25.10.2024

5. ELLIPTIC EQUATIONS

5.1. THE LAPLACE EQUATION. SETTING BOUNDARY VALUE PROBLEMS.

The Laplace equation arises in a number of physics problems that are far from each other. Let's look at some of them.

An established process of thermal conductivity or diffusion.

The equation of thermal conductivity in the three-dimensional case in the presence of stationary heat sources (2.17):

$$c\rho\frac{\partial u}{\partial t} = k\Delta u + F(x,y,z), \qquad (5.1)$$

or a similar diffusion equation. If a temperature (concentration) distribution has been established in the region under consideration, which does not change with time, that is, a stationary process, then equation (5.1) reduces to the *Poisson equation*:

$$\Delta u = -f(x,y,z), \qquad (5.2)$$

where

$$f(x,y,z) = \frac{F(x,y,z)}{k}.$$

If there are no heat sources in this area, $F(x,y,z)=0$, we obtain the Laplace equation:

$$\Delta u = 0. \qquad (5.3)$$

The electrostatic field.

The intensity of the electrostatic field is expressed in terms of a scalar potential

$$\vec{E} = -\mathrm{grad}\varphi .$$

Substituting tension into Maxwell's fourth equation:

$$\mathrm{div}\vec{E} = 4\pi\rho ,$$

where ρ is the volume charge density, we obtain the Poisson equation:

$$\Delta\varphi = -4\pi\rho ,$$

or, if there are no charges in the area under consideration, the potential of the electrostatic field obeys the Laplace equation.

Stationary currents in a homogeneous conductive medium.

The vector field of the density of stationary currents $\vec{j}(x,y,z)$, associated with the charge density ρ by the continuity equation

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\vec{j} = 0 ,$$

in the absence of current sources in the area under consideration, charges changing with time, obeys the equation

$$\mathrm{div}\vec{j} = 0 .$$

On the other hand, the current density is related to the electric field strength by Ohm's law in differential form

$$\vec{j} = \sigma\vec{E} ,$$

where σ is the specific conductivity.

Maxwell's second equation (the law of electromagnetic induction):

$$\mathrm{rot}\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

in the stationary case, it contains zero in the right part, therefore, a scalar potential φ can be introduced for the electric field:

$$\vec{E} = -\mathrm{grad}\varphi \, ,$$

which in the case of a homogeneous medium, $\sigma = const$, also obeys the Laplace equation.

Boundary value problems

So, it is required to find some physical quantity $u(x,y,z)$, which in the region of space V, bounded by the surface S, obeys the Laplace equation:

$$\Delta u = 0$$

or the Poisson equation:

$$\Delta u = -f(x, y, z) \, .$$

In this case, one of the boundary conditions must be set (that is, the boundary value problem is set). Let's look at the main ones.

The first boundary value problem, or *Dirichlet problem*, is posed if the unknown function itself is set at the boundary of the domain

$$u\big|_S = f_1(x, y, z) \, .$$

The second boundary value problem, or the Neumann problem, is posed if the derivative of an unknown function in the direction of the external normal is given at the boundary of the domain

$$\left.\frac{\partial u}{\partial n}\right|_S = f_2(x,y,z).$$

The third boundary value problem is posed if a combination of an unknown function and its derivative in the direction of the external normal is given at the boundary of the domain

$$\left.\left(\frac{\partial u}{\partial n} + hu\right)\right|_S = f_3(x,y,z).$$

In the case of a thermal conduction process, this corresponds to heat exchange with the external environment according to Newton's law.

In addition to the above classification, boundary value problems are also divided into internal ones (if the area V lies inside the surface S) and external (if the area V lies outside the surface S).

5.2. THE LAPLACE EQUATION IN CYLINDRICAL AND SPHERICAL COORDINATES. FUNDAMENTAL SOLUTIONS

In equations (5.2) and (5.3), the Laplace operator is represented in a rectangular Cartesian coordinate system. Depending on the specific geometry of the region V in which the process under study is considered, it is possible to represent the Laplace operator in the coordinate system most suitable for the corresponding region.

We derive the Laplace equation in cylindrical coordinates.

