Example 1.

To find a general solution of a quasi-linear inhomogeneous partial differential equation of the first order

$$x^2 u\frac{\partial u}{\partial x} + y^2 u\frac{\partial u}{\partial y} = x + y$$

<u>Solution:</u>

The characteristic system, which corresponds to this quasi-linear equation, in a symmetric form, has the form:

$$\frac{dx}{x^2 u} = \frac{dy}{y^2 u} = \frac{dz}{x + y}$$

The first integrated combination

$$\frac{dx}{x^2 u} = \frac{dy}{y^2 u}$$

after the reduction by u:

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

and integration, it gives

$$\frac{1}{x} - \frac{1}{y} = C_1.$$

To obtain another first integral, we make an integrable combination

$$\frac{dx - dy}{x^2 u - y^2 u} = \frac{du}{x + y}$$

Where from

$$\frac{d(x-y)}{u(x^2-y^2)}=\frac{du}{x+y}$$

or

$$\frac{d(x-y)}{x-y}=udu$$

After integration, we get

$$\ln|x-y|-\frac{u^2}{2}=C_2 .$$

Therefore, the general solution of this quasi-linear differential equation will be:

$$\Phi\left(\frac{1}{x}-\frac{1}{y},\ln|x-y|-\frac{u^2}{2}\right)=0 ,$$

where Φ is an arbitrary function. Since u is included in only one of the first integrals, the general solution can be written as

$$\ln|x-y|-\frac{u^2}{2}=F\left(\frac{1}{x}-\frac{1}{y}\right),$$

where F is an arbitrary function.

Example 2.

Find a general solution to a quasi-linear partial differential equation of the first order

$$xy\frac{\partial u}{\partial x}+(x-2u)\frac{\partial u}{\partial y}=yu .$$

Solution:

The characteristic system corresponding to this quasi-linear equation takes the form in a symmetric form:

$$\frac{dx}{xy} = \frac{dy}{x-2u} = \frac{du}{yu}$$

From the first integrable combination

$$\frac{dx}{xy} = \frac{du}{yu}$$

From the first integrable combination we obtain

$$\frac{u}{x} = C_1 .$$

To find the second independent integral of the characteristic system, we rewrite it as:

$$\begin{cases} x' = xy, \\ y' = x - 2u, \\ u' = yu. \end{cases}$$

Let's differentiate the second equation of the system

$$y'' = x' - 2u',$$

substitute instead of x' and y' their expressions from the first and third equations of the system:

$$y'' = x' - 2u' = xy - 2yu = y\left(x - 2u\right) = yy',$$

as a result, we obtain a second-order equation

$$y'' = yy'.$$

By replacing $y' = p$, where $p = p(y)$, $y'' = pp'$, we lower the order of the equation:

$$pp' = yp.$$

From where we have two equations: $p = 0$ and $p' = y$. The first one gives a trivial solution

$$y' = 0, \ \ y = const,$$

which we are not interested in. Solving the second equation, we get

$$p = \frac{y^2}{2} + \tilde{C}_2$$

or

$$y' = \frac{y^2}{2} + \tilde{C}_2$$

$$x - 2u = \frac{y^2}{2} + \tilde{C}_2$$

$$2x - 4u = y^2 + 2\tilde{C}_2$$

$$2x - 4u - y^2 = 2\tilde{C}_2 = C_2.$$

Thus, the general solution of the original quasi-linear differential equation is

$$\Phi\left(\frac{u}{x}, 2x - 4u - y^2\right) = 0,$$

where Φ is an arbitrary function.

Example 3.

Find a general solution to the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2.$$

Solution:

Let's make up the equations of characteristics:

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-x^2 - y^2}.$$

The first equation of this system can be solved separately from the second, since it does not contain u (the variable u, which stands in the left and right sides of this equation, is reduced). From equality

$$\frac{dx}{x} = \frac{dy}{y}$$

we get by integrating

$$\ln|x| = \ln|y| + \ln C,$$

where do we find the first integral of the system:

$$\frac{y}{x} = C_1.$$

The second equation of this system contains all three variables

To exclude the variable y, let's use the first integral found. Since the desired integral curve lies on one of the surfaces defined by the first integral found, at each point of this curve $y = C_1 x$ (the value of the constant C_1 is the same at all points of the desired integral curve, but may differ if you switch to another integral curve). Replace y with C_1x in the second equation, after the transformations we get

$$-(1 + C_1^2)x\,dx = u\,du.$$

Integrating, we find the dependence

$$(1 + C_1^2)x^2 + u^2 = C_2.$$

This ratio containing C_2 is not the first integral, since it also contains an arbitrary constant C_1. Given that for the found curve $C_1 = \frac{y}{x}$, we rewrite the ratio as

$$x^2 + y^2 + u^2 = C_2.$$

In this form of notation, the relation is performed for any of the integral curves, that is, it is the first integral.

