1.5. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF INTEGRAL EQUATIONS AND SYSTEMS

1.5.1.The Volterra equation of the second kind

Consider the Volterra linear integral equation of the second kind with a kernel $K(t)$ of the form

$$y(t) = f(t) + \int_0^t K(t-\tau)y(\tau)d\tau, \qquad (1.20)$$

where $K(t), f(t)$ - given functions, $y(t)$ - the desired function.

Let $y(t) \leftrightarrow Y(p)$, $f(t) \leftrightarrow F(p)$, $K(t) \leftrightarrow K^*(p)$. Passing to the images in equation (1.20) and using the convolution image property, we obtain the corresponding operator equation

$$Y(p) = F(p) + K^*(p)Y(p).$$

From here

$$Y(p) = \frac{F(p)}{1 - K^*(p)}.$$

The original for image $Y(p)$ has the desired solution to equation (1.20).

1.5.2. The Volterra equation of the first kind

Consider a linear integral Volterra equation of the first kind with a kernel $K(t)$ of the form

$$\int_0^t K(t-\tau)y(\tau)d\tau = f(t), \qquad (1.21)$$

where $K(t), f(t)$ - given functions, $y(t)$ - the desired function.

Let $y(t) \leftrightarrow Y(p)$, $f(t) \leftrightarrow F(p)$, $K(t) \leftrightarrow K^*(p)$. Then, applying the Laplace transform to equation (1.21), we obtain the operator equation.

$$K^*(p)Y(p) = F(p) \;\Rightarrow\; Y(p) = \frac{F(p)}{K^*(p)}.$$

The original for $Y(p)$ gives the desired solution to equation (1.21).

1.5.3.Systems of Volterra integral equations

Consider a system of Volterra integral equations of the form

$$y_i(t) = f_i(t) + \sum_{k=1}^{s} \int_0^t K_{ik}(t-\tau) y_k(\tau) d\tau \,, \; i = 1,2,\ldots s \,, \qquad (1.22)$$

where $K_{ik}(t)$, $f_i(t)$ - given functions, $i,k = 1,2,\ldots,s$.

Let

$$F_i(p) \leftrightarrow f_i(t) \,,\; K_{ik}^*(p) \leftrightarrow K_{ik}(t) \,,\; Y_i(p) \leftrightarrow y_i(t) \,.$$

Applying the Laplace transform to both parts of the equations (1.22), we obtain a system of operator equations

$$Y_i(p) = F_i(p) + \sum_{k=1}^{s} K_{ik}^*(p) Y_k(p) \,, \; i = 1,2,\ldots s \,, \qquad (1.23)$$

linear with respect to the images $Y_i(p)$. Solving the system (1.23), we find $Y_i(p)$, the originals for which will be the solution of the original system of integral equations (1.22).

What will be useful to us:

$\int\limits_0^{\infty} y(t)e^{-pt}\,dt = Y(p)$ - Laplace transform

$\int\limits_0^{\infty} y'(t)e^{-pt}\,dt = pY(p) - y(0)$ - Laplace transform of the derivative

$\int\limits_0^{\infty} y^{(n)}(t)e^{-pt}\,dt = p\int\limits_0^{\infty} y^{(n-1)}(t)e^{-pt}dt - y^{(n-1)}(0)$ - Laplace transform from high order derivatives

$\int\limits_0^{\infty}\left(\int\limits_0^{t} y(\tau)\,d\tau\right)e^{-pt}\,dt = \dfrac{Y(p)}{p}$ - Laplace transform from the integral

$\int\limits_0^{\infty}\left(\int\limits_0^{x} g(x-t)\,y(t)\,dt\right)e^{-px}\,dx = G(p)Y(p)$ - Laplace transform from a convolution type integral

Let we have $K(x-t)$ - the kernel of the integral operator (difference kernel).

The Volterra integral equation of the second kind looks like this:

$$y(x) = f(x) + \int\limits_0^{x} K(x-t)\,f(t)\,dt$$

Let's move everything to the left side:

$$y(x) - f(x) - \int\limits_0^{x} K(x-t)\,f(t)\,dt = 0$$

$$\int_0^\infty \left[y(x) - f(x) - \int_0^x K(x-t)\, f(t)\, dt \right] e^{-px} dx = 0$$

We have obtained three Laplace transformations:

$$\int_0^\infty y(x) e^{-px} dx = \int_0^\infty f(x) e^{-px} dx + \int_0^\infty \left(\int_0^x K(x-t)\, y(t)\, dt \right) e^{-px} dx$$

The image of the first integral $Y(p)$, the second is $F(p)$, and the third is a convolution type integral.

The integral equation turns into an algebraic equation for images:

$$Y(p) = F(p) + K^*(p) Y(p)$$

$$Y(p) = \frac{F(p)}{1 - K^*(p)} \ ,$$

where $F(p), K^*(p) -$ we know.

Example 1

Solve the integral equation

$$y(x) = \sin x + \int_0^x (x-t)y(t)dt .$$

Solution:

Let $y(x) \leftrightarrow Y(p)$.

