Aug. 28th. HW1.

1. Does the function $d(x, y) = \arctan|x-y|$ define the metric on $\mathbb{R}$?

No ; Pf: (1) identical : $d(x,y)=0 \Leftrightarrow \arctan|x-y|=0 \Leftrightarrow |x-y|=k\pi,\ k\in\mathbb{Z} \not\Leftrightarrow x=y$

the function does not satisfy the identity matrix.

2. Does the function $d(x, y) = \sqrt{|x-y|}$ define the metric on $\mathbb{R}$?

Pf: (1) identical : $d(x,y)=0 \Leftrightarrow \sqrt{|x-y|}=0 \Leftrightarrow |x-y|=0 \Leftrightarrow x=y$

(2) symmetric : $d(x,y)=\sqrt{|x-y|}=\sqrt{|y-x|}=d(y,x)$

(3) triangular : $d(x,y)=\sqrt{|x-y|} \le \sqrt{|x-z|+|z-y|} \le \sqrt{|x-z|}+\sqrt{|z-y|} = d(x,z)+d(y,z)$

$\Downarrow$

since $\left(\sqrt{|x-z|+|z-y|}\right)^2 \le \left(\sqrt{|x-z|}+\sqrt{|z-y|}\right)^2$

$\Leftrightarrow \quad 0 \le 2\sqrt{|x-z|\cdot|z-y|}$

3. Let (X, d) be a metric space. Let $Y = 2^X$ be a set of all subsets of X. Does the function

$$\rho(E, F) = \inf\{d(x, y) : x \in E,\ y \in F\}$$

define a metric on Y.

Pf: (1) identical : $\rho(E,F)=0 \Leftrightarrow, \exists x\in E,\ y\in F,\ d(x,y)=0 \Leftrightarrow x=y$ (since (X,d) is metric space)

(2) symmetric : $\rho(E,F)=\inf\{d(x,y): x\in E, y\in F\} = \inf\{d(y,x): x\in F, y\in E\} = \rho(F,E)$;

(3) triangular : $\rho(E,F)=\inf\{d(x,y): x\in E, y\in F\}$

consider $a,b,c,d\in X$. Let $E=\{a,b\}$ $F=\{c,d\}$. $G=\{b,c\}$

$\rho(E,F) = \min\{d(a,c), d(b,c), d(b,d), d(a,d)\}$. since $a\neq b\neq c\neq d$,

$d(a,c)\neq d(b,c)\neq d(b,d)\neq d(a,d)$. $\rho(E,F)>0$

However $\rho(E,G)+\rho(F,G) = d(b,b)+d(c,c)=0 < \rho(E,F)$

Thus, the function doesn't define metric. on Y

4. Prove that a sphere

$$S(a, r) = \{x \in \mathbb{R} : \rho(a, x) = r\}$$

is a closed set.

Pf: Assume the converse.

$\exists$ a limit point x_0 of S. s.t. $x_0 \notin S$. we assume $\rho(a,x_0)=r_0$. w.l.g. $r_0<r$

Suppose $V_{x_0}\setminus\{x_0\} = \{x\in\mathbb{R}: 0<\rho(x_0,x)\le \frac{|r-r_0|}{2}\}$

in this neighborhood $\rho(a,x) \le \rho(x_0,x)+\rho(x_0,a) = \frac{|r-r_0|}{2}+r_0 \le \frac{r+r_0}{2} < r$

(Actually, if. $r_0>r$, $\rho(a,x)\ge\rho(x_0,a)-\rho(x_0,x) = r_0-\frac{r_0-r}{2} = \frac{r_0+r}{2} > r$)

which means $S\cap V_{x_0}\setminus\{x_0\}=\phi$. x_0 is not a limit point.

5. Let H be the set of all real sequences $x = (x_1, x_2, x_3, \ldots)$ such that $|x_n| \leq 1$ for all $n \in \mathbb{N}$. For $x, y \in H$ let

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n| .$$

Prove that d is a metric on H.

Pf: 1) identical.

$d(x,y) = 0 \Leftrightarrow$ since $2^{-n}|x_n - y_n| \geq 0. \Leftrightarrow 2^{-n}|x_n - y_n| = 0 \Leftrightarrow x_n = y_n$, for all $n \Leftrightarrow x = y$.

2) symmetric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n| = \sum_{n=1}^{\infty} 2^{-n} |y_n - x_n| = d(y,x)$$

3) triangular

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n| \leq \sum_{n=1}^{\infty} 2^{-n}(|x_n - y_n| + |y_n - z_n|) = d(x,z) + d(z,y)$$

d is a metric on H.

Check d is correctly defined. $d(x,y) \leq 2 \sum_{n \geq 1} 2^{-n} < \infty$

Aug 30th. HW2

1. Let (X, d) be a metric space, $a \in X$, $r > 0$.
   (a) Prove that the sphere $S = \{x \in X : d(a, x) = r\}$ is the closed set.
   (b) Let $D \subset X$. Prove that $\partial D = \mathrm{cl}(D) \setminus \mathrm{int}(D)$ and that ∂D is closed.
   (c) Explain why $\emptyset$ and X are open and closed sets.

(a) Pf: It's sufficient to show that $S^c(a,r)$ is open set.

$\forall x_0 \in S^c(a,r)$, we have $\rho(a, x_0) \neq r$.

w.l.g. we assume $r > \rho(a, x_0)$.

then we can construct a open ball. $B(x_0, r_1) = \{x_0 \in S^c : \rho(x_0, x) < r_1\}$

where $r_1 = \dfrac{r - \rho(a, x_0)}{2}$

thus $B(x_0, r_1) \sqsubset S^c$. x_0 is interior point.

Since x_0 is arbitrary, S^c is open, S is closed.

(b) Pf: "$\sqsubset$" $\forall x_0 \in \partial D$ $\forall \varepsilon > 0$, $\dot{V}_{x_0}(\varepsilon) \cap D \neq \phi$

x_0 is a limit point. $x_0 \in \mathrm{Cl}\, D$.

$\dot{V}_{x_0}(\varepsilon) \cap X \setminus D \neq \phi$. no neighborhood completely contained in D.

$x_0 \notin \mathrm{int}\, D$. i.e. $x_0 \in \mathrm{Cl}\, D \setminus \mathrm{int}\, D$.

"$\sqsupset$" $\forall x_0 \in \mathrm{Cl}\, D \setminus \mathrm{int}\, D$

$x_0 \notin \mathrm{int}\, D \rightarrow$ every neighbor of x_0 contains points of $X \setminus D$.

1) $x_0 \in D$. $V_{x_0} \cap D \neq \phi$. have element x_0. at least.

2) $x_0 \in \{x \in X.\ x_0$ is limit point of $D\}$. $D \cap \dot{V}_{x_0} \neq \phi$. by def.

thus $x_0 \in \partial D$.

"Closeness" $\forall \{x_n\} \sqsubset \partial D$. $x_n \to a$. we need $a \in \partial D$.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, for $n > N$. $|x_n - a| < \frac{\varepsilon}{3}$.

since $x_n \in \partial D$. $\exists x_n^1 \in B(x_n, \frac{\varepsilon}{3}) \cap D$, $x_n^2 \in B(x_n, \frac{\varepsilon}{3}) \cap X \setminus D$.

For every ε, $|a - x_n^1| \leq |x_n^1 - x_n| + |x_n - a| = \frac{2\varepsilon}{3} < \varepsilon$ $x_n^1 \in B(a, \varepsilon) \cap D$.

$|a - x_n^2| < \varepsilon$. $x_n^2 \in B(a, \varepsilon) \cap X \setminus D$

thus any neighborhood of a contains both points of D and $X \setminus D$.

i.e. $a \in \partial D$.

(C) Pf: "ϕ is closed". ϕ has no limit point. so it all contained in ϕ.

"ϕ is close" $\Rightarrow$ $X \setminus \phi = X$ is open.

"X is closed" $X \setminus X = \phi$. no points outside of X.

impossible to have limit points outside of X.

"X is closed" $\Rightarrow$ $X \setminus X = \phi$ is open $\square$

2. Prove that $X = \mathbb{Q}$ with metric $d(x, y) = |x - y|$ is not complete. What is the reason for $\mathbb{R}$ with the same metric to be complete (refer to and formulate the theorem from the 1st semester).

Pf: consider a sequence $x_n = (1+\frac{1}{n})^n$.

we know $\lim_{n\to\infty} d(x_n, e) = 0$ it's conv. in $\mathbb{R}$. so it's Cauchy.

Since $e \notin \mathbb{Q}$. the sequence isn't conv. on $\mathbb{Q}$.

so $(\mathbb{Q}, d)$ not complete

Bolzano-Weierstrass: Every bounded sequence of real numbers contains convergent subsequence

discrete metric

$d(x,y) = \begin{cases} 0, & x=y \\ 1, & x \neq y. \end{cases}$

we can show every Cauchy sequence is constant.

3. Prove that a discrete space is always complete.

Pf: Take arbitrary discrete space D.

$\forall$ Cauchy's sequence $\{x_n\}_{n=1}^{\infty} \subset D$.

we can claim that $x_i \neq x_j$ when $j \neq i$.

(otherwise. $d(x_i, x_j) = 1 := \varepsilon_0$, the sequence will not be Cauchy's.)

Thus we have any $x_i \in \{x_n\}_{n=1}^{\infty}$ is the limit of $\{x_n\}_{n=1}^{\infty}$

Since $d(x_i, x_j) = 0$ $(i \neq j)$. $\{x_n\}_{n=1}^{\infty}$ is conv.

Sept. 1st HW3

1. Let (X, d) be a metric space.

   (a) Prove that any open set can be expressed as a union of some family of balls.

   (b) Prove that a set K is compact if from any cover by open balls there exists finite subcover.

Pf: (a) $\forall A$ is open. We need to check $A = \bigcup_n V_n$. for some $\{B_n\}$.

"$\supset$" for any $a \in A$. A is open. $\exists V_a$. s.t. $V_a \subset A$. thus. $\bigcup V_a \subset A$.

"$\subset$". $\forall a \in A$. $\exists V_a$. s.t. $a \in V_a \subset \bigcup V_a$. $A \subset \bigcup V_a$.

(b) Assume the converse: there exists a open cover $C = \{G_\alpha\}_{\alpha \in A}$. of K.

s.t. every subcover of C is infinite.

By (a). For every G_α. $\exists \{V_{\alpha i}\}_{i \in B}$. s.t. $\bigcup_{i \in B} V_{\alpha i} = G_\alpha$.

thus we have $K \subset \bigcup_{\alpha \in A} G_\alpha = \bigcup_{\alpha \in A.} (\bigcup_{i \in B} V_{\alpha i})$

the cover $\bigcup_{\alpha \in A} (\bigcup_{i \in B} V_{\alpha i})$ is cover by open ball. which has finite subcover.

Let a finite sequence $\{x_1, \dots x_k\} \subset \{\alpha_i\}_{i \in B.}^{\alpha \in A}$ s.t. $K \subset \bigcup^k V x_j$

For every fixed x_j $(1 \le j \le k)$, we can find a set. s.t. $V_{x_j} \subset G_{\alpha_j}$ For some $\alpha_j \in \{\alpha\}$

thus we have $K \subset \bigcup^k V_{x_i} \subset \bigcup^k G_{\alpha_j} \subset \bigcup G_\alpha$

2. Let $x^k = (x_1^k, x_2^k, \dots, x_n^k)$ be a sequence of points in $\mathbb{R}^n$. Prove that $\{x^k\}$ is convergent in Euclidean metric if and only if sequences of coordinates $\{x_j^k\}_{k=1}^\infty$ are convergent, $1 \le j \le n$, in $\mathbb{R}$.

"$\Rightarrow$" x^k is conv. i.e. $\exists a = \lim_{k \to \infty} x^k$ where $a = (a_1, a_2, \dots a_n)$.

Let $\varepsilon > 0$. $\exists N \in \mathbb{N}$. s.t. for any $k > N$.

$\left| \sum_{j=1}^n |a_j - x_j^k|^n \right|^{\frac{1}{n}} < \varepsilon^{\frac{1}{n}} \Rightarrow \sum_{j=1}^n |a_j - x_j^k| < \varepsilon$.

$\Rightarrow$ any fixed $i \in \mathbb{N}$ and $1 \le i \le n$,

$|a_j - x_j^k| < \varepsilon$. $\Rightarrow \lim_{k \to \infty} x_j^k = a_j$

thus the coordinate sequences $\{x_j^k\}_{k=1}^\infty$ are conv.

"$\Leftarrow$" $\exists \{a_1, \dots a_n \in \mathbb{R}$, s.t. $\lim_{k \to \infty} x_i^k = a_i$. for $1 \le i \le n$

i.e. let $\varepsilon > 0$. $\exists N_j \in \mathbb{N}$. for any $k > N_1$.

$|a_j - x_j^k|^n < (\frac{\varepsilon^{\frac{1}{n}}}{n})$

let $M = \max\{N_1, \dots N_{n-1}, N_n\}$. for any $k > M$.

$\left| \sum_{j=1}^n |a_j - x_j^k|^n \right|^{\frac{1}{n}} \le \left| \sum_{j=1}^n \frac{\varepsilon}{n} \right|^{\frac{1}{n}} < \varepsilon^{\frac{1}{n}} \le \varepsilon$

Let $a = (a_1, a_2 \dots a_n)$. ~~then~~ we have $\lim_{k \to \infty} x^k = a$ $\{x_k\}$ is conv.