The general solution of the first-order equation has the form (in an implicit form)

$$\Phi\left(\frac{y}{x}, x^2 + y^2 + u^2\right) = 0,$$

where Φ is an arbitrary differentiable function. It is possible to get an explicit solution from the last expression:

$$u = \pm\sqrt{f\left(\frac{y}{x}\right) - x^2 - y^2},$$

where f is an arbitrary differentiable function.

Comment

When finding the first integrals of a system written in symmetric form, derivative proportions are often used to obtain integrable combinations, for example

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}.$$

In the above example, we will rewrite the system in the form

$$\frac{x\,dx}{x^2} = \frac{y\,dy}{y^2} = \frac{u\,du}{-x^2-y^2},$$

then we will use the derived proportion by adding the numerators

and denominators of the first and second ratios:

$$\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{x\,dx}{x^2} = \frac{y\,dy}{y^2} = \frac{u\,du}{-x^2-y^2}.$$

Comparing the first relation with the last one, we get

$$x\,dx + y\,dy = -u\,du,$$

hence the first integral

$$x^2 + y^2 + u^2 = C_2.$$

FINDING A PARTICULAR SOLUTION (CAUCHY PROBLEM)

In order to single out one definite solution from the infinite set of solutions given by formula

$$\Phi\left(\psi_1\left(x,y,u\right),\psi_2\left(x,y,u\right)\right)=0,$$

it is necessary to find the function Φ included in the solution.

This can be done under additional conditions. We formulate the problem of finding a partial solution to equation

$$P\left(x,y,u\right)\frac{\partial u}{\partial x}+Q\left(x,y,u\right)\frac{\partial u}{\partial y}=R\left(x,y,u\right) \qquad (1)$$

— the Cauchy problem.

(*A general view of a first-order quasilinear equation with two independent variables, where $u=u\left(x,y\right)$ is the desired function; $P\left(x,y,u\right)$, $Q\left(x,y,u\right)$, $R\left(x,y,u\right)$ are continuous changes in the variables of the function in the considered area that do not vanish at the same time.*) We just wrote it down in this form.

Along some curve L_1 of the plane $\left(x,y\right)$, the values of the desired function are set

$$\begin{cases} y=f(x), \\ u=g(x), \end{cases}$$

where $f\left(x\right)$, $g\left(x\right)$ are differentiable functions. Find such a solution $u=u\left(x,y\right)$ of equation (1) in the vicinity of the line L_1 so that $u=u\left(x,f\left(x\right)\right)=g\left(x\right)$.

Geometric illustration of the Cauchy problem: through a space curve L with a continuously varying tangent (smooth curve), draw the integral surface of equation (1) (Fig. 1).

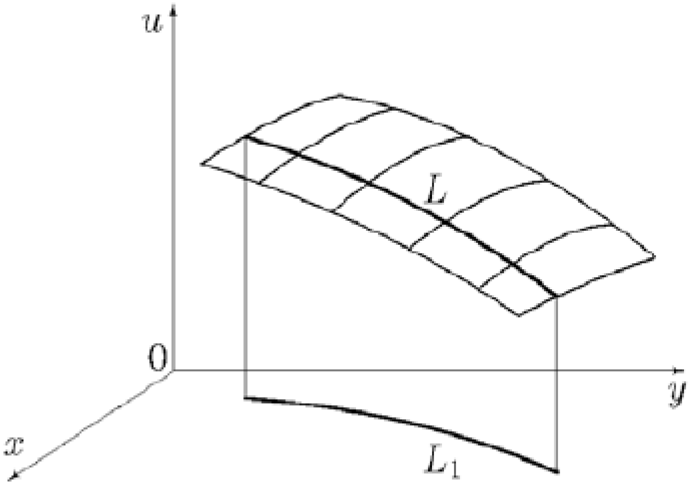

Fig.1.

