# The third homework.

**1. (8 points).**
On one shelf, there are 5 red books and 7 black books; on another shelf, there are 6 red books and 6 black books. A shelf is chosen at random (each with equal probability), and then two books are selected at random from the chosen shelf. Construct the distribution, plot the distribution function, and find the mathematical expectation and variance of the following random variables:
a) $X$ —- the number of **black books left** on the first shelf;
b) $Y$ —- the number of **black books taken**;
c)
$Z = 1$ if the number of black books on both shelves becomes equal, and
$Z = 0$ otherwise;
d)
$W = 2$, if the number of red and black books on the **first shelf** becomes equal;
$W = 1$, if the number of red and black books on the **second shelf** becomes equal;
$W = 0$, otherwise.
**2. (12 points)**
The random variable $X$ has the following probability density function:

$$f(x) = \begin{cases} a \cdot |x|, & x \in [-1, 1]; \\ b \cdot (x-1)^2, & x \in [1, 2]; \\ 0, & \text{otherwise.} \end{cases}$$

a) Find a,b, E(X), D(X), if $F(\frac{1}{2}) = \frac{1}{2}$,
b) Find the Excess kurtosis and skewness coefficient;
c) Find the mode, median, quartiles(I-st and III-rd) and quantiles of levels p=0.15, 0.27, 0.4, 0.6, 0.8;
d)**(6 points)** Find the distribution of the following random variables $(X+1)^2$, $(X-\frac{1}{2})^2$, $X^4$.

**3. (4 points)**
Describe the algorithm for simulating a random variable given the following density function:

a)

$$f_\xi(x) = \begin{cases} \frac{1}{x\,(1+\ln(x))^2}, & x \geq 1; \\ 0, & \text{else.} \end{cases}$$

b) Density of X in task 2.

**4. (10 points)**
a) Find the probability generating function for both types of the geometric distribution.
b) Prove that if the characteristic function of a random variable $X$ is real, i.e. $\phi_X(t) \in \mathbb{R}$, then the distribution of $X$ is symmetric.
c) Find $\mathbb{E}(X)$, $\mathrm{D}(X)$ if the characteristic function of the random variable $X$ is
$\phi_X(t) = \frac{\sin(at)}{at}$, $a \neq 0$.

d) Find $\mathbb{E}(X)$, $\mathrm{D}(X)$ if the characteristic function of the random variable $X$ is
$\phi_X(t) = \frac{4\cos(t)\sin^2(t/2)}{t^2}$.

e) Can the function $\cos(t^2)$ be a characteristic function of a random variable?
f)* Can the function $\cos^2(t)$ be a characteristic function of a random variable?

**5. (8 points)**

a) Show that if $X$ is a non-negative random variable and $\mathbb{E}(X)$ exists, then

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq t)\, dt.$$

b) Show that if $\mathbb{E}(X)$ exists, then

$$\mathbb{E}(X) = \int_0^\infty (1 - F(t) - F(-t))\ dt,$$

where $F$ is the distribution function of $X$.

c) At which point $u$ is the minimum of the following expression achieved, and why?

$$\mathbb{E}(X-u)^2$$

d)* At which point $u$ is the minimum of the following expression achieved, and why?

$$\mathbb{E}|X-u|$$

HW 3. (deadline Mar 21st).

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2022111221

**1. (8 points).**

On one shelf, there are 5 red books and 7 black books; on another shelf, there are 6 red books and 6 black books. A shelf is chosen at random (each with equal probability), and then two books are selected at random from the chosen shelf. Construct the distribution, plot the distribution function, and find the mathematical expectation and variance of the following random variables:

a) $X$ —- the number of **black books left** on the first shelf;

b) $Y$ —- the number of **black books taken**;

c)

$Z = 1$ if the number of black books on both shelves becomes equal, and

$Z = 0$ otherwise;

d)

$W = 2$, if the number of red and black books on the **first shelf** becomes equal;

$W = 1$, if the number of red ~~and~~ or black books on the ~~second~~ both. **shelf** becomes equal;

$W = 0$, otherwise.

