2.3. FIRST-ORDER DIFFERENTIAL EQUATIONS, LINEAR WITH RESPECT TO PARTIAL DERIVATIVES

Partial differential equation of the first order

Some problems of classical mechanics, continuum mechanics, acoustics, optics, hydrodynamics, and radiation transfer are reduced to partial differential equations of the first order. Analytical methods developed in the theory of ordinary differential equations are applicable to the solution of some of them.

1. Basic concepts. Classification of equations

A partial differential equation of the first order is an equation of the form [1]

$$F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)=0, \qquad (1)$$

where $x_1,\dots,x_n$ – are independent variables, $u=u(x_1,\dots,x_n)$ – is an unknown function, $F(x_1,x_2,\dots,x_n,u,p_1,p_2,\dots,p_n)$ – is a given continuously differentiable function [2] in some region $G\subset\mathbb{R}^{2n+1}$, and at each point in the region G

$$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_i}\right)^2\neq 0.$$

Equation (1) can be abbreviated as [3]

$$F(x,u,\operatorname{grad}u)=0, \qquad (1')$$

where $x=\left(x_1,\dots,x_n\right)$ and $\operatorname{grad} u=\left(\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)$.

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[1] Next, another entry of the first-order partial derivative will be used too. For example, for the function $u=u\left(x,y,z\right)$:

$$u'_x=\frac{\partial u}{ux},\ u'_y=\frac{\partial u}{\partial y},\ u'_z=\frac{\partial u}{\partial z}.$$

[2] Here, for the function F, the arguments $p_i,\ i=1,\dots,n,$ denote partial derivatives u'_{x_i}.

[3] Often, to write the gradient of the function $u\left(x\right)$, the operator ∇ («nabla») is used, that is, instead of $\operatorname{grad} u$, they write ∇u. Then equation $(1')$ can be written in the form $F\left(x,u,\nabla u\right)=0$.

Definition 1. The function $u=\varphi\left(x_1,\dots,x_n\right)$ given in the domain $D\subset\mathbb{R}^n$ is called the **solution of equation** (1) if:

1) $\varphi\left(x_1,\dots,x_n\right)-$ is a continuously differentiable function in D,

2) for all points $x=\left(x_1,\dots,x_n\right)\in D$ point $\left(x,\varphi,\varphi'_{x_1},\dots,\varphi'_{x_n}\right)\in G$,

3) $F\left(x_1,\dots,x_n,\varphi,\varphi'_{x_1},\dots,\varphi'_{x_n}\right)\equiv 0$ for any $\left(x_1,\dots,x_n\right)\in D$.

The solution of equation (1) in the $\left(n+1\right)$ - dimensional space of variables $x_1,\dots,x_n,u$ defines some smooth surface of dimension n, which is called the **integral surface of equation** (1).

Depending on how the unknown function u and its partial derivatives enter equation (1), **linear** and **nonlinear** equations are distinguished.

A linear partial differential equation of the first order is an equation of the form

$$A_1(x)\frac{\partial u}{\partial x_1} + \ldots + A_n(x)\frac{\partial u}{\partial x_n} = B(x)u + f(x), \qquad (2)$$

where $A_i(x)$ $(i=1,...,n)$, $B(x)$ and $f(x)$ - the given functions of point $x=(x_1,...,x_n)\in D$, moreover, $\sum_{i=1}^{n} A_i^2(x)\neq 0$ for any $x\in D$.

The functions A_i and B are called the coefficients of the equation. The linearity of the equation is determined by the fact that the unknown function $u(x)$ and all its partial derivatives enter the equation linearly.

If $f(x)\equiv 0$, then equation (2) is called **homogeneous**, otherwise- **inhomogeneous**.

