Chapter 4

FUNCTIONS OF SEVERAL VARIABLES

1.1. Areas of function definition. Lines and surfaces of the level.

Let two non-empty sets D and U. If each pair of real numbers (x,y) belonging to the set D, according to a certain rule, one and only one element u of U is matched, then it is said that the set D has a function f (or a mapping) with a set of values U.

At the same time, write $D \xrightarrow{f} U$, either $f: D \to U$ or $u = f(x,y)$.

The set D is called the domain of the function definition, and the set U, consisting of all numbers of the form $f(x,y)$, where $(x,y) \in D$, is the set of function values. The value of the function $u = f(x,y)$ at point $M(x_0,y_0)$ is called the private value of the function and is denoted $f(x_0,y_0)$ or $f(M)$.

The domain of definition of function $u = f(x,y)$ in the simplest cases is either a part of the plane bounded by a closed curve, and the points of this curve (the boundaries of the domain) may or may not belong to the domain of definition, or the entire plane, or a collection of several parts of the xOy plane.

The geometric image of function $u = f(x,y)$ in a rectangular coordinate system $Oxyu$ (graph of the function) is a certain surface.

Similarly, the function of any finite number of independent variables $u = f(x,y,z,...,t)$ is defined.

The line of the level of function $u = f(x,y)$ is called the line $f(x,y) = C$ on the plane of xOy, at the points of which the function retains a constant value $u = C$.

The surface of the level of function $u = f(x,y,z)$ is called the surface $f(x,y,z) = C$, at the points of which the function retains a constant value $u = C$.

Exercise 1.

Find the domain of definition of function $u = \sqrt{a^2 - x^2 - y^2}$.

Solution:

function u takes valid values under the condition $a^2 - x^2 - y^2 \geqslant 0$, that is, $x^2 + y^2 \leqslant a^2$. The area of definition of this function is a circle of radius a centered at the origin, including the boundary circle.

Exercise 2.

Find the domain of definition of function $u = \arcsin(x/y^2)$.

Solution:

This function is defined if $y \neq 0$ and $-1 \leqslant x/y^2 \leqslant 1$, that is, $-y^2 \leqslant x \leqslant y^2$. The domain of definition of a function is the part of the plane enclosed between two parabolas $y^2 = x$ and $y^2 = -x$, with the exception of point $O(0;\ 0)$.

Exercise 3.

Find the lines of the level of function $u = x^2 + y^2$.

Solution:

The equation of the family of lines of the level has the form $x^2 + y^2 = C\ (C > 0)$.. By giving C different real values, we get concentric circles centered at the origin.

Exercise 4.

Find the level surfaces of the function $u = x^2 + z^2 - y^2$..

Solution:

The equation of the family of level surfaces has the form $x^2 + z^2 - y^2 = C$.. If $C = 0$, then we get $x^2 + z^2 - y^2 = 0$- a cone, if $C > 0$, then $x^2 + z^2 - y^2 = C$ is a family of single-cavity hyperboloids, if $C < 0$, then $x^2 + z^2 - y^2 = C$ is a family of double-cavity hyperboloids.

1.2. Derivatives and differentials of functions of several variables.

1. *Partial derivatives of the first order.*

The first-order partial derivative of the function $z = f(x,y)$ with respect to the independent variable x is called the finite limit

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x,\ y) - f(x,\ y)}{\Delta x} = \frac{\partial z}{\partial x} = f'_x(x,\ y),$$

calculated at constant y.

The partial derivative of y is called the finite limit

$$\lim_{\Delta y \to 0} \frac{f(x,\ y+\Delta y) - f(x,\ y)}{\Delta y} = \frac{\partial z}{\partial y} = f'_y(x,\ y),$$

calculated at constant x.

For partial derivatives, the usual rules and formulas of differentiation are valid.

Exercise 1.

$$u = x^2 - 3xy - 4y^2 - x + 2y + 1.$$

Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Solution:

Considering y as a constant value, we get $\frac{\partial u}{\partial x} = 2x - 3y - 1$.

Considering x as a constant, we find $\frac{\partial u}{\partial y} = -3x - 8y + 2$.

Exercise 2.

$$z = e^{x^2+y^2}.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

$$\frac{\partial z}{\partial x} = e^{x^2+y^2}(x^2+y^2)'_x = 2xe^{x^2+y^2}, \quad \frac{\partial z}{\partial y} = e^{x^2+y^2}(x^2+y^2)'_y = 2ye^{x^2+y^2}.$$

Exercise 3.

$$\rho = u^4 \cos^2 \varphi.$$

Find $\frac{\partial \rho}{\partial u}$ and $\frac{\partial \rho}{\partial \varphi}$.

Solution:

$$\frac{\partial \rho}{\partial u} = 4u^3 \cos^2 \varphi, \quad \frac{\partial \rho}{\partial \varphi} = u^4 \cdot 2\cos\varphi(-\sin\varphi) = -u^4 \sin 2\varphi.$$

Exercise 4.

To show that the function $z = y\ln(x^2 - y^2)$ satisfies the equation
$\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{z}{y^2}$.

