Equations of Straight Line on Plane. Coordinate Form.

1 Revisit definition of the line

Definition. Let A and B be a pair of points. We call **straight line** (AB) (or just line) union of all segments containing both points A and B.

We also proved that for every two distant points A and B there is single and only single line (AB) that contains them both and there is single and only single segment containing them as endpoints.

2 General Assumptions

Assume that some plane α in the space $\mathbb{E}$ is chosen and fixed.

On this plane we choose arbitrary coordinate system: origin O, and a pair of unit vectors.

Vectorial parametric equation of this line is

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a}. \tag{1}$$

Here $\boldsymbol{r}_0$ is radius vector of arbitrary initial point $A \in a$, and $\boldsymbol{a} \parallel a$ is non-zero direction vector. t is parameter.

Vectorial normal equation of this line is

$$\boldsymbol{r} \cdot \boldsymbol{n} = D. \tag{2}$$

Here $\boldsymbol{n} \perp a$ is non-zero normal vector, and $D = \boldsymbol{r}_0 \cdot \boldsymbol{n}$, and $\boldsymbol{r}_0$ is radius vector of arbitrary initial point $A \in a$.

Let us now consider coordinates forms of equation of this line.

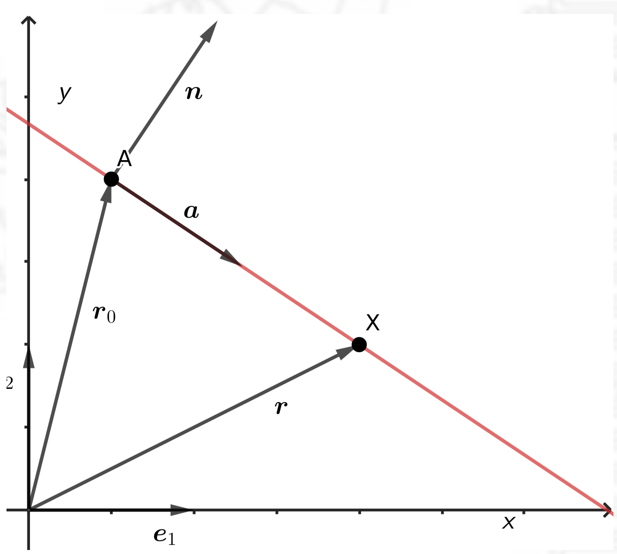

Figure 1: Derivation of equations of line

3 Coordinate parametric equations of a line on a plane

Let's determine the vectors $\boldsymbol{r}$, $\boldsymbol{r}_0$, and $\boldsymbol{a}$ in the vectorial parametric equation (1) through their coordinates:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad \boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

Separated writing of the equation for first and second coordinate form (1) yields us **coordinate parametric equations of a line on a plane**:

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \end{cases} \tag{3}$$

Condition on non-zero $\boldsymbol{a}$ now takes form that a_x *and* a_y *must not vanish simultaneously.*

Mechanical explanation for this system of equation is following.

We assume displacement of our point mass being expanded on coordinate axes. Thus, velocity vector is also expanded. For each component we need to write equation for varying coordinate.

Hence, we have two scalar motion equations for each coordinate. Their recombination restores vectorial displacement.

This direct connection of these two forms automatically yields us analogs for all dependencies and formulas we investigated for vectorial parametric equation.

Let us recap and modify that dependencies for right Cartesian coordinates.

1. Slope of the line $m = \dfrac{a_y}{a_x}$
2. If slope of the line expressed as $\dfrac{p}{q}$, $a_x = q$, $a_y = p$
3. y-intercept has expression

$$\begin{cases} x = 0 \\ y = y_0 - a_y \dfrac{x_0}{a_x} \end{cases}$$

4. x-intercept has expression

$$\begin{cases} x = x_0 - a_x \dfrac{y_0}{a_y} \\ y = 0 \end{cases}$$

5. Let a and a' be two lines expressed with parametric equations

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \end{cases}$$

and

$$\begin{cases} x' = x'_0 + a'_x\tau \\ y = y'_0 + a'_y t\tau \end{cases}$$

Tangent of the angle between these two lines has expression:

$$\tan\alpha = \frac{a_x a'_y - a'_x a_y}{a_x a'_x + a_y a'_y}$$

- For two parallel lines

$$a_x a'_y = a'_x a_y$$

- For two perpendicular lines

$$a_x a'_x = -a_y a'_y$$

6. Horizontal line has equations:

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 \end{cases}$$

or

$$\begin{cases} x = t \\ y = y_0 \end{cases}$$

7. Vertical line has equations:

$$\begin{cases} x = x_0 \\ y = y_0 + a_y t \end{cases}$$

or

$$\begin{cases} x = x_0 \\ y = t \end{cases}$$

Problem 1
Express coordinate parametric equation of a line crossing points $A(4, 12)$, and $B(6, 6)$

Solution
Suppose A be initial point. Thus, $x_0 = 4$, $y_0 = 12$.