| | red | black |
|---|---|---|
| I. | 5 | 7 |
| II. | 6 | 6 |

1) chose shelf

2) take 2 books

$\frac{19}{33}$. $x = 7$

$\frac{35}{132}$ $x = 6$

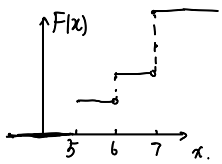
$\frac{7}{44}$ $x = 5$

35
21
56

a). discrete distribution $X$:

| X | 7 | 6 | 5 |
|---|---|---|---|
| P | $\frac{19}{33}$ | $\frac{35}{132}$ | $\frac{7}{44}$ |

$$F(x) = \begin{cases} 0 & x<5 \\ \frac{7}{44} & 5 \le x < 6 \\ \frac{56}{132} & 6 \le x < 7 \\ 1 & x \ge 7 \end{cases}$$




$P(X=7) = \frac{1}{2} + \frac{1}{2} \cdot \frac{C_5^2}{C_{12}^2} = \frac{19}{33}$ (chose shelf II or chose 2 red on I.)

$P(X=6) = \frac{1}{2} \cdot \frac{C_5^1 \cdot C_7^1}{C_{12}^2} = \frac{35}{132}$ (chose 1 red + 1 black on I.)

$P(X=5) = \frac{1}{2} \cdot \frac{C_7^2}{C_{12}^2} = \frac{7}{44}$ (chose 2 black on I).

$E(X) = \frac{19}{33} \times 7 + \frac{35}{132} \times 6 + \frac{7}{44} \times 5 = \frac{532 + 210 + 105}{132} = \frac{847}{132} \approx 6.42.$

$E(X^2) = \frac{19}{33} \times 49 + \frac{35}{132} \times 36 + \frac{7}{44} \times 25 = \frac{3724 + 1260 + 525}{132} = \frac{5509}{132} \approx 41.73.$

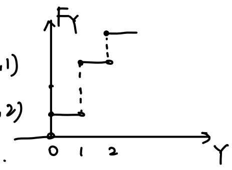
$D(X) = E(X^2) - (E(X))^2 = \frac{9779}{17424} \approx 0.56.$

b).

discrete distribution $Y$:

| Y | 0 | 1 | 2 |
|---|---|---|---|
| P | $\frac{25}{132}$ | $\frac{71}{132}$ | $\frac{3}{11}$ |

$$F_Y(y) = \begin{cases} 0 & y<0 \\ \frac{25}{132} & y \in [0,1) \\ \frac{8}{11} & y \in [1,2) \\ 1 & y \ge 2. \end{cases}$$




$P(Y=0) = \frac{1}{2} \cdot \frac{C_5^2}{C_{12}^2} + \frac{1}{2} \cdot \frac{C_6^2}{C_{12}^2} = \frac{10+15}{132} = \frac{25}{132}$

$P(Y=1) = \frac{1}{2} \cdot \frac{C_5^1 \cdot C_7^1}{C_{12}^2} + \frac{1}{2} \cdot \frac{C_6^1 \cdot C_6^1}{C_{12}^2} = \frac{35+36}{132} = \frac{71}{132}$

$P(Y=2) = \frac{1}{2} \cdot \frac{C_7^2}{C_{12}^2} + \frac{1}{2} \cdot \frac{C_6^2}{C_{12}^2} = \frac{21+15}{132} = \frac{36}{132} = \frac{3}{11}$

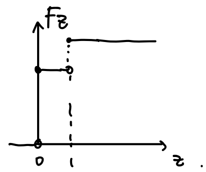
$E(Y) = \frac{25}{132} \cdot 0 + \frac{71}{132} \cdot 1 + \frac{3}{11} \cdot 2 = \frac{143}{132} \approx 1.08$

$E(Y) = \frac{25}{132} \cdot 0^2 + \frac{71}{132} \cdot 1^2 + \frac{3}{11} \cdot 2^2 = \frac{215}{132} \approx 1.63$

$D(Y) = E(Y) - (E(Y))^2 = 1.63 - \left(\frac{13}{12}\right)^2 \approx 0.46$

c).

$$F_Z(z) = \begin{cases} 0 & z<0. \\ \frac{97}{132} & z \in [0,1) \\ 1 & z \ge 1 \end{cases}$$