Solution:

We find $\frac{\partial z}{\partial x}=\frac{2xy}{x^2-y^2}; \quad \frac{\partial z}{\partial y}=\ln(x^2-y^2)-\frac{2y^2}{x^2-y^2}.$

Substitute the found expressions in the left part of the equation:

$$\frac{1}{x}\cdot\frac{2xy}{x^2-y^2}+\frac{1}{y}\left[\ln(x^2-y^2)-\frac{2y^2}{x^2-y^2}\right]=\frac{2y}{x^2-y^2}-\frac{2y}{x^2-y^2}+\frac{\ln(x^2-y^2)}{y}\equiv\frac{z}{y^2}.$$

We obtain the identity, that is, the function z satisfies this equation.

Exercise 5.

To show that the function $z=y^{y/x}\sin(y/x)$ satisfies the equation $x^2\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=yz.$

Solution:

We find

$$\frac{\partial z}{\partial x}=y^{y/x}\cdot\ln y\left(-\frac{y}{x^2}\right)\sin\left(\frac{y}{x}\right)+y^{y/x}\cos\left(\frac{y}{x}\right)\cdot\left(-\frac{y}{x^2}\right),$$

$$\frac{\partial z}{\partial y}=\left[y^{y/x}\cdot\ln y\left(\frac{1}{x}\right)+\frac{y}{x}\cdot y^{y/x-1}\right]\sin\left(\frac{y}{x}\right)+y^{y/x}\cos\left(\frac{y}{x}\right)\cdot\frac{1}{x}.$$

Substitute the found expressions in the left part of the equation:

$$-x^2\frac{y}{x^2}\cdot y^{y/x}\cdot\ln y\cdot\sin\left(\frac{y}{x}\right)-x^2\cdot\frac{y}{x^2}\cdot y^{y/x}\cos\left(\frac{y}{x}\right)+$$
$$+xy\cdot\frac{y}{x}\cdot y^{y/x-1}\cdot\sin\left(\frac{y}{x}\right)+xy\cdot\frac{1}{x}\cdot y^{y/x}\cdot\ln y\cdot\sin\left(\frac{y}{x}\right)+$$
$$+xy\cdot\frac{1}{x}\cdot y^{y/x}\cos\left(\frac{y}{x}\right)=yy^{y/x}\sin\left(\frac{y}{x}\right)\equiv yz.$$

We obtain the identity, that is, the function z satisfies this equation.

2. *Full differential.*

The total increment of the function $z=f(x,y)$ at point $M(x,y)$ is the difference $\Delta z=f(x+\Delta x,\ y+\Delta y)-f(x,\ y),$ where Δx and Δy are arbitrary increments of the arguments.

A function $z = f(x,y)$ is called **differentiable** at point (x,y) if at this point the total increment can be represented as $\Delta z = A\,\Delta x + B\,\Delta y + o(\rho),$ where $\rho = \sqrt{\Delta x^2 + \Delta y^2}.$

The full differential of the function $z = f(x,y)$ is the main part of the full increment Δz, linear with respect to the increments of the arguments Δx and Δy, that is, $dz = A\,\Delta x + B\,\Delta y.$.

The differentials of the independent variables coincide with their increments, that is, $dx = \Delta x$ and $dy = \Delta y$.

The full differential of the function $z = f(x,y)$ is calculated by the formula

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

Similarly, the total differential of the function of the three arguments $u = f(x,y,z)$ is calculated using the formula

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy + \frac{\partial u}{\partial z}\,dz.$$

For a sufficiently small $\rho = \sqrt{\Delta x^2 + \Delta y^2}.$, approximate equalities $\Delta z \approx dz;\ f(x+\Delta x,\ y+\Delta y) \approx f(x,\ y) + dz$ are valid for the differentiable function $z = f(x,y)$.

<u>Exercise 1.</u>

$$z = \operatorname{arctg} \frac{x+y}{x-y}.$$

Find $dz.$

<u>Solution:</u>

$$\frac{\partial z}{\partial x} = \frac{1}{1+\left(\frac{x+y}{x-y}\right)^2}\cdot\frac{-2y}{(x-y)^2} = -\frac{y}{x^2+y^2}, \quad \frac{\partial z}{\partial y} = \frac{1}{1+\left(\frac{x+y}{x-y}\right)^2}\cdot\frac{2x}{(x-y)^2} = \frac{x}{x^2+y^2}.$$

Therefore

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = \frac{x\,dy - y\,dx}{x^2+y^2}.$$

<u>Exercise 2.</u>

$u = x^{y^2 z}$.

Find du.

Solution:

We have $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$,

We find $\frac{\partial u}{\partial x} = y^2 z \cdot x^{y^2 z - 1}$, $\frac{\partial u}{\partial y} = x^{y^2 z} \cdot \ln x \cdot 2yz$, $\frac{\partial u}{\partial z} = x^{y^2 z} \cdot \ln x \cdot y^2$.

