Mathematical Analysis (3).
Midterm Exam. Fall 2023.
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Part 1. Metric spaces.

1. Metric spaces. Definition and examples.
2. Open sets. Definition and properties.
3. Limit points and closed sets. Definition and properties.
4. Limit of a sequence in a metric space. Dense sets.
5. Complete metric spaces. Definition. Completeness of $\mathbb{R}^n$.
6. Complete metric spaces. Completeness of $C[a,b]$ with uniform metric.
7. Complete metric spaces. Incompleteness of $C[a,b]$ with integral metric ρ_1.
8. Complete metric spaces. Theorem on the intersection of nested balls.
9. Baire theorem.
10. Compact sets. Properties of compact sets.
11. Sequential compactness.
12. Compactness in $\mathbb{R}^n$.
13. Limit of a function. Definition and properties.
14. Iterated limit of a function.
15. Continuity. Four equivalent definitions. Properties of continuous functions.
16. Characterization of continuity by preimages.
17. Continuous image of a compact set.
18. Cantor's theorem on uniform continuity.
19. Normed vector spaces. Examples. properties of a seminorm. Metric induced by the norm. Convergence in normed spaces. Arithmetic properties.

Part 2. Differential calculus of functions of several real variables.

20. Linear operators in Euclidean spaces.
21. Theorem on a norm of a linear operator.

22. Estimate for the norm of the linear operator between Euclidean spaces. Equivalence of norms in $\mathbb{R}^n$.
23. Differentiability of a function in Euclidean spaces. Uniqueness of the derivative operator. Examples.
24. Properties of the derivative (linearity, differentiation of a composition, product, scalar product).
25. Lagrange's theorem for a vector function.
26. Partial derivatives. Gradient. Formulas for Jacobi matrix and gradient. Derivative of a composition in coordinates.
27. Differentiability of a function with continuous partial derivatives.
28. Higher order partial derivatives. Schwarz's theorem on mixed derivatives.
29. Taylor-Lagrange theorem for multivariate functions.
30. Taylor-Peano theorem for multivariate functions. Taylor-Lagrange theorem in terms of differentials.
31. Extremal points of multivariate function.
32. Invertible linear operators.
33. Inverse function theorem.
34. Open mapping theorem.
35. Implicit function.
36. Lagrange's method for investigation of a function for the conditional extremum.

Part 3. Measure theory.

37. Definition of a semiring of subsets. Properties of semiring. Definition of a $\sigma-$Algebra. Properties of $\sigma-$algebra. $\sigma-$algebra generated by a family of sets.
38. Definition of measure and volume. First properties.
39. Continuity of a measure.
40. External measure. Definition and properties. Measurable sets.
41. Caratheodory theorem (without proof). Properties and uniqueness of standard extension.
42. Classic volume in $\mathbb{R}^n$. Definition and properties.
43. Lebesgue measure. Definition. Measurability of open sets.
44. Regularity of Lebesgue measure.

45. Approximation of measurable sets by Borel sets.
46. Lebesgue measure under smooth map. Transformation of Lebesgue measure under linear transform.
47. Example of non-measurable set.
48. Measurable function. Lebesgue sets. Properties of measurable functions
49. Measurability of a supremum, infimum and limit of a sequence of measurable functions.
50. Approximation of measurable function by simple and step functions..
51. Arithmetical properties of measurable functions.
52. Predicates that hold almost everywhere. Properties.
53. Convergence by the measure. Statements of Lebesgue, Riesz, Luzin theorems.

Part 4. Integral with respect to a measure.

54. Lebesgue integral. Monotonicity of Lebesgue integral.
55. B. Levy theorem.
56. Homogeneity of the integral.
57. Additivity of the integral.
58. Chebyshev inequality and its consequences.
59. Lebesgue theorem on dominated convergence.
60. Absolute continuity of the integral.
61. Lebesgue criteria for Riemann integrability. Comparison of Lebesgue and Riemann integrals.
62. Multiple and iterative integrals. Calculation of a measure by sections. Measure of a ball.
63. Measure of a product of two sets.
64. Measure of a graph and subgraph of a function.
65. Fubini and Tonelli theorems. Examples. Euler-Poisson integral.
66. Product measure.
67. General formula for the change of the variable. Change of the variable by diffeomorphism.

List of problems.

- Check the properties (completeness, separability, compactness) of metric/normed spaces, sets (closeness, openness, compactness, density).

- Limit of the function of several variables: proof of existence/absence, calculation, iterated limits.
- Investigate the continuity of a function.
- Calculation of partial derivative, gradient, Jacobi matrix. Proof of differentiability or nondifferentiability of a function.
- Calculation of Taylor's polynomial for a function of several variables.
- Calculation of partial derivatives of implicit function.
- Investigation of a function for extremal points.
- Investigation of an implicit function for extremal points.
- Conditional extremum.
- Change of the order of the integration in iterated integral.
- Calculation of the integral of multivariate function.
- Calculation of the area, volume (measure) of a set.