University of Waterloo

PMATH 370: Chaos and Fractals

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Lecture Notes

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Lecture 1

Introduction

Much of the material covered in this course involves the **iteration of functions**. For example, suppose that we are given a real-valued function f of a single real variable, i.e.,

$$f : \mathbb{R} \to \mathbb{R} \,. \tag{1}$$

Now let $x_0 \in \mathbb{R}$ be a starting point or "seed" and define the following iteration procedure,

$$
\begin{aligned}
x_1 &= f(x_0) \\
x_2 &= f(x_1) \\
&\vdots
\end{aligned}
\tag{2}
$$

which can be compactly expressed as follows,

$$x_{n+1} = f(x_n) \quad n = 0, 1, 2, \cdots \,. \tag{3}$$

Eq. 3 is an example of a **discrete dynamical system**. Without trying to get too involved in details, a dynamical system (DS) has two major components:

1. A relevant **space** – we'll call it X – in which the DS is defined, referred to as the **phase space**. In the example in Eq. (3), the phase space is the real line $\mathbb{R}$. We may also wish to focus our attention on a subset of $\mathbb{R}$, say $X = [a, b]$.

   Other examples of possible phase spaces are $\mathbb{R}^n$ (or subsets), the complex plane $\mathbb{C}$ (or subsets), the space of complex n-vectors $\mathbb{C}^n$, etc.. Later in this course, we'll be examining DS in $\mathbb{R}^2$ and $\mathbb{C}$.

2. A **mapping** of the space X to itself. In Eq. (3) the mapping is the function f. The mapping defines an **evolution** in the phase space, where points are mapped to (possibly) new points. The evolution in Eq. (3) is a **discrete** evolution, since it proceeds in discrete steps, i.e., a point x_n is mapped to a point x_{n+1}. The indices n and $n + 1$, which may be viewed as times, are discrete.

   This may be contrasted to the evolution of a solution $x(t)$ of a differential equation which proceeds in continuous time $t \in \mathbb{R}$. As such, a differential equation may be viewed as giving rise to a **continuous dynamical system**.

As we'll discuss later in the course, even simple discrete dynamical systems, because of their discrete nature, can exhibit rather fascinating behaviours as compared to their continuous counterparts. That being said, continuous dynamical systems can also exhibit remarkable and complicated behaviours.

Given a seed point $x_0 \in \mathbb{R}$, the set of points,

$$\mathcal{O}(x_0) = \{x_0, x_1, x_2, \cdots\} = \{x_n\}_{n=0}^{\infty} \tag{4}$$

determined by the iteration procedure in (3) is known as the **forward orbit** of x_0. (There can also be **backward orbits** – we'll discuss them later.) It can be viewed as the "future state" of the seed point x_0 according to the discrete evolution defined in Eq. (3). Of course, its evolution is determined by the mapping f.

Note that we may also express the points x_n in the forward orbit of x_0 in the following way:

$$\begin{aligned}
x_1 &= f(x_0) \\
x_2 &= f(x_1) = f(f(x_0)) = (f \circ f)(x_0) = f^{\circ 2}(x_0) \\
x_3 &= f(x_2) = f(f(x_1)) = \cdots = f^{\circ 3}(x_0) \\
&\vdots \\
x_n &= f^{\circ n}(x_0)
\end{aligned} \tag{5}$$

where

$$f^{\circ n}(x) = (f \circ f \circ \cdots \circ f)(x) \quad n \text{ times} \tag{6}$$

denotes the n-fold composition of the function f with itself. Note that $f^{\circ n}$ does **not** mean $f(x)$ raised to the nth power, i.e., $[f(x)]^n$. By definition,

$$f^{\circ(n+1)}(x) = f(f^{\circ n}(x)). \tag{7}$$

For convenience, we shall relax the notation: For the remainder of the course, unless otherwise indicated, we'll let $f^n(x)$ denote $f^{\circ n}(x)$.

One of the basic goals of dynamical systems theory is to understand the eventual or asymptotic behaviour of an iterative process such as Eq. (3). In other words, how do the iterates x_n behave as $n \to \infty$? There are many possibilites, including the following:

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1. They "blow up", i.e., their magnitudes become arbitrarily large," i.e., $|x_n| \to \infty$,

2. They approach a single point $\bar{x}$, i.e., $x_n \to \bar{x}$. (The point $\bar{x}$ may or may not depend on the starting point x_0.)

3. They approach an oscillatory or cyclic behaviour. For example, they may eventually alternate between two distinct values, p_1 and p_2. In this case we say that the sequence $\{x_n\}$ approaches the **two cycle** (p_1, p_2) where $f(p_1) = p_2$ and $f(p_2) = p_1$.

4. The cyclic behaviour may involve more points, e.g., an N-**cycle**, $(p_1, p_2, \cdots, p_N)$, where $N \geq 2$ and $f(p_1) = p_2$, $f(p_2) = p_3$, ..., $f(p_{N-1}) = p_N$ and $f(p_N) = p_1$.

5. The iterates x_n seem to demonstrate a kind of irregular, random-looking, behaviour.

We shall encounter examples of all of the above behaviours in this course.

Of particular interest are constant solutions to Eq. (3), i.e., the existence of a point $\bar{x} \in \mathbb{R}$ such that

$$x_0 = \bar{x} \quad \implies \quad x_n = \bar{x}, \quad n = 1, 2, \cdots . \tag{8}$$

Such a point $\bar{x}$ is a **fixed point** of the function f, i.e.,

$$f(\bar{x}) = \bar{x} . \tag{9}$$

One is interested to know if such solutions/fixed points are **stable** or **unstable**, i.e., if we pick an initial condition x_0 near $\bar{x}$, do the iterates x_n approach $\bar{x}$ (**stable** or **attractive** fixed point) or do they travel away from $\bar{x}$ (**unstable** or **repulsive** fixed point). In Case 2 above, the point $\bar{x}$ corresponds to an attractive fixed point.

In the case of nonlinear functions f, there is also the possibility of solutions that are periodic cycles, as mentioned in Cases 3 and 4. In those cases, the cycles were attractive. It is also possible that cycles are repulsive, i.e., if you start at a point x_0 close to one of the points p_i of an N-cycle, the iterates x_n move away from the N-cycle.

A brief pause: Why study discrete dynamical systems/iteration processes

This discussion will be very brief. Firstly, many models in applied mathematics are **discrete models** – you observe "something" at a time T and then come back and observe it at a time $T + \Delta T$. Furthermore, you have some knowledge that the "something" that is evolving in time is doing so in a discrete, as opposed to continuous, way. It can be argued that the modelling of animal populations, e.g., foxes, bears, in a suitably large area, can be done in such a discrete manner: Such animals have mating seasons and, with suitable assumptions on the availability of resources, e.g., food, the number of offspring born in a certain year will be proportional, in some way, to the number of males and females prior to mating season. The animal population is roughly constant throughout the season/year and certainly does not vary in a continuous manner (population is measured in integers). As such, it seems to make sense to try to relate the population P_N at year N to the population P_{N-1} of the previous year.

As we shall see a little later, even simple population models with some adjustable parameters can exhibit interesting and complicated dynamical behaviour.

Secondly, many, if not almost all, numerical procedures in applied mathematics/scientific computing involve some kind of iteration procedure. For example, as you may already have seen, the "numerical solution" of a differential equation (DE), i.e., the construction of good approximations to the solution $x(t)$ of a DE actually involves the iteration of some kind of numerical scheme (e.g., the so-called "Runge-Kutta" procedure). The performance of such procedures in fixed-point arithmetic (i.e., the use of a finite number of digits) can also complicate matters.

Thirdly, the mathematics itself associated with discrete dynamical systems is extremely interesting, leading to the concepts of "chaos" and "fractals" that will be investigated in this course.

Let us now consider some simple examples of discrete dynamical systems.

Example 1: $f(x) = \frac{1}{2}x$ on $X = \mathbb{R}$.

In this case, the iteration scheme in Eq. (3) is

$$x_{n+1} = \frac{1}{2}x_n. \tag{10}$$

If we start with an arbitrary seed point x_0, then the iteration scheme in Eq. (3) becomes

$$
\begin{aligned}
x_1 &= \frac{1}{2}x_0 \\
x_2 &= \frac{1}{2}x_1 = \frac{1}{4}x_0 \\
x_3 &= \frac{1}{2}x_1 = \frac{1}{8}x_0 \\
&\vdots \\
x_n &= \frac{1}{2^n}x_0.
\end{aligned}
\tag{11}
$$

From this result, we may make a few conclusions:

1. If $x_0 = 0$, then all $x_n = 0$. This is a consequence of the fact that $\bar{x} = 0$ is a fixed point of $f(x) = \frac{1}{2}x$, i.e., $f(0) = 0$.

2. If $x_0 > 0$, then $x_n \to 0^+$ as $n \to \infty$. (The x_n approach 0 monotonically, i.e., $\{x_n\}$ is a monotonically decreasing sequence.)

3. If $x_0 < 0$, then $x_n \to 0^-$ as $n \to \infty$. (The x_n approach 0 monotonically, i.e., $\{x_n\}$ is a monotonically increasing sequence with limit 0.

