1.3 Functions

1.3.1 The Concept of a Function (Mapping)

Let X and Y be certain sets. We say that there is a *function* defined on X with values in Y if, by virtue of some rule f, to each element $x \in X$ there corresponds an element $y \in Y$.

In this case the set X is called the *domain of definition* of the function. The symbol x used to denote a general element of the domain is called the *argument* of the function, or the *independent variable*. The element $y_0 \in Y$ corresponding to a particular value $x_0 \in X$ of the argument x is called the *value* of the function at x_0, or the value of the function at the value $x = x_0$ of its argument, and is denoted $f(x_0)$. As the argument $x \in X$ varies, the value $y = f(x) \in Y$, in general, varies depending on the values of x. For that reason, the quantity $y = f(x)$ is often called the *dependent variable*.

The set

$$f(X) := \{y \in Y \mid \exists x \left((x \in X) \wedge (y = f(x))\right)\}$$

of values assumed by a function on elements of the set X will be called the *set of values* or the *range* of the function.

The term "function" has a variety of useful synonyms in different areas of mathematics, depending on the nature of the sets X and Y: *mapping*, *transformation*, *morphism*, *operator*, *functional*. The commonest is *mapping*, and we shall also use it frequently.

If $A \subset X$ and $f : X \to Y$ is a function, we denote by $f|A$ or $f|_A$ the function $\varphi : A \to Y$ that agrees with f on A. More precisely, $f|_A(x) := \varphi(x)$ if $x \in A$. The function $f|_A$ is called the *restriction* of f to A, and the function $f : X \to Y$ is called an *extension* or a *continuation* of φ to X.

Example 1 The formulas $l = 2\pi r$ and $V = \frac{4}{3}\pi r^3$ establish functional relationships between the circumference l of a circle and its radius r and between the volume V of a ball and its radius r. Each of these formulas provides a particular function $f : \mathbb{R}_+ \to \mathbb{R}_+$ defined on the set $\mathbb{R}_+$ of positive real numbers with values in the same set.

Example 3 The mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ (the direct product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \mathbb{R}_t \times \mathbb{R}_x$ of the time axis $\mathbb{R}_t$ and the spatial axis $\mathbb{R}_x$) into itself defined by the formulas

$$x' = x - vt,$$

$$t' = t,$$

is the classical Galilean transformation for transition from one inertial coordinate system (x, t) to another system (x', t') that is in motion relative to the first at speed v.

$$x' = \frac{x - vt}{\sqrt{1 - (\frac{v}{c})^2}},$$

$$t' = \frac{t - (\frac{v}{c^2})x}{\sqrt{1 - (\frac{v}{c})^2}}.$$

This is the well-known (one-dimensional) *Lorentz*[12] *transformation*, which plays a fundamental role in the special theory of relativity. The speed c is the speed of light.

Example 4 The *projection* $\mathrm{pr}_1 : X_1 \times X_2 \to X_1$ defined by the correspondence $X_1 \times X_2 \ni (x_1, x_2) \overset{\mathrm{pr}_1}{\longmapsto} x_1 \in X_1$ is obviously a function. The second projection $\mathrm{pr}_2 : X_1 \times X_2 \to X_2$ is defined similarly.

Example 5 Let $\mathcal{P}(M)$ be the set of subsets of the set M. To each set $A \in \mathcal{P}(M)$ we assign the set $C_M A \in \mathcal{P}(M)$, that is, the complement to A in M. We then obtain a mapping $C_M : \mathcal{P}(M) \to \mathcal{P}(M)$ of the set $\mathcal{P}(M)$ into itself.

Example 6 Let $E \subset M$. The real-valued function $\chi_E : M \to \mathbb{R}$ defined on the set M by the conditions $(\chi_E(x) = 1 \text{ if } x \in E) \wedge (\chi_E(x) = 0 \text{ if } x \in C_M E)$ is called the *characteristic function* of the set E.

Example 7 Let $M(X; Y)$ be the set of mappings of the set X into the set Y and x_0 a fixed element of X. To any function $f \in M(X; Y)$ we assign its value $f(x_0) \in Y$ at the element x_0. This relation defines a function $F : M(X; Y) \to Y$. In particular, if $Y = \mathbb{R}$, that is, Y is the set of real numbers, then to each function $f : X \to \mathbb{R}$ the function $F : M(X; \mathbb{R}) \to \mathbb{R}$ assigns the number $F(f) = f(x_0)$. Thus F is a function defined on functions. For convenience, such functions are called *functionals*.

