Complex Analysis 2024. Homework 1.

1. Calculate the following complex numbers, indicate the real and imaginary parts, calculate their absolute value.

$$z_1 = (1-2i)(3-4i); \quad z_2 = (-2i)^7; \quad z_3 = i^{2024}; \quad z_4 = \frac{1+2i}{2-i}.$$

Solution.

$$z_1 = -7-10i, \quad \operatorname{Re} z_1 = -5;\ \operatorname{Im} z_1 = -10;\ |z_1| = 5\sqrt{5};$$

$$z_2 = 2^7 i, \quad \operatorname{Re} z_2 = 0;\ \operatorname{Im} z_1 = 2^7;\ |z_1| = 2^7;$$

$$z_3 = 1, \quad \operatorname{Re} z_3 = 1;\ \operatorname{Im} z_3 = 0;\ |z_4| = 1;$$

$$z_4 = i, \quad \operatorname{Re} z_4 = 0;\ \operatorname{Im} z_4 = 1;\ |z_4| = 1;$$

2. Describe and draw sets defined by the equations

(a) $\operatorname{Re} z = 2 \operatorname{Im} z$;
Solution. This equation describes a line

$$x = 2y.$$

(b) $|z-1-i| = 3$;
Solution. This equation describes a circle of radius 3 with center at $(1,1)$.

(c) $|z-2| = \operatorname{Re} z$.
Solution. This equation describes a parabola

$$y^2 - 4x + 4 = 0; \quad x = \frac{y^2}{4} + 1.$$

3. Prove the parallelogram identity

$$|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Solution. First

$$|z_1 + z_2|^2 = (z_1+z_2)\overline{z_1 + z_2} = (z_1+z_2)(\overline{z}_1+\overline{z}_2) = |z_1|^2+|z_2|^2+z_1\overline{z}_2+z_2\overline{z}_1 = |z_1|^2+|z_2|^2+2\,\mathrm{Re}($$

Then

$$\begin{aligned}
&|z_1 + z_2|^2 + |z_1 - z_2|^2 = \\
&\qquad |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z}_2) + |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\overline{z}_2) = \\
&\qquad\qquad 2(|z_1|^2 + |z_2|^2) \quad (2)
\end{aligned}$$

4. Prove that $z^{-1} = \bar{z}/|z|^2$.

Solution.

$$z(\bar{z}/|z|^2) = |z|^2/|z|^2 = 1.$$

Complex Analysis 2024. Homework 2.

1. In assumption that $f = u + iv$ is $\mathbb{C}-$differentiable prove that

$$f'(z_0) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Proof. A $\mathbb{C}-$differentiable function f satisfies Cauchy-Riemann identities

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

In this case

$$f'_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

□

2. Prove the fundamental property of exponent

$$e^{w+z} = e^w e^z, \ w, z \in \mathbb{C}.$$

Proof. Let $w = u + iv$ and $z = x + iy$, $x, v \in \mathbb{R}$. Then

$$e^{x+iy}e^{u+iv} = e^x(\cos y + i\sin x)e^u(\cos v + i\sin v) = \\ e^{x+u}(\cos y\cos v - \sin x\sin v + i(\sin y\cos v + \cos y\sin v)) = \\ e^{x+u}(\cos y + v + i\sin(y+v)) = e^{x+u+i(y+v)} = e^{w+z}.$$

□

3. Prove that

$$\overline{e^z} = e^{\bar{z}}.$$

Proof. If $z = x + iy$ then $\bar{z} = x - iy$ and

$$\overline{e^z} = \overline{e^x\cos y + e^x\sin y} = e^x\cos y - e^x\sin y = e^{x-iy} = e^{\bar{z}}.$$

□

4. Find all points at which the function $f(z) = |z|^2$ is differentiable. Find partial derivatives $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar{z}}$.

Proof. First, $f(z) = x^2 + y^2$ is $\mathbb{R}-$differentiable on $\mathbb{C}$,

$$f'_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = x - iy = \bar{z};$$

$$f'_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = x + iy = z.$$

$f'_{\bar{z}} = 0$ only if $z = 0$. Consequently, it f is $\mathbb{C}-$differentiable only at 0 (but not holomorphic!).

