Chapter I: Metric spaces

Motivation

Recall the following definitions and statements from mathematical analysis:

- A function $f\colon \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R}$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(x')| < \delta$ whenever $|x - x'| < \varepsilon$.
- A sequence $\{x_n\}$ of real numbers converges to $x \in \mathbb{R}$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $|x - x_n| < \varepsilon$ for any $n \geq N$.
- A sequence $\{x_n\}$ in $\mathbb{R}$ is a Cauchy sequence if $\lim |x_n - x_m| = 0$.
- Cauchy's criterion for convergence: Any Cauchy sequence converges in $\mathbb{R}$.
- A sequence $\{x_n\}$ in $\mathbb{R}$ is bounded if there exists $M \in \mathbb{R}$ such that $|x_n| \leq M$, or equivalently, if there exists $M' \in \mathbb{R}$ such that $|x_n - x_m| \leq M'$ for any $n, m \in \mathbb{N}$.
- Bolzano–Weierstrass theorem: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

All of them are based on the quantity $|x-y|, x, y \in \mathbb{R}$. What are properties of this function?

I. $|x - y| = 0$ iff $x = y$;

II. $|x - y| = |y - x|$ for any x, y;

III. $|x - y| \leq |x - z| + |z - y|$ for any x, y, z.

1 Definition and examples

Definition. A function $d\colon X \times X \to \mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$ is a **metric** (度量) (or a **a distance function** (距离函数)) on a set X if

I. $d(x, y) = 0$ iff $x = y$;

II. $d(x, y) = d(y, x)$ for any $x, y \in X$;

III. $d(x, y) \leq d(x, z) + d(z, y)$ for any $x, y, z \in X$ (*the triangle inequality*).

The pair (X, d), where d is a metric on X, is a **metric space** (度量空间).

Examples. 1. $X = \mathbb{R}, d(x, y) = |x - y|$. This is the default metric for $\mathbb{R}$.

2. $X = \mathbb{R}^n, d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ (*Euclidean distance*)

3. $X = \mathbb{R}^n, d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sum_{i=1}^n |x_i - y_i|$ (*Manhattan distance*)

4. $X = \mathbb{R}^n, d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max_{1 \leq i \leq n} |x_i - y_i|$ (*Chebyshev distance*)

5. $X = C([a, b]), d(f, g) = \int_a^b |f(t) - g(t)| dt$

6. Let X be an Euclidean space and $d(x, y) = ||x - y||$

7. Let X be the set of N-digit numbers and $d(n, m)$ be the number of distinct digits in n and m (*Hamming distance*)

8. Let $X = \{x, y, z, w\}$ and $d(x, y) = d(x, z) = d(y, z) = 2, d(x, w) = d(y, w) = d(z, w) = 1$ with $d(a, a) = 0, d(a, b) = d(b, a)$ for any $a, b \in X$.

9. Let X be any set and
$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases} \quad (\textbf{discrete metric} \text{ (离散度量)})$$

10. Let p be a fixed prime and for $n \in \mathbb{N}$ define $\nu_p(n) = k$, where $p^k \mid n$ and $p^{k+1} \nmid n$. Extend ν_p to $\mathbb{Q}$ by setting
$$\nu_p(\tfrac{a}{b}) = \begin{cases} \nu_p(a) - \nu_p(b) & \text{if } \frac{a}{b} \neq 0, \\ \infty & \text{if } \frac{a}{b} = 0 \end{cases} \quad (p\text{-}adic \ norm)$$
$X = \mathbb{Q}, d(x, y) = |x - y|_p$, where $|r|_p = p^{-\nu_p(r)}$ (*p-adic metric*)

11. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$. The relation $x \sim y$ if $x = -y$ is an equivalence relation on S^n and $[x] = \{x, -x\}$ is the equivalence class of $x \in S^n$. Let $X = \mathbb{P}^n = \{[x] \mid x \in S^n\}$ (*n-dimension projective space*) and $d([x], [y]) = \min(||x - y||, ||x + y||)$

12. If (X, d) is a metric space and $X' \subset X$ then $(X', d|_{X'})$ is a metric space called a **metric subspace** (度量子空间) of (X, d).

Exercise 1.1. *Which of the following functions are metrics on* $\mathbb{R}^2$*:*

i) $d((x_1, x_2), (y_1, y_2)) = \min(|x_1 - y_1|, |x_2 - y_2|)$

ii) $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|^2 + |x_2 - y_2|^2$

iii) $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + 2|x_2 - y_2|$

Exercise 1.2. *Prove that if d and d' are two metrics on X, then $d + d'$ and $\max(d, d')$ also are metrics on X.*

Exercise 1.3. *Let d be a metric on X and d' be a metric on X'. Prove that*
$$D((x, x'), (y, y')) = \max(d(x, y), d'(x', y')) \quad (\textit{product metric})$$
is a metric on $X \times X'$.

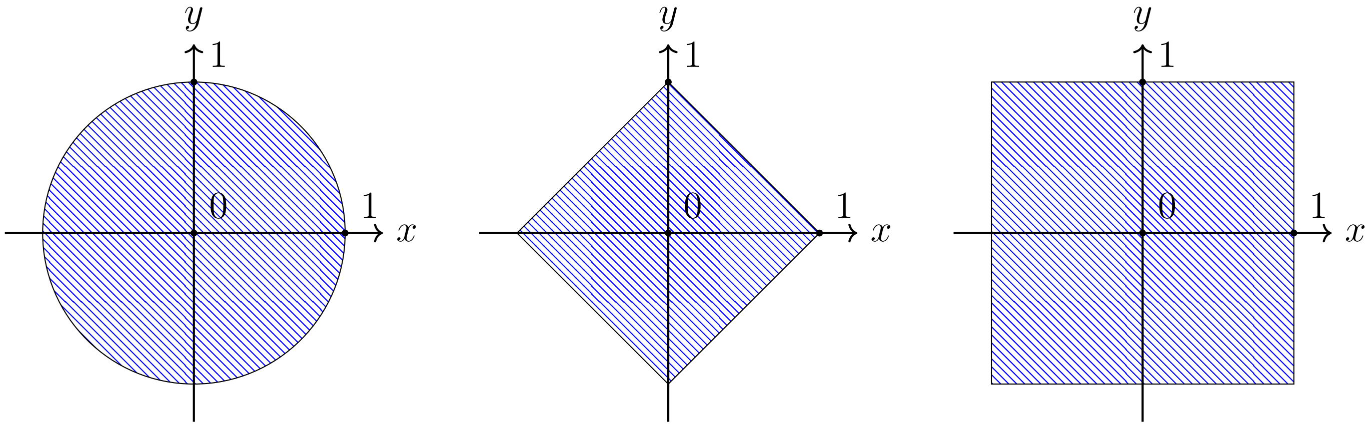
Exercise 1.4. *Let (X,d) be a metric space and $x,y,z,w \in X$. Show that*

$$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w).$$

Definition. Let (X,d) be a metric space. The **open ball** (开球) $B_r(x)$ with center $x \in X$ and radius $r > 0$ is the set

$$B_r(x) = \{y \in X \mid d(x,y) < r\}.$$

Examples. The open balls $B_1((0,0))$ in $\mathbb{R}^2$ for the Euclidean, Manhattan, and Chebyshev metrics, respectively.



Exercise 1.5. *Let (X,d) be a metric space and $x \in X$. Prove that $\cup_{r>0} B_r(x) = X$ and $\cap_{r>0} B_r(x) = \{x\}$.*

Exercise 1.6. *Let d be a metric on X and d' be a metric on X'. Prove that $B_r((x,x')) = B_r(x) \times B_r(x')$, where $X \times X'$ is equipped with the product metric (Exercise 1.3).*

2 Continuity and limits

Definition. Let (X,d) and (Y,d') be metric spaces. A map $f \colon X \to Y$ is **continuous** (连续的) at $x \in X$ if for any $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x), f(x')) < \varepsilon$ whenever $d(x,x') < \delta, x' \in X$. In other words, f is continuous at x if for any $\varepsilon > 0$ there is $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$.

The map f is **continuous** if it is continuous at each point of X.

Examples.

1. If $(X,d) = (Y,d')$, the map id_X is continuous.
2. If X has the discrete metric, any map $f \colon X \to Y$ is continuous.
3. If $X = Y = \mathbb{R}$ and X has the Euclidean metric, Y has the discrete metric, $\mathrm{id}_{\mathbb{R}}$ is not continuous at any point.

Exercise 2.1. *Show that the function $f(x) = x^2$*

i) $f \colon (\mathbb{Q}, |\cdot|) \to (\mathbb{Q}, |\cdot|_p)$ *is not continuous*

ii) $f\colon (\mathbb{Q}, |\cdot|_p) \to (\mathbb{Q}, |\cdot|)$ *is not continuous*

iii) $f\colon (\mathbb{Q}, |\cdot|_p) \to (\mathbb{Q}, |\cdot|_p)$ *is continuous*

Proposition 2.1. *Let* (X, d), (Y, d'), (Z, d'') *be metric spaces. Let* $f\colon X \to Y$ *be continuous at* $x \in X$ *and let* $g\colon Y \to Z$ *be continuous at* $f(x)$*. Then* $g \circ f\colon X \to Z$ *is continuous at* x*.*

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(x)$, there exists $\gamma > 0$ such that $g(B_\gamma(f(x))) \subset B_\varepsilon(g(f(x)))$. Since f is continuous at x, there exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\gamma((f(x))$. Thus $g(f(B_\delta(x))) \subset g(B_\gamma(f(x))) \subset B_\varepsilon(g(f(x)))$. □

Exercise 2.2. *Let* (X, d) *and* (Y, d') *be metric spaces and* $A \subset X, x \in A$*. Prove that if* $f\colon X \to Y$ *is continuous at* x *then* $f|_A\colon A \to Y$ *is continuous at* x*, where the metric on* A *is the restriction of* d*.*

Definition. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X **converges** (收斂) to $x \in X$ if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(x, x_n) < \varepsilon$ for any $n \geq N$ and $x = \lim x_n$ is the **limit** (极限) of $\{x_n\}$. In other words, $\lim x_n = x$ if and only for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $x_n \in B_\varepsilon(x)$ for any $n \geq N$. In yet other words, $\lim x_n = x$ if and only if $\lim d(x, x_n) = 0$.

Examples. 1. The sequence $\{p^n\}$ in $(\mathbb{Q}, |\cdot|_p)$ converges to 0.

2. The sequence $\{1/n\}$ in $(0, 1]$ does not converge.

Proposition 2.2. *Let* (X, d) *be a metric space. If a sequence* $\{x_n\}$ *in* X *has a limit, then it is unique.*

Proof. Assume that x, x' are two limits of $\{x_n\}$. Then there are $N, N' \in \mathbb{N}$ such that $d(x, x_n) < d(x, x')/2$ for any $n \geq N$ and $d(x', x_n) < d(x, x')/2$ for any $n \geq N'$. Assuming that $N \geq N'$ one has $d(x, x') \leq d(x, x_N) + d(x', x_N) < d(x, x')/2 + d(x, x')/2 = d(x, x')$, a contradiction. □

Exercise 2.3. *Show that every subsequence of a convergent sequence in a metric space converges to the same limit.*

Exercise 2.4. *Prove that a sequence* $\{x_n\}$ *in a discrete metric space* X *converges if and only if there are* $x \in X$ *and* $N \in \mathbb{N}$ *such that* $x_n = x$ *for all* $n \geq N$*.*

Exercise 2.5. *Which of the following sequences converge in* $\mathbb{Q}$ *with respect to the p-adic metric:*

i) $\{p^n/(p^n - 1)\}$

ii) $\{n!\}$

iii) $\{1/n\}$

Proposition 2.3. *Let* $(X, d), (Y, d')$ *be metric spaces. A map* $f\colon X \to Y$ *is continuous at* $x \in X$ *if and only if* $\lim f(x_n) = f(x)$ *for any sequence* $\{x_n\}$ *in* X *with* $\lim x_n = x$*.*

Proof. Suppose f is continuous at x and $\lim x_n = x$. For any $\varepsilon > 0$ there is δ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. If $\lim x_n = x$ then there is $N \in \mathbb{N}$ such that $x_n \in B_\delta(x)$ for any $n \geq N$. Then $f(x_n) \in f(B_\delta(x)) \subset B_\varepsilon(f(x))$ for any $n \geq N$.

Conversely, suppose $\lim f(x_n) = f(x)$ for any sequence $\{x_n\}$ in X such that $\lim x_n = x$. If f is not continuous at x, there is $\varepsilon > 0$ such that $f(B_{1/n}(x)) \not\subset B_\varepsilon(f(x))$ for any $n \in \mathbb{N}$. Let $y_n \in f(B_{1/n}(x)) \setminus B_\varepsilon(f(x))$ and $y_n = f(x_n), x_n \in B_{1/n}(x)$. Then $\lim x_n = x$ and $\{f(x_n)\}$ does not converge to $f(x)$, a contradiction. □

Exercise 2.6. *Let (X, d) be a metric space.*

i) *Let $y \in X$. Prove that $f \colon X \to \mathbb{R}, f(x) = d(x, y)$ is continuous.*

ii) *Prove that $d \colon X \times X \to \mathbb{R}$ is continuous, where $X \times X$ is equipped with the product metric (see Exercise 1.3)*

3 Open subsets

Definition. Let (X, d) be a metric space and $x \in X$. A subset A of X is **open** (开的) if for any $a \in A$ there is $\delta > 0$ such that $B_\delta(x) \subset A$.

Examples. 1. (a, b) and (a, ∞) are open in $\mathbb{R}$.

2. $[a, b]$ is not open in $\mathbb{R}$.

3. In a discrete metric space, any subset is open.

Proposition 3.1. *Let (X, d) be a metric space. Then*

1. *X and $\varnothing$ are open.*

2. *If $U_1, \ldots, U_k$ are open subsets of X, then $\cap_{1\leq i\leq k} U_i$ is also open.*

3. *If $\{U_\omega\}_{\omega\in\Omega}$ is a family of open subsets of X then $\cup_{\omega\in\Omega} U_\omega$ is also open.*

4. *$B_r(x)$ is open for any $r > 0, x \in X$.*

Proof. 1. Trival

2. Let $x \in \cap_{1\leq i\leq k} U_i$. For any $1 \leq i \leq k$, there is $\delta_i > 0$ such that $B_{\delta_i}(x) \subset U_i$. If $\delta_j \leq \delta_i$ for any $1 \leq i \leq k$, then $B_{\delta_j}(x) \subset B_{\delta_i}(x) \subset U_i$ for any $1 \leq i \leq k$, thus $B_{\delta_j}(x) \subset \cap_{1\leq i\leq k} U_i$.

3. Let $x \in \cup_{\omega\in\Omega} U_\omega$. Then $x \in U_\omega$ for a certain $\omega \in \Omega$ and there is $\delta > 0$ such that $B_\delta(x) \subset U_\omega \subset \cup_{\omega\in\Omega} U_\omega$.

4. Let $y \in B_r(x)$. It suffices to find an open ball centered at y that is contained in $B_r(x)$. Let $s = r - d(x, y)$. Then $s > 0$ and if $z \in B_s(y)$, then $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + s = r$, so that $z \in B_r(x)$.

□

Remark. If $\{U_\omega\}_{\omega\in\Omega}$ is a family of open subsets of X then $\cap_{\omega\in\Omega}U_\omega$ may not be open. For example, $\cap_{n\in\mathbb{N}}(-1/n, 1/n) = \{0\}$ in $\mathbb{R}$.

