Homework 1.

1. $I = \displaystyle\int \sqrt{1+\sin 2x}\,\mathrm{d}x \ (0 \le x \le \pi)$

$$I = \int \sqrt{(\cos x + \sin x)^2}\,\mathrm{d}x = \int |\cos x + \sin x|\ \mathrm{d}x = \begin{cases} \sin x - \cos x + C_1, & 0 \le x \le 3\pi/4, \\ -\sin x + \cos x + C_2, & 3\pi/4 \le x \le \pi. \end{cases}$$

$$I\left(\frac{3\pi}{4} - 0\right) = I\left(\frac{3\pi}{4} - 0\right) \Rightarrow \sqrt{2} + C_1 = -\sqrt{2} + C_2 \Rightarrow C_2 = 2\sqrt{2} + C_1,$$

$$I = \begin{cases} \sin x - \cos x + C_1, & 0 \le x \le 3\pi/4, \\ -\sin x + \cos x + 2\sqrt{2} + C_1, & 3\pi/4 \le x \le \pi. \end{cases}$$

2. $I = \displaystyle\int \frac{x\,\mathrm{d}x}{\sqrt{x^2+1+\sqrt{(1+x^2)^3}}}$

$$I = \frac{1}{2}\int \frac{\mathrm{d}(x^2+1)}{\sqrt{x^2+1+\sqrt{(1+x^2)^3}}} = \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}\sqrt{1+\sqrt{t}}} = \int \frac{\mathrm{d}(\sqrt{t})}{\sqrt{1+\sqrt{t}}} = \int \frac{\mathrm{d}u}{\sqrt{1+u}} = 2\sqrt{1+u} + C$$

$$= 2\sqrt{1+\sqrt{1+x^2}} + C$$

3. $I = \displaystyle\int \frac{x^2\,\mathrm{d}x}{(1-x)^{100}} = -\int \frac{(1-t)^2\,\mathrm{d}t}{t^{100}} = \int \left(-\frac{1}{t^{100}} + \frac{2}{t^{99}} - \frac{1}{t^{98}}\right)\mathrm{d}t$

$$= \frac{1}{99(1-x)^{99}} - \frac{1}{49(1-x)^{98}} + \frac{1}{97(1-x)^{97}} + C$$

4. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+e^x}} = \int \frac{\mathrm{d}x}{e^{x/2}\sqrt{e^{-x}+1}} = -2\int \frac{\mathrm{d}(e^{-x/2})}{\sqrt{(e^{-x/2})^2+1}} = -2\log\left(e^{-x/2} + \sqrt{e^{-x}+1}\right) + C$

$$= x - 2\log\left(1 + \sqrt{e^x+1}\right) + C.$$

The substitution $u = e^x$ is possible as well.

5. $I = \displaystyle\int \frac{xe^{\arctan x}\,\mathrm{d}x}{\sqrt{(1+x^2)^3}} = \int \frac{x}{\sqrt{1+x^2}}\,\mathrm{d}\left(e^{\arctan x}\right) = \frac{xe^{\arctan x}}{\sqrt{1+x^2}} - \int \frac{e^{\arctan x}}{\sqrt{(1+x^2)^3}}\,\mathrm{d}x = \frac{xe^{\arctan x}}{\sqrt{1+x^2}}$

$$-\int \frac{\mathrm{d}(e^{\arctan x})}{\sqrt{1+x^2}} = \frac{xe^{\arctan x}}{\sqrt{1+x^2}} - \frac{e^{\arctan x}}{\sqrt{1+x^2}} - I$$

$$I = \frac{(x-1)e^{\arctan x}}{2\sqrt{1+x^2}} + C$$

6. $\displaystyle\int \frac{\mathrm{d}x}{\sin x \cos x} = \int \frac{\mathrm{d}x}{\tan x \cos^2 x} = \int \frac{\mathrm{d}(\tan x)}{\tan x} = \log|\tan x| + C$

7. $\displaystyle\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}x}{2\sin x/2 \cos x/2} = \int \frac{\mathrm{d}(x/2)}{\sin x/2 \cos x/2} = \log|\tan(x/2)| + C$

8. $\displaystyle\int \frac{xe^x\,\mathrm{d}x}{(x+1)^2} = \int xe^x\,\mathrm{d}\left(\frac{-1}{x+1}\right) = \frac{-xe^x}{x+1} + \int e^x\,\mathrm{d}x$

9. $I = \displaystyle\int \sin(\log x)\,\mathrm{d}x = x\sin(\log x) - \int \cos(\log x)\,\mathrm{d}x = x\sin(\log x) - x\cos(\log x) - I$

$$I = \frac{x\sin(\log x) - x\cos(\log x)}{2}$$

10. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}(x+1)}\,\mathrm{d}x = \left[\sqrt{x} = t\,,\, \frac{\mathrm{d}x}{2\sqrt{x}} = \mathrm{d}t\right] = 2\int \frac{\arctan t\,\mathrm{d}t}{t^2+1} = 2\int \arctan t\,\mathrm{d}(\arctan t)$

$$= \arctan^2\sqrt{x} + C$$