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Analytic Geometry. Vectors

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First Look on Vectors

- We call vector a segment with assigned to it direction, so one ts endpoint considered as initial point and second endpoint considered as terminal of vector
- Reminder. We postulated before that there is one and only one segment having some pair of points as endpoints, and vice-versa there is one and only one pair of endpoints of given segment
- Physics gives us alternative approach to understanding vectors. Here we assign direction to some physical measure (e.g. force, speed, etc.)
- In mus be noted that some "directed" physical measures are not invariant against mirror image of spaces and are called pseudovectors (or axial vectors)
- More deep generalization of term vector is tensor
- Here we start discussion on most general features of vectors

Directed Segments I

- We start with assigning direction to particular segments (we call them particular vectors)
- Lets assign a direction to arbitrary segment AB and A becomes its initial, and B becomes its terminal point
- We say that vector points from A to B and utilize notation $\overrightarrow{AB}$ or $\vec{a}$
- Alternative notations may fortune in literature: lines over ($\bar{a}$) and under ()$\underline{a}$) the vector denote, bold font ($\boldsymbol{F}$)
- Thus, we consider vector as ordered pair of its initial and terminal points
- Relations ("lies on", "parallel", "forms angle", etc.) of the vector with segments, lines and planes are inherited from relations of parent segment with these object
- We will not separate vectors on plane and in space in this discussion

Directed Segments II

- We say that vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are codirected and write $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ if for any arbitrary points M and N shaping segments AM overlapping AB and CN overlapping CD, and $AM = BM$, length of segment MN limited by constant finite value. In common words, points M and N follow each other on "parallel courses"
- There is no need to enforce this definition with demand of parallelism of AB and CD. If these segments lay on crossing line, distance between two following points may decrease until lines common points, but will grow without any limitation after following this point
- If vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lay on parallel lines, but are not codirected, we say that they are anti-codirected and write $\overrightarrow{AB} \uparrow\downarrow \overrightarrow{CD}$

Directed Segments III

- Theorem 1: *there are two vectors, say $\overrightarrow{AB}$ and $\overrightarrow{CD}$, are codirected with third arbitrary vector, say $\overrightarrow{EF}$. Thus, given vectors are codirected*
- Proof: in given condition there are positive real numbers d_{AB} and d_{DC} and for any equal segments AM, CN, and EP containing B, D, and F respectively $MP < d_{AB}$ and $NP < d_{CD}$. Points M, N, and P shape triangle (or lay on a line as extreme case), thus $MN \leqslant MP + NP < d_{AB} + d_{CD}$. This is definition that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are codirected. $\square$
- We say vectors are equal if shaping them segments are equal (have equal length), and they are codirected. Notation: $\overrightarrow{AB} = \overrightarrow{CD}$
- Theorem 2: *there are two vectors, say $\overrightarrow{AB}$ and $\overrightarrow{CD}$, and for arbitrary vector $\overrightarrow{EF}$ $\overrightarrow{AB} = \overrightarrow{EF}$, and $\overrightarrow{CD} = \overrightarrow{EF}$. Thus, $\overrightarrow{AB} = \overrightarrow{CD}$*
- Proof: $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are codirected, as well as $\overrightarrow{CD}$ and $\overrightarrow{EF}$. Thus, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are codirected. Their lengths are equal with the same value, thus are equal to each other. This is definition of vector equality as it is. $\square$

Directed Segments IV

- Consequence: relation of vectors equality is reflexive, symmetric, and transitive
  - $\overrightarrow{a} = \overrightarrow{a}$
  - $\overrightarrow{a} = \overrightarrow{b}$ and $\overrightarrow{b} = \overrightarrow{a}$
  - By Theorem 2: if $\overrightarrow{a} = \overrightarrow{b}$, and $\overrightarrow{b} = \overrightarrow{c}$, then $\overrightarrow{a} = \overrightarrow{c}$
- Theorem 3: Consider vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ laying on different lines. These vectors are codirected if and only if they lay on parallel lines, and they lay in the same half-plane with respect to line AC.
- Proof:
  - Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lay on parallel lines, and they lay in the same half-plane with respect to line AC.
  - Consider arbitrary segments $AM = CN$, overlapping AB and CD respectively (thus, laying in the same half-plane with respect to AC).
  - Figure $AMNC$ resembles parallelogram for any AM and CN
  - Thus, $AC = MN$ and vectors are codirected. $\square$
  - Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ being codirected, and do not belong to the same line
  - Consider segment $CE || AB$, and laying in the same half-plane as CD with respect to line AC
  - $\overrightarrow{CD} \upuparrows \overrightarrow{AB}$ by definition

Directed Segments V

- Let angle $\angle DCE$ be ordinary
- Thus in the family of transverse segments $\{PM\}$, there $CP = CN$, distance between P and N grows without any limitation with growing CN.
- In the same time $MP = AC$ for any P(and N)
- Consider triangle MNP: $MN > NP - MP = NP - AC$, and NP grows without limitation with NP.
- Thus, $\overrightarrow{AB} \not\upuparrows \overrightarrow{CD}$ if angle $\angle CDE$ is ordinary.
- Therefore, CE overlaps CD, thus CD is parallel to AB and lies in the same half-pale with respect to AC. $\square$

- Particular case: laying on the same line vectors are codirected if and only if ray shaped by first vector contains second vector or vice-versa.

Directed Segments VI

- Theorem 4: For arbitrary vector $\overrightarrow{AB}$ and any point C there is one and only one vector $\overrightarrow{CD} = \overrightarrow{AB}$
- Proof:
  - For point C not laying on the line shaped with AB there is plane containing all points A, B, and C
  - We take point D from this plane to shape segment $CD || AB$, $CD = AB$, and laying in the same half-plane as AB with respect to line AC
  - Theorem 3 establishes that $\overrightarrow{AB} = \overrightarrow{CD}$
  - Uniqueness of the line parallel to given and containing specified distant point grants Uniqueness of$\overrightarrow{CD}$
  - For point C laying on the line shaped with AB there are two cases: $\overrightarrow{AC} \upuparrows \overrightarrow{AB}$, thus CD is continuation of AC, int opposite case CD overlaps AC
  - Uniqueness of $\overrightarrow{CD} = \overrightarrow{AB}$ is consequence of the uniqueness of the segment of given length starting from endpoint point of arbitrary ray.

Zero Vector I

Consider statement:

- For arbitrary vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ necessary and sufficient condition for $\overrightarrow{AB} = \overrightarrow{CD}$ is $\overrightarrow{AC} = \overrightarrow{BD}$
- Proof:
  - For the case there AB and CD lay on different lines condition $\overrightarrow{AB} = \overrightarrow{CD}$ is equal to the statement that AB and CD are sides of parallelogram.
  - Opposite pair of sides, AC and BD are also equal and parallel to each other and lie in the same half-plane wit respect to AB (and as well CD), thus $\overrightarrow{AC} \upuparrows \overrightarrow{BD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$.
  - For $\overrightarrow{AC} = \overrightarrow{BD}$ the same reasoning gives $\overrightarrow{AB} = \overrightarrow{CD}$. $\square$

For $A = C$ this statement is logically invalid, as we require now $\overrightarrow{AA} = \overrightarrow{BB}$.

