# Models and Methods of Tropical Mathematics

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# Tropical Mathematics: Introduction

- Tropical (idempotent) mathematics deals with the theory and application of semirings (semifields) with idempotent operations
- An operation is idempotent, if applied to operands of the same value, it returns this value as output (example: $\max(x, x) = x$)
- Methods of tropical mathematics find applications in many areas to provide new solutions to various old and novel problems in
    - *project scheduling, location analysis, decision making,*
    - *discrete event systems, neural networks, cryptographic protocols, pattern recognition and other fields*

- Tropical mathematics has its origins in 1960s in the works of R. A. Cuninghame-Green, B. Giffler, A. J. Hoffman, S. N. N. Pandit, N. N. Vorobyev, I. V. Romanovsky
- First researches concentrated on the ability to rename such operation as $\max$ into a generalized idempotent addition $\oplus$
- These formal tricks allowed one to replace the polish postfix notation by the standard infix notation: $\max(x, y) = x \oplus y$
- Moreover, after translation into the tropical language, many problems that are not linear in the ordinary sense became linear
- This offers a potential for the use of the concept of linearity and related results to study nonlinear problems

- If the operation is idempotent, it is not invertible, and hence a subtraction as the inversion of $\oplus$ is undefined in tropical algebra
- Because of lack of subtraction, most of the techniques available in linear mathematics cannot be translated into the tropical language
- This leads to the need to develop new approaches to the solution of tropical analogues of many traditional problems
- At the same time, tropical solutions normally appear to be less complicated than that in the conventional mathematics
- Application of methods of tropical mathematics can offer complete analytical solutions to a range of classical and new problems

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- References at http://www.math.spbu.ru/user/krivulin/
- Papers in open access archive arXiv at http://arxiv.org/a/krivulin_n_1

# Examples of Applications

- Temporal project scheduling in project management
- Minimax location on the plane and in multidimensional space
- Rating alternatives from pairwise comparisons in decision making

# Project Scheduling: Constraints and Objectives

- Project scheduling is aimed at the development of optimal schedules of activities in a project, subject to various constraints
- The scheduling objectives are usually set in terms of time-oriented criteria to optimize, such as makespan, lateness and tardiness
- In real-world problems other objectives can be added, taking into account the project cost, profit, resource allocation or consumption
- Scheduling constraints may include temporal constraints in the form of time bounds for and relationships between activities
- The constraints may be formulated as material and manpower resource requirements, budget limitations and others restrictions

## Temporal Project Scheduling Problems

- Project scheduling problems with constraints of different types may be rather complicated and even known to be NP-hard to solve
- Solution approaches involve methods of mixed integer linear programming, combinatorial and discrete optimization
- The temporal scheduling problems with only time-oriented objectives and constraints, can be formulated as linear programs
- These problems are solved using algorithms of linear programming which offer quite efficient numerical techniques
- Linear programming typically provides efficient numerical solutions, but does not allow to derive all solutions analytically
- In the framework of tropical algebra, many temporal project scheduling problems can be analytically solved in explicit form

## Start-Finish Relations

- Consider a project that involves $n$ activities (tasks, operations, jobs) performed in parallel, subject to a set of temporal constraints
- The start-finish relations specify the minimum allowed time lag between the start of one activity and finish of another
- Each activity finishes when all constraints for its finish are fulfilled
- For each activity $i = 1, \ldots, n$, the following notation is used:

$x_i$, *the unknown start time*;
$y_i$, *the unknown finish time*;
$a_{ij}$, *the given minimum possible time lag between the start of activity $j = 1, \ldots, n$ and finish of $i$ ($a_{ij} = -\infty$ if unspecified)*

- The start-finish constraints take the form of the following inequalities (where at least one inequality holds as an equality):

$$y_i \geq x_j + a_{ij}, \qquad i = 1, \ldots, n$$

## Scalar Representation of Model

- Combining all start-finish relations for activity $i$ yields the equation

$$y_i = \max(x_1 + a_{i1}, \ldots, x_n + a_{in}), \qquad i = 1, \ldots, n$$

- After replacing the operations $\max$ by $\oplus$ and $+$ by $\otimes$, we obtain

$$y_i = a_{i1} \otimes x_1 \oplus \cdots \oplus a_{in} \otimes x_n, \qquad i = 1, \ldots, n$$

- The multiplication sign $\otimes$, as usual, can be eliminated to write

$$y_i = a_{i1} x_1 \oplus \cdots \oplus a_{in} x_n, \qquad i = 1, \ldots, n$$

- The last equation is very similar to the ordinary linear expression

$$y_i = a_{i1} x_1 + \cdots + a_{in} x_n, \qquad i = 1, \ldots, n$$

## Vector Representation of Model

- We introduce the matrix and vectors

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

- The model is represented in the form of the vector equation

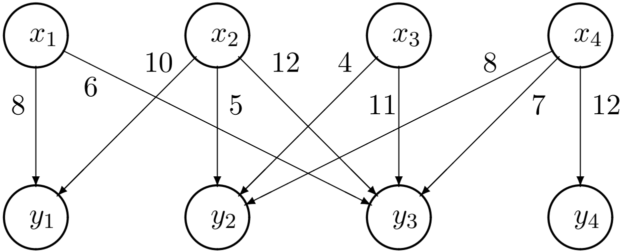
$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

- The vector equation corresponds the system of scalar equations

$$\begin{aligned} y_1 &= a_{11} \otimes x_1 \oplus \cdots \oplus a_{1n} \otimes x_n, \\ &\vdots \\ y_n &= a_{n1} \otimes x_1 \oplus \cdots \oplus a_{nn} \otimes x_n \end{aligned}$$

## Graph and Matrix of Project

- Example of the graph of a project




- The corresponding matrix of the project ( $\mathbb{0} = -\infty$ ):

$$\boldsymbol{A} = \begin{pmatrix} 8 & 10 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 5 & 4 & 8 \\ 6 & 12 & 11 & 7 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 12 \end{pmatrix}$$

## Due Dates

- Suppose that due dates are given for activity in the project, which specify the time by which the activities should be finished
- For each activity $i = 1, \ldots, n$, the following notation is used:

  $p_i$, *the given due date*

- Let us introduce the vector notation:

$$\boldsymbol{p} = \begin{pmatrix} p_1 & \ldots & p_n \end{pmatrix}^T$$

## Scheduling Problem

- Consider the problem to find the start time $x_i$ of each activity $i$, for which the completion time $y_i$ coincides with the due dates $p_i$
- The solution of the problem corresponds to solving the following vector equation (in terms of algebra with $\oplus = \max$ and $\otimes = +$)

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p} \qquad \text{(one-sided equation)}$$

## Start-Start Relations

- Consider a project with start-start relations that specify the minimum allowed time lag between the start time of two activities
- For each activity $i = 1, \ldots, n$, we use the following notation:

$x_i$, *the unknown start time*;
$b_{ij}$, *the given minimum possible time lag between the start of activity $j = 1, \ldots, n$ and start of $i$ ($b_{ij} = -\infty$ if unspecified)*

- The start-start relations are written as the inequalities

$$x_i \geq x_j + b_{ij}, \qquad i = 1, \ldots, n$$

- All relations for activity $i$ are combined into one inequality

$$x_i \geq \max(b_{i1} + x_1, \ldots, b_{in} + x_n) \quad \text{(in ordinary notation)},$$
$$x_i \geq b_{i1} x_1 \oplus \cdots \oplus b_{in} x_n \quad \text{(after replacing operations)}$$

## Vector Representation

- In matrix-vector notation, we have

$$\boldsymbol{B} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

## Scheduling Problem

- The problem of finding the start time $x_i$ for each $i$ to satisfy the start-start relations, corresponds to solving the inequality

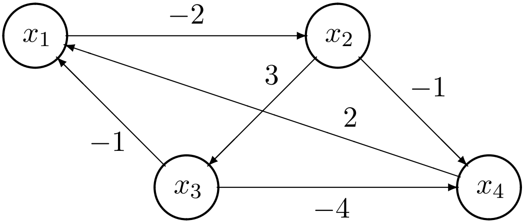
$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}$$

- If each activity starts immediately as soon as all its start-start relations are satisfied, the problem reduces to the equation

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{x} \qquad \text{(homogeneous two-sided equation)}$$

## Graph and Matrix of Project

- Example of the graph of a project




- The corresponding matrix of the project ( $\mathbb{0} = -\infty$ ):

$$
\boldsymbol{B} = \begin{pmatrix} 0 & -2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & 3 & -1 \\ -1 & \mathbb{0} & 0 & -4 \\ 2 & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix}
$$

## Release Dates

- Suppose that release dates are given for activities in the project, which specify the earliest allowed start time for each activity
- For each activity $i = 1, \ldots, n$, we additionally define

  $g_i$, *the given release date*

- The release date constraints take the form of inequalities

$$x_i \geq g_i, \qquad i = 1, \ldots, n$$

- The start-start relations and release dates yield the inequalities

$$x_i \geq \max(b_{i1} + x_1, \ldots, b_{in} + x_n, g_i) \quad \text{(in ordinary notation)},$$
$$x_i \geq b_{i1} x_1 \oplus \cdots \oplus b_{in} x_n \oplus g_i \quad \text{(after replacing operations)}$$

## Vector Representation

- We introduce the vector notation

$$\boldsymbol{g} = \begin{pmatrix} g_1 & \dots & g_n \end{pmatrix}^T$$

## Scheduling Problem

- Consider the problem to find the start time $x_i$ of each activity $i$ to satisfy both the start-start relations and release dates constraints
- The solution of the problem corresponds to solving the inequality

$$\boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}$$

- If each activity starts immediately as soon as all its start-start relations are satisfied, the problem reduces to the equation

$$\boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} = \boldsymbol{x} \qquad \text{(nonhomogenious two-sided equation)}$$

## Scheduling with Mixed Constraints

- Consider a project with a matrix $\boldsymbol{A}$ of start-finish relations and a vector $\boldsymbol{p}$ of due dates, which result in the constraint in the form

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p}$$

- Further assume that start-start constraints with a matrix $\boldsymbol{B}$ are also imposed, which yield the inequality constraint

$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}$$

## Scheduling Problem

- As scheduling problem of interest, one can consider the derivation of the vector $\boldsymbol{x}$ of start time, which satisfies the system

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{p}, \\ \boldsymbol{B}\boldsymbol{x} &\leq \boldsymbol{x} \end{aligned}$$

# Optimality Criteria

## Project Makespan

- Consider a project with constraints given by start-finish relations
- Suppose we need to minimize the project makespan (the overall duration of the project) as the optimality criterion for scheduling
- For each activity $i = 1, \ldots, n$, we use the following notation:

$x_i$, *the unknown start time*;

$y_i$, *the unknown finish time*;

$a_{ij}$, *the given minimum possible time lag between the start of activity $j = 1, \ldots, n$ and finish of $i$ ($a_{ij} = -\infty$ if unspecified)*

- Furthermore, we introduce the matrix and vectors

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{y} = (y_i)$$

- We use the obvious identity $\min(u,v) = -\max(-u,-v)$ to represent the overall duration of the project as the difference

$$\max_{1\leq i\leq n} y_i - \min_{1\leq i\leq n} x_i = \max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n}(-x_i)$$

- Consider the column vector $\boldsymbol{x}$ and define its conjugate row vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \boldsymbol{x}^- = \begin{pmatrix} -x_1 & \dots & -x_n \end{pmatrix}$$

- We also define the vector of arithmetic zeros and its conjugate as

$$\mathbf{1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{1}^- = \mathbf{1}^T = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}$$

- Consider the objective function representing the project makespan

$$\max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n}(-x_i)$$

- After replacing the operations $\max$ by $\oplus$ and $+$ by $\otimes$, we obtain

$$(y_1 \oplus \cdots \oplus y_n)((-x_1) \oplus \cdots \oplus (-x_n))$$

- In vector notation, taking into account that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we have

$$\mathbf{1}^T \boldsymbol{y}\boldsymbol{x}^{-}\mathbf{1} = \mathbf{1}^T \boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^{-}\mathbf{1}$$

## Scheduling Problem

- The problem is to derive a vector $\boldsymbol{x}$ of start time, which attains

$$\min_{\boldsymbol{x}} \quad \mathbf{1}^T \boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^{-}\mathbf{1}$$

## Maximum Deviation From Due Dates

- Consider a project with start-finish constraints and due dates
- Let us define the maximum deviation from due dates as the optimality criterion for scheduling, which has to be minimized
- For each activity $i = 1, \ldots, n$, we use the following notation:

$x_i$, *the unknown start time*;

$y_i$, *the unknown finish time*;

$a_{ij}$, *the given minimum possible time lag between the start of activity $j = 1, \ldots, n$ and finish of $i$ ($a_{ij} = -\infty$ if unspecified)*;

$p_i$, *the given due date*

- We introduce the matrix and vectors

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{y} = (y_i), \qquad \boldsymbol{p} = (p_i)$$

- We use the identity $|u| = \max(-u, u)$ to represent the maximum deviation of the elements of $\boldsymbol{y}$ from the elements of $\boldsymbol{p}$ as follows:

$$\begin{aligned}\max_{1\leq i\leq n}|y_i - p_i| &= \max_{1\leq i\leq n}\max(y_i - p_i, p_i - y_i)\\ &= \max\left(\max_{1\leq i\leq n}(y_i + (-p_i)), \max_{1\leq i\leq n}(p_i + (-y_i))\right)\end{aligned}$$

- Consider the vector of finish time and vector of due dates

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \qquad \boldsymbol{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

- For these two vectors, define their conjugate row vectors

$$\boldsymbol{y}^{-} = \begin{pmatrix} -y_1 & \dots & -y_n \end{pmatrix}, \qquad \boldsymbol{p}^{-} = \begin{pmatrix} -p_1 & \dots & -p_n \end{pmatrix}$$

- Consider the expression of the maximum deviation

$$\max(\max_{1\leq i\leq n}(y_i+(-p_i)),\max_{1\leq i\leq n}(p_i+(-y_i)))$$

- After replacing the operations $\max$ by $\oplus$ and $+$ by $\otimes$, we obtain

$$(y_1\otimes(-p_1)\oplus\cdots\oplus y_n\otimes(-p_n))\oplus(p_1\otimes(-y_1)\oplus\cdots\oplus p_n\otimes(-y_n))$$

- In vector notation, with the substitution $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$, we obtain

$$\boldsymbol{p}^{-}\boldsymbol{y}\oplus\boldsymbol{y}^{-}\boldsymbol{p}=\boldsymbol{p}^{-}\boldsymbol{A}\boldsymbol{x}\oplus(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$$

## Scheduling Problem

- The scheduling problem is to find a vector $\boldsymbol{x}$ that provides

$$\min_{\boldsymbol{x}}\quad\boldsymbol{p}^{-}\boldsymbol{A}\boldsymbol{x}\oplus(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$$

## Maximum Flowtime

- Consider a project with start-finish and start-start constraints
- We define the maximum flowtime (maximum total time, cycle time) of activities as the optimality criterion, which has to be minimized
- For each activity $i = 1, \ldots, n$, we use the following notation:

  $x_i$, *the unknown start time*;
  $y_i$, *the unknown finish time*;
  $a_{ij}$, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and finish of* $i$;
  $b_{ij}$, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and start of* $i$

- The flowtime of activity $i$ is given by the difference

$$y_i - x_i, \qquad i = 1, \ldots, n$$

- We introduce the matrices and vectors

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{B} = (b_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{y} = (y_i)$$

- Let us consider the maximum flowtime over all activities

$$\max(y_1 - x_1, \dots, y_n - x_n) \qquad \text{(in ordinary notation)}$$
$$y_1 \otimes (-x_1) \oplus \cdots \oplus y_n \otimes (-x_n) \qquad \text{(after replacing operations)}$$

- In vector notation, with the substitution $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we obtain

$$\boldsymbol{x}^{-}\boldsymbol{y} = \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

## Scheduling Problem

- The scheduling problem is to find a vector $\boldsymbol{x}$ that attains

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ \text{s. t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \end{aligned}$$

## Maximum Deviation of Finish Time

- Consider a project with start-finish and start-start constraints
- Suppose that the optimal schedule has to minimize the maximum deviation of finish time
- For each activity $i = 1, \ldots, n$, we use the following notation:

  $x_i$, *the unknown start time*;
  $y_i$, *the unknown finish time*;
  $a_{ij}$, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and finish of* $i$;
  $b_{ij}$, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and start of* $i$

- The maximum deviation of finish time over all activities is given by

$$\max_{1 \leq i \leq n} y_i - \min_{1 \leq i \leq n} y_i = \max_{1 \leq i \leq n} y_i + \max_{1 \leq i \leq n} (-y_i)$$

- We represent the maximum deviation of finish time as follows:

$$\max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n}(-y_i) \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{i=1}^{n} y_i \otimes \bigoplus_{j=1}^{n}(-y_j) \qquad \text{(after replacing operations)}$$

- In vector notation, with the substitution $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we have

$$\mathbf{1}^T \boldsymbol{y}\boldsymbol{y}^- \mathbf{1} = \mathbf{1}^T \boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^- \mathbf{1}, \qquad \mathbf{1} = (0,\dots,0)^T$$

## Scheduling Problem

- The problem is to find a vector $\boldsymbol{x}$ that provides the minimum

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \mathbf{1}^T \boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^- \mathbf{1}, \\ \text{s. t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \end{aligned}$$

# Location Analysis: Minimax Location Problem

- The problem is to locate a new point in a feasible area to minimize the maximum Chebyshev distance (with addends) to given points
- The Chebyshev distance (maximum or $l_\infty$ -metric) between two vectors $\boldsymbol{r} = (r_1, \ldots, r_n)^T$ and $\boldsymbol{s} = (s_1, \ldots, s_n)^T$ in $\mathbb{R}^n$ is given by

$$d(\boldsymbol{r}, \boldsymbol{s}) = \max_{1 \leq i \leq n} |r_i - s_i|$$

- Suppose there is a set of vectors $\boldsymbol{r}_k = (r_{1k}, \ldots, r_{nk})^T \in \mathbb{R}^n$ for all $k = 1, \ldots, m$ and a vector of addends $\boldsymbol{w} = (w_1, \ldots, w_m)^T \in \mathbb{R}^m$
- The location problem is to minimize the maximum distance (with addends) from a new vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ to the vectors $\boldsymbol{r}_k$ :

$$\min_{\boldsymbol{x}} \quad \max_{1 \leq k \leq m} (d(\boldsymbol{r}_k, \boldsymbol{x}) + w_k)$$