3. Prove that $\mathbb{R}^n$ is separable.

3. It's sufficient to show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

since $\mathbb{Q}^n = \mathbb{Q} \times \mathbb{Q} \times \cdots \times \mathbb{Q}$, $\mathbb{Q}$ is countable and n is a constant. the at most countable union of countable set is countable. $\mathbb{Q}^n$ is Countable.

it's equivalent to show $\mathbb{Q}$ is dense in $\mathbb{R}$.

We can firstly show $\forall r \in \mathbb{R}. \forall \varepsilon > 0. \exists q \in \mathbb{Q}. d(q, r) < \varepsilon$;

by Archimedes property. $\exists h > 0$. For any $r \in \mathbb{R}$

$\exists n \in \mathbb{N}$. $(n-1)h \le r < nh$.

Let $h = \varepsilon$. $q = nh$. $d(q, r) < \varepsilon$.

Thus we have $\mathbb{Q}$ is dense in $\mathbb{R}$.

otherwise, since $\bar{\mathbb{Q}} \neq \mathbb{R}$. $\mathbb{Q} \subset \mathbb{R}$, we have. $\{x \in \mathbb{R} \setminus \mathbb{Q} : x \text{ is limit point of } \mathbb{Q}\} \subsetneq \mathbb{R} \setminus \mathbb{Q}$

That's impossible. since $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. $V_{x_0} \cap \mathbb{Q} \neq \phi$.

(for any fixed $V_{x_0}(\varepsilon)$. we have $d(q, x_0) < \varepsilon$).

any $x \in \mathbb{R} \setminus \mathbb{Q}$ is limit point of $\mathbb{Q}$. thus. $\mathbb{Q}$ is dense in $\mathbb{R}$.

similarly we have $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

4. Consider a space of polynomials $\mathcal{P} = \left\{ p(x) = \sum_{k=1}^{n} a_k x^k : k \in \mathbb{N},\ a_k \in \mathbb{R} \right\}$ on $[a, b]$ with uniform metric

$$d(p, q) = \max_{x \in [a,b]} |p(x) - q(x)|, \quad p, q \in \mathcal{P}.$$

Prove that this space is separable. Check the completeness of this space.

Pf: Actually we can define arbitrary $p(x) \in \mathcal{P}$ as:

$$p(x) = (a_1, a_2, \dots, a_n, x) \in \mathbb{R}^k \times [a, b]$$

Also $\forall a = (a_1', a_2', \dots a_n', x') \in \mathbb{R}^k \times [a, b]$. we have some $p(x) = a$.

thus we match the space $\mathcal{P}$ with the space $\mathbb{R}^k \times [a, b]$.

Actually in (3) we have proved $\overline{\mathbb{Q}^k} = \mathbb{R}^k$. $\mathbb{Q}^k$ is countable.

Let $\mathbb{Q}_1 = \{x \in \mathbb{Q} \wedge x \in [a, b]\}$ we have $\mathbb{Q}_1$ is Countable

and similar in 3. we have $\bar{\mathbb{Q}}_1 = [a, b]$, thus $[a, b]$ is separable

thus we can countable dense set $\mathbb{Q}^k \times \mathbb{Q}_1$ of space $\mathbb{R}^k \times [a, b]$.

Consider $\mathcal{Q} = \left\{ p(x) = \sum_{k=1}^{n} a_k x^k : k \in \mathbb{N},\ a_k \in \mathbb{Q} \right\}$

$\mathcal{Q}$ is countable and dense in $\mathcal{Q}$.

Not complete. $\quad P_n(x) = \sum_{k=1}^{n} \frac{x^k}{k!} \qquad P_n(x) \rightrightarrows e^x \notin \mathcal{P}$

5. Find the limit of a sequence $\{x^n\}$ if

(a) $x^n = \left(\frac{1}{n}, \frac{n+1}{n}\right)$ in $\mathbb{R}^2$;

(b) $x^n = \left(\left(1+\frac{2}{n}\right)^{n/2}, e^{-n}\right)$ in $\mathbb{R}^2$;

(c) $f_n(x) = \arctan(nx)$ in $C[1,2]$.

5. 1) $\lim\limits_{n\to\infty} \frac{1}{n} = 0.$ $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1.$

By the problem 2. $\lim x^n = \lim (x_1^n, x_2^n) = (0,1)$.

2) $\lim\limits_{n\to\infty} (1+\frac{2}{n})^{\frac{n}{2}} = e$ $\lim\limits_{n\to\infty} e^{-n} = 0.$

$\lim\limits_{n\to\infty} x^n = (e,0).$

3). $\lim\limits_{n\to\infty} f_n(x) = \frac{\pi}{2}.$ $x \in [1,2].$

$\lim\limits_{n\to\infty} \sup\limits_{x\in[1,2]} |f_n(x) - \frac{\pi}{2}| = \lim \sup |f_n(2) - \frac{\pi}{2}| = \lim\limits_{n\to\infty} |\arctan 2n - \frac{\pi}{2}| = 0$

$\therefore \{f_n(x)\} \to \frac{\pi}{2}.$

$d(f_n, \frac{\pi}{2}) = \max\limits_{x\in[1,2]} |\frac{\pi}{2} - \arctan nx| = |\frac{\pi}{2} - \arctan 2n| \to 0.$

Sept. 6th. HW4.

1. Consider a space ℓ^∞ of bounded sequences. Prove that it is a Banach space with respect to the norm (check axioms of the norm and completeness)

x_n l^∞的元素　x_n^k 元素构成的列.

$$\|x\|_\infty = \sup |x_k|. \qquad \ell^\infty := \{\{x_n\} : \exists C,\ |x_n| < C,\ \forall n \geq 1\}.$$

$\sup |x_n - y_n|$

Pf: Check norm:

1) $\|x\|_\infty = 0 \iff \sup|x_k| = 0 \iff x_k = 0$ for every k. $\iff x = 0$

2) $p(\lambda x) = \sup|\lambda x_k| = \lambda \cdot \sup|x_k|$ (since $x_k \in l$ is bounded)

3) $p(x+y) = \sup|x_k + y_k| \leq \sup|x_k| + \sup|y_k| = p(x) + p(y)$

Check completeness of $(\ell^\infty, \|\cdot\|_\infty)$

$\forall \{x^n\}$ $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$. s.t. $n, m > N$. $d(x^n, x^m) = \sup_k |x_k^n - x_k^m| < \varepsilon$

$x^n = (x_1^n, x_2^n, \ldots x_k^n \ldots)$ $\quad x^m = (x_1^m, \ldots x_k^m, \ldots)$

$\{x_k^n\}$ are Cauchy sequences of $\mathbb{R}$. $\exists a_k$. $\lim_{n\to\infty} x_k^n = a_k$.

Let $a = (a_1, \ldots a_k \ldots)$. $\quad \lim_{n\to\infty} x^n = a$ $\forall \varepsilon, \exists N_1 \in \mathbb{N}$, $n > N_1$. $|x^n - a| < \varepsilon$.

We need to $a \in \ell^\infty$ to show the convergence.

$|a| \leq |a - x^n| + |x^n| < \varepsilon + C_1$ (since $|x^n| \in \ell^\infty$. $\exists C_1$. $|x^n| < C_1$)
$\leq 1 + C_1$.

Let $C := 1 + C_1$. $|a| < C$. thus $a \in \ell^\infty$. $\{x^n\}$ is conv.

2. Prove that ℓ^∞ is not separable.

Pf: Assume the converse. $\exists D = \{\{x_i^j\}, \ldots \{x_n^j\} \ldots\}$ $j \in \mathbb{N}$ s.t.
$x_{k_1}^{j_1} \neq x_{k_2}^{j_2}$ where $j_1 \neq j_2$ or $k_1 \neq k_2$ (otherwise, exclude both of them and construct new sequence)
for every sequence $\{x_i^j\}$, $i \in \mathbb{N}$. $\exists C_i$. $|x_i^j| < C_i$ for any $j \in \mathbb{N}$ and $\bar{D} = \ell^\infty$
By the construction. $D \subset \ell^\infty$. now we needs to check whether the set of limit points belongs to ℓ^∞
$\forall \{x^{j_0}\}$, a limit point of D. we have $\sup|x^{j_0} - x_i^j| < \varepsilon$. for some i.
$\sup|x^{j_0}| \leq \sup|x_i^j| + \varepsilon \leq C_i + \varepsilon < C_i + 1$ thus we have $\{x^{j_0}\} \in \ell^\infty$. $\bar{D} \subset \ell^\infty$
Now we need $\ell^\infty \subset \bar{D}$ $\quad$ let $\delta = \inf|x_{i_1}^{j_1} - x_{i_2}^{j_2}|$, $j_1, j_2, i_1, i_2 \in \mathbb{N}$. $\delta \neq 0$.
Consider a set $A = [x_n, x_n + \frac{\delta}{2}]$. since the set is bounded, $A \in \ell^\infty$.
since the set is not countable. $A \notin D$
For any $\{x_k^j\}$, $d(A, \{x_k^j\}) \geq d(\{x_n\}, \{x_k^j\}) - d(A, \{x_n\}) \geq \frac{\delta}{2}$.
Then, For neighborhood $V_A(\frac{\delta}{4})$. $\dot{V}_A(\frac{\delta}{4}) \cap D = \phi$. A is not limit point. $\Rightarrow \ell^\infty \neq \bar{D}$
Which means $\ell^\infty \neq \bar{D}$. Contradicts.

Consider: $C = \{\{x_n\} : x_n = 0 \text{ or } x_n = 1\}$
C is uncountable.
Assume $\exists \bar{Q} = \ell^\infty$
for every $x \in C$. $Q \cap B(x, \frac{1}{2})$ is not empty.
This implies Q is uncountable.

3. Consider a space ℓ^p of sequences such that $\ell^p := \{\{x_k\} : \sum_{k=1}^{\infty} |x_k|^p < \infty\}$.

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x|^p\right)^{1/p} < \infty.$$

Prove that $(X, \|\|_p)$ is a Banach space.

Pf: norm: 1) $\|x\|_p = 0 \Leftrightarrow \sum_{k=1}^{\infty} |x_k|^p = 0 \Leftrightarrow$ every $k \in \mathbb{N}$, $x_k = 0$ $\Leftrightarrow x = 0$

2) $\|\lambda x\|_p = \left(\sum_{k=1}^{\infty} |\lambda x_k|^p\right)^{\frac{1}{p}} = \left(|\lambda|^p \sum_{k=1}^{\infty} |x_k|\right)^{\frac{1}{p}} = |\lambda| \left(\sum_{k=1}^{\infty} |x_k|\right)^{\frac{1}{p}} = |\lambda|\, \|x\|_p$.

3) $\|x+y\|_p = \left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{\frac{1}{p}}$ (By Minkowski's inequality)

$= \|x\|_p + \|y\|_p$.

completeness: $\forall$ a Cauchy's sequence $\{x^1, \dots, x^n, \dots\}$

s.t.. for any $\varepsilon > 0$. $\exists N \in \mathbb{N}$. for any $m, n > N$. $d_p(x^n, x^m) < \varepsilon$.

$x^n = (x_1^n, x_2^n, \dots)$. for any sequence $\{x_k^n\}$. $k \in \mathbb{N}$.

It's Cauchy's on $\mathbb{R}$. thus it's conv. $\lim_{n\to\infty} x_k^n = a_k$.

Let $a = (a_1, \dots, a_k, \dots)$ it remains to check $a \in \ell_p$. i.e. $\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{\frac{1}{p}} < \infty$.

since $\lim_{n\to\infty} x_k^n = a_k$. $\exists N > n$. s.t. for any $n > N$. $|a_k - x_k^n| < \frac{1}{k^2}^{\frac{1}{p}}$

$$\left(\sum_{k=1}^{\infty} |a_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{\infty} |a_k - x_k^n|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |x_k^n|^p\right)^{\frac{1}{p}} \text{ (By Minkowski. inequality.)}$$

$$\leq \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{p}} + \left(\sum |x_k^n|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\frac{\pi}{6}\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{\infty} |x_k^n|^p\right)^{\frac{1}{p}}$$

since $x^n \in \ell^p$. $\left(\sum |x_k^n|^p\right)^{\frac{1}{p}} < \infty$. $\Rightarrow \left(\sum |a_k|^p\right)^{\frac{1}{p}} < \infty$. $a \in \ell^p$.

4. Consider the metric $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ on $\mathbb{R}$. Prove that $\rho(0, x)$ is not a norm.

Pf: $\rho(0, x) = \|x\| = \dfrac{|x|}{1+|x|}$

$$\|\lambda x\| = \frac{|\lambda x|}{1+|\lambda x|} = \lambda \frac{|x|}{1+|\lambda||x|} \neq \lambda \frac{|x|}{1+|\lambda x|} \quad (\text{when } \lambda \neq \pm 1)$$

Thus $\rho(0, x)$ doesn't satisfy the scalar multiplication rules.

5. Let $f, g \in C[a,b]$. Prove that the set $E = \{h \in C[a,b] : f \le h \le g\}$ is closed with respect to the uniform metric.

consider $f_n \in E$. $f_n \to f$. on $C[a,b]$. $f_n \rightrightarrows f$

if for every n. $h \le f_n \le g$.

then $h \le f \le g$.

