Real Analysis

family of subsets of X (always contain ϕ)

ring : $A, B \in$ it. $A \cup B$, $A \cap B$, $A \setminus B \in$ it.

semiring $\mathcal{P}$: 1) $\phi \in \mathcal{P}$ 2) $A, B \in \mathcal{P}$. $A \cap B \in \mathcal{P}$ 3) $A, B \in \mathcal{P}$, $A \setminus B = \bigcup_{n=1}^{N} C_n$, $C_n \in \mathcal{P}$ (C_n is mut. disj.)

algebra : 1) $\phi \in$ it 2) $A \in$ it. $A^c \in$ it 3) $A, B \in$ it. $A \cap B \in$ it. (equiv. $A \cup B \in$ it)

σ-algebra $\mathcal{A}$: 1) $\phi \in \mathcal{A}$ 2) $A \in \mathcal{A}$, $A^c \in \mathcal{A}$ 3) $A_k \in \mathcal{A}$, $\bigcup_{k\geq 1}^{\infty} A_k \in \mathcal{A}$ (equiv. $\bigcap_{k\geq 1}^{\infty} A_k \in \mathcal{A}$)

Borel

Borel σ-algebra : $\mathcal{B}_X$ in $(X, \mathcal{T})$. $\mathcal{B}^m$ in $\mathbb{R}^m$ minimal σ-algebra contain all open subsets of X. ($\mathbb{R}^m$).

Borel (sub)set : element in Borel σ-algebra. ($\triangle$ product of two Borel set is alway Borel).

Borel measure : a measure $\mu : \mathcal{B}_X \to [0, +\infty)$

measurable.

measurable set E : $E \in \mathcal{A}$ (measurable w.r.t $\mathcal{A}$)

measurable space : $(X, \mathcal{A})$

measurable function : (w.r.t. $\mathcal{A}$. $\mathcal{A}'$) : $(X, \mathcal{A})$. $(X', \mathcal{A}')$. $f: X \to X'$, $\forall E \in \mathcal{A}'$. $f^{-1}(E) \in \mathcal{A}$.

measure.

volume $\uparrow \not\downarrow$ measure μ } set X. semiring $\mathcal{P}$ on X. function $\mu : \mathcal{P} \to [0, +\infty)$

s.t. 1) $\mu\phi = 0$ 2) finite add. $\mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu A_n$ / countable add. $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu A_n$

regular measure. if $\forall E$ - measurable. $\forall \varepsilon > 0$ $\exists K$ - compact, G - open s.t. $K \subset E \subset G$. and $\mu(G \setminus E)$, $\mu(E \setminus K) < \varepsilon$.

image measure of μ under $f : (X, \mathcal{A}) \to (X', \mathcal{A}')$. $f_*(\mu)(Y) = \mu(f^{-1}(Y))$.

set.

μ-negligible $(X, \mathcal{A}, \mu)$. $Y \subset X$. Y is set of measure 0. (i.e. $\exists E \in \mathcal{A}$. s.t. $Y \subset E$, $\mu E = 0$). (particularly, $Y \in \mathcal{A}$ and negligible. $\mu Y = 0$)

Space

measurable space : $(X, \mathcal{A})$

measured space : $(X, \mathcal{A}, \mu)$

complete measured space : all μ-negligible sets are measurable.

A much concrete def. of integral.

To do: 可测函数判定.

Approximation function

1. simple function

Definition An $\mathbb{R}$-valued measurable function is called *simple* if the set of its values is finite.

sum. product. max. min.
linear combination.

If f is a simple function, there is a finite partition of X into measurable sets (we will call it *admissible* for f) such that f is constant on its elements. For instance, such a partition can be obtained as follows. Let $a_1, \ldots, a_N$ be all pairwise distinct values of f. Put $e_k = f^{-1}(\{a_k\})$. Obviously, these sets are measurable and form a partition of X that is admissible for f.

use it to approximation

可测函数的有收敛简单函数列.

Theorem (Approximation by simple functions) *Every non-negative measurable function $f : X \to \overline{\mathbb{R}}$ is the pointwise limit of an increasing sequence of non-negative simple functions f_n. If f is bounded, then we may assume that the sequence $\{f_n\}_{n \geqslant 1}$ converges uniformly on X.*

Corollary *Every measurable function f can be pointwise approximated by simple functions f_n satisfying the condition $|f_n| \leqslant |f|$.*

If f is bounded, then this approximation may be assumed uniform.

Lemma *Let f be a non-negative simple function, $\{A_j\}_{j=1}^{M}$, $\{B_k\}_{k=1}^{N}$ be admissible partitions for f, and a_j, b_k be the values of f on A_j and B_k, respectively. Then*

$$\sum_{j=1}^{M} a_j \mu(A_j) = \sum_{k=1}^{N} b_k \mu(B_k).$$

use it to define integrals:

Definition 1 Let f be a non-negative simple function, $\{A_j\}_{j=1}^{M}$ be an arbitrary admissible partition for f, and a_j be the value of f on A_j. The *integral* of f over a set $E \subset X$ is defined as

$$\sum_{j=1}^{M} a_j \mu(E \cap A_j) \tag{1}$$

and is denoted by $\int_E f \, d\mu$.