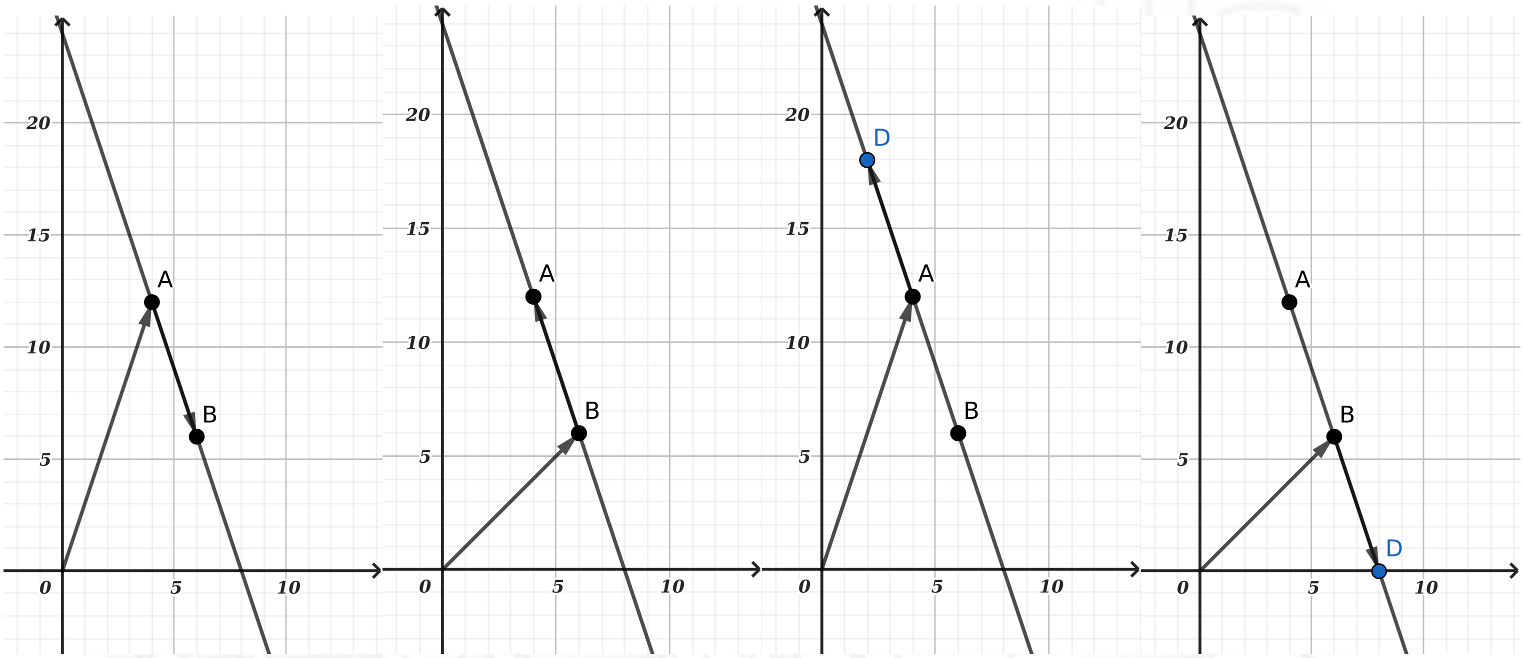

Figure 2: Building of coordinate parametric equations of line contains the same collection of possibilities as building of vectorial parametric equation

Suppose point will be moved from A to B with value of parameter $t = 1$

$$x_b = x_a + a_x$$
$$y_b = y_a + a_y$$

Thus, $a_x = x_b - x_a = 6 - 4 = 2$, $a_y = y_b - y_a = 6 - 12 = -6$

The system of equations is

$$\begin{cases} x = 4 + 2t \\ y = 12 - 6t \end{cases}$$

Alternatively, we can take point B as initial point:

$$\begin{cases} x = 6 + 2t \\ y = 6 - 6t, \end{cases}$$

or reverse direction of movement:

$$\begin{cases} x = 4 - 2t \\ y = 12 + 6t \end{cases}$$

4 Canonical equation of a line on a plane

Suppose in coordinate parametric equations

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \end{cases}$$

Both parameters governing direction of the line are non-zero: $a_x \neq 0$, and $a_y \neq 0$. It means that line is not parallel with both coordinate axes.

In this case the parameter t can be explicitly expressed through both x and y:

$$\begin{cases} t = \dfrac{x - x_0}{a_x} \\ t = \dfrac{y - y_0}{a_y} \end{cases}$$

Putting parameter t the same value for both equation yields **canonical equation** of a line on a plane:

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} \tag{4}$$

For line parallel with first coordinate axis canonical equation has form:

$$y = y_0 \tag{5}$$

For line parallel with second coordinate axis canonical equation has form:

$$x = x_0 \tag{6}$$

5 The equation of a line passing through two given points on a plane

Problem

Express canonical equation of the line passing through points $A(x_0, y_0)$, and $B(x_1, y_1)$, $A \neq B$

Solution

We express direction vector as $\overrightarrow{AB}$ with coordinates $\begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \end{pmatrix}$.

