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Analytic Geometry. Vector Algebra

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Vector Algebra

- We introduced terms of directed segments, vectors and coordinate system using fundamentals of geometry as base for out talk
- Now we bring these objects into accurate algebraic form of reasoning called **vector algebra**
- We overlook base definitions of vectors and operations with vectors as these definitions stay unchanged
- Given definitions and features of operations with vectors give us an option to write and transform linear vectorial expressions in pretty algebraic manner
- We start with explanation of number of elements in any introduced
- After this we discuss transformation of the bases
- And finally we generalize our operations with vectors for arbitrary skew-angular bases

Linear Combinations of Vectors I

- Assume that some set of n vectors $\boldsymbol{a}_1 , \ldots , \boldsymbol{a}_n$ is given
- We call it a collection of n vectors, a system of n vectors, or a family of n vectors either
- Now we can compose some vectorial expressions composed of these system of vectors:

$$\alpha_1 \boldsymbol{a}_1 + \ldots + \alpha_n \boldsymbol{a}_n = \sum_{k=1}^{n} \alpha_1 \boldsymbol{a}_1 = \boldsymbol{b}$$

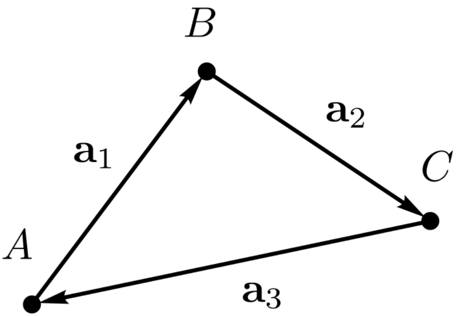
- We call such expression a **linear combination** of vectors $\boldsymbol{a}_1 , \ldots , \boldsymbol{a}_n$
- Real numbers $\alpha_1, \ldots , \alpha_n$ we call the **coefficients of a linear combination**
- Vector $\boldsymbol{b}$ we call the **value of a linear combination**
- There is no for us to multiply liner combination by any real number and to build linear combination with values previous-tier linear combinations of vectors $\boldsymbol{a}_1 , \ldots , \boldsymbol{a}_n$
- This process can be repeated several times

Linear Combinations of Vectors II

- However, upon expanding, upon applying features of multiply by real number, and upon collecting similar terms all such complicated vectorial expressions reduce to linear combinations of the vectors
- Each vectorial expression composed of vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ by means of the operations of addition and multiplication by numbers can be transformed to some linear combination of these vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$.
- We say two linear combinations are assumed to be **coinciding** if only difference between them is order of terms
- We say that linear combination is **trivial** if it is composed of only zero coefficients.
- In opposite case we say tat linear combination is **non-trivial**
- Linear combination is called **vanishing**or **equal with zero** if it is value is zero vectors
- Each trivial linear combination is equal to zero. However, the converse is not valid.

Linear Combinations of Vectors III

- Example:



- Consider $\triangle ABC$ and vectors $\overrightarrow{AB} = \boldsymbol{a}_1,\ \overrightarrow{BC} = \boldsymbol{a}_2,\ \overrightarrow{CA} = \boldsymbol{a}_3$
- Sum of these vectors is zero: $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3 = \mathbf{0}$
- We are free to assign coefficient of 1 to each vector in the expression $1 \cdot \boldsymbol{a}_1 + 1 \cdot \boldsymbol{a}_2 + 1 \cdot \boldsymbol{a}_3 = \mathbf{0}$
- This liear combination is **non-trivial** and **vanishing**

Linear Dependence and Linear Independence

- A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is called **linearly dependent** if there is a **non-trivial** linear combination of these vectors which is **equal to zero**
- A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is called **linearly independent** if there it is not linearly dependent
- System of three vectors from our example is example of linearly dependent system
- Linear dependence is a property of systems of vectors, it is not a property of linear combinations.
- Linear combinations are only tools for revealing the linear dependence
- Along with triviality and non-triviality property of dependence is invariant against transposition of element in system of vectors

