Homework 1.

1. $\displaystyle\int \sqrt{1+\sin 2x}\,\mathrm{d}x \ (0 \le x \le \pi) = \int |\sin x + \cos x|\,dx = |\sin x - \cos x| + C$ $\quad \begin{cases} \sin x - \cos x + C_1 & x \in [0, \frac{3\pi}{4}] \\ -\sin x + \cos x + C_1 & x \in [\frac{3\pi}{4}, \pi] \end{cases}$

2. $\displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{x^2+1+\sqrt{(1+x^2)^3}}} = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{(x^2+1)+\sqrt{(x^2+1)^3}}} \overset{x^2+1=t}{=\!=} \frac{1}{2}\int \frac{dt}{\sqrt{t+\sqrt{t^3}}} = \frac{1}{2}\int \frac{dt}{\sqrt{t}\cdot\sqrt{1+\sqrt{t}}} \underset{dt=2mdm}{\overset{\sqrt{t}=m}{=\!=}} \int \frac{m\cdot dm}{m\cdot\sqrt{1+m}} = \int \frac{dm}{\sqrt{1+m}} = 2\int \frac{dm}{2\sqrt{1+m}} = 2\sqrt{1+m} + C$

$= 2\sqrt{1+\sqrt{t}} + C$

$= 2\sqrt{1+\sqrt{1+x^2}} + C$

3. $\displaystyle\int \frac{x^2\,\mathrm{d}x}{(1-x)^{100}} = \frac{x^2}{99(1-x)^{99}} + \int 2x\cdot\frac{1}{(1-x)^{99}} = \frac{x^2}{99(1-x)^{99}} + 2\int\frac{d(1-x)}{(1-x)^{99}} = \frac{x^2}{99(1-x)^{99}} + 2\cdot\frac{1}{98}(1-x)^{-98} + C$

$= \frac{x^2}{99}(1-x)^{-99} + \frac{1}{49}(1-x)^{-98} + C$

4. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+e^x}} \underset{x=\ln(t^2-1)}{\overset{\sqrt{1+e^x}=t(t>1)}{=\!=}} \int \frac{2t\cdot dt}{t^2-1\cdot t} = 2\int\frac{dt}{t^2-1} = \int\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = \ln|\sqrt{1+e^x}-1| - \ln|\sqrt{1+e^x}+1| + C$

5. $\displaystyle\int \frac{xe^{\arctan x}\,\mathrm{d}x}{(1+x^2)^{3/2}} = e^{\arctan x}\cdot\frac{x}{\sqrt{1+x^2}} - \int e^{\arctan x}\cdot\frac{\sqrt{1+x^2}-\frac{2x\cdot x}{2\sqrt{1+x^2}}}{1+x^2} = e^{\arctan x}\cdot\frac{x}{\sqrt{1+x^2}} - \int e^{\arctan x}\cdot\frac{1}{\sqrt{1+x^2}(1+x^2)} = e^{\arctan x}\cdot\frac{x-1}{\sqrt{1+x^2}} - \int e^{\arctan x}\cdot\frac{x}{\sqrt{(1+x^2)^3}}$

$\therefore$ the original integral $= \frac{1}{2}\cdot e^{\arctan x}\cdot\frac{x-1}{\sqrt{1+x^2}}$

6. $\displaystyle\int \frac{\mathrm{d}x}{\sin x\cos x} = \int\frac{d(\sin x)}{\sin x(1-\sin^2 x)} \overset{\sin x = t}{=\!=} \int\frac{dt}{t(1-t)(1+t)} = \int\left(\frac{\frac{1}{2}}{1-t} + \frac{-\frac{1}{2}}{1+t} + \frac{1}{t}\right)dt = -\frac{1}{2}\ln|1+\sin x| - \frac{1}{2}\ln|1-\sin x| + \ln|\sin x| + C$

$= \ln|\tan x| + C$

7. $\displaystyle\int \frac{\mathrm{d}x}{\sin x} = \int\frac{dx}{2\cos\frac{x}{2}\sin\frac{x}{2}} = \int\frac{d\frac{x}{2}}{\cos\frac{x}{2}\sin\frac{x}{2}} = \ln|\tan\frac{x}{2}| + C$

8. $\displaystyle\int \frac{xe^x\,\mathrm{d}x}{(x+1)^2} = -\frac{x\cdot e^x}{(x+1)} + \int(x+1)\cdot\frac{e^x}{x+1}\,dx = \frac{e^x(x+1)-xe^x}{x+1} = \frac{e^x}{x+1} + C$

9. $\displaystyle\int \sin(\log x)\,\mathrm{d}x \underset{dt=\frac{1}{x}dx}{\overset{\log x = t}{=\!=}} \int e^t\sin t\,dt = e^t\sin t - \int e^t\cos t\,dt = e^t\sin t - e^t\cos t - \int e^t\sin t\,dt$

$\displaystyle\int e^t\sin t\,dt = \frac{e^t\sin t - e^t\cos t}{2} = \frac{x[\sin(\ln x) - \cos(\ln x)]}{2} + C$

10. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}(x+1)}\,\mathrm{d}x = 2\int\frac{\arctan\sqrt{x}}{(x+1)}\,d(\sqrt{x}) = 2\arctan^2\sqrt{x} - 2\int\frac{\arctan\sqrt{x}}{(x+1)}\,dx$

$\Leftrightarrow \displaystyle\int\frac{\arctan\sqrt{x}}{\sqrt{x}(x+1)}\,dx = \cancel{\tfrac{1}{2}}\arctan^2\sqrt{x} + C$

Homework 2.

1. $I = \displaystyle\int \frac{(x^3+1)\,\mathrm{d}x}{x^3 - 5x^2 + 6x}$

2. $\displaystyle\int \frac{x\,\mathrm{d}x}{x^3 - 1}$

3. $I = \displaystyle\int \frac{\mathrm{d}x}{x^6 + 1}$

4. For what condition $I = \displaystyle\int \frac{ax^2 + bx + c}{x^3(x+1)^2}\,\mathrm{d}x$ is rational?

5. For what condition $I = \displaystyle\int \frac{\alpha x^2 + 2\beta x + \gamma}{(x - x_1)^4}\,\mathrm{d}x$ is rational?

6. $I = \displaystyle\int \frac{2x^4 - 4x^3 + 24x^2 - 40x + 20}{(x-1)(x^2 - 2x + 2)^3}\,\mathrm{d}x$

1. $I = \displaystyle\int \frac{(x^3+1)\,dx}{x^3-5x^2+6x} = x + \int \frac{5x^2-6x+1}{x^3-5x^2+6x}\,dx$

$= x - \displaystyle\int \frac{(5x-1)(x-1)}{x(x-2)(x-3)}\,dx$

$5x^2-6x+1 = A(x-2)(x-3) + B(x-3)x + C(x-2)x.$

$\begin{cases} A+B+C=5 \\ 5A+3B+2C=6 \\ 6A=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{6} \\ B=-\frac{9}{2} \\ C=\frac{28}{3} \end{cases}$

$\therefore I = \displaystyle\int \frac{1}{6}\frac{dx}{x} + \left(-\frac{9}{2}\right)\frac{dx}{x-2} + \left(\frac{28}{3}\right)\frac{dx}{x-3} = \frac{1}{6}\ln|x| + \frac{28}{3}\ln|x-3| - \frac{9}{2}\ln|x-2| + C$

2. $\displaystyle\int \frac{x\,dx}{x^3-1} = \int \frac{x\,dx}{(x-1)(x^2+x+1)} = \int \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right)dx$

$A(x^2-x+1) + (Bx+C)(x-1) = x.$

$\begin{cases} A+B=0 \\ -A+C-B=1 \\ A-C=0 \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=1 \end{cases}$

$\displaystyle\int \left(\frac{1}{x-1} + \frac{-x+1}{x^2+x+1}\right)dx = \ln|x-1| + -\frac{1}{2}\int \frac{d(x^2+x+1)-3}{x^2+x+1}$

$= \ln|x-1| + -\frac{1}{2}\ln(x^2+x+1) + \frac{3}{2}\displaystyle\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$

$= \ln|x-1| + \left(-\frac{1}{2}\right)\ln(x^2+x+1) + \frac{\cancel{9}}{8}\,\frac{3}{2}\times\frac{4}{3}\displaystyle\int \frac{1}{\frac{4}{3}(x+\frac{1}{2})^2+1}\,dx$

$= \ln|x-1| + \left(-\frac{1}{2}\right)\ln(x^2+x+1) + \frac{9}{8}\cdot\frac{\sqrt{3}}{2}\displaystyle\int \frac{d\,\frac{2}{\sqrt{3}}(x+\frac{1}{2})}{[\frac{2}{\sqrt{3}}(x+\frac{1}{2})]^2+1}$

$= \ln|x-1| - \frac{1}{2}\ln(x^2+x+1) + \cancel{\frac{9\sqrt{3}}{16}}\,\sqrt{3}\arctan\frac{2}{\sqrt{3}}(x+\frac{1}{2})$

3. $I = \displaystyle\int \frac{dx}{x^6+1} = \int \frac{dx}{(x^2+1)(x^4-x^2+1)} = \int \frac{dx}{x^4-x^2+1}\,dx - \int \frac{x^2\,dx}{(x^2+1)(x^4-x^2+1)}$

$= \displaystyle\int \frac{dx}{x^4-x^2+1} - \frac{1}{3}\int \frac{dx^3}{(x^3)^2+1}$

$= \frac{1}{2}\displaystyle\int \frac{(x^2+1)\,dx}{x^4-x^2+1} - \frac{1}{2}\int \frac{x^2-1}{x^4-x^2+1} - \frac{1}{3}\arctan x^3.$

$= \frac{1}{2}\displaystyle\int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+1} - \frac{1}{2}\int \frac{d(x+\frac{1}{x})}{(x+\frac{1}{x})^2-3} - \frac{1}{3}\arctan x^3 = \frac{1}{2}\arctan(x-\frac{1}{x}) - \frac{1}{3}\arctan x^3 - \frac{1}{4\sqrt{3}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right| + C$

4. $I = \displaystyle\int \frac{ax^2+bx+c}{x^3(x+1)^2}$

$Q_1 = x^2(x+1) \qquad Q_2 = x(x+1)$

$I = \dfrac{Ax^2+Bx+C}{x^2(x+1)} + \displaystyle\int \frac{Dx+E}{x(x+1)}\,dx$

$\displaystyle\int \frac{Dx+E}{x(x+1)} = \int \left(\frac{F}{x} + \frac{G}{x+1}\right)dx.$

Thus. I is rational if and only if $F=G=0$

which means.

$\left(\dfrac{Ax^2+Bx+C}{x^2(x+1)}\right)' + \dfrac{Dx+E}{x^2(x+1)} = \dfrac{ax^2+bx+c}{x^3(x+1)^2}$

$\dfrac{x(x+1)(2Ax+B) - (x^2+2(x+1))(Ax^2+Bx+C) + (Dx+E)x(x+1)}{x^3(x+1)^2} = \dfrac{ax^2+bx+c}{x^3(x+1)^2}.$

$\begin{cases} 2B=a \\ -3C-B=b \\ -2C=c \end{cases}$ $\quad \frac{a}{2} = \frac{3c}{2} - b.$

$\Rightarrow$ a. b. c should such that $a+2b=3c$

5. $I = \dfrac{\alpha x^2+2\beta x+\gamma}{(x-x_1)^4} = \dfrac{Ax^2+Bx+C}{(x-x_1)^3} + \displaystyle\int \frac{D}{x-x_1}\,dx$

I is rational if and only if $D=0$.

$\dfrac{\alpha x^2+2\beta x+\gamma}{(x-x_1)^4} = \left(\dfrac{Ax^2+Bx+C}{(x-x_1)^3}\right)' = \dfrac{(2Ax+B)(x-x_1)^3 - 3(x-x_1)^2(Ax^2+Bx+C)}{(x-x_1)^6} = \dfrac{2Ax^2+(B-2Ax_1)x - x_1B - 3Ax^2 - 3Bx - 3C}{(x-x_1)^4}$

$\begin{cases} -A = \alpha. \\ B-2Ax_1-3B=\beta \\ -3C=\gamma \end{cases}$ $\quad \alpha.\beta.\gamma.x_1$ can always satisfied the condition of A.B.C

6. ① $I = \displaystyle\int \frac{2x^4-4x^3+24x^2-40x+20}{(x-1)(x^2-2x+2)^3}\,dx \overset{x-1=t}{=\!=} \int \frac{2(t+1)^2t^2+2(t+1)^2+20t^2}{t(t^2+1)^3}\,dt = \int \frac{2(t+1)^2}{t(t^2+1)^2}\,dt + \int \frac{20t}{(t^2+1)^3}\,dt$

② $\displaystyle\int \frac{2(t+1)^2}{t(t^2+1)^2}\,dt. = \int \frac{At+B}{(t^2+1)^2}\,dt + \int \frac{Ct+D}{t^2+1} + \int \frac{E}{t}\,dt. = \int \frac{4}{(t^2+1)^2}\,dt - \int \frac{2t}{t^2+1}\,dt + 2\int \frac{dt}{t^2+1}\,dt.$

$(At+B)t + (Ct+D)(t^3+t) + E(t^2+1)^2 = 2(t+1)^2$

$\begin{cases} C+E=0 \\ D=0 \\ A+C+2E=2 \\ B+D=4 \\ E=2 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=4 \\ C=-2 \\ D=0 \\ E=2 \end{cases}$

③ $\displaystyle\int \frac{1}{(t^2+1)^2}\,dt \overset{t=\tan s}{=\!=} \int \frac{\sec^2 s}{(\tan^2 s+1)^2}\,ds = \int \frac{ds}{\sec^2 s} = \int \cos^2 s\,ds$

$= \frac{1}{2}\displaystyle\int (1+\cos 2s)\,ds = \frac{s}{2} + \frac{\sin 2s}{4} + C = \frac{\arctan t}{2} + \frac{t}{2(t^2+1)} + C$

In conclusion

$I = -5\cdot(x^2-2x+2)^{-2} + 4\arctan(x-1) + \dfrac{2(x-1)}{(x^2-2x+2)} - \ln(x^2-2x+2) + C$

Homework 3.

1. $I=\int \dfrac{\sin 2x \,\mathrm{d}x}{4\cos^2 x+12\cos x-7}$

2. $I=\int \dfrac{\mathrm{d}x}{\sin^6 x+\cos^6 x}$

3. $I=\int \cot^6 x \,\mathrm{d}x$

4. $I=\int \sin^3 2x\cos^2 3x \,\mathrm{d}x$

5. $I=\int \dfrac{\mathrm{d}x}{\sin(x+a)\cos(x+b)}$

1. $I=\int \dfrac{\sin 2x\,dx}{4\cos^2x+12\cos x-7}=\int\dfrac{\sin 2x}{(2\cos x+7)(2\cos x-1)}dx=-\int\dfrac{2\cos x\,d(\cos x)}{(2\cos x+7)(2\cos x-1)}=-\frac{1}{4}\int\left(\dfrac{\cos x}{2\cos x-1}-\dfrac{\cos x}{2\cos x+7}\right)d\cos x$

$\overset{\cos x=t}{=}\frac{1}{4}\int\left(\frac{t}{2t+7}-\frac{t}{2t-1}\right)dt.=\frac{1}{4}\int\frac{1}{2}\left(\frac{2t+7}{2t+7}-\frac{7}{2t+7}\right)-\frac{1}{2}\left(\frac{2t-1}{2t-1}+\frac{1}{2t-1}\right)dt=-\frac{1}{8}\int\left(\frac{7}{2t+7}+\frac{1}{2t-1}\right)dt=-\frac{1}{8}\int\frac{\frac{7}{2}d(2t+7)}{2t+7}+\frac{\frac{1}{2}d(2t-1)}{2t-1}$

$=-\frac{7}{16}\ln(2\cos x+7)-\frac{1}{16}\ln|2\cos x-1|+C$

2. $I=\int\dfrac{dx}{\sin^6x+\cos^6x}=\int\dfrac{dx}{(\sin^2x+\cos^2x)(\sin^4x-\sin^2x\cos^2x+\cos^4x)}=\int\dfrac{dx}{\sin^4x-\sin^2x\cos^2x+\cos^4x}=\int\dfrac{dx}{\cos^4x-\cos2x\cdot\sin^2x}=\int\dfrac{dx}{\left(\frac{1+\cos2x}{2}\right)^2-\cos2x\left(\frac{1-\cos2x}{2}\right)}=\int\dfrac{dx}{\frac{3}{4}\cos^22x+\frac{1}{4}}$

$=4\int\dfrac{dx}{4\cos^22x+\sin^22x}=2\int\dfrac{d(2x)}{4\cos^22x+\sin^2x}=2\int\dfrac{\sec^22x\,d(2x)}{4+\tan^22x}=2\int\dfrac{d(\tan2x)}{4+\tan^22x}=\int\dfrac{d\left(\frac{\tan2x}{2}\right)}{1+\left(\frac{\tan2x}{2}\right)^2}=\arctan\left(\frac{\tan2x}{2}\right)+C$ ✓

