Mathematical analysis 2. Homework 30.

1. Find the pointwise limit function $f(x)$ of a sequence $\{f_n(x)\}$ on a set E, where

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad E = \mathbb{R}$$

2. Prove that a sequence $\{f_n(x)\}$ converges uniformly on E and find the limit function if

$$f_n(x) = n\sin(1/(nx)), E = [1; +\infty).$$

3. Check the uniform convergence of a sequence $\{f_n(x)\}$ and find the limit function on E if

$$f_n(x) = x^n - x^{n+2}, E = [0, 1].$$

1. i/ $x<0$. $f(x) = \lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{1}{nx+\frac{1}{nx}} = 0$

ii/ $x=0$ $f(x) = f_n(x) \equiv 0$

iii/ $x>0$ $f(x) = \lim\limits_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim\limits_{n\to\infty} \frac{1}{nx+\frac{1}{nx}} = 0$

thus we have the pointwise limit function $f(x) \equiv 0$ $(x \in \mathbb{R})$

2. proof: $f_n = n\cdot\sin(\frac{1}{nx}) \sim n\cdot\frac{1}{nx} = \frac{1}{x} := f(x)$

$$\left| n\sin\frac{1}{nx} - \frac{1}{x}\right| = \left| n\cdot\left[\frac{1}{nx} - \frac{1}{6}\cdot\left(\frac{1}{nx}\right)^3 + o\left(\frac{1}{n^3x^3}\right)\right) - \frac{1}{x}\right|$$

$$= \left|\frac{1}{6}\cdot\frac{1}{n^2x^3} + o\left(\frac{1}{n^2x^3}\right)\right| \to 0 \quad (n\to 0)$$

thus the limit function is $f(x) = \frac{1}{x}$, $x \in [1,+\infty)$

3. Solution: $f_n(x) = x^n(1-x^2)$

consider x^n. when $n\to\infty$, $x^n = \begin{cases} 1, & x=1 \\ 0, & x\in[0,1) \end{cases}$

$1-x^2 = 0$ when $x=1$.

thus. $f(x) \equiv 0$ when $x \in [0,1]$

the $\lim\limits_{n\to\infty} \sup|f_n(x) - f(x)| = \lim\limits_{n\to\infty} \sup[x^n - x^{n+2}]$

consider $f_n(x) = x^n - x^{n+2}$ $f_n'(x) = (n-(n+2)\cdot x^2)x^{n-1}$

$f_n(x)_{max} = f_n\left(\sqrt{\frac{n}{n+n}}\right)$

thus $\lim\limits_{n\to\infty} \sup [x^n - x^{n+2}] = \lim\limits_{n\to\infty} (\sqrt{\frac{n}{n+2}})^n - (\sqrt{\frac{n}{n+2}})^{n+2} = 0.$

by the Lemma, this is equivalent to $f_n \rightrightarrows f$ on $[0,1]$.

where $f(x) \equiv 0$ on $[0,1]$.

Mathematical analysis 2. Homework 31.

1. Check the uniform convergence of a sequence $\{f_n(x)\}$ and find the limit function on E_1 and E_2 if
$$f_n(x) = \sqrt{n}(\sqrt{1+nx} - \sqrt{nx}), \quad E_1 = (0,1), \quad E_2 = (1,+\infty).$$

2. Check the uniform convergence of a sequence $\{f_n(x)\}$ and find the limit function on E_1 and E_2 if
$$f_n(x) = \arcsin \frac{x^n}{1+x^n}, \quad E_1 = (0,1), \quad E_2 = (0,a), \ a < 1.$$

3. Check the uniform convergence of a sequence $\{f_n(x)\}$ and find the limit function on E_1 and E_2 if
$$f_n(x) = \frac{1}{x^2}\sqrt{1+\frac{x}{n}}, \quad E_1 = (0,1), \quad E_2 = (1,+\infty).$$

4. Let f be continuous on $\mathbb{R}$ and $a < b$. Prove that a sequence
$$f_n(x) = \sum_{k=1}^{n-1} \frac{1}{n} f\left(x + \frac{k}{n}\right)$$
converges uniformly on $[a,b]$.

1. $f_n(x) = n\cdot\left(\sqrt{\frac{1}{n}+x} - \sqrt{x}\right) = \dfrac{1}{\sqrt{x} + \sqrt{x+\frac{1}{n}}}$

① $x \in E_2$

Let $f(x) = \dfrac{1}{2\sqrt{x}}$

$$|f_n(x) - f(x)| = \left|\frac{\sqrt{x+\frac{1}{n}} - \sqrt{x}}{2\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x+\frac{1}{n}}\right)}\right| = \left|\frac{1}{n}\cdot\frac{1}{2\sqrt{x}\cdot\left(\sqrt{x}+\sqrt{x+\frac{1}{n}}\right)^2}\right| < \left|\frac{1}{n}\cdot\frac{1}{8x^{\frac{3}{2}}}\right|$$

$|f_n(x) - f(x)| < \frac{1}{n} \to 0$. (Assertion 4. in Lemma 1.3)

i.e $\forall \varepsilon > 0$. $\exists N = [\frac{1}{\varepsilon}]$. $\forall n > N$. $\forall x \in E_2$. $|f_n(x) - f(x)| < \frac{1}{n} < \frac{1}{N} \le \varepsilon$

$f_n(x) \rightrightarrows f(x)$ on E_2.

② $x \in E_1$. $f_n(x)$ is not pointwise convergent since

when $x \to 0$. $\sqrt{1+nx} - \sqrt{nx} \to 1$.

then $f_n(x) \to \infty$.

$f_n(x)$ is not uniformly converges on (0.1)

2、① $E_1 = (0,1)$. $x^n \to 0\ (n\to\infty)$.

