Real Analysis 2024. Homework 7.

1. For f in $L^1[a,b]$, define $\|f\| = \int_a^b x^2|f(x)|dx$. Show that this is a norm on $L^1[a,b]$.

   Proof. **(1)** $\|f\| \geq 0$ and if $\|f\| = 0$ then $x^2\,|f(x)| = 0$ for a.e. $x \in [a,b]$ and, hence, $|f(x)| = 0$ for a.e. $x \in [a,b]$.

   (2)
$$\|\alpha f\| = \int_a^b x^2|\alpha f(x)|dx = |\alpha| \int_a^b x^2|f(x)|dx = |\alpha|\,\|f\|$$

   (3)
$$\|f+g\| = \int_a^b x^2|f(x)+g(x)|dx \leq \int_a^b x^2|f(x)|dx + \int_a^b x^2|g(x)|dx = \|f\|+\|g\|$$

   □

2. Prove that $L^p[0,1] \neq L^q[0,1]$ if $p \neq q$.

   Proof. Suppose $p > q$. Let $f(x) = \frac{1}{x^{1/p}}$. Then $f \notin L^p([0,1])$ and $f \in L^q([0,1])$.

   □

3. Assume that $\mu(E) < \infty$. Prove that
$$\|f\|_\infty = \lim_{p\to+\infty} \|f\|_p$$
   for any measurable on E function f.

   Proof. Assume first that $\|f\|_\infty < +\infty$. Since
$$\|f\|_p \leq \mu(E)^{1/p}\,\|f\|_\infty$$
   then
$$\lim_{p\to+\infty} \|f\|_p \leq \|f\|_\infty\,. \tag{1}$$
   Let $\varepsilon > 0$ then $|f| > \|f\|_\infty - \varepsilon$ on a set F of a positive measure. In this case,
$$\|f\|_p \geq \mu(F)^{1/p}(\|f\|_\infty - \varepsilon)$$

and letting $p \to +\infty$ we see that

$$\lim_{p\to+\infty} \|f\|_p \geq \|f\|_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and by estimate (1) we see that

$$\lim_{p\to+\infty} \|f\|_p = \|f\|_\infty .$$

Assume now that $\|f\|_\infty = +\infty$. Then for every $M > 0$ $|f| > M$ on a set of a positive measure. Analogously to previous consideration we see that

$$\lim_{p\to+\infty} \|f\|_p \geq M.$$

Since $M > 0$ is arbitrary we obtain

$$\lim_{p\to+\infty} \|f\|_p = +\infty = \|f\|_\infty .$$

□

4. Suppose $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(E)$. Show that $f = 0$ a.e. if and only if

$$\int_E f \cdot g = 0 \text{ for all } g \in L^q(E).$$

Proof. If $f = 0$ a.e. then $fg = 0$ a.e. and

$$\int_E f \cdot g = 0.$$

Assume now that $f \in L^p$

$$\int_E f \cdot g = 0 \text{ for all } g \in L^q(E).$$

Let $f(x) = |f|\, e^{i\theta(x)}$ and $g(x) = e^{-i\theta(x)}\, |f|^{p-1}$. Then

$$\|g\|_q^q = \int_E |g|^q = \int_E |f|^{q(p-1)} \int_E |f|^p = \|f\|_p^p < \infty$$

and $g \in L^q(E)$. At the same time

$$0 = \int_E f \cdot g = \int_E |f|^p$$

and $f = 0$ a.e.

□

5. (The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f. For $1 \leq p < \infty$, suppose there is a function g in $L^p(E)$ such that for all $n, |f_n| \leq g$ a.e. on E. Prove that $f_n \to f$ in $L^p(E)$.

Proof. Since $|f_n| \leq g$ a.e. then $|f_n - f| \leq 2g$ a.e. and and $|f_n - f|^p \to 2^p |f|^p$ then by dominated convergence theorem

$$\int_E |f_n - f|^p \, d\mu \to \int_E |f - f|^p \, d\mu = 0.$$

and $f_n \to f$ in $L^p(E)$. □