Once again, we can summarize the behaviours in 2. and 3. as follows:

$$|x_n| \to 0 \quad \text{as} \quad n \to \infty. \tag{12}$$

We may summarize these behaviours in the following diagram.
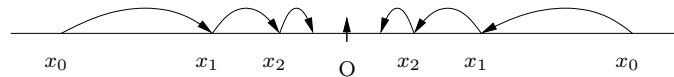
The vertical arrow at $\bar{x} = 0$ indicates that if you start at 0 you stay at 0. It is often desirable to simplify this diagram by removing all of the "bumps" and simply illustrating the net "flow" of points

toward or away from fixed points instead of particular points being mapped to particular points. In this case, the simplified diagram of the flow becomes the following:
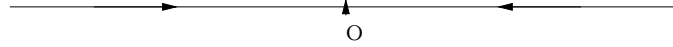
This diagram is the **phase diagram** or **phase portrait** associated with the dynamical system in Eq. (13). (The **phase space** associated with this dynamical system is the real line $\mathbb{R}$.) The portrait shows that the fixed point $\bar{x} = 0$ is an **attractive fixed point**. It remains fixed under the action of the mapping f. But if you start at any other point $x_0 \in \mathbb{R}$ then you move closer and closer to 0 under the iteration of f.

In some cases, it may be desirable to plot an actual orbit $\mathcal{O}(x_0)$ associated with a seed point x_0. The plot of iterates $\{x_n\}$ vs. n is often called a **time series**. The generic behaviour for a time series associated with this example (in the case $x_0 > 0$) is sketched below.
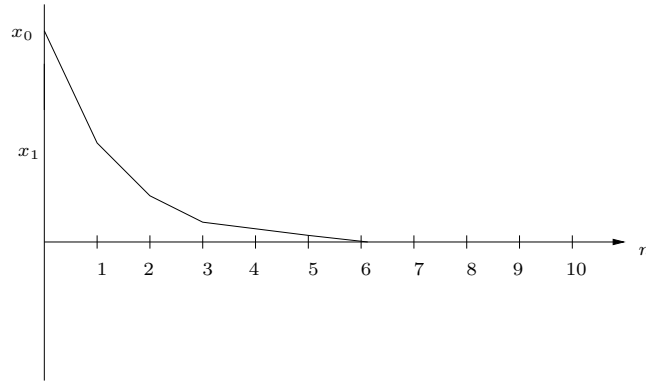
Example 2: $f(x) = 2x$ on $X = \mathbb{R}$.

In this case, the iteration scheme in Eq. (3) is

$$x_{n+1} = 2x_n \,. \tag{13}$$

If we start with an arbitrary seed point x_0, then the iteration scheme in Eq. (3) becomes

$$
\begin{aligned}
x_1 &= 2x_0 \\
x_2 &= 2x_1 = 4x_0 \\
x_3 &= 2x_1 = 8x_0 \\
&\vdots \\
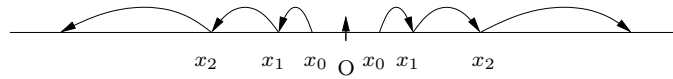x_n &= 2^n x_0 \,. \tag{14}
\end{aligned}
$$

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From this result, we may make a few conclusions:

1. If $x_0 = 0$, then all $x_n = 0$. This is a consequence of the fact that $\bar{x} = 0$ is a fixed point of $f(x) = 2x$, i.e., $f(0) = 0$.

2. If $x_0 > 0$, then $x_n \to \infty$ as $n \to \infty$.
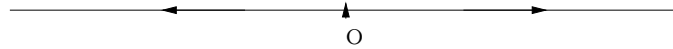
3. If $x_0 < 0$, then $x_n \to -\infty$ as $n \to \infty$.

Of course, we can combine the behaviours in 2. and 3. as follows:

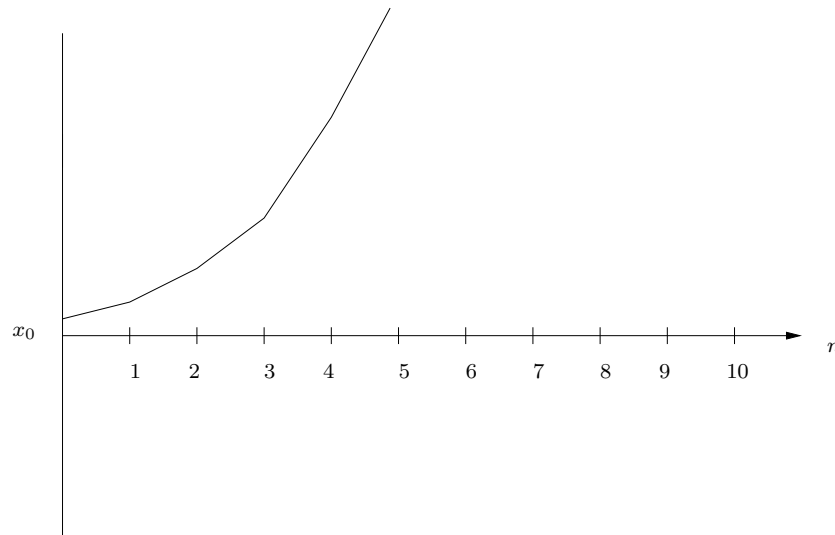$$|x_n| \to \infty \quad \text{as} \quad n \to \infty. \tag{15}$$

We may summarize these behaviours in the following diagram.

$$x_2 \qquad x_1 \quad x_0 \quad \text{O} \quad x_0 \ x_1 \qquad x_2$$

The simplified phase portrait for this system is as follows:

O

A generic time series for this example is sketched below.

In this case, the fixed point $\bar{x} = 0$ is a **repulsive fixed point**: If you start at a point x_0 close to 0 (it doesn't matter how close: $x_0 = 10^{-50}$ will work), then you move away from 0 under the iteration of f.

Example 3: $f(x) = -\frac{1}{2}x$ on $X = \mathbb{R}$.

In this case, the iteration scheme in Eq. (3) is

$$x_{n+1} = -\frac{1}{2}x_n. \tag{16}$$

If we start with an arbitrary seed point x_0, then the iteration scheme in Eq. (3) becomes

$$\begin{aligned}
x_1 &= -\frac{1}{2}x_0 \\
x_2 &= -\frac{1}{2}x_1 = \frac{1}{4}x_0 \\
x_3 &= \frac{1}{2}x_1 = -\frac{1}{8}x_0 \\
&\vdots \\
x_n &= (-1)^n \frac{1}{2^n}x_0.
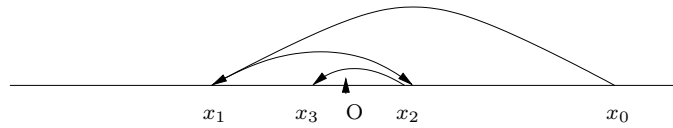\end{aligned} \tag{17}$$

From this result, we may make a few conclusions:

1. If $x_0 = 0$, then all $x_n = 0$. This is a consequence of the fact that $\bar{x} = 0$ is a fixed point of $f(x) = \frac{1}{2}x$, i.e., $f(0) = 0$.

2. For any $x_0 \neq 0$, $x_n \to 0$ but in an oscillatory manner.

We can also write that

$$|x_n| \to 0 \quad \text{as} \quad n \to \infty. \tag{18}$$

We may summarize these behaviours in the following diagram.


In this case, the fixed point $\bar{x} = 0$ is an **attractive fixed point**. If you start at any $x_0 \in \mathbb{R}$, then you are moved toward 0 under iteration of f (even if the motion is an oscillatory one – in this case a damped oscillatory motion toward 0).

The above diagram is much more complicated because of the flip-flops about the fixed point $\bar{x} = 0$. If we ignore the flip-flops and simply wish to summarize the flow in terms of where points are eventually going, then we obtain the following simplified diagram,

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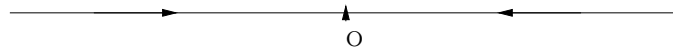
which is identical to the phase portrait of Example 1. In both cases, the fixed point $\bar{x} = 0$ is attractive. We ignore the oscillatory behaviour in this example and simply tabulate that points are mapped arbitrarily closely to the attractive fixed point.

A generic time series plot for this example is sketched below, showing the damped oscillatory nature of the iterates.
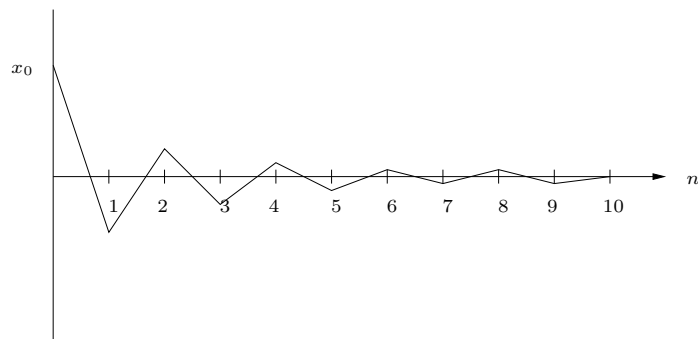
Lecture 2

The three simple examples studied in the previous lecture are particular cases of the more general function $f_a(x)$ defined as follows,

$$\boxed{f_a(x) = ax\,, \qquad a \in \mathbb{R}\,.}$$
(19)

Here, $a \in \mathbb{R}$ is a constant. Note that $f_a(x)$ is a **linear** function of x since

$$
\begin{aligned}
f_a(c_1 x_1 + c_2) &= a(c_1 x_1 + c_2 x_2) \\
&= c_1 a x_1 + c_2 a x_2 \\
&= c_1 a x_1 + c_2 a x_2 \\
&= c_1 f_a(x_1) + c_2 f_a(x_2)\,.
\end{aligned}
$$
(20)

The iteration process,

$$x_{n+1} = f_a(x_n) \quad n \geq 0\,,$$
(21)

becomes

$$x_{n+1} = a x_n\,, \quad n \geq 0\,.$$
(22)

The functions $\{f_a(x)\}$ comprise a **one-parameter family** of linear functions. The parameter a plays the role of a **multiplier**. Note that for all $a \in \mathbb{R}$, $\bar{x} = 0$ is a fixed point of $f_a(x)$, i.e.,

$$f_a(0) = a \cdot 0 = 0\,.$$
(23)

For $a \neq 1$, $\bar{x} = 0$ is the only fixed point of $f_a(x)$. In the special case $a = 1$, each point $x \in \mathbb{R}$ is a fixed point of $f_1(x)$ since

$$f_1(x) = 1 \cdot x = x\,.$$
(24)

But this is a rather special case.

