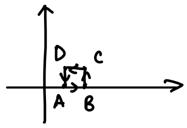
Dec. 10th. P275 曲线积分-II

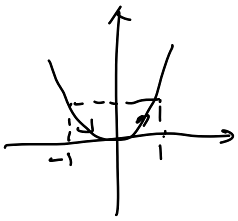
(1) $\int_L (x^2+y^2)dx + (x^2+y^2)dy$



$$I = \int_1^2 x^2 dx + \int_0^1 (4-y^2)dy - \int_1^2 (x^2+1)dx - \int_0^1 (1-y^2)dy.$$

$$= \frac{7}{3} + 4 - \frac{1}{3} - \frac{7}{3} - 1 - \frac{2}{3} = 2$$

(2) $\int_L (x^2-2xy)dx + (y^2-2xy)dy$ $\qquad P(x,f(x))dx + Q(x,f(x))f'(x)dx.$

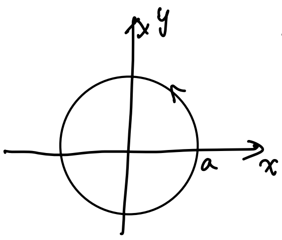


$$\int_L (x^2-2x^3)dx + (x^4-2x^3)\cdot 2x\,dx.$$

$$= \int_{-1}^{1} 2x^5 - 4x^4 - 2x^3 + x^2\,dx$$

$$= \frac{x^6}{3} - \frac{4x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\Big|_{-1}^{1} = \frac{-12+5}{15} \times 2 = -\frac{14}{15}.$$

(3) $\int_L \frac{(x+y)dx - (x-y)dy}{x^2+y^2}$ $\qquad \int P(x(t),y(t))\,x'(t)\,dt + Q(x(t),y(t))\,y'(t)\,dt.$



$$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$$

$$\int_D^{2\pi} \frac{-a^2(\sin t+\cos t)\cdot\sin t - a^2(\cos t - \sin t)\cos t}{a^2}\,dt$$

$$= \int_0^{2\pi} -1\,dt = -2\pi.$$

(4).

$\int_L y\,dx - x\,dy + (x^2+y^2)dz$ $\qquad L: \begin{cases} x = e^t \\ y = e^{-t} \\ z = a^t \end{cases}$ $\quad t: 1\to 0$

$$\int_1^0 e^{-t}\cdot e^t dt - e^t\cdot(-e^{-t})dt + (e^{2t}+e^{-2t})\cdot \ln a\cdot a^t dt.$$

$$= -2 + \ln a\int_1^0 e^{2t}\cdot a^t dt + e^{-2t}\cdot a^t dt.$$

$$= -2 + \ln a\cdot\left[\frac{e^{2t}\cdot a^t}{2+\ln a}\Big|_1^0 + \frac{e^{-2t}\cdot a^t}{\ln a - 2}\Big|_1^0\right]$$

$$= -2 + \ln a\cdot\left(\frac{1-ae^2}{\ln a+2} + \frac{1-ae^{-2}}{\ln a-2}\right)$$

(5). 直线: $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$ $\quad \int x\,dx+y\,dy+(x+y-1)\,dz$.

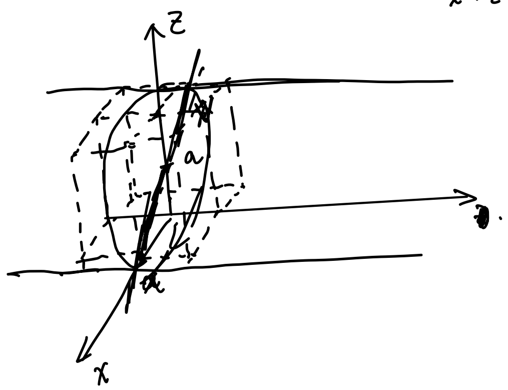
$$\begin{cases} z=3x-2 \\ y=2x-1 \end{cases}$$

$$\int_1^2 x\,dx+2(2x-1)\,dx+3(x+2x-1-1)\,dx$$

$$=\int_1^2 (14x-8)\,dx = 7x^2-8x\Big|_1^2 = 12+1=13$$

(6). $\int_L y\,dx+z\,dy+x\,dz$ $\quad L:\begin{cases} x^2+y^2+z^2=2az \\ x+z=a\ (a>0) \end{cases}$ $\quad x^2+y^2+(a-x)^2=2a(a-x)$ $\quad 2x^2+y^2=a^2$

$$\begin{cases} x=\frac{a\cos\varphi}{\sqrt{2}} \\ y=a\sin\varphi \end{cases}$$



$$\int_L y\,dx+z\,dy+x\,dz = \int_D^{2\pi} -a\sin\varphi\cdot\frac{1}{\sqrt{2}}a\sin\varphi\,d\varphi+\left(a-\frac{a\cos\varphi}{\sqrt{2}}\right)a\cos\varphi\,d\varphi+\frac{a\cos\varphi}{\sqrt{2}}\cdot\frac{a\sin\varphi}{\sqrt{2}}\,d\varphi$$

$$=\int_0^{2\pi} -\frac{1}{\sqrt{2}}a^2\sin^2\varphi\,d\varphi+a^2\cos\varphi\,d\varphi-\frac{1}{\sqrt{2}}a^2\cos^2\varphi\,d\varphi+\frac{a^2}{2}\sin\varphi\cos\varphi\,d\varphi$$

$$=-\sqrt{2}\pi a^2+\frac{a^2}{2}\int_0^{2\pi}\sin\varphi\,d(\sin\varphi)$$

$$=-\sqrt{2}\pi a^2$$

(7) $x=\frac{1}{\tan\alpha}y \Rightarrow \frac{1}{\tan^2\alpha}y^2+y^2+z^2=1. \Rightarrow \frac{1}{\sin^2\alpha}y^2+z^2=1$

