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Analytic Geometry. Vector Algebra

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Vector Algebra

- We introduced terms of directed segments, vectors and coordinate system using fundamentals of geometry as base for out talk
- Now we bring these objects into accurate algebraic form of reasoning called **vector algebra**
- We overlook base definitions of vectors and operations with vectors as these definitions stay unchanged
- Given definitions and features of operations with vectors give us an option to write and transform linear vectorial expressions in pretty algebraic manner
- We start with explanation of number of vectors in all introduced by us vectorial bases
- After this we discuss transformation of the bases
- And finally we generalize our operations with vectors for arbitrary skew-angular bases

Linear Combinations of Vectors I

- Assume that some set of n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is given
- We call it a collection of n vectors, a system of n vectors, or a family of n vectors either
- Now we can compose some vectorial expressions composed of these system of vectors:

$$\alpha_1 \boldsymbol{a}_1 + \ldots + \alpha_n \boldsymbol{a}_n = \sum_{k=1}^{n} \alpha_1 \boldsymbol{a}_1 = \boldsymbol{b}$$

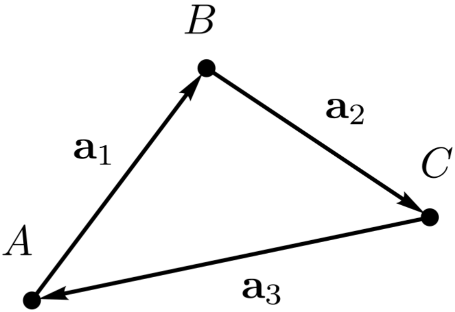
- We call such expression a **linear combination** of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$
- Real numbers $\alpha_1, \ldots, \alpha_n$ we call the **coefficients of a linear combination**
- Vector $\boldsymbol{b}$ we call the **value of a linear combination**
- There is no restriction for us to multiply liner combination by any real number and to build linear combination with values previous-tier linear combinations of given system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$
- This process can be repeated as many times as it required

Linear Combinations of Vectors II

- However, upon expanding, upon applying features of multiply by real number, and upon collecting similar terms all such complicated vectorial expressions reduce to linear combinations of the vectors
- Each vectorial expression composed of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ by means of the operations of addition and multiplication by numbers can be transformed to some linear combination of these vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.
- We say two linear combinations are assumed to be **coinciding** if only difference between them is order of terms
- We say that linear combination is **trivial** if it is composed of only zero coefficients.
- In opposite case we say that linear combination is **non-trivial**
- Linear combination is called **vanishing** or **equal with zero** if it is value is zero vectors
- Each trivial linear combination is equal to zero. However, the converse is not valid.

Linear Combinations of Vectors III

- Example:



- Consider $\triangle ABC$ and vectors $\overrightarrow{AB} = \boldsymbol{a}_1,\ \overrightarrow{BC} = \boldsymbol{a}_2,\ \overrightarrow{CA} = \boldsymbol{a}_3$
- Sum of these vectors is zero: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 = \mathbf{0}$
- We are free to assign coefficient of 1 to each vector in the expression $1 \cdot \boldsymbol{a}_1 + 1 \cdot \boldsymbol{a}_2 + 1 \cdot \boldsymbol{a}_3 = \mathbf{0}$
- This linear combination is **non-trivial** and **vanishing**

Linear Dependence and Linear Independence

- A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is called **linearly dependent** if there is a **non-trivial** linear combination of these vectors which is **equal to zero**
- A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is called **linearly independent** if it is not linearly dependent
- System of three vectors from our example is example of linearly dependent system
- Linear dependence is a property of systems of vectors, it is not a property of linear combinations.
- Linear combinations are only tools for revealing the linear dependence
- Along with triviality and non-triviality property of dependence is invariant against transposition of element in system of vectors

Linear Independence Criterion

- A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is linearly independent if and only if vanishing of a linear combination of these vectors implies its triviality.
- Proof:
  - Definition of linear independence means that there is no linear combination of these vectors being non-trivial and being equal to zero simultaneously
  - Indeed, the non-existence of a linear combination of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, being non-trivial and vanishing simultaneously means that a linear combination of these vectors is inevitably trivial whenever it is equal to zero.
  - In other words vanishing of a linear combination of these vectors implies triviality of this linear combination. $\square$
- Alternative form: A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is linearly independent if and only if non-triviality of a linear combination of these vectors implies that it is not equal to zero

Relation of Linear Dependence. Properties I

- The vector $\boldsymbol{b}$ is said to be **expressed** as a linear combination of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ if it is the value of some linear combination composed of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.
- For the sake of brevity the vector $\boldsymbol{b}$ is sometimes said to be **linearly expressed** through the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ or to be **expressed in a linear way through** $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.
- Key features of linear dependence
  1. A system of vectors comprising the null vector is linearly dependent;
  2. A system of vectors comprising a linearly dependent subsystem is linearly dependent itself;
  3. If a system of vectors is linearly dependent, then at least one of these vectors is expressed in a linear way through other vectors of this system;
  4. If a system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is linearly independent, while complementing it with one more vector $\boldsymbol{a}_{n+1}$ makes the system linearly dependent, then the vector $\boldsymbol{a}_{n+1}$ is linearly expressed through the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$. We cal such system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ **complete**
  5. If a vector $\boldsymbol{b}$ is linearly expressed through some m vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ and if each of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ is linearly expressed through some other n vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$, then the vector $\boldsymbol{b}$ is linearly expressed through the vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$.

Relation of Linear Dependence. Properties II

- Proof for (1): Suppose that a system of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ comprises zero vector. For the sake of certainty we can assume that $\boldsymbol{a}_k = \mathbf{0}$. Let's compose the following linear combination of the vectors our vector system:

$$0 \cdot \boldsymbol{a}_1 + \dots + 0 \cdot \boldsymbol{a}_{k-1} + 1 \cdot \boldsymbol{a}_k + 0 \cdot \boldsymbol{a}_{k+1} + \dots + 0 \cdot \boldsymbol{a}_n = \mathbf{0}$$

This linear combination is non-trivial since the coefficient of vector $\boldsymbol{a}_k$ is nonzero. And its value is equal to zero. Hence, system is linearly dependent. $\square$

Relation of Linear Dependence. Properties III

- Proof for (2): Since linear dependence is not sensible to the order in which the vectors in a system are enumerated, we can assume that first k vectors form linear dependent subsystem in it. Then there exists some non-trivial liner combination of these k vectors being equal to zero:

$$\alpha_1 \cdot \boldsymbol{a}_1 + \ldots + \alpha_k \cdot \boldsymbol{a}_k = \mathbf{0}$$

Let's expand this linear combination by adding other vectors with zero coefficients:

$$\alpha_1 \cdot \boldsymbol{a}_1 + \ldots + \alpha_k \cdot \boldsymbol{a}_k + 0 \cdot \boldsymbol{a}_{k+1} + 0 \cdot \boldsymbol{a}_n = \mathbf{0}$$

It is obvious that the resulting linear combination is nontrivial, and its value is equal to zero. Hence, system is linearly dependent. $\square$

Relation of Linear Dependence. Properties IV

- Proof for (3): Linear dependency means that in the equation

$$\alpha_1 \boldsymbol{a}_1 + ... + \alpha_n \boldsymbol{a}_n = \boldsymbol{0}$$

at least one coefficient, say α_k is nonzero. Since we add term $-\alpha_k \boldsymbol{a}_k$ to both sides of the equations and multiply it by $1/\alpha_k$ we obtain expression:

$$\boldsymbol{a}_k = \frac{-\alpha_1}{\alpha_k} \boldsymbol{a}_1 + \ldots \frac{-\alpha_{k-1}}{\alpha_k} \boldsymbol{a}_{k-1} + \frac{-\alpha_{k-+}}{\alpha_k} \boldsymbol{a}_{k-+} + \ldots + \frac{-\alpha_n}{\alpha_k} \boldsymbol{a}_n$$

Now we see that the vector α_k is linearly expressed through other vectors of the system.
□