Hence, $a_x = x_1 - x_0$, $a_y = y_1 - y_0$.

General form of canonical equation (4) now has form:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}. \tag{7}$$

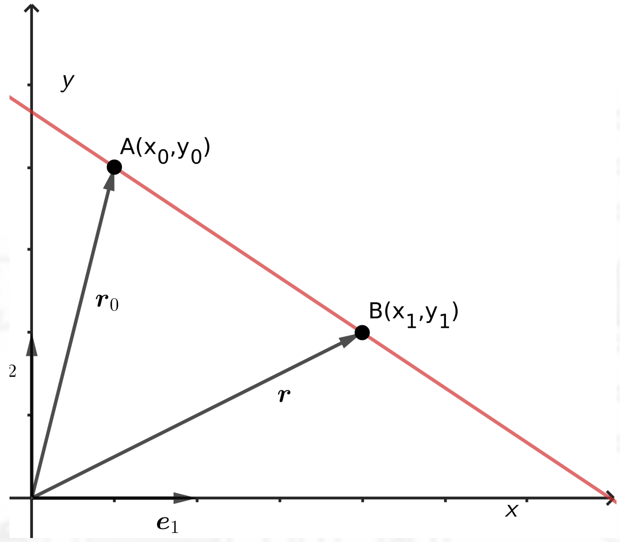

Figure 3: Line passing through pair of points

This equation corresponds with case $x_0 \neq x_1$, and $y_0 \neq y_1$.

Suppose $x_0 = x_1$, then we write the equation (6):

$$x = x_0 = x_1 \tag{8}$$

Suppose $y_0 = y_1$, then we write the equation (5):

$$y = y_0 = y_1 \tag{9}$$

The conditions $x_0 = x_1$ and $y_0 = y_1$ cannot be fulfilled simultaneously since the points A and B are distinct, i. e. $A \neq B$.

We call equation of line written in the form (7), or (8), or (9) the **equation of a line passing through two given points**

6 Line through the origin

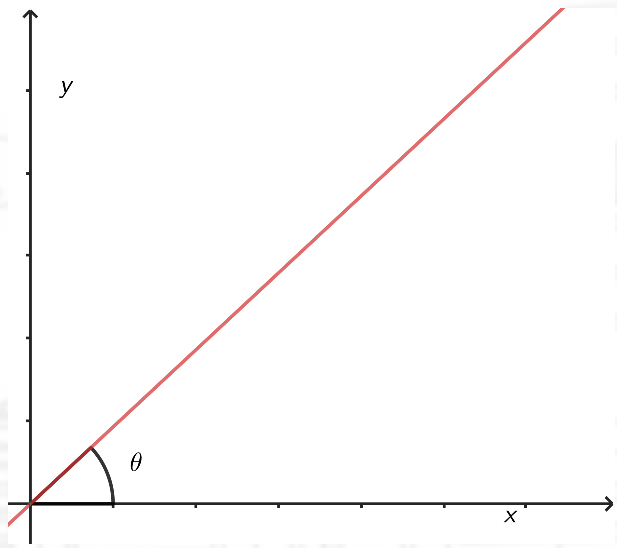

Figure 4: Line passing through the origin

Suppose point A in equation of a line passing through two given points (7) is the origin of coordinate system.

Equation in this case reduces to

$$\frac{x}{x_1} = \frac{y}{y_1}.$$

Explicit formula for y has form:

$$y = x\frac{y_1}{x_1}$$

Ratio $\frac{y_1}{x_1} = \frac{y_1 - 0}{x_1 - 0}$ is the slope m of a line passing through the origin and point $B(x_1, x_2)$. Thus, explicit equation for the line through the origin is

$$y = xm$$

7 Explicit equation for any line

Let us now express y as function of x in general case. We start with canonical equation (4):

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y}.$$

$$y - y_0 = (x - x_0)\frac{a_y}{a_x}$$

Since numbers a_x and a_y have meaning of the coordinates of direction vector, their relation resembles slope of the line: $m = \dfrac{a_y}{a_x}$

$$y - y_0 = (x - x_0)m \tag{10}$$

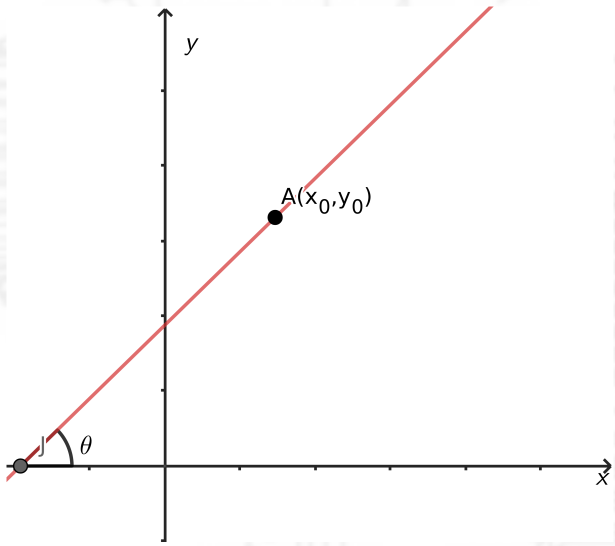

Figure 5: Line passing through the point A with determined slope

This explicit equation of the line we call **point-slope form** of equation of the line.

