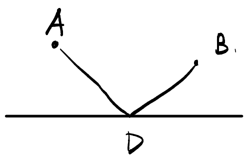
Optimization

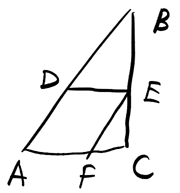
type
- smooth
- constraint

Heron's problem



find D. minimize $(AD + DB)$.

Euclid's problem



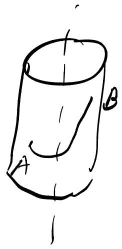
find ADEF (parallelogram). has max area.

Variation Sept 26th.

(1) $J(x) = \int_0^{\frac{\pi}{2}} \left((x')^2 - x^2 \right) dt$, $x(0) = 1$, $x(\frac{\pi}{2}) = 1$

(2) $J(x) = \int_0^1 \left(x(2t - x) \right) dt$, $x(0) = 1$, $x(1) = 2$.

(3) find geodesics on $x^2 + y^2 = R^2$.



find curve min length.

（A,B是任意两点）

(4) $\int_{(0,-1)}^{(1,0)} \left((x')^2 + x^2 + 4x\sin t \right) dt$.

(5) $J(x,y) = \int_{(0,0,0)}^{(1,1,1)} \left(x'^2 + y'^2 - 4ty' - 4y \right) dt$.

(6) $J(x,y) = \int_{(1,1,1)}^{(2,3,4)} \frac{1}{y} \sqrt{1 + (x')^2 + (y')^2}\, dt$.

use an integral of motion.

（拉格朗日方程与t无关时可用）

HW1 Sept 26th.

(1) $J(x) = \int_0^{\frac{\pi}{2}} \left((x')^2 - x^2 \right) dt$, $x(0) = 1$, $x(\frac{\pi}{2}) = 1$

Sol: $\mathcal{L} = (x')^2 - x^2$ independent with t.

thus $x' \mathcal{L}_{x'} - \mathcal{L} = \text{const.}$ i.e. $x' \cdot 2x' - \left[(x')^2 - x^2 \right] = C^2$ $(C > 0)$.

$\Rightarrow (x')^2 + x^2 = C^2$ $\left(\frac{dx}{dt}\right)^2 + x^2 = C^2$

$\Rightarrow$ let $x' = C \cdot \sin\theta$ $\Rightarrow$ then $x = C \cdot \cos\theta$ $\Rightarrow$ $\frac{-C \sin\theta \, d\theta}{dt} = C \cdot \sin\theta$.

$x = C \cdot \cos\theta$ $dx = -C \cdot \sin\theta \, d\theta$. $\theta = -t + \tilde{C}$

$x = C \cdot \cos(-t + \tilde{C})$ apply the boundary condition. we get $x(t) = \sqrt{2} \cos(-t + \frac{\pi}{4})$

check the Jacobi sufficient condition.

1) is stationary point and $\mathcal{L}_{x'x'} = 2 > 0$.

2) $P(t) = 1$ $Q(t) = 1$ $\Rightarrow$ $h'' + h = 0$ s.t. $h(0) = h(\frac{\pi}{2}) = 0$.

general sol is $h = A \cos t + B \sin t$. from $h(0) = 0 \Rightarrow A = 0$, from $h(\frac{\pi}{2}) = 0 \Rightarrow B = 0$ $\}$ $\Rightarrow$ only trivial sol.

no conjugate point. in $[0, \frac{\pi}{2}]$

$J[\sqrt{2}\cos(t - \frac{\pi}{4})] = \int_0^{\frac{\pi}{2}} 2\sin^2(t - \frac{\pi}{4}) - 2\cos^2(t - \frac{\pi}{4}) \, dt = -2 \int_0^{\frac{\pi}{2}} \cos(2t - \frac{\pi}{2}) . dt$.

$= -\sin(2t - \frac{\pi}{2}) \Big|_0^{\frac{\pi}{2}} = -2$ i.e $J_{min} = -2$.

(2) $J(x) = \int_0^1 \left(x(2t - x) \right) dt$, $x(0) = 1$, $x(1) = 2$.

Sol: apply the Euler - Lagrange equation $\mathcal{L}_x - \frac{d}{dt} \mathcal{L}_{x'} = 0$

$\Rightarrow$ $2t - 2x = 0$ $\Rightarrow$ $x = t$ no solution s.t. $x(0) = 1$. $x(1) = 2$

if both ends are free.