From Eq. (22), it is easy to see that the iterates are given by

$$
\begin{aligned}
x_1 &= a x_0 \\
x_2 &= a x_1 = a^2 x_0 \\
x_3 &= a x_2 = a^3 x_0 \\
&\vdots
\end{aligned}
$$
(25)

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which leads to the general result,

$$x_n = a^n x_0 .$$
(26)

Once again, if $x_0 = 0$, then all $x_n = 0$, a consequence of the fact that $x = 0$ is a fixed point of $f_a(x)$. From Eq. (25) it follows that

$$|x_n| = |a|^n |x_0| ,$$
(27)

from which we can determine all of the different dynamical behaviours of the iteration scheme which are dependent on the parameter a.

- **Case 1:** $|a| < 1$. From Eq. (27), we have that $|x_n| \to 0$ as $n \to \infty$, which implies that $x_n \to 0$, independent of the starting point x_0. This is sufficient to conclude that the fixed point $\bar{x} = 0$ is **attractive** (or stable).

  We can, however, say a little more about the way in which the x_n approach 0, as we did for the examples in the previous lecture, by dividing Case 1 into two subcases:

  - **Case 1(a):** $0 \leq a < 1$. In this case $x_n \to 0$ monotonically, i.e., the elements x_n do not "flip" about 0. In the special case that $a = 0$, $x_n = 0$ for all $n \geq 1$.

  - **Case 1(b):** $-1 < a < 0$. In this case, the elements x_n "flip" about 0 as they approach it, i.e., they exhibit a damped oscillatory motion toward 0.

  A generic phase portrait is sketched below.
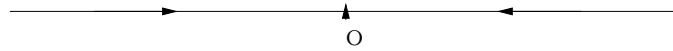
  Of course, it's going to look like the phase portrait we sketched for the mapping $f(x) = \dfrac{1}{2} x$ in the previous lecture.

- **Case 2:** $|a| > 1$. From Eq. (27), we have that $|x_n| \to \infty$ as $n \to \infty$, which implies that the fixed point $\bar{x} = 0$ is **repulsive** (or unstable). Once again, we can divide this case into two subcases:

  - **Case 2(a):** $a > 1$. In this case, the x_n do not "flip" about 0. If $x_0 > 0$, then $x_n > 0$ and $x_n \to \infty$. If $x_0 < 0$, then $x_n < 0$ and $x_n \to -\infty$.

  - **Case 2(b):** $a < -1$. In this case, the elements x_n "flip" about 0, i.e., oscillate in sign, as they grow in magnitude.

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A generic phase portrait is sketched below.
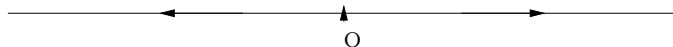
As expected, it looks like the phase portrait we sketched for the mapping $f(x) = \dfrac{2}{x}$ in the previous lecture.

There is one final case that we have not explored in previous examples:

- **Case 3:** $|a| = 1$. In this case $|x_n| = |x_0|$. This implies that the iterates x_n are at the same distance from $x = 0$ as x_0 is – they move neither closer to nor farther away from $x = 0$.

  - **Case 3(a):** $a = 1$. Here, $x_n = x_0$. You start at x_0 and stay there. This is a consequence of the fact, stated earlier, that each $x \in \mathbb{R}$ is a fixed point of the map $f_1(x) = x$.

  - **Case 3(b):** $a = -1$. In this case, $f_a(x) = f_{-1}(x) = -x$. Here, $x_0 = x_2 = x_4 = \cdots$ and $x_1 = x_3 = \cdots = -x_0$. You start at x_0, then move to $-x_0$ and then back to x_0. Each point is part of a two-cycle $(x_0, -x_0)$. (In the special case x_0, the two-cycle is a one-cycle, i.e., a periodic orbit $x_n = x_0$.)

    Let us examine this subcase a little further. Recall that if you start at x_0, you move to $-x_0$, and then back to x_0. This implies that two applications of f_{-1} keep you where you are. This is a consequence of the fact that for $f_{-1}(x) = -x$,

    $$g(x) = f_{-1}(f_{-1}(x)) = -(-x) = x\,. \tag{28}$$

    In other words, the function $g(x) = f_{-1}^{\circ 2}(x)$ is the identity map.

    The important idea here is that if you have a two-cycle (p_1, p_2) of a dynamical system, $x_{n+1} = f(x_n)$, then each element p_i is a fixed point of the composition $g(x) = f(f(x)) = f^{\circ 2}(x))$. This idea can be generalized to N-cycles, as we'll see later.

Now consider the following modification of the function $f_a(x)$ discussed above:

$$\boxed{f_{a,b}(x) = ax + b \quad a, b \in \mathbb{R}\,.} \tag{29}$$

We now have a **two-parameter family** of functions. The function $f_{a,b}(x)$ is composed of a term which is linear in x, namely, ax, and the translation b. Such functions are known as **affine functions**.

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Note, however, that in general, $f_{a,b}(x)$ is **not** a linear function of x:

$$
\begin{aligned}
f_{a,b}(c_1 x_1 + c_2) &= a(c_1 x_1 + c_2 x_2) + b \\
&= c_1 a x_1 + c_2 a x_2 + b \\
&= c_1 a x_1 + b + c_2 a x_2 + b - b \\
&= c_1 f_{a,b}(x_1) + c_2 f_{a,b}(x_2) - b .
\end{aligned}
\tag{30}
$$

This indicates that only in the case $b = 0$ is $f_{a,b}(x)$ a linear function. But this is the function $f_a(x)$ that we studied earlier.

The iteration procedure

$$
x_{n+1} = f_{a,b}(x_n) \quad n \geq 0 ,
\tag{31}
$$

becomes

$$
x_{n+1} = a x_n + b \quad n \geq 0 .
\tag{32}
$$

This process involves a multiplication by the scalar a followed by a translation by the scalar b.

If we start with an initial value, or "seed", $x_0 \in \mathbb{R}$, the the first few iterates x_n yielded by Eq. (32) are as follows,

$$
\begin{aligned}
x_1 &= a x_0 + b \\
x_2 &= a x_1 + b = a(a x_0 + b) + b \\
&= a^2 x_0 + ab + b \\
&= a^2 x_0 + b(a + 1) \\
x_3 &= a x_2 + b = a^3 x_0 + a^2 b + ab + b \\
&= a^3 x_0 + b(a^2 + a + 1).
\end{aligned}
$$

There is clearly a pattern here. The general expression for x_n seems to be

$$
\boxed{x_n = a^n x_0 + b S_n,}
\tag{33}
$$

where

$$
S_n = 1 + a + a^2 + \cdots + a^{n-1}.
\tag{34}
$$

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S_n is the partial sum of n terms of a geometric series with first term 1 and ratio a. If $a = 1$, then $S_n = n$. If $a \neq 1$, then

$$S_n = \frac{1 - a^n}{1 - a}. \tag{35}$$

Note: To be completely "rigorous," one would have to prove that the iterates given in Eq. (33) satisfy Eq. (32). This can easily be done by induction. First of all, for $n = 1$, Eq. (33) yields

$$x_1 = ax_0 + bS_1 = ax_0 + b, \tag{36}$$

so it is true. Now assume that it is true for $n = N$, i.e., S_N, where $N > 1$ and show that it is true for $n = N + 1$, i.e., S_{N+1}. There are several ways to do this. Here is one. Starting with Eq. (32),

$$
\begin{aligned}
x_{N+1} &= ax_N + b \\
&= a(a^N x_0 + bS_N) + b \quad \text{(by assumption -- true for } n = N) \\
&= a^{N+1} x_0 + b(aS_N + 1).
\end{aligned} \tag{37}
$$

But

$$
\begin{aligned}
aS_N + 1 &= a(1 + a + \cdots a^{N-1}) + 1 \\
&= a + a^2 + \cdots a^N + 1 \\
&= S_{N+1}.
\end{aligned} \tag{38}
$$

Substitution into the final line of Eq. (37) yields

$$x_{N+1} = a^{N+1} x_0 + bS_{N+1}, \tag{39}$$

which implies that Eq. (33) is true for $n = N + 1$. Therefore it is true for all $n \geq 1$. The proof is complete.

All of this being done, it is still a little cumbersome to determine the behaviour of iterates x_n using Eq. (33). As we saw in our previous example, it might help if we first looked for fixed points of the function $f_{a,b}(x) = ax + b$ and then go from there. In what follows, we shall sketch the main ideas and leave the details as an exercise.

A fixed point of $f_{a,b}(x)$ will satisfy the equation,

$$f_{a,b}(\bar{x}) = \bar{x}, \tag{40}$$

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which translates to

$$a\bar{x} + b = \bar{x} \quad \Longrightarrow \quad \bar{x} = \frac{b}{1-a}, \quad a \neq 1. \tag{41}$$

In other words, $f_{a,b}(x) = ax + b$ has a single (i.e., unique) fixed point,

$$\bar{x} = \frac{b}{1-a}, \quad a \neq 1. \tag{42}$$

Let's check this:

$$
\begin{aligned}
f_{a,b}(\bar{x}) &= a\bar{x} + b \\
&= \frac{ab}{1-a} + b \\
&= \frac{ab + b(1-a)}{1-a} \\
&= \frac{b}{1-a} \\
&= \bar{x}.
\end{aligned}
\tag{43}
$$

Bravo!

