

Measure and Integral. Examples.

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1 Calculation of the integral of a function of two variables.

Theorem 1.1. Assume that $\Omega \subset \mathbb{R}^2$ and $f : \Omega \rightarrow \mathbb{R}$ is such that

$$\iint_{\Omega} |f(x, y)| dx dy < +\infty.$$

Then

$$\iint_{\Omega} f(x, y) dx dy = \int_{\mathbb{R}} dx \int_{\Omega_1(x)} f(x, y) dy = \int_{\mathbb{R}} dy \int_{\Omega_2(y)} f(x, y) dx,$$

where

$$\Omega_1(x) = \{y : (x, y) \in \Omega\} \text{ and } \Omega_2(y) = \{x : (x, y) \in \Omega\}$$

are sections of the set Ω by variables x and y respectively.

Below we assume that function f is bounded and continuous a.e. or nonnegative.

1.1 Calculation methods

1.1.1 Calculation of double integral as iterated integral.

Let $\Omega = \{(x, y) : x_1 \leq x \leq x_2, y_1(x) \leq y \leq y_2(x)\}$. Then

$$\iint_{\Omega} f(x, y) dx dy = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} f(x, y) dy.$$

1.1.2 Change of the variable

Consider the C^1 -smooth change of variables

$$x = x(u, v), \quad y = y(u, v),$$

that transforms Ω' (in variables u, v, w) to Ω (in variables x, y). Assume that Jacobian of the change is not zero, that is

$$|J(u, v)| = \left| \det \begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix} \right| \neq 0.$$

Then

$$\iint_{\Omega} f(x, y) dx dy = \iint_{\Omega'} f(x(u, v), y(u, v)) |J(u, v)| du dv. \quad (1)$$

1.1.3 Example of changes and their Jacobians.

1. Affine change. Let $(x, y)^T = A(u, v)^T + b$, where A is 2×2 matrix, $\det A \neq 0$, and $b \in \mathbb{R}^2$. Then $|J(u, v)| = |\det A|$.

2. Polar change Let $r > 0$, $\varphi \in [0, 2\pi)$ and

$$x = r \cos \varphi, \quad y = r \sin \varphi.$$

Then $|J| = r$.

3. Changes of coordinates can be strange. Let $a, b > 0$, $\alpha > 0$, and

$$x = ar \cos^\alpha \varphi, \quad y = br \sin^\alpha \varphi,$$

where $r > 0$, $\varphi \in [0, 2\pi)$. Then

$$|J| = ab\alpha r \cos^{\alpha-1} \varphi \sin^{\alpha-1} \varphi.$$

1.2 Examples with solutions.

Problem 1. Change the order of integration in the following integral

$$I = \int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x, y) dy.$$

Solution. The domain of integration is

$$\Omega = \{(x, y) : 1 \leq x \leq 2, 2 - x \leq y \leq \sqrt{2x - x^2}\}.$$

If $(x, y) \in \Omega$ then

$$0 = \min_{x \in [1, 2]} (2 - x) \leq y \leq \max_{x \in [1, 2]} \sqrt{2x - x^2} = 1.$$

Let $0 \leq y \leq 1$ and $(x, y) \in \Omega$. Then

- $2 - x \leq y$ and $2 - y \leq x$.
- $y \leq 2x - x^2 = 1 - (x - 1)^2$ and $x \leq 1 + \sqrt{1 - y}$ (since $1 \leq 2 - y \leq x$).