$f = \arcsin 0 = 2\pi k$. $k \in \mathbb{Z}$. w.l.g. let $f=0$.

$$|f_n(x) - f(x)| = \left| \arcsin \frac{x^n}{1+x^n} \right| = \left| \frac{x^n}{1+x^n} + o\left(\frac{x^n}{1+x^n}\right)\right| \to 0.\ (n\to 0)$$

$f_n(x) \rightrightarrows f(x) \equiv 0$. on $(0,1)$.

② $E_2 = (0,a)$

$$f(x) = \begin{cases} 0, & x \in (0,1) \\ \frac{\pi}{6}, & x=1 \\ \frac{\pi}{2}. & x \in (1,a) \end{cases}$$

Check the uniform convergence when $x\in(1,a)$. ($x\in(0,1)$ has checked, $x=1$ trivial)

$$|f_n(x) - f(x)| = \left| \arcsin \frac{x^n}{1+x^n} - \frac{\pi}{2}\right|$$

$$\frac{x^n}{1+x^n} \to 1\ (n\to\infty). \qquad \left|\arcsin\frac{x^n}{1+x^n} - \frac{\pi}{2}\right| \to 0.$$

thus. $f_n \rightrightarrows f$, $f(x) = \begin{cases} 0, & x\in(0,1) \\ \frac{\pi}{6}. & x=1 \\ \frac{\pi}{2}, & x\in(1,a) \end{cases}$

3. ① $x \in E_2 = [1,+\infty)$.

Let $f(x) = \frac{1}{x^{\frac{3}{2}}}$

$$|f_n(x) - f(x)| = \left| \frac{1}{x^2}\sqrt{1+\frac{x}{n}} - \frac{1}{x^2}\cdot\sqrt{x}\right| < \frac{\sqrt{x}}{x^{\frac{3}{2}}}\left|\sqrt{\frac{1}{x}+\frac{1}{n}} - 1\right| < \left|\sqrt{1+\frac{1}{n}} - 1\right|$$

Let $\varepsilon_n = \sqrt{1+\frac{1}{n}} - 1$. $\varepsilon_n \to 0\ (n\to 0)$. by Lemma 1.3. $f_n \rightrightarrows f = \frac{1}{x^{\frac{3}{2}}}$ when $x\in E_2$

② $x\in E_1$ f_n is not uniform convergent

since $f_n > \frac{1}{x^2}$ which is unbound on $[0,1)$

4. $f\in C(\bar{\mathbb{R}}) \Rightarrow f\in R(\bar{\mathbb{R}})$ Let $g(x) = \int_x^{x+1} f(t)\,dt$.

Let partition $\{t_k\}_{k=0}^{n}$ $t_0 = x$ $t_n = x+1$ $t_k = x+\frac{k}{n}$. $\xi_k = f(x_k)$.

$$g(x) = \sum_{k=0}^{n-1} f(\xi_k)\Delta x = \frac{1}{n}\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right)$$

$$|f_n(x) - g(x)| = \frac{1}{n}\left|\sum_{k=0}^{n-1} f\left(x+\frac{k}{n}\right) - \sum_{k=1}^{n-1} f\left(x+\frac{k}{n}\right)\right| = \frac{1}{n}|f(x)|$$

$f(x)$ is bounded on $[a,b]$ (by continuity). $|f(x)|\le M$. Let $\varepsilon_n = \left\{\frac{M}{n}\right\} \to 0\ (n\to 0)$

By Lemma 1.3. $f_n \rightrightarrows g = \int_x^{x+1} f$.

Mathematical analysis 2. Homework 32.

1. Prove that $f_n(x) = \frac{1}{\sqrt{n}} \sin nx$ is uniformly convergent on $\mathbb{R}$, while a sequence $\{f_n'\}$ is divergent for every $x \in \mathbb{R}$.

2. Let $f \in C^1[a,b]$ and $f_n(x) = n(f(x+1/n) - f(x))$. Prove that $f_n \rightrightarrows f'$ on any segment $[a_1, b_1]$ such that $a < a_1 < b_1 < b$.

3. Check the uniform convergence of a sequence $\{f_n(x)\}$ on E_1, E_2 and E_3 if

$$\frac{1}{(nx-1)^2(x-n)^2+1}, \quad E_1 = [0,1],\ E_2 = [1,2],\ E_3 = [2, +\infty).$$

4. Recall that $\mathbb{Q}$ is countable and thus there exists a bijection $r : \mathbb{N} \to \mathbb{Q}$. Define the sequence $\{r_n\}$ by letting $r_n = r(n)$. Now define $f_n : \mathbb{R} \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} 1, & x \in \{r_1, r_2, \dots, r_n\}; \\ 0, & \text{otherwise.} \end{cases}$$

- Find the pointwise limit $f : \mathbb{R} \to \mathbb{R}$ of the sequence $\{f_n\}$.
- Is f_n Riemann integrable? Explain.
- Is f Riemann integrable? Explain.

1. (1). let $f(x) \equiv 0$ on $\mathbb{R}$.

$\left| f_n(x) - f(x) \right| = \left| \frac{1}{\sqrt{n}} \sin nx \right| \leq \frac{1}{\sqrt{n}}$. let $\varepsilon_n = \frac{1}{\sqrt{n}}$. $\varepsilon_n \to 0 \ (n \to 0)$

By Lemma 1.3. $f_n(x) \rightrightarrows 0$.