Pf: Assume the converse.

$\exists\, h_0 \notin E$. s.t. $E \cap \dot{V}h_0 \neq \phi$

Since $h_0 \notin E$. For some $x_0 \in [a,b]$. $h_0(x_0) > g$ or $h_0(x_0) < f$.

w.l.g. let $h_0(x_0) > g(x_0)$ since. $h_0, g \in C[a,b]$.

we can construct a neighborhood $V_{h_0}\left(\frac{h_0(x_0) - g(x_0)}{2}\right)$

s.t. $E \cap V_{h_0}\left(\frac{h_0(x_0) - g(x_0)}{2}\right) = \phi$.

which mean h_0 is not limit point of E.

6. Provide an example of a nested sequence of open subsets G_n of $\mathbb{R}$, $G_{n+1} \subset G_n$, such that $\bigcap_{k=1}^{\infty} G_k = \varnothing$.

$(0, \frac{1}{n})$

Ex: $G_n = (-1, -1 + \frac{1}{n})$

$-1 + \frac{1}{n+1} < -1 + \frac{1}{n}$. $G_{n+1} \subset G_n$.

if $\exists\, x \in \bigcap_{k=1}^{\infty} G_k$. $\Rightarrow x \in (-1, -1 + \frac{1}{n})$

Let $x = -1 + \varepsilon$. $(0 < \varepsilon < 1)$. for any ε, $\exists N$, $n > N$, $\varepsilon > \frac{1}{n}$

then $x \notin G_{N+1}$. contradicts to the assumption.

7. Let (X, d) be a complete metric space, $\{G_n\}$ a sequence of open subsets. Prove that $\operatorname{cl} G_{n+1} \subset G_n$ for every n then $\bigcap_{k=1}^{\infty} G_k \neq \varnothing$. Not true. $G_k = (k, +\infty)$

Pf: $G_{n+1} \subset \operatorname{Cl} G_{n+1} \subset G_n \subset \operatorname{Cl} G_n$, the sequence $\{\operatorname{Cl} G_n\}$ is nested.

By Thm. 3.8. (X,d) is complete $\Rightarrow \bigcap_{k=1}^{\infty} \operatorname{Cl} G_n \neq \phi$.

i.e. $\exists\, A \in \bigcap_{k=1}^{\infty} \operatorname{Cl} G_n$

i.e. $A \in \operatorname{Cl} G_n \subset G_{n-1}$ for every $n \in \mathbb{N}$.

i.e. $A \in \bigcap_{k=1}^{\infty} G_n$.

$\Rightarrow \bigcap_{k=1}^{\infty} G_n \neq \phi$.

$\{x^k\}$ elements are $\{x^1\}, \{x^2\}, \dots, \{x^n\}, \dots$

$x^1 = (x_1^1, x_2^1, \dots, x_n^1) \leftarrow$ sequence.

for every $n \in \mathbb{N}$, $a \in G_n \overset{?}{\Longrightarrow} a \in \bigcap_{n=1}^{\infty} G_n$.

Sept. 11th HW5

1. Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces and $f : D \subset X \to Y$, a be a limit point of D. Assume that Y is complete. Prove that the function f has a limit at a if and only if

$$\forall \varepsilon > 0\ \exists V_a\ \forall u, v \in \dot{V}_a \cap D\ \rho_Y(f(u), f(v)) < \varepsilon.$$

Pf "$\Rightarrow$" $\lim_{x\to a} f(x) = A$. , $x \in X$.

$\forall \varepsilon>0.\ \exists \delta>0.\ \forall x \in D\setminus\{a\}.\quad \rho_X(x, a) < \delta \Rightarrow \rho_Y(f(x), A) < \frac{\varepsilon}{3}$

let $V_a = \{x \in X, \rho_X(x, a) < \delta\}$. $u, v \in \dot{V}_a$.

$\rho_Y(f(u), f(v)) \le \rho_Y(f(u), A) + \rho_Y(f(v), A) = \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$.

"$\Leftarrow$" $\forall \varepsilon>0, \exists V_a$, consider a sequence $\{x_n\}_{n=1}^{\infty}$, $\lim x_n = a$.

s.t. $\exists N \in \mathbb{N}$, for any $m, n > N$. $x_m, x_n \in \dot{V}_a \cap D$.

thus. $\rho_Y(f(x_m), f(x_n)) < \varepsilon$

Thus. $\{f(x_n)\}_{n=1}^{\infty}$ is Cauchy's. Since Y is complete. $\exists A \in Y$. s.t. $\lim_{n\to\infty} f(x_n) = A$.

By Heine's def. a is a limit point of f.

2. Explain why is the completeness of Y necessary in this theorem.

The completeness of Y ensures the limit of sequence $\{f(x_n)\}_{n=1}^{\infty}$ is contained in set Y.

Otherwise, we can't prove the convergence of Cauchy sequence $\{f(x_n)\}$.

i.e. maybe $\lim_{n\to\infty} f(x_n) = A$ and $A \notin Y$. and by the uniqueness of limit point,

the conclusion will not hold.

$(\mathbb{R}^n, d)$. $d(x,y) = |x-y|$

$f(x) = \begin{cases} x, & x \ne 0 \\ 1, & x = 0. \end{cases}$

f has no limit at 0.

3. Consider $C[a, b]$ with uniform norm $\|f\| = \max_{x\in[a,b]} |f(x)|$. Prove that the operator $A : C[a, b] \to C[a, b]$ defined by

$$Af(x) = \int_a^x f$$

is bounded and find its norm.

Pf:

$\| A f(x) \| = \| \int_a^x f \| \le \int_a^x \|f\| \le (b-a) \max_{x\in[a,b]} |f(x)|$.

since $f(x) \in C[a,b]$. $f(x)$ is bounded on $[a,b]$. By Weierstrass thm.

Let $M = \max_{x\in[a,b]} |f(x)| < +\infty$.

thus. $\|A\| \le (b-a)\cdot M$. is bounded.

Actually we have $Af(x) \in C[a,b]$ since it's differentiable.

$\| A f(x) \| = \max_{x\in[a,b]} | \int_a^x f(x) |$

$\|Af\| = \sup_{\|x\|=1} \|Af(x)\| = \sup_{\|x\|=1} \max_{x\in[a,b]} \left| \int_a^x f \right| = (b-a)\|1\| = b-a$

4. Let $A \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^1)$. Then matrix (A) is a row vector $(a_1, \ldots, a_n)$. Prove that $\|A\|^2 = \sum_{k=1}^{n} a_k^2$.

Pf: "≤" let $x = (x_1, x_2, \ldots x_n)$

$$\|A(x)\|^2 = \left(\sum_{k=1}^{n} a_k x_k\right)^2 \leq \sum_{k=1}^{\infty} a_k^2 \quad \sum_{k=1}^{\infty} x_k^2 = \|x\| \cdot \sum_{k=1}^{\infty} a_k^2$$

$$\|A\| = \sup_{\|x\|=1} \|A(x)\| \leq \sup_{\|x\|=1} \sum_{k=1}^{\infty} a_k^2 \cdot \|x\| = \sum_{k=1}^{\infty} a_k^2.$$

"≤"

$$\|A\|^2 = \left(\sup_{\|x\|=1} \|A(x)\|\right)^2 = \sup_{\|x\|=1} (A(x))^2 = \sup_{\|x\|=1} \left(\sum_{k=1}^{\infty} a_k x_k\right)^2$$

why consider $\|A\|$.

$$\|Ax\| = \left\|\sum a_i x_i\right\| \leq \sqrt{\sum a_i^2} \cdot \sqrt{\sum x_i^2}$$

$x = (a_1 \ldots a_n)$

5. Consider linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Find the estimate for $\|A\|$ if both $\mathbb{R}^n$ and $\mathbb{R}^m$ are considered with norm $\|\|_\infty$.

Pf: denote $(A) = (a_{ij})_{1\leq i\leq m,\, 1\leq j\leq n}$.

$x = (x_1, x_2, \ldots x_n)^T$.

$$A_i(x) = \sum_{j=1}^{n} a_{ij} x_j \qquad A(x) = \begin{pmatrix} A_1(x) \\ \vdots \\ A_n(x) \end{pmatrix}$$

$\|A\| = \sup_{\|x\|=1} \|A(x)\|$. $\quad \|x\|_\infty = 1. \Rightarrow \max_{1\leq k\leq n} |x_k| = 1.$

we need the supremum. thus we can let $a_{ij} x_j = |a_{ij}|$

(i.e. $a_{ij} > 0$, $x_j = 1$. $a_{ij} < 0$. $x_j = -1$)

$$\|A(x)\| = \max_{1\leq i\leq m} |A_i(x)|.$$

thus. $\|A\| = \sup_{\|x\|=1} \left(\max_{1\leq i\leq m} |A_i(x)|\right) = \max_{1\leq i\leq m} \left|\sum_{j=1}^{n} |a_{ij}|\right|$

$$\|Ax\|_\infty = \max \left|\sum_{j=1}^{n} a_{kj} x_j\right|$$

$$\leq \max_{1\leq k\leq m} \sum_{j=1}^{n} |a_{kj}| \max |x_j|$$

$$= \|x\|_\infty \max_{1\leq k\leq m} \sum_{j=1}^{n} |a_{kj}|$$

Sept. 13th HW6.

1. Explain why the function f is differentiable and find its partial derivatives

   (a) $f(x,y) = x^2 + xy + y^2$;

   (b) $f(x,y,z) = z + \sin(xy^2)$;

   (c) $f(x,y,z) = \sqrt{x^2+y^2+z^2}$. Is this function differentiable at $(0,0,0)$?

(a) denote direction vector $h = (h_1, h_2)^T$.

$f(x+h_1, y+h_2) = f(x) + (2x+y,\ 2y+x)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + h_1^2 + h_2^2 + h_1 h_2$

we have $\frac{h_1^2+h_2^2+h_1h_2}{\|h\|} \to 0$ when $h \to 0_2$. $f'(x,y) = \begin{pmatrix} 2x+y \\ 2y+x \end{pmatrix}$, thus it's differentiable.

$f'_x = 2x+y \quad f'_y = 2y+x$

(b) $f'_x = y^2 \cos(y^2 x)$.

$f'_y = 2xy \cos(y^2 x)$.

$f'_z = 1$.

All partial derivatives are exist and continuous $\to$ f is differentiable

(c) $f'_x = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad f'_y = \frac{y}{\sqrt{x^2+y^2+z^2}} \qquad f'_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$

Denote $h = (h_1, h_2, h_3)^T$

$\lim_{t\to 0^+} \frac{f(0+th) - f(0)}{t} = (+1)\cdot h \qquad \lim_{t\to 0^-} \frac{f(0+th)-f(0)}{th} = (-1)\cdot h$.

the limit $t \to 0$ will not exist. f not diff at $(0,0,0)$

2. Consider the function

$$f(x,y) = \begin{cases} 1, & y = \sin x,\ x \neq 0; \\ 0, & y \neq \sin x \text{ or } y = x = 0. \end{cases}$$

Calculate partial derivatives of f at $(0,0)$. Is this function differentiable at $(0,0)$?

$f'_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x + 0, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$

$f'_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$.

when $x \to 0, y \to 0$ there exist some (x_k, y_k) s.t. $y_k = \sin x_k$.

thus, the limit $\lim_{t\to 0} \frac{f(0_2+th) - f(0)}{t}$ will not exist.

the function is not diff at $(0,0)$

$\lim_{x\to 0} f(x, \sin x) = 1$

3. Prove that function $f(x,y)=\sin(x+y)$ is Lipschitz, i.e. there exist $M>0$ such that

$$|f(x_1,y_1)-f(x_2,y_2)|\le M\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

Pf: $\forall (x_1,y_1),(x_2,y_2)\in D$. grad $f=(\cos(x+y), \sin(x+y))$

$$|f(x_1,y_1)-f(x_2,y_2)|^2=|\sin(x_1+y_1)-\sin(x_2+y_2)|^2$$
$$=\left|2\sin\left(\frac{(x_1+y_1)-(x_2+y_2)}{2}\right)\cos\left(\frac{x_1+y_1+x_2+y_2}{2}\right)\right|^2$$
$$=4\left|\sin\frac{(x_1-x_2)+(y_1-y_2)}{2}\right|^2$$
$$\le |(x_1-x_2)+(y_1-y_2)|^2 \le 2|(x_1-x_2)^2+(y_1-y_2)^2|$$
(since $|(x_1-x_2)-(y_1-y_2)|^2\ge 0$).

we take $M=\sqrt{2}$.

$$|f(x_1,y_1)-f(x_2,y_2)|\le\sqrt{2}\cdot\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

4. Find the domain and Jacobi matrix of the function

$$f(x,y)=\begin{pmatrix} xy; \\ \ln(1+x+y). \end{pmatrix}$$

1) $f(x,y): D\subset\mathbb{R}^2\to\mathbb{R}^2$.

we need: $1+x+y>0$.

$D=\{(x,y): x+y>-1\}$.