Integrability.

Definition Given an arbitrary measurable function f on a set E, we keep the notation introduced above and put

$$\int_E f \, d\mu = \int_E f_+ \, d\mu - \int_E f_- \, d\mu$$

if at least one of the integrals $\int_E f_\pm \, d\mu$ is finite. In this case, the function f is said to be *integrable* on E (with respect to the measure μ). If both integrals $\int_E f_\pm \, d\mu$ are finite, then f is *summable* on E (with respect to the measure μ).

Fourier series.

§1 Premise

range : $L^2(X, \mu)$ norm $\|\cdot\|$

operation : scalar product $\langle f, g\rangle = \int_X f\bar{g}\, d\mu$.

1) $\langle g, f\rangle = \overline{\langle f, g\rangle}$ $\|f\|^2 = \langle f, f\rangle$. $|\langle f, g\rangle| \leq \|f\|\|g\|$.

2) $f_n \to f$. $g_n \to g \Rightarrow \langle f_n, g_n\rangle \to \langle f, g\rangle$.

3) parallelogram identity $\|f\|^2 + \|g\|^2 = \|f+g\|^2 + \|f-g\|^2$

§2. Orthogonal

def. (function). $f, g \in L^2(X, \mu)$. $\langle f, g\rangle = 0$.

def. (system) $\{e_\alpha\}_{\alpha \in A}$. orthonormal $\|e_\alpha\| = 1$.

thm. $\{e_k\}_{k=1}^n$ - OS. $L = \text{span}\{e_1, e_2, \dots e_n\}$. $k = 1, 2, \dots n$.

$$C_k(f) = \frac{\langle f, e_k\rangle}{\|e_k\|}$$

$f - \sum_{k=1}^n C_k(f)\, e_k \perp v$, $\forall v \in L$

$\|f - \sum_{k=1}^n a_k e_k\|$ takes the minimum when $a_k = C_k(f)$. $\sum_{k=1}^n C_k(f)\, e_k$ is the best approximation in L.

Lm $\{e_n\}_{n \in \mathbb{N}}$ - OS. $\sum_{n=1}^\infty a_n e_n$ conv. iff $\sum_{n=1}^\infty |a_n|^2 \|e_n\|^2 < +\infty$

§3. Fourier.

def. (coefficient). $\{e_n\}_{n \in \mathbb{N}}$ - OS. $f \in L^2(X, \mu)$. $C_k(f) = \dfrac{\langle f, e_k\rangle}{\|e_k\|}$

def. (series of f). $\sum_{n=1}^\infty C_n(f)\, e_n$

For arbitrary $f \in L^2(X, \mu)$. it's Fourier series conv. in the norm (but not necessarily to f).

thm. Bessel's inequality. $\sum_{k=1}^\infty |C_k(f)|^2 \|e_k\|^2 \leq \|f\|^2$

Riesz-Fisher thm $\forall \{e_n\}_{n \in \mathbb{N}}$ - OS. $\forall f \in L^2(X, \mu)$. the Fourier series of f conv. in the norm

and $f = \sum_{n=1}^\infty C_n(f)\, e_n + h$. $h \perp e_n$ for all $n \in \mathbb{N}$. (Fourier series of $h = 0$).

§4. Basis.

<u>def.</u> $\{e_n\}_{n\in\mathbb{N}}$ - OS. if $\forall f\in L^2(X,\mu)$. f coincide with $\sum_{n=1}^{\infty} c_n(f)\, e_n$ a.e. (i.e. $h=0$ a.e.).

Parseval's Identity. if $\{e_n\}_{n\in\mathbb{N}}$ is basis, $\forall f,g\in L^2(X,\mu)$ $\langle f,g\rangle = \sum_{n=1}^{\infty} c_n(f)\,\overline{c_n(g)}\,\|e_n\|^2$

<u>def.</u> (complete family of functions). $\{f_\alpha\}_{\alpha\in A}\subset L^2(X,\mu)$.

if $f\in L^2(X,\mu)$. $f\perp f_\alpha$ for every $\alpha\in A$. $\Rightarrow$ $f=0$ a.e. (i.e. $\|f\|=0$).

<u>def.</u> (everywhere dense (set)). $\{f_\alpha\}_{\alpha\in A}$. $\forall f\in L^2(X,\mu)$. $\forall\varepsilon>0$. $\exists\, g=\sum_{k=1}^{n} C_k f_{\alpha_k}$. s.t. $\|f-g\|<\varepsilon$.

<u>Lm</u>. set of all l.c. of $\{f_\alpha\}_{\alpha\in A}$ is everywhere dense $\Rightarrow \{f_\alpha\}_{\alpha\in A}$ is complete

if $\{f_\alpha\}_{\alpha\in A}$ is orthogonal "$\Leftrightarrow$"

<u>Thm.</u> $\{e_n\}$ - OS. TFAE. (1) basis (2) complete (3) $\forall f\in L^2(X,\mu)$. Parseval's Identity. $\sum_{n=1}^{\infty}|c_n(f)|^2\,\|e_n\|^2 = \|f\|^2$ holds

§5. Property.