(2) $f_n'(x) = \frac{1}{\sqrt{n}} \cdot n \cos nx = \sqrt{n} \cos nx$. has no limit when $x \neq \frac{\pi}{2} + \pi k \ (k \in \mathbb{Z})$

$f_n'(x)$ diverges

2. $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}} = f'(x)$

$f_n(x)$ is pointwise convergent. on $[a_1, b_1] \subset [a,b]$.

w.l.g $n<m$

$\left| f_n(x) - f_m(x) \right| \leq \left| n f(x+\frac{1}{n}) - n f(x+\frac{1}{m}) + n f(x+\frac{1}{m}) - m f(x+\frac{1}{m}) + m f(x) - n f(x) \right|$

$\leq \left| (1-\frac{n}{m}) \dfrac{f(x+\frac{1}{n}) - f(x+\frac{1}{m})}{\frac{1}{n} - \frac{1}{m}} + (\frac{n}{m} - 1) \dfrac{f(x+\frac{1}{m}) - f(x)}{\frac{1}{m}} \right| \leq \left| \dfrac{f(x+\frac{1}{n}) - f(x+\frac{1}{m})}{\frac{1}{n} - \frac{1}{m}} - \dfrac{f(x+\frac{1}{m}) - f(x)}{\frac{1}{m}} \right|$

$= \left| f'(\xi_1) - f'(\xi_2) \right| \quad \xi_1 \in [x+\frac{1}{m}, x+\frac{1}{n}] \quad \xi_2 \in [x, x+\frac{1}{m}]$

$\forall x \in [a_1, b_1] \subset [a,b]$. $\forall \xi_1, \xi_2 \in [a,b]$.

Since $f' \in [a,b]$ $\forall \varepsilon > 0$ $\exists \delta > 0$. s.t. $| f(x) - f(x_0) | < \delta$. for any $|x - x_0| < \delta$

Let $\delta=1$. $\exists N\in\mathbb{N}$. s.t $\forall n,m>N$. $\forall x\in E$. $|\xi_1-\xi_2|<1$

$|f_n(x)-f_m(x)|<|f'(\xi_1)-f'(\xi_2)|<\varepsilon$.

By Cauchy's criterion $f_n(x)$ uniformly conv.

Since the limit is unique. $f_n \rightrightarrows f'$ on $\forall [a_1,b_1]\subset[a,b]$

3. $f_n=\dfrac{1}{(nx-1)^2(x-n)^2+1}$

① $E_2=[1,2]$. $\lim\limits_{n\to\infty} f_n=0$.

$$|f_n(x)-0|=\left|\frac{1}{(nx-1)^2(x-n)^2+1}\right|\le\frac{1}{(n-1)^2(n-2)^2-1}:=\varepsilon_n\to 0\ (n\to 0)$$

By Lemma 1.3. $f_n\rightrightarrows 0$ on $[1,2]$.

② E_1, E_3. f_n has no limit when $n\to\infty$.

f_n is not uniformly convergent on $[0,1]$ or $[2,+\infty)$.

4. (1) $\lim\limits_{n\to\infty} f_n(x)=\begin{cases}1. & x=\{r_n\}\\ 0, & \text{otherwise}\end{cases}\Rightarrow f=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$

(3) f is not integrable. (i.e. $D(x)$ is not integrable

For some partition, we can let $\xi_k=r_i$ then $\sigma(f_n,T,\xi)=b-a$ on $\forall[a,b]\subset\mathbb{R}$

also, we can let $\xi_k=p_i$, $p_i\in[x_k,x_{k+1}]\setminus\{r_n\}$, then $\sigma=0$.

the Riemann sums are different when ξ_k are different. f is not Riemann Integrable

(2) For any fixed n_0, Let $\max\{r(n)\}=q$, $1\le n\le n_0$.

thus we have $f_{n_0}(x)=\begin{cases}D(x) & , x\in[0,q]\\ 0 & , x>q.\end{cases}$

thus for any $f_{n_0}(x)$ is not integrable. i.e. f_n is not integrable

Mathematical analysis 2. Homework 33.

1. Let $f_n(x) = nx/(nx+1)$ for $n \in \mathbb{N}$ and $x \in [a,1]$ where $0 < a < 1$.
   - Prove directly that the sequence (f_n) is uniformly Cauchy.
   - If f is the uniform limit of (f_n), find $\int_a^1 f$ without computing f.

1.(1) $\forall \varepsilon > 0$. $\exists N = [\frac{1}{\varepsilon a^2}]$, for any $m, n > N$. $\exists x \in [a,1]$,

$$|f_n(x) - f_m(x)| = \left| \frac{nx}{nx+1} - \frac{mx}{mx+1} \right| = \left| \frac{nx - mx}{(nx+1)(mx+1)} \right| \leq \left| \frac{n-m}{nm \cdot a^2} \right|$$

$$\leq \left| \frac{n}{nma^2} \right| \leq \left| \frac{1}{ma^2} \right| < \varepsilon. \quad \text{thus } f_n(x) \text{ is uniformly convergent.}$$

(2) By the thm. Since $f_n \rightrightarrows f$ on $[a,1]$. $f \in R[a,1]$.

Thus we have $\int_a^1 f = \lim_{n\to\infty} \int_a^1 f_n = \lim_{n\to\infty} \int_a^1 \frac{nx}{nx+1}\,dx = \lim_{n\to\infty} x - \frac{\ln(nx+1)}{n} \Big|_a^1 = 1-a$

2. Consider the sequence of functions f_n on $A = [0, +\infty)$ defined as follows:

$$f_n(x) = \begin{cases} 1/n, & 0 \leq x \leq n^2, \\ 0, & x > n^2. \end{cases}$$

- Prove that f_n converges uniformly to $f = 0$ on A;
- For each fixed $n \in \mathbb{N}$ find the improper integral

$$I_n = \int_0^\infty f_n$$

and show that $\lim I_n = +\infty$.
- The results above seem to contradict Theorems 1.10 and 1.11. Explain why there is no contradiction.

(1) proof: $x > n^2$ is trivial, now consider $0 \leq x \leq n^2$

$|f_n(x) - f(x)| = |\frac{1}{n}| := \varepsilon_n$. $\varepsilon_n \to 0$ when $(n \to 0)$.

thus we have $f_n \rightrightarrows f$.