Relation of Linear Dependence. Properties V

- Proof for (4): Let's consider a linearly independent system of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ such that adding the next vector $\boldsymbol{a}_{n+1}$ to it makes it linearly dependent. There is some nontrivial linear combination of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n, \boldsymbol{a}_{n+1}$ being equal to zero:

$$\alpha_1 \boldsymbol{a}_1 + \ldots + \alpha_n \boldsymbol{a}_n + \alpha_{n+1} \boldsymbol{a}_{n+1} = \mathbf{0}$$

Suppose $\alpha_{n+1} = 0$. In this case we would get the nontrivial linear combination of n vectors being equal to zero:

$$\alpha_1 \boldsymbol{a}_1 + \ldots + \alpha_n \boldsymbol{a}_n = \mathbf{0}$$

This contradicts to the linear independence of the first n vectors$\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$.
Hence,$\alpha_{n+1} \neq 0$, and we can repeat the trick already used above:

$$\boldsymbol{a}_{n+1} = \frac{-\alpha_1}{\alpha_{n+1}} \boldsymbol{a}_1 + \ldots + \frac{-\alpha_n}{\alpha_{n+1}} \boldsymbol{a}_n . \square$$

Relation of Linear Dependence. Properties VI

- Proof for (5):

$$\boldsymbol{b} = \sum_{i=1}^{m} \alpha_i \boldsymbol{a}_i, \qquad \boldsymbol{a}_i = \sum_{j=1}^{n} \gamma_{ij} \boldsymbol{c}_j$$

$$\boldsymbol{b} = \sum_{i=1}^{m} \alpha_i \left(\sum_{j=1}^{n} \gamma_{ij} \boldsymbol{c}_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \alpha_i \gamma_{ij} \right) \boldsymbol{c}_j . \square$$

- Consequence: Any subsystem in a linearly independent system of vectors is linearly independent
- Steinitz Theorem: If the vectors $\boldsymbol{a}_1 , \ldots , \boldsymbol{a}_n$ are linearly independent and if each of them is linearly expressed through some other vectors $\boldsymbol{b}_1 , \ldots , \boldsymbol{b}_m$, then $m \geqslant n$.
- Steinitz Theorem is very important in studying multidimensional spaces. Here it is provided for reference

Linear Dependence for A Single Vector

- Suppose our system contains only single vector $\boldsymbol{a}_1$
- Linear dependence now means

$$\alpha_1 \boldsymbol{a}_1 = \mathbf{0}, \quad \alpha_1 \neq 0$$

- Thus, $\boldsymbol{a}_1 = \mathbf{0}$
- Suppose now $\boldsymbol{a}_1 = \mathbf{0}$
- Therefore $1 \cdot \boldsymbol{a}_1 = \mathbf{0}$, and this system of single vector is linearly dependent
- A system composed of a single vector $\boldsymbol{a}_1$ is linearly dependent if and only if this vector is zero vector.

Linear Dependence for A Pair of Vectors I

- Linear dependence now means

$$\alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2 = \boldsymbol{0}, \quad \alpha_1 \neq 0, \text{ or } \alpha_2 \neq 0$$

- Since the linear dependence is not sensitive to the order of vectors in a system, without loss of generality we can assume that $\alpha_1 \neq 0$, ad express $\boldsymbol{a}_1$:

$$\boldsymbol{a}_1 = \frac{\alpha_2}{\alpha_1} \boldsymbol{a}_2 = \beta_2 \boldsymbol{a}_2$$

- There is tree possibilities:
  - $\beta_2 > 0$, thus $\boldsymbol{a}_2 \upuparrows \boldsymbol{a}_1$
  - $\beta_2 < 0$, thus $\boldsymbol{a}_2 \uparrow\downarrow \boldsymbol{a}_1$
  - $\beta_2 = 0$, thus $\boldsymbol{a}_1 = \boldsymbol{0}$ and has undefined direction
- As a summary $\boldsymbol{a}_2 || \boldsymbol{a}_1$

Linear Dependence for A Pair of Vectors II

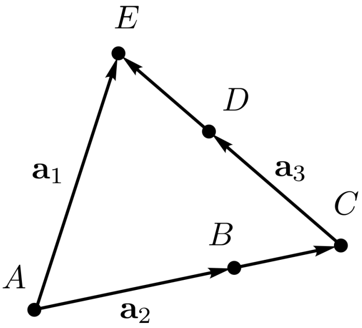
- Suppose $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are collinear.
- It means that there is real number β_2 and $\boldsymbol{a}_1 = \beta_2 \boldsymbol{a}_2$
- Thus there is non-trivial linear combination:

$$1 \cdot \boldsymbol{a}_1 + (-\beta_2) \cdot \boldsymbol{a}_2 = \mathbf{0}$$

- The existence of such a linear combination means that the vectors are linearly dependent. Thus, the converse proposition that the collinearity of two vectors implies their linear dependence is proved.
- A system of two vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is linearly dependent if and only if these vectors are collinear.

Linear Dependence for A Triplet of Vectors I

- Consider a system contained three of vectors: $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$



- Assume that it is linearly dependent
- Properties of linear dependency grant us expression:

$$\boldsymbol{a}_1 = \beta_2 \boldsymbol{a}_2 + \beta_3 \boldsymbol{a}_3$$

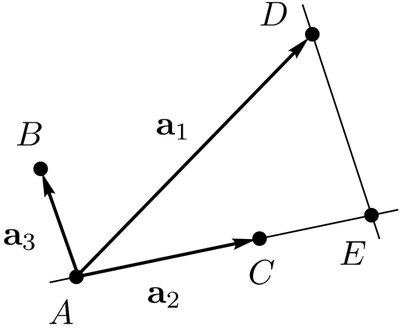
- Let A be arbitrary point and $\boldsymbol{a}_2 = \overrightarrow{AB}$, $\beta_2 \boldsymbol{a}_2 = \overrightarrow{AC}$
- Starting in point C we may build directed segments $\boldsymbol{a}_3 = \overrightarrow{CD}$, and $\beta_3 \boldsymbol{a}_3 = \overrightarrow{CE}$
- By the low of triangle of addition: $\boldsymbol{a}_1 = \overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE}$
- Provided illustration supposes $\beta_2 > 0$ and $\beta_3 > 0$. Plot corresponding illustrations for other possible combinations of signs for nonzero betas as **home assignment**

Linear Dependence for A Triplet of Vectors II

- Three points A, C, and E successfully shape a plane. And, as AB overlaps AC and CD overlaps CE, vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ are coplanar with this plane
- Taking any point A' not laying in this plane shapes family corresponding directed segments on parallel plane
- The linear dependence of three vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ implies their coplanarity.
- Here we assumed general case: $\boldsymbol{a}_1 \neq \mathbf{0}$, $\boldsymbol{a}_2 \neq \mathbf{0}$, and $\boldsymbol{a} \nparallel \boldsymbol{a}_2$.
- Consider special cases as **home assignment**

Linear Dependence for A Triplet of Vectors III

- The coplanarity of three vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ imply their linear dependence.
- Proof:
  - Let $\boldsymbol{a}_1 = \mathbf{0}$ and $\boldsymbol{a}_2 = \mathbf{0}$, thus first property of linear dependence is fulfilled
  - Let $\boldsymbol{a}_1 \neq \mathbf{0}$, $\boldsymbol{a}_2 \neq \mathbf{0}$, but $\boldsymbol{a} \parallel \boldsymbol{a}_2$. This means their linear dependence Thus we demonstrated linear dependent subsystem and second property of linear dependence is fulfilled.

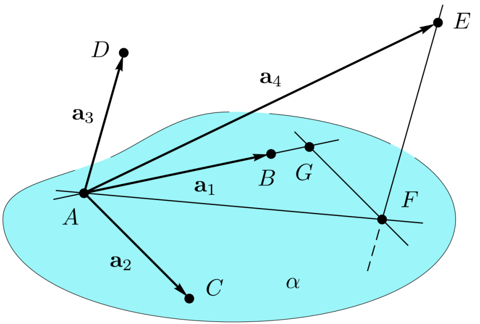


  - Consider general case
  - Let A be arbitrary point and $\overrightarrow{AB} = \boldsymbol{a}_3$, $\overrightarrow{AC} = \boldsymbol{a}_2$, $\overrightarrow{AD} = \boldsymbol{a}_1$
  - Coplanarity means that all four points A, B, C, and D lie on the same plane α
  - Consider two crossing lines: $A \in p$, $C \in p$, $D \in q$, $q \parallel AB$
  - Cross point of these lines is arbitrary point E: $\overrightarrow{AE} \parallel \overrightarrow{AC}$, and $\overrightarrow{ED} \parallel \overrightarrow{AB}$
  - $\boldsymbol{a}_1 = \overrightarrow{AD} = \overrightarrow{AE} + \overrightarrow{ED} = \beta_2 \boldsymbol{a}_2 + \beta_3 \boldsymbol{a}_3$
  - $1 \cdot \boldsymbol{a}_1 + (-\beta_2) \cdot \boldsymbol{a}_2 + (-\beta_3) \cdot \boldsymbol{a}_3 = \mathbf{0}.\square$
- A system of three vectors is linearly dependent if and only if these vectors are coplanar.

