Differential Geometry

Chapter 1. Concept of a Smooth Curve.

§1. Preliminary.

Def. 1.1 (product). operate in ordinary 3-dim Euclidean space ($\mathbb{E}$).

$\vec{i}, \vec{j}, \vec{k}$. right orthonormal basis. ; x, y, z. coordinates ; O origin ;

$\vec{a}(a_1, a_2, a_3)$, $\vec{b}(b_1, b_2, b_3)$, $\vec{c}(c_1, c_2, c_3)$ arbitrary vector.

(1) scalar product. $\langle \vec{a}, \vec{b} \rangle = \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

(2) vector product. $[\vec{a}, \vec{b}] = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(3) triple product. $(\vec{a}, \vec{b}, \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Def 1.2. (Topological). $M \subset \mathbb{E}$ M- arbitrary set

(1) Mapping. $f: M \to \mathbb{E}$. $\forall X \in M$. $f(X) \in \mathbb{E}$.

(2) Image. $f(M) = \{ Y \in \mathbb{E} : Y = f(X).\ \forall X \in M \}$.

(3) Topological mapping (synonym. Homeomorphism).

i/ 1-to-1. (bijection) ii/ f and f^{-1} cont.

(4). Homeomorphic set. : M and $f(M)$ under t.p. mapping are t.p. equivalent or homeomorphic set.

(5). Open set. $G \subseteq \mathbb{E}$ $\forall X \in G$. $\exists \varepsilon > 0$. s.t. $\forall Y \in \mathbb{E}$. $\mathrm{dist}(X, Y) < \varepsilon \Rightarrow Y \in G$.

(6) A neighborhood of the point. A neighborhood of the point $X \in \mathbb{E}$ is any open set containing this open.

decompose the set M into subset M' and M''

(7) Connected set. $M \subseteq \mathbb{E}$. $\nexists$ $G', G'' \subseteq \mathbb{E}$ and $M = M' \cup M''$ $(M' \cap M'' = \phi)$

s.t. $M' \subseteq G'$, $M' \cap G'' = \phi$.

$M'' \subseteq G''$, $M'' \cap G' = \phi$.

§2. Curves . . . .

Def. 1.3. (Elementary curve). A set $\gamma \subseteq \mathbb{E}$ is elementary curve if:

γ is the image of an open segment on the straight line. under an arbitrary t.p. mapping. (1-to-1. 在几何上体现为"无自交")

γ的连通性

Remark: elementary curve can be expressed with parametric equation:

$(a,b) \subset \mathbb{R}$. $t \in \mathbb{R}$. $t \in (a,b)$.
$$\begin{cases} x = f_1(t) \\ y = f_2(t) \\ z = f_3(t). \end{cases}$$

Def 1.4. (Simple curve). A set $\gamma \subseteq \mathbb{E}$. is simple curve if · γ is connected and $\forall X \in \gamma$. $\exists U_X \subseteq \mathbb{E}$ s.t. $\gamma \cap U_X$ is an elementary curve. ~ "locally elementary" 的拓扑空间.

空间上的邻域.

Thm 1.1. (The structure of a simple curve in the large).

A simple curve is homeomorphic to either an open interval or a circumference.

Remark: open interval / circumference $\xLeftrightarrow{\text{t.p. map}}$ simple curve. 也作为 simple curve 的等价定义.

Def 1.5. (Closed curve). A simple curve which is homeomorphic to a circumference is closed.

Def 1.6. A neighborhood of a point. X on a simple curve γ. is the common part of the curve γ. and "∩" an arbitrary neighborhood of point X. in $\mathbb{E}$.

△ "neighborhood on curve" 是取 neigh 和 curve 交集.

Remark: $\exists$ A neighborhood of each point of a simple curve, is an elementary curve

Lemma 1.2. Let g be an open interval or circumference. $\gamma = f(g)$. γ is simple curve. f is t.p. mapping. Let $\forall X \in g$. (U_X 指在 g 上的邻域).

$\forall U_X$. $f(U_X)$. is neighborhood of $f(X)$ on the curve γ.

$\forall U_{f(X)}$. $\exists V_X$. s.t. $f(V_X) = U_{f(X)}$.

Def 1.7. (Locally one-to-one mapping). A mapping f of a set M into a space is locally one to one mapping if $\forall X \in M$. $\exists U_X$. s.t. f on U_X is one-to-one

Def 1.8. (General curve). A set γ. in space is general curve if:

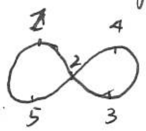
γ is the image of a simple curve under f. f - cont. and local one-to-one.
(notation: $\bar{\gamma}$.-simple curve. γ- general curve. f- t.p. mapping $f(\bar{\gamma}) = \gamma$).

Def 1.9. (Equivalence). Let. f_1, f_2. mappings. γ_1, γ_2 simple curves. They define one and the same general curve γ. if a t.p. mapping $g: \gamma_1 \to \gamma_2$. s.t.

$\forall A \in \gamma_1$: $f_1(A) = f_2(g(A))$.

$\forall B \in \gamma_2$: $f_1(g^{-1}(B)) = f_2(B)$.

e.g.



γ_1: $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5$. different general curve.

γ_2: $1 \mapsto 2 \mapsto 4 \mapsto 3 \mapsto 2 \mapsto 5$

Def 1.10. (converge). $\{f(X_n)\} \subseteq \gamma$. converges to $f(X) \in \gamma$. if the sequence of points. $X_n \in \bar{\gamma}$ conv. to $X \in \bar{\gamma}$.

Def. 1.11. A neighborhood of the point $f(X) \in \gamma$. is the image of any neighborhood of the point $X \in \bar{\gamma}$ under mapping f.

Lemma 1.3. Suppose. $f: \bar{\gamma} \to \gamma$. $f': \bar{\gamma}' \to \gamma$. γ is a general curve. f, f' are t.p. map.

$\bar{\gamma}, \bar{\gamma}'$ are simple.

f' 连续是必要的.
要构造 $f(X_n)$ 极限对应到 $\bar{\gamma}$.

The system of conv. sequence and neighborhood of points on γ. under mapping f and f' are the same (△用 f' 和 f 定义的收敛序列和邻域结构在 γ 上是完全一致的)

Lemma 1.4. Local investigation of any general curve is an investigation of the local simple curve.

Summary:

CURVE   elementary $\xrightarrow{\text{locally elementary}}$ simple $\xrightarrow{f.\ \text{loc. 1-1 \& cont}}$ general

$\uparrow f$: t.p map. $\uparrow f$. t.p. map. $\nearrow$ f. loc 1-1 & cont

SET (parameter)   open segment (straight)   open interval / circumference

Def 1.12. (Regular curve). An arbitrary general curve γ. is the regular curve if $\forall x \in \gamma$. $\exists U_x$. permits a regular parametrization.

i.e. $\begin{cases} x = f_1(t) \\ y = f_2(t) \\ z = f_3(t) \end{cases}$ $t \in (a,b)$ or $[a,b)$. f_1, f_2, f_3 are regular (def. k-times. cont. diff)

Def 1.13. (Smooth curve) A regular curve with $k=1$ is smooth.

Def. 1.14 (Analytic curve) An arbitrary general curve γ is analytic curve if: f_1, f_2, f_3. is analytic.

Thm 1.5. (sufficient condition for. regularity).

Let $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$ $t \in (a,b)$. x, y, z be regular function. This triplet is a regular parametrization of an curve γ if: $x'^2(t) + y'^2(t) + z'^2(t) > 0$. $\forall t \in (a,b)$.

Pf: Assume the converse. $\exists t_0 \in (a,b)$. $\forall \delta > 0$. $\exists t_1, t_2 \in O_\delta(t_0)$. $t_1 \neq t_2$:

$\begin{cases} x(t_1) = x(t_2) \\ y(t_1) = y(t_2) \\ z(t_1) = z(t_2) \end{cases}$ by mean value thm. $\exists \tau_1, \tau_2, \tau_3 \in [\min(t_1,t_2), \max(t_1,t_2)]$.

$x'(\tau_1) = y'(\tau_2) = z'(\tau_3) = 0$.

by arbitrariness. of δ. $t_1, t_2 \to t$. by cont. of x', y', z'.

$x'(t_0) = y'(t_0) = z'(t_0) = 0$.

Therefore. $x'^2(t_0) + y'^2(t_0) + z'^2(t_0) = 0$. contradicts. $\square$.

Procedure: Local Reduction of Variable.

For suitable choice of x, y, z. the simple curve maybe parametrised in a form:

$$\begin{cases} x = t \\ y = \varphi(t) \\ z = \psi(t) \end{cases} \quad t \in (a,b) \quad \Rightarrow \quad \begin{cases} y = \varphi(x) \\ z = \psi(x) \end{cases} \quad x \in (a,b)$$

Thm 1.6. (The 2D implicit function thm).

Suppose $F: \mathbb{R}^2 \to \mathbb{R}$ is a cont. diff. func. Let (x_0, y_0) be a point satisfy equation $F(x,y) = 0$. If $F_y(x_0, y_0) \neq 0$. then in a neighborhood of point (x_0, y_0). we can write $y = f(x)$. where $f: \mathbb{R} \to \mathbb{R}$. is regular. func.

这一f在$O_\delta(x_0)$上是唯一的

$0 = \frac{\partial F}{\partial y}(x_0, y_0) \cdot f_2'(t_0) + \frac{\partial F}{\partial x}(x_0, y_0) f_1'(t_0)$

注意. 在参数方程中 $F_y \neq 0$ 对应 x 的逆函数. $x = f_1(t)$. $f_1'(t) \neq 0$.

(因为 $\frac{\partial F}{\partial y} \neq 0$. y 可由x唯一决定.
$\frac{\partial F}{\partial y} \neq 0 \Rightarrow f_1'(t_0) \neq 0$. 否则 $f_1' = f_2' = 0$ 非正则

Thm 1.7. Suppose γ is a regular curve and $\gamma: \begin{cases} x = f_1(t) \\ y = f_2(t) \\ z = f_3(t) \end{cases}$ $t \in (a,b)$ or $t \in [a,b)$

is its regular parametrisation in a neighborhood of (x_0, y_0, z_0) (corresponding to t_0)

Suppose $f_1'(t_0) \neq 0$. In this case, in a sufficiently small neighborhood of that point, the curve γ. may be expressed in form $\begin{cases} y = \varphi(x) \\ z = \psi(x). \end{cases}$ $x \in O_\delta(t_0)$

Pf: $\exists \chi(x)$. χ - regular func. s.t. $\chi(x_0) = t_0$. (by thm 1.6.)

$\forall x \in O_\delta(t_0)$. $x = f_1(\chi(x))$. $\xrightarrow[\text{at } x_0]{\text{derivate}}$ $1 = f_1'(\underset{t_0}{\chi(x_0)}) \cdot \chi'(x_0)$. thus $\chi'(x_0) \neq 0$.

$\chi(x)$ is monotonic. and for sufficiently small δ. the map $t = \chi(x): O_\delta(x_0) \to t$. is tp.

∵连续可导.小邻域内不突变

Thus $\forall t \in \chi(O_\delta(x_0))$. the curve: $\begin{cases} y = f_2(\chi(x)) \\ z = f_3(\chi(x)) \end{cases}$ $x \in O_\delta(x_0)$. $\square$.

Def 1.15. (Plane curve). A curve is a plane curve if all of its point lie in a plane. (usually use xOy plane. otherwise use rotation & translation).

We shall say a plane curve is defined by the equation $\varphi(x, y) = 0$.

(satisfy $\varphi(x,y)=0 \underset{\times}{\overset{\Leftarrow}{\Rightarrow}}$ belongs to the curve). → 不同于参数定义，这一定义是不完整的

Thm 1.8 (Implicit equation of elementary plane curve).

Suppose $\varphi(x,y)$. is a regular func. s.t. $\varphi(x,y) = 0$. in set M.

Let $A = (x_0, y_0) \in M$. and. $\varphi_x^2(x_0, y_0) + \varphi_y^2(x_0, y_0) > 0$. Then this point has neighbor $\bar{A}$ s.t. $\bar{A} \cap M$. form a regular elementary curve.

根据隐函数定理. $\varphi(x,y)=0$ 在 $O_\delta(x_0) \times O_\delta(y_0)$ 唯一地由 $\psi(x)$ 给出

Pf: w.l.g. $\varphi_y(x_0, y_0) \neq 0$. $\exists \delta > 0$. $\varepsilon > 0$. ψ - regular s.t. $\forall x \in O_\delta(x_0)$: $\varphi(x, \psi(x)) = 0$ (thm. 1.6)

define $y = \psi(x)$, $x \in O_\delta(x_0)$. - elementary curve. ← 唯一性才说明定义的是初等曲线. (无分支).

Thm 1.9 (Implicit equation of elementary curve in general).

Suppose $\varphi(x,y,z)$. $\psi(x,y,z)$ · regular func. (w.r.t. x, y, z). s.t. $\varphi(x,y,z) = \psi(x,y,z) = 0$ in M.

Let $A = (x_0, y_0, z_0)$. $\in M$. and $J|_A = \begin{pmatrix} \varphi_x & \varphi_y & \varphi_z \\ \psi_x & \psi_y & \psi_z \end{pmatrix}\Bigg|_{(x_0, y_0, z_0)}$ has rank 2.

Then $\exists$ neighborhood $\bar{A}$. s.t. $\bar{A} \cap M$ form a regular elementary curve.

§3. Singular Points.

<u>Notation</u>: γ - regular plane curve P - a point on γ.

<u>Def 1.16</u> (Regular point on a plane curve). P is the regular point if the curve permits a regular parameterization. $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in $O_\delta(P)$. and $x'^2 + y'^2 \neq 0$. at P.

正则点. 参数化变量之间存在平滑关系.

<u>Def 1.17</u> (Singular point on a plane curve) P is the singular point if the regular parametrization $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in $O_\delta(P)$. with. $x' = y' = 0$ at P: for <u>arbitrary</u> regular parametrization of a regular curve.

△这个"任意"很重要. $\begin{cases} x = t^3 \\ y = t^7 \end{cases}$ $t \in (-1, 1)$. not singular. $\Rightarrow$ $\begin{cases} x = T \\ y = \pm |T|^{\frac{7}{3}} \end{cases}$

<u>Thm 1.10</u>. (Bürman - Lagrange thm.).

$\rightarrow z = f(w) = \sum_{n=1}^{\infty} f_n (w-a)^n$

Suppose. $z = f(w)$. f is analytic at a point a. and $f'(a) \neq 0$, then it's possible to invert (or solve) equation of w: $w = g(z)$ given by a power series.

$$g(z) = a + \sum_{n=1}^{\infty} g_n \frac{(z - f(a))^n}{n!}, \text{ where } g_n = \lim_{w \to a} \frac{d^{n-1}}{dw^{n-1}} \left[\left(\frac{w-a}{f(w) - f(a)} \right)^n \right]$$

reversion of series. 级数反转.

g是解析函数. 不仅仅是形式上反演级数; 定理还说明. 这个级数有非零收敛半径. $\exists O_r(f(a))$. z在内收敛.

<u>Coro</u>. If $f(s) = \sum_{n=1}^{\infty} f_n s^n$. Consider $T(z) = z f(T(z))$. the only sol. can be expressed as power series. $T(z) = \sum_{n=1}^{\infty} t_n z^n$. $t_n = \frac{1}{n} [s^{n-1}] (f(s))^n$

*对奇异性的调查等价于对一些特别的参数化后曲线的调查 ↓

<u>Lemma 1.11</u> Suppose γ is an analytic curve and O is a point on γ. With a suitable choice of coordinate axes, the curve may be parametrised in a neighborhood of O.

$$\begin{cases} x = a_1 t^{n_1} \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots \end{cases} \quad n_1 \leq m_1 \quad (n_1 \leq m_1 < m_2 < \cdots$$

<u>Pf</u>: Let O be origin. $\begin{cases} x = x(s) \\ y = y(s) \end{cases}$ with a property $x(0) = y(0) = 0$.
$\rightarrow$ arbitrary, analytic parametrisation

Let n_1, m_1 - order of the first n_1 / m_1 - non-zero derivates. of x and y.

w.l.g. $n_1 \leq m_1$ (if $n_1 > m_1$ change role of x and y).

$$\begin{cases} x = \bar{a}_1 s^{n_1} + \bar{a}_2 s^{n_2} + \cdots \\ y = \bar{b}_1 s^{m_1} + \bar{b}_2 s^{m_2} + \cdots \end{cases} \xrightarrow[\Downarrow]{t = s \left(\frac{\bar{a}_1 s^{n_1} + \bar{a}_2 s^{n_2} + \cdots}{\bar{a}_1 s^{n_1}} \right)^{\frac{1}{n_1}}} \begin{cases} x = a_1 t^{n_1} \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots \end{cases}$$

$s = t + d_1 t^{1+\alpha_1} + d_2 t^{1+\alpha_2}$.

$0 < \alpha_1 < \alpha_2 < \cdots$ d_i 是系数

Thm 1.12 Suppose an arbitrary analytic curve $\gamma: \begin{cases} x = a_1 t^{n_1} \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots \end{cases}, n_1 \le m_1.$

in a neighborhood of point O. O is singular (on γ) $\Leftrightarrow \exists m_k: n_1 \nmid m_k$

Pf: "$\Rightarrow$": (by converse-negative prop. if $\forall m_k$, $n_1 \mid m_k \Rightarrow O$ is regular)

n_1 is not even. if so. $\begin{cases} x(t) = x(-t) \\ y(t) = y(-t) \end{cases}$ for sufficient small t. the in $O_\delta(0)$. curve is symmetric. smooth.

let m_k be multiple of n_1 (odd). let $s = t^{n_1}$.

$$\gamma \to \begin{cases} x = a_1 s \\ y = b_1 s^{k_1} + b_2 s^{k_2} + \cdots \end{cases} \quad O: s=0 \quad \left.\frac{dx}{ds}\right|_{s=0} \neq 0. \text{ regular.}$$

"$\Leftarrow$" assume. $\exists k$. $m_k \nmid n_1$ and O is regular. Let $\begin{cases} x = f_1(\sigma) \\ y = f_2(\sigma) \end{cases}$ in $O_\delta(0)$.

描述的是同一个点

$$O: \sigma = \sigma_0. (f_1'(\sigma_0))^2 + (f_2'(\sigma_0))^2 \neq 0. \quad \frac{f_2(\sigma)}{f_1(\sigma)} = \frac{x(t)}{y(t)}$$

Then $\frac{x(t)}{y(t)} \xrightarrow[t \to 0]{\text{可导+洛必达}} \frac{f_2'(\sigma)}{f_1'(\sigma)}$. (而 $\frac{y(t)}{x(t)} = \frac{b_1}{a_1} t^{m_1 - n_1} + \cdots \frac{b_k}{a_k} t^{\overset{\geq 0}{m_k - n_1}}$, 有限)

Thus $f_1'(\sigma) \neq 0$. ($\Rightarrow$ 存在邻域使 f_1 可局部反解为 $\sigma = g(x)$. $y = f_2(\sigma) = f_2(g(x)) = \varphi(x)$).

