16.10.24

3. HYPERBOLIC EQUATIONS

3.1. THE STRING OSCILLATION EQUATION AND ITS SOLUTION BY THE D'ALEMBERT METHOD

The study of methods for solving boundary value problems for hyperbolic equations begins with the Cauchy problem for the equation of free vibrations of a string:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0\,, \tag{3.1}$$

$$\begin{cases} u(x,0) = \varphi(x), \\ \dfrac{\partial u(x,0)}{\partial t} = \psi(x). \end{cases} \tag{3.2}$$

3.1.1. D'ALEMBERT FORMULA

We transform equation (3.1) to a canonical form containing a mixed derivative. The equation of characteristics

$$\left[\frac{dx}{dt}\right]^2 - a^2 = 0$$

splits into two equations:

$$\frac{dx}{dt} - a = 0\,,\ \frac{dx}{dt} + a = 0\,,$$

the integrals of which are

$$x - at = C_1\,,\ x + at = C_2\,.$$

Now, assuming

$$\xi = x + at\ ,\ \ \eta = x - at\ ,$$

equation (3.1) is transformed to the form

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0\,. \tag{3.3}$$

The general solution of equation (3.3) is determined by the formula

$$u = f_1(\xi) + f_2(\eta)\ ,$$

where $f_1(\xi)$ and $f_2(\eta)$ are arbitrary functions. Returning to the variables x, t, we get

$$u = f_1(x + at) + f_2(x - at)\ . \tag{3.4}$$

The resulting solution depends on two arbitrary functions f_1 and f_2. It is called the *D'Alembert solution*.

Next, substituting formula (3.4) into (3.2), we will have

$$f_1(x) + f_2(x) = \varphi(x)\ , \tag{3.5}$$

$$af_1'(x) - af_2'(x) = \psi(x)\,, \tag{3.6}$$

from where, integrating the second equality (3.6), we get

$$f_1(x) - f_2(x) = \frac{1}{a}\int_{x_0}^{x} \psi(y)dy + C\ , \tag{3.7}$$

where x_0 and C are constants. From the formulas (3.5) and (3.7) we find

$$f_1(x)=\frac{1}{2}\left[\varphi(x)+\frac{1}{a}\int_{x_0}^{x}\psi(y)dy+C\right],$$

$$f_2(x)=\frac{1}{2}\left[\varphi(x)-\frac{1}{a}\int_{x_0}^{x}\psi(y)dy-C\right].$$

At the same time, taking into account the formula (3.4), we have

$$u(x,t)=\frac{1}{2}\left[\varphi(x+at)+\frac{1}{a}\int_{x_0}^{x+at}\psi(y)dy+C+\varphi(x-at)-\frac{1}{a}\int_{x_0}^{x-at}\psi(y)dy-C\right]$$

and finally we get the formula

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)dy\,. \qquad (3.8)$$

The formula (3.8) is called the *D'Alembert's formula.*

It is not difficult to verify that formula (3.8) satisfies equation (3.1) and initial conditions (3.2) given that $\varphi(x)\in C^2(R)$ and $\psi(x)\in C^1(R)$. Thus, the described method proves both the uniqueness and the existence of a solution to the problem.

Example 1

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},$$

if

$$u\big|_{t=0}=x^2,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=0\,.$$

<u>Solution:</u>

We know that $u_{tt} - a^2 u_{xx} = 0$.

Then $a = 1$.

We know that $\begin{cases} u(x;0) = \varphi(x) \\ \dfrac{\partial u}{\partial t}(x;0) = \psi(x) \end{cases}$

Since $u(x;0) = x^2$, then $\varphi(x) = x^2$.

Since $u'_t(x;0) = 0$, then $\psi(x) = 0$.

$$f_1 = \frac{1}{2}\left[x^2 + \frac{1}{1}\int_{x_0}^{x} 0dy + C\right] = \frac{1}{2}\left[x^2 + C\right]$$

$$f_2 = \frac{1}{2}\left[x^2 - C\right]$$

We know that $u = f_1(x - at) + f_2(x + at)$ and $a = 1$:

$$u = f_1(x - t) + f_2(x + t) = \frac{1}{2}\left[(x-t)^2 + C\right] + \frac{1}{2}\left[(x+t)^2 - C\right] =$$

$$= \frac{1}{2}\left[x^2 - 2xt + t^2 + x^2 + 2xt + t^2\right] = x^2 + t^2$$

We have $u = x^2 + t^2$.