3. $I=\int\cot^6x\,dx=\int\dfrac{\cos^6x}{\sin^6x}dx=-\int\dfrac{\cos^4x(1-\sin^2x)}{\sin^4x}d(\cot x)=-\int\cot^4x\left(1-\frac{1}{\cot^2x+1}\right)d\cot x=-\int\cot^4x\cdot\frac{\cot^2x}{\cot^2x+1}d\cot x=-\int\dfrac{\cot^4x(\cot^2x+1)-\cot^2x(\cot^2x+1)+\cot^2x+1-1}{\cot^2x+1}d\cot x=-\int(\cot^4x-\cot^2x+1)\,d\cot x+\int\frac{1}{\cot^2x+1}d\cot x.$

$=-\frac{1}{5}\cot^5x+\frac{1}{3}\cot^3x-\cot x+\arctan(\cot x)+C_1$

$=-\frac{1}{5}\cot^5x+\frac{1}{3}\cot^3x-\cot x-x+C$ ✓

4. $I=\int\sin^32x\cos^23x=\int\sin2x\left(\frac{1-\cos4x}{2}\right)\left(\frac{1+\cos6x}{2}\right)=\int\frac{\sin2x-\sin2x\cos4x}{2}\cdot\frac{1+\cos6x}{2}=\int\frac{\sin2x-\frac{1}{2}\sin6x+\frac{1}{2}\sin2x}{2}\cdot\frac{(1+\cos6x)}{2}dx$

$=\frac{1}{4}\int\left(\frac{3}{2}\sin2x-\frac{1}{2}\sin6x\right)(1+\cos6x)\,dx=\frac{1}{\cancel{4}\,8}\int\left(\cancel{\tfrac{3}{2}}\,3\sin2x+\cancel{\tfrac{3}{2}}\,3\sin2x\cos6x-\sin6x\,\cancel{+}-\frac{1}{2}\sin12x\right)dx$

$\cancel{\tfrac{3}{2}}\,3\sin2x\cos6x=\frac{3}{\cancel{4}\,2}(\sin8x-\sin4x)$

$\therefore I=-\frac{3}{16}\cos2x+\frac{1}{48}\cos6x+\frac{1}{192}\cos12x-\frac{3}{128}\cos8x+\frac{3}{64}\cos4x+C$

5. $I=\int\dfrac{dx}{\sin(x+a)\cos(x+b)}=\dfrac{1}{\cos(a-b)}\int\dfrac{\cos[(x+a)-(x+b)]}{\sin(x+a)\cos(x+b)}dx=\dfrac{1}{\cos(a-b)}\int\left(\dfrac{\cos(x+a)}{\sin(x+a)}+\dfrac{\sin(x+b)}{\cos(x+b)}\right)dx$

$=\dfrac{1}{\cos(a-b)}\ln\left|\dfrac{\sin(x+a)}{\cos(x+b)}\right|+C$ ✓

Homework 4.

1. $I = \displaystyle\int \frac{\mathrm{d}x}{\sqrt{x} + \sqrt[4]{x}}$

2. $I = \displaystyle\int \frac{\sqrt{x}+1}{x^2 - \sqrt{x}}\,\mathrm{d}x$

3. $I = \displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt[4]{x^3(a-x)}}, a > 0.$

4. $I = \displaystyle\int \frac{\mathrm{d}x}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}}$, $n \in \mathbb{N}$. (Hint: Look at the lecture example $\displaystyle\int \frac{\mathrm{d}x}{\sqrt[4]{(x-1)^3(x+2)^5}}$.)

1. $$I = \int \frac{dx}{\sqrt{x} + \sqrt[4]{x}} \xlongequal[dx = 4t^3 dt]{\sqrt[4]{x} = t > 0} \int \frac{4t^3\,dt}{t^2 + t} = \int \frac{4t^2}{t+1}\,dt = \int \frac{4t(t+1) - 4(t+1) + 4}{t+1}\,dt = \int \left(4t - 4 + \frac{4}{t+1}\right)dt = 2t^2 - 4t + 4\ln|t+1| + C = 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln|\sqrt[4]{x}+1| + C$$

2. $$I = \int \frac{\sqrt{x}+1}{x^2 - \sqrt{x}}\,dx \xlongequal{\sqrt{x} = t > 0} \int \frac{(t+1)\cdot 2t}{t^4 - t}\,dt = 2\int \frac{t+1}{(t-1)(t^2+t+1)}\,dt = 2\int \left(\frac{A}{t-1} + \frac{Bt+C}{t^2+t+1}\right)dt$$

$$A(t^2+t+1) + (Bt+C)(t-1) = t+1$$

$$\begin{cases} A+B = 0 \\ A-B+C = 1 \\ A-C = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{2}{3} \\ B = -\frac{2}{3} \\ C = -\frac{1}{3} \end{cases}$$

$$I = \frac{4}{3}\int \frac{1}{t-1}\,d(t-1) - \frac{2}{3}\int \frac{2t+1}{t^2+t+1}\,dt = \frac{4}{3}\ln|t-1| - \frac{2}{3}\ln|t^2+t+1| + C = \frac{4}{3}\ln|\sqrt{x}-1| - \frac{2}{3}\ln(x + \sqrt{x} + 1) + C$$

3. $$I = \int \frac{x\,dx}{\sqrt[4]{x^3(a-x)}}, \quad a > 0 \qquad x^3(a-x) > 0 \Rightarrow x < a.$$

$$= \int \sqrt[4]{\frac{x}{a-x}}\,dx$$

let $\sqrt[4]{\frac{x}{a-x}} = t$ $\quad x = \frac{at^4}{t^4+1}$ $\quad dx = \frac{4at^3(t^4+1) - 4t^3\cdot at^4}{(t^4+1)^2} = \frac{4at^3}{(t^4+1)^2}$

thus. $$I = \int \frac{4at^4}{(t^4+1)^2}\,dt = a\left(-\frac{1}{t^4+1}\cdot t + \int \frac{1}{t^4+1}\,dt\right)$$

$$\int \frac{1}{t^4+1}\,dt = \frac{1}{4\sqrt{2}}\int \frac{t+\sqrt{2}}{t^2+\sqrt{2}t+1}\,\cancel{dt} - \frac{1}{4\sqrt{2}}\int \frac{t-\sqrt{2}}{t^2-\sqrt{2}t+1}\,dt$$

$$= \frac{1}{4\sqrt{2}}\int \frac{(t+\frac{\sqrt{2}}{2})dt}{t^2+\sqrt{2}t+1} + \frac{1}{8}\int \frac{1}{(t+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\,dt - \frac{1}{4\sqrt{2}}\int \frac{t-\frac{\sqrt{2}}{2}}{t^2-\sqrt{2}t+1}\,dt + \frac{1}{8}\int \frac{1}{(t-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\,dt$$

$$I = \frac{-at}{t^4+1} + \frac{a}{4\sqrt{2}}\ln\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} + \frac{a}{2\sqrt{2}}\arctan(\sqrt{2}t+1) + \frac{a}{2\sqrt{2}}\arctan(\sqrt{2}t-1) + C$$

where $t = \sqrt[4]{\frac{x}{a-x}}$.

4. $$I = \int \frac{dx}{\sqrt[n]{(x-a)^{n+1}(x-b)^{n-1}}} \quad n \in \mathbb{N}.$$

$$= \int \sqrt[n]{\frac{x-b}{x-a}}\cdot\frac{1}{|(x-a)(x-b)|}$$

let $\sqrt[n]{\frac{x-b}{x-a}} = t$, $\quad x = \frac{at^n - b}{t^n - 1}$ $\quad x - a = \frac{a-b}{t^n-1}$ $\quad x - b = \frac{(a-b)t^n}{t^n-1}$

$$dx = \frac{-n(a-b)\cdot t^{n-1}}{(t^n-1)^2}$$

$$I = \int \frac{-n(a-b)t^n}{(t^n-1)^2}\cdot\frac{(t^n-1)^2}{(a-b)^2\cdot|t|^n}\,dt = \int \frac{-n\,t^n}{(a-b)|t|^n}\,dt$$

if $t > 0$. $I = \int \frac{-n}{a-b}\,dt = -\frac{n}{a-b}\cdot t = -\frac{n}{|a-b|}\sqrt[n]{\frac{x-b}{x-a}} + C_1$

if $t < 0$, n is odd. $I = \frac{n}{a-b}\sqrt[n]{\frac{x-b}{x-a}} + C_2$

n is even. impossible.

let $I(t\to 0^+) = I(t \to 0^-)$. $\Rightarrow$ $C_1 = C_2$.

Thus. $I = -\frac{n}{|a-b|}\left|\sqrt[n]{\frac{x-b}{x-a}}\right| + C_1$

Homework 5.

1. $I = \displaystyle\int \frac{(x^3 - 2)\,\mathrm{d}x}{\sqrt{x^2 + x + 1}}$

2. $I = \displaystyle\int \frac{\mathrm{d}x}{(x+1)^5\sqrt{x^2 + 2x}}$

3. Using the linear fractional substitution $x = \dfrac{\alpha t + \beta}{1 + t}$ find $I = \displaystyle\int \frac{\mathrm{d}t}{(x^2 - x + 1)\sqrt{x^2 + x + 1}}$.

4. $I = \displaystyle\int \frac{(x^2 + 1)\,\mathrm{d}x}{(x^2 - 1)\sqrt{x^4 + 1}}$

5. $I = \displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^4 + 2x^2 - 1}}$

1. $$\int \frac{(x^3-2)\,dx}{\sqrt{x^2+x+1}} = (Ax^2+Bx+C)\sqrt{x^2+x+1} + \lambda \int \frac{dx}{\sqrt{x^2+x+1}} \quad dx$$

$$\frac{x^3-2}{\sqrt{x^2+x+1}} = (2Ax+B)\sqrt{x^2+x+1} + \frac{(2x+1)(Ax^2+Bx+C)}{2\sqrt{x^2+x+1}} + \frac{\lambda}{\sqrt{x^2+x+1}}$$

$$\begin{cases} 2A + 2A = 1 \\ 2A+B+2B+A = 0 \\ 2A+B+2C+B = 0 \\ B+C+\lambda = -2 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = -\frac{1}{4} \\ C = 0 \\ \lambda = -\frac{7}{4}. \end{cases} \quad \begin{matrix} A = \frac{1}{3} \\ B = -\frac{5}{12} \\ C = -\frac{1}{24} \\ D = -\frac{25}{16} \end{matrix}$$

$$\int \frac{1}{\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}}\,dx = \frac{2}{\sqrt{3}}\int \frac{dx}{\sqrt{[\frac{2}{\sqrt{3}}(x+\frac{1}{2})]^2+1}} = \ln\left|\frac{2}{\sqrt{3}}(x+\tfrac{1}{2}) + \sqrt{\tfrac{4}{3}(x+\tfrac{1}{2})^2+1}\right| + C'$$

Thus, the original integral $= \frac{1}{4}(x^2-1)\sqrt{x^2+x+1} + \ln\left|\frac{2}{\sqrt{3}}(x+\frac{1}{2}) + \sqrt{\frac{4}{3}(x+\frac{1}{2})^2+1}\right| + C$

$$\left(\frac{x^3}{3} - \frac{5x}{12} - \frac{1}{24}\right)\sqrt{x^2+x+1} - \frac{25}{16}\ln\left|x + \frac{\sqrt{x^2+x+1}}{2}\right| + C$$

2. $$I = \int \frac{dx}{(x+1)^5\sqrt{x^2+2x}} \;\frac{t = \frac{1}{x+1}}{dx = -\frac{1}{t^2}}\; \int t^5 \cdot -\frac{1}{t^2} \cdot \frac{1}{\sqrt{\frac{1}{t^2}-1}}\,dt = \int \frac{-t^4}{\sqrt{1-t^2}}\,dt = \int (1+t^2)\sqrt{1-t^2}\,dt - \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\int (1+t^2)\sqrt{1-t^2}\,dt \;\frac{t = \sin\theta}{dt = \cos\theta\, d\theta.}\; \int (1+\sin^2\theta)\cos\theta \cdot \cos\theta\, d\theta. = \int \left(1+\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right) = \frac{1}{4}\int (3 + 2\cos 2\theta - \cos^2 2\theta)\,d\theta. = \frac{1}{4}\int \left(3 + 2\cos 2\theta - \frac{1+\cos 4\theta}{2}\right)d\theta. = \frac{5\theta}{8} + \frac{\sin 2\theta}{4} - \frac{1}{32}\sin 4\theta. + C$$

$$I = \int \frac{dx}{(x+1)^5\sqrt{x^2+2x}} \;\frac{\frac{1}{x+1} = t}{-\frac{1}{t^2}dt = dx}\; -\int \frac{t^4}{\sqrt{1-t^2}}\,dt = \int \frac{(1-t^2)(1+t^2)-1}{\sqrt{1-t^2}}\,dt = \int (1+t^2)\sqrt{1-t^2}\,dt - \int \frac{1}{\sqrt{1-t^2}}\,dt$$

$$\frac{t = \sin\theta}{dt = \cos\theta\, d\theta} \quad \int (1+\sin^2\theta)\cdot\cos^2\theta\, d\theta = \int \left(\frac{1+\cos 2\theta}{2} + \frac{1-\cos 4\theta}{8}\right)d\theta = \frac{5}{8}\theta + \frac{\sin 2\theta}{4} - \frac{\sin 4\theta}{32} + C$$

$$= \frac{5}{8}\arcsin t + \frac{1}{2}\cdot t\cdot\sqrt{1-t^2} + \frac{1}{4}t\sqrt{1-t^2} - \frac{1-t^2}{4} + C$$

$$I = -\frac{3}{8}\arcsin\frac{1}{x+1} + \frac{3}{8(x+1)}\sqrt{1-\frac{1}{(x+1)^2}} + \frac{1}{4}\sqrt{1-\frac{1}{(1+x)^2}}\;\frac{1}{(1+x)^3} + C$$

3. $$\int \frac{dx}{(x^2-x+1)\sqrt{x^2+x+1}} \qquad \text{Let } x = \frac{\alpha t+\beta}{t+1} \qquad \text{Let } t+1>0$$

$$x^2 - x + 1 = \frac{(\alpha t+\beta)^2 - (\alpha t+\beta)(t+1) + (t+1)^2}{(t+1)^2} = \frac{(\alpha^2-\alpha+1)t^2 + (2\alpha\beta-\alpha-\beta+2)t + \beta^2-\beta+1}{(t+1)^2}$$

$$x^2 + x + 1 = \frac{(\alpha t+\beta)^2 + (\alpha t+\beta)(t+1) + (t+1)^2}{(t+1)^2} = \frac{(\alpha^2+\alpha+1)t^2 + (2\alpha\beta+\alpha+\beta+2)t + \beta^2+\beta+1}{(t+1)^2}$$

$$\begin{cases} 2\alpha\beta-\alpha-\beta+2 = 0 \\ 2\alpha\beta+\alpha+\beta+2 = 0 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = -1 \end{cases} \text{ or } \begin{cases} \alpha = -1 \\ \beta = 1 \end{cases}, \text{ let } \begin{cases} \alpha = 1 \\ \beta = -1 \end{cases} \quad dx = \left(1 - \frac{2}{t+1}\right)' = \frac{2}{(t+1)^2}\,dt$$

$$\int \frac{\frac{2}{(1+t)^2}\,dt}{\frac{t^2+3}{(t+1)^2}\cdot\frac{\sqrt{3t^2+1}}{|t+1|}} = \int \frac{2(t+1)\,dt}{(t^2+3)\cdot\sqrt{3t^2+1}} = 2\int \frac{t\,dt}{(t^2+3)\sqrt{3t^2+1}} + 2\int \frac{dt}{(t^2+3)\sqrt{3t^2+1}}$$

let $u = (\sqrt{3t^2+1})'$.

$$du = \left(\frac{3t}{\sqrt{3t^2+1}}\right)'dt = \frac{3(\sqrt{3t^2+1}) - 3t\cdot\frac{3t}{\sqrt{3t^2+1}}}{(3t^2+1)}\,dt = \frac{3}{\sqrt{3t^2+1}} - \frac{3(3t^2+1)-3}{(3t^2+1)\sqrt{3t^2+1}} = \frac{3}{(3t^2+1)\sqrt{3t^2+1}}\,dt$$

$$\frac{du}{3-u^2} = \frac{dt}{\sqrt{3t^2+1}} \quad \therefore \int \frac{dt}{(t^2+3)\sqrt{3t^2+1}} = \int \frac{3(3-u^2)\,du}{(27-8u^2)(3-u^2)} = 3\int \frac{du}{27-8u^2} = \frac{1}{2\sqrt{3}}\int \left(\frac{1}{3\sqrt{3}-2\sqrt{2}u} + \frac{1}{3\sqrt{3}+2\sqrt{2}u}\right)du = \frac{1}{4\sqrt{6}}\ln\left|\frac{3\sqrt{3}+2\sqrt{2}u}{3\sqrt{3}-2\sqrt{2}u}\right| + C_1$$

Let $v = \sqrt{3t^2+1}$. $\quad dv = \dfrac{3t}{\sqrt{3t^2+1}}\,dt \qquad t^2+3 = \dfrac{v^2+8}{3}$