The L line can be defined in a more general way:

$$\begin{cases} x = \varphi(\sigma), \\ y = \varkappa(\sigma), \\ u = \chi(\sigma). \end{cases} \quad (2)$$

The first two of the equations (2) define the curve L_1 in parametric form, all three equations define the curve L in space (x,y,u), for which L_1 is a projection onto the plane (x,y). In this case, the condition

$$u(\varphi(\sigma), \varkappa(\sigma)) = \chi(\sigma)$$

must be fulfilled.

The geometric solution to the Cauchy problem is obvious:

a characteristic must be drawn through each point of a given line L.

The set of characteristics passing through all points of the line L form the desired integral surface (see Fig. 1).

A system of differential equations

$$\frac{dx}{P(x,y,u)} = \frac{dy}{Q(x,y,u)} = \frac{du}{R(x,y,u)}$$

can be integrated without knowledge of the integral surface. The general integral of the system

$$\begin{cases} \psi_1(x,y,u) = C_1, \\ \psi_2(x,y,u) = C_2 \end{cases}$$

defines a family of characteristics (depending on two parameters C_1 and C_2), which has the following property: one characteristic passes through each point of the domain where the conditions for the existence and uniqueness of the solution are met. Constants C_1 and C_2 are independent, each of them can be assigned any values; a two-parameter family of characteristics is obtained as a result of the intersection of each surface of one family with each surface of the other (Fig.2).

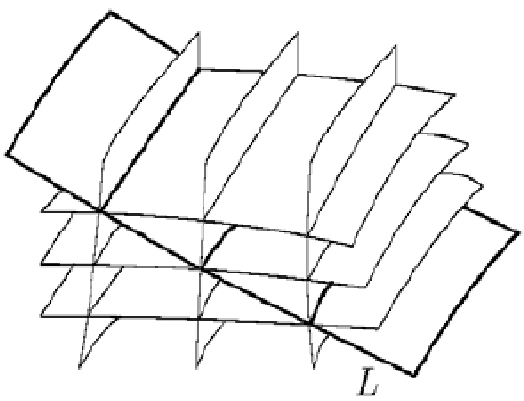

Fig.2.

Of all the surfaces defined by the first integrals, we need to leave only such pairs of surfaces whose intersection lines pass through the points of the line L; for this, we must learn how to select pairs C_1 and C_2 accordingly. In other words, between arbitrary constants C_1 and C_2 in the general integral of the system, it is necessary to establish some dependence $\Phi(C_1, C_2) = 0$.

A surface from the family $\psi_1(x,y,u) = C_1$ can be drawn through each point of the line L. Substituting the coordinates of this point, given as functions of the parameter σ, into the equation of the surface, we establish a relationship between the value of the constant C_1 defining the surface and the value of the parameter σ corresponding to the point of intersection of the line L with this surface:

$$C_1 = \psi_1(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_1(\sigma). \qquad (3)$$

A similar ratio

$$C_2 = \psi_2(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_2(\sigma) \qquad (4)$$

gives the dependence between the parameter σ of the point of the curve L and the constant C_2 of the surface of another family intersecting the line L at this point. Any pair of values C_1 and C_2 calculated by formulas (3) and (4) for one value σ will determine a pair of surfaces whose intersection line (characteristic) passes through the point of the line L corresponding to this value σ. Therefore, formulas (3) and (4) taken together define in parametric form the desired dependence between C_1 and C_2:

$$\begin{cases} C_1 = C_1(\sigma), \\ C_2 = C_2(\sigma). \end{cases} \quad (5)$$

Thus, the system of equations

$$\begin{cases} \psi_1(x, y, u) = C_1(\sigma), \\ \psi_2(x, y, u) = C_2(\sigma) \end{cases} \quad (6)$$

defines a family of characteristics passing through the line L. This family of characteristics, depending on one parameter σ, forms the desired integral surface (solution of the Cauchy problem).

Excluding the parameter σ from equations (5), can obtain a dependence of the form:

$$\Phi(C_1, C_2) = 0. \quad (7)$$

Accordingly, the solution of the Cauchy problem will be presented as

$$\Phi(\psi_1(x, y, u), \psi_2(x, y, u)) = 0. \quad (8)$$

To exclude σ, for example, equation (3) with respect to σ should be resolved and the expression $\sigma(C_1)$ should be substituted into the left part of the relation (4). This is possible when σ enters the left parts of equations (3) and (4).