Linear Dependence for More than Tree Vectors I

- Consider a system contained three of vectors: $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}_3$
- This system is linearly dependent
- Proof:
  - Suppose subsystem $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ is linearly independent. Opposite case means that general system is linearly independent too.
  - Thus, these vectors are non-coplanar and any pair of them is also linearly independent as well as any single vector.
  - $\boldsymbol{a}_1 \neq \mathbf{0}$, $\boldsymbol{a}_2 \neq \mathbf{0}$, $\boldsymbol{a}_2 \neq \mathbf{0}$, and $\boldsymbol{a} \nparallel \boldsymbol{a}_2$

Linear Dependence for More than Tree Vectors II



- Consider arbitrary point in space A
- We establish directed segments: $\overrightarrow{AB} = \boldsymbol{a}_1$, $\overrightarrow{AC} = \boldsymbol{a}_2$, $\overrightarrow{AD} = \boldsymbol{a}_3$, $\overrightarrow{AE} = \boldsymbol{a}_4$
- Points A, B, C shape a plane α
- $D \notin \alpha$, and $\boldsymbol{a}_3$ is not coplanar with α
- Consider $EF \parallel \boldsymbol{a}_3$, and $F \in \alpha$
- $\boldsymbol{a}_4 = \overrightarrow{AE} = \overrightarrow{AF} + \overrightarrow{FE} = \overrightarrow{AF} + \beta_3 \boldsymbol{a}_3$
- Let $FG \parallel \boldsymbol{a}_2$, and G is cross point with line established by $\boldsymbol{a}_1$
- $\overrightarrow{AF} = \overrightarrow{AG} + \overrightarrow{GF} = \beta_1 \boldsymbol{a}_1 + \beta_2 \boldsymbol{a}_2$
- $\boldsymbol{a}_4 = \beta_1 \boldsymbol{a}_1 + \beta_2 \boldsymbol{a}_2 + \beta_3 \boldsymbol{a}_3$
- $1 \cdot \boldsymbol{a}_4 + (-\beta_1) \cdot \boldsymbol{a}_1 + (-\beta_2) \cdot \boldsymbol{a}_2 + (-\beta_3) \cdot \boldsymbol{a}_3 = \mathbf{0}$. $\square$

Linear Dependence for More than Tree Vectors III

- Any system consisting of more than four vectors in the space is linearly dependent
- Proof: we have proven that any system of four vectors is linearly dependent. Thus, in any system of more than four system arbitrary subsystem of four vectors may be selected, and it is always linearly dependent. Therefore, system itself is linearly dependent

Linear Combinations and Bases. On a Line I

- Alternative notation for our abstract vector id **free vector**
- If we bring some restriction on location in space for a free vector, we say that it is **partially free vector**. E.g. vector must lay on specified line or plane
- Suppose a is arbitrary line in space
- Each vector parallel with a shapes directed segment laying on a with selected point as an initial point
- For more clear explanation we may restrict our set of vectors parallel with a with vectors laying on a

Linear Combinations and Bases. On a Line II

- For any vector $\boldsymbol{e}$ laying on a all vectors $\boldsymbol{x}$ laying on a are expressed with formula

$$\boldsymbol{x} = x\boldsymbol{e}$$

- Thus $\boldsymbol{e}$ and selected origin point shape basis along this line and x is coordinate of vector $\boldsymbol{x}$ and coordinate of point corresponding with its endpoint
- Adding of any other vector $\boldsymbol{f}$ parallel with that line shapes linearly dependent system of vectors
- We agree that length of $\boldsymbol{e}$ is unity without any leak of generalization

Linear Combinations and Bases. On a Line III

- Transformation this basis
  - Change of origin. Suppose we need to transit the origin into point O' with coordinate a with respect to "old" basis

$$x = x' + a$$
$$x' = x - a$$

  - Change the direction. Suppose we need to change direction of $\boldsymbol{e}$ to opposite one

$$x = -x'$$
$$x' = -x$$

  - Change the scale. Suppose there is invariant scale segment and length of our "unit vector" $\boldsymbol{e}$ is e with respect to this scale. For any codirected with $\boldsymbol{e}$ vector $\boldsymbol{e}'$ with length e' there is equation of change the scale

$$x = \frac{e'}{e} x'$$
$$x' = \frac{e}{e'} x$$

Linear Combinations and Bases. On a Line IV

- Some operations with coordinates on a line
  - Length of radius vectors

$$\boldsymbol{a} \mapsto (a)$$
$$\boldsymbol{a}^2 = a\boldsymbol{e} \cdot a$$
$$bvcte = a^2|\boldsymbol{a}| = |a|$$

  - Distance between points

$$A \mapsto \overrightarrow{OA} \mapsto (a)$$
$$B \mapsto \overrightarrow{OB} \mapsto (b)$$
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \mapsto (b - a)$$
$$AB = |b - a|$$

Example: $A \mapsto (-3)$, $B \mapsto (+4)$, thus $AB = 7$

Linear Combinations and Bases. On a Line V

- Split of the segment. Suppose points A and B have coordinates (x_1) and (x_2) respectively, $x_1 < x_2$ without any leak of generalization. Point C with coordinate $x_1 < x < x_2$ splits this segment. This relation of coordinates origins from axiom of measurement

$$\lambda = \frac{AC}{CB} = \frac{x - x_1}{x_2 - x}$$
$$\lambda(x_2 - x) = x - x_1$$
$$x_2 + x_1 = x(1 + \lambda)$$
$$x = \frac{x_2 + x_1}{1 + \lambda}$$

Particular case: coordinate of center. $\lambda = 1$, thus $x = \dfrac{x_2 + x_1}{2}$

Linear Combinations and Bases. On a Line VI

- Functional dependence of coordinates
  - Now we discuss only algebraic form dependence for coordinates, but this dependence also may have form of any dynamic system, discrete or continuous one
  - This dependence may have form of explicit equation for coordinate:

    $$F(x) = 0,$$

    or be some sequence:

    $$x_i = f(x_{i-1}), \quad i = 1, 2, .., \text{ and } x_0 \text{ is given}$$

    or be some parametrized equation:

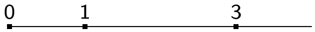
    $$x = f(t),$$

    t here is parameter with domain of arbitrary segment of real numbers' axis.

Linear Combinations and Bases. On a Line VII

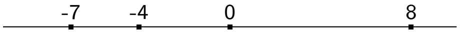
- Examples
  - $x^3 - 4x^2 + 3x = 0$

$$x_1 = 0$$
$$x^2 - 4x + 3 = 0$$
$$x_2 = 1; \quad x_3 = 3$$



  - Uniform linear motion: $x = tv + c$; $v = \text{const}$, $c = \text{const}$

$$x = 3t - 7, \quad t = 0, t = 1, t = 5$$



  - Geometric progression: $x_i = x_{i-1} \cdot q$, $q = \text{const}$
    As **home assignment** plot first 4 elements of geometric progression for $x_0 = 2$, $q = 1/2$

Linear Combinations and Bases. On a Line VIII

Problem for home assignment Transform coordinates on a line to
(1) increase scale twice ($e'/e = 2$), and
(2) with respect to original basis points with coordinates $x < -7$ became all positive and points with coordinates $x > -7$ become all negative.