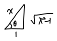
$$\int \frac{dv}{3}\cdot\frac{3}{v^2+8} = \int \frac{dv}{v^2+8} = \frac{2\sqrt{2}}{8}\arctan\frac{v}{2\sqrt{2}} + C_2$$

the original integral $= \dfrac{\sqrt{2}}{2}\arctan\dfrac{\sqrt{3t^2+1}}{2\sqrt{2}} + \dfrac{1}{2\sqrt{6}}\ln\left|\dfrac{3\sqrt{3}+2\sqrt{2}\cdot\sqrt{3t^2+1}}{3\sqrt{3}-2\sqrt{2}\cdot\sqrt{3t^2+1}}\right| + C$

where $t = \dfrac{x+1}{1-x}$ $(t>-1)$

4. $$I=\int\frac{(x^2+1)\,dx}{(x^2-1)\sqrt{x^4+1}}=\int\frac{\frac{(x^2+1)}{(x^2-1)^2}dx}{\sqrt{\frac{x^4+1}{x^4-2x^2+1}}}=\int\frac{\frac{x^2+1}{(x^2-1)^2}\,dx}{\sqrt{1+(\frac{\sqrt{2}x}{x^2-1})^2}}$$

First we consider $\int\frac{x^2+1}{(x^2-1)^2}\,dx\ \overset{x=\sec\theta}{\underset{dx=\sec\theta\cdot\tan\theta\,d\theta}{=\!=}}\int\frac{\sec^2\theta+1}{\tan^4\theta}\cdot\sec\theta\cdot\tan\theta\,d\theta=\int\frac{\frac{2\cos^2\theta+\sin^2\theta}{\cos^2\theta}}{\tan^4\theta}\sec\theta\tan\theta\,d\theta=\int\frac{(2\sec^2\theta-\tan^2\theta)\sec\theta\tan\theta}{\tan^4\theta}\,d\theta=-\frac{\sec\theta}{\tan^3\theta}+C_1=-\frac{x}{x^2-1}+C$



$$I=-\frac{1}{\sqrt{2}}\int\frac{d(\frac{\sqrt{2}x}{x^2-1})}{\sqrt{1+(\frac{\sqrt{2}x}{x^2-1})^2}}=-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}x}{x^2-1}+\sqrt{1+\frac{2x^2}{(x^2-1)^2}}\right|+C$$

5. $$I=\int\frac{dx}{x\sqrt{x^4+2x^2-1}}\ \overset{\sqrt{t}=\frac{1}{x}}{\underset{dx=-\frac{1}{2}t^{-\frac{3}{2}}dt}{=\!=}}=\int\frac{t^{\frac{1}{2}}\cdot(-\frac{1}{2})\cdot t^{-\frac{3}{2}}\,dt}{\sqrt{\frac{1}{t^2}+2\frac{1}{t}-1}}=-\frac{1}{2}\int\frac{dt}{\sqrt{1+2t-t^2}}=-\frac{1}{2}\int\frac{dt}{\sqrt{2-(1-t)^2}}=\frac{1}{2}\arcsin\frac{1-t}{\sqrt{2}}+C=\frac{1}{2}\arcsin\frac{x^2-1}{\sqrt{2}x^2}+C$$

Homework 6.

1. $I = \displaystyle\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}}$

2. $I = \displaystyle\int \sqrt{x^3+x^4}\,\mathrm{d}x$. Hint: After a change of variable, it would be useful to reduce the solution to the recursive integral $\displaystyle\int \frac{\mathrm{d}t}{(t^2-a^2)^n}$ and to prove the recursion formula for it. Look at the lecture example $\displaystyle\int \frac{\mathrm{d}t}{(t^2+1)^n}$.

3. $I = \displaystyle\int \frac{\mathrm{d}x}{\sqrt[3]{x}(1-\sqrt[6]{x})}$.

4. For what conditions the integral $I = \displaystyle\int \sqrt{1+x^n}\,\mathrm{d}x$, $n \in \mathbb{Q}$, is an elementary function?

5. $I = \displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{1+\sqrt[3]{x^2}}}$.

1. $I = \displaystyle\int \frac{dx}{\sqrt[4]{1+x^4}}$

let $x^{-4}+1 = t^4$, $t = \dfrac{\sqrt[4]{1+x^4}}{x}$, $dx = -\frac{1}{4}(t^4-1)^{-\frac{5}{4}} \cdot 4t^3\,dt$.

$$I = \int \frac{-t^3(t^4-1)^{-\frac{5}{4}}dt}{t\cdot(t^4-1)^{-\frac{1}{4}}} = \int \frac{-t^2}{t^4-1}dt = -\frac{1}{2}\int\left(\frac{1}{t^2-1}+\frac{1}{t^2+1}\right)dt = -\frac{1}{2}\left(\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + \arctan t\right)+C = -\frac{1}{4}\ln\left|\frac{\sqrt[4]{1+x^4}-x}{\sqrt[4]{1+x^4}+x}\right| + \frac{1}{2}\arctan\frac{\sqrt[4]{1+x^4}}{x} + C$$

2. $I = \sqrt{x^3+x^4}\,dx$

let $t = (x^{-1}+1)^{\frac{1}{2}}$ $\quad I = \displaystyle\int -\frac{2t}{(t^2-1)^2}\cdot t\cdot\frac{1}{(t^2-1)^2}dt = \int\frac{-2t^2}{(t^2-1)^4}dt = \int\frac{-2(t^2-1)}{(t^2-1)^4}dt - 2\int\frac{1}{(t^2-1)^4}dt = -2\left(\int\frac{1}{(t^2-1)^3}dt + \int\frac{1}{(t^2-1)^4}dt\right)$

$$\int\frac{1}{(t^2-1)^4}dt = \frac{t}{(t^2-1)^4} + 8\int\frac{t^2}{(t^2-1)^5}dt = \frac{t}{(t^2-1)^4} + 8\int\frac{1}{(t^2-1)^4}dt + 8\int\frac{1}{(t^2-1)^5}dt.$$

Let $\displaystyle\int\frac{1}{(t^2-1)^n} = I_n$. $\quad I_n = \dfrac{t}{(t^2-1)^n} + 2n(I_n + I_{n+1})$. $\quad I_{n+1} = \dfrac{1-2n}{2n}I_n - \dfrac{t}{(t^2-1)^n\cdot 2n}$

$I_2 = -\frac{1}{2}I_1 + \frac{t}{t^2-1} = \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - \frac{t}{2(t^2-1)}$

$I_3 = -\frac{3}{4}\left(\frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - \frac{t}{2(t^2-1)}\right) - \frac{t}{4(t^2-1)^2}$

$I_4 = -\frac{5}{6}\left(-\frac{3}{16}\ln\left|\frac{t+1}{t-1}\right| - \frac{3}{4}\cdot\frac{t}{2(t^2-1)} - \frac{t}{4(t^2-1)^2}\right) - \frac{t}{6(t^2-1)^3}$

$$\therefore\ I = -2\left[\left(-\frac{3}{16}+\frac{15}{96}\right)\ln\left|\frac{t+1}{t-1}\right| + \left(\frac{3}{8}-\frac{5}{16}\right)\frac{t}{t^2-1} + \left(-\frac{1}{4}+\frac{5}{24}\right)\frac{t}{(t^2-1)^2} - \frac{t}{6(t^2-1)^3}\right] + C$$

$$= \frac{1}{8}\ln\left|\sqrt{x+1}+\sqrt{x}\right| - \frac{1}{8}\sqrt{x(x+1)} - \frac{1}{12}\sqrt{x^4+x^3} + \frac{1}{3}\sqrt{x^5(x+1)} + C$$

3. $I = \displaystyle\int\frac{dx}{\sqrt[3]{x}(1-\sqrt[6]{x})}$

Let $t = \dfrac{1}{1-\sqrt[6]{x}}$ $\quad I = \displaystyle\int\frac{6\cdot(1-\frac{1}{t})^5\cdot\frac{1}{t^2}}{(1-\frac{1}{t})^2}\cdot t\,dt = 6\int\frac{(1-\frac{1}{t})^3}{t}dt = 6\int\left(\frac{1}{t}-\frac{3}{t^2}+\frac{3}{t^3}-\frac{1}{t^4}\right)dt = 6\left(\ln|t| + \frac{3}{t} - \frac{3}{2t^2} + \frac{1}{3t^3}\right) + C$

$$I = 6\ln\left|\frac{1}{1-\sqrt[6]{x}}\right| + 18(1-\sqrt[6]{x}) - 9(1-\sqrt[6]{x})^2 + 2(1-\sqrt[6]{x})^3 + C$$

4. $I = \displaystyle\int x^0(1+x^n)^{\frac{1}{2}}dx$

we have $m=0$, $p=\frac{1}{2}$.

by Chebyshev's Theorem, we can know that I is elementary fundmental

if 1° $\frac{m+1}{n}\in\mathbb{Z}$. i.e $\frac{1}{n}\in\mathbb{Z}$

2° $\frac{m+1}{n}+p\in\mathbb{Z}$ i.e. $\frac{n+2}{2n}\in\mathbb{Z}$.

5. $\displaystyle\int = \frac{x\,dx}{\sqrt{1+\sqrt[3]{x^2}}} \ \overset{t=\sqrt{1+x^{\frac{2}{3}}}}{\underset{dx=3(t^2-1)^{\frac{1}{2}}}{=}} \int\frac{(t^2-1)^{\frac{3}{2}}\cdot 3(t^2-1)^{\frac{1}{2}}\cdot t}{t}dt = 3\int(t^2-1)^2dt = \frac{3}{5}(1+x^{\frac{2}{3}})^{\frac{5}{2}} - 2(1+x^{\frac{2}{3}})^{\frac{3}{2}} + 3(1+x^{\frac{2}{3}})^{\frac{1}{2}} + C$

Homework 8.

2. proof $\geq$ and $\leq$

(1) $\forall x, y$ $f(x)-f(y) \leq \underbrace{\sup_{x\in D} f(x) - \inf_{y\in D} f(y)}_{\text{regard as upper bound}} \Rightarrow \sup(f(x)-f(y)) \leq \sup f(x) - \inf f(y)$

(2) $\sup(f(x)-f(y)) \geq f(x)-f(y) \Rightarrow f(y)+\sup(f(x)-f(y)) \geq f(x) \Rightarrow f(y)+\sup(f(x)-f(y)) \geq \sup f(x)$

$\inf f(y) \geq \sup f(x) - \sup(f(x)-f(y))$

1. Prove that $\sup\limits_{x,y\in D}(f(x)-f(y)) = \sup\limits_{x\in D} f(x) - \inf\limits_{y\in D} f(y)$.

利用x.y任意性. 上/下界定义.上/下界与确界关系

2. Find $\int_a^b x^m\,dx$, $0<a<b$, $m\neq -1$, using the definition of Riemann inegral.

3. Find $\int_{-1}^1 \sqrt[3]{x^2}\sin^5 x\,dx$, using the definition of Riemann integral.

4. Give an example of Riemann-integrable function such that it does not have a primitive.

1. Proof: let $\sup\limits_{x\in D} f(x) = M$ $\inf\limits_{y\in D} f(y) = m$.

By def. of supremum and infremum, $\forall \varepsilon>0$. $\exists x_1, y_1 \in D$. s.t. $\begin{cases} f(x_1) > M - \frac{\varepsilon}{2} \\ f(y_1) < m + \frac{\varepsilon}{2} \end{cases}$

thus $f(x_1)-f(y_1) > (M-m)-\varepsilon$.

Since ε is arbitrary. by the def of supremum. $\sup\limits_{x,y\in D}(f(x)-f(y)) = M-m = \sup\limits_{x\in D} f(x) - \inf\limits_{y\in D} f(y)$

$\{x_i\}$ be geometric progression. (等比级数).

2. Solution: let a partition $T=\{x_i\}_{i=0}^{n}$. s.t. $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ $x_k = aq^i$ $q=\sqrt[n]{\frac{b}{a}}$

we choose $\xi_i = x_i = a\cdot q^i$

$$\int_a^b x^m dx = \lim_{\lambda_T\to 0}\sum_{i=0}^{n-1} f(\xi_i)\Delta x_i = \lim_{\lambda_T\to 0}\sum_{i=0}^{n-1} a^m\cdot q^{mi}\cdot a\cdot(q^{i+1}-q^i)$$

$$= \lim_{\lambda_T\to 0} a^{m+1}\cdot(q-1)\cdot\sum_{i=0}^{n-1} q^{(m+1)i}$$

$$= \lim_{n\to\infty} a^{m+1}(q-1)\frac{q(1-q^{(m+1)(n-1)})}{1-q^{m+1}}$$

$$= \lim_{n\to\infty} a^{m+1}\frac{q\left(\frac{b^{m+1}}{a^{m+1}}\cdot\frac{1}{q^{m+1}}-1\right)}{\sum_{i=0}^{m} q^i} = \lim_{n\to\infty}\frac{\frac{1}{q^m}\cdot b^{m+1}-a^{m+1}}{\sum_{i=0}^{m} q^i}$$

$$= \frac{\frac{b^{m+1}}{q^m}-a^{m+1}}{\sum_{i=0}^{m} q^i} \qquad \text{since } q=\lim_{n\to\infty}\sqrt[n]{\frac{b}{a}} = 1$$

then $\int_a^b x^m dx = \frac{b^{m+1}-a^{m+1}}{m+1}$

3. Solution:

We choose partitions and tagged points that are symmetric with respect to the origin, since the integrand is odd, it follows that the Riemann sum is equal to 0

$\int_{-1}^{1} \sqrt[3]{x^2}\sin^5 x$

$i=n$

construct a partition T: $-1 = x_0 < x_1 < \cdots < x_{2n-1} < x_{2n} = 1$ $\quad x_i = \frac{i-n}{n}$

Let $\xi_i = x_i = \frac{i-n}{n}$ $\qquad \Delta x_i = x_{i+1} - x_i = \frac{1}{n}$.

$$\int_{-1}^{1} \sqrt[3]{x^2} \sin^5 x = \lim_{\lambda T \to 0} \sum_{i=0}^{i=2n-1} \left(\frac{i-n}{n}\right)^{\frac{2}{3}} \cdot \sin^5\left(\frac{i-n}{n}\right) \cdot \frac{1}{n}$$

we can find pair of points. $\quad (x_1, x_{2n-1}) \ldots \quad (x_{n-1}, x_{n+1})$.

s.t. $\quad f(x_{n-i}) + f(x_{n+i}) = 0 \quad 1 \le i \le n-1 \quad$ and $f(x_n) = 0$.

then $\quad \int_{-1}^{1} \sqrt[3]{x^2} \sin^5 x = \lim_{n \to \infty} 1 \cdot \sin^5(-1) \cdot \cancel{\frac{1}{n}} = 0$.

4. example : $\int_{-1}^{1} e^{-x^2} dx$. since $f(x) = e^{-x^2}$ is continuous on $[-1, 1]$.

and $\int e^{-x^2} dx$ is not a elementary function $\quad$ ✗

Any piece wise function with points of the first kind(jumps)

Homework 9.

1. Prove that $f(x) = \text{sign}\left(\sin\frac{\pi}{x}\right) \in \mathcal{R}[0,1]$. $\left\{ \begin{array}{l} \text{Def.} \\ \text{Darboux criterion} \\ \text{Lebesgue criterion} \end{array} \right.$

2. Prove that if $f, g \in \mathcal{R}[a,b]$, then $f + g \in \mathcal{R}[a,b]$.

3. Let f be differentiable and f' be bounded on $[0,1]$. Prove that

$$\int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) = O\left(\frac{1}{n}\right).$$

4. Give an example of the function defined on $[a,b]$, continuous on (a,b), which is not integrable on $[a,b]$.

1. $f(x)$ has discontinuity point $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$

and the $\omega_i \leq 2$ for any intervals extract from $[0,1]$.

$\forall \varepsilon > 0$, Consider $[\frac{\varepsilon}{5}, 1]$, the $f(x)$ is piecewise continuous for any fixed ε. on closed interval $[\frac{\varepsilon}{5}, 1]$.

thus there exist $\delta_1 > 0$. s.t for partition γ. if $\max|\lambda\gamma| < \delta_1$

we have $\sum_{i=0}^{n_1} \omega_i \Delta x_i < \frac{\varepsilon}{5}$.

Let $\delta = \min\{\delta_1, \frac{\varepsilon}{5}\}$. let a partition s.t. $0 = x_0 < x_1 \cdots < x_n = 1$

and $\max \Delta x_i < \delta$.

Then there exist $\exists x_j$, $x_j \leq \frac{\varepsilon}{5} < x_{j+1}$. $(1 \leq j \leq n-1)$.

$$\sigma = \sum_{i=0}^{n} \omega_i \Delta x_i = \sum_{i=0}^{j} \omega_i \Delta x_i + \sum_{i=j+1}^{n} \omega_i \Delta x_i$$

$$\leq 2 \cdot \sum_{i=0}^{j} \cdot \Delta x_i + \frac{\varepsilon}{5} = 2 \cdot x_{j+1} \cdot \frac{\varepsilon}{5} < 2 \cdot \frac{2\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon. \quad \square.$$

2. Since $f, g \in R[a,b]$.

For any $\varepsilon > 0$. there exist $\delta_1 > 0$. for any $\max \Delta x_i < \delta_1$. $\sum_{i=0}^{n_1} \omega_i(f) \Delta x_i < \frac{\varepsilon}{2}$.

