1. Count the number of three-digit numbers ijk have the property that $i \le j \le k$.

Solution: the number is three-digit, thus $i \neq 0$. since $i \le j \le k$. $j, k \neq 0$.

$\Rightarrow$ we only consider $1 \le i \le j \le k \le 9$

1) consider $i = j = k$. $C_9^1 = 9$ numbers(111 - 999).

2) consider two of them equal.

actually we just need select two number. the permutation preserve the order is unique.

$C_9^2 \times 2 = 72$ ($\times 2$ is defined which number doubled. e.g. for 1, 2 are selected. we have 112, 122)

3) consider three mutually different number. similarly, the order unique. $C_9^3 = 84$

result: $72 + 84 + 9 = 165$

2. How many 5−element subsets of $\{0, 1, \ldots, 9\}$ contain at least one odd element?

Solution $|S|$ 5-element subsets$| = C_{10}^5$

all even number subsets (5 - element) $= C_5^5$

result $= C_{10}^5 - C_5^5 = \dfrac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} - 1 = 251$

3. Determine the coefficient of x^3y^4z in the expansion of $(x + y^2 + z)^6$.

S: $P(3, 2, 1) = \dfrac{6!}{3!\,2!\,1!} = \dfrac{1 \times 2 \times 3 \times 4 \times 5}{2} = 60$

4. How many integers are there between 1 and 33000 that are not divisible by either 3 or 5, but are divisible by 11?

S: let $A = \{ x \mid (1 \le x \le 33000) \wedge (3 \nmid x) \}$

$B = \{ y \mid (1 \le x \le 33000) \wedge (5 \nmid x) \}$

$C = \{ z \mid (1 \le x \le 33000) \wedge (11 \mid x) \}$

$$|A \cap B \cap C| = 33000 - \underset{\neg A}{11000} - \underset{\neg B}{6600} - \underset{\neg C}{30000} + \underset{\neg A \wedge \neg B}{2200} + \underset{\neg A \wedge \neg C}{10000} + \underset{\neg A \wedge \neg B}{6000} - \underset{\neg A \wedge \neg B \wedge \neg C}{2000}$$

$= 1600$

5. There is an infinite number of cards on each of which some natural number is written. It is known that for any natural number n there are exactly n cards on which the divisors of this number are written. Prove that every natural number occurs on at least one card.

1) For prime, if the number not occurs. then it have n cards "1" (no other divisor)

thus. for a smaller prime. p. we have n cards "1". the condition for p asks exactly p cards are p's divisor

but we already have $n \ge p$ cards. contradictory.

2) For composite number. Assume n_0 doesn't occur. $n_0 = p_1^{a_1} \cdots p_t^{a_t}$ ($p_1 \ldots p_t$ are primes)

it's sufficient to prove that every natural number n needs to occur exact t times. t is the number of fraction s.t. $0 < \frac{m}{n} < 1$. m, n are co-prime.

6. How many words of length n can be formed from the letters x, y, z if the letter x has to occur an even number of times?

Method 1.

Let the result is A_n. the converse result(x. odd times) B_n.

for $A_{n+1} = 2A_n + B_n$

+y/z at the end +x at the end

$B_{n+1} = 2B_n + A_n$.

$A_{n+1} + B_{n+1} = 3(A_n + B_n)$, $A_1 = 2$ $B_1 = 1$

$A_{n+1} - B_{n+1} = A_n - B_n = A_1 - B_1 = 1$.

$\therefore A_n = \dfrac{3^n + 1}{2}$

(more precise result)

Method 2

S: x 0 time. 2^n

x 2 time $2^{n-2} \cdot (C_{n-1}^1 + C_{n-1}^2) = 2^{n-2} \cdot C_n^2$

x 4 time. $2^{n-4} \cdot (C_{n-3}^1 + 3C_{n-3}^2 + 3C_{n-3}^3 + C_{n-3}^4 = 2^{n-4} \cdot C_n^4$

x 2k time $2^{n-2k} \cdot C_n^{2k}$

1) if n is even.

$|S| = 2^n + C_n^2 \cdot 2^{n-2} + \cdots + C_n^n$

2) if n is odd.

