Complex Analysis 2024. Homework 1.

1. Calculate the following complex numbers, indicate the real and imaginary parts, calculate their absolute value.

$$z_1 = (1-2i)(3-4i); \quad z_2 = (-2i)^7; \quad z_3 = i^{2024}; \quad z_4 = \frac{1+2i}{2-i}.$$

$z_1 = 3 - 8 - 6i - 4i = -5 - 10i$ $\qquad$ $\mathrm{Re}\, z_1 = -5$ $\quad$ $\mathrm{Im}\, z_1 = -10$ $\quad$ $|z_1| = 5\sqrt{5}$

$z_2 = (-2i)^7 = -128 \cdot (-1)\, i = 128\, i$ $\qquad$ $\mathrm{Re}\, z_2 = 0$ $\quad$ $\mathrm{Im}\, z_2 = 128$ $\quad$ $|z_2| = 128$

$z_3 = (i^4)^{506} = 1$ $\qquad$ $\mathrm{Re}\, z_3 = 1$ $\quad$ $\mathrm{Im}\, z_3 = 0$ $\quad$ $|z_3| = 1$

$z_4 = \dfrac{1+2i}{2-i} = \dfrac{(1+2i)(2+i)}{(2-i)(2+i)} = \dfrac{2-2+5i}{5} = i$ $\qquad$ $\mathrm{Re}\, z_4 = 0$ $\quad$ $\mathrm{Im}\, z_4 = 1$ $\quad$ $|z_4| = 1$

2. Describe and draw sets defined by the equations

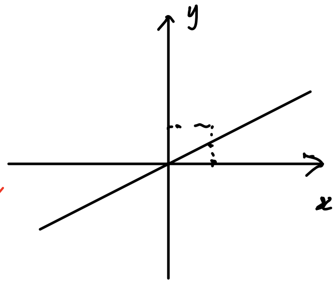
(a) $\mathrm{Re}\, z = 2\, \mathrm{Im}\, z$;

(b) $|z - 1 - i| = 3$;

(c) $|z - 2| = \mathrm{Re}\, z$.

(a) straight line

$x = 2y$

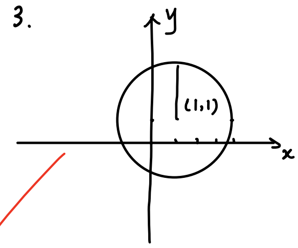


(b) circle center $(1,1)$ radius 3.

$z = x + yi$.

$|(x-1) + i(y-1)| = 3$.

$\Rightarrow (x-1)^2 + (y-1)^2 = 9$

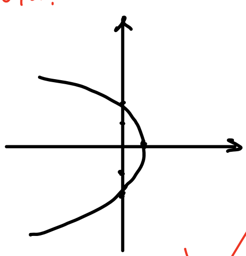


(c) $z = x + iy$. parabola.

$|(x-2) + iy| = x$

$\Rightarrow (x-2)^2 + y^2 = x^2$

$\Rightarrow y^2 = 4x - 4$.



3. Prove the parallelogram identity

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2).$$

Pf: $z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$ $\quad a_1, a_2, b_1, b_2 \in \mathbb{R}.$

$$|(a_1+a_2) + i(b_1+b_2)|^2 + |(a_1-a_2) + i(b_1-b_2)|^2$$

$$= (a_1+a_2)^2 - (b_1+b_2)^2 + 2(a_1+a_2)(b_1+b_2)i + (a_1-a_2)^2 - (b_1-b_2)^2 + 2(a_1-a_2)(b_1-b_2)i$$

$$= 2[(a_1^2 - b_1^2) + 2a_1b_1 i] + 2[(a_2^2 - b_2^2 + 2a_2b_2 i)]$$

$$= 2(|z_1|^2 + |z_2|^2)$$

4. Prove that $z^{-1} = \bar{z}/|z|^2$.

Pf: Let $z = x + iy$. $x, y \in \mathbb{R}$.

$$z^{-1} = \frac{1}{(x+iy)} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2}$$

$$\frac{\bar{z}}{z^2} = \frac{x-iy}{x^2+y^2} = z^{-1}$$

$z^{-1}(z) = 1$

$(z)(\bar{z}/|z|^2) = \frac{|z|^2}{|z|^2} = 1$

Complex Analysis 2024. Homework 2.

1. In assumption that $f = u + iv$ is $\mathbb{C}$−differentiable prove that

$$f'(z_0) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Pf: $f'(z_0) = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)$

f is $\mathbb{C}$-diff. then the C-R condition satisfied.

$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \Leftrightarrow \frac{\partial f}{\partial x}(z_0) = -i\frac{\partial f}{\partial y}(z_0)$

thus $\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right) = \frac{1}{2}\cdot\left(\frac{\partial f}{\partial x}(z_0) + \frac{\partial f}{\partial x}(z_0)\right) = \frac{\partial f}{\partial x}(z_0)$

$= \frac{i}{2}\cdot\left(-\frac{\partial f}{\partial y}(z_0) - \frac{\partial f}{\partial y}(z_0)\right) = -i\cdot\frac{\partial f}{\partial y}(z_0)$

$\frac{\partial f}{\partial x} = \frac{\partial(u+vi)}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$ similarly, $-i\frac{\partial f}{\partial y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$

2. Prove the fundamental property of exponent

$$e^{w+z} = e^w e^z,\ \ w, z \in \mathbb{C}.$$

Pf: Let $w = u_1 + iv_1$ $z = u_2 + iv_2$. $u_k, v_k \in \mathbb{R}$. $k = 1, 2$.

