Real Analysis 2024. Homework 1.

1. Prove that any open subset of a real line can be expressed as at most countable union of disjoint open intervals.

Pf: $\forall E \subset \mathbb{R}$. E is open. $\forall x \in E$. $\exists \varepsilon > 0$, $(x-\varepsilon, x+\varepsilon) \subset E$.

denote $a := \inf \{ y \in \mathbb{R} \mid y < x, (y, x) \subset E \}$

$b := \sup \{ z \in \mathbb{R} \mid z > x, (x, z) \subset E \}$

(a,b) is the largest open interval contains x.

in topology, we have proved E can be expressed as the at most countable union of open intervals.

i.e. $E = \bigcup_{n \geq 1} U_n$ for each U_n. $\forall x \in U_n$, we can find $(a_n, b_n) \subset E$ and contains x.

we claim that (a_n, b_n) is either overlap or disjoint.

thus we remove the overlaped intervals and rearrange them. as (a_{n_k}, b_{n_k})

$E = \bigcup_{k \geq 1} (a_{n_k}, b_{n_k})$

any open set. = union of connected component.

connected component disjoint or overlaped. each contains a rational numbers.

2. Prove that Borel $\sigma-$algebra in $\mathbb{R}$ can be generated by the family of open rays $\{(-\infty, a) : a \in \mathbb{R}\}$.

Pf: $\forall$ open set $U \in \mathcal{B}_{\mathbb{R}}$.

$(a,b) = \bigcup_{n=1}^{\infty} [a+\frac{1}{n}, b) = \bigcup_{n=1}^{\infty} (-\infty, b) \setminus (-\infty, a+\frac{1}{n}])$.

by Problem 1. $U = \bigcup_{n \geq 1} (a_n, b_n)$ (a_n, b_n) are disjoint open intervals.

$$(a_n, b_n) = (a_n, +\infty) \cup (-\infty, b_n) = \left[\mathbb{R} \setminus \bigcap_{m \geq 1}^{\infty} (-\infty, a_n + \tfrac{1}{m}) \right] \cup (-\infty, b_n)$$

thus, $U = \bigcup_{n \geq 1} \left[\mathbb{R} \setminus \bigcap_{m \geq 1}^{\infty} (-\infty, a_n + \frac{1}{m}) \cup (-\infty, b_n) \right]$

since $a_n + \frac{1}{m}$, $b_n \in \mathbb{R}$. $(-\infty, a_n + \frac{1}{m})$, $(-\infty, b_n)$ belongs to the family of open rays.

i.e. $\mathcal{B}_{\mathbb{R}}$ can be generated by $\{ (-\infty, a) \mid a \in \mathbb{R} \}$.

3. Let $(X, \mathcal{A})$ be measurable space, X' be a set, $\mathcal{A}'_{\min} = \{\varnothing, X'\}$ be a minimal $\sigma-$algebra on X'. Prove that every map $f : X \to X'$ is measurable.

Pf: $f^{-1}(\phi) = \phi \in \mathcal{A}$

$f^{-1}(X') = X \in \mathcal{A}$.

i.e. $\forall E \in \mathcal{A}'$, $f^{-1}(E) \in \mathcal{A}$. f is measurable.

Real Analysis 2024. Homework 2.

function meas. $\Rightarrow$ set meas.

1. Let (f_n) be a sequence of real-valued measurable functions on X. Then the set

$$E = \{x \in X \mid f_n(x) \text{ converges to some limit }\}$$

is measurable. (Hint: use Cauchy criterion).

Pf: by thm 1.6. $\limsup f_n$, $\liminf f_n$ are measurable function.

thus. the set $X(\limsup f_n = +\infty)$; $X(\liminf f_n = -\infty)$; $X(\liminf f_n \neq \limsup f_n)$ are measurable.

the set $E = X \setminus X(\limsup f_n = +\infty) \cup X(\liminf f_n = -\infty) \cup X(\liminf f_n \neq \limsup f_n)$

Thus. E is measurable.

write Cauchy criterion in terms of set operations. $E = \bigcap_{k\geq 1} \bigcup_{N \geq 1} \{x \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$.

func. $|f_n(x) - f_m(x)|$ measurable.

2. Assume that μ is a measure on $(X, \mathcal{A})$. Prove that

$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$$

for every $E, F \in \mathcal{A}$.

Pf: denote $E \cap F = U_1$. $E \setminus (E \cap F) = U_2$ $F \setminus (E \cap F) = U_3$

then we have $\mu(E) = \mu U_1 + \mu U_2$. $\mu(F) = \mu U_1 + \mu U_3$. (monotonicity of measure)

$\mu(E \cup F) = \mu U_2 + \mu U_3 + \mu U_1$

thus. $LHS = \mu U_2 + \mu U_3 + 2\mu U_1 = (\mu U_1 + \mu U_2) + (\mu U_1 + \mu U_3)$

$= \mu E + \mu F = RHS$.

3. Prove that Borel σ-algebra $\mathcal{B}_{\mathbb{R}^2}$ is the same as product σ-algebra $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

Pf: $\forall A \in \mathcal{B}_{\mathbb{R}^2}$, $A = \bigcup_{n=1}^{\infty} I_n \times I_n'$. $\{I_n\}\{I_n'\}$ are semi-open interval.

1) $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \{E \times E' \mid E, E' \in \mathcal{B}_{\mathbb{R}}\}$ (by def. of product σ-algebra)

by this definition, $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ contains all open sets of $\mathbb{R}^2$.

and Borel σ-algebra is the smallest σ-algebra contains all open sets of $\mathbb{R}^2$.

i.e. $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \supset \mathcal{B}_{\mathbb{R}^2}$

2) $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ and $\mathcal{B}_{\mathbb{R}^2}$ are both σ-algebra on $\mathbb{R}\times\mathbb{R} = \mathbb{R}^2$.

the product of two Borel sets is always Borel
$\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \equiv \mathcal{B}_{\mathbb{R}^2}$

denote the projection map $p_{1,2} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$. $p_1(x,y) = x$. $p_2(x,y) = y$.

for arbitrary σ-algebra $\mathcal{A}$. p_1, p_2 are measurable iff $p_1^{-1}(Y) = Y \times \mathbb{R} \in \mathcal{A}$.

$p_2^{-1}(Y') = \mathbb{R}\times Y' \in \mathcal{A}$. for any $Y, Y' \in \mathcal{B}_{\mathbb{R}}$

since $Y \times Y' = (Y\times\mathbb{R}) \cap (\mathbb{R}\times Y')$, $Y\times Y' \in \mathcal{A}$.

also since $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra contains all sets of the form $Y\times Y'$ where $Y, Y' \in \mathcal{B}_{\mathbb{R}}$. thus $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$ is the smallest σ-algebra s.t. p_1, p_2 are both meas.

(actually, this is remark 1.4.2.)

It remains to check p_1, p_2 are meas. w.r.t. $\mathcal{B}_{\mathbb{R}^2}$

$\forall Y \in \mathcal{B}_{\mathbb{R}}$. $p_1^{-1}(Y) = Y\times\mathbb{R}$.

denote a product σ-algebra of $\mathbb{R}^2$ as $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{A}_{min}$; $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{A}_{min} \subset \mathcal{B}_{\mathbb{R}^2}$

since $Y \in \mathcal{B}_{\mathbb{R}}$ and $\mathbb{R} \in \mathcal{A}_{min}$. $Y \times \mathbb{R} \in \mathcal{B}_{\mathbb{R}} \otimes \mathcal{A}_{min} \subset \mathcal{B}_{\mathbb{R}^2}$.

p_1 is measurable and p_2 similarly.

thus $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^2}$

4. Prove that a Cantor's set is a Borel set (see MA(3)).

Pf: $C = \bigcap_{n=1}^{\infty} C_n = [0,1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{k=0}^{3^m-1} \left(\frac{3k+1}{3^{m+1}}, \frac{3k+2}{3^{m+1}}\right)$ intersection of a nested sequence of closed sets

denote $E = \bigcup_{m=0}^{\infty} \bigcup_{k=0}^{3^m-1} \left(\frac{3k+1}{3^{m+1}}, \frac{3k+2}{3^{m+1}}\right)$

by the def. of σ-algebra, $C \in \mathcal{B}_{[0,1]}$ iff $E \in \mathcal{B}_{[0,1]}$

and E is the countable union of open intervals. thus $E \in \mathcal{B}_{[0,1]}$. thus $C \in \mathcal{B}_{[0,1]}$.