Therefore

$$du = y^2 z x^{y^2 z - 1}\,dx + 2yz \cdot x^{y^2 z} \cdot \ln x\,dy + y^2 x^{y^2 z} \cdot \ln x\,dz.$$

Exercise 3.

Calculate approximately $\sqrt{\sin^2 1{,}55 + 8e^{0{,}015}}$, , based on the function $z = \sqrt{\sin^2 x + 8e^y}$ for $x = \pi/2 \approx 1{,}571,\ y = 0$. .

Solution:

The desired number is the incremented value of the function z at $\Delta x = 0{,}021$, $\Delta y = 0{,}015$. Find the value of z for $x = \frac{\pi}{2}, y = 0$, we have $z = \sqrt{\sin^2(\pi/2) + 8e^0} = 3$. .

We find the increment of the function:

$$\Delta z \approx dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \frac{\sin 2x\,\Delta x + 8e^y\,\Delta y}{2\sqrt{\sin^2 x + 8e^y}} = \frac{8 \cdot 0{,}015}{6} = 0{,}02.$$

Hence $\sqrt{\sin^2 1{,}55 + 8e^{0{,}015}} \approx 3{,}02$. .

Exercise 4.

Calculate approximately $\operatorname{arctg}(1{,}02/0{,}95)$, , based on the function $z = \operatorname{arctg}(y/x)$ for $x = 1,\ y = 1$.

Solution:

The value of the function z at $x = 1,\ y = 1$. is $z = \operatorname{arctg}(1/1) = \pi/4 \approx 0{,}785$. . Let's find the increment of the function Δz for $\Delta x = -0{,}05,\ \Delta y = 0{,}02$:

$$\Delta z \approx dz = \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = -\frac{y\,\Delta x}{x^2+y^2} + \frac{x\,\Delta y}{x^2+y^2} = \frac{x\,\Delta y - y\,\Delta x}{x^2+y^2} = \frac{1\cdot 0{,}02 + 1\cdot 0{,}05}{2} = 0{,}035.$$

Therefore, $\operatorname{arctg}(1{,}02/0{,}95) = z + \Delta z \approx 0{,}785 + 0{,}035 = 0{,}82.$

3. Partial derivatives and differentials of higher orders.

Second-order partial derivatives of function $z = f(x,y)$ are called partial derivatives of its first-order partial derivatives.

Notation of partial derivatives of the second order:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x,\ y);\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x\,\partial y} = f''_{xy}(x,\ y);$$
$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y\,\partial x} = f''_{yx}(x,\ y);\quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f''_{yy}(x,\ y).$$

Similarly, partial derivatives of the third and higher orders are defined and denoted, for example:

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^3} = f'''_{xxx}(x,\ y);\quad \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{\partial^3 z}{\partial x^2\,\partial y} = f'''_{xxy}(x,\ y)$$

The so-called "mixed" derivatives, differing from each other only by the sequence of differentiation, are equal if they are continuous, for example,

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}.$$

A second-order differential from function $z = f(x,y)$ is called a differential from its full differential, that is, $d^2 z = d\,(dz)$.

Differentials of the third and higher orders are defined similarly:

$$d^3 z = d\,(d^2 z);\quad d^n z = d\,(d^{n-1} z).$$

. If x and y are independent variables and the function $f(x,y)$ has continuous partial derivatives, then higher-order differentials are calculated using formulas

$$d^2 z = \frac{\partial^2 z}{\partial x^2}\,dx^2 + 2\,\frac{\partial^2 z}{\partial x\,\partial y}\,dx\,dy + \frac{\partial^2 z}{\partial y^2}\,dy^2;$$
$$d^3 z = \frac{\partial^3 z}{\partial x^3}\,dx^3 + 3\,\frac{\partial^3 z}{\partial x^2\,\partial y}\,dx^2\,dy + 3\,\frac{\partial^3 z}{\partial x\,\partial y^2}\,dx\,dy^2 + \frac{\partial^3 z}{\partial y^3}\,dy^3.$$

Generally speaking, there is a symbolic formula

$$d^n z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^n z,$$

which formally it is revealed according to the binomial law.

Exercise 1.

$$z=y\ln x.$$

Find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x\,\partial y}, \frac{\partial^2 z}{\partial y^2}.$

Solution:

Let's find partial derivatives

$$\frac{\partial z}{\partial x}=\frac{y}{x};\quad \frac{\partial z}{\partial y}=\ln x.$$

.

Differentiating again we get:

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{y}{x}\right)=-\frac{y}{x^2};\quad \frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}(\ln x)=0;\quad \frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{1}{x}.$$

Exercise 2.

$$z=\ln \operatorname{tg}(y/x).$$

Find: $\frac{\partial^2 z}{\partial x\,\partial y}.$

Solution:

$$\frac{\partial z}{\partial x}=\frac{1}{\operatorname{tg}(y/x)}\cdot \sec^2(y/x)\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2}\cdot\frac{2}{\sin(2y/x)},$$

$$\frac{\partial^2 z}{\partial x\,\partial y}=-\frac{1}{x^2}\cdot\frac{2}{\sin(2y/x)}-\frac{y}{x^2}\cdot\frac{-2\cos(2y/x)\cdot(2/x)}{\sin^2(2y/x)}=$$

$$=\frac{2}{x^3\sin^2(2y/x)}\cdot(2y\cos(2y/x)-x\sin(2y/x)).\ \blacktriangle$$

Exercise 3.