$LHS = e^{(u_1+iv_1)+(u_2+iv_2)} = e^{(u_1+u_2)+i(v_1+v_2)} \underset{=}{\underline{\text{def. of exp.}}} e^{u_1+u_2}(\cos(v_1+v_2) + i\sin(v_1+v_2))$

$RHS = e^{u_1+iv_1}\cdot e^{u_2+iv_2} \underset{=}{\underline{\text{def of exp.}}} e^{u_1}(\cos v_1 + i\sin v_1)\, e^{u_2}(\cos v_2 + i\sin v_2)$

$= e^{u_1+u_2}[\cos v_1\cos v_2 - \sin v_1\sin v_2 + i(\sin v_1\cos v_2 + \cos v_1\sin v_2))$

$= e^{u_1+u_2}[\cos(v_1+v_2) + i\sin(v_1+v_2)]$

$LHS = RHS$

3. Prove that

$$\overline{e^z} = e^{\bar{z}}.$$

Pf: $z = a + bi$, $a, b \in \mathbb{R}$.

$LHS = \overline{e^z} = \overline{e^a(\cos b + i\sin b)} = e^a(\cos b - i\sin b)$

$RHS = e^{\bar{z}} = e^{a-bi} = e^a(\cos(-b) + i\sin(-b)) = LHS$

4. Find all points at which the function $f(z) = |z|^2$ is differentiable. Find partial derivatives $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar{z}}$.

Pf. Let $z = x + iy$ $\quad x, y \in \mathbb{R}$.

$f(z) = f(x,y) = x^2 + y^2$ which is diff. on $\mathbb{R}^2$. $f(z)$ is $\mathbb{R}$-diff on $\mathbb{C}$.

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = x + y$$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = x - y.$$

$f(z)$ is $\mathbb{C}$-diff iff $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$

thus $x_0 - y_0 = 0 \Rightarrow x_0 = y_0$.

i.e. $f(z)$ is $\mathbb{C}$-diff on $\{(x_0, x_0) \mid x_0 \in \mathbb{R}\}$

5. Prove that a function $f(z) = \bar{z}$ is not complex differentiable at any point.

Let $z = x + iy$

$f(z) = f(x,y) = x - iy$.

$u(x,y) = x$. $\quad v(x,y) = -y$. $\quad u, v$ are diff. $\quad$ thus it's $\mathbb{R}$-diff.

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(1+1) = 1$$

i.e. $\frac{\partial f}{\partial \bar{z}}(z) \equiv 1 \neq 0$. at any point. so $f(z)$ doesn't satisfy the C-R condition. $f(z)$ is not $\mathbb{C}$-diff at any point.

6. Calculate

$$z_1 = (1 + \sqrt{3}i)^9; \quad z_2 = (3 - 3i)^5; \quad z_3 = e^{(1+i)\frac{pi}{2}};$$

$e^{\frac{\pi}{2} + i\frac{\pi}{2}} = i e^{\frac{\pi}{2}}$

$|z_1| = 2$. $\quad z_1 = \left(2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right)^9 = 2^9 \cdot (\cos 3\pi + i\sin 3\pi) = -2^9$

$|z_2| = 3\sqrt{2}$ $\quad z_2 = \left(3\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)\right)^5 = 3^5 \cdot 2^2\sqrt{2}\left(\cos\frac{5}{4}\pi + i\left(\sin -\frac{5\pi}{4}\right)\right)$

$= 972\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\cdot\frac{\sqrt{2}}{2}\right) = -972 + 972i$

$z_3 = e^{-\frac{p}{2} + \frac{p}{2}i} = e^{-\frac{p}{2}}\left(\cos\frac{p}{2} + i\sin\frac{p}{2}\right)$

243 × 4.

972

7. How are numbers z_1 and z_2 related if $arg(z_1) = arg(z_2)$?

$z_1 = r_1 \cdot e^{i\varphi_1}$ $\qquad z_2 = r_2 \cdot e^{i\varphi_2}$

if $\varphi_1 = \varphi_2$. then $\frac{z_1}{z_2} = \frac{r_1}{r_2}$. z_1, z_2 has real ratio

Complex Analysis 2024. Homework 3.

1. Plot the path given by $\gamma(t)$

$$\gamma(t) = i + e^{it}, \quad 0 \le t \le \pi.$$

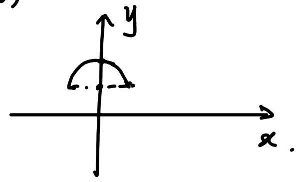
Plot it's image with respect to mapping $f(z) = (z-i)^3$;

$\gamma(t) = i + (\cos t + i\sin t) = \cos t + i(1+\sin t)$

$\begin{cases} x = \cos t \\ y = 1+\sin t \end{cases} \Rightarrow x^2 + (y-1)^2 = 1$

$y \ge 1, \ -1 \le x \le 1$

semi-circle.



$f(\gamma(t)) = (e^{it})^3 = e^{3ti} = \cos 3t + i\sin 3t.$

$t \in [0, \pi]$

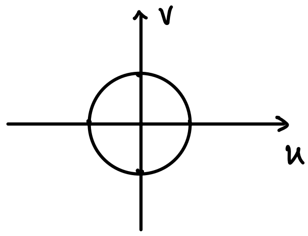
$\begin{cases} u = \cos 3t \\ v = \sin 3t \end{cases}$

circle

⇓ "一圈半"

$0 \le t \le \frac{2\pi}{3}.$ circle

$\frac{2\pi}{3} \le t \le \pi$ upper half of the circle.