5. Consider a sequence of measures μ_i on a $\sigma-$algebra $\mathcal{A}$ and a sequence of nonnegative numbers $\alpha_i \in \mathcal{A}$. For a set $E \in \mathcal{A}$ let

$$\mu(E) = \sum_{i=1}^{\infty} \alpha_i \mu_i(E).$$

Prove that μ is a measure on $\mathcal{A}$.

Pf: 1) $\mu\phi = \sum_{i=1}^{\infty} \alpha_i \mu_i(\phi) = \sum_{i=1}^{\infty} \alpha_i \cdot 0 = 0$

$n=1$ $i=1$ $n=1$

2) let $\{E_n\}_{n=1}^{\infty}$, $E_n \in \mathcal{A}$ and $E_i \cap E_j = \phi$ when $i \neq j$.

$$\mu\left(\bigcup_{n=1} E_n\right) = \sum_{i=1}^{\infty} \alpha_i \mu_i\left(\bigcup_{n=1}^{\infty} E_n\right) \xlongequal{\mu_i \text{ are measures}} \sum_{i=1}^{\infty} \alpha_i \left(\sum_{n=1}^{\infty} \mu_i E_n\right)$$

$$= \sum^{\infty} \sum^{\infty} \alpha_i \mu_i E_n = \sum^{\infty} \mu E_n$$

Another solution for 1. (Use Cauchy criterion

1. Let (f_n) be a sequence of real-valued measurable functions on X. Then the set
$$E = \{x \in X \mid f_n(x) \text{ converges to some limit } \}$$
is measurable. (Hint: use Cauchy criterion).

Pf: let $g_n = f_n \chi_E$. g_n is measurable as the product of measurable function.

and g_n converge pointwisely on X: $g(x)$ converges to the same limit as $f_n(x)$ when $x \in E$. and converges to 0 when $x \in X \setminus E$.

denote $\lim_{n\to\infty} g_n = g$. g is measurable function.

$E = \{x \mid g(x) \neq 0\} \cup \{x \mid \lim f_n(x) = 0\} =: E_1 \cup E_2$.

E_1 can be written as $E(g \neq 0)$. it's measurable.

and $E_2 = \{x \in E \mid \forall \varepsilon > 0, \exists N > 0 : \forall n > N. |f_n(x)| < \varepsilon\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n \geq N}^{\infty} \{x \mid |f_n(x)| < \frac{1}{k}\}$

since $\{x \mid |f_n(x)| < \frac{1}{k}\} = X(f_n > -\frac{1}{k}) \cap X(f_n < \frac{1}{k})$ is measurable.

F_2 is measurable. thus E is measurable.

Lemma Assume $a_{k,i} \geq 0$. $k, i \in \mathbb{N}$. Then $S_1 = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} a_{k,i} = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{k,i} = S_2$.

Real Analysis 2024. Homework 3.

1. Consider a function $f : [0,1] \to [0,\cancel{2}\,1]$ defined as following:

$$f(x) = \begin{cases} x + 1/2, & 0 \le x < 1/2; \\ x - 1/2, & 1/2 \le x \le 1. \end{cases}$$

Find the image of Lebesgue measure with respect to this transform.

$E \subset (0,1)$. E - meas.

$f^{-1}(E) = A_1 \cup A_2$.

$A_1 = \{x+\frac{1}{2} : x \in E \cap [0,\frac{1}{2}]\}$

$A_2 = \{x-\frac{1}{2} : x \in E \cap [\frac{1}{2},1)\}$

$f_*(\mu)(E) = \mu(f^{-1}(E)) = \mu A_1 + \mu A_2$

$= \mu(E \cap [0,\frac{1}{2}]) + \mu(E \cap [\frac{1}{2},1)]$

$= \mu E.$

$\forall Y \in \mathcal{B}_{[0,\cancel{2}1]}$

$Y = (Y \cap [0,1)) \cup (\cancel{Y \cap [1,2]})$

$f^{-1}(Y \cap [1,\cancel{2}]) = \phi$ or $\{\frac{1}{2}\}$ $\qquad \mu f^{-1}(\cancel{Y \cap [1,2]}) = 0$

since $Y \cap [0,1)$ is measurable w.r.t Lebesgue measure.

then $Y \cap [0,1) = \bigcup_{n=1}^{\infty} [a_n, b_n)$ $\quad [a_n,b_n)$ mutually disjoint

divide them into 3 groups. and rearrange them.

$0 \le a_{n_1} < b_{n_1} \le \frac{1}{2} \qquad 0 \le a_{n_2} < \frac{1}{2} < b_{n_2} \le 1 \qquad \frac{1}{2} \le a_{n_3} < b_{n_3} \le 1$

$f^{-1}(Y \cap [0,1)) = \bigcup_{n_1=1}^{\infty} [a_{n_1}+\frac{1}{2}, b_{n_1}+\frac{1}{2}) \ \bigcup_{n_3=1}^{\infty} [a_{n_3}-\frac{1}{2}, b_{n_3}-\frac{1}{2}) \ \bigcup_{n_2=1}^{\infty} [a_{n_2}+\frac{1}{2}, 1) \cup [0, b_{n_2}-\frac{1}{2})$

$\mu f^{-1}(Y \cap [0,1)) = \sum_{n_1 \in \mathbb{N}} (b_{n_1} - a_{n_1}) \sum_{n_3 \in \mathbb{N}} (b_{n_3} - a_{n_3}) \sum_{n_2 \in \mathbb{N}} [(\frac{1}{2} - a_{n_2}) + (b_n - \frac{1}{2} - 0)]$

$= \sum (b_{n_1} - a_{n_1}) \sum (b_{n_2} - a_{n_2}) \sum (b_{n_3} - a_{n_3}) = \sum (b_n - a_n)$

$= \mu(Y \cap [0,1)).$

thus, $\forall Y \in \mathcal{B}_{[0,2]}$. $\quad f_*(\mu)(Y) = \mu(Y \cancel{\cap [0,1)})$

the Lebesgue measure preserved.

2. Consider a measure defined as linear combination of Dirac measures

$$\mu = \delta_0 + \delta_1 + 2\delta_2$$

on $\mathcal{A}_{max}$ on $\mathbb{R}$. Consider a function $f(x) = x^2$. Prove that it is measurable map with respect to $\sigma-$algebra $\mathcal{A}_{max}$. Find $f_*(\mu)$.

Pf: 1) $\forall E \in \mathcal{A}_{max}$ $\quad f^{-1}(E) = \{\sqrt{x} \mid x \in E \text{ and } x \geq 0\} \subset \mathbb{R}$. thus $f^{-1}(E) \in \mathcal{A}_{max}$

f is measurable w.r.t. $\mathcal{A}_{max}$.