$P(Z=1) = \frac{1}{2} \cdot \frac{C_7^1 \cdot C_5^1}{C_{12}^2} \cdot \frac{1}{2} \cdot 0 = \frac{35}{132}$

$P(Z=0) = 1 - P(Z=1) = \frac{97}{132}$

| Z | 1 | 0 |
|---|---|---|
| P | $\frac{35}{132}$ | $\frac{97}{132}$ |

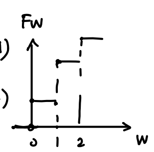
$E(Z) = \frac{35}{132} \approx 0.27$ $D(Z) = \frac{35}{132} - \left(\frac{35}{132}\right)^2 = 0.19$

d)

$P(W=2) = \frac{1}{2} \cdot \frac{C_7^2}{C_{12}^2} = \frac{21}{132} \approx 0.16$

| W | 0 | 1 | 2 |
|---|---|---|---|
| P | $\frac{10}{33}$ | $\frac{71}{132}$ | $\frac{7}{44}$ |

$$F_W(w) = \begin{cases} 0 & w<0 \\ \frac{10}{33} & w \in [0,1) \\ \frac{37}{44} & w \in [1,2) \\ 1 & w \ge 2 \end{cases}$$




$P(W=1) = \frac{1}{2} \cdot \frac{C_6^1 \cdot C_6^1}{C_{12}^2} + \frac{1}{2} \cdot \frac{C_5^1 \cdot C_7^1}{C_{12}^2} = \frac{71}{132} \approx 0.54$

$P(W=0) = 1 - P(W=2) - P(W=1) = \frac{10}{33} \approx 0.30$

$E(W) = \frac{10}{33} \cdot 0 + \frac{71}{132} \cdot 1 + \frac{21}{132} \cdot 2 = \frac{113}{132} \approx 0.86$ $E(W^2) = \frac{71}{132} \cdot 1^2 + \frac{21}{132} \cdot 2^2 = \frac{155}{132} = 1.17$

$D(W) = \frac{155}{132} - \left(\frac{113}{132}\right)^2 \approx 0.44$

**2. (12 points)**

The random variable $X$ has the following probability density function:

$$f(x) = \begin{cases} \frac{4}{5}\cdot |x|, & x \in [-1,1]; \\ \frac{3}{5}\cdot (x-1)^2, & x \in [1,2]; \\ 0, & \text{otherwise.} \end{cases}$$

a) Find a,b, E(X), D(X), if $F(\frac{1}{2}) = \frac{1}{2}$,
b) Find the Excess kurtosis and skewness coefficient;
c) Find the mode, median, quartiles(I-st and III-rd) and quantiles of levels p=0.15, 0.27, 0.4, 0.6, 0.8;

a) Use the normalization property. $\int_{-1}^{1} a|x|\,dx + \int_{1}^{2} b(x-1)^2 = 1.$

$\Rightarrow \quad ax^2\Big|_0^1 + \frac{b}{3}(x-1)^3\Big|_1^2 = a + \frac{b}{3} = 1.$

$F(\frac{1}{2}) = \int_{-1}^{0} a(-x)\,dx + \int_0^{\frac12} ax\,dx = \frac{5}{8}a \quad \Rightarrow \frac{5}{8}a = \frac{1}{2} \quad \Rightarrow a = \frac{4}{5}$ $\Bigg\} \Rightarrow \begin{cases} a = \frac{4}{5} \\ b = \frac{3}{5}. \end{cases}$

$E(X) = \int_{-\infty}^{\infty} x f(x) = \frac{3}{5}\int_1^2 x(x-1)^2\,dx = \frac{3}{5}\left(\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{1}{2}x^2\Big|_1^2\right) = \frac{3}{5}\cdot\left(4 - \frac{16}{3} + 2 - \frac{1}{4} + \frac{2}{3} - \frac{1}{2}\right) = \frac{7}{20}$

↑ in $[-1,1]$ $xf(x)$ is odd.