Linear Combinations and Bases. On a Plane I

- Let α be arbitrary plane in the space
- Consider free vectors parallel with α
- For these vectors we may yield directed segments originating from any point on our plane α and laying in this plane
- This establishes collection of partially free vectors **laying in** α.
- Observe a pair of non-collinear of these partially free vectors
- By the one hand it was proved that linear combination of **any** pair of non-collinear vectors is linearly independent, and adding of **any** third coplanar with that pair vector makes this vectors system linearly dependent
- By the other hand we demonstrated that any vector laying on a plane has it explicit expression with each pair of non-collinear vectors laying in the same plane
- We say that this system and corresponding basis is **complete** one

Linear Combinations and Bases. On a Plane II

- Let vectors of our basis be $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$
- Suppose O is arbitrary point on α and $\overrightarrow{OA} = \boldsymbol{e}_1$ and $\overrightarrow{OA} = \boldsymbol{e}_2$
- We assign O as an **origin** for our coordinate system
- For any vector $\boldsymbol{x}$ laying in α now we have explicit linear expression:
- $\boldsymbol{x} = x_1\boldsymbol{e}_2 + x_2\boldsymbol{e}_2$
- Real numbers x_1 and x_2 are coordinates of vector $\boldsymbol{x}$ and its endpoint with respect to constructed basis and may be distinguished as column or row matrix:

$$\boldsymbol{a} \mapsto \begin{pmatrix} a_1 & a_2 \end{pmatrix} \quad B \mapsto \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- Thus, we assign order of the vectors shaping the basis in addition, to keep all matrices corresponding
- That means that bases $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}'_1 = \boldsymbol{e}_2$, $\boldsymbol{e}'_2 = \boldsymbol{e}_1$ are non-equal one with other

Linear Combinations and Bases. In Space

- Situation in space is pretty similar
- We already proved both facts that system of three non-coplanar vectors is complete linearly independent (addition of any other vector makes it linearly dependent)
- By the other hand, expression of vector by three non-coplanar in space is explicit
- We also must fix origin O and order of vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in our basis and therefor obtain coordinates of any vector as linear combination of basis vectors
- $\boldsymbol{x} = x_1\boldsymbol{e}_2 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3$
- These three numbers may be considered as a row or column matrix

$$\boldsymbol{a} \mapsto \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \quad B \mapsto \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Review

- We have proven that single non-zero vector on a line, pair of non-collinear vectors on plane, and triplet of non-coplanar vectors in space is an exact number of vectors necessary to express any directed segment assigned to this carrier (line, plane, space)
- It is a good assumption to suspect that in multidimensional geometry number of such vectors will correspond with number of dimensions
- This theory introduces no additional restriction for relative position and length of vectors shaping these bases
- This general case of basis we call **skew-angular** basis

Dot Product and Skew-Angular Basis I

- Now we consider dot product in general case of a skew-angular basis
- Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be some arbitrary skew-angular basis
- The only condition on these vectors is their non-coplanarty and pairwise non-colinearity
- Consider vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with known coordinates:

$$\boldsymbol{a} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \quad \boldsymbol{b} = \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}$$

- we write "$=$" instead "$\mapsto$" here emphasize the fact that once a basis is fixed, vectors are uniquely identified with their coordinates.
- Other form to write a and b is

$$\boldsymbol{a} = \sum_{i=1}^{3} a^i \boldsymbol{e}_i \quad \boldsymbol{b} = \sum_{j=1}^{3} b^j \boldsymbol{e}_j$$

Dot Product and Skew-Angular Basis II

- Substituting that formulas into our dot product, we get

$$\boldsymbol{a} \cdot \boldsymbol{b} = \left(\sum_{i=1}^{3} a^i \boldsymbol{e}_i \right) \cdot \left(\sum_{j=1}^{3} b^j \boldsymbol{e}_j \right) = \sum_{i=1}^{3} \sum_{j=1}^{3} (a^i \boldsymbol{e}_i \cdot b^j \boldsymbol{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j (\boldsymbol{e}_i \cdot \boldsymbol{e}_j)$$

- The quantities $g_{i,j} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ depend only on a basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, namely on the lengths of the basis vectors and on the angles between them.

Dot Product and Skew-Angular Basis III

- This 9-element collection $\{g_{i,j}\}$ is usually arranged into a square matrix called **Gram matrix of a basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$** (Jørgen Pedersen Gram (27 June 1850 – 29 April 1916))

$$G = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

- Now we continue our expression for dot product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j g_{ij}$$

- This corresponds to the matrix expression:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} a^1 & a^2 & a^3 \end{pmatrix} \cdot \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \cdot \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} = \boldsymbol{a}^\top \cdot G \cdot \boldsymbol{b}$$

Dot Product and Skew-Angular Basis IV

- A square matrix A is called symmetric, if it is preserved under transposing, i. e. if the following equality is fulfilled: $A^\top = A$.
- Gram matrix is symmetric
- Proof:
  - Definition and properties of dot product yield $g_{i,j} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \boldsymbol{e}_j \cdot \boldsymbol{e}_i = g_{j,i}$
  - And $g_{j,i}$ are exactly elements of $G^\top$
- This symmetry sterilizes the difference between transposed and non-transposed columns in the generalized definition of the dot product

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{j=1}^{3} \sum_{i=1}^{3} a^j b^i g_{ji} = \sum_{j=1}^{3} \sum_{i=1}^{3} b^i a^j g_{ij}$$

$$\boldsymbol{b}^\top \cdot G \cdot \boldsymbol{a}$$

- Let us check that derived formulas for orthonormal basis are still valid

Dot Product and Skew-Angular Basis V

- $|\mathbf{e}_i| = 1$, and vectors are pairwise orthogonal

$$g_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$g_{ij} = \delta_{ij}$$

- In the orthonormal basis Gram matrix is exactly unity matrix:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\top \cdot G \cdot \mathbf{b} = \mathbf{a}^\top \cdot Id \cdot \mathbf{b} = \mathbf{a}^\top \cdot \mathbf{b}$$

Problem 1 There is skew-angular basis on a plane with unit basis vectors and angle $\frac{2}{3}\pi$ between main directions. Calculate Gram matrix and find length of sides and area for triangle having coordinates of edges in this basis $A(14, 3)$, $B(9, -2)$, $C(4, 1)$

Dot Product and Skew-Angular Basis. Problems Corner II

There is skew-angular basis on a plane with unit basis vectors and angle $\frac{2}{3}\pi$ between main directions. Calculate Gram matrix and find length of sides and area for triangle having coordinates of edges in this basis $A(14, 3)$, $B(9, -2)$, $C(4, 1)$

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

$$|\boldsymbol{e}_1| = |\boldsymbol{e}_2| = 1,\ \angle(\boldsymbol{e}_1, \boldsymbol{e}_1) = \varphi = \frac{2}{3}\pi$$

$$g_{11} = \boldsymbol{e}_1\boldsymbol{e}_1 = 1,\ g_{22} = \boldsymbol{e}_2\boldsymbol{e}_2 = 1,\ g_{12} = g_{21} = \boldsymbol{e}_1\boldsymbol{e}_2 = \cos\varphi = -1/2$$

$$G = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix}$$

Dot Product and Skew-Angular Basis. Problems Corner III

$$\overrightarrow{BA} = \boldsymbol{a} = \begin{pmatrix} 5 \\ 5 \end{pmatrix};\ \overrightarrow{AC} = \boldsymbol{b} = \begin{pmatrix} -10 \\ -2 \end{pmatrix};\ \overrightarrow{BC} = \boldsymbol{c} = \begin{pmatrix} -5 \\ 3 \end{pmatrix};$$

$AB = \sqrt{\overrightarrow{BA}\overrightarrow{BA}} = \sqrt{a^1 a^1 g_{11} + a^1 a^2 g_{12} + a^2 a^1 g_{21} + a^2 a^2 g_{22}} = \sqrt{25 - 25/2 - 25/2 + 25} = \sqrt{25} = 5$

$AC = \sqrt{\overrightarrow{AC}\overrightarrow{AC}} = \sqrt{b^1 b^1 g_{11} + b^1 b^2 g_{12} + b^2 b^1 g_{21} + b^2 b^2 g_{22}} = \sqrt{100 - 10 - 10 + 4} = \sqrt{84} = 2\sqrt{21}$