2) we know the Jacobi matrix will be the size of 2×2.

$$(f'(x,y))=\begin{pmatrix} f_1'(x,y) \\ f_2'(x,y)\end{pmatrix}=\begin{pmatrix} f_{1x}' & f_{1y}' \\ f_{2x}' & f_{2y}'\end{pmatrix}=\begin{pmatrix} y & x \\ \frac{1}{1+x+y} & \frac{1}{1+x+y}\end{pmatrix}$$

Sept. 18 th.

Mathematical analysis 3. Homework 7.

Find the limit $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$ and iterated limits $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y)$, $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)$ or prove that some of these limits do not exist.

1. $f(x,y) = \frac{y\sin(x)+x\sin y}{|x|+|y|}$, $a=b=0$;
2. $f(x,y) = \frac{\ln(1+x\sin(x+y))}{|x|+y^2}$, $a=b=0$;
3. $f(x,y) = \sin\frac{\pi x}{2x+y}$, $a=b=+\infty$;
4. $f(x,y) = \frac{x^y}{1+x^y}$, $a=+\infty$, $b=0+$.

1. $\lim\limits_{y\to 0} f(x,y) = \lim\limits_{y\to 0} \frac{x\sin y}{|x|+|y|} = 0$ $\quad \lim\limits_{x\to 0}\lim\limits_{y\to 0} = 0$

$\lim\limits_{x\to 0} f(x,y) = \lim\limits_{x\to 0} \frac{y\cdot\sin x}{|x|+|y|} = 0$

$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) \leqslant \lim\limits_{(x,y)\to(0,0)} \frac{2|x||y|}{|x|+|y|} = 0. \Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

2. $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y) = \lim\limits_{x\to 0} \frac{\ln(1+x\sin x)}{|x|} = \lim\limits_{x\to 0}\frac{x\sin x}{|x|} = 0.$

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y) = \lim\limits_{y\to 0} 0 = 0.$

$\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x\sin(x+y)}{|x|+y^2} = \lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{x(x+y)}{|x|+y^2} \leqslant \lim \frac{|x+y|\,|x|}{|x|} = 0$

3. $\lim\limits_{x\to+\infty}\lim\limits_{y\to+\infty} f(x,y) = \lim\limits_{x\to+\infty} 0 = 0.$

$\lim\limits_{y\to+\infty}\lim\limits_{x\to+\infty} f(x,y) = \lim\limits_{y\to+\infty} \sin\frac{\pi}{2} = 1$

$\lim\limits_{\substack{x\to+\infty\\ y\to+\infty}} f(x,y) \overset{y=kx}{=\!=} \lim\limits_{x\to+\infty} \frac{\pi x}{(2+k)x} = \frac{\pi}{2+k}.$ for any different k.

the result is different, thus the limit not exist.

4. $f(x,y)=\frac{x^y}{1+x^y}$, $a=+\infty$, $b=0+$.

$$\lim_{x\to+\infty}\lim_{y\to0^+} f(x,y)=\lim_{x\to+\infty}\frac{1}{2}=\frac{1}{2}.$$

$$\lim_{y\to0^+}\lim_{x\to+\infty} f(x,y)=\lim_{y\to0^+} 1 = 1.$$

since the iterated limits don't equal . the double limit not exist

Sept. 20th. HW8

1. Calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$ where

$$f(x,y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2), & x^2+y^2 \neq 0; \\ 0, & x=y=0. \end{cases}$$

$$\frac{\partial f}{\partial x} = \begin{cases} \frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2} \cdot y & x^2+y^2\neq 0 \\ 0 & x=y=0 \end{cases} \qquad \frac{\partial f}{\partial y} = \begin{cases} \frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2} \cdot x, & x^2+y^2 \neq 0 \\ 0 & , x=y=0 \end{cases}$$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \lim_{\Delta y \to 0} \frac{f'_x(0,\Delta y) - f'_x(0,0)}{\Delta y} = \frac{-\frac{\Delta y^4}{\Delta y^4}\cdot \Delta y}{\Delta y} = -1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \lim_{\Delta x \to 0} \frac{f'_y(\Delta x,0) - f'_y(0,0)}{\Delta x} = \frac{\frac{\Delta x^4}{\Delta x^4}\cdot \Delta x}{\Delta x} = 1$$

2. Calculate $\frac{\partial^3 f}{\partial x \partial y \partial z}$ if

$$f = \sqrt{xy^3z^5}.$$

S: $\frac{\partial f}{\partial x} = \frac{\sqrt{y^3z^5}}{2\sqrt{x}}$ $\qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{3}{4} \cdot \frac{\sqrt{yz^5}}{\sqrt{x}}$

$$\frac{\partial^3 f}{\partial x \partial y \partial z} = \frac{15}{8}\sqrt{\frac{yz^3}{x}}$$

3. Find second differential d^2u at $(0,0)$ if

$$u(x,y) = y \operatorname{arctg} \frac{x}{1+2y}.$$

$$d^2u = d\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}\cdot dy\right)$$

$$= d\left(\frac{\partial u}{\partial x}\right)dx + \frac{\partial u}{\partial x}\cdot dx^2 + d\left(\frac{\partial u}{\partial y}\right)\cdot dy + \frac{\partial u}{\partial y}\cdot dy^2$$

$$= \left(\frac{\partial^2 u}{\partial x^2}dx + \frac{\partial^2 u}{\partial x \partial y}dy\right)dx + \left(\frac{\partial^2 u}{\partial y \partial x}dx + \frac{\partial^2 u}{\partial y^2}dy\right)dy$$

$$= \frac{\partial^2 u}{\partial x^2}dx^2 + \frac{\partial^2 u}{\partial y^2}dy^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy \qquad ((x,y) \text{ cont at } (0.0))$$

$$= (2y+1)^2 y \cdot \frac{-2x}{((1+2y)^2+x^2)^2}dx^2 + \left[\frac{-2xy}{x^2+(1+2y)^2} + \frac{-2x}{(1+2y)^2+x^2} + 8xy\cdot\frac{2y+1}{(1+2y)^2+x^2}\right]dy^2$$

$$+2\left[\frac{1}{1+\left(\frac{x}{1+2y}\right)^2} + y\cdot\frac{-1}{\left(1+\left(\frac{x}{1+2y}\right)^2\right)^2}\cdot\frac{-2x}{(1+2y)^2}\right]dxy$$

$$= \cancel{1}\ 2\,dxy$$

4. Find second differential d^2u, where

$$u(x,y) = f(x^2+y^2, x^2-y^2, 2xy)$$

and $f \in C^2$.

$$u'_x = 2x \cdot \left[\frac{\partial f}{\partial(x^2+y^2)} + \frac{\partial f}{\partial(x^2-y^2)} + \frac{\partial f}{\partial(2xy)} \right]$$

$$u'_y = 2y \left[\frac{\partial f}{\partial(x^2+y^2)} + \frac{\partial f}{\partial(x^2-y^2)} + \frac{\partial f}{\partial(2xy)} \right]$$

$$u''_{xx} = 4x^2 \left[\frac{\partial^2 f}{\partial(x^2+y^2)^2} + \frac{\partial^2 f}{\partial(x^2-y^2)^2} + \frac{\partial^2 f}{\partial(2xy)^2} \right] + 2 \left[\frac{\partial f}{\partial(x^2+y^2)} + \frac{\partial f}{\partial(x^2-y^2)} + \frac{\partial f}{\partial(2xy)} \right]$$

$$u''_{yy} = 4y^2 \left[\frac{\partial^2 f}{\partial(x^2+y^2)^2} + \frac{\partial^2 f}{\partial(x^2-y^2)^2} + \frac{\partial^2 f}{\partial(2xy)^2} \right] + 2 \left[\frac{\partial f}{\partial(x^2+y^2)} + \frac{\partial f}{\partial(x^2-y^2)} + \frac{\partial f}{\partial(2xy)} \right]$$

$$u''_{xy} = 4xy \left[\frac{\partial^2 f}{\partial(x^2+y^2)^2} + \frac{\partial^2 f}{\partial(x^2-y^2)^2} + \frac{\partial^2 f}{\partial(2xy)^2} \right]$$

$$d^2u = 4 \left[\frac{\partial^2 f}{\partial(x^2+y^2)^2} + \frac{\partial^2 f}{\partial(x^2-y^2)^2} + \frac{\partial^2 f}{\partial(2xy)^2} \right] \cdot (x^2 dx^2 + y^2 dy^2 + 2xy \cdot dx \cdot dy)$$

$$+ 8 \left[\frac{\partial f}{\partial(x^2+y^2)} + \frac{\partial f}{\partial(x^2-y^2)} + \frac{\partial f}{\partial(2xy)} \right] (dx^2 + dy^2)$$

5. Prove that $\Delta u = 0$ if

$$u(x,y) = e^x(x\cos y - y\sin y).$$

Pf: $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial^2 y}$

$$= \frac{\partial}{\partial x}\left[e^x(x\cos y - y\sin y) + e^x(\cos y)\right] + \frac{\partial}{\partial y}\left[e^x x\cdot(-\sin y) - e^x \sin y - e^x \cos y \cdot y\right]$$

$$= e^x x\cos y + e^x \cos y - e^x y \sin y + e^x \cos y - \cos y \cdot e^x \cdot x - e^x \cos y - e^x \cos y + e^x \sin y \cdot y$$

$$= 0$$

6*. Find a function $\varphi(t)$ if the function $f = \varphi\left(\frac{x^2+y^2}{x}\right)$ satisfies Laplace equation $(x \neq 0)$ $\Delta f = 0$.

f'_x $\cancel{\varphi'_x} = \left(1 - \dfrac{y^2}{x^2}\right) \varphi'$ $\qquad$ $\varphi''_{xx} = \dfrac{2y^2}{x^3} \cdot f' + \left(1 - \dfrac{y^2}{x^2}\right)^2 f''$

f'_y $\cancel{\varphi'_y} = \left(\dfrac{2y}{x}\right) \varphi'$ $\qquad$ $\varphi''_{yy} = \dfrac{2}{x} \cdot f' + \dfrac{4y^2}{x^2} f''$

$$\Delta f = 0 \iff \frac{2(y^2+x^2)}{x^3} \cdot f' + \left(1 + \frac{y^2}{x^2}\right)^2 f'' = 0$$

$$\iff (2x \cdot f' + (x^2+y^2) f'')(x^2+y^2) = 0$$

① $x^2+y^2=0$

$x\neq 0$. impossible.

Let $\frac{x^2+y^2}{x}=t$.

② $2\cdot f'+\frac{(x^2+y^2)}{x}f''=0$

$2\varphi'(t)+t\cdot\varphi''(t)=0.$

$\Leftrightarrow\quad (t^2\varphi'(t))'=0.$

$\Leftrightarrow\quad t^2\varphi'(t)=C.$

$\Leftrightarrow\quad \varphi(t)=-\frac{C_1}{t}+C_2.$

Sept. 22nd HW9

1. Find extremal points of function f

(a) $f = \frac{x+y}{xy} - xy$;

(b) $f = x^3/3 + 3x^2e^y - e^{-y^2}$;

(c) $f = \sin x \sin y \sin(x+y)$, $x, y \in (0, \pi)$;

(d) $f = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$.

(a) $f'_x = -\frac{1}{x^2} - y.$

$f'_y = -\frac{1}{y^2} - x.$

Let $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases} \Rightarrow (-1,-1).$

$d^2f = \frac{2}{x^3}dx^2 + 2(-1)\cdot dx\,dy + \frac{2}{y^3}dy^2$

$d^2f(p) = -2\left(dx^2 + dx\,dy + dy^2\right)$ $d^2f(p)$ is negative. definite.

$(-1,-1)$ is strict maximum point.

(b). $f'_x = x^2 + 6x\cdot e^y$

$f'_y = 3x^2e^y + 2y\cdot e^{-y^2}$

$\begin{cases} x^2+6x\cdot e^y = 0 \\ 3x^2\cdot e^y + 2y\cdot e^{-y^2} = 0. \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases}$

$d^2f = (2x+6e^y)(dx)^2 + (6x\cdot e^y)(dx\cdot dy) + (3x^2\cdot e^y + 2\cdot e^{-y^2} - 4y^2e^{-y^2})dy^2$

$\det d^2f(p) = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix}$ p.d. $\rightarrow$ $(0,0)$ strictly minimum.

(C) $f'_x = \sin y(\sin x\cos(x+y) + \cos x\sin(x+y)) = \sin y\sin(2x+y)$

$f'_y = \sin x\sin(2y+x).$ $\Rightarrow P_1: \begin{cases} x = \frac{\pi}{3} \\ y = \frac{\pi}{3} \end{cases}$ $\Rightarrow P_2: \begin{cases} x = \frac{2\pi}{3} \\ y = \frac{2\pi}{3}. \end{cases}$

$d^2f = 2\sin y\cos(2x+y)\cdot(dx)^2 + 2\sin 2(x+y)\,dx\,dy + 2\sin x\cos(2y+x)(dy)^2$

$$\det d^2f(p_1) = \begin{vmatrix} -\sqrt{3} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\sqrt{3} \end{vmatrix}$$ not definite $\Rightarrow$ stationary point only.

$$\det d^2f(p_2) = \begin{vmatrix} \sqrt{3} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \sqrt{3} \end{vmatrix}$$ p.d. $\Rightarrow$ strict mim. point.

(d) $f = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$.

