Chapter 1 General Mathematical Concepts and Notation

$\boxed{8.22}$

bracket n.括号. criterion n.判据(原则). simultaneously adj.同时的 (/ɪəɪ/, /leɪ/)

equivalent adj.等价于(的) proof by contradiction 反证法.

P follows from L $\Leftrightarrow$ L implies P

priorities: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ "or" 可以覆盖 "both" 的情况

证明结束 □ 或 ■

定义 $:=$ 例: $f'(x) := \lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ $\exists!$ 唯一存在

$A \Rightarrow B$.

A \ B	0	1
0	1	1
1	0	1

推理的正确性 (人为规定).

$\boxed{8.23}$

elementary adj.初等的. property n.所有物.财产.财物.性质

assemblage n.集合 (~~ed~~ age v.使集合). dichotomy n.一分为二、二分法

definite adj.确定的. naive adj.缺乏经验的.幼稚的.

集合. distinguishable → 有区分 → 互异性

cardinality n.势(集合的元素数). (无限集势相同 ∞).

$\exists$ existance $\forall$ generalization (全称.概括)

subset 子集. inclusion relation. 被包含 → a proper subset 真子集. ($A \subset B$ is strict).

$A \subseteq B := \forall x((x \in A) \Rightarrow (x \in B))$

union 并集. intersection 交集. difference 差. $A \setminus B$ (从A去B)

complement 补集 (相对全集的差集)

$C_M A = M \setminus A = \bar{A}$ (M是清晰的) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ $\overline{A \cap B} = \bar{A} \cup \bar{B}$

unordered pair $\{A, B\}$

ordered pair (A, B)

$X \times Y := \{(x, y) \mid (x \in X) \wedge (y \in Y)\}$ Cartesian product. (笛卡尔乘积) A and B ≃ A定位.

first projection, second projection ⇒ coordinate(s) 坐标 direct product. $X \times Y$. 直积.

without loss of generality 不妨设

RHS. 式子右. LHS 式子左

$f^{-1}: Y \to X$. $f: X \to Y$.

if: $f: X \to Y$ $g: Y \to X$.

$g \circ f = \mathbb{1}_X$ $f \circ g = \mathbb{1}_Y$.

then: $g = f^{-1}$.

$h \circ (g \circ f) = (h \circ g) \circ f$.

Lemma

$(g \circ f = e_X) \Rightarrow (g \text{ is surjective}) \wedge (f \text{ is injective})$

Proof: 1° $\forall x \in X$, since $g \circ f = e_X$, it implies that

$(g \circ f)(x) = e_X(x)$, i.e., $g[f(x)] = x$. $X = e_X(X) = (g \circ f)(X) = g(f(X)) \subset g(Y)$

so g is surjective.

2° suppose $\forall x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$

since $g \circ f = e_X$ it follows that

$(g \circ f)(x_1) = e_X(x_1)$ $(g \circ f)(x_2) = e_X(x_2)$

i.e. $\cancel{(g \circ f)}\ g(f(x_1)) = x_1$. $g(f(x_2)) = x_2$.

since $f(x_1) = f(x_2)$. then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$

hence f is injective.

Proposition The mappings $f: X \to Y$ and $g: Y \to X$ are bijective and mutually inverse to each other if and only if $g \circ f = e_X$ and $f \circ g = e_Y$

"$\Leftarrow$" $g \circ f = e_X \Rightarrow (f \text{ is injective}) \wedge (g \text{ is surjective})$

$f \circ g = e_Y \Rightarrow (g \text{ is injective}) \wedge (f \text{ is surjective})$ $\Rightarrow$ In conclusion f and g are both bijective.

$\forall x \in X$ denote $y = f(x)$

$g \circ f = e_X \Rightarrow (g \circ f)(x) = e_X(x)$

i.e. $g(y) = x$.

similarly. $\forall y \in Y$. denote $x = f(y)$

we must have $f(x) = y$.

That's to say f and g are mutually inverse

"$\Rightarrow$" obviously.

Relation

$R \subset X \times Y$. $xRy \Leftrightarrow (x, y) \in R$. x is connected with y by the relation R.

diagonal. $\Delta = \{(a,b) \in X^2 \mid a = b\}$.

$a \Delta b$. $(a,b) \in \Delta$, i.e. $a = b$.

Cartesian product > Relation > Function

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Relation R (on X^2).

① aRa (reflexivity)

② $aRb \Rightarrow bRa$ (symmetry). $R = X^2$

③ $(aRb) \wedge (bRc) \Rightarrow aRc$ (transitivity)

满足①②③的 R: equivalence relation (如三角形全等. 相似关系 直线平行关系)

aRb. a,b等价 $\Leftrightarrow a \sim b$

$R \subsetneq X^2$, 有 $\exists x_1, x_2$. $(x_1, x_2) \notin R$.

① aRa

② $(aRb) \wedge (bRc) \Rightarrow (aRc)$

③ $(aRb) \wedge (bRa) \Rightarrow a \triangle b \Rightarrow a=b$. (antisymmetry)

①②③ partial ordering $a \preccurlyeq b$. (不一定所有元素间满足关系)

if $\forall a \forall b. ((aRb) \vee (bRa))$. any two elements of X are comparable. (relation R is called an ordering).

$X \Rightarrow$ linear ordering

Functions (特殊的 relation)

$(xRy_1) \wedge (xRy_2) = (y_1 = y_2)$	A relation R is said to be functional.
严格的函数定义. 切示定义	A functional relation is called a function.

graph 'n'同 = relation 集合论.

pre-image $f^{-1}(y)$ 映射 $f: X \to Y$ 中. y的原像的集合 (可以是空集). 不一定要双射才有

$\boxed{\text{proof}}$ injective 一般反证. 设单元素. 双元素集 $x_1 \neq x_2$. $\Rightarrow f(x_1) \neq f(x_2)$.

surject. $\forall y \in Y$, $f^{-1}(y) \neq \varnothing$. (相当于解 $y = \cdots$)

bijective (1) ~~$\forall y_1 = y_2$, s.t. $x_1 =$~~ $\forall f(x_1) = f(x_2)$. $\Rightarrow x_1 = x_2$.

