Each correctly solved task gives 0.5 points. In total, you can get 5 points for 10 exercises.

EXERCISES

1. Program the steepest descent and Newton algorithms using the backtracking line search, Algorithm 1 (see on the next page). Use them to minimize the Rosenbrock function (see on the next page). Set the initial step length $\alpha_0 = 1$ and print the step length used by each method at each iteration. First try the initial point $x_0 = (1.2, 1.2)^T$ and then the more difficult starting point $x_0 = (-1.2, 1)^T$.

2. Show that if $0 < c_2 < c_1 < 1$, there may be no step lengths that satisfy the Wolfe conditions.

3. Show that the one-dimensional minimizer of a strongly convex quadratic function is given by formula (2) on the next page.

4. Show that the one-dimensional minimizer of a strongly convex quadratic function

always satisfies the Goldstein conditions (see on the next page).

5. Prove that $\|Bx\| \geq \|x\|/\|B^{-1}\|$ for any nonsingular matrix B.

6. Consider the steepest descent method with exact line searches applied to the convex quadratic function (3) on the next page. Using the properties given in lecutre 3, show that if the initial point is such that $x_0 - x^*$ is parallel to an eigenvector of Q, then the steepest descent method will find the solution in one step.

7. Let Q be a positive definite symmetric matrix. Prove that for any vector x, we have

$$\frac{(x^T x)^2}{(x^T Q x)(x^T Q^{-1} x)} \geq \frac{4\lambda_n \lambda_1}{(\lambda_n + \lambda_1)^2},$$

where λ_n and λ_1 are, respectively, the largest and smallest eigenvalues of Q.

8. Compute the eigenvalues of the 2 diagonal blocks of (4) on the next page and verify that each block has a positive and a negative eigenvalue. Then compute the eigenvalues of Aand verify that its inertia is the same as that of B.

9. Describe the effect that the modified Cholesky factorization (5) on the next page would have on the Hessian $\nabla^2 f(x_k) = \mathrm{diag}(-2, 12, 4)$.

10. Consider a block diagonal matrix B with 1×1and2×2 blocks. Show that the eigenvalues and eigenvectors of B can be obtained by computing the spectral decomposition of each diagonal block separately.

Algorithm 1 (Backtracking Line Search).
Choose $\bar{\alpha} > 0$, $\rho \in (0, 1)$, $c \in (0, 1)$; Set $\alpha \leftarrow \bar{\alpha}$;
repeat until $f(x_k + \alpha p_k) \leq f(x_k) + c\alpha \nabla f_k^T p_k$
$\alpha \leftarrow \rho\alpha$;
end (repeat)
Terminate with $\alpha_k = \alpha$.

The Rosenbrock function: $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.

The Goldstein conditions: $f(x_k) + (1-c)\alpha_k \nabla f_k^T p_k \leq f(x_k + \alpha_k p_k) \leq f(x_k) + c\alpha_k \nabla f_k^T p_k$,

The Wolfe conditions:
$$f(x_k + \alpha_k p_k) \leq f(x_k) + c_1 \alpha_k \nabla f_k^T p_k, \tag{1a}$$
$$\nabla f(x_k + \alpha_k p_k)^T p_k \geq c_2 \nabla f_k^T p_k, \tag{1b}$$

with $0 < c_1 < c_2 < 1$.

If f is a convex quadratic, $f(x) = \frac{1}{2}x^T Q x - b^T x$, its one-dimensional minimizer along the ray $x_k + \alpha p_k$ can be computed analytically and is given by

$$\alpha_k = -\frac{\nabla f_k^T p_k}{p_k^T Q p_k}. \tag{2}$$

Let us consider the function $f(x) = \frac{1}{2}x^T Q x - b^T x$, (3)

where Q is symmetric and positive definite.

The matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 2 & 3 & 4 \end{bmatrix}$$

can be written in the form (3.51) with $P = [e_1, e_4, e_3, e_2]$,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{2}{3} & 1 & 0 \\ \frac{2}{9} & \frac{1}{3} & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & \frac{7}{9} & \frac{5}{9} \\ 0 & 0 & \frac{5}{9} & \frac{10}{9} \end{bmatrix}. \tag{4}$$

The Cholesky factorization $PAP^T + E = LDL^T = MM^T$ (5)
where E is a nonnegative diagonal matrix that is zero if A is sufficiently positive definite.