# Differential Geometry

## Part I.

*Differential Geometry of Smooth Curves*

2025

# Chapter 1

# The Concept of the Curve

## 1.1 The Smooth Curve

Differential geometry is that branch of mathematics which investigates geometric forms, primarily curves and surfaces, but also families of curves and surfaces, using methods of infinitesimal analysis. It is characteristic of differential geometry that it studies the properties of curves and surfaces "in the small", i.e., the properties of arbitrarily small pieces of curves and surfaces. Differential geometry arose and developed in close relationship with analysis, which itself grew, to a significant degree, out of geometric problems.

Many geometric concepts preceded the corresponding ideas in analysis. Thus, for example, the notion of a tangent preceded the concept of a derivative and the idea of area and volume preceded that of an integral.

A curve is one of the fundamental objects considered in differential geometry. In this class, we'll discuss the concept of curve to the extent required in the present course.

### Topological Mapping

Most of the notions provided here must be well known within the courses of Mathematical Analysis, Algebra, Analytic Geometry and Topology. We provide definitions of them here just for a recap and to avoid any discrepancies in the terminology.

Let us operate in ordinary three-dimensional Euclidean space ($\mathbb{E}$). $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ is an arbitrary right orthonormal basis of an arbitrary Cartesian coordinate system in $\mathbb{E}$. $x$, $y$, $z$ are coordinates and $O$ is the origin of that coordinate system.

Let $\boldsymbol{a}\left(a_1, a_2, a_3\right)$ and $\boldsymbol{b}\left(b_1, b_2, b_3\right)$ be arbitrary vectors expressed in mentioned coordinates.

**Definition** (Scalar product)**.** The scalar or dot product of these vectors is

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

**Definition** (Vector product)**.** The vector or cross product of these vectors is

$$[\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

**Definition** (Triple product)**.** The triple product of these vectors and vector $\boldsymbol{c}\left(c_1, c_2, c_3\right)$ is

$$(\boldsymbol{a} \cdot \boldsymbol{b} \cdot \boldsymbol{c}) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

Suppose $M \subset \mathbb{E}$ is an arbitrary set of points in space.

**Definition** (Mapping)**.** We say that $f$ is a given **mapping** of the set $M$ into space $\mathbb{E}$ if *an arbitrary* point $f(X) \in \mathbb{E}$ is assigned for *each* point $X \in M$.

**Definition** (Image)**.** The point $f(X) \in \mathbb{E}$ assigned to the point $X \in M$ is called the **image of the point** $X$.

The set of points consisting of the assigned images of all the points of the set $M$

$$f(M) = \{Y \in \mathbb{E} : Y = f(X), \forall X \in M\},$$

is called the **image of the set** $M$.

**Definition** (Topological mapping)**.** We say that mapping $f(M)$ is the **topological mapping** if conditions are fulfilled:

1. The images of distinct points are distinct (it is a one-to-one mapping)

$$\forall X \in M, X' \in M, X \neq X' : f(X) \neq f(X');$$

2. If $X \in M$ is an arbitrary point, and $X_n \in M$ is a sequence converging to $X$, then the sequence of points points $Y_n = f(X_n)$ which are the images of the pints $X_n$ converges, and moreover, it converges to the point $f(X)$, which is the image of the point $X$ (continuous mapping);

3. If $f(X)$ is an arbitrary point of the image $f(M)$ and $f(X_n) \in f(M)$ is a sequence which converges to $f(X)$, then the sequence of $X_n$ corresponding to the $f(X_n)$, converges, and it points converges to the point $X$ (bicontinuous mapping).

*Remark.* The fact that mapping is one-to-one yields the fact that there is the reverse mapping $f^{-1}$, therefore, these conditions may be expressed as

1. Mapping is one-to-one
2. Mapping and corresponding reverse mapping are continuous

**Definition.** (Homeomorphic sets) We say that a set $M$ and its image $f(M)$ under a topological mapping are *topologically equivalent* or *homeomorphic.*

**Definition** (Open set)**.** A set $G \subset \mathbb{E}$ is said to be **open** if for every point $X \in G$ we can find a positive real number $\varepsilon > 0$ such that all the points in space whose distances from $X$ are less than $\varepsilon$ also belong to $G$

$$\forall X \in G,\ \exists \varepsilon > 0 : \forall Y \in \mathbb{E} : dist(X, Y) < \varepsilon \implies Y \in G.$$

Obviously, a set consisting of an arbitrary number of open sets is open.

**Definition** (A neighborhood of the point)**.** A **neighbourhood of the point** $X \in \mathbb{E}$ is any open set containing this point.

*Remark.* In a metric space $\mathbb{M} = (M, d)$, a set $V$ is a neighborhood of a point $X$ if there exists an open ball with center $X$ and radius $\varepsilon > 0$, such that

$$O_\varepsilon(X) = \{Y \in M : dist(X, Y) < \varepsilon\}$$

is contained in $V$.

**Definition** (Connected set)**.** A set $M \subset \mathbb{E}$ is said to be **connected** if there **do not exist** two open sets $G' \subset \mathbb{E}$ and $G" \subset \mathbb{E}$ which decompose the set $M$ into two subsets $M'$ and $M''$, one of which belongs only to $G'$ and the other only to $G"$:

$$\nexists M' \subset \mathbb{E},\ M'' \subset \mathbb{E} :$$
$$M = M' \cup M'',$$
$$M' \subset G', M' \cap G'' = \varnothing,$$
$$M' \subset G', M' \cap G'' = \varnothing.$$

## Elementary Curve

**Definition** (Elementary curve)**.** We call a set $\gamma \in \mathbb{E}$ the **elementary curve** if this set is the image of an open segment on the straight line under an arbitrary topological mapping.

Suppose open interval $(a, b) \subset \mathbb{R}$ of parameter $t \in \mathbb{R}$ ($a < t < b$) expresses that segment.

To make this parametrisation clear, suppose we choose the Cartesian coordinates in space to express the underlined straight line with parametric equations

$$\begin{cases} x = \alpha_x t + \beta_x, \\ y = 0, \\ z = 0. \end{cases}$$

$a$ and $b$ are values of $t$ corresponding with the endpoints of the segment. We exclude them to make the segment open.

Therefore, we talk about points of the open interval $(a, b)$ while we mention corresponding points of the original *geometric* segment.

Let $f_1(t)$, $f_2(t)$, and $f_3(t)$ to be the coordinates of the point on the curve which corresponds to the point $t$ on the open interval.

**Definition** (Parametric equations of the elementary curve)**.** The system of equations

$$\begin{cases} x = f_1(t), \\ y = f_2(t), \\ z = f_3(t) \end{cases} \tag{1.1}$$

are called **the equations of the curve $\gamma$ in the parametric form**.

An elementary curve is defined uniquely by its equations in the parametric form. In this connection, then, we may speak about the definition of a curve by its equations.

## Simple Curve

**Definition** (Simple curve)**.** A set $\gamma \subset \mathbb{E}$ will be called a **simple curve** if this set is connected and each of its points $X$ has a neighborhood $\mathfrak{R} \subset \mathbb{E}$ such that the $\gamma \cap \mathfrak{R}$ is an elementary curve.

**Theorem 1.1.1** (The structure of a simple curve in the large)**.** *A simple curve is homeomorphic to either an open interval or a circumference.*

In a more detailed form, this proposition splits into two parts.

- The image of an open interval or circumference under a topological mapping into space is a simple curve.

- A simple curve is the image of an open interval or circumference under a topological mapping.

*Remark.* The property of a simple curve of being homeomorphic to an open interval or a circumference, indicated in the theorem 1.1.1, completely characterises the curve and, consequently, a simple curve can be defined employing this property.

**Definition** (Closed curve)**.** A simple curve which is homeomorphic to a circumference is said to be **closed**.

**Definition** (Neighborhood of the point on a simple curve)**.** A neighbourhood of a point $X$ on a simple curve $\gamma$ is the common part of the curve $\gamma$ and an arbitrary neighbourhood of the point $X$ in the space $\mathbb{E}$.

- There is a neighbourhood of each point of a *simple curve*, which is an *elementary curve.*

- We mention such an elementary neighbourhood in any ordinary talk about a neighbourhood of a point on a curve.

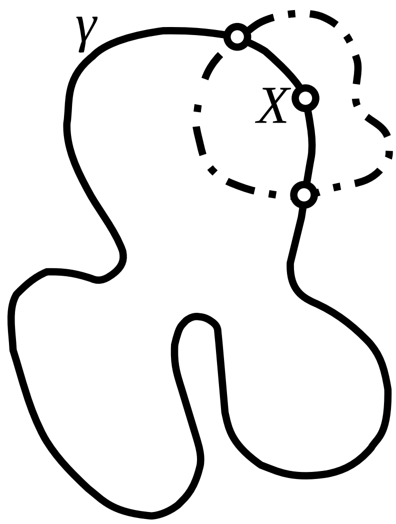


Figure 1.1. Neighborhood of the point $X$ on a simple curve $\gamma$

**Lemma 1.1.2.** *Suppose a simple curve $\gamma$ is the image of $g$, an open interval or a circumference, under an arbitrary topological mapping $f$.*

*Let $X \in g$ be an arbitrary point, and let $\omega \subset g$ be an arbitrary neighbourhood of $X$.*

*The image of $\omega$ under the mapping $f$ is a neighborhood of the point $f(X)$ on the curve $\gamma$.*

*Conversely, any neighbourhood of the point $f(X)$ can be obtained in this manner.*

*Proof.* First, straightforward application of neighbourhood definition yields that $\omega$ is an *open interval* or an *open arc of a circumference*.

Hence, the image of $\omega$ under the mapping $f$ is an *elementary curve*.

For each point $Y \in \mathbb{E}: \; f(Y) \in f(\omega)$ an open sphere

$$\Sigma_\delta^f(Y) = \{Z \in \mathbb{E} : dist(Z, f(Y)) < \delta\}, \delta > 0,$$

$$\Sigma_\delta^f(Y) \cap \gamma = f(\omega).$$

can be constructed around $f(Y)$ in virtue of the *bicontinuity* of the mapping $f$.

The set $G = \cup\Sigma_\delta^f(Y)$ consisting of all such open spheres $\Sigma_\delta^f(Y)$ is open. This $G$ contains only those points of the curve $\gamma$ which belong to the elementary curve $f(\omega)$. By the given definition, $f(\omega)$ is a *neighbourhood* of the point $f(X)$ on the curve. $\square$

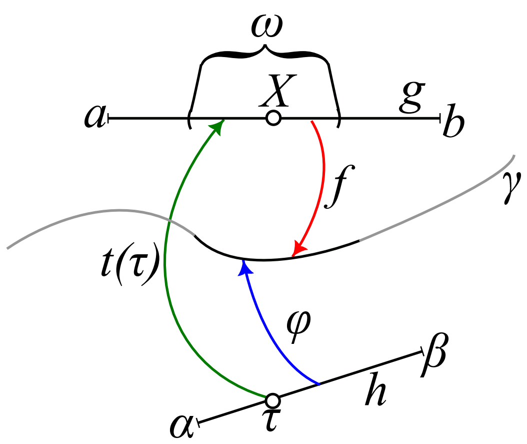


Figure 1.2. To the proof of lemma 1.1.2.

Suppose $f(\omega)$ is a neighbourhood of the point $f(X)$ on the curve $\gamma$. Since $f(\omega)$ is an elementary curve, it is the image of an open interval $h = (\alpha, \beta)$, $\tau \in h$ under an arbitrary topological mapping $\varphi$:

$$\varphi(h) = f(\omega).$$

Suppose that $g = (a, b)$, $t \in g$ is the open interval.

Each point $\tau \in h$ is assigned a definite point on the curve $\gamma$, and to the latter point there corresponds a definite point $t \in g$.

Thus, $t$ may be considered as a function of $\tau$: $t = t(\tau)$.

This function $t$ establishes *a topological mapping* of the open interval $h$ onto an open interval $g$. And the image of the open interval $h$ is the set $\omega$:

$$f^{-1}(\varphi(h)) = \omega.$$

Let us justify now that $\omega$ is an open interval. Because of the continuity of the function $t(\tau)$, if the points $t' < t''$ belong to the set $\omega$ then the closed interval $t' \leq t \leq t''$ also belongs to $\omega$. This is so because a continuous function $t(\tau)$ which assumes the values $t'$ and $t''$ also takes on all intermediate values. Thus, $\omega$ is an interval.

It remains to show that its endpoints do not belong to $\omega$. In fact, a neighbourhood of the point $f(X)$ on the curve $\gamma$ is a part of the curve belonging to some open set $G$. If $X$ belongs to $\omega$, i.e. if the image of $X$ belongs to $G$, then in virtue of the continuity of the mapping $f$ the images of all points on the interval $g$ which are sufficiently close to also belong to $G$.

It follows from this that $\omega$ is an open set and hence it is an open interval. □

## General Curve

**Definition** (Locally one-to-one mapping)**.** A mapping $f$ of a set $M$ into a space is said to be **locally one-to-one** if each of the points of $M$ has a neighbourhood in which the mapping $f$ is one-to-one.

**Definition** (General curve)**.** A set $\gamma$ of points in space will be called a **general curve** if this set is the image of a simple curve under a continuous and locally one-to-one mapping of it into space.

**Definition.** We shall say that the mapping $f_1$ of a simple curve $\gamma_1$ and the mapping $f_2$ of a simple curve $\gamma_2$ define **one and the same** general curve $\gamma$ if a one-to-one and bicontinuous (hence topological) correspondence $g$ can be established between the points of the curves $\gamma_1$ and $\gamma_2$ where the images of corresponding points on these curves *coincide* on the curve $\gamma$:

$$\begin{aligned} \forall A \in \gamma_1 :\ f_1(A) = f_2(g(A)), \\ \forall B \in \gamma_2 :\ f_1(g^{-1}(B)) = f_2(B). \end{aligned} \tag{1.2}$$

*Example.* Let us observe a general curve on the figure 1.3. This curve can be thought of as the image of a circumference under a topological mapping in two distinct ways.

Suppose a point moves on a circumference. Then its image moves along the curve. In this connection, the image-point, running along the curve, may assume

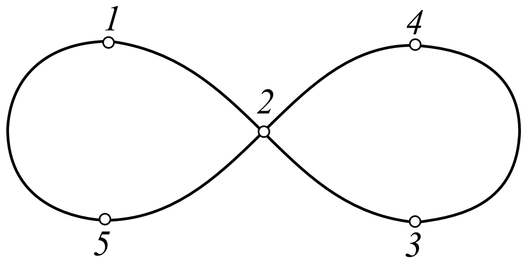

Figure 1.3. To the example of the general curve.

successively the positions $1 \mapsto 2 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 5$, but it can also trace out the curve in the order $1 \mapsto 2 \mapsto 4 \mapsto 3 \mapsto 2 \mapsto 5$. Mappings, corresponding to these courses, define distinct general curves, although as point sets they coincide.

## The Converge Sequences

Suppose a general curve $\gamma$ is the image under a topological mapping $f$ of the simple curve $\bar{\gamma}$ into space.

**Definition.** We shall say that a sequence of points $f(X_n) \in \gamma$, **converges** to the point $f(X) \in \gamma$, if the sequence of points $X_n \in \bar{\gamma}$ *converges* to the point $X \in \bar{\gamma}$.

**Definition.** *A neighborhood of the point* $f(X) \in \gamma$ is the image of any neighborhood of the point $X \in \bar{\gamma}$ under the mapping $f$.

**Lemma 1.1.3.** *Suppose an arbitrary alternative mapping $f'$ of the simple curve $\bar{\gamma}'$ defining the same general curve $\gamma$ is fixed.*

*An identical system of the convergent sequences and the same neighbourhoods of points on this curve $\gamma$ may be constructed with that arbitrary mapping as well as with the "original" mapping $f$.*

*Proof.* While $\bar{\gamma}$ and $\bar{\gamma}'$ are simple curves, a topological mapping between them may be established $g(\bar{\gamma}') = \bar{\gamma}$ in the manner to grant that $\forall X' \in \bar{\gamma}'\ f(g(X')) = f'(X')$.

Therefore, the images of corresponding convergent sequences on the curves $\bar{\gamma}$ and $\bar{\gamma}'$ define the same convergent sequence on the curve $\gamma$.

The images of the corresponding neighbourhoods of the corresponding points on the curves $\bar{\gamma}$ and $\bar{\gamma}'$ defines the same neighbourhood of the point on the general curve $\gamma$. □

**Corollary 1.1.4.** *Local investigation of any general curve is an investigation of the local simple curve.*

## Parameterisation of the General Curve

If we consider a simple curve, in particular an elementary curve, as a general curve, then the concept of convergence of points on it is equivalent to the concept of geometric convergence, and the concept of neighbourhood is equivalent to the concept of geometric neighbourhood, introduced for simple curves.

Since a general curve is the image of a simple curve under a locally one-to-one and continuous mapping, and a simple curve is the image of an open interval or a circumference under a topological mapping, a general curve is the image of an open interval or a circumference under a locally one-to-one and continuous mapping.

Such a mapping can be given analytically employing the equations

$$\begin{cases} x = f_1(t), \\ y = f_2(t), \\ z = f_3(t), \end{cases} \qquad t \in (a,b), \text{ or } t \in [a,b) \tag{1.3}$$

or

$$\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t), \end{cases} \qquad t \in (a,b), \text{ or } t \in [a,b) \tag{1.4}$$

**Definition.** The system of equations (1.3) or (1.4) expresses the equations of the curve in the parametric form.

## Regular Curve

**Definition** (Regular curve ($k$-times continuously differentiable curve))**.** We say that an arbitrary general curve $\gamma$ is the **regular curve** if each of the points of this curve has a neighbourhood which permits a regular parametrization.

I.e. functions $f_1$, $f_2$, $f_3$ in the equations (1.3) are regular ($k$-times continuously differentiable).

**Definition** (Smooth curve)**.** We say that a regular curve with $k = 1$ is **smooth**.

**Definition** (Analytic curve)**.** We say that an arbitrary general curve $\gamma$ is the **analytic curve** if functions $f_1$, $f_2$, $f_3$ in the equations (1.3) are analytic.

**Theorem 1.1.5** (Evidence for regularity)**.** *Suppose we are given a triplet of regular functions*

$$x = x(t), \quad y = y(t), \quad z = z(t), \quad t \in (a,b).$$

*This triplet is a regular parametrization of an arbitrary curve $\gamma$ in form (1.4) if condition*

$$x'^2(t) + y'^2(t) + z'^2(t) > 0, \quad t \in (a, b) \tag{1.5}$$

*is successfully satisfied.*

*The mentioned curve is the image of the open interval $a < t < b$ under a continuous and locally one-to-one mapping which assigns to the point $t$ on the open interval the point in space with coordinates $x(t)$, $y(t)$, $z(t)$.*

*Proof.* The assertion about one-to-oneness is the only statement to be justified.

Suppose this condition is not satisfied. Therefore, $\exists t_0 \in (a, b)$, and, so

$$\forall \delta > 0 \quad \exists t_1, t_2 \in O_\delta(t_0),\ t_1 \neq t_2 :$$

$$x(t_1) - x(t_2) = 0, \quad y(t_1) - y(t_2) = 0, \quad z(t_1) - z(t_2) = 0.$$

By the mean value theorem, we obtain from this that

$$x'(\tau_1) = 0, \quad y'(\tau_2) = 0, \quad z'(\tau_3) = 0$$

with $\tau_1 \in (t_1, t_2)$, $\tau_2 \in (t_1, t_2)$ and $\tau_3 \in (t_1, t_2)$.

Since $t_1$ and $t_2$ are arbitrarily close to $t_0$, continuity of the derivatives $x'(t)$, $y'(t)$, $z'(t)$ yields

$$x'(t_0) = 0, \quad y'(t_0) = 0, \quad z'(t_0) = 0.$$

Therefore

$$x'^2(t_0) + y'^2(t_0) + z'^2(t_)) = 0, \quad t_0 \in (a, b). \tag{1.6}$$

Contradiction! □

## Local Reduction of Variable

Certain *simple curves* may be parametrised in a form

$$\begin{cases} x = t, \\ y = \varphi(t), \\ z = \psi(t), \end{cases} \qquad t \in (a, b) \tag{1.7}$$

for a suitable choice of the $x$, $y$ and $z$ coordinate axes. These equations may be reduced as

$$\begin{cases} y = \varphi(x), \\ z = \psi(x), \end{cases} \qquad x \in (a, b). \tag{1.8}$$

When and how we can perform the same reduction for a general parametrisation (1.3) may be just in a local form?

**Theorem 1.1.6** (The 2D implicit function theorem (recap)). *Suppose $F : \mathbb{R}^2 \to \mathbb{R}$ is a continuously differentiable function. Let $(x_0, y_0)$ be a point satisfying equation $F(x, y) = 0$. If*

$$F_y(x_0, y_0) \neq 0,$$

*then in a neighborhood of the point $(x_0, y_0)$ we can write*

$$y = f(x),$$

*where $f : \mathbb{R} \to \mathbb{R}$ is a real function.*

**Theorem 1.1.7.** *Suppose $\gamma$ is a regular curve and (1.3) is its regular parametrisation in a neighbourhood of the point $(x_0, y_0, z_0)$, corresponding to parameter value $t_0$. Suppose $f_1'(t_0) \neq 0$.*

*In this case, in a sufficiently small neighbourhood of that point, the curve $\gamma$ may be expressed in the form*

$$\begin{cases} y = \varphi(x), \\ z = \psi(x), \end{cases} \qquad x \in O_\delta(t_0). \tag{1.9}$$

*Proof.* The implicit function theorem yields

$$\exists \chi(x), \text{ regular function: } \chi(x_0) = t_0,$$

and

$$\forall x \in O_\delta(t_0),\ x = f_1(\chi(x)).$$

Derivative of this equation yields at the point $x_0$

$$1 = f_1'(t_0)\chi'(x_0),$$

hence $\chi'(x_0) \neq 0$. Thus, $\chi(x)$ is monotonic and for sufficiently small $\delta$ the mapping of $O_\delta(x_0)$ onto $t$ axis with the equation $t = \chi(x)$ is topological.

Hence in the neighborhood $t \in \chi(O_\delta(x_0))$ the curve $\gamma$ has an expression:

$$\begin{cases} y = f_2(\chi(x)), \\ z = f_3(\chi(x)), \end{cases} \qquad x \in O_\delta(x_0). \tag{1.10}$$

□

## Plane Curve. Implicit Representation of a Curve

**Definition** (Plane curve)**.** A curve is said to be a **plane curve** if all of its points lie in a plane.

We shall assume that this plane is the $xOy$ plane. Otherwise we perform the corresponding rotation and translation of the coordinates.

**Definition.** We shall say that a plane curve is defined by the equation

$$\varphi(x, y) = 0 \tag{1.11}$$

expressing by this only the fact that the coordinates of points on The curve satisfies the given equation.

In this connection, there may exist points in the plane which satisfy the given equation but do not belong to the curve.

Thus, defining a curve employing the equation (1.11), in distinction to the parametric definition considered above, is incomplete. But still, some significant questions may be answered just with an analysis of this equation.

**Theorem 1.1.8** (Implicit equation of elementary plane curve)**.** *Suppose $\varphi(x, y)$ is a regular function satisfying equation (1.11) in each point of the arbitrary set $M$.*

*Let $A(x_0, y_0) \in M$, and for this point $A$*

$$\varphi_x^2 + \varphi_y^2 > 0.$$

*Then this point has a neighbourhood $\bar{A}$ such that all the points of the set $M$ belonging to it form a regular elementary curve.*

*Proof.* Suppose for differentness $\varphi_y(x_0, y_0) \neq 0$. The theorem on implicit function yields

$$\exists\, \delta > 0,\, \varepsilon > 0, \text{ and } \psi - \text{regular}$$

$$\forall x \in O_\delta(x_0) : \varphi(x, \psi(x)) = 0,$$

where these points are the only points of the rectangle $O_\delta(x_0) \times O_\varepsilon(y_0)$ satisfying mentioned equation. The elementary curve, about which we are talking in the theorem, is defined employing the equation

$$y = \psi(x), \quad x \in O_\delta(x_0).$$

□

**Theorem 1.1.9** (Implicit equation of elementary curve in general). *Suppose $\varphi(x, y, z)$ and $\psi(x, y, z)$ are regular functions of the arguments $x$, $y$, $z$.*

*Suppose $M$ is the set of points in space, satisfying the equations*

$$\varphi(x, y, z) = 0, \quad \psi(x, y, z),$$

*and $(x_0, y_0, z_0)$ is a point in this set at which the rank of the Jacobi matrix*

$$\begin{pmatrix} \varphi_x & \varphi_y & \varphi_z \\ \psi_x & \psi_y & \psi_z \end{pmatrix}$$

*equals two.*

*Then the point $(x_0, y_0, z_0)$ has a neighbourhood such that all the points of the set $M$ belonging to it form a regular elementary.*

Proof of this theorem is an analogous application of the implicit function theorem in its general form.

## 1.2 Regular and Singular Points

### Regular and Singular Points on Plane Curves

Suppose $\gamma$ is a *regular* plane curve and $P$ is a point on it.

**Definition** (Regular point on a plane curve). $P$ is the **regular point** if the curve permits a regular parameterisation

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \tag{1.12}$$

in a neighbourhood of this point satisfying the condition

$$x'^2 + y'^2 \neq 0 \tag{1.13}$$

at the point $P$.

**Definition** (Singular point on a plane curve). $P$ is the **singular point** if the parameterisation (1.12) with the condition in point $P$ (1.12) *does not exists.*

Hence,

$$x' = y' = 0 \tag{1.14}$$

at a *singular point* for an arbitrary regular parametrisation of a regular curve.

*Example.* Let us investigate a point $t = 0$ of the curve expressed with a parametrisation

$$\begin{cases} x = t^3 \\ y = t^7 \end{cases} \qquad t \in (-1, 1).$$

**Problem**

What is the nature of this point? Is it singular? What kind of parameterisation is it singular for? Demonstrate such parameterisation.

**Solution**

This curve has equal expression as

$$\begin{cases} x = \tau \\ y = \pm|\tau|^{\frac{7}{3}} \end{cases} \qquad \tau \in (-1, 1).$$

This point is regular for any regular parametrisation with the order of

## Condition a Point to be Singular

**Theorem 1.2.1** ( Bürman-Lagrange theorem(recap))**.** *Suppose $z$ is defined as a function of $w$ by an equation of the form*

$$z = f(w)$$

*where function $f$ is analytic at a point $a$ and $f'(a) \neq 0$.*

*Then it is possible to invert or solve the equation for $w$, expressing it in the form $w = g(z)$ given by a power series*

$$g(z) = a + \sum_{n=1}^{\infty} g_n \frac{(z - f(a))^n}{n!},$$

*where*

$$g_n = \lim_{w \to a} \frac{d^{n-1}}{dw^{n-1}} \left[ \left( \frac{w - a}{f(w) - f(a)} \right)^n \right].$$

The theorem further states that this series has a non-zero radius of convergence, i.e., $g(z)$ represents an analytic function of $z$ in a neighbourhood of $z = f(a)$. This is also called *reversion of series.*

Suppose $f(s) = \sum\limits_{n=1}^{\infty} f_n s^n$. The equation

$$T(z) = zf(T(z))$$

has the only valid solution

$$T(z) = \sum_{n=1}^{\infty} t_n z^n, \quad t_n = \frac{1}{n}[s^{n-1}](f(s))^n$$

**Lemma 1.2.2.** *Suppose $\gamma$ is an analytic curve and that $O$ is a point on $\gamma$. Then, with a suitable choice of coordinate axes, the curve may be parametrised so that its equations will have the form*

$$\begin{cases} x = a_1 t^{n_1}, \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots, \quad n_1 \leq m_1 \end{cases}$$

*in a neighbourhood of the point $O$.*

*Proof.* Let us apply $O$ as the origin and

$$\begin{cases} x = x(s) \\ y = y(s) \end{cases}$$

as an arbitrary analytic parametrisation with a property

$$x(0) = y(0) = 0.$$

We are doing so without any loss of generalisation as a simple change of coordinates will transform our parametrisation to this pretty form.

Suppose $n_1$ and $m_1$ are the orders of the first non-zero derivatives of $x$ and $y$. Again, without any loss of generalisation, we expect $n_1 \leq m_1$, if $n_1 > m_1$, we exchange the roles of $x$ and $y$.

$$\begin{cases} x = \bar{a}_1 s^{n_1} + \bar{a}_2 s^{n_2} + \cdots \\ y = \bar{b}_1 s^{m_1} + \bar{b}_2 s^{m_2} + \cdots \end{cases}$$

Let us introduce new parameter $t$ derived with the power expansion of the $x(s)$:

$$t = s \left( \frac{\bar{a}_1 s^{n_1} + \bar{a}_2 s^{n_2} + \cdots}{\bar{a}_1 s^{n_1}} \right)^{\frac{1}{n_1}}.$$

For this change of parameter, the parametrisation of an arbitrary neighbourhood of the point $O$ will have the form

$$\begin{cases} x = a_1 t^{n_1} \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots. \end{cases} \tag{1.15}$$

□

**Theorem 1.2.3.** *Suppose an arbitrary analytic curve $\gamma$ is defined by the parameterisation*

$$\begin{cases} x = a_1 t^{n_1}, \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots, \quad n_1 \le m_1 \end{cases} \tag{1.16}$$

*in a neighbourhood of the point $O$.*

*The necessary and sufficient condition that $O$ is a singular point on the curve $\gamma$ is that at least one of the $m_k$ is not divisible by $n_1$.*

*Proof. Necessity.*

First, neither $n_1$, nor $m_k$ $\forall k$ is not even.

If there is even power, then

$$x(t) = x(-t), \quad y(t) = y(-t)$$

for an arbitrary small $t$. Let all $m_k$ be multiples of the *odd* power $n_1$.

Let us change the variable:

$$s = t^{n_1}.$$

Now the curve parametrization (1.16) in a neighborhood of the point $O$ assumes the form:

$$\begin{cases} x = a_1 s, \\ y = b_1 s^{k_1} + b_2 s^{k_2} + \cdots . \end{cases}$$

Obviously, the point $O$ corresponding to the value $s = 0$ of the parameter is a regular point on the curve. □

*Proof. Sufficiency.*

Suppose $\exists k : m_k$ is not divisible by $n_1$, and $O$ is a regular point. Let

$$\begin{cases} x = f_1(\sigma) \\ y = f_2(\sigma), \end{cases}$$

be its parametrisation in a neighbourhood of the point $O$. Parameter $\sigma = \sigma_0$ corresponds with the point $O$:

$$[f_1'(\sigma_0)]^2 + [f_2'(\sigma_0)]^2 \ne 0.$$

$$\frac{f_2(\sigma)}{f_1(\sigma)} = \frac{y(t)}{x(t)},$$

and

$$\frac{y(t)}{x(t)} \xrightarrow[t\to 0]{} \frac{f_2'(\sigma_0)}{f_1'(\sigma_0)},$$

and is finite. Hence $f_1' \neq 0$ in $O$.

Previously, we justified that in this case,e in a neighbourhood of the point $O$ exists an implicit expression of the curve as

$$y = \varphi(x),$$

or

$$x = \psi(y).$$

$\varphi(x)$ and $\psi(y)$ are analytic functions here. With an expression of the analytic function with a power series, we yield

$$y = \varphi(x) = c_1 x + c_2 x^2 + \cdots$$

in a neighbourhood of the point $O$. Substituting $x = x(t)$, $y = y(t)$ we obtain

$$b_1 t^{m_1} + b_2 t^{m_2 + \cdots} = c_1 a_1 t^{n_1} + c_2 a_1^2 t^{2n_1} + \cdots ,$$

hence

$$\forall k : \quad m_k = k n_1.$$

Contradiction! □

**Theorem 1.2.4** (Condition a point to be singular). *Suppose an arbitrary analytic curve $\gamma$ is defined by the parametrisation*

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \tag{1.17}$$

*in a neighbourhood of the point $O$. $x(t)$ and $y(t)$ are arbitrary analytic functions here.*

*Suppose the first nonzero derivatives of the functions $x(t)$ and $y(t)$ have orders $n_1$ and $m_1$ respectively, where $n_1 < m_1$.*

*Then the point $O$ will be a singular point if $m_1$ is not divisible by $n_1$.*

We have seen that the answer to the question of whether a point on a curve is a singular point or a regular point, is equivalent to investigate some special parametrisation of the curve.

To obtain this parameterisation, it is sufficient to know how to expand the functions $x(t)$ and $y(t)$ of an arbitrary analytic definition of the curve in a power series of analytic functions

$$s = t\left(\frac{x(t)}{x^{n_1}(0)t^{n_1}}\right)^{1/n_1}.$$

The Bürman-Lagrange theorem, known from the theory of analytic functions, asserts that these expansions can be found.

Let us recap the example from the previous paragraph:

$$\begin{cases} x = t^3 \\ y = t^7 \end{cases} \qquad t \in (-1, 1).$$

We demonstrated that point $t = 0$ is regular concerning any regular parametrisation with order of regularity $k = 2$. But if we take into account analytic parameterisation, we will see that this point is *singular*!

## Turning Point

Suppose $O$ is a singular point on a curve $\gamma$. $n_1$ and $m_1$ are powers in parametrization

$$\begin{cases} x = a_1 t^{n_1}, \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots, \quad n_1 \leq m_1 \end{cases}$$

**Definition** (Turning point of the first kind)**.** If $m_1$ is odd and is not divisible by $n_1$, which is even, we say that $O$ is **the turning point of the first kind**.

Example of such a point for the curve

$$\begin{cases} x = t^2 \\ y = t^3 + t^4 + t^5 \end{cases}$$

is shown on the figure 1.4

**Definition** (Turning point of the second kind)**.** If $m_1$ and $n_1$ are even, we say that $O$ is **the turning point of the second kind**.

Example of such a point for the curve

$$\begin{cases} x = t^2 \\ y = t^4 + t^5 \end{cases}$$

is shown on the figure 1.5

In both examples, the axis $Ox$ is a tangent line of the curve. This important feature will be discussed a bit later.

Both definitions yield a corollary from the theorem 1.2.4:

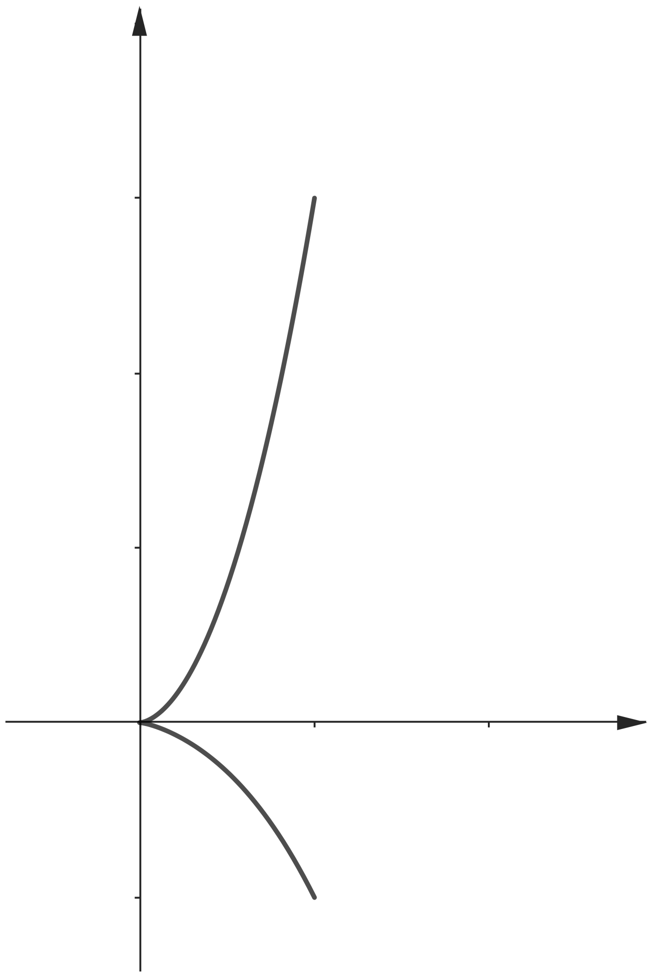

Figure 1.4. Turning point of the first kind

**Corollary 1.2.5.** *Suppose an arbitrary analytic curve $\gamma$ is defined by the parametrisation*

$$\begin{cases} x = x(t), \\ y = y(t) \end{cases} \tag{1.18}$$

*in a neighbourhood of the singular point $O$. $x(t)$ and $y(t)$ are arbitrary analytic functions here.*

*Suppose the first nonzero derivatives of the functions $x(t)$ and $y(t)$ have orders $n_1$ and $m_1$ respectively, where $n_1 < m_1$.*

*The point will be a turning point of the first kind if $n_1$ is even and $m_1$ is odd, and a turning point of the second kind if both $n_1$ and $m_2$ are even.*

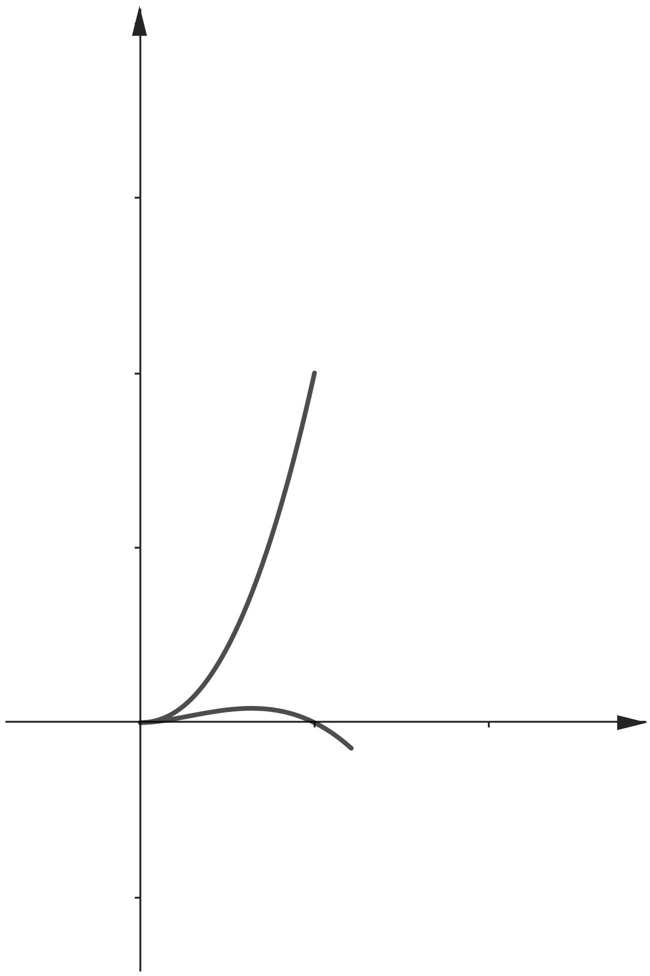

Figure 1.5. Turning point of the second kind

## Singular Points for the Analytic Curves Expressible by the Implicit Equations

Suppose a plane analytic curve $\gamma$ is defined employing the equation

$$\varphi(x, y) = 0,$$

$\varphi(x, y)$ is an analytic function of $x$ and $y$.