Linear Independence Criterion

- A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is linearly independent if and only if vanishing of a linear combination of these vectors implies its triviality.
- Proof:
  - Definition of linear independence means that there is no linear combination of these vectors being non-trivial and being equal to zero simultaneously
  - Indeed, the non-existence of a linear combination of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, being non-trivial and vanishing simultaneously means that a linear combination of these vectors is inevitably trivial whenever it is equal to zero.
  - In other words vanishing of a linear combination of these vectors implies triviality of this linear combination. $\square$
- Alternative form: A system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is linearly independent if and only if non-triviality of a linear combination of these vectors implies that it is not equal to zero

Realtion of Linear Dependence. Properties I

- The vector $\boldsymbol{b}$ is said to be **expressed** as a linear combination of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ if it is the value of some linear combination composed of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.
- For the sake of brevity the vector $\boldsymbol{b}$ is sometimes said to be **linearly expressed** through the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ or to be **expressed in a linear way through** $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.
- Key features of linear dependence
  1. A system of vectors comprising the null vector is linearly dependent;
  2. A system of vectors comprising a linearly dependent subsystem is linearly dependent itself;
  3. If a system of vectors is linearly dependent, then at least one of these vectors is expressed in a linear way through other vectors of this system;
  4. If a system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is linearly independent, while complementing it with one more vector $\boldsymbol{a}_{n+1}$ makes the system linearly dependent, then the vector $\boldsymbol{a}_{n+1}$ is linearly expressed through the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$;
  5. If a vector $\boldsymbol{b}$ is linearly expressed through some m vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ and if each of the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ is linearly expressed through some other n vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$, then the vector $\boldsymbol{b}$ is linearly expressed through the vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$.

Realtion of Linear Dependence. Properties II

- Proof for (1): Suppose that a system of vectors $\boldsymbol{a}_1\ ,\ .\ .\ .\ ,\ \boldsymbol{a}_n$ comprises zero vector. For the sake of certainty we can assume that $\boldsymbol{a}_k = \mathbf{0}$. Let's compose the following linear combination of the vectors our vector system:

$$0 \cdot \boldsymbol{a}_1 + \ldots + 0 \cdot \boldsymbol{a}_{k-1} + 1 \cdot \boldsymbol{a}_k + 0 \cdot \boldsymbol{a}_{k+1} + \ldots + 0 \cdot \boldsymbol{a}_n = \mathbf{0}$$

  This linear combination is non-trivial since the coefficient of vector $\boldsymbol{a}_k$ is nonzero. And its value is equal to zero. Hence, system is linearly dependent. $\square$

Realtion of Linear Dependence. Properties III

- Proof for (2): Since linear dependence is not sensible to the order in which the vectors in a system are enumerated, we can assume that first k vectors form linear dependent subsystem in it. Then there exists some non-trivial liner combination of these k vectors being equal to zero:

$$\alpha_1 \cdot \boldsymbol{a}_1 + \ldots + \alpha_k \cdot \boldsymbol{a}_k = \mathbf{0}$$

Let's expand this linear combination by adding other vectors with zero coefficients:

$$\alpha_1 \cdot \boldsymbol{a}_1 + \ldots + \alpha_k \cdot \boldsymbol{a}_k + 0 \cdot \boldsymbol{a}_{k+1} + 0 \cdot \boldsymbol{a}_n = \mathbf{0}$$

It is obvious that the resulting linear combination is nontrivial, and its value is equal to zero. Hence, system is linearly dependent. $\square$

Realtion of Linear Dependence. Properties IV

- Proof for (3): Linear dependency means that in the equation

$$\alpha_1 \boldsymbol{a}_1 + ... + \alpha_n \boldsymbol{a}_n = \mathbf{0}$$

at least one coefficients, say α_k is nonzero. Since we add term $-\alpha_k \boldsymbol{a}_k$ to both sides of the equations and multiply it by $1/\alpha_k$ we obtain expression:

$$\boldsymbol{a}_k = \frac{-\alpha_1}{\alpha_k}\boldsymbol{a}_1 + \dots \frac{-\alpha_{k-1}}{\alpha_k}\boldsymbol{a}_{k-1} + \frac{-\alpha_{k-+}}{\alpha_k}\boldsymbol{a}_{k-+} + \dots + \frac{-\alpha_n}{\alpha_k}\boldsymbol{a}_n$$

