

Loran Series

If $z = z_0$ is a singularity of a function f , then certainly f cannot be expanded in a power series with z_0 as its center. However, about an isolated singularity $z = z_0$, it is possible to represent f by a series involving both negative and nonnegative integer powers of $z - z_0$; that is,

$$f(z) = \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots \quad (1)$$

As a very simple example of (1) let us consider the function $f(z) = 1/(z-1)$. As can be seen, the point $z = 1$ is an isolated singularity of f and consequently the function cannot be expanded in a Taylor series centered at that point. Nevertheless, f can be expanded in a series of the form given in (1) that is valid for all z near 1 :

$$f(z) = \cdots + \frac{0}{(z - 1)^2} + \frac{1}{z - 1} + 0 + 0 \cdot (z - 1) + 0 \cdot (z - 1)^2 + \cdots \quad (2)$$

The series representation in (2) is valid for $0 < |z - 1| < \infty$.

Using summation notation, we can write (1) as the sum of two series

$$f(z) = \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (3)$$

The two series on the right-hand side in (3) are given special names. The part with negative powers of $z - z_0$, that is,

$$\sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k} = \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k} \quad (4)$$

is called the principal part of the series (1) and will converge for $|1/(z - z_0)| < r^*$ or equivalently for $|z - z_0| > 1/r^* = r$. The part consisting of the nonnegative powers of $z - z_0$,

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \quad (5)$$

is called the analytic part of the series (1) and will converge for $|z - z_0| < R$. Hence, the sum of (4) and (5) converges when z satisfies both $|z - z_0| > r$ and $|z - z_0| < R$, that is, when z is a point in an annular domain defined by $r < |z - z_0| < R$.

By summing over negative and nonnegative integers, (1) can be written compactly as

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

The principal part of the series (2) consists of exactly one nonzero term, whereas its analytic part consists of all zero terms. Our next example illustrates a series of the form (1) in which the principal part of the series also consists of a finite number of nonzero terms, but this time the analytic part consists of an infinite number of nonzero terms.

EXAMPLE 1 Series of the Form Given in (1)

The function $f(z) = \frac{\sin z}{z^4}$ is not analytic at the isolated singularity $z = 0$ and hence cannot be expanded in a Maclaurin series. However, $\sin z$ is an entire function, and we know that its Maclaurin series,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots$$

converges for $|z| < \infty$. By dividing this power series by z^4 we obtain a series for f with negative and positive integer powers of z :

$$f(z) = \frac{\sin z}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1}{3!z}}_{\text{principal part}} + \underbrace{\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \dots}_{\text{analytic part}} \quad (6)$$

The analytic part of the series in (6) converges for $|z| < \infty$. (Verify.) The principal part is valid for $|z| > 0$. Thus (6) converges for all z except at $z = 0$; that is, the series representation is valid for $0 < |z| < \infty$.

A series representation of a function f that has the form given in (6) is called a Laurent series or a Laurent expansion of f about z_0 on the annulus $r < |z - z_0| < R$.

EXAMPLE 2 Four Laurent Expansions

Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains,

- (a) $0 < |z| < 1$
- (b) $1 < |z|$
- (c) $0 < |z - 1| < 1$
- (d) $1 < |z - 1|$

In parts (a) and (b) we want to represent f in a series involving only negative and nonnegative integer powers of z , whereas in parts (c) and (d) we want to represent f in a series involving negative and nonnegative integer powers of $z - 1$.

(a) By writing

$$f(z) = -\frac{1}{z} \frac{1}{1-z}$$

we can use (6) of Section 6.1 to write $1/(1 - z)$ as a series:

$$f(z) = -\frac{1}{z} [1 + z + z^2 + z^3 + \dots]$$

The infinite series in the brackets converges for $|z| < 1$, but after we multiply this expression by $1/z$, the resulting series

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \dots$$

converges for $0 < |z| < 1$.