$z = \sin x \sin y.$

Find: $d^2 z.$

Solution:

$$\triangle \ \frac{\partial z}{\partial x} = \cos x \sin y, \ \frac{\partial z}{\partial y} = \sin x \cos y,$$
$$\frac{\partial^2 z}{\partial x^2} = -\sin x \sin y, \ \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos y; \ \frac{\partial^2 z}{\partial y^2} = -\sin x \sin y,$$
$$d^2 z = -\sin x \sin y \, dx^2 + 2 \cos x \cos y \, dx \, dy - \sin x \sin y \, dy^2. \ \blacktriangle$$

Exercise 4.

$z = x^2 y.$

Find: $d^3 z.$

Solution:

$$\triangle \ \frac{\partial z}{\partial x} = 2xy, \ \frac{\partial^2 z}{\partial x^2} = 2y, \ \frac{\partial^3 z}{\partial x^3} = 0, \ \frac{\partial z}{\partial y} = x^2, \ \frac{\partial^2 z}{\partial y^2} = 0,$$
$$\frac{\partial^3 z}{\partial y^3} = 0, \ \frac{\partial^3 z}{\partial x^2 \partial y} = 2, \ \frac{\partial^3 z}{\partial x \partial y^2} = 0;$$
$$d^3 z = 0 \cdot dx^3 + 3 \cdot 2dx^2 \, dy + 3 \cdot 0 \cdot dx \cdot dy^2 + 0 \cdot dy^3 = 6dx^2 \, dy. \ \blacktriangle$$

4. *Differentiation of complex functions*.

Let $z = f(x,y)$, where $x = \varphi(t)$, $y = \psi(t)$ and the functions $f(x,y)$, $\varphi(t)$, $\psi(t)$ are differentiable. Then the derivative of the complex function $z = f(\varphi(t),\psi(t))$ is calculated using the formula

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

If $z = f(x,y)$, where $y = \varphi(x)$, then the **full derivative** of z by x is found by the formula

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx}.$$

If $z = f(x,y)$, where $x = \varphi(\xi,\eta), y = \psi(\xi,\eta)$, then the partial derivatives are expressed as follows:

$$\frac{\partial z}{\partial \xi}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \xi}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \xi} \text{ и } \frac{\partial z}{\partial \eta}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial \eta}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial \eta}.$$

Exercise 1.

$z=e^{x^2+y^2}$, where $x=a\cos t,\ y=a\sin t.$

Find:

$\frac{dz}{dt}$.

Solution:

We have

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}=e^{x^2+y^2}\cdot 2x(-a\sin t)+e^{x^2+y^2}\cdot 2y(a\cos t)=$$
$$=2ae^{x^2+y^2}(y\cos t-x\sin t).$$

Let's express x and y through t, we get

$$\frac{dz}{dt}=2ae^{a^2}(a\sin t\cos t-a\cos t\sin t)=0.$$

Exercise 2.

$z=\ln(x^2-y^2)$, where $y=e^x$.

Find:

$\frac{\partial z}{\partial x}$, $\frac{dz}{dx}$.

Solution:

We have

$$\frac{\partial z}{\partial x}=\frac{2x}{x^2-y^2}.$$

Using the full derivative formula, we find

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}=\frac{2x}{x^2-y^2}-\frac{2ye^x}{x^2-y^2}=\frac{2(x-ye^x)}{x^2-y^2}.$$

5. *A derivative in this direction. The gradient of the function.*

The derivative of the function $z=f(x,y)$ at point $M(x,y)$ in the direction of vector $l=\overline{MM_1}$ is called the limit

$$\frac{\partial z}{\partial l}=\lim_{|MM_1|\to 0}\frac{f(M_1)-f(M)}{|MM_1|}=\lim_{\rho\to 0}\frac{\Delta z}{\rho},$$

where $\rho=\sqrt{\Delta x^2+\Delta y^2}$.

If the function $f(x,y)$ is differentiable, then the derivative in this direction is calculated by the formula

$$\frac{\partial z}{\partial l}=\frac{\partial z}{\partial x}\cos\alpha+\frac{\partial z}{\partial y}\sin\alpha,$$

where α is the angle formed by the vector and the Ox- axis.

In the case of a function of three variables $u=f(x,y,z)$, the derivative in this direction is determined similarly. The corresponding formula has the form

$$\frac{\partial u}{\partial l}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma,$$

where $\cos\alpha,\cos\beta,\cos\gamma$ are the guiding cosines of the vector l.

The gradient of the function $z=f(x,y)$ at point $M(x,y)$ is a vector with a beginning at point M, having its coordinates partial derivatives of the function z :

$$\operatorname{grad} z=\frac{\partial z}{\partial x}\mathbf{i}+\frac{\partial z}{\partial y}\mathbf{j}.$$

The gradient of the function and the derivative in the direction of vector l are related by the formula $\frac{\partial z}{\partial l}=grad\,z$.