□

5. Prove that a function $f(z) = \bar{z}$ is not complex differentiable at any point.

   Proof. First, $f(z) = x - iy$ is $\mathbb{R}-$differentiable on $\mathbb{C}$.

$$f'_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = 0;$$

$$f'_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 1 \neq 0.$$

   Consequently, Cauchy-Riemann condition is not satisfied at any point of $\mathbb{C}$. □

6. Calculate

$$z_1 = (1 + \sqrt{3}i)^9; \quad z_2 = (3 - 3i)^5; \quad z_3 = e^{(1+i)\frac{\pi}{2}};$$

$$z_1 = -2^9; \quad z_2 = -cdot3^5(1 - i); \quad z_3 = ie^{\pi/2}.$$

7. How are numbers z_1 and z_2 related if $arg(z_1) = arg(z_2)$?

   Solution. This equation implies that $z_1 = az_2$ for some $a > 0$.

Complex Analysis 2024. Homework 3.

1. Plot the path given by $\gamma(t)$

$$\gamma(t) = i + e^{it}, \quad 0 \leq t \leq \pi.$$

Plot it's image with respect to mapping $f(z) = (z - i)^3$;

Solution. This path circumscribes counterclockwise a upper half of the circle of radius 1 with center at i. The image circumscribes counterclockwise a circle of radius 1 around zero when $0 \leq t \leq 2\pi/3$ and upper half of the circle when $2\pi/3 \leq t \leq \pi$.

2. Find the image of the given line under the complex mapping $w = z^2$

(a) $\operatorname{Re} z = \operatorname{Im} z$;

(b) $\operatorname{Re} z = 3$;

Solution

(a) The image is a set of points $w = (x + ix)^2 = 2ix^2$. It is equal to the ray on imaginary line $\operatorname{Re} w = 0,\ \operatorname{Im} w > 0$.

(b) The image is a set of points $w = (3 + iy)^2 = 9 - y^2 + 6yi,\ y \in \mathbb{R}$. This is parabola $\operatorname{Re} w = 9 - \left(\frac{\operatorname{Im} w}{6}\right)^2$.

3. Calculate all values of

$$\sqrt[3]{-3 + 3i}; \quad \sqrt[5]{-1 + \sqrt{3}i}.$$

First

$$2^(1/6)3^{1/3}\left(\cos\frac{\pi}{4} + isin\frac{\pi}{4}\right);$$

$$2^{1/6}3^{1/3}\left(\cos\frac{11\pi}{12} + isin\frac{11\pi}{12}\right);$$

$$2^{1/6}3^{1/3}\left(\cos\frac{19\pi}{12} + isin\frac{19\pi}{12}\right).$$

second

$$\sqrt[5]{2}\left(\cos\left(\frac{2\pi}{15}\right) + i\sin\left(\frac{2\pi}{15}\right)\right);$$

$$\sqrt[5]{2}\left(\cos\left(\frac{8\pi}{15}\right)+i\sin\left(\frac{8\pi}{15}\right)\right);$$

$$\sqrt[5]{2}\left(\cos\left(\frac{14\pi}{15}\right)+i\sin\left(\frac{14\pi}{15}\right)\right);$$

$$\sqrt[5]{2}\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right);$$

$$\sqrt[5]{2}\left(\cos\left(\frac{28\pi}{15}\right)+i\sin\left(\frac{28\pi}{15}\right)\right);$$

4. Find the image of the domain $|z| < 8$, $\pi/2 < \arg z < 3\pi/4$ under each of the following principal nth root function ($k = 0$ in our definition)

$$f(z) = z^{1/3}; \quad f(z) = z^{1/2}.$$

Solution. 1) $|z| < 2,\ \pi/6 < \arg z < \pi/4$;
2) $|z| < 2\sqrt{2},\ \pi/4 < \arg z < 3\pi/8$;

Complex Analysis 2024. Homework 4.

1. Find all values of $\operatorname{Ln} z$ at points
$$z_1 = 1+i; \quad z_2 = -3; \quad z_2 = -1+i\sqrt{3}.$$

Solution. Since $1+i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}e^{i\frac{\pi}{4}}$ then
$$\operatorname{Ln}(1+i) = \frac{1}{2}\ln 2 + \frac{\pi}{4}i + 2\pi k i, \quad k \in \mathbb{Z};$$
Since $-3 = 3e^{i\pi}$ then
$$\operatorname{Ln}(-3) = \ln 3 + \pi(2k+1)i, \quad k \in \mathbb{Z};$$
Since $-3 = 3e^{i\pi}$ then
$$\operatorname{Ln}(-3) = \ln 3 + \pi(2k+1)i, \quad k \in \mathbb{Z};$$
Since $z_2 = -1+i\sqrt{3} = 2\left(-\frac{1}{2} + \frac{sqrt3}{2}i\right) = 2e^{i\frac{5\pi}{6}}$ then
$$\operatorname{Ln}(-1+i\sqrt{3}) = \ln 2 + \frac{5\pi}{6}i + 2\pi k i, \quad k \in \mathbb{Z}.$$