$\begin{cases} \mathcal{L}_{x'} = 0 \\ \mathcal{L}_x - \frac{d}{dt}\mathcal{L}_{x'} = 0 \end{cases}$ $\Rightarrow$ $x(t) = t$. $J_{max} = J[x(t) = t] = \int_0^1 t^2 dt = \frac{1}{3}$

no extremum for J on $[0,1]$ if both ends are free / right end free.

(3) find geodesics on $x^2+y^2=R^2$

Sol. parametrization $\vec{r}(\theta, z) = (R\cos\theta, R\sin\theta, z)$ $\quad \theta\in[0,2\pi)$. $z\in\mathbb{R}$.

by arc length. $(ds)^2 = R^2(d\theta)^2 + (dz)^2$

find geodesics is equivalent to find $\min \mathcal{L}[s,\theta,z,\theta',z'] = \int_{s_0}^{s_1} \sqrt{R^2\theta'^2+z'^2}\, ds$.

the Lagrangian: $\mathcal{L}(s,\theta',z',\theta,z) = \sqrt{R^2\theta'^2+z'^2}$ $\quad$ let $\mathcal{L}_1 = R^2\theta'^2+z'^2$

we claim that $\mathcal{L}_1$ will attain the extremum simultaneously with $\mathcal{L}$. so we consider $\mathcal{L}_1$ for convenience.

$\mathcal{L}_1$ independent with θ, z. thus.

$$\begin{cases} \frac{d}{ds}\frac{\partial\mathcal{L}_1}{\partial\theta'} = 0 \\ \frac{d}{ds}\frac{\partial\mathcal{L}_1}{\partial z'} = 0 \end{cases} \Rightarrow \begin{cases} R^2\theta''(s) = 0 \\ z''(s) = 0 \end{cases} \Rightarrow \begin{cases} \theta(s) = as+b \\ z(s) = cs+d \end{cases}$$

thus, the parametric equation of the geodesic is

$\vec{r}(s) = (R\cos(as+b), R\sin(as+b), cs+d)$. $\quad a,b,c,d\in\mathbb{R}$ $\quad$ s. is para. of arc length.

(4) $\int_{(0,-1)}^{(1,0)} ((x')^2 + x^2 + 4x\,\mathrm{sh}t)\, dt$.

Sol: $\mathcal{L}(t,x,x') = (x')^2 + x^2 + 4x\sinh t$.

$\mathcal{L}_x - \frac{d}{dt}\mathcal{L}_{x'} = 0 \Rightarrow 2x + 4\sinh t - 2x'' = 0 \Rightarrow x''-x = 2\sinh t$

i) solve $x''-x=0 \Rightarrow \lambda = \pm 1. \Rightarrow x = Ae^{t}+Be^{-t}$.

ii) solve $x''-x = 2\sinh t$.

$$\begin{cases} A'(t)e^{t} + B'(t)e^{-t} = 0 \\ A'(t)e^{t} - B'(t)e^{-t} = 2\sinh t \end{cases} \Rightarrow \begin{cases} A(t) = \frac{1}{2}[t+\frac{1}{2}e^{-2t}] + C_A \\ B(t) = -\frac{1}{2}[\frac{1}{2}e^{2t}-t] + C_B \end{cases} \Rightarrow \begin{aligned} x = &\tfrac{1}{2}[t\,2\cosh t - \sinh t] \\ &+ C_A e^{t} + C_B e^{-t} \end{aligned}$$

apply the condition $\begin{cases} x(1)=0 \\ x(0)=-1 \end{cases} \Rightarrow \begin{cases} C_A = -\frac{1}{4} \\ C_B = -\frac{3}{4} \end{cases} \Rightarrow x(t) = \frac{1}{2}[te^{t}+\frac{1}{2}e^{-t}-\frac{1}{2}e^{t}+te^{-t}-\frac{1}{2}e^{t}-\frac{3}{2}e^{-t}]$

$= \frac{1}{2}[2t\cosh t - 2\cosh t] = (t-1)\cosh t$

1) $\mathcal{L}_{x'x'} = 2 > 0$.

2) $P(t) = 1 \quad Q(t) = 1. \Rightarrow h''+h = 0$ s.t. $h(0) = h(1) = 0$

similarly as Problem (1). no non-trivial sol. Jacobi sufficient condition satisfied.

Thus, when $x(t) = (t-1)\cosh t$, J attains minimum.

