# MMTM TEST 42

**Problem 83.** Find in $\mathbb{R}_{\max,+}$ all solutions of the equation $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$, where :

$$\boldsymbol{A}=\begin{pmatrix}-1 & -2 & 1\\ 1 & -1 & 2\\ 2 & -1 & -1\end{pmatrix} \qquad \boldsymbol{b}=\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

**ANSWER:**

Firstly, we need to calculate the scalar $\varDelta$ to check the type of solution, that is:

$$(\boldsymbol{b}^{-}\boldsymbol{A})^{-}=\left((1\ \ 0\ \ -1)\begin{pmatrix}-1 & -2 & 1\\ 1 & -1 & 2\\ 2 & -1 & -1\end{pmatrix}\right)^{-}=(1\ \ -1\ \ 2)^{-}=\begin{pmatrix}-1\\ 1\\ -2\end{pmatrix}$$

$$\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}=\begin{pmatrix}-1 & -2 & 1\\ 1 & -1 & 2\\ 2 & -1 & -1\end{pmatrix}\begin{pmatrix}-1\\ 1\\ -2\end{pmatrix}=\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

$$\varDelta=\left(\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}\right)^{-}\boldsymbol{b}=(1\ \ 0\ \ -1)\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}=0=\mathbb{1}$$

Since the condition $\varDelta=\mathbb{1}$ holds, we conclude that the equation has solutions, including the maximal solution:

$$\boldsymbol{x}=(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

We can describe all solutions by finding all minimal sets of columns in the matrix $\boldsymbol{A}=(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3)$ that generate the vector $\boldsymbol{b}$ .Now we need to check if there are any columns could be dropped,we denote that:

$$\boldsymbol{A}_{(1)}=\begin{pmatrix}-2 & 1\\ -1 & 2\\ -1 & -1\end{pmatrix} \qquad \boldsymbol{A}_{(2)}=\begin{pmatrix}-1 & 1\\ 1 & 2\\ 2 & -1\end{pmatrix} \qquad \boldsymbol{A}_{(3)}=\begin{pmatrix}-1 & -2\\ 1 & -1\\ 2 & -1\end{pmatrix}$$

$$\varDelta_{(1)}=\left(\boldsymbol{A}_{(1)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(1)})^{-}\right)^{-}\boldsymbol{b}$$

$$=\left(\begin{pmatrix}-2 & 1\\ -1 & 2\\ -1 & -1\end{pmatrix}\left((1\ \ 0\ \ -1)\begin{pmatrix}-2 & 1\\ -1 & 2\\ -1 & -1\end{pmatrix}\right)^{-}\right)^{-}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}=\left(\begin{pmatrix}-2 & 1\\ -1 & 2\\ -1 & -1\end{pmatrix}\begin{pmatrix}-1\\ 2\end{pmatrix}\right)^{-}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

$$=(1\ \ 0\ \ 0)\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}=1\neq\mathbb{1}$$

$$\varDelta_{(2)}=\left(\boldsymbol{A}_{(2)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(2)})^{-}\right)^{-}\boldsymbol{b}$$

$$=\left(\begin{pmatrix}-1 & 1\\ 1 & 2\\ 2 & -1\end{pmatrix}\left((1\ \ 0\ \ -1)\begin{pmatrix}-1 & 1\\ 1 & 2\\ 2 & -1\end{pmatrix}\right)^{-}\right)^{-}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}=\left(\begin{pmatrix}-1 & 1\\ 1 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}-1\\ -2\end{pmatrix}\right)^{-}\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}$$

$$=(1\ \ 0\ \ -1)\begin{pmatrix}-1\\ 0\\ 1\end{pmatrix}=0=\mathbb{1}$$

$$\Delta_{(3)} = \left(\boldsymbol{A}_{(3)}\left(\boldsymbol{b}^{-}\boldsymbol{A}_{(3)}\right)^{-}\right)^{-}\boldsymbol{b}$$