Now we see that the vector α_k is linearly expressed through other vectors of the system.
$\square$

Realtion of Linear Dependence. Properties V

- Proof for (4): Let's consider a linearly independent system of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ such that adding the next vector $\boldsymbol{a}_{n+1}$ to it we make it linearly dependent. There is some nontrivial linear combination of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n, \boldsymbol{a}_{n+1}$ being equal to zero:

$$\alpha_1 \boldsymbol{a}_1 + \ldots + \alpha_n \boldsymbol{a}_n + \alpha_{n+1} \boldsymbol{a}_{n+1} = \mathbf{0}$$

Suppose $\alpha_{n+1} = 0$. In this case we would get the nontrivial linear combination of n vectors being equal to zero:

$$\alpha_1 \boldsymbol{a}_1 + \ldots + \alpha_n \boldsymbol{a}_n = \mathbf{0}$$

This contradicts to the linear independence of the first n vectors$\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.
Hence,$\alpha_{n+1} \neq 0$, and we can repeat the trick already used above:

$$\boldsymbol{a}_{n+1} = \frac{-\alpha_1}{\alpha_{n+1}} \boldsymbol{a}_1 + \ldots + \frac{-\alpha_n}{\alpha_{n+1}} \boldsymbol{a}_n . \square$$

Realtion of Linear Dependence. Properties VI

- Proof for (5):

$$\boldsymbol{b} = \sum_{i=1}^{m} \alpha_i \boldsymbol{a}_i, \qquad \boldsymbol{a}_i = \sum_{j=1}^{n} \gamma_{ij} \boldsymbol{c}_j$$

$$\boldsymbol{b} = \sum_{i=1}^{m} \alpha_i \left(\sum_{j=1}^{n} \gamma_{ij} \boldsymbol{c}_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \alpha_i \gamma_{ij} \right) \boldsymbol{c}_j . \square$$

- Consequence: Any subsystem in a linearly independent system of vectors is linearly independent
- Steinitz Theorem: If the vectors $\boldsymbol{a}_1 , \ldots , \boldsymbol{a}_n$ are linearly independent and if each of them is linearly expressed through some other vectors $\boldsymbol{b}_1 , \ldots , \boldsymbol{b}_m$, then $m \geqslant n$.
- Steinitz Theorem is very important in studying multidimensional spaces. Here it is provided for reference

Linear Dependence for A Single Vector

- Suppose our system contains only single vector $\boldsymbol{a}_1$
- Linear dependence now means

$$\alpha_1 \boldsymbol{a}_1 = \mathbf{0}, \quad \alpha_1 \neq 0$$

- Thus, $\boldsymbol{a}_1 = \mathbf{0}$
- Suppose now $\boldsymbol{a}_1 = \mathbf{0}$
- Therefore $1 \cdot \boldsymbol{a}_1 = \mathbf{0}$, and this system of single vector is linearly dependent
- A system composed of a single vector $\boldsymbol{a}_1$ is linearly dependent if and only if this vector is zero vector.

Linear Dependence for A Pair of Vectors I

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- Linear dependence now means

$$\alpha_1 \boldsymbol{a}_1 + \alpha_2 \boldsymbol{a}_2 = \mathbf{0}, \quad \alpha_1 \neq 0, \text{ or } \alpha_2 \neq 0$$

- Since the linear dependence is not sensitive to the order of vectors in a system, without loss of generality we can assume that $\alpha_1 \neq 0$, ad express $\boldsymbol{a}_1$:

$$\boldsymbol{a}_1 = \frac{\alpha_2}{\alpha_1} \boldsymbol{a}_2 = \beta_2 \boldsymbol{a}_2$$

- There is tree possibilities:
  - $\beta_2 > 0$, thus $\boldsymbol{a}_2 \upuparrows \boldsymbol{a}_1$
  - $\beta_2 < 0$, thus $\boldsymbol{a}_2 \uparrow\downarrow \boldsymbol{a}_1$
  - $\beta_2 = 0$ thus $\boldsymbol{a}_1 = \mathbf{0}$ and has undefined direction
- As a summary $\boldsymbol{a}_2 || \boldsymbol{a}_1$

