18.10.24

3. HYPERBOLIC EQUATIONS

3.2.2. THE INHOMOGENEOUS EQUATION

The following mixed problem is considered.

Task 2.

Let it be necessary to find a solution to the inhomogeneous equation of string vibrations:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t)\,,\ 0 < x < l\,,\ t > 0\,, \tag{3.26}$$

satisfying the initial and boundary conditions:

$$u(x,0) = \varphi(x)\,,\ \frac{\partial u(x,0)}{\partial t} = \psi(x)\,, \tag{3.27}$$

$$u(0,t) = 0\,,\ u(l,t) = 0\,,\ t \ge 0\,.$$

We will look for a solution to Task 2 in the form of a Fourier series expansion in x:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin\frac{k\pi}{l}x\,, \tag{3.28}$$

considering t as a parameter.

Let's imagine the function $f(x,t)$ as a Fourier series:

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) \sin\frac{k\pi}{l}x\,,$$

$$f_k(t)=\frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}xdx\,. \qquad (3.29)$$

Substituting series (3.28) and (3.29) into the original equation (3.26):

$$\sum_{k=1}^{\infty}\left[u_k''(t)+a^2\left(\frac{k\pi}{l}\right)^2 u_k(t)-f_k(t)\right]\sin\frac{k\pi}{l}x=0\,,$$

we see that it will be satisfied if all the expansion coefficients are equal:

$$u_k''(t)+a^2\left(\frac{k\pi}{l}\right)^2 u_k(t)=f_k(t)\,. \qquad (3.30)$$

To determine $u_k(t)$, we obtained an ordinary differential equation with constant coefficients.

Further, the initial conditions (3.27) give:

$$\varphi(x)=\sum_{k=1}^{\infty}u_k(0)\sin\frac{k\pi}{l}x\,,$$

$$\psi(x)=\sum_{k=1}^{\infty}u_k'(0)\sin\frac{k\pi}{l}x\,,$$

therefore,

$$u_k(0)=\frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}xdx,$$

$$u_k'(0)=\frac{2}{l}\int_0^l \psi(x)\sin\frac{k\pi}{l}xdx, \qquad (3.31)$$

$$\varphi_k=u_k(0),\quad \psi_k=u_k'(0).$$

The conditions (3.31) completely determine the solution (3.30):

$$u_k(t) = \varphi_k \cos\frac{ak\pi}{l}t + \frac{l}{ak\pi}\psi_k \sin\frac{ak\pi}{l}t + \frac{l}{ak\pi}\int_0^t f_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)d\tau. \quad (3.32)$$

Thus, the desired solution to Task 2, according to formulas (3.28) and (3.32), will be written in the form

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\varphi_k \cos\frac{ak\pi}{l}t + \frac{l}{ak\pi}\psi_k \sin\frac{ak\pi}{l}t + \frac{l}{ak\pi}\int_0^t f_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)d\tau\right\}\times$$
$$\times\sin\frac{k\pi}{l}x,$$

where the values $\varphi_k, \psi_k, f_k(\tau)$ are calculated by (3.31) and (3.29), respectively.

Example 5.

Find a solution to the boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 2b,\ 0 < x < l,\ t > 0,$$

$$u(x,0) = 0,\ \frac{\partial u(x,0)}{\partial t} = 0,$$

$$u(0,t) = 0,\ u(l,t) = 0,\ t \geq 0.$$

<u>Solution:</u>

Here

$$\varphi(x) = 0,\ t,\ \psi(x) = 0,\ f(x,t) = 2b,\ a = 1.$$

Therefore,

$\varphi_k = 0$, $\psi_k = 0$.

$$f_k(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}x\,dx.$$

$$f_k(t) = \frac{2}{l}\int_0^l 2b\sin\frac{k\pi}{l}x\,dx = -\frac{4b}{l}\frac{l}{k\pi}\cos\frac{k\pi}{l}x\Big|_0^l =$$

$$= -\frac{4b}{k\pi}\left[(-1)^k - 1\right] = \begin{cases} 0, & k = 2n, \\ \dfrac{8b}{k\pi}, & k = 2n+1. \end{cases}$$

Further

$$\int_0^t f_{2n+1}(\tau)\sin\frac{(2n+1)\pi}{l}(t-\tau)d\tau = \frac{8b}{(2n+1)\pi}\int_0^t \sin\frac{(2n+1)\pi}{l}(t-\tau)d\tau =$$

$$= \frac{8b}{(2n+1)\pi}\frac{l}{(2n+1)\pi}\cos\frac{(2n+1)\pi}{l}(t-\tau)\Big|_0^t = \frac{8bl}{(2n+1)^2\pi^2}\left[1-\cos\frac{(2n+1)\pi}{l}t\right].$$

