1. Solution: (1)$\forall \varepsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $|x_n| < \frac{\varepsilon}{2}$ when $n > N_1$.

Let $N_2 = \max\left\{N_1, \frac{2(|x_1| + |x_2| + \cdots + |x_{N_1}|)}{\varepsilon}\right\}$, when $n > N_2$,

$$\left|\frac{x_1 + x_2 + \cdots + x_n}{n} - 0\right|$$

$$\leqslant \frac{|x_1| + |x_2| + \cdots + |x_{N_1}| + |x_{N_1+1}| + \cdots + |x_n|}{n}$$

$$= \frac{|x_1| + |x_2| + \cdots + |x_{N_1}|}{n} + \frac{|x_{N_1+1}| + \cdots + |x_n|}{n}$$

$$< \frac{\varepsilon}{2} + \frac{n - N_1}{n} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, $\lim_{n \to \infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = 0$.

(2) Suppose $x_n = a + \alpha_n$, $y_n = b + \beta_n$, where $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \beta_n = 0$.Then

$$\frac{x_1 y_n + x_2 y_{n-1} + \cdots + x_n y_1}{n}$$

$$= \frac{(a + \alpha_1)(b + \beta_n) + (a + \alpha_2)(b + \beta_{n-1}) + \cdots + (a + \alpha_n)(b + \beta_1)}{n}$$

$$= ab + b \cdot \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} + a \cdot \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} + \frac{\alpha_1 \beta_n + \alpha_2 \beta_{n-1} + \cdots + \alpha_n \beta_1}{n}$$

It follows from (1) that

$\lim_{n \to \infty} \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} = \lim_{n \to \infty} \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} = 0$. Since $\lim_{n \to \infty} \beta_n = 0$, $\exists M > 0$ such that $\forall n \in \mathbb{N}$, it always holds that

$|\beta_n| \leqslant M$. Thus,

$$0 \leqslant \left|\frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n}\right| \leqslant \frac{|\alpha_1|\,|\beta_n| + |\alpha_2|\,|\beta_{n-1}| + \cdots + |\alpha_n|\,|\beta_1|}{n}$$

$$\leqslant M \cdot \frac{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|}{n}$$

Since $\lim\limits_{n\to\infty} |\alpha_n| = 0$, it follows from (1) that

$\lim\limits_{n\to\infty} \dfrac{|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|}{n} = 0$. Hence, it follows from Squeeze

Theorem that $\lim\limits_{n\to\infty} \left|\dfrac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n}\right| = 0$, i.e.,

$\lim\limits_{n\to\infty} \dfrac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n} = 0$. Hence,

$$b \cdot \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} + a \cdot \frac{\beta_1 + \beta_2 + \cdots + \beta_n}{n} + \frac{\alpha_1\beta_n + \alpha_2\beta_{n-1} + \cdots + \alpha_n\beta_1}{n} \to 0$$

($n \to \infty$), i.e., $\lim\limits_{n\to\infty} \dfrac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} = ab$.

2. Solution: Since the convergence of numerical series is independent from its first finite terms, we can assume that $\forall n \in \mathbb{N}^*$, it holds that

$b_n \cdot \dfrac{a_n}{a_{n+1}} - b_{n+1} \geqslant \delta > 0$. Hence, $\forall n \in \mathbb{N}^*$,

$$a_{n+1} \leqslant \frac{1}{\delta}(a_nb_n - a_{n+1}b_{n+1}),$$

$$\sum_{k=1}^{n} a_k \leqslant a_1 + \frac{1}{\delta}\sum_{k=2}^{n}(a_{k-1}b_{k-1} - a_kb_k) = a_1 + \frac{1}{\delta}(a_1b_1 - a_nb_n) < a_1 + \frac{a_1b_1}{\delta}$$

Hence, the numerical series $\sum\limits_{n=1}^{\infty} a_n$ converges.

3. Solution: Suppose that M,m are respectively the upper and lower bound of the infinite subset S of real numbers. Assume that S has no limit point, then $\forall x\in[m,M]$, there exists a neighborhood U_x such that $U_x\cap S$ is finite. Thus, $\{U_x\}_{x\in[m,M]}$ forms an open covering of the closed interval $[m,M]$. By the Finite Covering Theorem, there exists a finite sub-covering $\{U_{x_1},U_{x_2},\cdots,U_{x_N}\}$ of $[m,M]$, i.e., $S\subset[m,M]\subset\bigcup_{i=1}^{N}U_{x_i}$. However, $\bigcup_{i=1}^{N}U_{x_i}$ is finite, which contradicts with the fact that S is infinite. Hence, the assumption is false, i.e., S has limit point.