OR:

$$u(x,t) = \frac{\varphi(x + at) + \varphi(x - at)}{2},$$

where $a = 1,\ \varphi(x) = x^2$.

$$u(x,t)=\frac{(x+t)^2+(x-t)^2}{2}=\frac{x^2+2xt+t^2+x^2-2xt+t^2}{2}=$$
$$=\frac{2\left(x^2+t^2\right)}{2}=x^2+t^2$$

Let's check three conditions:

1) $\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}$

$$u_x=2x$$
$$u_{xx}=2$$
$$u_t=2t$$
$$u_{tt}=2$$

2) $u\big|_{t=0}=x^2$

$$u=x^2+0=x^2$$

3) $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0$

$$u_t=2t$$

$$u_t(0)=2\cdot 0=0$$

Example 2

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2},$$

if

$$u\big|_{t=0} = 0\,,\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = x\,.$$

<u>Solution:</u>

We know that $u_{tt} - a^2 u_{xx} = 0$.

Then $a = 2$.

We know that $\begin{cases} u(x;0) = \varphi(x) \\ \dfrac{\partial u}{\partial t}(x;0) = \psi(x) \end{cases}$

Since $u(x;0) = 0$, then $\varphi(x) = 0$.

Since $u'_t(x;0) = x$, then $\psi(x) = x$.

We know that

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)\,dy$$

and $a = 2$:

$$u(x,t) = \frac{1}{4}\int_{x-2t}^{x+2t} y\,dy = \frac{1}{8}y^2\Big|_{x-2t}^{x+2t} = \frac{1}{8}\left[(x+2t)^2 - (x-2t)^2\right] =$$

$$= \frac{1}{8}\left[x^2 + 4xt + 4t^2 - x^2 + 4xt - 4t^2\right] = \frac{1}{8}[8xt] = xt$$

We have $u(x,t) = xt$.

<u>Let's check three conditions:</u>

1) $\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$

$u_x = t$

$u_{xx} = 0$

$u_t = x$

$u_{tt} = 0$

2) $u\big|_{t=0} = 0$

$u = x \cdot 0 = 0$

3) $\frac{\partial u}{\partial t}\bigg|_{t=0} = x$

$u_t = x$

3.1.2. THE INHOMOGENEOUS EQUATION

Consider the Cauchy problem for an inhomogeneous oscillation equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)\,,\ x \in R\,,\ t > 0\,, \qquad (3.9)$$

$$u(x,0) = \varphi(x)\,,\ \frac{\partial u(x,0)}{\partial t} = \psi(x)\,,\ x \in R\,. \qquad (3.10)$$

It is easy to verify that the solution of the problem (3.9), (3.10) $u = u(x,t)$ is represented in the form

$$u = \upsilon + \omega, \qquad (3.11)$$

where υ is the solution of the Cauchy problem (3.1), (3.2), and ω is the solution of the following problem:

$$\begin{cases} \dfrac{\partial^2 \omega}{\partial t^2} = a^2 \dfrac{\partial^2 \omega}{\partial x^2} + f(x,t), \qquad x \in R, \quad t > 0, \\ \omega(x,0) = 0, \quad \dfrac{\partial \omega(x,0)}{\partial t} = 0, \quad x \in R. \end{cases} \tag{3.12}$$

Let $W(x,t;\tau)$ be the solution of the auxiliary Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 W}{\partial t^2} = a^2 \dfrac{\partial^2 W}{\partial x^2}, \qquad x \in R, \quad t > \tau, \\ W(x,t;\tau)\Big|_{t=\tau} = 0, \quad \dfrac{\partial W(x,0)}{\partial t}\Big|_{t=\tau} = f(x,\tau). \end{cases} \tag{3.13}$$

We show that the solution of the $\omega(x,t)$ problem (3.12) is determined by the formula

$$\omega(x,t) = \int_0^t W(x,t;\tau)d\tau\,, \tag{3.14}$$

where $W(x,t;\tau)-$ is the solution to the problem (3.13).

Really,

$$\omega(x,0) = 0\,, \quad \frac{\partial \omega(x,t)}{\partial t} = W(x,t;t) + \int_0^t \frac{\partial W(x,t;\tau)}{\partial t} d\tau\,,$$

therefore, $\dfrac{\partial \omega(x,0)}{\partial t} = 0$ by virtue of the initial condition (3.13).