We justified before that point $O(x_0, y_0)$ of the curve $\gamma$ is regular if

$$\varphi_x^2 + \varphi_y^2 \neq 0$$

at this point.

Therefore, the only points on the curve which *can be singular* ones are the points satisfying the condition

$$\varphi_x = \varphi_y = 0.$$

Without any loss of generality, let us assume that point $O$ is the origin. In a neighbourhood of this point, the curve permits parametrisation

$$\begin{cases} x = a_1 t^{n_1} \\ y = b_1 t^{m_1} + b_2 t^{m_2} + \cdots, \quad n_1 \leq m_1. \end{cases} \tag{1.19}$$

In order to determine whether the point $O$ is a singular point of the curve and to explain the nature of the singularity at this point, it is sufficient to know powers $n_1$, $m_k$ $\forall k$.

The identity

$$\varphi(x(t), y(t)) = 0 \tag{1.20}$$

is to be used to determine these numbers.

A few remarks must be made here.

- The numbers $n_1$, $m_k$ are not defined uniquely by the above identity:
  1. Change of the variable $t = s^p$ does not change the character of the parametrisation
  2. In the general case, several analytic curves which are *geometrically different* in large, even in an arbitrarily small neighbourhood of the point $O$, will satisfy the equation $\varphi(x, y) = 0$.
- Hence, the character of the singularity of the point $O$ on various curves will be distinct.

The investigation of the singular point $O$ for a curve, defined by the equation $\varphi(x, y) = 0$, must be understood in the sense of investigating the nature of the singularity of the point $O$ concerning every analytic curve, defined in a neighbourhood of the point by means of the equation $\varphi(x, y) = 0$.

## Singular Point of the Analytic Curve having non-Degenerate Quadratic Form in Power Series of the Parametrisation Expansion

We demand function $\varphi(x, y)$ may be expressed as

$$\varphi(x, y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2 + \cdots .$$

With a non-degenerate quadratic form in the mentioned expansion.

Three cases are distinguished here

1. $a_{20}a_{02} - a_{11}^2 > 0$;

2. $a_{20}a_{02} - a_{11}^2 < 0$;

3. $a_{20}a_{02} - a_{11}^2 = 0$.

With rotation of the axes, we can attain the case where the term containing $xy$ will be absent in the expansion of the function $\varphi(x, y)$ in a power series.

Let us substitute $x(t)$ and $y(t)$ into this expansion to obtain an identity in $t$:

$$a_{20}a_1^2t^{2n_1} + a_{02}b_1^2t^{2m_1} + \cdots = 0.$$

Suppose $n_1 < m_1$, hence there is only one term $a_{20}a_1^2$ corresponding with the lowest power of $t$, which is $2n_1$. This yields $a_{20} = 0$, which is impossible in both the first and second cases.

Hence, in this two cases we must assume $n_1 = m_1$ and therefore terms $a_{20}a_1^2t^{2n_1}$ and $a_{02}b_1^2t^{2m_1}$ have the lowest degree.

Let us take a look on the *first case*. The conditions

$$a_{20}a_{02} > 0$$

and

$$a_{20}a_1^2 + a_{02}b_1^2 = 0.$$

must be satisfied simultaneously, which is obviously impossible.

Therefore, there is no analytic curve with equation $\varphi(x, y) = 0$ containing point $O$.

It turns out, in this case, that in a sufficiently small neighbourhood of the point $O$ no points exist which are different from $O$ and satisfy the equation $\varphi(x, y) = 0$.

**Definition** (Isolated singular point)**.** Such a point $O$ is called the **isolated singular point**.

For the second case, we have two independent systems of solutions for $a_1$ and $b_1$ with an accuracy up to within an unessential factor

$$\begin{cases} a_1 = \sqrt{|a_{02}|} \\ b_1 = \sqrt{|a_{20}|}, \end{cases} \qquad \begin{cases} a_1 = \sqrt{|a_{02}|} \\ b_1 = -\sqrt{|a_{20}|}, \end{cases}$$

Now, if we begin with any system of values for $a_1$ and $b_1$, and $n_1 = m_1$, then the powers $m_k$, $k > 1$ and the coefficients $b_k$ are already uniquely determined by the equation $\varphi(x(t), y(t)) = 0$. It remains to check if these $m_k$ are multiples of $n_1 = m_1$. If it is so, there are two analytic curves, geometrically distinct in an arbitrarily small neighbourhood of the point 0 and satisfying the implicit equation of our curve. The point $O$ is a regular point for these curves, investigated separately.

**Definition** (Nodal point)**.** The point is still considered a singular point in the case under consideration, and it is called **a nodal point**.

Let us justify the statement about these two curves.

Let us express the function $\varphi$ in the form

$$\varphi(x, y) = Ax^2 + 2Bxy + Cy^2$$

where $A$, $B$ and $C$ are analytic functions of $x$ and $y$ having values $a_{20}$, 0 and $a_{02}$ in the point $O$ and therefore satisfying the condition

$$AC - B^2 < 0 \tag{1.21}$$

in a neighbourhood of point $O$.

Hence, in a small neighbourhood of point $O$

$$\varphi(x, y) = C(y - x\xi_1(x, y))(y - x\xi_2(x, y))$$

where $\xi_1$ and $\xi_2$ are roots of the equation

$$A + 2B\xi + C\xi^2 = 0.$$

Therefore, the locus of the equation $\varphi(x, y)$ in a neighbourhood of point $O$ consists of two analytic curves $y - x\xi_j(x, y) = 0$, $j = 1, 2$. For both equations

$$\frac{\partial}{\partial x}(y - x\xi_j(x, y))\bigg|_O = -\xi_i(0, 0) \neq 0.$$

Hence, $O$ is regular for both curves. $\square$

Finally, let us observe case 3.

Now we assume that $a_{20} = 0$ since $a_{20}a_{02} = 0$. Now we have expansion

$$\varphi(x, y) = a_{02}y^2 + a_{30}x^3 + \cdots,$$

and will assume $a_{30} \neq 0$.

This corresponds to the general case to the fact that the forms $\varphi_2 = a_{20}x^2 + 2a_{11}xy + a_{02}y^2$ and $\varphi_3 = a_{30}x^3 + \cdots + a_{03}y^3$ do not have common divisors.

Substituting $x(t)$ and $y(t)$ for $x$ and $y$ in the expansion of the function $\varphi(x, y)$, we note that the terms with the lowest powers of $t$ are now $a_{02}b_1^2t^{2m_1}$ and $a_{30}a_1^3t^{3n_1}$, hence $2m_1 = 3n_1$.

I.e. $m_1$ is not divisible by $n_1$. Consequently, the point is a singular point of the curve.

It turns out that if both $m_1$ and $n_1$ are assumed to be even, then all the $m_k$ turn out to be even, since they can be expressed linearly and homogeneously in terms of $m_1$ and $n_1$. But, as was noted above, $n_1$ and all the $m_k$ cannot be even. Therefore, only $n_1$ is even. This means that the singular point is a **turning point of the first kind**.

### Examples

#### Problem 1

Investigate properties of the point $(0,0)$ of the curve given with equation

$$x^3 + xy^2 - x^2 - y^2 = 0$$

#### Solution

First, $a_{20} = a_{02} = -1$, and we deal with the case one ($a_{20}a_{02} > 0$).
Second, let us factorise the equation:

$$x(x^2 + y^2) - (x^2 + y^2) = 0,$$

finally

$$(x^2 + y^2)(x - 1) = 0.$$

*Real* locus of this figure is the straight line $x = 1$ and the point $(0,0)$, which is an isolated singular point as it was justified above. See figure 1.6.

□

#### Problem 2

Sketch and investigate the locus of the equation (Lemniscate[1] of Bernoulli)

$$(x^2 + y^2)^2 - 2a^2(x^2 - y^2) = 0.$$

#### Solution

Lemniscate of Bernoulli is a well-known classic plane curve which is geometrically defined from two given points $F_1$ and $F_2$, known as *foci*, at a distance $2a$ from each other as the locus of points $P$ so that $PF_1 \cdot PF_2 = a^2$.

[1] Its name is from *lemniscatus*, which is Latin for "decorated with hanging ribbons".

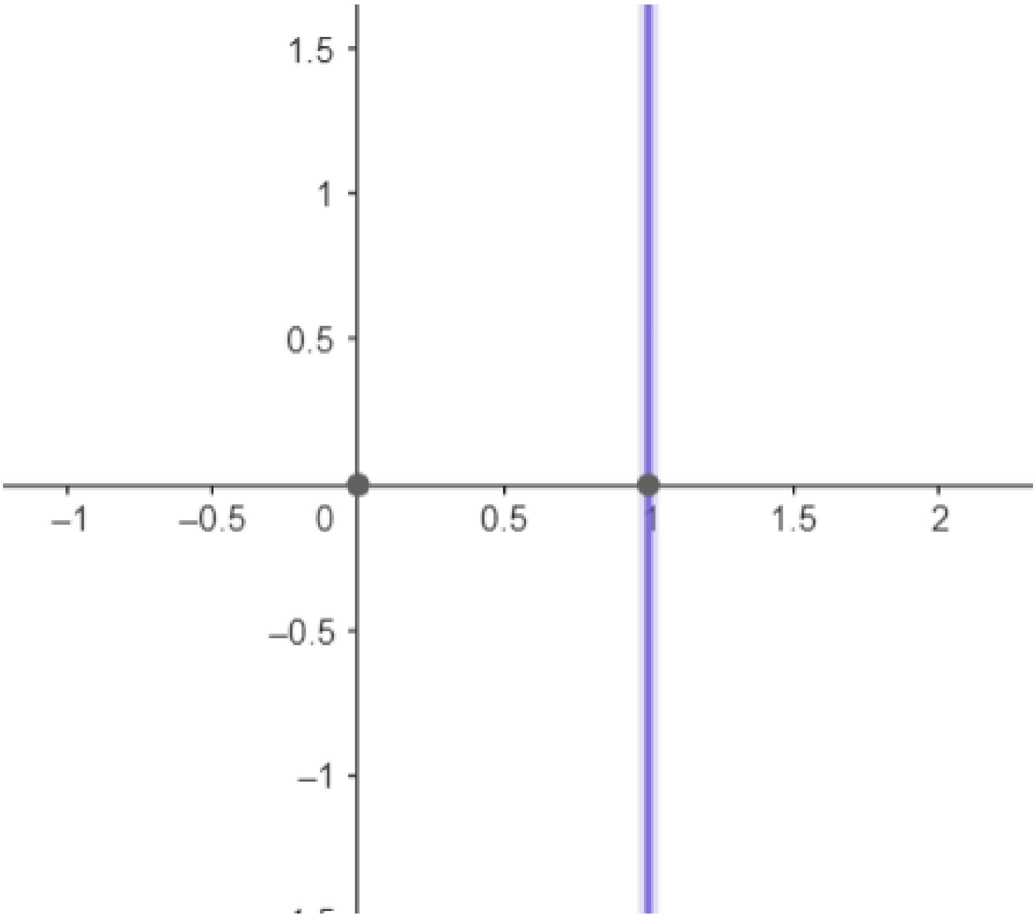

Figure 1.6. Locus of the equation $x^3 + xy^2 - x^2 - y^2 = 0$

To match this definition with the given equation, let us expand the expression in the left part of the equation and add the zero term $a^4 - a^4$ to it:

$$x^4 - 2a^2x^2 + a^4 + 2x^2y^2 + 2a^2y^2 + y^4 - a^4 = 0$$

After regrouping, we yield

$$(x^2 - a^2)^2 + y^4 + 2y^2(x^2 + a^2) = a^4,$$

or

$$(x - a)^2(x + a)^2 + y^4 + y^2(x^2 + a^2 - 2ax + x^2 + a^2 + 2ax) = a^4,$$

and finally

$$(x - a)^2(x + a)^2 + y^4 + y^2((x + a)^2 + (x - a)^2) = a^4,$$

with substitution $(x - a)^2 = A$, $(x - b)^2 = B$, and $y^2 = C$ we derive

$$AB + AC + BC + C^2 = (A + C)(B + C),$$

hence

$$((x-a)^2+y^2)((x+a)^2+y^2)=a^4.$$

The final equation yields a geometrical definition of the lemniscate of Bernoulli mentioned above.

With an ordinary change of coordinates

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

we derive the polar equation of this curve:

$$r = 2a^2\cos 2\varphi$$

This definition gives us a few nice approaches to construct the lemniscate. Here is one of them.

Let us place the foci into the points $F_1$ and $F_2$ and place point $O$ at the centre of the segment $F_1F_2$. We establish a circumference of the radius $\dfrac{a}{\sqrt{2}}$ around $F_1$ or $F_2$ and start sketching all possible secants $OPS$. The length of the segment $PS$ corresponds with the value of $r$ corresponding with angle $\varphi$.

Finally, let us observe a quadratic form of the equation, which is

$$-2a^2x^2+2a^2y^2$$

and corresponds with the second case in a neighbourhood of the point $O$.

Sketch on the figure 1.7 demonstrates the concept of sketching the lemniscate and nodal nature of the point $O$.

□

**Problem 3**

Investigate the $(0,0)$ point for the semi-cubic parabola

$$y^2-x^3=0$$

**Solution**

For this line quadratic form is just $y^2$, hence $a_{20}=0$, $a_{02}=1$, and we deal with case three.

$(0,0)$ is a turning point of the first kind. Figure 1.8 justifies this reasoning.

□

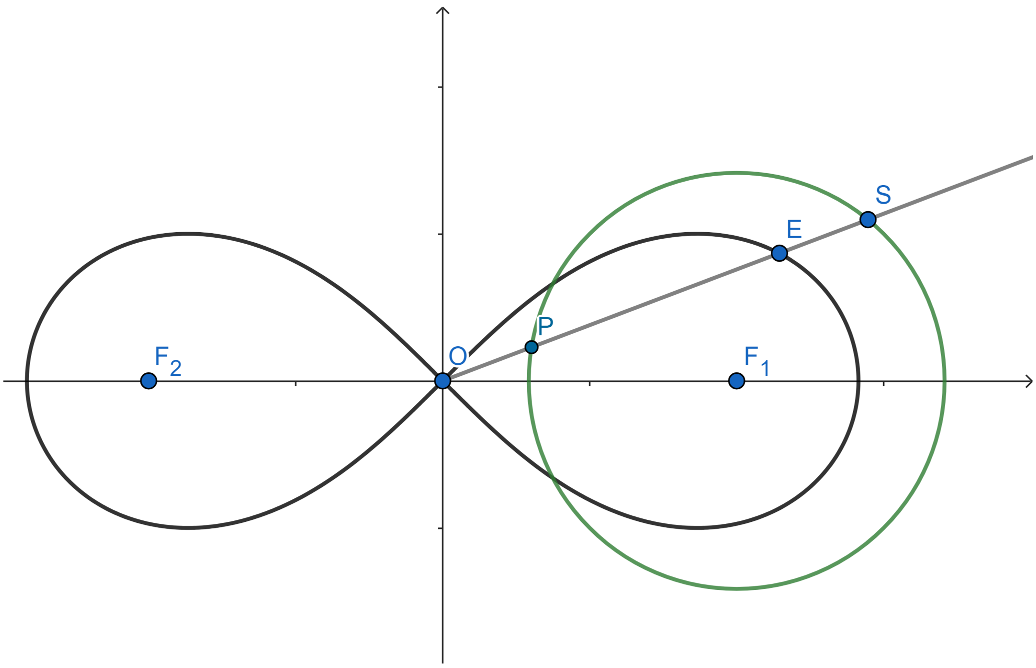


Figure 1.7. The lemniscate of Bernoulli

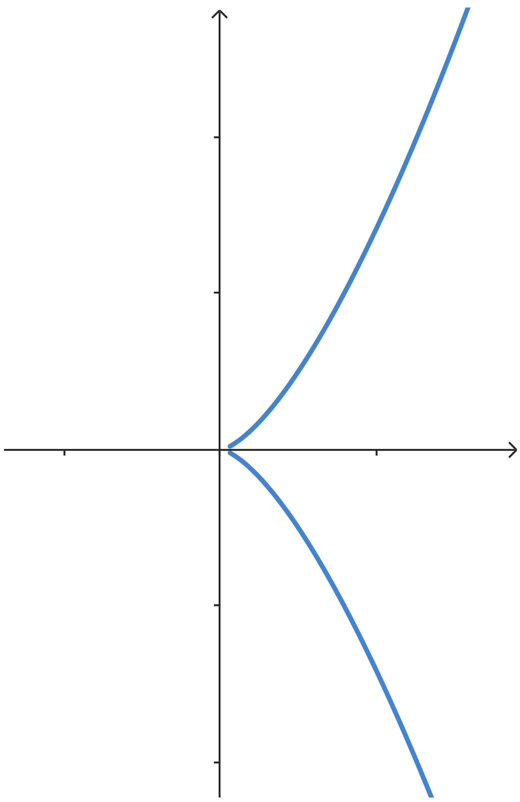

Figure 1.8. The semi-cubic parabola

## 1.3 The Asymptote

### Definition

Suppose $\gamma$ is a non-closed curve and

$$\begin{cases} x = x(t) \\ y = y(t), \end{cases} \qquad t \in (a, b)$$

is its parametrisation.

**Definition.** We say that **a curve tends to infinity from one side** if

$$x^2(t) + y^2(t) \to \infty$$

as

$$t \to a \textbf{ or } t \to b$$

**Definition.** We say that **a curve tends to infinity from both sides** if

$$x^2(t) + y^2(t) \to \infty$$

as

$$t \to a \textbf{ and } t \to b$$

The property of a curve to tend to infinity does not depend on its parametrisation.

Suppose the curve $\gamma$ tends to infinity, for example, as $t \to a$.

**Definition** (The Asymptote)**.** The straight line $g$ is called an **asymptote** to the curve $\gamma$ if the length $d(t)$ from a point on the curve $\gamma$ to the straight line $g$ tends to zero when $t \to a$.

### Explicit Case

**Theorem 1.3.1** (The necessary and sufficient that an asymptote exists)**.** *A necessary and sufficient condition for the curve $\gamma$ defined by the equations*

$$\begin{cases} x = x(t) \\ y = y(t), \end{cases} \qquad t \in (a, b)$$

*and tending to infinity as $t \to a$, have an asymptote is that*

1. *At least one of the ratios*

$$\frac{y(t)}{x(t)}, \quad \frac{x(t)}{y(t)}$$

*tends to a final limit as $t \to a$. For definiteness we assume* $\frac{y(t)}{x(t)} \to k$.

2. *The expression*

$$y(t) - kx(t)$$

*also tends to a definite limit, say $l$, as $t \to a$, provide the first condition is satisfied*

*If the mentioned conditions are satisfied, the equation of the asymptote is*

$$y - kx - l = 0$$

*Proof. Necessity.*

Suppose $g$ with equation $y - kx - l = 0$ is an asymptote of the $\gamma$.

The length from each point on a curve to that straight line may be expressed as

$$y(t) - kx(t) - l = Ad(t),$$

$A$ here as an arbitrary constant.

The condition that the $g$ is an asymptote is

$$y(t) - kx(t) - l \xrightarrow[t \to a]{} 0$$

It must be underlined that $x(t) \xrightarrow[t \to a]{} \infty$ as in opposite case the condition $x^2(t) + y^2(t) \to \infty$ yields that $y(t) - kx(t) - l$ is not limited. So we yield

$$\frac{y(t)}{x(t)} \to k$$

and

$$y(t) - kx(t) \xrightarrow[t \to a]{} l$$

□

*Proof. Sufficiency.* Conditions

$$y(t) - kx(t) \xrightarrow[t \to a]{} l \text{ and } \frac{y(t)}{x(t)} \xrightarrow[t \to a]{} k$$

yield us that

$$y(t) - kx(t) - l \xrightarrow[t \to a]{} 0$$

Hence, the length from the points on the curve and the mentioned straight line tends to zero as $t \to a$. □

The theorem provided permits an alternative statement.

**Theorem 1.3.2.** *A necessary and sufficient condition for the curve $\gamma$ defined by the equations*

$$\begin{cases} x = x(t) \\ y = y(t), \end{cases} \qquad t \in (a, b)$$

*and tending to infinity as $t \to a$, have an asymptote is that*

1. *Each of the two ratios*
$$\frac{x(t)}{\rho(t)}, \quad \frac{y(t)}{\rho(t)}$$
*where*
$$\rho(t) = \sqrt{x^2(t) + y^2(t)}$$
*tend to a limit.*

   *Suppose that*
$$\frac{x(t)}{\rho(t)} \to \alpha, \quad \frac{y(t)}{\rho(t)} \to \beta$$

2. *The expression*
$$-\beta x(t) + \alpha y(t)$$
*also tends to a definite limit as $t \to a$, provide the first condition is satisfied. If this limit is denoted by $p$, then the equation of the asymptote will be*
$$-\beta x + \alpha y - p = 0.$$

*Example.* Suppose the curve $\gamma$ is defined by the equation

$$y = \varphi(x), \quad x \in (a, b).$$

Which is the same thing as

$$\begin{cases} x = t \\ y = \varphi(t), \end{cases} \qquad t \in (a, b)$$

Suppose $\varphi(t) \xrightarrow[t \to a]{} \infty$.
With $t \to a$,

$$\frac{x(t)}{y(t)} = \frac{t}{\varphi(t)} \to k = 0, \quad x(t) - ky(t) = t - 0 \cdot \varphi(t) \to a,$$

Therefore, the curve has the asymptote expressed with the equation

$$x - a = 0.$$

Let us employ the alternative statement.

With $t \to a$,

$$\frac{t}{\sqrt{t^2 + \varphi^2(t)}} \to 0, \quad \frac{\varphi(t)}{\sqrt{t^2 + \varphi^2(t)}} \to 1, \quad -t + 0 \cdot \varphi(t) \to -a.$$

Therefore, he curve has the asymptote with the equation

$$x - a = 0.$$

Hence, both statements yield the same asymptote.

## Implicit Case

We now consider the problem of asymptotes to a curve defined by means of an equation in the implicit form $\varphi(x, y) = 0$.

As noted, the equation $\varphi(x, y) = 0$ defines a curve only in the sense that points on the curve satisfy the equation $\varphi(x, y) = 0$, but, generally speaking, these do not exhaust all points in the plane which have this property.

The problem of finding the asymptotes to a curve, defined employing the equation $\varphi(x, y) = 0$, is not completely defined. It turns out to be possible to only point out a set of lines which contain the asymptotes among them.

We shall restrict ourselves to the case of **algebraic curves** (i.e. the case where $\varphi(x, y) = 0$ is a polynomial in the variables $x$ and $y$).

### Parametric Form of Asymptotes

An asymptote can be expressed as:

$$\begin{cases} x = \bar{x} + \lambda u, \\ y = \bar{y} + \mu u, \end{cases}$$

where:

- $(\lambda, \mu)$ is the **direction vector** of the asymptote,
- $(\bar{x}, \bar{y})$ is a **reference point** (not necessarily on the curve),

- $u \to \infty$ parametrizes motion along the line.

A point $Q(u)$ on the curve approaching this asymptote satisfies:

$$x(u) = \bar{x} + \lambda u + \xi(u), \quad \xi(u) \xrightarrow{u\to\infty} 0,$$
$$y(u) = \bar{y} + \mu u + \eta(u), \quad \eta(u) \xrightarrow{u\to\infty} 0.$$

**Key Conditions for Asymptotes**

1. **Homogeneous Expansion of $\varphi$**
Decompose $\varphi$ into homogeneous components:

$$\varphi(x, y) = \varphi_n(x, y) + \varphi_{n-1}(x, y) + \cdots + \varphi_0(x, y),$$

where $\varphi_k$ contains all degree-$k$ terms.
2. **Substitute $(x(u), y(u))$ into $\varphi$**
Using homogeneity ($\varphi_k(\lambda u, \mu u) = u^k \varphi_k(\lambda, \mu)$):

$$\varphi(x(u), y(u)) = u^n \varphi_n(\lambda, \mu) + u^{n-1}\left(\varphi_{n-1}(\lambda, \mu) + \bar{x}\frac{\partial \varphi_n}{\partial \lambda} + \bar{y}\frac{\partial \varphi_n}{\partial \mu}\right) + \cdots .$$

3. **Dominant-Term Conditions**
For $\varphi(x(u), y(u)) = 0$ as $u \to \infty$:

- **Leading order ($u^n$ term)**:

$$\varphi_n(\lambda, \mu) = 0 \quad \text{(determines possible directions } (\lambda, \mu)).$$

- **Next order ($u^{n-1}$ term)**:

$$\varphi_{n-1}(\lambda, \mu) + \bar{x}\frac{\partial \varphi_n}{\partial \lambda} + \bar{y}\frac{\partial \varphi_n}{\partial \mu} = 0 \quad \text{(fixes } (\bar{x}, \bar{y})).$$

**Equation of the Asymptote**

The asymptote is the line:

$$\frac{x - \bar{x}}{\lambda} = \frac{y - \bar{y}}{\mu},$$

or equivalently (eliminating $\bar{x}, \bar{y}$ via the second condition):

$$x\frac{\partial \varphi_n}{\partial \lambda} + y\frac{\partial \varphi_n}{\partial \mu} + \varphi_{n-1}(\lambda, \mu) = 0.$$

## 1.4 Problems Corner

**Problem 1**

Let $O$ be a point on a circle of radius $a$, and a ray intersecting the circle at a varying point $A$ rotates around $O$. Mark off two line segments $AM = AN$ of the length $2a$ on this ray on two sides of $A$. Deduce the equation of a curve traced by the points $M$ and $N$.

**Solution**

Solution:

First, we introduce the coordinate system to place the circle centred at $C(a, 0)$, hence its equation:

$$(x-a)^2 + y^2 = a^2,$$

and $O$ is the origin. The ray from $O$ at angle $\theta$ has parametric equations:

$$\begin{cases} x = t\cos\theta, \\ y = tsin\theta. \end{cases}$$

Substituting them into the circle's equation yields:

$$(t\cos\theta - a)^2 + (t\sin\theta)^2 = a^2.$$

Simplifying:

$$t^2 - 2at\cos\theta = 0.$$

Solutions of this equation are

$$t = 0,$$

point $O$ and

$$t = 2a\cos\theta,$$

point $A$

Thus, $A$ has coordinates $(2a\cos 2\theta, 2a\cos\theta\sin\theta)$:

Let us locate $M$ and $N$ by displacing $A$ by $2a$ on the ray. So we add $\pm 2a\cos\theta$ to $x$ coordinate and $\pm 2a\sin\theta$ to $y$ coordinate.

Coordinates of $M$: $(2a\cos\theta(\cos\theta + 1), 2a\sin\theta(\cos\theta + 1))$

Coordinates of $N$: $(2a\cos\theta(cos\theta - 1), 2a\sin\theta(cos\theta - 1))$

This yields us polar equations of the curves. For $M$:

$$r = 2a(1 + \cos\theta)$$

for $N$:

$$r = 2a(1 + \cos\theta)$$

This is a cardioid and a reflected cardioid

These are standard equations of cardioids, where $M$ traces a cardioid opening to the right, and $N$ traces a cardioid opening to the left.

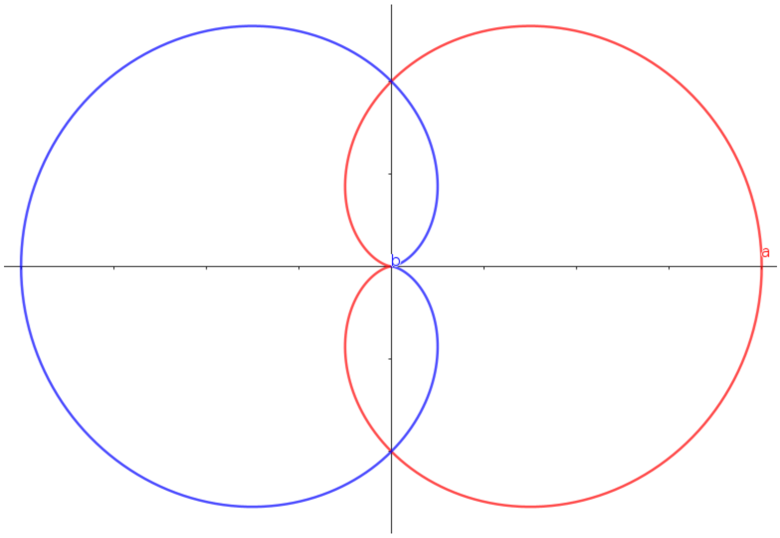


Figure 1.9. Cardioid (red) and reflected cardioid (blue).

**Problem 2**

Justify that astroid

$$x^{2/3} + y^{2/3} = a^{2/3}$$

is an analytic curve. Isolate its singular points. Discuss the types of these points.

**Solution**

First, we must justify that the curve adapts to the analytic parameterisation. Let us derive the equation for $y$:

$$y = \pm(a^{2/3} - x^{2/3})^{3/2}.$$

This expression is not analytic at points where $x^{2/3} = a^{2/3}$ (i.e., at the cusps), but elsewhere, we can find analytic branches.

It must also be noted that there is a parameterisation of the strid with analytic functions:

$$\begin{cases} x = a\cos^3\theta, \\ y = a\sin^3\theta. \end{cases}$$

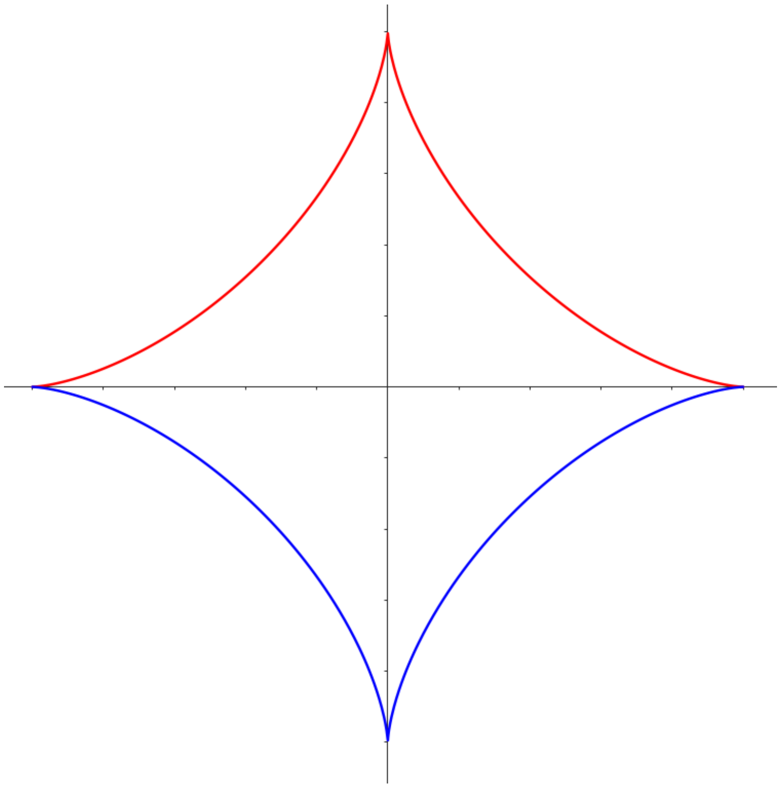

Figure 1.10. The sketch of astroid colours is assigned for the branches of the equation for $y$

Let us look for the singular points. As we have no candidates for the singular point, we must perform the general investigation. We find the derivatives by $\theta$ first:

$$\begin{cases} x' = -3a\cos^2\theta\sin\theta \\ y' = 3a\sin^2\theta\cos\theta \end{cases}$$

Now we apply the general definition of the singularity to these formulas:

$$\begin{cases} \cos^2\theta\sin\theta = 0 \\ \sin^2\theta\cos\theta = 0 \end{cases}$$

This yields $\theta = \dfrac{k\pi}{2}$ as the parameter corresponding to the singular points. Hence these points are $(0, a)$, $(0, -a)$, $(a, 0)$, $(-a, 0)$.

Let us investigate the nature of point $theta = 0$. Application of the second derivative at the parameterisation yields

$$\begin{cases} x'' = 3a(2\cos\theta\sin^2\theta - \cos^3\theta), \\ y'' = 3a(2\cos^2\theta\sin\theta - \sin^3\theta), \end{cases}$$

application of the third derivative yields

$$\begin{cases} x'' = 3a(7\cos^2\theta\sin\theta - 2\sin^3\theta), \\ y'' = -3a(7\cos\theta\sin^2\theta - 2\cos^3\theta). \end{cases}$$

With putting the $\theta = \dfrac{k\pi}{2}$ onto these formulate we yield $n_1 = 2$, and $m_1 = 3$, or $n_1 = 3$, and $m_1 = 2$. In the second case, we just reverse the order of variables as it was explained before. *Conclusion*: all investigated singular points are the turning points of the first kind.

**Problem 3**

Write the equation of the asymptote for the curve

$$\begin{cases} x = t \\ y = \dfrac{1}{1-t} \end{cases} \qquad t \in (-1, 1)$$

**Solution**

The presented curve is a branch of a hyperbola, and it tends to infinity with $t \to 1$.

In this case

$$\frac{x(t)}{y(t)} \to 0,$$

$$x(t) - 0 \cdot y(t) \to 1,$$

hence, the equation of the asymptote is

$$x - 1 = 0.$$

**Problem 4**

Write the equation of the asymptote for the tractrix

$$\begin{cases} x = a\sin t \\ y = a(\cos t + \ln\tan\frac{t}{2}) \end{cases}$$

**Solution**

*Remark.* The tractrix is a curve with the property that the segment of the tangent between the point of contact and some fixed line (actually, the asymptote) is constant.

First, term $\tan\frac{t}{2}$ yelds us a valid interval for $t$ $(0, \pi)$. These two points must be checked.

For $t \to 0^+$:

$$\frac{a\sin t}{a(\cos t + \ln\tan\frac{t}{2})} \to 0;$$

$$a\sin t \to 0;$$

Asymptote is

$$x = 0$$

For $t \to \pi^-$:

$$\frac{a\sin t}{a(\cos t + \ln\tan\frac{t}{2})} \to 0;$$

$$a\sin t \to 0;$$

Asymptote is

$$x = 0$$

So, the asymptote is $x = 0$.

**Problem 5**

Write the equation of the asymptote for the hyperbola:

$$x^2 - 3xy + 2y^2 + x + 1 = 0$$

**Solution**

By substituting the equation of the asymptote

$$\begin{cases} x = \bar{x} + \lambda u, \\ y = \bar{y} + \mu u \end{cases}$$

onto the equation of a hyperbola, we yield the major term:

$$\varphi_2(\lambda, \mu) = \lambda^2 - 2\lambda\mu + 2\mu^2$$

With demanding this $\varphi_2(\lambda, \mu) = 0$ we yield two independent systems of values masterd with $\lambda = 1$, $\mu = 1$, and $\lambda = 2$, $\mu = 1$.

Substituting the values we yield the lines:

$$-\bar{x} + \bar{y} + 1 = 0$$

and

$$\bar{x} - 2\bar{y} + 2 = 0$$

**Problem 6**

Write the equation of the asymptote for the folium of Descartes

$$x^3 + y^3 - 3axy = 0$$

**Solution**

The major term for this equation after substitution the asymptote into it is

$$\varphi_3 = \lambda^3 + \mu^3.$$

Equality $\varphi_3 = 0$ yields us a system of solutions mastered with $\lambda = 1$ and $\mu = -1$. Substituting these values, we yield for the linear term

$$x + y + a = 0.$$

This term is the equation of the asymptote.

# Chapter 2

# The Contact of Curves

Suppose $M$ and $\bar{M}$ are sets of points in space $\mathbb{E}$ having the point $O$ in common. Let $X$ be an arbitrary point in the set $\bar{M}$, $h(X)$ its distance from the set $M$ (the greatest lower bound of the distances of the points of the set $M$ from the point $X$) and $d(X)$ the distance of the point $X$ from the point $O$.

**Definition** (Contact with the set)**.** We say that the set $\bar{M}$ has **contact with the set** $M$ in the point $O$ if

$$\frac{h(X)}{(d(X))^{\alpha}} \xrightarrow[X \to O]{} 0 \quad (\alpha > 1).$$

## 2.1 Vector Functions of a Scalar Argument

In the following discussion, we shall make extensive use of vector analysis methods. In this connection, we first recall the definition of certain concepts.

**Definition** (Vector function)**.** Suppose $G$ is an arbitrary set of points on the real line, in a plane or in space. We say that a **vector function $\boldsymbol{f}$** is defined on the set $G$ if $\boldsymbol{f}$ assigns a vector $\boldsymbol{f}(X)$ to each point $X$ in $G$.

**Definition** (Limit of vector function)**.** We say that $\boldsymbol{f}(X) \xrightarrow[X \to X_0]{} \boldsymbol{a}$, if $|\boldsymbol{f}(X) - \boldsymbol{a}| \to 0$ when $X \to X_0$.