(2) 找另一个映射. 证明与之互逆.

Cardinality. (势)

等势的

The set X is said to be equipollent to the set Y if there exists a bijective mapping of X onto Y.

. equipollent is a equivalence relation. $(X \sim Y)$

card $X \leq$ card $Y :\Leftrightarrow \exists Z \subset Y$ (card X = card Z) → 可建立 X 到 Z 的双射.

* 无限集之间可以等势(如 $\mathbb{Z}$ 与偶数集).

* card $X \leq$ card Y. doesn't not exclude the inequality card $Y \leq$ card X. even when X is a proper subset of Y.

e.g. $x \to \frac{x}{1-|x|}$. $(-1,1) \to$ entire axis (以验证 bijective 得到 equipollent)

(有限集)

finite if it is not equipollent to any proper set of itself (Dedekind). otherwise. infinite.

properties:

1° (card $X \leq$ card Y) $\wedge$ (card $Y \leq$ card Z) $\Rightarrow$ card $X \leq$ card Z. (obvious).

no larger than

2° (card $X \leq$ card Y.) $\wedge$ (card $Y \leq$ card X) $\Rightarrow$ card X = card Y.

3° $\forall X \forall Y$. (card $X \leq$ card Y) $\vee$ (card $Y \leq$ card X).

$P(X)$ (幂集). the set of all subsets of the set X.

Theorem card $X <$ card $P(X)$

e.g. card $\phi <$ card $\{\{\phi\}\}$.

proof: "$\leq$" $P(X)$ contains all one-element subsets of X

"card $X \neq$ card $P(X)$."

assume card X = card $P(X)$.

then $\exists f: X \to P(X)$. f is a bijective.

consider $A = \{x \in X \mid x \notin f(x)\}$, $f(x) \in P(X)$.

{ We claim that A is nonempty, otherwise. $\forall x \in X$. we have $x \in f(x)$, then x must have more than one correspondening terms. contradicts with f is a bijective }

since $A \in P(X)$, then $\exists a \in X$. $f(a) = A$.

if $a \in A$. then $a \notin f(a) = A$. i.e $a \notin A$.

if $a \notin A$. then $a \in f(a) = A$. i.e. $a \in A$.

both are impossible.

Chapter 2 The Real Numbers

Date . .

(1) Axioms for Addition.

identity element 0

(2) Axioms for Multiplication.

neutral $1 \in (R \setminus 0)$

(3). Order Axioms.

~~的~~ $\leq$. linear Order $\forall x \in R. \forall y \in R. (x \leq y) \vee (y \leq x)$.

(4) the Axioms of Completeness. (CONTINUITY).

If X and Y are nonempty subsets of R having the property that $x \leq y$ for every $x \in X$. and every $y \in Y$. then there exists $c \in R$. s.t. $x \leq c \leq y$ for all $x \in X$. $y \in Y$.

Any set on which these axioms hold can be considered a concrete realization or model of the real numbers.

To any abstract system of axioms.

(1) consistent. set satisfying all the condition

(2) determine the mathematical object uniquely.

Some General Algebraic Properties of Real Numbers.

a). Consequences of the Addition Axioms.

1.° unique 0. $0_1 = 0_2 + 0_1 . = 0_2$ (⇒ 0₁的公理 ⇐ 0₂的公理.)

2°. unique negative. $x_1 = x_1 + 0 = x_1 + (x + x_2) = (x_1 + x) + x_2 = x_2$ ($x_1 . x_2$ 都是 x 负元).

3° the equation $a + x = b$ has the unique solution $x = b + (-a)$ (减法未定义).

b) Consequences of the Multiplication Axioms.

1° unique 1

2° ~~的~~ reciprocal x^{-1} unique

3° For $a \in R \setminus 0$. the equation $a \cdot x = b$ has the unique solution $x = b \cdot a^{-1}$

c) Consequences of Addition and Multiplication Axioms.

1°. $x \in \mathbb{R}$. $x \cdot 0 = 0$. if $x \in \mathbb{R} \setminus 0$. $x^{-1} \in \mathbb{R} \setminus 0$

2° $(x \cdot y = 0). \Rightarrow (x = 0) \vee (y = 0)$

3° $x \in \mathbb{R}$. $-x = (-1) \cdot x$.

⇑ 加法公理给定义 ⇑ 乘法&加法(1为乘法、-1来源于加法).

$\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

*根据 Addition. Multiplication 定义 (~~f~~ mapping). $a=b \Rightarrow a+c = b+c$. $bc = ac$.

4° $(-1)(-x) = x$.

5° $(-x)\cdot(-x) = x\cdot x$.

d). Consequence of Order Axioms.

~~⇒~~ 1° strict inequality.

For any $x\in\mathbb{R}, y\in\mathbb{R}$. precisely one of the relations holds: $x<y$, $x=y$, $y>x$.

W.l.g suppose $x\le y \Rightarrow (x<y)\vee(x=y)$.

$\Rightarrow (x\le y)\wedge(x\ne y)\wedge(y\le z) \Rightarrow (x\le z)\wedge(x\ne y)$ exclude $x=z$.

2° For any, $x,y,z\in\mathbb{R}$. $(x<y)\wedge(y\le z) \Rightarrow (x<z)$

$(x\le y)\wedge(y<z) \Rightarrow (x<z)$.

e) Combination Axioms.

1° $x,y,z,w\in\mathbb{R}$.

$(x<y) \Rightarrow (x+z)<(y+z)$

$(0<x) \Rightarrow (-x)<0)$

$(x\le y)\wedge(z\le w) \Rightarrow x+z\le y+w$.

$(x\le y)\wedge(z<w) \Rightarrow x+z<y+w$.