Let's replace the variables: $x = \rho cos\varphi,\ y = \rho sin\varphi,\ z = z$. From here we get

$$\frac{\partial u}{\partial \rho}=\frac{\partial u}{\partial x}\cos\varphi+\frac{\partial u}{\partial y}\sin\varphi\,,$$

$$\frac{\partial^2 u}{\partial \rho^2}=\frac{\partial^2 u}{\partial x^2}\cos^2\varphi+2\frac{\partial^2 u}{\partial x\partial y}\sin\varphi\cos\varphi+\frac{\partial^2 u}{\partial y^2}\sin^2\varphi\,,$$

$$\frac{\partial u}{\partial \varphi}=-\frac{\partial u}{\partial x}\rho\sin\varphi+\frac{\partial u}{\partial y}\rho\cos\varphi\,,$$

$$\frac{\partial^2 u}{\partial \varphi^2}=\frac{\partial^2 u}{\partial x^2}\rho^2\sin^2\varphi-2\frac{\partial^2 u}{\partial x\partial y}\rho^2\sin\varphi\cos\varphi+$$

$$+\frac{\partial^2 u}{\partial y^2}\rho^2\cos^2\varphi-\frac{\partial u}{\partial x}\rho\cos\varphi-\frac{\partial u}{\partial y}\rho\sin\varphi\,.$$

Consequently, the Laplace equation (5.3) in cylindrical coordinates for the function $u=u\left(\rho,\varphi,z\right)$ will take the form

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2}+\frac{\partial^2 u}{\partial z^2}=0\,.$$

If there is an axial symmetry in the problem, $u=u\left(\rho\right)$, the equation is greatly simplified

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{du}{d\rho}\right)=0\,.$$

Integrating, we obtain a solution for $\rho\neq 0$ in the form

$$u(\rho)=C_1\ln\rho+C_2\,. \qquad (5.4)$$

Solution (5.4) when choosing constants of the form

$$C_1=-1\,,\ C_2=0:$$

$$u(\rho)=\ln\frac{1}{\rho}\,,$$

It is called the *fundamental solution of the Laplace equation in cylindrical coordinates.*

Similarly, the Laplace equation in spherical coordinates for the function $u=u(r,\theta,\varphi)$ has the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\right]=0.$$

If the problem obviously has spherical symmetry: $u=u(r)$, the equation is also simplified

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right)=0$$

and it can be easily integrated, as a result we get at $r\neq 0$:

$$u(r)=\frac{C_1}{r}+C_2. \qquad (5.5)$$

Solution (5.5) when choosing constants of the form

$$C_1=1\,,\; C_2=0:$$

$$u(r)=\frac{1}{r}$$

It is commonly called the *fundamental solution of the Laplace equation in spherical coordinates.*

5.3. HARMONIC FUNCTIONS AND THEIR BASIC PROPERTIES

Definition. Let 's define a region of space $V\subset R^3$ bounded by the surface S. The function $u(x,y,z)$ is called harmonic in the domain V if:

1) the function $u(x,y,z)$ is continuous along with the first derivatives in the domain $\bar{V}$;

2) has continuous second derivatives inside the domain V;

3) satisfies the Laplace equation:

$$\Delta u(x,y,z)=\frac{\partial^2 u(x,y,z)}{\partial x^2}+\frac{\partial^2 u(x,y,z)}{\partial y^2}+\frac{\partial^2 u(x,y,z)}{\partial z^2}=0$$

for

$$\forall (x,y,z)\in V .$$

Harmonic functions have a number of remarkable properties, which we formulate in the form of three theorems.

Theorem 1 (on the absence of sources).

If $u(x,y,z)$ is a harmonic function in the region V bounded by the surface S, then

$$\oint_{\sigma}\frac{\partial u}{\partial n}d\sigma=0 , \qquad (5.6)$$

where σ is any closed surface lying entirely inside the surface S.

Why this property is called the absence of sources can be understood by the example of electrostatics. If u is the potential of the electrostatic field, the integral in the ratio (5.6) is the flux of the field strength vector through a closed surface. The equality of the flow through any closed surface to zero means that there are no charges in the V region — sources of the electrostatic field.

Theorem 2 (on the average value). If $u(x,y,z)$ is a function harmonic in the region V, and M_0 is a point inside V, then

$$u(M_0)=\frac{1}{4\pi a^2}\oint_{\Sigma_a} u d\sigma , \qquad (5.7)$$

where Σ_a is a sphere of radius a centered at point M_0, lying entirely in the region V.

That is, the value of the harmonic function in the center of the sphere is always equal to the average value over the surface of the sphere.

The mean value theorem can be formulated in another way. Writing down the formula (5.7) for a sphere of smaller radius $\rho < a$:

$$4\pi\rho^2 u(M_0) = \oint\limits_{\Sigma_\rho} u d\sigma$$

and integrating over ρ from 0 to a, we get

$$\frac{4}{3}\pi a^3 u(M_0) = \int\limits_{V_a} u dV ,$$

or

$$u(M_0) = \frac{1}{V_a} \int\limits_{V_a} u dV .$$

Thus, the value of the harmonic function in the center of the sphere is also the average in terms of the volume of the sphere.

Theorem 3 (the principle of maximum value). The function $u(x,y,z)$, which is harmonic in the region V with the boundary S, can reach its maximum and minimum values only on the surface S.