Linear Dependence for A Pair of Vectors II

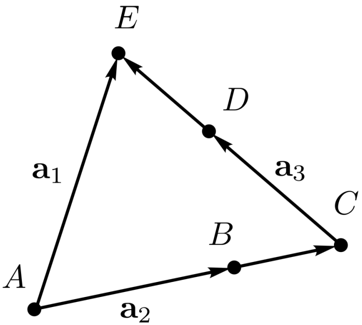
- Suppose $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are collinear.
- It means that there is real number β_2 and $\boldsymbol{a}_1 = \beta_2 \boldsymbol{a}_2$
- Thus there is non-trivial linear combination:

$$1 \cdot \boldsymbol{a}_1 + (-\beta_2) \cdot \boldsymbol{a}_2 = \mathbf{0}$$

- The existence of such a linear combination means that the vectors are linearly dependent. Thus, the converse proposition that the collinearity of two vectors implies their linear dependence is proved.
- A system of two vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is linearly dependent if and only if these vectors are collinear.

Linear Dependence for A Triplet of Vectors I

- Consider a system contained three of vectors: $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$



- Assume that it is linearly dependent
- Properties of linear dependency grant us expression:

$$\boldsymbol{a}_1 = \beta_2 \boldsymbol{a}_2 + \beta_3 \boldsymbol{a}_3$$

- Let A be arbitrary point and $\boldsymbol{a}_2 = \overrightarrow{AB}$, $\beta_2 \boldsymbol{a}_2 = \overrightarrow{AC}$
- Starting in point C we may build directed segments $\boldsymbol{a}_3 = \overrightarrow{CD}$, and $\beta_3 \boldsymbol{a}_3 = \overrightarrow{CE}$

- By the low of triangle of addition: $\boldsymbol{a}_1 = \overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CE}$
- Provided illustration supposes $\beta_2 > 0$ and $\beta_3 >$. Plot corresponding illustrations for other possible combinations of signs for nonzero betas as **home assignment**

Linear Dependence for A Triplet of Vectors II

- Three points A, C, and E successfully shape a plane. And, as AB overlaps AC and CD overlaps CE, vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ are coplanar with this plane
- Taking any point A' not laying in this plane shapes family corresponding directed segments on parallel plane
- The linear dependence of three vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ implies their coplanarity.
- Here ve assumed general case: $\boldsymbol{a}_1 \neq \mathbf{0}$, $\boldsymbol{a}_2 \neq \mathbf{0}$, and $\boldsymbol{a} \nparallel \boldsymbol{a}_2$.
- Consider special cases as **home assignment**

Linear Dependence for A Triplet of Vectors III

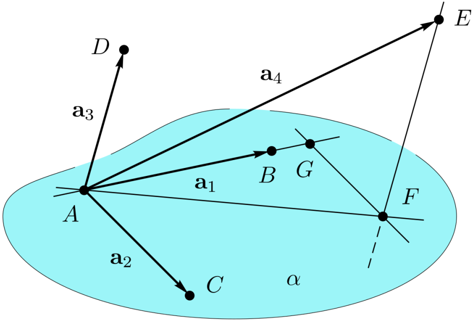
- The coplanarity of three vectors $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, and $\boldsymbol{a}_3$ imply their linear dependence.
- Proof:
  - Let $\boldsymbol{a}_1 = \mathbf{0}$ and $\boldsymbol{a}_2 = \mathbf{0}$, thus first property of linear dependence is fulfilled
  - Let $\boldsymbol{a}_1 \neq \mathbf{0}$, $\boldsymbol{a}_2 \neq \mathbf{0}$, but $\boldsymbol{a} \parallel \boldsymbol{a}_2$. This means their linera dependence Thus we demonstrated linear dependent subsystem and second property of linear dependence is fulfilled.
  - Consider general case
  - Let A be arbitrary point and $\overrightarrow{AB} = \boldsymbol{a}_3$, $\overrightarrow{AC} = \boldsymbol{a}_2$, $\overrightarrow{AD} = \boldsymbol{a}_1$
  - Coplanarity means that all four points A, B, C, and D lie on the same plane α
  - Consider two crossing lines: $A \in p$, $C \in p$, $D \in q$, $q \parallel AB$
  - Cross point of these lines is arbitrary point E $\overrightarrow{AE} \parallel \overrightarrow{AE}$, and $\overrightarrow{ED} \parallel \overrightarrow{AB}$
  - $\boldsymbol{a}_1 = \overrightarrow{AD} = \overrightarrow{AE} + \overrightarrow{ED} = \beta_2 \boldsymbol{a}_2 + \beta_3 \boldsymbol{a}_3$
  - $1 \cdot \boldsymbol{a}_1 + (-\beta_2) \cdot \boldsymbol{a}_2 + (-\beta_3) \cdot \boldsymbol{a}_3 = \mathbf{0}.\square$
- A system of three vectors is linearly dependent if and only if these vectors are coplanar.