2° $x,y,z\in\mathbb{R}$.

$(0<x)\wedge(0<y) \Rightarrow (0<xy)$

$(x<0)\wedge(y<0) \Rightarrow (0<xy)$

$(x<0)\wedge(0<y) \Rightarrow (xy<0)$

$(x<y)\wedge(0<z) \Rightarrow (xz<yz)$

$(x<y)\wedge(z<0) \Rightarrow (yz<xz)$.

3° $0<1$. 4° $(0<x) \Rightarrow (0<x^{-1})$ and $(0<x)\wedge(x<y) \Rightarrow (0<y^{-1})\wedge(y^{-1}<x^{-1})$,

positive: large ~~less~~ than 0

negative: less than 0

the Existence of a Least Upper (or Greatest Lower) Bound of a Set of Numbers.

Def2 A set $X\subset\mathbb{R}$ be bounded above/below.

$\exists c\in\mathbb{R}$, s.t. $x\le c$ ($x\ge c$) for all $x\in X$

Def3 A set that is bounded both above and below is called bounded.

(不一定存在) (smallest minimal). ($a\le x$)

Def4. An element $a\in X$. largest/maximal. element of X if $x\le a$. for all $x\in X$.

$(a \max X) := (a\in X \wedge \forall x\in X\, (x\le a))$.

the smallest number that bounds a set $X\subset\mathbb{R}$

Def5. $(s=\sup X) := \forall x\in X\, ((x\le s)\wedge(\forall s'<s\ \exists x'\in X\, (s'<x')))$. the supremum of X.

Def6. $(i=\inf X) := \forall x\in X\, ((i\le x)\wedge(\forall i'>i\ \exists x'\in X\, (x'<i')))$ infimum

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if exists $\max \mathbb{X}$, $\min \mathbb{X}$. $\max \mathbb{X} = \sup \mathbb{X}$ $\min \mathbb{X} = \inf \mathbb{X}$.

Lemma Every nonempty set of real numbers that is bounded from above (上确界原理) has a unique least upper bound. (The least upper bound principle).

proof: $Y = \{y \in \mathbb{R} \mid \forall x \in \mathbb{X} (x \le y)\} \ne \phi$
$\forall x \in \mathbb{X}, y \in Y$ s.t. $x \le y$.
By completeness axioms.
there exists $c \in \mathbb{R}$
s.t. $\forall x \in \mathbb{X}, y \in Y$. $x \le c \le y$
Hence $c = \sup \mathbb{X}$
(infimum similarly)

Lemma. ($\mathbb{X}$ bounded below) $\Rightarrow$ ($\exists! \inf \mathbb{X}$)

Dedekind's Theorem (戴德金分割. 数的模型).

If ~~…~~ $x \le y$ for all $x \in \mathbb{X}$ and $y \in Y$, then $\mathbb{X}$ is bounded above, Y is bounded below, and $\mathbb{X} \cup Y = \mathbb{R}$. ~~…~~ then $\sup \mathbb{X} = \inf Y$, either $\mathbb{X}$ has a maximal points or Y has a minimal points.

The Most Important Classes of Real Numbers

Def 1. A set $\mathbb{X} \subset \mathbb{R}$ is inductive if for each number $x \in \mathbb{X}$, it also contain $x+1$

* The intersection $\mathbb{X} = \bigcap_{\alpha \in A} X_\alpha$ of any family of inductive sets (↳ 归纳集) $\mathbb{X}_\alpha$ if not empty is an inductive set.

Def 2. The set of natural numbers (无0) is the smallest inductive set containing 1, that is, the intersection of all ~~insect~~ inductive sets that contain 1

$(E \subset \mathbb{N}) \wedge (1 \in E) \wedge (\forall x \in E\ (x \in E \Rightarrow (x+1) \in E)) \Rightarrow E = \mathbb{N}$

1° The sum ~~…~~ and product of natural numbers are natural numbers.

proof. denote by E the set of natural numbers n for which $(m+n) \in \mathbb{N}$. for all $m \in \mathbb{N}$
Then $1 \in E$ since $(m \in \mathbb{N}) \Rightarrow (m+1 \in \mathbb{N})$ for any $m \in \mathbb{N}$. If $n \in E$, that is, $(m+n) \in \mathbb{N}$, then $(n+1) \in E$ also, since $(m+(n+1)) = ((m+n)+1) \in \mathbb{N}$. By the principle of induction, $E = \mathbb{N}$. and we have proved that addition doesn't lead outside of $\mathbb{N}$

2° $(n \in \mathbb{N}) \wedge (n \ne 1) \Rightarrow ((n-1) \in \mathbb{N})$

3° For any $n \in \mathbb{N}$ the set $\{x \in \mathbb{N} \mid n < x\}$ contains a $\min\{x \in \mathbb{N} \mid n < x\} = n+1$

4° $(m \in \mathbb{N}) \wedge (n \in \mathbb{N}) \wedge (n < m) \Rightarrow (n+1 \le m)$

5. In any nonempty subset of the set of natural numbers there is a minimal element

① $1 \in E$ ② $E \subset \mathbb{N}$ ③ 若 $n \in E$, 则 $n+1 \in E$

Rational and Irrational Numbers.

Def3. The union of the set of ①natural numbers. ②the set of negatives of natural numbers, ③and zero is called the set of integers and is denoted by $\mathbb{Z}$

. proof $\in \mathbb{Z}$ ①/②/③ holds (proof addition do not lead outside of $\mathbb{Z}$. $\Rightarrow m-n \in \mathbb{Z}$).

$\mathbb{Z}$ is an Abelian group. the reciprocals (倒数) of the integers are not in $\mathbb{Z}$ (except the reciprocals of 1 and -1)

$m=k\cdot n$ $k,m,n\in\mathbb{Z}$, m is divisible by n. or a multiple of n. or n is a divisor of m

A number $p\in\mathbb{N}, p\neq 1$, is prime if it has no divisors in $\mathbb{N}$ except 1 and p.