Zero Vector II

Now we generalize our definition

- We call **vector** any ordered pair of points, distant or not
- For vector composed of a pair of distant pints we call first point int initial and second point terminal, and assign to the vector direction from initial to terminal point
- Vector composed of a pair of equal points we call **zero vector**
- We assign zero length to such vector and left its direction to be undefined

Vector as Abstract Object

- We call **(abstract) vector** the object assigned to the class of equal directed segments.
- This object keeps information on length and direction of any directed segment in specified class
- Vector in this sense resembles directed segment of specified length and direction established from any point
- We say that vector applied to point (or body)
- For example, vector $\boldsymbol{a}$ may be applied to point A, and shape directed segment $\overrightarrow{AB} = \boldsymbol{a}$. In the same time, application of this vector to distant point A'shapes new directed segment $\overrightarrow{A'B'} = \overrightarrow{AB} = \boldsymbol{a}$
- We will denote all zero vectors as $\mathbf{0}$
- We define vector with specification of its direction and length
- Direction may be specified with a ray "directing" vector
- Length has common sense with ordinary vectors and denoted as $|\boldsymbol{a}|$

Addition of Vectors I

- Consider translation of the point particle form point A to point B, and next to point C.
- Directed segments $\overrightarrow{AB}$ and $\overrightarrow{BC}$ represent these translations.
- As a final result we must consider translation from A to C and corresponding directed segment $\overrightarrow{AC}$
- It will be natural to consider $\overrightarrow{AC}$ as a sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
- If these directed segments correspond to (abstract) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, we may define sum $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ with $\boldsymbol{c}$ corresponding to AC by the procedure:
  - Select arbitrary point A
  - Apply $\boldsymbol{a}$ to A and obtain $\overrightarrow{AB} = \boldsymbol{a}$
  - Apply $\boldsymbol{b}$ to B and obtain $\overrightarrow{BC} = \boldsymbol{c}$
  - Construct $\boldsymbol{c} = \overrightarrow{AC}$
- Invariance of this definition against choose of A is a matter of proof

Addition of Vectors II

- Theorem: sum of vectors is invariant against point selected to establish directed segments.
- Proof:
  - Consider distant points A and A'. Definition of vector states $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{A'B'}$ and $\boldsymbol{b} = \overrightarrow{BC} = \overrightarrow{B'C'}$
  - Thus, $\overrightarrow{AA'} = \overrightarrow{BB'}$ and $\overrightarrow{BB'} = \overrightarrow{CC'}$
  - Transition: $\overrightarrow{AC} = \overrightarrow{A'C'}$. $\square$
- We call described approach **The triangle low of vectors addition**
- Key disadvantage: symmetry is not obvious and requires proof.
- **The parallelogram low of vectors addition**
  - Select arbitrary point A, and apply both $\boldsymbol{a}$ and $\boldsymbol{b}$ to it: $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{AC} = \boldsymbol{b}$
  - This triplet, A, B, C, allows shaping parallelogram $ABCD$
  - $\overrightarrow{AB} = \overrightarrow{CD} = \boldsymbol{a}$ and $\overrightarrow{AC} = \overrightarrow{BD} = \boldsymbol{b}$
  - $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{a} + \boldsymbol{b} = \overrightarrow{AC} + \overrightarrow{CD} = \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{c}$

Addition of Vectors III

- On a single line (as well as on parallel lines) sum of vectors depends on their direction:
  - for $\boldsymbol{a} \upuparrows \boldsymbol{b}$ direction will be the same and legths will sum
  - for $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b}$ direction corresponds to greater (by length) vector and less length will be subtracted from greater
- Given definition of addtion of vectors corresponds to definition of vector as abstract object, but not as particular directed segment
- Constructive manipulations with particular directed segments is just application of this definition

Commutative Group

Key features of addition operation

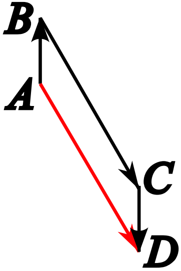
1. For any pair $\boldsymbol{a}$ and $\boldsymbol{b}$ there is vector $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$
2. For any pair $\boldsymbol{a}$ and $\boldsymbol{b}$ $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
3. $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$
4. $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$
5. For any vector $\boldsymbol{a}$ there is vector $-\boldsymbol{a}$ and $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$
6. Operation of subtraction may be defined for any pair of vectors: if $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, we define $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$ if $\boldsymbol{a} = \boldsymbol{c} + \boldsymbol{b}$

By this list of properties we may conclude that set of vectors form algebraic structure known as **commutative group** with operation of addition

Problem 1

1. Arbitrary body moved from point A to point B, later to point C, and finally to point D
2. Distances are folloving $AB = 1$cm, $BC = 3$cm, and $CD = 1$cm
3. $\angle ABC = 60°$, $\overrightarrow{AB} \upuparrows \overrightarrow{DC}$
4. Draw sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$ and find it's length.

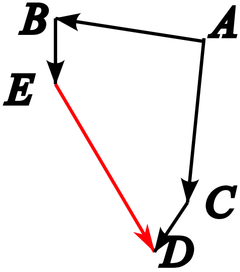
Problems Corner II



Desired figure is parallelogram, $|\overrightarrow{AD}| = |\overrightarrow{BC}|$ =3cm

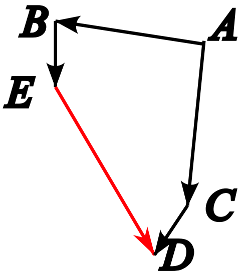
Problem 2

Problems Corner III



Express $\overrightarrow{ED}$ as sum of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BE}$, and $\overrightarrow{CD}$

Problems Corner IV



Express $\overrightarrow{ED}$ as sum of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BE}$, and $\overrightarrow{CD}$

$-\overrightarrow{BE} + (-\overrightarrow{AB}) + \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{ED}$

Problems Corner V

Home assignment

Problem 3

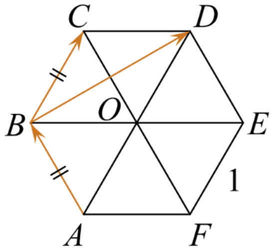
There is a triangle $\triangle ABC$ with equal sides. Plot and find length of vectors $\overrightarrow{AB} + \overrightarrow{AC}$, $\overrightarrow{AB} - \overrightarrow{AC}$. For calculations let length of side be $2\sqrt{3}$cm

Problem 4

In triangle $\triangle ABC$ $\angle A = 120^\circ$ $AB = 1$cm, $AC = 2$cm. Calculate $\overrightarrow{AB} + \overrightarrow{AC}$, $\overrightarrow{BA} + \overrightarrow{AC}$

Problem 5

There is right hexagon $ABCDEF$ (each side is equal) showed on the plot. O is center of the figure. Find $|\overrightarrow{AB}+\overrightarrow{BC}|$, $|\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{ED}|$, and $|\overrightarrow{OD}+\overrightarrow{DB}|$. Length of each side is 1.