$y=\sin\alpha\cos\varphi$
$z=\sin\varphi$.
$x=\cos\alpha\cos\varphi$.

$$\int_L (y-z)\,dx+(z-x)\,dy+(x-y)\,dz.$$

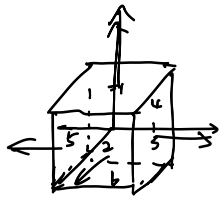
$$=\int_0^{2\pi}(\sin\alpha\cos\varphi-\sin\varphi)(-\cos\alpha\sin\varphi)\,d\varphi+(\sin\varphi-\cos\alpha\cos\varphi)(-\sin\alpha\sin\varphi)\,d\varphi$$
$$+(\cos\alpha\cos\varphi-\sin\alpha\cos\varphi)\cos\varphi\,d\varphi$$

$$=\int_0^{2\pi}\cos\alpha\sin^2\varphi\,d\varphi-\sin\alpha\sin^2\varphi\,d\varphi+\cos\alpha\cos^2\varphi\,d\varphi-\sin\alpha\cos^2\varphi\,d\varphi$$

$$=2\pi(\cos\alpha-\sin\alpha)$$

P276 曲面积分 II.

(1) $\iint_{\Sigma} (x+y)\,dy\,dz + (y+z)\,dz\,dx + (z+x)\,dx\,dy$



$\int_{-h}^{h} dy \int_{-h}^{h} (h+x)\,dx = 4h^3$

$-\int_{-h}^{h} dy \int_{-h}^{h} (-h+x)\,dx = 4h^3.$

$I = (4h^3 + 4h^3) \times 3 = 24h^3$

(2) $\iint_{\Sigma} yz\,dz\,dx \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

$\begin{cases} x = a\cos\varphi\cos\psi \\ y = b\sin\varphi\cos\psi \\ z = c\sin\psi \end{cases}$

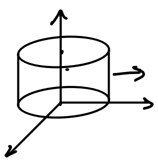
$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} abc^2 \sin\varphi\cos\psi\sin\psi\cos\psi\,(-\sin\varphi\cos\psi - \sin\psi\cos\varphi)\,d\psi.$

$= -abc^2\left[\int_0^{2\pi} \sin^2\varphi \int_0^{\frac{\pi}{2}} \sin\psi\cos^3\psi\,d\psi + \int_0^{2\pi} \sin\varphi\cos\varphi \int_0^{\frac{\pi}{2}} \sin^2\psi\cos^2\psi\,d\psi\right]$

$= -abc^2\left[-\frac{\pi}{4} + 0\right] = \frac{\pi}{4}abc^2$

(3) $\iint_{\Sigma} z\,dy\,dz + x\,dz\,dx + y\,dx\,dy.$ $\qquad x^2+y^2=1$

$z \geq 0, \quad z = 4.$



$\begin{cases} x = \cos\varphi \\ y = \sin\varphi \\ z = h \end{cases} \qquad \vec{n} = (0,0,-1)$

$\iint_{\Sigma} h\cdot\cos\varphi - \cos\varphi(-\sin\varphi) + \sin\varphi(-\sin\varphi)\cos\varphi.\ d\varphi.\ d\varphi.$

$= \int_0^4 h\,dh \int_0^{2\pi} \cos\varphi\,d\varphi + \int_0^4 dh \int_0^{2\pi} \sin\varphi\cos\varphi\,d\varphi. = 0.$

(4). $\iint_{\Sigma} 2x\,dy\,dz + 3\,dx\,dy.$ $\quad \Sigma:\ z = 4 - x^2 - y^2.\ z \geq 0.\ \downarrow.$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi. \end{cases} \qquad x^2+y^2+z-4=0.$

$\iint_{\Sigma} (4-r^2)\,r\cos\varphi\cdot(\sin\varphi\,dr + r\cos\varphi\,d\varphi)\cdot(-2r\,dr) + 3(\cos\varphi\,dr - r\sin\varphi\,d\varphi)(\sin\varphi\,dr + r\cos\varphi\,d\varphi)$

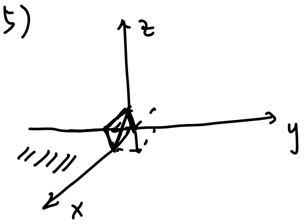
$= 2r^3(r^2-4)\cos^2\varphi\,d\varphi\,dr + 3r\cos^2\varphi\,d\varphi\,dr - 3r\sin^2\varphi\,d\varphi\,dr$

$\int_0^{2\pi} \cos^2\varphi\,d\varphi \int_0^2 2r^5 - 8r^3\,dr = -\frac{32}{3}\int_0^{2\pi} \frac{1+\cos 2\varphi}{2}\,d\varphi = -\frac{32}{3}\pi$ $\qquad$ why.

$\int_0^{2\pi} \cos 2\varphi\,d\varphi \int_0^2 3r\,dr = 6.\int_0^{2\pi} \sin 2\varphi = 0.$

$I = -\frac{32}{3}\pi.$

$$\begin{vmatrix} (4-r^2) r\cos\varphi & 0 & 3 \\ \cos\varphi & \sin\varphi & -2r \\ -r\sin\varphi & r\cos\varphi & 0 \end{vmatrix} dr\, d\varphi$$

(5) 