$|S| = 2^n + C_n^2 \cdot 2^{n-2} + C_n^4 \cdot 2^{n-4} + \cdots + 2 \cdot C_n^{n-1}$

(have trouble in reducing the result)

7. Find the number of permutations of $\{1, 2, \ldots, n\}$ with t fixed points.

if those t points need to be selected, result $= C_n^t A_{n-t}^{n-t}$

if those t points are given result $= A_{n-t}^{n-t}$,

8. Prove that $\phi(n^i) = n^{i-1}\phi(n)$ (Here ϕ is the Euler function).

Pf: assume that $n = p_1^{a_1} p_2^{a_2} \cdots p_t^{a_t}$ (p_i are distinct prime)

$\varphi(n) = n(1-\frac{1}{p_1})\cdots(1-\frac{1}{p_t}) = p_1^{a_1-1}(p_1-1)\cdots p_t^{a_t-1}(p_t-1)$

$n^i = p_1^{a_1 \times i} p_2^{a_2 \cdot i} \cdots p_t^{a_t \cdot i}$

$\varphi(n^i) = p_1^{a_1 \cdot i-1}(p_1-1)\cdots p_t^{a_t \cdot i-1}(p_t-1)$

$\varphi(n)\cdot n^{i-1} = (p_1^{a_1}\cdots p_t^{a_t})^{i-1}\cdot p_1^{a_1-1}(p_1-1)\cdots p_t^{a_t-1}(p_t-1)$

$= p_1^{a_1(i-1)+a_1-1}(p_1-1)\cdots p_t^{a_t(i-1)+a_t-1}(p_t-1)$

$= p_1^{a_1 \cdot i-1}(p_1-1)\cdots p_t^{a_t \cdot i-1}(p_t-1) = \varphi(n^i)$ $\square$

9. Let D_n be the divisor poset associated to n. Show that $\mu(x, y) = \mu(1, \frac{y}{x})$ for any $x, y \in D_n$.

Pf: 1) if $x=y$, $\mu(x,y) = 1 = \mu(1,1)$

2) if there is a chain s.t. $1|t_1, t_1|t_2, \ldots t_{p-1}|x, x|t_p, t_p|t_2, \ldots t_s|y$

we have $\mu(x,y) = -1$

it is clear that $\frac{y}{x} = t_i$. for some i. there also exists a chain $1|x, x|t_p \ldots t_{i-1}|t_i$

$\mu(1, \frac{y}{x}) = \mu(1, t_i) = -1 = \mu(x,y)$

3) if there is no chain between x, y. x, y not comparable in the poset

$\mu(x,y) = 0$. and $\frac{y}{x} \notin \mathbb{Z}$. $\mu(1, \frac{y}{x}) = 0 = \mu(x,y)$ $\square$

10. Prove the following equalities for Fibonacci numbers:

(a) $F_1 + \ldots + F_n = F_{n+2} - 1$

(b) $F_1^2 + \ldots + F_n^2 = F_n F_{n+1}$

(a). Pf by induction.

1). $n=1$. $F_1 = 1$. $F_{n+2} - 1 = F_3 - 1 = 1$. - ok

2) assume $n=k-1$ holds.

for $n=k$. $\sum_{i=1}^{k} F_i = F_k + \sum_{i=1}^{k-1} F_i = F_k + F_{k+1} - 1 = F_{k+2} - 1$. $\square$

(b) Also by induction

1) $n=1$. $F_1^2 = 1$. $F_1 F_2 = 1$

2) Assume that equation holds for $n=k-1$.

for $n=k$. $\sum_{i=1}^{k} F_i^2 = F_k^2 + F_k F_{k-1} = F_k(F_{k-1} + F_k) = F_k F_{k+1}$ $\square$

11. How many sequences $\{a_1, a_2, \ldots, a_{2n}\}$ consisting of 1 and -1 have the property that $a_1 + a_2 + \ldots + a_{2n} = 0$, and all partial sums a_1, $a_1 + a_2$, $\ldots$, $a_1 + a_2 + \ldots + a_{2n}$ are non-negative?