Vector Expansion

- In practical applications we often consider transitions and forces applications as distributed along directions being natural for the problem.
- We call that distributed parts **components** of vector
- We call process of this distribution **vector expansion**
- Vector may be restored by addition of all its components

Vector Expansion. Planar Case

- There are vector $\boldsymbol{v}$ laying on a plane, and arbitrary pair of crossing lines a, b laying in the same plane.
- Expansion of $\boldsymbol{v}$ into two components parallel to given lines is explicit operation for any vector $\boldsymbol{v}$.
- Proof:
  - Let O be cross point of a and b
  - Consider directed segment $\overrightarrow{OV} = \boldsymbol{v}$
  - Let $V \notin a$, and $V \notin b$.
  - Consider points A and B: $VB||a$ and $VA||b$
  - Figure $OAVB$ resembles parallelogram: $OA = BV$, $OB = AV$, $OA||BV$, $OB||AV$
  - Parallelogram low of vectors addition grants sum: $\overrightarrow{OV} = \overrightarrow{OA} + \overrightarrow{OB} = \boldsymbol{v}_a + \boldsymbol{v}_b = \boldsymbol{v}$, $\boldsymbol{v}_a||a$ and $\boldsymbol{v}_b||b$
  - Let $\boldsymbol{v}||a$. Now $\boldsymbol{v}_a = \boldsymbol{v}$ and $\boldsymbol{v}_b = \boldsymbol{0}$. And vice-versa: $\boldsymbol{v}||b$: $\boldsymbol{v}_a = \boldsymbol{0}$ and $\boldsymbol{v}_b = \boldsymbol{v}$.
  - Let $\boldsymbol{v}'_a||a$, and $\boldsymbol{v}'_b||b$ be alternative components of vector expansion
  - $\boldsymbol{v} = \boldsymbol{v}_a + \boldsymbol{v}_b = \boldsymbol{v}'_a + \boldsymbol{v}'_b \Rightarrow \boldsymbol{v}_a - \boldsymbol{v}'_a = \boldsymbol{v}'_b - \boldsymbol{v}_b$
  - $\boldsymbol{v}_a - \boldsymbol{v}'_a||a$, and $\boldsymbol{v}'_b - \boldsymbol{v}_b||b \Rightarrow \boldsymbol{v}_a - \boldsymbol{v}'_a = \boldsymbol{v}'_b - \boldsymbol{v}_b = \boldsymbol{0}$
  - $\boldsymbol{v}_a = \boldsymbol{v}'_a$ $\boldsymbol{v}_b = \boldsymbol{v}'_b$. $\square$

Vector Expansion. Plane and Crossing Line

- There are vector $\boldsymbol{v}$, and of plane *alpha* and crossing it line a.
- Expansion of $\boldsymbol{v}$ into two components parallel to given line and plane is explicit operation for any vector $\boldsymbol{v}$.
- Proof:
  - Let O be cross point of a and α
  - Consider directed segment $\overrightarrow{OV} = \boldsymbol{v}$
  - Consider plane β shaped by line a and line containing segment OV
  - Let b be cross line of α and β
  - Now
    - $OV \in \beta$, thus $\boldsymbol{v}||\beta$
    - $a \in \beta$, $b \in \beta$
    - O is cross point of a and b
    - There is single and only single vector expansion $\boldsymbol{v} = \boldsymbol{v}_a + \boldsymbol{v}_b$, $\boldsymbol{v}_a||a$, and $\boldsymbol{v}_b||b$ (thus, $\boldsymbol{v}_b||\alpha$)
  - Let $\boldsymbol{v}'_a||a$, and $\boldsymbol{v}'_b||\alpha$ be alternative components of vector expansion
  - $\boldsymbol{v} = \boldsymbol{v}_a + \boldsymbol{v}_b = \boldsymbol{v}'_a + \boldsymbol{v}'_b \Rightarrow \boldsymbol{v}_a - \boldsymbol{v}'_a = \boldsymbol{v}'_b - \boldsymbol{v}_b$
  - $\boldsymbol{v}_a - \boldsymbol{v}'_a||a$, and $\boldsymbol{v}'_b - \boldsymbol{v}_b||\beta \Rightarrow \boldsymbol{v}_a - \boldsymbol{v}'_a = \boldsymbol{v}'_b - \boldsymbol{v}_b = \mathbf{0}$
  - $\boldsymbol{v}_a = \boldsymbol{v}'_a$ $\boldsymbol{v}_b = \boldsymbol{v}'_b$. $\square$

Vector Expansion. Triplet of Lines

- There are vector $\boldsymbol{v}$, and triplet of lines a,b and c, not laying int ehe same plane
- Expansion of $\boldsymbol{v}$ into two components parallel to given lines is explicit operation for any vector $\boldsymbol{v}$.
- Proof:
  - We assume that lines have cross point O, or...
  - We replace one of the lines with parallel one crossing two another in their cross point O without any leak of generalization, as we discuss only parallelism here.
  - Let plane α be shaped by crossing lines b and c.
  - And line a crosses α int the point O
  - There is single and only single vector expansion $\boldsymbol{v} = \boldsymbol{v}_a + \boldsymbol{v}_\alpha$, $\boldsymbol{v}_a || a$, and $\boldsymbol{v}_\alpha || \alpha$
  - For vector $\boldsymbol{v}_\alpha$ there is single and only single expansion $\boldsymbol{v}_\alpha = \boldsymbol{v}_b + \boldsymbol{v}_c$, $\boldsymbol{v}_b || b$, and $\boldsymbol{v}_c || c$
  - Thus, there is single and only single vector expansion $\boldsymbol{v} = \boldsymbol{v}_a + \boldsymbol{v}_b + \boldsymbol{v}_b$, $\boldsymbol{v}_a || a$, $\boldsymbol{v}_b || b$, and $\boldsymbol{v}_c || c$. $\square$

Vector Expansion. Review

- Operation of vector expansion is explicit in each of following cases
  - Components of expansion parallel with crossing lines if vector is parallel with plane shaped by these lines
  - Components of expansion parallel with arbitrary plane and line crossing this plane
  - Components of expansion parallel with one of three lines not-parallel with any single plane in the same time

Vector Expansion. Operation of Addition

- Addition of vectors means addition of the components of its explicit expansion
- Proof:
  - Consider vectors $\boldsymbol{u}$, and $\boldsymbol{v}$, $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{w}$
  - We take a look only on case of three crossing lines a, b, and c as most general
  - For underlined vectors there is explicit expansions by given lines:
    - $\boldsymbol{v}=\boldsymbol{v}_a+\boldsymbol{v}_b+\boldsymbol{v}_c$, $\boldsymbol{v}_a||a$, $\boldsymbol{v}_b||b$, and $\boldsymbol{v}_c||c$
    - $\boldsymbol{u}=\boldsymbol{u}_a+\boldsymbol{u}_b+\boldsymbol{u}_c$, $\boldsymbol{u}_a||a$, $\boldsymbol{u}_b||b$, and $\boldsymbol{u}_c||c$
    - $\boldsymbol{w}=\boldsymbol{w}_a+\boldsymbol{w}_b+\boldsymbol{w}_c$, $\boldsymbol{w}_a||a$, $\boldsymbol{w}_b||b$, and $\boldsymbol{w}_c||c$
  - $\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}=(\boldsymbol{u}_a+\boldsymbol{v}_a)+(\boldsymbol{u}_b+\boldsymbol{v}_b)+(\boldsymbol{u}_c+\boldsymbol{v}_c)$
  - $(\boldsymbol{u}_a+\boldsymbol{v}_a)||a$, $(\boldsymbol{u}_b+\boldsymbol{v}_b)||b$, and $(\boldsymbol{u}_c+\boldsymbol{v}_c)||c$
  - As expansion is explicit, we may conclude:
  - $(\boldsymbol{u}_a+\boldsymbol{v}_a)=\boldsymbol{w}_a$, $(\boldsymbol{u}_b+\boldsymbol{v}_b)=\boldsymbol{w}_b$, and $(\boldsymbol{u}_c+\boldsymbol{v}_c)=\boldsymbol{w}_c$ $\square$

We can find sum of any vectors by adding their components in explicit and handy expansion.