*Remark.* Theorems on limits, analogous to limit theorems for scalar functions, hold for vector functions. Suppose $\boldsymbol{f}(X)$ and $\boldsymbol{g}(X)$ are vector functions and $\lambda(X)$ is scalar function for wich $\boldsymbol{f}(X) \xrightarrow[X \to X_0]{} \boldsymbol{a}$, $\boldsymbol{g}(X) \xrightarrow[X \to X_0]{} \boldsymbol{b}$ and $\lambda(X) \xrightarrow[X \to X_o]{} m$

then

$$\begin{aligned}
&\boldsymbol{f}(X) \pm \boldsymbol{g}(X) \to \boldsymbol{a} \pm \boldsymbol{b}, \\
&\lambda(X)\boldsymbol{f}(X) \to m\boldsymbol{a}, \\
&\boldsymbol{f}(X) \cdot \boldsymbol{g}(X) \to \boldsymbol{a} \cdot \boldsymbol{b}, \\
&\boldsymbol{f}(X) \times \boldsymbol{g}(X) \to \boldsymbol{a} \times \boldsymbol{b}.
\end{aligned}$$

**Definition** (vector function continuity)**.** The function $\boldsymbol{f}(X)$is said to be continuous at the point $X_0$ if $\boldsymbol{f}(X) \xrightarrow[X\to X_0]{} \boldsymbol{f}(X_0)$.

*Remark.* A simple consequence of properties of the limit is continuity of the following functions $\lambda(X)\boldsymbol{f}(X)$, $\boldsymbol{f}(X) \pm \boldsymbol{g}(X)$, $\boldsymbol{f}(X) \cdot \boldsymbol{g}(X)$, $\boldsymbol{f}(X) \times \boldsymbol{g}(X)$.

**Definition** (Derivative of the vector function)**.** Suppose $\boldsymbol{f}(t)$ is a vector function defined on a closed interval. We say that the vector function $\boldsymbol{f}$ has a **derivative** at the point $t$ on an open interval if exist

$$\lim_{h\to 0} \frac{\boldsymbol{f}(t+h) - \boldsymbol{f}(t)}{h} = \boldsymbol{f}'(t).$$

**Definition** ($k$-times differentiable)**.** A function, having continuous derivatives up to the $k$-th order inclusively, on the open interval $(a, b)$, is called a **$k$-times differentiable function** on this open interval.

**Definition** (Taylor formula)**.** The **Taylor formula** holds for vector functions. Namely, if $\boldsymbol{f}(t)$ is an $n$-times differentiable function, then

$$\boldsymbol{f}(t+\Delta t) = \boldsymbol{f}(t) + \Delta t \boldsymbol{f}'(t) + ... + \frac{\Delta t^n}{n!}(\boldsymbol{f}^{(n)}(t) + \boldsymbol{\varepsilon}(t, \Delta t)),$$

where $|\boldsymbol{\varepsilon}(t, \Delta t)| \xrightarrow[\Delta t\to 0]{} 0$.

**Definition.** The concept of integral in the Riemann sense for vector functions is introduced literally as in the case of scalar functions. The integral of a vector function possesses the usual properties. Namely, if $\boldsymbol{f}(t)$ is a vector function which is continuous on the closed interval $a \le t \le b$, $a < c < b$, $m$ is a constant, $\boldsymbol{r}$ is a constant vector, then

$$\int_a^b \boldsymbol{f}(t)dt = \int_a^c \boldsymbol{f}(t)dt + \int_c^b \boldsymbol{f}(t)dt,$$
$$\int_a^b m\boldsymbol{f}(t)dt = m\int_a^b \boldsymbol{f}(t)dt,$$
$$\int_a^b \boldsymbol{r}\cdot\boldsymbol{f}(t)dt = \boldsymbol{r}\cdot\int_a^b \boldsymbol{f}(t)dt,$$
$$\int_a^b \boldsymbol{r}\times\boldsymbol{f}(t)dt = \boldsymbol{r}\times\int_a^b \boldsymbol{f}(t)dt.$$

The formula

$$\frac{d}{dt}\int_a^x \boldsymbol{f}(t)dt = \boldsymbol{f}(x)$$

for the differentiation of a definite integral is valid.

*Remark.* Suppose $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are three vectors, not lying in one plane, and $\boldsymbol{r}(t)$ is a vector function defined on a segment. We shall define three scalar functions $x(t)$, $y(t)$, $z(t)$ by the condition

$$\boldsymbol{r}(t) = x(t)\boldsymbol{e}_1 + y(t)\boldsymbol{e}_2 + z(t)\boldsymbol{e}_3.$$

Hence, if the functions $x(t)$, $y(t)$, $z(t)$ are continuous or $\boldsymbol{r}$(t) differentiable, then the vector function is continuous respectively differentiable, and conversely.

In conclusion, we note that the parametric definition of a curve employing the equations

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

is equivalent to the definition of the curve employing one vector equation

$$\boldsymbol{r}(t) = x(t)\boldsymbol{e}_1 + y(t)\boldsymbol{e}_2 + z(t)\boldsymbol{e}_3,$$

where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are unit vectors having the directions of the coordinate axes $x$, $y$, $z$.

## 2.2 The Tangent to a Curve

Let $\gamma$ be a curve, $P$ be a point on $\gamma$, and let $g$ be a straight line passing through the point $P$. Let us take a point $Q$ on the curve and denote its distance from the point $P$ and from the line $g$ by $d$ and $h$ respectively (see Figure 2.1).

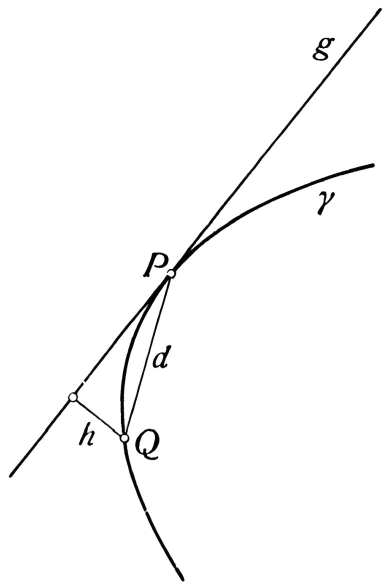


Figure 2.1. Tangent of the curve $\gamma$ at the point $P$.

**Definition** (Tangent)**.** We shall call the line $g$ **the tangent to the curve** $\gamma$ at the point $P$ if

$$\frac{h}{d} \xrightarrow[Q \to P]{} 0.$$

**Theorem 2.2.1** (Tangent uniqueness)**.** *A smooth curve $\gamma$ has a unique tangent at each point. If $\boldsymbol{r} = \boldsymbol{r}(t)$ is the vector equation of the curve, then the tangent at the point $P$ corresponding to the value t of the parameter, has the direction of the vector $\boldsymbol{r}'(t)$.*

*Proof.* Let us assume that the curve has a tangent $g$ at the point $P$ corresponding to the value t of the parameter. Suppose $\boldsymbol{\tau}$ is a unit vector having the same direction as the line $g$. The distance $d$ of the point $Q$, corresponding to the value $t + \Delta t$ of the parameter, from the point $P$ is equal to $|\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)|$. The distance $h$ of the point $Q$ from the tangent equals $|(\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)) \times \boldsymbol{\tau}|$. According to the definition of the tangent

$$\frac{h}{d} = \frac{|(\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)) \times \boldsymbol{\tau}|}{|\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)|} \xrightarrow[\Delta \boldsymbol{r} \to 0]{} 0.$$

But

$$\frac{|(\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)) \times \boldsymbol{\tau}|}{|\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)|} = \frac{|\frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t} \times \boldsymbol{\tau}|}{|\frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t}|} \to \frac{|\boldsymbol{r}'(t) \times \boldsymbol{\tau}|}{|\boldsymbol{r}'(t)|}.$$

From this it follows that $\boldsymbol{r}'(t) \times \boldsymbol{\tau} = 0$. This is possible only when the vector $\boldsymbol{\tau}$ has the same direction as the vector $\boldsymbol{r}'(t)$. Thus, if the tangent exists, then it has

the direction of the vector $\boldsymbol{r}'(t)$ and, consequently, it is unique.
The fact that the line $g$, passing through the point $P$ and having the same direction as the vector $\boldsymbol{r}'(t)$, is a tangent, is also true; for, as the preceding discussions show, for such a line we have

$$\frac{h}{d} = \frac{\left|(\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)) \times \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}\right|}{|\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)|} \to \frac{|\boldsymbol{r}'(t) \times \boldsymbol{r}'(t)|}{|\boldsymbol{r}'(t)|^2} = 0.$$

□

Knowing the direction of the tangent, it is not difficult to write its equation.

**Definition** (Equation of the Tangent)**.** If the curve is given employing the vector equation $\boldsymbol{r} = \boldsymbol{r}(t)$, then the position vector $\tilde{\boldsymbol{r}}$ of an arbitrary point on the tangent can be represented in the form

$$\tilde{\boldsymbol{r}} = \boldsymbol{r}(t) + \lambda \boldsymbol{r}'(t). \tag{2.1}$$

This is then the **equation of the tangent in the parametric form** (with parameter $\lambda$).

We shall write the equation of the tangent for some curves given in the analytic form.

- Suppose the curve is given by equations in the parametric form

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

  It is equivalent to the given equation in the vector form

$$\boldsymbol{r} = \boldsymbol{r}(t) = x(t)\boldsymbol{e}_1 + y(t)\boldsymbol{e}_2 + z(t)\boldsymbol{e}_3,$$

  where $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are unit vectors in the directions of the coordinate axes. Replacing the vector equation refeq:tangent by three scalar equations, we obtain the equations of the tangent, corresponding to the parametric form

$$\tilde{x} = x(t) + \lambda x'(t), \quad \tilde{y} = y(t) + \lambda y'(t), \quad \tilde{z} = z(t) + \lambda z'(t)$$

  or in the equivalent form

$$\frac{\tilde{x} - x(t)}{x'(t)} = \frac{\tilde{y} - y(t)}{y'(t)} = \frac{\tilde{z} - z(t)}{z'(t)}.$$

- The equation of the tangent in the case when the curve is given employing the equations
$$y = y(x), \quad z = z(x)$$
is easily gotten from the equation of the tangent for the case where the curve is given in the parametric form. It suffices to note that giving the curve is equivalent to giving it in the parametric form
$$x = t, \quad y = y(t), \quad z = z(t).$$
The equation of the tangent to the curve is written as
$$\tilde{x} - x = \frac{\tilde{y} - y(t)}{y'(t)} = \frac{\tilde{z} - z(t)}{z'(t)}$$
or in the equivalent form
$$\tilde{y} = y(x) + y'(x)(\tilde{x} - x), \quad \tilde{z} = z(x) + z'(x)(\tilde{x} - x).$$

- The equation of the tangent at the point $(x_0, y_0, z_0)$ to a curve given by the equations
$$\varphi(x, y, z) = 0, \quad \psi(x, y, z) = 0,$$
where the rank of the matrix
$$\begin{pmatrix} \varphi_x & \varphi_y & \varphi_z \\ \psi_x & \psi_y & \psi_z \end{pmatrix}$$
equals two. Suppose
$$x = x(t), \quad y = y(t), \quad z = z(t)$$
is any regular parametrisation of the curve in a neighbourhood of the point $(x_0, y_0, z_0)$. The equation of the tangent to the curve at this point is
$$\frac{\tilde{x} - x_0}{x_0'} = \frac{\tilde{y} - y_0}{y_0'} = \frac{\tilde{z} - z_0}{z_0'}. \tag{2.2}$$
We shall now compute $x_0'$, $y_0'$, $z_0'$.
Differentiating identities $\varphi(x(t), y(t), z(t)) \equiv 0$, $\psi(x(t), y(t), z(t)) \equiv 0$ with respect to $t$, we have
$$\begin{cases} \varphi_x x' + \varphi_y y' + \varphi_z z' = 0, \\ \psi_x x' + \psi_y y' + \psi_z z' = 0. \end{cases} \tag{2.3}$$

It follows that

$$\frac{x'}{\begin{vmatrix}\varphi_y & \varphi_z\\ \psi_y & \psi_z\end{vmatrix}} = \frac{y'}{\begin{vmatrix}\varphi_x & \varphi_z\\ \psi_x & \psi_z\end{vmatrix}} = \frac{z'}{\begin{vmatrix}\varphi_x & \varphi_y\\ \psi_x & \psi_y\end{vmatrix}}$$

and the equation of the tangent assumes the form

$$\frac{\tilde{x}-x_0}{\begin{vmatrix}\varphi_y & \varphi_z\\ \psi_y & \psi_z\end{vmatrix}} = \frac{\tilde{y}-y_0}{\begin{vmatrix}\varphi_x & \varphi_z\\ \psi_x & \psi_z\end{vmatrix}} = \frac{\tilde{z}-z_0}{\begin{vmatrix}\varphi_x & \varphi_y\\ \psi_x & \psi_y\end{vmatrix}}, \tag{2.4}$$

where the derivatives of $\varphi_x$, $\varphi_y$, ..., $\psi_z$ tangency $(x_0, y_0, z_0)$. If the curve lies in a plane and is defined by the equation $\varphi(x, y) = 0$, the equation of the tangent will be

$$\frac{\tilde{x}-x_0}{\varphi_x} = \frac{\tilde{y}-y_0}{\varphi_y}.$$

To derive this equation, it is sufficient to note that defining a curve in the $x$, $y$-plane by the equation $\varphi(x, y) = 0$ is equivalent to defining it in space employing the equations $\varphi(x, y) = 0$, $z = 0$.

**Definition** (Normal plane). The **normal plane** to a curve at the point $P$ is the plane which passes through the point $P$ and is perpendicular to the tangent at this point.

## 2.3 The Osculating Plane to a Curve

Suppose $\gamma$ is a curve and point $P \in \gamma$. $\alpha$ is a plane passing through the point $P$. Point $Q$ is arbitrary point on the curve $\gamma$, $dist(Q, \alpha) = h$ and $dist(Q, P) = d$ (Figure 2.2).

**Definition** (Osculating plane). The plane $\alpha$ will be called the **osculating plane** to curve $\gamma$ at the point $P$ if

$$\frac{h}{d^2} \xrightarrow[Q \to P]{} 0.$$

**Theorem 2.3.1.** *A regular curve $\gamma$ has an osculating plane at every point. In this connection, the osculating plane is either unique or any plane containing the tangent to the curve is an osculating plane.*
*If $\boldsymbol{r} = \boldsymbol{r}(t)$ is the equation of the curve $\gamma$, then the osculating plane at the point corresponding to the value $t$ of the parameter is parallel to the vectors $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$.*

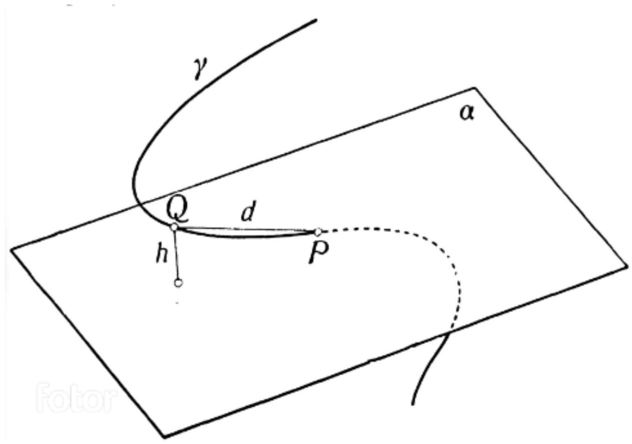


Figure 2.2. The osculating plane to the curve $\gamma$ at the point $P$.

*Proof.* Suppose $\alpha$ is an osculating plane to the curve $\gamma$ at the point $P$, corresponding to the value $t$ of the parameter. We shall denote the unit normal vector to the plane a by $\boldsymbol{e}$. The distance of the point $Q$, corresponding to the value $t + \Delta t$ of the parameter, from the plane $\alpha$ is

$$h = |\boldsymbol{e} \cdot (\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t))|.$$

The distance from this point to $P$ is

$$d = |\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t)|.$$

We have

$$\frac{h}{d^2} = \frac{|\boldsymbol{e} \cdot (\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t))|}{(\boldsymbol{r}(t + \Delta t) - \boldsymbol{r}(t))^2} = \frac{|\boldsymbol{e} \cdot (\boldsymbol{r}'(t)\Delta t + \frac{\boldsymbol{r}"(t)}{2}\Delta t^2 + \varepsilon_1 \Delta t^2)|}{(\boldsymbol{r}'(t)\Delta t + \varepsilon_2 \Delta t)^2} =$$
$$= \frac{|\frac{\boldsymbol{e} \cdot \boldsymbol{r}'(t)}{\Delta t} + \frac{\boldsymbol{e} \cdot \boldsymbol{r}"(t)}{2} + \varepsilon_1'|}{\boldsymbol{r}'^2(t) + \varepsilon_2'}$$

Since $h/d^2 \to 0$, $\varepsilon_1', \varepsilon_2' \to 0$ as $\Delta t \to 0$, and $|\boldsymbol{r}'(t)| \neq 0$, we have $\boldsymbol{e} \cdot \boldsymbol{r}'(t) = 0$, $\boldsymbol{e} \cdot \boldsymbol{r}"(t) = 0$. Thus, if the osculating plane exists, the vectors $\boldsymbol{r}'(t)$ and $\boldsymbol{r}"(t)$ are parallel to it.
To verify the fact that the osculating plane always exists, we take the plane $\alpha$, parallel to the vectors $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$ (we consider any plane to be parallel to the zero vector). Then $\boldsymbol{e} \cdot \boldsymbol{r}'(t) = \boldsymbol{e} \cdot \boldsymbol{r}"(t) = 0$ and, consequently,

$$\frac{h}{d^2} = \frac{|\varepsilon_1'|}{\boldsymbol{r}'^2(t) + \varepsilon_2'} \xrightarrow[\Delta t \to 0]{} 0.$$

Thus, the osculating plane exists at every point on the curve. Obviously, the osculating plane, being parallel to the vectors $\boldsymbol{r}'(t)$ and $\boldsymbol{r}"(t)$, will be unique if the vectors $\boldsymbol{r}'(t)$ and $\boldsymbol{r}"(t)$ are not parallel. But if these vectors are parallel (or the vector $\boldsymbol{r}"(t) = 0$), then any plane, drawn through the tangent to the curve, will be an osculating plane. □

### The Equation of the Osculating Plane

Suppose $\boldsymbol{r} = \boldsymbol{r}(t)$ is the vector equation of the curve and that $t$ is the value of the parameter which corresponds to the point $P$ on the curve. Suppose $\boldsymbol{r}'(t)$ and $\boldsymbol{r}''(t)$ are vectors which are not parallel at the point $P$. Then $\boldsymbol{r}'(t) \times \boldsymbol{r}"(t)$ will be the normal vector to the osculating plane. If $\tilde{\boldsymbol{r}}$ denotes the position vector of any point in the osculating plane at the point $P$, then the vectors $\tilde{\boldsymbol{r}} - \boldsymbol{r}(t)$ and $\boldsymbol{r}'(t) \times \boldsymbol{r}"(t)$ are orthogonal. It follows that the equation of the osculating plane is

$$(\tilde{\boldsymbol{r}} - \boldsymbol{r}(t), \boldsymbol{r}'(t) \times \boldsymbol{r}''(t)) = 0.$$

In the case when the curve is defined parametrically $x = x(t)$, $y = y(t)$, $z = z(t)$, we obtain from this equation the equation of the osculating plane in the form

$$\begin{vmatrix} \tilde{x} - x(t), & \tilde{y} - y(t), & \tilde{z} - z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0. \tag{2.5}$$

**Definition** (Normal)**.** Every straight line passing through a point on the curve perpendicular to the tangent is called a **normal** to the curve.

When the osculating plane is unique, two special straight lines are chosen from among these lines.

**Definition** (Principal normal)**.** The **principal normal** is the normal lying in the osculating plane.

**Definition** (Binormal)**.** The **binormal** is the normal perpendicular to the osculating plane.

## 2.4 The Contact of Curves

**Definition** ($C^n$-class curve)**.** A regular curve $\gamma$ is of class $C^n$ if:

- All component functions $x_k(t)$ $(k = 1, \dots, d)$ have continuous derivatives up to order $n$.
- The $n$-th derivative $\gamma^{(n)}(t)$ exists and is continuous.

## Definition

Suppose $\gamma$ and $\gamma'$ are elementary curves having a common point $O$. We choose the point $P \in \gamma'$ and denote $dist(P, \gamma) = h$, $dist(P, O) = d$ (Figure 2.3).

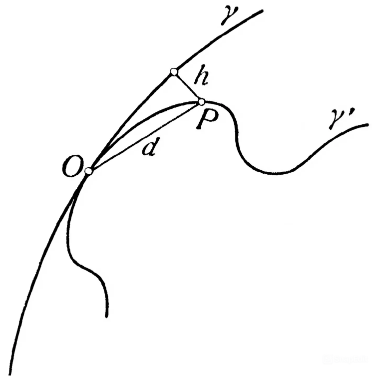


Figure 2.3. Contact of the curves $\gamma$ and $\gamma'$ at the point $O$.

**Definition** (Contact of order $n$)**.** We shall say that the curve $\gamma'$ has **contact of order** $n$ with the curve $\gamma$ at the point $O$ if

$$\frac{h}{d^n} \xrightarrow[P \to O]{} 0.$$

If $\gamma$ and $\gamma'$ are general curves and $\gamma, \gamma' \ni O$, we shall say that the curve $\gamma'$ has contact of order $n$ with the curve $\gamma$ at the point $O$ if an elementary neighborhood of $O$ on curve $\gamma'$ has $n$-th order contact with an elementary neighborhood of curve $\gamma$ at $O$.

*Remark.* We say that the intersection of curves is the 0-th contact.

## Examples of the Contacts

Contact order generalises tangency, distinguishing between mere intersection (0−th order) and shared geometry. It quantifies how well one curve approximates another near a point, which is essential for applications such as splines, robotics, and computer graphics.

**0th-Order Contact (“Intersection Only”)**

**Curves:**

- $\gamma : y = x^2$ (parabola)
- $\gamma' : y = x$ (line)

**At $O(0,0)$:**

The point $O(0,0)$ is the common point of intersection between the parabola $\gamma$ and the line $\gamma'$.

**Parameterization of $\gamma'$:**

The line $\gamma'$ can be parameterised as $P(t) = (t,t)$, where $t$ is a parameter. As $t$ varies, the point $P(t)$ moves along the line $\gamma'$.

**Calculation of $h$ (Distance from $P(t)$ to $\gamma$):**

The distance $h$ is defined as the minimal distance from the point $P(t)$ to the curve $\gamma$. Mathematically:

$$h = \inf_{(x,y)\in\gamma} \sqrt{(x-t)^2 + (y-t)^2}$$

To find this distance, we minimise the squared distance:

$$D = (x-t)^2 + (y-t)^2$$

Since $\gamma$ is the parabola $y = x^2$, we substitute $y$ with $x^2$:

$$D = (x-t)^2 + (x^2-t)^2$$

To minimize $D$, we find $x$ such that $\frac{dD}{dx} = 0$. However, for simplicity, we approximate $h$ for small $t$. For $t \to 0$, the point on $\gamma$ closest to $P(t) = (t,t)$ is approximately $(t,t^2)$. Therefore:

$$h \approx \sqrt{(t-t)^2 + (t^2-t)^2} = |t^2 - t|\sqrt{2}$$

For $t \to 0$, $|t^2 - t| \approx |t|$, so:

$$h \sim |t|\sqrt{2}$$

**Calculation of $d$ (Distance from $P(t)$ to $O$):**

The distance $d$ is the Euclidean distance from the point $P(t) = (t,t)$ to the origin $O(0,0)$:

$$d = \sqrt{t^2 + t^2} = |t|\sqrt{2}$$

**Evaluation of the Limit:**

The definition of $n$-th order contact requires:

$$\frac{h}{d^n} \xrightarrow[P \to O]{} 0$$

For **0th-order contact**, we check $n = 0$:

$$\frac{h}{d^0} = h \sim |t|\sqrt{2} \xrightarrow[t \to 0]{} 0$$

This confirms that $\frac{h}{d^0} \to 0$.

Next, we check $n = 1$:

$$\frac{h}{d^1} = \frac{|t|\sqrt{2}}{|t|\sqrt{2}} = 1 \not\to 0$$

Since $\frac{h}{d^1} \not\to 0$, the curves do not have **1st-order contact**.

**Geometric Interpretation:**

- **Intersection:** The curves intersect at $O(0,0)$.
- **Tangents:** The parabola $\gamma$ has a horizontal tangent (slope 0) at $O$, while the line $\gamma'$ has a slope of 1.
- **Conclusion:** Because the curves intersect but do not share a tangent, they exhibit **0th-order contact**.

**1st-Order Contact ("Shared Tangent")**

**Curves:**

- $\gamma : y = x^2$ (parabola)
- $\gamma' : y = 0$ (x-axis)

**At $O(0,0)$:**

The point $O(0,0)$ is the common point of intersection between the parabola $\gamma$ and the x-axis $\gamma'$.

**Parameterization of $\gamma'$:**

The x-axis $\gamma'$ can be parameterized as $P(t) = (t, 0)$, where $t$ is a parameter. As $t$ varies, the point $P(t)$ moves along the x-axis.

**Calculation of $h$ (Distance from $P(t)$ to $\gamma$):**

The distance $h$ is defined as the minimal distance from the point $P(t)$ to the curve $\gamma$. Mathematically:

$$h = \inf_{(x,y)\in\gamma} \sqrt{(x-t)^2 + (y-0)^2}$$

To find this distance, we minimise the squared distance:

$$D = (x-t)^2 + y^2$$

Since $\gamma$ is the parabola $y = x^2$, we substitute $y$ with $x^2$:

$$D = (x-t)^2 + x^4$$

To minimize $D$, we find $x$ such that $\frac{dD}{dx} = 0$. Differentiating $D$ with respect to $x$:

$$\frac{dD}{dx} = 2(x-t) + 4x^3$$

Setting $\frac{dD}{dx} = 0$:

$$2(x-t) + 4x^3 = 0 \implies x - t + 2x^3 = 0$$

For $t \to 0$, we can approximate $x \approx t$ because $2x^3$ becomes negligible. Substituting $x = t$, we get:

$$h \approx \sqrt{(t-t)^2 + (t^2)^2} = t^2$$

**Calculation of $d$ (Distance from $P(t)$ to $O$):**

The distance $d$ is the Euclidean distance from the point $P(t) = (t, 0)$ to the origin $O(0, 0)$:

$$d = \sqrt{t^2 + 0^2} = |t|$$

**Evaluation of the Limit:**

The definition of $n$-th order contact requires:

$$\frac{h}{d^n} \xrightarrow[P\to O]{} 0$$

For **1st-order contact**, we check $n = 1$:

$$\frac{h}{d^1} = \frac{t^2}{|t|} = |t| \xrightarrow[t\to 0]{} 0$$

This confirms that $\frac{h}{d^1} \to 0$.

Next, we check $n = 2$:

$$\frac{h}{d^2} = \frac{t^2}{t^2} = 1 \not\to 0$$

Since $\frac{h}{d^2} \not\to 0$, the curves do not have **2nd-order contact**.

**Geometric Interpretation:**

- **Intersection:** The curves intersect at $O(0, 0)$.
- **Tangents:** Both curves share the same tangent (horizontal line, slope 0) at $O$.
- **Curvatures:** The parabola $\gamma$ has nonzero curvature at $O$, while the x-axis $\gamma'$ has zero curvature. See the next chapter to clarify the notion of curvature.
- **Conclusion:** Because the curves share a tangent but have different curvatures, they exhibit **1st-order contact**.

**2nd-Order Contact ("Shared Curvature")**

**Curves:**

- $\gamma : y = x^2$ (parabola)
- $\gamma' : y = x^2 + x^3$ (cubic curve)

**At $O(0, 0)$:**

The point $O(0, 0)$ is the common point of intersection between the parabola $\gamma$ and the cubic curve $\gamma'$.

**Parameterization of $\gamma'$:**

The curve $\gamma'$ can be parameterized as $P(t) = (t, t^2+t^3)$, where $t$ is a parameter. As $t$ varies, the point $P(t)$ moves along the curve $\gamma'$.

**Calculation of $h$ (Distance from $P(t)$ to $\gamma$):**

The distance $h$ is defined as the minimal distance from the point $P(t)$ to the curve $\gamma$. Mathematically:

$$h = \inf_{(x,y)\in\gamma} \sqrt{(x-t)^2 + (y-(t^2+t^3))^2}$$

To find this distance, we minimize the squared distance:

$$D = (x-t)^2 + (y-(t^2+t^3))^2$$

Since $\gamma$ is the parabola $y = x^2$, we substitute $y$ with $x^2$:

$$D = (x-t)^2 + (x^2-(t^2+t^3))^2$$

To minimize $D$, we find $x$ such that $\frac{dD}{dx} = 0$. Differentiating $D$ with respect to $x$:

$$\frac{dD}{dx} = 2(x-t) + 2(x^2 - t^2 + t^3)(2x)$$

For $t \to 0$, we approximate $x \approx t$ because higher-order terms (e.g., $x^3$ or $t^3$) become negligible. Substituting $x = t$, we get:

$$h \approx \sqrt{(t-t)^2 + (t^2-(t^2+t^3))^2} = |t^3|$$

**Calculation of $d$ (Distance from $P(t)$ to $O$):**

The distance $d$ is the Euclidean distance from the point $P(t) = (t, t^2+t^3)$ to the origin $O(0,0)$:

$$d = \sqrt{t^2 + (t^2+t^3)^2}$$

Expand and simplify the expression for $d$:

$$d = \sqrt{t^2 + (t^2+t^3)^2} = \sqrt{t^2 + t^4 + 2t^5 + t^6}$$

As $t \to 0$, the terms $t^4$, $t^5$, and $t^6$ become much smaller than $t^2$ because:

$$t^4 \ll t^2, \quad t^5 \ll t^2, \quad t^6 \ll t^2$$

Higher-order terms (e.g., $t^4$, $t^5$, $t^6$) approach zero much faster than $t^2$. Thus, $t^2$ is the leading term that determines the behavior of $d$ as $t \to 0$.

The Euclidean distance $d$ is approximated as:

$$d = \sqrt{t^2 + (t^2 + t^3)^2} \sim |t| \quad \text{as } t \to 0$$

This approximation holds because $t^2$ is the dominant term in the expansion of $d$, and higher-order terms ($t^4$, $t^5$, $t^6$) become negligible for small $t$.

**Evaluation of the Limit:**

The definition of $n$-th order contact requires:

$$\frac{h}{d^n} \xrightarrow[P \to O]{} 0$$

For **2nd-order contact**, we check $n = 2$:

$$\frac{h}{d^2} = \frac{|t^3|}{t^2} = |t| \xrightarrow[t \to 0]{} 0$$

This confirms that $\frac{h}{d^2} \to 0$.

Next, we check $n = 3$:

$$\frac{h}{d^3} = \frac{|t^3|}{|t^3|} = 1 \not\to 0$$

Since $\frac{h}{d^3} \not\to 0$, the curves do not have **3rd-order contact**.

**Geometric Interpretation:**

- **Intersection:** The curves intersect at $O(0, 0)$.
- **Tangents:** Both curves share the same tangent (horizontal line, slope 0) at $O$.
- **Curvatures:** Both curves share the same curvature (second derivative) at $O$. See the next chapter to clarify the notion of curvature.
- **Conclusion:** Because the curves share both the tangent and curvature at $O$, they exhibit **2nd-order contact**.

**3rd-Order Contact**

**Curves:**

- $\gamma : y = e^x$ (exponential curve)
- $\gamma' : y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ (third-order Taylor approximation of $e^x$)

**At $O(0, 1)$:**

The point $O(0, 1)$ is the common point of intersection between the exponential curve $\gamma$ and its third-order Taylor approximation $\gamma'$.

**Parameterization of $\gamma'$:**

The curve $\gamma'$ can be parameterised as:

$$P(t) = \left(t, 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right)$$

Here, $t$ is a parameter, and as $t$ varies, the point $P(t)$ moves along the curve $\gamma'$.

**Calculation of $h$ (Distance from $P(t)$ to $\gamma$):**

The distance $h$ is defined as the minimal distance from the point $P(t) = \left(t, 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right)$ to the curve $\gamma : y = e^x$. Mathematically:

$$h = \inf_{(x,y)\in\gamma} \sqrt{(x - t)^2 + \left(y - \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right)\right)^2}$$

For small $t$, the point on the curve $\gamma$ closest to $P(t)$ is approximately $(t, e^t)$. This is because, for $t \to 0$, the horizontal distance $|x - t|$ is negligible compared to the vertical distance $|y - (1 + t + \frac{t^2}{2} + \frac{t^3}{6})|$. Thus, we can approximate $h$ by the **vertical distance**:

$$h \approx \left|e^t - \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right)\right|$$

- **Horizontal Distance is Small:** For $t \to 0$, the horizontal distance $|x - t|$ is negligible compared to the vertical distance $|y - (1 + t + \frac{t^2}{2} + \frac{t^3}{6})|$. Thus, the minimal distance $h$ is dominated by the vertical component.
- **Taylor Series Expansion:** The vertical distance can be computed explicitly using the Taylor series expansion of $e^t$, which is straightforward and avoids the need for solving optimisation problems.

To approximate $h$ for $t \to 0$, we use the Taylor series expansion of $e^t$:

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots$$

Subtracting the third-order Taylor approximation from $e^t$ gives the error term:

$$h \approx \left|\frac{t^4}{24}\right| = \frac{t^4}{24}$$

This shows that $h$ is proportional to $t^4$ as $t \to 0$.

**Calculation of $d$ (Distance from $P(t)$ to $O$):**

The distance $d$ is the Euclidean distance from the point $P(t) = \left(t, 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right)$ to the point $O(0, 1)$. Mathematically:

$$d = \sqrt{t^2 + \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6} - 1\right)^2} = \sqrt{t^2 + \left(t + \frac{t^2}{2} + \frac{t^3}{6}\right)^2}$$

Expand $\left(t + \frac{t^2}{2} + \frac{t^3}{6}\right)^2$:

$$\left(t + \frac{t^2}{2} + \frac{t^3}{6}\right)^2 = t^2 + t^3\left(\frac{1}{2} + \frac{1}{2}\right) + \text{higher-order terms}$$

For $t \to 0$, we can approximate:

$$\left(t + \frac{t^2}{2} + \frac{t^3}{6}\right)^2 \approx t^2 + t^3 + O(t^4)$$

Substitute the expansion back into $d$:

$$d = \sqrt{t^2 + (t^2 + t^3 + O(t^4))} = \sqrt{2t^2 + t^3 + O(t^4)}$$

Combine like terms:

$$d = \sqrt{2t^2 + t^3 + O(t^4)}$$

Ss $t \to 0$, the higher-order terms ($t^3$ and $O(t^4)$) become negligible compared to $t^2$. Thus, the dominant term in the expansion is $t^2$, and we can approximate:

$$d \approx \sqrt{2t^2} = \sqrt{2}|t|$$

For small $t$, $\sqrt{2}|t| \sim |t|$, since the constant factor $\sqrt{2}$ does not affect the limiting behavior as $t \to 0$, so:

$$d \sim |t|$$

**Evaluation of the Limit:**

The definition of $n$-th order contact requires:

$$\frac{h}{d^n} \xrightarrow[P \to O]{} 0$$

For **3rd-order contact**, we check $n = 3$:

$$\frac{h}{d^3} \sim \frac{t^4/24}{|t|^3} = \frac{|t|}{24} \xrightarrow[t \to 0]{} 0$$

This confirms that $\frac{h}{d^3} \to 0$.

Next, we check $n = 4$:

$$\frac{h}{d^4} \sim \frac{t^4/24}{|t|^4} = \frac{1}{24} \not\to 0$$

Since $\frac{h}{d^4} \not\to 0$, the curves do not have **4th-order contact**.

**Geometric Interpretation:**

- **Intersection:** The curves intersect at $O(0, 1)$.
- **Tangents:** Both curves share the same tangent (slope 1) at $O$.
- **Curvatures:** Both curves share the same curvature (second derivative) at $O$.
- **Third Derivatives:** Both curves share the same third derivative at $O$.
- **Conclusion:** Because the curves agree up to the third derivative at $O$, they exhibit **3rd-order contact**.

**Summary**

Therefore, we can give a summary of these examples:

- 0-th contact: Touch but cross.
- 1-st contact: Kiss and run parallel briefly.
- 2-nd contact: Oscillate together (like a circle and parabola).
- 3-rd+ contact: Nearly indistinguishable at microscopic scales.

**Theorem 2.4.1.** *Let $\gamma$ and $\gamma'$ be regular plane curves of class $C^n$, where:*

- *$\gamma$ is given implicitly by $\varphi(x, y) = 0$ with $\nabla\varphi = \varphi_x^2 + \varphi_y^2 \neq 0$ at $O(x_0, y_0)$,*
- *$\gamma'$ is parametrized by $(x(t), y(t))$ with $(x(t_0), y(t_0)) = O$.*

*Then $\gamma'$ has* ***contact of order*** *$n$ with $\gamma$ at $O$ if and only if the composite function $\Phi(t) := \varphi(x(t), y(t))$ satisfies:*

$$\Phi(t_0) = \Phi'(t_0) = \cdots = \Phi^{(n)}(t_0) = 0.$$

*Proof. Necessity*: Suppose $\gamma'$ has $n$-th order contact with $\gamma$ at $O$.

*1. Distance Scaling:* By definition, $\mathrm{dist}(P, \gamma) = h \sim d^n$ where $d = \mathrm{dist}(P, O)$. For $P = (x(t), y(t))$, Taylor expansion gives:

$$d = |t - t_0| \cdot \|\gamma'(t_0)\| + o(|t - t_0|).$$

The distance $h$ is the minimal distance from $P$ to $\gamma$, which (by the Implicit Function Theorem) scales as:

$$h = |\Phi(t)| \cdot \|\nabla\varphi(O)\|^{-1} + o(|\Phi(t)|).$$

Thus, $\frac{h}{d^n} \to 0$ implies $\Phi(t) = o(|t - t_0|^n)$, forcing all derivatives up to order $n$ to vanish.

*2. Derivative Calculation:* The chain rule yields:

$$\Phi'(t) = \varphi_x x'(t) + \varphi_y y'(t),$$

$$\Phi''(t) = \varphi_{xx}(x')^2 + 2\varphi_{xy}x'y' + \varphi_{yy}(y')^2 + \varphi_x x'' + \varphi_y y'',$$

and similarly for higher derivatives. At $t = t_0$, the vanishing of $\Phi, \Phi', \ldots, \Phi^{(n)}$ ensures the Taylor expansions of $\gamma$ and $\gamma'$ agree up to order $n$.