Since the integral included in the given equation is a convolution of two functions t and $y(t)$, then its image will be the product of images of these functions, that is $\frac{1}{p^2}Y(p)$. Applying the Laplace transform to the equation, we obtain the following operator equation:

$$Y(p) = \frac{1}{p^2+1} + \frac{1}{p^2}Y(p) .$$

His solution has the form

$$Y(p) = \frac{p^2}{(p^2-1)(p^2+1)} = \frac{p}{(p^2-1)}\frac{p}{(p^2+1)} .$$

Since

$$\frac{p}{\left(p^2-1\right)} \leftrightarrow \operatorname{ch} x , \quad \frac{p}{\left(p^2+1\right)} \leftrightarrow \cos x ,$$

then the original corresponding to the image $Y(p)$ is a convolution of two functions — $\operatorname{ch} x$ and $\cos x$:

$$y(x) = \int_0^x \operatorname{ch}(x-t)\cos t dt$$

Having calculated the integral, we get the desired solution:

$$y(x)=\frac{1}{2}\sin x+\frac{1}{4}e^{x}-\frac{1}{4}e^{-x}.$$

Example 2

Solve the integral equation

$$y(x)=\cos x+\int_{0}^{x}(x-t)y(t)dt;$$

<u>Solution:</u>

$$\cos x \leftrightarrow \frac{p}{p^2+1}$$

$$Y(p)=\frac{p}{p^2+1}+\frac{1}{p^2}Y(p)$$

$$Y(p)=\frac{p^3}{(p^2-1)(p^2+1)}=p\cdot\frac{p}{p^2-1}\cdot\frac{p}{p^2+1}$$

$$\frac{p}{p^2-1}\leftrightarrow ch(t)=f_1(t)$$

$$\frac{p}{p^2+1}\leftrightarrow \cos(t)=f_2(t)$$

We found the original corresponding to the image using the Duhamel integral.

Duhamel integral:

$$f(t)=f_1(0)f_2(t)+\int_{0}^{t}f_1'(\tau)f_2(t-\tau)d\tau$$

We have

$$y(t) = 1 \cdot cos(t) + \int_0^t sh(\tau) cos(t-\tau) d\tau$$

Using twice integration by parts in the integral, we have

$$\int sh(\tau) cos(\tau - t) d\tau = \frac{sh(\tau) sin(\tau - t) + ch(\tau) cos(\tau - t)}{2} + C =$$

$$= \frac{e^{-\tau}\left[\left(e^{2\tau} - 1\right) sin(\tau - t) + \left(e^{2\tau} + 1\right) cos(\tau - t)\right]}{2} + C$$

And

$$\int_0^t sh(\tau) cos(\tau - t) d\tau = \frac{e^{-t}\left(e^{2t} + 1\right)}{4} - \frac{cos(t)}{2}$$

$$y(t) = 1 \cdot cos(t) + \frac{e^{-t}\left(e^{2t} + 1\right)}{4} - \frac{cos(t)}{2} = \frac{e^{-t}\left(e^{2t} + 1\right)}{4} + \frac{cos(t)}{2}$$

Our solution is

$$y(x) = \frac{e^{-x}\left(e^{2x} + 1\right)}{4} + \frac{cos(x)}{2}$$

Example 3

Solve a system of integral equations:

$$\begin{cases} y(x) = e^x + \int\limits_0^x y(t)dt - \int\limits_0^x e^{(x-t)} z(t)dt, \\ z(x) = -x - \int\limits_0^x (x-t) y(t)dt - \int\limits_0^x z(t)dt. \end{cases}$$

<u>Solution:</u>

Let $y(x) \leftrightarrow Y(p)$, $z(x) \leftrightarrow Z(p)$.

We apply the Laplace transform to each equation of the system. Using the properties on the integration of the original and on convolution to construct images of the original equations, we obtain

$$\begin{cases} Y(p) = \dfrac{1}{p-1} + \dfrac{Y(p)}{p} - \dfrac{Z(p)}{p-1}, \\ Z(p) = -\dfrac{1}{p^2} - \dfrac{Y(p)}{p^2} - \dfrac{Z(p)}{p}. \end{cases}$$

Solving a system of algebraic equations, we find the images

$$Y(p) = \frac{1}{p-2},$$

$$Z(p) = -\frac{1}{p(p-2)} = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{p-2}\right),$$

which correspond to the originals:

$$y(x) = e^{2x},$$

$$z(x) = \frac{1}{2} - \frac{1}{2}e^{2x}.$$

Example 4

Solve the integral-differential equation

$$y''(x)+2y'(x)-2\int_0^x \sin(x-t)y'(t)dt=\cos x\ ,\ \ y(0)=y'(0)=0\,.$$

Solution:

Let $y(x)\leftrightarrow Y(p)$.

We apply the Laplace transform to a given equation:

$$p^2Y(p)+2pY(p)-2\frac{1}{p^2+1}pY(p)=\frac{p}{p^2+1}\,.$$

Solving the equation with respect to $Y(p)$, we obtain

$$Y(p)=\frac{1}{p(p+1)^2}=\frac{1}{p}-\frac{1}{(p+1)^2}-\frac{1}{p+1}\leftrightarrow y(x)=1-e^{-x}x-e^{-x}\,.$$

In that way

$$y(x)=1-e^{-x}x-e^{-x}\,.$$

Example 5

Solve the integral-differential equation

$$y''(x)-2y'(x)+y(x)+2\int_0^x \cos(x-t)y''(t)dt+$$

$$+2\int_0^x \sin(x-t)y'(t)dt=\cos x\ ,\ \ y(0)=y'(0)=0\,.$$

Solution:

Let $y(x) \leftrightarrow Y(p)$.

We apply the Laplace transform to a given equation:

$$p^2Y(p) - 2pY(p) + Y(p) + 2\frac{p}{p^2+1}p^2Y(p) + \frac{2}{p^2+1}pY(p) = \frac{p}{p^2+1}.$$

Solving the equation with respect to $Y(p)$, we obtain

$$Y(p) = \frac{p}{p^2+1}\frac{1}{p^2+1} \leftrightarrow y(x) = \int_0^x \cos(x-t)\sin t\,dt = \frac{x}{2}\sin x.$$

When switching to the original, the convolution image property was used.