2. Find the image of the given line under the complex mapping $w = z^2$

(a) $\operatorname{Re} z = \operatorname{Im} z$;

(b) $\operatorname{Re} z = 3$;

denote that $z = x + iy$.

(a) $\operatorname{Re} z = \operatorname{Im} z \Rightarrow x = y.$

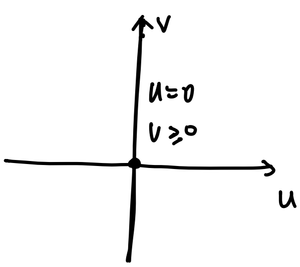
$w = (x + yi)$

$= x^2 - y^2 + 2xy\, i$

$\begin{cases} u = 0 \\ v = 2x^2 \ge 0 \end{cases}$

line : $u = 0$.

ray $\operatorname{Re} w = 0$ $\operatorname{Im} w > 0$

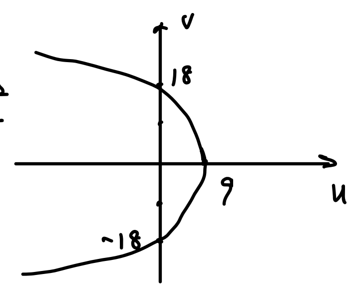


(b) $\operatorname{Re} z = 3 \Rightarrow z = 3 + iy.$

$w = (3+iy)^2 = 9 - y^2 + 6y\, i.$

$\begin{cases} u = 9 - y^2 \\ v = 6y \end{cases} \Rightarrow 36u = 324 - v^2$

parabola : $u = 9 - \frac{v^2}{36}$



3. Calculate all values of

$$\sqrt[3]{-3+3i}; \quad \sqrt[5]{-1+\sqrt{3}i}.$$

1). Let $w = \sqrt[3]{-3+3i}$

$$w^3 = -3+3i = 3\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$w = \sqrt[3]{3\sqrt{2}}\, e^{i\frac{\frac{3\pi}{4}+2\pi k}{3}}$$

$$w_0 = 3^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} e^{i\frac{\pi}{4}} \quad w_1 = 3^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} \cdot e^{i \cdot \frac{11}{12}\pi} \quad w_2 = 3^{\frac{1}{3}} \cdot 2^{\frac{1}{6}} \cdot e^{i\frac{19}{12}\pi}$$ ✓

2) Let $w = \sqrt[5]{-1+\sqrt{3}i}$

$$w^5 = -1+\sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right).$$

$$w_k = 2^{\frac{1}{5}} e^{i\frac{\frac{2\pi}{3}+2\pi k}{5}} \quad k = 0, 1, 2, 3, 4.$$ ✓

4. Find the image of the domain $|z| < 8$, $\pi/2 < \arg z < 3\pi/4$ under each of the following principal nth root function ($k = 0$ in our definition)

$$f(z) = z^{1/3}; \quad f(z) = z^{1/2}.$$

(1) $f(z) = z^{\frac{1}{3}}$ denote $z = re^{i\varphi}$. $\varphi \in [-\pi, \pi)$

$$f(re^{i\varphi}) = \sqrt[3]{r}\, e^{i\frac{\varphi}{3}} \qquad |r| < 8 \qquad \frac{\pi}{2} < \varphi < \frac{3\pi}{4}.$$

image: $\left\{ f(z) \mid |f(z)| < 2,\ \frac{\pi}{6} < \arg f(z) < \frac{\pi}{4} \right\}$. ✓

(2) $f(z) = z^{\frac{1}{2}}$ denote $z = re^{i\varphi}$ $\varphi \in [-\pi, \pi)$

$$f(re^{i\varphi}) = \sqrt[2]{r}\, e^{i\frac{\varphi}{2}} \qquad |r| < 8 \qquad \frac{\pi}{2} < \varphi < \frac{3\pi}{4}$$

image $\left\{ f(z) \mid |f(z)| < 2\sqrt{2},\ \frac{\pi}{4} < \varphi < \frac{3\pi}{8} \right\}$. ✓

Complex Analysis 2024. Homework 4.

1. Find all values of $\operatorname{Ln} z$ at points

$$z_1 = 1 + i; \quad z_2 = -3; \quad z_2 = -1 + i\sqrt{3}.$$

$\arg z_1 = \frac{\pi}{4}$. $\quad \arg z_2 = 0 \quad \arg z_3 = \frac{2\pi}{3}$

$\ln z_1 = \ln\sqrt{2} + \frac{\pi}{4} i + 2\pi k i$

$\ln z_2 = \ln 3 + \underline{2\pi k i} \times + \pi i = \ln 3 + (2k+1)\pi i.$

$\ln z_3 = \ln 2 + \frac{2\pi}{3} i + 2\pi k i$

2. Calculate

$$(1+i)^i; \quad 3^{2i/\pi}; \quad (ei)^{sqrt2}.$$

$(1+i)^i = e^{i\ln(1+i)} = e^{i(\ln\sqrt{2} + \frac{\pi}{4} i + 2k\pi i)} = e^{\ln\sqrt{2}\, i} \cdot e^{-\frac{\pi}{4} - 2k\pi}$

$= e^{-\frac{\pi}{4} - 2k\pi} \cdot (\cos \ln\sqrt{2} + i \sin \ln\sqrt{2})$

$3^{\frac{2i}{\pi}} = e^{\frac{2i\ln 3}{\pi}} = \cos\left(\frac{2\ln 3}{\pi}\right) + i \sin\frac{2\ln 3}{\pi}$