Consequently,

$$I = \int_0^1 dy \int_{2-y}^{1+\sqrt{1-y}} f(x, y) dx.$$

Problem 2. Calculate

$$I = \iint_{\substack{0 \leq x \leq \pi \\ 0 \leq y \leq \pi}} |\cos(x + y)| dx dy.$$

Solution. Applying Fubini (or Tonelli) theorem we see that

$$I = \int_0^\pi dx \int_0^\pi |\cos(x+y)| dy = 2 \int_0^\pi dx = 2\pi$$

since $|\cos t|$ is π -periodic and

$$\begin{aligned} I &= \int_0^\pi |\cos(x+y)| dy = \int_x^{x+\pi} |\cos t| dt = \int_\pi^{x+\pi} |\cos t| dt + \int_x^\pi |\cos t| dt = \\ &\quad \int_0^x |\cos t| dt + \int_x^\pi |\cos t| dt = \int_0^\pi |\cos t| dt = 2. \end{aligned}$$

Problem 3. Calculate the integral

$$I = \iint_{\pi^2 \leq x^2 + y^2 \leq 4\pi^2} \sin(\sqrt{x^2 + y^2}) dx dy.$$

Solution. Consider the polar change

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad r > 0, \quad \varphi \in [0, 2\pi).$$

Then, the jacobian of the change $J = r$, and

$$I = \iint_{\pi \leq r \leq 2\pi} \sin(r) r dr d\varphi = \int_0^{2\pi} d\varphi \int_\pi^{2\pi} r \sin r dr = 2\pi \cdot \int_\pi^{2\pi} r \sin r dr = -6\pi^2$$

since

$$\begin{aligned}\int_{\pi}^{2\pi} r \sin r dr &= \int_{\pi}^{2\pi} r \sin d(-\cos r) = \\ &= -r \cos r \Big|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \cos r dr = -3\pi + \sin r \Big|_{\pi}^{2\pi} = -3\pi.\end{aligned}$$

Problem 4. Let $a > 0$. Calculate the area of the set

$$\Omega = \{(x, y) : (x - y)^2 + x^2 \leq a^2\}.$$

Solution. Consider the linear change

$$u = x, \quad y = u - v.$$

Its Jacobian is equal to

$$|J| = \left| \det \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right| = 1.$$

Consequently,

$$S = \iint_{u^2+v^2 \leq a^2} dudv = \iint_{0 < r < a} r dr d\varphi = \int_0^{2\pi} d\varphi \int_0^a r dr = \pi a^2.$$

Problem 5. Calculate the area of the set

$$\Omega = \{(x, y) : (x^3 + y^3)^2 < x^2 + y^2, \quad x, y > 0\}.$$

Solution. Consider the polar change

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad \varphi \in \left(0, \frac{\pi}{2}\right), \quad r > 0.$$

Then the set Ω is defined by inequality

$$r^6(\cos^3 \varphi + \sin^3 \varphi)^2 \leq r^2,$$

or, equivalently,

$$0 < r < r_0(\varphi) = (\cos^3 \varphi + \sin^3 \varphi)^{-1/2}.$$

Consequently, area of Ω is equal to

$$\begin{aligned} S = \int_0^{\pi/2} d\varphi \int_0^{r_0(\varphi)} r dr &= \frac{1}{2} \int_0^{\pi/2} r_0^2(\varphi) d\varphi = \frac{1}{2} \int_0^{\pi/2} \frac{1}{\cos^3 \varphi + \sin^3 \varphi} d\varphi = \\ &= \frac{\pi}{6} + \frac{\sqrt{2}}{3} \ln(1 + \sqrt{2}). \end{aligned}$$

Problem 6. Calculate the area of the set

$$\Omega = \left\{ (x, y) : 1 \leq \sqrt{x} + \sqrt{y} \leq 2, \quad 1 \leq \frac{y}{x} \leq 4 \right\}.$$

Solution. Consider the change

$$u = \sqrt{x} + \sqrt{y}, \quad v = \sqrt{\frac{y}{x}}.$$

Expressing old variables via new ones we see that

$$x = \frac{u^2}{(1+v)^2}, \quad y = xv^2 = \frac{u^2v^2}{(1+v)^2}.$$

Hence,

$$|J| = \left| \det \begin{pmatrix} \frac{2u}{(1+v)^2} & \frac{-2u^2}{(1+v)^3} \\ \frac{2uv^2}{(1+v)^2} & \frac{2u^2v}{(1+v)^3} \end{pmatrix} \right| = \frac{4u^3v}{(1+v)^4}$$

and

$$\Omega' = \{(u, v) : 1 \leq u \leq 2, \quad 1 \leq v \leq 2\}.$$

Consequently, the area of Ω is equal to

$$S(\Omega) = \iint_{\Omega} dxdy = \int_1^2 4u^3 du \int_1^2 \frac{vdv}{(1+v)^4} = \frac{65}{108}.$$