*Sufficiency*: If $\Phi^{(k)}(t_0) = 0$ for $k = 0, \ldots, n$, then:

$$\Phi(t) = \frac{\Phi^{(n+1)}(t_0)}{(n+1)!}(t - t_0)^{n+1} + o(|t - t_0|^{n+1}).$$

Thus, $h \sim |t - t_0|^{n+1}$ while $d \sim |t - t_0|$, so $\frac{h}{d^n} \sim |t - t_0| \to 0$. □

Let us revisit previously introduced examples.

**0th-Order Contact Curves:**

- $\gamma : \varphi(x, y) = y - x^2 = 0$
- $\gamma' : (x(t), y(t)) = (t, t)$

**At $t_0 = 0$ ($O(0,0)$):**

$$\begin{aligned}
\varphi(t) &= t - t^2 \\
\varphi(0) &= 0 \quad \text{(intersection)} \\
\varphi'(t) &= 1 - 2t \Rightarrow \varphi'(0) = 1 \neq 0
\end{aligned}$$

Only $\varphi(0) = 0$ holds. By Theorem 2.3.1, contact is **exactly order 0**.
**1st-Order Contact Curves:**

- $\gamma : \varphi(x, y) = y - x^2 = 0$
- $\gamma' : (x(t), y(t)) = (t, 0)$

**At $t_0 = 0$:**

$$\begin{aligned}
\varphi(t) &= -t^2 \\
\varphi(0) &= 0 \\
\varphi'(t) &= -2t \Rightarrow \varphi'(0) = 0 \\
\varphi''(t) &= -2 \Rightarrow \varphi''(0) = -2 \neq 0
\end{aligned}$$

Derivatives vanish up to order 1. By Theorem 2.3.1, contact is **exactly order 1**.

**2nd-Order Contact Curves:**

- $\gamma : \varphi(x, y) = y - x^2 = 0$
- $\gamma' : (x(t), y(t)) = (t, t^2 + t^3)$

**At $t_0 = 0$:**

$$\begin{aligned}
\varphi(t) &= t^2 + t^3 - t^2 = t^3 \\
\varphi(0) &= 0 \\
\varphi'(t) &= 3t^2 \Rightarrow \varphi'(0) = 0 \\
\varphi''(t) &= 6t \Rightarrow \varphi''(0) = 0 \\
\varphi'''(t) &= 6 \Rightarrow \varphi'''(0) = 6 \neq 0
\end{aligned}$$

Derivatives vanish up to order 2. By Theorem 2.3.1, contact is **exactly order 2**.

**3rd-Order Contact Curves:**

- $\gamma : \varphi(x, y) = y - e^x = 0$

- $\gamma' : (x(t), y(t)) = \left(t, 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right)$

**At $t_0 = 0$:**

$$\varphi(t) = \left(1 + t + \frac{t^2}{2} + \frac{t^3}{6}\right) - e^t$$
$$\varphi(0) = 0$$
$$\varphi'(t) = 1 + t + \frac{t^2}{2} - e^t \Rightarrow \varphi'(0) = 0$$
$$\varphi''(t) = 1 + t - e^t \Rightarrow \varphi''(0) = 0$$
$$\varphi'''(t) = 1 - e^t \Rightarrow \varphi'''(0) = 0$$
$$\varphi^{(4)}(t) = -e^t \Rightarrow \varphi^{(4)}(0) = -1 \neq 0$$

Derivatives vanish up to order 3. By Theorem 2.3.1, contact is **exactly order 3**.

## 2.5 The Envelope of a Family of Curves, Depending on a Parameter

First, let us discuss a particular problem.

**Problem**
Find all curves $\gamma'$ that have 2nd-order contact with a given curve $\gamma'$ at a point $O$.
**Steps to solution**
The curves must agree up to their "bending" at $O$.
Let:

- $\gamma$ be defined implicitly by $\varphi(x, y) = 0$
- $\gamma'$ be parametrized by $(x(t), y(t))$ with $O = (x(0), y(0))$

By Theorem 2.3.1, $\gamma'$ has second-order contact with $\gamma$ at $O$ if:

$$\Phi(0) = \varphi(x(0), y(0)) = 0 \qquad \text{(0th-order)} \qquad (2.6)$$
$$\Phi'(0) = \frac{d}{dt}\varphi(x(t), y(t))\Big|_{t=0} = 0 \qquad \text{(1st-order)} \qquad (2.7)$$
$$\Phi''(0) = \frac{d^2}{dt^2}\varphi(x(t), y(t))\Big|_{t=0} = 0 \qquad \text{(2nd-order)} \qquad (2.8)$$

Now we are ready to solve for the parameters.

1. Choose a **family of candidate curves** $\gamma'$ (e.g., circles, polynomials).

2. For each member of the family:

   - Compute $\Phi(t) = \varphi(x(t), y(t))$
   - Enforce conditions (2.6)–(2.8)
   - Solve the resulting system for parameters

**Example: Circles and Parabola** For $\gamma : y = x^2$ at $O(0,0)$:

- **Family**: Circles centered on y-axis: $\psi(x,y) = x^2 + (y-b)^2 - b^2 = 0$

- **Parametrization**: $(x(t), y(t)) = (t, b - \sqrt{b^2 - t^2})$

- **Conditions**:

$$\begin{aligned}
\Phi(0) &= 0 \quad \text{(automatically satisfied)} \\
\Phi'(0) &= 0 \quad \Rightarrow \text{No new constraint} \\
\Phi''(0) &= \frac{1}{|b|} - 2 = 0 \quad \Rightarrow b = \pm\frac{1}{2}
\end{aligned}$$

- **Solution**: Two osculating circles with radius $\frac{1}{2}$.

**Key Notes**

- **Uniqueness**: For second-order contact, solutions typically form a *discrete set* (e.g., two circles for the parabola).

- **Curvature Matching**: All solutions must share $\gamma$'s curvature $\kappa$ at $O$.

- **Generalization**: For $n$-th order contact, require $\Phi^{(k)}(0) = 0$ for $k = 0, \ldots, n$.

- **Infinite Solutions:** If $\gamma$ is a line, all lines through $O$ have 2nd-order contact (since curvature is zero).

- **No Solutions**: If $\gamma$ has a cusp at $O$, no smooth $\gamma'$ can have 2nd-order contact. For example (Figure 2.4), consider the cuspidal cubic curve $\gamma$ and the horizontal line $\gamma'$:

$$\begin{aligned}
\gamma &: y^2 = x^3 \quad \text{(cusp at } O(0,0)) \\
\gamma' &: y = 0 \quad \text{(x-axis)}
\end{aligned}$$

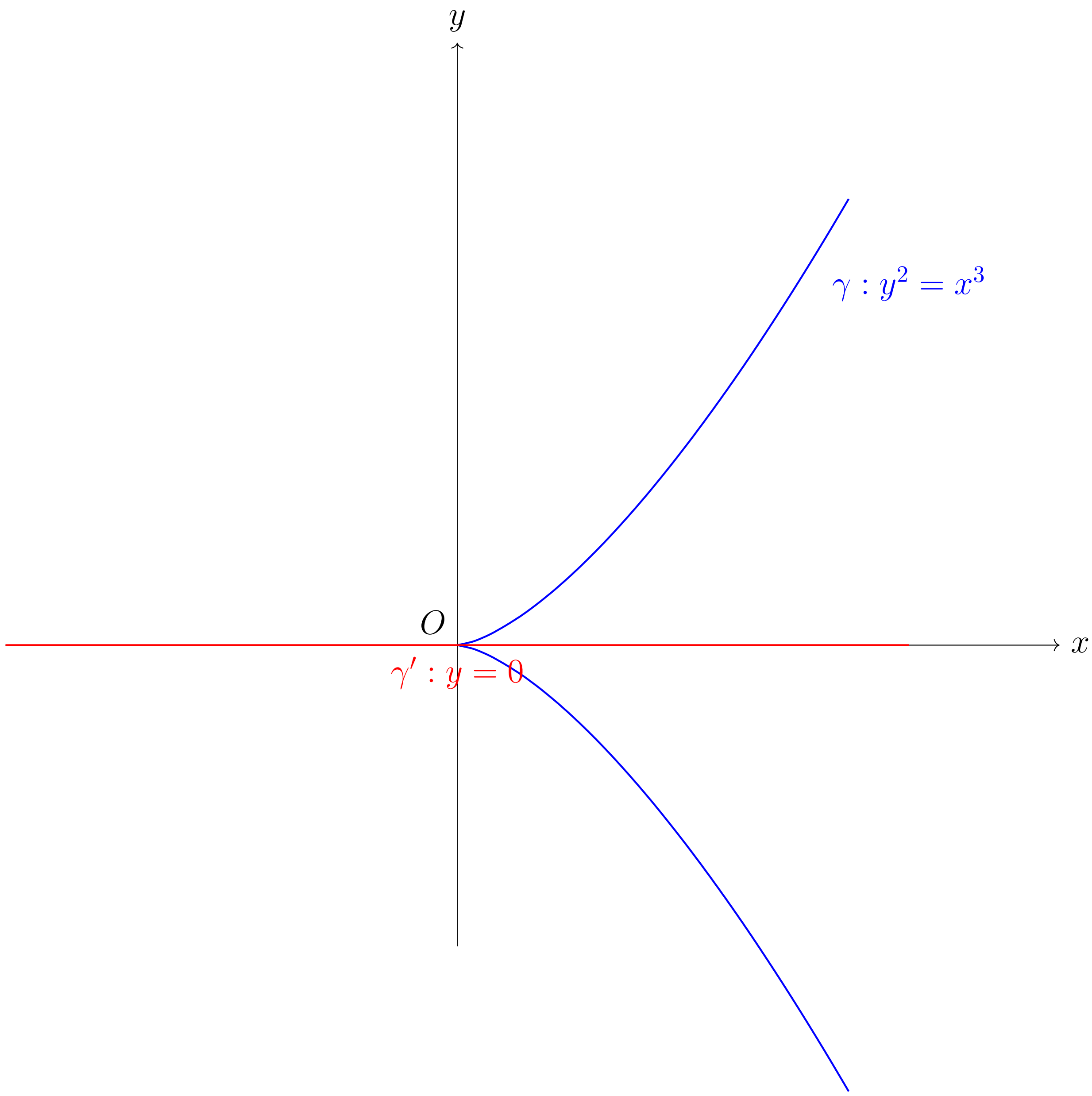


Figure 2.4. A cuspidal cubic curve

The problem of finding **all curves with second-order contact** generalises naturally to the study of **envelopes**—curves that are tangent to each member of a family of curves.

**Definition** (Envelope)**.** Suppose $S\{\gamma_\alpha\}$ is a family of smooth curves on a surface, depending on a parameter $\alpha$.

A smooth curve $\gamma$ is called an **envelope** of the family $S$ if

- for every point on the curve $\gamma$ it is possible to give a curve $\gamma_\alpha$ of the family which is tangent to the curve $\gamma$ at this point,

- for every curve $\gamma_\alpha$ of the family it is possible to give a point on the curve $\gamma$ at which the curve $\gamma_\alpha$ is tangent to $\gamma$,
- no curve of the family has a segment in common with the curve $\gamma$.

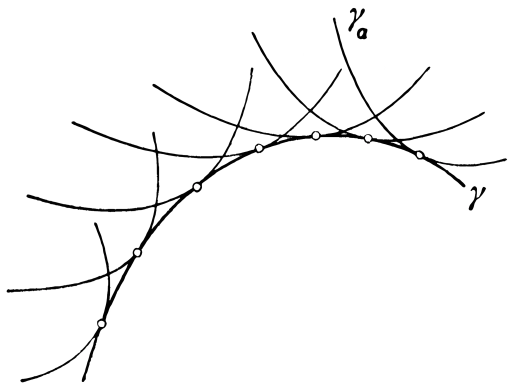


Figure 2.5. An envelope $\gamma$ of the family of smooth curves $S\{\gamma_\alpha\}$.

This arises when we consider:

- A *family* of candidate curves $\{\gamma_\alpha\}$ (e.g., circles of varying radii)
- The *envelope* as the curve tangent to all $\gamma_\alpha$, where each point of contact achieves (at least) first-order contact

*Example.* A smooth curve not having rectilinear arcs is the envelope of its tangents (Figure 2.6):

- A parabola-like curve (blue) representing the original smooth curve without any straight segments.
- Multiple tangent lines (light red) drawn at various points along the curve.
- One highlighted tangent (dark red) with its contact point marked.
- The curve is shown as the envelope of all its tangent lines.
- Each tangent touches the curve at exactly one point (1st-order contact).

**Theorem 2.5.1.** *Let $\{\gamma_\alpha\}$ be a $C^1$-family of plane curves defined by $\varphi(x, y, \alpha) = 0$, where $\varphi$ is $C^2$ and satisfies:*

- *Regularity: $\nabla_{x,y}\varphi = \varphi_x^2 + \varphi_y^2 \neq 0$ and $\varphi_\alpha \neq 0$ at all candidate points*

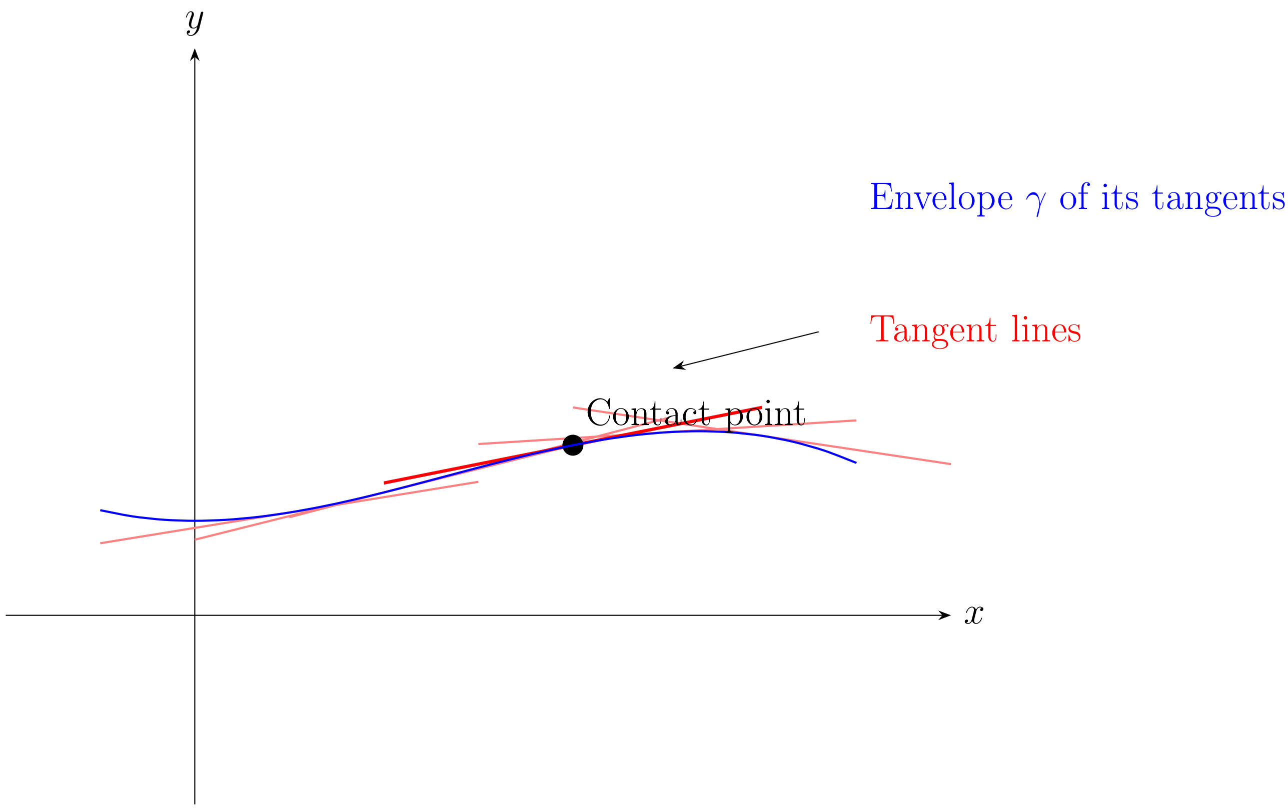


Figure 2.6. To example of the envelope

- *Transversality:* $\det \begin{pmatrix} \varphi_x & \varphi_y \\ \varphi_{x\alpha} & \varphi_{y\alpha} \end{pmatrix} \neq 0$

*Then the envelope* $\Gamma$ *is obtained by solving:*

$$\begin{cases} \varphi(x, y, \alpha) = 0 \\ \varphi_\alpha(x, y, \alpha) = 0, \end{cases} \tag{2.9}$$

*and eliminating* $\alpha$ *from these equations.*

*Remark.* The system of equations ( 2.9) can generally be satisfied by curves that are not envelopes.
For example, the equation of the envelope of the family of curves

$$(x - \alpha)^3 + (y - \alpha)^3 - 3(x - \alpha)(y - \alpha) = 0,$$

is satisfied by the line $x = y$, which, however, is not an envelope. This straight line consists of nodal points of the curves of the family (Figure 2.7). The theorem's transversality condition correctly excludes the degenerate solution at the origin.

*Proof. Necessity*:

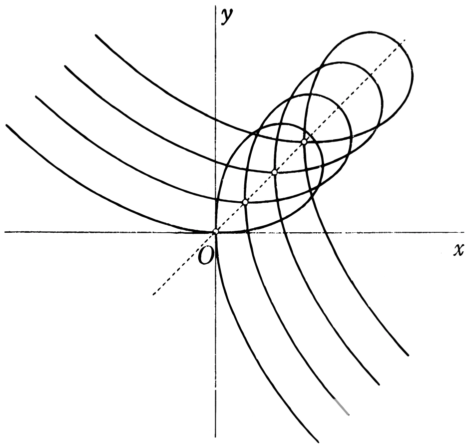


Figure 2.7

- Let $\Gamma$ be parametrized as $(x(s), y(s))$ with tangent vector $\mathbf{T}$.
- For each $s$, $\exists \alpha(s)$ such that $\gamma_{\alpha(s)}$ contacts $\Gamma$ at $(x(s), y(s))$.
- Total derivative along $\Gamma$:

$$\frac{d}{ds}\varphi(x(s), y(s), \alpha(s)) = \varphi_x x' + \varphi_y y' + \varphi_\alpha \alpha' = 0$$

- Since $\gamma_{\alpha(s)}$ is tangent to $\Gamma$, we have $\varphi_x x' + \varphi_y y' = 0$. Thus $\varphi_\alpha \alpha' = 0$.
- If $\alpha'(s) \neq 0$ (non-degenerate contact), then $\varphi_\alpha = 0$.

*Sufficiency*:

- Solutions to $\varphi = \varphi_\alpha = 0$ define a regular curve $\Gamma$ by the Implicit Function Theorem (using transversality).
- For fixed $\alpha$, the system $\varphi(x, y, \alpha) = \varphi_\alpha(x, y, \alpha) = 0$ gives a point where $\gamma_\alpha$ contacts $\Gamma$.
- The non-coincidence condition follows from $\varphi$ being $C^2$ and the regularity assumptions.

□

## 2.6 Problems Corner

### Problem 1

Find a curve expressed with an equation

$$\boldsymbol{r} = \boldsymbol{r}(t), \quad c < t < d,$$

with vector function $\boldsymbol{r}(t)$ satisfying condition

$$\boldsymbol{r}' = \lambda(t)\boldsymbol{a}$$

with $\lambda(t) > 0$ the continius function on $(c, d)$ and $\boldsymbol{a}$ arbitrary non-zero vector.

**Solution**
The given differential equation $\boldsymbol{r}'(t) = \lambda(t)\boldsymbol{a}$ has the solution:

$$\boldsymbol{r}(t) = \left(\int \lambda(t)\,dt\right)\boldsymbol{a} + \boldsymbol{C},$$

where $\boldsymbol{C}$ is a constant vector. Since $\lambda(t) > 0$ and is continuous, the antiderivative is strictly increasing.

**Geometric Interpretation**

- If $\int \lambda(t)\,dt$ is finite at both $c$ and $d$, the curve is a **line segment**.
- If the antiderivative diverges at one endpoint, the curve is a **ray**.
- If the antiderivative diverges at both endpoints, the curve is the **entire line**.

The direction of the curve is determined by $\boldsymbol{a}$, and the initial point is $\boldsymbol{C}$.

**Problem 2**
Deduce the equation of the tangent line and normal plane to the curve

$$\begin{cases} x = \cosh t \\ y = \sinh t \\ z = ct. \end{cases}$$

**Solution**
The curve is given by the vector function:

$$\mathbf{r}(t) = (\cosh t, \sinh t, ct).$$

The derivative of $\mathbf{r}(t)$ is:

$$\mathbf{r}'(t) = (\sinh t, \cosh t, c).$$

**1. Tangent Line at $t = t_0$:** The tangent line in parametric form is:

$$\mathbf{L}(s) = \mathbf{r}(t_0) + s\mathbf{r}'(t_0),$$

which expands to:

$$\mathbf{L}(s) = (\cosh t_0 + s\sinh t_0, \sinh t_0 + s\cosh t_0, ct_0 + sc).$$

The symmetric (canonical) equations of the tangent line are:

$$\frac{x-\cosh t_0}{\sinh t_0} = \frac{y-\sinh t_0}{\cosh t_0} = \frac{z-ct_0}{c}.$$

**2. Normal Plane at $t = t_0$:** The normal plane has $\mathbf{r}'(t_0)$ as its normal vector. Using the point-normal form:

$$\sinh t_0(x-\cosh t_0) + \cosh t_0(y-\sinh t_0) + c(z-ct_0) = 0.$$

Simplifying:

$$\sinh t_0\, x + \cosh t_0\, y + cz = \sinh t_0 \cosh t_0 + \cosh t_0 \sinh t_0 + c^2 t_0.$$

Using the identity $\sinh(2t_0) = 2\sinh t_0 \cosh t_0$, the equation becomes:

$$\sinh t_0\, x + \cosh t_0\, y + cz = \sinh(2t_0) + c^2 t_0.$$

**Problem 3**
Deduce the equations of the osculating plane to the helix

$$\begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 4t \end{cases}$$

at the point $t = 0$.

**Solution**
The helix is parameterised by:

$$\mathbf{r}(t) = (2\cos t, 2\sin t, 4t)$$

We need to find the osculating plane and normal line at $t = 0$.
**Compute the position vector at $t = 0$:**

$$\mathbf{r}(0) = (2\cos 0, 2\sin 0, 4\cdot 0) = (2, 0, 0)$$

**Compute the first derivative (tangent vector):**

$$\mathbf{r}'(t) = (-2\sin t, 2\cos t, 4)$$

At $t = 0$:

$$\mathbf{r}'(0) = (0, 2, 4)$$

**Compute the second derivative:**

$$\mathbf{r}''(t) = (-2\cos t, -2\sin t, 0)$$

At $t = 0$:

$$\mathbf{r}''(0) = (-2, 0, 0)$$

**Equation of the osculating plane:**

$$\begin{vmatrix} \tilde{x} - x(t), & \tilde{y} - y(t), & \tilde{z} - z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = 0.$$

Substituting the values at $t = 0$:

$$\begin{vmatrix} \tilde{x} - 2 & \tilde{y} - 0 & \tilde{z} - 0 \\ 0 & 2 & 4 \\ -2 & 0 & 0 \end{vmatrix} = 0.$$

Expanding the determinant along the first row:

$$(\tilde{x} - 2)\begin{vmatrix} 2 & 4 \\ 0 & 0 \end{vmatrix} - \tilde{y}\begin{vmatrix} 0 & 4 \\ -2 & 0 \end{vmatrix} + \tilde{z}\begin{vmatrix} 0 & 2 \\ -2 & 0 \end{vmatrix} = 0.$$

Computing the $2 \times 2$ determinants:

$$(\tilde{x} - 2)(2 \cdot 0 - 4 \cdot 0) - \tilde{y}(0 \cdot 0 - 4 \cdot (-2)) + \tilde{z}(0 \cdot 0 - 2 \cdot (-2)) = 0.$$

Simplifying:

$$0 - \tilde{y}(0 + 8) + \tilde{z}(0 + 4) = 0 \implies -8\tilde{y} + 4\tilde{z} = 0.$$

Dividing by 4:

$$-2\tilde{y} + \tilde{z} = 0 \quad \text{or equivalently} \quad \tilde{z} = 2\tilde{y}.$$

Thus, the equation of the osculating plane is:

$$2y - z = 0.$$

**Problem 4**

Deduce the equation of the principal normal and binormal to the curve

$$\begin{cases} x = t \\ y = t^2 \\ z = t^3 \end{cases}$$

at the point $t = 1$.

**Solution**
**Position vector int point of interst and its derivatives:**

$$\begin{aligned} \boldsymbol{r}(t) &= (t, t^2, t^3), \\ \boldsymbol{r}(1) &= (1, 1, 1); \\ \boldsymbol{r}'(t) &= (1, 2t, 3t^2), \\ \boldsymbol{r}'(1) &= (1, 2, 3); \\ \boldsymbol{r}''(t) &= (0, 2, 6t), \\ \boldsymbol{r}''(1) &= (0, 2, 6). \end{aligned}$$

**Direction vectors of binormal and principal normal:**
Suppose $\boldsymbol{n}$ is the normal vector of the osculating plane. Obviously, direction vector of binormal $\boldsymbol{b}$ corresponds with $n$:

$$\boldsymbol{b} = \alpha \boldsymbol{n}, \quad \alpha \neq 0.$$

The direction vector $\boldsymbol{p}$ of principal normal must be perpendicular both $n$ and tangent vector $\boldsymbol{r}'(t)$, hence

$$\boldsymbol{p} = \beta \boldsymbol{n} \times \boldsymbol{r'(t)}, \quad \beta \neq 0.$$

With known form of the equation of osculating plane (2.5), the expression for $\boldsymbol{n}$ and for $\boldsymbol{b}$ taking $\alpha = 1$ is

$$\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i}, & \boldsymbol{j}, & \boldsymbol{k} \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix} = \begin{vmatrix} \boldsymbol{i}, & \boldsymbol{j}, & \boldsymbol{k} \\ 1 & 2 & 3 \\ 0 & 2 & 6 \end{vmatrix} =$$
$$\boldsymbol{i}(2 \cdot 6 - 3 \cdot 2) - \boldsymbol{j}(1 \cdot 6 - 3 \cdot 0) + \boldsymbol{k}(1 \cdot 2 - 2 \cdot 0) = 6\boldsymbol{i} - 6\boldsymbol{j} + 2\boldsymbol{k}.$$

Particular case for the direction vector of binormal $\boldsymbol{p}$;

$$\boldsymbol{p} = \boldsymbol{b} \times \boldsymbol{r'(t)} = \begin{vmatrix} \boldsymbol{i}, & \boldsymbol{j}, & \boldsymbol{k} \\ 6 & -6 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -22\boldsymbol{i} - 16\boldsymbol{j} + 18\boldsymbol{k}$$

So we can take direction vectors as $\boldsymbol{b}(3, -3, 1)$ and $\boldsymbol{p}(11, 8, -9)$
**Canonical equation of binormal**

The binormal has direction vector $\boldsymbol{b}(3,-3,1)$ and initial point $\boldsymbol{r}(t)(1,1,1)$. Hence, its canonical equation is

$$\frac{x-1}{3}=\frac{y-1}{-3}=\frac{z-1}{1}$$

**Canonical equation of principal normal**

The principal normal has direction vector $\boldsymbol{p}(11,8,-9)$ and initial point $\boldsymbol{r}(t)(1,1,1)$. Hence, its canonical equation is

$$\frac{x-1}{11}=\frac{y-1}{8}=\frac{z-1}{-9}.$$

**Problem 5**

Find the contact order between $y=x^3$ and $y=0$ at $x=0$.

**Solution**

- **Fixed curve ($\gamma$)**: Define implicitly via $\varphi(x,y)=y-x^3=0$.
- **Variable curve ($\gamma'$)**: Parametrize as $(x(t),y(t))=(t,0)$.

$$\Phi(t):=\varphi(x(t),y(t))=\varphi(t,0)=0-t^3=-t^3.$$

$$\begin{aligned}
\Phi(0)&=0 \quad \text{(0th-order condition)},\\
\Phi'(t)&=-3t^2 \Rightarrow \Phi'(0)=0 \quad \text{(1st-order condition)},\\
\Phi''(t)&=-6t \Rightarrow \Phi''(0)=0 \quad \text{(2nd-order condition)},\\
\Phi'''(t)&=-6 \Rightarrow \Phi'''(0)=-6\neq 0 \quad \text{(3rd-order)}.
\end{aligned}$$

- All derivatives $\Phi^{(k)}(0)=0$ for $k=0,1,2$.
- The first non-zero derivative occurs at $k=3$.

By Theorem 2.3.1, the curves $y=x^3$ and $y=0$ have **exactly 2nd-order contact** at $(0,0)$.

- **0th-order**: Curves intersect at $(0,0)$.
- **1st-order**: Share horizontal tangent ($y=0$).

- **2nd-order**: Both have zero curvature at $x = 0$.
- **3rd-order**: Differ in their rate of curvature change (third derivatives: 6 vs. 0).

For $P(t) = (t, 0) \in \gamma'$:

$$\begin{aligned} h &= \text{dist}(P, \gamma) = |t^3|, \\ d &= \text{dist}(P, O) = |t|, \\ \frac{h}{d^2} &= \frac{|t^3|}{t^2} = |t| \to 0 \quad \text{(2nd-order)}, \\ \frac{h}{d^3} &= \frac{|t^3|}{|t^3|} = 1 \not\to 0 \quad \text{(not 3rd-order)}. \end{aligned}$$

This confirms the conclusion independently.

**Problem 6**

Find all circles that have second-order contact with the parabola $y = x^2$ at its vertex $(0, 0)$.

**Solution**

- **Parabola (fixed curve $\gamma$):** $\varphi(x, y) = y - x^2 = 0$
- **Circle (family $\gamma'$):** Center at $(0, b)$, radius $r$:

$$\psi(x, y) = x^2 + (y - b)^2 - r^2 = 0$$

- For contact at $(0, 0)$, the circle must pass through the origin:

$$0^2 + (0 - b)^2 = r^2 \implies r = |b|$$

We employ the bottom semicircle ($y \leq b$):

$$(x(t), y(t)) = (t, b - \sqrt{b^2 - t^2}), \quad t \in (-\epsilon, \epsilon)$$

Next, we apply contact conditions. Define $\Phi(t) = \varphi(x(t), y(t)) = \left(b - \sqrt{b^2 - t^2}\right) - t^2$.

For second-order contact at $t = 0$:

$$\begin{aligned}
&(0) \quad \Phi(0) = 0 \quad (\text{Automatic from } r = |b|) \\
&(1) \quad \Phi'(0) = \lim_{t \to 0} \frac{t}{\sqrt{b^2 - t^2}} - 2t = 0 \\
&(2) \quad \Phi''(0) = \lim_{t \to 0} \left( \frac{b^2}{(b^2 - t^2)^{3/2}} - 2 \right) = \frac{1}{|b|} - 2 = 0
\end{aligned}$$

From condition (2):

$$\frac{1}{|b|} = 2 \implies b = \pm\frac{1}{2}$$

*Answer:* these circles are

$$x^2 + \left(y \mp \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

These are the **osculating circles** to the parabola at its vertex:

- Radius $\frac{1}{2}$ matches the parabola's curvature $\kappa = 2$ at $(0, 0)$
- Centers lie on the y-axis at $(0, \frac{1}{2})$ and $(0, -\frac{1}{2})$

For $b = \frac{1}{2}$:

$$y = \frac{1}{2} - \sqrt{\frac{1}{4} - x^2} \approx x^2 + x^4 + \cdots \quad (\text{Agrees to second order})$$

**Definition** (Osculating circle)**.** Let $\gamma$ be a regular plane curve (order of regularity $k = 2$) parametrised by $\mathbf{r}(t)$, and let $P$ be a point on $\gamma$. The **osculating circle** to $\gamma$ at $P$ is the unique circle that satisfies:

1. **Contact of order** $\geq 2$ with $\gamma$ at $P$ (per Theorem 2.3.1), meaning:
   - It intersects $\gamma$ at $P$ (0th-order contact),
   - Shares the same tangent line as $\gamma$ at $P$ (1st-order contact),
   - Shares the same curvature as $\gamma$ at $P$ (2nd-order contact).
2. Its radius equals the **radius of curvature** $R = 1/\kappa$, where $\kappa$ is the curvature of $\gamma$ at $P$.
3. Its center lies on the **normal line** to $\gamma$ at $P$ in the concavity direction.

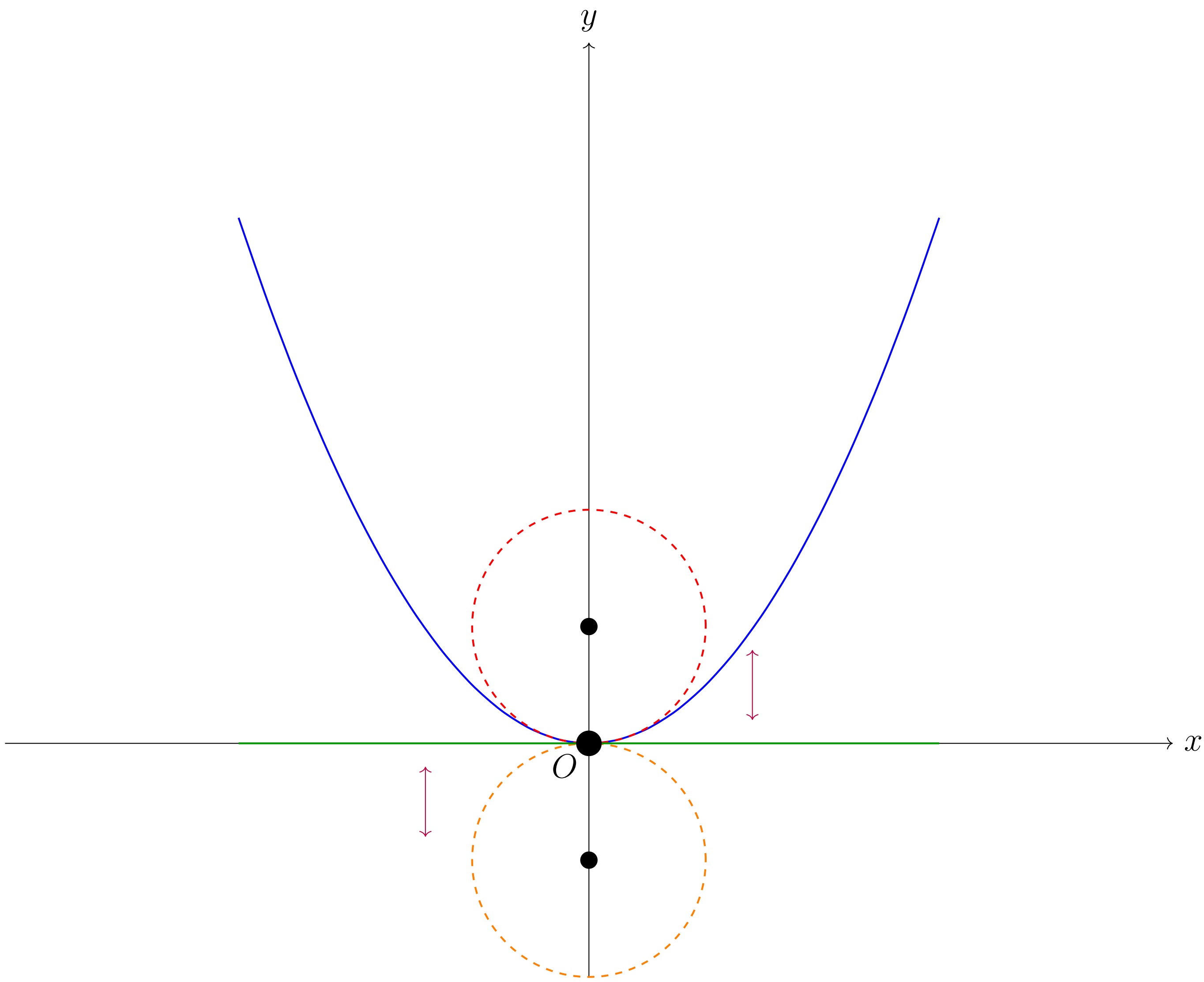


Figure 2.8. Visualisation of second-order contact between the parabola $y = x^2$ (blue) and its osculating circles at the vertex $O(0, 0)$.
**Key elements:**
1. Parabola $y = x^2$
2. Upper osculating circle (center $(0, 0.5)$, radius 0.5)
3. Lower osculating circle (center $(0, -0.5)$, radius 0.5)
4. Tangent line $y = 0$
5. Radius indicators showing $R = \frac{1}{2}$

The upper circle demonstrates physical second-order contact (matching position, tangent, and curvature), while the lower circle is a mathematical solution.

*Remark.* The exact meaning of this notion will be revealed while investigating the curvatures.

**Problem 7**
Given:

- A family of curves $\gamma_{\alpha_1,\ldots,\alpha_n}$ defined implicitly by $\varphi(x, y, \alpha_1, \ldots, \alpha_n) = 0$
- A regular $C^{n-1}$ curve $\gamma$ parametrized by $(x(t), y(t))$ with $\gamma(0) = O$

Find all parameter tuples $(\alpha_1, \ldots, \alpha_n)$ such that $\gamma_{\alpha_1,\ldots,\alpha_n}$ has $(n-1)$**-th order contact** with $\gamma$ at $O$. I.e. find a curve in the $\gamma_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ family that has $(n-1)$-th order contact with $\gamma$ at $O$.