Collinearity and Coplanarity

- We call vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ **collinear** and write $\overrightarrow{a} || \boldsymbol{b}$ if corresponding directed segments are parallel to arbitrary line
- We call a vector **coplanar** with arbitrary plane if corrsponding directed segments are parallel with the plane
- Zero vector **collinear** with any vector
- Zero vector **coplanar** with any plane

Product of Vector and Real Number I

- For given vector $\boldsymbol{a}$ and real number x we call their product vector denoted as $x\boldsymbol{a}$ with features:
  - $|x\boldsymbol{a}| = |x||\boldsymbol{a}|$
  - If $\boldsymbol{a} = \mathbf{0}$ or $x = 0$ (or both), then $x\boldsymbol{a} = \mathbf{0}$
  - If $x > 0$, then $x\boldsymbol{a} \upuparrows \boldsymbol{a}$
  - If $x > 0$, then $x\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$
- Theorem 1: There are vectors $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{b}$. Existence of real number x: $\boldsymbol{b} = x\boldsymbol{a}$ is necessary and sufficient condition for collinearity of $\boldsymbol{a}$ and $\boldsymbol{b}$. The number x is explicit for the pair of vectors.

Product of Vector and Real Number II

- Proof:
  - If $\boldsymbol{b} = x\boldsymbol{a}$, then $\boldsymbol{a}||\boldsymbol{b}$ by given definition of vectors parallelism and product of vector and number
  - Thus, $\boldsymbol{a} \upuparrows \boldsymbol{b}$, or $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b}$ or $\boldsymbol{b} = \boldsymbol{0}$
  - Consider $\boldsymbol{a}||\boldsymbol{b}$ and construct x
    - Let $\boldsymbol{b} = \boldsymbol{0} \Rightarrow x = 0$: $\boldsymbol{0} = 0\boldsymbol{a}$
    - Let $\boldsymbol{a} \upuparrows \boldsymbol{b} \Rightarrow x = \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|} > 0$: $|x\boldsymbol{a}| = |x||\boldsymbol{a}| = \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}|\boldsymbol{a}| = |\boldsymbol{b}|$ and $x\boldsymbol{a} \upuparrows \boldsymbol{a} \upuparrows \boldsymbol{b} \Rightarrow x\boldsymbol{a} \upuparrows \boldsymbol{b}$, thus $\boldsymbol{b} = x\boldsymbol{a}$
    - Let $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b} \Rightarrow x = -\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|} < 0$: $|x\boldsymbol{a}| = |x||\boldsymbol{a}| = \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}|\boldsymbol{a}| = |\boldsymbol{b}|$ and $x\boldsymbol{a} \uparrow\downarrow \boldsymbol{a} \uparrow\downarrow \boldsymbol{b} \Rightarrow x\boldsymbol{a} \upuparrows \boldsymbol{b}$, thus $\boldsymbol{b} = x\boldsymbol{a}$ $\square$

Note: build of anti-codirected counterpart of equal length for given vector resembles product of -1 and vector.

Product of Vector and Real Number III

Consider features of number-vector product

1. If $\boldsymbol{a} = \mathbf{0}$ or $x = 0$, then $x\boldsymbol{a} = \mathbf{0}$
2. $1 \cdot \boldsymbol{a} = \boldsymbol{a}$
3. $(-1) \cdot \boldsymbol{a} = -\boldsymbol{a}$
4. For arbitrary vector $\boldsymbol{a}$, and arbitrary real numbers x and y $x(y\boldsymbol{a}) = (xy)\boldsymbol{a}$
5. For arbitrary vector $\boldsymbol{a}$, and arbitrary real numbers x and y $(x + y)\boldsymbol{a} = x\boldsymbol{a} + y\boldsymbol{a}$
6. For arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and arbitrary real number x $x(\boldsymbol{a} + \boldsymbol{b}) = x\boldsymbol{a} + x\boldsymbol{b}$

Feature 1, 2, and 3 are direct consequence form definition, but features 4, 5, and 5 require some proof.
General steps for proof:

- Check zero-cases
- Check length of left and right sides
- Monitor redirection of vectors in left and right side with respect to original vectors

Product of Vector and Real Number IV

Proof for feature 4 ($x(y\boldsymbol{a}) = (xy)\boldsymbol{a}$):

- Let $\boldsymbol{a} = \mathbf{0}$. $x(y\mathbf{0}) = x\mathbf{0} = \mathbf{0}$, and $(xy)\boldsymbol{a} = (xy)\mathbf{0} = \mathbf{0}$
- Let $x = 0$ or $y = 0$ (or both are zero), thus $xy = 0$ and $(xy)\boldsymbol{a} = \mathbf{0}$. $0 \cdot (y\boldsymbol{a}) = 0 \cdot \boldsymbol{b} = \mathbf{0}$, or $x(0 \cdot \boldsymbol{a}) = x\mathbf{0} = \mathbf{0}$.
- Consider all-non-zero case
  - $|x(y\boldsymbol{a})| = |x||y\boldsymbol{a}| = |x||y||\boldsymbol{a}|$ and $|(xy)\boldsymbol{a}| = |xy||\boldsymbol{a}| = |x||y||\boldsymbol{a}|$
  - $xy > 0 \Rightarrow x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. Thus, $(xy)\boldsymbol{a} \upuparrows \boldsymbol{a}$
    - $x > 0$ and $y > 0$. Thus, $(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$, and $x\boldsymbol{b} \upuparrows \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \upuparrows y\boldsymbol{a} \upuparrows \boldsymbol{a}$, and $x(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$
    - $x < 0$ and $y < 0$. Thus, $(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$, and $x\boldsymbol{b} \uparrow\downarrow \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \uparrow\downarrow y\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$, and $x(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$
  - $xy > 0 \Rightarrow x < 0$ or $y < 0$ .Thus, $(xy)\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$
    - $x < 0$ and $y > 0$. Thus, $(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$, and $x\boldsymbol{b} \uparrow\downarrow \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \uparrow\downarrow y\boldsymbol{a} \upuparrows \boldsymbol{a}$, and $x(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$
    - $x > 0$ and $y < 0$. Thus, $(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$, and $x\boldsymbol{b} \upuparrows \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \upuparrows y\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$, and $x(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$
  - Left and right operations preserve length and direction of result vector. $\square$

Product of Vector and Real Number V

Proof for feature 5 ($(x+y)\boldsymbol{a} = x\boldsymbol{a} + y\boldsymbol{a}$):