$E(X^2) = 2\int_0^1 x^2\frac{4}{5}x\,dx + \frac{3}{5}\cdot\int_1^2 x^2(x-1)^2\,dx = \frac{2}{5}\cdot x^4\Big|_0^1 + \frac{3}{5}\left[\frac{x^5}{5} - \frac{1}{2}x^4 + \frac{1}{3}x^3\Big|_1^2\right] = \frac{2}{5} + \frac{31}{50} = \frac{51}{50}$

$D(X) = E(X^2) - (E(X))^2 = 1.02 - (0.35)^2 = 0.8975$

b) $a_1 = \frac{7}{20} \quad a_2 = \frac{51}{50}$

$a_3 = \int_{-\infty}^{\infty} x^3 f(x)\,dx = \frac{3}{5}\int_1^2 x^3(x-1)^2\,dx = \frac{3}{5}\left(\frac{64}{6} - \frac{64}{5} + 4 - \frac{1}{6} + \frac{2}{5} - \frac{1}{4}\right) = \frac{111}{100} = 1.11$

$a_4 = \int_{-\infty}^{\infty} x^4 f(x)\,dx = \frac{8}{5}\int_0^1 x^5\,dx + \frac{3}{5}\int_1^2 x^4(x-1)^2\,dx = \frac{4}{15} + \frac{3}{5}\left(-\frac{64}{21} + \frac{32}{5} - \frac{1}{7} + \frac{1}{3} - \frac{1}{5}\right) = \frac{4}{15} + \frac{117}{35}\cdot\frac{3}{5} = \frac{4}{15} + \frac{351}{175} \approx 2.27$

$\mu_3 = E[(X-E(X))^3] = a_3 - 3a_1\cdot a_2 + 3a_1^2\cdot a_1 - a_1^3 = 1.11 - 3\times 0.35\times 1.02 + 3\times 0.35^3 - 0.35^3 = 0.12475 \approx 0.12.$

$\mu_4 = E((X-E(X))^4] = a_4 - 4a_1a_3 + 6a_1^2\cdot a_2 - 4a_1^3\cdot a_1 + a_1^4 = 2.27 - 4\times 0.35\times 1.11 + 6\times 0.35^2\times 1.02 - 3\times 0.35^4 \approx 1.42$

Excess kurtosis. $\gamma_2 = \frac{\mu_4}{(\mu_2)^2} - 3 \approx -1.23$     skewness kurtosis $\gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{0.12}{(0.8975)^{3/2}} \approx 0.147$

c) i/ in $[-1,1]$. $\max f = \frac{4}{5}$   arg max $= -1$ or $1$.     Mode $= 1$ or $-1$

in $[1,2]$   $\max f = \frac{3}{5}$   arg max $= 2$.

ii/ $F(\frac{1}{2}) = \frac{1}{2}$     Median $= \frac{1}{2}$

iii/ find $F(x) = \frac{1}{4}$ and $\frac{3}{4}$.

$$F(x) = \begin{cases} \frac{2}{5} - \frac{2}{5}x^2, & x \in [-1,0] \\ \frac{2}{5} + \frac{2}{5}x^2, & x \in [0,1] \\ \frac{3}{5} + \frac{1}{5}x^3 - \frac{3}{5}x^2 + \frac{3}{5}x & x \in [1,2] \\ 1 & x \geq 2. \end{cases} \qquad F(0) = \frac{2}{5} \quad F(1) = \frac{4}{5}.$$

$\frac{1}{4} = \frac{2}{5} - \frac{2}{5}x^2 \quad \Rightarrow \quad x = -\sqrt{\frac{3}{8}} \approx -0.61$

$\frac{3}{4} = \frac{2}{5} + \frac{2}{5}x^2 \quad \Rightarrow \quad x = \sqrt{\frac{7}{8}} \approx 0.94$