$\Rightarrow y = \varphi(x)$ or $x = \psi(y)$. φ or ψ are analytic.

express as power series. $y = \varphi(x) = c_1 x + c_2 x^2 + \cdots$ in $O_\delta(0)$.

$\Rightarrow b_1 t^{m_1} + b_2 t^{m_2} + \cdots = c_1 a_1 t^{n_1} + c_2 a_1^2 t^{2n_1} + \cdots \Rightarrow \forall k.\ m_k = k n_1$. contradicts $\square$.

△给出一种奇异点判定：作参数化，使目标点转到原点，看幂次。

Thm 1.13. (Condition a point to be singular).

An arbitrary analytic curve γ defined by $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ in $O_\delta(0)$. (x, y analytic func)

The first nonzero derivatives of $x(t)$ and $y(t)$ have orders n_1 and m_1. ($n_1 < m_1$).

The point O will be singular if $n_1 \nmid m_1$

△解析函数在定义域内可展成泰勒级数.(前 n 阶导 和 多项式 n 阶系数 非零 对应).

$$\forall \begin{cases} x = x(t) \\ y = y(t) \end{cases} \xrightarrow{s = t\left(\frac{x(t)}{x^{(n_1)}(0) t^{n_1}}\right)^{\frac{1}{n_1}}} \begin{cases} x = x_{n_1}(0) s^{n_1} \\ y = y_{m_1}(0) s^{m_1} \end{cases}$$

幂次比 $\frac{m_1}{n_1}$ 决定了 curve 在 O 的性质.

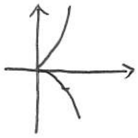
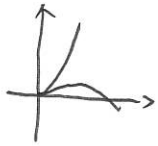
Def. 1.18 (Turning Point).

Suppose O is singular point on curve γ. with parametrization.

$$\gamma: \begin{cases} x = a_1 t^{n_1} \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots \end{cases} \quad (n_1 \le m_1 < m_2 \cdots).$$

If m_1 is odd, n_1 is even. O is the turning point of the 1st kind.

If m_1, n_1 are even, O is the turning point of the 2nd kind.

1st kind $\begin{cases} x = t^2 \\ y = t^3 + t^4 + t^5. \end{cases}$  2nd kind $\begin{cases} x = t^2 \\ y = t^4 + t^5 \end{cases}$ 

coro.: In notation above, $\gamma: \begin{cases} x = x(t) \\ y = y(t). \end{cases}$ x, y. analytic. n_1, m_1. first non-zero deri.

If n_1 is even. m_1 is odd. $\rightarrow$ O is turning point of 1st kind

If n_1, m_1 are even. $\rightarrow$ O is turning point of 2nd kind

Singular point expressible by Implicit func.

A plane analytic curve γ: $\varphi(x,y) = 0$ (φ is analytic of x and y).

$O(x_0, y_0)$. regular: $\varphi_x^2(x_0, y_0) + \varphi_y^2(x_0, y_0) \neq 0$.

singular: $\varphi_x = \varphi_y = 0$

Assume O is the origin. and apply the parametrization $\begin{cases} x = a_1 t^{n_1} \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots \end{cases}$

Aim to know n_1 and m_k. Solve $\varphi(x(t), y(t)) = 0$.

Remark: (1) n_1, m_k are not uniquely defined by $\varphi(x(t), y(t)) = 0$:

i) change variable $t = s^p$ not change characteristic of points. curve.

ii) A implicit func. $\varphi(x,y) = 0$ can express several different (geometrically) curves., even in small neighborhood of O.

(2). The singularity of O. on various curve will be distinct.

△隐函数奇点分析中，参数化可灵活选择，而O可能是满足$\varphi(x,y)=0$的不同曲线的公共奇点，可能要逐条分析

If $\varphi(x,y)$ has non-degenerated quadratic form. △注意…中是更高次项.

$$\varphi(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2 + \cdots \xrightarrow[\text{eliminate } xy.]{\text{with rotation on axes.}} a_{20}a_1^2 t^{2n_1} + a_{02}b_1^2 t^{2m_1} + \cdots = 0.$$

(let $n_1 \le m_1$. $2n_1$ is lowest power $\Rightarrow a_{20} = 0$. (所有幂次系数均 =0. 才满足隐函数方程. t^{2n_1}最低. 无法相消))

for 1) & 2). $n_1 = m_1$. $a_{20}a_{02} > 0$ or < 0. for 3). $n_1 < m_1$.

$\downarrow$

classify the quadratic.

1) $a_{20}a_{02} - a_{11}^2 > 0$ 2). $a_{20}a_{02} - a_{11}^2 < 0$ 3) $a_{20}a_{02} - a_{11}^2 = 0$.

O be the origin point $(0,0)$.

<u>Def 1.19</u> (isolated singular point). In above notation, and $\varphi(x,y)$ s.t. $a_{20}a_{02} - a_{11}^2 > 0$. we said O is a isolated singular point.

<u>Remark</u>: $a_{20}a_{02} > 0$ and $a_{20}a_1^2 + a_{02}b_1^2 = 0$ is impossible. no analytic curve $\varphi(x,y) = 0$ containing singular point O. (△"孤立"是邻域内没有点满足 $\varphi(x,y)=0$. 不仅是没有奇点满足 $\varphi(x,y)=0$).

consider 2). $\begin{cases} a_{20}a_{02} < 0. \\ a_{20}a_1^2 + a_{02}b_1^2 = 0 \end{cases}$ $\Rightarrow$ two sol. (independent, different analytic curve) i) $\begin{cases} a_1 = \sqrt{|a_{02}|} \\ b_1 = \sqrt{|a_{20}|} \end{cases}$ ii) $\begin{cases} a_1 = \sqrt{|a_{02}|} \\ b_1 = -\sqrt{|a_{20}|} \end{cases}$

For system i) or ii). use condition for a_1, b_1 and then find b_k, m_k.

<u>Def 1.20</u>. (Nodal point). In above notation and φ s.t. $a_{20}a_{02} - a_{11}^2 < 0$ and $\nexists k$. $n_1 = m_1 \nmid m_k$. We say O is a nodal point.

<u>Remark</u>: By solving i) and ii), we have 2. distinct analytic curve. O is regular points for these curve if we investigate separately. (but it's singular if we consider the family of curves).

consider 3) w.l.g. assume $a_{20} = 0$. then $\varphi(x,y) = a_{02}y^2 + a_{30}x^3 + \cdots$ $(a_{30} \ne 0)$ (since $\varphi_2 = a_{02}y^2$. $\varphi_3 = a_{30}x^3 + \cdots + a_{03}y^3$. no common divisor).

$2m_1 = 3n_1$ ($n_1 \nmid m_1$. O is singular).

by the prove of <u>Thm 1.12</u>. m_k, n_1 can't be all even.

$\Rightarrow$ only n_1 is even. O is turning point of 1st kind.

§4. Asymptotes. to a curve.

γ: non-closed curve with parametrization $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $t \in (a,b)$. $(*)$.

Def 1.21. We say a curve tends to infty from one side if $x^2(t) + y^2(t) \to +\infty$ as $t \to a$ or $t \to b$. - A curve tends to infty from both side if $x^2(t)+y^2(t) \to +\infty$ as $t \to a$ and $t \to b$.

(this property not depend on parametrization).

Def. 1.22. (asymptote). The straight line g is an asymptote to curve γ. if the length $d(t)$. from a point on the curve γ to g tend to 0. when $t \to a$. (suppose $\gamma \to$ infty. as $t \to a$).

* 显式参数方程下渐近线存在性判别：

Thm. 1.14. If curve $(*)$. tend to infty as $t \to a$. have an asymptote iff:

(1). At least. one of $\frac{y(t)}{x(t)}$. $\frac{x(t)}{y(t)} \xrightarrow[t\to a]{}$ finite limit. (w.l.g. $\frac{y(t)}{x(t)} \to k$).

(2). $y(t) - kx(t) \xrightarrow[t\to a]{}$ finite limit. (denote. $y(t) - kx(t) = l$).

Remark: if (1). (2) satisfied. the equation of asymptote: $y - kx - l = 0$.

Thm 1.15. An alternative condition of curve have asymptote.

denote $\rho(t) = \sqrt{x^2(t) + y^2(t)}$.

(1): $\frac{x(t)}{\rho(t)} \xrightarrow[t\to a]{} \alpha$. $\frac{y(t)}{\rho(t)} \xrightarrow[t\to a]{} \beta$. $\alpha, \beta \in \mathbb{R}$.

(2): $-\beta x(t) + \alpha y(t) \xrightarrow[t\to a]{} p$. $p \in \mathbb{R}$.

Remark: if (1), (2) satisfied. the equation of asymptote: $-\beta x + \alpha y - p = 0$

e.g. γ: $y = \varphi(x)$ $x \in (a,b)$. and $\varphi(x) \xrightarrow[x\to a]{} \infty$

$\Rightarrow \gamma: \begin{cases} x = t \\ y = \varphi(t). \end{cases}$ $t \in (a,b)$ $\qquad \frac{x(t)}{y(t)} = \frac{t}{\varphi(t)} \longrightarrow \frac{a}{+\infty} = 0.$

$x(t) - k y(t) = t - 0 \cdot \varphi(t) \to = a.$

the asymptote: $x = a$.

Implicit case.

有一些点满足方程但不在曲线上.

△ $\varphi(x,y)=0$ - not completely defined the curve.

restrict the case: algebraic curves. ($\varphi(x,y)=0$ is a polynomial in $x.y$.).

suppose. the equation of the asymptote in parametric form. $\begin{cases} x=\bar{x}+\lambda u \\ y=\bar{y}+\mu u. \end{cases}$ (λ,μ) 给出候选渐近线方向.

$Q(u)$ - on γ. closest to the asymptote. → 实际问题是先算候选渐近线和曲线距离.

$$Q(u):\begin{cases} x(u)=\bar{x}+\lambda u+\xi(u) \\ y(u)=\bar{y}+\mu u+\eta(u). \end{cases} \qquad \xi(u)\xrightarrow[u\to\infty]{}0 \qquad \eta(u)\xrightarrow[u\to\infty]{}0.$$

取极小值. (此时确定 $Q(u)$, $\bar{x}$, $\bar{y}$). 再表示出 $\xi(u) := x(u)-\bar{x}-\lambda u$. 和 $\eta(u)$ check $\xi(u)$, $\eta(u)$ 是否 $\to 0$.

$\varphi=\varphi_n(x,y)+\varphi_{n-1}(x,y)+\cdots+\varphi_0(x,y)$. φ_k - homogenuous poly. of deg k.

$$\Rightarrow \varphi(x(u).y(u)) = u^n\varphi_n(\lambda,\mu)+u^{n-1}\left[\bar{x}\frac{\partial\varphi_n(\lambda,\mu)}{\partial\lambda}+\bar{y}\frac{\partial\varphi_n(\lambda,\mu)}{\partial\mu}+\varphi_{n-1}(\lambda.\mu)\right]+O(u^{n-2})$$

$\varphi(x(u),y(u))=0 \Rightarrow$ the coefficient of $u^k\equiv 0$. △注意仅前2项是必须 $\equiv 0$ 的: 后面的可以调 ξ,η.

$$\Rightarrow \begin{cases} \varphi_n(\lambda.\mu)=0 \\ \bar{x}\dfrac{\partial\varphi_n}{\partial\lambda}+\bar{y}\dfrac{\partial\varphi_n}{\partial\mu}+\varphi_{n-1}(\lambda.\mu)=0 \end{cases}$$

→ 确定方向. $\lambda.\mu$.

→ 方向和截距 $\bar{x}.\bar{y}$.

△为什么前2项 ξ,η 不能调. u^n 里无 ξ,η. u^{n-1} 里 ξ,η 带系数,还是 $o(1)$

△多项式. 可设 ξ,η 形如 $\sum_{k=1}^{\infty}\frac{a_k}{u^k}$

△注意.这里. (λ,μ). 实际有意义的是比值. 可令 $\lambda=1$. ; $(\bar{x},\bar{y})$ 同理. 并不是4个未知数.

Chapter 2. Contact of Curves.

$M, \bar{M} \subseteq \mathbb{E}$. $O \in M \cap \bar{M}$. $\forall X \in \bar{M}$. $h(X) = \inf_{Y \in M} d(X, Y)$.
$d(X) = d(X, O)$.

Def 2.1. We say set $\bar{M}$ has contact with the set M in the point O.

if $\dfrac{h(X)}{(d(X))^{\alpha}} \xrightarrow[X \to O]{} 0 \quad (\alpha > 1)$.

Recall. def.

对G中的每一个点. 赋值一个向量

(1) Vector function: $\vec{f}: G \to \mathbb{R}^n$. G is a point set on real line/plane/space

(2) Limit: $\vec{f}(X) \xrightarrow[X \to X_0]{} \vec{a}$ if $|\vec{f}(X) - \vec{a}| \xrightarrow[X \to X_0]{} 0$.

property: $\forall \vec{f}, \vec{g}, \lambda$. s.t. $\vec{f}(X) \xrightarrow[X \to X_0]{} \vec{a}$ $\quad \vec{g}(X) \xrightarrow[X \to X_0]{} \vec{b}$ $\quad \lambda(X) \xrightarrow[X \to X_0]{} m$.

$\Rightarrow$ i/ $\vec{f}(X) \pm \vec{g}(X) \to \vec{a} \pm \vec{b}$ ii/ $\lambda(X)\vec{f}(X) \to m\vec{a}$

iii/ $\vec{f}(X) \cdot \vec{g}(X) \to \vec{a} \cdot \vec{b}$ iv/ $\vec{f}(X) \times \vec{g}(X) \to \vec{a} \times \vec{b}$

(3) continuous at X_0: $\vec{f}(X) \xrightarrow[X \to X_0]{} \vec{f}(X_0)$

(4) derivative: $\vec{f}'(t) = \lim\limits_{h \to 0} \dfrac{\vec{f}(t+h) - \vec{f}(t)}{h}$ $\quad$ * $\vec{f}$ defined on closed interval.
$\vec{f}$ has derivative at t on an open interval.

(5) k-times differential: $\vec{f}$ have cont. derivative up to k-th order on the open interval.

(6) Taylor formula: $\vec{f}$ is n-times diff.

$$\vec{f}(t+\Delta t) = \vec{f}(t) + \Delta t \cdot \vec{f}'(t) + \cdots + \frac{(\Delta t)^n}{n!}(\vec{f}^{(n)}(t) + \varepsilon(t, \Delta t)).$$

$|\varepsilon(t, \Delta t)| \xrightarrow[\Delta t \to 0]{} 0$.

(7). Riemann integral: $\vec{f}(t)$. cont. on $t \in [a,b]$. $a < c < b$, m is cont. $\vec{r}$. cont. vector.

then: i/ $\displaystyle\int_a^b \vec{f}(t)\,dt = \int_a^c \vec{f}(t)\,dt + \int_c^b \vec{f}(t)\,dt$. $\quad$ ii/ $\displaystyle\int_a^b m\vec{f}(t)\,dt = m\int_a^b \vec{f}(t)\,dt$.

iii/ $\displaystyle\int_a^b \vec{r} \cdot \vec{f}(t)\,dt = \vec{r} \cdot \int_a^b \vec{f}(t)\,dt$. $\quad$ iv/ $\displaystyle\int_a^b \vec{r} \times \vec{f}(t)\,dt = \vec{r} \times \int_a^b \vec{f}(t)\,dt$.

v/ $\dfrac{d}{dt}\displaystyle\int_a^x \vec{f}(t)\,dt = \vec{f}(x)$.

e.g. $\vec{r}=\vec{r}(t)$. $t\in(c,d)$. s.t. $r'(t)=\lambda(t)\cdot\vec{a}$. $\lambda(t)>0$. cont. on (c,d)

$\vec{r}(t)=\left(\int\lambda(t)\,dt\right)\vec{a}+\vec{c}$. $\Rightarrow$ direction determine by $\vec{a}$. initial point $\vec{c}$.

if $\int\lambda(t)\,dt.<\infty$. - line segment.

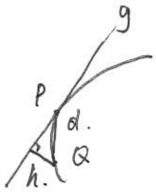
if $\int\lambda(t)\,dt$. diverge at 1 end point - ray.

if $\int\lambda(t)\,dt$ diverge at both end point - entire line.

§2. Tangent.

γ - curve. P - point on γ. g - straight line passing through point P.

$\forall$ point Q on curve γ. $d:=\operatorname{dist}(P,Q)$. $h:=\operatorname{dist}.(Q,g)$.



Def 2.2. The line g: the targent to the curve γ at P.
if $\frac{h}{d}\xrightarrow[Q\to P]{}0$.

Thm 2.1. A smooth curve γ. has. a unique tangent at each point.
$\vec{r}=\vec{r}(t)$, the tangent has direction of vector $\vec{r}'(t)$.

Def 2.3. position vector: $\tilde{\boldsymbol{r}}=\vec{r}(t)+\lambda\vec{r}'(t)$. $(*)$.
$(*)$ is the equation of the tangent in the parametric form (with param. λ).

Remark 1: suppose: $\vec{r}(t)=x(t)\vec{e_1}+y(t)\vec{e_2}+z(t)\vec{e_3}$.

$$\begin{cases}\tilde{x}=x(t)+\lambda x'(t)\\ \tilde{y}=y(t)+\lambda y'(t)\\ \tilde{z}=z(t)+\lambda z'(t)\end{cases}\Rightarrow\frac{\tilde{x}-x(t)}{x'(t)}=\frac{\tilde{y}-y(t)}{y'(t)}=\frac{\tilde{z}-z(t)}{z'(t)}.$$

Remark 2: consider tangent at (x_0,y_0,z_0) γ: $\begin{cases}\varphi(x,y,z)=0\\ \psi(x,y,z)=0\end{cases}$ (1).

where the rank $\begin{pmatrix}\varphi_x & \varphi_y & \varphi_z\\ \psi_x & \psi_y & \psi_z\end{pmatrix}\Big|_{(x_0,y_0,z_0)}=2$.

apply any regular parametrization $\begin{cases}x=x(t)\\ y=y(t)\\ z=z(t)\end{cases}$ in some $U(x_0,y_0,z_0)$.

diff (1). w.r.t. $t\Rightarrow\begin{cases}\varphi_x x'+\varphi_y y'+\varphi_z z'=0\\ \psi_x x'+\psi_y y'+\psi_z z'=0\end{cases}\Rightarrow\frac{x'}{\begin{vmatrix}\varphi_y & \varphi_z\\ \psi_y & \psi_z\end{vmatrix}}=\frac{y'}{\begin{vmatrix}\varphi_x & \varphi_z\\ \psi_x & \psi_z\end{vmatrix}}=\frac{z'}{\begin{vmatrix}\varphi_x & \varphi_y\\ \psi_x & \psi_y\end{vmatrix}}$

$\Rightarrow$ the tangent line: $\frac{\tilde{x}-x_0}{\begin{vmatrix}\varphi_y & \varphi_z\\ \psi_y & \psi_z\end{vmatrix}}=\frac{\tilde{y}-y_0}{\begin{vmatrix}\varphi_x & \varphi_z\\ \psi_x & \psi_z\end{vmatrix}}=\frac{\tilde{z}-z_0}{\begin{vmatrix}\varphi_x & \varphi_y\\ \psi_x & \psi_y\end{vmatrix}}$

Def 2.4. The **normal plane** to a curve at P is the plane which passes through the point P and is perpendicular to the tangent at this point.

