1.2. RESTORING THE ORIGINAL IMAGE

To find the original function for a given image, it requires knowledge of the tables of correspondence between the originals and images, the application of the properties of the Laplace transform, the decomposition of the image into the simplest fractions, the use of decomposition theorems.

Using the properties of the Laplace transform

First of all, it is necessary to bring the function to a simpler, "tabular" form.

If the denominator of a fraction contains a square trinomial, then the full square is allocated in it.

It is convenient to use the original integration theorem to find the original fraction $\frac{F(p)}{p^n}$ if the original $f(t)$ of the $F(p)$ image is known.

The presence of the $e^{-p\tau}, \tau > 0$ multiplier in image $F(p)$ indicates the need to apply the delay theorem.

If the image is represented as $F(p) = F_1(p)F_2(p)$ or $F(p) = pF_1(p)F_2(p)$ and the originals $f_1(t) \leftrightarrow F_1(p)$, $f_2(t) \leftrightarrow F_2(p)$ are known, then *Borel's theorem* and *Duhamel's integral* are used to find the original $f(t) \leftrightarrow F(p)$, respectively.

Duhamel's integral

If

$$f_1 * f_2 = \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau \leftrightarrow F_1(p) F_2(p)$$

then

$$f_1(t)\, f_2(0) + \int_0^t f_1(\tau)\, f_2'(t-\tau)\, d\tau \leftrightarrow pF_1(p) F_2(p).$$

Comment

1) Due to the symmetry of the convolution $f_1 * f_2 = f_2 * f_1$

$$f_1(t)\, f_2(0) + \int_0^t f_1(\tau)\, f_2'(t-\tau)\, d\tau = f_1(t)\, f_2(0) + \int_0^t f_1(t-\tau)\, f_2'(\tau)\, d\tau$$

2) Obviously

$$f_1(0)\, f_2(t) + \int_0^t f_1'(\tau) f_2(t-\tau)\, d\tau = f_1(0)\, f_2(t) + \int_0^t f_1'(t-\tau) f_2(\tau)\, d\tau$$

Example 6.

Find the original corresponding to the image (using the Duhamel integral):

a) $F(p) = \dfrac{2p^2}{\left(p^2+1\right)^2}$,

b) $F(p) = \dfrac{p^3}{\left(p^2+1\right)\left(p^2+4\right)}$,

c) $F(p)=\dfrac{2p}{(p-1)\left(p^2-2p-3\right)}$.

<u>Solution:</u>

a) Let's write an image in the form

$$\frac{2p^2}{\left(p^2+1\right)^2}=2p\cdot\frac{1}{p^2+1}\cdot\frac{p}{p^2+1}.$$

Since

$$\frac{1}{p^2+1}\leftrightarrow \sin t=f_1(t),\ \frac{p}{p^2+1}\leftrightarrow \cos t=f_2(t),$$

$$f_1(0)=\sin 0=0,\ f_1'(t)=\cos t,$$

then based on the Duhamel formula we have

$$2p\cdot\frac{1}{p^2+1}\cdot\frac{p}{p^2+1}\leftrightarrow 0+2\int_0^t \cos\tau\cos(t-\tau)\,d\tau=$$

$$=2\int_0^t\frac{\cos t+\cos(t-2\tau)}{2}d\tau=\left(\tau\cos t-\frac{1}{2}\sin(t-2\tau)\right)\Bigg|_0^t=t\cos t+\sin t.$$

b) Let's write an image in the form

$$\frac{p^3}{\left(p^2+1\right)\left(p^2+4\right)}=p\cdot\frac{p}{p^2+1}\cdot\frac{p}{p^2+4}.$$

$$\frac{p}{p^2+1}\leftrightarrow \cos t=f_1(t),\ \frac{p}{p^2+4}\leftrightarrow \cos 2t=f_2(t),$$

$$f_1(0)=\cos 0=1,\quad f_1'(t)=-\sin t.$$

Then we get

$$f(t)=f_1(0)f_2(t)+\int_0^t f_1'(\tau)f_2(t-\tau)d\tau=$$

$$=\cos 0\cdot\cos 2t+\int_0^t \cos'\tau\cdot\cos 2(t-\tau)d\tau=$$

$$=\cos 2t-\int_0^t \sin\tau\cos 2(t-\tau)d\tau=$$

$$=\cos 2t-\frac{1}{2}\int_0^t\left(\sin(3\tau-2t)+\sin(2t-\tau)\right)d\tau=$$

$$=\cos 2t+\frac{1}{6}\cos(3\tau-2t)\Big|_0^t-\frac{1}{2}\cos(2t-\tau)\Big|_0^t=\frac{4}{3}\cos 2t-\frac{1}{3}\cos t.$$

c) <u>HOMEWORK №1</u>

Example 7.