- Let $\boldsymbol{a} = \mathbf{0}$. $(x+y)\mathbf{0} = \mathbf{0}$, and $x\mathbf{0} + y\mathbf{0} = \mathbf{0}$
- Let $x + y = 0$, thus $x = (-1)\cdot y$. $0 \cdot \boldsymbol{a} = \mathbf{0}$, and $x\boldsymbol{a} + y\boldsymbol{a} = x\boldsymbol{a} + (-1)x\boldsymbol{a} = x\boldsymbol{a} - x\boldsymbol{a} = \mathbf{0}$
- Consider all-non-zero case and $xy > 0$.
  - $|x+y| = |x| + |y| \Rightarrow |(x+y)\boldsymbol{a}| = |(x+y)||\boldsymbol{a}| = (|x| + |y|)\boldsymbol{a} = |x||\boldsymbol{a}| + |y||\boldsymbol{a}|$
  - $x\boldsymbol{a} \upuparrows y\boldsymbol{a} \Rightarrow |x\boldsymbol{a} + y\boldsymbol{a}| = |x\boldsymbol{a}| + |y\boldsymbol{a}| = |x||\boldsymbol{a}| + |y||\boldsymbol{a}|$
  - $\mathrm{sign}(x) = \mathrm{sign}(y) = \mathrm{sign}(x+y) = s$
  - $(x+y)\boldsymbol{a} = s|x+y|\boldsymbol{a}$, and $x\boldsymbol{a} + y\boldsymbol{a} = s|x|\boldsymbol{a} + s|y|\boldsymbol{a}$, thus all $(x+y)\boldsymbol{a}$, $x\boldsymbol{a}$ and $y\boldsymbol{a}$ are (anti-)codirected with $\boldsymbol{a}$ in the same time and have the same direction.
- Consider all-non-zero case and $xy < 0$.
  - Let $\mathrm{sign}(-y) = sign(x+y)$ without leak of generalization
  - $(x+y)\boldsymbol{a} - y(a) = (x+y-y)\boldsymbol{a} = x\boldsymbol{a} \Rightarrow (x+y)\boldsymbol{a} = x\boldsymbol{a} + y\boldsymbol{a}$

Product of Vector and Real Number VI

Proof for feature 6 ($x(\boldsymbol{a}+\boldsymbol{b}) = x\boldsymbol{a}+x\boldsymbol{b}$)

- Let $x = 0$. $0 \cdot (\boldsymbol{a}+\boldsymbol{b}) = \mathbf{0}$, and $0 \cdot \boldsymbol{a} + 0 \cdot \boldsymbol{b} = \mathbf{0}+\mathbf{0} = \mathbf{0}$
- Let $\boldsymbol{a} \upuparrows \boldsymbol{b}$.
  - $\boldsymbol{a} \upuparrows \boldsymbol{b} \upuparrows \boldsymbol{a}+\boldsymbol{b}$, and $x\boldsymbol{a} \upuparrows x\boldsymbol{b} \Rightarrow x\boldsymbol{a} \upuparrows x\boldsymbol{b} \upuparrows x(\boldsymbol{a}+\boldsymbol{b})$
  - $|x(\boldsymbol{a}+\boldsymbol{b})| = |x||\boldsymbol{a}+\boldsymbol{b}| = |x|(|\boldsymbol{a}|+|\boldsymbol{b}|) = |x||\boldsymbol{a}|+|x||\boldsymbol{b}|$
  - $|x\boldsymbol{a}+x\boldsymbol{b}| = |x\boldsymbol{a}|+|x\boldsymbol{b}|| = |x||\boldsymbol{a}|+|x||\boldsymbol{b}|$
- Let $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b}$.
  - Let $(\boldsymbol{a}+\boldsymbol{b}) \upuparrows -\boldsymbol{b}$ without leak of generalization
  - $x(\boldsymbol{a}+\boldsymbol{b}) \upuparrows -x\boldsymbol{b}$
  - $x(\boldsymbol{a}+\boldsymbol{b}) + (-x\boldsymbol{b}) = x(\boldsymbol{a}+\boldsymbol{b}+(-\boldsymbol{b})) = x\boldsymbol{a} Rightarrow x(\boldsymbol{a}+\boldsymbol{b}) = x\boldsymbol{a}+x\boldsymbol{b}$

Product of Vector and Real Number VII

- General case $\boldsymbol{a} \nparallel \boldsymbol{b}$
  - Consider directed segments: $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$
  - Sum $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$ is diagonal of corresponding parallelogram $OADB$
  - Let $x > 0$. Directed segments $\overrightarrow{OA'} = x\boldsymbol{a}$ and $\overrightarrow{OB'} = x\boldsymbol{b}$ overlap $\overrightarrow{OA}$ and $\overrightarrow{OB}$ Sum $\overrightarrow{OD'} = \overrightarrow{OA'} + \overrightarrow{OB'} = x\boldsymbol{a} + x\boldsymbol{b}$ is diagonal of corresponding parallelogram $OA'D'B'$ similar with $OADB$
  - $\overrightarrow{OD} \upuparrows \overrightarrow{OD'}$, as opposite sides of parallelograms overlap
  - $OD' = xOD \Rightarrow |\overrightarrow{OD'}| = |x\overrightarrow{OD}| = |x(\boldsymbol{a} + \boldsymbol{b})| = |x\boldsymbol{a} + x\boldsymbol{b}|$ as parallelograms are similar
  - $x\overrightarrow{OA} + x\overrightarrow{OB} = x(\overrightarrow{OA} + \overrightarrow{OB}) \Rightarrow x\boldsymbol{a} + x\boldsymbol{b} = x(\boldsymbol{a} + \boldsymbol{b})$

Product of Vector and Real Number VIII

- Let $x = -1$. In terms of previous points $\overrightarrow{OA'} = \overrightarrow{OA}$, $\overrightarrow{OA'} = \overrightarrow{OB'}$, AA' and BB' shape crossing lines (O is cross point)
- $\angle AOB = \angle A'OB'$, thus $OADC$ and $OA'D'B'$ are equal parallelograms
- Segments $OD$and OD' lay on the single line and do not overlap, thus $\overrightarrow{OD} \uparrow\downarrow \overrightarrow{OD'}$, and $OD = OD'$ $\overrightarrow{OD'} = \overrightarrow{OA'} + \overrightarrow{OB'} = (-1) \cdot \overrightarrow{OA} + (-1) \cdot \overrightarrow{OB} = -\overrightarrow{OD} = (-1) \cdot (\overrightarrow{OA} + \overrightarrow{OB}) \Rightarrow$ $(-1) \cdot \overrightarrow{OA} + (-1) \cdot \overrightarrow{OB} = (-1) \cdot (\overrightarrow{OA} + \overrightarrow{OB})$
- $(-1) \cdot \boldsymbol{a} + (-1) \cdot \boldsymbol{b} = (-1) \cdot (\boldsymbol{a} + \boldsymbol{b})$
- Let $x < 0$. $x = -1 \cdot |x|$
- $x(\boldsymbol{a} + \boldsymbol{b}) = -1 \cdot |x|(\boldsymbol{a} + \boldsymbol{b}) = -1 \cdot (|x|\boldsymbol{a} + |x|\boldsymbol{b}) = -1 \cdot |x|\boldsymbol{a} + (-1) \cdot |x|\boldsymbol{b} = x\boldsymbol{a} + x\boldsymbol{b}$