**Solution**
**Summary of the solution method**
In agreement with the theorem we proved above, if the curve $\gamma$ has contact of order $n-1$ with the curve $\gamma_{\alpha_1,\alpha_2,\ldots,\alpha_n}$ at the point $O$, then

$$\varphi = 0, \quad \frac{d}{dt}\varphi = 0, \quad ..., \quad \frac{d^{(n-1)}}{dt^{(n-1)}}\varphi = 0$$

for the value of $t$ corresponding to the point $O$. From this system, we find the values of the parameters $\alpha_1$, $\alpha_2$, ..., $\alpha_n$, for which the curve $\gamma$ has the indicated property.
**Step 1: Define the Contact Function**

$$\Phi(t) := \varphi(x(t), y(t), \alpha_1, \ldots, \alpha_n)$$

**Step 2: Impose Contact Conditions**
For $(n-1)$-th order contact at $t = 0$:

$$\begin{aligned} \Phi(0) &= 0 && \text{(0th-order)} \\ \Phi'(0) &= 0 && \text{(1st-order)} \\ &\vdots \\ \Phi^{(n-1)}(0) &= 0 && \text{((}n-1\text{)-th order)} \end{aligned}$$

**Step 3: Solve the System**

- For polynomial families: Equate Taylor coefficients
- For implicit families: Use implicit differentiation
- Discard solutions where $\Phi^{(n)}(0) = 0$ (higher contact)

**Step 4: Verify Geometric Conditions**
Ensure:

- Curves are transverse (avoid degenerate contact)
- Curvature/torsion match for $n \geq 2$

*Remark*(Special Cases)

- **No solution**: If $\gamma$ has a cusp at $O$, no smooth $\gamma'$ may satisfy the conditions
- **Non-uniqueness**: For $n = 2$, expect discrete solutions (e.g., two osculating circles)
- **Envelope connection**: When $n = 1$, solutions form the envelope of the family (see below for details).

**Problem 8**
Find cubic $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3$ with 3rd-order contact to $y = \sin x$ at $x = 0$.

**Solution**

### Conditions for Third-Order Contact:

For the cubic to have third-order contact with $y = \sin x$ at $x = 0$, the following must hold:

$$\begin{cases} \sin 0 = \alpha_0 \\ \cos 0 = \alpha_1 \\ -\sin 0 = 2\alpha_2 \\ -\cos 0 = 6\alpha_3 \end{cases}$$

Solving these conditions:

$$\begin{cases} \alpha_0 = 0 \\ \alpha_1 = 1 \\ \alpha_2 = 0 \\ \alpha_3 = -\frac{1}{6} \end{cases}$$

Thus, the cubic curve is:

$$y = x - \frac{x^3}{6}$$

**Verification:**

To verify the order of contact, compute the derivatives of the difference $\sin x - \left(x - \frac{x^3}{6}\right)$ at $x = 0$:

$$\sin 0 - \left(0 - \frac{0^3}{6}\right) = 0$$

$$\cos 0 - \left(1 - \frac{3 \cdot 0^2}{6}\right) = 0$$

$$-\sin 0 - \left(0 - \frac{6 \cdot 0}{6}\right) = 0$$

$$-\cos 0 - (0 - 1) = 0$$

$$\sin 0 - (0 - 0) = 0$$

The fourth derivative is zero, so the cubic has **third-order contact** with $y = \sin x$ at $x = 0$.

**Problem 9**

Find the envelope of the family of straight lines which form a triangle $XOY$ of area $2a^2$ with the coordinate axes.

**Solution**

Each line forms a triangle with the axes, so its intercept form is:

$$\frac{x}{x_0} + \frac{y}{y_0} = 1 \quad \text{(where } x_0, y_0 \neq 0\text{)}.$$

The area of triangle $XOY$ is:

$$\frac{1}{2}|x_0 y_0| = 2a^2 \implies |x_0 y_0| = 4a^2.$$

We'll consider the positive case ($x_0 y_0 = 4a^2$), so:

$$y_0 = \frac{4a^2}{x_0}.$$

The family of lines can be written as:

$$\varphi(x, y, x_0) = \frac{x}{x_0} + \frac{x_0 y}{4a^2} - 1 = 0.$$

Here, $x_0$ serves as the parameter $\alpha$ of the family.

The envelope requires:

1. $\varphi(x, y, x_0) = 0,$

2. $\frac{\partial \varphi}{\partial x_0} = 0.$

Compute the partial derivative:

$$\frac{\partial \varphi}{\partial x_0} = -\frac{x}{x_0^2} + \frac{y}{4a^2} = 0.$$

Solve for $y$:

$$y = \frac{4a^2 x}{x_0^2}.$$

Now substitute back into $\varphi = 0$. From $\varphi = 0$:

$$\frac{x}{x_0} + \frac{x_0}{4a^2}\left(\frac{4a^2 x}{x_0^2}\right) - 1 = 0 \implies \frac{2x}{x_0} - 1 = 0.$$

Solve for $x_0$:

$$x_0 = 2x.$$

Substitute $x_0 = 2x$ into the expression for $y$:

$$y = \frac{4a^2 x}{(2x)^2} = \frac{a^2}{x}.$$

Eliminate the parameter $x_0$. The envelope is the curve:

$$xy = a^2.$$

This is a **hyperbola**.

*Verification*

- For any point $(x, y)$ on $xy = a^2$, the tangent line at $(x, y)$ has intercepts $2x$ and $2y$, forming a triangle of area $2a^2$.
- Example: When $a = 1$, the line $\frac{x}{2} + \frac{y}{2} = 1$ forms a triangle of area 2 and touches $xy = 1$ at $(1, 1)$.

**Problem 10**

Find the envelope of the family of straight lines on which the coordinate axes cut off a segment of constant length $a$.

**Solution**

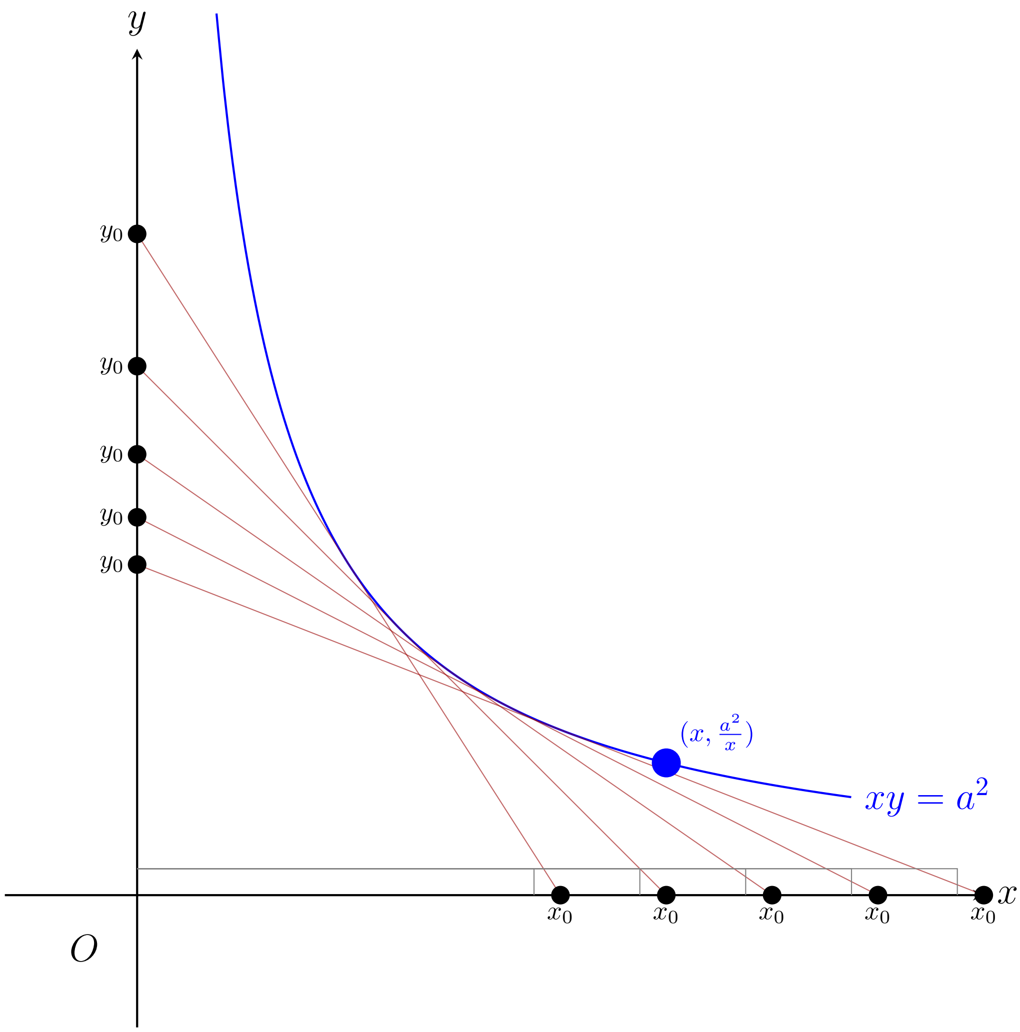


Figure 2.9. Envelope of lines forming triangles of area $2a^2$ with the axes. The hyperbola $xy = a^2$ (blue) is the envelope curve, with representative tangent lines (red) and their intercepts.

Each line has intercepts $(x_0, 0)$ and $(0, y_0)$ with the axes. The intercept length condition gives:

$$\sqrt{x_0^2 + y_0^2} = a \implies y_0 = \pm\sqrt{a^2 - x_0^2}.$$

The line equation in intercept form is:

$$\varphi(x, y, x_0) = \frac{x}{x_0} + \frac{y}{\sqrt{a^2 - x_0^2}} - 1 = 0.$$

We apply the envelope condition now. We require both $\varphi = 0$ and $\frac{\partial \varphi}{\partial x_0} = 0$:

$$\frac{\partial \varphi}{\partial x_0} = -\frac{x}{x_0^2} + \frac{y x_0}{(a^2 - x_0^2)^{3/2}} = 0.$$

Solving for $y$:

$$y = \frac{x(a^2 - x_0^2)^{3/2}}{x_0^3}.$$

Substitute $y$ into $\varphi = 0$:

$$\frac{x}{x_0} + \frac{x(a^2 - x_0^2)}{x_0^3} = 1 \implies x = \frac{x_0^3}{a^2}.$$

This gives the parametric equations:

$$x = \frac{x_0^3}{a^2}, \quad y = \frac{(a^2 - x_0^2)^{3/2}}{a^2}.$$

Using $x_0 = a \sin^{2/3} \theta$, we obtain the **astroid**:

$$x^{2/3} + y^{2/3} = a^{2/3}.$$

For any point $(x, y)$ on the astroid:

- The tangent line has intercepts $(x_0, 0)$ and $(0, y_0)$ satisfying $x_0^2 + y_0^2 = a^2$.
- Example: At $\theta = 45^\circ$, the tangent line $x + y = a/\sqrt{2}$ touches the astroid at $\left(\frac{a}{2^{3/2}}, \frac{a}{2^{3/2}}\right)$.

**Problem 11**

Find the envelope of the trajectories of a material point ejected from the origin of coordinates with initial velocity $v_0$.

**Solution**

For a projectile launched at angle $\theta$ in the gravity field expressed with acceliration $g$. We employ equations derived in the school course of physics:

$$x(t) = v_0 t \cos \theta, \quad y(t) = v_0 t \sin \theta - \frac{1}{2} g t^2$$

Eliminating $t$ gives the implicit form:

$$\varphi(x, y, \theta) = y - x \tan \theta + \frac{g x^2}{2 v_0^2} \sec^2 \theta = 0$$

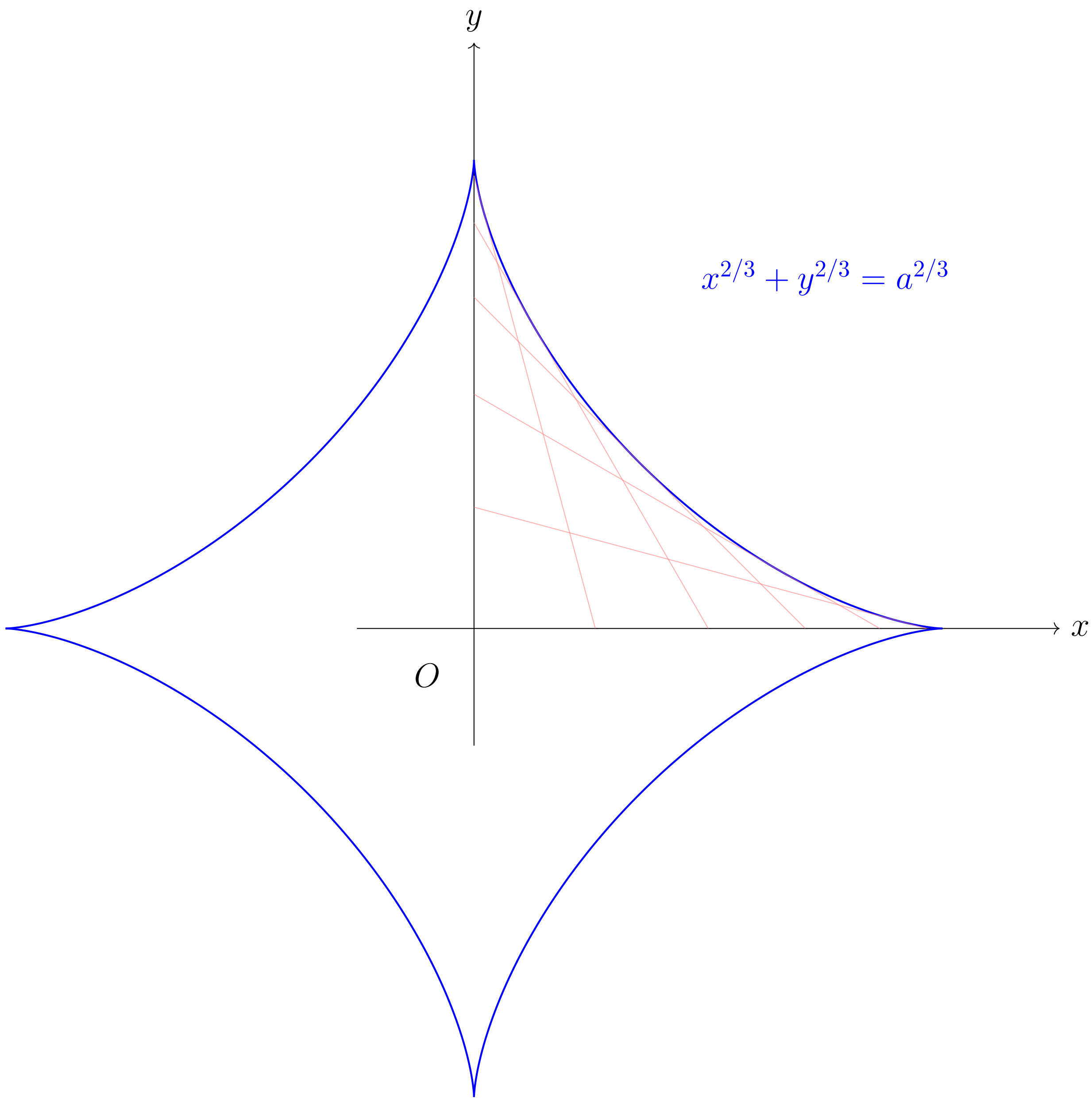


Figure 2.10. The astroid (blue) as the envelope of lines (red) with fixed intercept length $a$.

We require:

$$\varphi = 0$$

$$\frac{\partial \varphi}{\partial \theta} = -x \sec^2 \theta + \frac{gx^2}{v_0^2} \tan \theta \sec^2 \theta = 0$$

Simplifying:

$$\tan \theta = \frac{v_0^2}{gx}$$

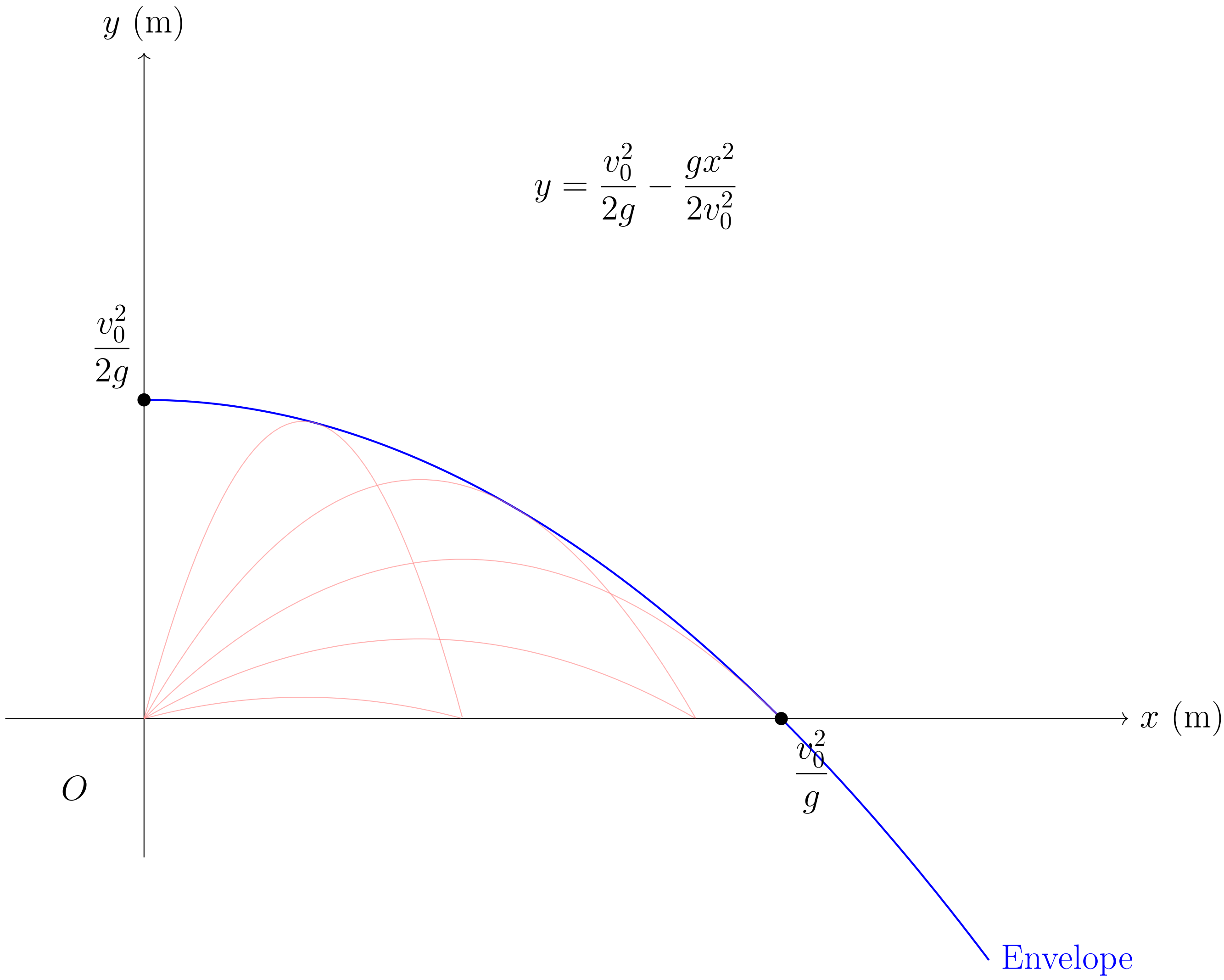


Figure 2.11. Envelope of projectile trajectories (blue) with initial velocity $v_0$. Sample trajectories (red) are shown for launch angles 15° to 75°.

Substitute back into $\varphi = 0$:

$$y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}$$

The envelope represents:

- The **maximum height** $\frac{v_0^2}{2g}$ when launched vertically ($\theta = 90°$)
- The **maximum range** $\frac{v_0^2}{g}$ when launched at 45°
- The boundary of all reachable points (no projectile can cross the blue curve)

**Problem 12**

What is the angle at which the curves $xy = a$ and $x^2 - y^2 = b$ intersect?

# Chapter 3

# The Concepts of Curvature and Torsion

## 3.1 The Arc Length of a Curve

Suppose $\gamma$ is an arbitrary curve, which is the image of an open interval $g$ or a circumference $k$ under a continuous and locally one-to-one mapping $\varphi$ into space.

**Definition** (Segment of the curve)**.** A **segment of the curve** $\gamma$ is the image of an arbitrary closed segment $\Delta$, belonging to that open interval $g$ (if the curve is parameterized by $g$) or to the arc $\kappa$ of the circumference $k$ (if the curve is parameterized by $k$) under the mapping $\varphi$.

The endpoints of the segment of the curve are the images of the endpoints of the open interval $\Delta$ or the endpoints of the arc $\kappa$ of the circumference.

We employ notation $\gamma|_{[a,c]}$ or $\gamma|_\Delta$ to express the segment corresponding with closed segments $[a, c]$ or $\Delta$.

**Theorem 3.1.1.** *Let $\varphi_1, \varphi_2$ be continuous and locally injective mappings on the open interval $g$ defining the same curve $\gamma = \varphi_1(g) = \varphi_2(g)$. If there exists a homeomorphism $\psi\colon g \to g$ such that $\varphi_1 = \varphi_2 \circ \psi$, then the sets of segments of $\gamma$ induced by $\varphi_1$ and $\varphi_2$ coincide.*

*Proof. Proof summary.* Since $\psi$ is a homeomorphism, it is strictly monotonic. For any closed interval $[a, b] \subseteq g$, $\psi([a, b])$ is a closed interval $[c, d] \subseteq g$ (preserving endpoints due to continuity and monotonicity). Thus:

$$\varphi_1([a, b]) = \varphi_2(\psi([a, b])) = \varphi_2([c, d]),$$

which shows that every segment of $\gamma$ induced by $\varphi_1$ corresponds to a segment induced by $\varphi_2$, and vice versa. The sets of segments are therefore identical.

Let us review this proof in detail.

*Step 1: Properties of $\psi$*

- Since $\psi$ is a homeomorphism on an open interval $(a, b)$, it is strictly monotonic (increasing or decreasing) by the Intermediate Value Theorem.

- $\psi$ maps closed intervals $[t_1, t_2] \subseteq (a, b)$ to closed intervals $[\psi(t_1), \psi(t_2)]$ or $[\psi(t_2), \psi(t_1)]$, depending on monotonicity.

- Endpoints are preserved: $\psi(\{t_1, t_2\}) = \{\psi(t_1), \psi(t_2)\}$.

*Step 2: Segment Correspondence*
For any segment $\varphi_1([t_1, t_2])$ induced by $\varphi_1$:

$$\begin{aligned}
\varphi_1([t_1, t_2]) &= (\varphi_2 \circ \psi)([t_1, t_2]) \\
&= \varphi_2(\psi([t_1, t_2])) \\
&= \varphi_2([\psi(t_1), \psi(t_2)]) \quad (\text{or } [\psi(t_2), \psi(t_1)) \text{ if decreasing})
\end{aligned}$$

Thus, every $\varphi_1$-*segment* equals some $\varphi_2$-*segment.*
*Step 3: Reverse Inclusion*
The converse follows by considering $\psi^{-1}$ (which exists since $\psi$ is bijective):

$$\begin{aligned}
\varphi_2([s_1, s_2]) &= \varphi_2(\psi(\psi^{-1}([s_1, s_2]))) \\
&= \varphi_1(\psi^{-1}([s_1, s_2])) \\
&= \varphi_1([\psi^{-1}(s_1), \psi^{-1}(s_2)])
\end{aligned}$$

*Conclusion*
The sets $\{\varphi_1([t_1, t_2]) \mid t_1, t_2 \in (a, b)\}$ and $\{\varphi_2([s_1, s_2]) \mid s_1, s_2 \in (a, b)\}$ are identical. $\square$

The Figure 3.1 visualises this theorem, showing two parametrisations of the same curve. The blue curve $\gamma$ is parametrized by both $\varphi_1$ (red, with parameters $t_i$) and $\varphi_2$ (green, with parameters $s_j$), where $\varphi_1 = \varphi_2 \circ \psi$. The parameter domains are shown below, with $\psi$ mapping between them. Corresponding points on $\gamma$ are connected by grey arrows to emphasise they represent the same geometric locations despite different parameter values.

Suppose $\tilde{\gamma}$ is a segment of the curve $\gamma$ and let $A$ and $B$ be its endpoints. We choose the points $A_0 \equiv A, A_1, ..., A_n \equiv B$, on the segment $\tilde{\gamma}$, proceeding from $A$ to $B$ in the sense that the point $A_i$ lies between $A_{i-1}$ and $A_{i+1}$. We join successive points $A_i$ and $A_{i+1}$ employing rectilinear segments. We then obtain the polygonal arc $\Gamma$ inscribed in the segment $\tilde{\gamma}$ of the curve.

**Definition** (Arc Length)**.** The least upper bound of the lengths of all possible polygonal arcs $\Gamma$, inscribed in the segment $\tilde{\gamma}$ of the curve, will be called the **arc length**, or simply the **arc**, of this segment.

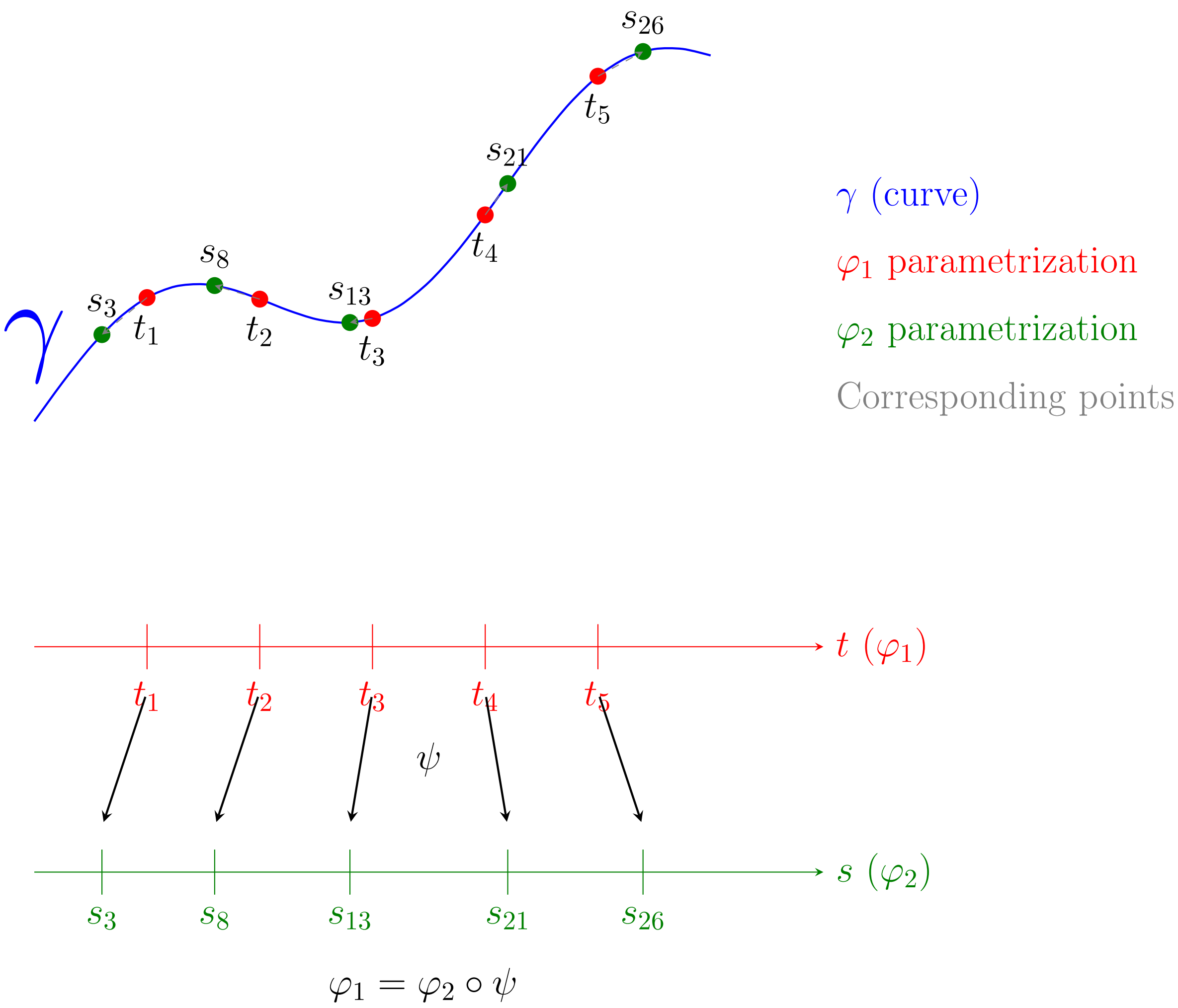


Figure 3.1. Visualisation of Theorem 3.1.1.

We use the notation $s(\tilde{\gamma})$ to denote the arc length of $\tilde{\gamma}$. This notation may be understood as a function mapping all segments of the curve to the real numbers, hence we talk about the *length function*.

See Figure 3.2 illustrating this definition. The smooth curve $\gamma$ (blue) is approximated by a polygonal arc $\Gamma$ (red dashed) whose vertices lie on $\gamma$. The arc length $s(\tilde{\gamma})$ of the curve segment is defined as the least upper bound of the lengths of all such possible inscribed polygonal approximations, where $\|\Gamma_i\|$ denotes the length of each linear segment. As the partition becomes finer ($\|\Gamma_i\| \to 0$), the sum of lengths converges to the true arc length.

**Definition** (Rectifiable Segment)**.** We shall say that the segment $\tilde{\gamma}$ of the curve $\gamma$ is rectifiable if the lengths of all possible polygonal arcs $\Gamma$, inscribed in the segment $\tilde{\gamma}$ of the curve, are uniformly bounded above by some finite number.

**Definition** (Rectifiable Curve)**.** We shall say that the curve $\gamma$ is **rectifiable** if each of its segments is rectifiable.

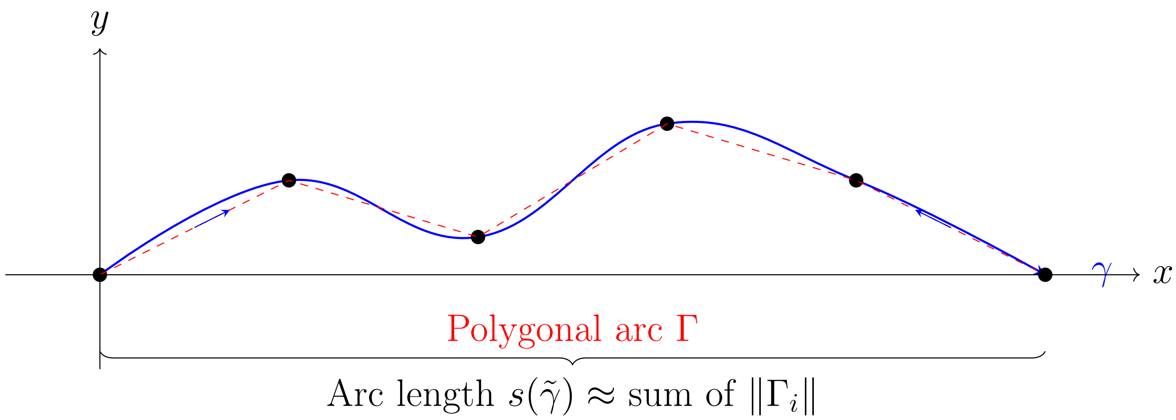


Figure 3.2. Approximation of arc length for a curve $\gamma$ using inscribed polygonal chains

**Theorem 3.1.2** (Fundamental Property of Rectifiable Curves)**.** *If the segment $A'B'$ of the curve $\gamma$ is a subset of the segment $AB$, and if the segment $AB$ is rectifiable, then the segment $A'B'$ is also rectifiable and the length of its arc $s(A'B')$ is less than the arc length $s(AB)$ of the segment $AB$.*

*Proof.* Let $\gamma$ be a rectifiable curve with segments $AB$ and $A'B'$ such that $A'B' \subseteq AB$. We have to prove the statements.

1. $A'B'$ is rectifiable.
2. $s(A'B') \leq s(AB)$, with strict inequality if $A'B' \subsetneq AB$ and $\gamma$ is non-constant on $AB \setminus A'B'$.

We prove each statement separately.

*1. Rectifiability of $A'B'$:*

Since $AB$ is rectifiable, the set of lengths of all polygonal approximations $\Gamma$ of $AB$ is bounded above by $s(AB)$. Any polygonal approximation $\Gamma'$ of $A'B'$ can be extended to a polygonal approximation $\Gamma$ of $AB$ by adding vertices. Thus:

$$\ell(\Gamma') \leq \ell(\Gamma) \leq s(AB),$$

where $\ell$ denotes the length of a polygonal chain. The lengths of all such $\Gamma'$ are bounded above by $s(AB)$, so $A'B'$ is rectifiable.

*2. Inequality $s(A'B') \leq s(AB)$:*

Let $\mathcal{P}_{AB}$ and $\mathcal{P}_{A'B'}$ be the sets of all polygonal approximations of $AB$ and $A'B'$, respectively. For every $\Gamma' \in \mathcal{P}_{A'B'}$, there exists $\Gamma \in \mathcal{P}_{AB}$ such that $\Gamma'$ is a subchain of $\Gamma$. Hence:

$$\ell(\Gamma') \leq \ell(\Gamma) \leq s(AB).$$

Taking the supremum over all $\Gamma' \in \mathcal{P}_{A'B'}$:

$$s(A'B') = \sup_{\Gamma' \in \mathcal{P}_{A'B'}} \ell(\Gamma') \leq s(AB).$$

*3. Strict Inequality when $A'B' \subsetneq AB$:*

Assume $\gamma$ is non-constant on $AB \setminus A'B'$. Then there exists a point $C \in AB \setminus A'B'$ where $\gamma$ is not locally constant. For any polygonal approximation $\Gamma'$ of $A'B'$, we can construct $\Gamma$ by adding $C$ as a vertex, ensuring:

$$\ell(\Gamma') < \ell(\Gamma) \leq s(AB).$$

Thus, $s(A'B')$ cannot equal $s(AB)$. □

*Example.* For the unit circle $\gamma(t) = (\cos t, \sin t)$:

- Let $AB = \gamma([0, \pi])$ (upper semicircle) with $s(AB) = \pi$.
- For $A'B' = \gamma([0, \pi/2])$ (first quadrant), $s(A'B') = \pi/2 < \pi$.

**Theorem 3.1.3** (Additivity of Arc Length for Rectifiable Curves)**.** *If $C$ is a point on the segment $AB$ of the curve which is distinct from both $A$ and $B$, and the segments $AC$ and $CB$ arc rectifiable, then the segment $AB$ is also rectifiable, and*

$$s(AC) + s(CB) = s(AB).$$

*Proof.* Let $\gamma\colon [a, b] \to \mathbb{E}$ be a continuous curve with arc length function $s$ (segment $AB$). For any point $c \in (a, b)$, if the restrictions $\gamma|_{[a,c]}$ (segmanr $AC$) and $\gamma|_{[c,b]}$ (segment $CB$) are rectifiable, we should justify:

1. $\gamma$ is rectifiable on $[a, b]$
2. $s(\gamma|_{[a,b]}) = s(\gamma|_{[a,c]}) + s(\gamma|_{[c,b]})$

*Part 1: Rectifiability of $\gamma$ on $[a, b]$*

- Let $\mathcal{P}_{ac}$ and $\mathcal{P}_{cb}$ be the sets of all polygonal approximations for $\gamma|_{[a,c]}$ and $\gamma|_{[c,b]}$ respectively.
- For any $\Gamma_{ac} \in \mathcal{P}_{ac}$ and $\Gamma_{cb} \in \mathcal{P}_{cb}$, their concatenation $\Gamma_{ab} = \Gamma_{ac} \cup \Gamma_{cb}$ is a polygonal approximation of $\gamma|_{[a,b]}$.
- The lengths satisfy:

$$\ell(\Gamma_{ab}) = \ell(\Gamma_{ac}) + \ell(\Gamma_{cb}) \leq s(\gamma|_{[a,c]}) + s(\gamma|_{[c,b]}) < \infty$$

- Thus, the lengths of all polygonal approximations of $\gamma|_{[a,b]}$ are bounded, proving rectifiability.

*Part 2: Equality of arc lengths*

We show both inequalities:

($\leq$) For any polygonal approximation $\Gamma_{ab}$ of $\gamma|_{[a,b]}$, we can refine it to include $c$ as a vertex (by the density of partitions), yielding:

$$\Gamma_{ab} = \Gamma_{ac} \cup \Gamma_{cb}$$

where $\Gamma_{ac}$ and $\Gamma_{cb}$ approximate $\gamma|_{[a,c]}$ and $\gamma|_{[c,b]}$ respectively. Thus:

$$\ell(\Gamma_{ab}) = \ell(\Gamma_{ac}) + \ell(\Gamma_{cb}) \leq s(\gamma|_{[a,c]}) + s(\gamma|_{[c,b]})$$

Taking supremum over all $\Gamma_{ab}$ gives:

$$s(\gamma|_{[a,b]}) \leq s(\gamma|_{[a,c]}) + s(\gamma|_{[c,b]})$$

($\geq$) For any $\varepsilon > 0$, choose:

- $\Gamma_{ac}$ with $\ell(\Gamma_{ac}) > s(\gamma|_{[a,c]}) - \varepsilon/2$
- $\Gamma_{cb}$ with $\ell(\Gamma_{cb}) > s(\gamma|_{[c,b]}) - \varepsilon/2$

Their concatenation satisfies:

$$\ell(\Gamma_{ac} \cup \Gamma_{cb}) > s(\gamma|_{[a,c]}) + s(\gamma|_{[c,b]}) - \varepsilon$$

Since $\Gamma_{ac} \cup \Gamma_{cb}$ is a valid approximation of $\gamma|_{[a,b]}$:

$$s(\gamma|_{[a,b]}) \geq s(\gamma|_{[a,c]}) + s(\gamma|_{[c,b]}) - \varepsilon$$

As $\varepsilon$ was arbitrary, the inequality holds.

Combining both inequalities proves the equality. □

## 3.2 The Arc Length of a Smooth Curve

**Theorem 3.2.1.** *A smooth curve $\gamma$ is rectifiable. If $\boldsymbol{r} = \boldsymbol{r}(t)$ is its smooth parametrization and $\tilde{\gamma}(a \leq t \leq b)$ is a segment of the curve $\gamma$ then the length of this segment is*

$$s(\tilde{\gamma}) = \int_a^b |\boldsymbol{r}'(t)| dt.$$

*Proof.* We prove both statements sequentially.