## Tropical Representation

- Scalar representation of the Chebyshev metric in terms of $\mathbb{R}_{\max,+}$

$$\begin{aligned} d(\boldsymbol{r},\boldsymbol{s}) &= \max_{1\leq i\leq n} |r_i - s_i| \\ &= \max_{1\leq i\leq n} \max(r_i - s_i, s_i - r_i) && \text{(in ordinary notation)} \\ d(\boldsymbol{r},\boldsymbol{s}) &= \bigoplus_{1\leq i\leq n} (s_i^{-1} r_i \oplus r_i^{-1} s_i) \\ &= \bigoplus_{1\leq i\leq n} s_i^{-1} r_i \oplus \bigoplus_{1\leq i\leq n} r_i^{-1} s_i && \text{(after replacing operations)} \end{aligned}$$

- Vector representation of the Chebyshev metric

$$d(\boldsymbol{r},\boldsymbol{s}) = \boldsymbol{s}^{-}\boldsymbol{r} \oplus \boldsymbol{r}^{-}\boldsymbol{s}$$

## Representation of Objective Function

- The objective function of the problem is written as

$$\max_{1\leq k\leq m}(d(\boldsymbol{r}_k,\boldsymbol{x})+w_k) \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{1\leq k\leq m} w_k(\boldsymbol{x}^{-}\boldsymbol{r}_k\oplus\boldsymbol{r}_k^{-}\boldsymbol{x})$$

$$=\bigoplus_{1\leq k\leq m} w_k\boldsymbol{x}^{-}\boldsymbol{r}_k\oplus\bigoplus_{1\leq k\leq m} w_k\boldsymbol{r}_k^{-}\boldsymbol{x} \qquad \text{(after replacing operations)}$$

- Consider a matrix that consists of the vectors $\boldsymbol{r}_k$ as columns

$$\boldsymbol{R}=(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_m)$$

- With this matrix, we can write

$$\bigoplus_{1\leq k\leq m} w_k\boldsymbol{r}_k=\boldsymbol{R}\boldsymbol{w}, \qquad \bigoplus_{1\leq k\leq m} w_k\boldsymbol{r}_k^{-}=\boldsymbol{w}^T\boldsymbol{R}^{-}$$

- Vector representation of the objective function

$$\bigoplus_{1\leq k\leq m} w_k(\boldsymbol{x}^{-}\boldsymbol{r}_k \oplus \boldsymbol{r}_k^{-}\boldsymbol{x}) = \boldsymbol{x}^{-}\boldsymbol{R}\boldsymbol{w} \oplus \boldsymbol{w}^T\boldsymbol{R}^{-}\boldsymbol{x}$$

- We introduce the vectors

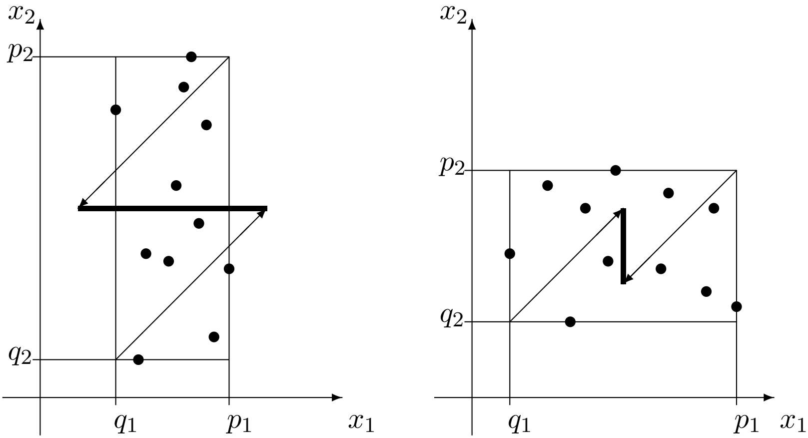
$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{w}, \qquad \boldsymbol{q}^{-} = \boldsymbol{w}^T\boldsymbol{R}^{-}$$

## Location Problem

- The problem is to find a vector $\boldsymbol{x}$ that attains the minimum

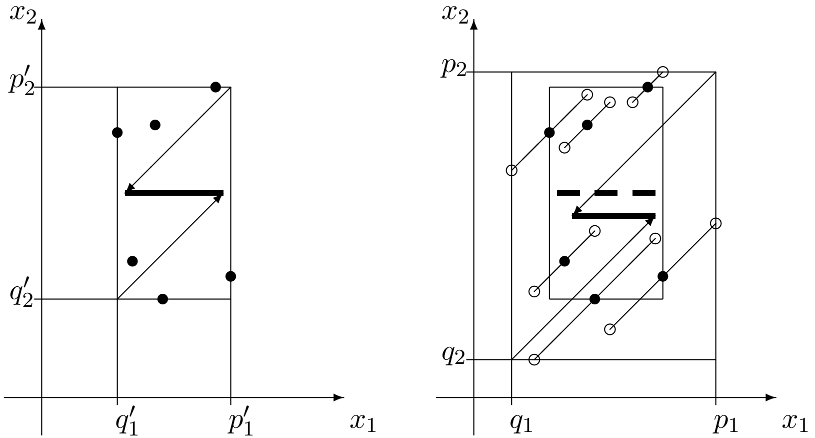
$$\min_{\boldsymbol{x}} \quad \boldsymbol{x}^{-}\boldsymbol{p} \oplus \boldsymbol{q}^{-}\boldsymbol{x}$$

## Solution of a problem with $w_k = 0$ in $\mathbb{R}^2$




- The solution is a segment on the line drawn across the minimal enclosing rectangle through the center points of its long sides

## Solution of a problem with $w_k > 0$ in $\mathbb{R}^2$




- Each given point $\boldsymbol{r}_k$ (left) is replaced with two points $w_k \boldsymbol{r}_k$ and $w_k^{-1} \boldsymbol{r}_k$ to produce a new minimum enclosing rectangle (right)

## Constrained Location Problem

- Suppose the following matrix and vectors are given:

$$\boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{n\times n}, \qquad \boldsymbol{g} = (g_i) \in \mathbb{R}^n, \quad \boldsymbol{h} = (h_i) \in \mathbb{R}^n$$

- The feasible location area is defined by the inequalities

$$\begin{aligned} b_{ij} + x_j &\leq x_i, \\ g_i \leq x_i &\leq h_i, \qquad i,j = 1,\dots,n \end{aligned}$$

- The feasible area is an intersection of the half-spaces given by $b_{ij} + x_j \leq x_i$, and the hyper-rectangle given by $g_i \leq x_i \leq h_i$

- The inequalities $b_{ij} + x_j \leq x_i$ for all $j = 1, \ldots, n$ combine into

$$\max_{1 \leq j \leq n} (b_{ij} + x_j) \leq x_i \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{1 \leq j \leq n} b_{ij} x_j \leq x_i \qquad \text{(after replacing operations)}$$

- Vector representation of constraints is of the form

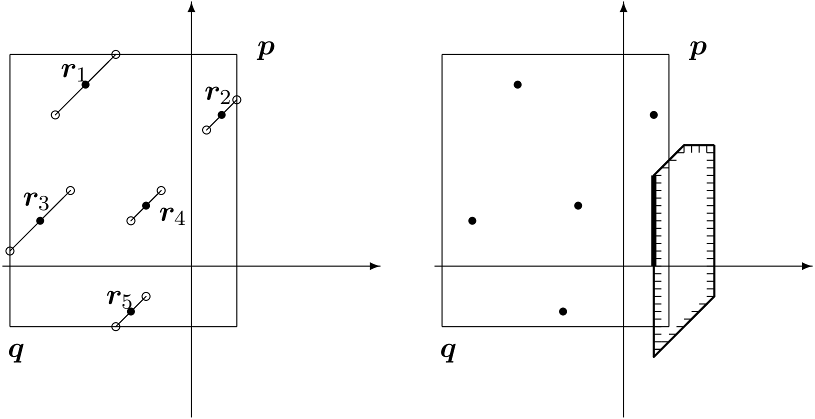
$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \qquad \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}$$

## Constrained Location Problem

- The problem is to find a vector $\boldsymbol{x}$ that attains the minimum

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{p} \oplus \boldsymbol{q}^{-}\boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h} \end{aligned}$$

## Solution to Constrained Problem in $\mathbb{R}^2$




- The minimal enclosing rectangle of a problem (left) and the solution to the problem under constraints (right)

# Decision Making: Ranking by Pairwise Comparisons

## Ranking by Pairwise Comparisons

- Consider a problem to evaluate ratings (scores, priorities, weights) of alternatives from the results of their pairwise comparisons
- Outcome of comparisons is given by a matrix $\boldsymbol{A} = (a_{ij})$, where $a_{ij}$ shows by how much times alternative $i$ is preferable than $j$
- A pairwise comparison matrix $\boldsymbol{A}$ is consistent if its entries are transitive to satisfy the condition $a_{ij} = a_{ik}a_{kj}$ for all $i, j, k$
- Each consistent matrix $\boldsymbol{A}$ has the entries $a_{ij} = x_i/x_j$ given by a positive vector $\boldsymbol{x} = (x_j)$ that entirely specifies the matrix $\boldsymbol{A}$
- If a comparison matrix $\boldsymbol{A}$ is consistent, its vector $\boldsymbol{x}$ (up to a positive factor) defines the individual ratings of alternatives

## Approximation Problem

- The pairwise comparison matrices which are encountered in real-world decision-making problems are usually inconsistent
- If a matrix $\boldsymbol{A}$ is inconsistent, approximation problem arises to find approximating consistent matrices $\boldsymbol{X} = (x_{ij})$ with $x_{ij} = x_i/x_j$
- The approximation with approximation error measured in linear scale involves optimization problems that are difficult to solve
- Evaluating the approximation error on a logarithmic scale may simplify the analysis and even provide a direct analytical solution
- As an example, one can consider log-Chebyshev approximation which uses the Chebyshev metric in logarithmic scale

## Log-Chebyshev Approximation

- For matrices $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{X} = (x_{ij})$, the log-Chebyshev distance with a logarithm to a base greater than $1$ is given by
$$d(\boldsymbol{A}, \boldsymbol{X}) = \max_{1 \leq i,j \leq n} |\log a_{ij} - \log x_{ij}|$$
- It follows from the monotonicity of logarithm that
$$d(\boldsymbol{A}, \boldsymbol{X}) = \log \max_{1 \leq i,j \leq n} \max \left\{ \frac{a_{ij}}{x_{ij}}, \frac{x_{ij}}{a_{ij}} \right\}$$
- Taking into account that $a_{ij} = 1/a_{ji}$ and $x_{ij} = x_i/x_j$, we have
$$d(\boldsymbol{A}, \boldsymbol{X}) = \log \max_{1 \leq i,j \leq n} \max \left\{ \frac{a_{ij} x_j}{x_i}, \frac{a_{ji} x_i}{x_j} \right\} = \log \max_{1 \leq i,j \leq n} \frac{a_{ij} x_j}{x_i}$$
- Since logarithm is monotone, the minimization of the logarithm is equivalent to minimizing its argument, which leads to the problem
$$\min_{\boldsymbol{x}} \max_{1 \leq i,j \leq n} \frac{a_{ij} x_j}{x_i}$$

## Tropical Representation

- Representation of the objective function in terms of max-algebra with addition $\oplus$ defined as $\max$ and multiplication $\otimes$ as usual

$$\max_{1\leq i,j\leq n} \frac{a_{ij}x_j}{x_i} \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{1\leq i,j\leq n} x_i^{-1}a_{ij}x_j \qquad \text{(after replacing operations)}$$

- Vector representation of the objective function

$$\bigoplus_{1\leq i,j\leq n} x_i^{-1}a_{ij}x_j = \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

## Pairwise Comparison Problem

- The problem is to find a vector $\boldsymbol{x}$ that provides the minimum

$$\min_{\boldsymbol{x}} \quad \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

## Constrained Rating

- Given a matrix $\boldsymbol{B} = (b_{ij})$ with nonnegative entries, suppose that the final ratings must satisfy the inequalities

$$b_{ij} x_j \leq x_i, \qquad i, j = 1, \ldots, n$$

- These constraints may require, for instance, that the rating of alternative $j$ must be at least in two times higher than that of $i$
- Combining the inequalities $b_{ij} x_j \leq x_i$ for $j = 1, \ldots, n$ gives

$$\max(b_{i1} x_1, \ldots, b_{in} x_n) \leq x_i \qquad \text{(in ordinary notation)}$$

$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \qquad \text{(after replacing operations)}$$

## Constrained Pairwise Comparison Problem

- The problem is to find a vector $\boldsymbol{x}$ that provides the minimum

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-} \boldsymbol{A} \boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \end{aligned}$$

# Tropical Algebra: Max-Algebra

- Max-algebra is the set of nonnegative reals $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$ with binary operations of addition $\oplus$ and multiplication $\otimes$
- Addition is defined as taking maximum

$$x \oplus y = \max\{x, y\} \qquad \forall x, y \in \mathbb{R}_+$$

- Addition possesses the idempotency property

$$x \oplus x = \max\{x, x\} = x \qquad \forall x \in \mathbb{R}_+$$

- Multiplication is defined as usual: $x \otimes y = x \times y$
- The neutral elements with respect to addition $\mathbb{0}$ and multiplication $\mathbb{1}$ coincide with the arithmetic zero $0$ and one $1$
- The multiplicative inverse and power have the usual meaning
- The additive inverse does not exists, and subtraction is undefined

## Examples

- The operations $\oplus$ and $\otimes$ are defined on nonnegative reals $\mathbb{R}_+$
- Addition:
$$2 \oplus 0 = 2 \qquad (\max(2,0) = 2)$$
$$1 \oplus 3 = 3 \qquad (\max(1,3) = 3)$$
- Multiplication:
$$1 \otimes 0 = 0 \qquad (1 \times 0 = 0)$$
$$2 \otimes (1/3) = 2/3 \qquad (2 \times (1/3) = 2/3)$$
- Exponentiation:
$$2^2 = 4 \qquad (2^2 = 4)$$
$$8^{1/3} = 2 \qquad (8^{1/3} = 2)$$
- Inversion:
$$1^{-1} = 1 \qquad (1^{-1} = 1)$$
$$2^{-1} = 1/2 \qquad (2^{-1} = 1/2)$$

## Max-Plus Algebra

- Max-plus algebra is the extended set of reals $\mathbb{R} \cup \{-\infty\}$ with binary operations of addition $\oplus$ and multiplication $\otimes$
- Addition is idempotent and defined as

$$x \oplus y = \max\{x, y\} \qquad \forall x, y \in \mathbb{R} \cup \{-\infty\}$$

- Multiplication is invertible and defined as arithmetic addition

$$x \otimes y = x + y \qquad \forall x, y \in \mathbb{R} \cup \{-\infty\}$$

- The neutral elements are given by

$$\mathbb{0} = -\infty, \qquad \mathbb{1} = 0$$

- For each $x \in \mathbb{R}$ its inverse $x^{-1}$ coincides with the opposite number $-x$ in the standard arithmetic
- The power $x^y$ corresponds to the arithmetic product $x \times y$

## Examples

- The operations $\oplus$ and $\otimes$ are defined on $\mathbb{R} \cup \{-\infty\}$
- Addition:
$$2 \oplus 0 = 2 \qquad (\max(2,0) = 2)$$
$$1 \oplus (-3) = 1 \qquad (\max(1,-3) = 1)$$
- Multiplication:
$$1 \otimes 0 = 1 \qquad (1 + 0 = 1)$$
$$2 \otimes (-3) = -1 \qquad (2 + (-3) = -1)$$
- Exponentiation:
$$2^2 = 4 \qquad (2 \times 2 = 4)$$
$$(-2)^{1/3} = -2/3 \qquad ((-2) \times (1/3) = -2/3)$$
- Inversion:
$$1^{-1} = -1 \qquad (1 \times (-1) = -1)$$
$$(-2)^{-1} = 2 \qquad ((-2) \times (-1) = 2)$$

## Idempotent Semifield

- Idempotent semifield is the algebraic system $\langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$
- The carrier set $\mathbb{X}$ includes the zero $\mathbb{0}$ and one $\mathbb{1}$, $\mathbb{0} \neq \mathbb{1}$
- The set $\mathbb{X}$ is closed under addition $\oplus$ and multiplication $\otimes$
- Both operations $\oplus$ and $\otimes$ are associative and commutative
- Multiplication $\otimes$ distributes over addition $\oplus$
- Addition is idempotent: $x \oplus x = x$ for all $x \in \mathbb{X}$
- Multiplication is invertible: for each $x \neq \mathbb{0}$ there exists inverse $x^{-1}$
- Idempotent addition induces a partial order on $\mathbb{X}$ by to the rule

$$x \leq y \qquad \text{if and only if} \qquad x \oplus y = y$$

## Idempotent Semifield (cont.)

- Integer powers are defined for each $x \neq \mathbb{0}$ and natural $n$ by

$$x^0 = \mathbb{1}, \quad x^n = x^{n-1} \otimes x, \quad x^{-n} = (x^{-1})^n, \quad \mathbb{0}^n = \mathbb{0}$$

- Algebraic completeness: the equation $x^n = a$ is solvable for each $a \in \mathbb{X}$ and natural $n$ (existence of rational exponents)
- Linear order: the partial order induced by idempotent addition by the rule $x \leq y \Longleftrightarrow x \oplus y = y$ is extendable to a total order
- Absorption rule: $x \otimes \mathbb{0} = \mathbb{0}$ for all $x \in \mathbb{X}$
- In what follows, the multiplication sign $\otimes$, as usual, is omitted

## Examples of Idempotent Semifields

- Max-algebra:
$$\mathbb{R}_{\max} = \langle \mathbb{R}_{+} \cup \{0\}, 0, 1, \max, \times \rangle$$
- Max-plus algebra:
$$\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle$$
- Min-algebra:
$$\mathbb{R}_{\min} = \langle \mathbb{R}_{+} \cup \{+\infty\}, +\infty, 1, \min, \times \rangle$$
- Min-plus algebra:
$$\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, + \rangle$$
- The semifields $\mathbb{R}_{\max,\times}$, $\mathbb{R}_{\max,+}$, $\mathbb{R}_{\min,\times}$, $\mathbb{R}_{\min,+}$ are isomorphic