$$= \left(\begin{pmatrix} -1 & -2 \\ 1 & -1 \\ 2 & -1 \end{pmatrix}\left((1\ \ 0\ \ -1)\begin{pmatrix} -1 & -2 \\ 1 & -1 \\ 2 & -1 \end{pmatrix}\right)^{-}\right)^{-}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \left(\begin{pmatrix} -1 & -2 \\ 1 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)^{-}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= (1\ \ 0\ \ -1)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = 0 = \mathbb{1}$$

since $\Delta_{(2)} = \Delta_{(3)} = \mathbb{1}$ ,the set of all columns in $\boldsymbol{A}$ is not minimal. since both $\boldsymbol{a_1}$ , $\boldsymbol{a}_2$ and $\boldsymbol{a_3}$ are not co-linear to $\boldsymbol{b}$ , the set $(\boldsymbol{a}_1, \boldsymbol{a_3})$ and $(\boldsymbol{a}_1, \boldsymbol{a}_2)$ cannot be further dropped,and are minimal. All the solutions of the 1-st kind equation given by :

$$\boldsymbol{x}_1 = \begin{pmatrix} x_1 = -1 \\ x_2 \leqslant 1 \\ x_3 = -2 \end{pmatrix} \qquad \boldsymbol{x}_2 = \begin{pmatrix} x_1 = -1 \\ x_2 = 1 \\ x_3 \leqslant -2 \end{pmatrix}$$

**Problem 84.** Find in $\mathbb{R}_{\min,+}$ all solutions of the equation $\boldsymbol{Ax} = \boldsymbol{x}$ , where :

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}$$

**ANSWER:**

Firstly,we need to check the trace to classify the type of solution. We have:

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & -1 & 1 \\ 3 & \mathbb{1} & 3 \\ 1 & \mathbb{1} & 2 \end{pmatrix} \quad \boldsymbol{A}^3 = \begin{pmatrix} \mathbb{1} & -1 & 1 \\ 3 & \mathbb{1} & 3 \\ 1 & \mathbb{1} & 2 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 2 \end{pmatrix}$$

Then we calculate the value of $Tr(\boldsymbol{A})$,that is:

$$Tr(\boldsymbol{A}) = tr\boldsymbol{A} \oplus tr\boldsymbol{A}^2 \oplus tr\boldsymbol{A}^3 = 1 \oplus \mathbb{1} \oplus \mathbb{1} = \mathbb{1}$$

Since the condition $Tr(\boldsymbol{A}) = \mathbb{1}$ holds,we claim that the equation has nontrivial solutions. Secondly,we calculate the Kleene star and Kleene plus matrices:

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2 \oplus \boldsymbol{A}^3$$

$$= \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix} \oplus \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix} \oplus \begin{pmatrix} \mathbb{1} & -1 & 1 \\ 3 & \mathbb{1} & 3 \\ 1 & \mathbb{1} & 2 \end{pmatrix} \oplus \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & -2 & 1 \\ 2 & \mathbb{1} & 3 \\ 1 & -1 & \mathbb{1} \end{pmatrix}$$

$$\boldsymbol{A}^+ = \boldsymbol{A}\boldsymbol{A}^*$$

$$= \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} \mathbb{1} & -2 & 1 \\ 2 & \mathbb{1} & 3 \\ 1 & -1 & \mathbb{1} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & -2 & 1 \\ 2 & \mathbb{1} & 3 \\ 1 & -1 & 1 \end{pmatrix}$$

Since the first two columns in the matrices coincide,and what's more,they are co-linear:

$$\begin{pmatrix} -2 \\ \mathbb{1} \\ -1 \end{pmatrix} = 2 \otimes \begin{pmatrix} \mathbb{1} \\ 2 \\ 1 \end{pmatrix}$$

Thus we can drop one of them, without loss of generality, we drop the second column, then we can write the solution of the equation as follows:

$$\boldsymbol{x} = u\begin{pmatrix}\mathbb{1}\\ 2\\ 1\end{pmatrix} \quad u > \mathbb{0} \quad or\ \boldsymbol{x} = \boldsymbol{0}$$

In terms of conventional algebra, the solutions(include the trivial one) are written as:

$$x_1 = 0 \quad x_2 = 2u \quad x_3 = u \quad u \in \mathbb{R}$$