$x - y + z = 1.$

$x \geq 0,\ y \leq 0,\ z \geq 0.$

$\vec{n} = (1, -1, 1)$

$x = 1 + y - z \geq 0.$

$z \leq y + 1.$

$z \geq 0.\quad y \leq 0.$

$dz\,dx = \cos\beta\, dS.$

$dy\,dz = \cos\alpha\, dS.$

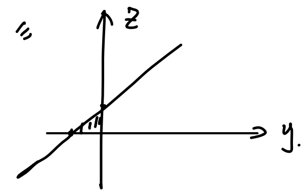
$dx\,dy = \cos\gamma\, dS.$

$\dfrac{dz\,dx}{dy\,dz} = -1$

$\dfrac{dx\,dy}{dy\,dz} = 1.$

$$\iint_\Sigma (f(x,y,z) + x)\, dy\,dz + (2f + y)\cdot(-1)\, dy\,dz. + (f + z)\, dy\,dz.$$

$$= \iint_\Sigma (x - y + z)\, dy\,dz = \iint_\Sigma dy\,dz = \frac{1}{2}.$$



(6) $\displaystyle\iint_\Sigma x^2 dy\,dz + y^2 dz\,dx + (z^2 + 5)\, dx\,dy \qquad \Sigma:\ z = \sqrt{x^2 + y^2}\ (0 \leq z \leq h).$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = r \end{cases} \Rightarrow \begin{vmatrix} r^2\cos^2\varphi & r^2\sin^2\varphi & r^2 + 5 \\ \cos\varphi & \sin\varphi & 1 \\ -r\sin\varphi & r\cos\varphi & 0 \end{vmatrix}$$

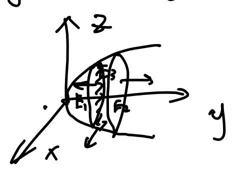
$$- \iint_\Sigma -r^3\cos^3\varphi - r^3\sin^3\varphi + (r^2 + 5) r.\ d\varphi\, dr.$$

$$= \iint_\Sigma r^3(\cos^3\varphi + \sin^3\varphi) - r^3 - 5r\ \ d\varphi\, dr.$$

$$① = \int_0^{2\pi} (\cos^3\varphi + \sin^3\varphi)\, d\varphi \int_0^h r^3\, dr = \frac{h^4}{4} \int_0^{2\pi} = 0$$

$$② = -2\pi \int_0^h r^3 + 5r\, dr = -2\pi \cdot \left[\frac{h^4}{4} + \frac{5h^2}{2}\right]$$

(7) $\iint_{\Sigma} \frac{e^{\sqrt{y}}}{\sqrt{z^2+x^2}}\, dz\, dx$ $\quad y = x^2+z^2 \quad y=1. \quad y=2$



Σ_1: $y=1$. $\quad x^2+z^2=1$.

$-\iint_{\Sigma_1} = -2e\pi$

$\iint_{\Sigma_2} = 2\sqrt{2}\, e^{\sqrt{2}}\pi$.

$x = R\cos\theta$
$y = R^2$
$z = R\sin\theta$

$$\begin{vmatrix} 0 & \frac{e^R}{R} & 0 \\ \cos\theta & 2R. & \sin\theta \\ -R\sin\theta & 0 & R\cos\theta \end{vmatrix}$$

$$-\iint \frac{e^R}{R}\cdot R\, dR\, d\theta = -2\pi\int_1^{\sqrt{2}} e^R\, dR. = -2\pi e^R \Big|_1^{\sqrt{2}} = -2\pi(e^{\sqrt{2}}-e)$$

(8) $\iint \frac{1}{x}dy\,dz + \frac{1}{y}dz\,dx + \frac{1}{z}dx\,dy$.

$$\begin{cases} x = a\cos\varphi\cos\psi \\ y = b\sin\varphi\cos\psi \\ z = c\sin\psi \end{cases}$$

$$\begin{vmatrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ -a\sin\varphi\cos\psi & b\cos\varphi\cos\psi & 0 \\ -a\cos\varphi\sin\psi & -b\sin\varphi\sin\psi. & c\cos\psi \end{vmatrix}$$

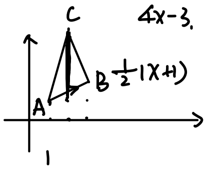
$$\iint \left(\frac{1}{x}\cdot bc\cos\varphi\cos^2\psi + \frac{1}{y}\, ac\sin\varphi\cos^2\psi + \frac{1}{z}\, ab\sin\psi\cos\psi\right) d\varphi\, d\psi.$$

$$= \frac{a2\pi}{bc} \cdot \int_{-\frac{\pi}{2}.}^{\frac{\pi}{2}} \cos\psi\, d\psi$$

$$= 4\pi\left(\frac{ab}{c} + \frac{bc}{a} + \frac{ac}{b}\right)$$

格林公式.

(1) $\int_L (x+y)^2 dx - (x^2+y^2) dy$ $\qquad \iint_D -2x-(2x+2y)\,dxdy = -\iint_D (4x+2y)\,dxdy$



$\begin{cases} 1\le x\le 2 \\ 4x-3\ge y\ge \frac{x+1}{2}. \end{cases}$ $\qquad \begin{cases} 2\le x\le 3 \\ -3x+11\ge y\ge \frac{x+1}{2} \end{cases}$

$$-\int_1^2 dx\int_{\frac{x+1}{2}}^{4x-3}(4x+2y)\,dy - \int_2^3 dx\int_{\frac{x+1}{2}}^{-3x+11}(4x+2y)\,dy$$

$$= -\int_1^2 4x(4x-3)+(4x-3)^2-4x\left(\frac{x+1}{2}\right)-\left(\frac{x+1}{2}\right)^2 dx$$

$$-\int_2^3 4x(-3x+11)+(-3x+11)^2-4x\left(\frac{x+1}{2}\right)-\left(\frac{x+1}{2}\right)^2 dx = -$$