$BC = \sqrt{\overrightarrow{BC}\overrightarrow{BC}} = \sqrt{c^1 c^1 g_{11} + c^1 c^2 g_{12} + c^2 c^1 g_{21} + c^2 c^2 g_{22}} = \sqrt{25 + 15/2 + 15/2 + 9} = \sqrt{49} = 7$

Dot Product and Skew-Angular Basis. Problems Corner IV

To find area it is required to find sines of angle between two sides. We start with cosines and utilize $\sin^2\alpha = 1 - \cos^2\alpha$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = a^1c^1g_{11} + a^1c^2g_{12} + a^2c^1g_{21} + a^2c^2g_{22} = -25 - 15/2 + 25/2 + 15 = 5$

$$\cos\alpha = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|vctBA||\overrightarrow{BC}|} = 5/5/7 = 1/7$$

$$|sin\alpha| = \sqrt{1 - cos^2\alpha} = \sqrt{\frac{48}{49}} = \frac{4}{7}\sqrt{3}$$

$$S = \frac{1}{2}|vctBA||\overrightarrow{BC}||sin\alpha| = \frac{1}{2} \cdot 5 \cdot 7 \cdot \frac{4}{7}\sqrt{3} = 10\sqrt{3}$$

Problem 2
There is skew-angular basis on a plane with unit basis vectors and angle $\frac{2}{3}\pi$ between main directions.
Plot triangle with edges $(3, 5)$, $(-4, 7)$, and $(5\frac{1}{2}, -3\frac{1}{2})$ in this coordinate system, and calculate length of each side

Cross Product and Skew-Angular Basis I

- Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be arbitrary skew-angular basis
- Consider pairwise cross product of two basis vectors

$$\boldsymbol{e_i} \times \boldsymbol{e_j} = C_{ij}^1 \boldsymbol{e}_1 + C_{ij}^2 \boldsymbol{e}_2 + C_{ij}^2 \boldsymbol{e}_3 = \sum_{k=1}^{3} C_{ij}^k \boldsymbol{e}_k$$

- This expansion contains three coefficients C_{ij}^k, and the indices i and j in it run independently over three values 1, 2, 3.
- This formula represents 9 equations and the total number of coefficients in it is equal to twenty-seven
- Each basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in the space is associated with a collection of twenty-seven constants C_{ij}^k determined uniquely by this basis through the expansions

$$\boldsymbol{e_i} \times \boldsymbol{e_j} = \sum_{k=1}^{3} C_{ij}^k \boldsymbol{e}_k$$

Cross Product and Skew-Angular Basis II

- These constants are called the **structural constants of the vector product in this basis**
- Unfortunately there is no matrix representation for these constants
- Later we will derive effective formulas for these constants
- Consider vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with known coordinates and so expansions

$$\boldsymbol{a} = \sum_{i=1}^{3} a^i \boldsymbol{e}_i \quad \boldsymbol{b} = \sum_{j=1}^{3} b^j \boldsymbol{e}_j$$

Cross Product and Skew-Angular Basis III

- Their cross product:

$$\boldsymbol{a} \times \boldsymbol{b} = \sum_{i=1}^{3} a^i \boldsymbol{e}_i \times \sum_{j=1}^{3} b^j \boldsymbol{e}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} ((a^i \boldsymbol{e}_i) \times (b^j \boldsymbol{e}_j)) =$$

$$\sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j (\boldsymbol{e}_i \times \boldsymbol{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j (\boldsymbol{e}_i \times \boldsymbol{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a^i b^j C_{ij}^k \boldsymbol{e}_k$$

- This general formula is called the *formula for calculating the vector product through the coordinates of vectors in a skew-angular basis*.
- For the right orthonormal basis we derived constants C_{ij}^k as Levi-Chivitta symbols
- For the left orthonormal basis these constants will be Levi-Chivitta symbols multiplied by -1

Mixed Product and Skew-Angular Basis I

- Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be some arbitrary skew-angular basis
- Consider mixed products involving vectors of this basis

$$c_{ijk} = (\boldsymbol{e_i} \boldsymbol{e_j} \boldsymbol{e_k})$$

- These quantities resemble *structural constants of the mixed product* in this basis.
- An important property of the structural constants c_{ijk} is their *complete skew symmetry* or *complete antisymmetry*

$$c_{ijk} = -c_{jik}, \quad c_{ijk} = -c_{ikj}, \quad c_{ijk} = -c_{kji}$$

- The following relationships are an immediate consequence of the property of complete antisymmetry of the structural constants of the mixed product c_{ijk}:

$$c_{ijj} = -c_{ijj}, \quad c_{iik} = -c_{iik}, \quad c_{iji} = -c_{iji}$$

- Thus $c_{ijk} = 0$ if any pair of indices is coinciding

Mixed Product and Skew-Angular Basis II

- As all indices are taken from 1, 2 and 3, following proposition is valid
- $c_{ijk} = c_{123}$, if (i, j, k) is even permutation of $(1, 2, 3)$
- $c_{ijk} = -c_{123}$, if (i, j, k) is odd permutation of $(1, 2, 3)$
- In general form

$$c_{ijk} = c_{123}\varepsilon_{ijk} = (\boldsymbol{e}_1\boldsymbol{e}_2\boldsymbol{e}_3)\varepsilon_{ijk}$$

- The only structural constant of the mixed product is signed (in other words oriented) volume of parallepiped shaped with basis vectors. Sign depends on orientation of basis vector and is $+$ for right basis and $-$ for left basis

Mixed Product and Skew-Angular Basis III

- Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be vectors with known coordinates with respect to our basis

$$\boldsymbol{a} = \sum_{i=1}^{3} a^i \boldsymbol{e}_i \quad \boldsymbol{b} = \sum_{j=1}^{3} b^j \boldsymbol{e}_j \quad \boldsymbol{c} = \sum_{k=1}^{3} c^k \boldsymbol{e}_k$$

- Their mixed product is

$$(\boldsymbol{abc}) = \left(\left(\sum_{i=1}^{3} a^i \boldsymbol{e}_i \right) \left(\sum_{j=1}^{3} b^j \boldsymbol{e}_j \right) \left(\sum_{k=1}^{3} c^k \boldsymbol{e}_k \right) \right)$$

Mixed Product and Skew-Angular Basis IV

- After application the properties of dot and cross product

$$(\boldsymbol{abc}) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}(a^i\boldsymbol{e}_i\, b^j\boldsymbol{e}_j\, c^k\boldsymbol{e}_k) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a^i b^j c^k(\boldsymbol{e_i e_j e_k}) =$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a^i b^j c^k c_{ijk} = c_{123}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a^i b^j c^k \varepsilon_{ijk} =$$

$$c_{123}\det\begin{pmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{pmatrix}$$

Constants of Products. Problems Corner I

Problem 3 There are two coordinate systems possessing the same origin: Cartesian one, and skew-angular one. Skew angular basis vectors expressed in Cartesian coordinates as
$\boldsymbol{e}_1 = (1, 1, 1)$, $\boldsymbol{e}_2 = (-2, -1, 1)$, $\boldsymbol{e}_3 = (0, 0, -3)$.
Find Gram matrix and oriented volume for this skew-angular basis

Constants of Products. Problems Corner II

There are two coordinate systems possessing the same origin: Cartesian one, and skew-angular one. Skew angular basis vectors expressed in Cartesian coordinates as $\boldsymbol{e}_1 = (1, 1, 1)$, $\boldsymbol{e}_2 = (-2, -1, 1)$, $\boldsymbol{e}_3 = (0, 0, -3)$.
Find Gram matrix and oriented volume for this skew-angular basis

$$G = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} = \begin{pmatrix} 1+1+1 & -2-1+1 & 0+0-3 \\ -2-1+1 & 4+1+1 & 0+0-3 \\ 0+0-3 & 0+0-3 & 0+0+9 \end{pmatrix} = \begin{pmatrix} 3 & -2 & -3 \\ -2 & 6 & -3 \\ -3 & -3 & 9 \end{pmatrix}$$

$$c_{123} = (\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3}) = \begin{vmatrix} 1 & 1 & 1 \\ -2 & -1 & 1 \\ 0 & 0 & -3 \end{vmatrix} = -3$$

