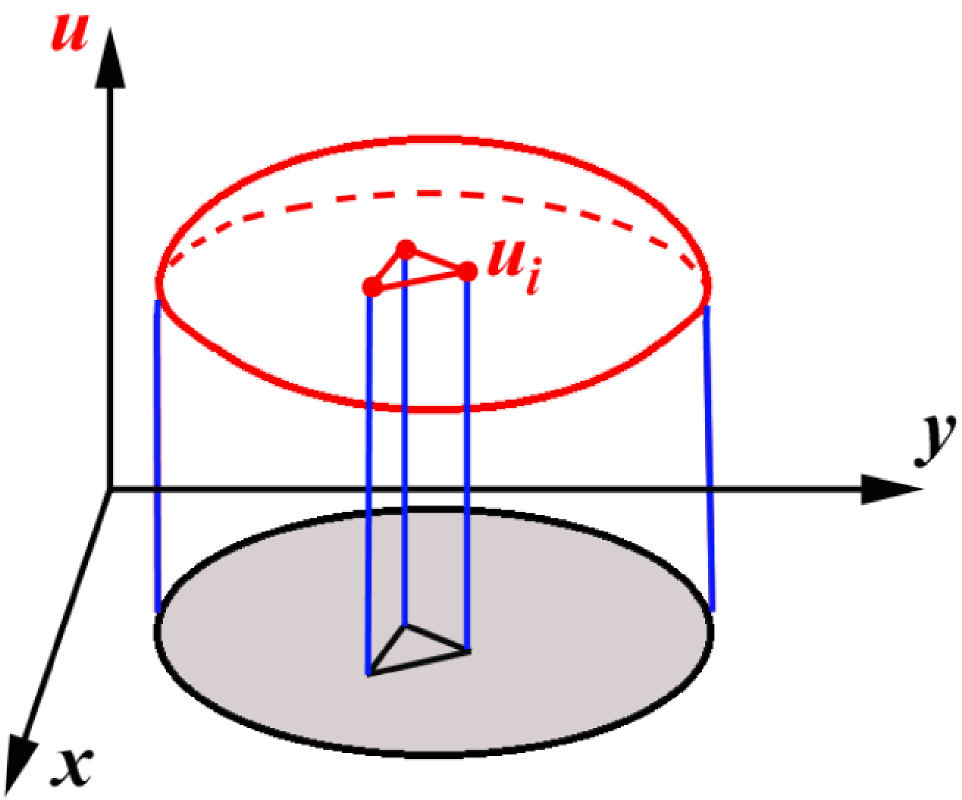
For a comparison, let us solve the same boundary-value problem using the shooting method (chapter 12):

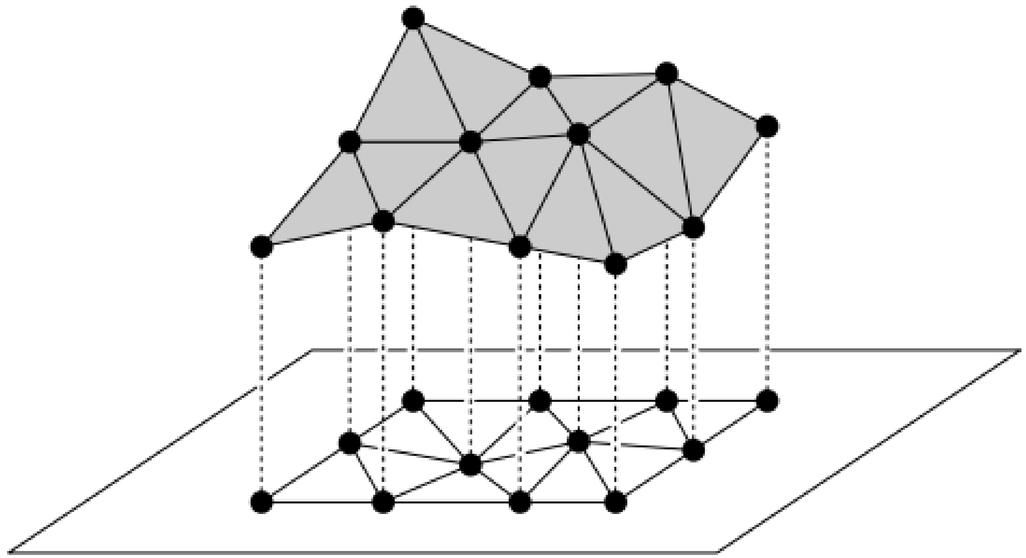

0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Method of weighted residuals for partial differential equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