2) $\forall E \in \mathcal{A}_{max}$ $\quad f_*(\mu)(E) = \mu(f^{-1}(E)) = \begin{cases} 0, & 0,1,2 \notin f^{-1}(E). \text{ i.e. } 0,1,4 \notin E. \\ 1, & 0 \in E \text{ or } 1 \in E. \times \quad 0 \in E \wedge 1,4 \notin E \text{ or } 1 \in E \wedge 0,4 \notin E \\ 2. & \{0,1\} \subset E \text{ or } 4 \in E. \quad \{0,1\} \subset E \wedge 4 \notin E \quad 0,1 \notin E \wedge 4 \in E \\ \cancel{3}. & \{0,4\} \subset E \text{ or } \{1,4\} \subset E \\ 4, & \{0,1,4\} \subset E \end{cases}$

$f_*(\mu) = \delta_0 + \delta_1 + 2\delta_4.$

3. Calculate the integral of function $f(x) = x^3$ on $[0, +\infty)$ with respect measure

$$\mu = \frac{\delta_1 + 2\delta_2 + 3\delta_3}{6}.$$

Solution: $\int_0^{+\infty} x^3 \, d\mu(x) = \frac{1}{6}(f(1) + 2f(2) + 3f(3)) = \frac{1}{6}(1 + 16 + 81) = \frac{49}{3}.$

we have $f \geq 0$. denote $\varphi = \sum_{k=1}^{\infty} C_k \chi_{A_k}$. $C_k \in [0, +\infty]$. A_k partitioned $[0, +\infty]$

$\int_0^{+\infty} f \, d\mu = \sup_{\varphi} \int_0^{+\infty} \varphi \, d\mu = \sup \sum C_k \cdot \mu(A_k \cap [0, +\infty))$. A_k partitioned $[0, +\infty]$.

denote $A_k = [a_k, b_{k-1})$. $\varphi \leq f$. i.e. $C_k \leq a_k^3$

Let $1 \in A_{k_1}$ $\quad 2 \in A_{k_2}$ $\quad 3 \in A_{k_3}$. (k_1, k_2, k_3 can be equal)

$$\sup \sum C_k \, \mu(A_k \cap [0, +\infty)) = \sup \left(C_{k_1} \cdot \frac{1}{6} + C_{k_2} \cdot \frac{1}{3} + C_{k_3} \cdot \frac{1}{2} \right)$$

$$= \sup \left(a_{k_1}^3 \cdot \frac{1}{6} + a_{k_2}^3 \cdot \frac{1}{3} + a_{k_3}^3 \cdot \frac{1}{2} \right)$$

by def. $\max a_{k_1} = 1$ $\quad \max a_{k_2} = 2$ $\quad \max a_{k_3} = 3$.

thus $\int_0^{+\infty} f \, d\mu = \sup \int_0^{+\infty} \varphi \, d\mu = 1 \cdot \frac{1}{6} + 2^3 \cdot \frac{1}{3} + 3^3 \cdot \frac{1}{2} = \frac{1}{6} + \frac{8}{3} + \frac{27}{2} = \frac{49}{3}$

4. Explain why the counting measure on $\mathbb{R}$ is not σ-finite.

define the counting measure $\mu(A) = \begin{cases} a. & a \text{ is the number of elements in } A. \\ +\infty & A \text{ has infinite elements.} \end{cases}$

If $\mathbb{R}$ is σ-finite w.r.t. to μ. i.e. $\exists\ X_n \in \mathcal{B}_{\mathbb{R}}$. s.t. $X = \bigcup_{n\geq 1}^{\infty} X_n$, $\mu X_n < +\infty$.

i.e. X_n has finite element.

The at most countable union of finite element is at most countable.

Which implies $\mathbb{R}$ is countable, which is contradictory.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let E_j, $1 \leq j \leq k$, be measurable sets. Prove that

$$\sum_{i=1}^{k} \mu E_i \leqslant \mu\left(\bigcup_{i=1}^{k} E_i\right) + \sum_{i<j} \mu\left(E_i \cap E_j\right).$$

Pf by induction

1). $n=2$ $\quad \mu E_1 \leq \mu E_1 + \sum_{j>1} \mu(E_1 \cap E_j)$ holds.

2) assume the formula holds fold $n = N-1$.

3) Consider $n = N$

$$\sum_{N} \mu E_k = \mu E_k + \sum_{N-1} \mu E_{k-1} \leq \mu E_N + \mu\left(\bigcup_{N-1} E_k\right) + \sum_{k<j}^{N-1} \mu(E_j \cap E_k)$$

by the previous hw. $\mu E_N + \mu\left(\bigcup_{N-1} E_k\right) = \mu\left(\bigcup_{N} E_k\right) + \mu\left(\left(\bigcup_{N-1} E_k\right) \cap E_N\right)$.

and $\left(\bigcup_{N-1} E_k\right) \cap E_N = \bigcup_{N-1} (E_k \cap E_N)$

$$\text{thus. } \sum_{N} \mu E_k \leq \mu\left(\bigcup_{N} E_k\right) + \mu\left(\left(\bigcup_{N-1} E_k\right) \cap E_N\right) + \sum_{k<j}^{N-1} \mu(E_j \cap E_k).$$

$$= \mu\left(\bigcup_{N} E_k\right) + \mu \bigcup_{N-1} (E_k \cap E_N) + \sum_{k<j}^{N-1} \mu(E_j \cap E_k).$$

$$\leq \mu\left(\bigcup_{N} E_k\right) + \sum_{k}^{N-1} \mu(E_k \cap E_N) + \sum_{k<j}^{N-1} \mu(E_j \cap E_k).$$

$$= \mu\left(\bigcup_{N} E_k\right) + \sum_{k<j}^{N} \mu(E_j \cap E_k)$$

In conclusion, the formula holds.

Real Analysis 2024. Homework 4.

1. Consider a counting measure on $\mathbb{N}$. Describe spaces of mesurable and of integrable functions.

denote μ as counting measure. $S(\mathbb{N})$. space of measurable function.
$L(\mathbb{N})$ space of integrable function.

$S(\mathbb{N})$ depends on σ-algebra on $\mathbb{N}$.

for example, for $\mathcal{A} = 2^{\mathbb{N}}$. any function $f: \mathbb{N} \to \mathbb{C} \in S(\mathbb{N})$

by def. of μ, $\int_{\mathbb{N}} |f| d\mu = \sum_{n\in\mathbb{N}} |f(n)|$

this space $\ell^1 = \{ \{c_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} |c_k| < +\infty \}$

$L(\mathbb{N}) = \{ f \mid \sum_{n\in\mathbb{N}} |f(n)| < +\infty$, i.e. the series abs. conv. $\}$.

2. Assume that $f_n \in L(E)$ is increasing sequence, $f_n \to f$ pointwise and $f \in L(E)$. Prove that $f_n \to f$ in $L(E)$.

Pf: We need to check $\|f_n - f\|_1 = \int_E (f_n - f) d\mu \underset{n\to\infty}{\longrightarrow} 0$.

$g_n = f - f_n$.

denote $\Phi(x) = \max\{ |f_1(x)|, f(x) \}$. $\Phi \in L(E)$ since $f_1, f \in L(E)$.

since $\langle f_n \rangle_{n=1}^{\infty}$ is increasing, $|f_n(x)| \leq \Phi(x)$. for all $n \in \mathbb{N}$ and $x \in E$.

Then apply the Lebesgue dominated conv. thm. $\int_E |f_n - f| d\mu \to 0$

i.e. $f_n \to f$ in $L(E)$

3. Let f be $\mu-$measurable on E and denote $E_t = E(|f| > t)$. Prove that

$$\mu E_t \leq \frac{1}{t^p} \int_E |f|^p \, d\mu.$$

$E_t = E(|f|^p > t^p)$.

Pf: To make sense of the proof, assume $t > 0$.

$$\int_E |f|^p d\mu \geq \int_{E_t} |f|^p d\mu \quad \text{(by monotonicity by sets)}$$

$$\geq \int_{E_t} t^p d\mu \quad (\text{on } E_t,\ |f| > t \text{ implies } |f|^p > t^p, \text{ then by monotonicity})$$

$$= t^p \cdot \mu E_t$$

$$\Rightarrow \mu E_t \leq \frac{1}{t^p} \int_E |f|^p d\mu.$$

4. Prove that a measure μ is σ−finite if and only if there exists a positive integrable function ($f > 0$ on X and $\int_X f d\mu < +\infty$).