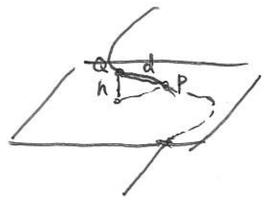
Remark: from parametric form of tangent. the normal plane at $t=t_0$.

$$x'(t_0)(x-x(t_0)) + y'(t_0)(y-y(t_0)) + z'(t_0)(z-z(t_0)) = 0.$$

§3. Osculating plane.

γ - curve. P - point on γ. α - plane pass through P.

$\forall Q \in \gamma$. $\mathrm{dist}(Q, \alpha) = h$. $\mathrm{dist}(Q, P) = d$.



Def 2.5. The plane α is **osculating plane** to curve γ at P if $\frac{h}{d^2} \xrightarrow[Q\to P]{} 0$.

Thm 2.2. A regular curve γ has an osculating plane at each point.
(osculating 要么唯一，要么是包含 tangent 的任意平面).

Remark: the osculating plane parallel to $\vec{r}'(t)$ and $\vec{r}''(t)$ → 不平行，则唯一.
平行，则包含切线任意平面.

Suppose $\vec{r}'(t)$, $\vec{r}''(t)$ not parallel at P. $\Rightarrow$ $\vec{r}'(t) \times \vec{r}''(t)$ normal vector of osculating

the equation of osculating plane: $(\tilde{r} - \vec{r}(t), \vec{r}'(t) \times \vec{r}''(t)) = 0$.

parametrically: $\begin{vmatrix} \tilde{x}-x(t) & \tilde{y}-y(t) & \tilde{z}-z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0.$

△ $\tilde{r}$ 位置向量，摆动平面(P的)任意一点.

→ t 的值与 P 对应，解 $\tilde{x}, \tilde{y}, \tilde{z}$.

Def 2.6. Every straight line passing through a point on the curve perpendicular to the tangent is **normal** to the curve.

principal normal: normal lying in the osculating plane.

binormal: normal perpendicular to the osculating plane.

△ 找 binormal: $\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}$ (就是 $\vec{r}'(t) \times \vec{r}''(t)$，摆动平面法向量).

canonical form: $\frac{x-x(t_0)}{b_x} = \frac{y-y(t_0)}{b_y} = \frac{z-z(t_0)}{b_z}$

principal: $\vec{p} = \vec{b} \times \vec{r}'(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_x & b_y & b_z \\ x'(t) & y'(t) & z'(t) \end{vmatrix}$ → 坐标值 $\Rightarrow \frac{x-x(t_0)}{p_x} = \frac{y-y(t_0)}{p_y} = \frac{z-z(t_0)}{p_z}$.

§4. Contact of curves.

Def 2.7. A regular curve γ is of C^n-class if:

(1) All component function $x_k(t)$ $(k \in [1:d])$ have cont. deri. up to order n.

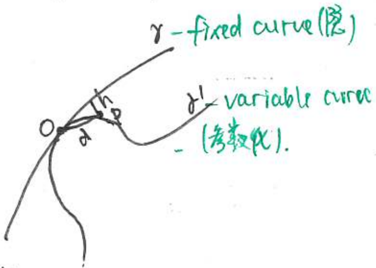
(2) The n-th deri. $\gamma^{(n)}(t)$ exists and is cont.

Def 2.8. γ, γ' - elementary curve. O - common point. choose $P \in \gamma'$.

denote $h := \mathrm{dist}(P, \gamma)$ ↙代入参数式. $d := \mathrm{dist}(P, O)$ ↙代入定点.

We say γ' has contact of order n with curve γ at point O

if $\frac{h}{d^n} \xrightarrow[P \to O]{} 0$.



Remark: if γ, γ' - general curve. we say γ' --- with curve γ.

if $\exists$ neighborhood of O on γ and γ' satisfy that condition.

Remark: the intersection of curves is 0-th contact.

△ "contact order" 衡量了2条曲线"共享"的几何属性的程度.

0th contact - intersection only   "touch but cross"

1th contact - shared tangent.   "kiss but run parallel briefly"

2th contact - shared curvature.   "oscillate together"

3+ contact.   "nearly indistinguishable"

Thm 2.3. Let γ, γ' regular plane curves of class C^n, $O(x_0, y_0)$. s.t.

- γ is given implicitly $\varphi(x,y) = 0$ with $\nabla\varphi|_{(x_0,y_0)} \neq 0$.
- γ' is parametrized $(x(t), y(t))$, with $t = t_0$ at O.

Then γ' has contact of order n with γ at O iff. $\Phi(t) := \varphi(x(t), y(t))$

satisfy: $\Phi(t_0) = \Phi'(t_0) = \cdots = \Phi^{(n)}(t_0) = 0$.

Pf: $\quad d = |t - t_0| \cdot \|\gamma'(t_0)\| + o(|t - t_0|)$

$\quad h = |\Phi(t)| \cdot \|\nabla\varphi(0)\|^{-1} + o(|\Phi(t)|)$

"$\Rightarrow$" $\frac{h}{d^n} \to 0 \Rightarrow |\Phi(t)| = o(|t - t_0|^n)$.

"$\Leftarrow$" $\Phi^{(k)}(t_0) = 0 \Rightarrow \Phi(t) = \frac{\Phi^{(n+1)}(t_0)}{(n+1)!}(t - t_0)^{n+1} + o(|t - t_0|^{n+1})$.

thus $h \sim |t - t_0|^{n+1}$ while $d \sim |t - t_0|$

Def 2.9. γ - regular curve. (regularity $k=2$). parametrised by $\vec{r}(t)$. P on γ.

The osculating circle to γ. at P. is the unique. circle s.t:

- contact of order ≥ 2. with γ at P. (intersect, tangent, curvature).
- it's radius. $R = \frac{1}{\kappa}$ (radius of curvature. curvature κ).
- it's center locates in the normal line to γ at P (in concavity direction) 凹面方向.

Method: Given a family of curve: $\gamma_{\alpha_1,\alpha_2,\dots\alpha_n}$ defined by $\varphi(x,y,\alpha_1,\alpha_2,\dots,\alpha_n)=0$.

A regular C^{n-1} curve γ parametrised by $(x(t),y(t))$. with $\gamma(0)=O$.

Find $\gamma_{\alpha_1,\alpha_2,\dots\alpha_n}$ with $n-1$-order contact to γ at O. (注意这里是定曲线参数化. 变曲线隐函数式).

step 1. define contact function $\Phi(t) := \varphi(x(t),y(t),\alpha_1,\alpha_2,\dots,\alpha_n)$.

step 2. impose contact condition.

$$\Phi(0)=\Phi'(0)=\Phi''(0)=\dots=\Phi^{(n-1)}(0)=0.$$ (注意这里是n个等式)

step 3. solve the system of equation. (if $\Phi^{(n)}(0)=0$. discard the sol.).

step 4. Verify geometric condition.

1. curves are transverse (no degenerate contact).
2. curvature / torsion. match (for $n \geq 2$).

Remark: (1). if γ has a cusp at O. no smooth $\gamma_{\alpha_1,\alpha_2,\dots\alpha_n}$.

(2). For $n=2$. Maybe $\exists$ discrete solution. (non-uniqueness of sol.)

(3) when $n=1$. solution form the envelope of the family.

Method. Find all curves γ' have 2nd-order contact with given γ. at point O.

γ: $\varphi(x,y)=0$. $\quad \gamma'$: $(x(t),y(t))$ with $O=(x(0),y(0))$

step 1. $\Phi(t)=\varphi(x(t),y(t))$

step 2. choose a family of candidate curve γ' (e.g. circle, parabola, ...).

step 3. Each family (with parameter). sol. $\begin{cases} \Phi(0)=0 \\ \Phi'(0)=0 \\ \Phi''(0)=0. \end{cases}$

Remark: (1) typically. sol. forms discrete set (e.g. symmetric 2 circle).

(2) γ is line. γ' can be all lines through O. (infty sol. $\kappa=0$).

(3) if γ has cusp at O. no smooth γ'. (e.g. $\gamma: y^2=x^3$. $(0,0)$).

§5. Envelope of a family of curves.

Def 2.10. Suppose $\{S_{\gamma_\alpha}\}$ is a family of smooth curve on a surface, depending on a parameter α. A smooth curve γ is an envelope of family S if:

- $\forall p \in \gamma$. $\exists \gamma_\alpha \in S$. γ_α tangent to γ at p.
- $\forall \gamma_\alpha \in S$. $\exists p \in \gamma$. γ_α tangent to γ at p.
- no curve of the family has a segment in common with the curve γ.

Remark: The envelope tangent to all γ_α; each point of contact has at least 1-order.

e.g. A smooth curve without rectilinear arcs (直线段弧). is the envelope of its tangent lines. and each tangent touches the curve at exactly 1 point (1st-order contact)

Thm 2.4. Let $\{\gamma_\alpha\}$. be a C^1-family of plane curve: $\varphi(x,y,\alpha)=0$. where φ is C^2-curve. s.t.

1/ $\Delta\varphi = \varphi_x^2 + \varphi_y^2 \neq 0$. and $\varphi_\alpha \neq 0$. 不是所有候选点. (regularity).

2/. $\det\begin{pmatrix} \varphi_x & \varphi_y \\ \varphi_{x\alpha} & \varphi_{y\alpha} \end{pmatrix} \neq 0$. (transversality).

Then the envelope T is the solution of $\begin{cases} \varphi(x,y,\alpha)=0 \\ \varphi_\alpha(x,y,\alpha)=0 \end{cases}$ (eliminating α).

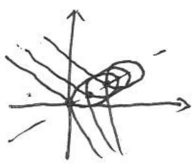
△这是一个等价条件, 可证明是包络T $\Leftrightarrow$ 满足方程组.

如: 解出的曲线其实是曲线的Nodal point.

△若仅有方程解. 可能存在解不是包络但满足方程组 (由于切向量退化. 因此需要1/, 2/条件才有等价关系)

e.g. γ_α: $(x-\alpha)^3+(y-\alpha)^3-3(x-\alpha)(y-\alpha)=0$.

$x=y$ is a sol. but not envelope.



$$\det\begin{pmatrix} \varphi_x & \varphi_y \\ \varphi_{x\alpha} & \varphi_{y\alpha} \end{pmatrix} = \begin{pmatrix} 3(x-\alpha)^2-3(y-\alpha) & 3(y-\alpha)^2-3(x-\alpha) \\ 6\alpha-6x+3 & 6\alpha-6y+3 \end{pmatrix} = 0 \text{ when } x=y.$$

Chapter 3. Curvature and Torsion.

§1. Arc length of a curve.

- γ - arbitrary curve. φ - locally 1-to-1 mapping. g - open interval / circumference
  $\varphi(g) = \gamma$.

Def 3.1. A segment of the curve γ, is the image of an arbitrary closed segment Δ, belonging to that open interval g (if the curve is parametrised by g). or image of the arc κ of circumference k (if the curve is parametrised by k). under mapping φ. The endpoint of the segment corr. to endpoint of Δ or κ.

notation: $\gamma|_{[a,c]}$ or $\gamma|_{\Delta}$.

Thm 3.1. (Uniqueness of the set of segments of the given curve) — 不随参数化方式所改变. 参数值不同, 几何位置是相同.

Let φ_1, φ_2 be cont. and locally 1-to-1 mapping on the open interval g defining same curve $\gamma = \varphi_1(g) = \varphi_2(g)$. If there exists a homeomorphism $\psi: g \to g$ s.t. $\varphi_1 = \varphi_2 \circ \psi$, then the sets of segments of γ induced by φ_1 and φ_2 coincide.

Pf: ψ - homeomorphism $\Rightarrow$ strictly monotonic

$\forall [a,b] \subseteq g$. $\psi([a,b]) = [c,d] \subseteq g$. (preserving the endpoint. by cont & mono.)

thus $\varphi_1([a,b]) = \varphi_2(\psi[a,b]) = \varphi_2([c,d])$.

every segment of γ induced by φ_1 corr. to a segment induced by φ_2. vice versa.

Suppose $\tilde{\gamma}$ is a segment of the curve γ, and let A and B be its endpoints.

Choose $A = A_0, A_1 \ldots A_n \equiv B$ on $\tilde{\gamma}$. proceeding from A to B.

Join A_i and A_{i+1} employing rectilinear segments.

$\Rightarrow$ the polygonal arc T inscribe in the segment $\tilde{\gamma}$ of the curve.

Def 3.2. (Arc length). The least upper bounded of the lengths of all possible polygonal arcs T, inscribed in the segment $\tilde{\gamma}$ of the curve, will be called the arc length (or arc) of segment $\tilde{\gamma}$

notation: $s(\tilde{\gamma})$. △可理解为函数. $s: \gamma$的所有线段 $\to$ 实数. "length function"

Def 3.3. The segment $\tilde{\gamma}$ of curve γ is rectifiable if the lengths of all possible polygonal arcs T (inscribed in the segment $\tilde{\gamma}$). are uniformly bounded by some finite number.

Def 3.4. The curve γ is rectifiable if each of its segments is rectifiable.

Thm 3.2. (Fundamental Property of rectifiable curve)
If $A'B'$, AB - segments of γ, $A'B' \subseteq AB$. AB is rectifiable. Then $A'B'$ also rectifiable and length $s(A'B') \leq s(AB)$.
(the equality holds iff. $A'B' = AB$).

Thm 3.3. (Additivity of arc length of rectifiable curve).
If C is a point on the segment AB of the curve which is distinct from both A and B, and the segment AC. CB are rectifiable. then AB also rectifiable,
$s(AC) + s(CB) = s(AB)$.

§2. Arc length of a smooth curve.

Thm 3.4. A smooth curve γ is rectifiable. If $\vec{r}=\vec{r}(t)$ is its smooth parametr. and $\tilde{\gamma}=\gamma|_{[a,b]}$ is a segment of γ. then: $S(\tilde{\gamma})=\int_a^b |\vec{r}'(t)|\,dt$.

Pf: 1/ rectifiability.

$\vec{r}(t)$. smooth. $\vec{r}'(t)$ exists and cont. on $[a,b]$.

$\exists M>0$. s.t. $|\vec{r}'(t)|\le M$. $\forall t\in[a,b]$.

$\forall$ partition $\mathbb{P}$ of $[a,b]$. $\ell(\mathbb{P})=\sum|\vec{r}(t_i)-\vec{r}(t_{i-1})|$.

by mean-value thm. $\exists\, \xi_i\in[t_{i-1},t_i]$. $|\vec{r}(t_i)-\vec{r}(t_{i-1})|\le \underbrace{\sum_{j=1}^{n}|r_j'(\xi_i^j)|}_{\le nM}(t_i-t_{i-1})$

$\Rightarrow \ell(P)\le n\cdot M(b-a)$.

2/ formula. $\vec{r}'$ is uni. cont. on $[a,b]$.

$\forall\varepsilon>0$. $\exists\delta>0$ s.t. $|t-s|<\delta\Rightarrow|\vec{r}'(t)-\vec{r}'(s)|<\varepsilon$.

consider partition $\mathbb{P}$ s.t. $\|\mathbb{P}\|<\delta$.

for each i. $\vec{r}(t_i)-\vec{r}(t_{i-1})=\vec{r}'(\tau_i)(t_i-t_{i-1})+\varepsilon_i(t_i-t_{i-1})$

since $\|\varepsilon_i\|<\varepsilon$ (*). $\|\vec{r}(t_i)-\vec{r}(t_{i-1})\|-|\vec{r}'(\tau_i)|\cdot|t_i-t_{i-1}|<\varepsilon\cdot|t_i-t_{i-1}|$.

summing over. $\left|\ell(P)-\sum_{i=1}^{n}|\vec{r}'(\tau_i)|(t_i-t_{i-1})\right|<\varepsilon(b-a)$.

As $\|\mathbb{P}\|\to0$, the Riemann sum conv. to the integral when $\varepsilon\to0$.

$$S(\gamma)=\lim_{\|P\|\to0}\ell(P)=\int_a^b|r'(t)|\,dt.$$

(*). by Taylor thm. $\|\varepsilon_i\|=\left\|\frac{1}{t_i-t_{i-1}}\int_{t_{i-1}}^{t_i}|\vec{r}'(t)-\vec{r}'(\tau_i)|\,dt\right\|\le\frac{1}{t_i-t_{i-1}}\int_{t_{i-1}}^{t_i}\varepsilon\,dt=\varepsilon$.

Def 3.5 (Natural Parametrization).

Let γ be a rectifiable curve with parametrization $\vec{r}(t)$ for $t\in I$. The natural parametrization of γ is the arc-length parametrization $\vec{r}(s)$ where:

1. s measures signed arc length from a base point $\vec{r}(t_0)$.

$$S(t)=\begin{cases}\int_t^{t_0}|\vec{r}'(\tau)|\,d\tau. & t\ge t_0\\ -\int_t^{t_0}|\vec{r}'(\tau)|\,d\tau. & t<t_0\end{cases}$$

2. the curve is "reparametrised" as $\vec{r}(s):=\vec{r}(t(s))$. $t(s)$ is the inverse of $s(t)$.
3. unit speed $\left|\frac{d\vec{r}}{ds}\right|\equiv1$ (wherever differentiable).

相当于 $k=\infty$，每一点都可展开为收敛的泰勒级数

Thm 3.5. γ - regular curve of class C^k / analytic, with $|\vec{r}\,'(t)| \neq 0$ everywhere.

Then: 1. It's natural parametrization $\vec{r}(s)$ exists and of same class (C^k or analytic)

2. It satisfies the unit speed condition. $\forall s$. $\left|\frac{d\vec{r}}{ds}\right| = 1$.

3. The arc length parameter s is given by $s(t) = \int_{t_0}^{t} |\vec{r}\,'(\tau)|\, d\tau$.

△自然参数化特征：1) s标记相等弧长间距. 2). 单位切向量.

3). 重新参数化下间距不变. → $\Delta s = s_2 - s_1$ 始终等于对应曲线弧长.