*Part 1: Rectifiability*

Since $\boldsymbol{r}(t)$ is $C^1$-class (smooth), $\boldsymbol{r}'(t)$ exists and is continuous on $[a, b]$. By the Extreme Value Theorem, there exists $M > 0$ such that:

$$|\boldsymbol{r}'(t)| \leq M \quad \forall t \in [a, b].$$

For any partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_n = b\}$, the length of the inscribed polygonal path satisfies:

$$\ell(\mathcal{P}) = \sum_{i=1}^{n} |\boldsymbol{r}(t_i) - \boldsymbol{r}(t_{i-1})|.$$

By the Mean Value Theorem applied to each component, there exist points $\xi_i \in (t_{i-1}, t_i)$ such that:

$$|\boldsymbol{r}(t_i) - \boldsymbol{r}(t_{i-1})| \leq \sum_{j=1}^{n} |r_j'(\xi_i^j)|(t_i - t_{i-1}) \leq n \cdot M(t_i - t_{i-1}).$$

Thus $\ell(\mathcal{P}) \leq nM(b - a)$, proving the lengths are bounded and $\gamma$ is rectifiable.

*Part 2: Arc Length Formula*

For any $\varepsilon > 0$, by uniform continuity of $\boldsymbol{r}'$, there exists $\delta > 0$ such that:

$$|t - s| < \delta \Rightarrow |\boldsymbol{r}'(t) - \boldsymbol{r}'(s)| < \varepsilon.$$

Choose a partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$. Then for each $i$:

$$\boldsymbol{r}(t_i) - \boldsymbol{r}(t_{i-1}) = \boldsymbol{r}'(\tau_i)(t_i - t_{i-1}) + \varepsilon_i(t_i - t_{i-1}),$$

where $|\varepsilon_i| < \varepsilon$ by Taylor's theorem. Therefore:

$$||\boldsymbol{r}(t_i) - \boldsymbol{r}(t_{i-1})| - |\boldsymbol{r}'(\tau_i)|(t_i - t_{i-1})| < \varepsilon(t_i - t_{i-1}).$$

Summing over $i$ gives:

$$\left| \ell(\mathcal{P}) - \sum_{i=1}^{n} |\boldsymbol{r}'(\tau_i)|(t_i - t_{i-1}) \right| < \varepsilon(b - a).$$

As $\|\mathcal{P}\| \to 0$, the Riemann sum converges to the integral while $\varepsilon \to 0$, proving:

$$s(\gamma) = \lim_{\|\mathcal{P}\| \to 0} \ell(\mathcal{P}) = \int_a^b |r'(t)| dt.$$

□

## Natural Parameterisation

**Definition** (Natural Parametrization)**.** Let $\gamma$ be a rectifiable curve with parametrisation $\mathbf{r}(t)$ for $t \in I$. The *natural parametrization* of $\gamma$ is the arc-length parametrization $\mathbf{r}(s)$ where:

1. The parameter $s$ measures signed arc length from a base point $\mathbf{r}(t_0)$:

$$s(t) = \begin{cases} \int_{t_0}^{t} |\mathbf{r}'(\tau)| d\tau & t \geq t_0 \\ -\int_{t}^{t_0} |\mathbf{r}'(\tau)| d\tau & t < t_0 \end{cases}$$

2. The curve is reparametrized as $\mathbf{r}(s) := \mathbf{r}(t(s))$, where $t(s)$ is the inverse of $s(t)$.

3. This yields unit rate: $\left|\frac{d\mathbf{r}}{ds}\right| = 1$ wherever differentiable.

**Theorem 3.2.2.** *Let $\gamma$ be a regular curve of class $C^k$ $(k \geq 1)$ or analytic, with $|r'(t)| \neq 0$ everywhere. Then:*

1. *Its natural parametrization $r(s)$ exists and is of the same class ($C^k$ or analytic)*

2. *It satisfies the unit speed condition:*

$$\left|\frac{dr}{ds}\right| = 1 \quad \forall s$$

3. *The arc length parameter $s$ is given by:*

$$s(t) = \int_{t_0}^{t} |r'(\tau)| d\tau$$

Comparison of arbitrary parametrisation versus natural parametrisation provided in the Figure 3.3. The natural parametrisation has: (1) parameters $s$ marking equal arc-length intervals, (2) unit tangent vectors, and (3) spacing invariant under reparametrization.

*Proof.* We prove both statements using the arc-length parametrisation construction.

*Regularity and Unit Speed*

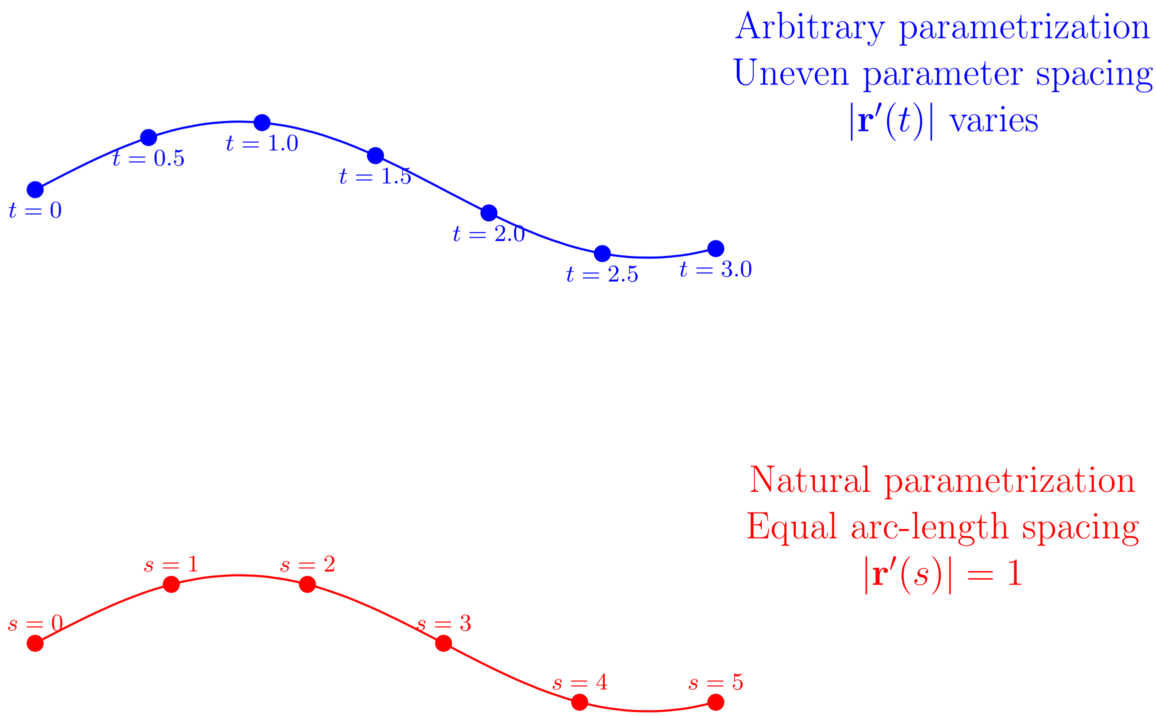


Figure 3.3. Comparison of arbitrary parametrization $\mathbf{r}(t)$ (top) versus natural parametrization $\mathbf{r}(s)$ (bottom)

1. Define the arc length function:

$$s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| d\tau$$

   Since $\mathbf{r}$ is $C^k$ and regular ($|\mathbf{r}'| \neq 0$), $s(t)$ is $C^k$ with $s'(t) = |\mathbf{r}'(t)| > 0$.

$$s'(t) = |\mathbf{r}'(t)|.$$

2. By the Inverse Function Theorem, the inverse $t(s)$ exists and is $C^k$. The natural parametrisation is the composition:

$$\mathbf{r}(s) := \mathbf{r}(t(s))$$

3. Differentiating shows unit speed:

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt}\frac{dt}{ds} = \frac{d\mathbf{r}}{dt}\frac{1}{ds/dt} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

   Thus $|\mathbf{r}'(s)| = 1$. The $C^k$ class follows from the chain rule.

*Parametrisation Independence*

Let $\tilde{\mathbf{r}}(\tilde{t})$ be another parametrization with $|\tilde{\mathbf{r}}'(\tilde{t})| = 1$. Then:

1. The arc length functions are related by:

$$s = \tilde{s} + C \quad \text{(constant)}$$

2. The reparametrization $\tilde{t}(s)$ is affine: $\tilde{t} = \pm s + C$

3. Thus, all natural parametrisations differ only by:

   - Base point ($C$)
   - Orientation ($\pm$)

$\square$

**Example: Helix Parametrization**

Consider the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$:

$$\begin{aligned}
\text{Original speed: } & |\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2} \\
\text{Arc-length: } & s(t) = \int_0^t \sqrt{2}\, d\tau = \sqrt{2}\, t \\
\text{Natural parametrization: } & \mathbf{r}(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) \\
\text{Tangent vector: } & \frac{d\mathbf{r}}{ds} = \left(-\frac{\sin\big(s/\sqrt{2}\big)}{\sqrt{2}}, \frac{\cos\big(s/\sqrt{2}\big)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \\
\text{Verification: } & \left|\frac{d\mathbf{r}}{ds}\right| = \sqrt{\frac{\sin^2 + \cos^2}{2} + \frac{1}{2}} = 1
\end{aligned}$$

**Corollary 3.2.3** (Global Regular Parametrization)**.** *Let $\gamma$ be a regular curve of class $C^k$ ($k \geq 1$) or analytic. Then:*

1. *$\gamma$ admits a global natural parametrization $\mathbf{r}(s)$ of the same class $C^k$ or analytic*

2. *Any reparametrization $\mathbf{r}(\varphi(t))$ with $\varphi\colon I \to \mathbb{R}$ being:*

   - *$C^k$ (respectively analytic)*
   - *Satisfying $\varphi'(t) \neq 0$ everywhere*

   *preserves the regularity and differentiability class*

Let us highlight the points of this corollary.

1. *Existence of Natural Parametrization*:

   - For a regular curve $\mathbf{r}(t) \in C^k$ $(k \geq 1)$ or analytic, the arc-length function:
   $$s(t) = \int_{t_0}^{t} |\mathbf{r}'(\tau)| d\tau$$
   is strictly increasing since $|\mathbf{r}'(t)| > 0$ (by regularity).
   - The inverse function $t(s)$ exists and inherits $C^k$/analytic properties via the **Inverse Function Theorem**, because:
   $$\frac{ds}{dt} = |\mathbf{r}'(t)| > 0$$

2. *Unit Speed Property*:

   - The natural parametrization $\mathbf{r}(s) := \mathbf{r}(t(s))$ satisfies:
   $$\left|\frac{d\mathbf{r}}{ds}\right| = \left|\frac{\mathbf{r}'(t)}{s'(t)}\right| = \frac{|\mathbf{r}'(t)|}{|\mathbf{r}'(t)|} = 1$$
   - This holds globally as $s(t)$ is bijective.

3. *Reparametrization Invariance*:

   - For any reparametrization $\sigma = \varphi(t)$ with $\varphi \in C^k$ (or analytic) and $\varphi'(t) \neq 0$:
   $$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{d\sigma}\varphi'(t) \implies \left|\frac{d\mathbf{r}}{d\sigma}\right| = \frac{|\mathbf{r}'(t)|}{|\varphi'(t)|} \neq 0$$
   - The composition $\mathbf{r} \circ \varphi^{-1}$ preserves:
     - Regularity ($|\frac{d\mathbf{r}}{d\sigma}| \neq 0$)
     - Differentiability class ($C^k$/analytic)

The natural parametrisation "standardises" the curve to unit speed, while regular reparametrizations preserve geometric properties but may change traversal speed.

## 3.3 Curvature and Torsion of a Curve

In the study of curves, we aim to understand their geometric properties beyond simple descriptions, such as length or position. Two fundamental concepts – curvature and torsion – arise naturally when we ask:

- How much does a curve bend? (Curvature)
- How much does a curve twist out of a plane? (Torsion)

### Curvature

Suppose $P$ is an arbitrary point on the regular curve $\gamma$ and $Q$ is a point on $\gamma$ near $P$. We denote the angle between the tangents drawn to the curve at $P$ and $Q$ by $\Delta\theta$ and the arc length of the segment $PQ$ of the curve by $\Delta s$ (Figure 3.4).

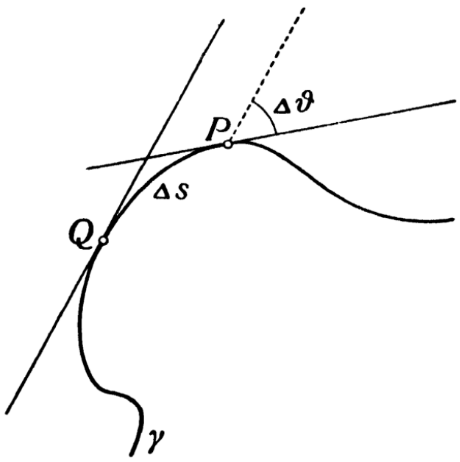


Figure 3.4

**Definition.** Curvature The **curvature** $k_1$ of a curve $\gamma$ at the point $P$ is the limit of the ratio $\dfrac{\Delta\theta}{\Delta s}$ as the point $Q$ approaches $P$.

$$k_1 = \lim_{P \to Q} \frac{\Delta\theta}{\Delta s}$$

**Theorem 3.3.1.** *Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be a regular, twice continuously differentiable ($C^2$) curve parametrized by arc length $s$. The **curvature** $k_1$ of the curve at a point $\boldsymbol{r}(s)$ is defined as the magnitude of the second derivative of the position vector with respect to $s$:*

$$k_1 = |\boldsymbol{r}''(s)|.$$

*Equivalently, the curvature is the rate of change of the unit tangent vector $\boldsymbol{\tau}(s) = \boldsymbol{r}'(s)$ with respect to arc length:*

$$k_1 = \left| \frac{d\boldsymbol{\tau}}{ds} \right|.$$

*Geometrically, the curvature measures how much the curve deviates from being a straight line at a given point. Specifically:*

- *If $k_1 = 0$ everywhere, the curve is a straight line.*
- *If $k_1 > 0$, the curve is bending, and the larger the value of $k_1$, the sharper the bend.*

*Proof.* Let $\boldsymbol{r}(s)$ be a regular curve parametrised by arc length $s$. Consider two nearby points $P$ and $Q$ on the curve, corresponding to arc lengths $s$ and $s + \Delta s$, respectively. Let:

- $\boldsymbol{\tau}(s) = \boldsymbol{r}'(s)$ be the unit tangent vector at $P$,
- $\boldsymbol{\tau}(s + \Delta s) = \boldsymbol{r}'(s + \Delta s)$ be the unit tangent vector at $Q$,
- $\Delta\theta$ be the angle between $\boldsymbol{\tau}(s)$ and $\boldsymbol{\tau}(s + \Delta s)$.

Since $\boldsymbol{\tau}(s)$ and $\boldsymbol{\tau}(s+\Delta s)$ are unit vectors, the angle $\Delta\theta$ between them satisfies:

$$|\boldsymbol{\tau}(s + \Delta s) - \boldsymbol{\tau}(s)| = 2\sin\left(\frac{\Delta\theta}{2}\right).$$

For small $\Delta\theta$, we have $\sin\left(\frac{\Delta\theta}{2}\right) \approx \frac{\Delta\theta}{2}$, so:

$$|\boldsymbol{\tau}(s + \Delta s) - \boldsymbol{\tau}(s)| \approx \Delta\theta.$$

Using the definition of the derivative, we can write:

$$\boldsymbol{\tau}(s + \Delta s) = \boldsymbol{\tau}(s) + \boldsymbol{\tau}'(s)\Delta s + \cdots .$$

Thus, the difference in tangent vectors is:

$$\boldsymbol{\tau}(s + \Delta s) - \boldsymbol{\tau}(s) = \boldsymbol{\tau}'(s)\Delta s + \text{higher-order terms.}$$

Taking the magnitude of both sides and ignoring higher-order terms (since $\Delta s \to 0$), we get:

$$|\boldsymbol{\tau}(s + \Delta s) - \boldsymbol{\tau}(s)| \approx |\boldsymbol{\tau}'(s)|\Delta s.$$

From Step 2, we know $|\boldsymbol{\tau}(s+\Delta s)-\boldsymbol{\tau}(s)| \approx \Delta\theta$, so:

$$\Delta\theta \approx |\boldsymbol{\tau}'(s)|\Delta s.$$

Dividing both sides by $\Delta s$ and taking the limit as $\Delta s \to 0$, we obtain:

$$k_1 = \lim_{\Delta s\to 0} \frac{\Delta\theta}{\Delta s} = |\boldsymbol{\tau}'(s)|.$$

Since $\boldsymbol{\tau}(s) = \boldsymbol{r}'(s)$, we have:

$$\boldsymbol{\tau}'(s) = \boldsymbol{r}''(s).$$

Thus, the curvature is:

$$k_1 = |\boldsymbol{r}''(s)|.$$

The curvature $k_1$ of a regular curve parametrised by arc length is given by $k_1 = |\boldsymbol{r}''(s)|$. This result connects the geometric interpretation of curvature (as the rate of change of the tangent vector) to its analytic definition.

□

*Remark.*

1. The curvature $k_1$ is an intrinsic property of the curve, meaning it does not depend on the choice of parametrisation.
2. For a curve not parametrised by arc length, the curvature can be computed using the formula:

$$k_1 = \frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3},$$

where $t$ is an arbitrary parameter.

## Normal and Binormal Vectors

If the curvature does not vanish at a given point on a curve, the vector $\boldsymbol{n} = (1/k_1)\boldsymbol{r}''(s)$ is a unit vector and lies in the osculating plane of the curve. Moreover, this vector is perpendicular to the tangent vector $\boldsymbol{\tau}$, so that $\boldsymbol{\tau}^2 = 1$ and, consequently $\boldsymbol{\tau}\cdot\boldsymbol{\tau}' = \boldsymbol{\tau}\cdot\boldsymbol{n}k_1 = 0$. Thus, this vector is directed along the principal normal to the curve. In the sequel, when we mention the unit vector on the principal normal to the curve, we shall have in mind the vector $\boldsymbol{n}$.

**Definition** (Unit Binormal Vector)**.** The vector $\boldsymbol{\tau}\times\boldsymbol{n} = \boldsymbol{b}$ is directed along the binormal of the curve. This vector will be called the **unit binormal vector** of the curve.

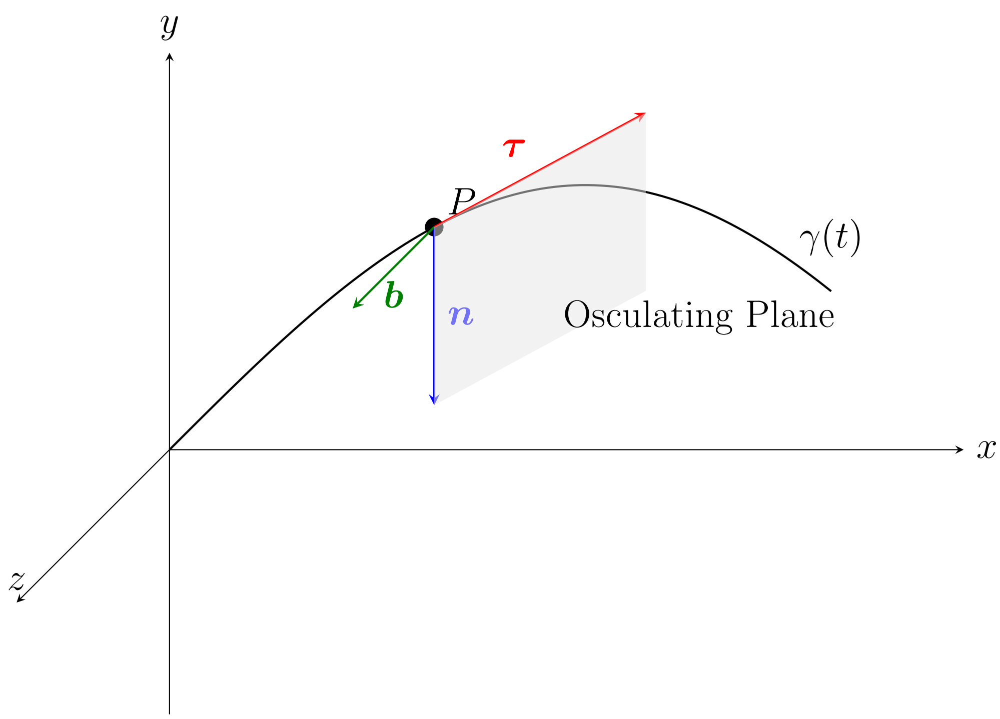


Figure 3.5. Normal and binormal vectors

## Curvature of a Curve Given Parametrically

Suppose the curve is given by the vector equation $\boldsymbol{r} = \boldsymbol{r}(t)$, where $t$ is an arbitrary parameter. We wish to express the curvature $k_1$ in terms of the derivatives of $\boldsymbol{r}$ concerning $t$.

### First Derivative Concerning Arc Length

The arc length $s$ is related to the parameter $t$ by the relation:

$$\frac{ds}{dt} = \left| \frac{d\boldsymbol{r}}{dt} \right| = |\boldsymbol{r}'(t)|.$$

The first derivative of $\boldsymbol{r}$ with respect to $s$ is:

$$\frac{d\boldsymbol{r}}{ds} = \frac{d\boldsymbol{r}}{dt}\frac{dt}{ds} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}.$$

### Second Derivative Concerning Arc Length

The second derivative of $\boldsymbol{r}$ with respect to $s$ is:

$$\frac{d^2\boldsymbol{r}}{ds^2} = \frac{d}{dt}\left(\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}\right)\frac{dt}{ds}.$$

Using the chain rule, we have:

$$\frac{d^2\boldsymbol{r}}{ds^2} = \frac{\boldsymbol{r}''(t)|\boldsymbol{r}'(t)| - \boldsymbol{r}'(t)\frac{\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|}}{|\boldsymbol{r}'(t)|^2}\frac{1}{|\boldsymbol{r}'(t)|}.$$

Simplifying, we obtain:

$$\frac{d^2\boldsymbol{r}}{ds^2} = \frac{\boldsymbol{r}''(t)|\boldsymbol{r}'(t)|^2 - \boldsymbol{r}'(t)(\boldsymbol{r}'(t)\cdot\boldsymbol{r}''(t))}{|\boldsymbol{r}'(t)|^4}.$$

**Curvature in Terms of $t$**

The curvature $k_1$ is the magnitude of the second derivative of $\boldsymbol{r}$ with respect to $s$:

$$k_1 = \left|\frac{d^2\boldsymbol{r}}{ds^2}\right|.$$

Substituting the expression for $\frac{d^2\boldsymbol{r}}{ds^2}$, we have:

$$k_1 = \frac{|\boldsymbol{r}''(t)\times\boldsymbol{r}'(t)|}{|\boldsymbol{r}'(t)|^3}.$$

*Example.* Suppose the curve is given parametrically by:

$$x = x(t), \quad y = y(t), \quad z = z(t).$$

The curvature $k_1$ of the curve is given by:

$$k_1^2 = \frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|^2}{|\boldsymbol{r}'(t)|^6}.$$

Expressed in terms of the derivatives of $x(t)$, $y(t)$, and $z(t)$, this becomes:

$$k_1^2 = \frac{\begin{vmatrix} x'' & y'' \\ x' & y' \end{vmatrix}^2 + \begin{vmatrix} y'' & z'' \\ y' & z' \end{vmatrix}^2 + \begin{vmatrix} z'' & x'' \\ z' & x' \end{vmatrix}^2}{(x'^2 + y'^2 + z'^2)^3}.$$

*Special Case: Plane Curve in the $x, y$-Plane*

If the curve lies entirely in the $x, y$-plane (i.e., $z(t) = 0$), the curvature simplifies to:

$$k_1^2 = \frac{(x''y' - y''x')^2}{(x'^2 + y'^2)^3}.$$

*Special Case: Plane Curve Given by $y = y(x)$* If the curve is given by $y = y(x)$, we can reparametrize it as $x = t$, $y = y(t)$. Then, the curvature becomes:

$$k_1^2 = \frac{y''^2}{(1 + y'^2)^3}.$$

Curvature of a curve given by the equations $x = x(t)$, $y = y(t)$, $z = z(t)$ is defined by

$$k_1^2 = \frac{\begin{vmatrix} x'' & y'' \\ x' & y' \end{vmatrix}^2 + \begin{vmatrix} y'' & z'' \\ y' & z' \end{vmatrix}^2 + \begin{vmatrix} z'' & x'' \\ z' & x' \end{vmatrix}^2}{(x'^2 + y'^2 + z'^2)^3}.$$

If the curve is a plane curve lying in the $x$, $y$-plane,

$$k_1^2 = \frac{(x''y' - y''x')^2}{(x'^2 + y'^2)^3}.$$

If the plane curve is given by the equation $y = y(x)$,

$$k_1^2 = \frac{y''^2}{(1 + y'^2)^3}.$$

*Remark.* Suppose we find that all the curves have curvature zero at all their points. We have $k_1 = |\boldsymbol{r}''(s)| = 0$. It follows that $\boldsymbol{r}''(s) = 0$ and, consequently, $\boldsymbol{r}(s) = \boldsymbol{a}s + \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors. Thus, a curve having curvature everywhere equal to zero is either a straight line or an open interval on a straight line.

## Absolute Torsion

Suppose $P$ is an arbitrary point on the curve $\gamma$ and $Q$ is a point on $\gamma$ near $P$. We denote the angle between the binormal vectors (or osculating planes) at $P$ and $Q$ by $\Delta\theta$, and the length of the segment $PQ$ on the curve by $\Delta s$.

**Definition** (Absolute torsion)**.** The **absolute torsion** $|k_2|$ of the curve $\gamma$ at the point $P$ is understood to be the limit of the ratio $\dfrac{\Delta\theta}{\Delta s}$ as $Q \to P$.

$$|k_2| = \lim_{P \to O} \frac{\Delta\theta}{\Delta s}$$

**Theorem 3.3.2.** *Let $\boldsymbol{r}(s)$ be a regular, three times continuously differentiable ($C^3$) curve parametrized by arc length $s$. The* ***absolute torsion*** *$|k_2|$ of the curve at a point $\boldsymbol{r}(s)$ is given by:*

$$|k_2| = \frac{|\boldsymbol{r}'(s) \cdot (\boldsymbol{r}''(s) \times \boldsymbol{r}'''(s))|}{|\boldsymbol{r}''(s)|^2}.$$

*Proof. Setup and Definitions:*

- Let $P$ and $Q$ be two nearby points on the curve $\gamma$, corresponding to arc lengths $s$ and $s + \Delta s$, respectively.
- The osculating plane at $P$ is spanned by $\boldsymbol{r}'(s)$ and $\boldsymbol{r}''(s)$.
- Let $\Delta\theta$ be the angle between the osculating planes at $P$ and $Q$.

*Approximation of $\Delta\theta$:*

- The angle $\Delta\theta$ is approximately equal to the angle between the binormal vectors at $P$ and $Q$.
- The binormal vector $\boldsymbol{b}(s)$ is proportional to $\boldsymbol{r}'(s) \times \boldsymbol{r}''(s)$.

*Change in the Binormal Vector:*

- The change in the binormal vector between $P$ and $Q$ is:

$$\Delta\boldsymbol{b} = \boldsymbol{b}(s + \Delta s) - \boldsymbol{b}(s).$$

- For small $\Delta s$, we have:

$$\boldsymbol{b}(s + \Delta s) \approx \boldsymbol{b}(s) + \boldsymbol{b}'(s)\Delta s.$$

- Thus, the magnitude of the change in the binormal vector is:

$$|\Delta\boldsymbol{b}| \approx |\boldsymbol{b}'(s)|\Delta s.$$

*Relationship Between $\Delta\theta$ and $\Delta\boldsymbol{b}$:*

- For small angles $\Delta\theta$, we have:

$$|\Delta\boldsymbol{b}| \approx \Delta\theta.$$

- Combining this with step 4:

$$\Delta\theta \approx |\boldsymbol{b}'(s)|\Delta s.$$

*Expression for $\boldsymbol{b}'(s)$:*

- Differentiate the binormal vector $\boldsymbol{b}(s)$ with respect to $s$:

$$\boldsymbol{b}'(s) = \frac{\boldsymbol{r}'(s) \times \boldsymbol{r}'''(s)}{|\boldsymbol{r}'(s) \times \boldsymbol{r}''(s)|}.$$

*Magnitude of $\boldsymbol{b}'(s)$:*

- Take the magnitude of $\boldsymbol{b}'(s)$:

$$|\boldsymbol{b}'(s)| = \frac{|\boldsymbol{r}'(s) \times \boldsymbol{r}'''(s)|}{|\boldsymbol{r}'(s) \times \boldsymbol{r}''(s)|}.$$

*Final Expression for $|k_2|$:*

- Substitute into the limit definition of torsion:

$$|k_2| = \lim_{Q \to P} \frac{\Delta\theta}{\Delta s} = |\boldsymbol{b}'(s)| = \frac{|\boldsymbol{r}'(s) \cdot (\boldsymbol{r}''(s) \times \boldsymbol{r}'''(s))|}{|\boldsymbol{r}''(s)|^2}.$$

□

## Sign of Torsion

The torsion $k_2$ measures how much the curve **twists out of the osculating plane** as it progresses. The **sign of** $k_2$ indicates the **direction** of this twisting relative to the tangent vector and the normal vector.

- **Positive Torsion** ($k_2 > 0$):
  - The curve twists **counterclockwise** relative to the osculating plane.
  - Geometrically, as you move along the curve, the **binormal vector** $\boldsymbol{b}(s)$ rotates in a **counterclockwise** direction around the tangent vector $\boldsymbol{\tau}(s)$.
- **Negative Torsion** ($k_2 < 0$):
  - The curve twists **clockwise** relative to the osculating plane.
  - Geometrically, as you move along the curve, the **binormal vector** $\boldsymbol{b}(s)$ rotates in a **clockwise** direction around the tangent vector $\boldsymbol{\tau}(s)$.

- **Zero Torsion** ($k_2 = 0$):
  - The curve does **not twist** out of the osculating plane.
  - The curve remains **planar** (lies entirely in a single plane).

The sign of the torsion $k_2$ is determined by the orientation of the **triple scalar product**:

$$k_2 = \frac{\boldsymbol{r}'(s) \cdot (\boldsymbol{r}''(s) \times \boldsymbol{r}'''(s))}{|\boldsymbol{r}''(s)|^2}.$$

- The **sign of the numerator** $\boldsymbol{r}'(s) \cdot (\boldsymbol{r}''(s) \times \boldsymbol{r}'''(s))$ determines the sign of $k_2$.
- If the triple scalar product is **positive**, $k_2 > 0$, and the curve twists counterclockwise.
- If the triple scalar product is **negative**, $k_2 < 0$, and the curve twists clockwise.

The binormal vector $\boldsymbol{b}(s)$ is defined as:

$$\boldsymbol{b}(s) = \boldsymbol{\tau}(s) \times \boldsymbol{n}(s),$$

where $\boldsymbol{\tau}(s)$ is the **unit tangent vector** and $\boldsymbol{n}(s)$ is the **unit normal vector**. The derivative of the binormal vector $\boldsymbol{b}'(s)$ is related to the torsion $k_2$ by:

$$\boldsymbol{b}'(s) = -k_2 \boldsymbol{n}(s).$$

The **sign of** $k_2$ determines the direction of $\boldsymbol{b}'(s)$:

- If $k_2 > 0$, $\boldsymbol{b}'(s)$ points in the **opposite direction** of $\boldsymbol{n}(s)$, indicating a **counterclockwise** twist.
- If $k_2 < 0$, $\boldsymbol{b}'(s)$ points in the **same direction** as $\boldsymbol{n}(s)$, indicating a **clockwise** twist.

This concept has simple visualisation

- **Positive Torsion** ($k_2 > 0$):
  - Imagine walking along the curve. If the curve twists to your **left**, the torsion is **positive**.

    - Example: A **right-handed helix**.

- **Negative Torsion ($k_2 < 0$)**:
    - If the curve twists to your **right**, the torsion is **negative**.
    - Example: A **left-handed helix**.

- **Zero Torsion ($k_2 = 0$)**:
    - If the curve does not twist at all, the torsion is **zero**.
    - Example: A **circle** or a **straight line**.

The **sign of the torsion** $k_2$ indicates the **direction of twisting** of the curve:

- $k_2 > 0$: Counterclockwise twist (e.g., right-handed helix).
- $k_2 < 0$: Clockwise twist (e.g., left-handed helix).
- $k_2 = 0$: No twist (e.g., planar curve).

## Torsion of a Parametric Curve

Suppose a curve defined parametrically by:

$$\boldsymbol{r}(t) = \big(x(t), y(t), z(t)\big).$$

1. Compute the first, second, and third derivatives of $x(t)$, $y(t)$, and $z(t)$:

$$\begin{aligned}
\boldsymbol{r}'(t) &= \big(x'(t), y'(t), z'(t)\big), \\
\boldsymbol{r}''(t) &= \big(x''(t), y''(t), z''(t)\big), \\
\boldsymbol{r}'''(t) &= \big(x'''(t), y'''(t), z'''(t)\big).
\end{aligned}$$

2. Compute the cross product $\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$:

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}.$$

3. Compute the dot product of $\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$ with $\boldsymbol{r}'''(t)$:

$$\big(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\big) \cdot \boldsymbol{r}'''(t).$$

4. Compute the magnitude of the cross product $\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$:

$$|\boldsymbol{r}' \times \boldsymbol{r}''| = \sqrt{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2}.$$

5. Compute the torsion $k_2(t)$ using the formula:

$$k_2(t) = \frac{\left(\boldsymbol{r}' \times \boldsymbol{r}''\right) \cdot \boldsymbol{r}'''}{|\boldsymbol{r}' \times \boldsymbol{r}''|^2}.$$

The torsion can also be expressed explicitly as:

$$k_2(t) = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{(y'z'' - z'y'')^2 + (z'x'' - x'z'')^2 + (x'y'' - y'x'')^2}$$

## Example

**Straight Line**

*Parametrization*

$$\boldsymbol{r}(t) = \boldsymbol{a} + t\boldsymbol{b}, \quad \boldsymbol{a}, \boldsymbol{b} \text{ constant}$$

*Properties*

- **Curvature:** $k_1 = 0$ (no bending)
- **Torsion:** $k_2 = 0$ (planar)

*Calculation*

$$\begin{aligned} \boldsymbol{r}'(t) &= \boldsymbol{b} \\ \boldsymbol{r}''(t) &= 0 \\ k_1 &= 0 \\ k_2 &= \text{undefined (by convention 0)} \end{aligned}$$

**Circle**

*Parametrization*

$$\boldsymbol{r}(t) = (R\cos t, R\sin t, 0), \quad R > 0$$

*Properties*

- **Curvature:** $k_1 = 1/R$ (constant bending)
- **Torsion:** $k_2 = 0$ (planar)

*Calculation*

$$\begin{aligned}
\boldsymbol{r}'(t) &= (-R\sin t, R\cos t, 0)\\
\boldsymbol{r}''(t) &= (-R\cos t, -R\sin t, 0)\\
|\boldsymbol{r}'(t)| &= R\\
\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) &= (0, 0, R^2)\\
k_1 &= \frac{R^2}{R^3} = \frac{1}{R}\\
k_2 &= 0 \quad \text{(since } \boldsymbol{r}''' \text{ is in the osculating plane)}
\end{aligned}$$

**Helix**
*Parametrization*

$$\boldsymbol{r}(t) = (R\cos t, R\sin t, ht), \quad R > 0, h \neq 0$$

*Properties*

- **Curvature:** $k_1 = \frac{R}{R^2+h^2}$ (constant)
- **Torsion:** $k_2 = \frac{h}{R^2+h^2}$ (constant)

*Calculation*

$$\begin{aligned}
\boldsymbol{r}'(t) &= (-R\sin t, R\cos t, h)\\
\boldsymbol{r}''(t) &= (-R\cos t, -R\sin t, 0)\\
\boldsymbol{r}'''(t) &= (R\sin t, -R\cos t, 0)\\
|\boldsymbol{r}'(t)| &= \sqrt{R^2 + h^2}\\
\boldsymbol{r}' \times \boldsymbol{r}'' &= (hR\sin t, -hR\cos t, R^2)\\
|\boldsymbol{r}' \times \boldsymbol{r}''| &= R\sqrt{R^2 + h^2}\\
k_1 &= \frac{R}{R^2 + h^2}\\
(\boldsymbol{r}' \times \boldsymbol{r}'') \cdot \boldsymbol{r}''' &= hR^2\\
k_2 &= \frac{h}{R^2 + h^2}
\end{aligned}$$

**Summary**
*Key Observations*

Table 3.1. Comparison of Curvature and Torsion

| **Curve** | $k_1$ | $k_2$ | **Geometric Interpretation** |
|---|---|---|---|
| Straight line | 0 | 0 | No bending, no twisting |
| Circle | $1/R$ | 0 | Constant bending, planar |
| Helix | $\dfrac{R}{R^2+h^2}$ | $\dfrac{h}{R^2+h^2}$ | Constant bending and twisting |

- Curvature measures deviation from straightness
- Torsion measures deviation from planarity
- Helix generalises the circle with added torsion
- When $h \to 0$, helix $\to$ circle ($k_2 \to 0$)
- When $R \to \infty$, helix $\to$ straight line ($k_1 \to 0$)

## 3.4 The Frenet-Serret Formulas

We are interested in seeking a way to completely describe the shape and behaviour of curves in space using arbitrary invariant quantities (properties that don't depend on the coordinate system). We want a coordinate-free description of the curve's shape. Parameters expressing the direction of the trajectory path, direction of the turn ("centripetal force"), and the tilt of the road ("banking angle") are good candidates to introduce such invariant expression.

### Frenet Frame

**Definition** (Frenet Frame)**.** Three half-lines, emanating from a point on the curve and having the directions of the vectors $\boldsymbol{\tau}$, $\boldsymbol{n}$, $\boldsymbol{b}$ are edges of a trihedron. This trihedron is called the **Frenet frame** or **natural trihedron** .

To investigate the properties of the curve in a neighbourhood of an arbitrary point $P$, it turns out to be convenient in many cases to choose a Cartesian system of coordinates, with the point $P$ on the curve as the origin of coordinates and the edges of the Frenet frame as the coordinate axes. Belo, we shall obtain the equation of a curve with such a choice of coordinate system.