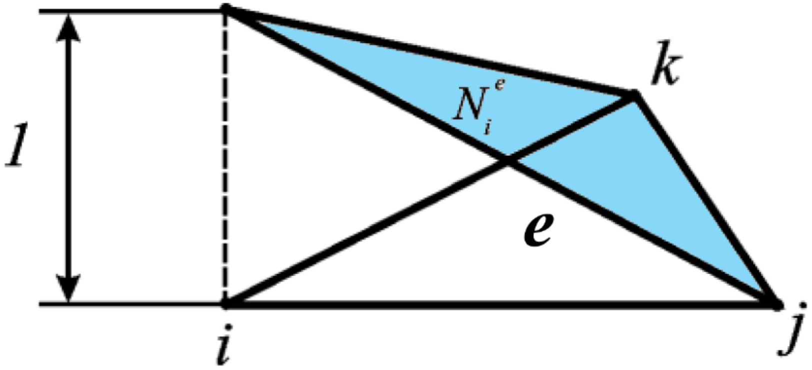
$$\bar{u}(x,y) = \sum_i N_i(x,y) \cdot u_i$$

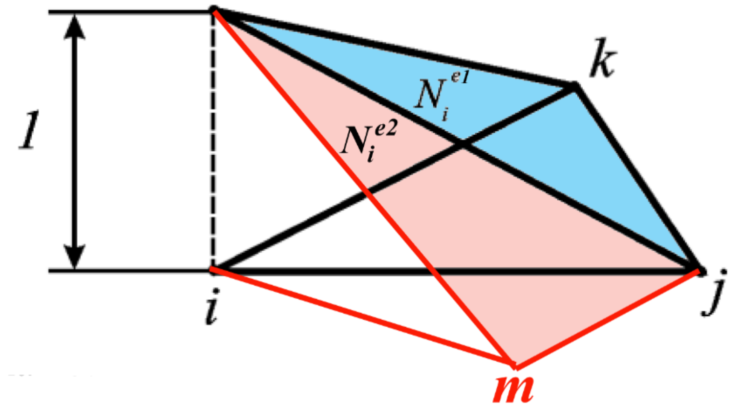




Domains of definition is divided into cells (typically triangles).

Basis functions N_i ***are pyramids*** constituted by inclined triangles





Approximate solution:

$$\bar{u}(x,y) = \sum_{i} N_i \cdot u_i \qquad N_i = \sum_{e} N_i^e \quad \textbf{\textit{piramid}}$$

summation goes over all nodes *i* and all elements N_i^{e1}, N_i^{e2}, N_i^{e3},... adjacent to each node.

The expression $\bar{u}(x,y)=\sum_i N_i(x,y)\cdot u_i$ can be inserted into integral relations obtained using weighted residuals, as shown below.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$$

Boundary condition： u=0 on the sides of rectangle D.

Integral relations:

$$\iint_D \left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}\right) N_i\,dxdy=\iint_D N_i f\,dxdy$$