→ 满足自然参数化条件的重新参数化.（平移，如通过线性变换 $s' = s + c$ 调整起点等）.

Coro 3.6. (Global regular parametrization).

Let γ be a regular curve of class C^k $(k \geq 1)$ or analytic. Then:

1. γ admits a global natural parametrization $\vec{r}(s)$. of the same class C^k / analytic.
2. Any reparametrization $\vec{r}(\varphi(t))$ with $\varphi: I \to \mathbb{R}$. is C^k / analytic. and $\varphi'(t) \neq 0$ $\forall t \in I$.
   (preserves the regularity and differentiable class)

△补充：若 $\vec{\alpha}(t) \neq 0$. 且可微.

(1) $|\vec{\alpha}(t)| = \text{const.} \iff \vec{\alpha} \cdot \vec{\alpha}\,' \equiv 0$.

(2) $\vec{\alpha}(t)$ 方向不变 $\iff \vec{\alpha}\,' \times \vec{\alpha} \equiv 0$.

(3). $\vec{\alpha}(t)$ 与某一固定方向垂直 $\implies (\vec{\alpha}, \vec{\alpha}\,', \vec{\alpha}\,'') \equiv 0$

$\impliedby$ $+ \ \vec{\alpha}\,'(t) \times \vec{\alpha}(t) \neq 0$. 处处.

§3. Curvature and Torsion.

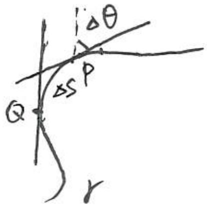
"curvature" – measures how much does a curve "bend"

"torsion" – measures how much does a curve "twist out of a plane".

γ – regular curve P – point on γ. Q – point on γ. near P.

$\Delta\theta$ – angle between the tangents drams to the curve at P and Q.

Δs – arc length of segment PQ of the curve.



<u>Def 3.6.</u> The curvature k_1 of a curve γ at the point P is the limit of the ratio $\frac{\Delta\theta}{\Delta s}$ as the point Q approaches P.

$$k_1 = \lim_{P \to Q} \frac{\Delta\theta}{\Delta s}$$

<u>Thm 3.7</u> Let $\vec{r} = \vec{r}(s)$ be a regular, twice continuously differentiable (C^2) curve parametrized by arc length s. The curvature of curve at a point $\vec{r}(s)$:

$$k_1 = |\vec{r}''(s)| = \left|\frac{d\vec{T}}{ds}\right| \qquad \vec{T}(s) = \vec{r}'(s), \text{ unit tangent vector.}$$

<u>Remark:</u> (1) k_1 measures "how much the curve deviates from being a straight line at a given point".

- $k_1 \equiv 0$ everywhere $\Longleftrightarrow$ the curve is a straight line. or an open interval on a straight line. $\to$ $\vec{r}''(s) \equiv 0$. $\vec{r}(s) = \vec{a}\cdot s + \vec{b}$
- If $k_1 > 0$. "bending". $k_1 \nearrow$. sharper the bend

(2) k_1 – not depend on the parametrisation.

(3). $$k_1 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$ t – is arbitrary parameter.

Suppose $k_1 \neq 0$ at a given point on the curve.

$\vec{n} = \frac{1}{k_1}\vec{r}''(s)$. unit vector and lies in the osculating plane.

$\vec{T} = \frac{1}{|\vec{r}'(s)|}\vec{r}'(s)$ unit tangent.

Fact: $\vec{n}\cdot\vec{T} = 0$. when $\|\vec{T}\| = 1$.

<u>Pf:</u> $\|T\| = 1$. $\vec{T}\cdot\vec{T} = 1$. $\Rightarrow \frac{d}{ds}(\vec{T}\cdot\vec{T}) = 2T\cdot T' = 0 \Rightarrow \vec{T}\cdot\vec{T}' = 0$

$\vec{T}\cdot(k_1\cdot\vec{n}) = 0 \Rightarrow \vec{T}\cdot\vec{n} = 0$

△ 对一般的 $\vec{r}''(t)\cdot\vec{r}'(t) \neq 0$ 仅对单位切向量才有此结论（事实上可放宽到 $\|\vec{T}\| = \text{const.}$）.
对 arc length s 恒成立 $|\vec{r}'(s)| \equiv 1. \Rightarrow \vec{r}'(s) \perp \vec{r}''(s)$

Def 3.7 The vector $\vec{b} = \vec{T} \times \vec{n}$ is the unit binormal vector of the curve.

Thm 3.8. $$k_1 = \frac{|\vec{r}''(t) \times \vec{r}'(t)|}{|\vec{r}'(t)|^3}$$

Pf: $$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| = |\vec{r}'(t)|.$$

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \cdot \frac{dt}{ds} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d}{dt}\left(\frac{\vec{r}'(t)}{|\vec{r}'(t)|}\right) \cdot \frac{dt}{ds} = \frac{\vec{r}''(t) \cdot |\vec{r}'(t)|^2 - \vec{r}'(t)\left(\vec{r}'(t) \cdot \vec{r}''(t)\right)}{|\vec{r}'(t)|^4}$$

$$\Rightarrow k_1 = \left|\frac{d^2\vec{r}}{ds^2}\right| = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}.$$

把$\vec{r}''$分解成$\parallel$和$\perp$于$\vec{r}'$分量

$|\vec{r}' \times \vec{r}''| = |\vec{r}'||\vec{r}''| \cdot \sin\langle \vec{r}', \vec{r}'' \rangle$. $|\vec{r}''_\perp| = |\vec{r}''| \sin\langle \cdot \rangle$

Remark: planar curve. (1) x, y plane $$k_1^2 = \frac{(x''y' - y''x')^2}{(x'^2 + y'^2)^3}$$

(2) $y = y(x)$ $$k_1^2 = \frac{(y'')^2}{(1 + y'^2)^3}$$

Suppose P - arbitrary point on γ; Q on γ near P;

$\Delta\theta$ - angle between binormal vectors.

Δs - length of segment PQ on the curve.

Def 3.8. The absolute torsion $|k_2|$ of the curve γ at the point P is:

$$|k_2| = \lim_{Q \to P} \frac{\Delta\theta}{\Delta s}.$$

Thm 3.9. Let $\vec{r}(s)$ be regular, C^3-curve, parametrized by arc length s. $|k_2|$ of the curve at a point $\vec{r}(s)$ is given by

$$|k_2| = \frac{|\vec{r}'(s) \cdot (\vec{r}''(s) \times \vec{r}'''(s))|}{\underbrace{|\vec{r}''(s)|^2}_{= k_1^2}}$$

$|r'(s)| \equiv 1$ by arc length. 一般式中是 $|\vec{r}''(t) \times \vec{r}'(t)|^2$

Pf: $|k_2| = |\vec{b}'(s)|$ $$\vec{b}(s) = \frac{\vec{r}'(s) \times \vec{r}''(s)}{|\vec{r}'(s) \times \vec{r}''(s)|}$$

$$\vec{b}'(s) = \frac{r' \times r'''}{|r''|} - \frac{(r' \times r'')}{|r''|^3}\left[(r' \times r'') \cdot (r' \times r''')\right]$$

$= r'' \cdot r'''$

$(a \times c) \cdot (a \times d) = c \cdot d - (a \cdot c)(a \cdot d)$

let $X = \dfrac{r' \times r'''}{|r''|}$ $\quad Y = \dfrac{(r' \times r'') \cdot (r'' \cdot r''')}{|r''|^3}$

$X \cdot b = Y \cdot b$. by $\vec{b}$'s direction $X = X_\parallel + X_\perp$. $X_\parallel = Y$. $X_\perp \perp b$.

$$\|b'\|^2 = \|X_\perp\|^2 = \|X\|^2 - \|Y\|^2 = \frac{\|r' \times r'''\|^2 \cdot \|r' \times r''\|^2}{|r''|^4} - \frac{(r''(s) \cdot r'''(s))^2}{|r''|^4}$$

$$= \frac{(r', r'', r''')^2}{|r''|^4}.$$

Δ 恒等式 $[a \cdot (b \times c)] = \|a \times c\|^2 \|b \times a\|^2 - [(b \cdot c) - (a \cdot c)(a \cdot b)]^2$

Remark: the sign of k_2 indicates the direction of the "twisting out of the osculating plane" relative to tangent and normal vector

$k_2 > 0$. - curve twists counterclockwise relative to osculating plane.

- moving along the curve. $\vec{b}(s)$ rotates counterclockwise around $\vec{T}(s)$.

$k_2 < 0$ - clockwise resp.

$k_2 = 0$ - not twist. ; lies entirely in a single plane.

$$k_2 = \frac{\vec{r}'(s)\cdot(\vec{r}''(s)\times\vec{r}'''(s))}{|\vec{r}''(s)|^2}$$

Remark: $\vec{b}(s) = \vec{T}(s)\times\vec{n}(s)$. $\vec{T}$ - unit tangent. $\vec{n}$ - unit normal vector.

$\Rightarrow \vec{b}'(s) = -k_2\,\vec{n}(s)$.

Suppose curve $\vec{r}(t) = (x(t), y(t), z(t))$.

$$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} \qquad (\vec{r}'(t)\times\vec{r}''(t))\cdot\vec{r}'''(t) = \begin{vmatrix} x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \\ x'''(t) & y'''(t) & z'''(t) \end{vmatrix}$$

$$k_2(t) = \frac{(\vec{r}'(t)\times\vec{r}''(t))\cdot\vec{r}'''(t)}{|\vec{r}'(t)\times\vec{r}''(t)|^2} = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{(y'z''-z'y'')^2+(z'x''-x'z'')^2+(x'y''-y'x'')^2}$$

Example.

curve		k_1	k_2
straight line	$\vec{r}(t) = \vec{a}+t\vec{b}$	0	0
circle	$\vec{r}(t) = (R\cos t, R\sin t, 0)$	$\frac{1}{R}$	0
helix	$\vec{r}(t) = (R\cos t, R\sin t, ht)$ $(R>0, h\neq 0)$	$\frac{R}{R^2+h^2}$	$\frac{h}{R^2+h^2}$

when $h\to 0$. helix $\to$ circle $(k_2\to 0)$

$R\to\infty$ helix $\to$ straight line $(k_1\to 0)$

the curvature in polar coordinates: $r = r(\theta)$

$$k_1 = \frac{\left|r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}{\left(r^2+\left(\frac{dr}{d\theta}\right)^2\right)^{3/2}}$$

§4 Frenet Frame.

Target: describe the shape and behaviour of curves using "invariant quantities" (don't depend on coordinates).

– direction of trajectory path, direction of turn, tilt of the road.

<u>Def 3.9</u> From a point on the curve, emanating 3 half-lines with direction $\vec{T}, \vec{n}, \vec{b}$. (edges of trihedron). This trihedron is Frenet frame or natural trihedron.

<u>Def 3.10.</u> Frenet - Serret Formula. $\begin{cases} \vec{T}' = k_1 \vec{n} \\ \vec{n}' = -k_1 \vec{T} - k_2 \vec{b} \\ \vec{b}' = k_2 \vec{n}. \end{cases}$

→ 找 $\vec{n}$. 除了从 $\vec{T}'$. 也可以从定义出发. 找与 $\vec{r}'(t) \times \vec{r}''(t)$ (找平面法向量垂直的向量. 在找平面内). 但这样会丢±

e.g. <u>helix</u> $\vec{r}(t) = (R\cos t, R\sin t, ht)$. $R>0$. $h \neq 0$.

$$\vec{T}(t) = \frac{(-R\sin t, R\cos t, h)}{\sqrt{R^2+h^2}} \quad \text{(unit tangent)}$$

$$\vec{n}(t) = (-\cos t, -\sin t, 0) \quad \text{(Normal)}.$$

$$\vec{b}(t) = \frac{(h\sin t, -h\cos t, R)}{\sqrt{R^2+h^2}} \quad \text{(Binormal)}.$$

$$k_1 = \frac{R}{R^2+h^2} \qquad k_2 = \frac{h}{R^2+h^2}$$

Suppose $\vec{r}(t) = \vec{r}(t_0) + \vec{r}'(t_0)(t-t_0) + \frac{1}{2}\vec{r}''(t_0)(t-t_0)^2 + \cdots$

→ arc length parametrization in neighborhood of P. ($s=0$ at P)

$$\vec{r}(s+\Delta s) = \left(\Delta s - \frac{k_1^2 \Delta s^3}{6} + \cdots\right)\vec{T} + \left(\frac{k_1 \Delta s^2}{2} + \frac{k_1' \Delta s^3}{6}\right)\vec{n} + \left(-\frac{k_1 k_2 \Delta s^3}{6} + \cdots\right)\vec{b} + \cdots$$

(substitution: $\vec{r}' = \vec{T}$, $\vec{r}'' = k_1 \vec{n}$, $\vec{r}''' = k_1' \vec{n} - k_1^2 \vec{T} - k_1 k_2 \vec{b}$).

Remark: for arc length s.

$$T(s) \cdot n(s) = n(s) \cdot b(s) = b(s) \cdot T(s) = 0.$$

§5. Natural Equation.

Thm 3.10 (Fundamental Theorem of Curves).

Let $k_1(s)$. $k_2(s)$. be smooth functions defined on an interval $I \subseteq \mathbb{R}$. with $k_1(s) > 0$ $\forall s \in I$. Then:

1). $\exists$ smooth. regular curve γ. parametrised by s as arc length, for which $k_1(s)$ is the curvature. $k_2(s)$ is the torsion at each point $\gamma(s)$.

2). The curve γ is uniquely determined up to rigid motion in space →刚性运动.
(i.e. if $\tilde{\gamma}$, γ both s.t. 1). $\tilde{\gamma}$ can be obtain from γ by translations and rotations).

Def 3.11. The system of equation $\begin{cases} k_1 = k_1(s) \\ k_2 = k_2(s) \end{cases}$ are natural equations of the curve.

Remark: the pair $(k_1(s), k_2(s))$ encode the intrinsic geometry property.
(not depend on the position / orientation in space)

Procedure. for given $k_1(s)$. $k_2(s)$ find the curve $\gamma(s)$

1). Solve the ode.

$$\frac{d}{ds}\begin{bmatrix} \vec{T}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & -k_2 \\ 0 & k_2 & 0 \end{bmatrix}\begin{bmatrix} \vec{T}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix}$$

$\Rightarrow \vec{T}.\ \vec{n}.\ \vec{b}$

2). construct the curve

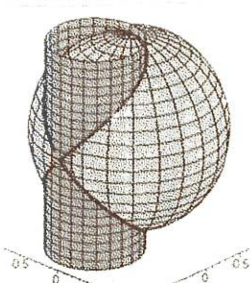
$$\vec{r}(s) = \int_0^s \vec{T}(u)\, du$$

*Def. (Viviani's curve / window). figure-eight-shaped space curve.

1) intersection of a sphere with a cylinder.

2). tangent to the sphere

3). passes through 2 poles (on a diameter) of the sphere



Chapter 4. Surface.

§1. Elementary surface.

<u>Def 4.1.</u> A planar region is an elementary region if it's an image of an open circle under an arbitrary topological mapping.

<u>Remark:</u> An elementary region is a "region homeomorphic" with a circle.

Property of elementary region:

1) connected 2) simple shaped (it has boundary that smooth and regular)
3) Finite area 4) open region. (not include it's boundary)

<u>Def 4.2.</u> A set Φ of points in space is an elementary surface if it's an image of elementary region in a plane under arbitrary tp. mapping.

<u>e.g.</u> elementary region $G: \{(u,v) \in \mathbb{R}^2 \mid u^2+v^2<1\}$

elementary surface $f: G \to \mathbb{E}$. $f(u,v) = (u, v, \sqrt{1-u^2-v^2})$

Φ - upper semisphere.

<u>Def 4.3.</u> Let Φ - elementary surface $\Phi = \{(x,y,z) \in \mathbb{E} \mid (x,y,z) \text{ s.t. } (*)\}$.

G - elementary region $G = \{(u,v) \in \mathbb{R}^2 \mid \cdots\}$

the system $\begin{cases} x = f_1(u,v) \\ y = f_2(u,v) \\ z = f_3(u,v) \end{cases}$ $(*)$, is the equation of the surface in the parametric form.

§2. Simple surface.

<u>Def 4.4.</u> A set Φ of point in space is a simple surface if this set is connected and every point $X \in \Phi$ has a neighborhood G_X s.t. $G_X \cap \Phi$ is an elementary surface.

<u>e.g.</u> A sphere - simple, not elementary.

<u>Prop 4.1.</u> If an closed set of points is deleted from any simple surface s.t. the remaining part is open and connected, then the remaining part will be also a simple surface.

Def 4.5. A simple surface is complete if the limit point of any conv. sequence of points on the surface is also a point on the surface.

e.g. sphere, paraboloid.

Def 4.6. If a simple complete surface is finite, then it's closed.

Def 4.7 A neighborhood of a point X on a simple surface is the common part of simple surface Φ and some neighborhood of point X in space.

e.g. For sphere Φ, the neighborhood of X on Φ is a disk.

Remark: Each point of the simple surface has neighborhood, which is elementary surf.

§3. General Surface

Def 4.8 A set of points Φ in space is a general surface if there exists a simple surface S and a mapping $f: S \to E$ s.t.:

1) f is cont. 2). f is locally 1-to-1. 3) $\Phi = f(S)$.

Def 4.9. The mapping f_1, f_2 of simple surface Φ_1, Φ_2 (respectively.) define the same general surface Φ if there exists homeomorphism $h: \Phi_1 \to \Phi_2$ such that: $f_1(p) = f_2(h(p)) \quad \forall p \in \Phi_1$

Suppose the general surface Φ is the image under t.p. mapping of a simple surface $\bar{\Phi}$.

Def 4.10. A sequence of points $f(X_n)$ on the surface Φ converges to the point $f(X)$ if the sequence of points X_n on the simple surface $\bar{\Phi}$ converges to the point X.

Def 4.11. A neighborhood of the point $f(X)$ on the surface Φ is image of an arbitrary neighborhood of the point X on the surface $\bar{\Phi}$ under the mapping f.

Remark: the convergence and neighborhood do not depend on mapping f.

§4. Regular surface.

Def 4.12 A surface is regular (k-times diff). if each point on the surface has a neighborhood, permits a regular parametrization $\begin{cases} x = f_1(u,v) \\ y = f_2(u,v) \\ z = f_3(u,v) \end{cases}$
where f_1, f_2, f_3 are regular func. defined in elementary region.
For $k=1$. the surface is smooth.

Def 4.13. A surface is analytic if it allows an analytic parametrization (f_1, f_2, f_3 analytic). in a sufficient small neighborhood of each point on surface.

Def 4.14. A point P on a surface Φ is regular point. if the surface permits a regular parametrization in neighborhood of P.

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \quad \text{and} \quad \operatorname{rank}\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}\Big|_P = 2.$$

Otherwise, P is singular point. (注意正则曲面上可以有奇异点，满足正则参数化. rank < 2).

