Real Analysis 2024. Homework 4.

1. Consider a counting measure on $\mathbb{N}$. Describe spaces of mesurable and of integrable functions.

   Any function $f : \mathbb{N} \to \mathbb{C}$ is a sequence of complex numbers $c_k = f(k)$. Any such function is measurable with respect to counting measure since the counting measure ν is defined on maximal $\sigma-$algebra $\mathcal{A}_{\max} = 2^{\mathbb{N}}$. The integral

$$\int_{\mathbb{N}} |f| \, d\nu = \sum_{k=1}^{\infty} |c_k|$$

   is finite iff a series $\sum_{k=1}^{\infty} c_k$ is absolutely convergent. This space is called

$$\ell^1 = \{\{c_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} |c_k| < \infty\}.$$

2. Assume that $f_n \in L(E)$ is increasing sequence, $f_n \to f$ pointwise and $f \in L(E)$. Prove that $f_n \to f$ in $L(E)$.

   Proof. Let $g_n = f - f_n \geq 0$. Then g_n is decreasing sequence and by monotone convergence theorem

$$\int_E |f - f_n| \, d\mu = \int_E g_n d\mu \to 0, \quad n \to \infty.$$

   □

3. Let f be $\mu-$measurable on E and denote $E_t = E(|f| > t)$. Prove that

$$\mu E_t \leq \frac{1}{t^p} \int_E |f|^p \, d\mu.$$

   Proof. Note that

$$E_t = E(|f|^p > t^p).$$

   Hence, by Chebyshev's inequality applied to $|f|^p$ we have

$$\mu E_t \leq \frac{1}{t^p} \int_E |f|^p \, d\mu.$$

   □

4. Prove that a measure μ is $\sigma-$finite if and only if there exists a positive integrable function ($f > 0$ on X and $\int_X f d\mu < +\infty$).

Proof. Assume first that μ is $\sigma-$finite. Then

$$X = \bigcup_{k=1}^{\infty} X_k, \quad \mu X_k < \infty.$$

Let

$$E_n = \left(\bigcup_{k=1}^{n} X_k \right) \setminus \left(\bigcup_{k=1}^{n-1} X_k \right), \ n > 1; \ E_1 = X_1.$$

Then E_n is measurable, $X = \bigcup\limits_{n=1}^{\infty} E_n$, and $c_n = \mu(E_n) < \infty$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \chi_{E_n} > 0.$$

Since $E_m \cap E_n$, $n \neq m$, then by monotone convergence theorem

$$\int_X f d\mu = \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Assume now that there exists function $f > 0$ such that $\int_X f d\mu < \infty$. Consider a set $X_k = X(f > 1/k)$. Then, by Chebyshev's inequality

$$\mu(X_k) \le k \int_X |f| < \infty.$$

And since $f(x) > 0$ for every $x \in X$ we see that

$$X = \bigcup_{k=1}^{\infty} X_k, \quad \mu X_k < \infty.$$

$\square$

5. Consider $f_n(x) = \frac{1}{n} \left(\frac{\sin nx}{x} \right)^2$. Prove that

   (a) $f \in L(0, \pi)$;

(b) $f_n(x) \to 0,\ n \to \infty$ for every $x \in (0, \pi)$;

(c) There is no such function $g \in L(0, \pi)$ such that $f_n(x) \leq g(x)$ for every $x \in (0, \pi)$ and every $n \in \mathbb{N}$.

Proof. (a)

$$\int_0^\pi |f_n(x)|\, dx = \frac{1}{n} \int_0^\pi \left(\frac{\sin nx}{x}\right)^2 dx = [x = t/n] =$$
$$\int_0^{n\pi} \left(\frac{\sin x}{x}\right)^2 dx < \int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx < +\infty.$$

(b) $f_n(x) \leq \frac{1}{nx^2} \to 0$, $n \to \infty$, $x \in (0, \pi)$.

(c) Assume the converse. In this case, by Lebesgue thm on dominated convergence, we must have $\int f_n dx \to 0$, $n \to \infty$. At the same time

$$\int f_n dx \to \int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx,\ n \to \infty.$$

□