Hence

$$u(x,t) = \sum_{n=0}^{\infty}\frac{8bl^2}{(2n+1)^3\pi^3}\left[1-\cos\frac{(2n+1)\pi}{l}t\right]\sin\frac{(2n+1)\pi}{l}x.$$

4. PARABOLIC EQUATIONS

4.1. ONE-DIMENSIONAL EQUATION OF THERMAL CONDUCTIVITY. SETTING BOUNDARY VALUE PROBLEMS

The process of temperature distribution in a rod, thermally insulated from the sides and thin enough that at any given time the temperature at all points of the cross section can be considered a single one, can be described by the function $u(x,t)$, representing the temperature in the cross section x at time t. This function $u(x,t)$ is the solution of the equation:

$$c\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+F(x,t),$$

called *the heat equation.*

Here $\rho(x), c(x), k(x) -$ are, respectively, the density, specific heat capacity and thermal conductivity coefficient of the rod at point x, and $F(x,t) -$ is the intensity of heat sources at point x at time t.

To identify the only one solution to the heat equation, it is necessary to attach the initial and boundary conditions to the equation.

The initial condition, unlike a hyperbolic equation, consists only in setting the values of the function u at the initial moment of time t_0:

$$u(x,t_0)=\varphi(x).$$

The main types of boundary conditions are boundary value problems of the first, second and third types.

The first boundary value problem is set if the temperature at the end of the rod $x = 0$ is maintained according to a certain law, for example:

$$u(0,t) = \mu(t) \text{ ,}$$

where $\mu(t)$ is a given function of time.

The second boundary value problem is posed if the heat flow q is set at the end of the rod $x = l$, for example:

$$q(l,t) = -k \frac{\partial u(l,t)}{\partial x} \text{ ,}$$

therefore, the boundary condition has the form

$$\frac{\partial u(l,t)}{\partial x} = \upsilon(t) = -\frac{1}{k} q(l,t) \text{ .}$$

In particular, in the case of a thermally insulated end, there is no heat flow through it, that is, $\upsilon(t) = 0$.

The third boundary value problem is formulated when heat exchange with the ambient occurs at the end of the rod $x = l$ according to *Newton's law:*

$$q(l,t) = H\left(u(l,t) - \theta(t)\right) \text{,}$$

where $\theta(t)$ is the ambient temperature, H is the heat exchange coefficient, that is, the amount of heat that has passed through a single section of the rod per unit time with a temperature change of one degree.

The boundary condition has the form

$$\frac{\partial u(l,t)}{\partial x} = -\lambda\left(u(l,t) - \theta(t)\right),$$

where $\lambda = \dfrac{H}{k}$.

Some limiting cases are also considered. For example, if the process of thermal conductivity is studied in a very long rod. For a short period of time, the influence of the temperature regime set at the boundary in the central part of the rod has a very weak effect, and the temperature in this area is mainly determined only by the initial temperature distribution. In problems of this type, it is usually assumed that the rod has an infinite length. Thus, a problem with initial conditions (the Cauchy problem) is posed on the temperature distribution on an infinite line: to find a solution to the thermal conductivity equation in the region $-\infty < x < \infty$ and $t \geq t_0$ satisfying the condition

$$u(x,t_0) = \varphi(x),\ -\infty < x < \infty,$$

where $\varphi(x)$ is a given function.

Similarly, if the section of the rod whose temperature we are interested in is located near one end and far from the other, then in this case the temperature is practically determined by the temperature regime of the near end and the initial conditions. In problems of this type, it is usually assumed that the rod is semi-infinite, and the coordinate measured from the end varies within $0 \leq x < \infty$. Let us give as an example the formulation of the first boundary value problem for a semi—infinite rod: to find a solution to the thermal conductivity equation in the region $-0 < x < \infty$ and $t \geq t_0$ satisfying the conditions

$$u(x,t_0) = \varphi(x)\,,\ 0 < x < \infty\,,$$
$$u(0,t) = \mu(t)\,,\ t \geq t_0\,,$$

where $\varphi(x)$ and $\mu(t)$ are given functions.

4.2. A METHOD FOR SEPARATING VARIABLES FOR THE EQUATION OF THERMAL CONDUCTIVITY. INSTANT POINT SOURCE FUNCTION

4.2.1. HOMOGENEOUS BOUNDARY VALUE PROBLEM

The following first boundary value problem is considered.