$\delta_2 > 0$. for any $\max \Delta x_i < \delta_2$ $\sum_{i=0}^{n_2} \omega_i(g) \Delta x_i < \frac{\varepsilon}{2}$.

Let $\delta = \min(\delta_1, \delta_2)$. for any $\max \Delta x_i < \delta$.

$\forall x_1, x_2$. $|f(x_1) - f(x_2) + g(x_1) - g(x_2)| \leq |f(x_1) - f(x_2)| + |g(x_1) - g(x_2)| \leq \omega(f, E) + \omega(g, E)$

thus. $\omega(f+g, E) \leq \omega(f, E) + \omega(g, E)$

$$\sum_{i=0}^{n} \omega_i(f+g) \Delta x_i = \sum_{i=0}^{n} \omega_i(f) \Delta x_i + \sum_{i=0}^{n} \omega_i(g) \Delta x_i < \varepsilon.$$

3. let $|f'| \leq A$ on $[0,1]$. $(A \in \mathbb{R})$

let a partition $\mathcal{T} = \left\{ \frac{k}{n} \right\}_{k=0}^{n}$, let $\xi_k = x_k = \frac{k}{n}$. $\Delta x = \frac{1}{n}$.

$$\sigma(f, \mathcal{T}, \xi) = \sum_{k=1}^{n} f(\xi_k) \cdot \Delta x_k = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

$$S_{\mathcal{T}}(f) - s_{\mathcal{T}}(f) = \sum_{k=1}^{n} (M_k - m_k) \Delta x_k$$

Since $|f'| \leq A$. then $M_k - m_k \leq A \cdot \Delta x_k$. (by Lagrange's formula.)

thus. $S_{\mathcal{T}}(f) - s_{\mathcal{T}}(f) \leq \frac{1}{n} \cdot n \cdot A \cdot \frac{1}{n} = \frac{A}{n} = O\left(\frac{1}{n}\right)$

$$\begin{cases} S_{\mathcal{T}}(f) \leq \sigma(f, \mathcal{T}, \xi) \leq s_{\mathcal{T}}(f) \\ S_{\mathcal{T}}(f) \leq \int_0^1 f(x)\,dx \leq s_{\mathcal{T}}(f) \end{cases} \Rightarrow \left| S_{\mathcal{T}}(f) - s_{\mathcal{T}}(f) \right| \geq \left| \int_0^1 f - \sigma(f, \mathcal{T}, \xi) \right|.$$

$$\left| \frac{\int_0^1 f(x) - \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)}{S_{\mathcal{T}}(f) - s_{\mathcal{T}}(f)} \right| \leq 1.$$

4. $f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & x \in (0,1). \end{cases}$ which is not bounded on $(0,1)$.
So it is not integrable.

Homework 10.

1. Prove that if $f \in \mathcal{R}[a,b]$ and $f(x) = 0$ at every point of continuity, then $\int_a^b f = 0$.

2. Integrating by parts find $I_n = \int_0^{\pi/4} \tan^{2n} x \, dx$.

3. $I = \int_0^{3/4} \frac{dx}{(x+1)\sqrt{x^2+1}}$.

4. Prove the formula $\frac{1}{2} + \sum_{k=1}^{n-1} \cos 2kx = \frac{\sin(2n-1)x}{2\sin x}$ and calculate $\int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} \, dx$.

1. Proof: Since $f \in R[a,b]$. $|f| \leq M$.

(1) if $f \in C[a,b]$. then $f \equiv 0$. $\int_a^b f = 0 \cdot (b-a) = 0$.

(2) if f has discontinuous point by Lebesgue. the set of discontinuity point is of measure zero

$\forall \varepsilon > 0$. $\exists \delta = \frac{\varepsilon}{N \cdot M}$. for any $\max \Delta x_i < \delta$.

we suppose that in partition $\{x_k\}_{k=0}^n$ s.t. $\max \Delta x_i < \delta$. Let $\xi_k \in [x_k, x_{k+1}]$

there are N number's ξ_k are discontinuity. point. Let them be $\{x_{k_i}\}_{i=1}^N$

$$\left| \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k \right| = \left| \sum_{k \neq k_i}^{n-1} 0 \cdot \Delta x_k + \sum_{i=1}^{N} f(\xi_{k_i}) \Delta x_k \right| \leq M \cdot \left| \sum_{i=1}^{N} \cdot \Delta x_k \right| < M \cdot N \cdot \delta = \varepsilon.$$

that is. $\lim_{\max \Delta x_k \to 0} \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k = 0$. i.e. $\int_a^b f = 0$

2. Solution. First. we need to find primitive of $\tan^{2n} x$.

$$F(x) = \int \tan^{2n} x \, dx = \int \tan^{2n-2} (\sec^2 x - 1) \, dx = \frac{1}{2n-1} \tan^{2n-1} x - \int \tan^{2n-2} dx$$

$$= \frac{1}{2n-1} \tan^{2n-1} x - \frac{1}{2n-3} \tan^{2n-3} x + \int \tan^{2n-4} dx$$

if n is even $\int \tan^{2n} x \, dx = \frac{1}{2n-1} \tan^{2n-1} x - \frac{1}{2n-3} \tan^{2n-3} x + \cdots + \frac{\tan^3 x}{3} - \tan x + x + C$

if n is odd. $\int \tan^{2n} x \, dx = \frac{1}{2n-1} \tan^{2n-1} x - \frac{1}{2n-3} \tan^{2n-3} x + \cdots + \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + C$

$$I_n = \int_0^{\frac{\pi}{4}} \tan^{2n} x \, dx = F\left(\frac{\pi}{4}\right) - F(0) = \begin{cases} \frac{\pi}{4} + \sum_{k=1}^{n} (-1)^k \cdot \frac{1}{2k-1} & n \text{ is even} \\ -\frac{\pi}{4} + \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{2k-1} & n \text{ is odd.} \end{cases}$$

3. $F(x)=\int \frac{dx}{(x+1)\sqrt{x^2+1}} \xlongequal[dx=-\frac{dt}{t^2}]{t=\frac{1}{x+1}} \int t\cdot \frac{-\frac{1}{t^2}}{\sqrt{(\frac{1}{t}-1)^2+1}}\,dt = -\int \frac{dt}{\sqrt{2t^2-2t+1}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{(t-\frac{1}{2})^2+\frac{1}{4}}}$

$= -\frac{1}{\sqrt{2}} \ln \left| t-\frac{1}{2}+\sqrt{(t-\frac{1}{2})^2+\frac{1}{4}} \right| + C.$

$\int_0^{\frac{3}{4}} \frac{dx}{(x+1)\sqrt{x^2+1}} = F(\frac{4}{7}) - F(1) = \frac{1}{\sqrt{2}} \ln \frac{\frac{1}{2}+\frac{\sqrt{2}}{2}}{\frac{1}{14}+\frac{\sqrt{50}}{14}} = \frac{1}{\sqrt{2}} \ln \frac{7(1+\sqrt{2})}{1+5\sqrt{2}}.$

$= \frac{4\sqrt{2}+9}{7}$

4. proof by induction

1) $n=1$. LHS $=\frac{1}{2}$ RHS $= \frac{\sin(2\times1-1)x}{2\sin x} = \frac{1}{2}$ $(\sin x \neq 0)$.

2) Assume $n=k$ the equation holds.

now consider $n=k+1$

RHS $= \frac{\sin(2k+1)x}{2\sin x} = \frac{\sin(2k-1)x\cdot\cos 2x + \sin 2x\cdot\cos(2k-1)x}{2\sin x}$

$= \left(\frac{1}{2}+\sum_{i=1}^{k-1}\cos 2ix\right)\cos 2x + \frac{\cos 2kx+\cos(2k-2)x}{2}.$

$= \frac{\cos 2x}{2} + \frac{\cos 0+\cos 4x}{2} + \frac{\cos 2x+\cos 6x}{2} + \cdots + \frac{\cos 2kx+\cos(2k-4)x}{2} + \frac{\cos(2k-2)x+\cos 2kx}{2}$

$= \frac{1}{2} + \sum_{i=1}^{k} \cos 2ix = $ LHS. $\square$.

Now we calculate. $F(x) = \int \frac{\sin(2n-1)x}{2\sin x}\,dx$

$F(x) = \int \left(\frac{1}{2}+\sum_{k=1}^{n-1}\cos 2kx\right)dx.$

$= \frac{x}{2} + \sum_{k=1}^{n-1} \frac{1}{2k}\sin(2k)x + C$

$\int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{2\sin x} = F(\frac{\pi}{2}) - F(0) = \frac{\pi}{4}.$

Homework 11.

1. $I=\displaystyle\int_0^{4\pi}\frac{\mathrm{d}x}{1+a\cos x},\ 0\le a<1.$

2. $I=\displaystyle\int_{0.5}^{31.5}\lfloor x\rfloor\,\mathrm{d}x.$

3. $\displaystyle\lim_{n\to\infty}S_n,\ S_n=\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}$

4. $\displaystyle\lim_{n\to\infty}S_n,\quad S_n=\sin\frac{\pi}{n}\sum_{k=1}^{n}\frac{1}{2+\cos\frac{\pi k}{n}}$

5. What number is greater

$$\int_0^{\pi/2}\sin^3 x\,\mathrm{d}x \text{ or } \int_0^{\pi/2}\sin^7 x\,\mathrm{d}x\ ?$$

1. Solution: since $a\in[0,1)$. $1+a\cos x>0$ for $x\in[0,4\pi]$

the function $f(x)=\dfrac{1}{1+a\cos x}$ · is continuous on $[0,4\pi]$

let. $F(x)=\displaystyle\int\frac{dx}{1+a\cos x}\ \frac{\tan\frac{x}{2}=t}{\cos x=\frac{1-t^2}{1+t^2}}\ \int\frac{\frac{2}{1+t^2}}{1+a\cdot\frac{1-t^2}{1+t^2}}dt=\int\frac{2\,dt}{1+t^2+a(1-t^2)}$

$$=\int\frac{2\,dt}{a+1+(1-a)t^2}=\frac{2}{a+1}\int\frac{dt}{1+\frac{1-a}{a+1}t^2}=\frac{2}{a+1}\sqrt{\frac{a+1}{1-a}}\int\frac{d(\sqrt{\frac{1-a}{a+1}}\,t)}{1+(\sqrt{\frac{1-a}{a+1}}\,t)^2}$$

$$=\frac{2}{\sqrt{1-a^2}}\cdot\arctan\sqrt{\frac{1-a}{a+1}}\cdot\tan\frac{x}{2}+C\Rightarrow \text{discontinuous point}: \pi, 3\pi$$

$$I=\int_0^{\pi-}\frac{dx}{1+a\cos x}+\int_{\pi+}^{3\pi-}\frac{dx}{1+a\cos x}+\int_{3\pi+}^{4\pi}\frac{dx}{1+a\cos x}=\frac{\pi}{\sqrt{1-a^2}}+\frac{2\pi}{\sqrt{1-a^2}}+\frac{\pi}{\sqrt{1-a^2}}=\frac{4\pi}{\sqrt{1-a^2}}$$

2. Solution: $f(x)=[x]$ is discontinuous on the integral points.

on $[0.5, 31.5]$. the discontinuous points are finite points

$$\int_{0.5}^{31.5}[x]\,dx=\int_{0.5}^{1}[x]\,dx+\int_1^2[x]\,dx+\cdots+\int_{30}^{31}[x]\,dx+\int_{31}^{31.5}[x]\,dx$$

$$=0+1+\cdots+30+31\cdot 0.5=480.5$$

(in every $x\in[N,N+1]$. $\int[x]\,dx=[x]\cdot x$)

3. $S_n = \sum_{k=1}^{n} \frac{1}{n+k} = \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}}$

Let $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} \frac{1}{1+\frac{k}{n}} = \sigma\left(\frac{1}{1+x}, \left\{\frac{k}{n}\right\}_{k=1}^{n}, \left\{\frac{k}{n}\right\}_{k=1}^{n}\right)$

$= \int_0^1 \frac{1}{1+x}\,dx = \ln|x+1| \Big|_0^1 = \ln 2.$

4. $\lim_{n\to\infty} S_n = \sigma\left(\frac{1}{2+\cos x}, \left\{\frac{\pi k}{n}\right\}_{k=1}^{n}, \left\{\frac{\pi k}{n}\right\}_{k=1}^{n}\right)$

$= \int_0^{\pi} \frac{1}{2+\cos x}\,dx$

since $2+\cos x \neq 0$ when $x \in [0,\pi]$. $f(x) = \frac{1}{2+\cos x}$ is continuous on $[0,\pi]$.

$F(x) = \int \frac{1}{2+\cos x}\,dx \xlongequal[\cos x = \frac{1-t^2}{1+t^2}]{t=\tan\frac{x}{2}} \int \frac{\frac{2}{1+t^2}}{2+\frac{1-t^2}{1+t^2}}\,dt = \int \frac{2}{3+t^2}\,dt$

$= 2 \cdot \frac{1}{3} \cdot \int \frac{1}{1+\left(\frac{t}{\sqrt{3}}\right)^2} = \frac{2\sqrt{3}}{3} \cdot \arctan\left(\frac{1}{\sqrt{3}} \tan\frac{x}{2}\right) + C$

$\lim_{n\to\infty} S_n = F(\pi-) - F(0) = \frac{2\sqrt{3}}{3} \cdot \frac{\pi}{2} = \frac{\sqrt{3}\pi}{3}$

5. $\int \sin^3 x\,dx = -\int (1-\cos^2 x)\,d(\cos x) = -\cos x + \frac{1}{3}\cos^3 x + C$

directedly use the strict monotonicity.

$\int_0^{\frac{\pi}{2}} \sin^3 x\,dx = 0 + \frac{2}{3} = \frac{2}{3}$

$\int \sin^7 x\,dx = -\int (1-\cos^2 x)^3\,d(\cos x) = -\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$

$\int_0^{\frac{\pi}{2}} \sin^7 x\,dx = 0 - \left(-1+1-\frac{3}{5}+\frac{1}{7}\right) = \frac{16}{35} < \frac{2}{3}$

thus. $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx < \int_0^{\frac{\pi}{2}} \sin^3 x\,dx.$

Homework 12.

1. Prove that for any $m, n \in \mathbb{N}$, $\quad m \neq n$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, \mathrm{d}x = 0.$$

2. Prove the inequality $\dfrac{1}{20\sqrt{2}} < \displaystyle\int_0^1 \frac{x^{19} dx}{\sqrt{1+x^2}} < \frac{1}{20}$.

3. $\displaystyle\int_0^1 x\sqrt{1+x}\, \mathrm{d}x$.

4. $J_n = \displaystyle\int_{-\pi}^{\pi} \cosh x \cos nx \, \mathrm{d}x, n \in \mathbb{N}$.

5. $J_{\alpha,n} = \displaystyle\int_0^1 x^\alpha \ln^n x \, \mathrm{d}x, \quad \alpha > 0, \quad n \in \mathbb{N}$.

1. proof: $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx = \int_{-\pi}^{\pi} \frac{\cos(m-n)x - \cos(m+n)x}{2} dx$

$$= \frac{1}{2}\left[\int_{-\pi}^{\pi} \cos(m-n)x \, dx - \int_{-\pi}^{\pi} \cos(m+n)x \, dx\right]$$

$$= \frac{1}{2(m-n)} \cdot \sin(m-n) \Big|_{-\pi}^{\pi} - \frac{1}{2(m+n)} \sin(m+n)x \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2(m-n)} \cdot (0-0) - \frac{1}{2(m+n)} (0-0) = 0.$$

2. $\displaystyle\frac{1}{20} = \frac{1}{20} x^{20} \Big|_0^1 = \int_0^1 x^{19} dx$

$$\frac{1}{20\sqrt{2}} = \frac{1}{20} \cdot \frac{x^{20}}{\sqrt{1+x^2}} \Big|_0^1 = \int_0^1 \frac{x^{19} dx}{\sqrt{1+x^2}} + \int_0^1 \frac{x^{20}}{20} \cdot -\frac{x}{\sqrt{(1+x^2)^2}} dx$$

$$= \int_0^1 \frac{x^{19}}{\sqrt{1+x^2}} dx - \int_0^1 \frac{x^{21}}{20\sqrt{(1+x^2)^2}} dx \text{ since } \int_0^1 \frac{x^{21} dx}{20\sqrt{(1+x^2)^2}} > 0.$$

then $\displaystyle\frac{1}{20\sqrt{2}} < \int_0^1 \frac{x^{19}}{\sqrt{1+x^2}}$

$\forall x \in [0,1]$. we have $\sqrt{1+x^2} > 1$. thus. $x^{19} > \dfrac{x^{19}}{\sqrt{1+x^2}}$ holds on $[0,1]$.

i.e. $\displaystyle\int_0^1 \frac{x^{19}}{\sqrt{1+x^2}} dx < \int_0^1 x^{19} dx = \frac{1}{20}$. $\square$.