## Isomorphism of Idempotent Semifields

$$
\begin{tikzcd}
\mathbb{R}_{\max,+} \arrow[r, shift left, "y=-x"] \arrow[d, shift left, "y=e^x"] & \mathbb{R}_{\min,+} \arrow[l, shift left, "y=-x"] \arrow[d, shift left, "y=e^x"] \\
\mathbb{R}_{\max} \arrow[r, shift left, "y=1/x"] \arrow[u, shift left, "y=\ln x"] & \mathbb{R}_{\min} \arrow[l, shift left, "y=1/x"] \arrow[u, shift left, "y=\ln x"]
\end{tikzcd}
$$

- Isomorphism of the semifields $\mathbb{R}_{\max,+}$ , $\mathbb{R}_{\min,+}$ , $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$

## Examples of Idempotent Semirings

- Max-min algebra:

$$\mathbb{R}_{\max,\min} = \langle \mathbb{R} \cup \{-\infty, +\infty\}, -\infty, +\infty, \max, \min \rangle$$

- Algebra defined on the set $\mathbb{X}$ of all subsets of a compact set $S$ :

$$\mathbb{X}_{\cup,\cap} = \langle \mathbb{X}, S, \emptyset, \cup, \cap \rangle$$

## Properties of Operations

- The extremal property of addition (majority law):

$$x \leq x \oplus y, \quad y \leq x \oplus y, \qquad \forall x, y \in \mathbb{X}$$

- The monotonicity of addition and multiplication:

$$x \leq y \quad \Longrightarrow \quad x \oplus z \leq y \oplus z, \quad xz \leq yz, \qquad \forall x, y, z \in \mathbb{X}$$

- The equivalence of inequalities:

$$x \oplus y \leq z \quad \Longleftrightarrow \quad x \leq z, \quad y \leq z, \qquad \forall x, y, z \in \mathbb{X}$$

- The monotonicity of powers:

$$x \leq y \quad \Longrightarrow \quad \begin{cases} x^q \geq y^q, & \text{if } q < 0; \\ x^q \leq y^q, & \text{if } q \geq 0; \end{cases} \qquad \forall x, y \in \mathbb{X} \setminus \{\mathbb{0}\}$$

## Binomial Identity

- A tropical analogue of binomial identity:

$$(x \oplus y)^\alpha = x^\alpha \oplus y^\alpha \qquad \forall x, y \in \mathbb{X}, \quad \alpha > 0$$

- Extension of the identity to $n$ terms:

$$(x_1 \oplus \cdots \oplus x_n)^\alpha = x_1^\alpha \oplus \cdots \oplus x_n^\alpha \qquad \forall x_1, \ldots, x_n \in \mathbb{X}, \quad \alpha \geq 0$$

- A tropical analogue of the inequality between arithmetic and geometric means:

$$x \oplus y \geq (xy)^{1/2}, \qquad \forall x, y \in \mathbb{X}$$

- Extension of the inequality to $n$ terms:

$$x_1 \oplus \cdots \oplus x_n \geq (x_1 \cdots x_n)^{1/n}, \qquad \forall x_1, \ldots, x_n \in \mathbb{X}$$

# Linear Function: Definition and Properties

- A tropical analogue of linear function $f : \mathbb{X} \to \mathbb{X}$ is given by

$$f(x) = ax \oplus b, \qquad a, b \in \mathbb{X}$$

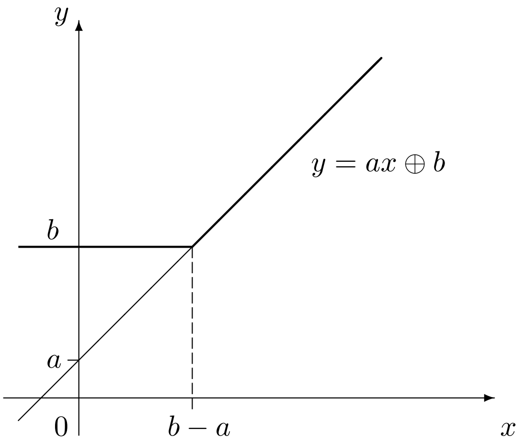
- If $b = \mathbb{0}$ the function is called homogeneous
- The additive property of the function:

$$f(x_1 \oplus x_2) = a(x_1 \oplus x_2) \oplus b = (ax_1 \oplus b) \oplus (ax_2 \oplus b) = f(x_1) \oplus f(x_2)$$

- As in the conventional algebra, we have

$$b = f(\mathbb{0}), \qquad a = \lim_{x \to \infty} x^{-1} f(x)$$

## Graph of Linear Function in $\mathbb{R}_{\max,+}$




- Graph of Linear Function in the framework of $\mathbb{R}_{\max,+}$

## Linear Equation in One Variable

- The general linear equation in one variable takes the form

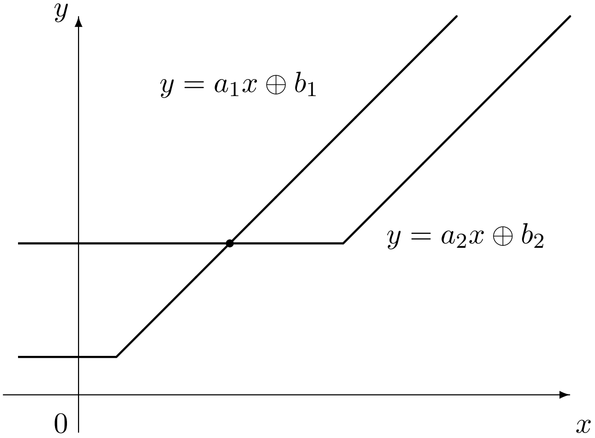
$$a_1 x \oplus b_1 = a_2 x \oplus b_2$$

- This equation cannot be reduced as follows:

$$ax = b$$

- In the framework of $\mathbb{R}_{\max,+}$, it can be solved graphically

## Graphical Solution of Linear Equation




- An example of the solution of linear equation

## Proposition

*The following statements hold:*

1. *If* $a_1 < a_2$ *and* $b_2 < b_1$ *, or* $a_2 < a_1$ *and* $b_1 < b_2$ *, then there is a unique solution*
$$x = (a_1 \oplus a_2)^{-1}(b_1 \oplus b_2);$$
2. *If* $a_1 \neq a_2$ *and* $b_1 \neq b_2$ *and both conditions of the previous case do not hold, then the equation has no solution;*
3. *If* $a_1 = a_2$ *and* $b_1 \neq b_2$ *, then the solution is given by the inequality*
$$x \geq a_1^{-1}(b_1 \oplus b_2);$$
4. *If* $a_1 \neq a_2$ *and* $b_1 = b_2$ *, then the solution is given by the inequality*
$$x \leq (a_1 \oplus a_2)^{-1} b_1;$$
5. *If* $a_1 = a_2$ *and* $b_1 = b_2$ *, then any* $x \in \mathbb{X}$ *is a solution*

## Vector Algebra

- The matrix and vector operations follow the standard rules, where the operations $+$ and $\times$ are replaced by $\oplus$ and $\otimes$
- Addition of vectors $\boldsymbol{a} = (a_j)$ and $\boldsymbol{b} = (b_j)$, and multiplication by scalar $x$ are given by the entrywise formulas

$$\{\boldsymbol{a} \oplus \boldsymbol{b}\}_j = a_j \oplus b_j, \qquad \{x\boldsymbol{a}\}_j = x a_j$$

- Zero vector has all components equal to $\mathbb{0}$ and it is denoted $\mathbf{0}$
- A vector without zero components is called regular
- For any nonzero column vector $\boldsymbol{a} = (a_j)$, its multiplicative conjugate transpose is the row vector $\boldsymbol{a}^{-} = (a_j^{-})$, where

$$a_j^{-} = \begin{cases} a_j^{-1}, & \text{if } a_j \neq \mathbb{0}; \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

## Examples

- Vector operations in the framework of $\mathbb{R}_{\max,+}$ :
- Vector addition

$$\begin{pmatrix} -1 \\ 0 \\ \mathbb{0} \end{pmatrix} \oplus \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

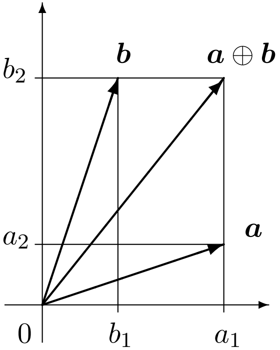
- Scalar multiplication

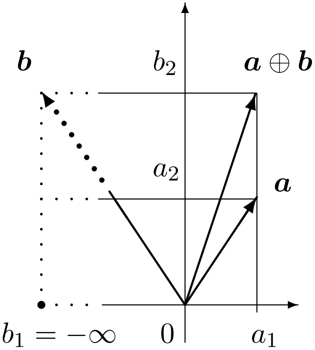
$$(-1) \begin{pmatrix} \mathbb{0} \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \mathbb{0} \\ 1 \\ -2 \end{pmatrix}$$

- Multiplicative conjugation

$$\begin{pmatrix} \mathbb{0} \\ -1 \\ 3 \end{pmatrix}^{-} = \begin{pmatrix} \mathbb{0} & 1 & -3 \end{pmatrix}$$

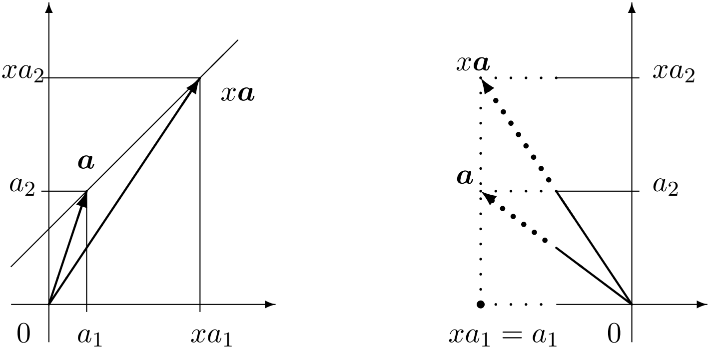
## Graphical Illustration of Vector Addition in $\mathbb{R}^2_{\max,+}$




- Addition of regular vectors (left) and with an irregular vector (right)
- Addition follows a Rectangle Rule instead of Parallelogram Rule

## Graphical Illustration of Scalar Multiplication in $\mathbb{R}^2_{\max,+}$




- Scalar multiplication of a regular vector (left) and of an irregular vector (right)

## Linear Dependence of Vectors

- Linear combination of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ with coefficients $x_1, \ldots, x_n \in \mathbb{X}$ is defined as the sum $x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n$
- A vector $\boldsymbol{b}$ is linearly dependent on vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, if there are scalars $x_1, \ldots, x_n \in \mathbb{X}$ such that

$$\boldsymbol{b} = x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n$$

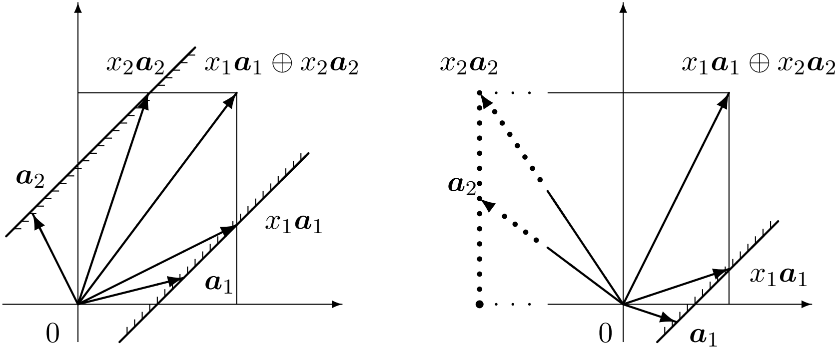
- Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear if $\boldsymbol{b} = x\boldsymbol{a}$ for some $x \in \mathbb{X}$
- The linear span of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is given by

$$\operatorname{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \{x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n | \ x_1, \ldots, x_n \in \mathbb{X}\}$$

  and forms a tropical linear space generated by the vectors
- Any vector $\boldsymbol{y}$ from this space is represented by the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ and vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ in the form

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

## Graphical Illustration of Linear Span of Vectors in $\mathbb{R}^2_{\max,+}$




- The linear span of two regular vectors is a band (left) and of regular and irregular vectors is a half-plane (right)

## Minimal Generating System

- If $\boldsymbol{b}$ is dependent on a system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, but independent of any its subsystem, the system is a minimal generating system for $\boldsymbol{b}$
- Let us verify that the representation of a regular vector as a linear combination of vectors of its minimal generating system is unique
- Suppose there are two different representations of the vector $\boldsymbol{b}$:

$$\boldsymbol{b} = x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n = x_1' \boldsymbol{a}_1 \oplus \cdots \oplus x_n' \boldsymbol{a}_n,$$

- Assume for definiteness that $x_i' < x_i$ for some $i = 1, \ldots, n$
- Then, $\boldsymbol{b} \geq x_i \boldsymbol{a}_i > x_i' \boldsymbol{a}_i$, which means that $x_i' \boldsymbol{a}_i$ does not affect $\boldsymbol{b}$
- Therefore, the vector $\boldsymbol{b}$ does not depend on the vector $\boldsymbol{a}_i$, which contradicts with the minimality of the system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$

## Matrix Algebra

- For conforming matrices $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$ and $\boldsymbol{C} = (c_{ij})$, and a scalar $x$, the matrix operations are given by

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{\boldsymbol{A}\boldsymbol{C}\}_{ij} = \bigoplus_{k} a_{ik} c_{kj}, \quad \{x\boldsymbol{A}\}_{ij} = x a_{ij}$$

- The zero matrix has all components equal to $\mathbb{0}$ and is denoted $\mathbf{0}$
- A matrix without zero columns (rows) is column (row) regular
- For any nonzero matrix $\boldsymbol{A} = (a_{ij})$, its multiplicative conjugate transpose is the matrix $\boldsymbol{A}^{-} = (a_{ij}^{-})$, where

$$a_{ij}^{-} = \begin{cases} a_{ji}^{-1}, & \text{if } a_{ji} \neq \mathbb{0}; \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

## Examples

- Matrix operations in the framework of $\mathbb{R}_{\max,+}$ :
- Matrix addition

$$\begin{pmatrix} -1 & 1 \\ 0 & -2 \\ 2 & \mathbb{0} \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 3 & \mathbb{0} \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \\ 2 & 0 \end{pmatrix}$$

- Matrix multiplication

$$\begin{pmatrix} 1 & -1 & \mathbb{0} \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & \mathbb{0} \\ 0 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 2 & 0 \end{pmatrix}$$

- Scalar multiplication

$$2\begin{pmatrix} -1 & 1 & \mathbb{0} \\ 0 & 2 & -2 \\ 1 & \mathbb{0} & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & \mathbb{0} \\ 2 & 4 & 0 \\ 3 & \mathbb{0} & -1 \end{pmatrix}$$

- Multiplicative conjugation

$$\begin{pmatrix} -1 & 1 & \mathbb{0} \\ 0 & 2 & -2 \\ 1 & \mathbb{0} & -3 \end{pmatrix}^{-} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -2 & \mathbb{0} \\ \mathbb{0} & 2 & 3 \end{pmatrix}$$

## Square Matrices

- The identity matrix has the usual diagonal form

$$I = \begin{pmatrix} \mathbb{1} & & \mathbb{0} \\ & \ddots & \\ \mathbb{0} & & \mathbb{1} \end{pmatrix}$$

- The identity matrix in max-plus algebra $\mathbb{R}_{\max,+}$

$$I = \begin{pmatrix} 0 & & -\infty \\ & \ddots & \\ -\infty & & 0 \end{pmatrix}$$

- The identity matrix in max-algebra $\mathbb{R}_{\max}$ has the conventional form with the arithmetic 1's on the diagonal and 0's elsewhere

## Square Matrices (cont.)

- Positive integer powers of a square matrix $\boldsymbol{A}$ indicates repeated (tropical) multiplication of the matrix by itself

$$\mathbf{0}^p = \mathbf{0}, \quad \boldsymbol{A}^0 = \boldsymbol{I}, \quad \boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{p-1}, \qquad \forall p \geq 1$$

- The entry $a_{ij}^{(k)}$ of the matrix $\boldsymbol{A}^k$ takes the form

$$a_{ij}^{(k)} = \bigoplus_{1 \leq i_1, \dots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$$

## Inverse Matrix

- A matrix $\boldsymbol{A}^{-1}$ is the inverse matrix for $\boldsymbol{A}$, if $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$
- A matrix is invertible if and only if it has only one nonzero entry in each row and column (proof by contradiction)
- The inverse matrix exists only for
  - *the strictly diagonal matrices (without zero diagonal entries),*
  - *the matrices obtained from the strictly diagonal by permutation of rows and/or columns*
- If a matrix $\boldsymbol{A}$ has an inverse, then $\boldsymbol{A}^{-} = \boldsymbol{A}^{-1}$
- Since the class of invertible matrices is very poor, conjugate transposition plays more important role than matrix inversion

# Linear Operators: Linear Equations

## Linear Equations

- Any $(m \times n)$-matrix $\boldsymbol{A}$ defines an operator from $\mathbb{X}^n$ to $\mathbb{X}^m$
- For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{X}^n$ and scalar $\alpha \in \mathbb{X}$, we have
  1. $\boldsymbol{A}(\boldsymbol{x} \oplus \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{A}\boldsymbol{y}$ *(additivity);*
  2. $\boldsymbol{A}(\alpha\boldsymbol{x}) = \alpha\boldsymbol{A}\boldsymbol{x}$ *(multiplicativity)*
- With these properties, the operator $\boldsymbol{A}$ is a linear operator
- The general linear equation in an unknown vector $\boldsymbol{x}$ is given by

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{C}\boldsymbol{x} \oplus \boldsymbol{d}$$

## Special Cases of General Equation

- One-sided equations:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{d}, \qquad \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{d}$$

- One-sided inequalities:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d}, \qquad \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{d}$$

- Two-sided equations:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}, \qquad \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$$

- Two-sided inequalities:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}, \qquad \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$$

# One-Sided Inequality: Definitions and Preliminaries

- Given an $(m \times n)$-matrix $\boldsymbol{A}$ and $m$-vector $\boldsymbol{b}$, the following inequality in an unknown $n$-vector $\boldsymbol{x}$ is called one-sided:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

- This inequality has the unknown vector $\boldsymbol{x}$ only on one side
- This one-sided inequality always has solutions; specifically, the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ obviously satisfies the inequality
- We obtain a solution of the inequality by applying properties of conjugate transposition and simple algebraic manipulations

