

# Home assignment 1

**Problem 1.** Find the locus of points for each of which the tangents drawn to the parabola are mutually perpendicular.

**Solution.**

Let consider equation of parabola in canonical form:

$$y^2 = 2px.$$

The equation of tangent line passing the point  $(x, y)$  is

$$yy^* = p(x + x^*).$$

Point  $(x^*, y^*)$  is the tangent point and satisfies the equation of parabola:

$$y^{*2} = 2px^*.$$

So, we can obtain the quadratic equation in unknown  $y^*$ :

$$y^{*2} - 2yy^* + 2px = 0,$$

that gives us two roots and two points of tangent.

The condition that two tangent lines intersect at  $(x, y)$  and perpendicular is:

$$y_1^*y_2^* = -p^2,$$

here  $y_{1,2}^*$  are the roots of quadratic equation. So, we have:

$$-p^2 = 2px \quad \Rightarrow \quad x = -\frac{p}{2},$$

the desired locus of points is the directrix of parabola. It also satisfies the condition  $D > 0$  for quadratic equation, since  $D = 4y^2 + 4p > 0$ .

**Problem 2.** Find the equation of an ellipse whose foci are  $(-3, 0)$  and  $(3, 0)$  and which touches the line  $x + y - 5 = 0$ .

### Solution.

Due to the location of focal axis we can write the equation of ellipse in canonical form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

$$c = \pm 3 \Rightarrow a^2 - b^2 = 9 \Rightarrow b^2 = a^2 - 9.$$

Let write the general equation of tangent to ellipse:

$$\frac{x_0x}{a^2} + \frac{y_0y}{a^2 - 9} = 1 \Rightarrow (a^2 - 9)x_0x + a^2y_0y - a^2(a^2 - 9) = 0.$$

We can write the proportion equations in form:

$$(a^2 - 9)x_0 = \lambda, \quad a^2y_0 = \lambda, \quad a^2(a^2 - 9) = 5\lambda.$$

From here

$$\lambda = \frac{a^2(a^2 - 9)}{5} \Rightarrow x_0 = \frac{a^2}{5}, \quad y_0 = \frac{a^2 - 9}{5}.$$

Point  $(x_0, y_0)$  lying on ellipse, so we have

$$\begin{aligned} \frac{a^4}{25a^2} + \frac{(a^2 - 9)^2}{25(a^2 - 9)} &= 1 \Rightarrow a^2 + a^2 - 9 = 25 \Rightarrow \\ a^2 = 17, \quad b^2 &= 8 \Rightarrow \\ \frac{x^2}{17} + \frac{y^2}{8} &= 1. \end{aligned}$$

**Problem 3.** Find the equations of two conjugate hyperbolas, knowing that the distance between the directrices of the first of them is 7.2 and the distance between the directrices of the second is 12.8.

**Solution.**

Let two conjugate hyperbolas have equations:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1.$$

Distance between two directrices can be found from formulas:

$$\Delta_1 = \frac{2a}{e_1} = \frac{2a^2}{c}, \quad \Delta_2 = \frac{2b}{e_2} = \frac{2b^2}{c}.$$

$$a^2 = \frac{c\Delta_1}{2}, \quad b^2 = \frac{c\Delta_2}{2}.$$

The value  $c$  would be the same for both hyperbolas.

$$\begin{aligned} a^2 + b^2 = c^2 &= \frac{c}{2}(\Delta_1 + \Delta_2) \quad \Rightarrow \quad c = \frac{\Delta_1 + \Delta_2}{2} = 10, \\ a^2 &= 36, \quad b^2 = 64, \\ \frac{x^2}{36} - \frac{y^2}{64} &= \pm 1. \end{aligned}$$

**Problem 4.** Show that the product of the distances from a point  $P$  on a hyperbola to the asymptotes has the same value for all  $P$ 's.

**Solution.**

Let consider the canonical equation of hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad x^2b^2 - y^2a^2 = a^2b^2.$$

Equations of asymptotes:

$$ay - bx = 0, \quad ay + bx = 0.$$

Let's find the distances from point  $P(x, y)$  to asymptotes:

$$\rho_1 = \frac{|ay - bx|}{\sqrt{a^2 + b^2}}, \quad \rho_2 = \frac{|ay + bx|}{\sqrt{a^2 + b^2}}.$$

$$\rho_1\rho_2 = \frac{|ay - bx| \cdot |ay + bx|}{a^2 + b^2} = \frac{|a^2y^2 - b^2x^2|}{a^2 + b^2} = \frac{a^2b^2}{a^2 + b^2}.$$

So,  $\rho_1\rho_2 = \text{const.}$   $\square$

**Problem 5.** Show that confocal ellipse and hyperbola (i.e. ellipse and hyperbola having common foci) intersect orthogonally. *Remark:* the angle at which the curves intersect is equal to the angle between the tangents to them drawn at the intersection point.

### Solution.

Here we can use the optical properties of conics and theorems on tangent lines to ellipse and hyperbola which bisect respectively the outer and inner angle at tangent point (see Fig. 1).

From figure  $2\alpha + 2\beta = 180^\circ$ , hence  $\alpha + \beta = 90^\circ$ .

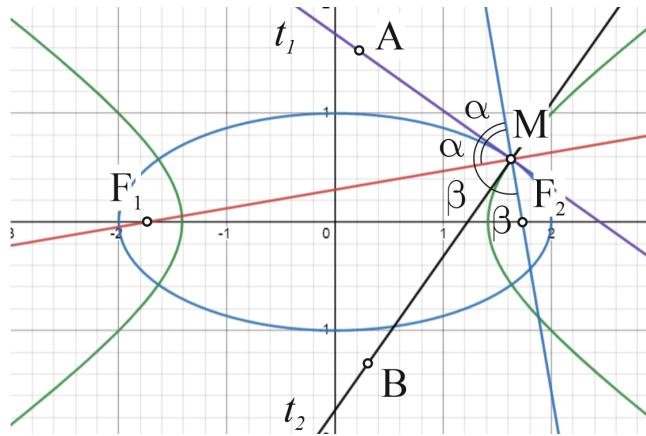


Figure 1