2 Calculation of the integral of a function of three variables.

Below we assume that function f is bounded and continuous a.e. or nonnegative.

2.1 Calculation methods

2.1.1 Calculation of triple integral as iterated integral.

Let

$$\Omega = \{(x, y, z) : x_1 \leq x \leq x_2, \quad y_1(x) \leq y \leq y_2(x), \\ z_1(x, y) \leq z \leq z_2(x, y)\}.$$

Then

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z) dz.$$

If $\Omega = \{(x, y, z) : x_1 \leq x \leq x_2, (y, z) \in S(x)\}$. Then

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \int_{x_1}^{x_2} dx \iint_{S(x)} f(x, y, z) dy dz.$$

In particular, the volume (with some restriction on Ω can be calculated as the integral of areas of sections, that is

$$\text{Vol}(\Omega) = \iiint_{\Omega} dx dy dz = \int_{x_1}^{x_2} \text{Area}(S(x)) dx.$$

These formulas can be generalized to \mathbb{R}^n , $n > 3$.

2.1.2 Change of the variable

Consider the C^1 -smooth change of variables

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w),$$

that transforms Ω' (in variables u, v, w) to Ω (in variables x, y, z). Assume that Jacobian of the change is not zero, that is

$$|J(u, v, w)| = \left| \det \begin{pmatrix} x'_u & x'_v & x'_w \\ y'_u & y'_v & y'_w \\ z'_u & z'_v & z'_w \end{pmatrix} \right| \neq 0.$$

Then

$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega'} f(x(u, v, w), y(u, v, w), z(u, v, w)) |J(u, v, w)| du dv dw. \quad (2)$$

2.1.3 Example of changes and their Jacobians.

1. Affine change. Let $(x, y, z)^T = A(u, v, w)^T + b$, where A is 3×3 matrix, $\det A \neq 0$, and $b \in \mathbb{R}^3$. Then $|J(u, v, w)| = |\det A|$.

2. Cylindric change Let $r > 0$, $\varphi \in [0, 2\pi)$ and

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z.$$

Then $|J| = r$.

3. Spheric change Let $r > 0$, $\varphi \in [0, 2\pi)$, $\psi \in [-\pi/2, \pi/2)$ and

$$x = r \cos \varphi \cos \psi, \quad y = r \sin \varphi \cos \psi, \quad z = r \sin \psi.$$

Then $|J| = r^2 \cos \psi$.

4. Changes of coordinates can be strange. Let $a, b, c > 0$, $\alpha, \beta > 0$, and

$$x = ar \cos^\alpha \varphi \cos^\beta \psi, \quad y = br \sin^\alpha \varphi \cos^\beta \psi, \quad z = cr \sin^\beta \psi,$$

where $r > 0$, $\varphi \in [0, 2\pi)$, $\psi \in [-\pi/2, \pi/2)$. Then

$$|J| = abc\alpha\beta r^2 \cos^{\alpha-1} \varphi \sin^{\alpha-1} \varphi \cos^{2\beta-1} \psi \sin^{\beta-1} \psi.$$

2.2 Examples with solutions.

Problem 1. Calculate the integral

$$I = \iiint_{\Omega} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}} dx dy dz.$$

$$\Omega = \left\{ (x, y, z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1 \right\}.$$

Solution. Consider the spherical change. Let $r > 0, \varphi \in [0, 2\pi), \psi \in [-\pi/2, \pi/2)$ and

$$x = ar \cos \varphi \cos \psi, \quad y = br \sin \varphi \cos \psi, \quad z = cr \sin \psi.$$