## Proposition (Properties of Conjugate Transposition)

*The following statements hold:*

1. *For any regular $n$-vector, the following inequality is valid:*

$$\boldsymbol{x}\boldsymbol{x}^{-} = \begin{pmatrix} x_1 x_1^{-1} & \dots & x_1 x_n^{-1} \\ \vdots & \ddots & \vdots \\ x_n x_1^{-1} & \dots & x_n x_n^{-1} \end{pmatrix} \geq \boldsymbol{I}$$

2. *For any nonzero vector $\boldsymbol{x}$, the following equality holds:*

$$\boldsymbol{x}^{-}\boldsymbol{x} = \bigoplus_{i:\, x_i \neq \mathbb{0}} x_i^{-1} x_i = \mathbb{1}$$

- Since all diagonal entries of the matrix $\boldsymbol{x}\boldsymbol{x}^{-}$ are equal to $\mathbb{1}$, and off-diagonal entries are greater than $\mathbb{0}$, we see that $\boldsymbol{x}\boldsymbol{x}^{-} \geq \boldsymbol{I}$
- The inequality $\boldsymbol{x}^{-}\boldsymbol{x} = \mathbb{1}$ is trivially holds for any nonzero $\boldsymbol{x}$

# Solution of One-Sided Inequality

- Given an $(m \times n)$-matrix $\boldsymbol{A}$ and $m$-vector $\boldsymbol{b}$, we start with the problem to find $n$-vectors $\boldsymbol{x}$ that satisfy the one-sided inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

**Lemma (Solution of One-Sided Inequality)**

*For any column-regular (without zero columns) matrix $\boldsymbol{A}$ and regular (w/o zero entries) vector $\boldsymbol{b}$, all solutions of the inequality are given by*

$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

## Proof

- Let us verify that the following inequalities are equivalent:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \qquad \boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

- Left multiplication of the first inequality by the matrix $(\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}$ and monotonicity of multiplication yield the result

$$(\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{b}$$

- It follows from the properties of conjugate transposition that

$$(\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{A} \geq \boldsymbol{I}, \qquad \boldsymbol{b}^{-}\boldsymbol{b} = \mathbb{1}$$

- After substitution, we obtain the second inequality as follows:

$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{b} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

- Left multiplication of the second inequality by $\boldsymbol{A}$ leads to the first:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} \leq \boldsymbol{b}\boldsymbol{b}^{-}\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \boldsymbol{b} \qquad \blacksquare$$

## Example in Two Dimensions

- Consider the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ with the matrix and vectors

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

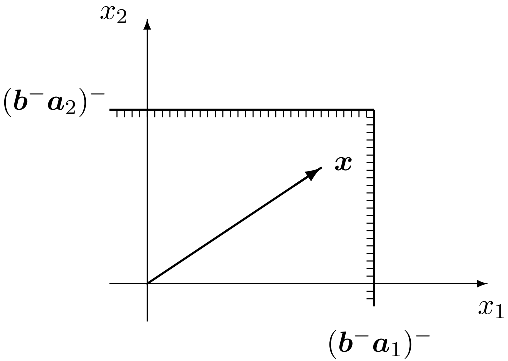
- We denote the columns of the matrix by small bold letters:

$$\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2), \qquad \boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \qquad \boldsymbol{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

- We assume that $a_{11}, a_{12}, a_{21}, a_{22} > \mathbb{0}$ and $d_1, d_2 > \mathbb{0}$
- All solutions of the inequality are given by

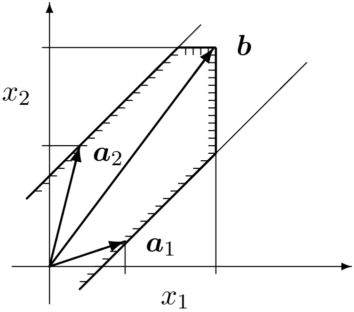
$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} (\boldsymbol{b}^{-}\boldsymbol{a}_1)^{-1} \\ (\boldsymbol{b}^{-}\boldsymbol{a}_2)^{-1} \end{pmatrix} = \begin{pmatrix} (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1} \\ (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1} \end{pmatrix}$$

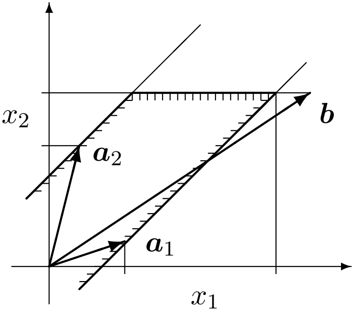
## Graphical Illustration of Solution to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ in $\mathbb{R}^2_{\max,+}$




- Solution of the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ with $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$, represented in the space of solution vectors $\boldsymbol{x}$ in Cartesian coordinates

## Graphical Illustration of Solution to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ in $\mathbb{R}^2_{\max,+}$




- Illustration of solutions in the space of columns in $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$
- Solutions are shown for the cases when $\boldsymbol{b}$ is inside (left) and outside (right) the linear span of the columns of $\boldsymbol{A}$

## Numerical Example

- Consider an inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ defined in $\mathbb{R}^3_{\max,+}$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

- To solve the inequality, we first calculate the product

$$\boldsymbol{b}^{-}\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}$$

- After conjugation of the obtained result, we arrive at the solution

$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$$

# One-Sided Equation: Definitions and Preliminaries

- Given an $(m \times n)$-matrix $\boldsymbol{A}$ and $m$-vector $\boldsymbol{b}$, the one-sided equation in an unknown $n$-vector $\boldsymbol{x}$ is defined as follows:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

- This equation has the unknown on one side and can be referred to as an equation of the first kind (by analogy with integral equations)
- Since the equation may have no (exact) solution, we concentrate on finding a best approximate solution in the sense of some metric
- We examine the distance between a vector and a tropical vector subspace, and then apply the result to solve the equation

## Generalized Metric

- We define the distance between regular vectors $\boldsymbol{x} = (x_i)$ and $\boldsymbol{y} = (y_i)$ by the following distance function:
$$d(\boldsymbol{x}, \boldsymbol{y}) = \bigoplus_i \left(x_i y_i^{-1} \oplus x_i^{-1} y_i\right) = \boldsymbol{y}^- \boldsymbol{x} \oplus \boldsymbol{x}^- \boldsymbol{y}$$
- If one of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is regular and the other is not, we put $d(\boldsymbol{x}, \boldsymbol{y}) = \infty$, where $\infty$ denotes an undefined value
- We observe that this function has its minimum value equal to $\mathbb{1}$
- In the context of $\mathbb{R}_{\max,+}$ (max-plus algebra), where $\mathbb{1} = 0$, the distance function $d$ coincides with the Chebyshev metric
$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_i |x_i - y_i| = \max_i \max(x_i - y_i, y_i - x_i)$$
- In $\mathbb{R}_{\max}$ (max-algebra), the function $d$ can be considered as a generalized metric that takes values in the interval $[1, \infty)$

## Distance Between Linear Span and Vector

- Consider the linear span of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, which is given by

$$\mathcal{A} = \mathrm{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \{x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n |\ x_1, \ldots, x_n \in \mathbb{X}\}$$

- With the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ and vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$, any vector $\boldsymbol{y} \in \mathcal{A}$ is represented as

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

- Define the distance between the linear span and a vector $\boldsymbol{b}$ as

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$$

## Proposition (Distance from Linear Span to Regular Vector)

*If the vector $\boldsymbol{b}$ is regular, then*

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\text{regular } \boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$$

## Proof

- Take a vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ such that $d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$ achieves its minimum
- If $\boldsymbol{y}$ is not regular, then the statement is true since $d(\boldsymbol{y}, \boldsymbol{b}) = \infty$
- Suppose $\boldsymbol{y} = (y_i)$ is regular, and assume the corresponding vector $\boldsymbol{x} = (x_j)$ to have a zero component, say $x_k = \mathbb{0}$
- We define the following index set and threshold value:

$$I = \{i | a_{ij} > \mathbb{0}\} \neq \emptyset, \qquad \varepsilon = \min\{a_{ij}^{-1} y_i | i \in I\} > \mathbb{0}$$

- We replace $x_k = \mathbb{0}$ by $x_k = \varepsilon$, and note that all components of $\boldsymbol{y}$ along with the minimum value of $d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$ remain unchanged ■

- Given a matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, we find the distance between the linear span of the columns in $\boldsymbol{A}$ and $\boldsymbol{b}$ by solving the problem

$$\min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$$

## Lemma (Evaluation of Distance)

*Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ regular vector. Define the scalar*

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}.$$

*Then, the distance between the linear span and vector $\boldsymbol{b}$ is given by*

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \Delta^{1/2},$$

*where the minimum is attained at*

$$\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

## Proof

- Assume both $\boldsymbol{A}$ and $\boldsymbol{b}$ to be regular, and consider the problem

$$\min_{\text{regular } \boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$$

- Substitution of the expression for the distance function yields

$$d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b}$$

- We take any regular vector $\boldsymbol{x}$, and denote the value of distance by

$$r = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b} > \mathbb{0}$$

- It follows from the extremal property of tropical addition that

$$r \geq \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x}, \qquad r \geq (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b}$$

## Proof (cont.)

- Let us solve with respect to $r$ the obtained system of inequalities

$$r \geq \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x}, \qquad r \geq (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b}$$

- The solution of the first inequality as a one-sided inequality yields

$$\boldsymbol{x} \leq r(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

- After left multiplication by $\boldsymbol{A}$ and conjugate transposition, we have

$$(\boldsymbol{A}\boldsymbol{x})^{-} \geq r^{-1}(\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}$$

- Substitution into the second inequality leads to the inequality

$$r \geq r^{-1}(\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = r^{-1}\Delta$$

- As a result, we obtain the lower bound for the objective function

$$r = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b} \geq \Delta^{1/2}$$

## Proof (cont.)

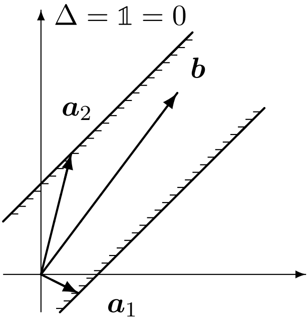
- Let us verify that the following lower bound is attainable:
$$r = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b} \geq \Delta^{1/2}$$

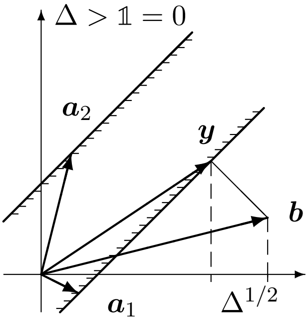
- Indeed, substitution of the vector $\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ gives
$$r = \Delta^{1/2}\boldsymbol{b}^{-}\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} \oplus \Delta^{-1/2}(\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \Delta^{1/2}$$

- Therefore, $\Delta^{1/2}$ is a strict (attainable) lower bound, and hence the minimum of the objective function which is the distance in question

- The vector $\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is a solution of the minimization problem that gives a closest vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ in the linear span ■

## Graphical Illustration of Evaluation of Distance in $\mathbb{R}^2_{\max,+}$




- Evaluation of the distance from a vector to a linear span
- Illustration is given for the cases when $\boldsymbol{b}$ is inside (left) and outside (right) the linear span of the columns of $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$

## Lemma (Linear Dependence)

*Let* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ *be vectors such that the matrix* $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ *is regular, and* $\boldsymbol{b}$ *be regular vector. Define the scalar*

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}.$$

*The vector* $\boldsymbol{b}$ *is linearly dependent on vectors* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ *if and only if*

$$\Delta = \mathbb{1}$$

## Proof

- From geometric viewpoint, a vector $\boldsymbol{b}$ is linearly dependent on $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ if $\boldsymbol{b}$ belongs to the linear span $\mathcal{A} = \operatorname{span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$
- By the lemma on evaluation of distance, the equality $\Delta = \mathbb{1}$ means that $\boldsymbol{b} \in \mathcal{A}$, whereas the inequality $\Delta > \mathbb{1}$ that $\boldsymbol{b} \notin \mathcal{A}$ ∎

## Linearly Independent System of Vectors

- A set of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is a linearly dependent system if at least one vector is linearly dependent on others
- Otherwise, this set forms a linearly independent system
- Two systems of vectors are equivalent systems if each vector of one system is linearly dependent on vectors of the other system
- Consider a system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ that may have dependent vectors
- To construct an equivalent independent system, we successively reduce the system until it becomes linearly independent
- We use a procedure that applies the criterion provided by the lemma on linear dependence to examine the vectors one by one
- The procedure removes a vector if it is linearly dependent on others, or leaves the vector in the system otherwise

# Solution of One-Sided Equation

- Given a matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, we consider the equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

**Theorem (Solution of One-Sided Equation)**

*Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ a regular vector. Define the scalar*

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}.$$

*Then, the following statements hold:*

1. *If $\Delta = \mathbb{1}$, then the equation has regular solutions including*

$$\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-};$$

2. *The above solution is the maximal solution, and it is unique if the columns in $\boldsymbol{A}$ form a minimal generating set for the vector $\boldsymbol{b}$*
3. *If $\Delta \neq \mathbb{1}$, then there are no regular solutions*

## Proof

- The fact that equality $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ holds for some $\boldsymbol{x}$ means that the vector $\boldsymbol{b}$ belongs to the linear span $\mathcal{A}$ of columns in the matrix $\boldsymbol{A}$
- It follows from the lemma on evaluation of distance that $\boldsymbol{b} \in \mathcal{A}$ if and only if the following condition holds:
$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \mathbb{1}$$
- As another consequence of the lemma, one can see that the regular solutions of the equation (if any exists) include the vector
$$\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$
- By the lemma on one-sided inequality, the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ is equivalent to $\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$, and thus this solution is maximal
- The uniqueness condition follows from unique representation of a vector as a linear combination of its minimal set of generators ■

## Example in Two Dimension

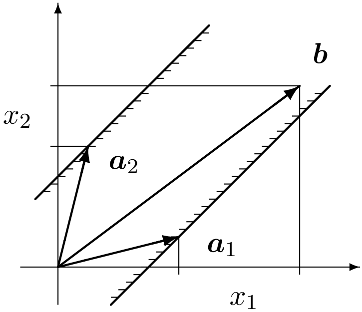
- Consider the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with the matrix and vectors

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Assume that $a_{11}, a_{12}, a_{21}, a_{22} > \mathbb{0}$ and $b_1, b_2 > \mathbb{0}$
- Suppose the condition $\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \mathbb{1}$ holds
- The maximal solution takes the form

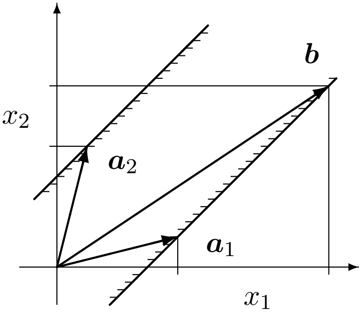
$$\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} (\boldsymbol{b}^{-}\boldsymbol{a}_1)^{-1} \\ (\boldsymbol{b}^{-}\boldsymbol{a}_2)^{-1} \end{pmatrix} = \begin{pmatrix} (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1} \\ (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1} \end{pmatrix}$$

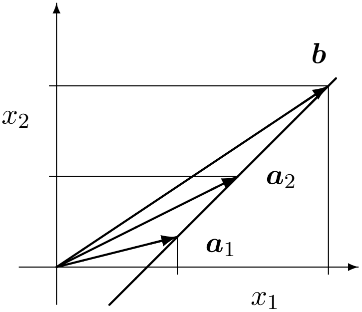
## Graphical Illustration of Unique Solution in $\mathbb{R}^2_{\max,+}$




- If the vector $\boldsymbol{b}$ is not collinear to any of the vectors $\boldsymbol{a}_1$ or $\boldsymbol{a}_2$, then the solution vector $\boldsymbol{x}$ of the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is unique

## Graphical Illustration of Nonunique Solutions in $\mathbb{R}^2_{\max,+}$




- If the vector $\boldsymbol{b}$ is collinear with only one vectors from $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (left), or with both vectors (right), the solution is nonunique

## Representation of Nonunique Solutions

- Suppose that the vector $\boldsymbol{b}$ is collinear to $\boldsymbol{a}_1$, but not to $\boldsymbol{a}_2$
- Then, the solution is any vector $\boldsymbol{x}$ with the components

$$x_1 = (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1},$$
$$x_2 \leq (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1}$$

- Assume both vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ to be collinear to each other
- In this case, there are two solution sets that consist of vectors $\boldsymbol{x}' = (x_1', x_2')^T$ and $\boldsymbol{x}'' = (x_1'', x_2'')^T$, where

$$x_1' = (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1}, \qquad x_1'' \leq (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1},$$
$$x_2' \leq (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1}; \qquad x_2'' = (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1}$$

## All Solutions of One-Sided Equation

- Let $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ be a matrix, $\boldsymbol{b}$ a vector, and $I$ be a subset of column indices of the matrix $\boldsymbol{A}$ such that $\boldsymbol{b} \in \operatorname{span}\{\boldsymbol{a}_i | i \in I\}$
- Then, any vector $\boldsymbol{x} = (x_i)$ with components

$$\begin{aligned} x_i &= (\boldsymbol{b}^{-}\boldsymbol{a}_i)^{-}, &\quad \text{if } i \in I; \\ x_i &\leq (\boldsymbol{b}^{-}\boldsymbol{a}_i)^{-}, &\quad \text{if } i \notin I \end{aligned}$$

  is a solution to the one-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
- To obtain all solutions to the equation, one has to find all minimal subsets of columns in $\boldsymbol{A}$ that generate the vector $\boldsymbol{b}$
- A generating subset of columns in the matrix $\boldsymbol{A}$ is minimal if it contains no proper subset that generates the vector $\boldsymbol{b}$
- To represent the solution for a minimal set given by $I$, we replace the equations in $\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ for $x_i$ with $i \notin I$ by inequalities

## Numerical Examples

- Consider an equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ defined in $\mathbb{R}^{3}_{\max,+}$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

- To verify the condition $\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \mathbb{1}$, we calculate

$$\boldsymbol{b}^{-}\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 \neq \mathbb{1} = 0$$

- Consider an equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ defined in $\mathbb{R}^3_{\max,+}$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ \mathbb{0} & -1 & 3 \\ 0 & \mathbb{0} & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbb{0} = -\infty$$

- We verify that the condition $\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \mathbb{1}$ is true:

$$\boldsymbol{b}^{-}\boldsymbol{A} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ \mathbb{0} & -1 & 3 \\ 0 & \mathbb{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} 2 & 0 & -1 \\ \mathbb{0} & -1 & 3 \\ 0 & \mathbb{0} & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 = \mathbb{1}$$