Def 4.15. A curve on a surface, all points of the curve are singular points of the surface, is singular curve.

Thm 4.2. Let G be an open subset of u,v-plane. let x, y, z be C^1-func. on G.
If Jacobian matrix has rank 2 everywhere in G. then $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ define a regular surface Φ. Moreover the mapping $\vec{r}: G \to \Phi$. given by $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$ is cont. and locally 1-to-1.

Thm 4.3 (Implicit definition of a surface)
Let $\varphi(x,y,z)$ be C^1-func. defined on open subset of $\mathbb{R}^3$. M be the set $\forall (x_0,y_0,z_0) \in M$. $\varphi(x_0,y_0,z_0) = 0$.
Suppose $(x_0,y_0,z_0) \in M$ and $\Delta\varphi = \varphi_x^2 + \varphi_y^2 + \varphi_z^2 \neq 0$.
Then $\exists\, U_{(x_0,y_0,z_0)} \subseteq M$. is a regular elementary surface.

Remark. denote $M: \{(x,y,z) \mid (x,y) \in U,\ z = f(x,y)\}$

§5. Parametrization of a Surface.

<u>Lemma 4.4.</u> Suppose $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. is a regular parametrization of a surface in $U_{Q(x_0,y_0)}$. (thus. $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}$ linearly independent since rank $(J|_Q) = 2$).

Let $\varphi(\alpha,\beta)$. $\psi(\alpha,\beta)$ be smooth func. s.t. $\begin{cases} u_0 = \varphi(\alpha_0,\beta_0) \\ v_0 = \psi(\alpha_0,\beta_0) \end{cases}$ $\left|\begin{vmatrix} \varphi_\alpha & \varphi_\beta \\ \psi_\alpha & \psi_\beta \end{vmatrix}\right|_{(\alpha_0,\beta_0)} \neq 0$.

Then then equation $\begin{cases} x = x(\varphi,\psi) \\ y = y(\varphi,\psi) \\ z = z(\varphi,\psi) \end{cases}$ defined a new parametrisation in $U(\alpha_0,\beta_0)$.

the mapping $\begin{cases} u = \varphi(\alpha,\beta) \\ v = \psi(\alpha,\beta) \end{cases}$ $(\alpha,\beta) \mapsto (u,v)$. is local. diffeomorphism

thus. $(x(\varphi,\psi), y(\varphi,\psi), z(\varphi,\psi))$ is smooth and the tangent vector remains linearly independent. ensure the parametrisation is regular.

△规定了参数化保正则性需要满足的条件.

§6. Singular points on regular surface

suppose $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$ regular parametrisation with rank $\left.\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_u \end{pmatrix}\right|_{Q(x_0,y_0)} < 2$.

denote $\vec{r}(u,v)$. $\quad$ rank$(J) < 2 \iff \vec{r}_u \times \vec{r}_v = 0$.

<u>Lemma 4.5.</u> Suppose $P(u,v)$ on surface near $Q(u_0,v_0)$.

Q is singular $\Rightarrow$ $\vec{r}_u \times \vec{r}_v |_Q = 0$. (necessary condition)

Moreover. if Q is singular. $\vec{\xi}(u,v) = \dfrac{\vec{r}_u(u,v) \times \vec{r}_v(u,v)}{|\vec{r}_u(u,v) \times \vec{r}_v(u,v)|}$ is undefined at Q.

the limit of $\vec{\xi}(u,v)$ as $P \to Q$. not exist. $\quad$ △反之不成立.

<u>Lemma 4.6.</u> Same notation as above. The point Q is regular if. $\vec{r}_u, \vec{r}_v$ l.i. at Q (i.e. $\vec{r}_u \times \vec{r}_v \neq 0$). The unit normal vector at Q is given by $\vec{\xi}(u,v) = \dfrac{\vec{r}_u(u,v) \times \vec{r}_v(u,v)}{|\vec{r}_u(u,v) \times \vec{r}_v(u,v)|}$

the limit of $\vec{\xi}(u,v)$ as $P \to Q$ is definite. ($\vec{\xi}(u,v)$ well-defined at Q).

§7. Singular points on implicit surface.

consider the surface $\varphi(x,y,z) = 0$.

<u>Def. 4.16.</u> A point $Q(x_0, y_0, z_0)$ is potentially singular if $\nabla\varphi(Q) = \vec{0}$.

<u>Remark</u>: this is an necessary, not sufficient condition for singularity.

$\varphi = x^2+y^2+z^2$. $\nabla\varphi = \vec{0}$ at $(0,0,0)$ isolated point (even not in the surface).

Second-order analysis. (Taylor expansion near Q).

$$\varphi(x,y,z) \approx \frac{1}{2}\Big[\varphi_{xx}(x-x_0)^2 + \varphi_{yy}(y-y_0)^2 + \varphi_{zz}(z-z_0)^2 + 2\varphi_{xy}(x-x_0)(y-y_0) + $$
$$+ 2\varphi_{xz}(x-x_0)(z-z_0) + 2\varphi_{yz}(y-y_0)(z-z_0)\Big]$$
$$= \sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j \qquad \xi_1 = x-x_0.\quad \xi_2 = y-y_0.\quad \xi_3 = z-z_0.$$

i) definite quadratic form:

(1). $\varphi = 0 \Leftrightarrow \xi_i = 0$. (only trivial sol.)

(2) Q is isolated point. (no surface near by).

e.g. $\varphi = x^2+y^2+z^2$ at $(0,0,0)$

ii) Indefinite quadratic form:

(1) Solution form a cone or similar singular surface.

(2) Q non-isolated singular point.

e.g. $\varphi = z^2 - (x^2+y^2)$. at $(0,0,0)$. (double zone at origin). the surface self-intersects.

Summary: at Q.

i) If $\nabla\varphi \neq \vec{0}$. the surface is regular near Q.

ii) If $\nabla\varphi = 0$. compute the quadratic form.

$A = (a_{ij})$ - Hessian matrix. $a_{ij} = \frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j}(Q)$.

All eigenvalues same sign $\Rightarrow$ isolated points.

mixed sign $\Rightarrow$ singular surface.

has 0 $\Rightarrow$ degenerated point.

§8. Tangent Plane to a Surface.

Φ - surface. Q, P - point on it. α - any plane passing through the point P.

$d := \text{dist}(Q, P)$. $\quad h := \text{dist}(Q, \alpha)$.

Def 4.17. The plane α is the tangent plane to surface Φ at point P if the ratio $\lim\limits_{Q \to P} \frac{h}{d} = 0$.

Thm 4.7. Let Φ be a smooth surface with parametrization $\vec{r}(u,v)$, $\vec{r}$ is C^1. At each point $P(u,v) \in S$, the tangent plane α exists and is uniquely spanned by $\vec{r}_u$ and $\vec{r}_v$. That is, tangent plane α is the set of all vectors of the form. $\vec{v} = a\vec{r}_u(u,v) + b\vec{r}_v(u,v)$. $a, b \in \mathbb{R}$. $\vec{r}_v, \vec{r}_u$ · l.i.

(hold for all regular points with smooth and non-degenerate parametrization)

Pf: formula: $h = \| \vec{r}(u+\Delta u, v+\Delta v) - \vec{r}(u,v) - \vec{r}_u(u,v)\Delta u - \vec{r}_v(u,v)\Delta v \|$.

$$d = \| \vec{r}(u+\Delta u, v+\Delta v) - \vec{r}(u,v) \|.$$

△若求曲面上一点的"normal line". 则方向为 $\vec{r}_u \times \vec{r}_v$. 过 $\vec{r}(u,v)$. 可写出直线方程

§9. Equations of Tangent Plane.

(1). for explicit surface $z = f(x,y)$.

tangent plane $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$.

(2) for implicit surface $F(x,y,z) = 0$.

tangent plane: $\nabla F(a,b,c) \cdot (x-a, y-b, z-c) = 0$.

$\left(\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right) \right)$.

"the normal to a surface" $\vec{n} = \nabla \vec{F}$. line: $\dfrac{x - x_0}{\frac{\partial F}{\partial x}} = \dfrac{y - y_0}{\frac{\partial F}{\partial y}} = \dfrac{z - z_0}{\frac{\partial F}{\partial z}}$

§10. Distance from a point to a surface.

Φ - surface. Q - point in space.

$$\mathrm{dist}(\Phi, Q) = \inf_{P \in \Phi} \mathrm{dist}.(P, Q).$$

<u>Lemma 4.8.</u> Suppose Φ - smooth curve : $F(x,y,z) = 0$. Let $O(x_0, y_0, z_0)$ be a point on Φ. $\nabla F(x_0, y_0, z_0) \neq \vec{0}$. $Q(x,y,z)$ - point in space near O. but not on Φ. denote $h := \mathrm{dist}(\Phi, Q)$.

If $Q \to O$. the ratio $\frac{F(x,y,z)}{h}$ approaches a finite. That is,

$$\lim_{Q \to O} \frac{F(x,y,z)}{h} = C \ (\text{const}, \neq 0).$$

<u>Remark</u>: $\lim_{Q \to O} \frac{F(x,y,z)}{h} = \|\nabla F(x_0, y_0, z_0)\|$ by the proof.

§11. Contact of surface and curve.

<u>Def 4.18.</u> Φ - smooth surface, γ - curve. intersects Φ at point O.

For any Q on γ. $h := \mathrm{dist}(Q, \Phi)$. $d :=$ arc length OQ.

We say γ has an nth-order contact with surface Φ. at point O.

if $\lim_{Q \to O} \frac{h}{d^n} = 0$.

<u>Thm 4.9.</u> Let S - elementary regular curve. γ - regular curve. have common point O.

Suppose : 1). S is locally describe by $F(x,y,z) = 0$ near O. F - smooth. $\nabla F|_O \neq \vec{0}$.

2). γ is locally parametrised by $\vec{\gamma}(t) = (x(t), y(t), z(t))$. near O s.t. the parametrization is regular and $\vec{\gamma}(t_0) = O$.

Then. the curve γ has contact of order n. with surface S at O. iff. the composition $F(t) = F(x(t), y(t), z(t))$ s.t. $F(t_0) = 0$, $\left.\frac{d^k F}{dt^k}\right|_{t=t_0} = 0$. $\forall k \in [1:n]$.

"$\Leftrightarrow$"

Pf: γ-regular. $\gamma'(t_0) \neq 0$. $|\vec{r}(t) - \vec{r}(t_0)| \underset{t \to t_0}{\sim} |t - t_0|$.

$$\frac{|F(Q)|}{h} \to \text{const.} \qquad \frac{F(x(t), y(t), z(t))}{h} \cdot \frac{h}{|r(t) - r(t_0)|^n} \underset{t \to t_0}{\longrightarrow} 0 . \quad (*)$$

expansion $F(x(t), y(t), z(t))$ at t_0. $(*)$ happens. iff $F(t_0) = F'(t_0) = \cdots = F^{(n)}(t_0) = 0$.

Example. S: $z = x^2 + y^2$ (paraboloid). curve: $\gamma(t) = (t, t, t^2 + t^3)$.

at $t = 0$. intersects. $F(t) = t^2 + t^2 - (t^2 + t^3) = t^2 - t^3$.

use thm4.8 $F(0) = 0$ $F'(0) = 0$ $F''(0) = -2$. $\Rightarrow$ 1st-order contact.

use def. $h \approx t^2$. $d \approx \sqrt{2}|t|$. (要做比，取主成分(最低阶)即可).

$$\lim_{t \to 0} \frac{h}{d} = 0 \qquad \lim_{t \to 0} \frac{h}{d^2} = \frac{1}{2} \neq 0. \quad \Rightarrow \text{1st-order contact.}$$

Def 4.19. The osculating sphere to a curve $\vec{r}(s)$ (arc length s). at point P. is the sphere that has 3rd-order contact with curve at P.

(This means the curve and sphere has same position, tangent. curvature and 3rd-derivative at P)

given: $\vec{r}(s)$ with Frenet frame $\{\vec{T}, \vec{n}, \vec{b}\}$. curvature $k_1(s)$ torsion $k_2(s)$.

sphere: $(\vec{r} - \vec{a})^2 = R^2$.

1st-order: $(\vec{r} - \vec{a}) \cdot \vec{T} = 0$ (center $\vec{a}$. lies in the normal plane).

2nd-order: $(\vec{r} - \vec{a}) \cdot \vec{n} = -\frac{1}{k_1}$

3rd-order: $(\vec{r} \cdot \vec{a}) \cdot \vec{b} = -\frac{k_1'}{k_1^2 k_2}$

$$\Rightarrow \vec{a} = \vec{r} + \frac{1}{k_1}\vec{n} + \frac{k_1'}{k_1^2 k_2}\vec{b}$$

$$R = \sqrt{\frac{1}{k_1^2} + \left(\frac{k_1'}{k_1^2 k_2}\right)^2}$$

Let S. regular (2-times diff.). surface. P - point on S. ↙切平面法向量.

U - paraboloid, containing P. U's axis parallel to the surface normal at P.

denote: $\forall Q \in S$. $d := dist(Q, P)$ $h := dist(Q, U)$.

Def. 4.20. The paraboloid U is osculating paraboloid of S at P. if $\lim_{Q \to P} \frac{h}{d^2} = 0$.

Remark: degenerate case (parabolic cyclinder / plane). are included.

Thm 4.10. At every point P of a regular surface S, there exists unique osculating paraboloid U. (U may degenerate into parabolic cyclinder or plane).

Remark: the coefficient of U are determined by 2nd-fundamental form of S at P.

↘ 2nd order Taylor coef.

For explicit surface $z = f(x, y)$. s.t. $f(0,0) = 0$. $\nabla f(0,0) = (0,0)$

(For arbitrary regular surface Φ of class C^2. $P \in \Phi$.

we just 1) place P at $(0,0,0)$ 2) Align the tangent at P with xy-plane

3). Orient the normal vector along the z-axis.).

$U: z = \frac{1}{2}(Ax^2 + 2Bxy + Cy^2)$ $A = \frac{\partial^2 f}{\partial x^2}$ $B = \frac{\partial^2 f}{\partial x \partial y}$ $C = \frac{\partial^2 f}{\partial y^2}$

degenerate case:

$A = B = C = 0$. plane $z = 0$.

$AC - B^2 = 0$. $(A^2 + B^2 + C^2 \neq 0)$. parabolic cylinder.

non-degenerate case

$AC - B^2 > 0$. elliptic paraboloid.

$AC - B^2$ hyperbolic paraboloid.

§12. Classification of Points on a Surface.

△ 对每一个点, osculating paraboloid 存在且唯一. 可以用其性质对曲面上的点进行分类.

		osculating paraboloid
elliptic point.	—	elliptical.
hyperbolic point.	—	hyperbolic
parabolic point.	—	parabolic cylinder (degenerates)
umbilical point.	—	plane (degenerates)

§13. Envelope of a Family of Surface.

<u>Def 4.21</u> Suppose $\{S_{\alpha,\beta}\}$ is a family of smooth surface depending on 1 or 2. parameters. A surface F is the envelope of the family if TFAS:

1. $\forall p \in F$. $\exists\ S_{\alpha,\beta} \in \{S_{\alpha,\beta}\}$. $S_{\alpha,\beta}$. tangent to F at point P.
2. $\forall S_{\alpha,\beta} \in \{S_{\alpha,\beta}\}$. $\exists p \in F$. $S_{\alpha,\beta}$ tangent to F at point P.
3. no surface in $\{S_{\alpha,\beta}\}$. has a region common with F.

To simplify, some auxiliary assumption:

$\forall p \in F$. F is the envelope, we can specify a region G_p., s.t.

(1) $\forall Q \in F$. Q near P. only one surface in $\{S_{\alpha,\beta}\}$. can be found having parameter belonging to G_p.

(2). $\vec{r}(u,v)$. - arbitrary smooth para. of F.; $a(u,v)$. $b(u,v)$ are parameter of surface (in $\{S_{\alpha,\beta}\}$, tangent to F at (u,v)). then $a(u,v)$, $b(u,v)$ are smooth func. of u, v.

<u>Thm 4.11.</u> (Envelope of a one-parameter family of surface).

Let $\{F_a\}$ be a family of smooth surfaces parametrized by a real parameter a, defined by implicit equ. $\varphi(x,y,z,a)=0$. φ - smooth of x, y, z, a.

If a smooth surface F is the envelope of the family, then $\forall (x,y,z)$ on F. $\exists$ a value of a. s.t.
$$\begin{cases} \varphi(x,y,z,a)=0 \\ \varphi_a(x,y,z,a)=0. \end{cases}$$
△ 联立消去 a. 解得包络面的隐式方程.

<u>Remark</u>: 1st equ. (x,y,z) lies on F_a.

2nd equ. F_a is tangent to envelop F. at (x,y,z)

<u>e.g.</u> A family of sphere: $(x-a)^2+y^2+z^2-R^2=0$.
$$\begin{cases} (x-a)^2+y^2+z^2-R^2=0 \\ -2(x-a)=0 \end{cases} \Rightarrow y^2+z^2=R^2.$$
注意这里不存在对x的限制. $x=a$. a是参数.

cyclinder of radius R centred along x-axis.

Thm 4.12. (Envelope of a two-parameter family of surface).

Let $\{G_{a,b}\}$. 2-parameter. smooth. surface. defined by $\varphi(x,y,z,a,b)=0$.

The envelope G of this family s.t. $\begin{cases} \varphi(x,y,z,a,b)=0 \\ \varphi_a(x,y,z,a,b)=0 \\ \varphi_b(x,y,z,a,b)=0 \end{cases}$

e.g. A family of sphere: $(x-a)^2+(y-b)^2+z^2=R^2$ R- fixed radius.

$\begin{cases} (x-a)^2+(y-b)^2+z^2=R^2 \\ -2(x-a)=0 \\ -2(y-b)=0 \end{cases} \Rightarrow z=\pm R.$ 2 planes.

§14. Envelope of a Family of Planes.

Suppose F. envelope of 1-parameter family plane $\{\Pi_a\}$.

Thm 4.13. If the 1-parameter family of planes. $\Pi_a: \varphi(x,y,z,a)=0$

The envelope F: $\begin{cases} \varphi(x,y,z,a)=0 \\ \varphi_a(x,y,z,a)=0 \end{cases}$

The envelope F is a developable surface, which is one of the following:

(1) cylindrical surface - $\{\Pi_a\}$. parallel to a fixed direction.

(2) conical surface - $\{\Pi_a\}$. pass through a fixed point.

(3) tangent surface to a space curve. - $\{\Pi_a\}$. tangent to a curve in space.

e.g. (1). cylindrical. $\Pi_a: x\cos a+y\sin a=1$, $\parallel$ z-axis F: $x^2+y^2=1$. - cylinder center along z-axis

(2) conical $\Pi_a: x\cos a+y\sin a+z\tan a=0$ through $(0,0,0)$

F: a cone, vertex at $(0,0,0)$

(3). tangent surface $\Pi_a: x\cos a+y\sin a+za=0$. tangent to helix.