4. Solution:

$$
\begin{aligned}
&\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\sin x)}{x^3}\\
&=\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\tan x)+\sin(\tan x)-\sin(\sin x)}{x^3}\\
&=\lim_{x\to 0}\frac{\tan(\tan x)-\sin(\tan x)}{x^3}+\lim_{x\to 0}\frac{\sin(\tan x)-\sin(\sin x)}{x^3}\\
&=\lim_{x\to 0}\frac{\frac{1}{2}\tan^3 x}{x^3}+\lim_{x\to 0}\frac{2\cos\left(\frac{\tan x+\sin x}{2}\right)\sin\left(\frac{\tan x-\sin x}{2}\right)}{x^3}\\
&=\frac{1}{2}+2\lim_{x\to 0}\frac{\frac{\tan x-\sin x}{2}}{x^3}=\frac{1}{2}+\lim_{x\to 0}\frac{\frac{1}{2}x^3}{x^3}=\frac{1}{2}+\frac{1}{2}=1
\end{aligned}
$$

5. Solution: “Sufficiency” Define $f(a)=f(a+0)$, $f(b)=f(b-0)$, then $f(x)$ is continuous on the closed interval $[a,b]$.

Hence, $f(x)$ is uniformly continuous on $[a,b]$, and consequently also uniformly continuous in (a,b).

"Necessity" Since $f(x)$ is uniformly continuous in (a,b), for $\forall\varepsilon>0$, $\exists\delta>0$ such that for any $x_1,x_2\in(a,b)$ and $|x_1-x_2|<\delta$, it holds that $|f(x_1)-f(x_2)|<\varepsilon$. In particular, if $x_1,x_2\in(a,a+\delta)$, it also holds that $|f(x_1)-f(x_2)|<\varepsilon$. By Cauchy Convergence Principle, $f(a+0)$ exists. Similarly, $f(b-0)$ also exists.

6. Solution: $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{\frac{1}{1+t^2}}{\frac{2t}{1+t^2}}=\dfrac{1}{2t}$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\mathrm{d}x}=\frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{-\frac{1}{2t^2}}{\frac{2t}{1+t^2}}=-\frac{1+t^2}{4t^3},\quad \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=1}=-\frac{1}{2}.$$

7. Solution: let $F(x)=f(x)\,(b-x)^a$, then $F(x)$ is continuous on $[a,b]$, differentiable in (a,b), and $F(a)=f(a)\,(b-a)^a=0$ $(f(a)=0)$, $F(b)=f(b)\,(b-b)^a=0$, by Rolle's Theorem, $\exists\,\xi\in(a,b)$ such that $F'(\xi)=0$, i.e.,

$f'(\xi)\,(b-\xi)^a+f(\xi)a(b-\xi)^{a-1}(-1)=0$, equivalently

$f(\xi)=\dfrac{b-\xi}{a}f'(\xi)$.

8. Solution: Without loss of generality, assume that $f(x)$ reaches its minimum value -1 at $x=x_0$, where $x_0 \in (0,1)$, then $f'(x_0)=0$.

By Taylor's Theorem,

$$f(0)=f(x_0)+\frac{f'(x_0)}{1!}(0-x_0)+\frac{f''(\xi_1)}{2!}(0-x_0)^2, \quad \xi_1 \in (0,x_0)$$

$$f(1)=f(x_0)+\frac{f'(x_0)}{1!}(1-x_0)+\frac{f''(\xi_2)}{2!}(1-x_0)^2, \quad \xi_2 \in (x_0,1)$$

i.e., $0=-1+\frac{f''(\xi_1)}{2!}x_0^2$, $0=-1+\frac{f''(\xi_1)}{2!}(1-x_0)^2$. Hence,

$f''(\xi_1)=\frac{2}{x_0^2}$, $f''(\xi_2)=\frac{2}{(1-x_0)^2}$. By the continuity of $f''(x)$,

$\exists\, \xi_3 \in [\xi_1,\xi_2]$ such that

$$f''(\xi_3)=\frac{1}{2}[f''(\xi_1)+f''(\xi_2)]=\frac{1}{x_0^2}+\frac{1}{(1-x_0)^2}.$$

Next, study the minimum value of the function $y=\frac{1}{x^2}+\frac{1}{(1-x)^2}$ in $(0,1)$.

$$y'(x)=\frac{-2}{x^3}+\frac{2}{(1-x)^3}=2\left[\frac{x^3-(1-x)^3}{x^3(1-x)^3}\right]=2\frac{(2x-1)(x^2-x+1)}{x^3(1-x)^3}$$

If $x \in \left(0,\frac{1}{2}\right)$, $y'(x)<0$, $y(x)$ is monotonically decreasing; if $x \in \left(\frac{1}{2},1\right)$, $y'(x)>0$, $y(x)$ is monotonically increasing. Thus, $y\left(\frac{1}{2}\right)=8$ is the minimum value. Hence,

$$M=\max_{0\leqslant x\leqslant 1} f''(x) \geqslant f''(\xi_3) \geqslant 8.$$