. the fundamental theorem of arithemtic Each natural number admits a representation as a product $n=p_1 \cdots p_k$, where $p_1, \dots, p_k$ are prime numbers. This representation is unique except for the order of the factors.

relative prime have no divisors except 1 and -1.

Def4. Numbers of the form $m\cdot n^{-1}$, where $m, n\in\mathbb{Z}$ are called rational

(表达形式不唯一. 需证明对运算不产生影响. $\frac{m_1}{n_1}=\frac{m_2}{n_2} \Leftrightarrow m_1 n_2 = m_2 n_1$ $(n_1 n_2 \neq 0)$ 判据.

Def5. The real numbers that are not rational are called irrational

the number $s\in\mathbb{R}$, s.t. $s>0 \wedge s^2=2$

By Pythagorean theorem, and irrationality of $\sqrt{2}$ is equivalent to the assertion that the diagonal and side of a square are (incommensurable 不可公度的)

Verify there exists a real number $s\in\mathbb{R}$ whose square equals 2. and that $s\notin\mathbb{Q}$.

algebraic. $a_0x^n+\dots+a_{n-1}x+a_n=0$. the root of an algebraic equation with (rational ~ equivalently $\mathbb{Z}$) cofficients

Otherwise, the number is called transcendental number

cardinality: algebraic number ~ rational number

transcendental (超越数) ~ real number

The Principle of Archimedes.

1° Any nonempty subset of natural numbers that is bounded from above contains a maximal element.

proof. If $E \subset \mathbb{N}$ is the subset in question.
By the least-upper-bound principle
$\exists! \sup E = s \in \mathbb{R}$.
By definition of least upper bound there is ~~a~~ a natural number $n \in E$ satisfying the condition
$s-1 < n \le s$.
But then $n = \max E$.
Since a natural number that is larger than n must be at least $n+1$ and $n+1 > s$ □

2° The set of natural numbers is not bounded above
(1) inductive set 角度 (2) 1° maximal element 角度).

3° Any nonempty subset of the integers that is bounded from above contains a ~~max~~. maximal element.

Proof: proof of 1° can be reacted verbatim (逐字的) (immedicate successor)

4° Any nonempty subset of the integers that is bounded below contains a minimal element.
Alternatively, one can pass to the negatives of the numbers and use what has been proved in 3° □.

5° The set of integers is unbounded above and unbound below

The principle of Archimedes.

For any fixed (确定的) positive number h and any real number x there exists a <u>unique</u> integer k such that $(k-1)h \le x < kh$ (用 h 度量 x)

proof: since $\mathbb{Z}$ is not bounded about
the set $\{n \in \mathbb{Z} \mid \frac{x}{h} < n\}$ is a nonempty subset of the integers that is bounded below
Then it contains a minimal element k, that is
$(k-1) \le \frac{h}{x} < k$.
Since $h > 0$, the inequalities are equivalent to those given in the statement of the principle Archimedes.

Corollary

1. For any positive number ε there exists a natural number n such that $0 < \frac{1}{n} < \varepsilon$.
2. If the number $x \in R$ is such that $0 \le x$ and $x < \frac{1}{n}$ <u>for all $n \in \mathbb{N}$</u>, then $x = 0$.
3. For any numbers $a, b \in \mathbb{R}$, such that $a < b$ there is a rational number $r \in \mathbb{Q}$. Such that $a < r < b$ (有理数是稠密的).

proof: we choose $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < b-a$
By the principle of Archimedes we can find a number $m \in \mathbb{Z}$ s.t. $\frac{m-1}{n} \le a < \frac{m}{n}$.
Then $b > \frac{m}{n}$
Since otherwise we would have $\frac{m-1}{n} \le a < b \le \frac{m}{n}$
$\Rightarrow$ then $\frac{1}{n} > b-a$
Thus $(r = \frac{m}{n} \in \mathbb{Q}) \wedge (a < \frac{m}{n} < b)$

4. For any number $x \in \mathbb{R}$ there exists a unique integer $k \in \mathbb{Z}$ s.t. $k \le x < k+1$

denote $[x]$, integer part of x. $\{x\} := x - [x]$. fractional part of x

$x = [x] + \{x\}$

Def 7. An open interval containing the point $x \in \mathbb{R}$. will be called a neighborhood of this point.

modulus / absolute value. $|x|$.

length. $(x-\delta, x+\delta)$. δ-neighborhood of x. Its length is 2δ.

distance. The distance between $x, y \in \mathbb{R}$. is the quantity $|x-y|$. (nonnegative).

$x, y, z \in \mathbb{R}$. then $|x-y| \leq |x-z| + |z-y|$, the so-called triangle inequality holds.

$|x+y| \leq |x|+|y|$. equality holds only when the numbers x and y are both negative or positive.

Basic Lemmas Connected with the completeness of the Real Numbers.

Def1. A function $f: \mathbb{N} \to X$ of a natural-number argument is called a sequence or, more fully, a sequence of elements of X.

Def2. Let $X_1, \dots X_n, \dots$ be a sequence of sets. If $X_1 \supset X_2 \supset \cdots \supset X_n \supset \cdots$ that is $X_n \supset X_{n+1}$ for all $n \in \mathbb{N}$. we say the sequence is nested.