(b) To obtain a series that converges for $1 < |z|$, we start by constructing a series that converges for $|1/z| < 1$. To this end we write the given function f as

$$f(z) = \frac{1}{z^2} \frac{1}{1 - \frac{1}{z}}$$

and again use (6) of Section 6.1 with z replaced by $1/z$:

$$f(z) = \frac{1}{z^2} \left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right]$$

The series in the brackets converges for $|1/z| < 1$ or equivalently for $1 < |z|$. Thus the required Laurent series is

$$f(z) = \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \dots$$

(c) This is basically the same problem as in part (a), except that we want all powers of $z - 1$. To that end, we add and subtract 1 in the denominator and use (7) of Section 6.1 with z replaced by $z - 1$:

$$\begin{aligned} f(z) &= \frac{1}{(1 - 1 + z)(z - 1)} \\ &= \frac{1}{z - 1} \frac{1}{1 + (z - 1)} \\ &= \frac{1}{z - 1} [1 - (z - 1) + (z - 1)^2 - (z - 1)^3 + \dots] \\ &= \frac{1}{z - 1} - 1 + (z - 1) - (z - 1)^2 + \dots \end{aligned}$$

The requirement that $z \neq 1$ is equivalent to $0 < |z - 1|$, and the geometric series in brackets converges for $|z - 1| < 1$. Thus the last series converges for z satisfying $0 < |z - 1|$ and $|z - 1| < 1$, that is, for $0 < |z - 1| < 1$.

(d) Proceeding as in part (b), we write

$$\begin{aligned}
f(z) &= \frac{1}{z-1} \frac{1}{1+(z-1)} = \frac{1}{(z-1)^2} \frac{1}{1+\frac{1}{z-1}} \\
&= \frac{1}{(z-1)^2} \left[1 - \frac{1}{z-1} + \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \dots \right] \\
&= \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \frac{1}{(z-1)^5} + \dots
\end{aligned}$$

Because the series within the brackets converges for $|1/(z-1)| < 1$, the final series converges for $1 < |z-1|$.

EXAMPLE 3 Laurent Expansions

Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for (a) $0 < |z-1| < 2$ and (b) $0 < |z-3| < 2$.

Solution

(a) As in parts (c) and (d) of Example 2, we want only powers of $z-1$ and so we need to express $z-3$ in terms of $z-1$. This can be done by writing

$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2} \frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2} \frac{1}{1-\frac{z-1}{2}}$$

and then using (6) of Section 6.1 with the symbol z replaced by $(z-1)/2$,

$$\begin{aligned}
f(z) &= \frac{-1}{2(z-1)^2} \left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} + \dots \right] \\
&= -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \dots
\end{aligned} \tag{16}$$

(b) To obtain powers of $z-3$, we write $z-1 = 2 + (z-3)$ and We now factor 2 from this expression

$$\begin{aligned}
f(z) &= \frac{1}{(z-1)^2(z-3)} = \frac{1}{z-3} \overbrace{[2+(z-3)]^{-2}}^{\text{binomial expansion}} \\
&= \frac{1}{4(z-3)} \left[1 + \frac{z-3}{2} \right]^{-2}
\end{aligned}$$

At this point we can obtain a power series for $\left[1 + \frac{z-3}{2}\right]^{-2}$ by using the binomial expansion,

$$f(z) = \frac{1}{4(z-3)} \left[1 + \frac{(-2)}{1!} \left(\frac{z-3}{2} \right) + \frac{(-2)(-3)}{2!} \left(\frac{z-3}{2} \right)^2 + \frac{(-2)(-3)(-4)}{3!} \left(\frac{z-3}{2} \right)^3 + \dots \right].$$

The binomial series in the brackets is valid for $|(z-3)/2| < 1$ or $|z-3| < 2$. Multiplying this series by $\frac{1}{4(z-3)}$ gives a Laurent series that is valid for $0 < |z-3| < 2$:

$$f(z) = \frac{1}{4(z-3)} - \frac{1}{4} + \frac{3}{16}(z-3) - \frac{1}{8}(z-3)^2 + \dots$$

EXAMPLE 4 A Laurent Expansion

Expand $f(z) = \frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0 < |z| < 1$.