(2) $I_n = \int_0^\infty f_n = \int_{n^2}^\infty 0\,dx + \int_0^{n^2} \frac{1}{n}\,dx = \frac{1}{n} \cdot x \Big|_0^{n^2} = n$

$\lim_{n\to\infty} n = +\infty$

(3)
1.10: The condition "$\{f_n\}$ be a sequence of Riemann integrable function $[a,b]$" not satisfied.

We consider the unbound interval $[0, +\infty)$.

1.11: For any fixed n. the condition $f_n \in C[a,b]$. not satisfied.

$x = n^2$ is a point of discontinuity.

3. Find a set of convergence (absolute and conditional) of functional series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(\ln x)^k}.$$

Solution: Let $S_n(x) = \sum_{k=1}^{n} \frac{(-1)^k}{(\ln x)^k} = \frac{-\frac{1}{\ln x}\left(1-\left(\frac{-1}{\ln x}\right)^n\right)}{1+\frac{1}{\ln x}} = \frac{\left(\frac{-1}{\ln x}\right)^n - 1}{\ln x + 1}$

Domain $x>0$ and $x \neq 1$

First check the absolute convergence

$\text{Let } S_n'(x) = \sum_{k=1}^{n} \left|\frac{1}{\ln x}\right|^k = \frac{\left|\frac{1}{\ln x}\right|\left(1-\left|\frac{1}{\ln x}\right|^n\right)}{1-\left|\frac{1}{\ln x}\right|} = \frac{1-\left|\frac{1}{\ln^n x}\right|}{|\ln x|-1}$

$x \in (0, \frac{1}{e})$. $\lim_{n\to\infty} S_n'(x) = -\frac{1}{\ln x+1}$ $\qquad x = e$ or $\frac{1}{e}$. $S_n(x) = n$. div.

$x \in (\frac{1}{e}, 1) \cup (1, e)$. $\lim_{n\to\infty} S_n'(x) \to +\infty$

$x \in (e, +\infty)$. $\lim_{n\to\infty} S_n'(x) = \frac{1}{\ln x - 1}$

thus. the series absolutely conv. to $-\frac{1}{\ln x+1}$ on $(0, \frac{1}{e})$

abs. conv. to $\frac{1}{\ln x-1}$ on $(e, +\infty)$

Second check the conditional convergence. we just need to consider $x \in (\frac{1}{e}, 1) \cup (1, e)$. and $x = e$ or $\frac{1}{e}$.

$S_n(x) = \frac{\left(\frac{-1}{\ln x}\right)^n - 1}{\ln x + 1}$ $\qquad \lim_{n\to\infty} S_n(x)$ not exists

$x = \frac{1}{e}$, $S_n(x) = \sum_{k=1}^{n} 1 = n \to$ div. ∞.

$x = 1$ $\qquad S_n(x) = \sum_{k=1}^{n} (-1)^n$ $\qquad \lim_{n\to\infty} S_n(x)$ not exists.

Thus. the absolutely conv. set is $(0, \frac{1}{e}) \cup (e, +\infty)$

4. Prove that a series

$$\sum_{k=1}^{\infty} \frac{kx}{1+k^3x^3}$$

doesn't converge uniformly on $(0, 1)$ using necessary condition for the uniform convergence.

Proof: Consider $u_k = \frac{kx}{1+k^3x^3}$

$\exists\, \varepsilon^* = \frac{1}{3}$. For any $k \in \mathbb{N}$. $\exists\, x = \frac{1}{k}$. s.t. $\left|u_k(\frac{1}{k}) - 0\right| = \left|\frac{1}{1+1} - 0\right| = \frac{1}{2} > \frac{1}{3}$.

Thus, $u_k \not\rightrightarrows 0$.

By necessary Condition of uniform convergence. the series not uni. conv. on $(0,1)$.

5. Prove that a series

$$\sum_{k=1}^{\infty} \frac{1}{n} \ln \frac{n+x}{n}$$

doesn't converge uniformly on $(1, +\infty)$ using Cauchy criteria.

$\exists \varepsilon_0 = \frac{\ln \frac{3}{2}}{2}$ $\forall N \in \mathbb{N}$, s.t. $\exists n > N$, $m = 2n$, $x = n$

$$|S_m - S_n| = \left| \sum_{k=n+1}^{2n} \frac{1}{k} \ln \frac{k+n}{k} \right| \geq \left| \ln \frac{3}{2} \sum_{k=n+1}^{2n} \frac{1}{k} \right| \geq \left| \ln \frac{3}{2} \cdot n \cdot \frac{1}{2n} \right| = \frac{\ln \frac{3}{2}}{2}$$

Thus. by Cauchy's Criteria, the series does not converges uniformly on $(1, +\infty)$

Mathematical analysis 2. Homework 34.

1. Prove that

$$\int_1^2 \left(\sum_{n=1}^{\infty} n e^{-nx} \right) = \frac{e}{e^2-1}.$$

1. Pf: Since $k e^{-kx} \in R[1,2]$. For any $k \in \mathbb{N}$.

We have

$$|u_k(x)| = \left| \frac{k}{e^{kx}} \right| \leq \frac{k}{e^k} := M_k. \quad x \in [1,2].$$

$$\lim_{k\to\infty} \frac{\frac{k+1}{e^{k+1}}}{\frac{k}{e^k}} = \lim_{n\to\infty} \frac{k+1}{k} \cdot \frac{1}{e} = \frac{1}{e} < 1. \quad \sum_{k=1}^{\infty} M_k \text{ conv.}$$

By the M-test. $\sum_{n=1}^{\infty} n e^{-nx}$ conv.