$Q_I \approx -0.612 \qquad Q_{III} \approx 0.94$

iv/ find $F(x) = 0.15,\ 0.27,\ 0.4,\ 0.6,\ 0.8$

we claim that

$-1 < Q_{0.15} < Q_{0.27} < 0 = Q_{0.4} < Q_{0.6} < 1 = Q_{0.8} < 2.$

$Q_{0.15} = F^{-1}(0.15) = -\sqrt{\frac{5}{8}} \approx -0.79$

$Q_{0.27} = F^{-1}(0.27) = -\sqrt{\frac{13}{40}} \approx -0.57$

$Q_{0.4} = F^{-1}(\frac{2}{5}) = 0$

$Q_{0.6} = F^{-1}(0.6) = \sqrt{\frac{1}{2}} \approx 0.71$

$Q_{0.8} = F^{-1}(\frac{4}{5}) = 1$

d)**(6 points)** Find the distribution of the following random variables $(X+1)^2$, $(X-\frac{1}{2})^2$, $X^4$.

i/ $g(x) = (x+1)^2$ is increasing on $[-1,+\infty)$    $g^{-1}(y) = \sqrt{y} - 1$, $x \in (0,9]$

$$f_Y(y) = |(g^{-1}(y))'|\, f_X(g^{-1}(y)) = \begin{cases} \frac{1}{2\sqrt{y}}\cdot\frac{4}{5}(1-\sqrt{y}) & y \in (0,1] \\ \frac{1}{2\sqrt{y}}\cdot\frac{4}{5}(\sqrt{y}-1) & y \in (1,4] \\ \frac{1}{2\sqrt{y}}\cdot\frac{3}{5}(\sqrt{y}-2)^2 & y \in [4,9] \\ 0 & y > 9. \end{cases} \Rightarrow f_Y(y) = \begin{cases} \frac{2}{5\sqrt{y}} - \frac{2}{5} & y \in (0,1] \\ \frac{2}{5} - \frac{2}{5\sqrt{y}} & y \in (1,4] \\ \frac{3}{10\sqrt{y}}(\sqrt{y}-2)^3 & y \in [4,9] \\ 0 & y > 9 \end{cases}$$

$$\Rightarrow F_{(X+1)^2}(y) = \begin{cases} \frac{4}{5}\sqrt{y} - \frac{2}{5}y, & y \in [0,1] \\ \frac{4}{5} + \frac{2}{5}y - \frac{4}{5}\sqrt{y}, & y \in [1,4] \\ \frac{4}{5} + \frac{1}{5}(\sqrt{y}-2)^3, & y \in [4,9] \\ 1 & y \geq 9. \end{cases}$$

ii) let $Z = (X-\frac{1}{2})^2$. $Z$ is piecewise cont. $[-1,\frac{1}{2}]$, $[\frac{1}{2},2]$.

$h(x) = (x-\frac{1}{2})^2$

$h_1^{-1}(z) = -\sqrt{z}+\frac{1}{2}$. $z \in [0,\frac{9}{4}]$ $h_1^{-1}(z) \in [-1,\frac{1}{2}]$

$h_2^{-1}(z) = \sqrt{z}+\frac{1}{2}$ $z \in [0,\frac{9}{4}]$ $h_2^{-1}(z) \in [\frac{1}{2},2]$

$$f_Z(z) = \sum_{i=1,2} |h_i^{-1}(z)'| \cdot f(h_i^{-1}(z)) = \frac{1}{2\sqrt{z}} \cdot \sum_{i=1,2} f(h_i^{-1}(z)) = \begin{cases} \frac{1}{2\sqrt{z}}\left(\frac{4}{5}(-\sqrt{z}+\frac{1}{2}) + \frac{4}{5}(\sqrt{z}+\frac{1}{2})\right) & z \in [0,\frac{1}{4}] \\ \frac{1}{2\sqrt{z}}\left(\frac{4}{5}(\sqrt{z}-\frac{1}{2}) + \frac{3}{5}(\sqrt{z}-\frac{1}{2})^2\right) & z \in [\frac{1}{4},\frac{9}{4}]. \end{cases}$$

$$= \begin{cases} \frac{2}{5\sqrt{z}} & z \in [0,\frac{1}{4}] \\ \frac{2}{5\sqrt{z}}(\sqrt{z}-\frac{1}{2}) + \frac{3}{10\sqrt{z}}(\sqrt{z}-\frac{1}{2})^2 & z \in [\frac{1}{4},\frac{9}{4}] \\ 0 & z \le 0 \text{ or } z > \frac{9}{4} \end{cases}$$