Then

$$\Omega' = \{(r, \varphi, \psi) : 0 < r < 1, \varphi \in [0, 2\pi), \psi \in [-\pi/2, \pi/2)\},$$

$$|J| = abcr^2 \cos \psi$$

and

$$\begin{aligned}
I &= abc \int_0^1 r^2 \sqrt{1-r^2} dr \int_{-\pi/2}^{\pi/2} \cos \psi d\psi \int_0^{2\pi} d\varphi = \\
&4\pi abc \int_0^1 r^2 \sqrt{1-r^2} dr = [r = \sin t] = \\
&4\pi abc \int_0^{\pi/2} \sin^2 t \cos^2 t dt = \pi abc \int_0^{\pi/2} \sin^2 2t dt = \\
&\pi abc \int_0^{\pi/2} \frac{1 - \sin 4t}{2} dt = \frac{\pi^2 abc}{4}
\end{aligned}$$

Problem 2. Calculate the volume of a set Ω bounded by surfaces

$$z = x + y, \ z = xy, \ x + y - 1, \ x = 0, \ y = 0.$$

Solution. Last three equations define in the plane O_{xy} the domain

$$S = \{(x, y) : x, y > 0, \ x + y < 1\}.$$

Notice that $xy \leq (x + y)$ for $(x, y) \in S$ since, in this case, $x(1 - y/2) + y(1 - x/2) \geq 0$.

Consequently,

$$\begin{aligned} \text{Vol}(\Omega) &= \iint_S (x + y - xy) dx dy = \int_0^1 dx \int_0^{1-x} (x + y - xy) dy = \\ &= \int_0^1 \left(xy + (1-x) \frac{y^2}{2} \right) \Big|_0^{1-x} dx = \int_0^1 \left(x(1-x) + \frac{(1-x)^3}{2} \right) dx = \frac{7}{24} \end{aligned}$$

Problem 3. Calculate the volume of a set Ω bounded by the surface defined by the equation

$$(x^2 + y^2 + z^2)^3 = 3xyz.$$

Solution. The set Ω consists of four parts of the equal volume that are situated in octants in which $xyz > 0$. Consider an octant $x, y, z > 0$ and the spherical change in this octant

$$x = r \cos \varphi \cos \psi, \quad y = r \sin \varphi \cos \psi, \quad z = r \sin \psi,$$

where $r > 0$, $\varphi \in (0, \pi/2)$, $\psi \in (0, \pi/2)$.

Then $|J| = r^2 \cos \psi$ and

$$\begin{aligned} \Omega' = \{(r, \varphi, \psi) : 0 < r^3 < 3 \cos(\varphi) \sin(\varphi) \cos^2(\psi) \sin(\psi), \\ \varphi \in (0, \pi/2), \psi \in (0, \pi/2)\}. \end{aligned}$$

Consequently, with notation $r_0^3 = 3 \cos(\varphi) \sin(\varphi) \cos^2(\psi) \sin(\psi)$, we see

that

$$\begin{aligned}
\text{Vol } \Omega &= 4 \int_0^{\pi/2} d\varphi \int_0^{\pi/2} \cos \psi d\psi \int_0^{r_0(\varphi, \psi)} r^2 dr = \\
&= \frac{4}{3} \int_0^{\pi/2} d\varphi \int_0^{\pi/2} r_0^3(\varphi, \psi) \cos(\psi) d\psi = \\
&= 4 \int_0^{\pi/2} \cos \varphi \sin \varphi d\varphi \int_0^{\pi/2} \cos^3 \psi \sin \psi d\psi = \\
&= 4 \frac{\sin^2 \varphi}{2} \left| \frac{(-\cos^4 \varphi)}{4} \right|_0^{\pi/2} = \frac{1}{2}
\end{aligned}$$