The gradient indicates the direction of the fastest growth of the function at a given point. The derivative $\frac{\partial z}{\partial l}$ in the direction of the gradient has the highest value equal to

$$\left(\frac{\partial z}{\partial l}\right)_{\text{наиб}}=|\operatorname{grad} z|=\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}.$$

In the case of function $u=f(x,y,z)$, the gradient of the function is

$$\operatorname{grad} u=\frac{\partial u}{\partial x}\mathbf{i}+\frac{\partial u}{\partial y}\mathbf{j}+\frac{\partial u}{\partial z}\mathbf{k}.$$

Exercise 1.

Find the derivative of the function $z=x^2-y^2$ at point $M(1;\ 1)$ in the direction of the vector l, which makes up the angle $\alpha=60^\circ$ with the positive direction of the Ox-axis.

Solution:

Find the values of partial derivatives at point M:

$$\frac{\partial z}{\partial x}=2x,\ \frac{\partial z}{\partial y}=-2y,\ \left(\frac{\partial z}{\partial x}\right)_M=2,\ \left(\frac{\partial z}{\partial y}\right)_M=-2.$$

Since $\cos\alpha=\cos 60^\circ=1/2,\ \sin\alpha=\sin 60^\circ=\sqrt{3}/2,$

then $\frac{\partial z}{\partial l}=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)=1-\sqrt{3}\approx-0{,}7.$

Exercise 2.

Find the derivative of the function $u=xy^2z^3$ at point $M(3;2;1)$ in the direction of the vector $\overline{MN}$, where $N(5;4;2)$.

Solution:

Let's find the vector $\overline{MN}$, and its guiding cosines:

$\overline{MN}=l=(5-3)i+(4-2)j+(2-1)k=2i+2j+k$;

$\cos\alpha=2/\sqrt{2^2+2^2+1^2}=2/3;\quad \cos\beta=2/3;\ \cos\gamma=1/3.$

Let's calculate the values of the partial derivatives at point M:

$$\frac{\partial u}{\partial x}=y^2z^3;\ \ \frac{\partial u}{\partial y}=2xyz^3;\ \ \frac{\partial u}{\partial z}=3xy^2z^2;\ \ \left(\frac{\partial u}{\partial x}\right)_M=4;\ \left(\frac{\partial u}{\partial y}\right)_M=12;\ \left(\frac{\partial u}{\partial z}\right)_M=36.$$

Therefore,

$$\frac{\partial u}{\partial l}=4\cdot\frac{2}{3}+12\cdot\frac{2}{3}+36\cdot\frac{1}{3}=22\frac{2}{3}.$$

Exercise 3.

Find the derivative of the function $z=\ln(x^2+y^2)$ at point $M(3;4)$ in the direction of the gradient of the function z.

Solution:

The vector l coincides with the gradient of the function $z=\ln(x^2+y^2)$ at point $M(3;\ 4)$ and is equal to $\operatorname{grad} z=\left(\frac{2x}{x^2+y^2}\right)_M \mathbf{i}+\left(\frac{2y}{x^2+y^2}\right)_M \mathbf{j}=\frac{6}{25}\mathbf{i}+\frac{8}{25}\mathbf{j}.$

Therefore,

$$\frac{\partial z}{\partial l}=|\operatorname{grad} z|=\sqrt{\left(\frac{6}{25}\right)^2+\left(\frac{8}{25}\right)^2}=\frac{2}{5}.$$

Exercise 4.

Find the magnitude and direction of the gradient of the function $u=tg\,x-x+3\sin y-\sin^3 y+z+ctg\,z$ at point $M(\frac{\pi}{4};\frac{\pi}{3};\frac{\pi}{2})$.

Solution:

Find the partial derivatives of

$$\frac{\partial u}{\partial x}=\sec^2 x-1,\ \frac{\partial u}{\partial y}=3\cos y-3\sin^2 y\cos y,\ \frac{\partial u}{\partial z}=1-\operatorname{cosec}^2 z$$

and calculate their values at point $M(\frac{\pi}{4};\frac{\pi}{3};\frac{\pi}{2})$:

$$\left(\frac{\partial u}{\partial x}\right)_M=2-1=1,\ \left(\frac{\partial u}{\partial y}\right)_M=3\cdot\frac{1}{2}-3\left(\frac{\sqrt{3}}{2}\right)^2\cdot\frac{1}{2}=\frac{3}{8},\ \left(\frac{\partial u}{\partial z}\right)_M=1-1=0.$$

Therefore,

$$(\operatorname{grad} u)_M=\mathbf{i}+\frac{3}{8}\mathbf{j};\quad |\operatorname{grad} u|_M=\sqrt{1^2+(3/8)^2}=\sqrt{73}/8;$$

$$\cos\alpha=\frac{1}{\sqrt{73}/8}=\frac{8}{\sqrt{73}};\quad \cos\beta=\sin\alpha=\frac{3}{\sqrt{73}}.\ \blacktriangle$$

6. *Differentiation of implicit functions.*

The derivative of the implicit function $y = y(x)$, given by the equation $F(x,y)=0$, where $F(x,y)$ is a differentiable function of variables x and y, can be calculated using the formula

$y' = -\frac{\partial F/\partial x}{\partial F/\partial y}$, provided $\frac{\partial F}{\partial y} \neq 0$.