$f'_x = -\frac{y^2}{4x^2}$ $\quad f'_y = \frac{y}{2x} - \frac{z^2}{y^2}$ $\quad f'_z = \frac{2z}{y} - \frac{2}{z^2}$

2. Prove that function $f = (y^2 - x)(y^2 - 2x)$ has minimum on every line passing through a point $(0,0)$ but has no minimum at $(0,0)$ as a function of two variables.

Pf: 1). ① $x=0$. $f = y^4$ $\quad f' = 4y^3$

we have $f'>0$ when $y>0$. $f'<0$. when $y<0$.

f has minimum at $(0,0)$.

② $y=kx$. $f=(k^2x^2-x)(k^2x^2-2x) = k^4x^4 - 3k^2x^3 + 2x^2$

$f' = 4k^4x^3 - 9k^2x^2 + 4x = x(2k^2x-1)(k^2x-4)$

In the neiborhood $V(0, \frac{1}{2k^2})$. we have $x>0$. $f'>0$ $\quad x<0$ $f'<0$.

f has minimum at $(0,0)$

2) $f'_x = -(y^2-2x) - 2(y^2-x) = 4x - 3y^2$

$f'_y = 2y(y^2-2x) + 2y(y^2-x) = 2y^3 - 6xy$.

$d^2f = 4 \cdot dx^2 + 2(-6y)(dx\,dy) + (6y^2-6x)(dy)^2$

$$\Delta_k = \begin{vmatrix} 4 & 0 \\ 0 & 0 \end{vmatrix} = 0$$

not p/n. d. $\Rightarrow$ f has no minimum at $(0,0)$

HW 20. Sept. 26th.

1. Find Taylor's decomposition of function f at (x_0, y_0) with residue $o(\rho^2)$, where $\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$.

   (a) $f = \frac{\cos x}{\cos y}$, $(x_0, y_0) = (0, 0)$;

   (b) $f = x^y$, $(x_0, y_0) = (1, 1)$;

(a) $$f(x,y) = f(0,0) + \left[\frac{\partial f(x_0,y_0)}{\partial x}dx + \frac{\partial f(x_0,y_0)}{\partial y}dy\right]$$

$$+\frac{1}{2}\left[\frac{\partial^2 f(x_0,y_0)}{\partial x^2}dx^2 + \frac{\partial^2 f(x_0,y_0)}{\partial y^2}dy^2 + 2\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}dxdy\right] + o(\rho^2)$$

$$= 1 + \frac{1}{2}\left[-x^2 + y^2\right] + o(\rho^2)$$

$$= 1 + \frac{y^2 - x^2}{2} + o(\rho^2)$$

(b) $$f(x,y) = f(1,1) + \left[\frac{\partial f(x_0,y_0)}{\partial x}dx + \frac{\partial f(x_0,y_0)}{\partial y}dy\right]$$

$$+\frac{1}{2}\left[\frac{\partial^2 f(x_0,y_0)}{\partial x^2}dx^2 + \frac{\partial^2 f(x_0,y_0)}{\partial y^2}dy^2 + 2\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}dxdy\right] + o(\rho^2)$$

$$\frac{\partial f}{\partial x} = y \cdot x^{y-1} \quad \frac{\partial f}{\partial y} = \ln x \cdot x^y \quad \frac{\partial^2 f}{\partial x^2} = y(y-1) \cdot x^{y-2}$$

$$\frac{\partial^2 f}{\partial y^2} = \ln^2 x \cdot x^y \quad \frac{\partial f}{\partial x \partial y} = \frac{1}{x} x^y + \ln x \cdot y \cdot x^{y-1} = x^{y-1}(1 + \ln x \cdot y)$$

$$f(x,y) = 1 + (x-1) + \frac{1}{2}\left[2(x-1)(y-1)\right] + o(\rho^2)$$

$$= 1 + (x-1) + (x-1)(y-1) + o(\rho^2)$$

2. Find Taylor's decomposition of the function

$$f = \ln(xy + z^2)$$

at $(x_0, y_0, z_0) = (0, 0, 1)$ with residue $o(\rho^2)$, where

$$\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

S: $\frac{\partial f}{\partial x} = \frac{y}{xy+z^2}$ $\quad \frac{\partial f}{\partial y} = \frac{x}{xy+z^2}$ $\quad \frac{\partial f}{\partial z} = \frac{2z}{xy+z^2}$

$\frac{\partial^2 f}{\partial x^2} = -\frac{y^2}{(xy+z^2)^2}$ $\quad \frac{\partial^2 f}{\partial y^2} = -\frac{x^2}{(xy+z)^2}$ $\quad \frac{\partial^2 f}{\partial z^2} = \frac{2(xy+z^2)-2z(2z)}{(xy+z)^2} = \frac{2(xy-z^2)}{(xy+z)^2}$

$\frac{\partial^2 f}{\partial x\partial y} = \frac{xy+z^2-xy}{(xy+z)^2} = \frac{z^2}{(xy+z^2)^2}$ $\quad \frac{\partial^2 f}{\partial x\partial z} = -\frac{2zy}{(xy+z^2)^2}$ $\quad \frac{\partial^2 f}{\partial y\partial z} = -\frac{2zx}{(xy+z^2)^2}$

$f = 0 + 2(z-1) + \frac{1}{2}\left[-2(z-1)^2 + 2xy\right] + o(\rho^2) = 2(z-1) - (z-1)^2 + xy + o(\rho^2)$

3. Find partial derivative of the first and second order of the implicit function u defined by the equation

$$u = x + \operatorname{arctg}\frac{y}{u - x}.$$

denote $F(x,y,u(x,y)) = x + \operatorname{arctg}\frac{y}{u-x} - u = 0.$

$$\frac{y^2}{(u-x)^2+y^2}$$

$\frac{\partial u}{\partial x} = -\frac{F'_x}{F'_u} = -\frac{1 + \frac{1}{1+\frac{y^2}{(u-x)^2}}\cdot\left(\frac{y}{(u-x)^2}\right)}{\frac{1}{1+\frac{y^2}{(u-x)^2}}\cdot\left(-\frac{y}{(u-x)^2}\right) - 1} = \frac{1+\frac{y^2}{(u-x)^2+y^2}}{1+\frac{y^2}{(u-x)^2+y^2}} = 1$

$\frac{\partial u}{\partial y} = -\frac{F'_y}{F'_u} = \frac{\frac{1}{1+\frac{y^2}{(u-x)^2}}\cdot\left(\frac{1}{u-x}\right)}{1+\frac{y^2}{(u-x)^2+y^2}} = \frac{u-x}{(u-x)^2+2y^2}$

$\frac{\partial^2 u}{\partial x^2} \equiv 0, \quad \frac{\partial^2 u}{\partial x\partial y} \equiv 0.$

$\frac{\partial^2 u}{\partial y^2} = -\left(\frac{F'_y}{F'_u}\right)'_y = \frac{\partial u}{\partial y}\cdot\left(\frac{1}{(u-x)^2+2y^2}\right) - (u-x)\cdot\frac{1}{[(u-x)^2+2y^2]^2}\cdot\left[2u\cdot\frac{\partial u}{\partial y} - 2x\frac{\partial u}{\partial y} + 4y\right]$

$= \frac{u-x}{[(u-x)^2+2y^2]^2}\left[1 - \frac{2(u-x)^2}{(u-x)^2+2y^2} + 4y\right]$

4. Find Taylor's decomposition of the implicit function u at $(x_0, y_0) = (1, 1)$ that is defined by the equation

$$u^3 - 2xu + y = 0, \;\; u(1,1) = 1,$$

with residue $o(\rho^2)$, where $\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

Denote $F(x, y, u) = u^3 - 2xu + y = 0.$

$$\frac{\partial u}{\partial x} = -\frac{F_x'}{F_u'} = \frac{2u}{3u^2 - 2x} \qquad \frac{\partial u}{\partial y} = -\frac{F_y'}{F_u'} = -\frac{1}{3u^2 - 2x}$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \cdot \frac{\frac{\partial u}{\partial x} - u\left(6u \cdot \frac{\partial u}{\partial x} - 2\right)}{[3u^2 - 2x]^2} = 2\,\frac{2u - 6u(2u) + 2u}{[3u^2 - 2x]^3} = 8\,\frac{u - 3u^2}{[3u^2 - 2x]^3}$$

$$\frac{\partial^2 u}{\partial y^2} = 0 . \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{6u \cdot \frac{\partial u}{\partial x} - 2}{(3u^2 - 2x)^2} = \frac{6u \cdot (2u) - 2(3u^2 - 2x)}{(3u^2 - 2x)^3} = \frac{6u^2 + 4x}{(3u^2 - 2x)^3}$$

$$u(x, y) = 1 + (2dx - dy) + \frac{1}{2}\left(-16x^2 + 20xy\right) + o(\rho^2)$$

$$= 1 + 2(x-1) - (y-1) - 8(x-1)^2 + 10(x-1)(y-1) + o(\rho^2)$$

HW 11. Oct. 8th.

1. Find extremal points of implicit functions defined by equation

$$2x^2 + 2y^2 + u^2 + 8yu - u + 8 = 0.$$

$25u^2 - u + 8 = 0$

$9u^2 - 16u^2 - u + 8 = 0.$

$$\frac{\partial u}{\partial x} = -\frac{F'_x}{F'_u} = -\frac{4x}{2u+8y-1} \qquad \frac{\partial u}{\partial y} = -\frac{F'_y}{F'_u} = -\frac{4y+8u}{2u+8y-1}$$

$$\begin{cases} -\dfrac{4x}{2u+8y-1} = 0 \Rightarrow x = 0 \\ -\dfrac{4y+8u}{2u+8y-1} = 0 \Rightarrow u = -\dfrac{1}{2}y \end{cases} \quad 1 \Rightarrow \begin{cases} u = 1 \\ x = 0 \\ y = -2 \end{cases} \quad \begin{cases} u = -\dfrac{8}{7} \\ x = 0 \\ y = \dfrac{16}{7} \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{4(2u+8y-1)}{(2u+8y-1)^2} + \frac{4x(-4x\cdot 2)}{(2u+8y-1)^3} = -\frac{16x^2 + 4(2u+8y-1)^2}{(2u+8y-1)^3}$$

$$= \frac{-4}{2u+8y-1} . \quad \frac{\partial^2 u}{\partial y^2} = \frac{-4}{2u+8y-1} . \quad \frac{\partial^2 u}{\partial x \partial y} = 0$$

$$K_1 = \begin{pmatrix} \frac{4}{15} & 0 \\ 0 & \frac{4}{15} \end{pmatrix} \qquad K_2 = \begin{pmatrix} -\frac{4}{15} & 0 \\ 0 & -\frac{4}{15} \end{pmatrix} \qquad -\frac{16}{7} + \frac{16}{7} \times 7 - 1$$

K_1, p.d $(0, -2)$. minimal extremum.

K_2. n.d. $(0, \frac{16}{7})$. maximal extremum

2. Find conditional extremum using Lagrange's function

(a) $u = xyz$, $\varphi = x^2 + y^2 + z^2 = 3$;

(b) $u = \ln(xy)$, $\varphi = x^3 + xy + y^3 = 0$;

(c) $u = x + 2y$, $\varphi = x^2 - 8y^2 = 8$

(a) the Jacobian $= \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) = (2x, 2y, 2z)$. can't $= 0$ since $x^2+y^2+z^2=3$.

$L(x) = xyz + \lambda(x^2+y^2+z^2-3) = 0$

$$\begin{cases} yz + 2\lambda x = 0 \\ xz + 2\lambda y = 0 \\ xy + 2\lambda z = 0 \end{cases} \Rightarrow \begin{cases} x^2 = y^2 = z^2 = 1. \quad \lambda = \pm\frac{1}{2} \\ x = y = z = 0 \ \ (\text{exclude}) \end{cases}$$

$d\varphi = x\,dx + y\,dy + z\,dz = 0.$

$$d^2L = (dx\ dy\ dz)\begin{pmatrix} 2\lambda & z & y \\ z & 2\lambda & x \\ y & x & 2\lambda \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$\Delta_2 = z^2 - 4\lambda^2 = 0.$

not definite.

(b) $\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = (3x^2+y,\ 3y^2+x) \neq 0.$

$$\begin{cases} 3x^2+y=0 \\ 3y^2+x=0 \end{cases} \Rightarrow \begin{cases} y=-3x^2 \\ x=-3y^2 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0. \end{cases}$$

$u=\ln(xy)$ implies $x\neq 0$ and $y\neq 0.$

$$L(x) = \ln(xy) + \lambda(x^3+xy+y^3)$$

$$\begin{cases} L'_x = \frac{1}{x} + \lambda(3x^2+y) = 0 \\ L'_y = \frac{1}{y} + \lambda(3y^2+x) = 0. \\ x^3+xy+y^3=0 \end{cases} \Rightarrow \begin{cases} x=-\frac{1}{2} \\ y=-\frac{1}{2}. \\ \lambda=8 \end{cases}$$

$$L''_{xx} = -\frac{1}{x^2}+6\lambda x \quad L''_{yy} = -\frac{1}{y^2}+6\lambda y \quad L''_{xy} = \lambda.$$

consider $\begin{pmatrix} -28 & 8 \\ 8 & -28 \end{pmatrix}$ n.g. $\Rightarrow$ $(-\frac{1}{2}, -\frac{1}{2})$. maximal extremum point.

$$d\varphi = 3x^2\,dx + x\,dy + y\,dx + 3y^2\,dy = 0$$

$$= (3x^2+y)\,dx + (3y^2+x)\,dy$$

$$d^2L = \left(-\frac{1}{x^2}+6\lambda x\right)dx^2 + \left(-\frac{1}{y^2}+6\lambda y\right)dy^2 + 2\lambda\,dx\,dy$$

$$= -\frac{dx^2}{x^2} - \frac{dy^2}{y^2} + \lambda(6x\,dx^2 + 6y\,dy^2 + 2dx\,dy)$$

$$= -\frac{dx^2}{x^2} - \frac{dy^2}{y^2}$$ $\Rightarrow$ n.d. maximal extremum point.