Suppose now that initial point in this equation in y-intercept $P(0, b)$. Substitution it into equation yields

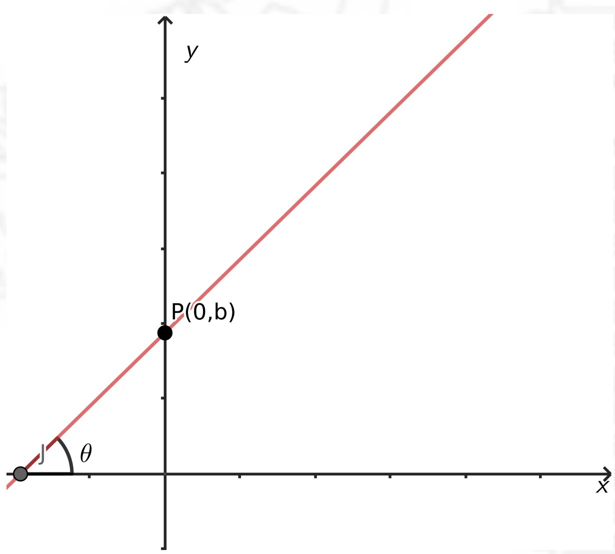

Figure 6: Slope-intercept form of line equation

$$y - b = mx$$

or

$$y = mx + b \tag{11}$$

This form of explicit equation we call **slope-intercept form**.

8 Double intercepts equation of a line on a plane

Suppose line expressed with two-points form of equation (7):

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}.$$

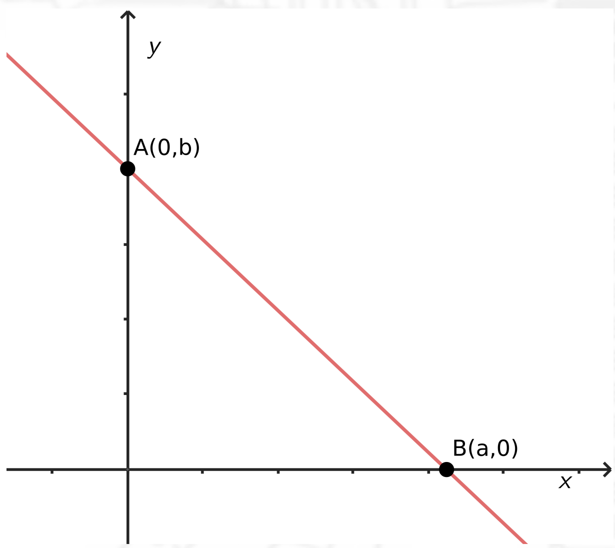

Figure 7: x- and y-intercepts

Let point first be y-intercept ($x_0 = 0$), and second point be x-intercept $y_1 = 0$

$$\frac{x}{x_1} = \frac{y - y_0}{-y_0}.$$
$$\frac{x}{x_1} = 1 - \frac{y}{y_0}$$

Putting $x_1 = a$ and $y_0 = b$, we yield **intercept form of the equation of line**:

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{12}$$

a and b here have meaning of distances till intercept points from the origin.

9 General equation of the line in Cartesian coordinates.

Suppose line in question expressed with vectorial normal equation (2):

$$\boldsymbol{r} \cdot \boldsymbol{n} = D. \tag{13}$$

Let coordinates of radius vector be $\begin{pmatrix} x \\ y \end{pmatrix}$, and coordinates of normal vector be $\left(x_n, y_n\right)$.

Dot product in Cartesian coordinates yields us:

$$xx_n + yy_n = D$$

Condition that normal vector is not-zero automatically restrict x_n and y_n be zeros in the same time.

Reordering of terms yields

$$xx_n = yy_n - D = 0$$

or

$$Ax + By + C = 0 \tag{14}$$

Here $A = x_n$, $B = x_n$, $C = -D$.

We call equation in form (14) the **general equation of the line**

Suppose at least $B \neq$. Explicit slope-intercept equation may be derived from this general form to reveal meaning of its parameters:

$$\begin{aligned} By &= -Ax - C \\ y &= \frac{-A}{B}x + \frac{-C}{B} \end{aligned}$$

Thus, slope of the line expressed with general equation is $m = -\frac{A}{B}$, y-intercept is $b = -\frac{C}{B}$

If $B = 0$, then

$$Ax + C = 0$$

or

$$x = -\frac{C}{A},$$

a line parallel to the y-axis.