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{4}{5}\sqrt{z} & z \in [0,\frac{1}{4}] \\ \frac{2}{5}(\sqrt{z}-\frac{1}{2})^2 + \frac{1}{5}(\sqrt{z}-\frac{1}{2})^3 + \frac{2}{5} & z \in [\frac{1}{4},\frac{9}{4}] \\ 1 & z > \frac{9}{4} \end{cases}$$

iii) $k(x) = x^4$. $W = X^4$. $k_1^{-1}(w) = \sqrt[4]{w}$ $w \in [0,16]$. $k_1^{-1}(w) \in [0,2]$

piecewise cont. $k_2^{-1}(w) = -\sqrt[4]{w}$ $w \in [0,1]$. $k_2^{-1}(w) \in [-1,0]$

$$f_W(w) = \sum_{i=1,2} |k_i^{-1}(w)'|\, f(k_i^{-1}(w)) = \frac{1}{4}w^{-\frac{3}{4}} \sum_{i=1,2} f(k_i^{-1}(w)) = \begin{cases} \frac{1}{4}w^{-\frac{3}{4}}\left(-\frac{4}{5}(\sqrt[4]{w}) + \frac{4}{5}\sqrt[4]{w}\right). & w \in (0,1] \\ \frac{1}{4}w^{-\frac{3}{4}}\left(\frac{3}{5}(\sqrt[4]{w}-1)^2\right) & w \in [1,16] \end{cases}$$

$$= \begin{cases} \frac{2}{5}w^{-\frac{1}{2}} & w \in (0,1]. \\ \frac{3}{20}w^{-\frac{3}{4}}(w^{\frac{1}{4}}-1)^2 & w \in (1,16]. \\ 0 & w \le 0 \text{ or } w > 16. \end{cases}$$

$$F_W(w) = \begin{cases} 0 & w \le 0 \\ \frac{4}{5}w^{\frac{1}{2}} & w \in (0,1] \\ \frac{1}{5}(w^{\frac{1}{4}}-1)^3 + \frac{4}{5}. & w \in (1,16] \\ 1 & w > 16 \end{cases}$$

**3. (4 points)**
Describe the algorithm for simulating a random variable given the following density function:

a)

$$f_\xi(x) = \begin{cases} \frac{1}{x(1+\ln(x))^2}, & x \ge 1; \\ 0, & \text{else.} \end{cases}$$

b) Density of X in task 2.

a) $\int_1^x \frac{dt}{t(1+\ln t)^2} = \int_1^x \frac{d(1+\ln t)}{(1+\ln t)^2} = -\frac{1}{1+\ln t}\Big|_1^x = 1 - \frac{1}{1+\ln x} = \frac{\ln x}{1+\ln x}$ $(x \ge 1)$

$F_\xi(x) = \begin{cases} 0 & x < 1. \\ \ln x\,(1+\ln x)^{-1} & x \ge 1 \end{cases}$ $F_\xi(1) = 0$ cont.

by Quantile Transformation thm. the random variable $\eta = F(\xi)$ has a uniform distribution on the interval $[0,1]$

$\xi = F^{-1}(\eta) = e^{\frac{\eta}{1-\eta}}$ has distribution $F_\xi$. when $\eta \sim U_{0,1}$.

b). consider the inverse function $F^{-1}(\eta) = \inf\{t \mid F(t) \ge \eta\}$.

$$F(x) = \begin{cases} \frac{2}{5} - \frac{2}{5}x^2. & x \in [-1,0] \\ \frac{2}{5} + \frac{2}{5}x^2. & x \in [0,1] \\ \frac{3}{5} + \frac{1}{5}x^3 - \frac{3}{5}x^2 + \frac{3}{5}x & x \in [1,2] \\ 1 & x \ge 2. \end{cases}$$

$$F^{-1}(\eta) = \begin{cases} -\sqrt{1-\frac{5}{2}\eta} & \eta \in [0,\frac{2}{5}) \\ \sqrt{\frac{5}{2}\eta - 1} & \eta \in [\frac{2}{5},\frac{4}{5}] \\ \sqrt[3]{5\eta-4} + 1 & \eta \in [\frac{4}{5},1] \end{cases}$$

$\xi = F^{-1}(\eta)$, $\eta \sim U_{0,1}$ has distribution function $F$.