(区间套定理) Cauchy-Cantor Principle For any nested sequence $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$
(闭) of closed intervals, there exists a point $c \in \mathbb{R}$ belonging to all of these intervals. If in addition it is known that for any $\varepsilon > 0$. there is an interval I_k whose length $|I_k|$ is less than ε, then c is the unique point common to all the intervals.

proof: Denote $I_n = [a_n, b_n]$, $n \in \mathbb{N}$
Since $\{I_n\}$ is nested, we have $a_n \le b_m$ ($\forall m \in \mathbb{N}$)
otherwise, $\exists n_0, m_0 \in \mathbb{N}$ s.t $a_{m_0} < b_{m_0} < a_{n_0} < b_{n_0}$
since $(I_{m_0} \subset I_{n_0}) \vee (I_{n_0} \subset I_{m_0})$, this is impossible
Let $X = \{a_n \mid n \in \mathbb{N}\}$ $Y = \{b_m \mid m \in \mathbb{N}\}$
By the Axiom of completeness, there exists $c \in \mathbb{R}$ s.t. $a_n \le c \le b_n$. $\forall n \in \mathbb{N}$ i.e. $c \in I_n$. $\forall n \in \mathbb{N}$
$c \in \bigcap_{n=1}^{\infty} I_n$

Assume there exists $c_1, c_2 \in \mathbb{R}$ such that
w.l.g. $c_1 < c_2$
let $\varepsilon = \frac{1}{2}(c_2 - c_1)$.
there exists I_k. s.t. $|I_k| < \varepsilon$
which contradicts with the fact.
$[c_1, c_2] \subset I_k$.
Hence. the assumption is false.
thus. c is unique
区间端点作 X. Y.

I is any system of nested closed intervals. then $\sup\{a \in \mathbb{R} \mid [a,b] \in I\} = \alpha \le \beta = \inf\{b \in \mathbb{R} \mid [a,b] \in I\}$
$[\alpha, \beta] = \bigcap_{[\alpha,\beta] \in I} [a,b]$

(有限覆盖定理). The Finite Covering Lemma

Def3. A system $S = \{X\}$ of sets X is said to cover a set Y if $Y \subset \bigcup_{X \in S} X$. that is, if every element $y \in Y$ belongs to at least one of the sets X in the system S

The Finite Covering Lemma. Every system of (都是内点) open intervals covering a closed interval
(Borel-Lebesgue Principle) contains a finite subsystem that covers the closed interval

Proof: Assume that I_0 can't be covered by finite open intervals from the system $S = \{X\}$
Then divide I_0 into two halves. at least one of them cannot be finitely covered denote it by I_1
Inductively, we obtain nested sequence of closed intervals $\{I_n\}$ By the nested interval Lemma, there exists unique $c \in \mathbb{R}$. s.t. $c \in \bigcap_{n=1}^{\infty} I_n \subset I_0$ Hence. there exists $X_0 \in S$ s.t. $c \in X_0$ Then there exist $\delta > 0$. s.t $(c-\delta, c+\delta) \subset X_0$.
Since $|I_n| = \frac{|I_0|}{2^n}$, we can find I_k s.t $|I_k| < \delta$. Then $I_k \subset (c-\delta, c+\delta) \subset X$
which contradicts with the definition of $\{I_k\}$
i.e I_0 can be finitely covered.

通过区间长度找到覆盖.

Def 4. A point $p \in \mathbb{R}$ is limit point of the set $X \subset \mathbb{R}$ if every neighborhood of the point contains an infinite subset of X.

"$\Leftarrow$" ( ) 缩小邻域. 排除已有点、得到新点、得到无穷多个点

$\Rightarrow$ This condition is obviously equivalent to the assertion that every neighborhood of p contains at least one point of X different from p itself.

If $X = \{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{N}\}$ the only limit point of X is the point $0 \in \mathbb{R}$

For the open interval (a,b). every point of the closed interval $[a,b]$ is a limmit point. and there are no others.

For set $\mathbb{Q}$ of rational numbers every point of $\mathbb{R}$ is limit point; for. as we know, every open interval of the real numbers contains rational numbers.

(有理数"稠密". 任何实数间都有有理数).

The limit Point Lemma (Bolzano-Weierstrass) Every bounded infinite set of real numbers has at least one limit point.

proof: Let X be the given subset of $\mathbb{R}$.

It follows from the definition of boundedness that X is contained in some closed interval $I \subset \mathbb{R}$

We should show that at least one point of I is a limit point of X

Assume that each point $x \in I$ would have a neighboorhood $U(x)$ that contains no elements of X or at most a finite number.

The totality of such neighborhood $\{U(x)\}$ constructed for the points $x \in I$ forms a covering of I by open interval $U(x)$

By the finite covering Lemma. we can extract the finite subsystem from $\{U(x)\}$ that covers I.

i.e. $X \subset I \subset \{U(x_i)\}$

However, there are only finite points of X in $U(x_i)$ and hence only finitely many in their union

$\Rightarrow X$ is a finite set. which... □

有限个区间. 有限个点 $\rightarrow$ 有限的 X

* The limit Point Lemma $\Rightarrow$ the axiom of completeness.

Proof: Let $X, Y \subset \mathbb{R}$. s.t. $\forall x \in X\ \forall y \in Y\ (x \le y)$

Assume that there is no such c that $x \le c \le y\ (c \in \mathbb{R})$

Let $a_1 \in X$. $b_1 \in Y$.

Consider the element $\frac{a_1+b_1}{2}$.

If $\frac{a_1+b_1}{2}$ s.t. $\cancel{\exists y \in Y}$ $\exists x \in X$. $x > \frac{a_1+b_1}{2}$, then let $a_2 = \frac{a_1+b_1}{2}$. $b_2 = b_1$.

Else $b_2 = \frac{a_1+b_1}{2}$ $a_2 = a_1$

By this way. we obtain squence of set $[a_n, b_n]$.

s.t. $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$

By the limit point lemma. there exist $c \in \mathbb{R}$. which is the limit point of $\{a_n \mid n \in \mathbb{N}\}$ $\Rightarrow \{a_n\}$ is infinite

(In fact. if $X = \{x\}$ is a singleton, then $a_n = x$, $n = 1, 2, \ldots$ In this case, we can let $c = x$, which obviously satisfy the requirement)

思想：构造良备 X. Y. $[a_n, b_n] \rightarrow \{a_n\}, \{b_n\}$.

