Homework: Optimal Control

July 2025

Problem 1.2

For the function $F = \sqrt{x^2 + y(y')^2}$, where y and y' are functions of x, find

$$\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial y'}, \quad \text{and} \quad \frac{dF}{dx}.$$

Solution

For simplicity, let $G = \sqrt{x^2 + y(y')^2}$, that is $F = \sqrt{G}$ then $\frac{\partial F}{\partial G} = \frac{1}{2\sqrt{G}} = \frac{1}{2F}$. 3 partial derivatives can be compute directedly,

- $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial G}\frac{\partial G}{\partial x} = \frac{x}{F}$
- $\frac{\partial F}{\partial y} = \frac{y(y')^2}{F}$
- $\frac{\partial F}{\partial y'} = \frac{2yy'}{F}$

Next, we calculate the total derivative $\frac{dF}{dx}$, by the chain rule:

$$\begin{aligned}
\frac{dF}{dx} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial F}{\partial y'} \cdot \frac{d(y')}{dx} \\
&= \frac{x}{F} + \frac{y(y')^2}{F} \cdot y' + \frac{2yy'}{F} \cdot y'' \\
&= \frac{x + y(y')^2 \cdot y' + 2yy'y''}{\sqrt{x^2 + y(y')^2}}
\end{aligned}$$

Problem 1.5

Consider the functional

$$S[y] \; = \; \int_0^X \left[(y')^2 - y^2\right] dx, \qquad 0 < X < \pi,$$

with boundary conditions

$$y(0) = 0, \qquad y(X) = 1.$$

1. Show that the Euler–Lagrange equation for $S[y]$ is
$$y'' + y = 0.$$
2. Solve this differential equation under the given boundary conditions and verify that the stationary function is
$$y(x) \; = \; \frac{\sin x}{\sin X}.$$

Solution

1. Given
$$S[y] = \int_0^X \left[(y')^2 - y^2\right] dx, \qquad y(0) = 0,\ y(X) = 1,\ 0 < X < \pi,$$
set $F(x, y, y') = (y')^2 - y^2$. The Euler–Lagrange condition
$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$
yields
$$\frac{d}{dx}(2y') - (-2y) = 0 \implies 2y'' + 2y = 0 \implies y'' + y = 0.$$

2. Solve $y'' + y = 0$ with $y(0) = 0$ and $y(X) = 1$.
The characteristic function is $\lambda^2 + 1 = 0$
The general solution is $y(x) = A\cos x + B\sin x$.
From $y(0) = 0$ we get $A = 0$; from $y(X) = 1$ we obtain $B \sin X = 1$. Therefore
$$y(x) = \frac{\sin x}{\sin X}.$$

Problem 2 The First Travel Problem

Let $A(a, y_A)$ lie in the region $y > 0$ (speed c_1) and $B(b, y_B)$ lie in the region $y < 0$ (speed c_2). With $y = y(x)$ the ray path, the travel–time functional is

$$J[y] = \int_a^b \frac{\sqrt{1 + y'(x)^2}}{v\big(y(x)\big)}\, dx, \qquad v(y) = \begin{cases} c_1, & y > 0, \\ c_2, & y < 0. \end{cases}$$

Solution

1. **Why the integral has this implicit solution.** Because the Lagrangian
$$L(y, y') = \frac{\sqrt{1 + y'^2}}{v(y)}$$
does *not* depend explicitly on x, the Euler–Lagrange equation $\frac{d}{dx}\big(\partial L/\partial y'\big) = \partial L/\partial y$ admits the Beltrami first integral
$$L - y'\frac{\partial L}{\partial y'} = C,$$
where C is a constant along extremals. Writing out the two terms and collecting,
$$\frac{\sqrt{1 + y'^2}}{v(y)} - y'\frac{y'}{v(y)\sqrt{1 + y'^2}} = \frac{1 + y'^2 - y'^2}{v(y)\sqrt{1 + y'^2}} = \frac{1}{v(y)\sqrt{1 + y'^2}} = C$$

2. **Why Straight-line segments inside each medium.** Inside a uniform medium $v(y) \equiv c_k$ ($k = 1$ or 2). From the boxed relation,
$$y'(x)^2 = \frac{1}{C^2 c_k^2} - 1 \quad \implies \quad y'(x) = \pm\sqrt{\frac{1}{C^2 c_k^2} - 1} \ = \text{ constant.}$$
A constant slope means a straight line. Thus the ray is *piecewise linear*; the direction changes only at the interface where v jumps while C remains fixed.