3. $\int_0^1 x\sqrt{1+x}\,dx \xlongequal[dx=2tdt]{t=\sqrt{1+x}} \int_1^{\sqrt{2}} (t^2-1)\cdot 2t\cdot t\,dt$

$= \int_1^{\sqrt{2}} (2t^4-2t^2)dt = \frac{2}{5}t^5 - \frac{2}{3}t^3 \Big|_1^{\sqrt{2}} = \frac{8\sqrt{2}}{5} - \frac{4\sqrt{2}}{3} - \frac{2}{5} + \frac{2}{3} = \frac{4\sqrt{2}+4}{15}$

4. $J_n = \int_{-\pi}^{\pi} \cosh x \cos nx\,dx.\quad n\in\mathbb{N}.$

$= \sinh x\cos nx \Big|_{-\pi}^{\pi} + n\int_{-\pi}^{\pi} \cosh x \sin nx\,dx$

$= \sinh x\cos nx \Big|_{-\pi}^{\pi} + n\sinh x \sin nx \Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} \cosh x\cos nx\,dx$

$(n+1)J_n = \sinh x\cos nx \Big|_{-\pi}^{\pi}$

$= \frac{e^{\pi}-e^{-\pi}}{2}\cos n\pi - \frac{e^{-\pi}-e^{\pi}}{2}\cos n\pi$

$= (e^{\pi}-e^{-\pi})\cos n\pi \quad n\in\mathbb{N}.$ $\qquad J_n = (-1)^n\frac{2\sinh\pi}{n^2+1}$

$\therefore J_n = \frac{(e^{\pi}-e^{-\pi})\cos n\pi}{n+1} \quad n\in\mathbb{N}.$

5. $J_{\alpha,n} = \int_0^1 x^{\alpha}\ln^n x\,dx,\ \alpha>0,\ n\in\mathbb{N}.$

$\lim_{x\to0^+} x^{\alpha}\ln^n x = 0 \qquad \text{Let } f(x) = \begin{cases} x^{\alpha}\ln^n x. & x\in(0,1] \\ 0 & x=0. \end{cases}$

$f(x)\in R[0,1]$., since $f(x)$ is continuous on $[0,1]$.

$J_{\alpha,n} = \frac{x^{\alpha+1}\ln^n x}{\alpha+1}\Big|_0^1 - \frac{n}{\alpha+1}\int_0^1 x^{\alpha}\ln^{n-1}x\,dx.$ (since $\alpha>0,\ \alpha\neq-1$)

$J_{\alpha,n} = (-1)\cdot\frac{n}{(\alpha+1)} J_{\alpha,n-1}.$

$J_{\alpha,n} = (-1)^n\cdot\frac{n!}{(\alpha+1)^n} J_{\alpha,0} = (-1)^n\cdot\frac{n!}{(\alpha+1)^n}\int_0^1 x^{\alpha}\,dx$

$= (-1)^n\cdot\frac{n!}{(\alpha+1)}\cdot\frac{x^{\alpha+1}}{\alpha+1}\Big|_0^1 = (-1)^n\frac{n!}{(\alpha+1)^{n+1}}.\quad n\in\mathbb{N}.\ \alpha>0.$

Homework 13.

1. a. Explain, why the equality $\int_{-1}^{1} \frac{d}{dx}\left(\arctan \frac{1}{x}\right) dx = \arctan \frac{1}{x}\Big|_{-1}^{1} = \frac{\pi}{2}$ is incorrect.

   b. Is the function $\frac{d}{dx}\left(\arctan \frac{1}{x}\right)$ integrable on $[-1,1]$?

   c. What is the correct answer for the integral, if any?

2. Let $f \in C[-a,a]$, prove that

   a. if f is even, then $\int_{-a}^{a} f = 2\int_{0}^{a} f$;

   b. if f is odd, then $\int_{-a}^{a} f = 0$.

3. Let $f \in C[0,1]$, prove that

   a. $\int_{0}^{\frac{\pi}{2}} f(\sin x)\, dx = \int_{0}^{\frac{\pi}{2}} f(\cos x)\, dx$,

   b. $\int_{0}^{\pi} x f(\sin x)\, dx = \frac{\pi}{2}\int_{0}^{\pi} f(\sin x)\, dx$.

1. a). the function is discontinuous on the point $x=0$.

b) Yes. consider the function $f(x) = \frac{d(\arctan\frac{1}{x})}{dx} = -\frac{1}{x^2}\cdot\frac{1}{\frac{1}{x^2}+1} = -\frac{1}{x^2+1}$

the $f(x)$ is continuous on $[-1,1]$

thus. the function is Riemann integrable

c). $\int_{-1}^{1} \frac{d(\arctan\frac{1}{x})}{dx} dx = \arctan\frac{1}{x}\Big|_{-1}^{0^-} + \arctan\frac{1}{x}\Big|_{0^+}^{1}$

$= -\frac{\pi}{2} + \frac{\pi}{4} + \left(\frac{\pi}{4} - \frac{\pi}{2}\right) = -\frac{\pi}{2}$

2. a) $f \in C[-a,a]$. suppose $F' = f$. $F \in C[-a,a]$

$\int_{-a}^{a} f = \int_{0}^{a} f + \int_{-a}^{0} f =$

for any partition $J = \{x_k\}_{k=0}^{n}$, $x_0 = 0$. $x_n = a$ of $[0,a]$.

$\int_0^a f = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k \underbrace{=}_{f \text{ is even}} \sum_{k=0}^{n-1} f(-\xi_k)\Delta x_k$

since $\xi_k \in [x_k, x_{k+1}]$, $-\xi_k \in [-x_{k+1}, -x_k]$

the partition $\{-x_k\}_{k=0}^{n}$ will correspond the interval $[-a,0]$.

$\Rightarrow \int_0^a f = \sum_{k=0}^{n-1} f(-\xi_k)\Delta x_k = \int_{-a}^{0} f$. i.e. $\int_{-a}^{a} f = 2\int_{-a}^{a}$

b) similar as a). $\int_0^a f = \sum_{k=0}^{n-1} f(\xi_k)\Delta x_k \underbrace{=}_{f \text{ is odd}} -\sum_{k=0}^{n-1} f(-\xi_k)\Delta x_k = -\int_{-a}^{0} f$

i.e. $\int_{-a}^{a} f = \int_0^a f - \int_0^a f = 0$.

3. $f \in C[0,1]$

a) $\displaystyle\int_0^{\frac{\pi}{2}} f(\sin x)\,dx \xlongequal[dx=\frac{1}{\sqrt{1-t^2}}dt]{\sin x = t \in [0,1]} \int_0^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt.$ $\quad \sim x = \arcsin x$

$\displaystyle\int_0^{\frac{\pi}{2}} f(\cos x)\,dx \xlongequal[dx=\frac{1}{\sqrt{1-s^2}}ds]{\cos(-x) = s \in [0,1]} \int_0^1 \frac{f(s)}{\sqrt{1-s^2}}\,ds. = \int_0^1 \frac{f(t)}{\sqrt{1-t^2}}\,dt$

thus $\displaystyle\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$

b). proof:

$\displaystyle\int_0^{\pi} x f(\sin x)\,dx \xlongequal{x-\frac{\pi}{2}=t} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t+\tfrac{\pi}{2}) f(\cos t)\,dt$

$\displaystyle= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t f(\cos t)\,dt + \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos t)\,dt$

$\displaystyle= \frac{\pi}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos t)\,dt$ (since the function $t f(\cos t)$ is odd

$\displaystyle= \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

Homework 14.

1. $I = \displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,\mathrm{d}x$. (Hint: Apply a problem from the previous homework.)

2. Find the integral $I = \displaystyle\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x}\,\mathrm{d}x$ if it exists. $\left(\text{Hint: Look at the lecture example } I = \displaystyle\int_0^{\pi} \frac{\sin nx}{\sin x}\,\mathrm{d}x.\right)$

3. Find $\displaystyle\lim_{x\to+\infty} \frac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$.

4. Find $\displaystyle\int_{-\pi/2}^{\pi/2} \sqrt[3]{\sin x}\,\mathrm{d}x$.

   (Hint: Apply a problem from the previous homework.)

1. Solution let $f(x) = \dfrac{x}{2-x^2}$ $x \in [0,1]$

$I = \int_0^{\pi} x f(\sin x)\,dx \overset{\text{by HW13.4.}}{=\!=} \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$

$\int \frac{\sin x}{1+\cos^2 x}\,dx = -\int \frac{d(\cos x)}{1+\cos^2 x} = -\arctan\cos x$

thus $\displaystyle\int_0^{\pi} \frac{\sin x\,dx}{1+\cos^2 x} = -\arctan\cos x \Big|_0^{\pi} = -\left(-\frac{\pi}{4}\right) + \frac{\pi}{4} = \frac{\pi}{2}$

$I = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$.

2. $\displaystyle\lim_{x\to\frac{\pi}{2}^+} \frac{\cos(2n+1)x}{\cos x} = \lim_{x\to\frac{\pi}{2}^-} \frac{\cos(2n+1)x}{\cos x} = (-1)^n(n+1)$.

$\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$

$\displaystyle\frac{\cos(2n+1)x}{\cos x} = \frac{e^{(2n+1)xi} + e^{-(2n+1)xi}}{e^{ix}+e^{-ix}} = e^{2nxi} - e^{2(n-1)xi} + \cdots + (-1)^n + \cdots + e^{-2nix}$

$= \cos 2nx - \cos 2(n-1)x + \cdots + (-1)^{n-1}\cos 2x + (-1)^n$

$\displaystyle\int_0^{\pi} \frac{\cos(2n+1)x}{\cos x} = \frac{1}{2n}\sin 2nx - \frac{1}{2(n-1)}\sin 2(n-1)x + \cdots + \frac{(-1)^{n-1}}{2}\sin 2x + (-1)^n x \Big|_0^{\pi}$

$= (-1)^n \pi$

3. $\lim_{x \to +\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt}$

Solution. By L^Hopitals rule

$$\lim_{x \to +\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt} = \lim_{x \to +\infty} \frac{2\int_0^x e^{t^2} dt \cdot e^{x^2}}{e^{2x^2}} = \lim_{x \to +\infty} \frac{2\int_0^x e^{t^2} dt}{e^{x^2}} = \lim_{x \to +\infty} \frac{2 \cdot e^{x^2}}{2x \cdot e^{x^2}} = \lim_{x \to +\infty} \frac{1}{x} = 0$$

4. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt[3]{\sin x}\, dx$

Solution: since the function $f(x) = \sqrt[3]{\sin x}$ is odd, and $\frac{\pi}{2} + (-\frac{\pi}{2}) = 0$.

By the proof of previous homework. we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt[3]{\sin x}\, dx = \int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x} + \int_{-\frac{\pi}{2}}^{0} \sqrt[3]{\sin x} = \int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x} - \int_0^{\frac{\pi}{2}} \sqrt[3]{\sin x} = 0$$

Homework 15.

1. Define the sign of the integral $I = \int_{-2}^{2} x^3 e^x \, dx$ via the mean value theorems.

2. Using the second mean value theorem estimate the integral $\int_0^{100} \frac{e^{-x}\, dx}{x+100}$ twice: (a) $f(x) = \frac{1}{x+100}$, $g(x) = e^{-x}$, (b) $g(x) = \frac{1}{x+100}$, $f(x) = e^{-x}$. Which estimate is more accurate?

3. Let $I = \int_a^b \frac{\cos x}{\sqrt{x}} \, dx, 0 < a < b$. Prove the inequality $-\frac{2}{\sqrt{a}} < I < \frac{2}{\sqrt{a}}$.

1. Solution: $I = \int_{-2}^{2} x^3 e^x = \int_0^2 x^3 e^x dx + \int_{-2}^{0} x^3 e^x dx$

since $0 < e^{-2} \leq e^x \leq 1.$ $x \in [-2, 0]$

$0 < 1 \leq e^x \leq e^2$ $x \in [0, 2]$.

$\therefore \exists\, \xi_1 \in [e^{-2}, 1]$. $\xi_2 \in [1, e^2]$ s.t. $I = \xi_1 \int_0^2 x^3 dx + \xi_2 \int_{-2}^{0} x^3 dx$

$f(x) = x^3$ is odd function. $I = (\xi_1 - \xi_2) \int_0^2 x^3$

Since $\xi_1 - \xi_2 > 0$ (ξ_1, ξ_2 both equal to 1 is impossible).

$\int_0^2 x^3 dx = \frac{x^4}{4} \Big|_0^2 = 4 > 0.$ thus, $I > 0$ $\square$.

2. (a) $I = \int_0^{\xi} \frac{dx}{x+100} + e^{-100} \int_{\xi}^{100} \frac{dx}{x+100} = \ln(x+100)\Big|_0^{\xi} + e^{-100} \ln(x+100)\Big|_{\xi}^{100}$

$= \ln(\xi+100) - 2\ln 10 + e^{-100} \ln 200 - e^{-100} \ln(\xi+100)$

$= e^{-100} \ln 200 - \ln 100 + (1 - e^{-100}) \ln(\xi+100).$ $\xi \in [0, 100]$.

margin of error : $\ln 2 - e^{-100} \ln 2 = (1 - e^{-100}) \ln 2.$

(b) $= \frac{1}{100} \int_0^{\xi} e^{-x} dx + \frac{1}{200} \int_{\xi}^{100} e^{-x} dx = \frac{1}{200} \left(2 \cdot -e^{-x}\Big|_0^{\xi} + -e^{-x}\Big|_{\xi}^{100} \right) = \frac{2 - e^{-\xi} - e^{-100}}{200}$

margin of error $= \frac{1 - e^{-100}}{200}$

Since $\frac{1}{200} < \ln 2.$ thus. estimate (b) is more accurate.

3. $g(x) = \frac{1}{\sqrt{x}}$ is decreasing on $[a,b] \subset \mathbb{R}_+$.

$\exists \mu \in [\frac{1}{\sqrt{b}}, \frac{1}{\sqrt{a}}]$. s.t. $I = \mu \cdot \int_a^b \cos x \, dx = \mu(\sin b - \sin a)$.

$-2 \leq \sin b - \sin a \leq 2$.

$\Rightarrow -2\mu \leq I \leq 2\mu$.

since $-\mu \geq -\frac{1}{\sqrt{a}}$ $\mu \leq \frac{1}{\sqrt{a}}$ $\Rightarrow -\frac{2}{\sqrt{a}} \leq -2\mu \leq I \leq 2\mu \leq \frac{2}{\sqrt{a}}$

Homework 16.

1. Prove that $\lim\limits_{n\to\infty}\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x = 0.$

   Hint: Represent the integral as $\displaystyle\int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,\mathrm{d}x + \int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x$ and estimate each integral.

2. Prove Young's inequality

$$a, b > 0, \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

   Hint: Consider the function $y(x) = x^{p-1}$ and find its inverse function. Represent ab, $\frac{a^p}{p}$, and $\frac{b^q}{q}$ as areas of rectangle and two curvilinear trapezoids and compare these areas.

3. Suppose f, g increase on $[a,b]$. Prove that

$$\frac{1}{b-a}\int_a^b fg \geq \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(\frac{1}{b-a}\int_a^b g\right).$$

   Hint: Apply Chebyshev's inequality for some auxiliary functions.

1. proof: $\forall \varepsilon > 0$. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,dx + \int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}} \sin^n x\,dx.$

$f(x) = \sin^n x$ is increasing. since $n \in \mathbb{N}$. let $g(x) = 1$.

thus. $\exists \xi_1 \in [0, \frac{\pi}{2} - \frac{\varepsilon}{2}]$ s.t. $\displaystyle\int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,dx = 0\cdot x\Big|_0^{\xi} + \sin^n(\tfrac{\pi}{2}-\tfrac{\varepsilon}{2})(\tfrac{\pi-\varepsilon}{2} - \xi_1)$

$\exists \xi_2 \in [\frac{\pi-\varepsilon}{2}, \frac{\pi}{2}]$ s.t. $\displaystyle\int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}} \sin^n x\,dx = \sin^n(\tfrac{\pi-\varepsilon}{2})(\xi_2 - \tfrac{\pi-\varepsilon}{2}) + (\tfrac{\pi-\varepsilon}{2} - \xi_2)$

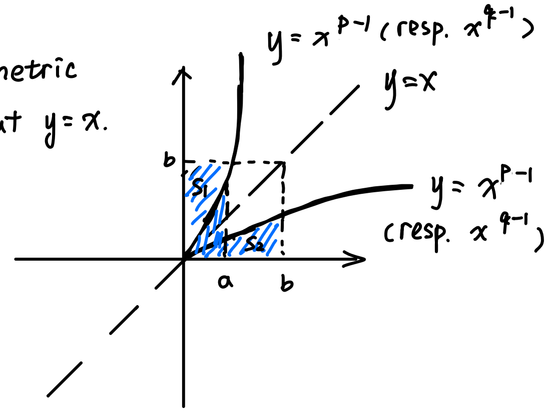
$\therefore \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \sin^n(\tfrac{\pi-\varepsilon}{2})(\xi_2 - \xi_1) + \tfrac{\pi-\varepsilon}{2} - \xi_2$

since $0 < \sin(\frac{\pi-\varepsilon}{2}) < 1$ $\quad \sin^n(\frac{\pi-\varepsilon}{2}) \to 0 \quad (n \to \infty)$

i.e. for any $\varepsilon > 0$. $\exists N \in \mathbb{N}$. s.t when $n > N$. $\sin^n(\frac{\pi-\varepsilon}{2}) < \dfrac{\varepsilon}{2(\xi_2 - \xi_1)}$

$\therefore \left|\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx\right| \leq \left|\sin^n(\tfrac{\pi-\varepsilon}{2})(\xi_2 - \xi_1)\right| + \left|\tfrac{\pi-\varepsilon}{2} - \xi_2\right| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$ $\square$.