F: tangent surface to the helix.

Chapter 5 First Quadratic Form of a surface.

Φ - regular surface. $\vec{r} = \vec{r}(u,v)$ - arbitrary regular surface.

$\vec{n}$ - unit normal vector at (u,v).

three quadratic form $d\vec{r}^{\,2}$, $-d\vec{r}\cdot d\vec{n}$, $d\vec{n}^{\,2}$

(1) $I = d\vec{r}^{\,2}$ - positive definite.

$d\vec{r}^{\,2} = 0 \Leftarrow d\vec{r} = \vec{r}_u du + \vec{r}_v dv = 0$. $\vec{r}_u \times \vec{r}_v \neq 0$. thus only if $du = dv = 0$.

denote $\vec{r}_u^{\,2} = E$. $\vec{r}_u \cdot \vec{r}_v = F$. $\vec{r}_v^{\,2} = G$. (coefficients of the 1st fundamental form)

△E.F.G本身会随参数而变化.

$I = d\vec{r}^{\,2} = (\vec{r}_u du + \vec{r}_v dv)^2 = E du^2 + 2F du dv + G dv^2$

↳ 1st fundamental form of the surface

§1. Length of a curve on a surface.

Def 5.1. Let Φ - simple surface. γ - curve. We say γ lies on Φ if every point of γ is also a point of Φ.

Let $P_0 \in \Phi \cap \gamma$. (suppose γ lies on Φ)

- Φ parametrised near P_0 as $\vec{r}_\Phi = \vec{r}_\Phi(u,v)$, with $P_0 := \vec{r}_\Phi(u_0, v_0)$.
- γ parametrised near P_0 as. $\vec{r}_\gamma = \vec{r}_\gamma(t)$. with $P_0 := \vec{r}_\gamma(t_0)$

For $|t-t_0| < \delta$. (sufficiently small). each $P(t)$. of γ lies in the parametrised neigh. of P_0 on Φ. Thus, $\exists!$ $u(t)$. $v(t)$ s.t. $\vec{r}_\gamma(t) := \vec{r}_\Phi(u(t). v(t))$

Def 5.2. The equation $\begin{cases} u = u(t) \\ v = v(t) \end{cases}$ is the equations of the curve γ on surface Φ.

Thm 5.1. (Regular curves on regular surface).

Φ - regular. with $\vec{r}_\Phi(u,v)$. γ - regular with $\vec{r}_\gamma(t)$.

Suppose in a neighborhood of $P \in \Psi$. s.t:

- $\vec{r}_\Phi$ is regular : $\vec{r}_u \times \vec{r}_v \neq 0$ for all (u,v) in a neighborhood of P.
- $\vec{r}_\gamma$ is regular : $\vec{r}_\gamma'(t) \neq 0$. for all t near t_0 where $\vec{r}_\gamma(t_0) = P$.

If γ lies on Φ. then locally $\vec{r}_\gamma(t) = \vec{r}_\Phi(u(t), v(t))$. and. $(u'(t))^2 + (v'(t))^2 \neq 0$.

Remark: this ensures the regularity of reparametrization. the curve $(u(t). v(t))$ in parameter domain is non-singular.

In above notation, the arc length of γ between points $P_0 = \vec{r}(u(t_0), v(t_0))$ and $P = \vec{r}(u(t), v(t))$ is:

$$s(t_0, t) = \int_{t_0}^{t} \left\| \frac{d\vec{r}}{dt} \right\| dt = \int_{t_0}^{t} \sqrt{E \cdot u'(t)^2 + 2F\, u'(t) v'(t) + G\, v'(t)^2}\, dt = \int_{\gamma(P_0, P)} |d\vec{r}(u,v)|\, dt$$

$$= \int_{\gamma(P_0, P)} \sqrt{I}.$$ △ 1st quadratic form 定义了曲面上的一个度量.

△要想求曲面上的曲线/弧长，则需要知道曲面的 E, F, G.

<u>Def 5.3.</u> $\sqrt{I}$ is the line element.

△ first quadratic form 不能唯一确定曲面；但一般地，∀2曲面，不存在一种参数化使其相等.

§2. Angle between curves on a surface

<u>Def 5.4.</u> A direction on a regular surface Φ, parametrised by $\vec{r}(u,v)$ is determined by a tangent vector in tangent plane at $P \in \Phi$: $d\vec{r} = \vec{r}_u\, du + \vec{r}_v\, dv$.

<u>Remark:</u> $(du : dv)$ – ratios of parameter changes (invariant under scaling).

$d\vec{r}$ – explicitly defines the direction.

<u>Def 5.5.</u> The angle θ between 2 direction $(du_1 : dv_1)$, $(du_2 : dv_2)$ at P is the angle between their tangent vectors:

$$d\vec{r}_1 = \vec{r}_u\, du_1 + \vec{r}_v\, dv_1 \qquad d\vec{r}_2 = \vec{r}_u\, du_2 + \vec{r}_v\, dv_2$$

using the 1st fundamental form.

$$d\vec{r}_1 \cdot d\vec{r}_2 = |d\vec{r}_1| \cdot |d\vec{r}_2| \cos\theta.$$

denote: $d\vec{r}_1 \cdot d\vec{r}_2 := I(d_1, d_2)$ $\quad d\vec{r}_1^{\,2} := I(d_1)$, $\quad d\vec{r}_2^{\,2} := I(d_2)$

$$\cos\theta = \frac{E\, du_1 du_2 + F(du_1 dv_2 + du_2 dv_1) + G\, dv_1 dv_2}{\sqrt{E\, du_1^2 + 2F du_1 dv_1 + G dv_1^2}\ \sqrt{E du_2^2 + 2F du_2 dv_2 + G dv_2^2}} = \frac{I(d_1, d_2)}{\sqrt{I(d_1)} \cdot \sqrt{I(d_2)}}$$

<u>Def 5.6.</u> Let Φ be regular, with $\vec{r}(u,v)$, γ - curve on Φ

- Geometrically: The curve γ has direction $(du : dv)$ at $P = \vec{r}(u,v)$, if its tangent vector at P parallel to $d\vec{r} = \vec{r}_u\, du + \vec{r}_v\, dv$.
- Parametrically: if γ, parametrised by $\vec{r}(t) = \vec{r}(u(t), v(t))$, its direction at P is $(u'(t) : v'(t))$. Since $\vec{r}'(t) = \vec{r}_u\, u'(t) + \vec{r}_v\, v'(t)$. △ $\begin{cases} u'(t) = u(t)\, dt \\ v'(t) = v(t)\, dt \end{cases}$ 2个定义一致.

Def5.7 (Angle between 2 curve on a surface).

$\gamma, \bar{\gamma}$ - regular curves on a surface. Φ parametrized by $\vec{r}(u,v)$, intersecting at a point $P = \vec{r}(u_0, v_0)$. The angle between γ and $\bar{\gamma}$ at P is the angle θ between their tangent vectors in tangent plane $T_P\Phi$.

If $\gamma - \vec{r}(u(t), v(t))$ $\quad \bar{\gamma} - \vec{r}(\bar{u}(T), \bar{v}(T))$.

there tangent vector : $T_1 = \vec{r}_u \cdot u'(t) + \vec{r}_v \cdot v'(t)$ $\quad T_2 = \vec{r}_u \bar{u}'(T) + \vec{r}_v \cdot \bar{v}'(T)$

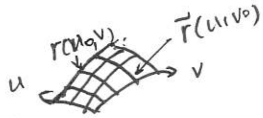
the angle : $\cos\theta = \dfrac{Eu'\bar{u}' + F(u'\bar{v}' + \bar{u}'v') + Gv'\bar{v}'}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2} \cdot \sqrt{E\bar{u}'^2 + 2F\bar{u}'\bar{v}' + G\bar{v}'^2}}$

(the angle not depend on parametrization of $\gamma . \bar{\gamma} . \Phi$).

§3. Coordinates Curve

Def5.8. Let Φ be a regular surface with $\vec{r}(u,v)$. $(u,v) \in$ open domain in $\mathbb{R}^2$.

- The u-curve (v-const. curve) is obtained by fixing $v = v_0$: $\vec{r}(u, v_0)$.
  (with tangent vector $\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$)
- The v-curve (u-const. curve) is obtained by fixing $u = u_0$: $\vec{r}(u_0, v)$
  (with tangent vector $\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$).



Property (1) the coordinate net is the grid formed by u- and v-curves.

(2) $\vec{r}_u . \vec{r}_v$. span. tangent plane at each plane.

(3). the curves are regular (no- self-intersections or cusps) if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$

(4). the coordinate net is orthogonal if $\vec{r}_u \cdot \vec{r}_v = 0$.

e.g. spherical coordinates $\vec{r}(\theta, \varphi)$.

θ-curve longitudes(经). φ- curve latitudes(纬). (球坐标. 经纬线相互⊥).

Thm 5.2. (Orthogonality of coordinate curve)

Φ - regular with $\vec{r}(u,v)$, 1st fundamental coef. E, F, G. TFAE:

(1) The coordinate curves are orthogonal.

(2). F vanishes identically. → $F(u,v) = \vec{r}_u \cdot \vec{r}_v = 0$.

(3). $\vec{r}_u, \vec{r}_v$ are orthogonal at every point. ↗ $\forall u,v \in D$ - parameter domain

(4). 1st fundamental form is diagonal: $I = E du^2 + G dv^2$.

Thm 5.3. (Existence of Orthogonal Parametrization)

Φ - regular of class C^k $(k \geq 1)$. $\forall P_0 = \vec{r}(u_0, v_0)$ on Φ.

$\mathcal{F}_1$ - $\forall$ smooth family of curves defined near P_0 by $\varphi(u,v) = \text{const.}$ with $\nabla\varphi|_{P_0} \neq \vec{0}$,

there exists: (1) $U_{P_0} \subset \Phi$.

(2). A regular C^k reparametrization $\vec{r}^*(s,t)$ of $\vec{r}|_U$ s.t.

i/ $\varphi = \text{const.}$ become s-curve (i.e. $t = \text{const.}$)

ii/ the orthogonal trajectories become t-curve (i.e. $s = \text{const.}$).

iii/ the new parametrization satisfy $F^* = 0$.

(the new 1st fundamental form: $I^* = E^* ds^2 + G^* dt^2$).

§4. Surface area.

F - smooth surface $G \subseteq F$. - region bounded by a finite number of p.w. smooth curves

$G \xrightarrow{\text{subregion}} \{g\} \xrightarrow[\text{to tangent plane}]{\text{projection}} \{\bar{g}\}$.

Def 5.9. The area of the region G on the surface F is defined as the limit of the sum of the areas $A(\bar{g})$, $A(G) = \lim\limits_{diam(g)\to 0} \sum A(\bar{g})$

Thm 5.4. (Surface area of a parametrized surface)

Let $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. - smooth para. of F.

Assume $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ (i.e. surface is regular). $G \subseteq F$. $\tilde{G}$ - corresponding region in the u-v plane. Then $A(G) = \iint\limits_{\tilde{G}} |\vec{r}_u \times \vec{r}_v| \, du\, dv$.

小区域 g. 投影在切平面上是平行四边形，面积为 $|\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$.

Remark: $|\vec{r}_u \times \vec{r}_v| = \vec{r}_u^{\,2} \vec{r}_v^{\,2} - (\vec{r}_u \cdot \vec{r}_v)^2 = EG - F^2$. $\rightarrow$ 面积仅由 1st-fundament form 决定

$$A(G) = \iint\limits_{\tilde{G}} \sqrt{EG - F^2} \, du\, dv.$$

§5. Conformal mapping.

Def 5.10. Let Φ_1 and Φ_2 be regular surfaces in $\mathbb{R}^3$. A mapping $f: \Phi_1 \to \Phi_2$ is conformal if:

(1) f is bijective. (2) f is smooth (3) f preserves angle between curves.

Remark of (3): $\forall \gamma_1, \gamma_2 \subseteq \Phi_1$, $\gamma_1 \cap \gamma_2 \ni p$. the angle θ between their tangent vectors at p. = the angle between $f(\gamma_1)$ and $f(\gamma_2)$ at $f(p)$.

Thm 5.5. Φ_1, Φ_2 - regular. $p_1 \in \Phi_1$. $p_2 \in \Phi_2$. with $\vec{r}_1(u,v)$ in U_{p_1}. $\vec{r}_2(u,v)$ in U_{p_2}. where $p_1 = \vec{r}_1(u_0, v_0)$ $p_2 = \vec{r}_2(u_0, v_0)$. in these parametrization.

$\frac{E_1}{E_2} = \frac{F_1}{F_2} = \frac{G_1}{G_2} = \lambda > 0$. then $f: \Phi_1 \to \Phi_2$ $f(\vec{r}_1(u,v)) = \vec{r}_2(u,v)$. is comformal (in U_{p_1}).

△若2参数化下，曲面的 1st fundament form 成比例(一个是另一个的常数倍. 可说明共形性.

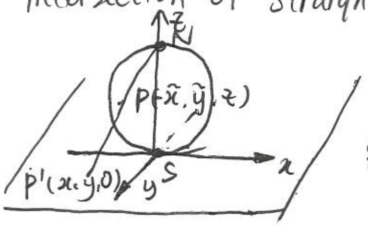
<u>Thm 5.6.</u> $\Phi_1 . \Phi_2$ - regular. $\forall P_1 \in \Phi_1$. $\forall P_2 \in \Phi_2$. Then there exists a conformal mapping of some U_{P_1} on Φ_1 onto some U_{P_2} on Φ_2.

△与5.5相比.仅说明在 $P_1 \cdot P_2$ 的邻域内这样的共形映射总是存在的.

§6. Stereographic Projection.

<u>Def 5.11.</u> Let S^2 be a sphere of radius R centered at the point $(0,0,R)$ in three-dimensional Euclidean space. The stereographic projection of S^2 onto its equatorial plane (xy-plane) is a mapping defined as:

1. Projection point: Let $N=(0,0,2R)$ - north pole; $S=(0,0,0)$ - south pole.
2. Mapping: $\forall p \in S$ and $p \neq N$. the stereographic projection p' is the intersection of straigh line pN and xy-plane.



3. formulas.

$$p': \begin{cases} x = \dfrac{2R\tilde{x}}{2R-\tilde{z}} \\ y = \dfrac{2R\tilde{y}}{2R-\tilde{z}} \end{cases} \qquad p: \begin{cases} \tilde{x} = \dfrac{4R^2 x}{x^2+y^2+4R^2} \\ \tilde{y} = \dfrac{4R^2 y}{x^2+y^2+4R^2} \\ \tilde{z} = \dfrac{2R(x^2+y^2)}{x^2+y^2+4R^2} \end{cases}$$

<u>Remark</u>: the stereographic projection is a way to "flatten" the sphere onto a plane.

equator (赤道). $\rightarrow$ circle with radius $2R$.

southern hemisphere $\rightarrow$ in the circle. northern $\rightarrow$ exterior

<u>Thm 5.7</u> The stereographic projection of a sphere S^2 of radius R. centre at $(0,0,R)$ onto the xy-plone is a conformal mapping. Specially:

- it preserve the angles between intersecting curves on the sphere.
(if $\gamma_1 . \gamma_2$ intersects at p. denote stereographic projection be f. $f(\gamma_1)$. $f(\gamma_2)$ intersects at $f(p)$. γ_1, γ_2 and $f(\gamma_1)$. $f(\gamma_2)$ has same angle)
- it's a bijective mapping between points on the sphere and points on the plane (except. N)
- 1st-quadratic form of the plane is a scalar multiple of the 1st quadratic form of the sphere ($ds^2_{plane} = dx^2+dy^2$ $\quad ds^2_{sphere} = d\tilde{x}^2+d\tilde{y}^2+d\tilde{z}^2 = \dfrac{16R^4(dx^2+dy^2)}{(x^2+y^2+4R^2)^2}$)

§7. Isometric Surfaces

Def 5.12. $\Phi_1 . \Phi_2$ is isometric if exists. bijective map $f: \Phi_1 \to \Phi_2$ that preserves the length of all curves on the surface.

($\forall \gamma: [a,b] \to \Phi_1$. the length of γ on Φ_1 = the length of $f\circ\gamma$ on Φ_2).

Remark1: length preservation. $\Rightarrow$ 1st fundamental form of 2 surface is same under f.

Remark2: isometric surface has same "intrinsic geometry"

Lemma 5.8. Let. Φ_1, Φ_2 - regular surfaces. $P_1 \in \Phi_1$. $P_2 \in \Phi_2$; $\vec{r}_1(u,v)$. $\vec{r}_2(u,v)$ be regular parametrization of the surfaces in neighborhoods of the points P_1. P_2. If the 1st-quadratic form of the surface (corr. to. $\vec{r}_1 . \vec{r}_2$). are identical. (1st-quadratic form 相等.

Then a mapping $f: U_{P_1} \to U_{P_2}$. (on Φ_1. Φ_2 respectively). which corresponding the points with same coordinates. u,v. is isometric.

△ f 将 $\vec{r}_1$ 和 $\vec{r}_2$ 中坐标同为 (u,v) 建立映射.

e.g. Φ_1: rectangular region $x \in (0,\frac{\pi}{2})$ $y \in (0,1)$ in xOy plane

Φ_2: region on the cylinder. $x^2+y^2=1$. $x>0$. $y>0$. $0<z<1$.

Φ_2 permits the parametrization $\begin{cases} x = \cos u \\ y = \sin u . \\ z = v . \end{cases}$ $\begin{matrix} u \in (0,\frac{\pi}{2}) \\ 0<v<1 . \end{matrix}$

$I = du^2 + dv^2$.

Lemma 5.9 (Parametrization under Isometry).

Let $\Phi_1 . \Phi_2$ be regular isometric surfaces. $\forall P_1 \in \Phi_1$. in U_{P_1} we have regular. $\vec{r}_1(u,v)$

Then there exists a regular $\vec{r}_2(u,v)$. of Φ_2 in some U_{P_2} (P_2 is the corr. point of P_1 under isometry) s.t.:

(1). The points on Φ_1 and Φ_2 under the isometry have the same coordinates (u,v).

(2). The 1st-fundamental forms Φ_1 and Φ_2 corr. to $\vec{r}_1$. and $\vec{r}_2$ are identical.

Remark: (1) $\Leftrightarrow \exists \varphi$. $\vec{r}_2(u,v) = \varphi(\vec{r}_1(u,v))$

(2) $\Leftrightarrow$ $E_1 = E_2$. $F_1 = F_2$. $G_1 = G_2$.

Thm 5.10. Let $\Phi_1 . \Phi_2$ regular. $\forall P_1 \in \Phi_1 . P_2 \in \Phi_2$.