$(ei)^{\sqrt{2}} = e^{\sqrt{2}}\, i^{\sqrt{2}} = e^{\sqrt{2}}\, e^{\sqrt{2}\ln i} = e^{\sqrt{2}} \cdot e^{\sqrt{2}(\frac{\pi}{2} i + 2k\pi i)} = e^{\sqrt{2}} \cdot e^{i(\frac{\sqrt{2}}{2}\pi + 2k\pi)} = e^{\sqrt{2}}(\cos\frac{\sqrt{2}}{2}\pi + i \sin\frac{\sqrt{2}}{2}\pi)$

3. Prove that the principal value of logarithm

$$\ln z = \ln|z| + i\arg z, \quad \arg z \in (-\pi \quad, \pi \quad),$$

is conformal in a slit domain $\mathbb{C}\backslash(-\infty, 0]$

Pf: Let $z = x + iy$. $\quad f(z) = \ln(z) = \ln\sqrt{x^2+y^2} + i \arctan\frac{y}{x}$.

$f(z)$ is $\mathbb{R}$-diff since $\ln\sqrt{x^2+y^2}$, $\arctan\frac{y}{x}$ is diff. on $\mathbb{C} \backslash (-\infty, 0]$

$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{x}{x^2+y^2} - \frac{x}{x^2+y^2}\right) = 0$, thus. $\ln z$ is $\mathbb{C}$-diff.

$f'(z) = \frac{\partial f}{\partial z}(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{x}{x^2+y^2}$

$f'(z) \neq 0$ since $x \neq 0$ $x^2+y^2 > 0$ on $\mathbb{C} \backslash (-\infty, 0]$.

thus $\ln z$ is conformal.

4. Prove that Jacobian of a conformal map $f : D \to G$ is equal to

$$J_f = |f'(z)|^2 .$$

$f(z) = f(x+iy) = u(x,y) + i\, v(x,y)$.

for conformal map, C-R condition holds. $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$

$$|f'(z)|^2 = \left|\frac{\partial f}{\partial z}(z)\right|^2 = \frac{1}{4}\left|\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right|^2 = \frac{1}{4}\left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right|^2$$

$$= \frac{1}{4}\left|2\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

$$Jf = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Thus we have $Jf = |f'(z)|^2$

5. Evaluate the integral $\int_\gamma |z|^2\, dz$, where γ is $x = t^2$, $y = 1/t$, $1 \le t \le 2$.

$\gamma(t) = (x(t), y(t)) = (t^2, \frac{1}{t})$. $t \in [1,2]$ $\quad u(t) = t^4 + \frac{1}{t^2}$ $\quad v(t) = 0$

$$\int_\gamma f\,dz = \int_\gamma u(t)\cdot x'(t)\,dt - v(t)\,y'(t)\,dt + i\int_\gamma u(t)y'(t)\,dt + v(t)\,x'(t)\,dt$$

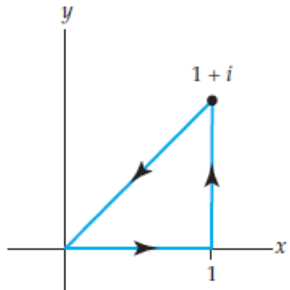
$$= \int_1^2 u(t)\,x'(t)\,dt + i\int_1^2 u(t)\,y'(t)\,dt$$

$$= \int_1^2 (t^4 + \tfrac{1}{t^2})\,2t\,dt + i\int_1^2 (t^4 + \tfrac{1}{t^2})\cdot(-\tfrac{1}{t^2})\,dt$$

$$= 2\left[\frac{t^6}{6} + \ln t\Big|_1^2\right] + i\int_1^2 (-t^2 - \tfrac{1}{t^4})\,dt = 21 + 2\ln 2 - \tfrac{21}{8}i$$

$x - yi$

6. Evaluate the integral $\oint_\gamma \bar{z}^2 dz$ along the contour γ given in the figure



divide γ into 3 parts: γ_1: $x: 0\to 1$, $y\equiv 0$ γ_2: $x\equiv 1$, $y: 0\to 1$ γ_3: $x=y$, $(1,1)\to(0,0)$

$$\oint_\gamma \bar{z}^2 dz = \oint_\gamma [(x^2-y^2)+i(-2xy)]\, d(x+iy)$$

$$= \oint_\gamma [(x^2-y^2)dx + 2xy\,dy] + i\oint_\gamma (x^2-y^2)dy + (-2xy)dx$$

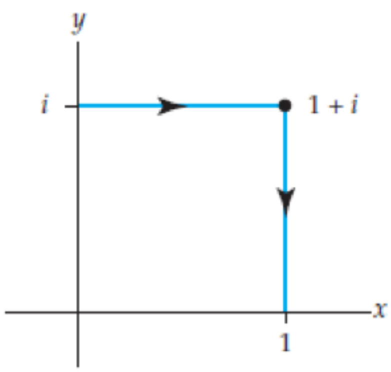
$$\oint_\gamma \bar{z}^2 dz = \oint_{\gamma_1+\gamma_2+\gamma_3} = \int_0^1 x^2 dx + \int_0^1 2\cancel{x}y\,dy + i\int_0^1 (1-y^2)dy + \oint_1^0 2x^2 dx + i\int_1^0 -2x^2 dx$$

$$= \frac{1}{3} + 1 + i\cdot\frac{2}{3} + \left(-\frac{2}{3}\right) + i\cdot\frac{2}{3} = \frac{2}{3} + i\frac{4}{3}$$ $1-\frac{1}{3}.$ $\left.\frac{2}{3}x^3\right|_1^0$