(2) $\int_L xy^2dx - x^2y\,dy = \iint_D -2xy-2xy\,dxdy = -4\iint_D r^2\sin\varphi\cos\varphi\cdot r\,dr\,d\varphi.$

$$= -\int_0^{2\pi}\sin\varphi\cdot d(\sin\varphi) = 0.$$

(3). $\int_L (x^2y\cos x+2xy\sin x-y^2e^x)dx+(x^2\sin x-2ye^x)dy.$

$$= \iint_D (2x\sin x+x^2\cos x-2ye^x-x^2\cos x-2x\sin x+2ye^x)\,dxdy = 0$$

(4) 令 L' 是 L 与 $-L$ 围成的闭合曲线. L' 是诱导负向. $|y|=\sin x.$

$$-\int_{L'} e^x[(1-\cos y)dx-(y-\sin y)dy] = -\iint_{L'} -e^xy\,dxdy.$$

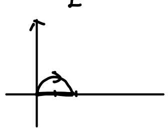
$$= \iint_{L'} e^x \overset{y}{\cancel{(1+\sin y)}}\,dxdy. = \int_0^1 e^x dx\int_{\overset{0}{\cancel{-\sin x}}}^{\sin x}\overset{y.}{\cancel{(1+\sin y)}}\,dy.$$

$$= \cancel{2}\int_0^1 e^x \overset{\frac{\sin^2 x}{2}}{\cancel{\sin x}}\,dx = e^x(\sin x-\cos x)\Big|_0^{\pi} = e^{\pi}-1$$

$$\int e^x\sin x = e^x\sin x - \int e^x\cos x = e^x\sin x-\left(e^x\cos x+\int e^x\sin x.\right.$$

$$\int_L\vec{f} = \frac{1}{2}\int_{L'}\vec{f} = \cancel{\frac{e^{\pi}-1}{2}} \quad \frac{e^{\pi}-1.}{5}$$

(5). $\int_L (x^2-y)dx+(x+\sin^2y)dy$

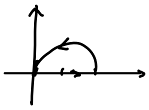


$L_1 = y=0.$

$$\int_{L+L_1} = -\iint_D -1-(-1) = 0$$

$$\int_L = \int_{L+L_1}+\int_{L_1} = \int_0^2 x^2dx = \frac{8}{3}$$

(6). $\int_L [e^x \sin y - b(x+y)]dx + (e^x\cos y - ax)\,dy$



$L_1: y=0. \quad x: 0 \to 2a.$

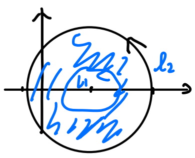
$\int_{L+L_1^-} = \iint (\cos y \cdot e^x - a) - (e^x \cos y - b)\, dx\,dy.$

$= \iint_D (b-a)\,dx\,dy = (b-a)\cdot \frac{a^2\pi}{2}$

$\int_{L_1^-} = \int_{L_1^-} -bx\,dx = b\int_0^{2a} x\,dx = 2a^2 b$

$\int_L = a^2 b(\frac{\pi}{2}+2) - \frac{a^3\pi}{2}.$

(7). $\int_L \frac{x\,dy - y\,dx}{4x^2+y^2} \xlongequal{L_1\ 4x^2+y^2=1} \int x\,dy - y\,dx = \int_0^{2\pi} \frac{1}{2}\,dt = \pi.$



$R>1$. 包含 $(0,0)$. 在 $(0,0)$. $\frac{\partial P}{\partial y}$ $\frac{\partial Q}{\partial x}$ 不存在.

不能直接用与路径无关.

但在此区满足格林公式使用条件, 且 $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

可取路径 l_1. $\int_{l_2+l_1^-} = \iint 0\,dx\,dy = 0$ $\quad \int_{l_2} = \int$

(8) $\int_L \frac{(x-y)\,dx + (x+4y)\,dy}{x^2+4y^2}$

$\frac{\partial Q}{\partial x} = \frac{x^2+4y^2 - 2x(x+4y)}{(x^2+4y^2)^2} = \frac{4y^2 - x^2 - 8xy}{(x^2+4y^2)^2}$

$\frac{\partial P}{\partial y} = \frac{-(x^2+4y^2) - 8y(x-y)}{(x^2+4y^2)^2} = \frac{4y^2 - x^2 - 8xy}{(x^2+4y^2)^2}$

$L_1: \quad x^2+4y^2 = 1.$

$\int_L = \int_{L_1} (x-y)\,dx + (x+4y)\,dy \xlongequal{\begin{cases} x = \cos t \\ y = \frac{1}{2}\sin t\end{cases}} \int_0^{2\pi} -(\cos t - \frac{1}{2}\sin t)\sin t + \frac{1}{2}\cos t(\cos t + 2\sin t)\,dt$

$= \int_0^{2\pi} \frac{1}{2}\,dt = \pi.$

(9). $\int_L \frac{e^x[(x\sin y - y\cos y)\,dx + (x\cos y - y\sin y)\,dy]}{x^2+y^2}$ $\qquad \sin(\cos t)$

$L_1: x^2+y^2 = 1$ 逆时针.

$\int_L = \int_{L_1} e^{\cos t}[-\sin t(\cos t \sin y - \sin t \cos y) + \cos t(\cos t \cos y + \sin t \sin y)\,dt]$

$= \int_{L_1} e^{r\cos t} \cos(r\sin t)\,dt.$

$r \to 0, \quad I = 2\pi.$

2.

① $S=\frac{1}{2}\int xdy-ydx=\frac{1}{2}a^2\int \cos^3t\cdot 3\sin^2t\cos t\,dt+\sin^3t\cdot 3\cos^2t\sin t\,dt$

$=\frac{3a^2}{2}\int_0^{2\pi}\cos^2t\sin^2t\,dt=\frac{3a^2}{2}\cdot 2\pi\cdot\frac{1}{8}=\frac{3a^2\pi}{8}$

② 交点 $(0,0)$. $(a,0)$.