For some U_{P_1} be mapped isometrically onto some U_{P_2} iff. there exists $\vec{r_1}, \vec{r_2}$ (regular parametrization) of $U_{P_1} . U_{P_2}$ s.t. the 1st-fundamental form of $\Phi_1 . \Phi_2$ w.r.t. $\vec{r_1} . \vec{r_2}$ are identical. (i.e. $E_1 = E_2$. $F_1 = F_2$. $G_1 = G_2$)

△是 5.8 和 5.9 的总结.

Coro 5.11. Let Φ_1 , Φ_2 be isometric surface then:

(1) Angles between corr. curves on Φ_1 and Φ_2 equals.

(2) Corr. regions on Φ_1 , Φ_2 have equal areas.

△保角. 保面积均由 1st-fundamental form 相等直接得出.

§8. Bending on surface.

Def 5.13. A bending of a surface is a continuous deformation of it under which lengths of curves on the surface remain invariant.

Chapter 6. Second Quadratic Form of a Surface.

Φ - regular surface. $\vec{r} = \vec{r}(u,v)$. regular parametrization of Φ.

$\vec{n}(u,v)$. unit normal vector to the surface at the point $P(u,v)$.

<u>Def 6.1.</u> The second fundamental form of the surface is a quadratic form defined as $II = -d\vec{r} \cdot d\vec{n} = L\,du^2 + 2M\,du\,dv + N\,dv^2$.

$L = -\vec{r}_u \cdot \vec{n}_u$. $\quad M = -\frac{1}{2}(\vec{r}_u \cdot \vec{n}_v + \vec{r}_v \cdot \vec{n}_u) \quad N = -\vec{r}_v \cdot \vec{n}_v$

§1. Coefficients of 2nd-quadratic form.

<u>Prop 6.1</u> The coefficients of 2nd-quadratic form can also be written as:

$L = \vec{r}_{uu} \cdot \vec{n}$. $\quad M = \vec{r}_{uv} \cdot \vec{n}$. $\quad N = \vec{r}_{vv} \cdot \vec{n}$.

Pf: since $d\vec{r} \cdot \vec{n} \equiv 0 \Rightarrow d(d\vec{r} \cdot \vec{n}) = d^2\vec{r} \cdot \vec{n} + d\vec{r} \cdot d\vec{n} = 0$.

$\Rightarrow II = d^2\vec{r} \cdot \vec{n} = \vec{r}_{uu} \cdot \vec{n}\,du^2 + 2(\vec{r}_{uv} \cdot \vec{n})\,du\,dv + \vec{r}_{vv} \cdot \vec{n}\,dv^2$.

<u>Prop 6.2.</u> Explicit formulas for coef:

$$L = \frac{(\vec{r}_{uu}, \vec{r}_u, \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|} \qquad \left(\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}, \quad \vec{r}_u \times \vec{r}_v = \sqrt{EG - F^2}\right).$$

$$M = \frac{(\vec{r}_{uv}, \vec{r}_u, \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|}$$

$$N = \frac{(\vec{r}_{vv}, \vec{r}_u, \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|}.$$

For explicit expression $z = z(x,y)$.

$$L = \frac{z_{xx}}{\sqrt{1+z_x^2+z_y^2}} \qquad M = \frac{z_{xy}}{\sqrt{1+z_x^2+z_y^2}}$$

$$N = \frac{z_{yy}}{\sqrt{1+z_x^2+z_y^2}}$$

The equation of the osculating paraboloid: $z = \frac{1}{2}(Lx^2 + 2Mxy + Ny^2)$.
(approximates the surface near a point. up to 2nd order).

△ 摆动抛物面, 由 2nd-quadratic form 决定, 衡量曲面的 "local behaviour"

△ 2nd quadratic form 衡量法向量的变化幅度. 给出了曲率的信息.

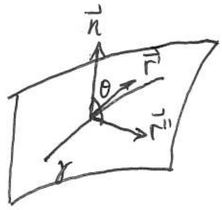
- $LN - M^2 > 0$ point is elliptic
  - < 0 point is hyperbolic.
  - $= 0$ point is parabolic or umbilic.

§2. Curvature of a Curve lying on a Surface.

Φ - regular with $\vec{r} = \vec{r}(u,v)$ - regular. γ be a regular curve on the surface that passes through the point $P(u,v)$ and has direction $(du:dv)$ at this point. $\vec{r} = \vec{r}(s)$. - natural of γ. $\vec{n}$ - the normal of the surface.
k - curvature of curve, θ - angle between principal normal to γ and normal to Φ.

$\vec{r}'' \cdot \vec{n} = k\cos\theta$. ($\vec{r}''$ directed along the principal normal to the curve)

$\vec{r}'' \cdot \vec{n} = (\vec{r}_{uu} \cdot n) u'^2 + 2(\vec{r}_{uv} \cdot \vec{n}) u'v' + (\vec{r}_{vv} \cdot \vec{n}) v'^2$



$$\Rightarrow k\cos\theta = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = \frac{\mathrm{II}}{\mathrm{I}}.$$

($k\cos\theta$. only depend on $P(u,v)$ and the direction of γ at P).

denote $k_0 = k\cos\theta$. (确定点,确定方向后,对任意曲线,k_0是常数.

Def 6.2. The quatity k_0 is normal curvatura of the surface in given direction $(du:dv)$. △对于将曲面与一个平面相交所得曲线的曲率.这个平面垂直于切平面.方向为$(du:dv)$.

△法曲率中的 $\vec{r}$ 是曲线和曲面参数方程的统一:

对曲面 $\vec{r} = \vec{r}(u,v)$; Φ上面的曲线可以用参数方程 $\begin{cases} u = u(s) \\ v = v(s) \end{cases}$ 表示.

因此γ的参数方程为 $\vec{r} = \vec{r}(u(s), v(s)) := \vec{r}(s)$.

γ的tangent vector: $\vec{r}'(s) = \frac{d\vec{r}}{ds} = \vec{r}_u \cdot \frac{du}{ds} + \vec{r}_v \frac{dv}{ds}$.

γ的principal normal: $\vec{r}''(s) = \frac{d\vec{r}'}{ds} = \vec{r}_{uu} \cdot \left(\frac{du}{ds}\right)^2 + 2\vec{r}_{uv} \cdot \left(\frac{du}{ds}\right) \cdot \left(\frac{dv}{ds}\right) + \vec{r}_{vv} \cdot \left(\frac{dv}{ds}\right)^2$

$+ \underbrace{\vec{r}_u \cdot \frac{d^2u}{ds^2} + \vec{r}_v \cdot \frac{d^2v}{ds^2}}$ → 在曲面切平面上.与曲面法向垂直,
在$\vec{r}'' \cdot \vec{n}$项中已相消.

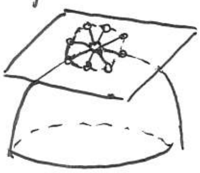
又因为 $|\vec{r}'(s)|^2 = E \cdot \left(\frac{du}{ds}\right)^2 + 2F\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) + G\left(\frac{dv}{ds}\right)^2 = 1. \Rightarrow \mathrm{I} = (ds)^2$.

有. $k\cos\theta = \frac{\mathrm{II}}{\mathrm{I}}$.

Thm 6.3. The normal curvatura of Φ at point $P(u,v)$ in direction $(du:dv)$ is equal to the normal curvature of the osculating paraboloid to Φ at P in the same direction.

△方向$(du:dv)$是切方向.要计算$(du:dv)$需要曲面上的曲线方程.

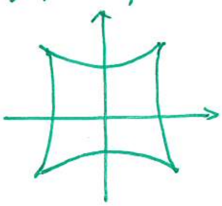
Def 6.3. Let $P(u,v)$ be a point on surface. k_0 be the normal curvature of the surface at P in direction $(du:dv)$. The indicatrix curvature (Dupin Indicatrix). is the geometric locus of points, by drawing from P in every direction $(du:dv)$. a segment of length. $\left|\frac{1}{k_0}\right|^{\frac{1}{2}}$.

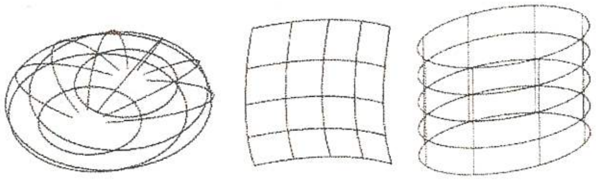


Thm 6.4. At point P on surface Φ. the indicatrix of curvature is:

- An ellipse — if P is elliptic point $(LN-M^2>0)$
- A pair of conjugate hyperbolas — if P is hyperbolic point $(LN-M^2<0)$
- A pair of parallel straight line — if P is parabolic point $(LN-M^2=0)$

△ conjugate hyperbolas.





corr. surface: (locally). ellipsoid-like $LN-M^2>0$ saddle-like $LN-M^2<0$ cylinder-like $LN-M^2=0$

Coro 6.5. Let Φ be a surface, $P(u,v)$ on Φ. Let U be the osculating paraboloid to Φ at P. Then the indicatrix of curvature of Φ at P equal to the indicatrix of curvature of U at P.

§3. Asymptotic Performance of Surfaces

<u>Def 6.4</u> A direction $(du:dv)$ on a regular surface at $P(u,v)$ is called an asymptotic direction if the normal curvature of the surface in this direction vanishes. (i.e. $L(du)^2 + 2M\,du\,dv + N(dv)^2 = 0$).

△求同时很解 du, dv.

<u>Remark</u>: elliptic point – no asymptotic directions. (II is definite)

hyperbolic point – two distinct asymptotic directions.

parabolic point – exactly one asymptotic direction (II is semidefinite)

umbilical point – every direction is asymptotic direction. (II, I fixed proportional)

<u>Def 6.5</u>. A curve on a surface is asymptotic curve if its tangent direction at every point is an asymptotic direction.

<u>Remark</u>: we have the diff. equation of asymptotic curve: $L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0$

<u>trivial e.g.</u> A straight line on the surface.

<u>Thm 6.6</u>. If γ is an asymptotic curve on surface S, then the osculating plane of γ is tangent to the surface at every point on γ.

这是2簇曲线，分别有切方向 $(du:dv) = (0:1)$ 和 $(1:0)$

<u>Thm 6.7</u>. The coordinate curves $u = \text{const}$ and $v = \text{const}$ are asymptotic iff $L = N = 0$.

Moreover, in some neighbourhood of hyperbolic point on the surface, it's always possible to find a parametrization, s.t. coordinates curves are asymptotic.

△在双曲点的邻域内，渐近方向总是存在且平缓变化。(即可以重新选取局部参数化 (u', v')，使 $L' = N' = 0$)

<u>Def 6.6</u>. $\forall P(u,v) \in \Phi$ – regular. $(du:dv)$ and $(\delta u:\delta v)$ are two directions on the surface at the point P, and g', g'' are straight lines passing through the point P, and having direction $(du:dv)$, $(\delta u:\delta v)$, respectively.

Then $(du:dv)$, $(\delta u:\delta v)$ are conjugate directions if the line g' and g'' are polar conjugate concerning the osculating paraboloid to the Φ at P.

<u>Remark</u>: "polar conjugate for osculating paraboloid" $\Longleftrightarrow L\,du\,\delta u + M(du\,\delta v + dv\,\delta u) + N\,dv\,\delta v = 0$.

<u>Remark</u>: The diameters of indicatrices curvature with direction $(du:dv)$ and $(\delta u:\delta v)$ are conjugate diameter.

Def 6.7 $\{\gamma_\alpha\}$. $\{\gamma_\beta\}$. two families of curves. on a surface. forming a net.
(that is, $\forall p \in \Phi$. $\exists! \tilde{\gamma}_\alpha \in \gamma_\alpha$. $\tilde{\gamma}_\beta \in \gamma_\beta$ passes through p). the net is **conjugate net** if the curves from different families have conjugate direction at each point.

Remark: if the coordinate net is conjugate net. $M = 0$ $\therefore$ conjugate direction $(du:0).(0:\delta v)$

Remark: Every asymptotic direction is self-conjugate.
i.e. $(du:dv)$ – asymptotic direction.
normal curvature $= 0 \Rightarrow Ldu^2 + 2Mdudv + Ndv^2 = 0$

Thm 6.8. $\forall p \in \Phi$. p is not an umbilical point. Φ can be parametrised s.t. the coordinate curves form a conjugate net.

△ 因此，一簇曲线可以是任意的（只要不是渐近的）. 总存在对应的另一簇曲线. 它们在 P 的邻域内可以被参数化成 "coordinate curve."

§4. Principal directions on a surface.

Def 6.8. The direction $(du:dv)$ on a surface is called a **principal direction** if the normal curvature of the surface in this direction attains an extremal value.

△ 几何角度. 这一方向与 indicatrix curvature 的轴对应

Remark: at each point, exactly 2 principal direction. they are (→ 除了 umbilical/spherical point.) orthogonal and conjugate.

$$I(d,\delta) = E\,du\,\delta u + F(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = 0 \quad \text{(orthogonality)}$$
$$II(d,\delta) = L\,du\,\delta u + M(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v = 0 \quad \text{(conjugacy)}$$

$\Longleftrightarrow$ a necessary and sufficient condition for $(du:dv)$ to be principal

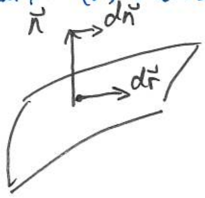
$$\begin{vmatrix} Edu + Fdv & Fdu + Gdv \\ Ldu + Mdv & Mdu + Ndv \end{vmatrix} = 0 \Longleftrightarrow \begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

Remark: principal direction not define in two case:
(1). umbilical point. $L = M = N = 0$, any direction is principal.
(2) Spherical point. indicatrix curvature is circle. $\frac{II}{I}$ proportional.
any direction is principal

Def 6.9. The normal curvatures corr. to the principal direction are principal curvature.
(也是法曲率的最大和最小值).

Thm 6.9 (Rodrigues's thm.). If (d) is a principal direction, then $d\vec{n} = -k\,d\vec{r}$.
k is the normal curvature of the surface in direction (d).
Conversely, if $d\vec{n} = \lambda\, d\vec{r}$ in the direction (d), then (d) is a principal direction.

Remark: (d) means $(du : dv)$.



$$d\vec{r} = \vec{r}_u\,du + \vec{r}_v\,dv.$$
$$d\vec{n} = \vec{n}_u\,du + \vec{n}_v\,dv$$

§5. Lines of Curvature.

Def 6.10. A curve on a surface is a line of curvature if its direction at every point aligns with a principle direction of the surface.

The D.E. of lines of curvature:

$$(EM - FL)\,du^2 + (EN - GL)\,du\,dv + (FN - GM)\,dv^2 = 0.$$

or.
$$\begin{vmatrix} dv^2 & -du\,dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

Remark: if the surface and the parametrization s.t.
coordinates curve $\Leftrightarrow$ lines of curvature, then $F=0$ and $M=0$.
△实际上，若 P 不是 spherical / umbilical (即 2个 principal direction 不相等).
在 U_p 上总存在使 coordinates curve 和 lines of curvature 等价的参数化.

Thm 6.10. If 2 surface intersects along some curve γ under a const. angle, and if this curve is line of curvature on one surface, then it will also be the line of curvature on the other surface.
△intersects under a const. angle $\Leftrightarrow$ $\vec{n}_1, \vec{n}_2$ 曲面法向量. $\vec{n}_1 \cdot \vec{n}_2 = \text{const}$. or $d(\vec{n}_1 \cdot \vec{n}_2) = 0$.

Coro 6.11. If a sphere (or a plane) intersects any surface at a const. angle then the intersection curve is a line of curvature.
△球面/平面上，任一曲线是 "lines of curvature".

§6. Mean and Gaussian curvature of a surface.

Thm 6.12. (Euler's formula for normal curvature).

Given a surface Φ and $p \in \Phi$, the normal curvature k_θ in any direction can be determined using principal curvatures k_1 and k_2 of Φ and angle θ ($\theta = \langle$ chosen direction, a principal direction $\rangle$).

Euler formula: $k_\theta = k_1 \cos^2\theta + k_2 \sin^2\theta$.

Thm 6.13. The principal curvature k_1, k_2 are the roots of the equation:

$$\begin{vmatrix} L - kE & M - kF \\ M - kF & N - kG \end{vmatrix} = 0.$$

△ 等价于求 $S = \mathrm{I}^{-1}\mathrm{II}$ 的特征值. $\frac{1}{(EG-F^2)}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} L & M \\ M & N \end{pmatrix}$. S-shape operator

Def. 6.11. The mean curvature H of a surface is $H = \frac{1}{2}(k_1 + k_2)$

Def 6.12. The Gaussian curvature K of a surface is $K = k_1 k_2$.

Formula: $H = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}$ $\quad K = \frac{LN - M^2}{EG - F^2}$.

Property (1) mean curvature (用 Euler formula 易证).

i) $\forall \theta$. $H = \frac{1}{2}(k_\theta + k_{\theta + \frac{\pi}{2}})$.

ii) $\forall \theta$ $H = \frac{1}{2\pi}\int_0^{2\pi} k_\theta \, d\theta$.

(2). Gaussian curvature: the sign classify the points.

$K > 0$ Elliptic point

$K < 0$ Hyperbolic point.

$K = 0$ $\begin{cases} H = 0. & \text{umbilical point.} \\ H \neq 0. & \text{parabolic point.} \end{cases}$

Def 6.13. (Spherical image).

$M \subseteq \Phi$. $\forall p \in M$. $\vec{n}(p)$ - unit normal vector. M' - spherical image (Gauss map):

$M' = \{ G(p) \text{ - tip of } \vec{n}(p) \} \subseteq S^2$ - unit sphere.

Thm 6.14. (Gauss theorem).

Let O be a point on a surface, and let G be a region on the surface. $O \in G$.

We have. $\dfrac{A(G')}{A(G)} \longrightarrow |K(O)|$ as "G shrinks to O".

Idea: G' (spherical image) $\vec{r} = \vec{n}(u,v)$. (unit. normal vector).

$$A(G') = \iint_G |\vec{n}_u \times \vec{n}_v| \, du\, dv \qquad \begin{cases} \vec{n}_u = -k_1 \vec{r}_u \\ \vec{n}_v = -k_2 \vec{r}_v \end{cases}$$

$$|\vec{n}_v \times \vec{n}_u| = |k_1 k_2| \, |\vec{r}_u \times \vec{r}_v| .$$

△ 特殊情况.(每个非脐点的邻域内可存在这样的参数系(u,v):).

r_u, r_v. 是彼此正交的主方向 $\Leftrightarrow$ 曲率线网是 u-曲线和 v-曲线网 $\Leftrightarrow$ $F = M = 0$.