Using the properties of the Laplace transform to find the original corresponding to the image:

a) $F(p)=\dfrac{p}{p^2-2p+26}$,

b) $F(p)=\dfrac{1}{p\left(p^2+4\right)}$,

c) $F(p)=\dfrac{e^{-p}}{p+1}$,

d) $F(p)=\dfrac{p}{(p^2+4)^2}$

<u>Solution:</u>

a) Let's transform the image by highlighting the full square in the denominator. To find the original, we will use the displacement theorem, the linearity property and the image table.

$$\frac{p}{p^2-2p+26}=\frac{(p-1)+1}{(p-1)^2+25}=\frac{p-1}{(p-1)^2+25}+\frac{1}{(p-1)^2+25}\leftrightarrow$$

$$\leftrightarrow e^t\left(\cos 5t+\frac{1}{5}\sin 5t\right)$$

b) From the table of images, we have $\frac{2}{p^2+4}\leftrightarrow \sin 2t$.

Using the linearity and integration properties of the original, we find

$$F(p)=\frac{1}{p\left(p^2+4\right)}=\frac{1}{2}\cdot\frac{1}{p}\cdot\frac{2}{p^2+4}\leftrightarrow$$

$$\frac{1}{2}\int_0^t \sin 2\tau d\tau=-\frac{1}{4}\cos 2\tau\Big|_0^t=\frac{1}{4}(1-\cos 2t).$$

You can also find the original by representing the original function as the sum of the simplest fractions,

$$F(p)=\frac{1}{4}\left(\frac{1}{p}-\frac{p}{p^2+4}\right)\leftrightarrow$$

$$\leftrightarrow\frac{1}{4}(1-\cos 2t)$$

c) From the table of images we have $\frac{1}{p+1}\leftrightarrow e^{-t}$. The presence of a multiplier e^{-p} indicates the need to apply the delay theorem. Therefore

$$\frac{e^{-p}}{p+1}\leftrightarrow e^{-(t-1)}\,\theta(t-1)$$

d) Let's write an image in the form

$$F(p)=\frac{p}{(p^2+4)^2}=\frac{1}{p^2+4}\cdot\frac{p}{p^2+4},$$

$$\frac{1}{p^2+4}\leftrightarrow \sin 2t,\ \frac{p}{p^2+4}\leftrightarrow \cos 2t\,.$$

Let's apply the image multiplication theorem (Borel's theorem)

$$F(p)\leftrightarrow \sin 2t * \cos 2t=\int_0^t \sin 2\tau \cos 2(t-\tau)d\tau=$$

$$=\frac{1}{2}\int_0^t(\sin(4\tau-2t)+\sin 2t)\,d\tau=\frac{1}{2}\left(-\frac{\cos(4\tau-2t)}{4}+\tau\cdot\sin 2t\right)\Bigg|_0^t=$$
$$=\frac{1}{2}\left(-\frac{\cos 2t}{4}+t\sin 2t+\frac{\cos(-2t)}{4}\right)=\frac{t}{2}\sin 2t.$$

1.2.1 The elementary method

Example 8.

Find the original corresponding to the image:

a) $F(p)=\dfrac{-5}{p(p-1)\left(p^2+4p+5\right)}$;

b) $F(p)=\dfrac{1}{p(p-1)\left(p^2+4\right)}$.

Solution:

a) Let's imagine $F(p)$ as the sum of elementary fractions:

$$F(p)=\frac{-5}{p(p-1)\left(p^2+4p+5\right)}=\frac{A}{p}+\frac{B}{p-1}+\frac{Cp+D}{p^2+4p+5}.$$