Exercise 3.1. *Prove that in a finite metric space, any subset is open.*

Exercise 3.2. *Let U be an open subset of a metric space (X, d). Show that $U \setminus \{y\}$ is also open for any $y \in U$.*

Exercise 3.3. *Let X, Y be two metric spaces and $f\colon X \to Y$. Let $X = A \cup B$ for open subsets $A, B \subset X$. Prove that if $f|_A$ and $f|_B$ are continuous then f itself is continuous.*

Proposition 3.2. *A subset U of a metric space (X, d) is open if and only if it is a union of open balls.*

Proof. If U is open then for any $x \in U$ there is $\delta_x > 0$ such that $B_{\delta_x}(x) \subset U$. Then $U = \cup_{x\in U}B_{\delta_x}(x)$. The inverse assertion follows from Proposition 3.1. □

Corollary 3.3. *Let X, Y be metric spaces. A subset of $X \times Y$ is open in the product metric if and only if it is a union of subsets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.*

Proof. Follows from Proposition 3.2 and Exercise 1.6. □

Exercise 3.4. *Let X, Y be two metric spaces and let $X \times Y$ be equipped with the product metric (Exercise 1.3).*

i) *Prove that $A \times B$ is open in $X \times Y$ if $A \subset X, B \subset Y$ are open.*

ii) *Prove that the projection map $\pi_X\colon X \times Y \to X, \pi_X((x, y)) = x$ is continuous.*

Hint. Use Exercise 1.6.

Proposition 3.4. *Let (X, d) be a metric space and $Y \subset X$ be its metric subspace. Then $A \subset Y$ is an open subset of Y if and only if $A = Y \cap B$, where B is an open subset of X.*

Proof. First notice that

$$B^Y_r(x) = \{y \in Y \mid d(x, y) < r\} = \{y \in X \mid d(x, y) < r\} \cap Y = B^X_r(x) \cap Y,$$

where B^X, B^Y are open balls in X, Y, respectively.

Let $x \in A$. If $A = Y \cap B$, where B is an open subset of X, then $x \in B$ and there is $\delta > 0$ such that $B^X_\delta(x) \subset B$. Then $B^Y_\delta(x) = B^X_\delta(x) \cap Y \subset B \cap Y = A$.

Conversely, if A is an open subset of Y then $A = \cup_{\omega\in\Omega}B^Y_\omega$ by Proposition 3.2, where B^Y_ω are open balls in Y. Then

$$A = \cup_{\omega\in\Omega}B^Y_\omega = \cup_{\omega\in\Omega}(B^X_\omega \cap Y) = (\cup_{\omega\in\Omega}B^X_\omega) \cap Y$$

and $\cup_{\omega\in\Omega}B^X_\omega$ is open in X. □

Example. Let $Y = [0,1] \cup [2,3]$ be a metric subspace of $\mathbb{R}$. Then $[0,1]$ is an open subset of Y since $[0,1] = (-1,1.5) \cap Y$ and $(-1,1.5)$ is open in $\mathbb{R}$.

Exercise 3.5. *Prove that in the above example $[0,1]$ is an open subset of Y using the definition of open subset.*

Proposition 3.5. *Let $(X,d),(Y,d')$ be metric spaces. Then $f\colon X \to Y$ is continuous if and only if for each open subset $U \subset Y$, the inverse image $f^{-1}(U)$ is an open subset of X.*

Proof. First, suppose f is continuous and U is open, $x \in f^{-1}(U)$. Choose $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subset U$. There exists $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x)) \subset U$ which implies $B_\delta(x) \subset f^{-1}(U)$.

Conversely, suppose that the inverse image of any open subset of Y is open. Let $x \in X$ and $\varepsilon > 0$. Then $f^{-1}(B_\varepsilon(f(x)))$ is open and there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$. Then $f(B_\delta(x)) \subset B_\varepsilon(f(x))$. □

Remark. The image of an open set A under a continuous map f may not be open. Consider $f\colon \mathbb{R} \to \mathbb{R}, f(x) = x^2$ and $A = (-1,1)$.

Definition. Two metrics on the same set X are **equivalent** (等价的) if the families of all open subsets of X with respect to them coincide.

Proposition 3.6. *Two metrics d and d' on X are equivalent if there are $c, C > 0$ such that $cd(x,y) \le d'(x,y) \le Cd(x,y)$ for any $x,y \in X$.*

Proof. First, we prove that any open subset defined by d' is open with respect to d provided that $d'(x,y) \le Cd(x,y)$ for any $x,y \in X$. The inequality implies that $B_r(x) \subset B'_{Cr}(x)$ where the former is the open ball with respect to d and the latter is the open ball with respect to d'. If U is open with respect to d' then for any $x \in U$ there is $\varepsilon > 0$ such that $B'_\varepsilon(x) \subset U$. Now $B_{\varepsilon/C}(x) \subset B'_\varepsilon(x) \subset U$.

Finally the inequality $cd(x,y) \le d'(x,y)$ implies $d(x,y) \le c^{-1}d'(x,y)$, whence any open subset defined by d is open with respect to d'. □

Corollary 3.7. *The Euclidean, Manhattan, and Chebyshev metrics on $\mathbb{R}^n$ are equivalent.*

Proof.

$$\max_{1\le i\le n} |x_i - y_i| \le \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \le \sum_{i=1}^{n} |x_i - y_i| \le n \max_{1\le i\le n} |x_i - y_i|$$

□

Exercise 3.6. *Let (X,d) be a metric space.*

i) *Prove that $d'(x,y) = \sqrt{d(x,y)}$ is a metric on X.*

ii) *Prove that d' is equivalent to d.*

Definition. A family of open subsets $\mathcal{B} = \{U_\omega\}_{\omega\in\Omega}$ of a metric space X is its **base** (拓扑基) if for each open $U \subset X$ and each $x \in U$ there is $U_\omega \in \mathcal{B}$ such that $x \in U_\omega \subset U$.

Examples. 1. The family of all open balls is a base

2. $\{B_{1/n}(q)\}_{n\in\mathbb{N},q\in\mathbb{Q}}$ is a base of $\mathbb{R}$

3. Any base of a discrete metric space X must contain all the singleton subsets of M.

Exercise 3.7. *Let $\mathcal{B}$ be a base of a metric space Y.*

1. *Prove that any open subset of Y is the union of elements of $\mathcal{B}$.*

2. *Let X be a metric spaces and $f\colon X \to Y$. Show that f is continuous if $f^{-1}(U)$ is open in X for any $U \in \mathcal{B}$.*

4 Closed subsets

Definition. A subset of a metric space is **closed** (闭的) if its complement is open.

Examples. 1. $[a, b]$ is closed in $\mathbb{R}$. In particular, $\{a\}$ is closed.

2. In a discrete metric space, any subset is closed.

3. A line in $\mathbb{R}^2$ with Euclidean metric is closed.

Proposition 4.1. *Let (X, d) be a metric space. Then*

1. *X and $\varnothing$ are closed.*

2. *If $U_1, \ldots, U_k$ are closed subsets of X, then $\cup_{1\le i\le k} U_i$ is also closed.*

3. *If $\{U_\omega\}_{\omega\in\Omega}$ is a family of closed subsets of X then $\cap_{\omega\in\Omega} U_\omega$ is also closed.*

Proof. Follows from Propostion 3.1 and the formulas

$$X \setminus \cup_{\omega\in\Omega} U_\omega = \cap_{\omega\in\Omega} X \setminus U_\omega, \quad X \setminus \cap_{\omega\in\Omega} U_\omega = \cup_{\omega\in\Omega} X \setminus U_\omega.$$

□

Exercise 4.1. *The closed ball with center $x \in X$ and radius $r > 0$ is defined as*

$$\overline{B}_r(x) = \{y \in X \mid d(x, y) \le r\}.$$

Prove that the closed ball is closed.

Exercise 4.2. *Let (X, d) be a metric space.*

i) *Prove that any open subset of X can be expressed as a union of closed balls.*

ii) *Prove that any closed subset of X can be expressed as an intersection of open sets.*

Exercise 4.3. *Let X, Y be two metric spaces and $A \subset X, B \subset Y$ be closed. Prove that $A \times B$ is closed in $X \times Y$ equipped with the product metric (Exercise 1.3).*

Hint. Use Exercise 1.6 and express $X \times Y \setminus A \times B$ as a union of open subsets.

Proposition 4.2. *Let $(X, d), (Y, d')$ be metric spaces. Then $f \colon X \to Y$ is continuous if and only if for each closed subset $F \subset Y$, the inverse image $f^{-1}(F)$ is a closed subset of X.*

Proof. Follows from Propostion 3.5 and the formula $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. □

Remark. The image of a closed set A under a continuous map f may not be open. Consider $f \colon \mathbb{R} \to \mathbb{R}, f(x) = e^x$ and $A = \mathbb{R}$.

Definition. Let A be a subset of a metric space (X, d). A point $x \in X$ is **adherent** (聚点) to A if $B_r(x) \cap A \neq \varnothing$ for all $r > 0$. The **closure** (闭包) $\overline{A}$ of A is the set of all points adherent to A. Clearly, $A \subset \overline{A}$.

The subset A is **dense** (稠密的) in X if $\overline{A} = X$.

Example. The closure of (a, b) in $\mathbb{R}$ is $[a, b]$.

Remark. The closure of the open ball $B_r(x)$ may not be the closed ball $\overline{B}_r(x)$. Consider an open ball of radius 1 in a discrete metric space.

Proposition 4.3. *If A is a subset of a metric space (X, d).*

1. *$\overline{A}$ is closed.*

2. *$\overline{A}$ is the intersection of all closed subsets containing A.*

Proof. 1. If $x \in X \setminus \overline{A}$ then there is $r > 0$ such that $B_r(x) \cap A = \varnothing$. For any $y \in B_r(x)$ one has $B_{r-d(x,y)}(y) \subset B_r(x) \subset X \setminus A$ which implies $y \notin \overline{A}$. Therefore $B_r(x) \subset X \setminus \overline{A}$.

2. It sufficed to prove that for any closed $F \subset X$ such that $A \subset F$ one has $\overline{A} \subset F$. Let $x \in X$ be adherent to A. If $x \notin F$ then $x \in X \setminus F$ which is open. Then there $r > 0$ such that $B_r(x) \subset X \setminus F \subset X \setminus A$. But since x is adherent to A, $B_r(x) \cap A \neq \varnothing$, a contradiction. Thus $x \in F$ and $\overline{A} \subset F$. □

Proposition 4.4. *Let (X, d) be a metric space and $A \subset X$. The following conditions are equivalent*

1. *A is dense in X*

2. *Any nonempty open subset of X contains a point from A*

3. *For any $x \in X$ and any $r > 0$ there is $a \in A$ such that $d(x, a) < r$.*

Proof.

$1 \Rightarrow 2$ Suppose A is dense and $U \subset X$ is nonempty and open. Choose $x \in U$, there is $\delta > 0$ such that $B_\delta(x) \subset U$. Since x is adherent to A, there is $a \in B_\delta(x) \cap A$, which yields $a \in U$.

$2 \Rightarrow 3$ Suppose that any nonempty open subset of X contains a point from A. If $x \in X$ then the open ball $B_r(x)$ contains $a \in A$ and $d(x, a) < r$.

$3 \Rightarrow 1$ Suppose that for any $x \in X, r > 0$ there is $a \in A$ such that $d(x, a) < r$. If $x \in X$ then any open ball $B_r(x)$ contains a point from A, therefore x is adherent to A and $\overline{A} = X$. □

Exercise 4.4. *Prove that if two metrics d and d' on X are equivalent then the closures of any subset A with respect to them are equal.*

Proposition 4.5. *Let F be a subset of a metric space (X, d). The following conditions are equivalent*

1. *F is closed*
2. *if $\{x_n\}$ is a sequence in F and $\lim x_n = x$ then $x \in F$*
3. *$F = \overline{F}$*

Proof.

$1 \Rightarrow 2$ Suppose F is closed, $\{x_n\}$ is a sequence in F and $\lim x_n = x$. If $x \in X \setminus F$ which is open then there exists $r > 0$ such that $B_r(x) \subset X \setminus F$. There is $N \in \mathbb{N}$ such that $x_n \in B_r(x)$ for any $n \geq N$, a contradiction.

$2 \Rightarrow 3$ If $x \in \overline{F}$ then for any $n \in \mathbb{N}$ there exists $x_n \in B_{1/n}(x) \cap F$. It gives a sequence $\{x_n\}$ in F which converges to x, thus $x \in F$.

$3 \Rightarrow 1$ Trivial since $\overline{F}$ is closed by Proposition 4.3. □

Exercise 4.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that its graph $\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}\}$ is a closed subset of $\mathbb{R}^2$.*

Exercise 4.6. *Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ for any subsets A, B.*

Exercise 4.7. *Find two $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.*

5 Completeness

Definition. A sequence $\{x_n\}$ in a metric space (X, d) is a **Cauchy sequence** (基本序列) if $\lim d(x_n, x_m) = 0$.

Proposition 5.1. *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X.*

1. *If $\{x_n\}$ converges then it is a Cauchy sequence*
2. *If $\{x_n\}$ is a Cauchy sequence that contains a convergent subsequence then $\{x_n\}$ converges.*

Proof. 1. Suppose that $\lim x_n = x$. Then $d(x_n, x_m) \le d(x_n, x) + d(x, x_m)$ and the right-hand side tends to zero as $n, m \to \infty$.

2. Suppose that a sequence $\{x_{n_k}\}$ converges to x. Let $\varepsilon > 0$. Choose $N > 0$ so large that $d(x_n, x_m) < \varepsilon/2$ whenever $n, m \ge N$. Choose $L > 0$ so large that $d(x_{n_j}, x) < \varepsilon/2$ whenever $j \ge L$. Set $M = \max(N, n_L)$. If $m, n_j \ge M$, then

$$d(x_m, x) \le d(x_m, x_{n_j}) + d(x_{n_j}, x) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which shows that $\lim x_n = x$. □

Definition. A metric space (X, d) is **complete** (完备的) if every Cauchy sequence in X converges. A subset Y of a metric space (X, d) is **complete** if the corresponding metric subspace $(Y, d|_Y)$ is complete.

Examples. 1. $\mathbb{R}$ is complete

2. Any discrete metric space is complete since every Cauchy sequence there is eventually constant, hence convergent.

3. $(0, 1)$ is not complete since the sequence $\{\frac{1}{n}\}$ does not converge in $(0, 1)$.

4. $\mathbb{Q}$ with the Euclidean metric is not complete since a sequence of rational approximations of $\sqrt{2}$ is a Cauchy sequence which does not converge in $\mathbb{Q}$.

5. $\mathbb{Q}$ with the p-adic metric is not complete as hinted by the following table ($p = 2$)

n	1	2	3	4	5	$\cdots$
x_n	1	-3	$-\frac{1}{3}$	$\frac{31}{3}$	$-\frac{3647}{93}$	
$x_n - x_{n-1}$		-4	$\frac{8}{3}$	$\frac{32}{3}$	$-\frac{1536}{31}$	
$x_n^2 + 7$	8	16	$\frac{64}{9}$	$\frac{1024}{9}$	$\frac{13361152}{8649}$	

Exercise 5.1. *If $\{x_n\}, \{y_n\}$ are Cauchy sequences in a metric space (X, d), show that the sequence $a_n = d(x_n, y_n)$ converges in $\mathbb{R}$.*

Exercise 5.2. *Let $X = \mathbb{N}, d(m, n) = |1/m - 1/n|$. Show that (X, d) is not complete.*

Exercise 5.3. *Show that a metric space with a finite number of elements is complete.*

Exercise 5.4. *Show that $\mathbb{Z}$ with p-adic metric is not complete.*

Hint. $1 + p + p^2 + \cdots = \frac{1}{1-p}$

Exercise 5.5. *Prove that the intersection of any collection of complete subsets of a metric space is complete.*

Proposition 5.2. *1. A closed subset of a complete metric space is complete.*

2. A complete subset Y of a metric space X is closed in X.

Proof. 1. Let Y be a closed subset of a complete space X and let $\{y_n\}$ be a Cauchy sequence in Y. Then $\{y_n\}$ is also a Cauchy sequence in X and there exists $x \in X$ such that $\lim y_n = x$. Since Y is closed, $x \in Y$.