$\mathbb{Q}$ 与 $\mathbb{R}$. $\mathbb{R}$中挖去"一个$\mathbb{R}\setminus\mathbb{Q}$, 一个区间两个区间

关注 $\frac{1}{n}$, $(\frac{1}{2})^n$"

If $\exists y_0 \in Y$ s.t. $c > y_0$.

there exists a_n,

s.t. $|a_n - c| < \frac{1}{2}(c - y_0)$

$-\frac{1}{2}c + \frac{1}{2}y_0 < a_n - c < \frac{1}{2}c - \frac{1}{2}y_0$.

$y_0 < \frac{c}{2} + \frac{y_0}{2} < a_n < \frac{3}{2}c - \frac{1}{2}y_0$.

$y_0 < a_n$. which causes a contradiction.

Hence $c \le y_0$

If $\exists x_0 \in X$. s.t. $c < x_0$.

there exist b_n, a_n.

$|a_n - c| < \frac{1}{3}|c - x_0|$ $\quad |a_n - b_n| < \frac{1}{3}|c - x_0|$

$|b_n - c| < |a_n - c| + |a_n - b_n|$

$< \frac{2}{3}\cancel{(c - x_0)}(x_0 - c)$

b $\frac{2}{3}c - \frac{2}{3}x_0 < b_n - c < \frac{2}{3}x_0 - \frac{2}{3}c$.

$\frac{5}{3}c - \frac{2}{3}x_0 < b_n < \frac{2}{3}x_0 + \frac{1}{3}c < x_0$

which causes a contradiction

Hence $c \ge x$.

In conclusion, $x \le c \le y$. □

2. Borel - Lebesgue Principle $\Rightarrow$ Axiom of Completeness.

Proof Suppose that we obtain exactly the same nested intervals as in 1.

Then $(a_1-1, b_1+1) \setminus [a_n, b_n] = (a_1-1, a_n) \cup (b_n, b_1+1)$.

i.e. we obtain a system of open intervals $\{(a_1-1, a_n), (b_n, b_1+1)\}$.

We claim that $\bigcap_{n=1}^{\infty}[a_n, b_n]$ is nonempty. then $\{c\} = \bigcap_{n=1}^{\infty}[a_n, b_n]$ (the uniqueness ...)

Otherwise. $\bigcap_{n=1}^{\infty}[a_n, b_n] = \phi$. then $\bigcup_{n=1}^{\infty}(a_1-1, b_1+1)\setminus[a_n, b_n] = (a_1-1, b_1+1)\setminus\bigcap_{n=1}^{\infty}[a_n, b_n] = (a_1-1, b_1+1)\setminus\phi$.

i.e. $\bigcup_{n=1}^{\infty}(a_1-1, b_1+1)\setminus[a_n, b_n] \supset [a_1, b_1]$. By the finite covering lemma, there exists a finite subsystem of open intervals which can also cover $[a_1, b_1]$

Which is contradictory, since the union of these finite open intervals is included in certain $(a_1-1, a_n) \cup (b_n, b_1+1)$ $(n=1,2,\cdots)$, which can't cover $[a_1, b_1]$. As in 1. we can show ...

3. the Nested Interval Lemma

Proof: Suppose we obtain ~~the~~ exactly the same nested intervals as in 1.

Then, by the Nested Interval Lemma.

there exists unique $c \in \mathbb{R}$. s.t. $\{c\} = \bigcap_{n=1}^{\infty}[a_n, b_n]$

and we can show that $\forall x \in X\ \forall y \in Y\ (x \le c \le y)$ as in 1.

2.4. Countable and Uncountable Sets

<u>Def1</u> A set X is countable if it is equipollent with the set $\mathbb{N}$ of natural numbers.

that is. $\operatorname{card} X = \operatorname{card} \mathbb{N}$. (可标号,建立mapping)

1) An infinte subset of a countable set is countable (如偶数集之于整数集).

2) The union of the sets of a finite or countable (可数无穷多个). system of ~~com~~ countable sets. is a countable set.

Proof 1). We should verify that every infinite subset E of $\mathbb{N}$ is equipollent with $\mathbb{N}$.

We construct a mapping $f: \mathbb{N} \to E$

There is a minimal element of $E_1 := E$, which we assign to the number $1 \in \mathbb{N}$. and denote $e_1 \in E_1$. The E_1 is infinite. $E_2 := E_1 \setminus \{e_1\} \neq \phi$. which we assign the number $2 \in \mathbb{N}$. and donote $e_2 \in E_2$. Similarly, since E is infinite, this construction can't terminate at any finite step with index.

~~Be~~ According to the principle of induction, ~~for $n \in \mathbb{N}$,~~ there is a $e_n \in E$ to each $n \in \mathbb{N}$. Hence, the f is injective.

Then we should verify $f(\mathbb{N}) = E$. Let $e \in E$. The set $\{n \in \mathbb{N} \mid n \le e\}$ is finite, and hence the subset of it $\{n \in E \mid n \le e\}$ is also finite. Let k be the number of elements in the latter set. Then by the construction $e = e_k$ Hence, f is surjective.

2). $X_m = \{x_m^1 \cdots x_m^n \cdots\}$. $X = \bigcup_{m \in \mathbb{N}} X_m$. $x_m^n \in X_m$. $\to$ 有对应 (m,n)

$(m,n) \mapsto \frac{(m+n-1)(m+n-2)}{2} + m$.

(统计(m,n)在队列中的排序).

Finite / countable → most countable

the union of an at most countable family of at most countable sets is at most countable.

至多可数个至多可数集的并形成的集合是至多可数的.