$\Leftrightarrow$ 两个基本形式: $\begin{cases} \mathrm{I} = E(du)^2 + G(dv)^2 \\ \mathrm{II} = k_1 E (du)^2 + k_2 G (dv)^2 \end{cases}$ $\Rightarrow$ 主曲率为 $\begin{cases} k_1 = \dfrac{L}{E} \\ k_2 = \dfrac{N}{G} \end{cases}$

Chapter 7. Fundamental Equations of the Theory of Surface.

§.1. Gauss Formula.

Gaussian curvature.

w.r.t. II. $K = \dfrac{LN - M^2}{EG - F^2}$

w.r.t. I. $K = $.

<u>coro7.1.</u> Isometric surfaces have same K. at corr. points.

K is invariant under "bending".

<u>coro7.2.</u> Developable surface $K=0$. everywhere.

(locally isometric to a plane).

<u>coro7.3.</u> I and II of a surface do not independent.

§2. Natural Frame.

$\{r_u, r_v, n\}$. - basis. - 与参数有关、且一般来说不正交.

$$\begin{cases} \vec{r}_{uu} = T^1_{11}\vec{r}_u + T^2_{11}\vec{r}_v + \lambda_{11}\vec{n} \\ \vec{r}_{uv} = T^1_{12}\vec{r}_u + T^2_{12}\vec{r}_v + \lambda_{12}\vec{n} \\ \vec{r}_{vv} = T^1_{22}\vec{r}_u + T^2_{22}\vec{r}_v + \lambda_{22}\vec{n}. \\ \vec{n}_u = \alpha_{11}\vec{r}_u + \alpha_{12}\vec{r}_v + \alpha_{10}\vec{n} \\ \vec{n}_v = \alpha_{21}\vec{r}_u + \alpha_{22}\vec{r}_v + \alpha_{20}\vec{n}. \end{cases}$$

- coefficients. α. λ.

(1) $\alpha_{10} = \alpha_{20} = 0$. $\vec{n}$. unit length. $\vec{n}\cdot\vec{n}_u = \frac{1}{2}(\vec{n}^2)_u = 0$.

(2). $\alpha_{11} = \dfrac{-LG + MF}{EG - F^2}$ $\alpha_{21} = \dfrac{NF - MG}{EG - F^2}$

$\alpha_{12} = \dfrac{LF - ME}{EG - F^2}$ $\alpha_{22} = \dfrac{-NE + MF}{EG - F^2}$

$n_u \cdot r_u + n \cdot r_{uu} = \dfrac{\partial (n\cdot r_u)}{\partial u} = 0$. $\Rightarrow n_u \cdot r_u = -L$.

$n_u \cdot r_v + n \cdot r_{uv} = \dfrac{\partial (n \cdot r_v)}{\partial u} = 0 \Rightarrow n_u \cdot r_v = -M$.

(3) $\lambda_{11} = L$. $\lambda_{12} = M$. $\lambda_{13} = N$

- Christoffel Symbols.

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2} E_u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \frac{1}{2} E_v. \end{cases}$$

$$\begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F = \frac{1}{2} E_v \\ \Gamma_{22}^1 F + \Gamma_{12}^2 G = \frac{1}{2} G_u \end{cases}$$

$$\begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \frac{1}{2} G_u. \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \frac{1}{2} G_v. \end{cases}$$

let $\Delta = EG - F^2$.

$$\Rightarrow \begin{cases} \Gamma_{11}^1 = \frac{1}{2\Delta}(GE_u - 2FF_u + FE_v) \\ \Gamma_{11}^2 = \frac{1}{2\Delta}(2EF_u - EE_v - FE_u) \end{cases}$$

$$\begin{cases} \Gamma_{12}^1 = \frac{1}{2\Delta}(GE_v - FG_u) \\ \Gamma_{12}^2 = \frac{1}{2\Delta}(EG_u - FE_v). \end{cases}$$

$$\begin{cases} \Gamma_{22}^1 = \frac{1}{2\Delta}(2GF_v - GG_u - FG_v) \\ \Gamma_{22}^2 = \frac{1}{2\Delta}(EG_v - 2FF_v + FG_u). \end{cases}$$

If $I = E\,du^2 + G\,dv^2$ (i.e. $F = 0$).

$$\begin{cases} \Gamma_{11}^1 = \frac{1}{2} \cdot \frac{E_u}{E}. & \Gamma_{11}^2 = -\frac{1}{2} \cdot \frac{E_v}{G}. \\ \Gamma_{12}^1 = \frac{1}{2} \cdot \frac{E_v}{E} & \Gamma_{12}^2 = \frac{1}{2} \cdot \frac{G_u}{G}. \\ \Gamma_{22}^1 = -\frac{1}{2} \cdot \frac{G_u}{E} & \Gamma_{22}^2 = \frac{1}{2} \cdot \frac{G_v}{G}. \end{cases}$$

Def 7.1. Gauss - Peterson - Codazzi condition.

$E\,du^2 + 2F\,du\,dv + G\,dv^2$, $L\,du^2 + 2M\,du\,dv + N\,dv^2$. two arbitrary quadratic forms. the first is positive definite. the following 3 conditions:

(1) Gauss formula.

$$\frac{LN - M^2}{EG - F^2} = K(E, F, G)$$

(2). 1st Peterson - Codazzi formula

$$(EG - 2FF + GE)(L_v - M_u) - (EN - 2FM + GL)(E_v - F_u) + \begin{vmatrix} E & E_u & L \\ F & F_u & M \\ G & G_u & N \end{vmatrix} = 0$$

(3) 2nd Peterson - Codazzi formula.

$$(EG - 2FF + GE)(M_v - N_u) - (EN - 2FM + GL)(F_v - G_u) + \begin{vmatrix} E & E_v & L \\ F & F_v & M \\ G & G_v & N \end{vmatrix} = 0.$$

Remark: these conditions ensure the compatibility of I and II. for the existence of a surface.

Peterson-Codazzi equation. (simplified using Christoffel symbol relate. L. M. N)

$$\frac{\partial L}{\partial v} - \frac{\partial M}{\partial u} = L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11}$$

$$\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12}$$

Thm 7.4. (Bonnet's thm.).

Let I, II. be two arbitrary quadratic forms. I is p.d. if coef. of I and II satisfy the Gauss - Peterson - Codazzi conditions. then there exists a surface unique up to its position in space. I. II are its first and second fundmental forms.

Chapter 8. Intrinstic Geometry of Surface.

"property" – only depend on the length of the curves on the surface.
(for regular surface, depend on "I").

"aspects" – lengths of the curve, angle between curves, areas of region, K.

§1. Geodesic curvature of a curve on a surface.

Φ – regular surface. γ on Φ. $\forall p \in \gamma$, α – tangent plane to Φ at P.

$\bar{\gamma}$ – projecting a small neighborhood of P on γ ($U_p \cap \gamma$) onto to α. (a curve on α)

Def 8.1. The geodesic curvature of γ at P is the curvature of $\bar{\gamma}$ at P.

Thm 8.1. (Meusnier's thm).

Suppose κ – geodesic curvature of γ on S. and k_0 – normal curvature.

The total curvature $k = \sqrt{\kappa^2 + k_0^2}$. (注意和曲率的定义区别,受参数化影响).

Remark: "orthogonal decomposition" – tangential (geodesic) and normal components.

Def 8.2. curvature vector $\vec{k}(t) := \frac{d^2\vec{r}}{dt^2} = \vec{r}_{uu}\left(\frac{du}{dt}\right)^2 + 2\vec{r}_{uv}\left(\frac{du}{dt}\frac{dv}{dt}\right) + \vec{r}_{vv}\left(\frac{dv}{dt}\right)^2 +$
$+ \vec{r}_u\left(\frac{d^2u}{dt^2}\right) + \vec{r}_v\left(\frac{d^2v}{dt^2}\right)$.

Remark: If γ – parametrised with arc length. $\|\vec{k}(s)\| =$ curvature.

$\vec{k}(t) = \vec{k}_{tan} + \vec{k}_{norm}$. $\quad \vec{k}_{norm} = (\vec{k}\cdot\vec{n})\cdot\vec{n}$

$\kappa = \|\vec{k}_{tan}\|$. $\quad k_0 = \|\vec{k}_{norm}\|$. $\quad k = \|\vec{k}\| = \sqrt{\|\vec{k}_{tan}\|^2 + \|\vec{k}_{norm}\|^2}$

Remark: $\kappa = k\cos\vartheta \qquad \vartheta = \langle$principal normals of γ, and $\bar{\gamma}\rangle$.

conclusion:

i) $\vec{r} = \vec{r}(s)$ – natural para. $\quad \kappa = (\vec{n}, \vec{r}', \vec{r}'')$

ii) $\vec{r} = \vec{r}(t)$ – arbitrary para. $\quad \kappa = \frac{1}{|\vec{r}_t|^{3/2}}(\vec{n}, \vec{r}_t, \vec{r}_{tt})$.

iii) $\vec{r} = \vec{r}(u,v)$ – surface para. $\begin{cases} u = u(t) \\ v = v(t) \end{cases}$ – γ.

$\kappa = \frac{\sqrt{EG - F^2}}{(Eu'^2 + 2Fu'v' + Gv'^2)^{3/2}}(u''v' - v''u' + Av' - Bu')$.

where $\begin{cases} A = \Gamma^1_{11}u'^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}v'^2 \\ B = \Gamma^2_{11}u'^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}v'^2 \\ C = Lu'^2 + 2Mu'v' + Nv'^2 \end{cases}$

§2. Geodesic curves on a surface.

Def 8.3. A curve on a surface is **geodesic curve** if its geodesic curvature vanishes at each of its points.

Prop 8.2. (1) Two surface tangent along γ. if γ is geodesic on one surface, then it's geodesic on the other

(2) γ is geodesic, then the osculating plane of γ is perpendicular to the tangent plane of the surface. at every point (where the curvature of γ does not vanish)

Remark: (2), is because. the principal normal of γ $\Leftrightarrow$ surface normal

D.E. of geodesics: $u''v' - v''u' + Av' - Bu' = 0$

(where $A = \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2$).

$B = \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2$.

若令 $v = v(u)$.

$v'' - Av' + B = 0$.

注意这里要用链导. 求 $\frac{dv}{du}$. $\frac{d^2v}{du^2}$.

处理 u'', u' 不能直接消去.

Thm 8.3. A unique geodesic can be drawn in any direction through every point on a regular surface.

$\forall\ P(u_0, v_0)$. direction $(u_0' : v_0')$.

$\begin{cases} u'' + A = 0 \\ v'' + B = 0 \end{cases} \longrightarrow$ let $u = u(t)$ $v = v(t)$ be the sol.

s.t. $\begin{cases} u(t_0) = u_0. & u'(t_0) = u_0' \\ v(t_0) = v_0. & v'(t_0) = v_0' \end{cases}$

$\Rightarrow$ 测地线的 ODE 组. 也可用于测地线求解.

§3. Semigeodesic Parametrization.

- one family of coordinates curves consists of geodesics, the other family of coordinates curve orthogonal to them.

Φ - regular. γ - regular on Φ. $P \in \gamma$. $\vec{r} = \vec{r}(u,v)$ arbitrary regular para. near P.

$\gamma: \begin{cases} u = u(t) \\ v = v(t). \end{cases}$ $v'(t_0) \neq 0$, at P.

Procedure: 1) resolve. $t = t(v)$ 2) define a family of curves S: $u = u(t(v)) + \underset{\text{const.}}{c}$

3). $v = $ const. curves from S. → S 中的曲线基本平行于 γ.

$u = $ const. curves orthogonal to them.

4). $\tilde{\gamma}: \begin{cases} u = u_0. \\ v = v(t) \end{cases}$ $\forall t$. on $\tilde{\gamma}$. draw $\tilde{\gamma}_t$ - a geodesic. $\doteq$

$v = v(u,t)$. $t \to t_0$. s.t. $v'' - Av' + B = 0$ (v' w.r.t. u).

Procedure: (5). resolve $v = v(u,t)$. for t near (u_0, v_0, t_0)

$t = \varphi(u,v)$. $\varphi_u^2 + \varphi_v^2 \neq 0$.

(6). the surface can be parametrised near P s.t.

$$\begin{cases} \varphi(u,v) = \text{const}. & (\text{对应 } u\text{-const 簇，是测地线}) \\ \perp \varphi(u,v) = \text{const}. & (\text{对应 } v\text{-const 簇．是与前一簇正交的坐标曲线}. \end{cases}$$

(7). For this parametrization. $\langle \vec{r}_u, \vec{r}_v \rangle = 0$. $F = 0$.

Thm 8.4. If $v = \text{const}$. are geodesics. then:

(1). E is independent of v. (i.e. $E_v = 0$)

(2) the Christoffel symbol $\Gamma_{11}^2 = -\frac{1}{2}\frac{E_v}{G} = 0$.

Remark: (1) since $E_v = 0$. let $d\tilde{u} = \sqrt{E(u)}\, du$. $I = E\,du^2 + G\,dv^2 \Longrightarrow I = d\tilde{u}^2 + G\,dv^2$

(2) geometrically, the length of geodesic segment $v = \text{const}$. between $\tilde{u} = c_1, c_2$ (对应两条正交轨线)
is exactly $|c_1 - c_2|$

△ 半测地坐标网优点：i) 简便的 I.

ii) 有直接几何意义的参数 ($\tilde{u}$)

§4. Shortest Curves on a surface.

Def 8.4. A curve γ on surface, joining points P. Q. is shortest curve if any other curve on the surface connecting P. Q has length greater / equal to that of γ.

Thm 8.5. (Local Minimality of Geodesics).

If γ is a geodesics passing through a point P. and R. S on γ. sufficiently close to P. then the segment RS of γ is the shortest curve connecting R and S on the surface.

Def* A coordinate system (u,v) is semigeodesic if:

i) u-curves geodesics. v-curves orthogonal to them.

ii) $I = du^2 + G(u,v)\,dv^2$.

initial condition: $\begin{cases} G(0,v) = 1. \\ G_u(0,v) = 0. \end{cases}$

§5. Gauss - Bonnet Thm.

<u>Def 8.5</u>. A curve γ is piecewise regular if it consists of finitely many regular (C^1) segment $\gamma_1, \dots, \gamma_n$ joined at vertices (where tangent vectors many jump).

<u>Thm 8.6</u>. (Gauss - Bonnet for a regular region).

Let G be a region on a regular surface Φ that is homeomorphic to a disc, bounded by a piecewise regular curve $\gamma = \gamma_1 \cup \dots \cup \gamma_n$.

If: i/ γ is positively oriented (counterclockwise)

ii/ κ be geodesic curvature of γ.

iii/ $\alpha_1, \dots \alpha_n$ be the interior angles at vertices of γ.

iv/ K be Gaussian curvature of Φ.

Then:
$$\sum_{k=1}^{n} \int_{\gamma_k} \kappa \, ds + \sum_{k=1}^{n} (\pi - \alpha_k) = 2\pi - \iint_G K \, d\sigma.$$

Moreover, if γ is regular (no vertices). $\oint_\gamma \kappa \, ds = 2\pi - \iint_G K \, d\sigma$.

<u>Def 8.6</u>. A region on a surface is geodesic triangle if it's bounded by three geodesic segments and is homeomorphic to a closed disc.

let α, β, γ be interior angles (exterior + interior $= \pi$).

$$\alpha + \beta + \gamma = \pi + \iint_T K \, d\sigma$$

if $K > 0$. RHS $> \pi$. (e.g. sphere)

$K < 0$ RHS $< \pi$. (e.g. hyperbolic plane)

$K = 0$ RHS $= \pi$.

§6. Surfaces with constant Gaussian Curvature.

Φ - a surface with const. K. (u,v) is a semigeodesic system.

$\vec{r} = \vec{r}(u,v)$. semigeodesic para. in U_p.

$$K = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}.$$

let $y(u) = \sqrt{G(u,v)}$. (fixed v). $y''(u) + K\,y(u) = 0$.

$K>0$.	$I = du^2 + \cos^2(\sqrt{K}u)\,dv^2$.	e.g. sphere - spherical geometry
$K<0$.	$I = du^2 + \cosh^2(\sqrt{K}u)\,dv^2$	e.g. pseudosphere - hyperbolic geometry
$K=0$	$I = du^2 + dv^2$	e.g. plane. - Euclidean geometry.

§7. Locally Isometry Thm.

Thm 8.7. (Local isometry of constant curvature surface)

All surface with the same const. K. are locally isometric.

Let Φ_1. Φ_2. with same const K. $\forall P_1 \in \Phi_1$. $P_2 \in \Phi_2$. $\forall$ unit direction $l_1 \in T_{P_1}\Phi_1$. $l_2 \in T_{P_2}\Phi_2$.

$\exists\, U_{1P_1}$, U_{2P_2}. s.t. (1) U_1 is isometric to U_2 ($\exists$ diffeomorphism preserves I).

(2) The differential of the isometry maps l_1 to l_2.

Remark: (2) means $\Rightarrow$ for $\varphi: U_1 \to U_2$. isometry. ($ds_1^2 = du^2 + G(u)\,dv^2 = ds_2^2$).

$$d\varphi\left(\frac{\partial}{\partial u}\Big|_{P_1}\right) = \frac{\partial}{\partial u}\Big|_{P_2} \quad d\varphi\left(\frac{\partial}{\partial v}\Big|_{P_1}\right) = \frac{\partial}{\partial v}\Big|_{P_2}.$$

§8. Bertrand's thm.

<u>Def 8.7.</u> Two curves γ_1 and γ_2 on a surface are parallel if they maintain const. distance from each other their lengths.

<u>Lemma 8.8.</u> Consider two nearby geodesics. $\gamma_1(s)$ and $\gamma_2(s)$ on surface with const. K. separates by a small distance $d(s)$. Then. $\frac{d^2 d}{ds^2} + Kd = 0$.
(s is arc length along the geodesics)

<u>Thm 8.9</u> (Bertrand's thm. const K.).
Let Φ be a surface with const. Gaussian curvature K. then:

(1) $K>0$. $\forall 2$ geodesics. — intersect or locally parallel. $d(s)$ maining const. angular dist.

(2) $K=0$ geodesics are straight lines in locally coordinates.
parallel geodesics remain const. Euclidean dist.

(3). $K<0$. $\forall 2$ geodesics. — diverge exponentially or asymptotic.

<u>Remark:</u> (1). $d(s) = d_0 \cos(\sqrt{K} s)$. (period $\frac{2\pi}{\sqrt{K}}$).

(2) $d(s) = C_1 s + C_2$.

(3). $d(s) = d_0 \cosh(\sqrt{-K} s)$.

<u>Coro 8.10.</u> $K = $ const. (several special case in $\mathbb{R}^3$)

(1). on a sphere ($K>0$). all geodesics (great circle) intersects twice

(2). in a plane ($K=0$). parallel lines never meet.

(3). in a hyperbolic plane ($K<0$). exists geodesics. neither intersects nor parallel.
(i.e. ultraparallel)