Pf: "$\Rightarrow$" μ is σ-finite i.e. $\exists \{X_n\}_{n=1}^{\infty}$. $\mu X_n < +\infty$. $\bigcup_{n=1}^{\infty} X_n = X$.

denote $A_n = \bigcup_{k=1}^{n} X_k$ thus we have an non-decreasing sequence of set $\{A_n\}_{n=1}^{\infty}$.

and denote $A_0 = \phi$

$$f(x) = \begin{cases} \frac{1}{2^n \cdot \mu(A_{n+1} \setminus A_n)}, & x \in A_{n+1} \setminus A_n. \ \mu(A_{n+1} \setminus A_n) \neq 0 \quad n = 0, 1, 2, \dots \\ \frac{1}{2^n}. & x \in A_{n+1} \setminus A_n. \ \mu(A_{n+1} \setminus A_n) = 0 \quad n = 0, 1, 2, \dots \end{cases}$$

$f > 0$ since $\frac{1}{2^n} > 0$ and $\mu \geqslant 0$

$$\int_X f d\mu = \sum_{n=0}^{\infty} \frac{1}{2^n \mu(A_{n+1} \setminus A_n)} \cdot \mu(A_{n+1} \setminus A_n) + 0 \leqslant \sum_{n=0}^{\infty} \frac{1}{2^n} \leqslant 2 < +\infty$$

"$\Leftarrow$" $\exists f > 0$ s.t. $\int_X f d\mu < +\infty$.

$\exists \varphi \leq f$. φ is simple $\varphi = \sum_{k=1}^{N} c_k A_k$. $\bigcup_{k=1}^{N} A_k = X$. A_k mutually disjoint.

thus $\int_X \varphi \, d\mu = \sum_{k=1}^{N} c_k \mu(A_k) < +\infty$. $\mu(A_k) < +\infty$.

thus X is finite, more obviously σ-finite

Consider a set $X_k = X(f > \frac{1}{k})$. by Chebyshev's $\mu(X_k) \leqslant k \int_X |f| < \infty$.

and since $f > 0$ for every $x \in X$. $X = \bigcup_{k=1}^{\infty} X_k$ $\mu X_k < +$

5. Consider $f_n(x) = \frac{1}{n}\left(\frac{\sin nx}{x}\right)^2$. Prove that

(a) $f_n \in L(0,\pi)$;

(b) $f_n(x) \to 0,\ n \to \infty$ for every $x \in (0,\pi]$;

(c) There is no such function $g \in L(0,\pi)$ such that $f_n(x) \le g(x)$ for every $x \in [0,\pi]$ and every $n \in \mathbb{N}$.

Pf: (a) it suffices to check that $\int_0^\pi f_n\,d\mu < +\infty$ i.e. $\int_0^\pi \frac{1}{n}\left(\frac{\sin nx}{x}\right)^2 dx. < +\infty$ for all $n \in \mathbb{N}$.

we have $\lim_{x\to 0} \frac{\sin nx}{nx} = 1$ and $\left|\frac{\sin nx}{nx}\right| \le 1$ on $(0,\pi)$.

thus $\int_0^\pi n \cdot \left(\frac{\sin nx}{nx}\right)^2 dx \le n\int_0^\pi dx = n\pi < +\infty.$

$f_n(x) \le \frac{1}{nx^2} \to 0 \quad (n \to 0)$

(b) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n}\cdot\left(\frac{\sin nx}{x}\right)^2 \le \lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{x^2} = 0$ for any fixed $x \in (0,\pi]$

(c) Assume function g exists and satisfies the above condition.

by (a)(b). denote $f(x) \equiv 0$ on $(0,\pi]$.

we have $f_n \to f$ when $n\to\infty$ for all $x \in (0,\pi]$. $f, f_n \in L(0,\pi)$

And by Lebesgue dominated conv. thm, we have $\int_0^\pi \frac{1}{n}\cdot\left(\frac{\sin nx}{x}\right)^2 dx \to 0.$

but since: $\int f_n\,dx \to \int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx.$

$$\int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx = -\frac{1}{x}\cdot\sin^2 x\,\Big|_0^{+\infty} + \int_0^{+\infty} \frac{2\sin x\cos x}{x}\,dx = \int_0^{+\infty} \frac{\sin 2x}{2x}\,d(2x) = \frac{\pi}{2}.$$

$$\lim_{n\to\infty}\int_0^\pi \frac{1}{n}\cdot\left(\frac{\sin nx}{x}\right)^2 dx = \lim_{n\to\infty} \cancel{n} \int_0^{n\pi} \frac{(\sin nx)^2}{(nx)^2}\,d(nx) \to +\infty. \times$$

thus it causes a contradiction.

Real Analysis 2024. Homework 5.

1. Let $p \in X$ and δ_p be Dirac measure. Prove that

$$\int_X f d\delta_p = f(p)$$

for every function $f : X \to \mathbb{R}$.

Pf: 1) f is characteristic function $f = \chi_A$.

$$\int_X f d\delta_p = \chi_A . \delta_p(A) = \begin{cases} 1 , & p \in A. \\ 0 , & p \in X \setminus A. \end{cases} = \chi_A(p) = f(p)$$

2) by linearity. the formula holds when f is simple function

3) $f \geq 0$. $\displaystyle\int_X f d\delta_p = \sup_{\substack{\varphi \leq f \\ \varphi \text{ is simple}}} \int \varphi \, d\delta_p = \sup_{\varphi} \int_X \sum_{k=1}^{N} C_k \chi_{A_k} d\delta_p$ where A_k mutually disjoint and $\bigcup_{k=1}^{N} A_k = X$.

$$= \sup \sum_{k=1}^{N} \int_{A_k} C_k \, d\delta_p = \sup \sum_{k=1}^{N} C_k \cdot \delta_p(A_k)$$

by construction of A_k. $\exists! j$, $1 \leq j \leq N$. s.t. $\delta_p(A_j) = 1$. and $\delta_p(A_k) = 0$ for $j \neq k$.

$$\int_X f d\delta_p = \sup C_j$$ since $\varphi \leq f$. $p \in A_j$. $\sup C_j = f(p)$.

4). f be any function. $f = f_+ - f_-$ $f_+ = \max\{f, 0\}$. $f_- = \max\{0, -f\}$.

and amply 3). $\int_X f d\delta_p = f(p)$.

2. Let $n \in \mathbb{N}$, $0 \le k \le 2^n - 1$. Consider an interval

$$\Delta_{k,n} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

$n \quad k$

$n, 2^n \; (0 - 2^{n+1}$

and let $f_{n,k}(x) = \chi_{\Delta_{k,n}}$ be a characteristic function of this interval. This defines a countable family of functions that can be considered as a sequence. For example,

$$g_1 = f_{1,0},\ g_2 = f_{1,1},\ g_3 = f_{2,0},\ g_4 = f_{2,1},\ g_5 = f_{2,2},\ g_6 = f_{2,3},\ g_7 = f_{3,0}, \ldots.$$

Let $m_n = 1 + 2 + \ldots + 2^{n-1} = 2^n - 1$ and for $m_n \le j < m_{n+1}$ we let $k = j - m_n$. Then the above sequnce is defined as follows:

$m_n = 2^n - 1 \qquad m_{n+1} = 2^{n+1} - 1$

$$g_j = f_{n, j-m_n}, \quad m_n \le j < m_{n+1}.$$

(a) Prove that a sequence g_j converges in $L^1[0,1]$.

(b) Indicate a subsequence that converges to the limit function a.e.

Pf: (a) $g_j = f_{n, j-m_n} = \chi_{\left[\frac{j-2^n+1}{2^{n+1}}, \frac{j-2^n+2}{2^{n+1}}\right)} \quad j \in \{2^n - 1, 2^n, \ldots, 2^{n+1} - 1\}$.

for $p, q \in \mathbb{N}$ and w.l.g, $p > q$.