If A $= 0$, then

$$By + C = 0$$

or

$$y = -\frac{C}{B},$$

a line parallel to the x-axis.

10 Parallel and perpendicular lines

Suppose two lines expressed with equations

$$Ax + By + C = 0$$

and

$$A'x + B'y + C' = 0$$

slopes of these lines are $m = -\frac{A}{B}$ and $m' = -\frac{A'}{B'}$.
For parallel lines slopes coincide, thus

$$-\frac{A}{B} = -\frac{A'}{B'}$$
$$\frac{A}{A'} = \frac{B}{B'}$$

For prerpendicular lines slopes are negative reciprocal:

$$-\frac{A}{B} = \frac{B'}{A'}$$
$$-AA' = BB'$$
$$AA' + BB' = 0$$

11 Normal equation of the line in Cartesian coordinates

A straight line is completely determined if the length of the perpendicular from the origin (0.0) to the line is known and if the angle which this perpendicular makes with the x-axis is known (actually this is a way to write valid equation in form (14)).

Let AB be the given line. Draw ON perpendicular to AB.

The perpendicular distance p from O to AB is taken as positive for all positions of AB, and ω is the angle from 0 to 2π which ON makes with the positive direction of the x-axis. Let the coordinates of intersection point C be (x_1, y_1).

Then $x_1 = p\cos(\omega)$, $y_1 = p\sin(\omega)$, and slope of line AB is negative reciprocal of the slope to its perpendicular:

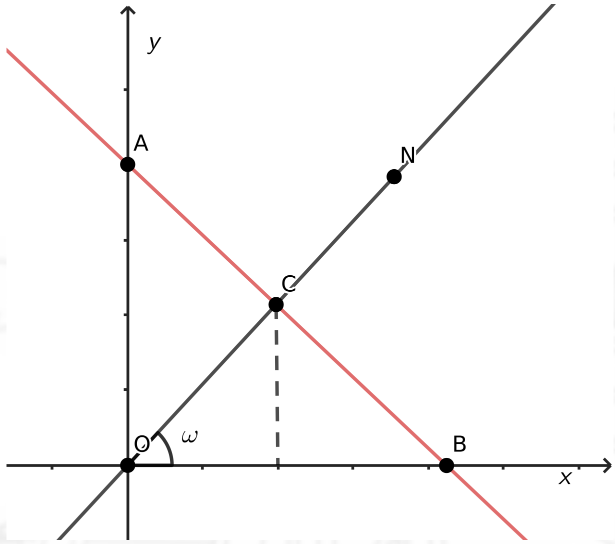

Figure 8: Deriving equation of line in normal form

$$m = -\frac{1}{\tan\omega} = -\cot\omega = -\frac{\cos\omega}{\sin\omega}$$

For any point of the line $X(x, y)$ point-slope form of line equation now can be written:

$$y - y_1 = -\cot\omega(x - x_1)$$

or

$$\begin{aligned} y - p\sin\omega &= -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega) \\ y\sin\omega - p\sin^2\omega &= -x\cos\omega + p\cos^2\omega \\ x\cos\omega + y\sin\omega &= p(\sin^2\omega + \cos^2\omega) \end{aligned}$$

$$x\cos\omega + y\sin\omega - p = 0 \tag{15}$$

This equation (15) we call **the normal form of the equation of a straight line in Cartesian coordinates**

12 Reduction to normal form

Suppose general equation

$$Ax + By + C = 0$$

and normal equation

$$x\cos\omega + y\sin\omega - p = 0$$

express the same line.

Thus, there is constant ratio k:

$$\frac{\cos\omega}{A} = \frac{\sin\omega}{B} = \frac{-p}{C}.$$

Then

$$Ak = \cos\omega, \quad Bk = \sin\omega, \quad Ck = -p.$$

Squaring and adding the first two,

$$\cos^2\omega + \sin^2\omega = k^2(A^2 + B^2),$$

or

$$k^2(A^2 + B^2) = 1$$
$$k^2 = \frac{1}{A^2 + B^2}$$
$$k = \frac{1}{\pm\sqrt{A^2 + B^2}}$$

Substituting for k,

$$\cos\omega = \frac{A}{\pm\sqrt{A^2 + B^2}}, \quad \sin\omega = \frac{B}{\pm\sqrt{A^2 + B^2}}, \quad -p = \frac{C}{\pm\sqrt{A^2 + B^2}}.$$

Hence, the normal form of $Ax + By + C = 0$ is

$$\frac{A}{\pm\sqrt{A^2 + B^2}}x + \frac{B}{\pm\sqrt{A^2 + B^2}}y + \frac{C}{\pm\sqrt{A^2 + B^2}} = 0$$

where the sign before the radical is chosen opposite to that of C. If $C = 0$, the sign before the radical is chosen the same as that of B.

13 Distance from line to a point

To find the perpendicular distance d from line L to point (x_1, y_1) we establish line L_1 through (x_1, y_1) and parallel to L.