7. Evaluate the integral $\oint_\gamma (z^2 - z + 2)dz$ along the contours γ given in the figures (see page 2) close or not ?

denote $z = x+iy$.

$$\oint_\gamma (x^2-y^2+2xy\cdot i - x - iy + 2)(dx+idy) = \int_\gamma (x^2-y^2-x+2)dx + (y-2xy)dy$$
$$+i\int_\gamma (x^2-y^2-x+2)dy - (y-2xy)dx$$

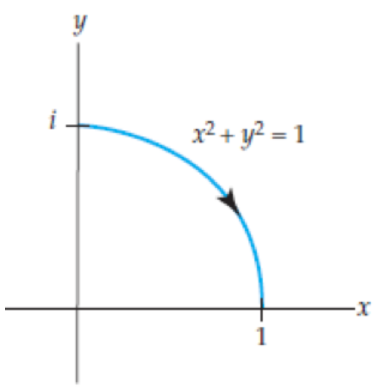


(1) $\gamma: \gamma_1 \cup \gamma_2.$ $\gamma_1: x: 0\to 1$, $y\equiv 1$ $\gamma_2: x\equiv 1$, $y: 1\to 0$

$$\gamma_1: \int_0^1 (x^2-x+1)dx - i\int_0^1 (1-2x)\,dx = \frac{1}{3}-\frac{1}{2}+1-0 = \frac{5}{6}$$

$$\gamma^2: \int_1^0 (y-2y)dy + i\int_1^0 (2-y^2)dy = \frac{1}{2} + i\left(-\frac{5}{3}\right)$$

$$\gamma: \int_{\gamma_1+\gamma_2} f\,dz = \frac{4}{3} + i\left(-\frac{5}{3}\right)$$



(2) γ: $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$ $t: \frac{\pi}{2} \to 0.$

$$\gamma: \int_{\frac{\pi}{2}}^0 (\cos 2t - \cos t + 2)(-\sin t)\,dt + (\sin t - \sin 2t)\cos t\,dt$$
$$+ i\int_{\frac{\pi}{2}}^0 (\cos 2t - \cos t + 2)(\cos t)\,dt - (\sin t - \sin 2t)(-\sin t)\,dt$$

$$= \int_{\frac{\pi}{2}}^0 -\sin 3t + \sin 2t - 2\sin t\,dt. + i\int_{\frac{\pi}{2}}^0 \cos 3t - \cos 2t + 2\cos t\,dt.$$

$$= \frac{1}{3} - 1 + 2 + i\left(-\frac{1}{3}+2\right) = \frac{4}{3} + i\left(-\frac{5}{3}\right) = \frac{4}{3} + i\left(-\frac{5}{3}\right)$$

8. Prove Cauchy-Goursat theorem for rectangles.

Let f is $\mathbb{C}$-diff on D. Then for every rectangles $\square \subset D$ with its boundary $\int_{\partial\square} f(z)\,dz = 0$.

Pf: Assume $\exists\, \square_0 \subset D$. s.t. $\left|\int_{\partial\square_0} f dz\right| = M > 0$. partition $\square_0$ into 4 equal rectangle $\int_{\partial\square_0} f dz = \sum_{i=1}^{4} \int_{\partial\square_{0i}} f dz$

$\exists\, \square_1$. s.t. $\left|\int_{\partial\square_1} f dz\right| \geq \frac{M}{4}$ partition $\square_1$ into 4 equal rectangle and repeat the procedure.

$\{\square_n\}_{n=1}^{\infty}$, $\left|\int_{\partial\square_n} f dz\right| \geq \frac{M}{4^n}$. $\overline{\square}_{n+1} \subset \overline{\square}_n$ and $\bigcap_{n=1}^{\infty} \square_n = \{z_0\}$. $z_0 \in D$ is unique.

Then apply $\mathbb{C}$-differentiability at z_0. $\forall \varepsilon > 0\ \exists \delta > 0$. s.t. in $U_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}$

$f(z) = f(z_0) + f'(z_0)(z - z_0) + \alpha(z)(z - z_0)$ $\quad \alpha(z) < \varepsilon$ for every $z \in U$.

$\exists N \in \mathbb{N}$, for $n > N$, $\overline{\square}_n \subset U$. $\int_{\partial\square_n} f dz = \int_{\partial\square_n} f(z_0)\,dz + \int_{\partial\square_n} f'(z_0)(z - z_0)\,dz + \int_{\partial\square_n} \alpha(z)(z - z_0)\,dz$.

The 1st and 2nd parts are integration on closed curve with constant and polynomial, it's 0.

$\left|\int_{\partial\square_n} f dz\right| = \left|\int_{\partial\square_n} \alpha(z)(z - z_n)\,dz\right| \leq \varepsilon \left|\int_{\partial\square_n} |z - z_0|\,dz\right| \leq \varepsilon \cdot |\partial\square_n|^2$, where $|\partial\square_n|$ is the perimeter.

$|\partial\square_n| = \frac{|\partial\square_0|}{2^n}$. $\Rightarrow \frac{M}{4^n} \leq \left|\int_{\partial\square_n} f dz\right| \leq \varepsilon \cdot \frac{|\partial\square_0|^2}{4^n}$ $\Rightarrow M \leq \varepsilon \cdot |\partial\square_0|^2$, thus $M \to 0$ (since $\varepsilon \to 0$)

which contradicts with the assumption.