$\|g_p - g_q\|_1 = \int |g_p - g_q|_1 \, d\mu \le \mu \cdot \Delta_{n, p-m_n} + \mu \Delta_{n, q-m_n} \le \frac{1}{2^{n-1}} \to 0$

thus $\{g_j\}$ is Cauchy's in $L^1[0,1]$.

$\int_0^1 g_j \, dx = \int_0^1 f_{n, j-m_n} \, dx = \frac{1}{2^n} \to 0.$

by the completeness of $L^1[0,1]$. $\{g_j\}$ conv. in $L^1[0,1]$

(b) since $\int_{[0,1)} |g_j| \, d\mu = \mu \Delta_{n, j-m_n} = \frac{1}{2^n} \to 0$.

the limit function $g = \lim_{j \to \infty} g_j$ is $g = 0$.

subsequence $\{g_{j'}\}$ when $j' = m_{n+1} - 1 = 2^{n+1} - 2$, i.e. $\{g_{2^{n+1}-2}\} = \{\chi_{[\frac{2^n-1}{2^n}, 1)}\}$.

which conv. to 0 a.e.

3. Let $\mu(X) < +\infty$, $f_n \in L^1(X,\mu))$, and $f_n \rightrightarrows f$ on X. Prove that $f \in L^1(X,\mu)$ and $\|f_n - f\|_1 \xrightarrow[n\to\infty]{} 0$.

Pf: $f_n \rightrightarrows f$. i.e. $\forall \varepsilon>0$. $\exists N \in \mathbb{N}$. $|f_n - f| < \varepsilon$ for all $x \in X$ and $n > N$.

fixed $N_1 > N$. $|f_{N_1} - f| < \varepsilon \Rightarrow |f| - |f_{N_1}| \le |f_{N_1} - f| < \varepsilon \Rightarrow |f| < |f_{N_1}| + \varepsilon$.

$$\int_X |f|\, d\mu < \int_X [|f_{N_1}| + \varepsilon]\, d\mu \le \int_X |f_{N_1}|\, d\mu + \varepsilon \cdot \mu X.$$

since $f_{N_1} \in L^1(X,\mu)$ and $\mu(X) < +\infty$, $\int_X |f|\, d\mu < +\infty$. i.e. $f \in L^1(X,\mu)$

$\forall \varepsilon' > 0$. $\exists N' \in \mathbb{N}$. $|f_n - f| < \frac{\varepsilon'}{\mu X}$ for all $x \in X$ and $n > N'$

$$\|f_n - f\|_1 = \int_X |f_n - f|\, d\mu \le \frac{\varepsilon'}{\mu X} \mu X < \varepsilon' \quad \text{for } n > N'$$

thus $\|f_n - f\|_1 \xrightarrow[n\to+\infty]{} 0$.

By monotonicity of integral.

$\|f_n - f\|_1 \le \mu(X) \sup_{x\in X} |f_n(x) - f(x)| \to 0 \quad (n \to +\infty)$

4. Provide an example of a set and a measure such that $\mu(X) = +\infty$, $f_n, f \in L^1(X,\mu))$, and $f_n \rightrightarrows f$ on X but $\|f_n - f\|_1 \not\to 0$.

$X = [1, +\infty)$. μ be Lebesgue measure.

$$f_n = \begin{cases} \frac{1}{n}, & x \in [1, n] \\ 0, & x > n. \end{cases}$$

$f_n = \frac{1}{n} \chi_{[n, 2n)}$ $\quad \|f_n\|_1 = 1 \not\to 0$.

$\int |f_n|\, d\mu = \int_1^n \frac{1}{n}\, dx = 1 - \frac{1}{n} < +\infty$. $\quad f_n \in L^1(X,\mu)$

$f_n \rightrightarrows f \equiv 0$. $\forall \varepsilon > 0$. $\exists N = [\frac{1}{\varepsilon}]$, for any $n > N$. $|f_n - f| \le \frac{1}{n} < \varepsilon$

but $\int_X |f_n - f|\, d\mu = \int_X |f_n|\, d\mu = 1 . \ne 0$.

Real Analysis 2024. Homework 6.

$E_k = \{(x,y) \in [-1,1]^2 : \frac{1}{k+1} < |x|, |y| < \frac{1}{k}\}$ $\int_{E_k} \frac{1}{|x|+|y|} \le 2(k+1)\mu_2(E_k) \le \frac{8(k+1)}{k^2(k+1)^2} \le \frac{8}{k^3}$

1. Consider a space $(\mathbb{R}^2, \mathcal{B}_2, \mu_2)$ and a function $f(x,y) = \frac{1}{|x|+|y|}$ prove that $f \in L^1([-1,1]^2)$.

$\int_{[-1,1]^2} \frac{dx\,dy}{|x|+|y|} \le 8 \sum_{k=1}^{\infty} \frac{1}{k^3} < +\infty.$

Pf: it suffices to check $\int_{[-1,1]^2} \frac{1}{|x|+|y|} d\mu < +\infty$

$$\int_{[-1,1]^2} \frac{1}{|x|+|y|} d\mu = \int_{-1}^{1} dx \int_{-1}^{1} \frac{1}{|x|+|y|} dy = 4 \int_0^1 dx \int_0^1 \frac{1}{x+y} dy = 4 \int_0^1 \ln \frac{x+1}{x} dx$$

$$= 4 \int_0^1 \ln\left(1+\frac{1}{x}\right) dx \le 4 \int_0^1 \ln 2 = 4\ln 2 < +\infty$$

2. Consider $\mathbb{N}$ with counting measure. Describe space $L^p(\mathbb{N})$.

$L^p(\mathbb{N}) = \left\{ \{c_k\}_{k=1}^{\infty} : \|\{c_k\}\|_p = \left(\boxed{\sum_{k=1}^{\infty}} |c_k|^p < +\infty\right)^{\frac{1}{p}} \right\}$

积分直接变求和，体现 counting measure.

$f: \mathbb{N} \to \mathbb{C}$. for $f \in L^p(\mathbb{N})$, we need $\int |f|^p d\mu < +\infty$

denote a family of set $\{X_1, X_2, \dots X_n\}$ $X_i = \{x_{1i}, x_{2i}, \dots x_{mi}\} \subset \mathbb{N}$.

$A = \{a_1, \dots a_n\} \subset \mathbb{C}$. $|a_i| < +\infty$.

$f(x) = a_i$. for $x \in X_i$.

$f(x) = 0$ for $x \in \mathbb{N} \setminus \bigcup X_i$

i.e. f has finite value in finite set.

3. Prove Hölder's inequality for simple functions using Hölder's inequality for finite sums. Apply theorem on monotone sequence to obtain Hölder's inequality for nonnegative measurable functions.

Pf: Hölder's inequality for finite sums: $a_i, b_i \ge 0$, $p > 1$ $\frac{1}{p} + \frac{1}{q} = 1$ $\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}$

$(X, \mathcal{A}, \mu)$ $f, g \in S(X)$. $f, g: X \to [0, +\infty]$.

1) f, g is simple. $f = \sum_{k=1}^{N_1} c_k \chi_{X_k}$ $g = \sum_{j=1}^{N_2} d_j \chi_{Y_j}$. $c_k, d_j \in [0, +\infty)$.