2. Suppose $x \in X$ is adherent to Y. For any $n \in \mathbb{N}$ there is $y_n \in Y \cap B_{1/n}(x)$ which yields a sequence $\{y_n\}$ in Y that converges to x. By Proposition 5.1, $\{y_n\}$ is a Cauchy sequence in X; hence it is also a Cauchy sequence in Y. Since Y is complete, $\{y_n\}$ converges to some $y \in Y$. Proposition 2.2 now gives $x = y \in Y$. Thus $Y = \overline{Y}$ is closed. □

Theorem 5.3. *Let A be a dense subset of a metric space (X, d). Suppose that every Cauchy sequence in A converges in X. Then X is complete.*

Proof. We need to show that every Cauchy sequence $\{x_n\}$ in X converges in X. For any $n \in \mathbb{N}$ pick up $y_n \in A$ such that $d(x_n, y_n) < 1/n$. First, we show that $\{y_n\}$ is Cauchy.

Fix $\varepsilon > 0$. Let $N' \in \mathbb{N}$ be such that $d(x_n, x_m) < \varepsilon/3$ for all $n, m \geq N'$. Put $N = \max(N', 3/\varepsilon)$. Then

$$d(y_n, y_m) \leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < 1/n + \varepsilon/3 + 1/m \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for all for all $n, m \geq N$ whence $\{y_n\}$ is a Cauchy sequence. Let $y \in X$ be its limit.

Let $M' \in \mathbb{N}$ be such that $d(y_n, y) < \varepsilon/2$ for all $n \geq M'$. Put $M = \max(M', 2/\varepsilon)$. Then

$$d(x_n, y) \leq d(x_n, y_n) + d(y_n, y) < 1/n + \varepsilon/2 \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all for all $n \geq M$ whence $\{x_n\}$ tends to y. □

Theorem 5.4 (Banach Fixed Point Theorem)**.** *Let (X, d) be a non-empty complete metric space with a map $T\colon X \to X$ such that*

$$d(T(x), T(y)) \leq q d(x, y)$$

for a certain $q \in [0, 1)$ and all $x, y \in X$. Then T admits a unique fixed point $x^* \in X$ (i.e. $T(x^*) = x^*$). Furthermore, $x^* = \lim T^{(n)}(x)$ for an arbitrary $x \in X$.

Proof. Let $x \in X$ and define a sequence $\{x_n\}$ in X where $x_n = T^{(n)}(x)$. It follows by induction on n that $d(x_{n+1}, x_n) \leq q^n d(x_1, x_0)$ for all $n \in \mathbb{N}$. Then we show that $\{x_n\}$ is a Cauchy sequence. Indeed, for $m > n$

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\ &\leq q^{m-1} d(x_1, x_0) + q^{m-2} d(x_1, x_0) + \cdots + q^n d(x_1, x_0) \\ &= \sum_{k=n}^{m-1} q^k d(x_1, x_0) < \sum_{k=n}^{\infty} q^k d(x_1, x_0) = \frac{q^n}{1-q} d(x_1, x_0). \end{aligned}$$

By completeness of (X, d), the sequence $\{x_n\}$ has a limit $x^* \in X$. Furthermore, if $\varepsilon > 0$ then $d(T(x), T(y)) < q\varepsilon$ for any $x, y \in X$ with $d(x, y) < \varepsilon$. It implies that T is continuous, so

$$x^* = \lim x_n = \lim T(x_{n-1}) = T(\lim x_{n-1}) = T(x^*),$$

and x^* must be a fixed point of T. Lastly, T cannot have more than one fixed point, since for any pair of fixed points y and z one has $d(y,z) = d(T(y),T(z)) < qd(y,z)$ whence $d(y,z) = 0$ and $y = z$. $\square$

Remarks. 1. Banach fixed point theorem may not be valid if X is not complete. Consider $f\colon (0,1] \to (0,1]$, $f(x) = x/2$.

2. A fun application of Banach fixed point theorem. Pick a map of your city and drop it on the floor. Then the theorem implies that there exists a unique point on the map which sits exactly above its origin.

Theorem 5.5 (Baire Category Theorem). *Let $\{U_n\}$ be a sequence of dense open subsets of a complete metric space (X,d). Then $\cap U_n$ is also dense in X.*

Proof. Let $x \in X$ and $\varepsilon > 0$. It suffices to find $y \in B_\varepsilon(x)$ that belongs to $\cap U_n$. Set $r_0 = \varepsilon/2$ and $x_0 = x$. Since U_1 is dense in X, there exists $x_1 \in U_1 \cap B_{r_0}(x_0)$. Since $U_1 \cap B_{r_0}(x_0)$ is open, there exists $0 < r_1 < r_0/2$ such that $B_{2r_1}(x_1) \subset U_1 \cap B_{r_0}(x_0)$.

As U_2 is dense in X, there exists $x_2 \in U_2 \cap B_{r_1}(x_1)$. The set $U_2 \cap B_{r_1}(x_1)$ is open, and there exists $0 < r_2 < r_1/2$ such that $B_{2r_2}(x_2) \subset U_2 \cap B_{r_1}(x_1)$.

Continuing in this manner, we obtain a sequence $\{x_n\}$ in X and a sequence $\{r_n\}$ of positive numbers such that $r_n < r_{n-1}/2$ and $B_{2r_n}(x_n) \subset U_n \cap B_{r_{n-1}}(x_{n-1})$. It follows that

$$B_{r_n}(x_n) \subset B_{2r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1}) \subset \cdots \subset B_{r_0}(x_0) \subset B_{2r_0}(x_0) = B_\varepsilon(x).$$

Hence $x_m \in B_{r_n}(x_n)$ if $m > n$, so that $d(x_m, x_n) < r_n < r_0/2^n$. Consequently $\{x_m\}$ is a Cauchy sequence and by completeness of X, there exists $y \in X$ such that $x_n \to y$. Now for all $m > n$,

$$d(y,x_n) \le d(y,x_m) + d(x_m,x_n) < d(y,x_m) + r_n.$$

Letting $m \to \infty$, one concludes that $d(y,x_n) \le r_n$, which implies $y \in B_{2r_n}(x_n) \subset U_n$ for every n, thus $y \in \cap U_n$. Moreover, $d(x,y) = d(x_0,y) \le r_0 = \varepsilon/2 < \varepsilon$ which concludes the proof. $\square$

Remark. The hypothesis of completeness in the Baire Category Theorem is crucial. For example, arrange the rational numbers in a sequence $\{s_n\}$ and set $U_n = \mathbb{Q} \setminus \{s_n\}$. Then each U_n is a dense open subset of $\mathbb{Q}$. However, $\cap U_n$ is empty.

Corollary 5.6. *Let $\{A_n\}$ be a sequence of closed subsets in a complete metric space (X,d). Assume that $\cup A_n$ contains an open ball. Then there exists k such that A_k contains an open ball.*

Proof. Denote $U_n = X \setminus A_n$. Assume that every A_n contains no open balls. Then for any $x \in X$ and $r > 0$ there is $a \in B_r(x)$ such that $a \notin A_n$. Then $a \in U_n$ which implies that U_n is dense. By definition, U_n is open, and it follows from Baire Category Theorem that $\cap U_n$ is dense. But $X \setminus \cap U_n = \cup A_n$ contains an open ball, which is a contradiction. $\square$

Example. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ be a continuous function such that $f(nx) \to 0$ for any $x > 0$. We will prove that $\lim_{x\to\infty} f(x) = 0$.

Fix $\varepsilon > 0$ and put

$$S_n = \{x > 0 \mid |f(mx)| \le \varepsilon \text{ for all}\, m \ge n\}.$$

Then $S_n = \cap_{m\geq n}\frac{1}{m}f^{-1}([-\varepsilon,\varepsilon])$ is closed and $\cup S_n = \mathbb{R}_{>0}$. Corollary 5.6 implies that $(a,b)\subset S_k$ for some $k\in\mathbb{N}$. Thus $|f(x)|\leq\varepsilon$ for any

$$x \in T = (ka, kb) \cup ((k+1)a, (k+1)b) \cup ((k+2)a, (k+2)b) \cup \cdots$$

The sequence $(k+n+1)/(k+n)$ is decreasing and tends to 1, whence there is $N\in\mathbb{N}$ such that $(k+n+1)/(k+n) < b/a$ for all $n\geq N$. Then T contains (M,∞) for some $M>0$. Therefore $|f(x)|\leq\varepsilon$ for any $x>M$ and $\lim_{x\to\infty}f(x)=0$.

6 Completion

Proposition 6.1. *Let (X,d) be a metric space and let S be the set of Cauchy sequences in X. Let $\sim$ be a relation in S defined by $\{x_i\}\sim\{y_i\}$ if $\lim d(x_i,y_i)=0$. Then*

1. *The relation $\sim$ is an equivalence relation.*

2. *Let $\hat{X}$ denote the set of equivalence classes of S and let $[s]$ denote the equivalence class of $s=\{x_i\}\in S$. The map $D([s],[t])=\lim d(x_i,y_i)$, where $s=\{x_i\}, t=\{y_i\}$, is correctly defined and defines a metric on $\hat{X}$.*

3. *For $x\in X$, define $[x]$ to be the equivalence class of the constant sequence $x,x,\ldots$. The map $\varphi\colon X\to\hat{X}, \varphi(x)=[x]$ is injective and $D([x],[y])=d(x,y)$.*

4. *$\varphi(X)$ is dense in $\hat{X}$, in particular, if $s=\{x_i\}\in S$, then $\varphi(x_i)\to[s]$.*

5. *The metric space $(\hat{X},D)$ is complete.*

Proof. 1. Only the transitivity property is non-trivial. If $\{x_i\}\sim\{y_i\}$ and $\{y_i\}\sim\{z_i\}$ then $\lim d(x_i,y_i)=\lim d(y_i,z_i)=0$. Since $d(x_i,z_i)\leq d(x_i,y_i)+d(y_i,z_i)$, the sequence $\{d(x_i,z_i)\}$ tends to 0 and $\{x_i\}\sim\{z_i\}$.

2. Let $\{x_i\},\{y_i\}\in S$. Then $|d(x_i,y_i)-d(x_j,y_j)|\leq d(x_i,x_j)+d(y_i,y_j)$ (see Exercise 1.4), whence $d(x_i,y_i)$ is a Cauchy sequence in $\mathbb{R}$ which converges.

Further, let $\{x_i\}\sim\{x'_i\},\{y_i\}\sim\{y'_i\}$. Then

$$d(x'_i,y'_i)-d(x_i,x'_i)-d(y'_i,y_i)\leq d(x_i,y_i)\leq d(x'_i,y'_i)+d(x_i,x'_i)+d(y'_i,y_i).$$

Since $d(x_i,x'_i),d(y'_i,y_i)\to 0$, one gets $\lim d(x'_i,y'_i)=\lim d(x_i,y_i)$.

It remains to verify that D is a metric on $\hat{X}$. Only the triangle inequality is non-trivial. Let $\{x_i\},\{y_i\},\{z_i\}\in S$. Then passage to limits in the inequality $d(x_i,y_i)+d(y_i,z_i)\geq d(x_i,z_i)$ gives the desired result.

3. By definition, $D([x],[y])=\lim d(x,y)=d(x,y)$. This also implies the injectivity of φ.

4. For any $\varepsilon>0$ there is $N\in\mathbb{N}$ such that $d(x_i,x_j)<\varepsilon$ whenever $i,j\geq N$. Then $D([x_i],[s])=\lim_{j\to\infty}d(x_i,x_j)\leq\varepsilon$ for any $i\geq N$.

5. By Theorem 5.3, it is enough to prove that any Cauchy sequence in $\varphi(X)$ converges in $\hat{X}$. If $\{[x_i]\}$ is a Cauchy sequence for $x_i\in X$, then $s=\{x_i\}\in S$ since $D([x_i],[x_j])=d(x_i,x_j)$ for all $i,j\in\mathbb{N}$. Finally, $[x_i]\to[s]$. $\square$

Definition. The metric space $(\hat{X}, D)$ is called the **completion** (完备化) of (X, d).

Remark. Imagine X as a 2-dimensional flat spot. If X is not complete, then this spot has tiny pinpricks in it which represent missing points (points that will eventually appear in X). To produce the completion of X, we have to find a way to close up these pinpricks, but our only resource is X itself. Make infinite number of copies of X and stack it over X one over another. Now every sequence $\{x_n\}, x_n \in X$ corresponds to a vertical path up to the "top" of the stack. The Cauchy sequences are those paths that have a vertical asymptote. Then two Cauchy sequences (vertical paths) are equivalent if their vertical asymptotes coincides. Thus $\hat{X}$ can be considered as the set of all such asymptotes which has no pinpricks.

Example. One way to define $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the Euclidean metric.

Exercise 6.1. *Prove that φ is a bijection if X is complete.*

Definition. Let $(X, d), (Y, d')$ be metric space. A map $f\colon X \to Y$ is **isometry** (等距同构) if it is surjective and $d(x, x') = d'(f(x), f(x'))$ for any $x, x' \in X$.

Exercise 6.2. *Prove that any isometry is bijective.*

Proposition 6.2. *Let (Y, d) be a complete metric space and $X \subset Y$. Then $\varphi\colon X \to \hat{X}, \varphi(x) = [x]$ can be extended to an isometry $\overline{\varphi}\colon \overline{X} \to \hat{X}$, i.e., $\overline{\varphi}(x) = \varphi(x)$ for any $x \in X$.*

Proof. If $y \in \overline{X}$, there is a sequence $s = \{x_n\}$ in X which converges to y. Clearly $s \in S$ and we define $\overline{\varphi}(y) = [s]$. Check first that $\overline{\varphi}$ is correctly defined. Let $s' = \{x'_n\}$ is another sequence in X that converges to y. Then $d(x_n, x'_n) \to 0$ whence $s \sim s'$ and $[s] = [s']$. If $x \in X$ then the constant sequence $x, x, \ldots$ tends to x and $\overline{\varphi}(x) = [x] = \varphi(x)$.

Let $[s] \in \hat{X}$, where $s = \{x_n\} \in S$. Since Y is complete, there is $y \in \overline{X}$ such that $\{x_n\} \to y$. and by definition $\overline{\varphi}(y) = [s]$. It shows that $\overline{\varphi}$ is surjective.

Finally, let $y, y' \in \overline{X}$ and sequences $s = \{x_n\}, s' = \{x'_n\}$ in X converge to y, y', respectively. Then $D(\overline{\varphi}(y), \overline{\varphi}(y')) = D([s], [s']) = \lim d(x_n, x'_n) = d(y, y')$. □

Remark. 1. The above proposition implies that the completion of a subspace can be identified with its closure. The completion is a more general notion which can be defined without an ambient space, while the closure requires an ambient space.

2. The above proposition is not valid if Y is not complete. For example, consider $Y = (-1, 1)$ and $X = (0, 1)$. Here $\overline{X} = [0, 1)$ which is not complete. On the other hand, one can easily see that if there is an isometry between two metric spaces and one of them is complete then another is also complete.

Exercise 6.3. *Let (Y, d) be a metric space and $X \subset Y$. Prove that there is an isometry $\psi\colon \overline{X} \to Z$ for a certain $Z \subset \hat{X}$.*

Definition. Let $(X, d), (Y, d')$ be metric spaces. A map $f\colon X \mapsto Y$ is **uniformly continuous** (一致连续的) if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d'(f(x), f(y)) < \varepsilon$ whenever $x, y \in X$ satisfy $d(x, y) < \delta$. Evidently every uniformly continuous function is continuous.

Example. The function $f(x) = x^2$ is uniformly continuous on $[0,1]$ but not uniformly continuous on $\mathbb{R}$. Indeed, for any $\varepsilon > 0$ one can take $\delta = \varepsilon/2$, then $|x - y| < \delta$ implies $|x^2 - y^2| < \varepsilon$ since $\max\limits_{x,y\in[0,1]} |x + y| = 2$. On the other hand, for any $\delta > 0$ one can find $x, y \in \mathbb{R}$ such that $x - y = \delta/2, x + y = 4\varepsilon/\delta$ whence $x^2 - y^2 = 2\varepsilon > \varepsilon$.