Solution By partial fractions we can rewrite f as

$$f(z) = \frac{8z+1}{z(1-z)} = \frac{1}{z} + \frac{9}{1-z}$$

Then by (6) of Section 6.1 ,

$$\frac{9}{1-z} = 9 + 9z + 9z^2 + \dots$$

The foregoing geometric series converges for $|z| < 1$, but after we add the term $1/z$ to it, the resulting Laurent series

$$f(z) = \frac{1}{z} + 9 + 9z + 9z^2 + \dots$$

is valid for $0 < |z| < 1$.

In the preceding examples the point at the center of the annular domain of validity for each Laurent series was an isolated singularity of the function f . A re-examination of Theorem 6.10 shows that this need not be the case.

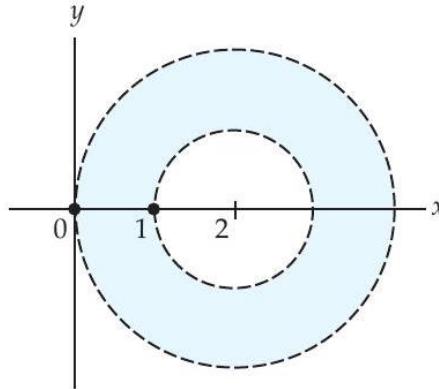


Figure 6.9 Annular domain for Example 5

[†] For α real, the binomial series $(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \dots$ is valid for $|z| < 1$.

EXAMPLE 5 A Laurent Expansion

Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$.

Solution The specified annular domain is shown in Figure 6.9. The center of this domain, $z = 2$, is the point of analyticity of the function f . Our goal now is to find two series involving integer powers of $z - 2$, one converging for $1 < |z - 2|$ and the other converging for $|z - 2| < 2$. To accomplish this, we proceed as in the last example by decomposing f into partial fractions:

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z) \quad (17)$$

Now,

$$\begin{aligned} f_1(z) &= -\frac{1}{z} = -\frac{1}{2+z-2} \\ &= -\frac{1}{2} \frac{1}{1+\frac{z-2}{2}} \\ &= -\frac{1}{2} \left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \dots \right] \\ &= -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \dots \end{aligned}$$

This series converges for $|(z-2)/2| < 1$ or $|z-2| < 2$. Furthermore,

$$\begin{aligned} f_2(z) &= \frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{z-2} \frac{1}{1+\frac{1}{z-2}} \\ &= \frac{1}{z-2} \left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \dots \right] \\ &= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \dots \end{aligned}$$

converges for $|1/(z-2)| < 1$ or $1 < |z-2|$. Substituting these two results in (17) then gives

$$f(z) = \dots - \frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2} - \frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \dots$$

This representation is valid for z satisfying $|z-2| < 2$ and $1 < |z-2|$; in other words, for $1 < |z-2| < 2$.

EXAMPLE 6 A Laurent Expansion

Expand $f(z) = e^{3/z}$ in a Laurent series valid for $0 < |z| < \infty$.

Solution From (12) of Section 6.2 we know that for all finite z , that is, $|z| < \infty$,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \quad (18)$$

We obtain the Laurent series for f by simply replacing z in (18) by $3/z$, $z \neq 0$,

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \dots \quad (19)$$

This series (19) is valid for $z \neq 0$, that is, for $0 < |z| < \infty$.

Remarks

(i) In conclusion, we point out a result that will be of special interest to us in Sections 6.5 and 6.6. Replacing the complex variable s with the usual symbol z , we see that when $k = 1$, formula (8) for the Laurent series coefficients yields $a_{-1} = \frac{1}{2\pi i} \oint_C f(z)dz$, or more important,

$$\oint_C f(z)dz = 2\pi i a_{-1} \quad (20)$$

(ii) Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is the Laurent series; that is, the series we obtain is unique.