Higher-order derivatives of an implicit function can be found by sequentially differentiating the specified formula, while considering y as a function of x.

Similarly, the partial derivatives of an implicit function of two variables $z = \varphi(x,y)$, given by the equation $F(x,y,z)=0$, where $F(x,y,z)$ is a differentiable function of variables x,y and z, can be calculated using the formulas $\frac{\partial z}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z}$, $\frac{\partial z}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$ under the condition $\frac{\partial F}{\partial z} \neq 0$.

Exercise 1.

$$\cos(x+y)+y=0.$$

Find: y'.

Solution:

$F(x,y)=\cos(x+y)+y.$

Let's find $\frac{\partial F}{\partial x} = -\sin(x+y)$, $\frac{\partial F}{\partial y} = -\sin(x+y)+1.$

Hence $y' = -\frac{-\sin(x+y)}{1-\sin(x+y)} = \frac{\sin(x+y)}{1-\sin(x+y)}.$

Exercise 2.

$$y - \sin y = x.$$

Find: y' and y''.

Solution:

$F(x, y) = y - \sin y - x.$

We have $\frac{\partial F}{\partial x} = -1$, $\frac{\partial F}{\partial y} = 1 - \cos y = 2\sin^2\frac{y}{2}$, where $y' = -\frac{-1}{2\sin^2(y/2)} = \frac{1}{2}\operatorname{cosec}^2\frac{y}{2}$.

Let's find the second derivative:

$$y'' = \frac{1}{2}\cdot 2\operatorname{cosec}\frac{y}{2}\left(-\operatorname{cosec}\frac{y}{2}\operatorname{ctg}\frac{y}{2}\right)\cdot\frac{1}{2}y' = -\frac{1}{4}\operatorname{cosec}^4\frac{y}{2}\operatorname{ctg}\frac{y}{2}.$$

Exercise 3.

$$z^3 - 3xyz = a^3.$$

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

$$F(x, y, z) = z^3 - 3xyz - a^3.$$

Let's find $\frac{\partial F}{\partial x} = -3yz,\ \frac{\partial F}{\partial y} = -3xz,\ \frac{\partial F}{\partial z} = 3z^2 - 3xy$.

Then $\frac{\partial z}{\partial x} = -\frac{-3yz}{3z^2 - 3xy} = \frac{yz}{z^2 - xy};\ \frac{\partial z}{\partial y} = -\frac{-3xz}{3z^2 - 3xy} = \frac{xz}{z^2 - xy}$.

Exercise 4.

$$xyz = x + y + z.$$

Find dz.

Solution:

We know $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$, .

At first, we find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial x} = \frac{yz - 1}{xy - 1},\ \frac{\partial z}{\partial y} = -\frac{xz - 1}{xy - 1}.$$

Therefore

$$dz = -\frac{1}{xy - 1}\left[(yz - 1)\,dx + (xz - 1)\,dy\right].$$

1.3. **Tangent plane and normal to the surface.**

A tangent plane to a surface at point M is a plane passing through a point M of the surface if the angle between this plane and the secant passing through point M to any point M_1 of the surface tends to zero when M_1 tends to M.

The tangent plane to the surface at point M contains tangents to all curves drawn on the surface through the point M.

The normal to the surface at point M is a straight line passing through M perpendicular to the tangent plane at this point.

If the surface is given by the equation $F(x,y,z)=0$ and at point $M(x_0,y_0,z_0)$ the partial derivatives of $\left(\frac{\partial F}{\partial x}\right)_M, \left(\frac{\partial F}{\partial y}\right)_M, \left(\frac{\partial F}{\partial z}\right)_M$ are finite and do not vanish at the same time, then the equation of the tangent plane to the surface at point $M(x_0,y_0,z_0)$ is written as $\left(\frac{\partial F}{\partial x}\right)_M (x-x_0)+\left(\frac{\partial F}{\partial y}\right)_M (y-y_0)+\left(\frac{\partial F}{\partial z}\right)_M (z-z_0)=0$, and the equation of the normal to the surface at the same point is written as

$$\frac{x-x_0}{\left(\frac{\partial F}{\partial x}\right)_M}=\frac{y-y_0}{\left(\frac{\partial F}{\partial y}\right)_M}=\frac{z-z_0}{\left(\frac{\partial F}{\partial z}\right)_M}.$$

If the equation of the surface is given explicitly: $z=f(x,y)$, where the partial derivatives $\left(\frac{\partial z}{\partial x}\right)_M$ *and* $\left(\frac{\partial z}{\partial y}\right)_M$ at point $M(x_0,y_0,z_0)$ are finite (and can be equal to zero at the same time), then the equation of the tangent plane at point M is written as

$$z-z_0=\left(\frac{\partial z}{\partial x}\right)_M (x-x_0)+\left(\frac{\partial z}{\partial y}\right)_M (y-y_0),$$

and the equation of the normal as

$$\frac{x-x_0}{\left(\frac{\partial z}{\partial x}\right)_M}=\frac{y-y_0}{\left(\frac{\partial z}{\partial y}\right)_M}=\frac{z-z_0}{-1}.$$

Equality to zero, for example, $\left(\frac{\partial z}{\partial x}\right)_M$, means that the tangent plane is parallel to the Ox- axis, and the normal lies in the plane $x=x_0$.