- Since the condition $\Delta = \mathbb{1}$ holds, we conclude that the equation has solutions, including the maximal solution

$$\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

- We can describe all solutions by finding all minimal sets of columns in the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ that generate the vector $\boldsymbol{b}$
- If all columns in $\boldsymbol{A}$ form the minimal generating set (no column can be dropped), then the vector $\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is unique solution
- To see if we can drop a column, say the first column, to have $\boldsymbol{b} \in \operatorname{span}(\boldsymbol{a}_2, \boldsymbol{a}_3)$, we need to verify the condition

$$\Delta_{(1)} = (\boldsymbol{A}_{(1)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(1)})^{-})^{-}\boldsymbol{b} = \mathbb{1}, \qquad \boldsymbol{A}_{(1)} = (\boldsymbol{a}_2, \boldsymbol{a}_3)$$

- If $\Delta_{(1)} = \mathbb{1}$, we further verify that $\boldsymbol{b} \in \operatorname{span}(\boldsymbol{a}_2)$ and $\boldsymbol{b} \in \operatorname{span}(\boldsymbol{a}_3)$

- We form with the matrices

$$\boldsymbol{A}_{(1)} = \begin{pmatrix} 0 & -1 \\ -1 & 3 \\ \mathbb{0} & 1 \end{pmatrix}, \quad \boldsymbol{A}_{(2)} = \begin{pmatrix} 2 & -1 \\ \mathbb{0} & 3 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_{(3)} = \begin{pmatrix} 2 & 0 \\ \mathbb{0} & -1 \\ 0 & \mathbb{0} \end{pmatrix}$$

- We check whether $\Delta_{(i)} = (\boldsymbol{A}_{(i)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(i)})^{-})^{-}\boldsymbol{b} = \mathbb{1}$ for $i = 1, 2, 3$:

$$\boldsymbol{b}^{-}\boldsymbol{A}_{(1)} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 3 \\ \mathbb{0} & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}_{(1)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(1)})^{-} = \begin{pmatrix} 0 & -1 \\ -1 & 3 \\ \mathbb{0} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\Delta_{(1)} = (\boldsymbol{A}_{(1)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(1)})^{-})^{-}\boldsymbol{b} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 = \mathbb{1}$$

- In the same way, we obtain

$$\Delta_{(2)} = (\boldsymbol{A}_{(2)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(2)})^{-})^{-}\boldsymbol{b} = 0 = \mathbb{1},$$
$$\Delta_{(3)} = (\boldsymbol{A}_{(3)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(3)})^{-})^{-}\boldsymbol{b} = 1 \neq \mathbb{1}$$

- Since $\Delta_{(1)} = \Delta_{(2)} = \mathbb{1}$, the set of all columns in $\boldsymbol{A}$ is not minimal
- Taking into account that both $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are not collinear to $\boldsymbol{b}$, the set $(\boldsymbol{a}_2, \boldsymbol{a}_3)$ cannot be further reduced, and hence is minimal
- By the same argument, we conclude that $(\boldsymbol{a}_1, \boldsymbol{a}_3)$ is a minimal set
- All solutions of the equation form two subsets given by

$$\begin{aligned} x_1 &\leq -1, \qquad & x_1 &= -1, \\ x_2 &= 1, & x_2 &\leq 1, \\ x_3 &= -2, & x_3 &= -2 \end{aligned}$$

# Two-Sided Inequality: Definitions and Preliminaries

- Given an $(n \times n)$-matrix $\boldsymbol{A}$, the following inequality in an unknown $n$-vector $\boldsymbol{x}$ is called two-sided:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

- This inequality has the unknown vector $\boldsymbol{x}$ on both sides
- This two-sided inequality always has solutions; specifically, the trivial solution $\boldsymbol{x} = \mathbf{0}$ obviously satisfies the inequality
- We obtain a solution of the inequality by applying a tropical analogue of matrix determinant and Kleene (star) matrix operator

## Trace and Determinant of Matrix

- The trace of a square matrix $\boldsymbol{A} = (a_{ij})$ of order $n$ is given by

$$\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn} = \bigoplus_{i=1}^{m} a_{ii}$$

- For any matrix $\boldsymbol{A} = (a_{ij})$ of order $n$, a tropical analogue of the matrix determinant is a trace function of matrix powers defined as

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n = \bigoplus_{k=1}^{n} \operatorname{tr} \boldsymbol{A}^k$$

- The determinant is the sum of cyclic products of matrix entries

$$\operatorname{Tr}(\boldsymbol{A}) = \bigoplus_{1 \leq k \leq n} \bigoplus_{1 \leq i_1, \dots, i_k \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$$

## Kleene Star Operator

- For any square matrix $\boldsymbol{A}$, a Kleene star operator is defined which maps the matrix $\boldsymbol{A}$ into the infinite some of integer powers
$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2 \oplus \cdots = \bigoplus_{k \geq 0} \boldsymbol{A}^k$$

## Lemma (Extremal Property of Kleene Star)

*For any $(n \times n)$-matrix $\boldsymbol{A}$ with $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the next statements hold:*

1. *For any integer $k \geq 0$, the following inequality is valid:*
$$\boldsymbol{A}^k \leq \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1};$$
2. *The Kleene star matrix reduces to the finite sum of powers*
$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} = \bigoplus_{k=0}^{n-1} \boldsymbol{A}^k$$

## Proof

- We verify that if $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, then for all integers $k \geq 0$, we have
$$\boldsymbol{A}^k \leq \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}$$
- The entries of the power $\boldsymbol{A}^k$ is defined by entries in $\boldsymbol{A} = (a_{ij})$ as
$$\{\boldsymbol{A}^k\}_{ij} = \bigoplus_{1 \leq i_1, \dots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$$
- Consider a product under summation and denote it by
$$P = a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$$
- We rearrange multipliers to write $P = P_{\mathsf{c}} P_{\mathsf{a}}$, where $P_{\mathsf{c}}$ consists of cyclic subproducts of $P$ and $P_{\mathsf{a}}$ does not have cyclic subproducts
- We first extract from $P$ all cyclic subproducts of length $1$ (of the form $a_{kk}$), then the subproducts of length $2$ ($a_{kl} a_{lk}$) and so on
- We continue this until the subproducts of length $n$ are extracted

## Proof (cont.)

- Since any cyclic product of length from $1$ to $n$ is not greater than $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, we see that the inequality $P_{\mathsf{c}} \leq \mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ is valid
- After extracting all cyclic products in $P$, we denote the remaining subproduct by $P_{\mathsf{a}}$ to represent the original product as $P = P_{\mathsf{c}} P_{\mathsf{a}}$
- We note that $P_{\mathsf{a}}$ is acyclic with a length not exceeding $n-1$
- Since each product of length $l \leq n-1$, starting from index $i$ and ending with $j$ is bounded from above by $\{\boldsymbol{A}^l\}_{ij}$, we have
$$P_{\mathsf{a}} \leq \{\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}\}_{ij}$$
- As a results, we arrive at the upper bound for $P$:
$$a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = P = P_{\mathsf{c}} P_{\mathsf{a}} \leq \{\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}\}_{ij}$$
- This bound holds for all products under summation, and therefore
$$\{\boldsymbol{A}^k\}_{ij} \leq \{\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}\}_{ij} \qquad \blacksquare$$

# Solution of Two-Sided Inequality

- Given a $(n \times n)$-matrix $\boldsymbol{A}$, we solve the problem to find $n$-vectors $\boldsymbol{x}$ that satisfy the two-sided inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

**Theorem**

*The following statements hold:*

1. *If* $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, *then all solutions of the inequality are given by*

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}, \qquad \boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1},$$

   *where* $\boldsymbol{u}$ *is a vector of parameters;*

2. *If* $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, *the inequality has only trivial solution* $\boldsymbol{x} = \boldsymbol{0}$

## Proof of Statement 1

- Let us show that under the condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the vector $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ satisfies the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with any vector $\boldsymbol{u}$
- Indeed, since $\boldsymbol{A}\boldsymbol{A}^{*} = \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n} \leq \boldsymbol{A}^{*}$, we have

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}^{*}\boldsymbol{u}) = (\boldsymbol{A}\boldsymbol{A}^{*})\boldsymbol{u} \leq \boldsymbol{A}^{*}\boldsymbol{u} = \boldsymbol{x}$$

- Suppose now that $\boldsymbol{x}$ is a solution of the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$, and verify that the equality $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ holds for some vector $\boldsymbol{u}$
- Left multiplication of two-sided inequality by $\boldsymbol{A}$ yields the inequality $\boldsymbol{A}^{k}\boldsymbol{x} \leq \boldsymbol{x}$ for all integers $k \geq 1$, and therefore,

$$\boldsymbol{A}^{*}\boldsymbol{x} = (\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1})\boldsymbol{x} \leq \boldsymbol{x}$$

- Because $\boldsymbol{A}^{*} \geq \boldsymbol{I}$, the inequality $\boldsymbol{A}^{*}\boldsymbol{x} \geq \boldsymbol{x}$ is valid as well
- Both inequalities result in the equality $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ with $\boldsymbol{u} = \boldsymbol{x}$ ∎

## Example in Two Dimensions

- Consider the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with the matrix and vector

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Suppose that the condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ holds
- We calculate the matrix

$$\boldsymbol{A}^2 = \begin{pmatrix} a_{11}^2 \oplus a_{12}a_{21} & a_{11}a_{12} \oplus a_{12}a_{22} \\ a_{21}a_{11} \oplus a_{22}a_{21} & a_{12}a_{21} \oplus a_{22}^2 \end{pmatrix}$$

- Consider the condition

$$\mathrm{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{A}^2 = a_{11} \oplus a_{22} \oplus a_{12}a_{21} \leq \mathbb{1}$$

- It follows from this condition, that the next inequalities are valid:

$$a_{11} \leq \mathbb{1}, \qquad a_{22} \leq \mathbb{1}, \qquad a_{12}a_{21} \leq \mathbb{1}$$

- Since $a_{11}, a_{22} \leq \mathbb{1}$, the Kleene star matrix takes the form

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix} \oplus \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}$$

- All solutions of the two-sided inequality are given by

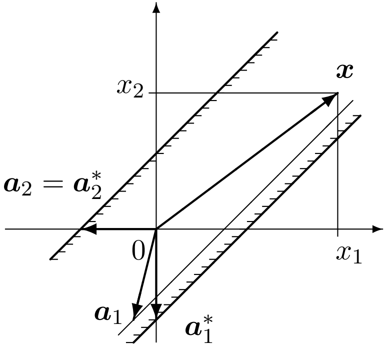
$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

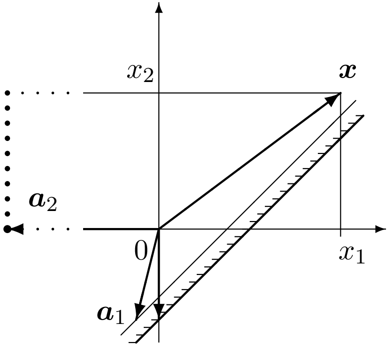
  where $\boldsymbol{u}$ is a vector of parameters

- In scalar form, the solution is written as

$$x_1 = u_1 \oplus a_{12} u_2,$$
$$x_2 = a_{21} u_1 \oplus u_2$$

## Graphical Illustration of Solution to $Ax \leq x$ in $\mathbb{R}^2_{\max,+}$




- Solutions of a two-sided inequality for a matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$ without (left) and with (right) zero entries

## Numerical Example

- Consider an inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ defined in $\mathbb{R}^3_{\max,\times}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix}$$

- To verify the existence condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, we calculate

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix},$$

$$\boldsymbol{A}^3 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}$$

- Since $\mathrm{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{A}^2 \oplus \operatorname{tr} \boldsymbol{A}^3 = 1 = \mathbb{1}$, the condition holds

- Calculation of the Kleene star matrix $\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2$ yields

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix}$$

- All solutions of the two-sided inequality are given by

$$\boldsymbol{x} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad u_1, u_2, u_3 \geq 0$$

- In terms of conventional algebra, the solution is written as

$$x_1 = \max\left(u_1, \frac{1}{2}u_2, 2u_3\right), \qquad x_2 = \max(2u_1, u_2, 4u_3),$$
$$x_3 = \max\left(\frac{1}{3}u_1, \frac{1}{6}u_2, u_3\right)$$

## Representation of Generating Matrix

- Consider an inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with a matrix $\boldsymbol{A}$ of order $n$
- Suppose that $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ and examine the solution defined by the Kleene matrix $\boldsymbol{A}^{*} = (\boldsymbol{a}_{1}^{*}, \ldots, \boldsymbol{a}_{n}^{*})$ and vector $\boldsymbol{u} = (u_{1}, \ldots, u_{n})^{T}$ as

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u} = u_{1}\boldsymbol{a}_{1}^{*} \oplus \cdots \oplus u_{n}\boldsymbol{a}_{n}^{*}$$

- This representation means that each solution is a linear combination of columns $\boldsymbol{a}_{1}^{*}, \ldots, \boldsymbol{a}_{n}^{*}$, which generate all solutions
- If a column in $\boldsymbol{A}^{*}$ is linearly dependent on others, it can be removed from the set of generators without losing solutions
- To eliminate dependent columns, we apply the procedure of constructing an equivalent linear independent system of vectors

## Numerical Example

- Consider the solution in the last example, generated by the matrix

$$\boldsymbol{A}^{*} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix}$$

- Since the first and second columns are collinear, one of them, say the second, can be removed to represent the solution as

$$\boldsymbol{x} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1/3 & 1 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1, u_2 \geq 0$$

- In terms of standard algebra, the solution is written as

$$x_1 = \max(u_1, 2u_2), \quad x_2 = \max(2u_1, 4u_2), \quad x_3 = \max\left(\frac{1}{3}u_1, u_2\right)$$

# Two-Sided Equation: Definitions and Preliminaries

- Given an $(n \times n)$-matrix $\boldsymbol{A}$, the following equation in an unknown $n$-vectors $\boldsymbol{x}$ is called a two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$$

- The equation has the unknown on both sides and can be called an equation of the second kind (by analogy with integral equations)
- This equation is also referred to as the Bellman equation
- The two-sided equation always has the trivial solution $\boldsymbol{x} = \boldsymbol{0}$
- Existence conditions for nontrivial solutions can be represented in terms of the function (determinant) $\mathrm{Tr}(\boldsymbol{A}) = \mathrm{tr}\,\boldsymbol{A} \oplus \cdots \oplus \mathrm{tr}\,\boldsymbol{A}^n$
- We describe all solutions in a parametric form that is based on calculation of the Kleene star matrix $\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}$

## Proposition (Solution Set of Two-Sided Equation)

*The set of solutions of the two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ is closed under vector addition and scalar multiplication*

## Proof

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ and $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{y}$, and $\alpha$ and $\beta$ are scalars, then for the vector $\boldsymbol{z} = \alpha\boldsymbol{x} \oplus \beta\boldsymbol{y}$, we have

$$\boldsymbol{A}\boldsymbol{z} = \boldsymbol{A}(\alpha\boldsymbol{x} \oplus \beta\boldsymbol{y}) = \alpha\boldsymbol{A}\boldsymbol{x} \oplus \beta\boldsymbol{A}\boldsymbol{y} = \alpha\boldsymbol{x} \oplus \beta\boldsymbol{y} = \boldsymbol{z} \quad \blacksquare$$

- We now can conclude that the set of solutions is a tropical vector space, which can be described by its generating matrix
- Below, we show how this generating matrix can be constructed

## Kleene Star and Kleene Plus Matrices

- For any square matrix $\boldsymbol{A}$, the Kleene Star and Kleene Plus matrices are defined as infinite sums given by

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^{2} \oplus \cdots, \qquad \boldsymbol{A}^{+} = \boldsymbol{A}\boldsymbol{A}^{*} = \boldsymbol{A} \oplus \boldsymbol{A}^{2} \oplus \cdots$$

- It follows from the extremal property of the Kleene star that if $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, then for any $k \geq 0$, the following inequality holds:

$$\boldsymbol{A}^{k} \leq \boldsymbol{A}^{*}$$

- As a result, when $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the infinite sums become finite to define the Kleene star and Kleene plus matrices in the form

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}, \qquad \boldsymbol{A}^{+} = \boldsymbol{A}\boldsymbol{A}^{*} = \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n}$$

## Proposition

*If the condition* $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ *holds, then the following equality is valid:*

$$\boldsymbol{I} \oplus \boldsymbol{A}^{+} = \boldsymbol{A}^{*}$$

## Proof

- Since $\boldsymbol{A}^{k} \leq \boldsymbol{A}^{*}$ for all integers $k > 0$, we immediately obtain
$$\boldsymbol{I} \oplus \boldsymbol{A}^{+} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} \oplus \boldsymbol{A}^{n} = \boldsymbol{A}^{*} \oplus \boldsymbol{A}^{n} = \boldsymbol{A}^{*} \qquad \blacksquare$$

## Remarks

- If the equality $\boldsymbol{I} \oplus \boldsymbol{A}^{+} = \boldsymbol{A}^{*}$ holds, then $\boldsymbol{A}^{+} \leq \boldsymbol{A}^{*}$
- In the matrices $\boldsymbol{A}^{*} = (a_{ij}^{*})$ and $\boldsymbol{A}^{+} = (a_{ij}^{+})$, the corresponding entries $a_{ij}^{*}$ and $a_{ij}^{+}$ coincide except for diagonal entries
- The diagonal entries satisfy the conditions $a_{ii}^{*} = \mathbb{1}$ and $a_{ii}^{+} \leq \mathbb{1}$

## Proposition

*If the condition* $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ *is valid, then the following statements hold:*

1. *The Kleene matrices* $\boldsymbol{A}^{*} = (\boldsymbol{a}_i^{*})$ *and* $\boldsymbol{A}^{+} = (\boldsymbol{a}_i^{+})$ *have common columns that coincide;*
2. *The equality* $\boldsymbol{a}_i^{*} = \boldsymbol{a}_i^{+}$ *holds if and only if* $a_{ii}^{(m)} = \mathbb{1}$*, where* $a_{ii}^{(m)}$ *is a diagonal entry in the matrix* $\boldsymbol{A}^{m} = (a_{ij}^{(m)})$ *for some* $m = 1, \ldots, n$