Constants of Products. Problems Corner III

Problem 4

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Find their dot product if their coordinates in skew-angular basis described in problem 3 are $(1, 1, 1)$ and $(1, 2, 3)$

$$G = \begin{pmatrix} 3 & -2 & -3 \\ -2 & 6 & -3 \\ -3 & -3 & 9 \end{pmatrix}$$

Constants of Products. Problems Corner IV

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Find their dot product if their coordinates in skew-angular basis described in problem 3 are $(1, 1, 1)$ and $(1, 2, 3)$

$$G = \begin{pmatrix} 3 & -2 & -3 \\ -2 & 6 & -3 \\ -3 & -3 & 9 \end{pmatrix}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j g_{ij} =$$

$$= 1 \cdot 1 \cdot 3 + 1 \cdot 2 \cdot (-2) + 1 \cdot 3 \cdot (-3) +$$
$$+1 \cdot 1 \cdot (-2) + 1 \cdot 2 \cdot 6 + 1 \cdot 3 \cdot (-3) +$$
$$= 1 \cdot 1 \cdot (-3) + 1 \cdot 2 \cdot (-3) + 1 \cdot 3 \cdot 9$$
$$= 3 - 4 - 9 - 2 + 12 - 9 - 3 - 6 + 27 = 9$$

Constants of Products. Problems Corner V

Problem 5

Three vectors have length 1, 2, and 5 respectively. Angle between each pair of vectors is $\pi/3$. Find Gram matrix and oriented volume for basis shaped by this vectors

Problem 6

For basis in problem 4 find dot product for vectors with coordinates

$(1, 1, 1)$ and $(1, 2, 3)$

$(2, 1, 2)$ and $(3, 2, 3)$

$(0, 1, 1)$ and $(0, 0, 3)$

Linear Vector Space I

- Before we continue study of features of vectorial products let us introduce some fundamental definitions
- Let $\mathbb{K}$ be numeric field (field of rational ($\mathbb{Q}$), real ($\mathbb{R}$), or complex ($\mathbb{C}$)) numbers
- A set V equipped with binary operation of addition and with the operation of multiplication by numbers from the field $\mathbb{K}$, is called a **linear vector space** over the field $\mathbb{K}$, if the following conditions called **linear vector space axioms** are fulfilled:
  1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$ for all $\boldsymbol{u}, \boldsymbol{v} \in V$;
  2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{v} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{v})$ for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$;
  3. there is an element $\mathbf{0} \in V$ such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$ for all $\boldsymbol{u} \in V$; any such element is called a **zero element**;
  4. for any $\boldsymbol{u} \in V$ and for any zero element $\mathbf{0}$ there is an element $\boldsymbol{u}'; \in V$ such that $\boldsymbol{u} + \boldsymbol{u}' = \mathbf{0}$; it is called an opposite element for $\boldsymbol{u}$;
  5. $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{v} + \alpha\boldsymbol{u}$ for any number $\alpha \in \mathbb{K}$, and for any two elements $\boldsymbol{u}, \boldsymbol{v} \in V$;
  6. $(\alpha + \beta) \cdot \boldsymbol{u} = \alpha \cdot \boldsymbol{u} + \beta \cdot \boldsymbol{u}$ for any two numbers $\alpha, \beta \in \mathbb{K}$ and for any element $\boldsymbol{u}; \in V$
  7. $\alpha \cdot (\beta \cdot \boldsymbol{u}) = (\alpha \cdot \beta) \cdot \boldsymbol{u}$ for any two numbers $\alpha, \beta \in \mathbb{K}$ and for any element $\boldsymbol{u}; \in V$
  8. $1 \cdot \boldsymbol{u} = \boldsymbol{u}$ for the number $1 \in \mathbb{K}$ and for any element $\boldsymbol{u}; \in V$.

Linear Vector Space II

- Linear algebra studies these linear vector spaces
- Element of that space called vector
- Following features of linear vector space are direct consequence of this definition
  9. zero vector $\mathbf{0} \in V$ is unique;
  10. for any vector $\boldsymbol{u} \in V$ the opposite vector $\boldsymbol{u}' \in V$ is unique;
  11. the product of the number $0 \in \mathbb{K}$ and any vector $\boldsymbol{u} \in V$ is equal to zero vector: $0 \cdot \boldsymbol{u} = \mathbf{0}$;
  12. the product of an arbitrary number $\alpha \in \mathbb{K}$ and zero vector is equal to zero vector: $\alpha \cdot \mathbf{0} = \mathbf{0}$;
  13. the product of the number $-1 \in \mathbb{K}$ and the vector $\boldsymbol{u} \in V$ is equal to the opposite vector: $(-1) \cdot \boldsymbol{u} = \boldsymbol{u}'$.
- A non-empty subset $U \subset V$ in a linear vector space V over a numeric field $\mathbb{K}$ is called a subspace of the space V if:
  1. from $\boldsymbol{u}_1, \boldsymbol{u}_2 \in U$ it follows that $\boldsymbol{u}_1 + \boldsymbol{u}_2 \in U$;
  2. from $\boldsymbol{u} \in U$ it follows that $\alpha \boldsymbol{u} \in U$ for any number $\alpha \in \mathbb{K}$.

Linear Vector Space III

- Let V be a linear vector space over a numeric field $\mathbb{K}$. A numeric function $y = f(\boldsymbol{v}, \boldsymbol{w})$ with two arguments$\boldsymbol{u}, \boldsymbol{v} \in V$ and with the values in the field $\mathbb{K}$ is called a **bilinear form** if
  1. $f(\boldsymbol{v}_1 + \boldsymbol{v}_2, (\boldsymbol{w}) = f(\boldsymbol{v}_1, \boldsymbol{w}) + f(\boldsymbol{v}_2, \boldsymbol{W})$ for any two $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$;
  2. $f(\alpha \cdot \boldsymbol{v}, \boldsymbol{w}) = \alpha f(\boldsymbol{v}, \boldsymbol{w})$ for any $\boldsymbol{v} \in V$ and for any $\alpha \in \mathbb{K}$;
  3. $f(\boldsymbol{v}, \boldsymbol{w}_1 + \boldsymbol{v}2) = f(\boldsymbol{v}, \boldsymbol{w}_1) + f(\boldsymbol{v}, \boldsymbol{w}_2)$ for any two $\boldsymbol{w}_1, \boldsymbol{w}_2 \in V$;
  4. $f(\boldsymbol{v}, \alpha \cdot \boldsymbol{w}) = \alpha f(\boldsymbol{v}, \boldsymbol{w})$ for any $\boldsymbol{v} \in V$ and for any $\alpha \in mathbbK$.
- A bilinear form $f(\boldsymbol{v}, \boldsymbol{w})$ is called a **symmetric bilinear form** if $f(\boldsymbol{v}, \boldsymbol{w}) = f(\boldsymbol{w}, \boldsymbol{v})$
- A bilinear form $f(\boldsymbol{v}, \boldsymbol{w})$ is called a **skew-symmetric bilinear form** if $f(\boldsymbol{v}, \boldsymbol{w}) = -f(\boldsymbol{w}, \boldsymbol{v})$
- For any bilinear forms there are operations of **symmetrization** and **alteration**:

$$f_+(\boldsymbol{v}, \boldsymbol{w}) = \frac{f(\boldsymbol{v}, \boldsymbol{w}) + f(\boldsymbol{w}, \boldsymbol{v})}{2}$$

$$f_-(\boldsymbol{v}, \boldsymbol{w}) = \frac{f(\boldsymbol{v}, \boldsymbol{w}) - f(\boldsymbol{w}, \boldsymbol{v})}{2}$$

$$f(\boldsymbol{v}, \boldsymbol{w}) = f_+(\boldsymbol{v}, \boldsymbol{w}) + f_-(\boldsymbol{v}, \boldsymbol{w})$$

Linear Vector Space IV

- The expansion of a given bilinear form $f(\boldsymbol{v}, \boldsymbol{w})$ into the sum of a symmetric and a skew-symmetric bilinear forms is unique.
- A numeric function $y = g(\boldsymbol{v})$ with one vectorial argument $\boldsymbol{v} \in V$ is called a quadratic form in a linear vector space V if $g(\boldsymbol{v}) = f(\boldsymbol{v}, \boldsymbol{v})$ for some bilinear form $f(\boldsymbol{v}, \boldsymbol{w})$.