Exercise 1.

The surface is given: $z=x^2-2xy+y^2-x+2y$, . Make up the equation of the tangent plane and the equation of the normal to the surface at point $M(1;\ 1;\ 1)$.

Solution:

Find the partial derivatives $\frac{\partial z}{\partial x}=2x-2y-1$ и $\frac{\partial z}{\partial y}=-2x+2y+2$ and their values at point $M(1;\ 1;\ 1)$.: $\left(\frac{\partial z}{\partial x}\right)_M=-1,\ \left(\frac{\partial z}{\partial y}\right)_M=2.$

The equation is tangent to the plane: $z-1=-(x-1)+2(y-1)$, or $x-2y+z=0.$

The normal equation is: $(x-1)/(-1)=(y-1)/2=(z-1)/(-1).$

Exercise 2.

To draw tangent planes parallel to the plane $x+y+z=1$ to the surface $x^2+2y^2+3z^2=11$.

Solution:

$F(x,\ y,\ z)=x^2+2y^2+3z^2-11.$

Find partial derivatives: $\frac{\partial F}{\partial x}=2x,\ \frac{\partial F}{\partial y}=4y,\ \frac{\partial F}{\partial z}=6z.$.

From the condition of parallelism of the tangent plane and the given plane, it follows that $(\partial F/\partial x)/1=(\partial F/\partial y)/1=(\partial F/\partial z)/1$, or $(2x)/1=(4y)/1=(6z)/1.$

By adding surfaces $x^2+2y^2+3z^2=11$ to these equations, we find the coordinates of the tangent points: $M_1(\sqrt{6};\ \sqrt{6}/2;\ \sqrt{6}/3)$ and $M_2(-\sqrt{6};\ -\sqrt{6}/2;\ -\sqrt{6}/3)$.. Therefore, the equations of tangent planes have the form $1\cdot(x\pm\sqrt{6})+1\cdot(y\pm\sqrt{6}/2)+1\cdot(z\pm\sqrt{6}/3)=0$,, that is, $x+y+z+11/\sqrt{6}=0$ and $x+y+z-11/\sqrt{6}=0.$

1.4. The extremum of a function of two independent variables.

1. *The extremum of the function.*

The function $z=f(x,y)$ has a **maximum (minimum)** at point $M_0(x_0,y_0)$ if the value of the function at this point is greater (less) than the value at any other point $M(x,y)$ of some neighborhood of point M_0, that is, $f(x_0,y_0)>f(x,y)$ (respectively, $f(x_0,y_0)<f(x,y)$) for all points $M(x,y)$ satisfying the condition $|M_0M|<\delta$, where δ is a sufficiently small positive the number.

The maximum or minimum of a function is called its **extremum.** The point M_0 at which the function has an extremum is called the **extremum point.**

If the differentiable function $z = f(x,y)$ reaches an extremum at point $M_0(x_0,y_0)$, then its partial derivatives of the first order at this point are zero, that is, $\frac{\partial f(x_0, y_0)}{\partial x}=0, \frac{\partial f(x_0, y_0)}{\partial y}=0$. **(necessary extreme conditions).**

The points at which the partial derivatives are zero are called **stationary points.** Not every stationary point is an extremum point.

Let $M_0(x_0,y_0)$ be the stationary point of the function $z = f(x,y)$. Denote

$$A=\frac{\partial^2 f(x_0, y_0)}{\partial x^2}, \quad B=\frac{\partial^2 f(x_0, y_0)}{\partial x\, \partial y}, \quad C=\frac{\partial^2 f(x_0, y_0)}{\partial y^2}$$

and make up the discriminant $\Delta = AC - B^2$. .

Then:

- if $\Delta > 0$, then the function has an extremum at point M_0, namely a maximum at $A<0$ (or $C<0$) and a minimum at $A>0$ (or $C>0$).
- if $\Delta < 0$, then there is no extremum at the point M_0. **(sufficient conditions for the presence or absence of an extremum)**.
- if $\Delta = 0$, then further investigation is required (doubtful case).

<u>Exercise 1.</u>

Find the extremum of the function: $z = x^2 + xy + y^2 - 3x - 6y.$.

<u>Solution:</u>

We find partial derivatives of the first order:

$$\frac{\partial z}{\partial x} = 2x + y - 3, \frac{\partial z}{\partial y} = x + 2y - 6 .$$

Using the necessary conditions of the extremum, we find the stationary points:

$$\begin{cases} 2x + y - 3 = 0, \\ x + 2y - 6 = 0, \end{cases}$$

where $x=0, y=3;\ M(0;\ 3).$.