So what happens when $a = 1$? In this case, the function $f_{a,b}(x)$ is given by

$$f_{1,b}(x) = x + b. \tag{44}$$

The associated iteration procedure is given by

$$x_{n+1} = x_n + b. \tag{45}$$

In other words, if you are at position x_n, you move a distance $|b|$ (right if $b > 0$ and left if $b < 0$) to arrive at x_{n+1}. If $b \neq 0$, no point "remains still." In other words, there are no fixed points. The only way that you can stay at the same point is if $b = 0$. But in this case, the function $f_{1,0}(x)$ is

$$f_{1,0}(x) = x. \tag{46}$$

But this is the **identity map** on $\mathbb{R}$ – it leaves points unchanged. The associated iteration equation is

$$x_{n+1} = x_n. \tag{47}$$

If you start at an $x_0 \in \mathbb{R}$, you remain at x_0. In other words, **every point $x \in R$ is a fixed point of** $f_{1,0}(x) = x$!

Let's do one more thing before leaving this example. For the case $a \neq 1$, i.e., when $f_{a,b}(x)$ has a fixed point $\bar{x}$, given in Eq. (42), make the change of variable,

$$y_n = x_n - \bar{x}, \quad n \geq 0, \tag{48}$$

where the x_n are the iterates associated with Eq. (32) for a given seed point x_0. For $n \geq 0$, y_n represents the relative position of the iterate x_n with respect to the fixed point $\bar{x}$. Now rewrite this equation as

$$x_n = y_n + \bar{x}, \tag{49}$$

and substitute it into both sides of Eq. (32),

$$y_{n+1} + \bar{x} = a(y_n + \bar{x}) + b, \tag{50}$$

or

$$y_{n+1} + \bar{x} = ay_n + a\bar{x} + b, \tag{51}$$

But recall that $\bar{x}$ is the fixed point of $f_{a,b}(x)$, i.e., $\bar{x} = a\bar{x} + b$. Therefore, Eq. (51) becomes

$$\boxed{y_{n+1} = ay_n \quad n \geq 0.} \tag{52}$$

But we've already analyzed this iteration procedure in the previous example – just using a different letter for the iterates. We'll summarize the results as follows,

1. **Case 1:** $|a| < 1$. The iterates $y_n \to 0$.

2. **Case 2:** $|a| > 1$. For $y_0 \neq 0$, $|y_n| \to \infty$.

3. **Case 3:** $|a| = 1$. Then $|y_n| = |y_0|$.

Let's examine Case 1 in a little more detail. Recalling that $y_n = x_n - \bar{x}$, the fact that the iterates $y_n \to 0$ implies that

$$y_n = x_n - \bar{x} \to 0 \quad \implies \quad x_n \to \bar{x}, \quad \text{as} \quad n \to \infty. \tag{53}$$

In other words, all iterates x_n from our original iteration procedure approach the fixed point $\bar{x}$ of $f_{a,b}(x)$ as $n \to \infty$. This implies that the fixed point $\bar{x}$ is **attractive** for $|a| < 1$. A generic phase portrait for this behaviour is shown below.
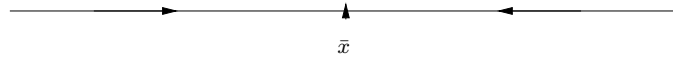
We see that it is simply the phase portrait for the mapping $f_a(x) = ax$ with $|a| < 1$ **shifted** from the point $x = 0$, the fixed point of $f_a(x)$, to $x = \bar{x}$, the fixed point of $f_{a,b}(x) = ax + b$. (This makes sense – recall that the y_n are the relative positions of the x_n with respect to the fixed point $\bar{x}$.)

It won't be surprising that for Case 2, i.e., $|a| > 1$, the fixed point $\bar{x}$ is **repulsive**. A generic phase portrait is shown below.
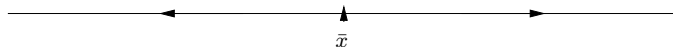
Later, we'll see that a graphical analysis of this function will help to make the above discussion clearer.

Let us now consider a simple nonlinear example.

Example: $f(x) = x^2$. The associated iteration scheme is

$$x_{n+1} = x_n^2 . \tag{54}$$

Once again, we begin by looking for fixed points. In this case, one can see, by inspection, that $f(x) = x^2$ has two fixed points, $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$, i.e., $f(0) = 0$ and $f(1) = 1$. But let's just do things a little more formally to make sure that we have found all the fixed points.

$$f(x) = x \quad \Longrightarrow \quad x^2 = x \quad \Longrightarrow \quad x^2 - x = 0 \quad \Longrightarrow \quad x(x-1) = 0 . \tag{55}$$

The solutions of the final equation are, of course, 0 and 1, so we have found all the fixed points of $f(x) = x^2$.

We can now try to express the iterates x_n in terms of x_0 as we did for the linear maps studied above. But the situation is a little more complicated. Given a seed point $x_0 \in \mathbb{R}$, the first few iterates are as follows,

$$
\begin{aligned}
x_1 &= x_0^2 \\
x_2 &= x_1^2 = (x_0^2)^2 = x_0^4 \\
x_3 &= x_2^2 = (x_0^4)^2 = x_0^8 \\
x_4 &= x_3^2 = (x_0^8)^2 = x_0^{16} .
\end{aligned}
\tag{56}
$$

You should be able to see the pattern, i.e.,

$$x_n = (x_0)^{2^n} . \tag{57}$$

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In other words, x_n is obtained by raising x_0 to the 2^nth power. Now take absolute values of both sides to obtain the following,

$$|x_n| = |x_0|^{2^n} . \tag{58}$$

From this we can immediate conclude the following:

1. If $|x_0| = 1$, then $|x_n| = 1$. We won't go into any more detail right now.

2. If $|x_0| < 1$, then $|x_n| \to 0$ as $n \to \infty$.

3. If $|x_0| > 1$, then $|x_n| \to \infty$ as $n \to \infty$.

This suggests that there are two separate regions with different dynamical behaviour, i.e., behaviour of the iterates x_n:

- Region 1: $R_1 = \{x \in \mathbb{R} \mid |x| < 1\}$. If $x_0 \in R_1$, then $x_n \to 0$.

- Region 2: $R_2 = \{x \in \mathbb{R} \mid |x| > 1\}$. If $x_0 \in R_2$, then $|x_n| \to \infty$.

These two regions are separated by the points $x = 1$ and $x = -1$. On the basis of this information, we could start to sketch a possible phase portrait. But if we "play around" with some more possible initial conditions, we can get some additional information about the dynamics of this iteration procedure:

1. If we choose $x_0 > 1$, then $x_1 = x_0^2 > x_0$, $x_2 = x_1^2 > x_1$, etc., and we can conclude that $x_n \to \infty$ as $n \to \infty$.

2. If we choose x_0 between 0 and 1, i.e., $0 < x_0 < 1$, then $x_1 = x_0^2 < x_0$, etc., so that $x_n \to 0^+$ as $n \to \infty$.

3. If $x_0 = -1$, then $x_1 = x_0^2 = 1 = \bar{x}_2$ and $x_n = 1$ for all $n \geq 1$. Since x_0 is mapped to the periodic (fixed) point $\bar{x}_2$, it is known as a **preperiodic** or **eventually periodic/fixed point**.

4. If $-1 < x_0 < 1$, then $x_1 = x_0^2 > 0$ and lies in the interval $(0, 1)$. From 2. above, it follows that all higher iterates x_n, $n \geq 1$ lie in the interval $(0, 1)$ and $x_n \to 0^+$.

5. If $x_0 < -1$, then $x_1 = x_0^2 > 1$. From 1. above, it follows that $x_n \to \infty$.

From these observations, the dynamics of this iteration scheme may be summarized as follows:
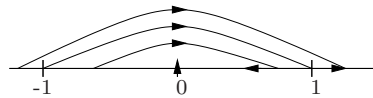
Clearly, the existence of more than one fixed point, made possible by the nonlinearity of the function $f(x)$, makes the situation more complicated, and therefore interesting. We'll see very shortly how a graphical analysis of the iteration procedure will allow us to see things much more quickly.

Example: $f(x) = \cos x$, which defines the discrete dynamical system,

$$x_{n+1} = \cos(x_n), \quad n = 0, 1, \cdots. \tag{59}$$

Here, we simply present the results of a numerical experiment. (Some ways to perform such numerical experiments will be presented at the end of this Lecture.)

If we choose $x_0 = 1$ and perform the iteration in Eq. (59), we obtain the following results (in class, only a few iterates were presented, and to only six-digit accuracy). To ten digits:

n	x_n
0	1.0000000000
1	0.5403023059
2	0.8575532158
3	0.6542897905
4	0.7934803587
5	0.7013687736
10	0.7442373549
20	0.7391843998
30	0.7390870427
40	0.7390851699
50	0.7390851339
60	0.7390851332
70	0.7390851332

For $n > 60$, there is no change in the first 10 decimal digits – all changes are occurring at higher-order decimal positions. It appears that the iterates are converging to a point $\bar{x} \cong 0.7390851332$ which could be the fixed point of the cosine function, i.e.,

$$f(\bar{x}) = \cos \bar{x} = \bar{x}. \tag{60}$$

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Here we simply state that if we start with any $x_0 \in \mathbb{R}$ (yes, even 10^{40}!), the iterates x_n converge to this fixed point. A little later, we'll see why this happens from an analysis of the graph of $f(x)$.

Appendix: Some methods of performing numerical experiments in this course

This section is directed to those students who have little or no experience with computational methods, if only to demonstrate the fundamental idea of "loops" as well as how to obtain a plot of some numerical results. Those who have some kind of scientific computational background can ignore this section. Here, we shall simply present a few very simple programs in different languages that you can use (and modify) to produce some numerical/graphical results.