Linear Dependence for More than Tree Vectors I

- Consider a system contained three of vectors: $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$, and $\boldsymbol{a}$

  This system is linearly dependent
- Proof:
  - Suppose subsystem $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ is linearly independent. Opposite case means that general system is linearly independent too.
  - Thus, these vectors are non-coplanar and any pair of them is also linearly independent as well as any single vector.
  - $\boldsymbol{a}_1 \neq \mathbf{0}$, $\boldsymbol{a}_2 \neq \mathbf{0}$, $\boldsymbol{a} \neq \mathbf{0}$, and $\boldsymbol{a} \nparallel \boldsymbol{a}_2$

Linear Dependence for More than Tree Vectors II



- Consider arbitrary point in space A
- We establish directed segments: $\overrightarrow{AB} = \boldsymbol{a}_1$, $\overrightarrow{AC} = \boldsymbol{a}_2$, $\overrightarrow{AD} = \boldsymbol{a}_3$, $\overrightarrow{AE} = \boldsymbol{a}_4$
- Points A, B, C shape a plane α
- $D \notin \alpha$, and $\boldsymbol{a}_3$ is not coplanar with α
- Consider $EF \parallel \boldsymbol{a}_3$, and $F \in \alpha$
- $\boldsymbol{a}_4 = \overrightarrow{AE} = \overrightarrow{AF} + \overrightarrow{FE} = \overrightarrow{AF} + \beta_3 \boldsymbol{a}_3$
- Let $FG \parallel \boldsymbol{a}_2$, and G is cross point with line established by $\boldsymbol{a}_1$
- $\overrightarrow{AF} = \overrightarrow{AG} + \overrightarrow{GF} = \beta_1 \boldsymbol{a}_1 + \beta_2 \boldsymbol{a}_2$
- $\boldsymbol{a}_4 = \beta_1 \boldsymbol{a}_1 + \beta_2 \boldsymbol{a}_2 + \beta_3 \boldsymbol{a}_3$
- $1 \cdot \boldsymbol{a}_4 + (-\beta_1) \cdot \boldsymbol{a}_1 + (-\beta_2) \cdot \boldsymbol{a}_2 + (-\beta_3) \cdot \boldsymbol{a}_3 = \mathbf{0}.$ $\square$

Linear Dependence for More than Tree Vectors III

- Any system consisting of more than four vectors in the space is linearly dependent
- Proof: we have proven that any system of four vectors is linearly dependent. Thus, in any system of more than four system arbitrary subsystem of four vectors may be selected, and it is always linearly dependent. Therefore, system itself is linearly dependent

Linear Combinations and Bases. Analytic Geometry of Line I

- Alternative notation for our abstract vector id **free vector**
- If we bring some restriction on location in space for a free vector, we say that it is **partially free vector**. E.g. vector must lay on specified line or plane
- Suppose a is arbitrary line in space
- Each vector parallel with a shapes directed segment laying on a with selected point as an initial point
- For more clear explanation we may restrict our set of vectors parallel with a with vectors laying on a