$\{X_k\}_{k=1}^{N_1}$. $\{Y_j\}_{j=1}^{N_2}$ mutually disjoint, partition X. and belongs to $\mathcal{A}$.

denote $A_{kj} = X_k \cap Y_j$. A_{kj} mutually disjoint and $\bigcup_{k=1}^{N_1} A_{kj} = Y_j$ $\bigcup_{j=1}^{N_2} A_{kj} = X_k$.

thus we get a new partition for $X = \bigcup_{k=1}^{N_1} \bigcup_{j=1}^{N_2} A_{kj}$

rearrange the set, denote $N = C_{\max(N_1, N_2)}^{\min\{N_1, N_2\}}$. $f = \sum_{k=1}^{N} c_k \chi_{X_k}$ $g = \sum_{k=1}^{N} d_k \chi_{X_k}$

$$\int_X fg \, d\mu = \int_X \sum_{k=1}^{N} c_k \chi_{X_k} \cdot \sum_{k=1}^{N} d_k \chi_{X_k} \, d\mu = \sum_{k=1}^{N} c_k d_k (\mu X_k)^2$$

$$\le \left(\sum_{k=1}^{N} c_k^p \mu X_k\right)^{\frac{1}{p}} \left(\sum_{k=1}^{N} d_k^q \mu X_k\right)^{\frac{1}{q}} = \left(\int f^p d\mu\right)^{\frac{1}{p}} \left(\int g^q d\mu\right)^{\frac{1}{q}}$$

$f \in L^p(X,\mu)$ $g \in L^q(X,\mu)$ consider increasing simple function $\{\varphi_n\}$ $\{\psi_n\}$.

$\varphi_n \le \varphi_{n+1}$ $f(x) = \lim\limits_{n\to+\infty} \varphi_n(x)$, $x \in X$

$\psi_n \le \psi_{n+1}$ $g(x) = \lim\limits_{n\to+\infty} \psi_n(x)$, $x \in X$.

Then $\varphi_n(x)\psi_n(x) \le \varphi_{n+1}(x)\psi_{n+1}(x)$, $fg(x) = \lim\limits_{n\to+\infty} \varphi_n(x)\psi_n(x)$

By Levy. thm. (2.5). $\int_X |fg|\,d\mu = \lim\limits_{n\to+\infty}\int_X \varphi_n\psi_n\,d\mu \le \lim\limits_{n\to+\infty}\left(\int_X |\varphi_n|^p d\mu\right)^{\frac{1}{p}}\left(\int_X |\psi_n|^q d\mu\right)^{\frac{1}{q}} = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}\left(\int_X |g|^q\right)^{\frac{1}{q}}$

4. Prove the General Lebesgue Dominated Convergence Theorem. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $[g_n]$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that

$$|f_n| \leq g_n \text{ on } E \text{ for all } n.$$

Prove that if

$$\lim_{n\to\infty} \int_E g_n = \int_E g < \infty,$$

then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Pf: we have $f_n \pm g_n \to f \pm g$ pointwisely a.e. on E.

notice that $|f_n| \leq g_n$ thus $f_n + g_n \geq 0$. $g_n - f_n \geq 0$.

then apply Fatou's thm for function for nonnegative function $f_n + g_n \geq 0$. $g_n - f_n \geq 0$.

$$\int_E f + \int_E g = \int_E (f+g) \leq \underline{\lim} \int_E (f_n + g_n) = \underline{\lim} \int_E f_n + \underline{\lim} \int_E g_n = \underline{\lim} \int_E f_n + \int_E g$$

Hence $\int_E f \leq \underline{\lim} \int f_n$. similarly $\overline{\lim} \int f_n \leq \int_E f$.

Which means the upper and lower limit coincides. i.e. $\lim\limits_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$.

5. Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

Pf: "$\Rightarrow$" $\int_E |f - f_n| \to 0$

$\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$, for $n > N$ $\int_E |f - f_n| < \varepsilon$.

by triangle inequality. $\left| \int_E |f| - \int_E |f_n| \right| = \int_E \big| |f| - |f_n| \big| \, d\mu \leq \int_E |f - f_n| \, d\mu < \varepsilon$.

i.e. $\int_E |f| = \lim\limits_{n\to\infty} \int_E f_n$

"$\Leftarrow$" denote $h_n = |f - f_n|$ $g_n = |f| + |f_n|$ $|h_n| \leq g_n$ for all n.

Consider the limit case. $h_n \to h \equiv 0$. a.e. on E.

$\lim\limits_{n\to\infty} \int g_n = \int |f| + \lim \int |f_n| = 2 \int |f| \, d\mu < \infty$ (by integrability of f on E).

then by general Lebesgue donimated conv. thm.

$\lim\limits_{n\to\infty} \int_E h_n = \int_E h = 0$, $\lim\limits_{n\to\infty} \int |f_n - f| = 0$, i.e. $\int_E |f_n - f| \to 0$.

6. For a measurable function f on $[1,\infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. Is it true that f is integrable over $[1,\infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? Is it true that f is integrable over $[1,\infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

1). Not true

$$f(x) = \begin{cases} 1 & x \in [n, n+\frac{1}{2}),\ n \in \mathbb{N} \\ -1 & x \in [n+\frac{1}{2}, n+1).\ n \in \mathbb{N}. \end{cases}$$

$a_n \equiv 0.$
$\Downarrow$
both conv. and abs conv.

$$\int_0^{+\infty} |f|\, dx = +\infty$$

Let $f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \chi_{[n,n+1)}$.

the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \int_n^{n+1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \chi_{[n,n+1)}\, d\mu = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

the series conv. to 0 but the integral even not exists.

since $\int_{[1,+\infty)} f_+ \ .\ \int_{[1,+\infty)} f_+ = +\infty$

2) Not true.

Let $f = \sum_{n=1}^{\infty} \chi_{[n,n+\frac{1}{2})} - \sum_{n=1}^{\infty} \chi_{[n+\frac{1}{2},n+1)}$

$a_n = \int_n^{n+1} f\, d\mu = \mu \chi_{[n,n+\frac{1}{2})} - \mu \chi_{[n+\frac{1}{2},n+1)} = 0$ $\quad \sum_{n=1}^{\infty} a_n$ conv.

but we also have $\int_{[1,+\infty)} f_+\, d\mu\ ,\ \int_{[1,+\infty)} f_-\, d\mu = +\infty$

Real Analysis 2024. Homework 7.

1. For f in $L^1[a,b]$, define $\|f\| = \int_a^b x^2|f(x)|dx$. Show that this is a norm on $L^1[a,b]$.

1). positive definite : $\|f\| = 0 \Leftrightarrow \int_a^b x^2 |f(x)| dx = 0$.

$\Leftrightarrow$ since $x^2|f(x)| \geq 0$. the integral $=0$ iff $x^2|f(x)| = 0$.

$x^2 \geq 0$, $|f(x)| \geq 0$ and $x^2 \equiv 0$ is impossible on $[a,b]$. thus $f \equiv 0$.

2). scalar multiplication. $\forall \alpha \in \mathbb{C}$.

$\|\alpha f\| = \int_a^b x^2 |\alpha f(x)| = \int_a^b x^2 |\alpha| \cdot |f(x)| dx = |\alpha| \int_a^b x^2 |f(x)| = |\alpha| \cdot \|f\|$.

3) triangular inequality. $f, g \in L^1[a,b]$

$\|f+g\| = \int_a^b x^2 |f+g| dx \leq \int_a^b x^2 (|f| + |g|) dx = \int_a^b x^2 |f| dx + \int_a^b x^2 |g| dx = \|f\| + \|g\|$

by numerical triangular inequality
and monotonicity of integral

2. Prove that $L^p[0,1] \neq L^q[0,1]$ if $p \neq q$.

Pf: w.l.g. let $p > q$. denote $a = \frac{p-q}{2}$ consider the function $f = x^{-\frac{1}{q+a}}$ on $[0,1]$.

$\int |f|^q d\mu = \int_0^1 x^{-\frac{q}{q+a}} = \frac{q+a}{a} x^{\frac{a}{q+a}} \Big|_0^1 = \frac{q+a}{a} < +\infty$ $\left(\frac{a}{q+a} > 0\right)$. $f \in L^q[0,1]$

$\int |f|^p d\mu = \int_0^1 x^{-\frac{p}{q+\frac{1}{2}}} = \frac{q+a}{-p+q+a} x^{\frac{-p+q+a}{q+a}} \Big|_0^1$

since $-p+q+a = \frac{p+q}{2} - p = \frac{q-p}{2} < 0$, $x^{\frac{-p+q+a}{q+a}} \to +\infty$ when $x \to 0$

$f \notin L^p[0,1]$ thus $L^p[0,1] \neq L^q[0,1]$

3. Assume that $\mu(E) < \infty$. Prove that

$$\|f\|_\infty = \lim_{p\to+\infty} \|f\|_p$$

for any measurable function f.