Integration by parts gives the expression **(compare (2´)) :**

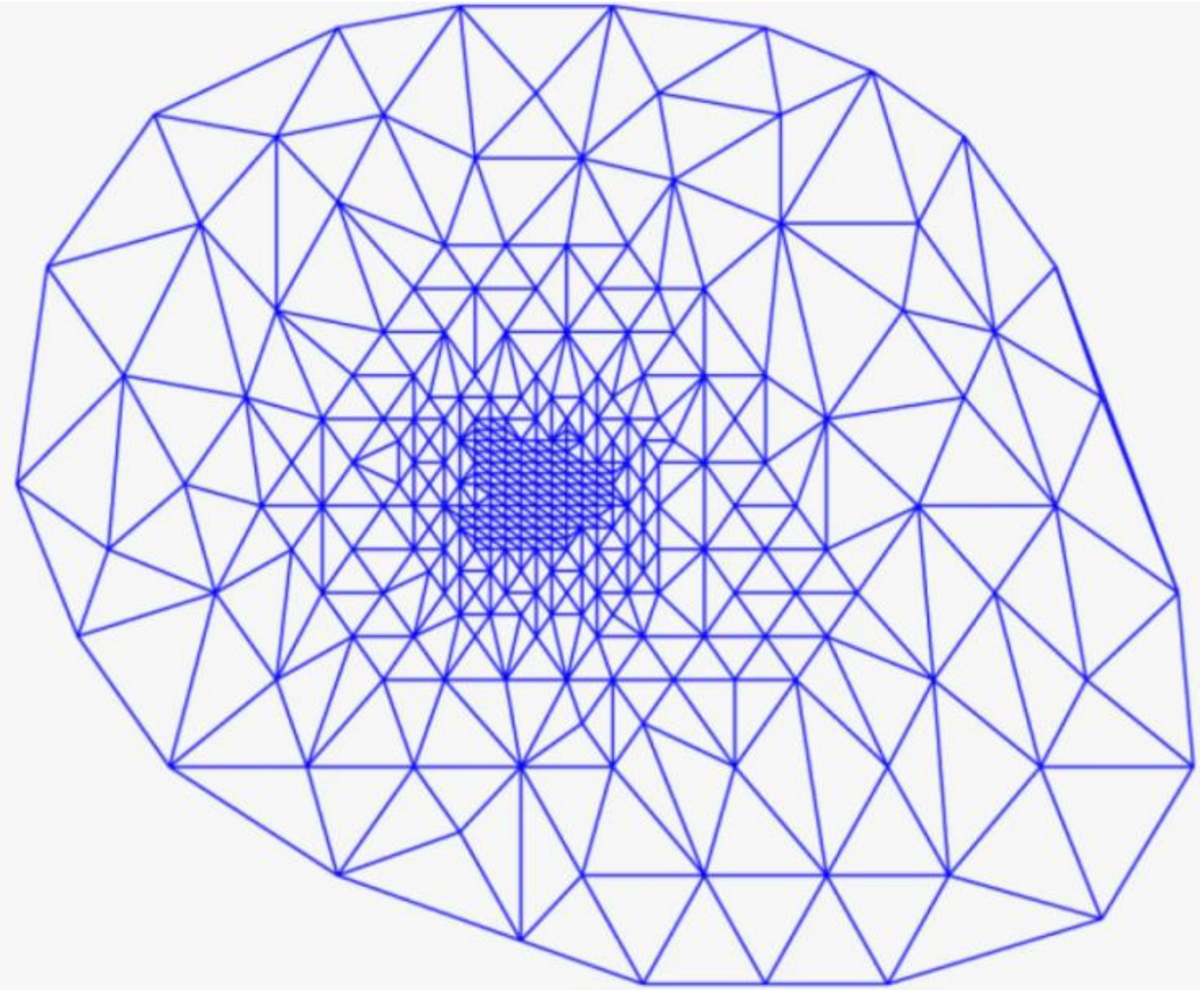
$$-\iint_D \left(\frac{\partial \bar{u}}{\partial x}\frac{\partial N_i}{\partial x} + \frac{\partial \bar{u}}{\partial y}\frac{\partial N_i}{\partial y} \right) dxdy = \iint_D f N_i \, dxdy$$

$$\bar{u} = \Sigma N_j u_j$$

$$\boxed{\sum_i u_j K_{ij} = b_i}$$

Therefore, we arrive at a system of algebraic equations for finding u_i .

Triangulation of the domain can be made with a few developed methods and codes.