A simple program in FORTRAN

In general style, FORTRAN is quite similar to a number of other programming languages such as BASIC and even MATLAB. The following program generated the iterates of the dynamical system,

$$x_{n+1} = \cos\left(x_n\right), \tag{61}$$

presented earlier in this lecture. (Only some of the iterates were included in the list presented earlier.)

```
c  program iter.f
c  simple program to perform iteration
c  x(n+1)=f(x(n))
c  for pmath 370 course
      z0=0.0d0
      z1=1.0d0
      x=1.0
      write(6,6000) 0,x
c  this is the basic loop (here, it computes iterates from x(1) to x(20))
      do 10 i=1,20
      xp = dcos(x)
      x=xp
      write(6,6000) i,x
```

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```
  10 continue
6000 format(i5,f15.10)
     stop
     end
```

The output of this program is as follows:

```
 0    1.0000000000
 1    0.5403023059
 2    0.8575532158
 3    0.6542897905
 4    0.7934803587
 5    0.7013687736
 6    0.7639596829
 7    0.7221024250
 8    0.7504177618
 9    0.7314040424
10    0.7442373549
11    0.7356047404
12    0.7414250866
13    0.7375068905
14    0.7401473356
15    0.7383692041
16    0.7395672022
17    0.7387603199
18    0.7393038924
19    0.7389377567
20    0.7391843998
```

Some simple programs in MATLAB

Here is a simple MATLAB program (stored in the file "iter1.m") that also generates iterates of the dynamical system

$$x_{n+1} = \cos(x_n). \tag{62}$$

```
%%  iter1.m
%%  first in a series of simple matlab programs to examine
%%  iteration sequences  x(n+1)=f(x(n))
%%  here, f(x)=cos(x)
    close all;
    clear all;
%% display numbers to 15 decimal digits
%% (otherwise, they are displayed only to 4 digits
    format long
    x(1)=1.0;
    niter=20;
    for i=1:niter,
    x(i+1)=cos(x(i));
    i+1,x(i+1)
    end;
%% the built-in MATLAB routine "plot" will plot the iterates as a time series
    plot(x);
```

If you now type "iter1" (assuming that MATLAB is set to your local directory), the following output will be shown on your screen (some blank lines have been removed in order to save space):

```
>> iter1

i =

     2
ans =

  0.540302305868140
i =
```

```
        3
ans =

   0.857553215846393
i =

        4
ans =

   0.654289790497779
i =

        5
ans =

   0.793480358742566
i =

        6
ans =

   0.701368773622757
i =

        7
ans =

   0.763959682900654
i =

        8
ans =

   0.722102425026708
i =

        9
ans =

   0.750417761763761
i =

       10
ans =

   0.731404042422510
```

```
i =

    11

ans =

  0.744237354900557

i =

    12

ans =

  0.735604740436347

i =

    13

ans =

  0.741425086610109

i =

    14

ans =

  0.737506890513243

i =

    15

ans =

  0.740147335567876

i =

    16

ans =

  0.738369204122323

i =

    17

ans =

  0.739567202212256

i =

    18

ans =
```

```
    0.738760319874211

i =

    19

ans =

    0.739303892396906

i =

    20

ans =

    0.738937756715344

i =

    21

ans =

    0.739184399771494

>>
```

The above results agree with those obtained from the FORTRAN program presented earlier. The following plot, showing the "damped oscillatory" convergence of the iterates to the fixed point $\bar{x} \cong 0.739184$ is also produced:
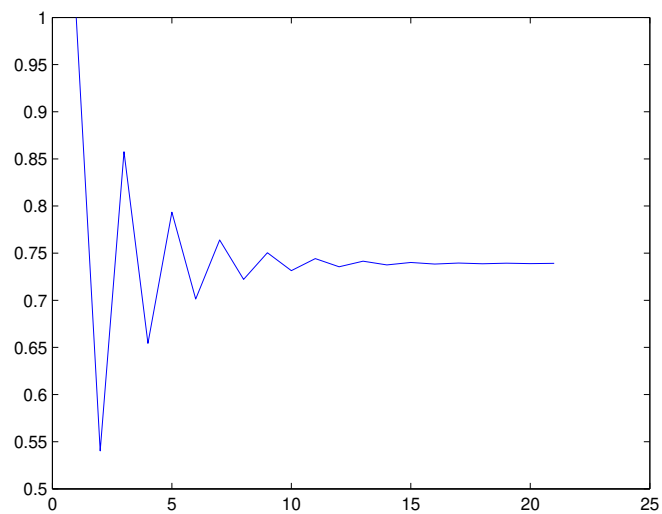
Some simple programs in MAPLE

Here is a simple MAPLE program that generates iterates of the dynamical system

$$x_{n+1} = \cos\left(x_n\right).\tag{63}$$