## Proof

- If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, the off-diagonal entries in $\boldsymbol{A}^{*}$ and $\boldsymbol{A}^{+}$ coincide
- The condition $\mathrm{Tr}(\boldsymbol{A}) = \mathrm{tr}\,\boldsymbol{A} \oplus \cdots \oplus \mathrm{tr}\,\boldsymbol{A}^{n} = \mathbb{1}$ means that the equality $\mathrm{tr}\,\boldsymbol{A}^{m} = \mathbb{1}$ is valid for at least one $m = 1, \ldots, n$
- The last equality holds if and only if $a_{ii}^{(m)} = \mathbb{1}$ for some index $i$
- In this case, we have $a_{ii}^{*} = a_{ii}^{+} = \mathbb{1}$, and thus $\boldsymbol{a}_i^{*} = \boldsymbol{a}_i^{+}$ ∎

## Matrix $A^\times$

- In order to describe solutions of the two-sided equation in a compact vector form, we introduce a matrix $\boldsymbol{A}^\times$ as follows
- Let $\boldsymbol{A}$ be a square $(n \times n)$-matrix such that $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$
- Let $\boldsymbol{A}^*$ and $\boldsymbol{A}^+$ be the Kleene star and Kleene plus matrices for $\boldsymbol{A}$ with columns $\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_n^*$ and $\boldsymbol{a}_1^+, \ldots, \boldsymbol{a}_n^+$ respectively
- We define a matrix $\boldsymbol{A}^\times$ of the same size as $\boldsymbol{A}$ with the columns

$$\boldsymbol{a}_i^\times = \begin{cases} \boldsymbol{a}_i^*, & \text{if } \boldsymbol{a}_i^* = \boldsymbol{a}_i^+; \\ \boldsymbol{0}, & \text{if } \boldsymbol{a}_i^* \neq \boldsymbol{a}_i^+; \end{cases}, \qquad i = 1, \ldots, n$$

- If $\mathrm{Tr}(\boldsymbol{A}) \neq \mathbb{1}$, we put $\boldsymbol{A}^\times = \boldsymbol{0}$

# Solution of Two-Sided Equation

- Given a $(n \times n)$-matrix $\boldsymbol{A}$, we solve the problem of finding $n$-vectors $\boldsymbol{x}$ that satisfy the two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$$

**Lemma (Solution of Two-Sided Equation)**

*If the condition $\operatorname{Tr}(\boldsymbol{A}) = \mathbb{1}$ holds, then any vector given by*

$$\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{v}, \qquad \boldsymbol{v} > \boldsymbol{0},$$

*satisfies the two-sided equation*

## Proof

- If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then the matrices $\boldsymbol{A}^*$ and $\boldsymbol{A}^+$ have common columns that are the same, say columns $\boldsymbol{a}_i^* = \boldsymbol{a}_i^+$
- Since the equality $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^+$ always holds, we can write

$$\boldsymbol{A}\boldsymbol{a}_i^* = \boldsymbol{a}_i^+ = \boldsymbol{a}_i^*,$$

  which means that the column $\boldsymbol{a}_i^*$ satisfies the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$
- We observe that all common columns of the matrices $\boldsymbol{A}^*$ and $\boldsymbol{A}^+$ form nonzero columns in the matrix $\boldsymbol{A}^\times$
- The vector $\boldsymbol{x} = \boldsymbol{A}^\times \boldsymbol{v}$ for any vector $\boldsymbol{v} > \boldsymbol{0}$ is a linear combination of columns in $\boldsymbol{A}^\times$, and thus satisfies the two-sided equation ■

## Irreducible Matrices

- A matrix $\boldsymbol{A}$ is reducible if simultaneous row-column permutations can put it into a block-triangular form, and irreducible otherwise
- The lower triangular normal form of a matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ \boldsymbol{A}_{s1} & \boldsymbol{A}_{s2} & \dots & \boldsymbol{A}_{ss} \end{pmatrix},$$

where $\boldsymbol{A}_{ii}$ is either an irreducible or zero matrix for all $i = 1, \dots, s$

## Proposition

*If a matrix $\boldsymbol{A}$ is irreducible, then any nontrivial solution $\boldsymbol{x} \neq \mathbf{0}$ of the two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ has no zero entries*

## Proof

- Consider the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ with an irreducible matrix $\boldsymbol{A}$
- Suppose that a nontrivial vector $\boldsymbol{x} = (x_i)$ is a solution of the equation, and verify that $\boldsymbol{x}$ does not have zero entries
- Let $\boldsymbol{x}$ have one zero entry, $x_k = \mathbb{0}$, whereas $x_j > \mathbb{0}$ for all $j \neq k$
- The scalar equation corresponding to row $k$ in $\boldsymbol{A}$ takes the form

$$a_{k1}x_1 \oplus \cdots \oplus a_{kn}x_n = \mathbb{0}$$

- Since $x_j > \mathbb{0}$ for $j \neq k$, the equation holds only if $a_{kj} = \mathbb{0}$, $j \neq k$
- By swapping rows $1$ and $k$, and columns $1$ and $k$, we obtain a matrix with a zero block in the first row, which is a contradiction
- The assumption that the solution vector $\boldsymbol{x}$ has more than one (but not all) zero entries is examined in an analogues way ■

## Proposition (Existence of Nontrivial Solutions)

*The two-sided equation* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ *with irreducible matrix* $\boldsymbol{A}$ *has nontrivial solutions if and only if the condition* $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ *holds*

## Proof

- The sufficiency of the condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ follows from the lemma on the solution of two-sided equation
- To verify the necessity of the condition, assume that $\boldsymbol{x}$ is a nontrivial solution, and show that then $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$
- Let us take an arbitrary cyclic sequence of indices $i_0, \ldots, i_m$, where $i_m = i_0$ and $1 \leq m \leq n$
- It follows from the equations $a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n = x_i$ for all $i$ that

$$a_{i_0 i_1} x_{i_1} \leq x_{i_0}, \qquad a_{i_1 i_2} x_{i_2} \leq x_{i_1}, \quad \ldots, \quad a_{i_{m-1} i_m} x_{i_m} \leq x_{i_{m-1}}$$

## Proof (cont.)

- Consider the inequalities

$$a_{i_0 i_1} x_{i_1} \leq x_{i_0}, \qquad a_{i_1 i_2} x_{i_2} \leq x_{i_1}, \quad \ldots, \quad a_{i_{m-1} i_m} x_{i_m} \leq x_{i_{m-1}}$$

- Side-by-side multiplication of inequalities yields

$$a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{m-1} i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \leq x_{i_0} x_{i_1} \cdots x_{i_{m-1}} = x_{i_1} x_{i_2} \cdots x_{i_m}$$

- By reducing by the common factor $x_{i_1} \cdots x_{i_m} \neq \mathbb{0}$, we obtain

$$a_{i_0 i_1} \cdots a_{i_{m-1} i_m} \leq \mathbb{1}$$

- Considering an arbitrary choice of $i_0, \ldots, i_{m-1}$, we have

$$\operatorname{tr} \boldsymbol{A}^m \leq \mathbb{1}, \qquad m = 1, \ldots, n$$

- As a result, the following inequality holds:

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n \leq \mathbb{1}$$

## Proof (cont.)

- It remains to verify that the inequality $\mathrm{Tr}(\boldsymbol{A}) \geq \mathbb{1}$ holds as well
- It follows from the scalar equations $a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n = x_i$ that for any index $i$, there is an index $j$ such that $a_{ij}x_j = x_i$
- Let us take an arbitrary index $i_0$ and construct a sequence $i_0, i_1, i_2, \ldots$ by choosing indices that satisfy the equalities

$$a_{i_0 i_1} x_{i_1} = x_{i_0}, \qquad a_{i_1 i_2} x_{i_2} = x_{i_1}, \qquad \ldots$$

- We select a cyclic subsequence $i_l, \ldots, i_{l+m}$ with $i_l = i_{l+m}$, $m \leq n$
- After side-by-side multiplication of equalities that correspond to the subsequence, and reduction by $x_{i_l} \cdots x_{i_{l+m}} \neq \mathbb{0}$, we obtain

$$a_{i_l i_{l+1}} \cdots a_{i_{l+m-1} i_{l+m}} = \mathbb{1}$$

- As a consequence of the last equality, we have

$$\mathrm{Tr}(\boldsymbol{A}) \geq \mathrm{tr}\, \boldsymbol{A}^m \geq a_{i_l i_{l+1}} \cdots a_{i_{l+m-1} i_{l+m}} = \mathbb{1} \qquad \blacksquare$$

## Complete Solution of Equation with Irreducible Matrix

- A complete solution of the two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ with an irreducible matrix is provided by the next statement

### Theorem (Complete Solution)

*Let $\boldsymbol{A}$ is an irreducible matrix. Then, the following statements hold:*

1. *If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then all regular solutions are given by*

$$\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{u},$$

   *where $\boldsymbol{u} > \boldsymbol{0}$ is a vector of parameters;*

2. *If $\mathrm{Tr}(\boldsymbol{A}) \neq \mathbb{1}$, then the equation has only trivial solution $\boldsymbol{x} = \boldsymbol{0}$*

## Example in Two Dimensions

- Consider the equation $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with the matrix and vector

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Suppose that the existence condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ holds
- We calculate the matrix

$$\boldsymbol{A}^2 = \begin{pmatrix} a_{11}^2 \oplus a_{12}a_{21} & a_{11}a_{12} \oplus a_{12}a_{22} \\ a_{21}a_{11} \oplus a_{22}a_{21} & a_{12}a_{21} \oplus a_{22}^2 \end{pmatrix}$$

- Consider the existence condition and represent it as follows:

$$\mathrm{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{A}^2 = a_{11} \oplus a_{22} \oplus a_{12}a_{21} = \mathbb{1}$$

- As a consequence of this condition, we have the inequalities

$$a_{11} \leq \mathbb{1}, \qquad a_{22} \leq \mathbb{1}, \qquad a_{12}a_{21} \leq \mathbb{1}$$

- Since $a_{11}, a_{22} \leq \mathbb{1}$, the Kleene matrices take the form

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix} \oplus \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix},$$

$$\boldsymbol{A}^{+} = \boldsymbol{A} \oplus \boldsymbol{A}^{2} = \boldsymbol{A}\boldsymbol{A}^{*} = \begin{pmatrix} a_{11} \oplus a_{12}a_{21} & a_{12} \end{pmatrix}$$

- To obtain the solution $\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{v}$, we need to derive the matrix $\boldsymbol{A}^{\times}$
- If $a_{11} = \mathbb{1}$, $a_{22} < \mathbb{1}$ and $a_{12}a_{21} < \mathbb{1}$, then we have

$$\boldsymbol{A}^{+} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & a_{12}a_{21} \oplus a_{22} \end{pmatrix}, \qquad \boldsymbol{A}^{\times} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ a_{21} & \mathbb{0} \end{pmatrix}$$

- By removing the second column of $\boldsymbol{A}^{\times}$, we write the solution as

$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} \\ a_{21} \end{pmatrix} v, \quad v \in \mathbb{X}$$

- If $a_{11} < \mathbb{1}$, $a_{22} = \mathbb{1}$ and $a_{12}a_{21} < \mathbb{1}$, then we have

$$\boldsymbol{A}^{*} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}, \qquad \boldsymbol{A}^{+} = \begin{pmatrix} a_{11} \oplus a_{12}a_{21} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}$$

- From the matrices $\boldsymbol{A}^{*}$ and $\boldsymbol{A}^{+}$, we obtain the generating matrix

$$\boldsymbol{A}^{\times} = \begin{pmatrix} \mathbb{0} & a_{12} \\ \mathbb{0} & \mathbb{1} \end{pmatrix}$$

- The corresponding solution can be written as

$$\boldsymbol{x} = \begin{pmatrix} a_{12} \\ \mathbb{1} \end{pmatrix} v, \quad v \in \mathbb{X}$$

- Provided that at least one of the conditions $a_{11} = a_{22} = \mathbb{1}$ and $a_{12}a_{21} = \mathbb{1}$ is satisfied, then we obtain

$$\boldsymbol{A}^{\times} = \boldsymbol{A}^{+} = \boldsymbol{A}^{\times} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}$$

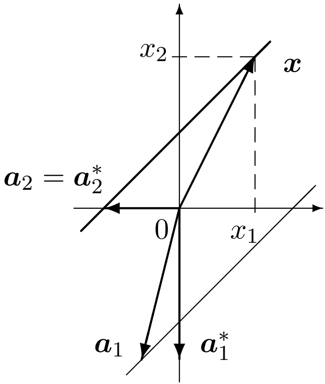
- In the case when $a_{12}, a_{21} \neq \mathbb{1}$, the solution is given by

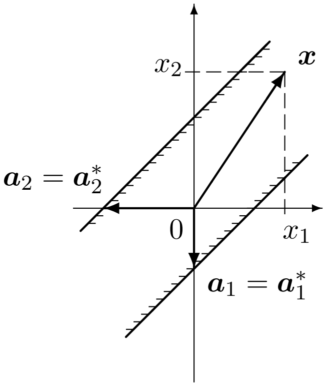
$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix} \boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{X}^{2}$$

- Under the condition $a_{12} = a_{21} = \mathbb{1}$, we have the solution

$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} v, \quad v \in \mathbb{X}$$

## Graphical Illustration of Solution to $Ax = x$ in $\mathbb{R}^2_{\max,+}$




- Examples of the solution generated by one column (left) and solution given by the linear span of both columns of $\boldsymbol{A}$ (right)

## Numerical Example

- Consider an equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ defined in $\mathbb{R}^3_{\max,\times}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix}$$

- To verify the existence condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, we calculate

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}, \qquad \boldsymbol{A}^3 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}$$

- Since $\mathrm{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{A}^2 \oplus \operatorname{tr} \boldsymbol{A}^3 = 1 = \mathbb{1}$, the condition holds

- Calculation of the Kleene star and Kleene plus matrices yields

$$\boldsymbol{A}^* = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix}, \qquad \boldsymbol{A}^+ = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}$$

- Since the first two columns in the matrices coincide, we obtain

$$\boldsymbol{A}^+ = \begin{pmatrix} 1 & 1/2 & \mathbb{0} \\ 2 & 1 & \mathbb{0} \\ 1/3 & 1/6 & \mathbb{0} \end{pmatrix}$$

- All solutions of the two-sided equation are given by

$$\boldsymbol{x} = \begin{pmatrix} 1 & 1/2 \\ 2 & 1 \\ 1/3 & 1/6 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1, u_2 \geq 0$$

- Consider the generating matrix of the solution

$$\begin{pmatrix} 1 & 1/2 \\ 2 & 1 \\ 1/3 & 1/6 \end{pmatrix}$$

- Since both columns in the matrix are collinear, we can drop one of them, say the second, to write the solution as

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 1/3 \end{pmatrix} u, \qquad u > \mathbb{0}$$

- In terms of conventional algebra, the solution is written as

$$x_1 = u, \qquad x_2 = 2u, \qquad x_3 = \frac{1}{3}u$$

## Nonhomogeneous Two-Sided Equation

- Given an $(n \times n)$-matrix $\boldsymbol{A}$ and $n$-vector $\boldsymbol{b}$, the following equation is called nonhomogeneous two-sided equation:

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$$

- The equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ is a homogeneous two-sided equation

**Lemma**

*The nonhomogeneous equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ with irreducible matrix $\boldsymbol{A}$ has solutions if and only if at least one of the following conditions hold:*

1. $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$;
2. $\boldsymbol{b} = \boldsymbol{0}$.

*If the equation has solutions, then $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b}$ is its minimal solution*

## Proof (Sufficiency)

- Under the condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the iterations of the equation yield

$$\begin{aligned}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b}) \oplus \boldsymbol{b} = \boldsymbol{A}^2\boldsymbol{x} \oplus (\boldsymbol{I} \oplus \boldsymbol{A})\boldsymbol{b} \\ = \boldsymbol{A}^3\boldsymbol{x} \oplus (\boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2)\boldsymbol{b} = \cdots = \boldsymbol{A}^n\boldsymbol{x} \oplus \boldsymbol{A}^*\boldsymbol{b}\end{aligned}$$

- As a result, the equation reduces to that in the equivalent form

$$\boldsymbol{A}^n\boldsymbol{x} \oplus \boldsymbol{A}^*\boldsymbol{b} = \boldsymbol{x}$$

- As a consequence of the last equation, we have the inequality

$$\boldsymbol{x} \geq \boldsymbol{A}^*\boldsymbol{b}$$

- Let us verify that the vector $\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b}$ is a solution of the equation

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{A}(\boldsymbol{A}^*\boldsymbol{b}) \oplus \boldsymbol{b} = (\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^n)\boldsymbol{b} = \boldsymbol{A}^*\boldsymbol{b} = \boldsymbol{x}$$

- Taking into account the above inequality, this solution is minimal
- Note that if $\boldsymbol{b} = \boldsymbol{0}$, the equation always has a solution $\boldsymbol{x} = \boldsymbol{0}$ ∎

## General Solution of Equation

- The set of all possible solutions of an equation (inequality) is called the general solution of the equation (inequality)
- The general solution of the two-sided inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ is given in parametric form by

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{X}^{n}$$

- The general solution of the homogeneous two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ with irreducible matrix is given in parametric form by

$$\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{v}, \qquad \boldsymbol{v} \in \mathbb{X}^{n}$$

- Every single solution of an equation (inequality) is referred to as a particular solution of the equation (inequality)

## Lemma

*Let $\boldsymbol{u}$ be the minimal (particular) solution of a nonhomogeneous equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ with irreducible matrix $\boldsymbol{A}$ and $\boldsymbol{v}$ be the general solution of the homogeneous equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$.*

*Then, the general solution of the nonhomogeneous equation is given by*

$$\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$$

## Proof

- Suppose $\boldsymbol{u}$ is a solution of the nonhomogeneous equation, and $\boldsymbol{v}$ is a solution of the homogeneous equation
- Then $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$ is a solution of the nonhomogeneous equation as

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{A}(\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{b} = (\boldsymbol{A}\boldsymbol{u} \oplus \boldsymbol{b}) \oplus (\boldsymbol{A}\boldsymbol{v}) = \boldsymbol{u} \oplus \boldsymbol{v} = \boldsymbol{x}$$