If the entire curve L lies, for example, on the surface $\psi_1(x,y,u)=C_1^0$, then the ratio

$$\psi_1(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_1^0$$

cannot be resolved relative to the σ parameter. But then this surface itself $\psi_1(x,y,u) = C_1^0$ is the integral surface in the Cauchy problem.

Finally, if a given curve L lies simultaneously on two surfaces of different families, then it itself is a characteristic, and the parameter σ is not included in any of the relations (3), (4). The Cauchy problem becomes indefinite, since each characteristic belongs to an infinite set of integral surfaces. Indeed, if only the constants C_1^0 and C_2^0 satisfy equation

$$\Phi(C_1, C_2) = 0,$$

where Φ is an arbitrary differentiable function, then equation

$$\Phi(\psi_1(x,y,u), \psi_2(x,y,u)) = 0$$

defines an integral surface passing through the line L. Thus, countless integral surfaces can be drawn through a given line L.

Example 4.

Find the integral surface of the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2,$$

passing through the curve

$$\begin{cases} x = a, \\ u = \sqrt{y^2 + a^2}, \end{cases}$$

where a is a constant.

<u>Solution:</u>

In Example 3, the first integrals of the equations of characteristics are found,

$$\frac{y}{x} = C_1, \qquad x^2 + y^2 + u^2 = C_2.$$

The first equation defines a set of planes passing through the Oz -axis, the second one defines spheres of different radii centered at the origin. The characteristics of the original partial differential equation can be represented as meridians on spheres.

In the equations of a given curve, we take y as an independent variable.

Substituting the coordinates of points into the first integrals of the system, we obtain

$$\begin{cases} C_1 = \dfrac{y}{a}, \\ C_2 = a^2 + y^2 + (y^2 + a^2). \end{cases}$$

From here

$$C_2 = 2a^2(1 + C_1^2).$$

Replacing C_1 and C_2 in this dependence with the functions on the left sides of the first integrals, we find the desired solution

$$x^2 + y^2 + u^2 = 2a^2 \frac{x^2 + y^2}{x^2}.$$

Example 5.

Find the integral surface of the equation

$$(x^2 - y^2 - u^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 2xu,$$

passing through the curve $L: \; x = 0, y = 2a\cos t, u = 2a\sin t$.

<u>Solution:</u>

Integrating a system of equations

$$\frac{dx}{x^2 - y^2 - u^2} = \frac{dy}{2xy} = \frac{du}{2xu}.$$

From the second equation

$$\frac{dy}{y} = \frac{du}{u}$$

we get the first integral

$$\frac{u}{y} = C_1.$$

To obtain another first integral, we present the system as

$$\frac{x\,dx}{x^2 - y^2 - u^2} = \frac{y\,dy}{2y^2} = \frac{u\,du}{2u^2},$$

and then we apply the derivative proportion, adding up all the numerators and all the denominators of the relations:

$$\frac{x\,dx + y\,dy + u\,du}{x^2 + y^2 + u^2} = \frac{dy}{2y}.$$

Thus, an integrable combination is obtained

$$\frac{d(x^2 + y^2 + u^2)}{x^2 + y^2 + u^2} = \frac{dy}{y},$$

where from

$$\frac{x^2+y^2+u^2}{y}=C_2.$$

The first found integrals define planes passing through the Ox axis and spheres centered on the Oy axis passing through the origin. The characteristics will be circles passing through the origin, for which the Ox axis is tangent.

The desired surface will be obtained by rotating a circle of radius a around the Ox axis, touching the axis at the origin. Let's find the equation of this surface. Substituting the coordinates of the points of a given line, expressed in terms of t, into the first integrals, we obtain

$$C_1=\operatorname{tg} t, \qquad C_2=\frac{2a}{\cos t}.$$

Excluding t, we find the relationship between C_1 and C_2:

$$4a^2(C_1^2+1)=C_2^2,$$

and then the solution of the Cauchy problem

$$4a^2(u^2+y^2)=(x^2+y^2+u^2)^2.$$

In spherical coordinates $(x=r\sin\theta,\ y=r\cos\theta\cos\varphi,\ u=r\cos\theta\sin\varphi)$ the equation of the surface takes the form $r=2a\cos\theta$.