2. $y(x) = x^{p-1}$ $\quad y^{-1}(x) = x^{q-1}$ two function is symmetric about $y = x$.



$\dfrac{a^p}{p} + \dfrac{b^q}{q} = \displaystyle\int_0^a x^{p-1}\,dx + \int_0^b x^{q-1}\,dx$

w.l.g. suppose $b > a$. by the geometric properties and symmetry. $\displaystyle\int_0^b x^{q-1}\,dx = S_2 = S_1.$

$\dfrac{a^p}{p} + \dfrac{b^q}{q} = S_3 + S_2 = S_1 + S_3 \leq S_{\text{rectangle } ab} = ab$ (by the figure)

the equality holds. if and only if $a^{p-1} = b$ i.e. $a^p = b^q$

3. Prove: $\frac{1}{b-a}\int_a^b fg \geqslant \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(\frac{1}{b-a}\int_a^b g\right)$

Let $g_1 = -g$, g is increasing, g_1 is decreasing on $[a,b]$

By Chebyshev's inequality.

$$\frac{1}{b-a}\int_a^b fg_1 \leqslant \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(\frac{1}{b-a}\int_a^b g_1\right)$$

$$-\frac{1}{b-a}\int_a^b fg \leqslant \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(-\frac{1}{b-a}\int_a^b g\right).$$

$$\Rightarrow \frac{1}{b-a}\int_a^b fg \geqslant \left(\frac{1}{b-a}\int_a^b f\right)\left(\frac{1}{b-a}\int_a^b g\right) \qquad \square$$

Homework 16.

1. Prove that $\lim\limits_{n\to\infty}\int_0^{\frac{\pi}{2}} \sin^n x\,dx = 0$.

   Hint: Represent the integral as $\int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,dx + \int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}} \sin^n x\,dx$ and estimate each integral.

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   Hint: Consider the function $y(x) = x^{p-1}$ and find its inverse function. Represent ab, $\frac{a^p}{p}$, and $\frac{b^q}{q}$ as areas of rectangle and two curvilinear trapezoids and compare these areas.

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   Hint: Apply Chebyshev's inequality for some auxiliary functions.

1. proof: $\forall \varepsilon > 0$. $\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,dx + \int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}} \sin^n x\,dx$.

$f(x) = \sin^n x$ is increasing. since $n \in \mathbb{N}$. let $g(x) = 1$.

thus. $\exists \xi_1 \in [0, \frac{\pi}{2}-\frac{\varepsilon}{2}]$ s.t. $\int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,dx = 0 \cdot x\Big|_0^{\xi} + \sin^n(\frac{\pi}{2}-\frac{\varepsilon}{2})(\frac{\pi-\varepsilon}{2} - \xi_1)$

$\exists \xi_2 \in [\frac{\pi-\varepsilon}{2}, \frac{\pi}{2}]$ s.t. $\int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}} \sin^n x\,dx = \sin^n(\frac{\pi-\varepsilon}{2})(\xi_2 - \frac{\pi-\varepsilon}{2}) + (\frac{\pi-\varepsilon}{2} - \xi_2)$

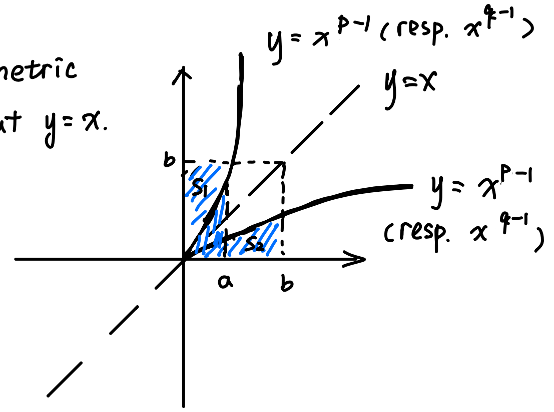
$\therefore \int_0^{\frac{\pi}{2}} \sin^n x\,dx = \sin^n(\frac{\pi-\varepsilon}{2})(\xi_2 - \xi_1) + \frac{\pi-\varepsilon}{2} - \xi_2$

since $0 < \sin(\frac{\pi-\varepsilon}{2}) < 1$ $\quad \sin^n(\frac{\pi-\varepsilon}{2}) \to 0$ $(n \to \infty)$

i.e. for any $\varepsilon > 0$. $\exists N \in \mathbb{N}$. s.t when $n > N$. $\sin^n(\frac{\pi-\varepsilon}{2}) < \frac{\varepsilon}{2(\xi_2 - \xi_1)}$

$\therefore \left|\int_0^{\frac{\pi}{2}} \sin^n x\,dx\right| \le \left|\sin^n(\frac{\pi-\varepsilon}{2})(\xi_2 - \xi_1)\right| + \left|\frac{\pi-\varepsilon}{2} - \xi_2\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □.

2. $y(x) = x^{p-1}$ $\quad y^{-1}(x) = x^{q-1}$ two function is symmetric about $y = x$.



$\frac{a^p}{p} + \frac{b^q}{q} = \int_0^a x^{p-1}\,dx + \int_0^b x^{q-1}\,dx$

w.l.g. suppose $b > a$. by the geometric properties and symmetry. $\int_0^b x^{q-1}\,dx = S_2 = S_1$.

$\frac{a^p}{p} + \frac{b^q}{q} = S_3 + S_2 = S_1 + S_3 \le S_{\text{rectangle } ab} = ab$ (by the figure)

the equality holds. if and only if $a^{p-1} = b$ i.e. $a^p = b^q$

3. Prove: $\frac{1}{b-a}\int_a^b fg \geqslant \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(\frac{1}{b-a}\int_a^b g\right)$

Let $g_1 = -g$, g is increasing, g_1 is decreasing on $[a,b]$

By Chebyshev's inequality.

$$\frac{1}{b-a}\int_a^b fg_1 \leqslant \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(\frac{1}{b-a}\int_a^b g_1\right)$$

$$-\frac{1}{b-a}\int_a^b fg \leqslant \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(-\frac{1}{b-a}\int_a^b g\right).$$

$$\Rightarrow \frac{1}{b-a}\int_a^b fg \geqslant \left(\frac{1}{b-a}\int_a^b f\right)\left(\frac{1}{b-a}\int_a^b g\right) \qquad \square$$

Homework 17.

1. Prove the identity $I = \int_0^{\sin^2 x} \arcsin\sqrt{t}dt + \int_0^{\cos^2 x} \arccos\sqrt{t}dt = \frac{\pi}{4}$.

2. Prove the inequality $\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$.

3. Solve the equation: $\int_{\log 2}^{x} \frac{dt}{\sqrt{e^t-1}} = \frac{\pi}{6}$.

1. $I = \int_0^{\sin^2 x} \arcsin\sqrt{t}\,dt + \int_0^{\cos^2 x} \left(\frac{\pi}{2} - \arcsin\sqrt{t}\right) dt$

$= \frac{\pi}{2}\cdot\cos^2 x + \int_{\cos^2 x}^{\sin^2 x} \arcsin\sqrt{t}\,dt$. (w.l.g. we suppose $\sin x, \cos x > 0$).

$\int \arcsin\sqrt{t}\,dt = t\arcsin\sqrt{t} - \int \frac{t\cdot\frac{1}{2\sqrt{t}}}{\sqrt{1-t}}dt$

$\int \frac{\sqrt{t}}{\sqrt{1-t}}dt \overset{\sqrt{1-t}=y.}{=\!=\!=} -\int \frac{\sqrt{1-y^2}}{y}\cdot 2y\,dy = -2\int\sqrt{1-y^2}\,dy$

$= -2\left[\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\arcsin y\right] + C_1 = -2\cdot\left[\frac{\sqrt{1-t}}{2}\cdot\sqrt{t} + \frac{\arcsin\sqrt{1-t}}{2}\right] + C_1$

$\therefore \int \arcsin\sqrt{t}\,dt = t\arcsin\sqrt{t} + \frac{\sqrt{1-t}}{2}\cdot\sqrt{t} + \frac{\arcsin\sqrt{1-t}}{2} + C.$

$\therefore \int_{\cos^2 x}^{\sin^2 x}\arcsin\sqrt{t}\,dt = -\left[\cos^2 x\left(\frac{\pi}{2}-x\right) + \frac{|\sin x\cos x|}{2} + \frac{x}{2}\right] + \left[\sin^2 x\cdot x + \frac{|\sin x\cos x|}{2} + \frac{\frac{\pi}{2}-x}{2}\right]$

$= -(\cos^2 x+\sin^2 x)x + \left(\frac{x}{2}+\frac{x}{2}\right) - \cos^2 x\cdot\frac{\pi}{2} + \frac{\pi}{4}. = -\cos^2 x\cdot\frac{\pi}{2} + \frac{\pi}{4}.$

$I = \int_{\cos^2 x}^{\sin^2 x}\arcsin\sqrt{t}\,dt + \cos^2 x\cdot\frac{\pi}{2} = \frac{\pi}{4}.$ □.

2. proof: $\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2}\Big|_0^1 = \frac{\pi}{6}.$

$\int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\arcsin\frac{x}{\sqrt{2}}\Big|_0^1 = \frac{\pi}{4\sqrt{2}}$

Since $x\in[0,1]$. $0 < x^3 \leq x^2 \leq 1$. (the equality holds only when $x=1$.)

thus. $\frac{\pi}{6} = \int_0^1\frac{dx}{\sqrt{4-x^2}} < \int_0^1\frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1\frac{dx}{\sqrt{4-2x^2}} = \frac{\pi}{4\sqrt{2}}$ □.

3. $\displaystyle\int_{\ln 2}^{x} \frac{dt}{\sqrt{e^t-1}} \xlongequal[e^t=u^2+1]{\sqrt{e^t-1}=u.} \int_1^{\sqrt{e^x-1}} \frac{1}{u}\cdot\frac{2u}{u^2+1}\,du. = 2\int_1^{\sqrt{e^x-1}} \frac{1}{u^2+1}\,du.$

$\displaystyle = 2\arctan u \Big|_1^{\sqrt{e^x-1}} = \frac{\pi}{6}.$

$\Rightarrow \arctan\sqrt{e^x-1} = \frac{\pi}{3} \quad \Rightarrow x = 2\ln 2.$

Homework 19.

1. Prove that if $\int_a^b f, \int_a^b g$ converge, $\alpha, \beta \in \mathbb{R}$, then $\int_a^b (\alpha f + \beta g)$ converge and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

2. Prove that if $\int_a^b f$ diverges, and $\int_a^b g$ converges, then $\int_a^b (f+g)$ diverges.

3. Find $\int_0^1 \log x \, \mathrm{d}x$

4. Find $I = \int_{-1}^1 f(x)\,\mathrm{d}x, \quad f(x) = \begin{cases} 1/x, x < 0, \\ 1/\sqrt{x}, x > 0. \end{cases}$

1. Proof: by Cauchy's criterion. (w.l.g. we consider the function on $[a,b)$).

$\forall \varepsilon > 0$. $\exists b_0 \in [a,b)$. $\forall b_1, b_2 \in [b_0, b)$ $\left|\int_{b_2}^{b_1} f\right| < \frac{\varepsilon}{2\alpha}$ (w.l.g. $b_1 > b_2$)

$\exists c_0 \in [a,b)$ $\forall c_1, c_2 \in [c_0, b)$ $\left|\int_{c_2}^{c_1} g\right| < \frac{\varepsilon}{2\beta}$ (w.l.g. $c_1 > c_2$)

Let $d_0 = \max\{b_0, c_0\}$. $\forall d_1, d_2 \in [d_0, b)$ (w.l.g. $d_1 > d_2$)

By linear property in Riemann integral

$$\left|\int_{d_2}^{d_1} \alpha f + \beta g\right| = \left|\alpha \int_{d_2}^{d_1} f + \beta \int_{d_2}^{d_1} g\right| \le \alpha \cdot \left|\int_{d_2}^{d_1} f\right| + \beta \left|\int_{d_2}^{d_1} g\right| < \alpha \cdot \frac{\varepsilon}{2\alpha} + \beta \cdot \frac{\varepsilon}{2\beta} = \varepsilon.$$

By Cauchy's Criterion. $\int_a^b \alpha f + \beta g$ is convergent.

W.l.g. assume $f, g \in R_{loc}[a,b)$.

$$\int_a^b \alpha f + \beta g = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} \alpha f + \beta g \xlongequal[\text{in Riemann integral}]{\text{by linear property}} \alpha \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f + \beta \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} g$$

$$= \alpha \int_a^b f + \beta \int_a^b g.$$

(if $b = +\infty$. let $\int_a^b \alpha f + \beta g = \lim_{B \to +\infty} \int_a^B \alpha f + \beta g$, then use the linear property in Riemann integral similarly).

2. By Cauchy's criterion Since $\int_a^b f$ is div.
$\exists 2\varepsilon_0 > 0.\ \forall b_0 \in [a,b).\ \exists b_1, b_2 \in [b_0, b)\quad \left|\int_{b_2}^{b_1} f\right| \geq 2\varepsilon_0.$

since $\int_a^b g$ is convergent.

Let $\varepsilon = \varepsilon_0.\ \exists b_0' \in [a,b).\ \exists b_1', b_2' \in [b_0', b).\ \left|\int_{b_2'}^{b_1'} g\right| < \varepsilon_0.$

thus $\exists \varepsilon_0 > 0.\ \forall b_0'' \in [b_0', b).\ \exists b_1'', b_2'' \in [b_0', b).$

$$\left|\int_{b_2''}^{b_1''} f+g\right| \geq \left|\int_{b_2''}^{b_1''} f\right| - \left|\int_{b_2''}^{b_1''} g\right| > 2\varepsilon_0 - \varepsilon_0 > \varepsilon_0.$$

thus. $\int_{b_0}^b f+g$ is divergent. Since $[b_0, b) \subset [a,b)$. $\int_a^b f+g$ is divergent.

3. $\int_0^1 \ln x\, dx. = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \ln x\, dx$

$\int \ln x\, dx = x\ln|x| - x + C.$

$$\int_0^1 \ln x\, dx = -1 + \lim_{\varepsilon \to 0^+} (\varepsilon - \varepsilon \ln|\varepsilon|) = -1$$

4.
$$I = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\, dx + \lim_{\varepsilon \to 0^-} \int_{-1}^{\varepsilon} f(x)\, dx.$$

$\int \frac{1}{x} dx = \ln|x| + C \qquad \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + C$

$$I = \lim_{\varepsilon \to 0^+} (2 - 2\sqrt{\varepsilon}) + \lim_{\varepsilon \to 0^-} (\ln|\varepsilon| - 0) = -$$

Homework 21.

1. Check whether the integral converges or diverges $\int_0^1 \frac{|\log x|}{x^\alpha}\,dx, \alpha \in \mathbb{R}$.

2. By Cauchy's criterion prove the divergence of $\int_0^1 \sin^2\left(\frac{1}{1-x}\right)\frac{dx}{1-x}$

3. Check whether the integral converges or diverges. Check whether it absolutely converges or diverges.

$$I = \int_0^1 \sin\left(\frac{1}{1-x}\right)\frac{dx}{1-x}$$

4. Check whether the integral converges or diverges depending on $m, n \in \mathbb{N}$ $I = \int_{+0}^{+\infty} \frac{P_m(x)}{P_n(x)}\,dx$, where $P_m(x)$ and $P_n(x)$ relatively prime polynomials of order m and n respectively.

1. First consider the $\int_0^1 \frac{dx}{x^\alpha} = \begin{cases} \alpha = 1. \ \ln x \,\big|_\varepsilon^1 & \to \text{div} \\ \alpha \neq 1. \ \frac{x^{1-\alpha}}{1-\alpha}\Big|_\varepsilon^1 & \Rightarrow \begin{matrix} 1-\alpha < 0 & \text{div} \\ 1-\alpha > 0 & \text{conv.} \end{matrix} \end{cases}$

the integral $\int_0^1 \frac{dx}{x^\alpha}$ conv when $\alpha < 1$ div when $\alpha \geq 1$

since $\lim\limits_{x\to 0^+} \frac{|\ln x| \cdot x^{-\alpha}}{x^{-\alpha}} = +\infty$. by comparison test. $\int_0^1 \frac{|\ln x|}{x^\alpha}$ div. when $\alpha \geq 1$

Now consider $\alpha < 1$.

$$\int \frac{\ln x}{x^\alpha}\,dx = \frac{x^{1-\alpha}}{1-\alpha}\ln x - \frac{x^{1-\alpha}}{1-\alpha}.$$

$$\int_0^1 \frac{|\ln x|}{x^\alpha}\,dx = \lim_{\varepsilon\to 0^+} \frac{x^{1-\alpha}}{1-\alpha}(1-\ln x)\Big|_\varepsilon^1 = \frac{1}{1-\alpha} \quad (\text{since } \alpha < 1).$$

That is, the integral diverges when $\alpha \geq 1$. converges when $\alpha < 1$.