We find the values of the second-order partial derivatives at the point M:

$$\frac{\partial^2 z}{\partial x^2}=2,\ \frac{\partial^2 z}{\partial y^2}=2,\ \frac{\partial^2 z}{\partial x\,\partial y}=1$$

We make up the discriminant $\Delta=AC-B^2=2\cdot 2-1=3>0;\ A>0.$.

Therefore, at the point $M(0;3)$, the given function has a minimum. The value of the function at this point is $z_{\min}=-9.$.

Exercise 2.

Find the extremum of the function $z=\frac{1}{2}xy+(47-x-y)\left(\frac{x}{3}+\frac{y}{4}\right)$.

Solution:

We find partial derivatives of the first order:

$$\frac{\partial z}{\partial x}=-\frac{1}{12}y-\frac{2}{3}x+\frac{47}{3},\ \frac{\partial z}{\partial y}=-\frac{1}{2}y-\frac{1}{12}x+\frac{47}{4}.$$

Using the necessary conditions of the extremum, we find the stationary points:

$$\begin{cases}-\frac{1}{12}y-\frac{2}{3}x+\frac{47}{3}=0,\\ -\frac{1}{2}y-\frac{1}{12}x+\frac{47}{4}=0,\end{cases}$$

or

$$\begin{cases}8x+\ \ y=188,\\ x+6y=141.\end{cases}$$

Hence $x=21,\ y=20;$, the stationary point $M\ (21;\ 20).$.

Let's find the values of the second derivatives at the point M :

$$\frac{\partial^2 z}{\partial x^2}=-\frac{2}{3},\frac{\partial^2 z}{\partial y^2}=-\frac{1}{2},\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{12}.$$

Then $\Delta=AC-B^2=(-2/3)(-1/2)-(-1/12)^2=1/3-1/144>0.$

Since $A<0$, the function has a maximum at point $M\ (21;\ 20).$: $z_{\max}=282.$.

2. *Conditional extremum. The largest and smallest values of a function in a closed domain.*

The conditional extremum of a function $z = f(x,y)$ is called the extremum of this function, achieved under the condition that the variables x and y are connected by the equation $\varphi(x,y)=0$ **(coupling equation).**

Finding a conditional extremum can be reduced to examining the usual extremum of the so-called **Lagrange function** $u = f(x,y)+\lambda\varphi(x,y)$, where λ is an indefinite constant multiplier.

The necessary conditions for the extremum of the Lagrange function have the form:

$$\begin{cases} \dfrac{\partial u}{\partial x}=\dfrac{\partial f}{\partial x}+\lambda\dfrac{\partial \varphi}{\partial x}=0, \\ \dfrac{\partial u}{\partial y}=\dfrac{\partial f}{\partial y}+\lambda\dfrac{\partial \varphi}{\partial y}=0, \\ \varphi(x, y)=0. \end{cases}$$

From this system of three equations, the unknowns , x,y *and* λ can be found.

In order to find the **largest or smallest values of a function** in a closed domain, it is necessary:

1) find stationary points located in a given area and calculate the values of the function at these points.

2) find the largest and smallest values of the function on the lines forming the boundary of the region.

3) choose the largest and smallest of all the values found.

Exercise 1.

Find the extremum of the function $z=xy$, provided that x and y are connected by the equation $2x+3y-5=0$.

Solution:

Consider the Lagrange function $u=xy+\lambda(2x+3y-5)$..

We have $\dfrac{\partial u}{\partial x}=y+2\lambda,$. $\dfrac{\partial u}{\partial y}=x+3\lambda.$

From the system of equations (necessary conditions of the extremum)

$$\begin{cases} y+2\lambda=0, \\ x+3\lambda=0, \\ 2x+3y-5=0 \end{cases}$$

we find $\lambda=-5/12,\ x=5/4,\ y=5/6.$.

It is not difficult to see that at point $(5/4;5/6)$ the function $z=xy$ reaches the highest value of $z_{\max}=25/24.$

Exercise 2.

Of all right-angled triangles with a given area S, find one whose hypotenuse has the least value.

Solution:

Let x and y be the legs of a triangle, and z be the hypotenuse.

Since $z^2=x^2+y^2$, the problem boils down to finding the smallest value of the function x^2+y^2, provided that x and y are connected by the equation $xy/2=S$, that is, $xy-2S=0.$. Consider the function $u=x^2+y^2+\lambda(xy-2S)$ and find its partial derivatives $\frac{\partial u}{\partial x}=2x+\lambda y,\ \frac{\partial u}{\partial y}=2y+\lambda x.$.

Since $x>0, y>0$, then from the system of equations $\begin{cases} 2x+\lambda y=0, \\ 2y+\lambda x=0, \\ xy/2=S \end{cases}$ we get the solution $\lambda=-2,\ x=y=\sqrt{2S}.$.

Thus, the hypotenuse matters if the legs of the polygon are equal to each other.