**4. (10 points)**

a) Find the probability generating function for both types of the geometric distribution.

b) Prove that if the characteristic function of a random variable $X$ is real, i.e. $\phi_X(t) \in \mathbb{R}$, then the distribution of $X$ is symmetric.

c) Find $\mathbb{E}(X)$, $\mathrm{D}(X)$ if the characteristic function of the random variable $X$ is $\phi_X(t) = \frac{\sin(at)}{at}$, $a \neq 0$.

d) Find $\mathbb{E}(X)$, $\mathrm{D}(X)$ if the characteristic function of the random variable $X$ is $\phi_X(t) = \frac{4\cos(t)\sin^2(t/2)}{t^2}$.

e) Can the function $\cos(t^2)$ be a characteristic function of a random variable?

f)* Can the function $\cos^2(t)$ be a characteristic function of a random variable?

a). $\mathrm{Geom}_1(p)$.

$$G_{X_1}(t) = \sum_{k=1}^{\infty} p(1-p)^{k-1} t^k = \frac{p}{1-p}\sum_{k=1}^{\infty}((1-p)t)^k = \frac{p}{1-p}\cdot\frac{(1-p)t}{1-(1-p)t} = \frac{pt}{1-t(1-p)} \qquad t\in[0,1]$$

$\mathrm{Geom}_2(p)$

$$G_{X_1}(t) = \sum_{k=0}^{\infty} p(1-p)^k t^k = p\cdot\frac{1}{1-(1-p)t} = \frac{p}{1-t(1-p)} \qquad t\in[0,1]$$

b) $$\int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \int_{-\infty}^{\infty}\cos(tx) f(x)\,dx + i\int_{-\infty}^{\infty}\sin(tx) f(x)\,dx. \in \mathbb{R}.$$

it implies $\int_{-\infty}^{\infty}\sin(tx) f(x)\,dx = 0$ for any $t$. thus $f(x)$ must be even (a.e, least). i.e. the distribution is symmetric.

c) $$E(X) = \frac{\varphi_X'(0)}{i} = \frac{1}{ia}\cdot\lim_{t\to0}\frac{at\cos at-\sin at}{t^2} = \frac{1}{ia}\cdot\lim_{t\to0}\frac{a\cos at - a^2t\sin at - a\cos at}{2t} = -\frac{a}{i}\lim_{t\to0}\frac{\sin at + at\cos at}{2} = 0.$$

$$E(X^2) = \frac{\varphi_X''(0)}{-1} = -\frac{1}{a}\lim_{t\to0}\frac{-a^2t^3\sin at - 2t(at\cos at - \sin at)}{t^4} = \frac{1}{a}\lim_{t\to0}\frac{a^2t^2\sin at + 2at\cos at - 2\sin at}{t^3} = \frac{1}{a}\lim_{t\to0}\frac{2ta^2\sin at + a^3t^2\cos at + 2(-a^2t\sin at)}{3t^2}$$

$$= \frac{1}{a}\lim_{t\to0}\frac{a^3\cos at}{3} = \frac{a^2}{3}.$$

$$D(X) = E(X^2) - (E(X))^2 = \frac{a^2}{3}. \qquad E(X) = 0$$

d) $$\varphi_X(t) = \frac{4\cos t\sin^2\frac{t}{2}}{t^2} = \frac{2\cos t(1-\cos t)}{t^2}.$$

$$E(X) = \frac{1}{i}\cdot\varphi_X'(0) = \frac{2}{i}\cdot\lim_{t\to0}\frac{(-\sin t + \sin 2t)\cdot t - 2(\cos t - \cos^2 t)}{t^3} = \frac{4}{2i}\lim_{t\to0}\frac{(-t+\frac{1}{6}t^3+2t-\frac{1}{6}(2t)^3)t - 2(1-\frac{t^2}{2})(\frac{t^2}{2}) + o(t^3)}{t^3} = 0$$