```
with(plots);
x:=array(1..110);
x[1]:=1.0;
for i from 1 to 19 do
x[i+1]:=cos(x[i]);
od;
listplot(x,view=[1..20,0..1]);
```

The following results are displayed on the screen:

$$x_1 \quad := \quad 1.0$$

$$x_2 \quad := \quad 0.5403023059$$

$$x_3 \quad := \quad 0.8575532158$$

$$x_4 \quad := \quad 0.6542897905$$

$$x_5 \quad := \quad 0.7934803587$$

$$x_6 \quad := \quad 0.7013687737$$

$$x_7 \quad := \quad 0.7639596829$$

$$x_8 \quad := \quad 0.7221024250$$

$$x_9 \quad := \quad 0.7504177618$$

$$x_{10} \quad := \quad 0.7314040424$$

$$x_{11} \quad := \quad 0.7442373549$$

$$x_{12} \quad := \quad 0.7356047404$$

$$x_{13} \quad := \quad 0.7414250866$$

$$x_{14} \quad := \quad 0.7375068905$$

$$x_{15} \quad := \quad 0.7401473356$$

$$x_{16} \quad := \quad 0.7383692041$$

$$x_{17} \quad := \quad 0.7395672022$$

$$x_{18} \quad := \quad 0.7387603199$$

$$x_{19} \quad := \quad 0.7393038924$$

$$x_{20} \quad := \quad 0.7389377567$$

These are in agreement with the previous computations. The following plot is also produced:

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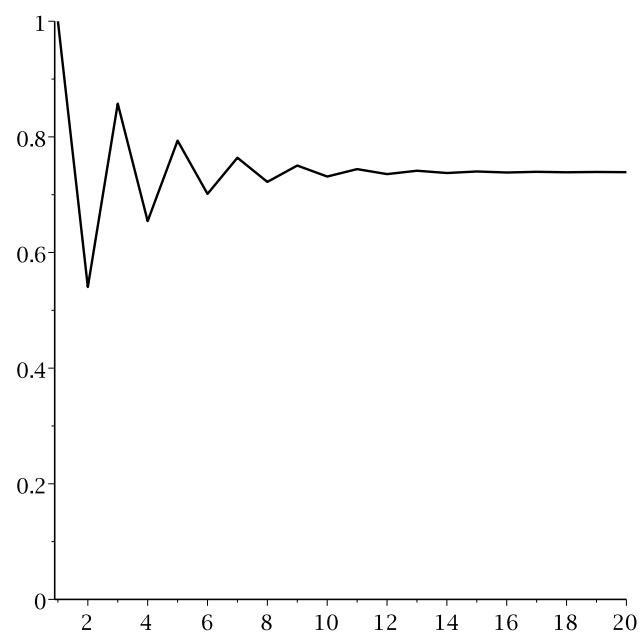
Lecture 3

## Graphical methods of analyzing the iteration scheme $x_{n+1} = f(x_n)$

We wish to represent the iteration process defined by

$$x_{n+1} = f(x_n), \tag{64}$$

where $f : \mathbb{R} \to \mathbb{R}$, graphically. We begin with a "seed" $x_0 \in \mathbb{R}$ and insert it into the "f machine". The output $x_1 = f(x_0)$ is then reinserted into the "f machine" to produce $x_2 = f(x_1)$, etc. How do we do this graphically? First start with a plot of the graph of $f(x)$ along with the line $y = x$, as shown in the figure on the next page. Now pick a starting point x_0 on the x-axis. Getting $x_1 = f(x_0)$ is easy: You simply find the point $(x_0, f(x_0))$ that has on the graph of $f(x)$ above (or below) the point $x = x_0$. In other words, travel upward (or downward) from $(x_0, 0)$ to $(x_0, f(x_0))$. We now have $x_1 = f(x_0)$. How do we input x_1 into the "f machine" to find $x_2 = f(x_1)$? First, we have to find where x_1 lies on the x-axis. We do this by travelling from the point $(x_0, f(x_0))$ horizontally to the line $y = x$: the point of intersection will be $(f(x_0), f(x_0)) = (x_1, x_1)$. We are now sitting directly above (or below) the point $(x_1, 0)$ on the x-axis, which is patiently waiting to be input into $f(x)$ to produce $x_2 = f(x_1)$. We can travel from (x_1, x_1) to $x_1, 0)$ and then back up (or down) to $(x_1, f(x_1)) = (x_1, x_2)$. From here, we once again travel to the line $y = x$, intersecting it at (x_2, x_2). From here, we travel to the curve $y = f(x)$ to intersect it at (x_2, x_3), etc.. The procedure is illustrated below.
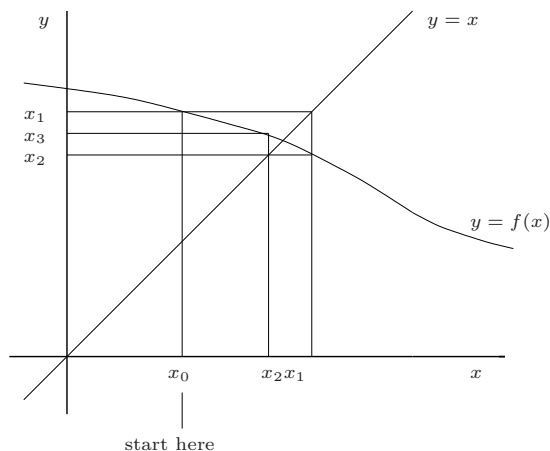
When you have performed this procedure a few times, you will see that including all the lines from intersection points (x_i, x_i) on the line $y = x$ and intersection points (x_i, x_{i+1}) on $y = f(x)$ to the x- and y-axis is unnecessary. In fact, these lines clutter up the figure. We have removed them to produce the figure below, in which the iteration process is presented in a much clearer way.
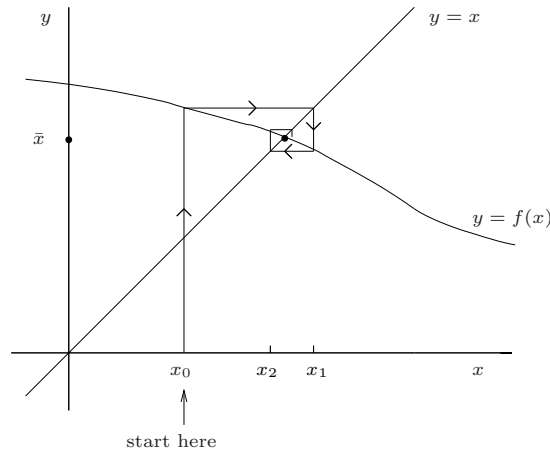
For rather obvious reasons, figures such as this one are called "cobweb diagrams". It appears as if the iterates x_0, x_1, x_2, x_3 are jumping back and forth, yet "zeroing in" on the point $(\bar{x}, \bar{x})$ at which the graph of f intersects the line $y = x$. Such a point of intersection must be a fixed point of f since it implies that $f(\bar{x}) = \bar{x}$. Of course, not all fixed points are attractive as the one in this diagram: The graphical procedure outlined above will give us some insight into what makes a fixed point attractive or repulsive.

Let us repeat this important concept: If the line $y = x$ intersects the graph $y = f(x)$ of a function $f(x)$ at $x = \bar{x}$, then $\bar{x}$ is a fixed point of $f(x)$. The reason: If the intersection takes place at $x = \bar{x}$, then

$$y = \bar{x} \quad \text{and} \quad y = f(\bar{x}), \tag{65}$$

which implies that

$$\bar{x} = f(\bar{x}). \tag{66}$$
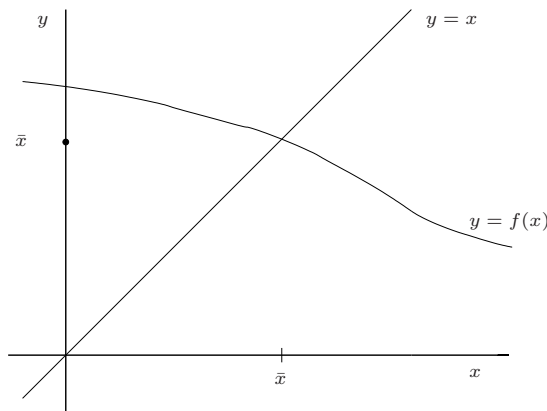
We now examine some simple dynamical systems using this graphical approach, starting with the

linear systems studied in Lectures 1 and 2.

In Examples 1-3 below, we consider the dynamical system,
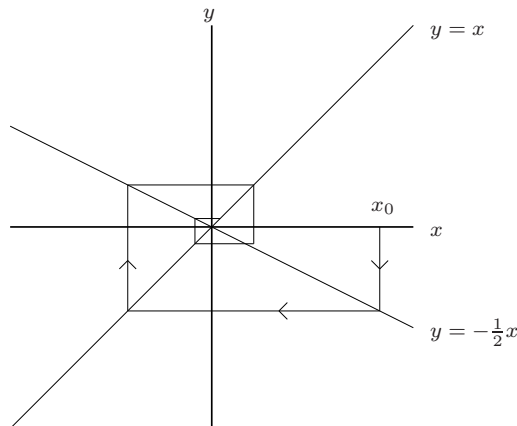
$$x_{n+1} = ax_n, \quad n \geq 0. \tag{67}$$

We know that the solution to this difference equation is

$$x_n = a^n x_0. \tag{68}$$
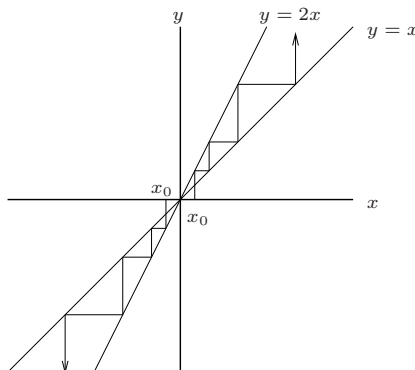
Example 1: $x_{n+1} = ax_n$, with $0 < a < 1$. For (i) $x_0 > 1$ or (ii) $x_0 < 1$, the graphical method shows the monotonic approach of the iterates $x_n = a^n x_0$ toward the fixed point $x = 0$, in agreement with the results obtained earlier.
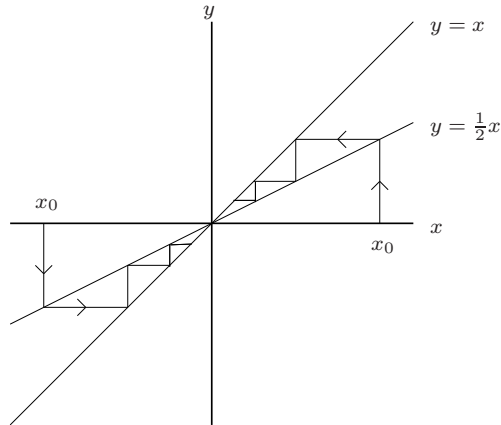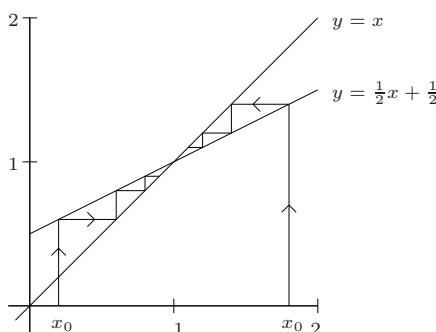
Example 2: $x_{n+1} = ax_n$, with $-1 < a < 0$. The iterates x_n oscillate about $x = 0$, $x_n = (-1)^n |a|^n x_0$, with $x_n \to 0$ as $n \to \infty$:

Example 3: $x_{n+1} = ax_n$, $a > 1$. The iterates $x_n = a^n x_0$ travel away from the fixed point 0:

Example 4: $x_{n+1} = ax_n + b$, with $0 < a < 1$, $b > 0$.

In this example, the two straight lines, $y = x$ and $y = ax + b$, can intersect at only one point, $x = \bar{x}$¡ the fixed point of $f(x) = ax + b$ which, as we found in the previous lecture, is $\bar{x} = \dfrac{b}{1-a}$. Since $|a| < 1$, this fixed point is **attractive**, as proved in the previous lecture: For any $x_0 \in \mathbb{R}$, $x_n \to \bar{x}$ as $n \to \infty$. Note that this picture looks like that of Example 1, with the fixed point translated from $x = 0$ to $x = \dfrac{b}{1-a}$. Recall that we discussed this idea in the previous lecture when we examined the "change of variable,"

$$y_n = x_n - \bar{x}, \quad n \geq 0. \tag{69}$$

The reader is encouraged to examine graphically the iteration of $f(x) = ax + b$ for the cases i) $-1 < a < 0$, ii) $a = -1$ and iii) $a < -1$.

We now examine a few simple nonlinear iteration processes using the graphical method described above. The graphs of nonlinear functions $f(x)$ are not necessarily straight lines – in other words, they can "bend". As such $f(x)$ can have more than one fixed point. Each fixed point $\bar{x}_i$ can behave