$$g(\boldsymbol{v}) = f(\boldsymbol{v}, \boldsymbol{v}) = f_+(\boldsymbol{v}, \boldsymbol{v})$$
$$f_-(\boldsymbol{v}, \boldsymbol{v}) = -f_-(\boldsymbol{v}, \boldsymbol{v}) = 0$$

- There is one and only one symmetric bilinear form $f(\boldsymbol{v}, \boldsymbol{w})$ generating specified quadratic form $g(\boldsymbol{v})$. These forms connected with **recovery formula**

$$f(\boldsymbol{v}, \boldsymbol{w}) = \frac{g(\boldsymbol{v} + \boldsymbol{w}) - (g(\boldsymbol{v}) + g(\boldsymbol{w}))}{2}$$

Linear Vector Space V

- Proof:

$$
\begin{aligned}
g(\boldsymbol{v}+\boldsymbol{w}) = f(\boldsymbol{v}+\boldsymbol{w},\boldsymbol{v}+\boldsymbol{w}) = f(\boldsymbol{v},\boldsymbol{v}) + f(\boldsymbol{v},\boldsymbol{w}) + f(\boldsymbol{w},\boldsymbol{v}) + f(\boldsymbol{w},\boldsymbol{w}) = \\
f(\boldsymbol{v},\boldsymbol{v}) + f(\boldsymbol{w},\boldsymbol{w}) + 2f(\boldsymbol{v},\boldsymbol{w}) = g(\boldsymbol{v}) + g(\boldsymbol{w}) + 2f(\boldsymbol{v},\boldsymbol{w}) \\
2f(\boldsymbol{v},\boldsymbol{w}) = g(\boldsymbol{v}+\boldsymbol{w}) - (g(\boldsymbol{v}) + g(\boldsymbol{w}))
\end{aligned}
$$

- It is natural to bring no difference between these two forms and use common letter g to denote them both
- Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in a linear vector space V are called orthogonal to each other with respect to the quadratic form g if $g(\boldsymbol{v},\boldsymbol{w}) = 0$
- A quadratic form g in a space V over the field of real numbers $\mathbb{R}$ is called a positive form if $g(\boldsymbol{v}) > 0$ for any nonzero vector $\boldsymbol{v} \in V$

Linear Vector Space VI

- A **Euclidean vector space** is a linear vector space V over the field of real numbers $\mathbb{R}$ which is equipped with some fixed positive quadratic form g.
- Let (V, g) be a Euclidean vector space.
- There are many positive quadratic forms in the linear vector space V, however, only one of them is associated with V so that it defines the *structure of Euclidean space in V*
- Two Euclidean vector spaces (V, g_1) and (V, g_2) with $g_1 \neq g_2$ coincide as linear vector spaces, but they are different when considered as Euclidean vector spaces.
- The square root of $g(\boldsymbol{v})$ is called the norm or the length of a vector $\boldsymbol{v}$.

Linear Vector Space VII

- Symmetric bilinear form generated by this quadratic denotes **scalar product** in Euclidean space

$$(\boldsymbol{u}|\boldsymbol{v}) = g(\boldsymbol{u}, \boldsymbol{v})$$

- This scalar product has the following properties corresponding with properties of symmetric bilinear forms:
  1. $(\boldsymbol{v}_1 + \boldsymbol{v}_2|\boldsymbol{w}) = (\boldsymbol{v}_1|\boldsymbol{w}) + (\boldsymbol{v}_2|\boldsymbol{W})$ for any two $\boldsymbol{v}_1, \boldsymbol{v}_2 \in V$;
  2. $(\alpha \cdot \boldsymbol{v}|\boldsymbol{w}) = \alpha(\boldsymbol{v}|\boldsymbol{w})$ for any $\boldsymbol{v} \in V$ and for any $\alpha \in \mathbb{K}$;
  3. $(\boldsymbol{v}|\boldsymbol{w}_1 + \boldsymbol{v}2) = (\boldsymbol{v}|\boldsymbol{w}_1) + (\boldsymbol{v}|\boldsymbol{w}_2)$ for any two $\boldsymbol{w}_1, \boldsymbol{w}_2 \in V$;
  4. $(\boldsymbol{v}|\alpha \cdot \boldsymbol{w}) = \alpha(\boldsymbol{v}|\boldsymbol{w})$ for any $\boldsymbol{v} \in V$ and for any $\alpha \in mathbbK$.
  5. $(\boldsymbol{v}|\boldsymbol{w}) = (\boldsymbol{w}|\boldsymbol{v})$ for any two $\boldsymbol{v}, \boldsymbol{w} \in V$;
  6. $|\boldsymbol{v}|^2 = (\boldsymbol{v}|\boldsymbol{v})$ for any $\boldsymbol{v} \in V$. $|\boldsymbol{v}| = 0$ implies $\boldsymbol{v} = \mathbf{0}$
- The following two additional properties of the scalar product require some proof
  - The Cauchy-Bunyakovsky-Schwarz inequality: $|(\boldsymbol{v}|\boldsymbol{w})| \leqslant |\boldsymbol{v}||\boldsymbol{w}|$ for all $\boldsymbol{v}, \boldsymbol{w} \in V$
  - The triangle inequality: $|\boldsymbol{v} + \boldsymbol{w}| \leqslant |\boldsymbol{v}| + |\boldsymbol{w}|$ for all $\boldsymbol{v}, \boldsymbol{w} \in V$

Linear Vector Space VIII

- Proof for Cauchy-Bunyakovsky-Schwarz inequality $|(\boldsymbol{v}|\boldsymbol{w})| \leqslant |\boldsymbol{v}||\boldsymbol{w}|$ for all $\boldsymbol{v}, \boldsymbol{w} \in V$:
  - For non-zero vectors $\boldsymbol{v}, \boldsymbol{w} \in V$ consider numeric function

$$f(\alpha) = |\boldsymbol{v} + \alpha \boldsymbol{w}|^2$$

  - Properties of scalar product yield polynomial nature of $f(\alpha)$

$$f(\alpha) = |\boldsymbol{v} + \alpha \boldsymbol{w}|^2 = (\boldsymbol{v} + \alpha \boldsymbol{w}|\boldsymbol{v} + \alpha \boldsymbol{w}) = (\boldsymbol{v}|\boldsymbol{v}) + 2\alpha(\boldsymbol{v}|\boldsymbol{w}) + \alpha^2(\boldsymbol{w}|\boldsymbol{w})$$

  - Property 6 yields 0 as lower bound for $f(\alpha)$: $f(\alpha) \geqslant 0$

Linear Vector Space IX

- To find actual minimum of this function we calculate derivative of it and let it be zero:

$$
\begin{aligned}
f'(\alpha) = 2(\boldsymbol{v}|\boldsymbol{w}) + 2\alpha(\boldsymbol{w}|\boldsymbol{w}) &= 0 \\
\alpha_{min} &= -\frac{(\boldsymbol{v}|\boldsymbol{w})}{(\boldsymbol{w}|\boldsymbol{w})} \\
f_{min} = f(\alpha_{min}) = \frac{|\boldsymbol{v}|^2|\boldsymbol{w}|^2 - (\boldsymbol{v}|\boldsymbol{w})^2}{|\boldsymbol{w}|^2} &\geqslant 0 \\
|\boldsymbol{v}|^2|\boldsymbol{w}|^2 - (\boldsymbol{v}|\boldsymbol{w})^2 &\geqslant 0 \\
(\boldsymbol{v}|\boldsymbol{w})^2 &\leqslant |\boldsymbol{v}|^2|\boldsymbol{w}|^2 \\
|(\boldsymbol{v}|\boldsymbol{w})| &\leqslant |\boldsymbol{v}||\boldsymbol{w}| \\
&\square
\end{aligned}
$$

Linear Vector Space X

- Proof for triangle inequality: $|\boldsymbol{v} + \boldsymbol{w}| \leqslant |\boldsymbol{v}| + |\boldsymbol{w}|$ for all $\boldsymbol{v}, \boldsymbol{w} \in V$
  - We consider the square of the norm for the sum $\boldsymbol{v} + \boldsymbol{w}$

$$|\boldsymbol{v} + \boldsymbol{w}|^2 = (\boldsymbol{v} + \boldsymbol{w}|\boldsymbol{v} + \boldsymbol{w}) = |\boldsymbol{v}|^2 + 2(\boldsymbol{v}|\boldsymbol{w}) + |\boldsymbol{w}|^2$$