thus $\int_1^2 \sum_{n=1}^{\infty} n e^{-nx} dx = \sum_{n=1}^{\infty} \int_1^2 n e^{-nx} dx = \sum_{n=1}^{\infty} -e^{-nx} \Big|_1^2 = \sum_{n=1}^{\infty} e^{-n} - e^{-2n}$

$$= \sum_{n=1}^{\infty} e^{-n} - \sum_{n=1}^{\infty} e^{-2n} = \lim_{n\to\infty} \frac{e^{-1}(1-e^{-n})}{1-e} - \lim_{n\to\infty} \frac{e^{-2}(1-e^{-2n})}{1-e^{-2}}$$

$$= \frac{1}{e-1} - \frac{1}{e^2-1} = \frac{e}{e^2-1}$$

2. Let $\sum_{k=1}^{\infty} |a_k| < \infty$. Prove that

$$\sum_{k=1}^{\infty} a_k \sin(kx)$$

converges uniformly on $\mathbb{R}$.

Pf: Denote $u_k(x) = a_k \sin kx$

$$|u_k(x)| \leq |a_k| \cdot |\sin kx| \leq |a_k|. \quad \sum_{k=1}^{\infty} |a_k| < \infty$$

By Weierstrass M-test. the series uniformly conv.

3. Check the uniform convergence of a series on $[0, +\infty)$

$$\sum_{n=1}^{\infty} \left(\sin \frac{\sqrt{x}}{x+n} \right)^3$$

1° $x=0$. The series equals to 0.

2° $x>0$

since $\frac{\sqrt{x}}{x+n} \to 0$ when $n \to 0$ (For any fixed x).

thus we have $\sin \frac{\sqrt{x}}{x+n} \leq \frac{\sqrt{x}}{x+n} = \frac{1}{\sqrt{x} + \frac{n}{\sqrt{x}}} \leq \frac{1}{2\sqrt{n}}$.

i.e. $\left(\sin \frac{\sqrt{x}}{x+n} \right)^3 \leq \frac{1}{8 n^{\frac{3}{2}}} := M_n. \quad \sum_{n=1}^{\infty} M_n < +\infty$

By Weierstarss M-test. the series conv.

4. Check the uniform convergence of a series on E_1 and E_2

$$\sum_{n=1}^{\infty} e^{-n\left(x^2+2\sin x\right)}, \quad E_1=(0;1], \quad E_2=[1;+\infty)$$

Pf: 1° $x \in E_2$. $e^{-n(x^2+2\sin x)} \le e^{-n} := M_n$.

$\lim_{n\to\infty} \frac{e^{-(n+1)}}{e^{-n}} = \frac{1}{e} < 1$. $\sum_{n=1}^{\infty} M_n < +\infty$. By Weierstarss M-test. the series uni. conv.

2° $x \in E_1$. $\sup_{x\in[0,1]} \left|e^{-n(x^2+2\sin x)}\right| \le e^{-n\cdot\frac{1}{n}} = e^{-1} \ne 0$.

Thus. $x \in E_1$. the series not uniformly conv.

5. Check the uniform convergence of a series on $[0,1]$

$$\sum_{n=1}^{\infty} x^n(1-x)$$

Pf, when $x=1$. $\sum_{n=1}^{\infty} 1^n(1-1)=0$

when $x=0$ $\sum_{n=1}^{\infty} 0^n(1-0)=0$.

when $x \in (0,1)$.

$\sum_{n=1}^{\infty} x^n(1-x) = \sum_{n=1}^{\infty} x^n - \sum_{n=1}^{\infty} x^{n+1} = \lim_{n\to\infty} \frac{x(1-x^n)}{1-x} + \lim_{n\to\infty} \frac{x^2(1-x^n)}{1-x} = x$

Thus, $\sum_{n=1}^{\infty} x^n(1-x) \rightrightarrows f = \begin{cases} x, & x\in[0,1) \\ 0, & x=1. \end{cases}$

Mathematical analysis 2. Homework 35.

1. Find radius of convergence of the following series

   (a) $\sum\limits_{k=1}^{\infty}\left(\frac{z}{\sqrt{k}}\right)^k$

   (b) $\sum\limits_{k=1}^{\infty}\left(\frac{i+1}{2}\right)^k z^{2k}$

1. (a) Apply the Cauchy-Adamar formula

$$\frac{1}{R} = \varlimsup_{n\to\infty}\sqrt[n]{\left(\frac{1}{\sqrt{n}}\right)^n} = \varlimsup_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$$

thus we have $R=+\infty$.

(b) $\sqrt{\frac{1+i}{2}} = \pm\frac{\sqrt{1+\sqrt{2}}}{2} \pm \frac{\sqrt{\sqrt{2}-1}}{2} i := c$

$$\sum_{k=1}^{\infty}\left(\frac{1+i}{2}\right)^k z^{2k} = \sum_{k=1}^{\infty}\left(\pm\frac{\sqrt{1+\sqrt{2}}}{2} \pm \frac{\sqrt{\sqrt{2}-1}}{2} i\right)^{2k} \cdot z^{2k}.$$

Apply the Cauchy-Adamar formula

$$\frac{1}{R} = \lim_{2k\to\infty}\sqrt[2k]{|c|^{2k}} = \frac{\sqrt{2}}{2} \qquad \text{thus } R=\sqrt{2}$$

2. Find radius of convergence of the following series and investigate the convergence and absolute convergence at the endpoints of the following series

   (a) $\sum\limits_{k=1}^{\infty}\frac{1}{\sqrt[3]{k}}\left(\frac{x-1}{3}\right)^k$;

   (b) $\sum\limits_{k=1}^{\infty}\frac{3-\sqrt{k}}{\sqrt{k^2+k+1}}(x-2)^k$.