Consequence. If the functions u and U are harmonic in the domain V, continuous in the closed domain $\bar{V}$, and the inequality $u \le U$ takes place on the surface S, then it persists everywhere in the domain V.

5.4. SOLVING THE DIRICHLET PROBLEM FOR A CIRCLE BY THE FOURIER METHOD

Let the area V be a circle of radius R centered at a point with coordinates (0,0):

$$V = O_R = \left\{(x,y) \mid x^2 + y^2 < R^2\right\},$$
$$\bar{V} = \bar{O}_R = \left\{(x,y) \mid x^2 + y^2 \le R^2\right\}.$$

The Dirichlet problem for the Laplace equation in the circle O_R is considered.

Find a function $u(x,y)$ that is twice continuously differentiable in the domain of O_R , continuous in the domain of $\bar{O}_R$, satisfying the Laplace equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ , } (x,y) \in O_R$$

and taking the set values at the boundary of the circle $x^2 + y^2 = R^2$:

$$u(x,y) = f(x,y) \text{ , } x^2 + y^2 = R^2 .$$

In polar coordinates: $x = \rho cos\varphi,\ y = \rho sin\varphi$, the Dirichlet problem for the Laplace equation in a circle is formulated as follows.

Find the function $u(\rho,\varphi)$ satisfying the Laplace equation inside the circle $\rho < R$:

$$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} = 0 \qquad (5.8)$$

and taking the set values at the boundary of the circle $\rho = R$:

$$u(R,\varphi) = f(\varphi) . \qquad (5.9)$$

We are looking for a solution to this problem in the form of a product:

$$u(\rho,\varphi) = \mathrm{P}(\rho)\Phi(\varphi) \text{ ,} \qquad (5.10)$$

substituting which into equation (5.8), we have

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\mathrm{P}}{d\rho}\right)\Phi(\varphi) + \frac{1}{\rho^2}\frac{d^2\Phi}{d\varphi^2}\mathrm{P}(\rho) = 0 .$$

Dividing both parts of this equation by $\frac{1}{\rho^2}\mathrm{P}(\rho)\Phi(\varphi)$, we get

$$\frac{\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)}{P(\rho)}=-\frac{\frac{d^2\Phi}{d\varphi^2}}{\Phi(\varphi)}. \qquad (5.11)$$

The right side of equality (5.11) is a function of only variable φ, and the left side is only ρ so the right and left sides of equality (5.11) retain a constant value when changing their arguments. It is convenient to denote this value by λ, that is

$$\frac{\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)}{P(\rho)}=-\frac{\frac{d^2\Phi}{d\varphi^2}}{\Phi(\varphi)}=\lambda.$$

It follows that the function $P(\rho)$ is the solution of the equation

$$\rho\frac{d}{d\rho}\left(\rho\frac{dP}{d\rho}\right)-\lambda P(\rho)=0, \qquad (5.12)$$

and for the function $\Phi(\varphi)$, we get the eigenvalue problem:

$$\begin{cases}\frac{d^2\Phi}{d\varphi^2}+\lambda\Phi(\varphi)=0,\\ \Phi(\varphi)=\Phi(\varphi+2\pi).\end{cases} \qquad (5.13)$$

Here, the periodicity condition of the function $\Phi(\varphi)$ is a consequence of the periodicity of the desired solution $u(\rho,\varphi)$ for the variable j with a period of 2π.

Nonzero periodic solutions to the problem (5.13) exist only for eigenvalues $\lambda=k^2$, $k=0,1,2,...$ and have the form

$$\Phi(\varphi)=A_k\cos k\varphi+B_k\sin k\varphi,$$

where A_k and B_k are arbitrary constants.

From (5.12) for the function $P(\rho)$ at $\lambda=k^2$, we obtain the equation

$$\rho^2\frac{d^2P}{d\rho^2}+\rho\frac{dP}{d\rho}-k^2P=0. \qquad (5.14)$$

We will look for partial solutions to this equation in the form

$$P(\rho) = \rho^{\alpha}, \ \alpha = \text{const} .$$

Substituting this function in (5.14), we get

$$\alpha = \pm k .$$

Therefore,

$$P(\rho) = \rho^{k}$$

or

$$P(\rho) = \rho^{-k} .$$

The second of these solutions should be discarded, since at $\rho = 0$ the function $P(\rho) = \rho^{-k}$ is not harmonic in the circle $\rho < R$.