(5) $$J(x,y) = \int_{(0,0,0)}^{(1,1,1)} (x'^2 + y'^2 - 4ty' - 4y)\, dt.$$

$$\begin{cases} \mathcal{L}_x - \frac{d}{dt}\mathcal{L}_{x'} = 0 \\ \mathcal{L}_y - \frac{d}{dt}\mathcal{L}_{y'} = 0 \end{cases} \Rightarrow \begin{cases} x'' = 0 \\ y'' = 0 \end{cases} \Rightarrow \begin{cases} x = at+b \\ y = ct+d \end{cases} \xrightarrow[(0,0,0)]{(1,1,1)} \begin{cases} x = t \\ y = t. \end{cases}$$

$\mathcal{L}_{x'x'} = 2 > 0$ $\quad \mathcal{L}_{y'y'} = 2 > 0$ $\quad \mathcal{L}_{x'y'} = 0.$ $\quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ positive definite

$$Q(t) = \frac{1}{2}\begin{pmatrix} \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{xy} & \mathcal{L}_{yy} \end{pmatrix} - \frac{1}{2}\frac{d}{dt}\begin{pmatrix} \mathcal{L}_{xx'} & \mathcal{L}_{xy'} \\ \mathcal{L}_{yx'} & \mathcal{L}_{yy'} \end{pmatrix} = 0$$

$$Qh - \frac{d}{dt}(Ph') = 0 \Rightarrow \frac{d}{dt}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} u' = 0. \Rightarrow u'' = \begin{pmatrix} (u)_1'' \\ (u)_2'' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} (u_1)_1 \\ (u_1)_2 \end{pmatrix} = \begin{pmatrix} a_{11}t + b_{11} \\ a_{12}t + b_{12} \end{pmatrix} \quad u_2 = \begin{pmatrix} (u_2)_1 \\ (u_2)_2 \end{pmatrix} = \begin{pmatrix} a_{21}t + b_{21} \\ a_{22}t + b_{22} \end{pmatrix}$$

(6) $$J(x,y) = \int_{(1,1,1)}^{(2,3,4)} \frac{1}{y}\sqrt{1 + (x')^2 + (y')^2}\, dt.$$

Sol: since the Lagrangian independent with t, use the integral of motion.

$$x'\mathcal{L}_{x'} + y'\mathcal{L}_{y'} - \mathcal{L} = 0 \quad \text{i.e.} \quad x' \cdot \frac{x'}{y\sqrt{1+x'^2+y'^2}} + y' \cdot \frac{y'}{y\sqrt{1+x'^2+y'^2}} - \frac{\sqrt{1+x'^2+y'^2}}{y} = \text{Const.}$$

$$\Rightarrow \frac{1}{y\sqrt{1+x'^2+y'^2}} = C$$

since $\mathcal{L}$ independent with x, by E-L equation $\frac{d}{dt}\mathcal{L}_{x'} = 0$ i.e. $\frac{d}{dt}\left[\frac{x'}{y\sqrt{1+x'^2+y'^2}}\right] = 0$

$$\Rightarrow \frac{d}{dt}[Cx'] = 0 \Rightarrow x = at+b \xrightarrow[(2,3)]{(1,1)} x = 2t-1$$

thus we have $\frac{1}{y\sqrt{5+y'^2}} = c \Rightarrow y^2 \cdot (5+y'^2) = \frac{1}{c^2} \Rightarrow y'^2 = \frac{1}{c^2y^2} - 5. \Rightarrow y' = \pm\sqrt{\frac{1}{c^2y^2} - 5}$

$$\frac{dy}{dt} = \pm\sqrt{\frac{1}{c^2y^2} - 5} \Rightarrow \frac{dy}{\sqrt{\frac{1}{c^2y^2} - 5}} = \pm dt. \Rightarrow -\frac{1}{5}\sqrt{\frac{1}{c^2} - 5y^2} = \pm t + \tilde{c} \Rightarrow y = \sqrt{\frac{1}{5c^2} - 5(t-\tilde{c})^2}$$

$$\xrightarrow[(2,4)]{(1,1)} y = \sqrt{21 - 5(t-3)^2} \quad \text{thus} \begin{cases} x(t) = 2t-1 \\ y(t) = \sqrt{21 - 5(t-3)^2} \end{cases} t \in [1,2] \quad J \text{ attains minimum.}$$