1) card $\mathbb{Z}$ = card $\mathbb{N}$ ($\mathbb{Z} \to \mathbb{N}, 0, -\mathbb{N}$, 并 → countable).

direct product.
2) card $\mathbb{N}^2$ = card $\mathbb{N}$. (前已证 $\mathbb{N}^2$ 与 $\mathbb{N}$ 某无穷子集等势 → countable)

3) card $\mathbb{Q}$ = card $\mathbb{N}$. ($\mathbb{Q} \to \mathbb{Q}^+ \ \mathbb{Q}^- \ 0$. $\frac{m}{n}$. $\mathbb{Q} \to \mathbb{N} \times \mathbb{N}$

4) The set of algebraic numbers is countable.

card $\mathbb{Q} \times \mathbb{Q}$ = card $\mathbb{N}$. By the inductive principle. card $\mathbb{Q}^k$ = card $\mathbb{N}$.

element $r \in \mathbb{Q}^k$ is an ordered set $(r_1, \dots r_k)$ of k rational sets.

$x^k + r_1 x^{k-1} + \dots + r_k = 0$.

Thus there are as many different algebraic equations of degree k as there are different ordered sets $(r_1, \dots r_k)$. that is, a countable sets.

连续统.

Def 2. The set $\mathbb{R}$ is also called number continuum and its cardinality the cardinality of the continuum.

Theorem (Cantor) card $\mathbb{N}$ < card $\mathbb{R}$.

Proof: we can show that even the closed interval $[0,1]$ is an uncountable set. (取不包含,排除再取).

If it is countable, i.e it can be writen in $x_1, x_2 \dots x_n \dots$ On the closed interval $[0,1] = I_0$, take ~~that~~ the point x_1. s.t. $(I_1 \neq \phi \ (I_1 \subset I_0)) \wedge (x_1 \notin I_1)$. On the I_1, take the point x_2, let $(I_2 \neq \phi) \wedge (I_2 \subset I_1)$, s.t. $x_2 \notin I_2$; If there is closed interval I_n ($I_n \neq \phi$).

Then ~~there~~ exists $|I_{n+1}| \neq 0$., $x_{n+1} \notin I_{n+1}$.

By the nested interval lemma there exists a $c \in [0,1]$, it belongs to intervals $I_0 \dots I_n \dots$

But this point of the closed interval $I_0 = [0,1]$ cannot be any point of the $x_1, \dots x_n \dots$

Corollary. 1) $\mathbb{Q} \neq \mathbb{R}$. and irrational number exists.

2) The exists transcendental numbers. since the set of algebraic numbers is countable

Continuum Hypothesis [Undecidable] There is no intermediate cardinality between countable sets and sets having cardinality of continuum.

(Can't be proved or disproved. ~~the~~ neither the hypothesis nor its negation contradicts ~~condict~~ the stardard axiom of the set theory.

(similar to the way in which Euclid's fifth postulate on parallel lines is independent of the other axiom of geometry.

Chapter 3 Limits

Def 1. A function $f: \mathbb{N} \to X$. a: sequence(序列). numerical sequence (数列)

Def 2. The number A is called the limit of the sequence $\{x_n\}$.
if for every $\varepsilon > 0$ there exists an index N such that $|x_n - A| < \varepsilon$ for all $n > N$

$(\lim_{n\to\infty} x_n = A) := \forall \varepsilon > 0\ \exists N \in \mathbb{N}.\ \forall n \in N(|x_n - A| < \varepsilon)$ 有时写作 $N(\varepsilon)$. 有一定关系. 非函数.

"ε-N" (by 柯西).

推导: $n > N \Rightarrow |x_n - A| < \varepsilon$. 分析: 根据给定 ε. 找到 n.

内点 → 找邻域也在其中.

neighborhood: a open interval contains the element A. ⇒ the neighborhood of A.

点集拓扑定义: Every neighborhood $V(A)$ of A there exists an index N s.t. all terms of the sequence having index (下标) larger than N belong to the neighborhood $V(A)$.

$(\lim_{n\to\infty} x_n = A) := \forall V(A)\ \exists N \in \mathbb{N}\ \forall n > N\ (x_n \in V(A))$.

* A sequence ~~have~~ has limit → convergent (收敛). not → divergent (发散).

Verify: The sequence $1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7} \cdots$ $(x_n = n^{(-1)^n})$. is divergent.

Proof: If A were the limit of this sequence any neighborhood of A contain all but a finite number of terms of the sequence.
a number $A \neq 0$. ~~for~~ if $\varepsilon = \frac{|A|}{2} > 0$. (A为中心, ε为半径) all the terms of the sequence of the form $\frac{1}{2k+1}$ for which $\frac{1}{2k+1} < \frac{|A|}{2}$ lie outside the ε-neighborhood. of A.
Also. the number 0. there are infinitely many terms of the sequence lying outside the 1-neighborhood of 0.

Theorem (Squeeze Theorem)

Suppose the sequences $\{x_n\}$. $\{y_n\}$, and $\{z_n\}$ are such that $x_n \le y_n \le z_n$ for all $n \in N \in \mathbb{N}$
If the seqaences $\{x_n\}$ and $\{z_n\}$ both converage to the same limit, then the sequence $\{y_n\}$ also converges to that limit.

e.g. A numerical sequence $\{a_n\}$. s.t. $a_n > 0$. for any $n \in \mathbb{N}$. and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = q\ (0 \le q < 1)$

Prove that $\lim_{n\to\infty} a_n = 0$ (正项级数比值有极限. 级数收敛)

$\sum_{n=1}^{\infty} a_n$. 级数 series

Proof: Let $\varepsilon = \frac{|1-q|}{2}$ ← 已知极限才可以任取.
there exists $N \in \mathbb{N}$.
s.t. $\left|\frac{a_{n+1}}{a_n} - q\right| < \varepsilon$. when $n > N$

$0 < \frac{a_{n+1}}{a_n} < \frac{1+q}{2} < 1$.