2. $\int_0^1 \sin^2\frac{1}{1-x} \cdot \frac{dx}{1-x} \overset{\frac{1}{1-x}=t}{=\!=\!=} \int_1^{+\infty} \frac{\sin^2 t}{t}\,dt = \int_1^{+\infty} \frac{1-\cos 2t}{2t}\,dt.$

$\overset{2t=s}{=} \frac{1}{2}\int_2^{+\infty} \frac{1-\cos s}{s}\,ds.$

For $\int_2^{+\infty} \frac{1}{s}\,ds$. $F(x) = \int_2^x \frac{1}{s}\,ds = \ln\frac{x}{2}$

$\exists \varepsilon_0 = \ln 2$. $\forall b_0 \in [2, +\infty)$ $\exists b_1, 3b_1 \in [b_0, +\infty)$. s.t. $\left|\int_{b_1}^{3b_1} \frac{1}{s}\,ds\right| = \ln 3 > \ln 2$.

By Cauchy's Criterion. $\int_2^{+\infty} \frac{1}{s}$ div.

For $\int_2^{+\infty} \frac{\cos s}{s}$. $F(x) = \int_2^x \cos s\,ds = \sin x - \sin 2 \leq 2$. bounded. $\lim\limits_{s\to+\infty} \frac{1}{s} = 0$

by Dirichlet's test. the integral $\int_2^{+\infty} \frac{\cos s}{s}$ conv.

Thus. $I = \frac{1}{2}\int_2^{+\infty} \frac{1-\cos s}{s}\,ds$ is divergent.

3. $I \xlongequal{t=\frac{1}{1-x}} \int_1^{+\infty} \frac{\sin t}{t}\, dt$

$F(x) = \int_1^x \sin t\, dt = -\cos t\Big|_1^x \leq 2.$ $F(x)$ is bounded.

$\lim\limits_{t\to+\infty} \frac{1}{t} = 0.$ By Dirichlet's test. $\int_1^{+\infty} \frac{\sin t}{t}\, dt$ is convergent.

since $\dfrac{\frac{\sin^2 t}{t}}{\frac{|\sin t|}{t}} = |\sin t| \leq 1$, that is bounded.

i.e. $\frac{\sin^2 t}{t} = O\left(\frac{|\sin t|}{t}\right)$

By 2. $\int_1^{+\infty} \frac{\sin^2 t}{t}\, dt$ diverges. $\int_1^{+\infty} \frac{|\sin t|}{t}$ divegeres.

In conclusion. I is conv. but not abs. conv.

4. 1) $m>n$. $\lim\limits_{x\to+\infty} \frac{P_m(x)}{P_n(x)} = +\infty$ the integral div.

2) $m=n$ $\lim\limits_{x\to+\infty} \frac{P_m(x)}{P_n(x)} = c$ ($c\in\mathbb{R}$. is the specific value of the coefficient of leading terms)

Since $c\neq 0$. the integral div.

3) $m<n$. $\lim\limits_{x\to+\infty} \frac{P_m(x)}{P_n(x)} = 0.$

if $P_n(x)=0$ has real root $t\in\mathbb{R}$. we have $P_n(x) = (x-t)^p P_{n_1}(x)$.

since $P_n(x)$. $P_m(x)$ are co-prime. $P_m(t)\neq 0$.

$$\lim_{x\to t} \left[(x-t)^p \frac{P_m(x)}{P_n(x)} \right] = \lim_{x\to t} \left[(x-t)^p \frac{P_m(x)}{(x-t)^p P_{n_1}(x)} \right] = \frac{P_m(t)}{P_{n_1}(t)} \neq 0$$

thus the integral $\int_0^t \frac{dx}{(x-t)^p}$ and $\int_0^t \frac{P_m(x)}{P_n(x)}$ div. / conv. simultaneously.

$$\int_0^t \frac{dx}{(x-t)^p} = \begin{cases} p=1 & \lim\limits_{\varepsilon\to0^+} \ln|x-t|\Big|_0^{t-\varepsilon} = +\infty \quad \text{div.} \\ p>1 & \lim\limits_{\varepsilon\to0^+} \frac{(x-t)^{1-p}}{1-p}\Big|_0^{t-\varepsilon} = \infty \quad \text{div.} \end{cases}$$

thus. $\int_0^{+\infty} \frac{P_m(x)}{P_n(x)}$ diverges when $P_n(x)=0$ has root(s) on $[0,+\infty)$

if $P_n(x)$ has no roots. on $[0,+\infty)$

we have $\lim\limits_{x\to+\infty} \left(x^{n-m} \frac{P_m(x)}{P_n(x)} \right) = c$. the integral. $\int_0^{+\infty} \frac{P_m(x)}{P_n(x)} dx$, $\int_0^{+\infty} \frac{1}{x^{n-m}} dx$. by comparison test.

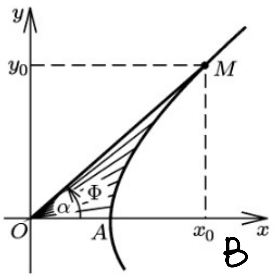
converges simultaneously only if $n-m>1$

Thus. the integral converges only if $n>m+1$ and $P_n(x)$ has no real root(s) on $[0,+\infty)$

Homework 22.

1. Find the area of the figure bounded by the parabola $y = 6x - x^2 - 7$ and by the line $y = x - 3$.

2. There is a point $M(x_0, y_0)$ on the hyperbola $x^2 - y^2 = a^2$. Find the area of the curvilinear triangle OAM.



3. Find the area of the figure bounded by the curve $x = a \sin t \cos^2 t,\ y = a \cos t \sin^2 t,\ 0 \le t \le \pi/2$.

1. Solution: first find the intersection points

$$\begin{cases} y = 6x - x^2 - 7 \\ y = x - 3 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -2 \end{cases} \quad \begin{cases} x = 4 \\ y = 1 \end{cases}$$

$$S = \int_1^4 [(6x - x^2 - 7) - (x-3)]\,dx = \int_1^4 5x - x^2 - 4\,dx$$

$$= \int_1^4 \frac{5}{2}x^2 - \frac{1}{3}x^3 - 4x \Big|_1^4 = \frac{8}{3} + \frac{1}{3} + 4 - \frac{5}{2} = \frac{11}{2}$$

2. let $B(x_0, 0)$. we can rewrite the hyperbola. in first quordrant

$y = \sqrt{x^2 - a^2}$ $(x \geqslant a,\ y \geqslant 0)$.

the area of curvilinear triangle ABM equals

$$S_{ABM} = \frac{1}{2}\int_a^{x_0} \sqrt{x^2 - a^2}\,dx = \frac{x}{4}\sqrt{x^2 - a^2} + \frac{a^2}{4}\ln\left(x + \sqrt{x^2 - a^2}\right)\Big|_a^{x_0} = \frac{x_0}{4}\sqrt{(x_0^2 - a^2)} + \frac{a^2}{4}\ln\frac{x_0 + \sqrt{x_0^2 - a^2}}{a}$$

the $S_{OAM} = S_{OOMB} - S_{ABM} = \frac{1}{2}x_0 y_0 - S_{ABM} = \frac{x_0}{4}\sqrt{(x_0^2 - a^2)} - \frac{a^2}{4}\ln\frac{x_0 + \sqrt{x_0^2 - a^2}}{a}.$

3. $$S = \frac{1}{2}\int_0^{\frac{\pi}{2}} [x(t)\,y'(t) - x'(t)\,y(t)]\,dt$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} a^2 \sin t \cos^2 t \left[-\sin t \cdot \sin^2 t + \cos t \cdot 2\sin t \cos t\right]$$

$$- a^2 \cos t \sin^2 t \left[\cos t \cdot \cos^2 t + \sin t \cdot 2\cos t(-\sin t)\right]\,dt$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \left[2\left(\sin^2 t\cos^4 t + \cos^2 t\sin^4 t\right) - \sin^4 t\cos^2 t - \sin^2 t\cos^4 t \right] dt$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \sin^2 t\cos^2 t\, dt = \frac{a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^2 2t}{2}\, dt = \frac{a^2}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4t)\, dt$$

$$= \frac{a^2}{8}\cdot\left(t - \frac{\sin 4t}{4} \Big|_0^{\frac{\pi}{2}} \right) = \frac{a^2\pi}{16}$$

Homework 23.

1. Find the perimeter of curvilinear triangle bounded by an arc of the circle $x^2 + y^2 = 2$ and the graphic of the function $y = \sqrt{|x|}$.

2. Find the length of the arc $y = \cosh x$ from $A(0, 1)$ to $B(b, \cosh b)$.

3. Find the length of the arc

$$x = t \sin t, \quad y = t \cos t, \quad z = \frac{2t\sqrt{2t}}{3}$$

from $A(0, 0, 0)$ to $B(0, 2\pi, 8\pi\sqrt{\pi}/3)$.

4. Find the volume of the solid which is a result of the revolving of the disk $(x - a)^2 + y^2 \le a^2$ around Oy.

1. First find the intersection points

$$\begin{cases} y^2 = |x| \quad (y>0). \\ x^2+y^2 = 2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases} \quad \begin{cases} x = -1 \\ y = 1 \end{cases}$$

the curvilinear triangle is symmetric about y-axis.

$$S = 2\int_0^1 \sqrt{2-x^2} - \sqrt{x} = 2\int_0^1 \sqrt{2-x^2} - 2\int_0^1 \sqrt{x} = 2\left[\frac{x}{2}\sqrt{2-x^2} + \frac{\sqrt{2}}{2} \arcsin\frac{x}{\sqrt{2}}\Big|_0^1\right.$$

$$\left. - \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1\right] = 2\left[\left(\frac{1}{2} + \frac{\sqrt{2}\pi}{8}\right) - \frac{2}{3}\right] = \frac{\sqrt{2}\pi}{4} - \frac{1}{3} = \frac{3\sqrt{2}\pi - 4}{12}$$

2. $y' = \sinh x \quad y \in C^1[0, b]$

$$S_\gamma = \int_0^b \sqrt{1 + \sinh^2 t}\, dt = \int_0^b |\cosh t|\, dt = \sinh t\Big|_0^b = \frac{e^b - e^{-b}}{2}$$

3. At A, $t=0$, at B, $t = 2\pi$

since $x' = t\sin t = \sin t + t\cos t$, smooth.

$y' = t\cos t = \cos t - t\sin t$, smooth.

$z' = \left(\frac{2\sqrt{2}\cdot t^{\frac{3}{2}}}{3}\right)' = \sqrt{2}\cdot t^{\frac{1}{2}}$

$$S_\gamma = \int_0^{2\pi} \sqrt{(\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 + 2t}\, dt$$

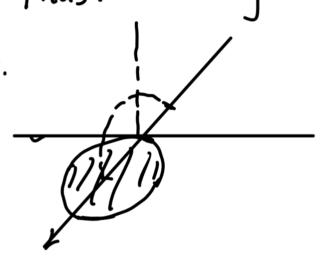
$$= \int_0^{2\pi} \sqrt{t^2 + 1 + 2t}\, dt$$

$$= \int_0^{2\pi} (t+1)\, dt = \frac{t^2}{2} + t\Big|_0^{2\pi}$$

$$= 2\pi^2 + 2\pi$$

Thus. the length $= 2\pi(\pi+1)$

4.



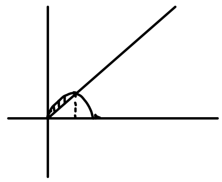
while revolving around Oy. the length of the curve of the centre of a circle. is the circumference of the circle with radius $r = a$.

$$V = S_{disk} \cdot S_\gamma = \pi a^2 \cdot 2\pi a = 4\pi^2 a^3$$

Homework 24.

1. The figure D is bounded by $y = x$, $y = 2x - x^3$, $x \geq 0$. Find the volume of the solid which is a result of the revolving of the figure around Ox.

1. Solution:



first find the intersection point $(x \geq 0)$

$$\begin{cases} y = x \\ y = 2x - x^3 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 1 \\ y = 1. \end{cases}$$

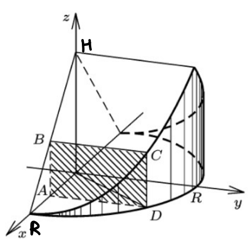
$$V = \pi \int_0^1 (2x - x^3)^2 dx - \pi \int_0^1 x^2 dx$$

$$= \pi \left[\frac{x^7}{7} - \frac{4x^5}{5} + \frac{4x^3}{3} \Big|_0^1 - \frac{x^3}{3} \Big|_0^1 \right] = \pi \left(\frac{1}{7} - \frac{4}{5} + \frac{4}{3} - \frac{1}{3} \right) = \frac{12\pi}{35}$$

2. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = R^2$ and the planes

$$y = 0, \ z = 0, \ \frac{x}{R} + \frac{z}{H} - 1 = 0, \ \frac{x}{R} - \frac{z}{H} + 1 = 0.$$

Use the picture as a hint.



$$\frac{x}{R} + \frac{z}{H} = 1$$

$$z = H - \frac{Hx}{R}$$

Solution: let $A(x, 0, 0)$, $B(x, 0, H - \frac{H}{R}x)$ $D(x, \sqrt{R^2 - x^2}, 0)$

Thus we have $S(x) = |AB| \cdot |AD| = (H - \frac{H}{R}x)(\sqrt{R^2 - x^2})$

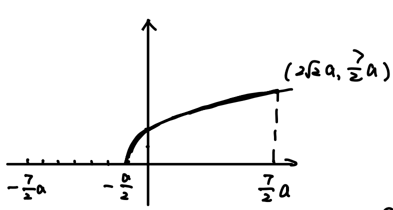
the solid is symmetric about plane yOz.

$$V = \int_0^R (H - \frac{H}{R}x)(\sqrt{R^2 - x^2})\, dx = \frac{H}{R} \int_0^R (R - x)\sqrt{R^2 - x^2}\, dx \ \underline{\underline{x = R\cos\theta}} \ \frac{H}{R} \int_0^{\frac{\pi}{2}} R(1 - \cos\theta) \cdot R^2 \sin^2\theta \cdot d\theta$$

$$= HR^2 \int_0^{\frac{\pi}{2}} (1 - \cos\theta) \sin^2\theta \, d\theta = HR^2 \left(\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta - \int_0^{\frac{\pi}{2}} \sin^2\theta \, d\sin\theta \right)$$

$$= HR^2 \left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4} \Big|_0^{\frac{\pi}{2}} - \frac{1}{3} \sin^3\theta \Big|_0^{\frac{\pi}{2}} \right) = HR^2 \left(\frac{\pi}{4} - \frac{1}{3} \right)$$

3. Fine the area of a surface of revolution for the arc of the parabola $2ax = y^2 - a^2$, $0 \leq y \leq 2\sqrt{2}a$ around Ox.



Solution: since $y \geq 0$. and $a > 0$.

$y = \sqrt{a^2 + 2ax}$ $\quad (x \geq -\frac{a}{2})$.

$y' = \dfrac{a}{\sqrt{2ax + a^2}}$ $\quad$ smooth on $(-\frac{a}{2}, \frac{7}{2}a]$

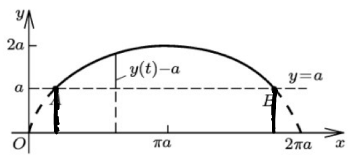
$$S_\gamma = \int_{-\frac{a}{2}}^{\frac{7}{2}a} \sqrt{1 + \frac{a^2}{2ax + a^2}}\, dx \ \underline{\underline{t = \sqrt{1 + \frac{a}{2x + a}}}} \ a \int_{\frac{3}{2\sqrt{2}}}^{\infty} t \cdot \frac{t}{(t^2 - 1)^2}\, dt = \left(\frac{9}{2} - \frac{3\ln 2}{4} \right) a$$

$$S = 2\pi S_\gamma = \left(9\pi - \frac{3\ln 2}{2} \pi \right) a$$

4. The line $y = a$ intersects the arc of the cycloid

$$x = a(t - \sin t), \quad y = a(1 - \cos t), 0 \le t \le 2\pi.$$

at the points A and B. Find the area of a surface of revolution for the arc AB around the line $y = a$.



Use the picture as a hint.

Solution: First we find the intersection point $A\left(\frac{\pi}{2}a, a\right)$ $B\left(\frac{3\pi}{2}a, a\right)$ w.l.g $a>0$.

$x' = a(1-\cos t)$. $y' = a\sin t$. $x, y \in C^1\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

$$S = 2\pi \int_\gamma = 2\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\, dt = 2\sqrt{2}\pi a \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sqrt{1-\cos t}\, dt$$

$$= 4\pi a \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left|\sin\frac{t}{2}\right| dt = 16\pi a \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\frac{t}{2}\, d\frac{t}{2} = 16\pi a \cdot \frac{\sqrt{2}}{2} = 8\sqrt{2}\pi a$$

Homework 25.

1. Prove item (b)-(e) of the following

   Theorem (Functions of bounded variations and arithmetic operations) Let $f, g \in V[a,b]$, then

   (a) $f + g \in V[a,b]$,

   (b) $fg \in V[a,b]$,

   (c) $\alpha f \in V[a,b] (\alpha \in \mathbb{R})$,

   (d) $|f| \in V[a,b]$,

   (e) if $\inf_{x\in[a,b]} |g(x)| > 0$, then $\frac{f}{g} \in V[a,b]$.