3. **Why Snell's law obtained from the first integral.**
   Let φ_k be the angle between the ray and the x–axis in medium k. Since $\tan\varphi_k = y'$, we have
$$\cos\varphi_k = \frac{1}{\sqrt{1+\tan^2\varphi_k}} = \frac{1}{\sqrt{1+y'^2}}.$$
   Substitute this into the boxed integral for each medium:
$$\frac{\cos\varphi_1}{c_1} = C = \frac{\cos\varphi_2}{c_2}.$$
   Define the usual angles to the normal: $\alpha_k = \frac{\pi}{2} - \varphi_k$, hence $\sin\alpha_k = \cos\varphi_k$. Eliminate C to obtain
$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{c_1}{c_2},$$
   which is Snell's law.

Problem 3 The Second Travel Probelm

Drive a car on a straight road for $t \in [0, T]$, starting from rest at the origin. Let $s(t)$ be position, $v(t)$ velocity, and $\alpha(t) \in [-1, 1]$ the "normalized acceleration" (positive for throttle, negative for brake). Fuel is consumed only while accelerating. Maximise the performance index
$$P[\alpha] = -\int_0^T \frac{k}{2}\max\bigl(0, \alpha(t)\bigr)^2\, dt \;+\; s(T)$$
subject to the dynamics
$$\dot{s}(t) = v(t), \qquad s(0) = 0, \qquad \dot{v}(t) = \ell\,\alpha(t), \qquad v(0) = 0.$$

Solution

1. **Why Form of the Hamiltonian.**
   Converting the Bolza problem to the Pontryagin framework, the Mayer term $s(T)$ contributes the costate terminal condition, while the running cost contributes $-\frac{k}{2}\max(0, \alpha^2)$ inside the Hamiltonian. With state $x = (s, v)$ and costate $p = (p_1, p_2)$, the Hamiltonian is therefore
$$H(x, p, \alpha) = p_1 v + p_2\,\ell\,\alpha \;-\; \frac{k}{2}\max(0, \alpha^2), \qquad \text{to be maximised w.r.t. } \alpha.$$

2. **Why Adjoint equations and terminal conditions.**
   Pontryagin's Maximum Principle (PMP) states $\mathbf{p}^* = -\partial H/\partial x$ and $\mathbf{p}(T) = \nabla_{x(T)}$. Since the terminal payoff depends only on $s(T)$,
$$p_1^*(T) = 1, \qquad p_2^*(T) = 0.$$
   Furthermore
$$\dot{p}_1^* = -\frac{\partial H}{\partial s} = 0, \quad \dot{p}_2^* = -\frac{\partial H}{\partial v} = -p_1^*.$$

3. **Why Explicit costate solution $\mathbf{p}^*(t) = (1, T - t)$.**
   From $\dot{p}_1^* = 0$ and $p_1^*(T) = 1$ we have $p_1^*(t) \equiv 1$.
   Integrating $\dot{p}_2^* = -1$ with the terminal condition $p_2^*(T) = 0$ yields $p_2(t) = T - t$.

4. **Why over $\alpha > 0$ we have that maximum ?**
   For $\alpha < 0$ the fuel term vanishes, but $(T - t)\ell\alpha < 0$ because $T - t > 0$; hence braking never maximises H. Restricting to $\alpha \geq 0$ we must maximise the concave quadratic, denote by
$$\phi(\alpha) = (T - t)\ell\,\alpha - \frac{k}{2}\alpha^2.$$
   Its stationary point is obtained from $\phi'(\alpha) = 0$, giving $\alpha^{\text{opt}} = \ell(T - t)/k$.

5. **Why Optimal control $\alpha^*(t) = \min\bigl(\ell(T-t)/k,\, 1\bigr)$.**
   The control is constrained to $0 \le \alpha \le 1$ when accelerating. Hence the maximiser found in 4 must be clipped at the upper bound 1:

   $$\alpha^*(t) = \min\Bigl(\frac{\ell}{k}\,(T-t),\, 1\Bigr).$$

   Equivalently, "floor it" with full throttle until $t_0 = T - k/\ell$, then reduce acceleration linearly to zero as $t \to T$. This is the fuel-optimal strategy demanded by the PMP analysis.