The normal equation of L is

$$x\cos\omega + y\sin\omega - p = 0,$$

and the equation of L_1 is

$$x\cos\omega + y\sin\omega - (p + d) = 0,$$

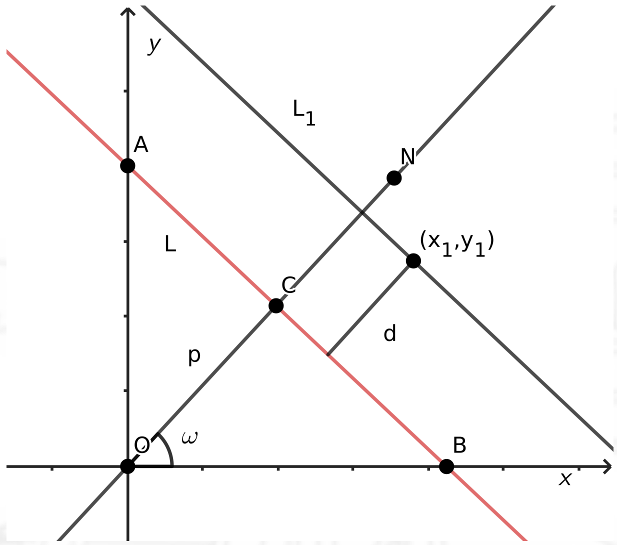

Figure 9: Distance from line to a point

as the lines are parallel.

Since the coordinates of (x_1, y_1) satisfy the equation for L_1,

$$x_1 \cos\omega + y_1 \sin\omega - (p + d) = 0,$$

Solving for d,

$$d = x_1 \cos\omega + y_1 \sin\omega - p$$

If L_1 and the origin are on opposite sides of the line L, the distance d is positive: if they are on the same side of the line L, d is negative.

14 Problems corner

Problem 1

Find the equation of the line passing through point $(-2,3)$ and perpendicular to the line $2x-3y+6=0$.

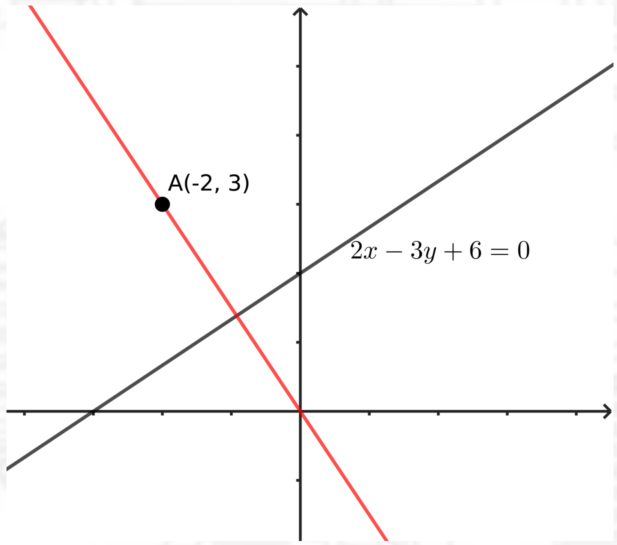

Figure 10: Perpendicular line passing through designated point

Solution

If two lines are perpendicular, the slope of one of the lines is the negative reciprocal of the slope of the other line. Slope of $2x-3y+6=0$, which is expressed in the general form $Ax+By+C=0$, is $m=-\dfrac{A}{B}=\dfrac{2}{3}$. Hence, the slope of the required line is $m'=-\dfrac{3}{2}$

Let (x,y) be any other point on the required line through $(-2,3)$ with slope $m'=-\dfrac{3}{2}$. Then we can write equation in slope-point form:

$$y-3=(x+2)m'$$
$$y-3=-\frac{3}{2}(x+2)$$

Simplifying

$$3x+2y=0.$$

Problem 2

Find the equation of the line which is the perpendicular bisector of the segment connecting points $(7,4)$ and $(-1,-2)$.

Solution

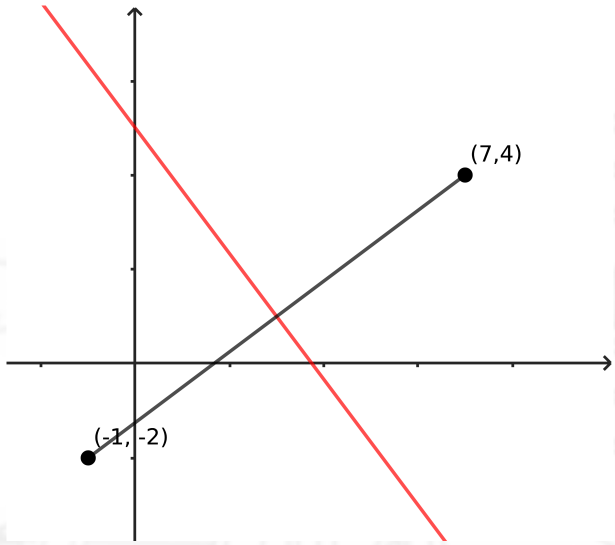

Figure 11: Perpendicular bisector

This line may be expressed directly form the condition "is the perpendicular bisector" by letting these lengths be equal. But here we explicitly start with equation of line and reveal its parameters.