Thus, according to (5.10), partial solutions of equation (5.8) can be written as follows:

$$u_k(\rho, \varphi) = \rho^k \left(A_k \cos k\varphi + B_k \sin k\varphi \right), \ k = 0,1,2,\ldots .$$

Since equation (5.8) is linear and homogeneous, the sum of the solutions is also a solution that can be represented as a series:

$$u(\rho, \varphi) = \sum_{k=0}^{\infty} u_k(\rho, \varphi) = \sum_{k=0}^{\infty} \rho^k \left(A_k \cos k\varphi + B_k \sin k\varphi \right).$$

Fulfilling the boundary condition (5.9), we obtain

$$u(\rho, \varphi) = \sum_{k=0}^{\infty} R^k \left(A_k \cos k\varphi + B_k \sin k\varphi \right) = f(\varphi) . \qquad (5.15)$$

Decompose the function $f(\varphi)$ into a Fourier series:

$$f(\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\varphi + b_k \sin k\varphi , \qquad (5.16)$$

where are the Fourier coefficients of the function $f(\varphi)$:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt\,,\ a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt\,, \tag{5.17}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt\,,\ k = 1,2,\dots\,. \tag{5.18}$$

Comparing the series (5.15) with the series (5.16), we get

$$A_0 = \frac{a_0}{2}\,,\ A_k = \frac{a_k}{R^k}\,,\ B_k = \frac{b_k}{R^k}\,.$$

Thus, the solution of the Dirichlet problem for the Laplace equation in the circle $\rho < R$ has the form

$$u(\rho,\varphi) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(\frac{\rho}{R}\right)^k \left(a_k \cos k\varphi + b_k \sin k\varphi\right), \tag{5.19}$$

where a_0, a_k, b_k are determined by formulas (5.17) and (5.18).

Theorem. Let the function $f(\varphi)$ be continuous on the boundary of the circle $\rho = R$. Then the sum of the series (5.19) with coefficients defined by formulas (5.17), (5.18) is the solution of the problem (5.8)–(5.9).

Example 1

Find the harmonic function $u(x,y)$ inside the unit circle, which takes the values x^2 at its boundary.

Solution:

The problem is reduced to solving the equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

if

$$x^2+y^2=\rho^2<1$$

on condition

$$u(x,y)=x^2,$$

if

$$x^2+y^2=\rho^2=1.$$

In polar coordinates $x=\rho cos\varphi, y=\rho sin\varphi$, we obtain the following Dirichlet problem: find the function $u(\rho,\varphi)$ satisfying the Laplace equation inside the circle $\rho<1$:

$$\Delta u=0$$

and taking on the boundary of the circle $\rho=1$ values

$$u(1,\varphi)=\cos^2\varphi.$$

According to (5.19), the decision will take the form

$$u(\rho,\varphi)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\rho^k\left(a_k\cos k\varphi+b_k\sin k\varphi\right). \quad (5.20)$$

The coefficients a_0, a_k, b_k are determined by formulas (5.17) and (5.18).

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 t\,dt=\frac{2}{\pi}\int_0^{\pi}\frac{1+\cos 2t}{2}dt=\frac{1}{\pi}\left(t+\frac{1}{2}\sin 2t\right)\Bigg|_0^{\pi}=1.$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 t \;\cos kt\,dt = \frac{2}{\pi}\int_0^{\pi} \frac{1+\cos 2t}{2}\cos kt dt =$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\cos kt + \cos 2t \cos kt\right)dt =$$

$$= \frac{1}{\pi}\left[\frac{1}{k}\sin kt\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\left(\cos t(k-2) + \cos t(k+2)\right)dt\right] =$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{k-2}\sin t(k-2) + \frac{1}{k+2}\sin t(k+2)\right)\Big|_0^{\pi}\right] = 0, \quad k \neq 2.$$

At $k = 2$:

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 t \cos 2t dt = \frac{1}{\pi}\int_0^{\pi}\left(1+\cos 2t\right)\cos 2t\, dt =$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\sin 2t\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi}\left(1+\cos 4t\right)d\,t\right] = \frac{1}{2\pi}\left(t + \frac{1}{4}\sin 4t\right)\Big|_0^{\pi} = \frac{1}{2}.$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 t \sin kt dt = 0 .$$

Substituting the coefficients found in (5.20), we obtain a solution to the problem:

$$u(\rho,\varphi) = \frac{1}{2} + \frac{\rho^2}{2}\cos 2\varphi .$$

Examples of possible exam assignments

3. HYPERBOLIC EQUATIONS (FOURIER METHOD)
$$\begin{cases} u_{tt} = a^2 u_{xx} \qquad (*) \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \frac{l}{8}\sin\frac{3\pi x}{l} \\ u_t(x,0) = 0 \end{cases}$$
4. PARABOLIC EQUATIONS
$$\begin{cases} u_t = a^2 u_{xx} \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \frac{Ax(l-x)}{l^2} \end{cases}$$