2. Prove that the function $f(x) = \begin{cases} x^2 \cos \frac{\pi}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$ has a bounded variation on $[0,1]$.

3. Is it true that is $|f| \in V[a,b]$, then $f \in V[a,b]$? Prove or give a counterexample.

1. (a) For any partition of $[a,b]$

$$\sum_{k=0}^{n-1}|(f+g)(x_{k+1})-(f+g)(x_k)| = \sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)+g(x_{k+1})-g(x_k)| < \sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)|+\sum_{k=0}^{n-1}|g(x_{k+1})-g(x_k)|$$

$$\underset{\text{by def of sup.}}{\leq} \sup\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)| + \sup\sum_{k=0}^{n-1}|g(x_{k+1})-g(x_k)| = \bigvee_a^b f + \bigvee_a^b g$$

since $f.g$ are bounded. $f+g$ is bounded

(b) for any partition

$$\sum_{k=0}^{n-1}|(f\cdot g)(x_{k+1})-(fg)(x_k)| < \sum_{k=0}^{n-1}|f(x_{k+1})g(x_{k+1})-f(x_k)g(x_{k+1})| + \sum_{k=0}^{n-1}|f(x_k)g(x_{k+1})-f(x_k)g(x_k)|$$

since $f.g \in V[a,b]$, then $\exists M_1, M_2$, $\forall x \in [a,b]$ $|f(x)| \leq M_1$. $|g(x)| \leq M_2$

So we continue the inequality one.

$$\leq M_2 \cdot \sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)| + M_1 \sum_{k=0}^{n-1}|g(x_{k+1})-g(x_k)| \underset{\text{def of sup}}{\leq} M_2 \bigvee_a^b f + M_1 \bigvee_a^b g < +\infty$$

(c) for any partition

$$\sum_{k=0}^{n-1}|(\alpha f)(x_{k+1})-(\alpha f)(x_k)| = \sum_{k=0}^{n-1}|\alpha\cdot f(x_{k+1}) - \alpha\cdot f(x_k)| = \alpha\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)|$$

$$\underset{\text{def of sup}}{\leq} \alpha \bigvee_a^b f. < +\infty \underset{\text{def. of sup.}}{\Longrightarrow} \bigvee_a^b \alpha f < +\infty$$

(d) for any partition.

$$\sum_{k=0}^{n-1}||f|(x_{k+1}) - |f|(x_k)|$$

$$\leq \sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)| = \bigvee_a^b f < +\infty$$

if $f(x_{k+1})$, $f(x_k)$ keep the same sign.

$$||f|(x_{k+1}) - |f|(x_k)| = |f(x_{k+1}) - f(x_k)|$$

if $f(x_{k+1})$, $f(x_k)$ have opposite sign.

$$||f|(x_{k+1}) - |f|(x_k)| = |f(x_k) + f(x_{k+1})| \leq |f(x_{k+1}) - f(x_k)|$$

(e) $g \in V[a,b]$.

Since $\inf_{x \in [a,b]} |g(x)| > 0. \Rightarrow g(x) \neq 0$ Let $\inf_{x \in [a,b]} |g(x)| = A$ $(A \in \mathbb{R})$

for any partition

$$\sum_{k=0}^{n-1} \left| \frac{1}{g(x_{k+1})} - \frac{1}{g(x_k)} \right| \leq \frac{1}{A^2} \sum_{k=0}^{n-1} |g(x_k) - g(x_{k+1})| = \frac{\overset{b}{\underset{a}{V}} g}{A^2} < +\infty$$

then function $\frac{1}{g} \in V[a,b]$

by the proof of (b). we have $\frac{f}{g} \in V[a,b]$.

2. $\lim_{x \to 0^+} x^2 \cos \frac{\pi}{x} = 0$. $f(x)$ is continuous on $[0,1]$

for any partition $\{x_k\}_{k=0}^{\infty}$

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{n-1} \left| x_{k+1}^2 \cos \frac{\pi}{x_{k+1}} - x_{k+1}^2 \cos \frac{\pi}{x_k} \right| + \left| x_{k+1}^2 \cos \frac{\pi}{x_k} - x_k^2 \cos \frac{\pi}{x_k} \right|$$

$$< \sum_{k=0}^{n-1} |x_{k+1}^2| + |x_{k+1}^2 - x_k^2| \leq \sum_{k=0}^{n-1} \left(|x_{k+1}^2| + |x_{k+1} + x_k| \cdot |x_{k+1} - x_k| \right)$$

$$< 2 \sum_{k=0}^{n-1} |x_{k+1}| + |x_{k+1} - x_k| = 2 \cdot \Big(\underbrace{\sum_{k=0}^{n-1} x_{k+1}}_{①} + \underbrace{\sum_{k=0}^{n-1} |x_{k+1} - x_k|}_{②} \Big)$$

$\sup ① = \int_0^1 x \, dx = \frac{1}{2}$

$\sup ③ = \overset{1}{\underset{0}{V}} g$ $\quad g(x) = x = 2x - x$. $g(x)$ is bounded variation

thus we have $\sup \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| < \frac{1}{2} + \overset{1}{\underset{0}{V}} g < +\infty$

3. Let the partition $\{x_k\}_{k=0}^{n}$ $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$.

where x_i such that. $\forall \varepsilon > 0$. $\operatorname{sgn} f(x_i - \varepsilon) + \operatorname{sgn} f(x_i + \varepsilon) = 0$.

i/ $f \equiv 0$. then $\overset{b}{\underset{a}{V}} f = 0$

ii/ $f \not\equiv 0$. then $|f| = \begin{cases} f, & x \in [x_{2t}, x_{2t+1}] \\ -f & x \in [x_{2t+1}, x_{2t+2}] \end{cases}$, w.l.g $\begin{cases} f, & x \in [x_{2t+1}, x_{2t+2}] \\ -f, & x \in [x_{2t}, x_{2t+1}] \end{cases}$

$$\overset{2t}{\underset{2t-1}{V}} f = -\overset{2t}{\underset{2t-1}{V}} |f|$$

$\overset{b}{\underset{a}{V}} f < +\infty$

$$\overset{b}{\underset{a}{V}} f = \sum_{i=0} \overset{2i+1}{\underset{2i}{V}} f + \sum_{i=0} \overset{2i+2}{\underset{2i+1}{V}} f = \sum_{i=0} \overset{2i+1}{\underset{2i}{V}} |f| + \sum_{i=0} \overset{2i+2}{\underset{2i+1}{V}} -|f|$$, by additivity of bounded variation

Homework 26.

1. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}$ converges, find the sum.

2. $\sum_{n=1}^{\infty} \left(\frac{2n^2-3}{2n^2+1}\right)^{n^2}$

3. Prove that the series $\sum_{n=1}^{\infty} \frac{1}{(1+i)^n}$ converges, find the sum.

4. Find the sum $\sum_{n=1}^{\infty} (\sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n})$.

1. proof: denote $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$

$$\lim_{n\to\infty} \frac{\frac{1}{n(n+1)(n+2)(n+3)}}{\frac{1}{n(n+1)}} = \lim_{n\to\infty} \frac{1}{(n+2)(n+3)} = 0$$

by comparison test. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges implies $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)}$ converges

$$\frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{3}\left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{1}{1\times2\times3} - \frac{1}{(n+1)(n+2)(n+3)}\right) = \frac{1}{18}$$

2. $$\left(\frac{2n^2-3}{2n^2+1}\right)^{n^2} = e^{n^2 \ln\left(1-\frac{4}{2n^2+1}\right)}$$

$$\lim_{n\to\infty} e^{n^2 \ln\left(1-\frac{4}{2n^2+1}\right)} = \lim_{n\to\infty} e^{-\frac{4n^2}{2n^2+1}} = e^{-2} \neq 0$$

the series diverges

3. $$\frac{1}{(1+i)^n} = \begin{cases} \frac{1-i}{2}\cdot(-4)^{-k}, & n = 4k+1 \\ -\frac{i}{2}\cdot(-4)^{-k}, & n = 4k+2 \\ -\frac{1+i}{4}\cdot(-4)^{-k}, & n = 4k+3 \\ (-4)^{-k+1}, & n = 4k+4 \end{cases}$$

$k \geq 0,\ k \in \mathbb{Z}.$

let $\{n_j\}_{j=0}^{\infty}$ be a strictly incresing sequence s.t.

$$A_j = a_{4j+1} + a_{4j+2} + a_{4j+3} + a_{4j+4} = \left(-\frac{5i}{4}\right)(-4)^{-j}$$

Since each group of the series has 4 terms,

Then by grouping thm. $\sum_{n=1}^{\infty} a_k = \sum_{j=0}^{\infty} A_j = \sum_{j=0}^{\infty} (-\frac{5i}{4}) \cdot (-4)^{-j} = (-\frac{5i}{4}) \cdot \sum_{j=0}^{\infty} (-4)^{-j}$

$$= (-\frac{5i}{4}) \cdot \lim_{j \to \infty} \frac{1[1-(-\frac{1}{4})^{j+1}]}{1+\frac{1}{4}} = -i$$

4. $\sum_{n=1}^{\infty} \sqrt{n+2} - 2\sqrt{n+1} + \sqrt{n} = \sum_{n=1}^{\infty} (\sqrt{n+2} - \sqrt{n+1}) - (\sqrt{n+1} - \sqrt{n})$

$$= \lim_{n \to \infty} (\sqrt{n+2} - \sqrt{n+1}) - (\sqrt{2} - 1)$$

$$= \lim_{n \to \infty} \frac{1}{\sqrt{n+2} + \sqrt{n+1}} + 1 - \sqrt{2} = 1 - \sqrt{2}.$$

Homework 27.

1. Prove that if $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$, where $a_n \in \mathbb{R}, b_n \in \mathbb{R}$ converge, then $\sum_{n=1}^{\infty} |a_n b_n|$, $\sum_{n=1}^{\infty} (a_n + b_n)^2$ converge.

2. Whether the series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n(n+1)}$ converges or diverges.

3. Prove that if $\lim_{n\to\infty} n a_n = a \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

4. Whether the series $\sum_{n=1}^{\infty} \frac{e^n + n^4}{3^n + \log^2(n+1)}$ converges or diverges.

1. For any $n \in \mathbb{N}$

$|a_n b_n| \leq \frac{a_n^2 + b_n^2}{2}$

By linearity of series. $\sum_{n=1}^{\infty} \frac{a_n^2 + b_n^2}{2}$ conv.

Thus we have $\sum_{n=1}^{\infty} |a_n b_n|$ conv.

$$\sum_{n=1}^{\infty} (a_n + b_n)^2 = \sum_{n=1}^{\infty} a_n^2 + b_n^2 + 2a_n b_n \leq \sum_{n=1}^{\infty} a_n^2 + b_n^2 + 2|a_n b_n|$$

by linearity of series. the sum of convergent series also converges.

Thus. $\sum_{n=1}^{\infty} (a_n + b_n)^2$ conv.

2. convergent.

proof: consider the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1.$

$\frac{\cos^2 n}{n(n+1)} \leq \frac{1}{n(n+1)}$ for all $n \in \mathbb{N}$.

thus. by Comparision test. the series conv.

3. consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = a \neq 0.$ $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} a_n$ div. or conv. simultaneously.

$\sum_{n=1}^{\infty} a_n$ div. since $\sum_{n=1}^{\infty} \frac{1}{n}$ div.

4. consider the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{e^n}{3^n}$

denote the original series by $\sum_{n=1}^{\infty} a_n$.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^n+n^4}{e^n} \cdot \frac{3^n}{3^n+\log^2(n+1)} = 1.$ two series conv/div simultaneously.

$K_{b_n} = \overline{\lim_{n\to\infty}} \sqrt[n]{(\frac{e}{3})^n} = \frac{e}{3} < 1.$ $\sum_{n=1}^{\infty} b_n$ conv. (by Cauchy's test).

Thus $\sum_{n=1}^{\infty} \frac{e^n+n^k}{3^n+\log^2(n+1)}$ conv.

Homework 28.

1. Whether the series $\sum_{n=1}^{\infty} \frac{2n^2+5n+1}{\sqrt{n^6+3n^2+2}}$ converges or diverges.

2. Whether the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges or diverges.

3. Whether the series $\sum_{n=1}^{\infty} \left(1-\cos\frac{\pi}{\sqrt[3]{n^2}}\right)$ converges or diverges. $2\sin^2\frac{\pi}{2\sqrt[3]{n^2}}$

4. Whether the series $\sum_{n=1}^{\infty} \left(1-\sqrt[3]{\frac{n-1}{n+1}}\right)^{\alpha}$ converges or diverges.

1. div.

Pf: Consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{2n^2+5n+1}{\sqrt{n^6+3n+2}}}{\frac{n^2}{n^3}} = \lim_{n\to\infty} \frac{2n^2+5n+1}{n^2} \cdot \sqrt{\frac{n^6}{n^6+3n+2}} = 2.$$

the original series div./conv. simultaneously to the $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series)

which is divergent

2. since $a_n \geq 0$ for all n.

$$D = \lim_{n\to\infty} \frac{(n+1)^2 \cdot e^{-(n+1)^3}}{n^2 \quad e^{-n^3}} = \lim_{n\to\infty} \frac{1}{e^{3n^2+3n+1}} = 0. < 1.$$

By d'Alembert's test, the series conv.

3. convergent. $1-\cos\frac{\pi}{\sqrt[3]{n^2}} \geq 0$ for all $n \in \mathbb{N}$.

$$K = \lim_{n\to\infty} \sqrt[n]{1-\cos\frac{\pi}{\sqrt[3]{n^2}}} = 0. \quad \left(\frac{\pi}{\sqrt[3]{n^2}} \to 0 \ , \ \cos\frac{\pi}{\sqrt[3]{n^2}} \to 1\right).$$

By Cauchy's test, the series converges.

4. ① $\alpha = 0$. $\sum_{n=1}^{\infty} 1$ div

② $\alpha < 0$. $\lim_{n\to\infty} \left(1-\sqrt[3]{\frac{n-1}{n+1}}\right)^{\alpha} = \infty$ div.

③ $\alpha > 0$. consider $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\alpha}$

$$\lim_{n\to\infty}\left(\frac{\frac{1}{n}}{1-\sqrt[3]{\frac{n-1}{n+1}}}\right)^{\alpha}=\lim_{n\to\infty}\left(\frac{1}{n}\cdot\frac{1+\sqrt[3]{\frac{n-1}{n+1}}+\sqrt[3]{\left(\frac{n-1}{n+1}\right)^2}}{\left(1-\frac{n-1}{n+1}\right)}\right)=\frac{3}{2}$$

thus, two series div. / conv. simultaneously.

we know that $\sum_{n=1}^{\infty}\frac{1}{n^{\alpha}}$ conv. when $\alpha>1$ div. when $0<\alpha<1$

(famous p-series).

In conclusion $\alpha\leq 1$ the series div.

$\alpha>1$ the series conv.

Homework 29.

1. Whether the series $\sum_{n=1}^{\infty} \frac{a^n}{n!}$, $a > 0$ converges or diverges.

2. Whether the series $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$ converges or diverges.

3. Whether the series $\sum_{n=1}^{\infty} n^5 \left(\frac{3n+2}{4n+3}\right)^n$ converges or diverges.

4. Whether the series $\sum_{n=1}^{\infty} \left(\frac{3n}{n+5}\right)^n \left(\frac{n+2}{n+3}\right)^{n^2}$ converges or diverges.

1. Let $\frac{a^n}{n!} = a_n \geq 0$ (for all $n \in \mathbb{N}$)

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{a}{n+1} = 0$. By d'Alembert's test. the series converges

2. Let $\frac{3^n n!}{n^n} = a_n \geq 0$ (for all $n \in \mathbb{N}$)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3(n+1)}{(n+1)^n \cdot (n+1)} \cdot n^n = 3 \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = 3 \lim_{n\to\infty} e^{n \ln\left(\frac{n}{n+1}\right)}$$

$$= 3 \lim_{n\to\infty} e^{n \cdot \left(\frac{n}{n+1} - 1\right)} = 3\, e^{\lim_{n\to\infty} -\frac{n}{n+1}} = \frac{3}{e} > 1$$

By By d'Alembert's test, the series div.

3. $\lim_{n\to\infty} \sqrt[n]{n^5 \cdot \left(\frac{3n+2}{4n+2}\right)^n} = \lim_{n\to\infty} \left(\sqrt[n]{n}\right)^5 \cdot \left(\frac{3n+2}{4n+2}\right) = \frac{3}{4} < 1.$

By Cauchy's test. the series conv.

4. $$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n}{n+5}\right)^n \left(\frac{n+2}{n+3}\right)^{n^2}} = \lim_{n\to\infty} \frac{3n}{n+5} \cdot \left(\frac{n+2}{n+3}\right)^n$$

$$= 3 \lim_{n\to\infty} e^{n \ln \frac{n+2}{n+3}} = 3 \lim_{n\to\infty} e^{n \cdot \left(\frac{n+2}{n+3} - 1\right)} = \frac{3}{e} > 1.$$

By Cauchy's test. the series div.