(c) $L(x) = x+2y+\lambda(x^2-8y^2-8)$

$$\begin{cases} L'_x = 1+2\lambda x=0 \\ L'_y = 2-16\lambda y=0 \\ x^2-8y^2-8=0 \end{cases} \Rightarrow \begin{matrix} \lambda=\pm\frac{1}{8} \\ \lambda=\frac{1}{8}.\ \ P=(-4,1) \\ \lambda=-\frac{1}{8}.\ \ P=(4,-1) \end{matrix}$$

$$L''_{xx} = 2\lambda. \quad L''_{yy} = -16\lambda \quad L''_{xy} = 0$$

$$d^2L = 2\lambda\,dx^2 - 16\lambda\,dy^2$$

$d\varphi = 2x\,dx - 16y\,dy = 0. \Rightarrow dx = -2dy$

$d^2L = 2\lambda \cdot (-2dy)^2 - 16\lambda\, dy^2 = -8\lambda\, dy^2$

$\lambda = \frac{1}{8}$. n.d. $(-4, 1)$. maximal extremum

$\lambda = -\frac{1}{8}$. p.d. $(4, -1)$ minimum extremum

3. Investigate the function u for conditional extremum and find out if the Lagrange's method can be applied to this problem

$$u = x^4 + y^4, \quad (x-1)^3 - y^2 = 0$$

$\frac{\partial\varphi}{\partial x} = 3x^2 - 6x + 3 \qquad \frac{\partial\varphi}{\partial y} = -2y$

cannot apply the Lagrange's method $(1, 0)$.

$L(X) = x^4 + y^4 + \lambda[(x-1)^3 - y^2]$

$$\begin{cases} L'_x = 4x^3 + 3\lambda(x^2 - 2x + 1) = 0 \\ L'_y = 4y^3 + \lambda(-2y) = 0 \\ (x-1)^3 - y^2 = 0 \end{cases} \Rightarrow$$

① $y = 0$. $x = 1$.

② $y^2 = \frac{\lambda}{2}$. $\Rightarrow 4x^3 + 6(x-1)^2 = 0$.

since $x \geq 1$. no solution

then the Lagrange method can't work.

Oct. 9th. HW12

1. Change the order of integration in the double integral

$$I = \int_0^{2\pi} dx \int_0^{\sin x} f(x, y) dy$$

$$\begin{cases} 0 \le x \le 2\pi \\ 0 \le y \le \sin x. \end{cases} \Rightarrow \quad \begin{matrix} 0 \le y \le 1. \\ \arcsin y \le x \le 2\pi. \end{matrix}$$

$$I = \int_0^1 dy \int_{\arcsin y}^{2\pi} f(x, y)\, dx.$$

2. Calculate

$$\iint_\Omega \left(x^2 + y^2\right) dxdy,$$

where Ω is a parallelogram bounded by lines $y = x$, $y = x + a$, $y = a$ and $y = 3a$, $(a > 0)$.

S: take the change: $\begin{matrix} u = y - x \\ v = y. \end{matrix} \Rightarrow \begin{cases} x = v - u \\ y = v \end{cases} \quad |J| = \left| \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \right| = 1$

$\Omega' = \{(u, v),\ u \in [0, a],\ v \in [a, 3a]\}$

$$\iint_\Omega (x^2+y^2)\, dxdy = \iint_{\Omega'} \left[(v-u)^2 + v^2\right] du\, dv. = \int_0^a du \int_a^{3a} (2v^2 - 2uv + u^2)\, dv$$

$$= 14a^4$$ ✓

3. Convert Cartesian coordinates to polar coordinates in the integral

$$\int_0^2 dx \int_x^{x\sqrt{3}} f\left(\sqrt{x^2 + y^2}\right) dy.$$

$\begin{cases} x = r\cos\varphi \\ y = \sqrt{3} r \sin\varphi \end{cases} \quad \varphi \in [0, 2\pi) \quad J = \sqrt{3} r$ $\quad \Rightarrow$ 正常设. $\begin{cases} x = \\ y = \end{cases}$

$\Omega = \{(x, y),\ 0 \le x \le 2,\ 0 \le y \le 2\sqrt{3}\}.$ $\quad \frac{\pi}{4} \le \varphi < \frac{\pi}{3} \quad 0 < r < \frac{2}{\cos\varphi}$

$$\int_0^2 dx \int_x^{x\sqrt{3}} f\left(\sqrt{x^2 + y^2}\right) dy. = \int_0^{2\pi} d\varphi \int_0^{2\sqrt{3}} f(2r) \cdot \sqrt{3} r\, dr$$ ×

4. Calculate

$$\iint_\Omega (x + y) dxdy,$$

where Ω is a set bounded by curves $y^2 = 2x$, $x + y = 4$, $x + y = 12$.

let $\begin{cases} u = x + y \\ v = y \end{cases}$ $\quad 4 \le u \le 12.$ $\quad |J| = \left| \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \right| = 1$

since $y^2 \le 2x$ $\quad \Omega' =$

$\Rightarrow v^2 \le 2(u - v) \Rightarrow v^2 + 2v + 1 \le 2u + 1$

$\Rightarrow -\sqrt{2u+1} - 1 \le v \le \sqrt{2u+1} - 1 \Rightarrow -6 \le v \le 4.$ × $\quad \frac{8156}{15}$

$$\iint_\Omega (x+y)\, dx\, dy = \iint_{\Omega'} u\, du\, dv = \int_4^{12} du \int_{-6}^{4} u\, dv = \int_4^{12} 10u\, du = 640$$

5. Calculate the integral considering the polar change

$$\iint_{\Omega} \sqrt{a^2 - x^2 - y^2}\, dx dy,$$

where the set Ω bounded by the loop of the lemniscate

$$\left(x^2 + y^2\right)^2 = a^2\left(x^2 - y^2\right), \quad x \geqslant 0.$$

Consider the polar change:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \varphi \in [0, 2\pi). \quad r > 0 \qquad |J| = r.$$

$$r^4 \leqslant a^2 r^2 \cdot \cos 2\varphi. \Rightarrow r^2 \leqslant a^2 \cos 2\varphi.$$

$$\iint_{\Omega} \sqrt{a^2 - x^2 - y^2}\, dx\, dy = \iint_{\Omega'} \sqrt{a^2 - r^2} \cdot r\, dr\, d\varphi = \int_0^{2\pi} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} \sqrt{a^2 - r^2}\, r\, dr.$$

$$= \int_0^{2\pi} d\varphi \cdot -\frac{1}{3}\left(a^2 - r^2\right)^{\frac{3}{2}} \Big|_0^{a\sqrt{\cos 2\varphi}} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{a^3}{3}\left(1 - (1 - \cos 2\varphi)^{\frac{3}{2}}\right) d\varphi$$

$$= \frac{1}{6}\pi a^3 - \frac{a^3}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - \cos 2\varphi)^{\frac{3}{2}}\, d\varphi. = \frac{\pi a^3}{6} - \frac{2\sqrt{2}a^3}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^3\varphi\, d\varphi. = \frac{\pi a^3}{6}$$

6. Prove that Jacobian of the spherical change of the coordinates is equal to $J = r^2 \cos\psi$.

Pf: Consider the spherical change:

$$\begin{cases} x = r\cos\varphi\cos\psi \\ y = r\sin\varphi\cos\psi \\ z = r\sin\psi \end{cases} \qquad \begin{array}{l} \varphi \in [0, 2\pi) \\ \psi \in [-\frac{\pi}{2}, \frac{\pi}{2}). \end{array}$$

$$J = \begin{vmatrix} x'_r & x'_\varphi & x'_\psi \\ y'_r & y'_\varphi & y'_\psi \\ z'_r & z'_\varphi & z'_\psi \end{vmatrix} = \begin{vmatrix} \cos\varphi\cos\psi & -r\sin\varphi\cos\psi & -r\cos\varphi\sin\psi \\ \sin\varphi\cos\psi & r\cos\varphi\cos\psi & -r\sin\varphi\sin\psi \\ \sin\psi & 0 & r\cos\psi \end{vmatrix}$$

$$= r^2\left(\sin\psi \cdot \begin{vmatrix} -\sin\varphi\cos\psi & -\cos\varphi\sin\psi \\ \cos\varphi\cos\psi & -\sin\varphi\sin\psi \end{vmatrix} + \cos\psi \begin{vmatrix} \cos\varphi\cos\psi & -\sin\varphi\cos\psi \\ \sin\varphi\cos\psi & \cos\varphi\cos\psi \end{vmatrix}\right)$$

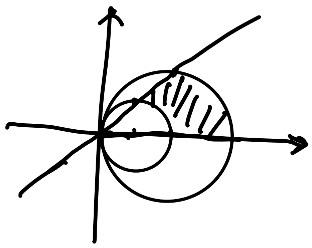
$$= r^2\left(\sin^2\psi\cos\psi + \cos^3\psi\right) = r^2\cos\psi.$$

Oct. 11th. HW13

1. Considering polar coordinates calculate the area bounded by the lines

$$x^2+y^2=2x, \quad x^2+y^2=4x, \quad y=x, \quad y=0.$$

$\frac{x^2+y^2}{x}.$



$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi. \end{cases}$ $r>0$ $|J| = r$

$\begin{cases} x^2+y^2=2x \\ x^2+y^2=4x \end{cases} \Rightarrow 2\cos\varphi \le r \le 4\cos\varphi$

$y \le x \Rightarrow \tan\varphi \le 1.$

$y \ge 0 \Rightarrow \varphi \ge 0.$ $\varphi \in [0, \frac{\pi}{4}].$

$\Omega' = \{(r,\varphi): \varphi \in [0,\frac{\pi}{4}],\ r \in [2\cos\varphi \le r \le 4\cos\varphi]\}$

$$\iint_{\Omega'} \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} r\,dr = \int_0^{\frac{\pi}{4}} 6\cos^2\varphi\, d\varphi = 3\int_0^{\frac{\pi}{4}} [1+\cos 2\varphi]\, d\varphi$$

$$= 3 \cdot \varphi + \frac{\sin 2\varphi}{2} \Big|_0^{\frac{\pi}{4}} = \frac{3\pi}{4} + \frac{3}{2}$$

2. Calculate the area of the set Ω bounded by a curve

$$(x^2+y^2)^2 = 8(x^2-y^2), \quad x^2+y^2 \ge 4.$$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $r>0$ $|J| = r$

$\begin{cases} (x^2+y^2)^2 = 8(x^2-y^2) \Rightarrow r^2 = 8\cos 2\varphi \\ x^2+y^2 \ge 4. \Rightarrow r \ge 2 \end{cases}$

$\Omega' = \{(r,\varphi): -\frac{\pi}{6} \le \varphi \le \frac{\pi}{6},\ 2 \le r \le \sqrt{8\cos 2\varphi}\}$

$$\iint_{\Omega'} r\,dr\,d\varphi = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\varphi \int_2^{2\sqrt{2\cos 2\varphi}} r\,dr = 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (4\cos 2\varphi - 2)\, d\varphi = 2\Big[2\sin 2\varphi - 2\varphi \Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\Big]$$

$$= 4\sqrt{3} - \frac{4\pi}{3}$$

3. Calculate the area of the set Ω bounded by curves

$$x^2 = y, \quad x^2 = 4y, \quad x^3 = y^2, \quad x^3 = 2y^3.$$

$u = \frac{x^2}{y}$ $v = \frac{x^3}{y^2}$

$1 \le u \le 4$

$1 \le v \le 2$. $J(x,y) \begin{pmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ \frac{3x^2}{y^2} & -\frac{2x^3}{y^3} \end{pmatrix}$

$= -\frac{x^4}{y^4}$

$\frac{x^2}{4} \le y \le \min\{x^2, x^{\frac{3}{2}}, 2^{\frac{1}{3}}x\}$.

denote $g(x) = \min\{x^2, x^{\frac{3}{2}}, 2^{\frac{1}{3}}x\}$

$$g(x) = \begin{cases} x^2, & x \in [0,1] \\ x^{\frac{3}{2}}, & x \in [1, 2^{\frac{2}{3}}] \\ \sqrt[3]{2}\,x, & x \in [2^{\frac{2}{3}}, 2^{\frac{7}{3}}] \end{cases}$$

$$S = \int_0^1 (x^2 - \frac{x^2}{4})dx + \int_1^{2^{\frac{2}{3}}} (x^{\frac{3}{2}} - \frac{x^2}{4})dx + \int_{2^{\frac{2}{3}}}^{2^{\frac{7}{3}}} (2^{\frac{1}{3}}x - \frac{x^2}{4})dx.$$

$$= \frac{1}{4} + \frac{2}{5}(2^{\frac{5}{3}} - 1) - \frac{1}{4} + \frac{17}{3} - 2^{\frac{2}{3}}$$

$$= \frac{17}{3} - \frac{2}{5} - \frac{2^{\frac{2}{3}}}{5} = \frac{79}{15} - \frac{2^{\frac{2}{3}}}{5}$$