$$E(X^2) = -\varphi_X''(0) = -2\cdot\lim_{t\to0}\left(\frac{(1-\frac{t^2}{2}+\frac{t^4}{4!})(\frac{t^2}{2}-\frac{t^4}{4!})}{t^2} + o(t^4)\right)''$$

$$= -2\cdot\lim_{t\to0}\left(\frac{1}{2} + -\frac{7}{24}t^2 + O(t^4)\right)'' = -2\cdot\left(-\frac{7}{24}\right)\cdot 2 = \frac{7}{6}$$

$$D(X) = E(X^2) - E(X) = \frac{7}{6}.$$

e) $\varphi(t) = \cos(t^2)$ can't be characteristic func. because $\varphi(t) < 0$ may happen.

i.e. don't satisfy Non-negative definiteness

f*) $\cos^2 t$ is cont. and non-negative definite. s.t $\cos^2(0) = 1$.

By Bochner-Khinchin's Thm. it can be.

**5. (8 points)**

a) Show that if $X$ is a non-negative random variable and $\mathbb{E}(X)$ exists, then

$$\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \geq t)\, dt.$$

b) Show that if $\mathbb{E}(X)$ exists, then

$$\mathbb{E}(X) = \int_0^\infty (1 - F(t) - F(-t))\, dt,$$

where $F$ is the distribution function of $X$.

c) At which point $u$ is the minimum of the following expression achieved, and why?

$$\mathbb{E}(X-u)^2$$

d)* At which point $u$ is the minimum of the following expression achieved, and why?

$$\mathbb{E}|X-u|$$

a). Pf: $X$ - non-negative. $\mathbb{E}(X)$ exist. $\int_0^\infty x f(x)\, dx < \infty$

$$\mathbb{E}(X) = \int_0^\infty x f(x)\, dx = \int_0^\infty \left(\int_0^x dt\right) f(x)\, dx = \int_0^\infty \int_0^x f(x)\, dx\, dt = \int_0^\infty \left(\int_t^\infty f(x)\, dx\right) dt$$

$$= \int_0^\infty \mathbb{P}(X \geq t)\, dt$$

b) Pf:

$$\mathbb{E}(X) = \int_{-\infty}^\infty x f(x)\, dx = \int_0^\infty x f(x)\, dx - \int_0^\infty x f(-x)\, dx.$$

$$= \int_0^\infty \mathbb{P}(X \geq t)dt - \int_0^\infty \mathbb{P}(-X \geq t)\, dt = \int_0^\infty (1 - \mathbb{P}(X \leq t))\, dt - \int_0^\infty \mathbb{P}(X \leq -t)\, dt$$

$$= \int_0^\infty [1 - F(t)]\, dt - \int_0^\infty F(-t)\, dt = \int_0^\infty [1 - F(t) - F(-t)]\, dt$$

c) Sol: $\mathbb{E}[(X-u)^2] = \mathbb{E}[X^2] - 2u\mathbb{E}[X] + u^2$

let $g(u) = u^2 - 2u\mathbb{E}[X] + \mathbb{E}[X^2] \Rightarrow g'(u) = 2u - 2\mathbb{E}[X] \Rightarrow g_{min} = g(\mathbb{E}[X])$.

$g''(u) = 2 > 0$

when $u = \mathbb{E}[X]$. it has minimum

d).

$$\mathbb{E}|X-u| = \int_{-\infty}^\infty |x-u| f(x)\, dx = \int_{-\infty}^u u f(x)\, dx - \int_{-\infty}^u x f(x)\, dx + \int_u^\infty x f(x)\, dx - \int_u^\infty u f(x)\, dx$$

$$\frac{d\,\mathbb{E}|X-u|}{du} = \int_{-\infty}^u f(x)\, dx - \int_u^\infty f(x)\, dx = \mathbb{P}(X \leq u) - \mathbb{P}(X > u)$$

when $X$ takes median, the derivative comes to 0.

and $X > X_{0.5}$ $\mathbb{P}(X \leq u) - \mathbb{P}(X > u) > 0$ thus it's the minimal point.

$X < X_{0.5}$ $\mathbb{P}(X \leq u) - \mathbb{P}(X > u) < 0$