   (c) $\sum\limits_{k=1}^{\infty}2^k x^{k^2}$

Solution: (a) Let $x-1=z$.

$$\sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k}}\left(\frac{x-1}{3}\right)^k = \sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k}\cdot 3^k} z^k$$

$$R = \lim_{k\to\infty}\frac{\sqrt[3]{k+1}\cdot 3^{k+1}}{\sqrt[3]{k}\ 3^k} = 3\lim_{k\to\infty}\sqrt[3]{\frac{k+1}{k}} = 3.$$

when $R=3$. we need to check the convergence when $x=-2$ or $x=4$

when $x=4$. $\sum\limits_{k=1}^{\infty}\frac{1}{\sqrt[3]{k}}\cdot 1^k = \sum\limits_{k=1}^{\infty}\frac{1}{\sqrt[3]{k}} \geq \sum\limits_{k=1}^{\infty}\frac{1}{k}$ which is div.

By comparison test. the series div.

$$x=-2 \quad \sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k}}\cdot(-1)^k = \sum_{k=2}^{\infty}(-1)^{k-1}\cdot\frac{1}{\sqrt{k-1}} + (-1)$$

since $\frac{1}{\sqrt{k-1}}$ is decreasing. $\frac{1}{\sqrt{k-1}} \to 0\ (k\to+\infty)$.

By Leibniz's test, the series conv.

$$\sum_{k=1}^{\infty}\left|\frac{1}{\sqrt[3]{k}}\cdot(-1)^k\right| = \sum_{k=1}^{\infty}\frac{1}{\sqrt[3]{k}} \quad \text{div.}$$

In conclusion, $R=3$. when $x=-2$. conditional conv.

$x=4$. div.

(b) Let $x-2=k$. $C_k=\frac{3^{-\sqrt{k}}}{\sqrt{k^2+k+1}}$

$$\lim_{k\to\infty}\frac{C_k}{C_{k+1}}=\lim_{k\to\infty}\frac{3^{\sqrt{k+1}}}{3^{\sqrt{k}}}\cdot\frac{\sqrt{k^2+3k+3}}{\sqrt{k^2+k+1}}=\lim_{k\to\infty}3^{\sqrt{k+1}-\sqrt{k}}=1$$

$R=1$.

when $x=3$. $\sum_{k=1}^{\infty}\frac{1}{3^{\sqrt{k}}\cdot\sqrt{k^2+k+1}}\leq\sum_{k=1}^{\infty}\frac{1}{3^{\ln k}\cdot\sqrt{k^2+k+1}}\leq\sum_{k=1}^{\infty}\frac{1}{e^{\ln k}\sqrt{k^2+k+1}}\leq\sum_{k=1}^{\infty}\frac{1}{k^2}$ conv.

$x=1$ $\sum_{k=1}^{\infty}\left|\frac{(-1)^k}{3^{\sqrt{k}}\cdot\sqrt{k^2+k+1}}\right|=\sum_{k=1}^{\infty}\frac{1}{3^{\sqrt{k}}\cdot\sqrt{k^2+k+1}}$

In conclusion. $R=1$. the series absolutely conv. at endpoints $x=1$ and $x=3$.

(c) $\sum_{k=1}^{\infty}2^k x^{k^2}=\sum_{k=1}^{\infty}(\sqrt[k]{2})^{k^2}\cdot x^{k^2}$

$\frac{1}{R}=\lim_{k\to\infty}\sqrt[k]{\sqrt[k]{2}}=1$. $R=1$.

when $x=1$. $\sum_{k=1}^{\infty}2^k$ div.

$x=-1$ $\sum_{k=1}^{\infty}2^k(-1)^{k^2}=\sum_{k=1}^{\infty}2^k(-1)^k$ (since the square doesn't change the odevity.)

$=\sum_{k=1}^{\infty}(-2)^k=\frac{(-2)(1-(-2)^k)}{3}=\frac{2}{3}[(-2)^k-1]$ div.

In conclusion. $R=1$. the series div. at the end points.

3. Let

$$l=\liminf\left|\frac{a_k}{a_{k+1}}\right|,\quad L=\limsup\left|\frac{a_k}{a_{k+1}}\right|,$$

and R be a radius of convergence of a real power series $\sum_{k=1}^{\infty}a_kx^k$. Prove that $l\leq R\leq L$.

Pf: denote $\left|\frac{a_k}{a_{k+1}}\right|=b_k$. $R=\lim_{k\to\infty}b_k$ (By Lemma 3.8)

we have $l=\underline{\lim}\, b_k$ $L=\overline{\lim}\, b_k$. (By Lemma 3.5)

1° if $R\in[0,+\infty)$

Let a subsequence $\{b_k\}$, by def. of limsup and liminf.

we have $l\leq\lim_{k\to\infty}b_{n_k}\leq L$, Since b_k has a limit. any subsequence

$\{b_{n_k}\}$ s.t. $\lim_{k\to\infty}b_{n_k}=R\Rightarrow l\leq R\leq L$.

2° if $R=+\infty$ By the existence of lower limit. we have $l\leq R$.

and by the def of supremum. $b_n\leq\sup_{x\geq n}b_k$, then pass the limit (by the preserve of the limit)

we have $R\leq L$. i.e. $L=+\infty$.

In conclusion. we have $l\leq R\leq L$

Mathematical analysis 2. Homework 36.

1. Assume that a series $\sum\limits_{k=1}^{\infty} a_k x^k$ converges for every $x \in (0,1)$,

$$\lim_{k\to\infty} k a_k = 0, \quad \lim_{x\to 1-} \sum_{k=1}^{\infty} a_k x^k = A.$$

Then

$$\sum_{k=1}^{\infty} a_k = A.$$

Pf: Denote. $\delta_n = \max\limits_{k \geqslant n} |k a_k|$ $\quad \lim\limits_{n\to\infty} \delta_n = 0$. $\{\delta_n\}$ is decreasing.