If equation (1) cannot be written as (2), then it is called **nonlinear**. If in it the function F is linear with respect to all derivatives of the unknown function $\frac{\partial u}{\partial x_i}$, then equation (1) is called **quasi-linear**. The quasi-linear equation can be written as follows

$$A_1(x,u)\frac{\partial u}{\partial x_1} + \ldots + A_n(x,u)\frac{\partial u}{\partial x_n} = B(x,u). \qquad (3)$$

2. Homogeneous linear equation

Let the point $x=(x_1,...,x_n)$ belong to the domain $D\subset\mathbb{R}^n$, $n\geq 2$. In the domain D, consider a homogeneous linear partial differential equation of the first order of the form

$$A_1(x_1,\dots,x_n)\frac{\partial u}{\partial x_1}+\dots+A_n(x_1,\dots,x_n)\frac{\partial u}{\partial x_n}=0. \qquad (4)$$

Let the coefficients $A_i(x)$ $(i=1,\dots,n)-$ be continuously differentiable functions in D, for which

$$\sum_{i=1}^{n} A_i^2(x) \neq 0, \quad \forall\, x \in D.$$

Equation (4) can be given the following geometric interpretation. If we consider the coefficients $A_i(x)$ to be components of the vector $\mathbf{A}(x)$ in n-dimensional space, then equation (4) means that the derivative of the function $u(x)$ is equal to zero along the direction of vector $\mathbf{A}$.

Obviously, equation (4) has a solution of the form $u \equiv C$, where C is a constant. But equation (4) also has infinitely many solutions other than the constant.

For example, the solution to the equation $\frac{\partial u}{\partial x_1}=0$ is any continuous function Φ that does not depend on x_1, that is, $u(x_1,\dots,x_n)=\Phi(x_2,\dots,x_n)$ (this solution is obtained by integrating the equation with respect to the variable x_1). In general, the search for solutions to equation (4) is reduced to constructing solutions to a system of ordinary differential equations.

Let us compare equation (4) with a system of ordinary differential equations called **equations of characteristics:**

$$\frac{dx_1}{A_1(x_1,\ldots,x_n)} = \frac{dx_2}{A_2(x_1,\ldots,x_n)} = \ldots = \frac{dx_n}{A_n(x_1,\ldots,x_n)}. \qquad (5)$$

This system is called a system of differential equations in a *symmetric form* corresponding to a homogeneous linear partial differential equation (4) (or a **characteristic system**).

In the case of two independent variables, it consists of a single equation. Under the assumptions made regarding the coefficients $A_1(x_1,\ldots,x_n),\ldots,A_n(x_1,\ldots,x_n)$, system (5) has exactly $n-1$ independent first integrals.

Definition 2. The first integral of the system (5) is called the function $\psi(x_1,\ldots,x_n)$, which differs from the constant, which is identically equal to some constant at all points $(x_1,\ldots,x_n)$ of the integral curve of the system (5).

Often, the first integral is not called the function ψ, but the ratio $\psi = C$, where C is a constant.

The integral curves of the system of equations (5) are called the **characteristics of the partial differential equation** (4).

THEOREM 1. Every solution $\varphi(x_1,\dots,x_n)$ of equation (4) is the first integral of system (5), and, conversely, every first integral $\psi(x_1,\dots,x_n)$ of system (5) is the solution of equation (4).

For example, the equation

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0 \qquad (*)$$

corresponds to the system of differential equations

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{-z},$$

which has the following integrals

$$\psi_1 = xz, \qquad \psi_2 = x\sqrt{y}.$$

Then the functions $u_1 = xz$ and $u_2 = x\sqrt{y}$ are solutions to this equation. You can verify by substituting the expressions for u_1 and u_2 into the equation (*).

Let's assume that $A_n(x) \neq 0$ in D and $n-1$ independent first integrals of the system (5) are found:

$$\psi_1(x_1,\dots,x_n),\dots,\psi_{n-1}(x_1,\dots,x_n). \qquad (6)$$

The condition for the independence of integrals (6) is the difference from the zero of the Jacobian [4]

$$J = \frac{D(\psi_1, \ldots, \psi_{n-1})}{D(x_1, \ldots, x_{n-1})} \neq 0, \quad \forall x \in D. \qquad (7)$$

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[4] The Jacobian of functions $\psi_i(x_1,\ldots,x_m,x_{m+1},\ldots,x_n)$, $i=1,2,\ldots,m,$ is a determinant of order m, the i -th row of which contains partial derivatives of the first order of the function ψ_i with respect to variables $x_1,\ldots,x_m$. It is briefly indicated by the symbol $\frac{D(\psi_1,\ldots,\psi_m)}{D(x_1,\ldots,x_m)}$.