2. Calculate
$$(1+i)^i; \quad 3^{2i/\pi}; \quad (ei)^{sqrt2}.$$

Solution.
$$(1+i)^i = e^{i\operatorname{Ln}(1+i)} = e^{i(\frac{1}{2}\ln 2 + \frac{\pi}{4}i + 2\pi k i)} = e^{-\frac{\pi}{4} - 2\pi k}\left(\cos\frac{\ln 2}{2} + i\sin\frac{\ln 2}{2}\right).$$
$$3^{2i/\pi} = e^{\frac{2\operatorname{Ln}(3)}{\pi}i} = e^{\frac{2\ln 3 + 2\pi k i}{\pi}i} = e^{-2k}\left(\cos\frac{\ln 9}{\pi} + i\sin\frac{\ln 9}{\pi}\right).$$

3. Prove that the principal value of logarithm
$$\ln z = \ln|z| + i\arg z, \quad \arg z \in (-\pi, \pi),$$
is conformal in a slit domain $\mathbb{C}\backslash(-\infty, 0]$.

Solution. We already know from the previous homeworks that
$$\ln(z)' = \frac{1}{z} \neq 0, \ z \neq 0.$$
Hence, $\ln z$ is conformal in every disk that doesn't contain 0 and, hence, in domain $\mathbb{C}\backslash(-\infty, 0]$.

4. Prove that Jacobian of a conformal map $f : D \to G$ is equal to

$$J_f = |f'(z)|^2 .$$

Solution. Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then

$$J_f = \left|\det \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix}\right| = \left|u'_x v'_y - v'_x u'_y\right| = (u'_x)^2 + (v'_x)^2 = |u'_x + iv'_x|^2 = |f'_x|^2 = |f'_z|^2$$

since f satisfies Cauchy-Riemann equations

$$u'_x = v'_y, \quad u'_y = -v'_x;$$

$$f'_z = \frac{1}{2}(f'_x - if'_y) = \frac{1}{2}((u'_x + v'_y) + i(v'_x - u'_y)) = u'_x + iv'_x.$$

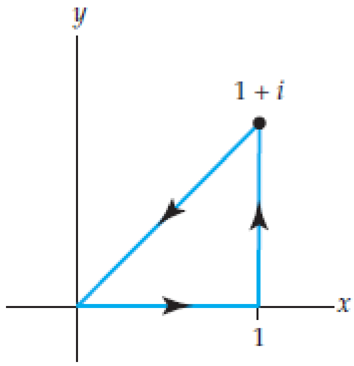
5. Evaluate the integral $\int_\gamma |z|^2 \, dz$, where γ is $x = t^2$, $y = 1/t$, $1 \le t \le 2$.

Solution.

$$\int_\gamma |z|^2 \, dz = \int_1^2 \left(t^4 + t^{-2}\right)(2t - it^{-2})dt = \int_1^2 (2t^5 + 2t^{-1})dt - i \int_1^2 (t^2 + t^{-4})dt =$$