$y=\sqrt{ax}-x$

$S=\frac{1}{2}\int xdy-ydx=\frac{1}{2}\int_0^a x(\sqrt{a}\cdot\frac{1}{2\sqrt{x}}-1)-(\sqrt{ax}-x)dx$

$=\frac{1}{2}\int_0^a\frac{\sqrt{a}}{2}\cdot\sqrt{x}-x-\sqrt{a}\sqrt{x}+x\,dx=-\frac{\sqrt{a}}{4}\int_0^a\sqrt{x}.$

$=\frac{\sqrt{a}}{4}\quad\frac{2}{3}x^{\frac{3}{2}}\Big|_0^a=\frac{a^2}{6}$

③ $(0,0)\to(2\pi a,0)$. $t\ 0\to 2\pi$ 逆时针负向

$S=\frac{1}{2}\int xdy-ydx=\frac{1}{2}\int_0^{2\pi}[a^2(t-\sin t)\cdot\sin t-a^2(1-\cos t)(1-\cos t)]dt$

$=\frac{a^2}{2}\int_0^{2\pi}(t\sin t-2+2\cos t)dt$

$=-2\pi a^2+\frac{a^2}{2}\int_0^{2\pi}t\sin t\,dt=-3a^2\pi.$ $\qquad S=3a^2\pi.$

3.(1) $\int_{(0,0)}^{(1,1)}(x-y)(dx-dy)=\int_{(0,0)}^{(1,1)}(x-y)dx+(y-x)dy$

$\frac{\partial P}{\partial y}=-1\qquad\frac{\partial Q}{\partial x}=-1.$

$u'_x=x-y\qquad u=\frac{1}{2}x^2-xy+c(y)$

$u'_y=y-x\qquad u=\frac{1}{2}y^2-xy+c(x).$

$\Rightarrow u=\frac{1}{2}(x^2+y^2)-xy+C.$

$I=u(1,1)-u(0,0)=0$

(2) $\int_{(2,1)}^{(1,2)} \varphi(x)\,dx + \psi(y)\,dy.$

$$\frac{\partial P}{\partial y} = 0. \qquad \frac{\partial Q}{\partial x} = 0.$$

C, B, A

$$\int_{(2,1)}^{(1,2)} \varphi(x)\,dx + \psi(y)\,dy = \int_2^1 \varphi(x)\,dx + \int_1^2 \psi(y)\,dy.$$

$$= \int_1^2 (\psi(t) - \varphi(t))\,dt.$$

(3)

$$\int_{(1,0)}^{(6,8)} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial P}{\partial y} = x \cdot \frac{-\frac{1}{2}\cdot 2y}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} \qquad \frac{\partial Q}{\partial x} = y \cdot \frac{-\frac{1}{2}\cdot 2x}{(x^2+y^2)^{\frac{3}{2}}} = \frac{\partial P}{\partial y}$$

$$\int_1^6 \frac{x\,dx}{x} + \int_0^8 \frac{y\,dy}{\sqrt{36+y^2}} = 5 + \frac{1}{2}\int_0^8 \frac{d(y^2+36)}{\sqrt{36+y^2}} = 5 + \frac{1}{2}\cdot 2\sqrt{36+y^2}\Big|_0^8 = 9.$$

(4). $\frac{\partial P}{\partial y} = -2x\sin y + 2y\cos x \qquad \frac{\partial Q}{\partial x} = 2y\cos x - 2x\sin y.$

$u'_x = 2x\cos y + y^2\cos x \qquad u = x^2\cos y + y^2\sin x + c(y)$

$u'_y = 2y\sin x - x^2\sin y \qquad u = y^2\sin x + x^2\cos y + c(x).$

$$u = x^2\cos y + y^2\sin x + C$$

6. $\frac{\partial P}{\partial y} = 2x. \qquad \frac{\partial Q}{\partial x} = 2x. \qquad Q = x^2 + c(y).$

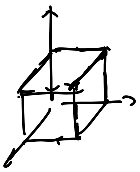
$$\int_{(0,0)}^{(t,1)} 2xy\,dx + Q(x,y)\,dy = \int_0^1 Q(0,y)\,dy = \int_0^1 Q(0,y)\,dy + t^2$$

$$\int_{(0,0)}^{(1,t)} 2xy\,dx + Q(x,y)\,dy = \int_0^t Q(0,y)\,dy + \int_0^1 2tx\,dx = \int_0^t Q(0,y)\,dy + t$$

$$\int_1^t Q(0,y)\,dy = t^2 - t \qquad Q(y) = 2y-1$$

$$Q(x,y) = x^2+2y-1.$$

8. (1) $\iint_{\Sigma} x^2\,dy\,dz + y^2\,dz\,dx + z^2\,dx\,dy.$



$$2\iiint (x+y+z)\,dx\,dy\,dz = 3a^4$$

(2). $3\iiint dx\,dy\,dz = 3V = 24\cdot 1\times1\times\frac{1}{2}\times1\times\frac{1}{3} = 4.$

u, v, w

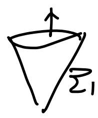
$x = \frac{u+v}{2}$

$y = \frac{v+w}{2}$

$z = \frac{u+w}{2}$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad \frac{1}{2}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{4} = \frac{1}{4}$$

(3). $\iint_{\Sigma_1+\Sigma_2} (x^2\cos\alpha + y^2\cos\beta + z^2\cos\gamma)\,dS$



$$= 2\iiint (x+y+z)\,dx\,dy\,dz.$$

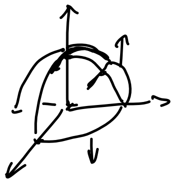
$$= 2\int_0^{2\pi} d\varphi \int_0^h dz \int_0^z \big(r(\cos\varphi+\sin\varphi)+z\big)\,r\,dr.$$

$$= 4\pi \int_0^h z\,dz \int_0^z r\,dr = 4\pi\int_0^h \frac{1}{2}z^3\,dz = \frac{\pi h^4}{2}$$

$$\iint_{\Sigma_1} z^2\,dx\,dy = h^2\cdot\pi h^2 = \pi h^4.$$

$$\iint_{\Sigma_1} = -\frac{\pi h^4}{2}$$

(4) $\iint x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$



$$\iint_{\Sigma_1+\Sigma_2} = 3\iiint dx\,dy\,dz = 2\pi R^3$$

$$\iint_{\Sigma_2^-} = 0$$

$$\iint_{\Sigma_1} = 2\pi R^3.$$

Exam

10 tasks, $\times 5$ points.