Pf: denote the set $E_f = \{x \mid f(x) > \|f\|_\infty\}$. $\mu E_f > 0$.

$\forall \varepsilon > 0$. denote a set $E_\varepsilon = \{x \mid |f(x) - \|f\|_\infty| < \varepsilon\}$.

we claim that $E_\varepsilon > 0$. otherwise $\|f\|_\infty + \varepsilon$ become the new essential bound.

on $E_\varepsilon \setminus E_f$. $\|f\|_\infty - \varepsilon < f(x) \le \|f\|_\infty$.

by monotonicity of integral. $\| \|f\|_\infty - \varepsilon \|_p \le \|f\|_p \le \|f\|_\infty$ on $E_\varepsilon \setminus E_f$.

i.e. $(\mu E_\varepsilon)^{\frac{1}{p}}(\|f\|_\infty - \varepsilon) \le (\int_{E_\varepsilon} |f|^p)^{\frac{1}{p}} \le (\int_E |f|^p)^{\frac{1}{p}} = \|f\|_p \le \|f\|_\infty$ on E.

$p \to +\infty$. $(\mu E_\varepsilon)^{\frac{1}{p}} \to 1$ $\quad \|f\|_\infty - \varepsilon \le \lim_{p\to+\infty} \|f\|_p \le \|f\|_\infty$. on E

since $\mu E < +\infty$, and $\varepsilon > 0$ is arbitrary. $\lim_{p\to+\infty} \|f\|_p = \|f\|_\infty$

$\forall \varepsilon > 0$. then denote $F = \{x \mid |f(x)| > \|f\|_\infty - \varepsilon\}$ $\mu F > 0$

In this case $\|f\|_p \ge \mu(F)^{\frac{1}{p}} (\|f\|_\infty - \varepsilon)$ $\quad$ $(\int (\|f\|_\infty - \varepsilon) d\mu)^{\frac{1}{p}}$

$p \to +\infty$ $\quad \lim_{p\to+\infty} \|f\|_p \ge \|f\|_\infty - \varepsilon$

$\varepsilon \to 0$ $\quad \lim_{p\to+\infty} \|f\|_p \ge \|f\|_\infty$

4. Suppose $1 \le p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(E)$. Show that $f = 0$ a.e. if and only if

$$\int_E fg = 0 \text{ for all } g \in L^q(E).$$

Pf: "$\Rightarrow$" by Hölder's inequality,

$$\int_E |fg| \le \left(\int_E |f|^p\right)^{\frac{1}{p}} \left(\int_E |g|^q\right)^{\frac{1}{q}} \le 0 \cdot \left(\int_E |g|^q\right)^{\frac{1}{q}}.$$

$g \in L^q(E)$ implies $\left(\int_E |g|^q\right)^{\frac{1}{q}} \le \left(\int_E |g|^q\right)^{\frac{1}{q}} < +\infty$ $\quad f(x) = |f| e^{i\theta(x)}$

Let $g(x) = e^{-i\theta(x)} |f|^{p-1}$ $\quad \|g\|_q^q = \int_E |g|^q = \int_E |f|^{q(p-1)} = \|f\|_p^p < \infty$

thus $\int_E |fg| = 0. \Rightarrow \int_E fg = 0.$ $\quad g \in L^q(E)$ $\quad 0 = \int_E f \cdot g = \int_E |f|^p$

"$\Leftarrow$" assume $\exists E_1 \subset E$. s.t. $f \ne 0$ and $E_1 > 0$ w.l.g. $f > 0$ on E_1.

let $g = \chi_{E_1} \in L^p(E)$

$\int_E fg \ge \int_{E_1} fg \, d\mu = \int_{E_1} f \, d\mu > 0$ which contradicts with $\int_E fg = 0$.

5. (The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f. For $1 \le p < \infty$, suppose there is a function g in $L^p(E)$ such that for all n, $|f_n| \le g$ a.e. on E. Prove that $f_n \to f$ in $L^p(E)$.

Pf: $|f_n| \le g \Rightarrow |f_n|^p \le g^p$ since $p \in [1, +\infty)$ pass the limit $|f|^p \le g^p$

and $|f_n|^p + g^p \ge 0$. $g^p - |f_n|^p$, by Fatou's thm.

$$\int f^p + \int g^p = \int (f^p + g^p) \le \underline{\lim} \int (g^p + f_n^p) = \int g^p + \underline{\lim} \int f_n^p. \Rightarrow \int f^p \le \underline{\lim} \int f_n^p$$

$$\int g^p - \int f^p = \int (g^p - f^p) \le \underline{\lim} \int (g^p - f_n^p) = \int g^p - \overline{\lim} \int f_n^p = \int g^p + \overline{\lim} \int_E (-f_n^p)$$

$$\Rightarrow \int f^p \le \underline{\lim} \int f_n^p \le \overline{\lim} \int f_n^p \le \int f^p \Rightarrow \lim \int f_n^p = \int f^p$$

denote $F_n = |f_n - f|$. $F_n \le 2g$ and $F_n \to 0$.

apply the previous result. $\lim \int F_n^p = \int 0^p = 0$

i.e. $\lim \int_E |f - f_n|^p = 0$

$|f_n| \le g$. $|f_n - f| \le 2g$. $|f_n - f|^p \to |f - f|^p$

by dominated conv. thm. $\int_E |f_n - f|^p d\mu \to \int_E |f - f|^p d\mu = 0$. and $f_n \to f$ in $L^p(E)$.

Real Analysis 2024. Homework 8.

1. Find condition on p under which the integral is finite

$$\iint_E \frac{dxdy}{(x^2+y^2)^p}. \quad E = \{(x,y) : 0 < x < 1,\ 0 < y < x^2\}.$$

Solution: consider the polar change. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad \begin{cases} 0<x<1 \\ 0<y<x^2 \end{cases} \Rightarrow \begin{cases} \theta \in [0, \frac{\pi}{4}] \\ \tan\theta\sec\theta < r < \sec\theta \end{cases}$

$$\iint_E \frac{r}{(r^2)^p}\, dr d\theta = \int_0^{\frac{\pi}{4}} d\theta \int_{\tan\theta\sec\theta}^{\sec\theta} r^{1-2p}\, dr.$$

1) $p=1$ $\quad \int_0^{\frac{\pi}{4}} \ln|\cot\theta|\, d\theta = -\int_0^{\frac{\pi}{4}} \ln|\tan\theta|\, d\theta \quad \frac{\tan\theta = t}{d\theta = \frac{1}{t^2+1}} \quad -\int_0^1 \ln|t| \cdot \frac{1}{t^2+1}\, dt = -\ln|t| \cdot \arctan t \Big|_0^1 + \int_0^1 \frac{\arctan t}{t}\, dt.$

$$-\ln|t| \cdot \arctan t = 0 + \lim_{t\to 0^+} \ln t \cdot \arctan t = \lim_{t\to 0^+} t \ln t = \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} = \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} = 0.$$

$$\int_0^1 \frac{\arctan t}{t}\, dt = \int_0^1 \frac{t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - o(t^5)}{t}\, dt = \int_0^1 1 - \frac{1}{3}t^2 + \frac{1}{5}t^4 - o(t^4)\, dt \leq \int_0^1 1 - \frac{1}{3}t^2 + \frac{1}{5}t^4 < +\infty$$

2) $p \neq 1$. $\quad \int_0^{\frac{\pi}{4}} \frac{1}{2-2p} \cdot r^{2-2p} \Big|_{\tan\theta\cdot\sec\theta}^{\sec\theta} = \frac{1}{2-2p} \int_0^{\frac{\pi}{4}} \frac{1}{\cos^{2-2p}} - \left(\frac{\sin\theta}{\cos^2\theta}\right)^{2-2p} d\theta.$

$\frac{\sqrt{2}}{2} \leq \cos\theta \leq 1$, on $[0, \frac{\pi}{4}]$. thus $\int_0^{\frac{\pi}{4}} \cos^{2p-2}\theta < +\infty.$

and $\int_0^{\frac{\pi}{4}} \sin\theta^{2-2p}$ will be finite/infinite simultaneously. with $\int_0^{\frac{\pi}{4}} \left(\frac{\sin\theta}{\cos^2\theta}\right)^{2-2p} d\theta.$

Since $\lim_{\theta\to 0} (\cos^2\theta)^{2-2p} = 1. \neq 0.$ and then apply the comparation test.