And finally:

$$\frac{\partial^2 \omega}{\partial t^2} - a^2 \frac{\partial^2 \omega}{\partial x^2} = \frac{\partial W(x,t;\tau)}{\partial t}\Big|_{t=\tau} + \int_0^t \frac{\partial^2 W(x,t;\tau)}{\partial t^2} - a^2 \frac{\partial^2 W(x,t;\tau)}{\partial x^2} d\tau = f(x,t)\,.$$

The solution of the problem (3.13) is determined by the D’Alembert’s formula:

$$W(x,t;\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi . \qquad (3.15)$$

Now, using the formulas (3.8), (3.11), (3.14) and (3.15), we find that the solution of the initial problem (3.9), (3.10) is given by the formula

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)dy+\frac{1}{2a}\int_{0}^{t}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau .$$

Example 3

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+x\sin t ,$$

if

$$u\big|_{t=0}=\sin x , \ \frac{\partial u}{\partial t}\bigg|_{t=0}=\cos x , \ x\in R .$$

<u>Solution:</u>

$$u_{tt}-a\,u_{xx}=f(x;t)$$

$$a=1$$

$$f(x;t)=x\cdot\sin t$$

$$u\big|_{t=0}=\sin x : u(x;0)=\sin x, \quad \text{so} \quad \varphi(x)=\sin x$$
$$u_t'\big|_{t=0}=\cos x : u_t'(x;0)=\cos x, \quad \text{so} \quad \psi(x)=\cos x$$

Since $u(x;t)=\upsilon(x;t)+\omega(x;t)$, we will find $\upsilon(x;t)$ and $\omega(x;t)$.

Let $\upsilon(x;t)=\upsilon_1+\upsilon_2$.

$$\upsilon_1(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}$$

$$\upsilon_1=\frac{1}{2}\big(\varphi(x+t)+\varphi(x-t)\big)=\frac{1}{2}\big(\sin(x+t)+\sin(x-t)\big)$$

$$\upsilon_2=\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\,dy=\frac{1}{2}\int_{x-t}^{x+t}\cos y\,dy=\frac{1}{2}\sin y\Big|_{x-t}^{x+t}=$$

$$=\frac{1}{2}\big(sin(x+t)-sin(x-t)\big)$$

$$\upsilon(x;t)=\frac{1}{2}\big(\sin(x+t)+\sin(x-t)\big)+\frac{1}{2}\big(sin(x+t)-sin(x-t)\big)=$$

$$=\sin(x+t)$$

We have $\upsilon(x;t)=\sin(x+t)$.

Now let's find $\omega(x;t)$.

Since $f(x;t)=x\cdot\sin t$, then

$$\omega=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi;\tau)\,d\xi\,d\tau=\frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}\xi\cdot\sin\tau\,d\xi\,d\tau=\frac{1}{2}\int_0^t\sin\tau\,d\tau\,\frac{\xi^2}{2}\Big|_{x-t+\tau}^{x+t-\tau}=$$

$$=\frac{1}{4}\int_0^t\sin\tau\Big[(x+t-\tau)^2-(x-t+\tau)^2\Big]d\tau=$$

$$=\frac{1}{4}\int_0^t\sin\tau\big[(x+t-\tau+x-t+\tau)(x+t-\tau-x+t-\tau)\big]d\tau=$$

$$=x\int_0^t\sin\tau\cdot(t-\tau)\,d\tau=x(t-\sin t)$$

We get

$$u = \sin(x+t) + xt - x\sin t$$

3.1.3. THE CONTINUATION METHOD

The first boundary value problem

The first boundary value problem for the oscillation equation on a half-line with a homogeneous boundary condition is set as follows: find a solution to the oscillation equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \ x > 0, \ t > 0, \qquad (3.16)$$

satisfying the boundary condition

$$u(0,t) = 0, \ t > 0 \qquad (3.17)$$

and the initial conditions

$$u(x,0) = \varphi(x), \ \frac{\partial u(x,0)}{\partial t} = \psi(x), \ x \geq 0. \qquad (3.18)$$

Let's add the conjugation conditions

$$\varphi(0) = 0, \ \psi(0) = 0$$

to ensure the continuity of the functions $u(x,t)$ and $\frac{\partial u(x,t)}{\partial t}$ at zero.

Let's define the functions $\varphi(x)$ and $\psi(x)$ in an odd way on the entire line by specifying new functions Φ and Ψ:

$$\Phi(x)=\begin{cases}\varphi(x), & x\ge 0,\\ -\varphi(-x), & x<0,\end{cases}$$

$$\Psi(x)=\begin{cases}\psi(x), & x\ge 0,\\ -\psi(-x), & x<0.\end{cases}$$

Consider a modified Cauchy problem:

$$\begin{cases}\dfrac{\partial^2 U}{\partial t^2}=a^2\dfrac{\partial^2 U}{\partial x^2}, & -\infty<x<\infty, \quad t>0,\\ U(x,0)=\Phi(x), & \dfrac{\partial U(x,0)}{\partial t}=\Psi(x).\end{cases}$$