To find A,B,C,D we have the equation

$$A(p-1)\left(p^2+4p+5\right)+Bp\left(p^2+4p+5\right)+(Cp+D)p(p-1)=-5\,.$$

Substituting different values of p, we obtain a system for determining the coefficients

$$p=0: \quad -5A=-5, \qquad p=1: \quad 10B=-5,$$

$$p=-1: \quad -4A-2B-2(-C+D)=-5,$$

$$p=-2: \quad -3A-2B+6(-2C+D)=-5.$$

We find the coefficients:

$$A=1, \quad B=-\frac{1}{2}, \quad C=-\frac{1}{2}, \quad D=-\frac{3}{2}.$$

$$F(p)=\frac{1}{p}-\frac{1}{2}\cdot\frac{1}{p-1}-\frac{1}{2}\cdot\frac{p+3}{p^2+4p+5}=$$

$$=\frac{1}{p}-\frac{1}{2}\frac{1}{p-1}-\frac{1}{2}\left(\frac{p+2}{(p+2)^2+1}+\frac{1}{(p+2)^2+1}\right)\leftrightarrow$$

$$1-\frac{1}{2}e^{t}-\frac{1}{2}e^{-2t}(\cos t+\sin t).$$

b) <u>HOMEWORK №2</u>

1.2.2 The conversion formula. Decomposition theorems

Theorem 1 (Riemann-Mellin).

Let the function $f(t)$ be the original with the growth index α_0, and $F(p)$ be its image. Then at any point t of the continuity of the original $f(t)$, the Riemann-Mellin formula is valid

$$f(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}F(p)e^{pt}dp \qquad (1.6)$$

where integration is performed along any straight line $\operatorname{Re} p = b$, $b > \alpha_0$, and the integral is understood in the sense of the principal value.

Equality takes place at every point at which $f(t)$ is continuous. At the point t_0, which is the point of discontinuity of the 1st kind of the function $f(t)$, the right side of the Riemann-Mellin formula is equal to

$$\frac{1}{2}\left(f(t_0 - 0) + f(t_0 + 0)\right).$$

The Riemann-Mellin formula (1.6) is the inverse of the formula $F(p) = \int\limits_0^{+\infty} f(t) e^{-pt}\, dt$ and is called *the inverse Laplace transform*.

The direct application of the conversion formula to restore the original $f(t)$ from the $F(p)$ image is difficult. Decomposition theorems are usually used to find the original.

Theorem 2 (the first decomposition theorem).

If the function $F(p)$ in the vicinity of point $p = \infty$ can be represented as a Laurent series (point $p = \infty$ is the zero of the function $F(p)$ and $F(p)$ is analytic in the vicinity of this point)

$$F(p) = \sum_{k=0}^{\infty} \frac{c_k}{p^{k+1}} = \frac{c_0}{p} + \frac{c_1}{p^2} + \frac{c_2}{p^3} + \ldots,$$

then the function

$$f(t)=\sum_{k=0}^{\infty} c_k \cdot \frac{t^k}{k!}=c_0+c_1t+c_2\cdot\frac{t^2}{2!}+\ldots, \quad t\geq 0$$

is the original with the image $F(p)$.

Example 9.

Find the original corresponding to the image using the first decomposition theorem:

a) $F(p)=\dfrac{p}{p^2+1}$,

b) $F(p)=\dfrac{1}{p\left(p^4+1\right)}$,

c) $F(p)=\dfrac{1}{p}e^{\frac{1}{p^2}}$.

<u>Solution:</u>

a) Decompose the function $F(p)$ into a Laurent series

$$\frac{p}{p^2+1}=\frac{1}{p\left(1+\frac{1}{p^2}\right)}=\frac{1}{p}\left(1-\frac{1}{p^2}+\frac{1}{p^4}-\cdots\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{2n+1}}, \quad |p|>1.$$

Since $\dfrac{1}{p^{2n+1}}\leftrightarrow\dfrac{t^{2n}}{(2n)!}$, then according to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{2n+1}}\leftrightarrow\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}=\cos t=f(t).$$

b) Decompose the function $F(p)$ into a Laurent series

$$\frac{1}{p\left(p^4+1\right)}=\frac{1}{p^5\left(1+\frac{1}{p^4}\right)}=\frac{1}{p^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n+5}},\quad |p|>1\,.$$

Since $\frac{1}{p^{4n+5}}\leftrightarrow\frac{t^{4(n+1)}}{(4(n+1))!}$, then according to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n+5}}\leftrightarrow\sum_{n=0}^{\infty}\frac{(-1)^n t^{4(n+1)}}{(4(n+1))!}=f(t)$$

c) Using the power series expansion of the function

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

we obtain

$$\frac{1}{p}e^{\frac{1}{p^2}}=\frac{1}{p}\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n}}=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}}\,.$$

According to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}}\leftrightarrow\sum_{n=0}^{\infty}\frac{t^{2n}}{n!(2n)!}=f(t)\,.$$

$$F(p)=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}}\leftrightarrow\sum_{n=0}^{\infty}\frac{t^{2n}}{n!(2n)!}=f(t)$$

Theorem 3 (the second decomposition theorem).