Vector Expansion along Given Directions I

We start with proving some statements

- Theorem 2: Let vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ be not collinear and α be a plane both $\boldsymbol{a}$ and $\boldsymbol{b}$ coplanar with. For any vector $\boldsymbol{x}$ coplanar with α there is explicit representation

$$\boldsymbol{x} = x_a\boldsymbol{a} + x_b\boldsymbol{b}$$

  Here x_a and x_b are real numbers.
- Proof:
  - Consider directed segments established from the same point O:
  - $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$
  - These segments shape lines a and b crossing in point O
  - Thus, there is explicit expansion $\boldsymbol{x} = \boldsymbol{x}_a + \boldsymbol{x}_b$, $\boldsymbol{x}_a || a$, and thus $\boldsymbol{x}_a || \boldsymbol{a}$, $\boldsymbol{x}_b || b$, and thus $\boldsymbol{x}_b || \boldsymbol{b}$,
  - By theorem 1, there is explicit correspondence: $\boldsymbol{x}_a = x_a\boldsymbol{a}$, and $\boldsymbol{x}_b = x_b\boldsymbol{b}$
  - Thus, there is explicit representation $\boldsymbol{x} = x_a\boldsymbol{a} + x_b\boldsymbol{b}$. $\square$

Vector Expansion along Given Directions II

- Theorem 3: Let vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be not coplanar with any single plane and not collinear pairwise. For any vector $\boldsymbol{x}$ there is explicit representation

$$\boldsymbol{x} = x_a \boldsymbol{a} + x_b \boldsymbol{b} + x_c \boldsymbol{c}$$

  Here x_a, x_b, and x_c are real numbers.
- Proof:
  - Consider directed segments established from the same point O:
  - $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, and $\overrightarrow{OC} = \mathbf{c}$
  - These segments shape lines a, b, and c crossing in point O
  - Thus, there is explicit expansion $\boldsymbol{x} = \boldsymbol{x}_a + \boldsymbol{x}_b + \boldsymbol{x}_c$, $\boldsymbol{x}_a || a$, and thus $\boldsymbol{x}_a || \boldsymbol{a}$, $\boldsymbol{x}_b || b$, and thus $\boldsymbol{x}_b || \boldsymbol{b}$, $\boldsymbol{x}_c || c$, and thus $\boldsymbol{x}_c || \boldsymbol{c}$,
  - By theorem 1, there is explicit correspondence: $\boldsymbol{x}_a = x_a \boldsymbol{a}$, $\boldsymbol{x}_b = x_b \boldsymbol{b}$, and $\boldsymbol{x}_c = x_c \boldsymbol{c}$
  - Thus, there is explicit representation $\boldsymbol{x} = x_a \boldsymbol{a} + x_b \boldsymbol{b} + x_c \boldsymbol{c}$. $\square$

Coordinate System I

Proved statements grant introduction of following fundamental term

- We call pair or triplet of numbers $\{x_*\}$ ($*$ may be a, or b, or c) introduced in theorem 2 or 3 the **coordinates** if vector $\boldsymbol{x}$ with respect to the **basis** $\boldsymbol{a}$, $\boldsymbol{b}$ or $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$
- Theorem 2 (revisited): Let vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ be not collinear and α be a plane both $\boldsymbol{a}$ and $\boldsymbol{b}$ coplanar with. For any vector $\boldsymbol{x}$ coplanar with α there are explicit coordinates with respect to the basis $\boldsymbol{a}$, $\boldsymbol{b}$
- Theorem 3 (revisited): Let vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be not coplanar with any single plane and not collinear pairwise. For any vector $\boldsymbol{x}$ there are explicit coordinates with respect to the basis $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$
- It must be specially noted that only condition for basis vectors now is non-collinearity for planar case and non-coplanarity and pairvize non-collinearity for space case.

Coordinate System II

- Theorem 4:
  - Addition of vectors equal with addition of their coordinates in arbitrary basis
  - Product of vector and real number equal with product of this and coordinates of the vector in arbitrary basis
- Proof is consequence of properties of sum of vectors and product of vector and real number.
- Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be arbitrary basis
- Consider vectors $\boldsymbol{x} = x_a\boldsymbol{a} + x_b\boldsymbol{b} + x_c\boldsymbol{c}$, and $\boldsymbol{y} = y_a\boldsymbol{a} + y_b\boldsymbol{b} + y_c\boldsymbol{c}$
  - $\boldsymbol{x} + \boldsymbol{y} = x_a\boldsymbol{a} + x_b\boldsymbol{b} + x_c\boldsymbol{c} + y_a\boldsymbol{a} + y_b\boldsymbol{b} + y_c\boldsymbol{c} = x_a\boldsymbol{a} + y_a\boldsymbol{a} + x_b\boldsymbol{b} + y_b\boldsymbol{b} + x_c\boldsymbol{c} + y_c\boldsymbol{c} = (x_a + y_a)\boldsymbol{a} + (x_b + y_b)\boldsymbol{b} + (x_c + y_c)\boldsymbol{c}$
  - As expansion is explicit, numbers $x_a + y_a$, $x_b + y_b$, and $x_c + y_c$ resemble components of $\boldsymbol{x} + \boldsymbol{y}$
- Consider real number x and vector $\boldsymbol{y} = y_a\boldsymbol{a} + y_b\boldsymbol{b} + y_c\boldsymbol{c}$
  - $x\boldsymbol{y} = x(y_a\boldsymbol{a} + y_b\boldsymbol{b} + y_c\boldsymbol{c}) = xy_a\boldsymbol{a} + xy_b\boldsymbol{b} + xy_c\boldsymbol{c}$
  - As expansion is explicit, numbers xy_a, xy_b, and xy_c resemble components of $x\boldsymbol{y}$

Angle Between Vectors

- Consider vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Establish directed segments from the same point O: $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. We distinguish angle $\angle AOB$ as angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Consider point O' distant form O and directed segments $\overrightarrow{O'A'} = \boldsymbol{a}$ and $\overrightarrow{O'B'} = \boldsymbol{b}$. $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{OB} = \overrightarrow{O'B'}$. $\overrightarrow{AB} = \overrightarrow{A'B'} = \boldsymbol{b} - \boldsymbol{a}$. Thus $\triangle AOB = \triangle A'O'B'$, and corresponding angles are equal
- Thus angle between vectors depends on only their relative direction
- We will write $\angle(\boldsymbol{a}, \boldsymbol{b})$
- We call vectors shaping right angle **orthogonal**

Dot Product I

- Consider vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
- We call number $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos \angle(\boldsymbol{a}, \boldsymbol{b})$ the **dot product** (or scalar product) of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
- If $\boldsymbol{a} = \mathbf{0}$, or $\boldsymbol{b} = \mathbf{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- If $\boldsymbol{b} = \boldsymbol{a}$, then we write $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{a}^2 = |\boldsymbol{a}|^2$
- For orthogonal not zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\cos \angle(\boldsymbol{a}, \boldsymbol{b}) = 0$, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- Key features of dot product
  1. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  2. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and number x: $(x\boldsymbol{a}) \cdot \boldsymbol{b} = x(\boldsymbol{a} \cdot \boldsymbol{b})$
     - Particular case: $(-\boldsymbol{a}) \cdot \boldsymbol{b} = -\boldsymbol{a} \cdot \boldsymbol{b}$
  3. For vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$: $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