Find a solution to a homogeneous equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}\,,\ 0 < x < l\,,\ 0 < t \leq T\,, \tag{4.1}$$

satisfying the initial condition

$$u(x,0) = \varphi(x)\,,\ 0 \leq x \leq l \tag{4.2}$$

and homogeneous boundary conditions

$$u(0,t) = 0\,,\ u(l,t) = 0\,,\ 0 \leq t \leq T\,. \tag{4.3}$$

We are looking for a solution to this problem in the form of a product

$$u(x,t) = X(x)T(t)\,,$$

substituting it into equation (4.1), we have

$$X(x)T'(t) = a^2 X''(x)T(t).$$

Dividing both parts of this equation by $a^2 X(x)T(t)$, we get

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}. \qquad (4.4)$$

The right side of equality (4.4) is a function of only the variable x, and the left side is only t, so the right and left sides of equality (4.4) retain a constant value when changing their arguments. It is convenient to denote this value by $-\lambda$, that is,

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

$$X''(x) + \lambda X(x) = 0,$$

$$T'(t) + \lambda a^2 T(t) = 0.$$

The general solutions of these equations have the form

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

$$T(t) = Ce^{-a^2\lambda t},$$

where $A, B, C -$ are arbitrary constants, and the function $u(x,t)$ is

$$u(x,t) = \left(A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\right)Ce^{-a^2\lambda t}.$$

Constants A and B can be found using the boundary conditions (4.3) of the problem. Since

$$T(t) \not\equiv 0,$$

then

$$X(0)=0, \; X(l)=0.$$

$$X(0)=A=0,$$

$$X(l)=A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l=0,$$

that is,

$$A=0 \text{ и } B\sin\sqrt{\lambda}l=0.$$

From where

$$\sqrt{\lambda}=\frac{k\pi}{l}, \; k=1,2,\ldots.$$

So,

$$X(x)=B\sin\frac{k\pi}{l}x.$$

The values $\lambda=\frac{k^2\pi^2}{l^2}$ found are called *eigenvalues* for a given boundary value problem, and the functions $X(x)=B\,sin\frac{k\pi}{l}x$ are called *eigenfunctions.*

When the values of λ are found, we get

$$T(t)=Ce^{-\frac{a^2k^2\pi^2}{l^2}t},$$

$$u_k(x,t)=a_k e^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi}{l}x,\ k=1,2,\dots .$$

Since equation (4.1) is linear and homogeneous, the sum of the solutions is also a solution that can be represented as a series:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}a_k e^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi}{l}x .$$

In this case, the solution must satisfy the initial condition (4.2):

$$u(x,0)=\sum_{k=1}^{\infty}a_k\sin\frac{k\pi}{l}x=\varphi(x) .$$

If the function $\varphi(x)$ decomposes into a Fourier series in the interval $(0,l)$ according to the sine, then

$$a_k=\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi .$$

Thus, the solution of the heat equation can be represented as the sum of an infinite series:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi e^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi}{l}x . \qquad (4.5)$$

Theorem. Let $\varphi(x) \in C^1([0,l]), \varphi(0) = \varphi(l) = 0$. Then there is a unique solution to the problem (4.1)–(4.3), which is represented as an absolutely and uniformly converging series (4.5).

The solution (4.5) can be represented as

$$u(x,t) = \int_0^l G(x,\xi,t)\varphi(\xi)d\xi \ ,$$

where the function

$$G(x,\xi,t) = \frac{2}{l}\sum_{k=1}^{\infty} e^{-\frac{a^2k^2\pi^2}{l^2}t} \sin\frac{k\pi x}{l} \sin\frac{k\pi\xi}{l} \ ,$$

is introduced, called the *instantaneous point source function.*

The physical meaning of the function $G(x,\xi,t)$ is that, as a function of the argument x, it represents the temperature distribution in the rod $0 \le x \le l$ at time t, if at $t = 0$ the temperature was zero, and at this moment at point $x = \xi$ a certain amount of heat Q was instantly released, and at the ends of the rod is constantly maintained the temperature is zero.

Example 1

A thin homogeneous rod $0 \le x \le l$ is given , the side surface of which is thermally insulated. Find the temperature distribution $u(x,t)$ in the rod if the ends of the rod are maintained at zero temperature, and the initial temperature $u(x,0) = u_0 = const$.

Solution:

The problem is reduced to solving the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \; 0 < x < l, \; t > 0$$

under conditions

$u(x,0) = u_0 = \text{const}$,

$u(0,t) = u(l,t) = 0$.

Let's calculate:

$$\frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi = \frac{2}{l}\int_0^l u_0 \sin\frac{k\pi}{l}\xi d\xi = -\frac{2u_0}{k\pi}\cos\frac{k\pi}{l}\xi\bigg|_0^l =$$

$$= -\frac{2u_0}{k\pi}\left((-1)^k - 1\right) = \begin{cases} \dfrac{4u_0}{k\pi}, & k = 2n+1, \\ 0, & k = 2n. \end{cases}$$

Then the solution will take the form (according to the formula (4.5)):

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}e^{-\frac{a^2(2n+1)^2\pi^2}{l^2}t}\sin\frac{(2n+1)\pi}{l}x.$$