Problem 4. Calculate the volume of a set Ω bounded by the surface defined by the equation

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1, \quad x, y, z > 0, \quad a, b, c > 0.$$

Solution. Consider an octant $x, y, z > 0$ and the change of the variable in this octant

$$x = ar \cos^4 \varphi \cos^4 \psi, \quad y = br \sin^4 \varphi \cos^4 \psi, \quad z = cr \sin^4 \psi,$$

$$r > 0, \quad \varphi \in (0, \pi/2), \quad \psi \in (0, \pi/2).$$

Then

$$|J| = 16abcr^2 \cos^3 \varphi \sin^3 \varphi \cos^7 \psi \sin^3 \psi,$$

$$\Omega' = \{(r, \varphi, \psi) : r < 1, \varphi, \psi \in (0, \pi/2)\}$$

and

$$\text{Vol } \Omega = 16abc \int_0^1 r^2 dr \int_0^{\pi/2} \sin^3 \varphi \cos^3 \varphi d\varphi \int_0^{\pi/2} \cos^7 \psi \sin^3 \psi d\psi =$$

$$\frac{16abc}{3} \frac{1}{12} \frac{1}{40} = \frac{abc}{90}.$$

Problem 5. Calculate the integral

$$I = \iiint_{\Omega} \sqrt{x^2 + y^2 + z^2} dx dy dz,$$

where

$$\Omega = \{(x, y, z) : x^2 + y^2 + z^2 < z\}.$$

Solution. Consider the spherical change

$$x = r \cos \varphi \cos \psi, \quad y = r \sin \varphi \cos \psi, \quad z = r \sin \psi,$$

where $r > 0$, $\varphi \in [0, 2\pi)$, $\psi \in [-\pi/2, \pi/2]$.

Then $|J| = r^2 \cos \psi$ and

$$\Omega' = \{(r, \varphi, \psi) : r < \sin \psi, \varphi \in [0, 2\pi), \psi \in [0, \pi/2)\}.$$

Consequently,

$$I = \int_0^{2\pi} d\varphi \int_0^{\pi/2} \cos \psi d\psi \int_0^{\sin \psi} r^3 dr = 2\pi \int_0^{\pi/2} \frac{\sin^4 \psi}{4} \cos \psi d\psi = \frac{\pi}{10} \sin^5 \psi \Big|_0^{\pi/2} = \frac{\pi}{10}.$$

Problem 6. Calculate the volume of a set Ω bounded by surfaces defined by equations

$$z = x^2 + y^2, \quad z = 2(x^2 + y^2), \quad xy = 1, \quad xy = 2, \quad x = 2y, \quad 2x = y.$$

Solution. Consider the change

$$u = \frac{z}{x^2 + y^2}, \quad v = xy, \quad w = \frac{x}{y}.$$

Then, old variables are expressed via new variables as

$$x = \sqrt{vw}, \quad y = \sqrt{v/w} = uv(w + 1/w)$$

and

$$|J| = \left| \det \begin{pmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ 0 & \frac{1}{2}\sqrt{\frac{1}{vw}} & \frac{-1}{2}\sqrt{\frac{v}{w^3}} \\ v(w+1/w) & u(w+1/w) & uv(1-1/w^2) \end{pmatrix} \right| = \frac{v}{2w}(w+1/w).$$

Consequently,

$$\Omega' = \{(u, v) : 1 \leq u \leq 2, 1 \leq v \leq 2\}$$

and

$$\text{Vol } \Omega = \frac{1}{2} \int_1^2 du \int_1^2 v dv \int_{1/2}^2 (w + 1/w) \frac{dw}{w} = 9/4. \quad (3)$$

Solution 2. The volume of Ω can be calculated as the volume of a figure lying between graphs of two functions $z = x^2 + y^2$ and $z = 2(x^2 + y^2)$

$$\text{Vol } \Omega = \iint_{\substack{1 \leq xy \leq 2 \\ 1/2 \leq x/y \leq 2}} (2(x^2 + y^2) - (x^2 + y^2)) dx dy = \iint_{\substack{1 \leq xy \leq 2 \\ 1/2 \leq x/y \leq 2}} (x^2 + y^2) dx dy$$