Then consider the integral $\int_0^{\frac{\pi}{4}} (\sin\theta)^{2-2p}\, d\theta$

$p<1$. $2-2p > 0$. $\quad (\sin\theta)^{2-2p} < 1$ on $[0, \frac{\pi}{4}]$, finite.

$p>1$ since $\frac{\sqrt{2}}{2}x < \sin x < x$ on $(0, \frac{\pi}{4})$. $\Rightarrow \frac{1}{x} < \frac{1}{\sin x} < \frac{\sqrt{2}}{x}$ on $(0, \frac{\pi}{4})$.

$\frac{1}{x}$, $\frac{\sqrt{2}}{x}$ will be infinite/finite simultaneously. $\int_0^{\frac{\pi}{4}} \left(\frac{1}{x}\right)^{2p-2} dx. \Rightarrow$ finite. iff $p < \frac{3}{2}$.

thus integral is finite iff $p < \frac{3}{2}$

$$I = \iint_E \frac{dx\,dy}{(x^2+y^2)^p} = \int_0^1 dx \int_0^{x^2} \frac{dy}{(x^2+y^2)^p} \quad \frac{y = x^2 t}{dy = x^2 dt} \quad \int_0^1 dx \int_0^1 \frac{x^2\, dt}{(x^2 + x^4 t^2)^p} = \int_0^1 \frac{dx}{x^{2p-2}} \int_0^1 \frac{dy}{(1+t^2x^2)^p}$$

$\frac{1}{2^p} \leq \frac{1}{(1+t^2x^2)^p} \leq 1 \quad 0 \leq t, x \leq 1. \qquad \frac{1}{2^p} \leq \frac{1}{(1+t^2x^2)^p} \leq 1. \qquad \frac{1}{2^p}\int_0^1 \frac{dx}{x^{2p-2}} \leq I \leq \int_0^1 \frac{dx}{x^{2p-2}}$

2. Find condition on p under which the integral is finite

$$\iint_{x+y>1} \frac{\sin x \sin y}{(x+y)^p} dxdy.$$

$\begin{cases} u=x \\ v=x+y \end{cases}$ $J=1.$ $I=\int_{v>1} \frac{|\sin u \sin(v-u)|}{v^p} du\, dv.$

Caution: the integrability of f is equivalent to integrability of $|f|$!

$F(u)=\int_1^{+\infty} \frac{|\sin(v-u)|}{v^p} dv.$ p.d. and has period $\to \pi$.

$I=\int_{-\infty}^{+\infty} F(u)|\sin u| du = \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} F(u)|\sin u| du.$

Δ: $\begin{cases} u=x+y \\ v=x-y \end{cases}$ $|J|^{-1}=2$, $|J|=\frac{1}{2}$.

$$\iint_{x+y>1} \left|\frac{\sin x \sin y}{(x+y)^p}\right| dxdy = \frac{1}{2}\iint_{u>1} \left|\frac{\sin\frac{u+v}{2}\sin\frac{u-v}{2}}{u^p}\right| du = \frac{1}{4}\iint_{\substack{u>1\\ v\in\mathbb{R}}} \left|\frac{\cos v-\cos u}{u^p}\right| du\,dv$$

$$\leq \frac{1}{4} \lim_{\substack{A\to+\infty\\ n\to+\infty}} \int_1^A du \int_{-n\pi}^{n\pi} \left|\frac{\cos v-\cos u}{u^p}\right| dv = \frac{1}{4}\lim_{\substack{A\to+\infty\\ n\to+\infty}} 2n \int_1^A \frac{|\cos u|}{u^p} du = \frac{1}{4}\int_1^{+\infty} \frac{|\cos u|}{u^p} du \cdot \lim_{n\to+\infty} 2n\pi.$$

$\Downarrow$

since $\left|\frac{\cos v-\cos u}{u^p}\right| \geq 0.$ if $\lim_{n\to+\infty}\int_{-n\pi}^{n\pi} \left|\frac{\cos v-\cos u}{u^p}\right|$ div. then $\int_{-\infty}^{+\infty}\left|\frac{\cos v-\cos u}{u^p}\right|$

$\int_1^{+\infty} |\cos u|$ is bounded. $\int_1^{+\infty} \frac{|\cos u|}{u^p} du$ either finite and $\neq 0$ or infinite.

thus. $\frac{1}{4}\int_1^{+\infty} \frac{|\cos u|}{u^p} du \cdot \lim_{n\to+\infty} 2n\pi. \to +\infty$ thus the integral is infinite for any $p\in\mathbb{R}$.

3. Find condition on p under which the integral is finite

$$\int_E \int \frac{dxdydz}{|x+y-z|^p}, \quad E=[-1,1]^3. \to$$ divide into 2 equal parts by plane $z=x+y$.

$I = 2\int_{\substack{-1\leq x,y\leq 1\\ x+y\leq 1}} dx\,dy \int_{x+y}^1 \frac{dz}{(z-x-y)^p}$

iff $p<1$. $\int_{x+y}^1 \frac{dz}{(z-x-y)^p} = \frac{1}{1-p}\frac{1}{(1-x-y)^{p-1}}$ conv.

$I=\frac{2}{1-p}\int_{\substack{-1\leq x,y\leq 1\\ x+y\leq 1}} (1-x-y)^{1-p} dx\,dy$

$\{-1\leq x,y\leq 1;\ x+y\leq 1\}.$

consider the variable change

$\begin{cases} x+y-z=u \\ y=v \\ z=w. \end{cases} \Rightarrow \begin{cases} x=u-v+w \\ y=v \\ z=w \end{cases}$

$$\int_{-1}^1 dv \int_{-1}^1 dw \int_{-1+v-w}^{1+v-w} \frac{1}{u^p} du.$$

if $p=1$. $\int_{-1}^1 dv \int_{-1}^1 \ln\left|\frac{1+v-w}{-1+v-w}\right| dw \xlongequal[|J|=\frac{1}{2}]{\begin{cases} a=v-w \\ b=v+w \end{cases}} \frac{1}{2}\int_{-1}^1 db \int_{-1}^1 \ln\left|\frac{1+a}{1-a}\right| da = (a+1)\ln\frac{a+1}{1-a}\Big|_{-1}^1 - \int_{-1}^1 (1-a)da$

$(a+1)\ln\frac{a+1}{a-1}\Big|_{-1}^1 = \lim_{a\to 1^-}(a+1)\cdot\ln\frac{a+1}{1-a} + \lim_{a\to -1^+}(a+1)\ln\frac{1-a}{a+1} = +\infty$ $\int_{-1}^1 (1-a)\,da < +\infty.$ infinite.

if $p\neq 1$. $\int_{-1}^1 dv \int_{-1}^1 dw \cdot u^{1-p}\Big|_{-1+v-w}^{1+v-w} = \int_{-1}^1 dv \int_{-1}^1 (1+v-w)^{1-p} - (-1+v-w)^{1-p}\, dw$

by thm 4.5. we have $p<1$ finite $p>1$ infinite.

thus integral is finite iff $p<1$.