For random 1, -1. the sum back to 0. exactly have C_{2n}^{n} ways. after 2n number

Firstly, we can consider a sequence. $\{\underbrace{1, -1, 1, -1, \ldots\ldots 1, -1}_{2n.}\}$

12. Solve the following recurrence relation: $a_n = 3a_{n-1} - 2a_{n-2} + 3^n$, and $a_0 = -3, a_1 = 6$.

$$a_n - a_{n-1} = 2(a_{n-1} - a_{n-2}) + 3^n \quad (n \geq 2)$$

$$a_n - a_{n-1} - 3^{n+1} = 2(a_{n-1} - a_{n-2} - 3^n)$$

let $b_n = a_n - a_{n-1} - 3^{n+1}$

$b_n = 2b_{n-1}$. $b_1 = 0$. $b_n \equiv 0$.

$\Rightarrow$ i.e. $a_n - a_{n-1} = 3^{n+1}$.

$$a_n - a_0 = 3^{n+1} + \cdots + 3^2$$

$$a_n = 3^{n+1} + \cdots 3^2 - 3 = \frac{9(3^n - 1)}{2} - 3 = \frac{3^{n+2} - 15}{2}$$

13. Find the generating function and the explicit formula for the sequence L_n, where $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.

S: Assume the generating function $f(x) = L_0 + L_1 x + L_2 x^2 + \cdots + L_3 x^3$

$$(x + x^2) f(x) = (x + x^2)(2 + x + 3x^2 + 4x^3 + 7x^4 + \cdots)$$

$$= 2x + x^2 + 3x^3 + 4x^4 + \cdots + 2x^2 + x^3 + 3x^4 + 4x^5 + \cdots$$

$$= 2x + 3x^2 + 4x^3 + 7x^4 + \cdots = (x - 2) + f(x)$$

$$f(x) = \frac{2 - x}{1 - x - x^2}$$

$$f(x) = (2 - x)\left[\frac{1}{\sqrt{5}x_1}\left(1 + \frac{x}{x_1} + \frac{x^2}{x_1^2} + \cdots\right) - \frac{1}{\sqrt{5}x_2}\left(1 + \frac{x}{x_2} + \frac{x^2}{x_2^2} + \cdots\right)\right]$$

where $x_1 = \frac{\sqrt{5} - 1}{2}$ $\quad x_2 = \frac{-\sqrt{5} - 1}{2}$

$$L_n = \frac{(-1)^n}{\sqrt{5}}\left(\left(\frac{-1+\sqrt{5}}{2}\right)^n - \left(\frac{-1-\sqrt{5}}{2}\right)^n\right) + 2\frac{(-1)^n}{\sqrt{5}}\left(\left(\frac{-1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{-1+\sqrt{5}}{2}\right)^{n+1}\right)$$

14. Suppose that $F(x)$ is the generating function of a sequence a_n. Write the generating function of the sequence $\sum_{k=0}^{n} k a_k$.

S: Assume the function we want is $G(x)$.

$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

$$F'(x) = a_1 + 2a x_2 + \cdots + n a_n x^{n-1}$$

$$G(x) = 0 + a_1 x + (a_1 + 2a_2)x^2 + (a_1 + 2a_2 + 3a_3)x^3 + \cdots$$

$$= F(x) x + F' x^2 + \cdots$$

$$= F'(x) \cdot (x + x^2 + \cdots) = F'(x) \frac{x}{1 - x}$$

15. Count the number of permutations in S_7 with at least 4 cycles.

S: 4 cycles. each cycles contains elements:

1) (4 1 1 1) $|\sigma_1| = \frac{7!}{4} = 1260$

2) (3 2 1 1) $|\sigma_2| = \frac{7!}{3 \cdot 2} = 840$

3) (2 2 2 1) $|\sigma_3| = \frac{7!}{2^3} = 630$

5 cycles

4) (3 1 1 1 1) $|\sigma_4| = \frac{7!}{3} = 1680$

5) (2 2 1 1 1) $|\sigma_5| = \frac{7!}{2^2} = 1260$

6 cycles

6) (2 1 1 1 1 1) $|\sigma_6| = \frac{7!}{2} = 2520$

7 cycles (actually no cycles) $\sigma_7 = 7! = 5040$

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