Example 8 Let Γ be the set of curves lying on a surface (for example, the surface of the earth) and joining two given points of the surface. To each curve $\gamma \in \Gamma$ one can assign its length. We then obtain a function $F : \Gamma \to \mathbb{R}$ that often needs to be studied in order to find the shortest curve, or as it is called, the *geodesic* between the two given points on the surface.

Example 9 Consider the set $M(\mathbb{R}; \mathbb{R})$ of real-valued functions defined on the entire real line $\mathbb{R}$. After fixing a number $a \in \mathbb{R}$, we assign to each function $f \in M(\mathbb{R}; \mathbb{R})$ the function $f_a \in M(\mathbb{R}; \mathbb{R})$ connected with it by the relation $f_a(x) = f(x + a)$. The function $f_a(x)$ is usually called the *translate* or *shift* of the function f by a. The mapping $A : M(\mathbb{R}; \mathbb{R}) \to M(\mathbb{R}; \mathbb{R})$ that arises in this way is called the *translation* of *shift operator*. Thus the operator A is defined on functions and its values are also functions $f_a = A(f)$.

A particle in motion is located at some point of the space $\mathbb{R}^3$ having coordinates $(x(t), y(t), z(t))$ at each instant t of time. Thus the motion of a particle can be interpreted as a mapping $\gamma : \mathbb{R} \to \mathbb{R}^3$, where $\mathbb{R}$ is the time axis and $\mathbb{R}^3$ is three-dimensional space.

If a system consists of n particles, its configuration is defined by the position of each of the particles, that is, it is defined by an ordered set $(x_1, y_1, z_1; x_2, y_2, z_2; \ldots; x_n, y_n, z_n)$ consisting of $3n$ numbers. The set of all such ordered sets is called the *configuration space* of the system of n particles. Consequently, the configuration space of a system of n particles can be interpreted as the direct product $\mathbb{R}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3 = \mathbb{R}^{3n}$ of n copies of $\mathbb{R}^3$.

Example 12 The kinetic energy K of a system of n material particles depends on their velocities. The total mechanical energy of the system E, defined as $E = K + U$, that is, the sum of the kinetic and potential energies, thus depends on both the configuration q of the system and the set of velocities v of its particles. Like the configuration q of the particles in space, the set of velocities v, which consists of n three-dimensional vectors, can be defined as an ordered set of $3n$ numbers. The ordered pairs (q, v) corresponding to the states of the system form a subset Φ in the direct product $\mathbb{R}^{3n} \times \mathbb{R}^{3n} = \mathbb{R}^{6n}$, called the *phase space* of the system of n particles (to be distinguished from the configuration space $\mathbb{R}^{3n}$).

The total mechanical energy of the system is therefore a function $E : \Phi \to \mathbb{R}$ defined on the subset Φ of the phase space $\mathbb{R}^{6n}$ and assuming values in the domain $\mathbb{R}$ of real numbers.

In particular, if the system is isolated, that is, no external forces are acting on it, then by the law of conservation of energy, at each point of the set Φ of states of the system the function E will have the same value $E_0 \in \mathbb{R}$.

1.3.2 Elementary Classifification of Mappings

When a function $f : X \to Y$ is called a mapping, the value $f(x) \in Y$ that it assumes at the element $x \in X$ is usually called the *image* of x.

The *image* of a set $A \subset X$ under the mapping $f : X \to Y$ is defined as the set

$$f(A) := \big\{ y \in Y \mid \exists x \, \big((x \in A) \wedge \big(y = f(x) \big) \big) \big\}$$

consisting of the elements of Y that are images of elements of A.

The set

$$f^{-1}(B) := \big\{ x \in X \mid f(x) \in B \big\}$$

consisting of the elements of X whose images belong to B is called the *pre-image* (or *complete pre-image*) of the set $B \subset Y$ (Fig. 1.6).

A mapping $f : X \to Y$ is said to be

surjective (a mapping of X *onto* Y) if $f(X) = Y$;
injective (or an *imbedding* or *injection*) if for any elements x_1, x_2 of X

$$\big(f(x_1) = f(x_2)\big) \Rightarrow (x_1 = x_2),$$

that is, distinct elements have distinct images;
bijective (or a *one-to-one correspondence*) if it is both surjective and injective.

If the mapping $f : X \to Y$ is bijective, that is, it is a one-to-one correspondence between the elements of the sets X and Y, there naturally arises a mapping

$$f^{-1} : Y \to X,$$

defined as follows: if $f(x) = y$, then $f^{-1}(y) = x$ that is, to each element $y \in Y$ one assigns the element $x \in X$ whose image under the mapping f is y. By the surjectivity of f there exists such an element, and by the injectivity of f, it is unique. Hence the mapping f^{-1} is well-defined. This mapping is called the *inverse* of the original mapping f.