## Proof (cont.)

- Let $x$ be any solution of the nonhomogeneous eqution
- We verify that $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$, where $\boldsymbol{u}$ is the minimal solution of the nonhomogeneous, and $\boldsymbol{v}$ a solution of the homogeneous equation
- Under the condition $\mathrm{Tr}(\boldsymbol{A}) \neq \mathbb{1}$, the homogeneous equation has only trivial solution, and then $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$, where $\boldsymbol{u} = \boldsymbol{x}$, $\boldsymbol{v} = \boldsymbol{0}$
- Assume now that the condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ is satisfied
- Put $\boldsymbol{u} = \boldsymbol{A}^*\boldsymbol{b}$, a minimal solution of the nonhomogeneous equation
- Then, $\boldsymbol{x} \geq \boldsymbol{A}^*\boldsymbol{b} = \boldsymbol{u}$, and hence, there is a vector $\boldsymbol{v}'$ which complements $u$ to $x$ as follows:

$$\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}'$$

## Proof (cont.)

- Since $\boldsymbol{u} = \boldsymbol{A}^{*}\boldsymbol{b}$, we can write

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{u} \oplus \boldsymbol{v}') = \boldsymbol{A}\boldsymbol{A}^{*}\boldsymbol{b} \oplus \boldsymbol{A}\boldsymbol{v}'$$

- Substitution into the homogeneous equation yields

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = (\boldsymbol{I} \oplus \boldsymbol{A}\boldsymbol{A}^{*})\boldsymbol{b} \oplus \boldsymbol{A}\boldsymbol{v}' = \boldsymbol{A}^{*}\boldsymbol{b} \oplus \boldsymbol{A}\boldsymbol{v}' = \boldsymbol{u} \oplus \boldsymbol{A}\boldsymbol{v}'$$

- Therefore, with $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{v}'$, the equality $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$ remains valid
- Further substitution $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{A}\boldsymbol{v}'$ leads to the result

$$\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = (\boldsymbol{I} \oplus \boldsymbol{A}\boldsymbol{A}^{*})\boldsymbol{b} \oplus \boldsymbol{A}^{2}\boldsymbol{v}' = \boldsymbol{A}^{*}\boldsymbol{b} \oplus \boldsymbol{A}^{2}\boldsymbol{v}' = \boldsymbol{u} \oplus \boldsymbol{A}^{2}\boldsymbol{v}'$$

- We can continue substitutions, and then conclude that the equality $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$ holds for any vector $\boldsymbol{v} = \boldsymbol{A}^{m}\boldsymbol{v}'$ for all integers $m \geq 0$
- As a result, this equality is valid for the vectors $\boldsymbol{A}^{*}\boldsymbol{v}'$ and $\boldsymbol{A}^{+}\boldsymbol{v}'$

## Proof (cont.)

- Let us take the vector $\boldsymbol{v}' = (v_i')$ with the entries
$$v_i' = \begin{cases} x_i, & \text{if } u_i < x_i; \\ \mathbb{0}, & \text{if } u_i = x_i; \end{cases} \qquad i = 1, \dots, n$$
- We see that $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}'$, and the inequality $\boldsymbol{v}' \leq \boldsymbol{v}$ holds for any vector $\boldsymbol{v}$ such that $\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$ (that is, $\boldsymbol{v}'$ is the minimal vector)
- In particular, $\boldsymbol{v}' \leq \boldsymbol{A}\boldsymbol{v}'$, which after left multiplication by $\boldsymbol{A}^*$, yields
$$\boldsymbol{A}^*\boldsymbol{v}' \leq \boldsymbol{A}^+\boldsymbol{v}'$$
- Since the opposite inequality $\boldsymbol{A}^*\boldsymbol{v}' \geq \boldsymbol{A}^+\boldsymbol{v}'$ always holds, we have
$$\boldsymbol{A}^*\boldsymbol{v}' = \boldsymbol{A}^+\boldsymbol{v}'$$
- It remains to put $\boldsymbol{v} = \boldsymbol{A}^*\boldsymbol{v}'$, and then write
$$\boldsymbol{A}\boldsymbol{v} = \boldsymbol{A}\boldsymbol{A}^*\boldsymbol{v}' = \boldsymbol{A}^+\boldsymbol{v}' = \boldsymbol{A}^*\boldsymbol{v}' = \boldsymbol{v}$$

  which means that $\boldsymbol{v}$ is a solution of homogeneous equation ■

- Given a $(n \times n)$-matrix $\boldsymbol{A}$ and $n$-vector $\boldsymbol{b}$, we find $n$-vectors $\boldsymbol{x}$ that satisfy the nonhomogeneous two-sided equation

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$$

- Combining the lemmas on the existence of solutions and general solution of nonhomogeneous equation yields the next statements

### Theorem

*Suppose that the nonhomogeneous equation with irreducible matrix has solutions, and let $\boldsymbol{x}$ be the general solution of the equation.*

*Then, the following statements hold:*

1. *If $\mathrm{Tr}(\boldsymbol{A}) < \mathbb{1}$, then there is a single solution $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b}$;*
2. *If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b} \oplus \boldsymbol{A}^{\times}\boldsymbol{v}$ for any vector $\boldsymbol{v} \in \mathbb{X}^{n}$;*
3. *If $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, then the equation has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ (when $\boldsymbol{b} = \boldsymbol{0}$)*

## Example in Two Dimensions

- Consider the equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ with

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- Suppose that $a_{11}, a_{12}, a_{21}, a_{22} > \mathbb{0}$
- Let us calculate the Klene matrix $\boldsymbol{A}^*$ and then the vector $\boldsymbol{A}^*\boldsymbol{b}$:

$$\boldsymbol{A}^* = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}, \qquad \boldsymbol{A}^*\boldsymbol{b} = \begin{pmatrix} b_1 \oplus a_{12}b_2 \\ a_{21}b_1 \oplus b_2 \end{pmatrix}$$

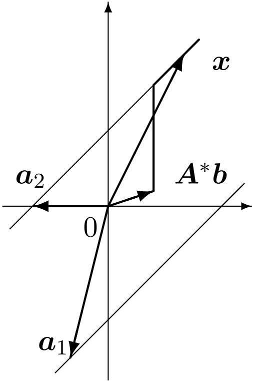
- If $\mathrm{Tr}(\boldsymbol{A}) < \mathbb{1}$, then

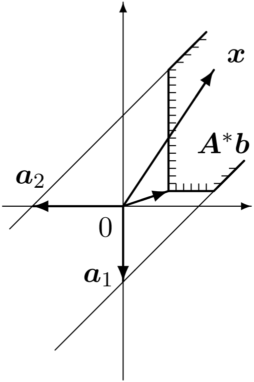
$$\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b}$$

- If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then

$$\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b} \oplus \boldsymbol{A}^\times \boldsymbol{v}, \qquad \boldsymbol{v} \in \mathbb{X}^2$$

## Illustration of Solution to $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ in $\mathbb{R}^2_{\max,+}$ if $\mathrm{Tr}(\boldsymbol{A}) = 0$




- Examples of solutions when the vector $\boldsymbol{b}$ is outside the solution set of homogeneous equation (left), and inside the set (right)

## Solution of Two-Sided Inequalities

- Suppose that a $n$-matrix $\boldsymbol{A}$ and $n$-vector $\boldsymbol{b}$ are given
- By analogy with two-sided equations, the following inequality can be referred to as a homogeneous two-sided inequality:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

- In the same way, the following inequality can be referred to as a nonhomogeneous two sided inequality

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$$

- We now show that these inequalities can be solved by converting into corresponding two-sided equations using auxiliary variables

- Given a $(n \times n)$-matrix $\boldsymbol{A}$, the problem is to find $n$-vectors $\boldsymbol{x}$ that satisfy the homogeneous two-sided inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

## Lemma

*The general solution of the homogeneous two-sided inequality with irreducible matrix is given by the following statements.*

1. *If* $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$*, then* $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ *for any vector* $\boldsymbol{u}$*;*
2. *If* $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$*, then there is only trivial solution* $\boldsymbol{x} = \boldsymbol{0}$

## Proof

- If a vector $\boldsymbol{x}$ solves the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$, it is also a solution of the following equation in two unknown vectors $\boldsymbol{x}$ and $\boldsymbol{u}$:
$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{u} = \boldsymbol{x}$$
- For each fixed $\boldsymbol{u}$, this is a nonhomogeneous equation in $\boldsymbol{x}$

## Proof (cont.)

- Consider the obtained nonhomogeneous equation (where $\boldsymbol{b} = \boldsymbol{u}$)

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{u} = \boldsymbol{x}$$

- After solving this equation with respect to $\boldsymbol{x}$, we have the solution

$$\boldsymbol{x} = \begin{cases} \boldsymbol{A}^{*}\boldsymbol{u}, & \text{if } \mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}; \\ \boldsymbol{A}^{*}\boldsymbol{u} \oplus \boldsymbol{A}^{\times}\boldsymbol{v} & \text{if } \mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}; \end{cases}$$

  where $\boldsymbol{u}$ and $\boldsymbol{v}$ are any vectors of parameters
- Since nonzero columns in the matrix $\boldsymbol{A}^{\times}$ coincide with the same columns in $\boldsymbol{A}^{*}$, we can combine both solutions as

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$$

- If $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, the equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{u} = \boldsymbol{x}$ can have only trivial solution $\boldsymbol{x} = \boldsymbol{0}$ which requires that $\boldsymbol{u} = \boldsymbol{0}$ ∎

## Lemma

*The nonhomogeneous inequality* $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$ *with irreducible matrix has solutions if and only if at least one of the following conditions hold:*

1. $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ *;*
2. $\boldsymbol{b} = \mathbf{0}$ *.*

*If the equation has solutions, then* $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b}$ *is its minimal solution*

## Theorem

*Suppose that the nonhomogeneous equation with irreducible matrix has solutions, and let* $\boldsymbol{x}$ *be the general solution of the equation.*

*Then, the following statements hold:*

1. *If* $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ *, then* $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ *for any vector* $\boldsymbol{u} \geq \boldsymbol{b}$ *;*
2. *If* $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$ *, then the inequality has only the trivial solution* $\boldsymbol{x} = \mathbf{0}$ *(when* $\boldsymbol{b} = \mathbf{0}$ *)*

## Proof of Theorem

- We use an auxiliary variable $\boldsymbol{u}$ to transform the inequality $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$ into the equation in both $\boldsymbol{x}$ and $\boldsymbol{u}$ in the form

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \oplus \boldsymbol{u} = \boldsymbol{x}$$

- We solve the equation with respect to $\boldsymbol{x}$ as a nonhomogeneous equation where $\boldsymbol{b}$ is replaced by $\boldsymbol{b} \oplus \boldsymbol{u}$, which yields the result

$$\boldsymbol{x} = \boldsymbol{A}^{*}(\boldsymbol{b} \oplus \boldsymbol{u})$$

- Since the vector $\boldsymbol{u}$ contributes to the solution only when it has entries greater than in $\boldsymbol{b}$, we can represent the solution as

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{b} \quad \blacksquare$$

## Solution of Systems of Equations

- Given regular matrices $\boldsymbol{A}$ and $\boldsymbol{C}$, and a regular vector $\boldsymbol{b}$, consider a system of two-sided and one-sided equations

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{x}, \\ \boldsymbol{C}\boldsymbol{x} &= \boldsymbol{b} \end{aligned}$$

- We suppose that each equality alone is solvable (otherwise the system has no soluition)
- For simplicity, we assume the matrix $\boldsymbol{A}$ to be irreducible
- The general solution of the first equation takes the form

$$\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{v}, \qquad \boldsymbol{v} \in \mathbb{X}^{n}$$

- Consider the general solution $\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{v}$ of the first equation

- Substitution into the second equation leads to one-sided equation in the unknown vector $\boldsymbol{v}$, which takes the form

$$\boldsymbol{C}\boldsymbol{A}^{\times}\boldsymbol{v} = \boldsymbol{b}$$

- If the last equation is solvable, then its maximal solution of this equation is given by $\boldsymbol{v} = (\boldsymbol{b}^{-}\boldsymbol{C}\boldsymbol{A}^{\times})^{-}$

- The corresponding solution of the original system takes the form

$$\boldsymbol{x} = \boldsymbol{A}^{\times}(\boldsymbol{d}^{-}\boldsymbol{C}\boldsymbol{A}^{\times})^{-}$$

# Eigenvalues and Eigenvectors: Introduction

- A scalar $\lambda$ is an eigenvalue of an $(n \times n)$-matrix $\boldsymbol{A}$, if there exists a nonzero $n$-vector $\boldsymbol{x}$ such that the following equality holds:

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$

- Any nonzero vector $\boldsymbol{x}$ that satisfies this equality, is an eigenvector of the matrix $\boldsymbol{A}$, which corresponds to the eigenvalue $\lambda$
- The set of all eigenvectors of a matrix $\boldsymbol{A}$ together with the zero vector form a tropical eigenspace of $\boldsymbol{A}$
- We examine the eigenvalue problem in the context of tropical analogues of the characteristic polynomial and equation of matrix
- We reduce the eigenvector problem to a two-sided equation

# Characteristic Polynomial and Equation

- Given an $(n \times n)$-matrix $\boldsymbol{A}$, we define a function of scalar $\lambda$ as

$$\chi_{\boldsymbol{A}}(\lambda) = \mathrm{Tr}(\lambda^{-1}\boldsymbol{A})$$

- We call the function $\chi_{\boldsymbol{A}}(\lambda)$ the characteristic polynomial of $\boldsymbol{A}$
- The characteristic equation of the matrix $\boldsymbol{A}$ is given by

$$\mathrm{Tr}(\lambda^{-1}\boldsymbol{A}) = \mathbb{1}$$

- For any matrix $\boldsymbol{A}$ and scalar $\lambda$, we use the following notation:

$$\boldsymbol{A}_{\lambda} = \lambda^{-1}\boldsymbol{A}, \qquad \boldsymbol{A}_{\lambda}^{\times} = (\boldsymbol{A}_{\lambda})^{\times}$$

## Polynomial Function

- A tropical polynomial in one variable $x$ is defined as follows:

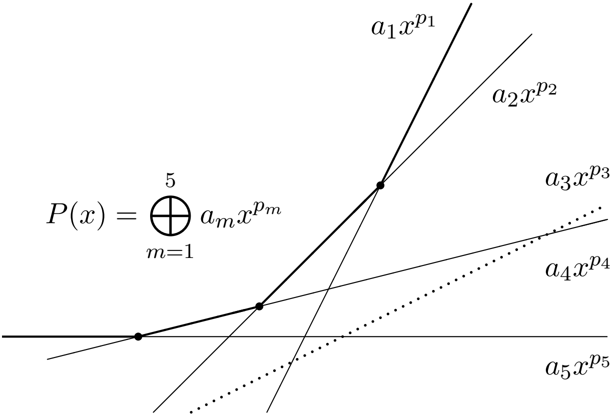
$$P(x) = \bigoplus_{m=1}^{n} a_m x^{p_m}, \qquad p_m \in \mathbb{Q}, \qquad p_m \geq 0, \qquad m = 1, \dots, n$$

- In terms of max-plus algebra $\mathbb{R}_{\max,+}$, the polynomial is given by

$$P(x) = \max_{1 \leq m \leq n} (p_m \times x + a_m)$$

- As it follows from this representation, any polynomial in $\mathbb{R}_{\max,+}$ is a piecewise linear convex nondecreasing function

## Graphical Illustration in $\mathbb{R}_{\max,+}$




- Graph of a polynomial as a piecewise linear convex function

## Solution of Polynomial Equation

- Consider a tropical polynomial in one variable $x$, defined as

$$P(x) = a_0 \oplus \bigoplus_{m=1}^{n} a_m x^m$$

### Lemma

*Suppose that* $a_0 < \mathbb{1}$ *and* $a_m \neq \mathbb{0}$ *for at least one* $m = 1, \ldots, n$*.*
*Then, the polynomial equation*

$$P(x) = \mathbb{1}$$

*has a unique solution that is given by*

$$x = \left( \bigoplus_{m=1}^{n} a_m^{1/m} \right)^{-1}$$

## Proof

- Let us examine (in the context of max-plus algebra) the function

$$P(x) = a_0 \oplus \bigoplus_{m=1}^{n} a_m x^m$$

- First, we observe that $P(x)$ is a continuous function
- Since $a_0 < \mathbb{1}$ and $a_m \neq \mathbb{0}$ for at least one $m > 0$, the function $P(x)$ takes both values which are greater and less than $\mathbb{1}$
- Therefore, a solution $x > \mathbb{0}$ of the equation $P(x) = \mathbb{1}$ exists
- The function $P(x)$ is monotone, and hence the solution is unique

## Proof (cont.)

- If $x > \mathbb{0}$ is a solution, then the following inequalities hold (with at least one inequality holding as an equality):
$$a_m x^m \leq \mathbb{1}, \qquad m = 1, \ldots, n$$
- The solution of these inequalities with respect to $x^{-1}$ yields
$$x^{-1} \geq a_m^{1/m}, \qquad m = 1, \ldots, n$$
- By combining these inequalities into one inequality, we have
$$x^{-1} \geq a_1 \oplus \cdots \oplus a_n^{1/n} = \bigoplus_{m=1}^{n} a_m^{1/m}$$
- Considering that $x^{-1} = a_m^{1/m}$ for at least one $m$, we replace the inequality sign by an equality sign, and then obtain
$$x = \left( \bigoplus_{m=1}^{n} a_m^{1/m} \right)^{-1} \qquad \blacksquare$$

# Eigenvalue of Irreducible Matrix

**Theorem (Eigenvalue of Irreducible Matrix)**

*A scalar $\lambda \neq \mathbb{0}$ is an eigenvalue of a irreducible matrix $\boldsymbol{A}$ if and only if $\lambda$ is a solution of the characteristic equation for $\boldsymbol{A}$*

**Proof**

- Let us represent the equality $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ as the two-sided equation

$$\boldsymbol{A}_\lambda \boldsymbol{x} = \boldsymbol{x}, \qquad \boldsymbol{A}_\lambda = \lambda^{-1}\boldsymbol{A}$$

- By the lemma on existence of nontrivial solutions, this equation has a nontrivial solution if and only if the following condition holds:

$$\mathrm{Tr}(\boldsymbol{A}_\lambda) = \mathrm{Tr}(\lambda^{-1}\boldsymbol{A}) = \mathbb{1}$$

- This means that $\lambda$ satisfies the characteristic equation of $\boldsymbol{A}$ ■