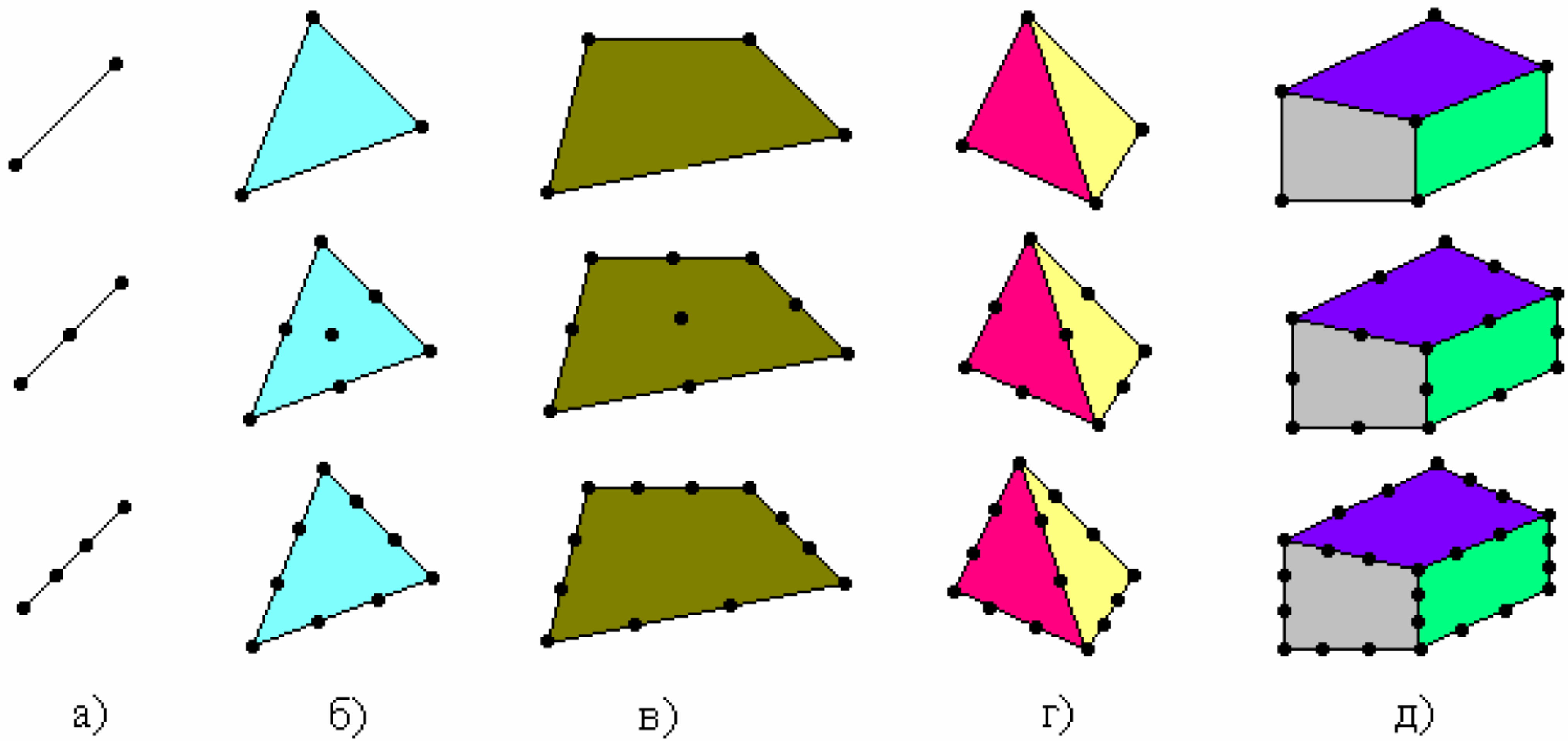
Triangulation in Scilab: see example

Triangulation in Mathlab: use subroutine "delaunay"

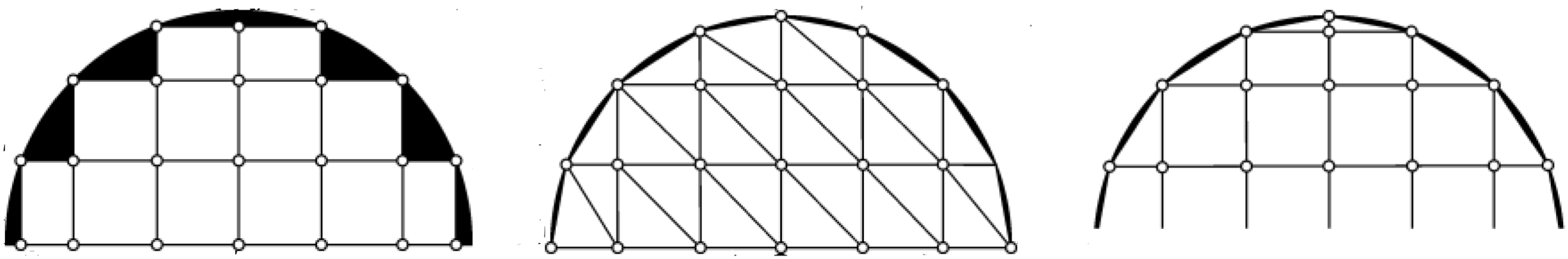
```
n = 16; % parameter
[x,y]= meshgrid ( linspace (0 ,1 ,n )); % 2D array of vertices
x=x (:); y=y (:); % array -> vector
e2p = delaunay (x,y); % Delaunay triangulation
npoint = size (x ,1); % # points
nelement = size (e2p ,1); % # elements
% plot the decomposition
triplot (e2p ,x,y,'o-','Color','b','MarkerFaceColor','r')
```

Types of finite elements which are foundations of basis functions N_i:

1D **2D** **2D** **3D** **3D**



On the approximation of a curvilinear boundary:



а б в

Finite-Difference **Weighted Residuals** **Weighted Residuals**

Benefits of the Weighted Residuals method (as compared to finite-difference method):

1) **easier handling curvilinear boundaries,**
2) **mesh can be easily refined in subdomains where solution changes abruptly; this will improve the accuracy of the solution.**