For any fixed $N \in \mathbb{N}$

we consider $\left|\sum\limits_{k=1}^{N} a_k - A\right| \leqslant \left|\sum\limits_{k=1}^{N} a_k - \sum\limits_{k=1}^{\infty} a_k x^k\right| + \left|\sum\limits_{k=1}^{\infty} a_k x^k - A\right|$

$= \left|\sum\limits_{k=1}^{N} a_k (1-x^k)\right| + \left|\sum\limits_{k=N+1}^{\infty} \frac{|k a_k| x^k}{k}\right| + \left|\sum\limits_{k=1}^{\infty} a_k x^k - A\right|$

$1 - x^k = (1-x)(1+\cdots+x^{k-1}) < k(1-x)$

$\left|\sum\limits_{k=1}^{N} a_k (1-x^k)\right| \leqslant \sum\limits_{k=1}^{N} |k \cdot a_k| \cdot |1-x| \leqslant N \cdot \delta_1 (1-x)$

$\left|\sum\limits_{k=N+1}^{\infty} \frac{|k a_k| \cdot x^k}{k}\right| \leqslant \frac{\delta_{N+1}}{N+1} \left|\sum\limits_{k=N+1}^{\infty} x^k\right| \leqslant \frac{\delta_{N+1}}{N+1} \cdot \frac{1}{1-x} \cdot x^N \leqslant \frac{\delta_{N+1}}{N} \cdot \frac{1}{1-x}$

$\forall \varepsilon > 0$. $\exists N$. s.t. for any $n > N$. $\delta_n < \frac{\varepsilon^2}{2}$. and. when $x = 1 - \frac{\varepsilon}{N}$. $\left|\sum\limits_{k=1}^{\infty} a_k x^k - A\right| < \frac{\varepsilon}{2}$ holds

Thus, $\left|\sum\limits_{k=1}^{N} a_k - A\right| < \delta_1 \cdot \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = (1+\delta_1)\,\varepsilon. \to 0.$

i.e. when $N \to \infty$, $\sum\limits_{k=1}^{\infty} a_k = A$.

2. Find a sum of power series

(a) $\sum\limits_{k=1}^{\infty} k z^k$;

(b) $\sum\limits_{k=1}^{\infty} (k+1)(k+2) x^k$

(a). $R = \lim\limits_{k\to\infty} \frac{k}{k+1} = 1. \quad \Rightarrow |z| < 1.$

Let $f(z) = \sum\limits_{k=1}^{\infty} k z^k = z \sum\limits_{k=1}^{\infty} k z^{k-1}$

$F(z) = \int \sum\limits_{k=1}^{\infty} k z^{k-1} dz = \sum\limits_{k=1}^{\infty} \int k z^{k-1} dz = \sum\limits_{k=1}^{\infty} z^k = z \cdot \sum\limits_{k=0}^{\infty} z^k = z \cdot \frac{1}{1-z}$

$f(z) = z \cdot F'(z) = \frac{z}{(1-z)^2} \qquad |z| < 1$

(b) $R = \lim\limits_{k\to+\infty} \frac{(k+1)(k+2)}{(k+2)(k+3)} = 1$

Let $f(x) = \sum\limits_{k=0}^{\infty} (k+1)(k+2) x^k = \sum\limits_{k=1}^{\infty} \left((k+2) x^{k+1}\right)' := F'(x)$

Let $g'(x) = F(x)$.

$g(x) = g(0) + \int_0^x F(t)\,dt = \sum\limits_{k=1}^{\infty} x^{k+2} = x^3 \sum\limits_{k=0}^{\infty} x^k = x^3 \frac{1}{1-x}$

$f(x) = g^{(2)}(x) = \left[\frac{3x^2(1-x)+x^3}{(1-x)^2}\right]' = \left[\frac{3x^2-2x^3}{(1-x)^2}\right]' = \frac{(6x-6x^2)(1-x)^2 + 2(1-x)(3x^2-2x^3)}{(1-x)^4} = \frac{2x^3-6x^2+6x}{(1-x)^3} \qquad |x| < 1$

3. Find Mc'Laurent series of a function and find radius of convergence

(a) $(1+x)e^{-x}$;

(a) Solution: consider $f'(x) = -x e^{-x} = -x \cdot \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-x)^{k+1}}{k!}$

$$f(x) = f(0) + \int_0^x f'(t)\,dt = 1 + \int_0^x \sum_{k=0}^{\infty} \frac{(-1)^{k+1} \cdot (x)^{k+1}}{k!}$$

$$= 1 + \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^{k+1} x^{k+1}}{k!}$$

$$= 1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} \cdot \frac{x^{k+2}}{k+2}$$

$R = \lim_{k\to\infty} \left|(-1)\frac{(k+1)!}{k!}\right| = 1.$ Mc'Laurent series: $f(x) = 1 + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!(k+2)} \cdot x^{k+2}$

4. Recall the Mc'Laurent's series for arcsin

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!(2n+1)} x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!} x^{2n+1}, \quad |x| < 1.$$

Check that this series (1) converges for $x = \pm 1$. What is the sum at those points?

(1) $x = 1$. $\sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!} = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!\,(2n+1)} = \sum_{n=1}^{\infty} \frac{(2n)!}{(2^n \cdot n!)^2 \cdot (2n+1)}$

$$= \sum_{n=1}^{\infty} \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n} \cdot e^{\theta_n}}{2^{2n} \cdot 2\pi n \cdot \left(\frac{n}{e}\right)^{2n} \cdot (e^{\theta_n'})^2\,(2n+1)} = \sum_{n=1}^{\infty} \frac{e^{\theta_n}}{\sqrt{\pi n}\,(2n+1) \cdot (e^{\theta_n'})^2}$$

the series $< \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$, which is converges

$x = -1$ $\sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!} (-1)^{2n+1} = -\sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!}$ conv.