We shall now express the derivatives of the vectors $\boldsymbol{\tau}$, $\boldsymbol{n}$, $\boldsymbol{b}$ concerning arc length of the curve again in terms of $\boldsymbol{\tau}$, $\boldsymbol{n}$, $\boldsymbol{b}$. We have

$$\boldsymbol{\tau}' = \boldsymbol{r}'' = k_1\boldsymbol{n}.$$

To obtain $\boldsymbol{b}'$, let us recall that the vector $\boldsymbol{b}'$ is parallel to $\boldsymbol{n}$ and that $\boldsymbol{b}' \cdot \boldsymbol{n} = -k_2$. It follows that

$$\boldsymbol{b}' = -k_2\boldsymbol{n}.$$

Finally,

$$\boldsymbol{n}' = (\boldsymbol{b} \times \boldsymbol{\tau})' = \boldsymbol{b}' \times \boldsymbol{\tau} + \boldsymbol{b} \times \boldsymbol{\tau}' = k_2\boldsymbol{n} \times \boldsymbol{\tau} + k_1\boldsymbol{b} \times \boldsymbol{n} = -k_1\boldsymbol{\tau} + k_2\boldsymbol{b}.$$

**Definition** (Frenet-Serret formulas)**.** The system of equations

$$\begin{cases} \boldsymbol{\tau}' = k_1\boldsymbol{n}, \\ \boldsymbol{n}' = -k_1\boldsymbol{\tau} + k_2\boldsymbol{b}, \\ \boldsymbol{b}' = -k_2\boldsymbol{n} \end{cases}$$

are called the **Frenet-Serret formulas**.

The Frenet-Serret formulas are a set of differential equations that describe how the Frenet frame (a set of three orthonormal vectors: Tangent, Normal, and Binormal) evolves along a smooth curve in three-dimensional space. These formulas are fundamental in differential geometry because they relate the curvature and torsion of a curve to the derivatives of the Frenet frame vectors.

The formulas provide a complete local description of how a curve behaves in space. If you figured out the curvature $k_1(s)$ and torsion $k_2(s)$k2 as functions of arc length $s$, you can reconstruct the curve (up to rigid motions). The formulas are independent of the coordinate system, making them intrinsic properties of the curve.

These formulas have applications in physics (e.g., motion of particles), engineering (e.g., design of roads and roller coasters), and computer graphics (e.g., modelling curves and surfaces).

## Geometric Interpretation of the Frenet-Serret Formulas

### Curvature ($k_1$)

- The first formula, $\boldsymbol{\tau}' = k_1\boldsymbol{n}$, states that the rate of change of the tangent vector $\boldsymbol{\tau}$ is proportional to the normal vector $\boldsymbol{n}$.

- **Interpretation**: The curve bends in the direction of the normal vector, and the curvature $k_1$ measures the strength of this bending.
  - If $k_1 = 0$, there is no bending, and the curve is a straight line.
  - If $k_1 > 0$, the curve bends, and higher $k_1$ corresponds to sharper bends.

### Torsion ($k_2$)

- The third formula, $\boldsymbol{b}' = -k_2\boldsymbol{n}$, states that the rate of change of the binormal vector $\boldsymbol{b}$ is proportional to the normal vector $\boldsymbol{n}$.
- **Interpretation**: The curve twists out of the osculating plane (the plane spanned by $\boldsymbol{\tau}$ and $\boldsymbol{n}$), and the torsion $k_2$ measures the strength of this twisting.
  - If $k_2 = 0$, the curve lies entirely in a plane (no twisting).
  - If $k_2 > 0$, the curve twists out of the plane.

### Normal Vector

- The second formula, $\boldsymbol{n}' = -k_1\boldsymbol{\tau} + k_2\boldsymbol{b}$, describes how the normal vector changes.
- **Interpretation**: The normal vector changes due to both bending ($k_1\tau$) and twisting ($k_2 b$).

### Frenet frame Example: The Helix

The helix is chosen because it exhibits both curvature (bending) and torsion (twisting).

*Parametrization*

$$\boldsymbol{r}(t) = (R\cos t, R\sin t, ht), \quad R > 0,\ h \neq 0$$

*Frenet Frame*

$$\begin{aligned}
\boldsymbol{\tau}(t) &= \frac{(-R\sin t, R\cos t, h)}{\sqrt{R^2+h^2}} \quad \text{(Tangent)}\\
\boldsymbol{n}(t) &= (-\cos t, -\sin t, 0) \quad \text{(Normal)}\\
\boldsymbol{b}(t) &= \frac{(h\sin t, -h\cos t, R)}{\sqrt{R^2+h^2}} \quad \text{(Binormal)}
\end{aligned}$$

*Curvature and Torsion*

$$k_1 = \frac{R}{R^2 + h^2}, \quad k_2 = \frac{h}{R^2 + h^2}$$

**Verification Frenet-Serret formulas**

*Verification of* $\boldsymbol{\tau}' = k_1 \boldsymbol{n}$

$$\begin{aligned} \boldsymbol{\tau}' &= \frac{(-R\cos t, -R\sin t, 0)}{R^2 + h^2}, \\ k_1 \boldsymbol{n} &= \frac{R}{R^2 + h^2}(-\cos t, -\sin t, 0) = \boldsymbol{\tau}'. \quad \checkmark \end{aligned}$$

*Verification of* $\boldsymbol{b}' = -k_2 \boldsymbol{n}$

$$\begin{aligned} \boldsymbol{b}' &= \frac{(h\cos t, h\sin t, 0)}{R^2 + h^2}, \\ -k_2 \boldsymbol{n} &= -\frac{h}{R^2 + h^2}(-\cos t, -\sin t, 0) = \frac{h}{R^2 + h^2}(\cos t, \sin t, 0) = \boldsymbol{b}'. \quad \checkmark \end{aligned}$$

*Note on Torsion Sign*
For the helix:

$$k_2 = \frac{h}{R^2 + h^2},$$

which ensures $\boldsymbol{b}' = -k_2 \boldsymbol{n}$ matches the Frenet-Serret formulas.

*Remark.* The sign of $k_2$ determines the handedness of the helix: $k_2 > 0$ corresponds to a right-handed helix (like a standard screw), while $k_2 < 0$ gives a left-handed helix.

## Expansion of the Parameterisation

The Taylor series expansion of a curve's parameterisation provides a powerful tool for understanding its local geometry. By relating the derivatives to the Frenet frame, we can derive essential geometric quantities such as curvature and torsion. This approach is particularly useful for analysing curves in differential geometry and applications such as physics and computer graphics.

Given a curve $\boldsymbol{r}(t) = (x(t), y(t), z(t))$, we can analyse its local behaviour around a fixed point $P = \boldsymbol{r}(t_0)$ using a Taylor series expansion. This expansion

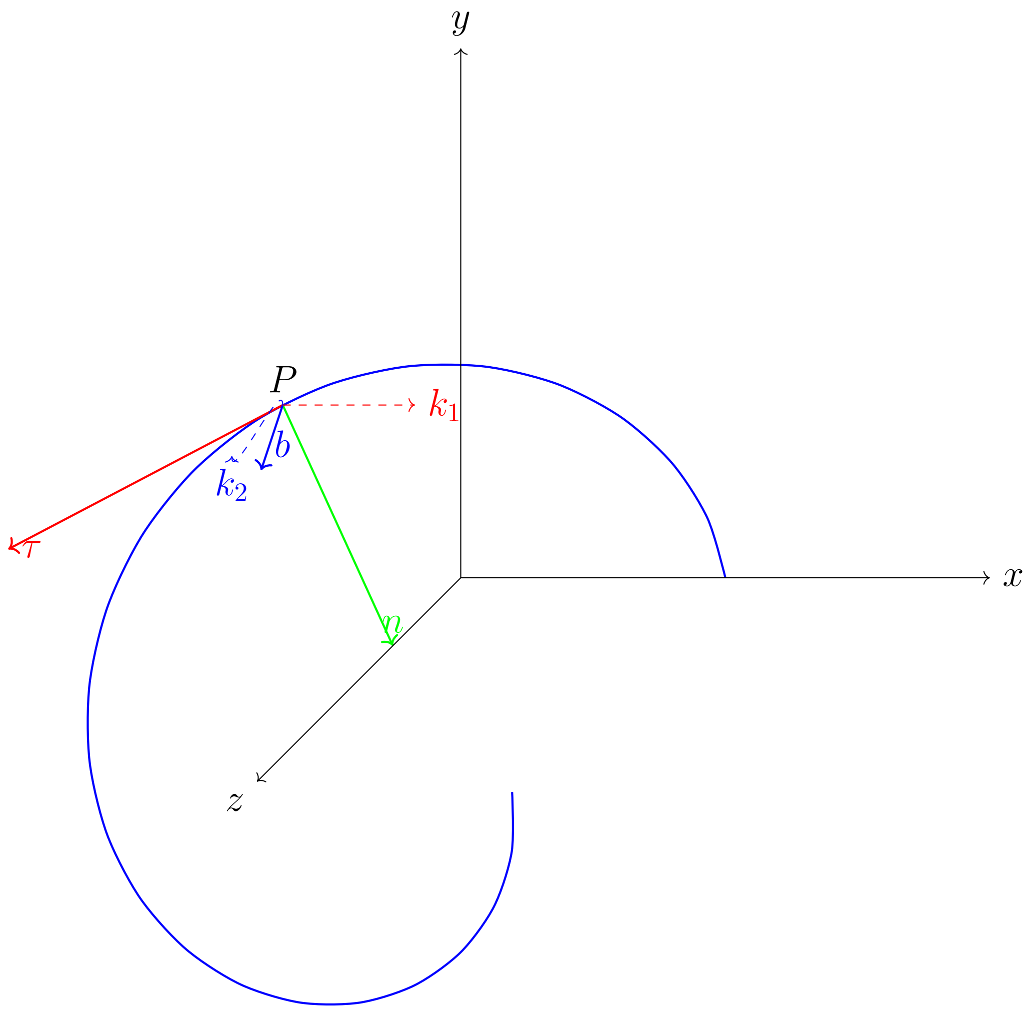


Figure 3.6. Frenet frame for helix

provides insights into the geometric properties of the curve, such as its tangent, normal, and binormal vectors, as well as its curvature and torsion.

The Taylor series expansion of $\boldsymbol{r}(t)$ around $t_0$ is:

$$\boldsymbol{r}(t) = \boldsymbol{r}(t_0) + \boldsymbol{r}'(t_0)(t - t_0) + \frac{1}{2}\boldsymbol{r}''(t_0)(t - t_0)^2 + \cdots ,$$

where $\boldsymbol{r}'(t_0)$ and $\boldsymbol{r}''(t_0)$ are the first and second derivatives of $\boldsymbol{r}(t)$ evaluated at $t_0$.

The first and second derivatives of $\boldsymbol{r}(t)$ are closely related to the Frenet frame:

- **Tangent Vector**: The first derivative $\boldsymbol{r}'(t_0)$ is proportional to the tangent vector $\boldsymbol{\tau}$. For an arc-length parametrization, $\boldsymbol{r}'(t_0) = \boldsymbol{\tau}$.

- **Normal Vector**: The second derivative $\boldsymbol{r}''(t_0)$ encodes information about curvature. Specifically, $\boldsymbol{r}''(t_0) = k_1\boldsymbol{n}$, where $k_1$ is the curvature and $\boldsymbol{n}$ is the normal vector.

The Taylor series expansion provides a local approximation of the curve around $t_0$:

- The linear term $\boldsymbol{r}'(t_0)(t - t_0)$ describes the tangent direction of the curve.
- The quadratic term $\frac{1}{2}\boldsymbol{r}''(t_0)(t - t_0)^2$ describes the bending of the curve, corresponding to the curvature $k_1$ and normal vector $\boldsymbol{n}$.

Let us now introduce the natural (arc length) parameterisation $\boldsymbol{r}(s + \Delta s)$ in a neighbourhood of $P$, corresponding to the arc $s$ along the axes of the Frenet frame at this point (at $P$ $s = 0$). We have

$$\boldsymbol{r}(s + \Delta s) = \boldsymbol{r}(s) + \Delta s \boldsymbol{r}'(s) + \frac{\Delta s^2}{2}\boldsymbol{r}''(s) + \frac{\Delta s^3}{6}\boldsymbol{r}'''(s) + \dots .$$

But at the point $P$, $\boldsymbol{r} = 0$, $\boldsymbol{r}' = \boldsymbol{\tau}$, $\boldsymbol{r}'' = k_1\boldsymbol{n}$, $\boldsymbol{r}''' = k_1'\boldsymbol{n} - k_1^2\boldsymbol{\tau} + k_1k_2\boldsymbol{b}$, and so on. Thus,

$$\begin{aligned}\boldsymbol{r}(s + \Delta s) = \Big(\Delta s - \frac{k_1^2\Delta s^3}{6} + \dots\Big)\boldsymbol{\tau} + \\ + \Big(\frac{k_1\Delta s^2}{2} + \frac{k_1'\Delta s^3}{6} + \dots\Big)\boldsymbol{n} + \\ + \Big(\frac{k_1k_2\Delta s^3}{6} + \dots\Big)\boldsymbol{b} + \cdots .\end{aligned}$$

## 3.5 Natural Equations

We see that to expand the function $\boldsymbol{r}(s + \Delta s)$ as a power series in $\Delta s$ it is sufficient to know the curvature and torsion of the curve as functions of the arc $s$. This gives the basis for assuming that the curvature and torsion determine the curve to some extent. And indeed, we do have the following valid theorem.

**Theorem 3.5.1** (Fundamental Theorem of Curves)**.** *Let $k_1(s)$ and $k_2(s)$ be smooth functions defined on an interval $I \subseteq \mathbb{R}$, with $k_1(s) > 0$ for all $s \in I$. Then:*

1. *There exists a regular, smooth curve $\gamma$ parametrized by parameter $s$ as arc length, for which $k_1(s)$ is the curvature and $k_2(s)$ is the torsion at each point $\gamma(s)$.*
2. *The curve $\gamma$ is uniquely determined up to rigid motions in space. That is, if $\tilde{\gamma}$ is another curve with the same curvature $k_1(s)$ and torsion $k_2(s)$, then $\tilde{\gamma}$ can be obtained from $\gamma$ by a combination of translations and rotations in space.*

*Proof. Existence of the Frenet Frame*

Consider the Frenet-Serret system of differential equations:

$$\begin{cases} \boldsymbol{\tau}'(s) = k_1(s)\,\boldsymbol{n}(s), \\ \boldsymbol{n}'(s) = -k_1(s)\,\boldsymbol{\tau}(s) + k_2(s)\,\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = -k_2(s)\,\boldsymbol{n}(s), \end{cases}$$

where $\boldsymbol{\tau}(s)$, $\boldsymbol{n}(s)$, and $\boldsymbol{b}(s)$ are the tangent, normal, and binormal vectors, respectively.

Given initial conditions $\boldsymbol{\tau}(s_0)$, $\boldsymbol{n}(s_0)$, and $\boldsymbol{b}(s_0)$ to be an orthonormal frame

The Picard-Lindelöf theorem guarantees the existence and uniqueness of solutions to systems of ordinary differential equations (ODEs) with smooth coefficients and given initial conditions.

Here, the functions $k_1(s)$ and $k_2(s)$ are smooth, and the initial conditions $\boldsymbol{\tau}(s_0)$, $\boldsymbol{n}(s_0)$, and $\boldsymbol{b}(s_0)$ form an orthonormal frame. Thus, the system has a unique solution for $\boldsymbol{\tau}(s)$, $\boldsymbol{n}(s)$, and $\boldsymbol{b}(s)$.

*Orthonormality of the Frame*

We show that the vectors $\boldsymbol{\tau}(s)$, $\boldsymbol{n}(s)$, and $\boldsymbol{b}(s)$ remain orthonormal for all $s$. Specifically:

$$\boldsymbol{\tau}(s)\cdot\boldsymbol{n}(s) = 0, \quad \boldsymbol{n}(s)\cdot\boldsymbol{b}(s) = 0, \quad \boldsymbol{b}(s)\cdot\boldsymbol{\tau}(s) = 0,$$

and

$$|\boldsymbol{\tau}(s)| = 1, \quad |\boldsymbol{n}(s)| = 1, \quad |\boldsymbol{b}(s)| = 1.$$

Suppose $\boldsymbol{\tau}(s)$, $\boldsymbol{n}(s)$, $\boldsymbol{b}(s)$ is the solution of this system satisfying the initial conditions $\boldsymbol{\tau}(s_0) = \boldsymbol{\tau}_0$, $\boldsymbol{n}(s_0) = \boldsymbol{n}_0$, $\boldsymbol{b}(s_0) = \boldsymbol{b}_0$, where $\boldsymbol{\tau}_0$, $\boldsymbol{n}_0$, $\boldsymbol{b}_0$ are three orthonormal vectors whose triple product equals 1: $(\boldsymbol{\tau}_0, \boldsymbol{n}_0, \boldsymbol{b}_0) = 1$.

We shall show that the vectors $\boldsymbol{\tau}(s)$, $\boldsymbol{n}(s)$, $\boldsymbol{b}(s)$ are unique and mutually perpendicular for arbitrary $s$, and $(\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b}) = 1$. To this end, we shall compute $(\boldsymbol{\tau}^2)'$, $(\boldsymbol{n}^2)'$, $(\boldsymbol{b}^2)'$, $(\boldsymbol{\tau}\cdot\boldsymbol{n})'$, $(\boldsymbol{n}\cdot\boldsymbol{b})'$, $(\boldsymbol{b}\cdot\boldsymbol{\tau})'$. Making use of the equations of the system, we obtain the following expressions for these derivatives:

$$\begin{aligned} (\boldsymbol{\tau}^2)' &= 2k_1\boldsymbol{\tau}\cdot\boldsymbol{n}, \quad (\boldsymbol{\tau}\cdot\boldsymbol{n})' = k_1\boldsymbol{n}^2 - k_1\boldsymbol{\tau}^2 - k_2\boldsymbol{\tau}\cdot\boldsymbol{b}, \\ (\boldsymbol{n}^2)' &= -k_1\boldsymbol{\tau}\cdot\boldsymbol{n} + k_2\boldsymbol{n}\cdot\boldsymbol{b}, \quad (\boldsymbol{n}\cdot\boldsymbol{b})' = -k_2\boldsymbol{n}^2 + k_2\boldsymbol{b}^2 - k_1\boldsymbol{\tau}\cdot\boldsymbol{b}, \\ (\boldsymbol{b}^2)' &= -2k_2\boldsymbol{n}\cdot\boldsymbol{b}, \quad (\boldsymbol{b}\cdot\boldsymbol{\tau})' = k_1\boldsymbol{n}\cdot\boldsymbol{b} + k_2\boldsymbol{\tau}\cdot\boldsymbol{n}. \end{aligned}$$

If we consider these equations as a system of differential equations for $\boldsymbol{\tau}^2$, $\boldsymbol{n}^2$, $\boldsymbol{b}^2$, $\boldsymbol{\tau}\cdot\boldsymbol{n}$, $\boldsymbol{n}\cdot\boldsymbol{b}$, $\boldsymbol{b}\cdot\boldsymbol{\tau}$, we note that it is satisfied by the set of values $\boldsymbol{\tau}^2 = 1$, $\boldsymbol{n}^2 = 1$, $\boldsymbol{b}^2 = 1$, $\boldsymbol{\tau}\cdot\boldsymbol{n} = 0$, $\boldsymbol{n}\cdot\boldsymbol{b} = 0$, $\boldsymbol{b}\cdot\boldsymbol{\tau} = 0$. On the other hand, this system is satisfied by the $\boldsymbol{\tau}^2 = \boldsymbol{\tau}^2(s)$, $\boldsymbol{n}^2 = \boldsymbol{n}^2(s)$, ..., $\boldsymbol{b}\cdot\boldsymbol{\tau} = \boldsymbol{b}(s)\cdot\boldsymbol{\tau}(s)$. Both these solutions coincide for $s = s_0$ and consequently, they coincide identically according to the theorem on the uniqueness of the solution. Hence, for all $s$ we have

$$\boldsymbol{\tau}^2(s) = 1, \quad \boldsymbol{n}^2(s) = 1, \quad ..., \quad \boldsymbol{b}(s) \cdot \boldsymbol{\tau}(s) = 0.$$

We shall show that $(\boldsymbol{\tau}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)) = 1$. Since $\boldsymbol{\tau}$, $\boldsymbol{n}$, $\boldsymbol{b}$ are mutually perpendicular unit vectors, we have $(\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b}) = \pm 1$. The triple product $(\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b})$ depends continuously on $s$, it equals $+1$ when $s = s_0$, and therefore it is equal to 1 for all $s$.

*Construction of the Curve*

Integrate $\boldsymbol{\tau}(s) = \boldsymbol{r}'(s)$ to obtain the position vector $\boldsymbol{r}(s)$ of the curve:

$$\boldsymbol{r}(s) = \int_{s_0}^{s} \boldsymbol{\tau}(u)\, du.$$

First, we note that the parametrisation of the curve $\gamma$ is the natural parametrisation. The arc length of the segment $s_0 s$ of the curve $\gamma$ equals

$$\int_{s_0}^{s} |\boldsymbol{r}'(s)| ds = \int_{s_0}^{s} |\boldsymbol{\tau}(s)| ds = s - s_0.$$

By definition, the curvature $k_1(s)$ is the magnitude of the derivative of the tangent vector:

$$k_1(s) = |\boldsymbol{\tau}'(s)| = |k_1(s)\boldsymbol{n}(s)| = k_1(s),$$

since $|\boldsymbol{n}(s)| = 1$. The torsion $k_2(s)$ is given by:

$$|k_2(s)| = |-\boldsymbol{b}'(s) \cdot \boldsymbol{n}(s)| = |-(k_2(s)\boldsymbol{n}(s)) \cdot \boldsymbol{n}(s)| = |-k_2(s)|,$$

since $\boldsymbol{n}(s) \cdot \boldsymbol{n}(s) = 1$. Thus, the curve $\boldsymbol{r}(s)$ has the prescribed curvature and torsion.

*Uniqueness Up to Rigid Motions*

Suppose $\tilde{\gamma}$ is another curve with the same curvature $k_1(s)$ and torsion $k_2(s)$. Then its Frenet frame satisfies the same system of differential equations with the same initial conditions. By uniqueness of solutions, $\boldsymbol{\tau}(s) = \tilde{\boldsymbol{\tau}}(s)$, $\boldsymbol{n}(s) = \tilde{\boldsymbol{n}}(s)$, and $\boldsymbol{b}(s) = \tilde{\boldsymbol{b}}(s)$ for all $s$. Integrating $\boldsymbol{\tau}(s) = \tilde{\boldsymbol{\tau}}(s)$ gives $\boldsymbol{r}(s) = \tilde{\boldsymbol{r}}(s) + \boldsymbol{c}$, where $\boldsymbol{c}$ is a constant vector. Thus, $\gamma$ and $\tilde{\gamma}$ differ only by a rigid motion.

□

Theorem 3.5.1 states that given smooth functions $k_1(s)$ and $k_2(s)$ defined on an interval $I \subseteq \mathbb{R}$, with $k_1(s) > 0$ for all $s \in I$, there exists a unique (up to rigid motions) regular, smooth curve $\gamma : I \to \mathbb{R}^3$ for which $k_1(s)$ is the curvature and $k_2(s)$ is the torsion.

These two functions, $k_1(s)$ and $k_2(s)$, completely determine the shape of the curve in space, up to rigid motions (translations and rotations). This is the essence of Theorem 3.5.1.

**Definition** (Natural Equations of the Curve)**.** The system of equations

$$\begin{cases} k_1 = k_1(s), \\ k_2 = k_2(s) \end{cases}$$

are called the **natural equations** of the curve.

The pair $(k_1(s), k_2(s))$ is called the **natural equations** of the curve because they encode the intrinsic geometric properties of the curve, independent of its position or orientation in space.

## Example: A Curve with Variable Curvature and Torsion

Consider a curve parametrised by arc length $s$ with the following natural equations:

$$k_1(s) = \frac{1}{1+s^2}, \quad k_2(s) = \frac{s}{1+s^2}.$$

Here:

- The **curvature** $k_1(s)$ decreases as $s$ increases, starting at $k_1(0) = 1$ and approaching $k_1(s) \to 0$ as $s \to \infty$.
- The **torsion** $k_2(s)$ starts at $k_2(0) = 0$, increases to a maximum, and then decreases as $s \to \infty$.

Such curves appear in real-world scenarios, such as:

- **Roller Coaster Design:** The curve could model a roller coaster track that starts with sharp turns (high curvature and torsion) and gradually becomes smoother as the ride progresses.
- **Road Design:** It could represent a winding mountain road that becomes straighter as it descends into a valley.
- **Robotics:** In robotics, such curves can be used to design paths for robotic arms that gradually reduce their movement complexity.

### Solve the Frenet-Serret Equations

The **Frenet-Serret equations** for this curve are:

$$\begin{cases} \boldsymbol{\tau}'(s) = k_1(s)\,\boldsymbol{n}(s), \\ n'(s) = -k_1(s)\,\boldsymbol{\tau}(s) + k_2(s)\,\boldsymbol{b}(s), \\ b'(s) = -k_2(s)\,\boldsymbol{n}(s), \end{cases}$$

where:

- $\boldsymbol{\tau}(s)$ is the tangent vector,
- $\boldsymbol{n}(s)$ is the normal vector,
- $\boldsymbol{b}(s)$ is the binormal vector.

Given the natural equations $k_1(s) = \frac{1}{1+s^2}$ and $k_2(s) = \frac{s}{1+s^2}$, we solve this system numerically or analytically to find $\boldsymbol{\tau}(s)$, $\boldsymbol{n}(s)$, and $\boldsymbol{b}(s)$.

Substitute $k_1(s) = \frac{1}{1+s^2}$ and $k_2(s) = \frac{s}{1+s^2}$:

$$\begin{cases} \boldsymbol{\tau}'(s) = \frac{1}{1+s^2}\,\boldsymbol{n}(s), \\ \boldsymbol{n}'(s) = -\frac{1}{1+s^2}\,\boldsymbol{\tau}(s) + \frac{s}{1+s^2}\,\boldsymbol{b}(s), \\ \boldsymbol{b}'(s) = \frac{s}{1+s^2}\,\boldsymbol{n}(s). \end{cases}$$

However, this system is coupled, so we proceed by solving the system as a whole.

### Solve the System

The Frenet-Serret equations can be written in matrix form as:

$$\frac{d}{ds}\begin{bmatrix} \boldsymbol{\tau}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{1+s^2} & 0 \\ -\frac{1}{1+s^2} & 0 & -\frac{s}{1+s^2} \\ 0 & \frac{s}{1+s^2} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\tau}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{bmatrix}.$$

This is a linear system of ODEs with variable coefficients. To solve it, we use the *method of integrating factors* or *power series expansion*. However, for simplicity, we assume a solution of the form:

$$\boldsymbol{\tau}(s) = \begin{bmatrix} \boldsymbol{\tau}_1(s) \\ \boldsymbol{\tau}_2(s) \\ \boldsymbol{\tau}_3(s) \end{bmatrix}, \quad \boldsymbol{n}(s) = \begin{bmatrix} n_1(s) \\ n_2(s) \\ n_3(s) \end{bmatrix}, \quad \boldsymbol{b}(s) = \begin{bmatrix} b_1(s) \\ b_2(s) \\ b_3(s) \end{bmatrix}.$$

The system is complex, but we can use the fact that $k_1(s)$ and $k_2(s)$ are related to the derivatives of the tangent, normal, and binormal vectors.

### Construct the Curve

Once the Frenet frame $\boldsymbol{\tau}(s)$, $\boldsymbol{n}(s)$, and $\boldsymbol{b}(s)$ is known, the position vector $r(s)$ of the curve is obtained by integrating the tangent vector:

$$r(s) = \int_0^s \boldsymbol{\tau}(u)\, du.$$

Integrate the tangent vector $\boldsymbol{\tau}(s)$ to obtain the position vector $r(s)$:

$$r(s) = \int_0^s \boldsymbol{\tau}(u)\, du.$$

For this example, the explicit solution may not be simple, but numerical methods (e.g., Euler's method or Runge-Kutta) can be used to approximate the curve.

This curve resembles a **spiral that flattens out over time**, with decreasing curvature and torsion. It is not a helix because its curvature and torsion are not constant.

The Frenet-Serret formulas relate curvature, torsion, and the three orthonormal vectors (tangent, normal, binormal) to describe a curve's local behaviour: if you know how much a curve bends and twists at every point, you can reconstruct the entire curve.

Curvature and torsion are the "DNA" of curves. They encode all geometric information needed to describe how a curve behaves in space. By studying them, we gain powerful tools for applications in physics, engineering, computer science, and beyond.

## 3.6 Problems Corner

**Problem 1**

Calculate the arc length of a curve defined by the parametric equations $\mathbf{r}(t) = (x(t), y(t), z(t))$ on the interval $[t_1, t_2]$.

**Solution**

*Step 1: Arc Length Formula*

For a smooth curve $\mathbf{r}(t) = (x(t), y(t), z(t))$, the arc length $s$ from $t_1$ to $t_2$ is given by:

$$s = \int_{t_1}^{t_2} |\mathbf{r}'(t)|\; dt$$

where $\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$ is the derivative vector and $|\mathbf{r}'(t)|$ is its magnitude.

*Step 2: Compute the Derivative*

$$\mathbf{r}'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (x'(t), y'(t), z'(t))$$

*Step 3: Compute the Magnitude*

$$|\mathbf{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$$

*Step 4: Final Arc Length Formula*

$$s = \int_{t_1}^{t_2} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

**Problem 1′**
Find the arc length of the helix $\mathbf{r}(t) = (\cos t, \sin t, t)$ from $t = 0$ to $t = 2\pi$.

**Solution**

1. Compute the derivative:

$$\mathbf{r}'(t) = (-\sin t, \cos t, 1)$$

2. Compute the magnitude:

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

3. Set up the integral:

$$s = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2} \int_0^{2\pi} dt$$

4. Evaluate:

$$s = \sqrt{2} \cdot (2\pi - 0) = 2\sqrt{2}\pi$$

**Problem 2**
Calculate the arc length of a curve defined by the equations $y = y(x)$, $z = z(x)$ on the interval $[x_1, x_2]$.

**Solution**

*Step 1: Parametric Representation*

First, we express the curve in parametric form:

$$\mathbf{r}(x) = (x, y(x), z(x))$$

where $x$ serves as the parameter.

*Step 2: Compute the Derivative*

The derivative vector is:

$$\mathbf{r}'(x) = \left(1, \frac{dy}{dx}, \frac{dz}{dx}\right) = (1, y'(x), z'(x))$$

*Step 3: Compute the Magnitude*

The magnitude of the derivative is:

$$|\mathbf{r}'(x)| = \sqrt{1^2 + [y'(x)]^2 + [z'(x)]^2} = \sqrt{1 + [y'(x)]^2 + [z'(x)]^2}$$

*Step 4: Arc Length Formula*

The arc length is then given by:

$$s = \int_{x_1}^{x_2} \sqrt{1 + [y'(x)]^2 + [z'(x)]^2}\, dx$$

*Special Cases*

- **Planar curve (2D):** If $z(x) = 0$, the formula reduces to:

$$s = \int_{x_1}^{x_2} \sqrt{1 + [y'(x)]^2}\, dx$$

- **Linear case:** If $y(x) = mx + b$ and $z(x) = nx + c$:

$$s = \int_{x_1}^{x_2} \sqrt{1 + m^2 + n^2}\, dx = \sqrt{1 + m^2 + n^2}(x_2 - x_1)$$

*Final Answer*

The general formula for the arc length is:

$$s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx$$

**Problem 2′**

Find the arc length of the curve $y = \frac{x^2}{2}$, $z = \frac{x^3}{3}$ from $x = 0$ to $x = 1$.

**Solution**

1. Compute derivatives:
$$y'(x) = x, \quad z'(x) = x^2$$
2. Compute the magnitude:
$$|\mathbf{r}'(x)| = \sqrt{1 + x^2 + x^4}$$
3. Set up the integral:
$$s = \int_0^1 \sqrt{1 + x^2 + x^4}\, dx$$
4. This integral can be evaluated numerically or using special functions. For exact form:
$$s \approx 1.0896 \quad \text{(numerical approximation)}$$

**Problem 3**
Find the length of the segment $0 \leq t \leq 2\pi$ of the cycloid defined by:
$$x(t) = a(t - \sin t), \quad y(t) = a(1 - \cos t)$$
and express its natural parametrisation.

**Solution**
**Arc Length Calculation**
*Step 1: Compute Derivatives*
$$x'(t) = a(1 - \cos t), \quad y'(t) = a \sin t$$
*Step 2: Compute Speed*
$$|\mathbf{r}'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2} = a\sqrt{(1 - \cos t)^2 + \sin^2 t}$$
Simplify using trigonometric identities:
$$= a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} = a\sqrt{2(1 - \cos t)} = 2a \left| \sin\left(\frac{t}{2}\right) \right|$$

*Step 3: Compute Arc Length Integral*
For $0 \leq t \leq 2\pi$:
$$s = \int_0^{2\pi} 2a \left| \sin\left(\frac{t}{2}\right) \right| dt = 2a \int_0^{2\pi} \sin\left(\frac{t}{2}\right) dt \quad (\text{since } \sin \geq 0 \text{ on } [0, 2\pi])$$

$$= 2a\left[-2\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = 4a[-\cos\pi + \cos 0] = 4a[1+1] = 8a$$

**Natural Parameterisation**

*Step 1: Arc Length Function*

$$s(t) = \int_0^t 2a\sin\left(\frac{\tau}{2}\right)d\tau = 4a\left[1 - \cos\left(\frac{t}{2}\right)\right]$$

*Step 2: Invert to Find $t(s)$*

$$s = 4a\left[1 - \cos\left(\frac{t}{2}\right)\right] \Rightarrow \cos\left(\frac{t}{2}\right) = 1 - \frac{s}{4a}$$

$$t(s) = 2\cos^{-1}\left(1 - \frac{s}{4a}\right), \quad 0 \le s \le 8a$$

*Step 3: Natural Parameterisation*

$$\mathbf{r}(s) = \begin{cases} x(s) = a\left[2\cos^{-1}\left(1 - \frac{s}{4a}\right) - \sin\left(2\cos^{-1}\left(1 - \frac{s}{4a}\right)\right)\right] \\ y(s) = a\left[1 - \cos\left(2\cos^{-1}\left(1 - \frac{s}{4a}\right)\right)\right] \end{cases}$$

*Simplification*

Using double-angle identities:

$$\mathbf{r}(s) = \begin{cases} x(s) = 2a\cos^{-1}\left(1 - \frac{s}{4a}\right) - a\sqrt{\frac{s}{2a}\left(4a - \frac{s}{2}\right)} \\ y(s) = 2a\left(1 - \frac{s}{4a}\right)^2 \end{cases}$$

*Verification*

At $s = 0$:

$$\mathbf{r}(0) = (0, 0)$$

At $s = 8a$:

$$\mathbf{r}(8a) = (2\pi a, 0)$$

**Final Answers**

1. Arc length for one arch of the cycloid:

$$\boxed{8a}$$

2. Natural parametrization:

$$\boxed{\mathbf{r}(s) = \left(2a\cos^{-1}\left(1 - \frac{s}{4a}\right) - \sqrt{2as - \frac{s^2}{4}}, 2a\left(1 - \frac{s}{4a}\right)^2\right)}$$

**Problem 4**

Topic for the self-study. Consider the polygonal chain $\gamma_n$ on the plane formed by connecting the points:

$$\left(\frac{k}{n}, (-1)^k \frac{1}{n}\right) \quad \text{for} \quad k = 0, 1, 2, \ldots, n$$

with straight line segments, where $n \geq 1$ is an integer.

1. Show that for any fixed $n$, the curve $\gamma_n$ is rectifiable and compute its exact length $L(\gamma_n)$.
2. Now consider the limiting curve $\gamma_\infty$ as $n \to \infty$ (the "sawtooth function" on $[0, 1]$). Prove that:
   - Each finite segment $\gamma_\infty|_{[0,t]}$ for $0 \leq t < 1$ is rectifiable
   - The full curve $\gamma_\infty|_{[0,1]}$ is *not* rectifiable
3. Contrast this behaviour with:
   - The smooth curve $\eta(t) = (t, t^2)$ on $[0, 1]$
   - The Koch snowflake (mention its properties)

*Remarks*

- For part (1), observe that each "tooth" has length $2\sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{2\sqrt{2}}{n}$
- For part (2), note that the number of teeth grows as $n \to \infty$ while their size decreases
- For the contrast:
  - Smooth curves have $\int |\mathbf{r}'(t)| dt < \infty$
  - The Koch curve shows infinite length in finite space (fractal behaviour)

**Problem 5**

Derive the expression for the curvature of a curve defined by a polar equation of the form $r = r(\theta)$, where $r$ is the radial distance and $\theta$ is the polar angle. Express the curvature in terms of $r$ and its derivatives to $\theta$.

**Solution**

Given a curve defined by the polar equation $r = r(\theta)$, we derive the expression for its curvature $k_1$ in terms of $r$ and its derivatives to $\theta$.