Theorem 6.3. *Let (X, d) and (Y, d') be metric spaces and $f\colon X \to Y$ be uniformly continuous. If Y is complete, then there exists a unique uniformly continuous map $\hat{f}\colon \hat{X} \to Y$ such that $\hat{f} \circ \varphi = f$.*

Proof. We start with proofs of the following facts:

1. If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y and thus converges. Indeed, for any $\varepsilon > 0$ there is $\delta > 0$ such that $d'(f(x_n), f(x_m)) < \varepsilon$ if $d(x_n, x_m) < \delta$. And there is $N \in \mathbb{N}$ such that $d(x_n, x_m) < \delta$ for $n, m \geq N$. Thus $d'(f(x_n), f(x_m)) < \varepsilon$ for $n, m \geq N$.

2. If two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X are equivalent then $\lim f(x_n) = \lim f(y_n)$. Indeed, if $d(x_n, y_n) \to 0$ then $d'(f(x_n), f(y_n)) \to 0$ and since two former sequences converge, their limits are equal.

The above facts show that the following definition is correct: $\hat{f}([s]) = \lim f(x_n)$ for $s = \{x_n\} \in S$. If $x \in X$ then the constant sequence $x, x, \ldots$ tends to x and $\varphi(x) = [x]$ and $\hat{f}([x])$ is the limit of the constant sequence $f(x), f(x), \ldots$ which is $f(x)$.

If $\varepsilon > 0$, there is $\delta > 0$ such that $d'(f(x), f(y)) < \varepsilon/2$ if $d(x, y) < \delta$. Let $s = \{x_n\}, t = \{y_n\} \in S$. For $\Delta = D([s],[t]) = \lim d(x_n, y_n)$, there is $N \in \mathbb{N}$ such that $|\Delta - d(x_n, y_n)| < \delta/2$ for $n \geq N$. Thus if $\Delta < \delta/2$ then $d(x_n, y_n) \leq \Delta + |\Delta - d(x_n, y_n)| < \delta$ and $d'(f(x_n), f(y_n)) < \varepsilon/2$ for $n \geq N$. Then $d'(\hat{f}([s]), \hat{f}([t])) = d'(\lim f(x_n), \lim f(y_n)) = \lim d'(f(x_n), f(y_n)) \leq \varepsilon/2 < \varepsilon$, which proves that $\hat{f}$ is uniformly continuous.

It remains to prove the uniqueness of $\hat{f}$. If $\tilde{f}\colon \hat{X} \to Y$ is continuous and $\tilde{f} \circ \varphi = f$ then for any $s = \{x_n\} \in S$ one has $\varphi(x_i) \to [s]$. Thus $\tilde{f}([s]) = \lim \tilde{f}(\varphi(x_i)) = \lim \hat{f}(\varphi(x_i)) = \hat{f}([s])$. □

Remark. The completeness of Y and uniform continuity of f in the above theorem are necessary: consider $f_1\colon \mathbb{Q} \to \mathbb{Q}, f_1(x) = x$ and $f_2\colon (0,1] \to \mathbb{R}, f_2(x) = 1/x$.

7 Bounded, totally bounded and separable metric spaces

Definition. Let (X, d) be a metric space. A family $\{U_\omega\}_{\omega\in\Omega}$ of open subsets of X is an **open cover** (开覆盖) of X (or the family $\{U_\omega\}_{\omega\in\Omega}$ **covers** (覆盖) X) if $\cup U_\omega = X$. A **subcover** (子覆盖) of a cover $\{U_\omega\}_{\omega\in\Omega}$ is a cover $\{U_\omega\}_{\omega\in\Omega'}$ for some $\Omega' \subset \Omega$.

The metric space X is **totally bounded** (全有界的) if for each $\varepsilon > 0$, there exists a finite number of open balls of radius ε that cover X and **bounded** (有界的) if there exists $R > 0$ such that $d(x, x') < R$ for any $x, x' \in X$. A subset Y of a metric space (X, d) is **totally bounded/bounded** if the corresponding metric subspace $(Y, d|_Y)$ is totally bounded/bounded.

Examples. 1. $[0,1]$ or $(0,1)$ are both bounded and totally bounded

2. $\mathbb{R}$ is neither bounded nor totally bounded

3. An infinite set X with discrete metric is bounded but not totally bounded for open balls with of radius ≤ 1

Exercise 7.1. *Let A, B be subsets of a metric space (X, d).*

i) *Prove that $A \cup B$ is bounded if A and B are bounded.*

ii) *Prove that $A \cup B$ is totally bounded if A and B are totally bounded.*

Exercise 7.2. *Let X be the set of binary sequences $\{x_n\}$ with each $x_n \in \{0, 1\}$. Define a metric on X by $d(\{x_n\}, \{y_n\}) = \sum_{n \geq 1} |x_n - y_n|/2^n$. Prove that (X, d) is totally bounded.*

Hint. For $\varepsilon > 0$, choose $m \in \mathbb{N}$ such that $\varepsilon > 1/2^m$ and consider the set of the sequences $\{y_n\} \in X$ such that $y_n = 0$ for all $n > m$.

Exercise 7.3. *Prove that the completion of a totally bounded metric space X is totally bounded.*

Hint. For $\varepsilon > 0$, choose $x_1, \ldots, x_n \in X$ such that $X = \cup_{j=1}^{n} B_{\varepsilon/2}(x_j)$ and show that $\hat{X} = \cup_{j=1}^{n} B_{\varepsilon}(\varphi(x_j))$ using the fact that $\varphi(X)$ is dense in $\hat{X}$ (see Proposition 6.1).

Proposition 7.1. *1. A totally bounded metric space is bounded.*

2. A subspace of totally bounded metric space is totally bounded.

3. A subset $X \subset \mathbb{R}^n$ is totally bounded if and only if it is bounded.

4. In a totally bounded metric space every sequence has a Cauchy subsequence.

Proof. 1. Let X be totally bounded. Choose $x_1, \ldots, x_m \in X$ such that the balls $B_1(x_j), 1 \leq j \leq m$, cover X. Put $R = 2 + \max_{1 \leq i,j \leq m}\{d(x_i, x_j)\}$. Then for any $x, x' \in X$ one has

$$d(x, x') \leq d(x, x_i) + d(x_i, x_j) + d(x_j, x') \leq R,$$

if $x \in B_1(x_i), x' \in B_1(x_j)$.

2. Let Y be a subspace of the totally bounded space X and let $\varepsilon > 0$. There are $x_1, \ldots, x_n \in X$ such that the balls $B_{\varepsilon/2}(x_j)$ cover X. One can assume that $B_{\varepsilon/2}(x_1), \ldots, B_{\varepsilon/2}(x_m)$ have common points with Y, while $B_{\varepsilon/2}(x_{m+1}), \ldots, B_{\varepsilon/2}(x_n)$ do not meet Y. Then $Y \subset \cup_{j=1}^{m} B_{\varepsilon/2}(x_j)$. Let y_j be any point in $B_{\varepsilon/2}(x_j) \cap Y, 1 \leq j \leq m$. The triangle inequality shows that $B_{\varepsilon/2}(x_j) \subset B_{\varepsilon}(y_j)$, whence the open balls $B_{\varepsilon}^{Y}(y_j) = B_{\varepsilon}(y_j) \cap Y, 1 \leq j \leq m$ in the subspace Y cover Y.

3. Suppose that $X \subset \mathbb{R}^n$ is bounded by R. Then X is contained in a hypercube of the form $T = [-R, R] \times \cdots \times [-R, R]$ for some $R > 0$. It suffices to prove that T is totally bounded. Let $\varepsilon > 0$ and $R' = \sqrt{n}R/\varepsilon$. The hypercube $T' = [-R', R'] \times \cdots \times [-R', R']$ is contained in the union of the open balls of radius $\sqrt{n}$ with centers in the points with integer coordinates inside T'. Indeed, put

$$y_i = \begin{cases} [x_i], & \text{if } x_i \geq 0 \\ [x_i] + 1, & \text{if } x_i < 0 \end{cases}$$

Then $-R' \leq y_i \leq R'$ and $0 \leq x_i - y_i < 1$ for all $1 \leq i \leq n$ whence $\sqrt{\sum_{i=1}^n (x_i - y_i)^2} < \sqrt{n}$. Clearly the number of these balls is finite. Scaling this construction by $\varepsilon/\sqrt{n}$ we obtain that T is contained in the union of a finite number of open balls of radius ε. Therefore T is covered by the intersections of these balls of radius ε with T (which are balls of radius ε in T).

4. Let (X, d) be a totally bounded metric space and $\{x_j\}$ be a sequence in X. We shall use a diagonalization procedure. By induction, we construct sequences $\{x_j^{(k)}\}, k \geq 1$, with the following properties:

(i) $\{x_j^{(1)}\} = \{x_j\}$.

(ii) $\{x_j^{(k)}\}$ is a subsequence of $\{x_j^{(k-1)}\}, k \geq 2$.

(iii) $\{x_j^{(k)}\}$ is contained in a ball of radius $1/k, k \geq 2$.

Assume that $k \geq 2$ and that we already have the sequence $\{x_j^{(k-1)}\}$. Let $B_1, \ldots, B_n$ be open balls of radius $1/k$ that cover X. There must exist at least one ball, say B_m, such that $\{x_j^{(k-1)}\} \in B_m$ for infinitely many $j \geq 1$. Denote by $\{x_j^{(k)}\}$ the corresponding subsequence of $\{x_j^{(k-1)}\}$.

Now set $y_n = x_n^{(n)}, n \geq 1$. Clearly $\{y_n\}$ is a subsequence of the original sequence $\{x_j\}$. Furthermore, the sequence $y_k, y_{k+1}, \ldots$ is a subsequence of $\{x_j^{(k)}\}$, whence $d(y_n, y_m) < 2/k$ for $n, m \geq k$. Thus $\{y_n\}$ is a Cauchy sequence. $\square$

Definition. A set is **countable**(可数的) if there exists an injective function from it into $\mathbb{N}$. A metric space (X, d) is **separable** (可分的) if it has a countable dense subset. A subset Y of a metric space (X, d) is **separable** if the corresponding metric subspace $(Y, d|_Y)$ is separable.

Example. 1. $\mathbb{R}$ is separable with $\mathbb{Q}$ being its countable dense subset

2. $\mathbb{R}$ with discrete metric is not separable

Exercise 7.4. *Let X be the set of sequences in $[0, 1]$. Define a metric on X by $d(\{x_i\}, \{y_i\}) = \sup |x_i - y_i|$. Prove that (X, d) is not separable.*

Hint. If $\{y^{(k)}\}$ is a dense subset of X, where $y^{(k)} = \{y_i^{(k)}\}$, consider $\{x_i\} \in X$ defined by

$$x_i = \begin{cases} y_i^{(i)} + \frac{1}{2}, & \text{if } y_i^{(i)} \leq \frac{1}{2} \\ y_i^{(i)} - \frac{1}{2}, & \text{if } y_i^{(i)} > \frac{1}{2} \end{cases}$$

Proposition 7.2. *1. A subspace of a separable metric space is separable.*

2. A totally bounded metric space is separable.

3. A metric space is separable if and only if it has a countable base.

Proof. 1. Suppose that (X, d) is a metric space with a dense subset $\{x_j\}_{j\in\mathbb{N}}$ and let $Y \subset X$. Consider the pairs of indices (j, n) such that $B_{1/n}(x_j) \cap Y \neq \varnothing$. Pick up any y_{jn} from this intersection and prove that the set $S = \{y_{jn}\}$ is dense in Y. Fix $\varepsilon > 0$ and choose $n \geq 2\varepsilon^{-1}$. For any $y \in Y$ there is x_j such that $d(x_j, y) < 1/n$. Then $y \in B_{1/n}(x_j) \cap Y$ and there is $y_{jn} \in S$ such that $d(y_{jn}, x_j) < 1/n$. It gives

$$d(y_{jn}, y) \leq d(y_{jn}, x_j) + d(x_j, y) < 1/n + 1/n \leq \varepsilon.$$

2. If (X, d) is totally bounded, for any $n \in \mathbb{N}$ there are open balls $B_{1/n}(x_1^{(n)}), \ldots, B_{1/n}(x_{m_n}^{(n)})$ that cover X. Then the set of its centers $S = \{x_k^{(n)}\}_{n\geq 1, 1\leq k\leq m_n}$ is countable. Fix $\varepsilon > 0$ and choose $n \geq \varepsilon^{-1}$. For any $x \in X$ there is $x_k^{(n)}$ such that $d(x_k^{(n)}, x) < 1/n \leq \varepsilon$ which shows that S is dense.

3. Suppose that (X, d) is a separable metric space and $\{x_j\}_{j\in\mathbb{N}}$ is its dense subset. Consider the countable family of open sets

$$\mathcal{B} = \{B_{1/n}(x_j) \mid j \geq 1, n \geq 1\}.$$

Let U be an open subset of X and $x \in U$. For some $n \geq 1$, one has $B_{2/n}(x) \subset U$. Choose j so that $d(x_j, x) < 1/n$. Then $x \in B_{1/n}(x_j) \subset B_{2/n}(x) \subset U$.

Conversely, let $\{U_n\}_{n\in\mathbb{N}}$ be a countable base of X. Choose $x_n \in U_n$. Then every nonempty open subset of X contains a point of the set $\{x_n\}$, so that it is dense in X, and X is separable. □

Exercise 7.5. *Show that if a discrete metric space is separable then it is countable.*

Hint. The only dense subset of a discrete metric space X is X.

Exercise 7.6. *Let $(X, d), (Y, d')$ be metric spaces and $f\colon X \to Y$ be a continuous surjective map. Prove that Y is separable if X is separable.*

Hint. If D is a countable dense subset of X, show that $f(D)$ is countable and dense in Y.

Proposition 7.3. *Every open cover of a separable metric space has a countable subcover.*

Proof. Suppose a metric space X is separable. Let $\{U_\omega\}_{\omega\in\Omega}$ be an open cover of X and $\mathcal{B}$ be a countable base of X. Let $\mathcal{C}$ be the subfamily of $\mathcal{B}$ consisting of the sets $V \in \mathcal{B}$ such that $V \subset U_\omega$ for some $\omega \in \Omega$. We claim that $\mathcal{C}$ is a cover of X. Indeed, if $x \in X$, then $x \in U_\omega$ for some $\omega \in \Omega$. Then there is $V \in \mathcal{B}$ such that $x \in V \subset U_\omega$ whence $V \in \mathcal{C}$.

Now for each $V \in \mathcal{C}$, select $\omega(V) \in \Omega$ such that $V \subset U_{\omega(V)}$. Then $\{U_{\omega(V)} \mid V \in \mathcal{C}\}$ covers X. Since $\mathcal{B}$ is countable, so is $\mathcal{C}$, so that the $U_{\omega(V)}$'s form a countable subcover of X. □

Exercise 7.7. *Show that every open subset of $\mathbb{R}$ is a countable union of open intervals.*

Hint. Use Proposition 7.3.

8 Compactness

Theorem 8.1. *The following are equivalent for a metric space (X, d):*

1. *Every open cover of X has a finite subcover.*
2. *Every sequence in X has a convergent subsequence.*
3. *X is totally bounded and complete.*

Proof.

$1 \Rightarrow 2$ Let $\{y_n\}$ be a sequence in X. Assume that for each $x \in X$, there exists $\varepsilon(x) > 0$ such that only finitely many terms of the sequence $\{y_n\}$ lie in $B_{\varepsilon(x)}(x)$. The set of these open balls $\{B_{\varepsilon(x)}(x) \mid x \in X\}$ forms an open cover of X. Therefore, there is a finite subcover

$$X = B_{\varepsilon(x_1)}(x_1) \cup \cdots \cup B_{\varepsilon(x_m)}(x_m).$$

Each of these balls contains only a finite number of $\{y_n\}$, thus there is only a finite number of the elements of $\{y_n\}$, an absurdity. This contradiction shows that there exists $x \in X$ such that each ball $B_\varepsilon(x)$ contains infinitely many terms of the sequence $\{y_j\}$.