By Alel's thm. we have proved that $S(R)$ conv.

thus $S(1) = \lim_{x\to 1^-} \arcsin x = \frac{\pi}{2}$.

$S(-1) = \lim_{x\to -1^+} \arcsin x = -\frac{\pi}{2}$

Mathematical analysis 2. Homework 37.

1. Find Maclaurin series and its radius of convergence for the function

$$f(x) = \int_0^x \frac{t^2dt}{\sqrt{1+t^2}}.$$

Solution: $f(x) = \int_0^x t^2 \cdot \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \cdot (t^2)^k dt = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \int_0^x t^{2k+2} dt = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \cdot \frac{x^{2k+3}}{2k+3}$

$$R = \lim_{k\to\infty} \frac{\binom{-\frac{1}{2}}{k} \cdot \frac{1}{2k+3}}{\binom{-\frac{1}{2}}{k+1} \cdot \frac{1}{2k+5}} = \left| \frac{k+1}{(-\frac{1}{2} - k - 1 + 1)} \right| = 1$$

2. Find the radius of convergence and the sum of the power series

$$S(x) = \sum_{n=1}^{\infty} \frac{(2x)^{4n+1}}{4n+1}.$$

Solution: $S(x) = \sum_{n=1}^{\infty} \frac{2^{4n+1}}{4n+1} x^{4n+1}$. $\qquad S'(x) = 2\sum_{n=1}^{\infty} \frac{(2x)^{4n}}{4n}$

$$R = \lim_{n\to\infty} \left| \frac{2^{4n+1}}{2^{4n+5}} \cdot \frac{4n+5}{4n+1} \right| = \frac{1}{16}$$

$$S(x) = S(0) + \int_0^x \sum_{n=1}^{\infty} (2t)^{4n} d2t = \int_0^x \frac{(2t)^4}{1-(2t)^4} d2t = \int_0^x (-1) + \frac{1}{2} \cdot \left(\frac{1}{1-(2t)^2} + \frac{1}{1+(2t)^2}\right) d(2t)$$

$$= -2x + \frac{1}{4} \ln\left|\frac{2x+1}{2x-1}\right| + \frac{1}{2} \arctan x + C$$

3. Find the integral using the power series decomposition

$$\int_0^{\infty} \frac{xdx}{1+e^x}.$$

$$\int_0^{\infty} \frac{x\,dx}{1+e^x} = \int_0^{\infty} \frac{1}{1+e^{-x}} \cdot xe^{-x} dx = \int_0^{\infty} (1+e^{-x})^{-1}\, x \cdot e^{-x} dx = \int_0^{\infty} \sum_{k=0}^{\infty} (-1)^k \cdot (e^{-x})^k x \cdot e^{-x}$$

$$= \sum_{k=0}^{\infty} \int_0^{\infty} e^{-(k+1)x} x \cdot (-1)^k dx = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(k+1)^2} = \frac{\pi}{8}$$

it remains to check the uni. conv.

$$\left| \frac{x}{1+e^x} - \sum_{k=0}^{N} (-1)^k x \cdot e^{-(k+1)x} \right| = \left| \sum_{k=N+1}^{\infty} (-1)^k x \cdot e^{-(k+1)x} \right| \leqslant \left| \sum_{k=N+1}^{\infty} \frac{x}{e^x} \cdot e^{-kx} \right| \leqslant \left| \sum_{k=N+1}^{\infty} e^{-kx} \right|$$

which tends to 0. when $N \to +\infty$.

thus, we have $\int_0^{\infty} \frac{x\,dx}{1+e^x} = \frac{\pi}{8}$

4. Prove that the function $y(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$ is a solution of differential equation $y' = xy$.

$$y' = \sum_{k=0}^{\infty} \left(\frac{x^{2k}}{2^k k!}\right)' = \sum_{k=1}^{\infty} \frac{x^{2k-1}}{2^{k-1} \cdot (k-1)!}$$

$$xy = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2^k k!} \;\underset{i=k+1}{\overline{\overline{\text{change the index}}}}\; \sum_{i=1}^{\infty} \frac{x^{2i-1}}{2^{i-1} \cdot (i-1)!}$$

thus we have LHS = RHS. i.e. $xy = y'$. when $y(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$

5. Find Maclaurin series at 0 of the solution of the following differential equation:

$$(1-x^2)y'' - xy' = 0,\ y(0)=0,\ y'(0)=1.$$

Pf: Let $y=\sum_{k=0}^{\infty} a_k x^k \quad y'=\sum_{k=1}^{\infty} k a_k x^{k-1} \quad y''=\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = 2a_2+\sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k$

$$(1-x^2)y''-xy'=0$$

$$\Leftrightarrow \quad 2a_2+\sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=2}^{\infty}k(k-1)a_k x^k-\sum_{k=1}^{\infty}k a_k x^k=0$$

$$\Rightarrow \quad a_2=0. \quad (k+2)(k+1)a_{k+2}=k^2 a_k.$$

$$a_0=y(0)=0. \qquad a_1=\frac{y'(0)}{1!}=1.$$

$$a_{k+2}=\frac{k^2}{(k+2)(k+1)}a_k$$

$$a_{k+4}=\frac{(k+2)^2}{(k+3)(k+4)}\cdot\frac{k^2}{(k+2)(k+1)}=\frac{k^2(k+2)}{(k+3)(k+4)(k+1)}a_k \quad \cdots$$

$$y=x+\frac{1^2}{3!}x^3+\frac{1^2\cdot 3^2}{5!}x^5+\cdots+\frac{((2n-1)!!)^2}{(2n+1)!}x^{2n+1}+\cdots$$

$$=x+\sum_{n=1}^{\infty}\frac{((2n-1)!!)^2}{(2n+1)!}x^{2n+1}$$