9. Suppose that a function f is holomorphic in domain D and that G is a domain bounded by a simple closed smooth path γ such that $\overline{G} \subset D$. Assume, moreover, that f' is continuous in D. Prove, using Green's formula that

$$\int_\gamma f dz = 0.$$

Pf: f is holomorphic. $\Rightarrow$ $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

denote $f = u + iv$. $dz = dx + idy$

$\int_\gamma f dz = \int_\gamma (u dx - v dy) + i \int_\gamma (u dy + v dx)$

since γ is closed and f is smooth. which implies $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exists.

$\int_\gamma (u dx - v dy) = \iint_G \frac{\partial u}{\partial y} - \left(-\frac{\partial v}{\partial x}\right) dx\,dy = 0$

(by C-R condition)

$\int_\gamma (u dy + v dx) = \iint_G \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right) dx\,dy = 0$

Complex Analysis 2024. Homework 5.

1. Assume that f is holomorphic in domain D and $\arg f(z)$ is constant. Prove that f is constant in D.

Pf: denote $f(z) = u(x,y) + i\,v(x,y)$. $x,y \in \mathbb{R}$. $u,v: \mathbb{R}^2 \to \mathbb{R}$.

if $u^2+v^2=0$. obviously $f \equiv 0$. if $u^2+v^2 \neq 0$,

$$\arg f(z) = \arccos \frac{u}{\sqrt{u^2+v^2}} = \arcsin \frac{v}{\sqrt{u^2+v^2}}$$

$$\arg f'_x = -\frac{|v|}{u^2+v^2} \cdot u'_x + \frac{|u|}{u^2+v^2} \cdot v'_x$$

$$\arg f'_y = -\frac{|v|}{u^2+v^2} \cdot u'_y + \frac{|u|}{u^2+v^2} \cdot v'_y .$$

$\arg f(z)$ is constant implies $\arg f'_x \equiv \arg f'_y \equiv 0 \Rightarrow \begin{cases} -|v| \cdot u'_x + |u| \cdot v'_x = 0 \\ -|v| \cdot u'_y + |u| \cdot v'_y = 0 \end{cases}$

since $f \in H(D)$. by C-R equation $\begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases} \Rightarrow \begin{cases} |v|\,u'_x + |u|\,u'_y = 0 \\ -|v|\,u'_y + |u|\,u'_x = 0 \end{cases}$

this homogeneous l.s. has determinant $-(v^2+u^2) \neq 0. \Rightarrow$ unique solution $u'_x = u'_y = 0$.

similarly. $v'_x = v'_y = 0$. thus $v(x,y)$. $u(x,y)$ is constant, i.e. f is constant.

2. Assume that f is holomorphic in domain D and

$$A \operatorname{Im} f(z) + B \operatorname{Re} f(z) + C = 0, z \in D,$$

for some real constants A, B, C. Prove that f is constant.

$B(Rf + iIf) - iA(Rf + iIf) = C$

$g = (B - iA) f(z) + C$

$\operatorname{Re} g = A \operatorname{Im} f + B \operatorname{Re} f + C = 0.$

Lemma $\operatorname{Re} g = 0 \to g = 0.$

Pf: denote $f(z) = f(x+iy)$ $\operatorname{Im} f(z) = v(x,y)$, $\operatorname{Re} f(z) = u(x,y)$

$Av + Bu + C = 0$

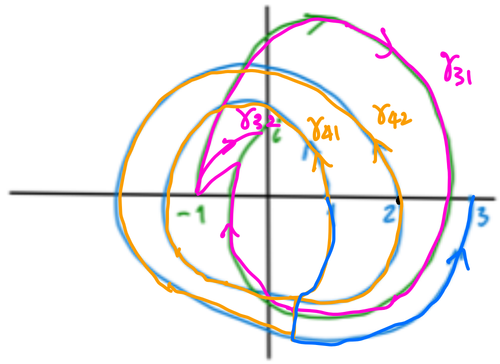
take the partial derivative $\begin{cases} Av'_x + Bu'_x = 0 \\ Av'_y + Bu'_y = 0 \end{cases}$

by C-R equation, $\begin{cases} Bu'_x - Au'_y = 0 \\ Au'_x + Bu'_y = 0 \end{cases} \Rightarrow D = \begin{vmatrix} B & -A \\ A & B \end{vmatrix} = B^2 + A^2.$

if $A^2+B^2 \neq 0$, $u'_x = u'_y = 0$ (similar as Problem 1). and $v'_x = v'_y = 0$ similarly.

thus v, u is constant. thus f is constant (if $A = B = 0$ not holds)

3. Calculate the integral $\int_\gamma \frac{dz}{z}$ along the following paths:



(a) $\gamma_1 = e^{it}, \quad t \in [0, 4\pi]$;

(b) $\gamma_2 = e^{-it}, \quad t \in [0, 2\pi]$;

(c) See paths γ_3 (green) γ_4 (blue) in the picture below.