Now we construct a convergent subsequence of $\{y_j\}$ as follows. Choose n_1 so that $y_{n_1} \in B_1(x)$. Since $y_n \in B_{1/2}(x)$ for infinitely many n, we can choose $n_2 > n_1$ such that $y_{n_2} \in B_{1/2}(x)$. Continuing in this manner, we find $n_1 < n_2 < \cdots < n_i < \cdots$ such that $y_{n_i} \in B_{1/i}(x)$. Then $\{y_{n_i}\}$ is a subsequence of $\{y_n\}$ that converges to x.

$2 \Rightarrow 3$ Let $\{x_n\}$ be a Cauchy sequence in X. It has a convergent subsequence. By Proposition 5.1, the sequence $\{x_n\}$ itself converges. Hence X is complete.

Now let $\varepsilon > 0$ and $y_1 \in X$. If $B_\varepsilon(y_1) \neq X$, let y_2 be any point in $X \setminus B_\varepsilon(y_1)$. Having chosen $y_1, \ldots, y_n$, we let y_{n+1} be any point in $X \setminus (\cup_{j=1}^n B_\varepsilon(y_j))$, provided the union does not coincide with X; otherwise we stop. The points $y_1, y_2, \ldots$ constructed in this manner satisfy $d(y_j, y_i) \geq \varepsilon, j \neq i$. If the procedure does not terminate, then the sequence $\{y_n\}$ has no convergent subsequence. Otherwise, when the choice of y_{n+1} is impossible, X is covered by n open balls of radius ε.

$3 \Rightarrow 2$ Follows from Proposition 7.1

$3 \Rightarrow 1$ By Proposition 7.3 every open cover of X has a countable subcover. It suffices to show then that if $\{U_n\}$ is a sequence of open subsets of X that cover X, then there is $m \in \mathbb{N}$ such that $X = U_1 \cup \cdots \cup U_m$.

Suppose that this is not the case, that is, that $U_1 \cup \cdots \cup U_m \neq X$ for all m. For each m, let x_m be any point in $X \setminus (U_1 \cup \cdots \cup U_m)$. The sequence $\{x_m\}$ has a subsequence that converges to some point $x \in X$. Since $x_n \in X \setminus (U_1 \cup \cdots \cup U_m)$ for all $n \geq m$ and $X \setminus (U_1 \cup \cdots \cup U_m)$ is closed, $x \in X \setminus (U_1 \cup \cdots \cup U_m)$. Since this is true for all m, one has $x \in X \setminus (\cup U_n)$, a contradiction. $\square$

Definition. A metric space is **compact**(紧的) if it satisfies one of the above equivalent conditions. A subset Y of a metric space (X, d) is **compact** if the corresponding metric subspace $(Y, d|_Y)$ is compact.

Corollary 8.2. *A compact metric space is separable.*

Proof. Follows from Proposition 7.2. □

Corollary 8.3. *A compact metric subspace is closed.*

Proof. Follows from Proposition 5.2. □

Examples.

1. $\mathbb{R}$ and (a,b) are not compact, $[a,b]$ is compact
2. Any finite metric space is compact
3. $\mathbb{R}$ with $d(x,y) = \min\{|x-y|, 1\}$ is not compact since $\{x_n\}, x_n = n$ has no convergent subsequence
4. The closed ball $\overline{B}_1(0)$ in the metric space of continuous functions on $[0,1]$ with $d(f,g) = \sup\limits_{x\in[0,1]} |f(x)-g(x)|$ is not compact. Consider continuous $f_n, n \in \mathbb{N}$ such that

$$f_n(x) = \begin{cases} 0, & \text{if } x \in [0, 1/(n+1)] \\ 0, & \text{if } x \in [1/n, 1] \\ 1, & \text{for some } x \in [1/(n+1), 1/n] \end{cases}$$

Then $d(f_n, f_m) = 1$ if $n \neq m$ and $\{f_n\}$ has no convergent subsequence.

Exercise 8.1. *Prove that a closed subset of a compact metric space is compact.*

Exercise 8.2. *Prove that a discrete metric space is compact only if it is finite using each of the three definitions of a compact metric space.*

Exercise 8.3. *Let $(X,d), (Y,d')$ be metric spaces. Prove that $X \times Y$ equipped with the product metric (Exercise 1.3) is compact if and only if both X and Y are compact.*

Hint. If $\{(x_n, y_n)\}$ is a sequence in $X \times Y$, choose a convergent subsequence $\{x_{n_k}\}$ and then a convergent subsequence $\{y_{n_{k_i}}\}$. If $\{(x_n)\}$ is a sequence in X, consider a sequence $\{(x_n, y)\}$ for some $y \in Y$.

Exercise 8.4. *Let $(X,d), (Y,d')$ be metric spaces and $f\colon X \to Y$ be continuous. Show that if X is compact, then the image of a closed subset $A \subset X$ under f is closed.*

Theorem 8.4 (Heine-Borel Theorem)**.** *A subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. A subset of $\mathbb{R}^n$ is bounded iff it is totally bounded (Proposition 7.1) and closed iff it is complete (Proposition 5.2). □

Remark. A closed and bounded (even totally bounded) subset in an arbitrary metric space may not be compact. Consider $X = (0,1) \cup \{2\}$ with Euclidean metric and its subset $A = (0,1)$.

Proposition 8.5. *Let $(X,d),(Y,d')$ be metric spaces and $f\colon X\to Y$ be continuous and surjective. If X is compact then Y is compact.*

Proof. Let y_n be a sequence in Y. Consider a sequence $\{x_n\}$ in X such that $f(x_n)=y_n, n\in\mathbb{N}$. If $\{x_{n_i}\}$ is its convergent subsequence then $\{y_{n_i}\}$ is a convergent subsequence of y_n. □

Corollary 8.6. *The projective plane $\mathbb{P}^2$ is compact.*

Proof. The map $f\colon S^2\to\mathbb{P}^2, f(x)=[x]$ is continuous and surjective, S^2 is compact by Heine-Borel Theorem. □

Proposition 8.7. *A continuous real-valued function on a compact metric space assumes its maximum and minimum values.*

Proof. Follows from Proposition 8.5 and Theorem 8.4. □

Theorem 8.8. *Let $(X,d),(Y,d')$ be metric spaces and $f\colon X\to Y$ be continuous. If X is compact then f is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous. Then there exist $\varepsilon>0$ and sequences $\{x_n\},\{y_n\}\in X$ such that $d(x_n,y_n)<1/n$ and $d'(f(x_n),f(y_n))\geq\varepsilon$. Choose a subsequence $x_{n_k}\to x\in X$. Since $d(x_{n_k},y_{n_k})\to 0$, we also obtain $y_{n_k}\to x$. Consequently $f(x_{n_k})\to f(x)$ and $f(y_{n_k})\to f(x)$, so that $d'(f(x_{n_k}),f(y_{n_k}))\leq d'(f(x_{n_k}),f(x))+d'(f(x),f(y_{n_k}))\to 0$. This contradiction establishes the theorem. □

Exercise 8.5. *Let (X,d) be metric space and A be its compact subset.*

i) *Prove that for any $x\in X$ there exists $a\in A$ such that $d(x,a)\leq d(x,a')$ for any $a'\in A$.*

ii) *Prove that there are $a,b\in A$ such that $d(a,b)\geq d(a',b')$ for any $a',b'\in A$.*

Hint. Use Exercise 2.6. For ii) use also Exercise 8.3.

Exercise 8.6. *Let (X,d) be a compact metric space. Prove that for each open cover $\{U_\omega\}_{\omega\in\Omega}$ of X there is $\varepsilon>0$ such that each open ball $B_\varepsilon(x)$ is contained in one of U_ω.*

Hint. Suppose the result is false. Then for every $n\in\mathbb{N}$ there is $B_{1/n}(x_n)$ which is not contained in any U_ω. Choose a convergent subsequence of $\{x_n\}$ and consider U_ω containing its limit.

9 Connectedness and path connectedness

Definition. A metric space X is **connected** (连通的) if the only two subsets of X that are simultaneously open and closed are X and $\varnothing$. A metric space is **disconnected**(不连通的) if it is not connected. A subset Y of a metric space (X,d) is **connected/disconnected** if the corresponding metric subspace $(Y,d|_Y)$ is connected/disconnected.

Proposition 9.1. *Let X be a metric space. Then the following are equivalent.*

1. *X is disconnected.*

2. *There exist non-empty open subsets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \varnothing$.*

3. *There exists a continuous surjective map $f\colon X \to \{0,1\}$, where the latter metric space is equipped with the discrete metric.*

Proof.

$1 \Leftrightarrow 2$ If $A \neq X, \varnothing$ is open and closed then $A, X \setminus A$ are non-empty open and $A \cup (X \setminus A) = X, A \cap (X \setminus A) = \varnothing$.

If $U, V \subset X$ are non-empty open and $U \cup V = X, U \cap V = \varnothing$ then $U \neq X, \varnothing$ is open and closed.

$3 \Leftrightarrow 2$ If $f\colon X \to \{0,1\}$ is surjective and continuous, then $U = f^{-1}(\{0\})$ and $V = f^{-1}(\{1\})$ are non-empty and open (since $\{0\}$ and $\{1\}$ are open in $\{0\} \cup \{1\}$.

If $U, V \subset X$ are non-empty open and $U \cup V = X, U \cap V = \varnothing$, define $f : X \to \{0,1\}$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in U \\ 1, & \text{if } x \in V \end{cases}$$

and it is clear that f is surjective and continuous. □

Exercise 9.1. *Show that a metric space X is disconnected iff there exist non-empty closed subsets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \varnothing$.*

Examples. 1. Discrete metric space is disconnected iff it contains more than one point

2. $\mathbb{R} \setminus \{0\}$ is disconnected

3. $\mathbb{Q}$ is disconnected since $\mathbb{Q} \cap [\sqrt{2}, \infty) = \mathbb{Q} \cap (\sqrt{2}, \infty)$ is both open and closed

Proposition 9.2. *Open, closed and half-open intervals in $\mathbb{R}$ are connected.*

Proof. Let X be an interval and suppose that $X = U \cup V$, where U and V are nonempty and closed and $U \cap V = \varnothing$. Then there are $x \in U, y \in V$, one can assume $x < y$. Since $V \cap [x, y]$ is closed, there is $z = \min\{v \in [x, y] : v \in V\}$ and $z \in V$. Since any $u \in [x, z)$ belongs to U and U is closed, it follows that $z \in U$, a contradiction. □

Exercise 9.2. *Prove that $\mathbb{Z}$ in p-adic metric is disconnected.*

Hint. Show that $\mathbb{Z} = \cup_{n=0}^{p-1} B_1(n)$ and these open balls do not overlap.

Proposition 9.3. *Let $f\colon X \to Y$ be a continuous surjective map of metric spaces, and suppose X is connected. Then Y is connected.*

Proof. If Y is disconnected, there is a continuous surjective map $g\colon Y \to \{0,1\}$. Then $g \circ f\colon X \to \{0\} \cup \{1\}$ is a continuous surjective map, a contradiction. □

Proposition 9.4. *Let X be a metric space and $U, A \subset X$. Suppose in addition that U is connected and $U \subset A \subset \overline{U}$. Then A is connected.*

Proof. Let $f\colon A \to \{0,1\}$ be continuous. Since U is connected and $f|_U$ is continuous (Exercise 2.2), either $f(U) = \{0\}$ or $f(U) = \{1\}$. Without loss of generality suppose that $f(U) = \{0\}$. If $a \in A \subset \overline{U}$, there is a sequence $\{x_n\}$ in U converging to a. Since f is continuous, $0 = f(x_n) \to f(a)$, therefore $f(a) = 0$. Thus $f(A) = 0$, which shows that A is connected. □

Lemma 9.5. *Let X be a metric space, and let $\{A_\omega\}_{\omega\in\Omega}$ be a collection of connected subsets of X such that $A_\omega \cap A_{\omega'} \neq \varnothing$ for all $\omega, \omega' \in \Omega$. Then, $A = \cup_\Omega A_\omega$ is connected.*

Proof. Let $f\colon A \to \{0,1\}$ be continuous. Since A_ω is connected for all $\omega \in \Omega$, we have $f(A_\omega) = \varepsilon_\omega$, where ε_ω is either 0 or 1. However, for any two $\omega, \omega' \in \Omega$, we have $A_\omega \cap A'_\omega \neq \varnothing$, and this implies that $\varepsilon_\omega = \varepsilon_{\omega'}$. Hence, f must be constant on A and it cannot be surjective. Thus A is connected. □

Exercise 9.3. *Let X be a metric space, and let $\{A_\omega\}_{\omega\in\Omega}$ be a collection of connected subsets of X such that for any $\omega, \omega' \in \Omega$ there are $\omega_1 = \omega, \omega_2, \ldots, \omega_n = \omega' \in \Omega$ such that $A_{\omega_i} \cap A_{\omega_{i+1}} \neq \varnothing$ for any $1 \leq i \leq n-1$. Prove that $A = \cup_\Omega A_\omega$ is connected.*

Theorem 9.6. *Let X, Y be metric spaces, and let $X \times Y$ be equipped with the product metric. Then $X \times Y$ is connected if and only if X and Y are connected.*

Proof. If $X \times Y$ is connected, then both X and Y are connected, since the projections $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ are continuous by Exercise 3.4.

Suppose now that X and Y are connected. Fix $a \in X$ and consider the sets $B = \{a\} \times Y, C_y = X \times \{y\} \subset X \times Y$ for $y \in Y$. First notice that B and C_y are connected subspaces of $X \times Y$. For B, it follows from the continuity of $f\colon Y \to \{a\} \times Y, f(y) = (y,a)$ and Proposition 9.3.

Now, for each $y \in Y$, we have $C_y \cap B = \{(a,y)\} \neq \varnothing$, and hence the subset $U_y = B \cup C_y$ is connected by Lemma 9.5. Consider the collection $\{U_y\}_{y\in Y}$. For any $y \neq y' \in Y$, we have $U_y \cap U_{y'} = B \neq \varnothing$, and hence $\cup_{y\in Y} U_y = X \times Y$ is connected again by Lemma 9.5. □

Corollary 9.7. $\mathbb{R}^n$ *with the Euclidean metric is connected.*

Proof. Theorem 9.6 implies that $\mathbb{R}^n$ with the Chebyshev metric is connected. The Euclidean metric is equivalent to it by Corollary 3.7. □

Proposition 9.8. *Let X, Y be metric spaces and $f\colon X \to Y$ be a continuous map. If X is connected then the graph $\Gamma_f = \{(x, f(x)) \mid x \in X\}$ is a connected subspace of $X \times Y$.*

Proof. Follows from Proposition 9.3 since $\varphi\colon X \to \Gamma_f, \varphi(x) = (x, f(x))$ is continuous and surjective. □

Exercise 9.4. *Prove that a circle in $\mathbb{R}^2$ is connected.*

Definition. Let X be a metric space, $x \in X$. A **connected component**(连通分支) of x is a connected subset $U \subset X$ such that any connected $V \subset X$ which contains x is contained in U.

Proposition 9.9. *Let X be a metric space.*

1. *A connected component of x is unique and equals the union of all connected subsets containing x.*

2. *Let $x, x' \in X$. The connected components of x and x' are equal iff $x, x' \in W$ for some connected subset $W \subset X$.*

3. *The connected components partition X, i.e., any two connected components are disjoint or coincide.*

4. *Connected components are closed.*

Proof. 1. Let U be the union of all connected subsets containing x. Lemma 9.5 implies that U is connected. If $x \in V$ and V is connected then $V \subset U$.

If U' is a connected component of x then $U \subset U'$ since U is connected and $x \in U$. Clearly $U' \subset U$ since U' is connected and $x \in U$, whence $U' = U$.

2. Suppose that $x, x' \in W$ for some connected subset $W \subset X$. Let U, U' be the connected components of x, x', respectively. If $x \in V$ and $V \subset X$ is connected then the subset $V \cup W$ is connected by Lemma 9.5, contains x' and thus is contained in U'. Therefore $V \subset U'$, whence $U \subset U'$. By symmetricity, $U' \subset U$ which yields $U = U'$. The inverse statement is trivial.