$$21 + \ln 4 - \frac{21}{8} i.$$

6. Evaluate the integral $\oint_\gamma \overline{z}^2 dz$ along the contour γ given in the figure

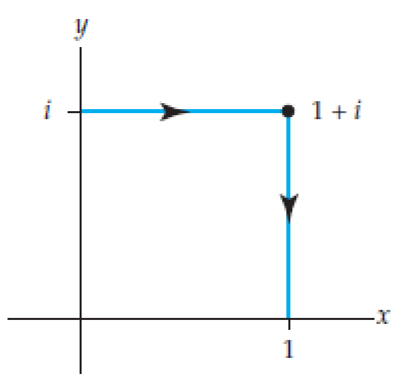


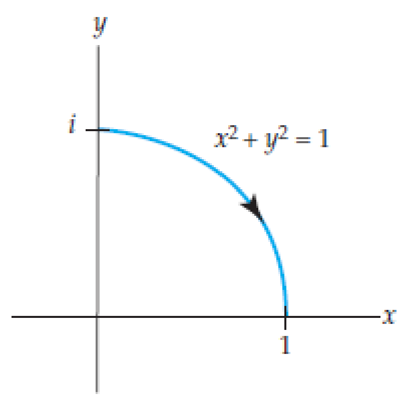
$$\oint_\gamma \overline{z}^2 dz = \underbrace{\int_0^1 (x - i0)^2 dx}_{z=x,\ dz=dx} + \underbrace{\int_0^1 \overline{(0+iy)}^2 i dy}_{z=1+iy;\ dz=idy} + \underbrace{\int_0^1 (\overline{(1+i)(1-t)})^2(-(1+i))dt}_{z=(1-t)(1+i);\ dz=-(1+i)dt} =$$

$$\int_0^1 x^2 dx - i\int_0^1 y^2 dy - (1-i)^2(1+i)\int_0^1 (1-t)^2 dt =$$

$$\frac{1}{3}(1 - i + 2(1-i)) = 1 - i.$$

7. Evaluate the integral $\int_\gamma (z^2 - z + 2)dz$ along the contours γ given in the figures (see page 2)





First contour

$$\int_\gamma (z^2 - z + 2)dz = \underbrace{\int_0^1 ((t+i)^2 - (t+i) + 2)dt}_{z=t+i,\ dz=dt} +$$

$$\underbrace{\int_0^1 ((1+(1-t)i)^2 - (1+(1-t)i) + 2)(-i)dt}_{z=1+i(1-t),\ dz=-idt} = \frac{5}{6} - \frac{5}{3}i.$$

Also we can notice that $F'(z) = z^2 - z + 2$ for $F(z) = z^3/3 - z^2/2 + 2z$ and for both contours

$$\int_\gamma (z^2 - z + 2)dz = F(1) - F(i) = 11/6 - (-i/3 + 1/2 + 2i) = 5/6 - 5i/3.$$

8. Prove Cauchy-Goursat theorem for rectangles.

   Solution. The proof is obtained word by word from the proof of Cauchy-Goursat theorem for triangles by dividing the rectangles into four equal parts.

9. Suppose that a function f is holomorphic in domain D and that G is a domain bounded by a simple closed smooth path γ such that $\overline{G} \subset D$. Assume, moreover, that f' is continuous in D. Prove, using Green's formula that

$$\int_\gamma f dz = 0.$$

   Solution. By Green's formula and Cauchy-Riemann identities we see that

$$\int_\gamma f dz = \int_\gamma (u+iv)dx + (-v+iu)dy) = \int_G ((-v+iu)'_x - (u+iv)'_y)dxdy =$$

$$\int_G ((u'_y - v'_x) + i(u'_x + v_y))dxdy = 0$$

Complex Analysis 2024. Homework 5.

1. Assume that f is holomorphic in domain D and $\arg f(z)$ is constant. Prove that f is constant in D.

   Proof. □

2. Assume that f is holomorphic in domain D and

$$A \operatorname{Im} f(z) + B \operatorname{Re} f(z) + C = 0, z \in D,$$

   for some real constants A, B, C. Prove that f is constant.

   Proof. Consider a function

$$g = (B - iA)f(z) + C.$$

   Then g is holomorphic

$$\operatorname{Re} g = A \operatorname{Im} f(z) + B \operatorname{Re} f(z) + C = 0$$

   and, consequently, $g = iC_1$ in domain D for some real constant C_1. Hence,

$$f(z) = \frac{iC_1 - C}{B - iA}.$$

   □

3. Calculate the integral $\int_\gamma \frac{dz}{z}$ along the following paths:

   (a) $\gamma_1 = e^{it}, \quad t \in [0, 4\pi]$;

   (b) $\gamma_2 = e^{-it}, \quad t \in [0, 2\pi]$;

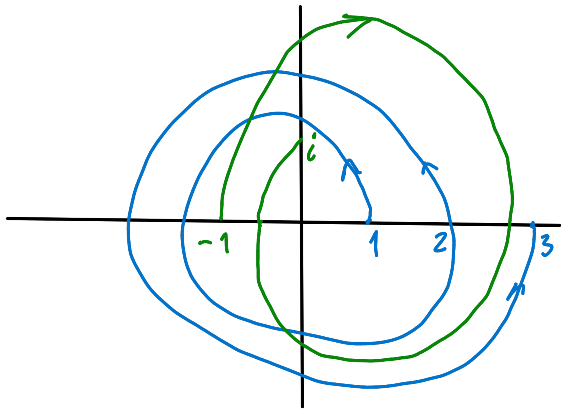
   (c) See paths γ_3 (green) γ_4 (blue) in the picture below.