(1). rearrangement preserves.

Lemma *Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal system and $\omega:\mathbb{N}\to\mathbb{N}$ be a bijection. Then the series*

(a) $\sum_{n=1}^{\infty} a_n e_n$ *and*

(b) $\sum_{k=1}^{\infty} a_{\omega(k)} e_{\omega(k)}$

converge simultaneously and, in the case of convergence, their sums are equal.

(2). in product space.

10.1.7 Let $\{e_k\}_{k\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be orthogonal systems in the spaces $\mathscr{L}^2(X,\mu)$ and $\mathscr{L}^2(Y,\nu)$, respectively. We use these systems to construct an OS $\{h_{k,n}\}_{k,n\in\mathbb{N}}$ in the space $\mathscr{L}^2(X\times Y,\mu\times\nu)$ by putting

$$h_{k,n}(x,y)=e_k(x)g_n(y)\quad (x\in X,\ y\in Y).$$

Theorem *If orthogonal systems $\{e_k\}_{k\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are complete, then the system $\{h_{k,n}\}_{k,n\in\mathbb{N}}$ is also complete.*

§6. Method and Exercise.

(1) Prove some OS is basis.

Prove it's everywhere dense $\Leftrightarrow$ complete $\Leftrightarrow$ basis.

some approximation thm. in Chapter 9.

Theorem 4 *Let $1\leqslant p<+\infty$, $f\in\widetilde{\mathscr{L}}^p(\mathbb{R}^m)$ and $\varepsilon>0$. Then there is a trigonometric polynomial T such that $\|f-T\|_p<\varepsilon$.*

Theorem *For $1\leqslant p<+\infty$, every function f in $\mathscr{L}^p(X)$ can be approximated (in the $\mathscr{L}^p$-norm) as closely as desired by a function in $C_0^\infty(\mathbb{R}^m)$.*

Corollary *Let $X\subset\mathbb{R}^m$ be a bounded measurable set, $1\leqslant p<+\infty$, and $f\in\mathscr{L}^p(X)$. For every $\varepsilon>0$, there is a polynomial P such that $\|f-P\|_p<\varepsilon$.*

Corollary *Let $1\leqslant p<+\infty$, and let f be a measurable function defined on $\mathbb{R}^m$ and 2π-periodic in each variable. If $\int_{(-\pi,\pi)^m}|f(x)|^p\,dx<+\infty$, then*

$$\int_{(-\pi,\pi)^m}|f(x)-f(x-h)|^p\,dx \xrightarrow[h\to 0]{} 0.$$

收敛. $(X, \mathcal{A}, \mu)$. $\{f_n\}_{n\in\mathbb{N}} \in S(X)$.

1) pointwise conv. a.e.

2) conv. w.r.t. /in measure

3) conv. in norm

4) uniformly conv.

关系:

$$\begin{tikzcd}
& \boxed{\text{a.e. uni. conv.}} \arrow[dl, shift right] \arrow[dr] & \\
\boxed{\text{a.e. conv.}} \arrow[ur, shift right, "{\mu X<+\infty;\ f_n, f<+\infty \text{ a.e.}}"'] \arrow[rr, "\mu X<+\infty"] & & \boxed{\text{conv. in meas.}}
\end{tikzcd}$$

1. & 4. **Theorem** (Egorov[3]) *Let $f_n, f \in \mathcal{L}^0(X, \mu)$, and let $f_n \xrightarrow[n\to\infty]{\text{a.e.}} f$. If $\mu(X) < +\infty$, then $f_n \xrightarrow[n\to\infty]{} f$ almost uniformly on X.*

2. & 3. **Theorem** *Let $1 \leqslant p < +\infty$ and $f_n \in \mathscr{L}^p(X, \mu)$ for all $n \in \mathbb{N}$.*

(a) *If $f \in \mathscr{L}^p(X, \mu)$ and $\|f_n - f\|_p \xrightarrow[n\to\infty]{} 0$, then $f_n \xrightarrow[n\to\infty]{} f$ in measure.* (L^p内, 依范数收敛 $\Rightarrow$ 依测度收敛)

(b) *If $f_n \xrightarrow[n\to\infty]{} f$ in measure or almost everywhere and $|f_n(x)| \leqslant g(x)$ almost everywhere for all n, where $g \in \mathscr{L}^p(X, \mu)$, then $f \in \mathscr{L}^p(X, \mu)$ and $\|f - f_n\|_p \xrightarrow[n\to\infty]{} 0$.*

1. & 2. **Theorem** (F. Riesz[2]) *Every sequence that converges in measure contains a subsequence that converges almost everywhere to the same limit.*

$g \in L^0(X)$. + $\mu X<+$ $\Rightarrow$