3. If two connected components have a common point x then both of them are the connected components of x and thus coincide.

4. The closure of a connected component U is connected by Proposition 9.4, hence coincide with U. □

Remark. A connected component may not be open. Consider $X = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$. The connected component of 0 is $\{0\}$ which is not open in X.

Examples.

1. The connected components of $\mathbb{R} \setminus \{0\}$ are $(-\infty, 0)$ and $(0, \infty)$.

2. The connected components of a discrete metric space are all the singleton subsets.

Exercise 9.5. *Let X be a metric space. For $x, y \in X$, define $x \sim y$ if $x, y \in V$ for some connected $V \subset X$. Prove that $\sim$ is an equivalence relation on X and the equivalence class of $x \in X$ coincides with its connected component.*

Exercise 9.6. *Prove that a non-empty connected subset of a metric space that is both open and closed is a connected component.*

Hint. Let A is connected, open and closed, $x \in A$. For a connected subset V show that $A \cap V$ and $V \setminus A$ are open in V.

Definition. Let X be a metric space. A continuous map $\varphi\colon [0, 1] \to X$ is a **path** (道路) in X that connects the point $\varphi(0)$ to the point $\varphi(1)$. A metric space X is **path-connected**(道路连通的) if, for each $x, y \in X$, there is a path connecting x to y. A subset Y of a metric space (X, d) is **path-connected** if the corresponding metric subspace $(Y, d|_Y)$ is path-connected.

Examples. 1. Open, closed and half-open intervals in $\mathbb{R}$ are path-connected.

2. $\mathbb{R} \setminus \{0\}$ is not path-connected.

Proposition 9.10. *Let X be a metric space. The relation on X*

$$x \sim y \text{ if there is a path connecting } x \text{ to } y$$

is an equivalence relation.

Proof. First, $\varphi \colon [0,1] \to X, \varphi(t) = x$ is a path connecting x to x.

Second, if $\varphi \colon [0,1] \to X$ is a path connecting x to y then $\psi(t) = \varphi(1-t)$ is a path connecting y to x.

Finally, let $\varphi, \psi \colon [0,1] \to X$ be paths connecting x to y and y to z, respectively. Then

$$\rho(t) = \begin{cases} \varphi(2t) & \text{if } 0 \le t \le 1/2, \\ \psi(2t-1) & \text{if } 1/2 < t \le 1 \end{cases}$$

is a path connecting x and z. □

Exercise 9.7. i) *Prove that a circle in $\mathbb{R}^2$ is path-connected.*

ii) *Prove that a sphere in $\mathbb{R}^3$ is path-connected.*

Exercise 9.8. *Let X be a metric space, and let $\{A_\omega\}_{\omega \in \Omega}$ be a collection of path-connected subsets of X such that $A_\omega \cap A_{\omega'} \neq \varnothing$ for all $\omega, \omega' \in \Omega$. Then, $A = \cup_\Omega A_\omega$ is path-connected.*

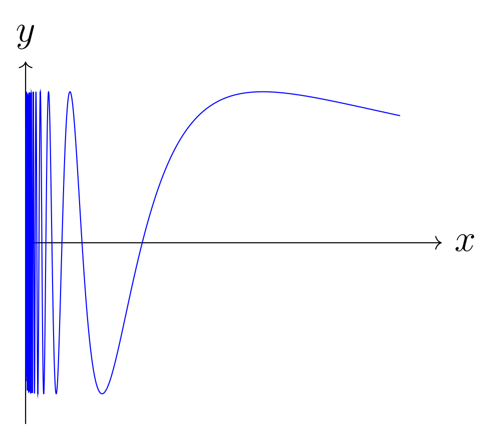
Proposition 9.11. *Let X be a path-connected metric space. Then X is connected.*

Proof. Let $f \colon X \to \{0,1\}$ be a continuous surjective map. Pick up $x \in f^{-1}(0), y \in f^{-1}(1)$. Since X is path-connected, there is a path $\varphi \colon [0,1] \to X$ with $\varphi(0) = x, \varphi(1) = y$. Then the composition $f \circ \varphi \colon [0,1] \to \{0,1\}$ is both continuous and surjective, which is impossible since [0,1] is connected by Proposition 9.2. □

Example (The topologist's sine curve). Let

$$\Gamma = \{(x, \sin(1/x)) \mid x > 0\} \subset \mathbb{R}^2, \quad A = \{(0,y) \mid -1 \le y \le 1\}$$

and $B = \overline{\Gamma} = \Gamma \cup A$.



Then Γ is connected by Proposition 9.8 and B is connected by Proposition 9.4.

We show that B is not path-connected. Let $\varphi\colon [0,1] \to B$ be a path with $\varphi(0) = (0,0), \varphi(1) = (1/\pi, 0)$. Denote $f\colon [0,1] \to \mathbb{R}, f = \pi_1 \circ \varphi$, where $\pi_1(x,y) = x$. Since $f(0) = 0$ and $f(1) = 1/\pi$, the Intermediate Value Theorem yields $t_1 \in (0,1)$ such that $f(t_1) = 2/3\pi$. Similarly we can then find $t_2 \in (0, t_1)$ such that $f(t_2) = 2/5\pi$. Iterating the process we get a decreasing sequence of positive numbers $\{t_n\}$ such that $f(t_n) = 2/(2n+1)\pi$. Let $t \in [0,1]$ be its limit. Finally, since φ is continuous, $\varphi(t_n) \to \varphi(t)$. On the other hand, $\varphi(t_n) = (2/(2n+1)\pi, a_n) \in \Gamma$, hence $a_n = \sin((2n+1)\pi/2) = (-1)^n$. Thus $\varphi(t_n) = (2/(2n+1)\pi, (-1)^n)$, which does not converge, hence a contradiction.

Proposition 9.12. *Let $f\colon X \to Y$ be a continuous surjective map of metric spaces, and suppose X is path-connected. Then Y is path-connected.*

Proof. Let $y_1, y_2 \in Y$. Choose $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$. Let $\varphi\colon [0,1] \to X$ be a path connecting x_1 and x_2. Then $f \circ \varphi\colon [0,1] \to Y$ is a path connecting y_1 and y_2. □

Corollary 9.13. *The projective plane $\mathbb{P}^2$ is path-connected.*

Proof. The map $f\colon S^2 \to \mathbb{P}^2, f(x) = [x]$ is continuous and surjective, S^2 is path-connected. □

Proposition 9.14. *If a nonempty open subset of $\mathbb{R}^n$ is connected then it is path-connected.*

Proof. Let X be a nonempty open subset of $\mathbb{R}^n$ and pick $x_0 \in X$. We want to show there is a path in X from x_0 to every point in X. Set

$$U = \{x \in X \mid \text{ there is a path in } X \text{ from } x_0 \text{ to } x\}.$$

This is a nonempty subset of X since $x_0 \in U$. We will show that U is open. Suppose $x \in U$. Since X is open, there is $B_r(x) \subset X$. Since $B_r(x)$ is path-connected, there is a path connecting x_0 to any point in $B_r(x)$. Thus $B_r(x) \subset U$ and U is open.

Consider

$$V = X \setminus U = \{y \in X \mid \text{ there is no path in } X \text{ from } x_0 \text{ to } y\}.$$

If $V \neq \varnothing$ and $y \in V$, then there is $B_s(y) \subset X$. Since all the points of $B_s(y)$ can be connected to y by a path in $B_s(y) \subset X$, no point in $B_s(y)$ can be connected to x_0 and $B_s(y) \subset V$. Therefore V is open in X which contradicts the connectedness of X. Thus $V = \varnothing$ and X is path-connected. □

Chapter II: Topological spaces

Motivation

Recall the following definitions from theory of metric spaces:

- A subset of a metric space is closed if its complement is open.

- The closure $\overline{A}$ of A is the intersection of all closed subsets containing A (Proposition 4.3).
- A map $f\colon X \to Y$ is continuous if for each open subset $U \subset Y$, the inverse image $f^{-1}(U)$ is an open subset of X.
- A subset A is dense in X if $\overline{A} = X$.
- X is separable if it has a countable dense subset.
- X is compact if every open cover of X has a finite subcover.
- X is connected if the only two subsets of X that are simultaneously open and closed are X and $\varnothing$.

All of them are based on the notion of open set. Properties of open sets are given in Proposition 3.1:

I. X and $\varnothing$ are open.

II. If $U_1, \dots, U_k$ are open subsets of X, then $\cap_{1\le i\le k} U_i$ is also open.

III. If $\{U_\omega\}_{\omega\in\Omega}$ is a family of open subsets of X then $\cup_{\omega\in\Omega} U_\omega$ is also open.

10 Basic definitions and examples

Definition. Let X be a set. A **topology**(拓扑) on X is a collection $\mathcal{T}$ of subsets of X satisfying the following conditions

I. $X, \varnothing \in \mathcal{T}$.

II. If $U_1, \dots, U_k \in \mathcal{T}$, then $\cap_{1\le i\le k} U_i \in \mathcal{T}$.

III. If $\{U_\omega\}_{\omega\in\Omega}$ is a collection of elements of $\mathcal{T}$ then $\cup_{\omega\in\Omega} U_\omega \in \mathcal{T}$.

A **topological space**(拓扑空间) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a topology on X.

Examples. 1. If (X, d) is a metric space then open subsets form a topology on X.

2. Let X be any set and $\mathcal{T}$ be the collection of all subsets of X (**discrete** topology 离散拓扑).
3. Let X be any set and $\mathcal{T} = \{\varnothing, X\}$ (**indiscrete** topology 平庸拓扑).
4. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$.
5. Let $X = \mathbb{R}$ and $\mathcal{T}$ be the collection of subsets of $\mathbb{R}$ which, in addition to $\varnothing$ and $\mathbb{R}$, contains all the subsets of the form (x, ∞) for $x \in \mathbb{R}$.

6. Let X be any set and $\mathcal{T}$ be the collection of subsets of X which, in addition to $\varnothing$ and X, contains all the subsets whose complements are finite (*finite compliment topology*)

7. Let $X = (\mathbb{R} \setminus \{0\}) \cup \{p, q\}$, where p, q denote two extra elements. Let $\mathcal{T}$ consist of all the subsets of the following form:

   - U, where U is an open subset of $\mathbb{R}$ such that $0 \notin U$
   - $(U \setminus \{0\}) \cup \{p\}, (U \setminus \{0\}) \cup \{q\}, (U \setminus \{0\}) \cup \{p, q\}$, where U is an open subset of $\mathbb{R}$ such that $0 \in U$

   (*the line with two origins*)

Remark. Equivalent metrics define the same topology.

Exercise 10.1. *Let $(X, \mathcal{T})$ be the line with two origins.*

1. *Check that $(X, \mathcal{T})$ is a topological space.*

2. *Prove there is no open sets $A, B \subset X$ such that $p \in A, q \in B$ and $A \cap B = \varnothing$.*

Exercise 10.2. *Let $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topologies on X?*

1. *$\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}$;*

2. *$\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}$;*

3. *$\varnothing, X, \{a, c, d\}, \{b, c, d\}$.*

Definition. Let $(X, \mathcal{T})$ be a topological space. The elements of $\mathcal{T}$ are called **open**(开的) subsets. An **open neighborhood**(开邻域) of a point is any open set containing this point. A family of open subsets $\mathcal{B} = \{U_\omega\}_{\omega \in \Omega}$ of X is its **base**(基) if for each open $U \in \mathcal{T}$ and each $x \in U$ there is $U_\omega \in \mathcal{B}$ such that $x \in U_\omega \subset U$.

Proposition 10.1. *(cf. Proposition 3.4)) Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Define*

$$\mathcal{T}' = \{U \cap A \mid U \in \mathcal{T}\}.$$

Then $\mathcal{T}'$ is a topology on A.

Proof. First, $A = X \cap A, \varnothing = \varnothing \cap A \in \mathcal{T}'$. Further, if $U_1, \ldots, U_k \in \mathcal{T}$, one has $U_1 \cap \cdots \cap U_k \in \mathcal{T}$ and hence

$$(U_1 \cap A) \cap \cdots \cap (U_k \cap A) = (U_1 \cap \cdots \cap U_k) \cap A \in \mathcal{T}'.$$

Finally, if $U_\omega \in \mathcal{T}, \omega \in \Omega$ then $\cup_{\omega \in \Omega} U_\omega \in \mathcal{T}$ and

$$\cup_{\omega \in \Omega}(U_\omega \cap A) = (\cup_{\omega \in \Omega} U_\omega) \cap A \in \mathcal{T}'.$$

$\square$

Definition. The topology $\mathcal{T}'$ is called the **relative topology**(相对拓扑) on A, or one says that $\mathcal{T}'$ is **induced**(由...诱导) by the topology $\mathcal{T}$.

Exercise 10.3. *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ be equipped with the relative topology $\mathcal{T}'$ induced by $\mathcal{T}$. Show that on $B \subset A$ the relative topology induced by $\mathcal{T}'$ coincides with the relative topology induced by $\mathcal{T}$.*

Proposition 10.2. *(cf. Corollary 3.3) Let $(X, \mathcal{T}), (Y, \mathcal{S})$ be topological spaces. Define*

$$\mathcal{R} = \left\{ \bigcup_{\omega\in\Omega} (U_\omega \times V_\omega) \mid U_\omega \in \mathcal{T}, V_\omega \in \mathcal{S} \right\}.$$

Then $\mathcal{R}$ is a topology on $X \times Y$.

Proof. First, $X \times Y, \varnothing = \varnothing \times \varnothing \in \mathcal{R}$ since $X, \varnothing \in \mathcal{T}, Y, \varnothing \in \mathcal{S}$. Clearly the union of a collection of elements from $\mathcal{R}$ is again in $\mathcal{R}$. Finally, if $U_\omega, U'_{\omega'} \in \mathcal{T}$ and $V_\omega, V'_{\omega'} \in \mathcal{S}$ for $\omega \in \Omega, \omega' \in \Omega'$ then $U_\omega \cap U'_{\omega'} \in \mathcal{T}, V_\omega \cap V'_{\omega'} \in \mathcal{S}$ for $\omega \in \Omega, \omega' \in \Omega'$ and

$$\begin{aligned}\left(\bigcup_{\omega\in\Omega}(U_\omega \times V_\omega)\right) \bigcap \left(\bigcup_{\omega'\in\Omega'}(U'_{\omega'} \times V'_{\omega'})\right) &= \bigcup_{\omega\in\Omega,\omega'\in\Omega'} \left((U_\omega \times V_\omega) \bigcap (U'_{\omega'} \times V'_{\omega'})\right) \\ &= \bigcup_{\omega\in\Omega,\omega'\in\Omega'} ((U_\omega \cap U'_{\omega'}) \times (V_\omega \cap V'_{\omega'})) \in \mathcal{R}.\end{aligned}$$

□

Definition. The topology $\mathcal{R}$ is called the **product topology**(积拓扑) on $X \times Y$.

Remark. The sets $\{U_\omega \times V_\omega \mid U_\omega \in \mathcal{T}, V_\omega \in \mathcal{S}\}$ form a base of the product topology.

Exercise 10.4. *Let X, Y be discrete topological spaces. Is the product topology on $X \times Y$ discrete?*

11 Closed sets and continuity

Definition. Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is **closed**(闭的) if $X \setminus A \in \mathcal{T}$.

Proposition 11.1. *Let $(X, \mathcal{T})$ be a topological space. Then*

1. *X and $\varnothing$ are closed.*
2. *If $F_1, \ldots, F_k$ are closed subsets of X, then $\cup_{1\le i\le k} F_i$ is also closed.*
3. *If $\{F_\omega\}_{\omega\in\Omega}$ is a family of closed subsets of X then $\cap_{\omega\in\Omega} F_\omega$ is also closed.*

Proof. Similar to Proposition 4.1. □

Exercise 11.1. *Let X be a topological space and $Y \subset X$. Prove that $F' \subset Y$ is closed in the relative topology iff $F' = F \cap Y$ for some closed $F \subset X$.*

Definition. Let X be a topological space and $A \subset X$. A point $a \in X$ is an **adherent** point (聚点) for A if all open neighborhoods of a intersect A. The **closure** $\overline{A}$ of A is the set of the adherent points of A.