Thus the property of two mappings of being inverses is reciprocal: if f^{-1} is inverse for f, then f is inverse for f^{-1}.

1.3.3 Composition of Functions and Mutually Inverse Mappings

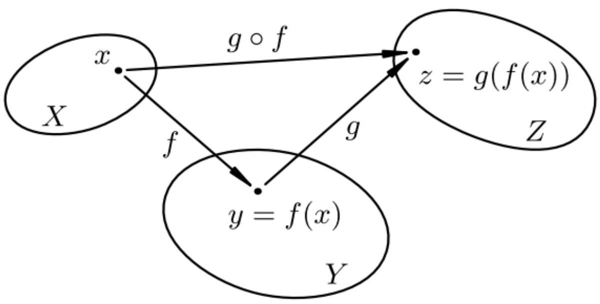
If the mappings $f : X \to Y$ and $g : Y \to Z$ are such that one of them (in our case g) is defined on the range of the other (f), one can construct a new mapping

$$g \circ f : X \to Z,$$

whose values on elements of the set X are defined by the formula

$$(g \circ f)(x) := g\big(f(x)\big).$$

The compound mapping $g \circ f$ so constructed is called the *composition* of the mapping f and the mapping g (in that order!).



The operation of composition sometimes has to be carried out several times in succession, and in this connection it is useful to note that it is associative, that is,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

We further note that even when both compositions $g \circ f$ and $f \circ g$ are defined, in general

$$g \circ f \neq f \circ g.$$

The mapping $f : X \to X$ that assigns to each element of X the element itself, that is $x \overset{f}{\longmapsto} y$, will be denoted e_X and called the *identity mapping* on X.

Lemma

$$(g \circ f = e_X) \Rightarrow (g \text{ is surjective}) \wedge (f \text{ is injective}).$$

Proposition *The mappings $f : X \to Y$ and $g : Y \to X$ are bijective and mutually inverse to each other if and only if $g \circ f = e_X$ and $f \circ g = e_Y$.*

1.3.4 Functions as Relations. The Graph of a Function

a. Relations

Definition 1 A *relation* $\mathcal{R}$ is any set of ordered pairs (x, y).

The set X of first elements of the ordered pairs that constitute $\mathcal{R}$ is called the *domain of definition* of $\mathcal{R}$, and the set Y of second elements of these pairs the *range of values* of $\mathcal{R}$.

Thus, a relation can be interpreted as a subset $\mathcal{R}$ of the direct product $X \times Y$. If

Instead of writing $(x, y) \in \mathcal{R}$, we often write $x\mathcal{R}y$ and say that x *is connected with y by the relation $\mathcal{R}$.*

Example 13 The diagonal

$$\Delta = \{(a, b) \in X^2 \mid a = b\}$$

is a subset of X^2 defining the relation of equality between elements of X. Indeed, $a\Delta b$ means that $(a, b) \in \Delta$, that is, $a = b$.

$a\mathcal{R}a$ (reflexivity);

$a\mathcal{R}b \Rightarrow b\mathcal{R}a$ (symmetry);
$(a\mathcal{R}b) \wedge (b\mathcal{R}c) \Rightarrow a\mathcal{R}c$ (transitivity).

A relation $\mathcal{R}$ having the three properties just listed, that is, reflexivity,[17] symmetry, and transitivity, is usually called an *equivalence relation*. An equivalence relation is denoted by the special symbol $\sim$, which in this case replaces the letter $\mathcal{R}$. Thus, in the case of an equivalence relation we shall write $a \sim b$ instead of $a\mathcal{R}b$ and say that a is *equivalent* to b.

A relation between pairs of elements of a set X having these three properties is usually called a *partial ordering* on X. For a partial ordering relation on X, we often write $a \preceq b$ and say that b *follows* a.

If the condition

$$\forall a\ \forall b\ \big((a\mathcal{R}b) \vee (b\mathcal{R}a)\big)$$

holds in addition to the last two properties defining a partial ordering relation, that is, any two elements of X are comparable, the relation $\mathcal{R}$ is called an *ordering*, and the set X with the ordering defined on it is said to be *linearly ordered*.

b. Functions and Their Graphs

A relation $\mathcal{R}$ is said to be *functional* if

$$(x\mathcal{R}y_1) \wedge (x\mathcal{R}y_2) \Rightarrow (y_1 = y_2).$$

A functional relation is called a *function*.

The *graph* of a function $f : X \to Y$, as understood in the original description, is the subset Γ of the direct product $X \times Y$ whose elements have the form $(x, f(x))$. Thus

$$\Gamma := \big\{(x, y) \in X \times Y \mid y = f(x)\big\}.$$