Linear Combinations and Bases. Analytic Geometry of Line II

- For any vector $\boldsymbol{e}$ laying on a all vectors $\boldsymbol{x}$ laying on a are expressed with formula

$$\boldsymbol{x} = x\boldsymbol{e}$$

- Thus $\boldsymbol{e}$ and selected origin point shape basis along this line and x is coordinate of vector $\boldsymbol{x}$ and coordinate of point corresponding with its endpoint
- Adding of any other vector $\boldsymbol{f}$ parallel with that line shapes linearly dependent system of vectors
- We agree that length of $\boldsymbol{e}$ is unity without any leak of generalization

Linear Combinations and Bases. Analytic Geometry of Line III

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- Transformation this basis
  - Change of origin. Suppose we need to transit the origin into point O' with coordinate a with respect to "old" basis
$$x = x' + a$$
$$x' = x - a$$
  - Change the direction. Suppose we need to change direction of $\boldsymbol{e}$ to opposite one
$$x = -x'$$
$$x' = -x$$
  - Change the scale. Suppose there is invariant scale segment and length of our "unit vector" $\boldsymbol{e}$ is e with respect to this scale. For any codirected with $\boldsymbol{e}$ vector $\boldsymbol{e}'$ with length e' there is equation of change the scale
$$x = \frac{e'}{e}x'$$
$$x' = \frac{e}{e'}x$$

Linear Combinations and Bases. Analytic Geometry of Line IV

- Some operations with coordinates on a line
  - Length of radius vectors

$$\boldsymbol{a} \mapsto (a)$$
$$\boldsymbol{a}^2 = a\boldsymbol{e} \cdot a$$
$$bvcte = a^2 |\boldsymbol{a}| = |a|$$

  - Distance between points

$$A \mapsto \overrightarrow{OA} \mapsto (a)$$
$$B \mapsto \overrightarrow{OB} \mapsto (b)$$
$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} \mapsto (b - a)$$
$$AB = |b - a|$$

Example: $A \mapsto (-3)$, $B \mapsto (+4)$, thus $AB = 7$

Linear Combinations and Bases. Analytic Geometry of Line V

- Split of the segment. Suppose points A and B have coordinates (x_1) and (x_2) respectively, $x_1 < x_2$ without any leak of generalization. Point C with coordinate $x_1 < x < x_2$ splits this segment. This relation of coordinates origins from axiom of measurement

$$\lambda = \frac{AC}{CB} = \frac{x - x_1}{x_2 - x}$$
$$\lambda(x_2 - x) = x - x_1$$
$$x_2 + x_1 = x(1 + \lambda)$$
$$x = \frac{x_2 + x_1}{1 + \lambda}$$

Particular case: coordinate of center. $\lambda = 1$, thus $x = \dfrac{x_2 + x_1}{2}$

- Functional dependence of coordinates
  - Now we discuss only algebraic form dependence for coordinates, but this dependence also may have form of any dynamic system, discrete or continuous one
  - This dependence may have form of explicit equation for coordinate:

    $$F(x) = 0,$$

    or be some sequence:

    $$x_i = f(x_{i-1}), \quad i = 1, 2, .., \text{ and } x_0 \text{ is given}$$

    or be some parametrized equation:

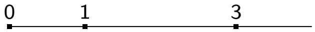
    $$x = f(t),$$

    t here is parameter with domain of arbitrary segment of real numbers' axis.

Linear Combinations and Bases. Analytic Geometry of Line VII

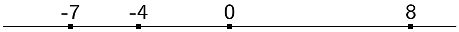
- Examples
  - $x^3 - 4x^2 + 3x = 0$

$$x_1 = 0$$
$$x^2 - 4x + 3 = 0$$
$$x_2 = 1; \quad x_3 = 3$$



  - Uniform linear motion: $x = tv + c$; $v = \text{const}$, $c = \text{const}$

$$x = 3t - 7, \quad t = 0, t = 1, t = 5$$



  - Geometric progression: $x_i = x_{i-1} \cdot q$, $q = \text{const}$
    As **home assignment** plot first 4 elements of geometric progression for $x_0 = 2$, $q = 1/2$

Problem for home assignment Transform coordinates on a line to (1) increase scale twice ($e'/e = 2$), and with respect to original basis points with coordinates $x < -7$ became all positive and points with coordinates $x > -7$ become all negative.