$\frac{y^4}{x^4}$

$\begin{cases} 2u + 3v = -4 \\ -2u - \cancel{4v} = 8 \end{cases}$

$-v = 4$

$v = -4$ $u = 4$

4. Let $a, b > 0$. Calculate the area of the set bounded by a curve

$$\sqrt[4]{\frac{x}{a}} + \sqrt[4]{\frac{y}{b}} = 1; \ x, y > 0.$$

$u^4 \cdot v^{-4}\, du\, dv$

$\int_1^4 u^4 du \cdot \int_1^2 v^{-4} dv$

$= \frac{4^5 - 1}{5} \cdot -\frac{1}{3}v^{-3}$

$= \frac{4^5 - 1}{5} \quad -\frac{1}{3}(\frac{1}{8} - 1)$

$= \frac{1023}{5} \cdot \frac{7}{24}$

$\begin{cases} u = \sqrt[8]{\frac{x}{a}}, & x = au^8 \\ v = \sqrt[8]{\frac{y}{b}}, & y = bv^8 \end{cases}$ $J = 64ab\, u^7 v^7$

$\begin{cases} u = r\cos\varphi \\ v = r\sin\varphi. \end{cases}$ $\varphi \in [0, 2\pi)$. × $r \in [0,1]$

$$\iint_\Omega f \cdot dx\,dy = \cancel{256}\,64\, ab \int_0^{\frac{\pi}{2}} \cos^7\varphi \sin^7\varphi\, d\varphi \int_0^1 r^{15} dr$$

$$= 16ab \int_0^{\frac{\pi}{2}} \cos^7\varphi \sin^7\varphi\, d\varphi$$

$$= 16ab \int_0^{\frac{\pi}{2}} \cos^7\varphi\, (\cos^2\varphi - 1)^3\, d\cos\varphi$$

$$= 16ab \cdot \frac{1}{14}\cos^{14}\varphi - \frac{1}{4}\cos^{12}\varphi + \frac{3}{10}\cos^{10}\varphi - \frac{1}{8}\cos^8\varphi \Big|_0^{\frac{\pi}{2}}$$

$$= 16ab \cdot (-\frac{1}{14} + \frac{1}{4} - \frac{3}{10} + \frac{1}{8})$$

$$= \frac{\cancel{1}2}{3\cancel{5}} ab \quad \frac{1}{70} ab$$

Oct. 13th HW14

1. Calculate the volume of the set bounded by the following surfaces

$$z = x^2 + y^2,\ x^2 + y^2 = x,\ x^2 + y^2 = 2x,\ z = 0.$$

Consider the cylindric change $r > 0$. $|J| = r$.

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z. \end{cases} \quad \begin{array}{l} 0 \le z \le r^2 \\ \Rightarrow \cos\varphi \le r \le 2\cos\varphi \end{array} \quad \varphi \in [0, 2\pi)$$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r \cdot dr \int_0^{r^2} dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r^3\, dr$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \cdot \left(\frac{r^4}{4} \Big|_{\cos\varphi}^{2\cos\varphi} \right) = \frac{15}{2} \int_0^{\frac{\pi}{2}} \cos^4\varphi\, d\varphi = \frac{3\pi}{16} \cdot \frac{15}{2} = \frac{45}{32}\pi.$$

2. Let $a, b, c > 0$. Calculate the volume of the set bounded by the following surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c},\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b},\ z = 0.$$

$$\begin{cases} \frac{x}{a} = r\cos\varphi \\ \frac{y}{b} = r\sin\varphi \\ \frac{z}{c} = h. \end{cases} \qquad |J| = \begin{vmatrix} a\cos\varphi & -ar\sin\varphi & 0 \\ b\sin\varphi & br\cos\varphi & 0 \\ 0 & 0 & c \end{vmatrix} = abc \cdot r$$

$$\Rightarrow \quad 0 \le h \le r^2$$

$$V = abc \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_0^{\sin\varphi + \cos\varphi} r\, dr \int_0^{r^2} dh$$

$$= abc \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_0^{\sqrt{2}\sin(\varphi + \frac{\pi}{4})} r^3\, dr = abc \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^4(\varphi + \frac{\pi}{4})\, d\varphi$$

$$= \frac{3\pi}{8} abc$$

3. Consider all rearrangements of order of integration in

$$V= \int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)dz$$

$$\begin{cases} \sqrt{x^2+y^2} \le z \le 1 \\ -\sqrt{1-x^2} \le y \le \sqrt{1-x^2} \\ -1 \le x \le 1. \end{cases}$$

the set is bounded by $x^2+y^2=z^2$, $z=1$. $x^2+y^2=1$.

$$V= \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz = \int_0^1 dz \int_{-z}^{z} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)\,dx.$$

$$= \int_{-1}^{1} dy \int_y^1 dz \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f\,dx = \int_{-1}^{1} dx \int_x^1 dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f\,dy$$

$$= \int_0^1 dz \int_{-z}^{z} dx \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f\,dy$$

4. Calculate

$$\iiint_\Omega \sqrt{x^2+y^2}dxdydz,$$

where Ω is bounded by the surfaces

$$x^2+y^2=z^2, \quad z=1.$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z. \end{cases} \qquad J = |r|.$$

$$\int_0^{2\pi} d\varphi \int_0^1 r^2 dr \int_r^1 dz. = \int_0^{2\pi} d\varphi \int_0^1 (1-r)r^2\,dr = \frac{\pi}{6}$$

HW15. Oct. 16th.

1. Calculate $F'(t)$ if

$$F(t) = \iiint\limits_{x^2+y^2+z^2 \le t^2} f(x^2+y^2+z^2)\,dxdydz,$$

where f is a differentiable function.

Consider $\begin{cases} x = r\cos\varphi\cos\psi \\ y = r\sin\varphi\cos\psi \\ z = r\sin\psi. \end{cases}$

$$F(t) = \int_0^t f(r^2)\cdot r^2\,dr \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi\, d\psi$$

$$= 4\pi. \int_0^t f(r^2)\cdot r^2\,dr$$

$$F'(t) = 4\pi.\ f(r^2).r^2 = 4\pi\,(x^2+y^2+z^2)\,f(x^2+y^2+z^2)$$ ✓

2. Let $a > 0$. Calculate the volume of a set bounded by surfaces

$$az = x^2+y^2, \quad z = \sqrt{x^2+y^2}.$$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z. \end{cases} \Rightarrow$ $0 \le z \le a$. (since the set needs to be bounded)

$az = r^2$

$z \le r \le \sqrt{az}$

$$V = \int_0^{2\pi} d\varphi \int_0^a dz \int_z^{\sqrt{az}} r\,dr.$$

$$= \pi \int_0^a az - z^2\,dz = \pi.\ \frac{az}{2} - \frac{z^3}{3}\Big|_0^a = \frac{\pi a^3}{6}.$$ ✓

3. Calculate the volume of a set bounded by the surface

$$x^2+y^2+z^4 = 1.$$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi. \\ z = \sqrt{|t|} \end{cases} \Rightarrow |J_1| = \frac{|r|}{2\sqrt{|t|}}. \Rightarrow r^2+t^2 = 1 \Rightarrow \begin{cases} r = a\cos\theta \\ t = a\sin\theta. \end{cases} \Rightarrow |J_2| = a.$

$$V = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\theta \int_0^1 \cdot a \cdot \frac{|a\cos\theta|}{2\sqrt{a\sin\theta}} \, da$$

$r>0$

$\sqrt{\sin\theta}$.

$$= 4\pi \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{|\sin\theta|}} d\theta \int_0^1 a^{\frac{3}{2}} da.$$

$$\frac{2}{5}\pi \int_0^{2\pi} \frac{\cos\theta}{\sqrt{|\sin\theta|}} d\theta$$

$$= \frac{8}{5}\pi \int_0^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{\sin\theta}} d\theta = \frac{\cancel{16}^{\,8}\pi}{5} \int_0^{\frac{\pi}{2}} d(\sqrt{\sin\theta}) = \frac{\cancel{16}^{\,8}\pi}{5}$$

4. Calculate

$$I = \iiint_\Omega \sqrt{x^2+y^2+z^2}\,dxdydz,$$

where $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leqslant z\}$.

$$\begin{cases} x = r\cos\varphi\cos\psi \\ y = r\sin\varphi\cos\psi \\ z = r\sin\psi \end{cases} \Rightarrow \quad |J| = r^2\cos\psi.$$

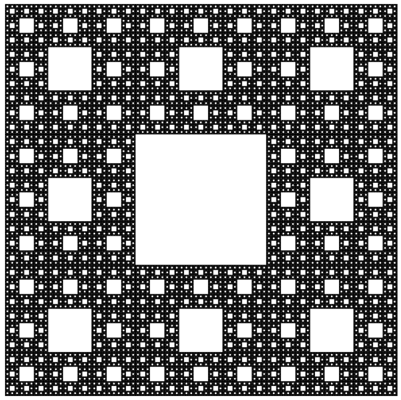
$$r^2 \leq r\sin\psi \Rightarrow r \leq \sin\psi.$$

$$I = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi \, d\psi \int_0^{\sin\psi} r^3 dr.$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin^4\psi \cos\psi \, d\psi = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \sin^4\psi \, d(\sin\psi) = \frac{\pi}{10}.$$

Mathematical analysis 3. Homework 16.

1. Consider a square $[0,1]^2$. The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum. The obtained set S is called Sierpinski carpet. Calculate the Lebesgue measure of S.



2. Can the unbounded Lebesgue measurable subset of $\mathbb{R}$ have finite positive Lebesgue measure?

3. Assume that E_1 and E_2 are $\mu-$measurable. Prove that $E_1 \cup E_2$ are measurable and

$$\mu(E_1 \cup E_2) = \mu E_1 + \mu E_2 - \mu(E_1 \cap E_2).$$

4. Assume that $A_k \subset [0,1]$ are Lebesgue measurable sets such that

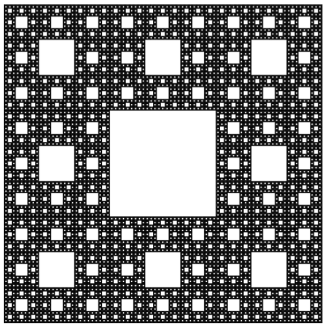
$$mA_1 + mA_2 + \ldots + mA_n > n - 1.$$

Prove that $m\left(\bigcap_{k=1}^{n} A_k\right) > 0$.

5. Assume that $E \subset \mathbb{R}$ is measurable and has positive measure. Prove that there exist $x, y \in E$ such that $x - y \in \mathbb{Q}$.

HW16.

1. Consider a square $[0,1]^2$. The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum. The obtained set S is called Sierpinski carpet. Calculate the Lebesgue measure of S.



$\mu[0,1]^2 = (1-0)(1-0) = 1.$

Then calculate the removed part. the removed cells are cubic.

its edge and quantity satisfy: edge $\frac{1}{3^n}$. quantity 8^{n-1}. $n \geq 1$

$$\mu S' = \sum_{n=1}^{\infty} \left(\frac{1}{3^n}\right)^2 \cdot 8^{n-1} = \sum_{n=1}^{\infty} \frac{1}{9} \cdot \left(\frac{8}{9}\right)^{n-1} = \frac{1}{9} \cdot \frac{1-\left(\frac{8}{9}\right)^{n-1}}{1-\frac{8}{9}} = 1.$$

the remaining part. $S = [0,1]^2 \setminus S'$

$$\mu S = \mu[0,1]^2 - \mu S' = 0.$$

2. Can the unbounded Lebesgue measurable subset of $\mathbb{R}$ have finite positive Lebesgue measure?

Yes. consider Lebesgue measurable subset $Q \cup [0,1]^n$

Q can be expressed as countable union of single point set.

the single point set has measure 0.

$$\mu Q = \bigcup_{n=1}^{\infty} \mu\left\{\frac{p}{q}\right\}_n = 0. \quad \mu(Q \cup [0,1]^n) = \mu Q + \mu[0,1]^n = 0+1 = 1.$$

3. Assume that E_1 and E_2 are $\mu-$measurable. Prove that $E_1 \cup E_2$ are measurable and

$$\mu(E_1 \cup E_2) = \mu E_1 + \mu E_2 - \mu(E_1 \cap E_2).$$

Pf: since E_1, E_2 μ-measurable $E_1 \in \mathcal{A}$. $E_2 \in \mathcal{A}$.

$E_1 \cup E_2 \in \mathcal{A}$.

$E_1 \cup E_2 = E_1 \cup (E_2 \setminus E_1 \cap E_2)$

$E_1 \cap (E_2 \setminus E_1 \cap E_2) \neq \phi$.

by countable additivity. $\mu(E_1 \cup E_2) = \mu E_1 + \mu E_2 - \mu(E_1 \cap E_2)$

4. Assume that $A_k \subset [0,1]$ are Lebesgue measurable sets such that

$$mA_1 + mA_2 + \ldots + mA_n > n-1.$$

Prove that $m\left(\bigcap_{k=1}^{n} A_k\right) \geq 0.$

Pf: prove its converse-negative proposition.