9 practice.

5 by Rotkevich → 1) partial derivate. e.g. $\frac{\partial^4 z}{\partial^2 x\,\partial^2 y}$

2) Check differentiability 3) external points

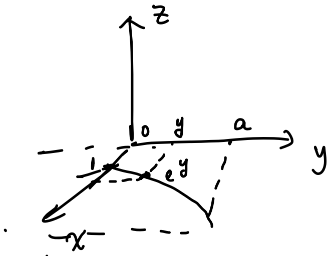
4) Calculate integral (triple) 5) conditional extremum

4 surface integrals / curve.

1 thm. (5 Question ≤ 3 lines from 2nd parts.)

(5). $\iint 2(1-x^2)\,dy\,dz + 8xy\,dz\,dx - 4zx\,dx\,dy$

$\Sigma: x = e^{\sqrt{y^2+z^2}}$



(6).

$\Sigma_1: z = x^2+y^2.$

$\Sigma_2: z = 1, \quad x^2+y^2 \leq 1.$

$$\iint_{\Sigma_1+\Sigma_2} = \iiint_V 3\,dx\,dy\,dz \qquad \begin{cases} z = z \\ x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$= 3\int_0^1 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{z}} r\,dr = 3\pi\int_0^1 z\,dz = \frac{3}{2}\pi.$$

$$\iint_{\Sigma_1} = \iint_{\Sigma_2} - \frac{3}{2}\pi = \iint_S dx\,dy - \frac{3}{2}\pi = -\frac{\pi}{2}$$

(7). $\iint_\Sigma \frac{ax\,dydz+(a+z)^2dxdy}{\sqrt{x^2+y^2+z^2}}$ $(a>0)$. $\Sigma: z=-\sqrt{a^2-x^2-y^2}$

Σ_1 $z=0$. $x^2+y^2\le a$. 上側.

$$\iint_\Sigma = \frac{1}{a}\iint_\Sigma ax\,dydz+(a+z)^2dxdy =$$

$$\frac{1}{a}\iint_{\Sigma+\Sigma_1^-} = -\frac{1}{a}\iiint_V (3a+2z)\,dxdydz = -2\pi a^3 - \frac{2}{a}\cdot 2\pi\int_{-\frac{\pi}{2}}^{0}\sin\varphi\cos\varphi\int_0^a r^3\,dr$$
$$= -\frac{3}{2}\pi a^3$$

$$\iint_{\Sigma_1} = \frac{1}{a}\iint_\Sigma ax\,dydz+(a+z)^2dxdy = a\iint dxdy = a^3\pi.$$

$$\iint_\Sigma = \iint_{\Sigma+\Sigma_1^-} + \iint_{\Sigma_1} = -\frac{1}{2}\pi a^3$$

(8). $$\iint_\Sigma \frac{x\,dydz+y\,dzdx+z\,dxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}} =$$

$$\frac{\partial P}{\partial x} = \frac{(x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3} = \frac{(x^2+y^2+z^2)(y^2+z^2-2x^2)}{(x^2+y^2+z^2)^{\frac{2}{2}}}$$

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}=0. \quad (x,y,z)\neq 0.$$

(a) Let $\Sigma_1: x^2+y^2+z^2=\frac{1}{2}$

$$\iint_{\Sigma_-+\Sigma_1} = 0, \qquad \iint_\Sigma = \iint_{\Sigma_1} = 24\iiint dxdydz = 4\pi$$

(b) Let $\Sigma_2: \frac{(x-2)^2}{16}+\frac{(y-1)^2}{9}\le 1$ $z=0$. 上側 $\quad \Sigma_3: \{x^2+y^2+z^2=\varepsilon^2\}$ 上側.

$$\iint_{\Sigma+\Sigma_{2-}+\Sigma_{3-}} = 0.$$

去掉 $x^2+y^2\le\varepsilon^2,\ z\ge 0.$

$$\iint_{\Sigma} = \iint_{\Sigma_2+\Sigma_3} = \iint_{\Sigma_3} \frac{x\,dydz + y\,dzdx + z\,dxdy}{\varepsilon^3} = \frac{1}{\varepsilon^2}\iint dS = \frac{1}{\varepsilon^2}\cdot 4\pi\varepsilon^2\cdot\frac{1}{2} = 2\pi$$

trial $\quad x^2=y^2+z^2. \quad x=\sqrt{y^2+z^2}$

$(2x,\ 2y,\ 2z-2a).$

$$\int_C (z-x^2-y)\,dx + (x+y+z)\,dy + (y+2x+z^2)\,dz.$$

$$y^2+\left(z-\frac{a}{2}\right)^2=\frac{a^2}{4}.$$

$$= \iint_S -dzdx + 2\,dxdy$$

$z=\frac{a}{2}+r\cos\theta$

$$= \iint -\left(-\frac{y}{x}\right) + 2\left(-\frac{z}{x}\right) dydz = \iint \frac{y-2z}{\sqrt{y^2+z^2}}\,dydz$$

$y=r\sin\theta$

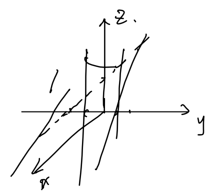
$$= \iint \frac{\qquad}{\sqrt{r^2+ar\cos\theta+\frac{a^2}{4}}}$$

12. Stokes.

(1) $\int y\,dx + z\,dy + x\,dz.$ $\quad \vec{n} = \left(\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}\right).$

$$= \iint \begin{vmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} dS = -\sqrt3 \iint dS = -\sqrt3\,\pi a^2$$