The subset A is **dense**(稠密的) in X if $\overline{A} = X$. In other words, A is dense in X if any open set in X has a point from A.

Proposition 11.2. (cf. Proposition 4.3) *Let A be a subset of a topological space X. Then*

1. *$\overline{A}$ is closed.*

2. *$\overline{A}$ is the intersection of all closed subsets containing A.*

Proof. 1. For any $x \in X \setminus \overline{A}$ there is an open neighborhood $U_x \subset X \setminus A$. If $a \in \overline{A} \cap U_x$ then a is an adherent point of A which has an open neighborhood with empty intersection with A, which is impossible. Thus $U_x \subset X \setminus \overline{A}$ and $X \setminus \overline{A} = \cup_{x \in X \setminus \overline{A}} U_x$ is open.

2. It suffices to prove that for any closed $F \subset X$ such that $A \subset F$ one has $\overline{A} \subset F$. Let $x \in X$ be adherent to A. If $x \notin F$ then $x \in X \setminus F$ which is an open neighborhood in $X \setminus A$. and thus does not intersect A, a contradiction. Thus $x \in F$ and $\overline{A} \subset F$. $\square$

Exercise 11.2. *Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$. Find the closure of $\{a\}$ in $(X, \mathcal{T})$.*

Definition. (cf. Proposition 3.5) Let X, Y be topological spaces. Then $f \colon X \to Y$ is **continuous**(连续的) if for each open subset $U \subset Y$, the inverse image $f^{-1}(U)$ is an open subset of X.

Remark. We do not define continuity at a point.

Examples.

1. Let X be $\mathbb{R}$ with indiscrete topology and Y be $\mathbb{R}$ with the Euclidean topology. Then $\mathrm{id}_{\mathbb{R}} \colon X \to Y$ is not continuous and $\mathrm{id}_{\mathbb{R}} \colon Y \to X$ is continuous.

2. Let $X = \mathbb{R}$ with finite compliment topology. Then $\exp \colon X \to X$ is continuous and $\sin \colon X \to X$ is not continuous.

Exercise 11.3. *Let $X = \mathbb{R}$ and $\mathcal{T}$ be the collection of subsets of $\mathbb{R}$ which, in addition to $\varnothing$ and $\mathbb{R}$, contains all the subsets of the form (x, ∞) for $x \in \mathbb{R}$. Which of the following maps from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T})$ are continuous:*

i) *$f(x) = x^2$;*

ii) *$f(x) = x^3$;*

iii) *$f(x) = -x^3$.*

Proposition 11.3. (cf. Proposition 2.1) *Let X, Y, Z be topological spaces. Let $f \colon X \to Y$ and $g \colon Y \to Z$ be continuous. Then $g \circ f \colon X \to Z$ is continuous.*

Proof. Follows from the formula $(g \circ f)^{-1}(A) = f^{-1}(g^{-1}(A))$. □

Proposition 11.4. (cf. Proposition 4.2) *Let X, Y be topological spaces. Then $f\colon X \to Y$ is continuous if and only if for each closed subset $F \subset Y$, the inverse image $f^{-1}(F)$ is a closed subset of X.*

Proof. Follows from the formula $X \setminus f^{-1}(F) = f^{-1}(Y \setminus F)$. □

Exercise 11.4. (cf. Exercise 2.2) *Let X, Y be topological spaces and $f\colon X \to Y$ be a continuous map.*

1. *For $A \subset X$, prove that $f|_A\colon A \to Y$ is continuous, where A is equipped with the relative topology.*
2. *If $B \subset Y$ and $\operatorname{Im} f \subset B$, prove that $f\colon X \to B$ is continuous, where B is equipped with the relative topology.*

Exercise 11.5. (cf. Exercise 3.4) *Let X, Y be two topological spaces and let $X \times Y$ be equipped with the product topology. Prove that the projection map $\pi_X\colon X \times Y \to X, \pi_X((x, y)) = x$ is continuous.*

12 Compactness

Definition. Let X be a topological space. A family $\{U_\omega\}_{\omega\in\Omega}$ of open subsets of X is an **open cover**(开覆盖) of X if $\cup U_\omega = X$. A **subcover**(子覆盖) of a cover $\{U_\omega\}_{\omega\in\Omega}$ is a cover $\{U_\omega\}_{\omega\in\Omega'}$ for some $\Omega' \subset \Omega$.

A topological space is **compact**(紧的) if every its open cover has a finite subcover. A subset of a topological space is **compact** if it is compact in the relative topology.

Examples.

1. Any finite and any indiscrete topological space are compact.
2. Let $X = \mathbb{R}$ with finite compliment topology. Then $(X, \mathcal{T})$ is compact.
3. Let $X = \mathbb{R}$ and $\mathcal{T}$ be the collection of subsets of $\mathbb{R}$ which, in addition to $\varnothing$ and $\mathbb{R}$, contains all the subsets of the form (x, ∞) for $x \in \mathbb{R}$. Then $(X, \mathcal{T})$ is not compact.
4. Let $X = \mathbb{R} \cup \{p\}$ and $\mathcal{T}$ contain X and all the subsets of $\mathbb{R}$. Then $(X, \mathcal{T})$ is compact but has no countable dense subset (cf. Corollary 8.2).
5. Let $X = \{a, b\}$ and $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}\}$. Then $A = \{a\}$ is compact but not closed (cf. Corollary 8.3).

Exercise 12.1. *Let $X = [-1, 1]$ and $\mathcal{T}$ be the collection of all subsets of $[-1, 1]$ which either do not contain 0 or contain $(-1, 1)$.*

1. *Show that $(X, \mathcal{T})$ is a topological space.*

2. Show that $(X, \mathcal{T})$ is compact.

Exercise 12.2. *Let X be a topological space. Show that if $A, B \subset X$ are compact then $A \cup B$ is compact.*

Exercise 12.3. *Let X be a topological space and $B \subset X$ be equipped with the relative topology. Show that if $A \subset B$ is compact as a subset of B then A is compact as a subset of X.*

Exercise 12.4. *Let X be the real line with two origins. Show that $(0,1] \cup \{p\}$ and $(0,1] \cup \{q\}$ are compact, but their intersection $(0,1]$ is not.*

Hint. The topology induced from X to $(0,1]$ coincides with the Euclidean topology.

Proposition 12.1. (cf. Exercise 8.1) *Let X be a compact topological space and $A \subset X$. If A is closed then A is compact.*

Proof. Let $\{V_\omega\}_{\omega\in\Omega}$ be an open cover of A. Then for any $\omega \in \Omega$ one has $V_\omega = U_\omega \cap A$ for some open subset $U_\omega \subset X$. Now $\{U_\omega\}_{\omega\in\Omega}$ together with $X \setminus A$ form an open cover of X. Choose its finite subcover $\{U_\omega\}_{\omega\in\Omega'}$ together with or without $X \setminus A$. In both cases $\{V_\omega\}_{\omega\in\Omega'}$ is a finite subcover of $\{V_\omega\}_{\omega\in\Omega}$. □

Proposition 12.2. (cf. Proposition 8.5) *Let X, Y be topological spaces and $f\colon X \to Y$ be continuous and surjective. If X is compact then Y is compact.*

Proof. If $\{U_\omega\}_{\omega\in\Omega}$ is an open cover of Y then $\{f^{-1}(U_\omega)\}_{\omega\in\Omega}$ is an open cover of X. Choose its finite subcover $\{f^{-1}(U_\omega)\}_{\omega\in\Omega'}$, then $\{U_\omega\}_{\omega\in\Omega'}$ is a finite subcover of $\{U_\omega\}_{\omega\in\Omega}$. □

Lemma 12.3. *Let Y be a topological space and let $\mathcal{B}$ be a base for the topology of Y. If every open cover of Y by sets in $\mathcal{B}$ has a finite subcover, then Y is compact.*

Proof. Let $\{U_\omega\}_{\omega\in\Omega}$ be an open cover of Y. For each $y \in Y$, choose $V_y \in \mathcal{B}$ and $\omega \in \Omega$ such that $y \in V_y \subset U_\omega$. The family $\{V_y \mid y \in Y\}$ forms an open cover of Y by sets in $\mathcal{B}$. Choose $V_{y_1}, \dots, V_{y_n}$ that cover Y. Then $U_{\omega_1}, \dots, U_{\omega_n}$, where $V_{y_i} \subset U_{\omega_i}, 1 \le i \le n$, cover Y. Hence Y is compact. □

Proposition 12.4. (cf. Exercise 8.3) *Let X, Y be topological spaces. Then that $X \times Y$ equipped with the product topology is compact if and only if both X and Y are compact.*

Proof. First, suppose that $X \times Y$ is compact. Clearly, the projection map $\pi_X\colon X \times Y \to X$ is continuous and surjective. Hence X is compact by Proposition 12.2.

Now, let X, Y be compact and $\mathcal{C}$ be an open cover of $X \times Y$ by open sets of the form $U \times V$, where U is open in X and V is open in Y. By Lemma 12.3, it suffices to show that the cover $\mathcal{C}$ has a finite subcover.

Fix $z \in Y$. The coordinate slice $X \times \{z\}$ is compact. Hence there are $U_1 \times V_1, \dots, U_m \times V_m$ in $\mathcal{C}$ such that their intersections with $X \times \{z\}$ cover it. We can assume that $z \in V_j$ for all $1 \le j \le m$ for otherwise $(U_j \times V_j) \cap (X \times \{z\})$ can be deleted and the remaining sets still cover $X \times \{z\}$. Then $V(z) = V_1 \cap \dots \cap V_m$ is an open neighborhood of z in Y. Furthermore, $X \times V(z)$ is covered by the intersections of $\dot{U}_j \times V_j, 1 \le j \le m$ with $X \times V(z)$. Indeed, let $(x, y) \in X \times V(z)$. Then $(x, z) \in U_j \times V_j$ for some $1 \le j \le m$ whence $x \in U_j$ and $(x, y) \in U_j \times V_j$.

Now $\{V(z)\}_{z\in Y}$ is an open cover of Y and there exist $z_1, \dots, z_k \in Y$ such that $Y = V(z_1) \cup \dots \cup V(z_k)$. Then $X \times Y = (X \times V(z_1)) \cup \dots \cup (X \times V(z_1))$. Since each $X \times V(z_i), 1 \le i \le k$ can be covered by a finite number of sets in $\mathcal{C}$, their finite aggregate covers $X \times Y$. □

13 Connectedness and path-connectedness

Definition. A topological space X is **connected**(连通的) if the only two subsets of X that are simultaneously open and closed are X and $\varnothing$. A topological space is **disconnected**(不连通的) if it is not connected. A subset is **connected/disconnected** if it is connected/disconnected with respect to the relative topology.

Proposition 13.1. (cf. Proposition 9.1) *Let X be a topological space. Then the following are equivalent.*

1. *X is disconnected.*
2. *There exist non-empty open subsets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \varnothing$.*
3. *There exists a continuous surjective map $f\colon X \to \{0,1\}$ where the latter topological space is equipped with the discrete topology.*

Proof. Similar to Proposition 9.1. □

Examples. 1. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$. Then $(X, \mathcal{T})$ is connected.

2. Let X be an infinite set with finite compliment topology. Then $(X, \mathcal{T})$ is connected.
3. Let Y be a topological space, $X = Y \cup \{p\}$ and $\mathcal{T}$ contain X and all the open subsets of Y. Then $(X, \mathcal{T})$ is connected.

Exercise 13.1. *Show that the line with two origins is connected.*

Proposition 13.2. (cf. Proposition 9.3) *Let $f\colon X \to Y$ be a continuous surjective map of topological spaces, and suppose X is connected. Then Y is connected.*

Proof. Similar to Proposition 9.3. □

Proposition 13.3. (cf. Proposition 9.4) *Let X be a topological space and $U, A \subset X$. Suppose in addition that U is connected and $U \subset A \subset \overline{U}$. Then A is connected.*

Proof. Let $f\colon A \to \{0,1\}$ be continuous. Since U is connected and $f|_U$ is continuous (Exercise 11.4), either $f(U) = \{0\}$ or $f(U) = \{1\}$. Without loss of generality suppose that $f(U) = \{0\}$. Let $a \in A$ and assume that $f(a) = 1$. Then $f^{-1}(\{1\})$ is an open neighborhood of $a \in \overline{U}$ and thus intersects U, a contradiction. Thus $f(A) = 0$, which shows that A is connected. □

Lemma 13.4. (cf. Lemma 9.5) *Let X be a topological space, and let $\{A_\omega\}_{\omega \in \Omega}$ be a collection of connected subsets of X such that $A_\omega \cap A_{\omega'} \neq \varnothing$ for all $\omega, \omega' \in \Omega$. Then, $A = \cup_\Omega A_\omega$ is connected.*

Proof. Similar to Lemma 9.5. □

Theorem 13.5. (cf. Theorem 9.6) *Let X, Y be topological spaces, and let $X \times Y$ be equipped with the product topology. Then $X \times Y$ is connected if and only if X and Y are connected.*

Proof. Similar to Theorem 9.6. $\square$

Proposition 13.6. (cf. Proposition 9.8) *Let X, Y be topological spaces and $f: X \to Y$ be a continuous map. If X is connected then the graph $\Gamma_f = \{(x, f(x)) \mid x \in X\}$ is a connected subset of $X \times Y$ equipped with the product topology.*

Proof. Similar to Proposition 9.8. It suffices to check that the map $\varphi: X \to X \times Y, \varphi(x) = (x, f(x))$ is continuous. Let $W = \cup_{\omega\in\Omega}(U_\omega \times V_\omega)$, where U_ω are open in X, V_ω are open in Y. Then

$$\varphi^{-1}(W) = \bigcup_{\omega\in\Omega} \varphi^{-1}(U_\omega \times V_\omega) = \bigcup_{\omega\in\Omega}(U_\omega \cap f^{-1}(V_\omega))$$

is open in X. $\square$

Definition. Let X be a topological space, $x \in X$. A **connected component**(连通分支) of x is a connected subset $U \subset X$ such that any connected $V \subset X$ which contains x is contained in U.

Proposition 13.7. (cf. Proposition 9.9) *Let X be a topological space.*

1. *A connected component of x is unique and equals the union of all connected subsets containing x.*

2. *Let $x, x' \in X$. The connected components of x and x' are equal iff $x, x' \in W$ for some connected subset $W \subset X$.*

3. *The connected components partition X, i.e., any two connected components are disjoint or coincide.*

4. *Connected components are closed.*

Proof. Similar to Proposition 9.9. $\square$

Exercise 13.2. *Let X, Y be topological spaces and $f: X \to Y$ be a bijection such that both f and f^{-1} are continuous. Show that if C is the connected component of $x \in X$ then $f(C)$ is the connected component of $f(x)$.*

Definition. Let X be a topological space. A continuous map $\varphi: [0,1] \to X$ is a **path**(道路) in X that connects or the point $\varphi(0)$ to the point $\varphi(1)$. A topological space X is **path-connected**(道路连通的) if, for each $x, y \in X$, there is a path connecting x to y. A subset is **path-connected** if it is path-connected with respect to the relative topology.