  - Cauchy-Bunyakovsky-Schwarz inequality provides effective estimation for the right side:

$$|\boldsymbol{v} + \boldsymbol{w}|^2 = |\boldsymbol{v}|^2 + 2(\boldsymbol{v}|\boldsymbol{w}) + |\boldsymbol{w}|^2 \leqslant |\boldsymbol{v}|^2 + |\boldsymbol{w}|^2 + 2|\boldsymbol{v}||\boldsymbol{w}| = (|\boldsymbol{v}| + |\boldsymbol{w}|)^2$$
$$|\boldsymbol{v} + \boldsymbol{w}| \leqslant |\boldsymbol{v}| + |\boldsymbol{w}|$$

Linear Vector Space XI

- Therefore, we actually build with vectors in space, on plane, and on line, and begin to study a *Euclidean space* $\mathbb{E}$ *over the field of real numbers* $\mathbb{R}$
- Ordered triplets (in a space), or pairs (on a plane) of coordinates , or single (on a line) coordinate are mathematical representation for our vectors

Features of Structural Constants of Products I

- Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be some arbitrary skew-angular basis
- Consider mixed products involving vectors of this basis

$$c_{ijk} = (\boldsymbol{e_i e_j e_k}) = \boldsymbol{e}_i \cdot \boldsymbol{e_j} \times \boldsymbol{e_k} = \boldsymbol{e}_i \cdot \sum_{q=1}^{3} C_{jk}^{q} \boldsymbol{e}_q = \sum_{q=1}^{3} C_{jk}^{q} (\boldsymbol{e_i} \cdot \boldsymbol{e_q}) = \sum_{q=1}^{3} C_{jk}^{q} g_{iq}$$

- Identity $c_{ijk} = c_{jki}$ derives more canonical writing

$$c_{ijk} = \sum_{q=1}^{3} C_{ij}^{q} g_{qk}$$

- To reverse this formula we need to study some additional properties of the Gram matrix

Features of Structural Constants of Products II

- Determinant of Gram matrix is a square of the oriented volume shaped with basis vectors

$$\det G = (\boldsymbol{e}_1 \boldsymbol{e}_2 \boldsymbol{e}_2)^2$$

- To prove this fact we first need to provide collection of four lows connecting Kronneker and Levi-Chivita symbols known as **contraction formulas**

Contraction Formula One I

$$\varepsilon^{mnp}\varepsilon_{ijk} = \begin{vmatrix} \delta_i^m & \delta_j^m & \delta_k^m \\ \delta_i^n & \delta_j^n & \delta_k^n \\ \delta_i^p & \delta_j^p & \delta_k^p \end{vmatrix}$$

- Proof:
  - Let us denote right value of this formula as f_{ijk}^{mnp}
  - Properties of transpositions of columns in determinants yield
    $f_{ijk}^{mnp} = -f_{jik}^{mnp}$
    $f_{ijk}^{mnp} = -f_{ikj}^{mnp}$
    $f_{ijk}^{mnp} = -f_{kji}^{mnp}$
  - Thus,

$$f_{ijk}^{mnp} = \begin{cases} 0 & \text{if there are coinciding values of the indices } i, j, k; \\ f_{123}^{mnp} & \text{for even permutation } (i, j, k) \text{ of } (1, 2, 3); \\ -f_{123}^{mnp} & \text{for odd permutation } (i, j, k) \text{ of } (1, 2, 3) \end{cases}$$

$$f_{ijk}^{mnp} = f_{123}^{mnp}\varepsilon_{ijk}$$

Contraction Formula One II

- Consider now f_{123}^{mnp}. Any transposition of upper indexes means transposition of rows of the matrix, thus also
  $f_{123}^{mnp} = -f_{123}^{nmp}$;
  $f_{123}^{mnp} = -f_{123}^{mpn}$;
  $f_{123}^{mnp} = -f_{123}^{pnm}$;
- Thus,

$$f_{123}^{mnp} = \begin{cases} 0 & \text{if there are coinciding values of the indices } m, n, p; \\ f_{123}^{123} & \text{for even permutation } (m, n, p) \text{ of } (1, 2, 3); \\ -f_{123}^{123} & \text{for odd permutation } (n, n, p) \text{ of } (1, 2, 3) \end{cases}$$

$$f_{123}^{mnp} = f_{123}^{123} \varepsilon^{mnp}$$

Contraction Formula One III

- Therefore,

$$f_{ijk}^{mnp} = f_{123}^{123} \varepsilon_{ijk} \varepsilon^{mnp}$$

$$f_{123}^{123} = \begin{vmatrix} \delta_1^1 & \delta_2^1 & \delta_3^1 \\ \delta_1^2 & \delta_2^2 & \delta_3^2 \\ \delta_1^3 & \delta_2^3 & \delta_3^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$f_{ijk}^{mnp} = \varepsilon_{ijk} \varepsilon^{mnp} \square$$

Contraction Formula Two

$$\sum_{k=1}^{3} \varepsilon^{mnk} \varepsilon_{ijk} = \begin{vmatrix} \delta_i^m & \delta_j^m \\ \delta_i^n & \delta_j^n \end{vmatrix}$$

- Proof:
  - Each term of the sum has form:

$$\varepsilon^{mnk} \varepsilon_{ijk} = \begin{vmatrix} \delta_i^m & \delta_j^m & \delta_k^m \\ \delta_i^n & \delta_j^n & \delta_k^n \\ \delta_i^k & \delta_j^k & 1 \end{vmatrix} = \delta_i^k \begin{vmatrix} \delta_j^m & \delta_k^m \\ \delta_j^n & \delta_k^n \end{vmatrix} - \delta_k^j \begin{vmatrix} \delta_i^m & \delta_k^m \\ \delta_i^n & \delta_k^n \end{vmatrix} + \begin{vmatrix} \delta_i^m & \delta_j^m \\ \delta_i^n & \delta_j^n \end{vmatrix}$$

  - Factors δ_i^k and δ_k^j yield 1 only for $k = i$ and for $k = j$
  - Summation yields

$$\sum_{k=1}^{3} \varepsilon^{mnk} \varepsilon_{ijk} == \sum_{k=1}^{3} \delta_i^k \begin{vmatrix} \delta_j^m & \delta_k^m \\ \delta_j^n & \delta_k^n \end{vmatrix} - \sum_{k=1}^{3} \delta_k^j \begin{vmatrix} \delta_i^m & \delta_k^m \\ \delta_i^n & \delta_k^n \end{vmatrix} + \sum_{k=1}^{3} \begin{vmatrix} \delta_i^m & \delta_j^m \\ \delta_i^n & \delta_j^n \end{vmatrix} =$$

$$\begin{vmatrix} \delta_j^m & \delta_i^m \\ \delta_j^n & \delta_i^n \end{vmatrix} - \begin{vmatrix} \delta_i^m & \delta_j^m \\ \delta_i^n & \delta_j^n \end{vmatrix} + 3 \begin{vmatrix} \delta_i^m & \delta_j^m \\ \delta_i^n & \delta_j^n \end{vmatrix} = \begin{vmatrix} \delta_i^m & \delta_j^m \\ \delta_i^n & \delta_j^n \end{vmatrix} \square$$

Contraction Formula Three

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon^{mjk}\varepsilon_{ijk} = 2\delta_i^m$$

- Proof:

$$\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon^{mjk}\varepsilon_{ijk} = \sum_{j=1}^{3}\begin{vmatrix}\delta_i^m & \delta_j^m \\ \delta_i^j & 1\end{vmatrix} = \sum_{j=1}^{3}(\delta_i^m - \delta_j^m\delta_i^j) = \sum_{j=1}^{3}\delta_i^m - \sum_{j=1}^{3}\delta_j^m\delta_i^j = 3\delta_i^m - \delta_i^m = 3\delta_i^m$$

Contraction Formula Four

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon^{ijk}\varepsilon_{ijk} = 6$$

- Proof:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\varepsilon^{ijk}\varepsilon_{ijk} = 2\sum_{i=1}^{3}\delta_i^i = 6$$