(2) $\begin{cases} x^2+y^2=1 \\ z=y+3. \end{cases} \Rightarrow$

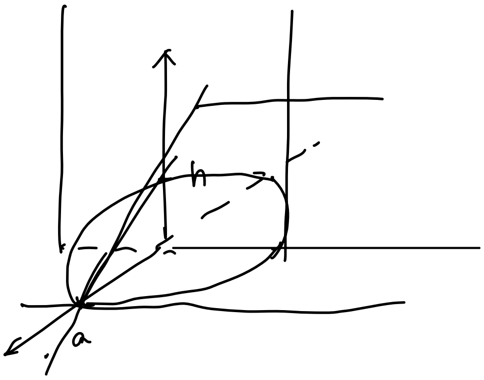


$\vec{n_0} = \left(0, -\frac{1}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$$= \iint_{\Sigma} \begin{vmatrix} 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 5x & -2y \end{vmatrix} = \iint -2\cdot 0 - 3\frac{1}{\sqrt2} + 5\cdot\frac{1}{\sqrt2} = \sqrt2 \iint dS. = \sqrt2\cdot\sqrt2\cdot 1\cdot\pi = 2\pi.$$

(3) $n = \left(\frac{1}{a}, 0, \frac{1}{h}\right)$, $\quad \vec{n_0} = \left(\frac{1}{\sqrt{1+\frac{a^2}{h^2}}}, 0, \frac{1}{\sqrt{1+\frac{h^2}{a^2}}}\right)$

$$= \iint_{\Sigma} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} = \iint \left[(-2)\cdot\frac{1}{\sqrt{1+\frac{a^2}{h^2}}} + (-2)\frac{1}{\sqrt{1+\frac{h^2}{a^2}}}\right] dS.$$



$$= -2\left(\frac{1}{a\sqrt{\frac{1}{h^2}+\frac{1}{a^2}}}+\frac{1}{h\sqrt{\frac{1}{a^2}+\frac{1}{h^2}}}\right)\cdot a\cdot\sqrt{a^2+h^2}\cdot\pi$$

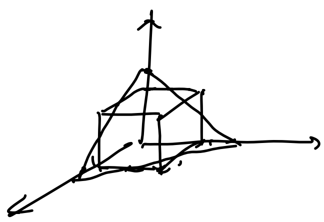
$$= -2\left(\frac{a+h}{\sqrt{a^2+h^2}}\right)\cdot a\cdot\sqrt{a^2+h^2}\cdot\pi.$$

$$= -2a\pi(a+h)$$

(4).

$$\int_L = \iint_\Sigma \begin{vmatrix} \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2-z^2 & z^2-x^2 & x^2-y^2 \end{vmatrix} dS = \frac{1}{\sqrt3}\iint_\Sigma -2y-2z-2z-2x-2x-2y = -\frac{4}{\sqrt3}\iint_\Sigma (x+y+z)\,dS$$

$$= -\frac{4}{\sqrt3}\cdot\frac{3}{2}\iint_\Sigma dS = -2\sqrt3\iint dS$$

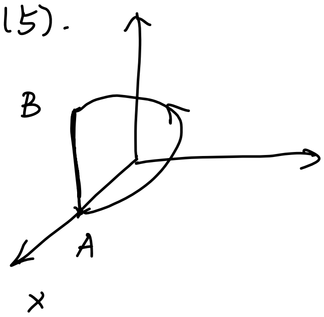


$$= -2\sqrt3\left(\frac{\sqrt3}{4}\cdot\left(\frac{3}{2}\sqrt2\right)^2-3\,\frac{\sqrt3}{4}\cdot\left(\frac{\sqrt2}{2}\right)^2\right)$$

$$= -2\sqrt3\cdot\frac{\sqrt3}{4}\cdot\left(\frac{9}{2}-3\cdot\frac{1}{2}\right)$$

$$= -\frac{3}{2}\cdot 3 = -\frac{9}{2}.$$

(5).



$$L_{BA}: \begin{cases} x=a \\ y=0 \\ 0\le z\le h. \end{cases}$$

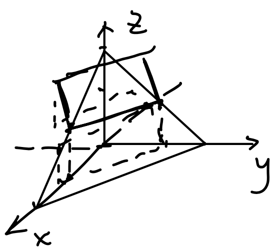
$$\int_{L_{BA}+L} = \iint_\Sigma \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix} = 0.$$

$$\int_L = \int_{L_{AB}} = \int_0^h z^2\,dz = \frac{h^3}{3}$$

(6). $\int (y^2-z^2)\,dx+(2z^2-x^2)\,dy+(3x^2-y^2)\,dz$

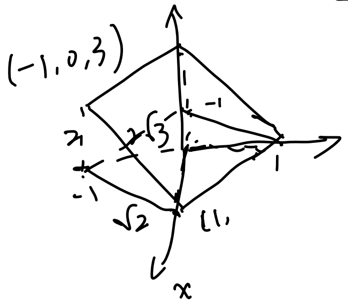
$$L: \begin{cases} x+y+z=2 \\ |x|+|y|=1 \end{cases}$$

$$= \iint \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2-z^2 & 2z^2-x^2 & 3x^2-y^2 \end{vmatrix}$$



$$= \iint -2y-4z-2z-6x-2x-2y$$

$$= -\frac{1}{\sqrt3}\iint\left[4(x+y+z)+2z+4x\right]dS =$$



$$= -\frac{1}{\sqrt{3}} \iint 8 + 2(2-x-y) + 4x \,]\, dS = -\frac{1}{\sqrt{3}} \cdot 12 \cdot 2\sqrt{3} + 2\iint (y-x)\,dxdy$$

$$= -24 + 2\iint (y-x)\,dxdy. = -24$$

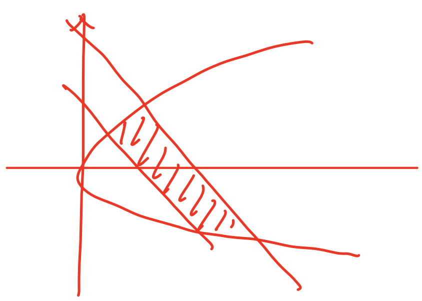
(1) $(1,2)$ $(2,2)$ $(1,1)$. $\quad grad = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