## Lemma (Evaluation of Eigenvalue)

*Every irreducible matrix $\boldsymbol{A}$ has a unique eigenvalue that is given by*

$$\lambda = \bigoplus_{m=1}^{n} \operatorname{tr}^{1/m}(\boldsymbol{A}^m)$$

## Proof

- The characteristic polynomial for $\boldsymbol{A}_\lambda = \lambda^{-1}\boldsymbol{A}$ takes the form

$$\operatorname{Tr}(\boldsymbol{A}_\lambda) = \operatorname{Tr}(\lambda^{-1}\boldsymbol{A}) = \bigoplus_{m=1}^{n} \operatorname{tr}(\lambda^{-m}\boldsymbol{A}^m) = \bigoplus_{m=1}^{n} \lambda^{-m} \operatorname{tr} \boldsymbol{A}^m$$

- The characteristic equation for $\boldsymbol{A}_\lambda$ is then represented as

$$\bigoplus_{m=1}^{n} \lambda^{-m} \operatorname{tr} \boldsymbol{A}^m = \mathbb{1}$$

## Proof (cont.)

- Consider the characteristic equation

$$\bigoplus_{m=1}^{n} \lambda^{-m} \operatorname{tr} \boldsymbol{A}^{m} = \mathbb{1}$$

- This equation takes the form of the polynomial equation

$$P(x) = a_0 \oplus \bigoplus_{m=1}^{n} a_m x^m = \mathbb{1},$$

  where $x = \lambda^{-1}$, $a_0 = \mathbb{0}$, $a_m = \operatorname{tr} \boldsymbol{A}^m$ for all $m = 1, \ldots, n$

- Since the matrix $\boldsymbol{A}$ is irreducible, we have $\operatorname{Tr}(\boldsymbol{A}) > \mathbb{0}$, and thus the inequality $a_m = \operatorname{tr} \boldsymbol{A}^m > \mathbb{0}$ is valid for some $m$

- An application of the lemma on the unique solution of the polynomial equation $P(x) = \mathbb{1}$ completes the proof ■

## Lemma (Evaluation of Eigenvector)

*If a matrix $\boldsymbol{A}$ has eigenvalue $\lambda > \mathbb{0}$, then the vector $\boldsymbol{x} = \boldsymbol{A}_{\lambda}^{\times}\boldsymbol{v}$ for any regular vector $\boldsymbol{v}$ is an eigenvector of $\boldsymbol{A}$ corresponding to $\lambda$*

## Proof

- The eigenvector of the matrix $\boldsymbol{A}$, which corresponds to the eigenvalue $\lambda$, satisfies the equation

$$\boldsymbol{A}_{\lambda}\boldsymbol{x} = \boldsymbol{x}$$

- Since $\lambda$ is an eigenvalue, it satisfies the characteristic equation

$$\mathrm{Tr}(\lambda^{-1}\boldsymbol{A}) = \mathbb{1}$$

- By applying the lemma about the solution of two-sided equation to $\boldsymbol{A}_{\lambda}\boldsymbol{x} = \boldsymbol{x}$, we obtain the eigenvectors in the form $\boldsymbol{x} = \boldsymbol{A}_{\lambda}^{\times}\boldsymbol{v}$ ■

## Eigenvalues and Eigenvectors

- Consider the eigenvalue of a matrix $\boldsymbol{A} = (a_{ij})$ of order $n$

$$\lambda = \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(\boldsymbol{A}^m)$$

- Representation in terms of the entries in the matrix $\boldsymbol{A}$ yields

$$\lambda = \bigoplus_{m=1}^{n} \bigoplus_{1 \leq i_0,\dots,i_{m-1} \leq n} \left(a_{i_0 i_1} \cdots a_{i_{m-1} i_0}\right)^{1/m}$$

- In the framework of max-plus algebra $\mathbb{R}_{\max,+}$, we have

$$\lambda = \max\left(a_{11}, \dots, a_{nn}, \frac{a_{12} + a_{21}}{2}, \dots, \frac{a_{n-1,n} + a_{n,n-1}}{2}, \dots\right)$$

- The eigenspace of the matrix $\boldsymbol{A}$, which corresponds to the eigenvalue $\lambda$, consists of vectors $\boldsymbol{x} = \boldsymbol{A}_\lambda^\times \boldsymbol{v}$ for all $\boldsymbol{v} \neq \boldsymbol{0}$

# Example in Two Dimensions

- Consider the matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Suppose that $a_{11}, a_{12}, a_{21}, a_{22} > \mathbb{0}$
- We calculate the eigenvalue of $\boldsymbol{A}$ by the formula

$$\lambda = a_{11} \oplus \sqrt{a_{12} a_{21}} \oplus a_{22}$$

- Next, we obtain the generating matrix for eigenvectors

$$\boldsymbol{A}_{\lambda} = \lambda^{-1} \boldsymbol{A} = \begin{pmatrix} \lambda^{-1} a_{11} & \lambda^{-1} a_{12} \\ \lambda^{-1} a_{21} & \lambda^{-1} a_{22} \end{pmatrix}$$

- We form the Kleene star and Kleene plus matrices for the matrix

$$\boldsymbol{A}_\lambda = \begin{pmatrix} \lambda^{-1}a_{11} & \lambda^{-1}a_{12} \\ \lambda^{-1}a_{21} & \lambda^{-1}a_{22} \end{pmatrix}$$

- Taking into account that $a_{11} \leq \lambda$ and $a_{22} \leq \lambda$, we obtain

$$\boldsymbol{A}_\lambda^* = \boldsymbol{I} \oplus \boldsymbol{A}_\lambda = \begin{pmatrix} \mathbb{1} & \lambda^{-1}a_{12} \\ \lambda^{-1}a_{21} & \mathbb{1} \end{pmatrix}$$

- Furthermore, we calculate the matrix

$$\boldsymbol{A}_\lambda^+ = \boldsymbol{A}_\lambda \boldsymbol{A}_\lambda^* = \begin{pmatrix} \lambda^{-1}a_{11} \oplus \lambda^{-2}a_{12}a_{21} & \lambda^{-1}a_{12} \\ \lambda^{-1}a_{21} & \lambda^{-2}a_{12}a_{21} \oplus \lambda^{-1}a_{22} \end{pmatrix}$$

(in particular, $\lambda^{-1}a_{21} \oplus \lambda^{-1}a_{22}\lambda^{-1}a_{21} = \lambda^{-1}a_{21}$)

- Let us consider various assumptions on columns in $\boldsymbol{A}_\lambda^*$ and $\boldsymbol{A}_\lambda^+$
- The matrices $\boldsymbol{A}_\lambda^*$ and $\boldsymbol{A}_\lambda^+$ have the same first column if

$$\lambda^{-1}a_{11} \oplus \lambda^{-2}a_{12}a_{21} = \mathbb{1},$$
$$\lambda^{-2}a_{12}a_{21} \oplus \lambda^{-1}a_{22} < \mathbb{1}$$

- It follows from these relations that $\lambda^{-2}a_{12}a_{21} < \mathbb{1}$, and therefore,

$$\lambda = a_{11} > \sqrt{a_{12}a_{21}} \oplus a_{22}$$

- As a result, the matrix $\boldsymbol{A}_\lambda^\times$ can be reduced to one column

$$\begin{pmatrix} \mathbb{1} \\ \lambda^{-1}a_{21} \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ a_{11}^{-1}a_{21} \end{pmatrix}$$

- If only the second columns in $\boldsymbol{A}_\lambda^\ast$ and $\boldsymbol{A}_\lambda^+$ coincide, then

$$\lambda^{-1}a_{11} \oplus \lambda^{-2}a_{12}a_{21} < \mathbb{1},$$
$$\lambda^{-2}a_{12}a_{21} \oplus \lambda^{-1}a_{22} = \mathbb{1}$$

- Therefore, $\lambda = a_{22} > a_{11} \oplus \sqrt{a_{12}a_{21}}$, and hence $\boldsymbol{A}_\lambda^\times$ reduces to

$$\begin{pmatrix} \lambda^{-1}a_{12} \\ \mathbb{1} \end{pmatrix} = \begin{pmatrix} a_{12}a_{22}^{-1} \\ \mathbb{1} \end{pmatrix}$$

- Finally, both columns coincide if the following equalities are valid:

$$\lambda^{-1}a_{11} \oplus \lambda^{-2}a_{12}a_{21} = \mathbb{1},$$
$$\lambda^{-2}a_{12}a_{21} \oplus \lambda^{-1}a_{22} = \mathbb{1}$$

- This statement is true when $\lambda = a_{11} = a_{22}$ and/or $\lambda = \sqrt{a_{12}a_{21}}$
- In this case, the matrix $\boldsymbol{A}_\lambda^\times$ has two columns,

$$\boldsymbol{A}_\lambda^\times = \begin{pmatrix} \mathbb{1} & \lambda^{-1}a_{12} \\ \lambda^{-1}a_{21} & \mathbb{1} \end{pmatrix}$$

- Consider the matrix obtained

$$\boldsymbol{A}_\lambda^\times = \begin{pmatrix} \mathbb{1} & \lambda^{-1}a_{12} \\ \lambda^{-1}a_{21} & \mathbb{1} \end{pmatrix}$$

- The columns in $\boldsymbol{A}_\lambda^\times$ are linearly dependent if

$$\begin{aligned} \mathbb{1} &= \mu\lambda^{-1}a_{12}, \\ \lambda^{-1}a_{21} &= \mu \end{aligned}$$

- The solution of the system with respect to $\lambda$ yields $\lambda = \sqrt{a_{12}a_{21}}$
- Therefore, if $\lambda = a_{11} = a_{22} > \sqrt{a_{12}a_{21}}$, then

$$\boldsymbol{A}_\lambda^\times = \begin{pmatrix} \mathbb{1} & a_{22}^{-1}a_{12} \\ a_{11}^{-1}a_{21} & \mathbb{1} \end{pmatrix}$$

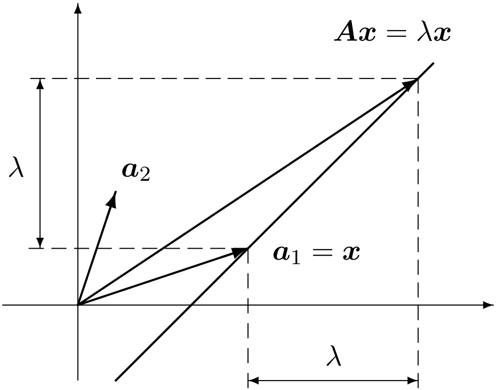
- If $\lambda = \sqrt{a_{12}a_{21}}$, then the matrix $\boldsymbol{A}_\lambda^\times$ can be reduced to

$$\begin{pmatrix} \mathbb{1} \\ a_{12}^{-1/2}a_{21}^{1/2} \end{pmatrix}$$

## Summary of Results

- If $a_{11} > \sqrt{a_{12}a_{21}}$ and $a_{11} > a_{22}$, then the eigenvalue is given by
$$\lambda = a_{11},$$
and the eigenvector coincides with the first column in $\boldsymbol{A}$,
$$\boldsymbol{x} = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$$
- If $a_{22} > a_{11}$ and $a_{22} > \sqrt{a_{12}a_{21}}$, then the eigenvalue is equal to
$$\lambda = a_{22},$$
and the eigenvector coincides with the second column
$$\boldsymbol{x} = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

## Graphical Illustration of Eigenelements in $\mathbb{R}^2_{\max,+}$



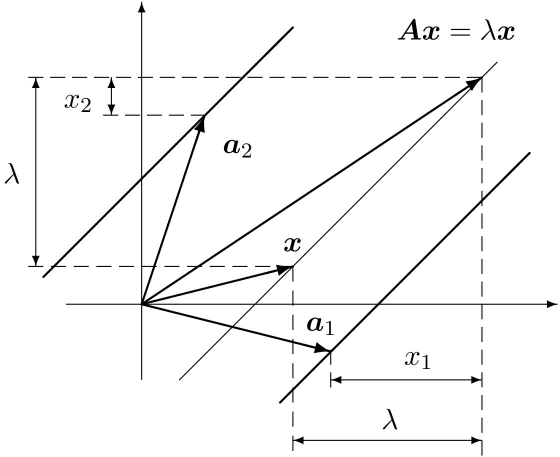

- Eigenvalue and eigenvector when $a_{11} > \sqrt{a_{12}a_{21}}$, $a_{11} > a_{22}$

## Summary of Results (cont.)

- If $a_{11} = a_{22} > \sqrt{a_{12}a_{21}}$, then the eigenvalue is given by
$$\lambda = a_{11},$$
and the eigenvectors coincide with both columns of the matrix,
$$\boldsymbol{x}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \qquad \boldsymbol{x}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

- Under the condition $\sqrt{a_{12}a_{21}} \geq a_{11}, a_{22}$, there is one eigenvalue
$$\lambda = \sqrt{a_{12}a_{21}},$$
and one eigenvector
$$\boldsymbol{x} = \begin{pmatrix} a_{12}^{1/2} \\ a_{21}^{1/2} \end{pmatrix}$$

## Graphical Illustration of Eigenelements in $\mathbb{R}^2_{\max,+}$




- Eigenvalue and eigenvector when $\sqrt{a_{12}a_{21}} \geq a_{11}, a_{22}$

## Numerical Example

- Let us find in $\mathbb{R}^3_{\min,+}$ the eigenvalue and eigenvectors of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 3 & 4 \\ 5 & 2 & 3 \end{pmatrix}$$

- We start with calculating the powers

$$\boldsymbol{A}^2 = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 3 & 4 \\ 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 \\ 3 & 3 & 4 \\ 5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 5 \\ 5 & 4 & 7 \\ 5 & 5 & 6 \end{pmatrix},$$

$$\boldsymbol{A}^3 = \begin{pmatrix} 4 & 3 & 5 \\ 5 & 4 & 7 \\ 5 & 5 & 6 \end{pmatrix} \begin{pmatrix} 2 & 1 & 4 \\ 3 & 3 & 4 \\ 5 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 5 & 7 \\ 7 & 6 & 8 \\ 7 & 6 & 9 \end{pmatrix}$$

- Evaluation of the eigenvalue yields

$$\lambda = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr}^{1/2}(\boldsymbol{A}^2) \oplus \operatorname{tr}^{1/3}(\boldsymbol{A}^3) = 2 \oplus \frac{1}{2}4 \oplus \frac{1}{3}6 = 2$$

- We form the matrix

$$\boldsymbol{A}_\lambda = \lambda^{-1}\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$$

- Furthermore, we calculate the powers

$$\boldsymbol{A}_\lambda^2 = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}_\lambda^3 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix}$$

- The Kleene star and Kleene plus matrices for $\boldsymbol{A}_\lambda$ take the form

$$\boldsymbol{A}_\lambda^* = \boldsymbol{I} \oplus \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{A}_\lambda^+ = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 2 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

- The generating matrix for eigenvectors is given by (with $\mathbb{0} = -\infty$)

$$\boldsymbol{A}_\lambda^\times = \begin{pmatrix} 0 & -1 & \mathbb{0} \\ 1 & 0 & \mathbb{0} \\ 1 & 0 & \mathbb{0} \end{pmatrix}$$

- Both nonzero columns are collinear and generate the same eigenvectors

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} u, \qquad u \neq \mathbb{0}$$

# Irreducible and Reducible Matrices

- A matrix $\boldsymbol{A}$ is called reducible if simultaneous permutations of its rows and columns transform it into a block-triangular normal form
- Otherwise, the matrix $\boldsymbol{A}$ is referred to as irreducible
- The lower block-triangular normal form of a matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \\ \boldsymbol{A}_{s1} & \boldsymbol{A}_{s2} & \dots & \boldsymbol{A}_{ss} \end{pmatrix},$$

  where $\boldsymbol{A}_{ii}$ is an irreducible square matrix for all $i = 1, \dots, s$
- We denote by $\lambda_i$ an eigenvector of the diagonal block $\boldsymbol{A}_{ii}$

## Further Results on Eigenvalues (without proof)

- Any square matrix $\boldsymbol{A}$ of order $n$ has at least one eigenvalue, which is called the spectral radius of $\boldsymbol{A}$ and given by

$$\lambda = \bigoplus_{m=1}^{n} \operatorname{tr}^{1/m}(\boldsymbol{A}^m)$$

- If the matrix $\boldsymbol{A}$ is irreducible, it has no other eigenvalues
- A reducible matrix may have more than one eigenvalues
- Each eigenvalue of a reducible matrix given in the block-triangular form is one of the eigenvalues $\lambda_1, \ldots, \lambda_s$ of the diagonal blocks
- However, some of the eigenvalues of diagonal blocks of an reducible matrix may not be eigenvalues of the matrix
- The spectral radius of the matrix is the maximum eigenvalue:

$$\lambda = \lambda_1 \oplus \cdots \oplus \lambda_s$$

## Examples in Two Dimensions

- Consider the diagonal matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \mathbb{0} \\ \mathbb{0} & a_{22} \end{pmatrix}$$

- Suppose that $a_{11}, a_{22} > \mathbb{0}$
- The matrix $\boldsymbol{A}$ has two eigenvalues

$$\lambda_1 = a_{11}, \qquad \lambda_2 = a_{22}$$

- Corresponding eigenvectors take the form

$$\boldsymbol{x}_1 = \begin{pmatrix} \mathbb{1} \\ \mathbb{0} \end{pmatrix}, \qquad \boldsymbol{x}_2 = \begin{pmatrix} \mathbb{0} \\ \mathbb{1} \end{pmatrix}$$

- Consider the lower-triangular matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \mathbb{0} \\ a_{21} & a_{22} \end{pmatrix}$$

- Assume that the conditions $a_{11}, a_{21}, a_{22} > \mathbb{0}$ hold
- If $a_{11} \geq a_{22}$ , then there are two eigenvalues

$$\lambda_1 = a_{11}, \qquad \lambda_2 = a_{22}$$

- The corresponding eigenvectors are given by

$$\boldsymbol{x}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \qquad \boldsymbol{x}_2 = \begin{pmatrix} \mathbb{0} \\ a_{22} \end{pmatrix}$$

- Under the condition $a_{11} = a_{22}$, there is one eigenvalue

$$\lambda = a_{11}$$

and two eigenvectors

$$\boldsymbol{x}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \qquad \boldsymbol{x}_2 = \begin{pmatrix} \mathbb{0} \\ a_{22} \end{pmatrix}$$

- If $a_{11} < a_{22}$, the matrix has one eigenvalue

$$\lambda = a_{22}$$

and one eigenvector

$$\boldsymbol{x} = \begin{pmatrix} \mathbb{0} \\ a_{22} \end{pmatrix}$$