Midpoint (x_0, y_0) of the segment it

$$x_0 = \frac{x_1 + x_2}{2} = \frac{7-1}{2} = 3$$
$$y_0 = \frac{y_1 + y_2}{2} = \frac{4-2}{2} = 1$$

Slope of the segment $m' = \dfrac{4+2}{7+1} = \dfrac{3}{4}$. Hence, slope of the line is negative reciprocal of it $m = -\dfrac{4}{3}$.

Let (x, y) be any other point on the required, line through $(3, 1)$ with slope $m = -\dfrac{4}{3}$.

$$y - 1 = -\frac{4}{3}(x - 3)$$

Simplifying

$$4x + 3y - 15 = 0.$$

Problem 3

Determine the particular value of the parameter k so that:
(a) $3kx + 5y + k - 2 = 0$ passes through point $(-1, 4)$;
(b) $4x - ky - 7 = 0$ has the slope 3;
(c) $kx - y = 3k - 6$ has the x-intercept 5.

Solution

For (a)

Substituting $x = -1$, $y = 4$: $3k(-1) + 5 \cdot 4 + k - 2 = 0$, $2k = 18$, $k = 9$

For (b)

Employing the given form $Ax + By + C = 0$, slope $m = -\dfrac{A}{B} = -\dfrac{4}{-k} = 3$, thus $k = \dfrac{4}{3}$

For (c)

When $y = 0$, $x = \dfrac{3k-6}{k} = 5$. Then, $3k - 6 = 5k$, $k = -3$

Problem 4

Find the equations of the lines which have the slope $-3/4$ and form with the coordinate axes a triangle of area 24 square units.

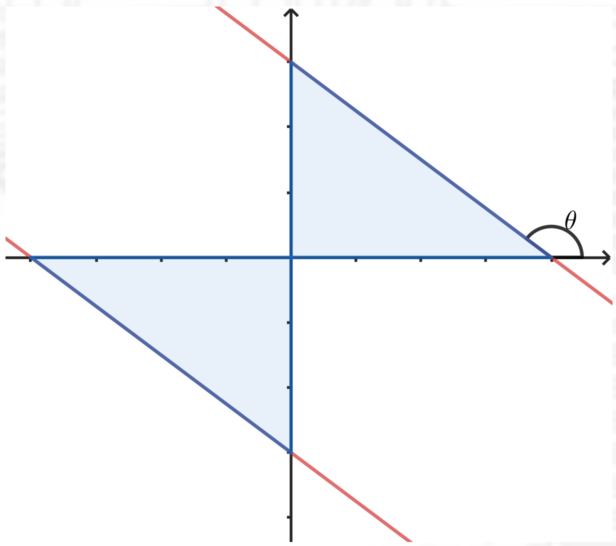

Figure 12: Triangles shaped with lines of slope $-3/4$

Solution

A line with slope $-\dfrac{3}{4}$ and y-intercept b is given by equation:

$$y = -\frac{3}{4}x + b$$

For $x = 0$ $y_x = b$, and for $y = 0$ $x_y = \dfrac{4}{3}b$

Area of triangle $S = \dfrac{1}{2}x_y y_x = \dfrac{1}{2}\dfrac{4}{3}b^2 = \dfrac{2}{3}b^2 = 24$ Hence,

$$\begin{aligned} 2b^2 &= 24 \cdot 3 \\ b^2 &= 36 \\ b &= \pm 6 \end{aligned}$$

Required equations are

$$y = -\frac{3}{4}x \pm 6,$$

or in general form

$$\begin{aligned} 3x + 4y + 24 &= 0 \\ 3x + 4y - 24 &= 0 \end{aligned}$$

Problem 5

A point (x, y) moves so that it is numerically twice as far from the line $x = 5$ as from the line $y = 8$. Find the equation of its locus.

Solution

Distance form the line $x = 5$ means value $|x - 5|$, and distance form the line $y = 8$ means value $|y - 8|$. In algebraic form for the proposition " it is numerically twice as far from the line $x = 5$ as from the line $y = 8$" we replace that moduli with $\pm$:

$$x - 5 = \pm 2(y - 8)$$

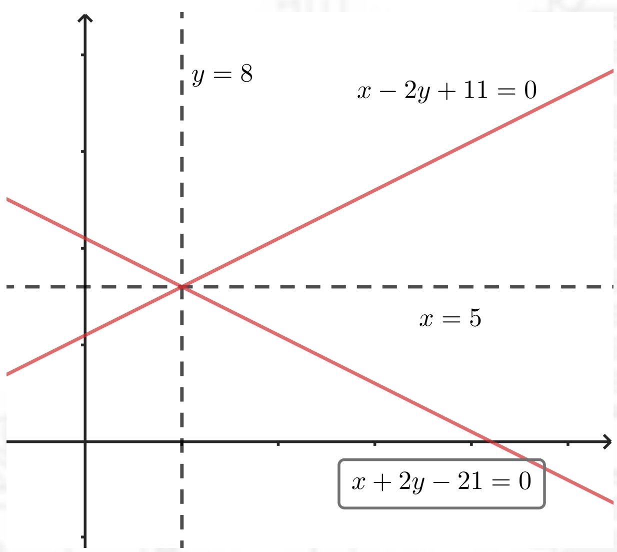

Figure 13: Locus of points laying in equal distance twice far from vertical than horizontal