HW2 Sept 30th

(2) $I(x) = \int_{(0,0)}^{(\frac{\pi}{4},-1)} (x^2 - x'^2 + 10x\sin 2t)\, dt$

Sol: E-L equation: $\mathcal{L}_x - \frac{d}{dt}\mathcal{L}_{x'} = 0 \Rightarrow 2x + 10\sin 2t + 2x'' = 0$

solve $x'' + x = -5\sin 2t$

$\lambda = \pm i \quad \lambda_0 = 2i$ general sol: $x_g(t) = A\cos t + B\sin t$.

find particular sol: $x_p = a\cos 2t + b\sin 2t$
$x_p'' = -4a\cos 2t - 4b\sin 2t$ $\Rightarrow \begin{cases} -3a = 0 \\ -3b = -5 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = \frac{5}{3} \end{cases}$

thus $x = A\cos t + B\sin t + \frac{5}{3}\sin 2t \xrightarrow[x(\frac{\pi}{4}) = -1]{x(0)=0} \begin{cases} A = 0 \\ B = -\frac{8\sqrt{2}}{3} \end{cases}$

$\mathcal{L}_{x'x'} = -2 < 0$. when $x = -\frac{8\sqrt{2}}{3}\sin t + \frac{5}{3}\sin 2t$, $I(x)$ attains maximum.

(3) $\int_0^{(T,T)} ((x')^2 + x)\, dt$

Sol: agrangian independent with t.

$x' \cdot 2x' - ((x')^2 + x) = \text{const} \Rightarrow x'^2 = x + C \Rightarrow \frac{dx}{dt} = \pm\sqrt{x+C}$

$\Rightarrow \frac{dx}{\pm\sqrt{x+C}} = dt \Rightarrow t = \pm 2\sqrt{x+C} + \tilde{C} \xrightarrow[(T,T)]{(0,0)} \begin{cases} \tilde{C} = 0 \\ C = \frac{T^2 - 4T}{4} \end{cases}$

$\Rightarrow x = \frac{t^2}{4} + T - \frac{T^2}{4}$

$\mathcal{L}_{x'x'} = 2 > 0$. when $x(t) = \frac{t^2}{4} + T - \frac{T^2}{4}$, $I(x)$ attains minimum.

(4). $\int_{(0,1,0)}^{(\pi,1,-1)} (2xy + (x')^2 - (y')^2 + y^2 - 2x^2)\, dt$

Sol: $\begin{cases} \mathcal{L}_x - \frac{d}{dt}\mathcal{L}_{x'} = 0 \\ \mathcal{L}_y - \frac{d}{dt}\mathcal{L}_{y'} = 0 \end{cases} \Rightarrow \begin{cases} 2y - 4x - 2x'' = 0 \\ 2x + 2y + 2y'' = 0 \end{cases} \Rightarrow \begin{cases} x'' + 2x - y = 0 & ① \\ y'' + y + x = 0 & ② \end{cases}$

from ① $y = x'' + 2x$ thus ② $\Rightarrow x^{(4)} + 3x^{(2)} + 3x = 0$. thus $\lambda^4 + 3\lambda^2 + 3 = 0$ is characteristic equ.

(5) $J(t,x,x') = \int_{(0,0)}^{(1,\frac{\pi}{2})} e^{t}\sqrt{1+(x')^2}\, dt.$

Sol: $\frac{d}{dt}\mathcal{L}_{x'} = 0 \Rightarrow \frac{e^{t}\cdot x'}{\sqrt{1+(x')^2}} = C \Rightarrow (e^{2t}-c^2)(x')^2 = c^2.$

$\Rightarrow x' = \pm\frac{c}{\sqrt{e^{2t}-c^2}} \Rightarrow dx = \pm\frac{c\,dt}{\sqrt{e^{2t}-c^2}} \xrightarrow[dt=\frac{du}{u}]{e^{t}=u} dx = \pm\frac{c\cdot du}{u\sqrt{u^2-c^2}}$

$x(t) = \tilde{C} \pm \arccos(ce^{-t})$, but no solution s.t. $\begin{cases} x(0)=0 \\ x(1)=\frac{\pi}{2}. \end{cases}$

The variation problem has no solution.

(6) $J(t,x,x') = \int_{(1,3)}^{(2,4)} (t^2xx' + tx^2)\, dt$

Sol: $\mathcal{L}_x - \frac{d}{dt}\mathcal{L}_{x'} = 0 \Rightarrow t^2x' + 2tx - 2tx - x't^2 = 0$ the E-L equation holds for any x

That is $J \equiv$ const for any x s.t. $\begin{cases} x(1)=3 \\ x(2)=4. \end{cases}$