Dot Product II

- Proof:
  1. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
     - This is a simple derivative from definition
  2. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and number x: $(x\boldsymbol{a}) \cdot \boldsymbol{b} = x(\boldsymbol{a} \cdot \boldsymbol{b})$
     - Case if $x = 0$, or (and) $\boldsymbol{a} = \mathbf{0}$, or (and) $\boldsymbol{b} = \mathbf{0}$ appears be trivial
     - Let $x > 0$, thus $x\boldsymbol{a} \upuparrows \boldsymbol{a}$, and $|x| = x$. Therefore, $\angle(\boldsymbol{a}, \boldsymbol{b}) = \angle(x\boldsymbol{a}, \boldsymbol{b})$, and $(x\boldsymbol{a}) \cdot \boldsymbol{b} = |x\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(x\boldsymbol{a}, \boldsymbol{b}) = x|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(\boldsymbol{a}, \boldsymbol{b}) = x(\boldsymbol{a} \cdot \boldsymbol{b})$
     - Let $x < 0$, thus $x\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$, and $|x| = -x$. Therefore, $\angle(x\boldsymbol{a}, \boldsymbol{b}) = \pi - \angle(\boldsymbol{a}, \boldsymbol{b})$, and $\cos\angle(x\boldsymbol{a}, \boldsymbol{b}) = -\cos\angle(\boldsymbol{a}, \boldsymbol{b})$, and $(x\boldsymbol{a}) \cdot \boldsymbol{b} = |x\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(x\boldsymbol{a}, \boldsymbol{b}) = -x|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot (-\cos\angle(\boldsymbol{a}, \boldsymbol{b})) = x(|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(\boldsymbol{a}, \boldsymbol{b})) = x(\boldsymbol{a} \cdot \boldsymbol{b})$

Dot Product III

3. For vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$: $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
   - We star with deriving two supplementary equations:

$$(\boldsymbol{a}+\boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a}\cdot\boldsymbol{b}$$
$$(\boldsymbol{a}-\boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{a}\cdot\boldsymbol{b}$$
$$(\boldsymbol{a}+\boldsymbol{b})^2 + (\boldsymbol{a}-\boldsymbol{b})^2 = 2(\boldsymbol{a}^2 + \boldsymbol{b}^2)$$

   - In the case $\boldsymbol{a} = \boldsymbol{0}$, or $\boldsymbol{b} = \boldsymbol{0}$ the equations ere successfully valid.
   - Consider directed segments $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Thus, $\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{AB}$
   - Now we can apply theorem of cosines for this triangle and definition of the dot product:

$$AB^2 = OA^2 + OB^2 - 2\cdot OA\cdot OB\cdot\cos\angle O$$
$$(\boldsymbol{ab})^2 = |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 - 2\cdot|\boldsymbol{a}|\cdot|\boldsymbol{b}|\cos(\boldsymbol{ab})$$
$$(\boldsymbol{a}+\boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a}\cdot\boldsymbol{b}$$

   - Consider $\boldsymbol{b'} = -\boldsymbol{b}$. $\boldsymbol{a}\cdot\boldsymbol{b'} = \boldsymbol{a}\cdot(-\boldsymbol{b}) = -\boldsymbol{a}\cdot\boldsymbol{b}$
   - $(\boldsymbol{a}+\boldsymbol{b})^2 = (\boldsymbol{a}-\boldsymbol{b'})^2 = \boldsymbol{a}^2 + \boldsymbol{b'}^2 - 2\boldsymbol{a}\cdot\boldsymbol{b'} = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a}\cdot\boldsymbol{b}$
   - Third formula we obtain with summarizing two proved

Dot Product IV

- Now we proceed with desired feature $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- For vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ consider combinations:
- $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$
- $\boldsymbol{q} = \boldsymbol{a} + \boldsymbol{c}$
- $(\boldsymbol{p} + \boldsymbol{q})^2 + (\boldsymbol{p} - \boldsymbol{q})^2 = 2(\boldsymbol{p}^2 + \boldsymbol{q}^2)$
- $(\boldsymbol{p} + \boldsymbol{q})^2 = (2\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}))^2 = 4\boldsymbol{a}^2 + (\boldsymbol{b} + \boldsymbol{c})^2 + 4\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c})$
- $(\boldsymbol{p} - \boldsymbol{q})^2 = (\boldsymbol{b} - \boldsymbol{c})^2 = 2(\boldsymbol{b}^2 + \boldsymbol{c}^2) - (\boldsymbol{b} + \boldsymbol{c})^2$
- $(\boldsymbol{p} + \boldsymbol{q})^2 + (\boldsymbol{p} - \boldsymbol{q})^2 = 4\boldsymbol{a}^2 + 4\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c}) + 2(\boldsymbol{b}^2 + \boldsymbol{c}^2)$
- $2(\boldsymbol{p}^2 + \boldsymbol{q}^2) = 2(\boldsymbol{a} + \boldsymbol{b})^2 + 2(\boldsymbol{a} + \boldsymbol{c})^2 = 4\boldsymbol{a}^2 + 2\boldsymbol{b}^2 + 4\boldsymbol{ab} + 2\boldsymbol{c}^2 + 4\boldsymbol{ac}$
- Comparing this two equation we obtain described

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

$\square$

Some Applications of Dot Product I

- **Work** in mechanics
- Let material point is affected with force $\boldsymbol{F}$ which caused arbitrary displacement described with $\boldsymbol{s}$
- $W = \boldsymbol{F} \cdot \boldsymbol{s} = Fs \cos \angle(\boldsymbol{F}, \boldsymbol{s})$
- Let material point be affected with a pair of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$
- $W = (\boldsymbol{F}_1 + \boldsymbol{F}_2) \cdot \boldsymbol{s} = \boldsymbol{F}_1 \cdot \boldsymbol{s} + \boldsymbol{F}_2 \cdot \boldsymbol{s}$
- We call a **parallelepiped** a three-dimensional figure formed by six opposed parallelograms
- Consider parallelepiped shaped with triplet of non-coplanar directed segments established from arbitrary point O: $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OC} = \boldsymbol{c}$
- It's diagonal resembles sum of these vectors $\boldsymbol{d} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$
- Length of diagonal:$d^2 = \boldsymbol{d}^2 = (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})^2$

Some Applications of Dot Product II

- We call an **axis** a line with predicted direction. In other words it is aggregate of all vectors parallel with arbitrary line and codirected
- We usually associate axis with vector $\boldsymbol{e}$, $|\boldsymbol{e}| = 1$, and call it **unit vector**
- Consider vector a and axis directed with vector $\boldsymbol{e}$
- $\overrightarrow{AB} = \boldsymbol{a}$, and $A'B'$ is a projection of segment AB on a line corresponding to given axis
- We will call signed length of segment $A'B'$ the **projection of vector on axis**.
- We take sign '$+$' if $\overrightarrow{A'B'} \upuparrows \boldsymbol{e}$, and sign '-' in opposite case
- $p_{\boldsymbol{e}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{e} = |\boldsymbol{a}| \cos(\boldsymbol{a}, \boldsymbol{e})$

Dot Product and Coordinate System I

- Previously we expanded a vector with a basis containing pair (on plane) or triplet (in space) of arbitrary vectors satisfying some conditions
- These vectors represent two (or three) axes govering two (three) **main directions**
- It is natural to represent our basis with corresponding set of unit vectors and talk about coordinates with respect to unit vectors
- This approach brought us non leak of generalization as ve may represent any vector shaping the axis as product of the corresponding unit vector and positive real number

Dot Product and Coordinate System II

- Theorem 1: let unit vectors shaping arbitrary basis be pairwise orthogonal. Dot product for vectors expanded with respect to these unit vectors has form for planer case