Consider the change

$$v = xy, \quad w = \frac{x}{y}.$$

Then

$$x = \sqrt{vw}, \quad y = \sqrt{v/w}, \quad x^2 + y^2 = v(w + 1/w)$$

and

$$|J(u, v, w)| = \left| \det \begin{pmatrix} \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{1}{vw}} & \frac{-1}{2}\sqrt{\frac{v}{w^3}} \end{pmatrix} \right| = \frac{1}{2w}.$$

Consequently,

$$\text{Vol } \Omega = \int_1^2 v dv \int_{1/2}^2 (w + 1/w) \frac{dw}{2w} = 9/4.$$

3 Calculation of the integral of a function of n variables.

3.1 Calculation methods

3.1.1 Calculation of multiple integral as iterated integral.

Let

$$\Omega = \left\{ x = (x_1, \dots, x_n) : a_1 \leq x_1 \leq b_1, \quad a_2(x_1) \leq x_2 \leq b_2(x_1), \dots, \right. \\ \left. a_n(x_1, \dots, x_{n-1}) \leq x_n \leq b_n(x_1, \dots, x_{n-1}) \right\}.$$

Then

$$\int_{\Omega} f(x_1, \dots, x_n) dx_1 \dots dx_n = \int_{a_1}^{b_1} dx_1 \int_{a_2(x_1)}^{b_2(x_1)} dx_2 \dots \int_{a_n(x_1, \dots, x_{n-1})}^{b_n(x_1, \dots, x_{n-1})} f dx_n.$$

If

$$\Omega = \left\{ x \in (x_1, \dots, x_n) : (x_1, \dots, x_k) \in \Omega_1, (x_{k+1}, \dots, x_n) \in \Omega(x_1, \dots, x_k) \right\}$$

Then

$$\int_{\Omega} f(x_1, \dots, x_n) dx_1 \dots dx_n = \int_{\Omega_1} dx_1 \dots dx_k \int_{\Omega(x_1, \dots, x_k)} f dx_{k+1} \dots dx_n.$$

3.2 Spherical change in \mathbb{R}^n .

Let

$$\begin{aligned} x_1 &= r \cos \varphi_1, \\ x_2 &= r \sin \varphi_1 \cos \varphi_2, \\ &\dots \\ x_{n-1} &= r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-2} \cos \varphi_{n-1}, \\ x_n &= r \sin \varphi_1 \sin \varphi_2 \dots \sin \varphi_{n-1} \sin \varphi_{n-1} \end{aligned}$$

Then the Jacobian of

$$J = r^{n-1} \sin^{n-2} \varphi_1 \sin^{n-3} \varphi_2 \dots \sin \varphi_{n-2}$$

Problem 1. Calculate

$$I = \int_0^1 \int_0^1 \cdots \int_0^1 (x_1 + x_2 + \dots + x_n)^2 dx_1 dx_2 \dots dx_n$$

.

Solution. Notice that

$$\int_0^1 \int_0^1 \cdots \int_0^1 x_k^2 dx_1 dx_2 \dots dx_n = \int_0^1 x_k^2 dx_k = \frac{1}{3}$$

and

$$\int_0^1 \int_0^1 \cdots \int_0^1 x_k x_j dx_1 dx_2 \dots dx_n = \int_0^1 x_k dx_k \int_0^1 x_j dx_j = \frac{1}{4}.$$

Hence

$$I = \sum_{k=1}^n \int_0^1 \int_0^1 \cdots \int_0^1 x_k^2 dx_1 dx_2 + 2 \sum_{k < j} \int_0^1 \int_0^1 \cdots \int_0^1 x_k x_j dx_1 dx_2 = \frac{n}{3} + \frac{n(n-1)}{2}$$