Let the function $F(p)$ of the complex variable p be analytic in the entire plane, with the exception of a finite number of isolated singularity points $p_1, p_2, \dots, p_n$, located in the half-plane $\operatorname{Re} p < \alpha_0$.

If $\lim\limits_{p\to\infty} F(p) = 0$, and $F(p)$ is absolutely integrable along any vertical line $\operatorname{Re} p = b,\ b > \alpha_0$, then $F(p)$ is an image, and the original $f(t)$ corresponding to the image $F(p)$ is determined by the formula

$$F(p) \leftrightarrow \sum_{k=1}^{n} \operatorname*{Res}_{p=p_k} \left[F(p) e^{pt} \right] = f(t)$$

If p_k is a pole of order m_k, then

$$\operatorname*{Res}_{p=p_k} \left[F(p) e^{pt} \right] = \lim_{p\to p_k} \left\{ \frac{d^{m_k-1}}{dp^{m_k-1}} \left((p-p_k)^{m_k} F(p) e^{pt} \right) \right\} =$$
$$= \sum_{j=0}^{m_k-1} \frac{t^{m_k-1-j}}{j!(m_k-1-j)!} \lim_{p\to p_k} \left\{ \frac{d^j}{dp^j} \left((p-p_k)^{m_k} F(p) \right) \right\}.$$

If $F(p) = \dfrac{P(p)}{Q(p)}$ is a rational regular irreducible fraction, p_k are poles of the order m_k, $(k = 1,2,\dots,n)$ of the function $F(p)$, then the original $f(t)$ corresponding to the image $F(p)$ is determined by the formula

$$F(p) \leftrightarrow$$

$$\sum_{k=1}^{n} \frac{1}{(m_k-1)!} \lim_{p\to p_k} \left\{ \frac{\mathrm{d}^{m_k-1}}{\mathrm{d}p^{m_k-1}} \left((p-p_k)^{m_k} F(p) e^{pt} \right) \right\} = f(t). \tag{1.7}$$

In particular, if $p_1, p_2, \dots, p_n$ are the simple poles of $F(p)$, then the function

$$f(t)=\sum_{k=1}^{n}\frac{P(p_k)}{Q'(p_k)}e^{p_k t} \qquad (1.8)$$

Example 10.

Find the original corresponding to the image

$$F(p)=\frac{p^2+2}{p^3-p^2-6p}.$$

<u>Solution:</u>

Since $p^3-p^2-6p=p(p-3)(p+2)$, the function $F(p)$ has three simple poles: $p_1=0$, $p_2=3$, $p_3=-2$. Let's construct the corresponding original using the formula (1.8):

$$f(t)=\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=0}+\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=3}+\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=-2}=-\frac{1}{3}+\frac{11}{5}e^{3t}+\frac{3}{5}e^{-2t}.$$

Example 11.

Using the second decomposition theorem, find the original corresponding to the image

a) $$F(p)=\frac{p^2+p-1}{(p-2)(p-5)(p+4)}$$

b) $$F(p)=\frac{p-1}{(p+1)\left(p^2+4\right)}$$

<u>Solution:</u>

a) Function $F(p)=\dfrac{p^2+p-1}{(p-2)(p-5)(p+4)}$ has simple poles (zeros of the denominator) $p_1=2, p_2=5, p_3=-4$. Let's denote

$$P(p)=p^2+p-1, \quad Q(p)=p^3-3p^2-18p+40,$$

$$Q'(p)=3p^2-6p-18.$$

Then for $p_1=2$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_1=2}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_1=2}=-\frac{5}{18},$$

for $p_2=5$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_2=5}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_2=5}=\frac{29}{27},$$

for $p_3=-4$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_3=-4}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_3=-4}=\frac{11}{54}.$$

Therefore, according to the formula (1.8)

$$F(p)=\frac{p^2+p-1}{(p-2)(p-5)(p+4)}\leftrightarrow$$

$$-\frac{5}{18}e^{2t}+\frac{29}{27}e^{5t}+\frac{11}{54}e^{-4t}=\frac{1}{54}\left(11e^{-4t}+58e^{5t}-15e^{2t}\right)=f(t).$$

b) HOMEWORK №3