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differently, i.e. one repulsive, the other attractive, so that the dynamics of the iteration process $x_{n+1} = f(x_n)$ can be quite complicated. We illustrate with a few examples.

Example 5: $f(x) = x^2$, i.e. $x_{n+1} = x_n^2$, on $\mathbb{R}$. This example was also examined in the previous lecture.
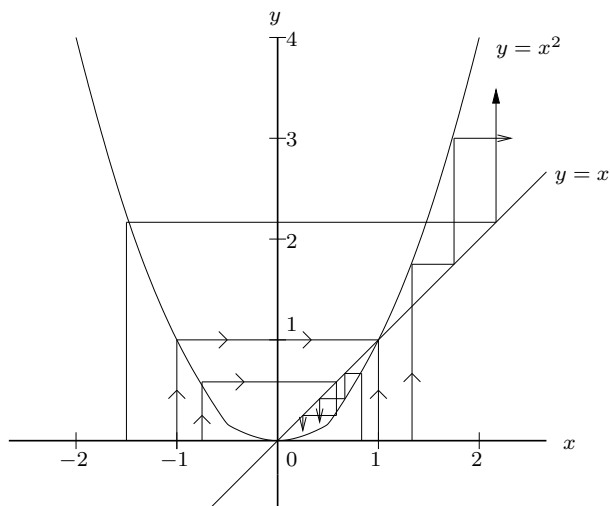
Clearly, the fixed points of $f(x)$ are $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. Thus, if $x_0 = 0$, then $x_n = 0$. If $x_0 = 1$, then $x_n = 1$. Let's now see what else is revealed by the graphical method:

1. If we begin with a point $x_0 \in (0, 1)$, then $x_1 = x_0^2 < x_0$. Likewise $x_0 > x_1 > x_2, \ldots, x_n > x_{n+1}, \ldots$ and $x_n \to 0$ as $n \to \infty$. (In other words, if we keep squaring a number starting in (0,1), we approach zero.)

2. If $x_0 \in (-1, 0)$, then $x_1 = x_0^2 \in (0, 1)$ and the above process is repeated, i.e. $x_n \to 0$ as $n \to \infty$.

3. If $x_0 = 1$, then $x_n = 1$. for $n \geq 1$.

4. If $x_0 > 1$, then $x_1 = x_0^2 > x_0$ and $x_{n+1} > x_n$, so that $x_n \to +\infty$ as $n \to \infty$.

5. If $x_0 < -1$, then $x_1 = x_0^2 > 1$ and $x_n \to +\infty$ as $n \to \infty$.

We can summarize all of the above results in the following phase portrait:

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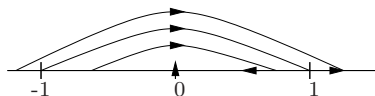
This is the same phase portrait that we obtained in the previous lecture without the graphical method.

A few other observations:

1. For all points $x_0 \in (-1, 1)$, $x_n \to 0$. We say that $\bar{x}_1 = 0$ is an **attractive** fixed point and that the interval $I = (-1, 1)$ is its **basin of attraction**.

   More generally, if $\bar{x}$ is an attractive fixed point of a map $f(x)$, we denote and define its **basin of attraction** as follows,

   $$W(\bar{x}) = \{x \in \mathbb{R} \mid f^n(x) \to \bar{x} \quad \text{as} \quad n \to \infty\}. \tag{70}$$

2. The fixed point $\bar{x}_2 = 1$ is **repulsive** since all points in a neighbourhood of $\bar{x}_2$, $J = (1 - \epsilon, 1 + \epsilon)$ leave J after a finite number of iterations. (Those to the left of 1 go to 0; those to the right of 1 go to $+\infty$.)

3. The point $x = -1$ is mapped to the fixed point $\bar{x}_2 = 1$, and is called a **preperiodic point** since it remains at $\bar{x}_2 = 1$ in future iterations.

4. For all other initial conditions, $x_0 \in (-\infty, -1) \cup (1, \infty) = \mathbb{R} - [0, 1]$, the iterates $x_n \to +\infty$ as $n \to \infty$.

5. Let's now return to the case $x_0 \in (0, 1)$. We saw, both graphically as well as numerically, that $x_0 > x_1 > x_2 > \cdots, x_n > x_{n+1} \cdots$. In other words, the sequence $\{x_n\}_{n=0}^{\infty}$ is a **decreasing sequence**. (In fact, it is a **strictly decreasing sequence** – we'll review sequences shortly.) In more complicated situations, it might not be so clear – graphically or numerically – to be able to conclude that the sequence is behaving in this way. This is where a little theoretical analysis can help. Here we shall simply state that the example we have just examined is a particular case of the following general result:

   Let $f : [a, b] \to [a, b]$ and suppose that $f(x) < x$ for all $x \in (a, b)$. (In other words, **the graph of $f(x)$ lies below the line** $y = x$.) Then for any point $x \in (a, b)$, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is a **decreasing sequence**, i.e., $f^{n+1}(x) < f^n(x)$.

   The proof of this result is very short and simple, and will be left as an exercise.

Hint: Let $x \in (a, b)$. Then, by hypothesis $f(x) < x$. Also by hypothesis, $y = f(x) \in (a, b)$. Therefore $f(y) < y$, etc..

6. Well, let's not stop there. Let's return to the case $x_0 > 1$. We saw that $x_0 < x_1 < \cdots$, i.e., $x_{n+1} > x_n$. This is a particular case of the following general result:

Let $f : [a, b] \to [a, b]$ and suppose that $f(x) > x$ for all $x \in (a, b)$. (In other words, **the graph of $f(x)$ lies above the line $y = x$**.) Then for any point $x \in (a, b)$, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is an **increasing sequence**, i.e., $f^{n+1}(x) < f^n(x)$.

The proof of this result is quite similar in form to that of the earlier result.

Example 6: $f(x) = x^2 + 1$, i.e. $x_{n+1} = x_n^2 + 1$, on $\mathbb{R}$.
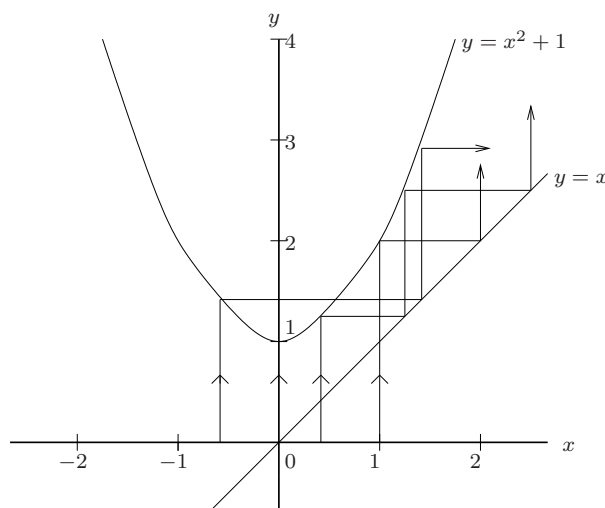
Note that $f(x)$ has no (real) fixed points. For $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$, $x_n \to +\infty$. For $x_0 > 0$, $x_1 < x_2 < x_3 \ldots, x_n \to +\infty$. For $x_0 < 0$, $x_1 > 0$, so that $x_n \to +\infty$. In summary, for all $x_0 \in \mathbb{R}$, $x_n \to +\infty$ as $n \to \infty$.

Note that if the above parabola is translated downward by a sufficient amount, the graph of f will intersect with the line $y = x$ to produce a fixed point. As the graph is translated downward even further, this fixed point evolves into two fixed points. The dynamics associated with this quadratic family $f_c(x) = x^2 + c$ can become quite complicated and will be the subject of later sections.

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Example 7: $f(x) = \sqrt{x}$, $x \geq 0$, i.e. $x_{n+1} = \sqrt{x_n}$ on $\mathbb{R}_+ = [0, \infty)$.
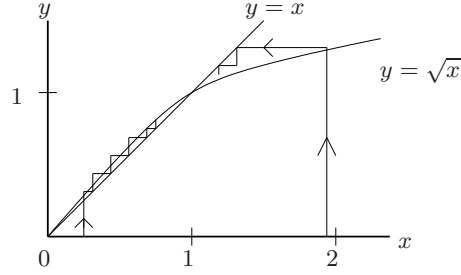
There are two fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. For $x_0 \in (0, 1)$, $x_1 = \sqrt{x_0} > x_0$, $x_2 > 1, \ldots$ so that $x_n \to 1$ as $n \to \infty$. For $x_0 > 1$, $x_1 = \sqrt{x_0} < x_0, \ldots x_{n+1} = \sqrt{x_n} < x_n$, so that $x_n \to 1$ as $n \to \infty$. Thus, $\bar{x}_2 = 1$ is an attractive fixed point and its basin of attraction if $I = (0, \infty)$. (In other words, if you start with a positive number x_0 and keep taking square roots, you approach 1 in the limit.)

We end this section with a more interesting example, as a motivation for later studies.

Example 8: $f(x) = \frac{16}{5}x(1 - x)$, $x \in [0, 1]$
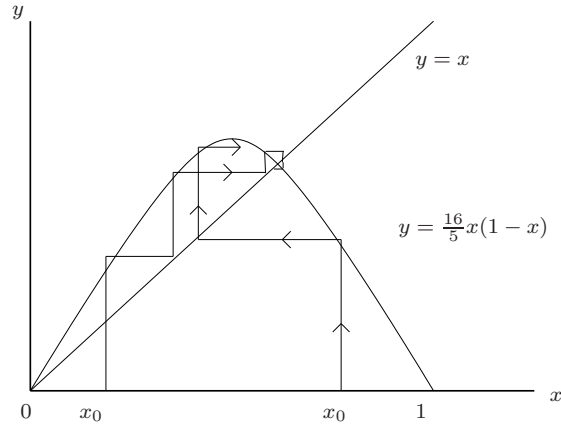