Assume $m\left(\bigcap_{k=1}^{n} A_k\right) = 0$ Prove $mA_1 + mA_2 + \cdots + mA_n \leq n-1$

Let $n=2$ $mA_1 + mA_2 = m(A_1 \cup A_2) + m(A_1 \cap A_2)$ (by Problem 3)

$m(A_1 \cup A_2) \leq m[0,1] = 1.$ $m(A_1 \cap A_2) = 0$ $mA_1 + mA_2 \leq 1.$

Assume $n = k-1$ for any $A_k \subset [0,1]$ s.t. $m\left(\bigcap^{k-1} A_n\right) = 0$.

we have $mA_1 + mA_2 + mA_3 + \cdots + mA_{k-1} \leq k-2.$

$n=k$ $\sum_{n=1}^{k-2} mA_n + m(A_{k-1} \cup A_k) + m(A_{k-1} \cap A_k) \leq 1 + \sum_{n=1}^{k-2} mA_n + m(A_{k-1} \cap A_k)$

since $A_{k-1} \cap A_k \subset [0,1]$. denote $A'_{k-1} = A_{k-1} \cap A_k$

thus $\bigcap_{n=1}^{k'-1} A_n = \left(\bigcap_{n=1}^{k-2} A_n\right) \cap (A_{k-1} \cap A_k) = \bigcap_{n=1}^{k} A_n$ thus $m\left(\bigcap_{n=1}^{k'-1} A_n\right) = 0$

$\sum_{n=1}^{k-2} mA_n + m(A_{k-1} \cap A_k) \leq k-2$ by hypothesis $\Rightarrow \sum_{n=1}^{k} A_n \leq k-2+1 = k-1.$

5. Assume that $E \subset \mathbb{R}$ is measurable and has positive measure. Prove that there exist $x, y \in E$ such that $x - y \in \mathbb{Q}$.

Assume $x - y \in \mathbb{R} \setminus \mathbb{Q}$ for all $x, y \in E$.

consider it's fractional

Let $\mu E = \varepsilon > 0$. $E \subset [0,1]$, $q \in [0,1]$

$E_q = \{x + q \mid x \in E\}$. $\mu E = \mu E_n$. for all $n \in \mathbb{N}$.

E_q are mutually disjoint (every elements has irration distance).

$\bigcup E_q \subset [0,2]$

$\mu \bigcup E_q = \infty \cdot \varepsilon = \infty$.

$\mu [0,2] = 2$. this contradicts to the monotone.

Mathematical analysis 3. Homework 17.

1. Assume that $f : [a, b] \to \mathbb{R}$ is differentiable. Prove that f' is a (Lebesgue) measurable function.

Pf: denote $f_n(x) = \dfrac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$. since f is diff. $f_n \to f'$ $(n \to \infty)$.

now it suffices to check. f_n is measurable.

since f is measurable, on $[a,b]$ (f is cont. on $[a,b]$. $[a,b]$ is measurable).

$f_n(x) = n(f(x+\frac{1}{n}) - f(x)) \in S([a,b])$. by Arithmetic properties.

of measurable function

2. Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Prove that f is measurable iff $f^{-1}(\{-\infty\})$, $f^{-1}(\{+\infty\})$ are measurable sets and f is measurable on Y.

Pf: "$\Rightarrow$" f is measurable.

$Y = X(f < +\infty) \cup X(f > -\infty) = \bigcup_{n=1}^{\infty} (X(f > -n) \cup X(f < -n))$ is measurable.

$\Rightarrow f|_Y$ is measurable.

$f^{-1}(\{-\infty\}) = X(f = -\infty) = \bigcap_{n=1}^{\infty} (f < n)$. measurable.

$f^{-1}(\{+\infty\}) = X(f = +\infty) = \bigcap_{n=1}^{\infty} (f > n)$ measurable.

"$\Leftarrow$" $\forall a \in \mathbb{R}$. $X(f > a) = Y(f|_Y > a) \cup f^{-1}(\{+\infty\})$. f is measurable.

3. Let $\{f_n\}$ is a sequence of measurable functions on X. prove that

$$E = \{x : \lim f_n(x) \text{ exists}\}$$

is a measurable set.

Pf: $f_n \in S(X)$. $\overline{\lim} f_n$, $\underline{\lim} f_n \in S(X)$

define a function $f : X \to \mathbb{R}$.

s.t. $\forall x \in X$, $f(x) = \overline{\lim} f_n(x) - \underline{\lim} f_n(x)$, $f \in S(X)$.

$E = X(f = 0) = X(f \leq 0) \cap X(f \geq 0)$. E is measurable

4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone then f is measurable.

Pf: f is monotone.

the number of point of discontinuity is at most countable

thus, denote $E_1 = \{x \in \mathbb{R} \mid f(x) \text{ is discontinous at } x\}$.

$\mu E_1 = 0$ (E_1 is at most countable).

the set $\mathbb{R} \setminus E_1$ is measurable and $f|_{\mathbb{R}\setminus E_1}$ is continuous.

$f \in S(\mathbb{R}\setminus E_1)$. $\mathbb{R} \setminus (\mathbb{R}\setminus E_1) = E_1$ is a set of measure 0.

thus we have $f \in S(\mathbb{R})$.

5*. Assume that μ is a Borel measure. Prove that for any finite family of segments $I_1, \ldots, I_n$ there exists a subfamily $\{J_1, \ldots, J_m\}$ of mutually disjoint intervals such that

$n \geq m$.

$$\sum_{k=1}^{m} \mu(J_k) \geq \frac{1}{2}\mu\left(\sum_{k=1}^{n} I_k\right).$$

$\exists\ I_1, \ldots I_n$. $\forall$ subfamily of $\sum \mu(J_k) < \frac{1}{2}\mu(\sum I_k)$.

$\mu\left(\sum_{k=1}^{n} I_k\right)$

Mathematical analysis 3. Homework 18.

1. Calculate Lebesgue integral $\int_0^1 f(x)dx$, where

$$f(x) = \begin{cases} x^2, & x \in \left(\frac{1}{3}, +\infty\right) \setminus \mathbb{Q}; \\ x^2, & x \in \left(-\infty, \frac{1}{3}\right) \setminus \mathbb{Q}; \\ 0, & x \in \mathbb{Q}. \end{cases}$$

2. Construct a sequence of integrable functions $f_n : [0,1] \to \mathbb{R}$ such that
   - The limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists a.e.;
   - $\lim\limits_{n\to\infty} \int\limits_0^1 f_n \to \int\limits_0^1 f$;
   - There is no integrable function G such that $|f_n| \leq G$ a.e. for every $n \in \mathbb{N}$.
3. Provide an example of function f such that the integral $\int_0^1 f$ exists as improper integral while f is not Lebesgue integrable.
4. Assume that the integral $\int_E fg dx$ exists and is finite for every $g \in L(E)$. Prove that f is bounded a.e.

HW 18.

1. Calculate Lebesgue integral $\int_0^1 f(x)dx$, where

$$f(x) = \begin{cases} x^2, & x \in \left(\frac{1}{3}, +\infty\right) \setminus \mathbb{Q}; \\ x^2, & x \in \left(-\infty, \frac{1}{3}\right) \setminus \mathbb{Q}; \\ 0, & x \in \mathbb{Q}. \end{cases}$$

Pf: let $g(x) = x^2$, $x \in \mathbb{R}$.

we have $g = f$, $x \in \mathbb{R} \setminus \mathbb{Q}$. and $\mu\mathbb{Q} = 0$. thus $f \sim g$ on $\mathbb{R}$.

on $[0,1]$, $d\mu(x) = dx$.

$\int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = \int_0^1 x^2 dx = \frac{1}{3}$.

2. Construct a sequence of integrable functions $f_n : [0,1] \to \mathbb{R}$ such that

- The limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists a.e.;
- $\lim\limits_{n\to\infty} \int\limits_0^1 f_n \to \int\limits_0^1 f$;
- There is no integrable function G such that $|f_n| \le G$ a.e. for every $n \in \mathbb{N}$.

$$f_n = \begin{cases} \frac{n}{n-1}, & x \in \mathbb{Q} \\ n, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \qquad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ \infty, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

f_n is Lebesgue integrable since $f_n \sim f' \equiv n$. since $\mu\mathbb{Q} = 0$.

no G since $f_n \xrightarrow[n\to\infty]{} \infty$.

3. Provide an example of function f such that the integral $\int_0^1 f$ exists as improper integral while f is not Lebesgue integrable.

function $f(x)=\begin{cases} \frac{1}{x-\frac{1}{2}} & x\in[0,\frac{1}{2})\cup(\frac{1}{2},1] \\ 1 & x=\frac{1}{2}. \end{cases}$ $f\in R_{loc}[a,b], \forall [a,b]\subset[0,1]$

$$\int_0^1 f\,d\mu = \int_0^1 f_+\,d\mu + \int_0^1 f_-\,d\mu.$$

$$= \int_{\frac{1}{2}}^1 f\,dx + \int_0^{\frac{1}{2}} f\,dx.$$

which both tends to infinite. not Lebesgue integrable

4. Assume that the integral $\int_E fg\,dx$ exists and is finite for every $g \in L(E)$. Prove that f is bounded a.e.

Pf: Assume the converse. $\exists E_1\subset E$, f is unbounded on E_1. $\mu E_1>0$.

denote. $E_{11}, E_{12}\subset E_1$. s.t. $\mu E_{11}, \mu E_{12}>0$. $E_{11}\cup E_{12}=E_1$. $E_{11}\cap E_{12}=\emptyset$.

Let $g(x)=\begin{cases} \operatorname{sgn}(f(x)), & x\in E_{11} \\ -\operatorname{sgn}(f(x)), & x\in E_{12}. \\ 1, & x\in E\setminus E_1. \end{cases}$

$g(x)\in L(E)$ since $|g(x)|\in L(E)$ (Indeed, $|g(x)|\sim g'$, $g'\equiv 1$).

$$\left|\int_E fg\,dx\right| = \left|\int_{E\setminus E_1} fg\,dx\right| + \left|\int_{E_1} fg\,dx\right| \geqslant \left|\int_{E_1} fg\,dx\right|$$

$$\int_{E_1} fg\,dx = \int_{E_1}(fg)_+\,dx - \int_{E_1}(fg)_-\,dx$$

$$= \int_{E_{11}}|f|\,dx - \int_{E_{12}}|f|\,dx.$$

since f is unbounded. both integral $\to\infty$.

Thus $\int_E fg\,dx$ not exists. which is contradictory. $\square$.

Nov. 6th.

1. $\lim_{\alpha\to 0}\int_0^1 \sqrt{1+\alpha^2x^4}dx$;

$$= \int_0^1 \lim_{\alpha\to 0} \sqrt{1+\alpha^2x^4}\ dx$$

$$= \int_0^1 \lim_{\alpha\to 0} 1+\frac{1}{2}\alpha^2x^4\ dx.$$

$$= \int_0^1 1\ dx = 1.$$

3. $\lim_{\alpha\to 1}\int_2^4 \frac{xdx}{1+x^2+\alpha^6}$;

$$= \frac{1}{2}\lim_{\alpha\to 1}\int_2^4 \frac{d(x^2)}{1+x^2+\alpha^6}$$

$$= \frac{1}{2}\lim_{\alpha\to 1} \ln|1+x^2+\alpha^6|\Big|_2^4.$$

$$= \frac{1}{2}\lim_{\alpha\to 1} \ln\left|\frac{17+\alpha^6}{5+\alpha^6}\right| = \frac{1}{2}\ln 3.$$

5. $\lim_{\alpha\to 0}\int_0^\pi x\cos(1+\alpha)xdx$

$$\int x\cos(1+\alpha)x\,dx = \frac{x\sin(1+\alpha)x}{1+\alpha} - \int \frac{\sin(1+\alpha)x}{1+\alpha}dx.$$

$$= \frac{x\sin(1+\alpha)x}{1+\alpha} + \frac{\cos(1+\alpha)x}{(1+\alpha)^2}. + C$$

$$\lim_{\alpha\to 0}\int_0^\pi x\cos(1+\alpha)x\,dx = \lim_{\alpha\to 0} \frac{\cos(1+\alpha)\pi}{(1+\alpha)^2} - \frac{1}{(1+\alpha)^2} = -2$$

1. $I(\alpha)=\int_0^1 \sin(\alpha x)dx$;

$$I'(\alpha) = \int_0^1 x\cos\alpha x\ dx = \int_0^1 \frac{x\sin\alpha x}{\alpha} + \frac{\cos\alpha x}{\alpha^2}\ dx$$

$$= \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2}$$

3. $I(\alpha)=\int_1^2 e^{\alpha x^2}\frac{dx}{x}$;

$$I'(\alpha) = \int_1^2 \frac{1}{x}\cdot x^2\cdot e^{\alpha x^2}dx = \frac{1}{2\alpha}\int_1^2 e^{\alpha x^2}d(\alpha x^2).$$

$$= \frac{1}{2\alpha}\left(e^{4\alpha} - e^{2\alpha}\right).$$