Examples. 1. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\}$. Then $(X, \mathcal{T})$ is path-connected. For example, a path connecting a and b can be given by $\varphi: [0,1] \to X$,

$$\varphi(t) = \begin{cases} a & \text{if } t \in [0, 1), \\ b & \text{if } t = 1 \end{cases}$$

2. Let Y be a topological space, $X = Y \cup \{p\}$ and $\mathcal{T}$ contain X and all the open subsets of Y. Then $(X, \mathcal{T})$ is path-connected. For example, a path connecting $y \in Y$ and p can be given by $\varphi\colon [0,1] \to X$,

$$\varphi(t) = \begin{cases} y & \text{if } t \in [0,1), \\ p & \text{if } t = 1 \end{cases}$$

Exercise 13.3. *Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$. Show that $(X, \mathcal{T})$ is path-connected.*

Exercise 13.4. *Show that the line with two origins is path-connected.*

Hint. Let $X = (\mathbb{R} \setminus \{0\}) \cup \{p, q\}$ be the line with two origins. Then $\varphi\colon [0,1] \to X$,

$$\varphi(t) = \begin{cases} 2t - 1 & \text{if } t \neq \frac{1}{2}, \\ p & \text{if } t = \frac{1}{2} \end{cases}$$

is a path connecting -1 and 1.

Exercise 13.5. *Prove that the product of two path-connected topological spaces with product topology is path-connected. Do not forget to prove that the path you choose is continuous.*

Proposition 13.8. (cf. Proposition 9.11) *Let X be a path-connected topological space. Then X is connected.*

Proof. Similar to Proposition 9.11. □

Proposition 13.9. (cf. Proposition 9.12) *Let $f\colon X \to Y$ be a continuous surjective map of topological spaces, and suppose X is path-connected. Then Y is path-connected.*

Proof. Similar to Proposition 9.12. □

14 Quotient topological space

Let $(X, \mathcal{T})$ be a topological space, and let $\sim$ be an equivalence relation on X. Let $\overline{X}$ be the set of equivalence classes, and let $\pi\colon X \to \overline{X}$ be the map that sends $x \in X$ to its equivalence class $[x]$. Define

$$\overline{\mathcal{T}} = \{U \subset \overline{X} \mid \pi^{-1}(U) \in \mathcal{T}\}.$$

Lemma 14.1. *The collection $\overline{\mathcal{T}}$ defines a topology on $\overline{X}$.*

Proof. First, $\varnothing = \pi^{-1}(\varnothing) \in \mathcal{T}$ and $X = \pi^{-1}(\bar{X})$.

Further, if $\{U_\omega\}_{\omega \in \Omega} \subset \overline{\mathcal{T}}$ then

$$\pi^{-1}\left(\bigcup_{\omega \in \Omega} U_\omega\right) = \bigcup_{\omega \in \Omega} \pi^{-1}(U_\omega) \in \mathcal{T}.$$

Similarly, if $\{U_1, \ldots, U_n\} \subset \overline{\mathcal{T}}$ then

$$\pi^{-1}\left(\bigcap_{i=1}^{n} U_i\right) = \cap_{i=1}^{n} \pi^{-1}(U_i) \in \mathcal{T}$$

which completes the proof. $\square$

Definition. The topological space $(\overline{X}, \overline{\mathcal{T}})$ is called the **quotient space**(商空间) of $(X, \mathcal{T})$ by the equivalence relation $\sim$, and the map $\pi \colon X \to \overline{X}$ is called the **quotient map**(商映射).

Exercise 14.1. *Let $X = [0,1]$ and $x \sim y$ if $x, y \in [0, 1/3]$ or $x, y \in (1/3, 2/3]$ or $x, y \in (2/3, 1]$. Describe explicitly the topology on the quotient set of X by $\sim$.*

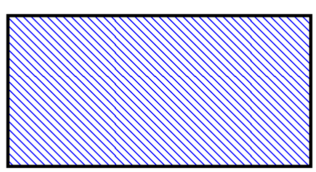
Proposition 14.2. *Let $(X, \mathcal{T})$ be a topological space, and let $\sim$ be an equivalence relation on X.*

1. *$\pi \colon X \to \overline{X}$ is continuous.*

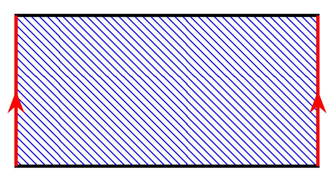
2. *If $f \colon \overline{X} \to Y$ is a map to some topological space $(Y, \mathcal{S})$, then f is continuous if and only if $f \circ \pi$ is continuous.*

Proof. 1. Follows from the definition of $\overline{\mathcal{T}}$.

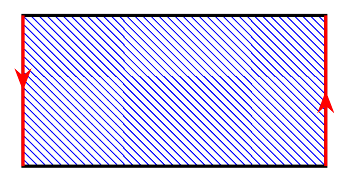
2. Suppose $f \circ \pi$ is continuous. Then $\pi^{-1}(f^{-1}(V)) = (f \circ \pi)^{-1}(V) \in \mathcal{T}$ for any $V \in \mathcal{S}$ whence $f^{-1}(V) \in \overline{\mathcal{T}}$ and thus f is continuous. The inverse implication is obvious. $\square$



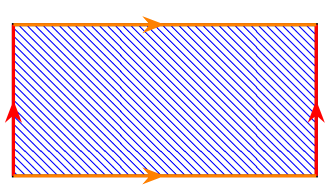
Stripe



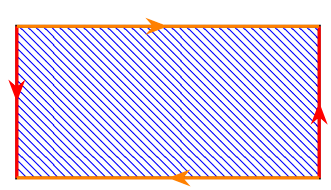
Cylinder



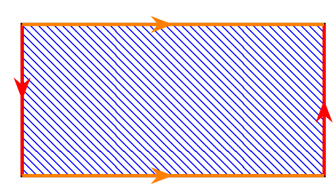
Möbius strip



Torus



Projective plane



Klein bottle

Corollary 14.3. *The quotient space of a compact/connected/path-connected topological space is compact/connected/path-connected. In particular, torus, Klein bottle, and Möbius strip are compact and path-connected.*

Exercise 14.2. *Let X, Y be topological spaces and $\sim$ be an equivalence relation on X. Let $f \colon X \to Y$ be a map such that $f(x) = f(x')$ for any $x \sim x', x, x' \in X$. Then*

i) *There exists a unique map* $\overline{f}\colon \overline{X} \to Y$ *such that* $f = \overline{f} \circ \pi$ *where* $\overline{X}$ *is the quotient space of* X *with respect to* $\sim$

ii) *If* f *is continuous then* $\overline{f}$ *is continuous*

iii) *If* f *is open (i.e. the image of every open subset of* X *is open in* Y*) then* $\overline{f}$ *is open*

Exercise 14.3. *The relation* $x \sim x'$ *if* $x - x' = 2\pi k, k \in \mathbb{Z}$ *is an equivalence relation on* $\mathbb{R}$*. Let* $\overline{\mathbb{R}}$ *be the quotient space of* $\mathbb{R}$ *with respect to* $\sim$ *and* $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ *be the circle of radius 1 with center at* 0*. Let* $f\colon \mathbb{R} \to S$ *be defined by* $f(t) = (\cos t, \sin t)$*.*

i) *Check that* f *satisfies the condition of Exercise 14.2 and thus defines the map* $\overline{f}\colon \overline{\mathbb{R}} \to S$ *satisfying* $f = \overline{f} \circ \pi$*.*

ii) *Check that* f *is continuous and open*

iii) *Prove that* $\overline{f}$ *is a bijection*

iv) *Prove that* $\overline{f}$ *is continuous and its inverse is also continuous.*

Hint. Use Exercise 14.2. The continuity of $\overline{f}^{-1}$ follows from the fact that $\overline{f}$ is an open map.

Exercise 14.4. *The relation* $(x, y) \simeq (x', y')$ *if* $x^2 + y^2 = x'^2 + y'^2 = 1$ *and* $x = -x', y = -y'$ *is an equivalence relation on the closed disc* $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ *which defines the quotient space* $\overline{D}$*. Let* $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$ *be the unit sphere and* $\mathbb{P}^2$ *be its quotient space by the equivalence relation* $(x, y, z) \sim (x', y', z')$ *if* $(x, y, z) = -(x', y', z')$*. Let* f *be the orthogonal projection from* S *to* $\overline{D}$ *defined by*

$$f((x, y, z)) = \begin{cases} \{(x, y)\} & \text{if } z > 0, \\ \{(-x, -y)\} & \text{if } z < 0, \\ \{(x, y), (-x, -y)\} & \text{if } z = 0 \end{cases}$$

i) *Check that* f *is correctly defined, satisfies the condition of Exercise 14.2, and thus defines the map* $\overline{f}\colon \mathbb{P}^2 \to \overline{D}$ *satisfying* $f = \overline{f} \circ \pi$*.*

ii) *Check that* f *is continuous and open*

iii) *Prove that* $\overline{f}$ *is a bijection*

iv) *Prove that* $\overline{f}$ *is continuous and its inverse is also continuous.*

15 Homeomorphism

Definition. Let X, Y be topological spaces. A map $f\colon X \to Y$ is a **homeomorphism**(同胚映射) if it is bijective and both f and f^{-1} are continuous. Two topological spaces X and Y are **homeomorphic**(同胚的) ($X \cong Y$) if there exists a homeomorphism $f\colon X \to Y$.

Remark. If f is bijective and continuous then f^{-1} may not be continuous. Consider $X = [0,1) \cup [2,3], Y = [0,2]$ and

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1, \\ x-1 & \text{if } 2 \le x \le 3 \end{cases}.$$

Proposition 15.1. *1. id_X is a homeomorphism*

2. If $f\colon X \to Y$ is a homeomorphism then $f^{-1}\colon Y \to X$ is a homeomorphism

3. If $f\colon X \to Y$ is a homeomorphism, $g\colon Y \to Z$ is a homeomorphism then $g \circ f\colon X \to Z$ is a homeomorphism

Proof. The third part follows from Proposition 11.3, the rest is trivial. □

Corollary 15.2. *The relation $\cong$ is an equivalence relation*

Proposition 15.3. *Let X, Y be homeomorphic topological spaces. Then*

1. *If X is compact then Y is compact*
2. *If X is connected then Y is connected*
3. *If X is path-connected then Y is path-connected*
4. *The number of connected components of X and Y is equal*

Proof. Follows from Propositions 12.2, 13.2, 13.9, and Exercise 13.2. □

Proposition 15.4. *Let X, Y be topological spaces and $f\colon X \to Y$ be a homeomorphism. If $A \subset X$ then $f|_A\colon A \to f(A)$ is a homeomorphism, where A and $f(A)$ are equipped with the relative topology.*

Proof. Follows from Exercise 11.4. □

Examples. 1. $(0,1) \cong \mathbb{R}$

$f\colon (0,1) \to \mathbb{R}, f(x) = \tan(x/\pi + 1/2)$

2. Exercise 14.3: a quotient space of $\mathbb{R}$ is homeomorphic to a circle

3. Exercise 14.4: a quotient space of the closed disc is homeomorphic to $\mathbb{P}^2$

4. $\Gamma_\varphi = \{(t, \varphi(t)) \in \mathbb{R}^2 \mid t \in [a,b]\} \cong [a,b]$ if $\varphi\colon [a,b] \to \mathbb{R}$ is continuous

   $f\colon [a,b] \to \Gamma_\varphi, f(t) = (t, \varphi(t))$

5. The region bounded by the graph of a continuous positive function over a closed interval and the x-axis $\Sigma_\varphi = \{(x,y) \in \mathbb{R}^2 \mid x \in [a,b], y \in [0, \varphi(x)]\}$ is homeomorphic to the strip $[a,b] \times [0,1]$

   $f\colon [a,b] \times [0,1] \to \Sigma_\varphi, f((t,s)) = (t, s\varphi(t))$

6. A convex polygon is homeomorphic to a circle

   Let p be a point inside the polygon P and C be a circle with center p. Then any ray with initial point p intersects both P and C once. This defines a continuous one-to-one correspondence between P and C whose inverse is also continuous.

7. $\mathbb{R}^2 \cong D = \{x \in \mathbb{R}^2 \mid ||x|| < 1\}$

   $f\colon \mathbb{R}^2 \to D, f(x) = \frac{x}{||x||+1}$

8. Punctured sphere is homeomorphic to a plane

   One can use a stereographic projection from the punctured point to a plane perpendicular to the diameter through this point.

9. A punctured plane is homeomorphic to the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$

   Both spaces can be viewed as a disjoint union of concentric circles indexed by $\mathbb{R}$ in case of the cylinder and $(0, \infty)$ in case of the punctured plane. One can use a homeomorphism between $\mathbb{R}$ and $(0, \infty)$ to relate the circles.

10. The square without a vertex $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x, y \leq 1\} \setminus \{(1, 1)\}$ is homeomorphic to a closed half-plane

    First, a square without a vertex is homeomorphic to a disk without a boundary point. Second, the disk without a boundary point $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1, y < 1\}$ is homeomorphic via the stereographic projection from $(0, 1)$ to the semi-open strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y < 1\}$. Third, the map $\varphi((x, y)) = (x, \frac{y}{1-y})$ gives a homeomorphism from the semi-open strip $\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y < 1\}$ to the closed half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$.

11. $(0, 1) \ncong (0, 1]$

    Removal of any point in $(0, 1)$ makes it disconnected while removal of 1 in $(0, 1]$ leaves it connected.

12. $\mathbb{P}^2 \ncong \mathbb{R}^2$

    $\mathbb{P}^2$ is compact and $\mathbb{R}^2$ is not

13. $\mathbb{R} \ncong \mathbb{R}^2$

    Removal of any point in $\mathbb{R}$ makes it disconnected while removal of any point in $\mathbb{R}^2$ leaves it connected.

14. Letter Y is not homeomorphic to letter I

    Removal of the central point in Y leaves 3 connected components while removal of any point in I leaves 1 or 2 connected components

15. Digit 8 is not homeomorphic to digit 9

    A *cut-point* is a point of a connected topological space such that its removal causes the resulting space to be disconnected.

Digit 8 has a unique cut-point and digit 9 has infinitely many cup-points

16. Letter A is not homeomorphic to letter P

    There are two point in A removal of which leaves 4 connected components while removal of any two points in P leaves 2 or 3 connected components

17. A sphere is not homeomorphic to a torus

    Removal of any continuous image of a circle on the sphere makes it disconnected while removal the equatorial circle on the torus leaves it connected.

18. An open cylinder is not homeomorphic to an open Möbius strip

    Removal of any subset homeomorphic to a circle in the cylinder makes it disconnected. On the other hand, there is a circle you can remove from any Möbius strip which leaves the it connected.

19. $\mathbb{R}^2 \not\cong \mathbb{R}^3$

    A topological space X is *simply connected* if it is path-connected, and whenever $p\colon [0,1] \to X$ and $q\colon [0,1] \to X$ are two paths with the same start and endpoint $p(0) = q(0), p(1) = q(1)$, then p can be continuously deformed into q while keeping both endpoints fixed. Formally, there exists a continuous map $F\colon [0,1] \times [0,1] \to X$ such that $F(t,0) = p(t), F(t,1) = q(t)$ for any $t \in [0,1]$.

    Removal of any point in $\mathbb{R}^2$ makes it not simply connected while removal of any point in $\mathbb{R}^3$ leaves it simply connected.

20. A twice-punctured plane is not homeomorphic to a punctured plane

    A *loop* in X is a path with the same start and endpoint which is called its *base-point*. The *product* of two loops γ, γ' with the same base-point is defined as the loop $\gamma \cdot \gamma'\colon [0,1] \to X$, where

    $$(\gamma \cdot \gamma')(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ \gamma'(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

    Then for any two loops γ, γ' in the punctured plane, the loop $\gamma \cdot \gamma'$ can be continuously deformed to the loop $\gamma' \cdot \gamma'$, while in the twice-punctured plane two simple loops around each poles do not satisfy this property.

Exercise 15.1. *Give a formal proof that $[0,1)$ and $[0,\infty)$ are homeomorphic*

Exercise 15.2. *Give a formal proof that a punctured circle and $\mathbb{R}$ are homeomorphic*

Exercise 15.3. *Explain why a twice-punctured sphere is homeomorphic to an open cylinder*

Exercise 15.4. *Prove that the image of a cut-point under a homeomorphism is a cut-point*

Exercise 15.5. *Give a formal proof that the letters X and Y are not homeomorphic*

Exercise 15.6. *Give a formal proof that the letters H and Y are not homeomorphic*