(a). $\int_{\gamma_1} \frac{dz}{z} = \int_0^{4\pi} \frac{i \cdot e^{it}}{e^{it}} dt = 4\pi i.$

(b) $\int_{\gamma_2} \frac{dz}{z} = \int_0^{2\pi} \frac{-i \cdot e^{-it}}{e^{-it}} dt = -2\pi i.$

(c) 1) by the independence of path. $\gamma_3 = \gamma_{31} + \gamma_{32}$. $\quad \gamma_{32}: \gamma(t) = e^{-it}$. $t \in [\frac{\pi}{2}, \pi]$

$$\int_{\gamma_{31}} \frac{dz}{z} = \oint_C \frac{dz}{z} = -2\pi i$$

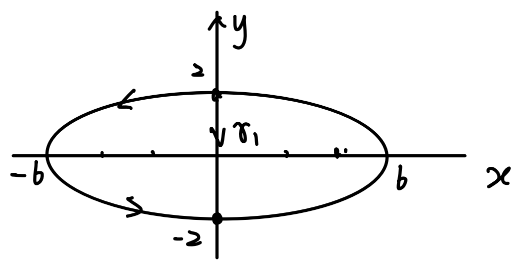
C is a circle centered at $(0,0)$, contained in domain bounded by γ_{31}, orientation is clockwise.

$$\int_{\gamma_3} = \int_{\gamma_{31}} + \int_{\gamma_{32}} = -2\pi i - \frac{\pi}{2} i = -\frac{5}{2}\pi i.$$

2) similarly as 1). $\gamma_4 = \gamma_{41} + \gamma_{42} + \gamma_{43}$. $\quad \gamma_{41}, \gamma_{42}$ is closed, $\quad \gamma_{43}: \gamma(t) = t$, $t \in [1,3]$

$$\int_{\gamma_4} = \int_{\gamma_{41}} + \int_{\gamma_{42}} + \int_{\gamma_{43}} = 4\pi i + \ln 3$$

4. Calculate $\int_\gamma dz$, where γ is the left half of the ellipse $\frac{1}{36}x^2 + \frac{1}{4}y^2 = 1$ from $z = 2i$ to $z = -2i$ to $z = -2i$.



1) denote the path $\gamma: [\frac{\pi}{2}, \frac{3\pi}{2}] \to \mathbb{C}$. $\gamma(t) = 3\cos t + 2\sin t\, i$.

$$\int_\gamma dz = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [-3\sin t + 2\cos t\, i] dt = -3\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin t\, dt + 2i \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos t\, dt$$

$$= 0 + -4i = -4i.$$

2) $f \equiv 1$ is holomorphic on $\mathbb{C}$. $\quad \gamma_1: [2,-2] \to \mathbb{C}$. $\gamma(t) = it$

$\mathbb{C}$ is simple connected, then γ, γ_1 is homotopic

$$\int_\gamma = \int_{\gamma_1} = \int_2^{-2} i\, dt = -4i.$$

5. Calculate $\int_\gamma \left(z+\frac{1}{z}\right)dz$, where γ is a circle $|z|=2$ oriented counter-clockwise.

$\gamma: [0,2\pi] \to \mathbb{C}. \qquad \gamma(t) = 2\cdot e^{it}$

$$\int_0^{2\pi}\left(2e^{it}+\frac{1}{2e^{it}}\right)\cdot 2ie^{it}\,dt = \int_0^{2\pi}\left(4i\cdot e^{2it}+i\cdot 2\pi\right)dt = 2\pi i + 0 = 2\pi i$$

6. Let $f(z)=c_0+c_1z+\cdots+c_nz^n$ be a polynomial with $c_k\in\mathbb{R}$. Show that

$$\int_{-1}^{1} f(x)^2dx \le \pi\int_0^{2\pi}\left|f\left(e^{i\theta}\right)\right|^2\frac{d\theta}{2\pi} = \pi\sum_{k=0}^{n} c_k^2.$$

Hint. For the first inequality, apply Cauchy-Goursat's theorem to the function $f(z)^2$ separately on the top half and the bottom half of the unit disk.

$$\int_{-1}^{1} f^2(x)\,dx + \int_0^{\pi} f^2(z)\,dz = 0$$

$$-\int_{-1}^{1} f^2(x)\,dx + \int_\pi^{2\pi} f^2(z)\,dz = 0$$

$$2\int_{-1}^{1} f(x)^2dx \le \int_0^{2\pi} |f(z)|^2|dz|$$

$$\le \int_0^{2\pi} |f(e^{i\theta})|^2\,d\theta$$

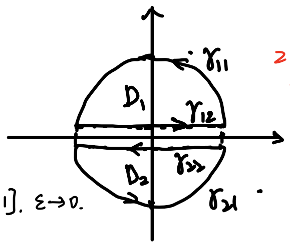
$$= 2\pi\int_0^{2\pi} |f(e^{i\theta})|^2\,\frac{d\theta}{2\pi}.$$

Pf: $f(z)\in H(\mathbb{C})$

denote the path : $\gamma:[0,2\pi]\to\mathbb{C}.\quad \gamma(\theta)=e^{i\theta}.$

and we have $\begin{cases}\gamma_{11}(\theta)=e^{i\theta} & \theta\in(0,\pi) \qquad \gamma_{12}(t)=t+i\varepsilon.\\ \gamma_{21}(\theta)=e^{i\theta} & \theta\in(\pi,2\pi). \qquad \gamma_{22}(t)=-t-i\varepsilon\end{cases}$

$t\in[-1,1].\ \varepsilon\to 0.$



thus domain D_1. D_2 is bounded by $\gamma_{11}.\gamma_{12}$ and γ_{21},γ_{22} respectively. $f^2(z)\in H(D_1)$ and $H(D_2)$.

$$\begin{cases}\int_{\gamma_{11}} f^2(z)\,dz + \int_{\gamma_{12}} f^2(z) = 0.\\ \int_{\gamma_{21}} f^2(z)\,dz + \int_{\gamma_{22}} f^2(z) = 0.\end{cases}$$

$$\int_\gamma = \int_{\gamma_{11}} + \int_{\gamma_{21}} = -\left(\int_{\gamma_{12}}+\int_{\gamma_{22}}\right) = -\int_{-1}^{1} f^2(t+i\varepsilon)\cdot dt + \int_{-1}^{1} f^2(-t-i\varepsilon)\,dt. = 2\int_{-1}^{1} f^2(x)\,dx.$$

$$2\int_{-1}^{1} f^2(x)\,dx = \int_\gamma f^2(z)\,dz = \int_0^{2\pi} f^2(e^{i\theta})\cdot ie^{i\theta}\,d\theta \le \int_0^{2\pi} |f^2(e^{i\theta})|\,d\theta.$$

$$= 2\pi\int_0^{2\pi} |f^2(e^{i\theta})|\cdot\frac{d\theta}{2\pi}.$$ which proves the first inequality.