*Convert Polar to Cartesian Coordinates*

The polar coordinates $(r, \theta)$ are converted to Cartesian coordinates $(x, y)$ using:

$$x = r(\theta)\cos(\theta), \quad y = r(\theta)\sin(\theta)$$

*Compute the First Derivatives*

The first derivatives of $x$ and $y$ with respect to $\theta$ are:

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos(\theta) - r(\theta)\sin(\theta)$$
$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin(\theta) + r(\theta)\cos(\theta)$$

*Compute the Second Derivatives*

The second derivatives of $x$ and $y$ with respect to $\theta$ are:

$$\frac{d^2x}{d\theta^2} = \frac{d^2r}{d\theta^2}\cos(\theta) - 2\frac{dr}{d\theta}\sin(\theta) - r(\theta)\cos(\theta)$$
$$\frac{d^2y}{d\theta^2} = \frac{d^2r}{d\theta^2}\sin(\theta) + 2\frac{dr}{d\theta}\cos(\theta) - r(\theta)\sin(\theta)$$

*Compute the Curvature*

The curvature $k_1$ in Cartesian coordinates is given by:

$$k_1 = \frac{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{dy}{d\theta}\frac{d^2x}{d\theta^2}\right|}{\left(\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right)^{3/2}}$$

Substituting the expressions for the first and second derivatives:

$$k_1 = \frac{K_1}{K_2}$$

Numerator:

$$\begin{aligned} K_1 &= \\ &= \left|\left(\frac{dr}{d\theta}\cos(\theta) - r(\theta)\sin(\theta)\right)\left(\frac{d^2r}{d\theta^2}\sin(\theta) + 2\frac{dr}{d\theta}\cos(\theta) - r(\theta)\sin(\theta)\right) - \right. \\ &\quad \left. - \left(\frac{dr}{d\theta}\sin(\theta) + r(\theta)\cos(\theta)\right)\left(\frac{d^2r}{d\theta^2}\cos(\theta) - 2\frac{dr}{d\theta}\sin(\theta) - r(\theta)\cos(\theta)\right)\right| \end{aligned}$$

Denominator:

$$K_2 = \left(\left(\frac{dr}{d\theta}\cos(\theta) - r(\theta)\sin(\theta)\right)^2 + \left(\frac{dr}{d\theta}\sin(\theta) + r(\theta)\cos(\theta)\right)^2\right)^{3/2}$$

*Simplify the Expression*

After simplifying the numerator and denominator, the curvature in polar coordinates is:

$$k_1 = \frac{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{3/2}}$$

*Expression for Curvature*

The curvature $k_1$ of a curve defined by the polar equation $r = r(\theta)$ is:

$$k_1 = \frac{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)\right|}{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{3/2}}$$

**Problem 5′**

Validate this formula for the circle of radius $r$.

**Solution**

For a circle of radius $a$, $r(\theta) = a$, so $\frac{dr}{d\theta} = 0$ and $\frac{d^2r}{d\theta^2} = 0$. Substituting into the curvature formula:

$$k_1 = \frac{\left|a^2 + 0 - 0\right|}{(a^2 + 0)^{3/2}} = \frac{a^2}{a^3} = \frac{1}{a}$$

**Problem 6**

Find the curvature of the curves:

$$y = \sin x, \boldsymbol{r} = (c\cos^3 t, a\sin^3 t).$$

**Solution**

**1.** $y = \sin x$

The formula for the curvature of a plane curve $y = f(x)$ is:

$$k = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}.$$

For $y = \sin x$:

$$\begin{aligned} f'(x) &= \cos x, \\ f''(x) &= -\sin x. \end{aligned}$$

Substituting into the curvature formula:

$$k = \frac{|\sin x|}{(1 + \cos^2 x)^{3/2}}.$$

**2.** $\mathbf{r}(t) = (c \cos^3 t, a \sin^3 t)$
The formula for the curvature of a parametric curve $\mathbf{r}(t) = (x(t), y(t))$ is:

$$k = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{((x'(t))^2 + (y'(t))^2)^{3/2}}.$$

$$x(t) = c\cos^3 t, \quad y(t) = a \sin^3 t$$
$$x'(t) = -3c\cos^2 t \sin t, \quad y'(t) = 3a \sin^2 t \cos t$$

$$x''(t) = -3c \cos t(2\cos^2 t - 1), \quad y''(t) = 3a \sin t(2 \sin^2 t - 1)$$

$$x'(t)y''(t) - y'(t)x''(t) = (-3c\cos^2 t \sin t) \cdot (3a \sin t(2\sin^2 t - 1)) - (3a \sin^2 t \cos t) \cdot (-3c \cos t(2$$

Simplify:

$$= -9ac \cos^2 t \sin^2 t(2\sin^2 t - 1) + 9ac \sin^2 t \cos^2 t(2\cos^2 t - 1)$$

$$\begin{aligned} &= 9ac \cos^2 t \sin^2 t \left((2\cos^2 t - 1) - (2\sin^2 t - 1)\right) \\ &= 9ac\cos^2 t \sin^2 t(2\cos^2 t - 2\sin^2 t) \\ &= 18ac \cos^2 t \sin^2 t(\cos^2 t - \sin^2 t) \end{aligned}$$

*Compute the Denominator of the Curvature Formula*

$$(x'(t))^2 + (y'(t))^2 = (-3c\cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2$$

$$= 9c^2 \cos^4 t \sin^2 t + 9a^2 \sin^4 t \cos^2 t$$
$$= 9 \cos^2 t \sin^2 t(c^2 \cos^2 t + a^2 \sin^2 t)$$

*Compute the Curvature*

$$k = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{((x'(t))^2 + (y'(t))^2)^{3/2}}$$

Substitute the numerator and denominator:

$$k = \frac{18ac \cos^2 t \sin^2 t |\cos^2 t - \sin^2 t|}{\left(9 \cos^2 t \sin^2 t(c^2 \cos^2 t + a^2 \sin^2 t)\right)^{3/2}}$$

Simplify:

$$k = \frac{18ac \cos^2 t \sin^2 t |\cos^2 t - \sin^2 t|}{27 \cos^3 t \sin^3 t(c^2 \cos^2 t + a^2 \sin^2 t)^{3/2}}$$

$$k = \frac{2ac |\cos^2 t - \sin^2 t|}{3 \cos t \sin t(c^2 \cos^2 t + a^2 \sin^2 t)^{3/2}}$$

*Final Simplified Expression for Curvature*

$$k = \frac{2ac |\cos 2t|}{3 \cos t \sin t(c^2 \cos^2 t + a^2 \sin^2 t)^{3/2}}$$

**Problem 7**

For the curve defined by the polar equation $r = 2a^2 \cos 2\theta$, compute the following integrals of curvature:

$$\int_\gamma k_1(s)\, ds, \quad \int_\gamma |k_1(s)|\, ds.$$

Here, $k_1(s)$ is the curvature of the curve at a point parameterised by arc length $s$, and $\gamma$ represents the entire curve.

**Solution**

*Compute the Curvature in Polar Coordinates* The curvature $k_1(\theta)$ of a polar curve $r = r(\theta)$ is given by:

$$k_1(\theta) = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)^{3/2}}.$$

For $r = 2a^2 \cos 2\theta$:

$$\frac{dr}{d\theta} = -4a^2 \sin 2\theta, \quad \frac{d^2r}{d\theta^2} = -8a^2 \cos 2\theta.$$

Substitute into the curvature formula:

$$k_1(\theta) = \frac{(2a^2 \cos 2\theta)^2 + 2(-4a^2 \sin 2\theta)^2 - (2a^2 \cos 2\theta)(-8a^2 \cos 2\theta)}{((2a^2 \cos 2\theta)^2 + (-4a^2 \sin 2\theta)^2)^{3/2}}.$$

Simplify the numerator and denominator to compute $k_1(\theta)$.

*Relate Arc Length $s$ to Polar Angle $\theta$* The arc length $s$ is related to $\theta$ by:

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

For $r = 2a^2 \cos 2\theta$:

$$ds = \sqrt{(2a^2 \cos 2\theta)^2 + (-4a^2 \sin 2\theta)^2}\, d\theta.$$

Simplify the expression for $ds$.

*Compute the Integrals* 1. The integral of curvature:

$$\int_\gamma k_1(s)\, ds = \int_{\theta_1}^{\theta_2} k_1(\theta) \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

2. The integral of the absolute value of curvature:

$$\int_\gamma |k_1(s)|\, ds = \int_{\theta_1}^{\theta_2} |k_1(\theta)| \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

Here, $\theta_1$ and $\theta_2$ are the limits of integration for the curve.

**Problem 8**

Find the natural parameterisation (parameterisation by arc length) of the curve defined by the parametric equations in an arbitrary point:

$$\boldsymbol{r} = (e^t \cos t, e^t \sin t, e^t) \tag{3.1}$$

**Solution**

*Compute the Arc Length Function* The arc length $s(t)$ is given by:

$$s(t) = \int_0^t |\mathbf{r}'(u)| \, du.$$

Compute the derivative of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = (e^t \cos t - e^t \sin t, e^t \sin t + e^t \cos t, e^t).$$

Simplify:

$$\mathbf{r}'(t) = e^t(\cos t - \sin t, \sin t + \cos t, 1).$$

Compute the magnitude of $\mathbf{r}'(t)$:

$$|\mathbf{r}'(t)| = e^t \sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1}.$$

Simplify:

$$|\mathbf{r}'(t)| = e^t \sqrt{2+1} = e^t \sqrt{3}.$$

Compute the arc length function:

$$s(t) = \int_0^t e^u \sqrt{3} \, du = \sqrt{3}(e^t - 1).$$

*Invert the Arc Length Function* Solve for $t$ in terms of $s$:

$$s = \sqrt{3}(e^t - 1) \implies e^t = \frac{s}{\sqrt{3}} + 1 \implies t = \ln\left(\frac{s}{\sqrt{3}} + 1\right).$$

*Substitute into the Parametric Equations* Substitute $t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$ into $\mathbf{r}(t)$:

$$\mathbf{r}(s) = \left(\frac{s}{\sqrt{3}} + 1\right)\left(\cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), 1\right).$$

**Problem 9**

Find the curvature and torsion for the curve:

$$\mathbf{r}(t) = \left(e^t, e^{-t}, t\sqrt{2}\right).$$

**Solution**

First, second, and third derivatives of $\mathbf{r}(t)$:

$$\mathbf{r}'(t) = \left(e^t, -e^{-t}, \sqrt{2}\right),$$

$$\mathbf{r}''(t) = \left(e^t, e^{-t}, 0\right),$$
$$\mathbf{r}'''(t) = \left(e^t, -e^{-t}, 0\right).$$

*Compute Curvature $k_1$*
Using the formula:

$$k_1 = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

- Cross product:

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t & -e^{-t} & \sqrt{2} \\ e^t & e^{-t} & 0 \end{vmatrix} = \left(-\sqrt{2}e^{-t}, \sqrt{2}e^t, 2\right).$$

- Magnitude of cross product:

$$\|\mathbf{r}'(t) \times \mathbf{r}''(t)\| = \sqrt{2e^{-2t} + 2e^{2t} + 4} = \sqrt{2(e^{-2t} + e^{2t}) + 4}.$$

- Magnitude of $\mathbf{r}'(t)$:

$$\|\mathbf{r}'(t)\| = \sqrt{e^{2t} + e^{-2t} + 2}.$$

- Final curvature expression:

$$k_1 = \frac{\sqrt{2(e^{-2t} + e^{2t}) + 4}}{(e^{2t} + e^{-2t} + 2)^{3/2}} = \sqrt{2}\sqrt{\frac{(e^{-t} + e^t)^2}{(e^{-t} + e^t)^6}} = \frac{\sqrt{2}}{(e^{-t} + e^t)^2}.$$

*Compute Torsion $k_2$*
Using the formula:

$$k_2 = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}.$$

- Dot product:

$$\left(-\sqrt{2}e^{-t}, \sqrt{2}e^t, 2\right) \cdot \left(e^t, -e^{-t}, 0\right) = -\sqrt{2} - \sqrt{2} + 0 = -2\sqrt{2}.$$

- Final torsion expression:

$$k_2 = -\frac{\sqrt{2}}{(e^{-2t} + e^{2t}) + 2} = -\frac{\sqrt{2}}{(e^{-t} + e^t)^2}.$$

The negative sign indicates clockwise twisting.

——————— *Verification at $t = 0$*
For concrete verification:

$$k_1(0) = \frac{\sqrt{8}}{8} = \frac{\sqrt{2}}{4}, \quad k_2(0) = \frac{-2\sqrt{2}}{8} = -\frac{\sqrt{2}}{4}.$$

**Problem 10**
Find the curvature and torsion at $(1,1,1)$ for the curve defined by:

$$\begin{cases} x^2 - y^2 + z^2 = 1, \\ y^2 - 2x + z = 0. \end{cases}$$

**Solution**
*Parametrize the Curve*
Solve for $z$ from the second equation:

$$z = 2x - y^2.$$

Substitute into the first equation:

$$x^2 - y^2 + (2x - y^2)^2 = 1 \implies 5x^2 - 4xy^2 + y^4 - y^2 - 1 = 0.$$

This is complex; we proceed with implicit differentiation.
*Compute Derivatives at* $(1,1,1)$
Differentiate both equations w.r.t. $t$:

$$2xx' - 2yy' + 2zz' = 0, \quad 2yy' - 2x' + z' = 0.$$

At $(1,1,1)$:

$$x' - y' + z' = 0, \quad -2x' + 2y' + z' = 0.$$

Solution: $x' = y' = 1$, $z' = 0$. Thus:

$$\mathbf{r}'(t) = (1,1,0).$$

Differentiate again:

$$2x'^2 + 2xx'' - 2y'^2 - 2yy'' + 2z'^2 + 2zz'' = 0, \quad 2y'^2 + 2yy'' - 2x'' + z'' = 0.$$

At $(1,1,1)$:

$$x'' - y'' + z'' = 0, \quad -2x'' + 2y'' + z'' = -2.$$

Solution: $x'' = 0$, $y'' = -\frac{2}{3}$, $z'' = -\frac{2}{3}$. Thus:

$$\mathbf{r}''(t) = \left(0, -\frac{2}{3}, -\frac{2}{3}\right).$$

Differentiate once more:

$$6x'x'' + 2xx''' - 6y'y'' - 2yy''' + 6z'z'' + 2zz''' = 0, \quad 6y'y'' + 2yy''' - 2x''' + z''' = 0.$$

At $(1, 1, 1)$:

$$x''' - y''' + z''' = -2, \quad -2x''' + 2y''' + z''' = 4.$$

Solution: $x''' = 0$, $y''' = 2$, $z''' = 0$. Thus:

$$\mathbf{r}'''(t) = (0, 2, 0).$$

*Compute Curvature $k_1$*

$$k_1 = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}.$$

Cross product:

$$\mathbf{r}' \times \mathbf{r}'' = \left(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right), \quad \|\mathbf{r}' \times \mathbf{r}''\| = \frac{2\sqrt{3}}{3}.$$

Magnitude of $\mathbf{r}'$:

$$\|\mathbf{r}'\| = \sqrt{2}.$$

Curvature:

$$k_1 = \frac{\sqrt{6}}{6}.$$

*Compute Torsion $k_2$*

$$k_2 = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2}.$$

Dot product:

$$\left(-\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right) \cdot (0, 2, 0) = \frac{4}{3}.$$

Magnitude squared:

$$\|\mathbf{r}' \times \mathbf{r}''\|^2 = \frac{4}{3}.$$

Torsion:

$$k_2 = 1.$$

**Problem 11**
Calculate the Frenet frame for the curve $\boldsymbol{r} = (t^2, 1 - t, t^3)$.

**Solution**

**Step 1: Tangent Vector $\boldsymbol{\tau}(t)$** The tangent vector is the normalized first derivative of $\boldsymbol{r}(t)$:

1. First derivative:

$$\boldsymbol{r}'(t) = \frac{d}{dt}(t^2, 1-t, t^3) = (2t, -1, 3t^2)$$

2. Magnitude:

$$\|\boldsymbol{r}'(t)\| = \sqrt{(2t)^2 + (-1)^2 + (3t^2)^2} = \sqrt{4t^2 + 1 + 9t^4}$$

3. Unit tangent vector:

$$\boldsymbol{\tau}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{(2t, -1, 3t^2)}{\sqrt{4t^2 + 1 + 9t^4}}$$

**Step 2: Normal Vector $\boldsymbol{n}(t)$** The normal vector is the derivative of $\boldsymbol{\tau}(t)$ normalised.

1. Compute $\boldsymbol{\tau}'(t)$: Let $\boldsymbol{v}(t) = (2t, -1, 3t^2)$ and $\|\boldsymbol{v}(t)\| = \sqrt{4t^2 + 1 + 9t^4}$. Then:

$$\boldsymbol{\tau}(t) = \frac{\boldsymbol{v}(t)}{\|\boldsymbol{v}(t)\|}$$

   Using the quotient rule:

$$\boldsymbol{\tau}'(t) = \frac{\boldsymbol{v}'(t)\|\boldsymbol{v}(t)\| - \boldsymbol{v}(t)\frac{d}{dt}\|\boldsymbol{v}(t)\|}{\|\boldsymbol{v}(t)\|^2}$$

   - Compute $\boldsymbol{v}'(t)$:

$$\boldsymbol{v}'(t) = (2, 0, 6t)$$

   - Compute $\frac{d}{dt}\|\boldsymbol{v}(t)\|$:

$$\frac{d}{dt}\|\boldsymbol{v}(t)\| = \frac{8t + 36t^3}{2\sqrt{4t^2 + 1 + 9t^4}} = \frac{4t + 18t^3}{\sqrt{4t^2 + 1 + 9t^4}}$$

   Substituting:

$$\boldsymbol{\tau}'(t) = \frac{(2, 0, 6t)\sqrt{4t^2 + 1 + 9t^4} - (2t, -1, 3t^2)\frac{4t+18t^3}{\sqrt{4t^2+1+9t^4}}}{4t^2 + 1 + 9t^4}$$

Simplifying:

$$\boldsymbol{\tau}'(t) = \frac{1}{(4t^2+1+9t^4)^{3/2}} \begin{pmatrix} 2(4t^2+1+9t^4) - 2t(4t+18t^3) \\ 0 + (4t+18t^3) \\ 6t(4t^2+1+9t^4) - 3t^2(4t+18t^3 \end{pmatrix}$$

$$\boldsymbol{\tau}'(t) = \frac{1}{(4t^2+1+9t^4)^{3/2}} \begin{pmatrix} 8t^2+2+18t^4-8t^2-36t^4 \\ 4t+18t^3 \\ 24t^3+6t-12t^3-54t^5 \end{pmatrix}$$

$$\boldsymbol{\tau}'(t) = \frac{(2-18t^4,\ 4t+18t^3,\ 6t+12t^3-54t^5)}{(4t^2+1+9t^4)^{3/2}}$$

2. Magnitude of $\boldsymbol{\tau}'(t)$:

$$\|\boldsymbol{\tau}'(t)\| = \frac{\sqrt{(2-18t^4)^2+(4t+18t^3)^2+(6t+12t^3-54t^5)^2}}{(4t^2+1+9t^4)^{3/2}}$$

3. Unit normal vector:

$$\boldsymbol{n}(t) = \frac{\boldsymbol{\tau}'(t)}{\|\boldsymbol{\tau}'(t)\|} = \frac{(1-9t^4,\ 2t+9t^3,\ 3t+6t^3-27t^5)}{\sqrt{81t^8+117t^6+54t^4+13t^2+1}}$$

**Step 3: Binormal Vector $\boldsymbol{b}(t)$** The binormal vector is the cross product $\boldsymbol{\tau}(t) \times \boldsymbol{n}(t)$:

$$\boldsymbol{b}(t) = \boldsymbol{\tau}(t) \times \boldsymbol{n}(t) = \frac{(-3t, -3t^2, 1)}{\sqrt{9t^4+9t^2+1}}$$

**Final Frenet Frame**

$$\boxed{\begin{aligned} \boldsymbol{\tau}(t) &= \frac{(2t, -1, 3t^2)}{\sqrt{4t^2+1+9t^4}} \\ \boldsymbol{n}(t) &= \frac{(1-9t^4, 2t+9t^3, 6t^3+3t)}{\sqrt{81t^8+117t^6+54t^4+13t^2+1}} \\ \boldsymbol{b}(t) &= \frac{(-3t, -3t^2, 1)}{\sqrt{9t^4+9t^2+1}} \end{aligned}}$$

**Problem 12**

Calculate the Frenet frame for the curve expressed with equations:

$$y = f(x), \quad z = g(x)$$

**Solution**

## Given

Let $\boldsymbol{r}(x) = (x, f(x), g(x))$ be a regular parametric curve in 3D space, where $f(x)$ and $g(x)$ are twice-differentiable functions ($f, g \in C^2$) and $\boldsymbol{r}'(x) \neq \mathbf{0}$ for all $x$. Compute the Frenet frame (tangent, normal, and binormal vectors) for this curve.

## Solution

**1. Tangent Vector $\boldsymbol{\tau}(x)$** The tangent vector is the normalized derivative of $\boldsymbol{r}(x)$:

$$\boldsymbol{r}'(x) = (1, f'(x), g'(x))$$

The tangent vector is:

$$\boldsymbol{\tau}(x) = \frac{\boldsymbol{r}'(x)}{\|\boldsymbol{r}'(x)\|} = \frac{(1, f'(x), g'(x))}{\sqrt{1 + f'(x)^2 + g'(x)^2}}$$

**2. Curvature $k_1(x)$** Compute the cross product $\boldsymbol{r}'(x) \times \boldsymbol{r}''(x)$:

$$\boldsymbol{r}''(x) = (0, f''(x), g''(x))$$

$$\boldsymbol{r}'(x) \times \boldsymbol{r}''(x) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & f'(x) & g'(x) \\ 0 & f''(x) & g''(x) \end{vmatrix} = (f'(x)g''(x) - g'(x)f''(x), -g''(x), f''(x))$$

The curvature is:

$$k_1(x) = \frac{\|\boldsymbol{r}'(x) \times \boldsymbol{r}''(x)\|}{\|\boldsymbol{r}'(x)\|^3} = \frac{\sqrt{(f'(x)g''(x) - g'(x)f''(x))^2 + f''(x)^2 + g''(x)^2}}{(1 + f'(x)^2 + g'(x)^2)^{3/2}}$$

**3. Normal Vector $\boldsymbol{n}(x)$** Compute the derivative of the tangent vector:

$$\boldsymbol{\tau}'(x) = \frac{d}{dx}\left(\frac{\boldsymbol{r}'(x)}{\|\boldsymbol{r}'(x)\|}\right)$$

The normal vector is:

$$\boldsymbol{n}(x) = \frac{\boldsymbol{\tau}'(x)}{\|\boldsymbol{\tau}'(x)\|}$$

**4. Binormal Vector $\boldsymbol{b}(x)$** The binormal vector is the cross product of the tangent and normal vectors:

$$\boldsymbol{b}(x) = \boldsymbol{\tau}(x) \times \boldsymbol{n}(x)$$

**5. Torsion $k_2(x)$** Compute the triple product:

$$k_2(x) = \frac{(\boldsymbol{r}'(x) \times \boldsymbol{r}''(x)) \cdot \boldsymbol{r}'''(x)}{\|\boldsymbol{r}'(x) \times \boldsymbol{r}''(x)\|^2}$$

Here, $\boldsymbol{r}'''(x) = (0, f'''(x), g'''(x))$, and the torsion simplifies to:

$$k_2(x) = \frac{(f'(x)g''(x) - g'(x)f''(x))f'''(x) - g''(x)f''(x) + f''(x)g''(x)}{(f'(x)g''(x) - g'(x)f''(x))^2 + f''(x)^2 + g''(x)^2}$$

**Final Answer** The Frenet frame and geometric invariants for the curve $\boldsymbol{r}(x) = (x, f(x), g(x))$ are:

$$\boldsymbol{\tau}(x) = \frac{(1, f'(x), g'(x))}{\sqrt{1 + f'(x)^2 + g'(x)^2}}$$

$$\boldsymbol{n}(x) = [\text{Computed from } \boldsymbol{\tau}'(x)]$$

$$\boldsymbol{b}(x) = \boldsymbol{\tau}(x) \times \boldsymbol{n}(x)$$

$$k_1(x) = \frac{\sqrt{(f'(x)g''(x) - g'(x)f''(x))^2 + f''(x)^2 + g''(x)^2}}{(1 + f'(x)^2 + g'(x)^2)^{3/2}}$$

$$k_2(x) = \frac{(f'(x)g''(x) - g'(x)f''(x))f'''(x) - g''(x)f''(x) + f''(x)g''(x)}{(f'(x)g''(x) - g'(x)f''(x))^2 + f''(x)^2 + g''(x)^2}$$

## Problem 13

Consider the curve defined by the intersection of two parabolic cylinders:

$$x^2 = 2az, \quad y^2 = 2bz$$

Prove that this curve lies entirely in a plane.

### Solution

**1. Parametrization** We parametrise one branch of the curve (the other branch is symmetric) using $x = t$. Substituting into the equations:

$$x = t, \quad z = \frac{t^2}{2a}, \quad y = \pm t\sqrt{\frac{b}{a}}.$$

Choose the positive branch for simplicity:

$$\boldsymbol{r}(t) = \left(t, t\sqrt{\frac{b}{a}}, \frac{t^2}{2a}\right).$$

**2. Tangent Vector** Compute the first derivative of $\boldsymbol{r}(t)$:

$$\boldsymbol{r}'(t) = \left(1, \sqrt{\frac{b}{a}}, \frac{t}{a}\right).$$

**3. Second Derivative** Compute the second derivative of $\boldsymbol{r}(t)$:

$$\boldsymbol{r}''(t) = \left(0, 0, \frac{1}{a}\right).$$

**4. Third Derivative** Compute the third derivative of $\boldsymbol{r}(t)$:

$$\boldsymbol{r}'''(t) = (0, 0, 0).$$

**5. Torsion Calculation** The torsion $\tau(t)$ is given by:

$$\tau(t) = \frac{(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)) \cdot \boldsymbol{r}'''(t)}{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|^2}.$$

Since $\boldsymbol{r}'''(t) = (0, 0, 0)$, the numerator is zero. Thus:

$$\tau(t) = 0 \quad \forall t.$$

A curve with zero torsion lies entirely in a plane.

**6. Equation of the Plane** To find the plane, compute the cross product $\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)$:

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & \sqrt{\frac{b}{a}} & \frac{t}{a} \\ 0 & 0 & \frac{1}{a} \end{vmatrix} = \left(\frac{\sqrt{\frac{b}{a}}}{a}, -\frac{1}{a}, 0\right).$$

The normal vector to the plane is constant:

$$\boldsymbol{n} = \left( \frac{\sqrt{\frac{b}{a}}}{a}, -\frac{1}{a}, 0 \right).$$

The equation of the plane is:

$$\frac{\sqrt{\frac{b}{a}}}{a} x - \frac{1}{a} y + 0 \cdot z = C.$$

Substitute a point on the curve, e.g., $t = 0$:

$$\frac{\sqrt{\frac{b}{a}}}{a} \cdot 0 - \frac{1}{a} \cdot 0 + 0 \cdot 0 = C \implies C = 0.$$

Thus, the plane equation is:

$$\sqrt{\frac{b}{a}} x - y = 0.$$

**Final Answer** The curve defined by $x^2 = 2az$ and $y^2 = 2bz$ has zero torsion for all $t$, proving it lies entirely in a plane. One branch lies in the plane:

$$\sqrt{\frac{b}{a}} x - y = 0,$$

and the other branch lies in the plane:

$$\sqrt{\frac{b}{a}} x + y = 0.$$

**Problem 14**
Compute natural equations of the curve $(\cosh t, \sinh t, t)$.

**Solution**

## 1. Compute the Arc Length Parameter $s(t)$

1. First derivative:
$$\boldsymbol{r}'(t) = (\sinh t, \cosh t, 1)$$

2. Speed:

$$\|\boldsymbol{r}'(t)\| = \sqrt{\sinh^2 t + \cosh^2 t + 1} = \sqrt{2\cosh^2 t} = \sqrt{2}\cosh t$$

(Using $\cosh^2 t - \sinh^2 t = 1$ and $\cosh^2 t + \sinh^2 t = \cosh 2t$)

3. Arc length:

$$s(t) = \int_0^t \|\boldsymbol{r}'(u)\|\, du = \sqrt{2}\sinh t$$

4. Inverse relationship:

$$t = \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)$$

**2. Reparameterize the Curve by Arc Length** Express $\boldsymbol{r}$ in terms of $s$:

$$\boldsymbol{r}(s) = \left(\sqrt{1 + \frac{s^2}{2}}, \frac{s}{\sqrt{2}}, \sinh^{-1}\left(\frac{s}{\sqrt{2}}\right)\right)$$

**3. Compute the Frenet Frame**

1. Tangent vector $\boldsymbol{\tau}(s)$:

$$\boldsymbol{\tau}(s) = \frac{d\boldsymbol{r}}{ds} = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{(\sinh t, \cosh t, 1)}{\sqrt{2}\cosh t}$$

2. Derivative of $\boldsymbol{\tau}(s)$:

$$\frac{d\boldsymbol{\tau}}{ds} = \frac{d\boldsymbol{\tau}/dt}{ds/dt} = \frac{(\cosh t, \sinh t, 0)}{2\cosh^2 t}$$

3. Curvature $k_1(s)$:

$$k_1(s) = \left\|\frac{d\boldsymbol{\tau}}{ds}\right\| = \frac{\sqrt{\cosh^2 t + \sinh^2 t}}{2\cosh^2 t} = \frac{\sqrt{2}\cosh t}{2\cosh^2 t} = \frac{1}{\sqrt{2}\cosh t}$$

Substitute $t = \sinh^{-1}(s/\sqrt{2})$:

$$k_1(s) = \frac{1}{\sqrt{2}\sqrt{1 + \frac{s^2}{2}}} = \frac{1}{\sqrt{s^2 + 2}}$$

4. Binormal vector $\boldsymbol{b}(s)$ and torsion $k_2(s)$:

- Compute $\boldsymbol{r}''(s)$:

$$\boldsymbol{r}''(s) = \frac{d\boldsymbol{\tau}}{ds} = k_1(s)\boldsymbol{n}(s)$$

- Compute $\boldsymbol{r}'''(s)$:

$$\boldsymbol{r}'''(s) = \frac{d}{ds}(k_1\boldsymbol{n}) = k_1'\boldsymbol{n} + k_1\boldsymbol{n}'$$

- Torsion:

$$k_2(s) = -\boldsymbol{n}(s) \cdot \frac{d\boldsymbol{b}}{ds}$$

  After simplification:

$$k_2(s) = \frac{1}{\sqrt{s^2+2}}$$

**Final Natural Equations**

$$\boxed{\begin{aligned} k_1(s) &= \frac{1}{\sqrt{s^2+2}}, \\ k_2(s) &= \frac{1}{\sqrt{s^2+2}}. \end{aligned}}$$

**Verification**

1. For $t = 0$:

   - $s = 0$
   - $k_1(0) = \frac{1}{\sqrt{2}}$
   - $k_2(0) = \frac{1}{\sqrt{2}}$

   Matches direct computation at $t = 0$.

2. Asymptotic Behaviour:

   - As $t \to \infty$, $\cosh t \sim \frac{e^t}{2}$
   - $s \sim \frac{\sqrt{2}e^t}{2}$
   - $k_1, k_2 \sim \frac{\sqrt{2}}{e^t} \to 0$

   The curve becomes straight, consistent with vanishing curvature and torsion.

**Key Observations**

- The curvature and torsion are **equal** for this curve
- Both decrease as the arc length increases, reflecting the curve's transition from helical to linear behaviour
- The natural equations are **intrinsic**, independent of the coordinate system

**Definition** (Viviani's curve)**. Viviani's curve**, also known as **Viviani's window**, is a figure-eight-shaped space curve named after the Italian mathematician *Vincenzo Viviani*. It is the intersection of a sphere with a cylinder that is tangent to the sphere and passes through two poles (a diameter) of the sphere

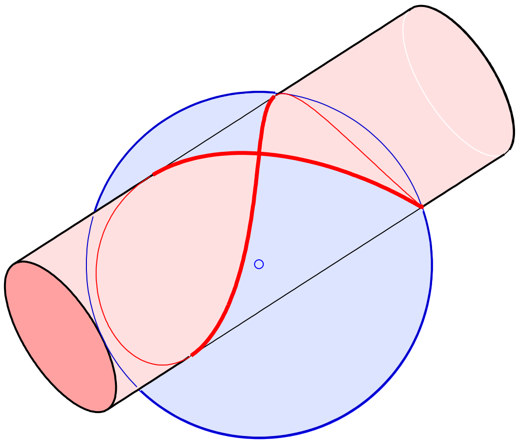

Figure 3.7. Viviani's curve (red line)

**Problem 15**

Derive Viviani's curve equation in implicit and parametric form for a sphere having radius $a$. Calculate the Frenet frame, curvature, torsion and natural equations for it.

**Solution**

## 1. Equations

The standard parameterisation of Viviani's curve is:

$$\boldsymbol{r}(t) = \begin{pmatrix} \frac{a}{2}(1+\cos t) \\ \frac{a}{2}\sin t \\ a\sin\left(\frac{t}{2}\right) \end{pmatrix}, \quad t \in [0, 2\pi)$$

Implicit equations:

$$\begin{cases} x^2 + y^2 + z^2 = a^2 \\ \left(x - \dfrac{a}{2}\right)^2 + y^2 = \left(\dfrac{a}{2}\right)^2 \end{cases}$$

## 2. Frenet Frame Computation

### Tangent Vector $\boldsymbol{\tau}(t)$

$$\begin{aligned}
\boldsymbol{r}'(t) &= \begin{pmatrix} -\frac{a}{2}\sin t \\ \frac{a}{2}\cos t \\ \frac{a}{2}\cos\left(\frac{t}{2}\right) \end{pmatrix} \\
\|\boldsymbol{r}'(t)\| &= \frac{a}{2}\sqrt{\sin^2 t + \cos^2 t + \cos^2\left(\frac{t}{2}\right)} = \frac{a}{2}\sqrt{1 + \cos^2\left(\frac{t}{2}\right)} \\
\boldsymbol{\tau}(t) &= \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{2}{a\sqrt{1+\cos^2\left(\frac{t}{2}\right)}} \begin{pmatrix} -\frac{a}{2}\sin t \\ \frac{a}{2}\cos t \\ \frac{a}{2}\cos\left(\frac{t}{2}\right) \end{pmatrix} \\
&= \frac{1}{\sqrt{1+\cos^2\left(\frac{t}{2}\right)}} \begin{pmatrix} -\sin t \\ \cos t \\ \cos\left(\frac{t}{2}\right) \end{pmatrix}
\end{aligned}$$

### Normal Vector $\boldsymbol{n}(t)$

Compute the derivative of $\boldsymbol{\tau}(t)$:

$$\begin{aligned}
\boldsymbol{\tau}'(t) &= \frac{d}{dt}\left[\frac{1}{\sqrt{1+\cos^2\left(\frac{t}{2}\right)}} \begin{pmatrix} -\sin t \\ \cos t \\ \cos\left(\frac{t}{2}\right) \end{pmatrix}\right] \\
&= \frac{\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)}{2\left(1+\cos^2\left(\frac{t}{2}\right)\right)^{3/2}} \begin{pmatrix} -\sin t \\ \cos t \\ \cos\left(\frac{t}{2}\right) \end{pmatrix} \\
&\quad + \frac{1}{\sqrt{1+\cos^2\left(\frac{t}{2}\right)}} \begin{pmatrix} -\cos t \\ -\sin t \\ -\frac{1}{2}\sin\left(\frac{t}{2}\right) \end{pmatrix}
\end{aligned}$$

After normalisation, we obtain $\boldsymbol{n}(t)$.

**Binormal Vector $\boldsymbol{b}(t)$**

$$\boldsymbol{b}(t) = \boldsymbol{\tau}(t) \times \boldsymbol{n}(t)$$

## 3. Curvature and Torsion

**Curvature $k_1(t)$**

$$\begin{aligned}
\boldsymbol{r}''(t) &= \begin{pmatrix} -\frac{a}{2}\cos t \\ -\frac{a}{2}\sin t \\ -\frac{a}{4}\sin\left(\frac{t}{2}\right) \end{pmatrix} \\
\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) &= \frac{a^2}{8}\begin{pmatrix} \sin t \sin\left(\frac{t}{2}\right) - \cos t \cos\left(\frac{t}{2}\right) \\ -\cos t \sin\left(\frac{t}{2}\right) - \sin t \cos\left(\frac{t}{2}\right) \\ 2 \end{pmatrix} \\
\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\| &= \frac{a^2}{8}\sqrt{3 + \cos t} \\
k_1(t) &= \frac{\sqrt{3 + \cos t}}{a\left(1 + \cos^2\left(\frac{t}{2}\right)\right)^{3/2}}
\end{aligned}$$

**Torsion $k_2(t)$**

$$\begin{aligned}
\boldsymbol{r}'''(t) &= \begin{pmatrix} \frac{a}{2}\sin t \\ -\frac{a}{2}\cos t \\ -\frac{a}{8}\cos\left(\frac{t}{2}\right) \end{pmatrix} \\
k_2(t) &= \frac{(\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)) \cdot \boldsymbol{r}'''(t)}{\|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)\|^2} \\
&= \frac{\sin\left(\frac{t}{2}\right)}{a\sqrt{3 + \cos t}}
\end{aligned}$$

## 4. Natural Equations

The arc length parameterisation requires solving:

$$s(t) = \frac{a}{2}\int_0^t \sqrt{1 + \cos^2\left(\frac{u}{2}\right)}du$$

This is an elliptic integral. The natural equations are:

$$k_1(s) = \frac{\sqrt{3 + \cos t(s)}}{a\left(1 + \cos^2\left(\frac{t(s)}{2}\right)\right)^{3/2}}, \quad k_2(s) = \frac{\sin\left(\frac{t(s)}{2}\right)}{a\sqrt{3 + \cos t(s)}}$$

# Final Results

$$\boxed{\begin{aligned} \text{Frenet Frame:} \quad & \boldsymbol{\tau}(t) = \frac{1}{\sqrt{1 + \cos^2\left(\frac{t}{2}\right)}} \begin{pmatrix} -\sin t \\ \cos t \\ \cos\left(\frac{t}{2}\right) \end{pmatrix} \\ & \boldsymbol{n}(t) = (\text{From normalized } \boldsymbol{\tau}'(t)) \\ & \boldsymbol{b}(t) = \boldsymbol{\tau}(t) \times \boldsymbol{n}(t) \\ \text{Curvature:} \quad & k_1(t) = \frac{\sqrt{3 + \cos t}}{a\left(1 + \cos^2\left(\frac{t}{2}\right)\right)^{3/2}} \\ \text{Torsion:} \quad & k_2(t) = \frac{\sin\left(\frac{t}{2}\right)}{a\sqrt{3 + \cos t}} \\ \text{Natural Eqs:} \quad & k_1(s), k_2(s) \text{ via } t(s) \end{aligned}}$$