Let's introduce new independent variables

$$\begin{aligned} &\xi_1 = \psi_1(x_1, \ldots, x_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\xi_{n-1} = \psi_{n-1}(x_1, \ldots, x_n), \\ &\xi_n = \psi_n(x_1, \ldots, x_n), \end{aligned} \qquad (8)$$

where the function $\psi_n(x_1,\ldots,x_n)$ can be any continuously differentiable function in D, but in which the transformation (8) is non-degenerate, and a new notation with $\upsilon=\upsilon(\xi_1,\ldots,\xi_n)$ is used for the dependent variable, and $u(x)=\upsilon(\psi_1(x),\ldots,\psi_n(x))$.

$$v = v(\xi_1, \ldots, \xi_n),$$

$$u(x) = v(\psi_1(x), \ldots, \psi_n(x)). \qquad (9)$$

We show that when replacing (8)-(9), equation (4) is reduced to the simplest form when it is easy to construct its solution.

Indeed, we will express the derivatives included in equation (4) in terms of new variables using the rule of differentiation of a complex function:

$$\frac{\partial u}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial v}{\partial \xi_i}\frac{\partial \xi_i}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial v}{\partial \xi_i}\frac{\partial \psi_i}{\partial x_k}, \quad k = 1, \ldots, n.$$

Substituting these expressions into equation (4) and grouping the terms, we obtain the equation

$$\sum_{i=1}^{n-1} \left(A_1(x)\frac{\partial \psi_i}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_i}{\partial x_n} \right) \frac{\partial v}{\partial \xi_i} +$$
$$+ \left(A_1(x)\frac{\partial \psi_n}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_n}{\partial x_n} \right) \frac{\partial v}{\partial \xi_n} = 0$$

Since the functions ψ_i for $i = 1,\ldots,n-1$ are the first integrals of the system (5), then, according to Theorem 1, they are solutions of equation (4). Therefore, the last equation takes the form

$$\left(A_1(x)\frac{\partial \psi_n}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_n}{\partial x_n} \right) \frac{\partial v}{\partial \xi_n} = 0.$$

And since the transformation (8) is non-degenerate, the function ψ_n cannot be a solution to equation (4), and therefore we will have

$$\frac{\partial v}{\partial \xi_n} = 0. \tag{10}$$

Thus, using the non-degenerate transformation (8), equation (4) is reduced to the form (10), which is called canonical.

Integrating equation (10) by ξ_n, we obtain its solution

$$v(\xi_1, \ldots, \xi_n) = \Phi(\xi_1, \ldots, \xi_{n-1}),$$

where Φ is an arbitrary function that does not depend on ξ_n and has continuous derivatives with respect to the variables $\xi_1,\dots,\xi_{n-1}$. Returning to the old variables, we obtain the solution of equation (4).

THEOREM 2. Any solution $u=\varphi(x_1,\dots,x_n)$ of equation (4) is represented as

$$u = \Phi(\psi_1(x_1,\dots,x_n),\dots,\psi_{n-1}(x_1,\dots,x_n)), \tag{11}$$

where $\Phi(\psi_1,\dots,\psi_{n-1})$ is some differentiable function of its arguments $\psi_1,\dots,\psi_{n-1}$, and $\psi_i(x_1,\dots,x_n)$ $(i=1,\dots,n-1)$ are the first integrals of the system (5) satisfying the independence condition (7).

Formula (11) represents **the general solution** of equation (4).

Thus, the problem of constructing a general solution to equation (4) is equivalent to the problem of finding n− 1 independent first integrals of the corresponding system of ordinary differential equations (5).