Hence, the locus of the point is the two straight lines:

$$x - 2y + 11 = 0$$

and

$$x + 2y - 21 = 0$$

or

$$(x - 2y + 11)(x + 2y - 21) = 0$$

Problem 6

Find the equations of the lines through $(4,-2)$ and at a perpendicular distance p of 2 units from the origin.

Solution

The equation of the system of lines through $(4,-2)$ and with arbitrary slope m is

$$y+2=m(x-4),$$
or
$$mx-y-(4m+2)=0$$

Normal form for this equation is

$$\frac{mx-y-(4m+2)}{\pm\sqrt{m^2+1}}=0$$

Thus, $p=\dfrac{4m+2}{\pm\sqrt{m^2+1}}=2$, or

$$(4m+2)^2=4(m^2+1)$$
$$4(4m^2+4m+1)=4(m^2+1)$$
$$3m^2+4m=0$$
$$m=0;\quad m=\frac{-4}{3}$$

The required equations are

$$y+2=1,$$

and

$$y+2=-\frac{4}{3}(x-4)$$
$$4x+3y-10=0.$$

Problem 7

Find the equations of the lines parallel to the line $12x-5y-15=0$ and at a perpendicular distance from it numerically equal to 4.

Solution

First, we bring our equation to normal form:

$$\pm\sqrt{A^2+B^2}=\pm\sqrt{144+25}=\pm\sqrt{169}=\pm13$$

This yields expression for the distance from our line to point $P(x', y')$ in 4 units:

$$\frac{12x' - 5y' - 15}{13} = \pm 4$$

We fixed $\pm$ before 13 with described in the up rule, but added it before 4 to express lines laying by both sides of line in the question.

Simplifying and dropping primes, the required equations are

$$12x - 5y - 67 = 0$$

and

$$12x - 5y + 37 = 0.$$

While we recap deriving of the general line equation we remember that only free term is connected with initial point. Thus, parallel transposition keeps terms by x and y as is.

Problem 8

Given the triangle $A(-2, 1)$, $B(5, 4)$, $C(2, -3)$, determine the length of the altitude through A, and the area of the triangle.

Solution

We call *altitude of a triangle* is a line segment through a vertex and perpendicular to a line containing the base (the side opposite the vertex).

If we calculated length of altitude h and its base a, area of triangle is $\frac{1}{2}ha$

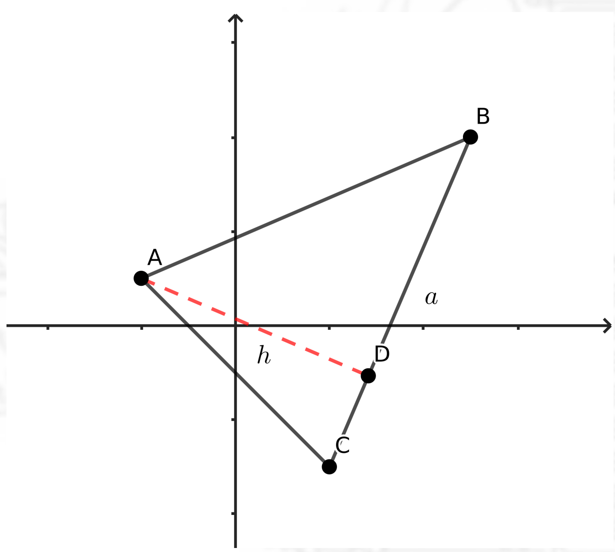

Figure 14: Altitude of triangle

Thus, perpendicular distance from the vertex to opposite side of triangle is exactly length of its altitude.

First, we write equation of BC:

$$\frac{y+3}{x-2} = \frac{4+3}{5-2} 7x - 3y - 23 = 0$$

Normalization multiply: $\sqrt{A^2+B^2} = \sqrt{49+9} = \sqrt{58}$ Distance from BC to A:

$$d = \frac{7(-2)+3(-1)-23}{\sqrt{58}} = \frac{-40}{\sqrt{58}}$$

Sign of this distance plays no role for us as we search some "effective" parameter.

Length of BC:

$$BC = \sqrt{(5-2)^2+(4+3)^2} = \sqrt{58}$$

Area of triangle:

$$S = \frac{1}{2}\sqrt{58}\frac{40}{\sqrt{58}} = 20$$