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

  and for space case

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

  Here a_* and b_* are coordinates of vector in given basis
- Proof of planar case:
  - Let $\boldsymbol{e}_1$, and $\boldsymbol{e}_1$ be these unit vectors
  - $|\boldsymbol{e}_1| = {\boldsymbol{e}_1}^2 = |\boldsymbol{e}_2| = {\boldsymbol{e}_2}^2 = 1,\ \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = (a_1 \boldsymbol{e}_1 + a_2 \boldsymbol{e}_2) \cdot (b_1 \boldsymbol{e}_1 + b_2 \boldsymbol{e}_2) = a_1 b_1 {\boldsymbol{e}_1}^2 + a_1 b_2 \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 + a_2 b_1 \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 + a_2 b_2 {\boldsymbol{e}_2}^2 = a_1 b_1 + a_2 b_2$ $\square$

Direction Cosines

- Consider basis in space with unit vectors $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, $\boldsymbol{e_3}$, and arbitrary vector $\boldsymbol{a}$ laying on an axis shaped with unit vector $\boldsymbol{e}$
- $\boldsymbol{e} \cdot \boldsymbol{e_i} = \cos \angle(\boldsymbol{e}, \boldsymbol{e_1}) = \cos \alpha_i, \; i = 1, 2, 3$
- These angles α_i define direction of the axis with unit vector $\boldsymbol{e}$
- We call them **direction cosines**
- Coordinates of the vector $\boldsymbol{e}$ are $\cos \alpha_1$, $\cos \alpha_2$, $\cos \alpha_3$
- $\boldsymbol{e}^2 = 1 = \cos^2 \alpha_1 + \cos^2 \alpha_2 + \cos^2 \alpha_3$

Cartesian Coordinate System in Space I

- Consider point N laying on a plane α and distant arbitrary point O.
- We choose a pair of crossing in point O lines shaping right angle
- On these lines we may define a pair of unit vectors, say $\boldsymbol{i}$ and $\boldsymbol{j}$, thus shape two orthogonal axes
- We can establish perpendiculars to these lines from point N, say NX and NY
- Points X and Y are **projections** of point N
- Observe directed segments $\overrightarrow{OX} = x\boldsymbol{i}$ and $\overrightarrow{OY} = y\boldsymbol{j}$
- Real numbers x and y determine position of point N with respect to fixed basis $\boldsymbol{i}$, $\boldsymbol{j}$ and initial point O. Signs of these numbers explicitly define necessary half-planes.
- In counter direction. Consider basis $\boldsymbol{i}$, $\boldsymbol{j}$ and initial point O
- We take real numbers x and y and build vectors $x\boldsymbol{i}$, $y\boldsymbol{j}$
- Consider sum and corresponding directed segment starting in point O $x\boldsymbol{i} + y\boldsymbol{j} = \overrightarrow{OP}$
- Thus, number x and y define position of point P
- Uniqueness of vector expansion grants uniqueness of such representation

Cartesian Coordinate System in Space II

-
- Consider arbitrary point M in space and arbitrary plane α with defined coordinate system as we're done before
- Let's establish perpendicular from point M to these point MM_α
- We call pint M_α with **projection** of point M to plane α
- Distance $|MM_alpha| = z$ defines position of point M to within half-space
- This indefiniteness avoided with fixation of half-spaces assigned with '$+$' and '-' signs. This means that we fixed unit vector $\boldsymbol{k}$ parallel with perpendicular, and say that point M has positive z if $\overrightarrow{M_\alpha M} \upuparrows \boldsymbol{k}$.
- We assign $z = 0$ for points laying on α
- Now position of M successfully defined with **ordered** triplet (x, y, z)
- For any triplet (x, y, z) we fix point on plane α with two initial coordinates and as a second step explicitly fix position and distance from the plane with third coordinate

Cartesian Coordinate System in Space III

- Does third coordinate play specific role? No!
- Consider three lines in space, crossing in a single point O, and perpendicular pairwise.
- Arbitrary system of three orthogonal axes may be established along these lines
- We may take any pair of these lines to shape plane α
- Axes along line in tat place will grant coordinate system on it
- Third line will be perpendicular with α and fixed axes on it will grant third coordinate direct

Cartesian Coordinate System in Space IV

- Finally, to obtain coordinates of any point M with respect to arbitrary coordinate system, we establish perpendiculars from point to each axis and took as coordinates of point length of the segment on the axis from center to endpoint of perpendicular. We take sign '$+$' if underlined directed segment from center to projection is codirected with axis and '-' in opposite case.
- To bind arbitrary point we establish directed segments along coordinate axes in proper direction depending on a sign, and move them to ensure that terminal of first is initial point for second and initial point of third is terminal point of second. Terminal point of third is desired point.

Cartesian Coordinate System in Space V

- Canonical right and left Cartesian coordinate system:
  - Center in point O
  - Plane α is horizontal for observer
  - First coordinate axis directed horizontally to observer and denoted as 'x'
  - Second coordinate axis directed horizontally to the right and denoted as 'y'
  - Second coordinate axis directed vertically from down to up and denoted as 'z'
  - Mnemonic rule 1. Take regular clockwise screw. Srewing it into plane α with direction of positive z means rotation of point on it's hat taken on positive half of x to positive half of y
  - Mnemonic rule 2. With your right hand. (1) Thumbs up. (2) Finger gun. (3) Shape right angle with index and middle finger. Look into the barrel or finger gun. Index finger is x, middle finger is y, and thumb finger is z
  - Reverse of any axis allows to screw counterclockwise screw. We call such coordinates left.

Affine Coordinate System in Space I

- To introduce Cartesian coordinate system we should take "threefold" $O\boldsymbol{ijk}$ satisfying:
  - O is center of coordinate system
  - $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = 1$
  - $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0$
- For any vector $\boldsymbol{r}$ we assign coordinates $x = \boldsymbol{i} \cdot \boldsymbol{r}$, $y = \boldsymbol{j} \cdot \boldsymbol{r}$, $z = \boldsymbol{k} \cdot \boldsymbol{r}$
- $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$
- In other hand, we allowed to take arbitrary "threefold" $O\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3}$ with only condition on non-coplanarity and pairwise non-collinearity on these vectors
- Any vector $\boldsymbol{r}$ now has explicit expansion $\boldsymbol{r} = x_1\boldsymbol{e_1} + x_2\boldsymbol{e_2} + x_3\boldsymbol{e_3}$
- Numbers x_1, x_2, x_3 describe position of the terminal point of directed segment $\overrightarrow{OM} = \boldsymbol{r}$, thus this triplet resembles coordinates of point M with respect to "threefold" $O\boldsymbol{e_1}\boldsymbol{e_2}\boldsymbol{e_3}$
- **Nota Bene!** We do not demand any restrictions on the lengths of $\boldsymbol{e_i}$

Affine Coordinate System in Space II

- How we determine x_i?
- We establish directed segments, say $\overrightarrow{OA_1} = \boldsymbol{e_1}$, $\overrightarrow{OA_2} = \boldsymbol{e_2}$, and $\overrightarrow{OA_3} = \boldsymbol{e_3}$
- These segments establish system of tree axes
- Suppose M lies on one of these axes, say shaped with $\boldsymbol{e_1}$
- $x_1 = s \cdot |\overrightarrow{OA_1}|/|\boldsymbol{e_1}|$ $s = 1$ if $\overrightarrow{OA_1} \upuparrows \boldsymbol{e_1}$, and $s = -1$ in opposite case. In other words, x_1 is signed length of segment OM with respect to scale e_1
- Suppose M lies on a plane α shaped with a pair of vectors from the threefold, say $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$