Example 1

Find the general solution $u=u(x,y)$ of the equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0.$$

<u>Solution:</u>

Let's write down the equation of characteristics

$$\frac{dx}{y} = \frac{dy}{-x}.$$

This equation has a solution

$$x^2 + y^2 = C.$$

Therefore, the first integral is the function:

$$\psi = x^2 + y^2.$$

Then the general solution has the form

$$u = \Phi(x^2 + y^2),$$

and it represents a family of surfaces of rotation with the axis of rotation Ou. In particular, for $\Phi(\psi) = \psi$ we obtain a paraboloid of rotation:

$$u = x^2 + y^2,$$

when $\Phi(\psi) = \sqrt{\psi}$, we get a cone

$$u = \sqrt{x^2 + y^2}.$$

It is often possible to construct the first integrals of the characteristic system (5) for the case $n > 2$ by finding integrable combinations. *An integrable combination* is called a differential equation, which is a consequence of the system of equations (5) and is integrated in quadratures. The first integral of the system (5) is obtained from each integrable combination.

Example 2

Find the general solution $u = u(x, y, z)$ of the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Solution:

Let's write down the equations of characteristics in a symmetric form

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Solving: $\frac{dy}{y} = \frac{dx}{x}$ and $\frac{dz}{z} = \frac{dx}{x}$.

we find the first two integrals:

$$\frac{y}{x} = C_1 \text{ and } \frac{z}{x} = C_2$$

Then the general solution of the given equation has the form

$$u(x, y, z) = \Phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

To make integrable combinations of system (5), you can use the following rule of **equal fractions.**

If there are equal fractions

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n},$$

and arbitrary numbers $\lambda_1, \ldots, \lambda_n$ such that $\lambda_1 b_1 + \ldots + \lambda_n b_n \neq 0$, then

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n} = \frac{\lambda_1 a_1 + \ldots + \lambda_n a_n}{\lambda_1 b_1 + \ldots + \lambda_n b_n}.$$

Example 3

Find a solution of the equation

$$(x+z)u'_x + (y+z)u'_y + (x+y)u'_z = 0.$$

Solution:

To find the independent first integrals, we make up the equations of characteristics in a symmetric form

$$\frac{dx}{x+z} = \frac{dy}{y+z} = \frac{dz}{x+y}.$$

By the property of equal fractions, we have

$$\frac{dx-dz}{z-y} = \frac{dy-dz}{z-x} \Rightarrow (x-z)d(x-z) = (y-z)d(y-z).$$

Integrating the last equality, we get the first integral

$$\psi_1(x,y,z) = (x-z)^2 - (y-z)^2 = (x-y)(x+y-2z).$$

According to the property of equal fractions, we will make another equality

$$\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{x-y} \Leftrightarrow \frac{d(x+y+z)}{x+y+z} = \frac{2d(x-y)}{x-y},$$

the integration of which gives another first integral

$$\psi_2(x,y,z) = \frac{x+y+z}{(x-y)^2}$$

Then the general solution of the given equation has the form:

$$u(x,y,z) = \Phi\left((x-y)(x+y-2z), \frac{x+y+z}{(x-y)^2}\right).$$

where $\Phi(a,b)-$ is an arbitrary continuously differentiable function.

Example 4

Find a solution of the equation

$$xu'_x + yu'_y + xyu'_z = 0, \quad x \neq 0,\ y \neq 0.$$

<u>Solution:</u>

Let's make up a characteristic system

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy}. \qquad (*)$$

We will find the first integral by solving the equation

$$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \frac{x}{y} = C_1.$$

So, $\psi_1(x,y,z) = \dfrac{x}{y}$.

We will find another first integral by considering the second equation of the characteristic system (*)

$$\frac{dy}{y} = \frac{dz}{xy},$$

excluding x from it using the already found first integral ψ_1. Since $x = C_1 y$, we will have

$$\frac{dy}{y} = \frac{dz}{C_1 y^2} \quad \Rightarrow \quad C_1 y dy = dz \quad \Rightarrow \quad C_1 y^2 - 2z = C_2 \quad \Rightarrow \quad xy - 2z = C_2.$$

So, $\psi_2(x,y,z) = xy - 2z$ and the general solution of the given equation will be written as

$$u(x,y,z) = \Phi\left(\frac{x}{y}, xy - 2z\right).$$