f has two fixed points: $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{11}{16}$ (Exercise). Clearly, if $x_0 = 0$, then $x_n = 0$. If x_0 is near 0, then the first few iterates x_1, x_2 are observed to travel away from 0 toward fixed point $\bar{x}_2$. The points $(x_n, f(x_n))$ travel "up the hump" until they reach roughly the height $\bar{x}_2$, whereupon they are directed toward the right side of the "hump". Graphical methods are insufficient to account for the long term behaviour of the x_n. If we magnify the graph of $f(x)$ near its fixed point $(\bar{x}_2, \bar{x}_2) = \left(\frac{11}{16}, \frac{11}{16}\right)$ and examine the "cobweb nature" of the iteration procedure $x_{n+1} = f(x_n)$, we find that a point x_0 near $\bar{x}_2 = \frac{11}{16}$ is sent by the function f to a point $x_1 = f(x_0)$ on the other side of $\bar{x}_2 = \frac{11}{16}$ and farther away from it, i.e. $|x_1 - \bar{x}_2| > |x_0 - \bar{x}_2|$. In other words, $\bar{x}_2 = \frac{11}{16}$ is a repulsive fixed point: If we choose

$x_0 = \bar{x}_2$, then $x_n = \bar{x}_2$, i.e. we remain at $\bar{x}_2$. However, if we choose an x_0 "ϵ-close" to $\bar{x}_2$, with $\epsilon > 0$, the iterates x_n travel away from $\bar{x}_2$, no matter how small an ϵ is chosen.

So what happens to the iterates x_n? Numerically, we find that they approach the two-cycle $(p_1, p_2) \cong (0.799, 0.513)$ as $n \to \infty$.

This is not the end of the story for this example. Even though the fixed point $\bar{x}_2$ is unstable, there is still an infinity of points that get mapped to $\bar{x}_2$ after a finite number of iterations and then stay at $\bar{x}_2$. To determine these points, we simply iterate "backwards", i.e. find graphically the point y_1 such that $f(y_1) = \bar{x}_2$, then find graphically the point y_2 such that $f(y_2) = y_1$ so that $f^2(y_2) = \bar{x}_2$, etc. In this way, we find an infinite set of **preperiodic** points y_n, $n = 1, 2, 3, \ldots$ such that $f^n(y_n) = \bar{x}_2$. Note that $y_n \to 0$ as $n \to \infty$.

In other words, these points can be found at arbitrarily small distances from the (repulsive) fixed point $\bar{x}_1 = 0$.

But that's not all! There's an infinity of points on the other side of the hump near $x = 1$ that also get mapped to $\bar{x}_2$ after a finite number of iterations. We can find them by going "right" instead of left in our "backwards" iteration procedure pictured above. In the figure below, we have found graphically the point z_1 near $n = 1$ such that $f(z_1) = y_1$ so that $f^2(z_1) = \bar{x}_2$ as well as the point z_2 such that $f(z_2) = y_2$ so that $f^3(z_2) = \bar{x}_2$. In this way, we find another infinite set of **preperiodic** points z_n, $n = 1, 2, 3, \ldots$ such that $f^{n+1}(z_n) = \bar{x}_2$. Note also that these points $z_n \to 1$ as $n \to \infty$.
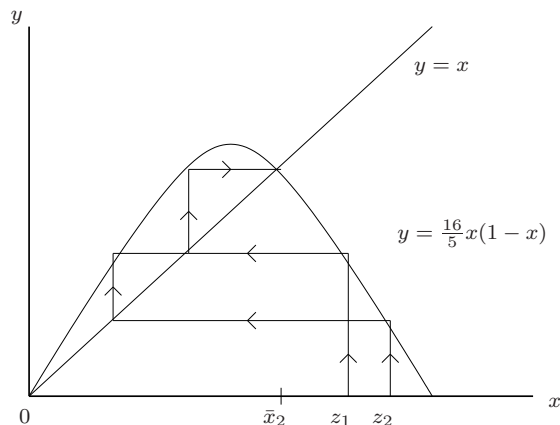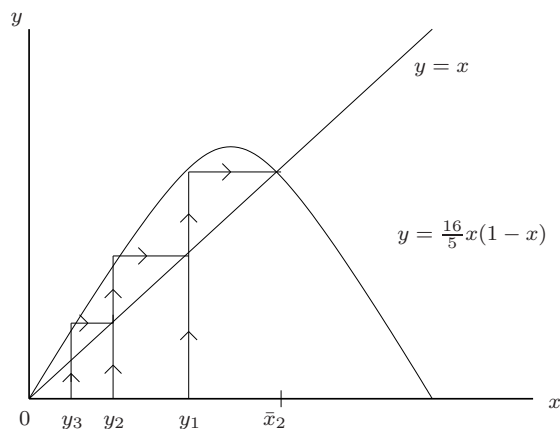
The reader is invited to devise an algorithm to run this iteration procedure "backwards" to numerically generate the sequences $\{y_n\}$ and $\{z_n\}$.

Determining the nature of fixed points of nonlinear mappings

We have examined the nature of fixed points for affine (degree one polynomial) mappings of the form,

$$f(x) = ax + b, \tag{71}$$

on several occasions. For $a \neq 1$, $\bar{x} = \dfrac{b}{1-a}$ is the unique fixed point of f, i.e. $f(\bar{x}) = \bar{x}$. If $|a| < 1$, $\bar{x}$ is attractive; if $|a| > 1$, $\bar{x}$ is repulsive. When $a = -1$, $\bar{x}$ is neither attractive nor repulsive and is referred to as **neutral** or **indifferent**. The attractivity of repulsivity of $\bar{x}$ as determined by the multiplier "a" can be viewed geometrically, as we show below for the case $a > 0$:
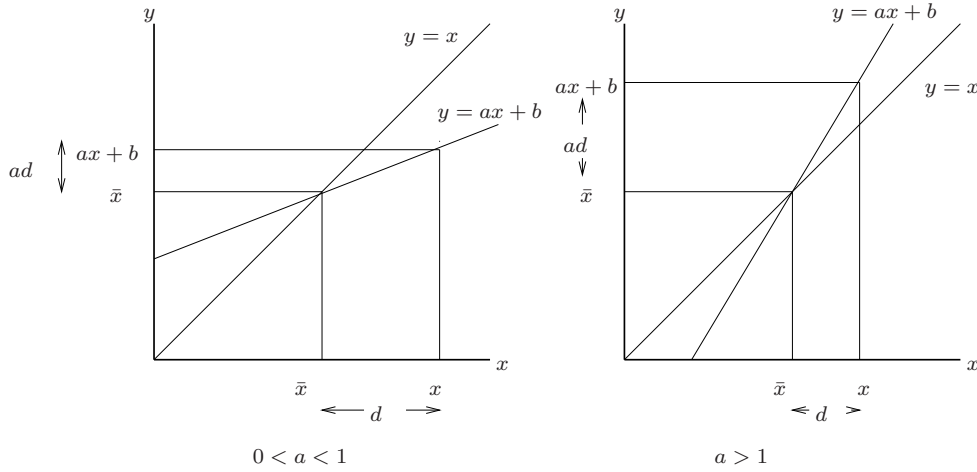
In each of the above cases, pick an $x \neq \bar{x}$ and let d denote the distance between $\bar{x}$ and x, i.e. $d = |x - \bar{x}|$. Then the distance between $f(x) = ax + b$ and $f(\bar{x}) = \bar{x} = a\bar{x} + b$ is

$$
\begin{aligned}
|f(x) - \bar{x}| &= |ax + b - a\bar{x} - b| \\
&= |a|\,|x - \bar{x}| \\
&= |a|d. \tag{72}
\end{aligned}
$$

If $|a| < 1$, then $f(x)$ is closer to $\bar{x}$ than x is. If $|a| > 1$, then $f(x)$ is farther from $\bar{x}$ than x is. This is a simple consequence of the slope a of the line $y = ax + b$. If we now repeat the procedure, then

$$
\begin{aligned}
|f^2(x) - \bar{x}| &= |a|\,|f(x) - \bar{x}| \\
&= |a|^2\,|x - \bar{x}| \tag{73}
\end{aligned}
$$

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and, in general

$$|f^n(x) - \bar{x}| = |a|^n |x - \bar{x}|. \tag{74}$$

If $|a| < 1$, then $|f^n(x) - \bar{x}| \to 0$ as $n \to \infty$, implying that the points $x_n = f^n(x)$ converge to $\bar{x}$ as $n \to \infty$. If $|a| > 1$, then $|f^n(x) - \bar{x}| \to \infty$ as $n \to \infty$. The equalities in (72)-(74) are a consequence of the fact that the graph of $f(x) = ax + b$ is a **straight line** with constant slope a for all $x \in \mathbb{R}$.

Appendix: A brief review of sequences and their properties

As seen in the example of the previous section, we shall be interested in the qualitative behaviour of sequences $\{x_k\}$ that are obtained by the iteration of a function f. Of primary concern is whether such sequences converge to a limit, oscillate or diverge to infinity.

For this purpose, it is important to recall some fundamental concepts involving sequences. Below are listed the major properties of sequences. We highly recommend that you consult a calculus textbook for more details.

Given an infinite sequence of real numbers $\{a_n\}_{n=0}^{\infty}$, the sequence is said to be:

1) **increasing** if $a_n \leq a_{n+1}$ for all n, **strictly increasing** if $a_n < a_{n+1}$, for all n,

2) **decreasing** if $a_n \geq a_{n+1}$ for all n, **strictly decreasing** if $a_n > a_{n+1}$ for all n,

3) **monotone** if either (strictly) increasing or (strictly) decreasing,

4) **alternating** if a_n and a_{n+1} have opposite signs for all n, i.e. $a_n a_{n+1} < 0$,

   oscillating if $b_n = a_{n+1} - a_n$ and $b_{n+1} = a_{n+2} - a_{n+1}$ have opposite signs for all n,

5) **bounded above** if there exists a constant $A \in \mathbb{R}$ such that $a_n \leq A$ for all n,

   bounded below if there exists a constant $B \in \mathbb{R}$ such that $a_n \geq B$ for all n,

   bounded if it is bounded both above and below,

6) **convergent** if $\lim_{n\to\infty} a_n = L$ exists, i.e. is finite.

It is also very important to review the meaning of the limit of a sequence:

"$\lim_{n\to\infty} a_n = L$" means: Given any $\epsilon > 0$, there exists an $N > 0$ (or more explicitly $N(\epsilon)$) such that $|a_n - L| < \epsilon$ for all $n \geq N$.

Note that this is a very strong requirement on the sequence elements as illustrated below:
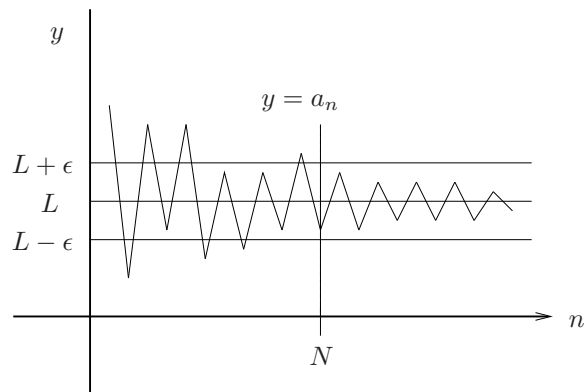
Not even one element a_n can be outside the "ribbon" $L - \epsilon < y < L + \epsilon$ for $n \geq N$.