(2) $$\frac{\partial f}{\partial \vec{v}}\Big|_{(1,2)} = \begin{pmatrix} \frac{\partial f}{\partial x}\Big|_{(1,2)} \\ \frac{\partial f}{\partial y}\Big|_{(1,2)} \end{pmatrix} (\vec{e_1}, \vec{e_2}).$$

4. Calculate

$$\iint_\Omega (x+y)dxdy,$$

where Ω is a set bounded by curves $y^2 = 2x$, $x+y=4$, $x+y=12$.



$$\begin{cases} x+y=u \\ y=v. \end{cases} \qquad u-v \qquad \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$-\sqrt{2u+1}-1 \le v \le \sqrt{2u+1}-1$$

$$\begin{cases} v^2 = 2(u-v) \\ 4 \le u \le 12. \end{cases} \qquad v^2+2v+1 = 2u+1.$$

$$\iint u\,dudv = \int_4^{12} u\,du \int_{-\sqrt{2u+1}-1}^{\sqrt{2u+1}-1} dv$$

$$= 2\int_4^{12} u\cdot\sqrt{2u+1}\,du.$$

$$\int u\sqrt{2u+1} = \int \frac{(2u+1)\sqrt{2u+1} - \sqrt{2u+1}}{2}\,du$$

$$= \int_4^{12} (2u+1)^{\frac{3}{2}}\,du - \int_4^{12} (2u+1)^{\frac{1}{2}}\,du$$

$$= \frac{1}{5}(2u+1)^{\frac{5}{2}}\Big|_4^{12} - \frac{1}{3}(2u+1)^{\frac{3}{2}}\Big|_4^{12}.$$

$$= \frac{5^5-3^5}{5} - \frac{5^3-3^3}{3} = \frac{8156}{15}$$

3. Consider all rearrangements of order of integration in

$$V = \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)dz$$

$$\int_{-1}^{1} dx \int_{|x|}^{1} dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} dy$$

$-1 \le x \le 1$

$\sqrt{1-x^2} \le y \le -\sqrt{1-x^2}$

$\sqrt{x^2+y^2} \le z \le 1$

$-1 \le x \le 1$

$x^2+y^2 \le 1$

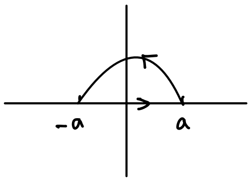
$x^2+y^2 \le z^2 \le 1$

$x^2 \le 1$

$$\int_{-1}^{1} dy \int_{|y|}^{1} dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} dx \quad \int_{0}^{1} dz \int_{-z}^{z} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} dx$$

$$\int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \int_{\sqrt{x^2+y^2}}^{1} dz \quad \int_{0}^{1} dz \int_{-z}^{z} dx \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} dy$$

3. $\oint_L xy^2\,dy - x^2y\,dx$ $\qquad L_1: y=0 \quad (-a,0)\to(a,0).$



$$\oint_{L+L_1} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$
$$= \iint (x^2+y^2)\,dx\,dy$$
$$= \int_0^{\pi} d\varphi \int_0^a r^3\,dr = \frac{\pi a^4}{4}$$

4. $\lim\limits_{\rho\to 0} \frac{1}{\pi\rho^2} \iint\limits_{x^2+y^2\le\rho^2} f(x,y)\,d\sigma.$

6. $f(t) = \int_1^{t^2} e^{-x^2}\,dx \qquad \int_0^1 t\,dt \int_1^{t^2} e^{-x^2}\,dx$

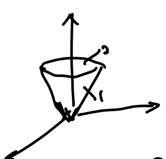
(1) $\quad z = \sqrt{a^2-x^2-y^2}$

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{a^2-x^2-y^2}} \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{a^2-x^2-y^2}}$$

$$\iint_S x+y+\sqrt{a^2-x^2-y^2}\,dS$$
$$= \iint_S \left(x+y+\sqrt{a^2-x^2-y^2}\right)\cdot\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$
$$= a\iint (x+y)\cdot\frac{1}{\sqrt{a^2-x^2-y^2}}\,dx\,dy + a\iint dx\,dy = \pi a^3$$
$$= \iint r(\sin\theta+\cos\theta)\frac{r}{\sqrt{a^2-r^2}}\,dr\,d\theta$$
$$= \int_0^{2\pi}(\sin\theta+\cos\theta)\,d\theta \int_0^a \frac{r^2}{\sqrt{a^2-r^2}}\,dr = 0$$

(2) $\iint\limits_S (x^2+y^2)\,dS \qquad S: \sqrt{x^2+y^2} \le z \le 1$

$S_1: z^2 = x^2+y^2 \qquad z\in[0,1].$



$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

S_1 $\sqrt{2}\iint r^3\,dr\,d\theta = \sqrt{2}\cdot\frac{1}{4}\cdot 2\pi = \frac{\sqrt{2}\pi}{2}$

S_2: $\iint r^3\,dr\,d\theta = \frac{\pi}{2}$

(3) $\iint \frac{dS}{x^2+y^2} \qquad S: x^2+y^2=R^2,\ z=0.\ z=H.$

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \\ z = h \end{cases} \qquad E = R^2 \quad F = 0 \qquad G = 1.$$

$$\iint \frac{1}{R^2}\cdot R\,d\theta\,dh = 2\pi\cdot H\cdot\frac{1}{R}$$