Complex Analysis. Sample problems.

1. Describe and sketch the subsets of points of $\mathbb{C}$ that satisfy the following inequalities

   (a) $|1+z| < |1-z|$;

   (b) $|z| > 1 - \operatorname{Re} z$;

   (c) $\operatorname{Re}(z(1-i)) < \sqrt{2}$;

   (d) $\operatorname{Re} \frac{1}{z} = \frac{1}{2}$;

2. Suppose $f(z)$ is differentiable at point $z_0 = x_0 + iy_0$. Let

$$u(x,y) = \operatorname{Re} f(x+iy), \ v(x,y) = \operatorname{Im} f(x+iy).$$

   Prove that

   (a) $f'(z_0) = u'_x(x_0,y_0) + iv'_x(x_0,y_0)$.

   (b) $f'(z_0) = v'_y(x_0,y_0) - iu'_y(x_0,y_0)$.

   (c) $f'(z_0) = u'_x(x_0,y_0) - iu'_y(x_0,y_0)$.

   (d) $f'(z_0) = v'_y(x_0,y_0) + iv'_x(x_0,y_0)$.

   (e) $|f'(z_0)|^2 = u'^2_x + u'^2_y = u'^2_x + v'^2_x = u'^2_y + v'^2_y = v'^2_x + v'^2_y$.

3. Suppose $f \in H(D)$ and

$$\operatorname{Re} f(z) = F(\operatorname{Im} f(z))$$

   for some strictly monotonic smooth function $F : \mathbb{R} \to \mathbb{R}$. Prove that $f(z) \equiv \text{const}$.

4. Describe the image of the path $\gamma(t) = e^{it}$, $0 \le t \le \pi$, by the function $\frac{1}{2}\left(z + \frac{1}{z}\right)$;

5. Find the image $f(E)$ if

$$f(z) = z^4; \quad E = \{z \in \mathbb{C} : |z| < 1, \ \operatorname{Im} z > 0\}.$$

6. Find Laurent series for function

$$\frac{1}{(z^2-1)(z^2+4)}$$

   centered at 0 convergent in

$$D = \{z : |z| > 2\}.$$

7. Find Laurent series for function

$$f(z) = \frac{1}{z(z-3)^2}$$

   on $V_{1,2} = \{0 < |z-1| < 2\}$.

8. Find Laurent series of a function

$$f(z) = z^2 \sin \pi \frac{z+1}{z}$$

   in annulus $V_{0,\infty} = \{z \in \mathbb{C} : 0 < |z| < \infty\} = \mathbb{C} \backslash \{0\}$.

9. Establish the type of singularity at $z = 1$ of the following functions:

   (a) $\frac{z^2-3z+2}{z^2-5z+4}$

   (b) $\frac{z^2-3z+2}{z^2-2z+1}$

   (c) $(z-1)e^{\frac{1}{z-1}}$

10. Establish the type of singularity at $z = \infty$ for following functions

    (a) $\frac{z^2+2}{z^{10}+2}$

    (b) $z^4 \cos \frac{1}{z}$;

    (c) $z^2 e^{-2z}$.

11. Calculate residues

    (a) $\operatorname*{res}_{z=\infty} \frac{\sin z}{z^2}$;

    (b) $\operatorname*{res}_{z=\pi/4} \frac{\cos z}{z-\pi/4}$;

    (c) $\operatorname*{res}_{z=0} \frac{e^{z^2}}{z^{2n+1}}, \quad n \in \mathbb{N}$;

12. Calculate residues at all finite singular points

    (a) $\frac{z^2}{1+z^4}$;

    (b) $\frac{1}{\sin z^2}$.

13. Calculate residues at all finite singular points and at ∞

    (a) $\frac{1+z^8}{z^6(z+2)}$;

    (b) $\sin z \sin \frac{1}{z}$;

    (c) $\frac{\cos z}{(z^2+1)^2}$.

14. Calculate integrals assuming that ∂D is oriented counterclockwise with respect to D

    (a) $\int_{\partial D} \frac{\cos z}{z^2-\pi^2} dz, \quad D = \{z \in D : |z-1| < 2\}$;

    (b) $\int_{\partial D} \frac{dz}{(z-1)^2(z^2+1)}, \quad D = \{z \in D : |z-1| < 1\}$;

    (c) $\int_{\partial D} \frac{z}{z+3} e^{1/3z} dz, \quad D = \{z \in D : |z| > 4\}$;

    (d) $\int_{\partial D} \frac{z^3}{z+1} dz, \quad D = \{z \in D : |z| > 4\}$;

15. Check the character of isolated branch point $z = 0$ of analytic in $0 < |z| < \infty$ functions

    (a) $f(z) = \sqrt{\ln(1+z)}$,
    $\ln(1+z)|_{z=0} = 0$;

    (b) $f(z) = z^{\sqrt{2}}$;

    (c) $\sqrt{z} + \sqrt[3]{z}$.

16. Find all singular points of analytic function $\sqrt[3]{z(1-z)^2}$ and determine their kind.

17. Find all singular points of analytic function $\ln(z + \sqrt{z^2+1})$ and determine their kind.

18. Calculate integrals

(a) $\int_{-\infty}^{\infty} \frac{x^4+1}{x^6+1} dx.$

(b) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)^2}, \quad a > 0, b > 0.$

(c) $\int_{-\infty}^{\infty} \frac{(x+1)e^{-3ix}}{x^2-2x+5} dx$

(d) $\int_{-\infty}^{\infty} \frac{x \cos x}{x^2-2x+10} dx;$

(e) $\int_{0}^{\infty} \frac{\cos x}{(x^2+4)^3} dx.$

(f) $\int_{-\pi}^{\pi} \frac{\sin^2 \varphi d\varphi}{1-2a\cos\varphi+a^2}, \quad a > 1;$

(g) $\int_{0}^{\infty} \frac{dx}{(x+1)\sqrt{x}};$

(h) $\int_{0}^{\infty} \frac{x^{\alpha-1}}{1+\sqrt[3]{x}} dx, \quad 0 < \operatorname{Re} \alpha < \frac{1}{3}.$