In this case, to find $U(x,t)$, we can apply the D'Alembert formula:

$$U(x,t)=\frac{\Phi(x+at)+\Phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(y)dy\,.$$

Let's take the function $U(x,t)$ as the function we need $u(x,t)$ for $x\ge 0, t\ge 0$. Obviously, conditions (3.16) and (3.18) for $x\ge 0, t\ge 0$ are fulfilled immediately — this follows from the definition of the functions $\Phi(x)$ and $\Psi(x)$. The fulfillment of condition (3.17) follows from the following transformations:

$$u(0,t)=U(0,t)=\frac{\Phi(at)+\Phi(-at)}{2}+\frac{1}{2a}\int_{-at}^{at}\Psi(y)dy\,.$$

Due to the odd number of functions $\Phi(x)$ and $\Psi(x)$, the first and second terms vanish, which gives the fulfillment of condition (3.17).

We express $\Phi(x)$ and $\Psi(x)$ through the original functions $\varphi(x)$ and $\psi(x)$, respectively:

$$x \ge at\ ,\ \begin{cases} \Phi(x+at) = \varphi(x+at), \\ \Phi(x-at) = \varphi(x-at), \\ \Psi(y) = \psi(y), \quad y \in [x-at, x+at]; \end{cases}$$

$$x < at\ ,\ \begin{cases} \Phi(x+at) = \varphi(x+at), \\ \Phi(x-at) = -\varphi(at-x). \end{cases}$$

Now let's write down an auxiliary formula for solving the first boundary value problem:

at $x < at$,

$$\int_{x-at}^{x+at} \Psi(y)dy = \int_{x-at}^{0} \Psi(y)dy + \int_{0}^{x+at} \Psi(y)dy =$$

$$= \int_{x-at}^{0} -\psi(-y)dy + \int_{0}^{x+at} \psi(y)dy =$$

{let's put $-y = y$ }

$$= \int_{at-x}^{0} \psi(y)dy + \int_{0}^{x+at} \psi(y)dy = \int_{at-x}^{x+at} \psi(y)dy\,.$$

Then the general formula will be as follows:

$$u(x,t) = \begin{cases} \dfrac{\varphi(x+at)+\varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(y)dy, & x \ge at, \\ \dfrac{\varphi(x+at)-\varphi(at-x)}{2} + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} \psi(y)dy, & x < at. \end{cases}$$

The second boundary value problem

The second boundary value problem for the equation of oscillations on a half-line with a homogeneous boundary condition is set as follows: find a solution to the equation of oscillations

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \; x > 0, \; t > 0, \qquad (3.19)$$

satisfying the boundary condition

$$\frac{\partial u(0,t)}{\partial x} = 0, \; t \ge 0, \qquad (3.20)$$

and the initial conditions:

$$u(x,0) = \varphi(x), \; \frac{\partial u(x,0)}{\partial t} = \psi(x), \; x \ge 0. \qquad (3.21)$$

We will act in the same way as in the previous case, however, only an even continuation will suit us here:

$$\Phi(x) = \begin{cases} \varphi(x), & x \ge 0, \\ \varphi(-x), & x < 0, \end{cases}$$

$$\Psi(x) = \begin{cases} \psi(x), & x \ge 0, \\ \psi(-x), & x < 0. \end{cases}$$

The new Cauchy problem and the solution for it according to the D’Alembert formula will look the same as in the previous case:

$$U(x,t) = \frac{\Phi(x+at) + \Phi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(y)dy.$$

Similarly, let the function $u(x,t)=U(x,t)$ for $x>0,\ t>0$. Then the fulfillment of conditions (3.19) and (3.21) is again obvious. Let's check the condition (3.20). Differentiating the D'Alembert formula and using the fact that the derivative of an even function is odd, we get

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial U(0,t)}{\partial x}=\frac{\Phi'(at)+\Phi'(-at)}{2}+\frac{1}{2a}\left[\Psi(at)-\Psi(-at)\right].$$

From the odd $\Phi'(t)$ and the parity $\Psi(t)$, it can be seen that both terms are equal to zero.

The general formula is obtained similarly:

$$u(x,t)=\begin{cases}\dfrac{\varphi(x+at)+\varphi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(y)dy, & x\geq at,\\[2ex] \dfrac{\varphi(x+at)+\varphi(at-x)}{2}+\dfrac{1}{2a}\left[\displaystyle\int_{0}^{at-x}\psi(y)dy+\int_{0}^{x+at}\psi(y)dy\right], & x<at.\end{cases}$$