   Answers.

   (a) $4\pi i$;

   (b) $-2\pi i$;

   (c) $\int_{\gamma_3} \frac{dz}{z} = -\frac{5\pi}{2} i$;



(d) $\int_{\gamma_4} \frac{dz}{z} = \ln 3 + 4\pi i$.

4. Calculate $\int_\gamma dz$, where γ is the left half of the ellipse $\frac{1}{36}x^2 + \frac{1}{4}y^2 = 1$ from $z = 2i$ to $z = -2i$.

   Solution.
$$\int_\gamma dz = z|_{2i}^{-2i} = -4i.$$

5. Calculate $\int_\gamma \left(z + \frac{1}{z}\right) dz$, where γ is a circle $|z| = 2$ oriented counter-clockwise.

   Solution.
$$\int_\gamma z dz = 0$$
   since $f(z) = z$ is holomorphic in $\mathbb{C}$ and γ is closed path;
$$\int_\gamma \frac{dz}{z} = 2\pi i.$$
   Hence,
$$\int_\gamma \left(z + \frac{1}{z}\right) dz = 2\pi i.$$

6. Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial with $c_k \in \mathbb{R}$. Show that
$$\int_{-1}^{1} f(x)^2 dx \le \pi \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \pi \sum_{k=0}^{n} c_k^2.$$
Hint. For the first inequality, apply Cauchy-Goursat's theorem to the function $f(z)^2$ separately on the top half and the bottom half of the unit disk.

Proof.
$$\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \frac{1}{2} \sum_{k,j=1}^{n} \int_0^{2\pi} c_k c_j e^{i(k-j)\theta} d\theta = \pi \sum_{k=0}^{n} c_k^2.$$
Since $\int_0^{2\pi} c_k c_j e^{i(k-j)\theta} d\theta = 2\pi \delta_{k,j}$.
Then integral over the top half is equal to 0 and
$$\int_{-1}^{1} f(x)^2 dx + \int_0^{\pi} f(z)^2 dz = 0.$$
Then integral over the lower half is equal to 0 and
$$-\int_{-1}^{1} f(x)^2 dx + \int_{\pi}^{2\pi} f(z)^2 dz = 0.$$
Hence,
$$2 \int_{-1}^{1} f(x)^2 dx \le \int_0^{2\pi} |f(z)|^2 |dz| \, 2\pi \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 d\theta.$$
□

7. Show that an analytic function $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)dz = 0$ for every closed path γ in D.

Proof. The sufficiency is obvious. Assume now that $\int_\gamma f(z)dz = 0$ for every closed path γ in D. Fix $z_0 \in D$ and for $z \in D$
$$F(z) = \int_\gamma f(z)dz$$

for some path γ that connects z_0 and z.

The definition doesn't depend on path γ. Assume that γ_1 and γ_2 connect z_0 and z. Then the compund path $\lambda = \gamma_1 \cup \gamma_2^{-1}$ is closed and

$$\int_{\gamma_1} f dz - \int_{\gamma_2} f dz = \int_{\gamma_1 \cup \gamma_2} f dz = 0.$$

To prove that F is differentiable consider z and $\delta > 0$ such that $B(z, \delta) \subset D$. Then

$$\frac{F(z+w) - F(z)}{w} = \frac{1}{w} \int_z^{z+w} f(\xi) d\xi \to f(z), \;\; w \to 0,$$

where the integral $\int_z^{z+w}$ is consedred over a segment that connects z and $z + w$. □

8. Show that

$$\left| \oint_{|z|=R} \frac{\log z}{z^2} dz \right| \le 2\sqrt{2}\pi \frac{\log R}{R}, \quad R > e^{\pi}.$$

Proof. Consider a parametrization $z = Re^{it}$ $-\pi \le t \le \pi$. Then

$$|\log z| = \sqrt{\log^2 R + t^2} \le \sqrt{2} \log R, \quad R > e^{\pi}$$

and

$$\left| \oint_{|z|=R} \frac{\log z}{z^2} dz \right| \le 2\pi R \frac{\sqrt{2} \log R}{R^2}.$$

□

9*. Show that if D is a bounded domain with smooth boundary, then

$$\int_{\partial D} \bar{z} dz = 2i \operatorname{Area}(D).$$

Proof. To prove this apply Green's formula. □