For the second equality,

$$\pi\int_0^{2\pi}|f^2(e^{i\theta})|\frac{d\theta}{2\pi} = \pi\int_0^{2\pi}\left|\sum_{m,j=0,\,m\neq j}^{m+j} C_mC\cdot e^{i(m+j)\theta} + \sum_{k=0}^{n} C_k^2 e^{2ik\theta}\right|\frac{d\theta}{2\pi} = \pi\int_0^{2\pi}\sum_{k=0}^{n} C_k^2\,|e^{2ik\theta}|\,d\theta.$$

(by the orthogonality of trigonometric function).

$$= \pi\cdot\sum_{k=0}^{n} C_k^2\int_0^{2\pi} 1\cdot\frac{d\theta}{2\pi} = \pi\cdot\sum_{k=0}^{n} C_k^2$$

7. Show that an analytic function $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)dz = 0$ for every closed path γ in D.

Pf: "$\Rightarrow$" denote the antiderivative by F. s.t. $F'(z) = f(z)$. $z \in D$.

$\Phi(t) = F(\gamma(t))$ is an antiderivative for any path $\gamma : [a,b] \to D$.

$\forall$ closed path $\gamma : I \to D$. $\gamma(a) = \gamma(b) = z_0$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt = \Phi(b) - \Phi(a)$$

$$= F(\gamma(b)) - F(\gamma(a)) = F(z_0) - F(z_0) = 0.$$

"$\Leftarrow$" denote Φ as the antiderivative along path γ.

by NL-formula. $\int_\gamma f(z)\,dz = \Phi(b) - \Phi(a) = 0. \iff F_n(\gamma(b)) = F_1(\gamma(a))$

since γ is closed, then $F_n(\gamma(b)) = F_n(\gamma(a))$, thus $F_n(\gamma(a)) = F_1(\gamma(a))$.

since $F_n = F_1 + c$. $c \in \mathbb{C}$. the above formula implies $c = 0$.

i.e. F_1 is the global antiderivative of f in D.

8. Show that

$$\left| \oint_{|z|=R} \frac{\log z}{z^2} dz \right| \le 2\sqrt{2}\pi \frac{\log R}{R}, \quad R > e^\pi.$$

$z = Re^{it}$ $t \in [-\pi, \pi]$

$|\ln z| = \sqrt{\ln^2 R + t^2} \le \sqrt{2}\ln R.$ $R > e^\pi$ $\ln R > \pi$

$2\pi \frac{\sqrt{2}\ln R}{R^2|e^{2it}|} R \cdot |ie^{it}| dt. = 2\sqrt{2}\pi \frac{\log R}{R}.$

Pf:

$$\oint_{|z|=R} \frac{\log z}{z^2} dz = \int_0^{2\pi} \frac{\ln R + it + 2\pi k i}{R^2 \cdot e^{2it}} R \cdot i \cdot e^{it}\,dt = \frac{1}{R}\int_0^{2\pi} (i\ln R - t - 2\pi k)(\cos t - i\sin t)\,dt$$

$$= \frac{1}{R}\int_0^{2\pi} (\ln R \sin t - t\cos t - 2\pi k\cos t) + i(\ln R\cos t + t\sin t + 2\pi k\sin t)\,dt$$

$$= \frac{1}{R}\int_0^{2\pi} (-t\cos t + i t\sin t)\,dt$$

$$\left| \oint_{|z|=R} \frac{\log z}{|z|^2} dz \right| \le \frac{1}{R}\int_0^{2\pi} \left| -t\cos t + i t\sin t \right| dt$$

$$= \frac{1}{R}\int_0^{2\pi} t\,dt = \frac{2\pi^2}{R} < \frac{2\pi\log R}{R} < \frac{2\sqrt{2}\pi\log R}{R}$$

$\Uparrow$

$R < e^\pi$

$\pi < \ln R.$

9*. Show that if D is a bounded domain with smooth boundary, then

$$\int_{\partial D} \bar{z} dz = 2i \operatorname{Area}(D).$$

$$\int_{\partial D} (x-iy)\, d(x+iy) = \int_{\partial D} x\,dx + y\,dy + i\int_{\partial D} -y\,dx + x\,dy$$

D is bounded. ∂D is smooth. by Green's formula.

$$i\int_{\partial D} -y\,dx + x\,dy = i\iint_D 1-(-1)\, dx\,dy = 2i\iint_D dx\,dy = 2i \operatorname{Area}(D).$$

$$\int_{\partial D} x\,dx + y\,dy = 0 \quad \text{since} \quad \frac{\partial(x)}{\partial y} = \frac{\partial(y)}{\partial x} = 0.$$

thus $\displaystyle\int_{\partial D} \bar{z}\,dz = 2i \operatorname{Area}(D)$