Multiple from both side

$0 < \frac{a_n}{a_{N+1}} < \left(\frac{1+q}{2}\right)^{n-N-1}$

$\Rightarrow 0 < a_n < \frac{a_{N+1}}{\left(\frac{1+q}{2}\right)^{N+1}} \left(\frac{1+q}{2}\right)^n$

Since $0 < \frac{1+q}{2} < 1$. it follows that $\lim_{n\to\infty} \left(\frac{1+q}{2}\right)^n = 0$.

a. General Properties

Thm 1) An ultimately constant (最终常值.) sequence converges
2) Any neighborhood of the limit of sequence contain all but finite number of terms of the sequence.
3) a convergent sequence cannot have two different limits.
4) a convergent sequence is bounded.

取 ε. 有邻域. 在有限项中取最值.

$\lim_{n\to\infty} x_n = A$

$|x_n| = |x_n - A + A| \le |x_n - A| + |A| < 1 + |A|$

b) Passage to the Limit and the Arithmetic Operations.（极限与四则运算可交换次序）. 运算法则.

$\lim_{n\to\infty} x_n = A$. $\lim_{n\to\infty} y_n = B$.

"multiply". 插项: $|x_n y_n - AB| = |x_n y_n - Ay_n + Ay_n - AB|$

$$\le |x_n - A| \cdot |y_n| + |A| \cdot |y_n - B|$$

let M a bound of $\{y_n\}$. $n \ge N$ 时. $|y_n| \le M$. $\le |x_n - A| \cdot |M| + |y_n - B| \cdot |A|$ $\quad \frac{\varepsilon}{2|A|}$

"division" $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{B}$

proof: $|y_n - B| < \frac{B}{2}$. (when $n \in N$)

$|y_n| = |B - B + y_n| \ge |B| - |y_n - B| > \frac{B}{2}$

$\left|\frac{1}{y_n} - \frac{1}{B}\right| = \frac{|y_0 - B|}{|y_n| \cdot |B|} < \frac{|y_n - B|}{\frac{1}{2}|B|^2}$ $\quad |y_n - B| < \frac{2\varepsilon}{|B|^2}$

Proof: $||a| - |b|| \le |a-b|$

$|a| = |a-b| + |b| \le |a-b| + |b|$

$\begin{cases} |a| - |b| \le |a-b| \\ |b| - |a| \le |a-b| \end{cases}$ 对换.

Useful property of this proval.

(1) $\lim_{n\to\infty} x_n = A \iff x_n = A + \alpha$. $\alpha \to 0$ $(n \to \infty)$

(2) $|x_n| \le M$. $\lim_{n\to\infty} y_n = 0 \Rightarrow \lim_{n\to\infty} (x_n \cdot y_n) = 0$ (无穷小量也是有限量).

apply: $\left|\frac{1}{y_n} - \frac{1}{B}\right| = \left|\frac{1}{B+\beta} - \frac{1}{B}\right| = \left|\frac{-\beta}{B(B+\beta)}\right|$ $\quad |B+\beta| \ge |B| - |\beta| \ge \frac{|B|}{2}$

c) Passage to the limit and Inequalities

Thm 3 $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n = B$. If $A < B$. then exists an index $N \in \mathbb{N}$.
such that $x_n < y_n$ for all $n \in \mathbb{N}$.

Corollay. $\lim_{n\to\infty} x_n = A$ and $\lim_{n\to\infty} y_n B$. If there exists N such that for all $n > N$.

a) $x_n > y_n \Rightarrow A \ge B$. ($\frac{2}{n}$ & $\frac{1}{n}$).

b) $x_n \ge y_n \Rightarrow A \ge B$.

c) $x_n > B \Rightarrow A \ge B$ ($y_n = B$. constant sequence)

d) $x_n \ge B \Rightarrow A \ge B$.

Cauchy Criterion.

Def 7. A sequence $\{x_n\}$ is called a fundamental or Cauchy sequence. if for any $\varepsilon > 0$.
there exists an index $N \in \mathbb{N}$. s.t. $|x_m - x_n| < \varepsilon$ whenever $(n > N) \wedge (m > N)$

(不需要预先知极限)

Cauchy's convergence criterion A numerical sequence converges. $\Leftrightarrow$ it's a Cauchy sequence

"$\Leftarrow$" If $\{x_n\}$ is finite-valued. then $\{x_n\}$ must be ultimately constant. Hence it's convergent

If $\{x_n\}$ is infinite-valued, then. for $\varepsilon = 1$. there exists $N_1 \in \mathbb{N}$ s.t. $|x_n - x_m| < \varepsilon = 1$ ~~when~~
when $n, m > N_1$. $\forall n > N_1$. $|x_n| = |x_n - x_{N_1+1} + x_{N_1+1}| \le |x_n - x_{N_1+1}| + |x_{N_1+1}| < 1 + |x_{N_1+1}|$

set $M = \max\{|x_1|, |x_2|, \dots, 1 + |x_{N_1+1}|\}$, then $\forall n \in \mathbb{N}$, we have $|x_n| \le M$

By the limit Point lemma. there exists $A \in \mathbb{R}$. such that A is the ~~po~~ limit point of $\{x_n\}$

$\forall \varepsilon > 0$. there exists $N_2 \in \mathbb{N}$. s.t. $|x_n - x_m| < \frac{\varepsilon}{2}$ when $n, m > N_2$

Since A is the limit points. there exists x_{n_0} such that $|x_{n_0} - A| < \frac{\varepsilon}{2}$ and $n_0 > N_2$.

Then when $n > N_2$. $|x_n - A| \le |x_n - x_{n_0}| + |x_{n_0} - A| = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

* Use nested interval principle: $a_n := \inf_{k \ge n} x_k$. $b_n := \sup_{k \ge n} x_k$.

e.g. $x_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$. ($\sum_{n=1}^{\infty}\frac{1}{n}$. 调和级数. divergent)

Since $|x_{2n}-x_n| = \frac{1}{n+1}\cdots+\frac{1}{n+n} > n\cdot\frac{1}{2n} = \frac{1}{2}$

for all $n\in\mathbb{N}$. we can find $n, 2n$. By Cauchy criterion implies this sequence does not have a limit. ($\exists\varepsilon>0.\ \forall N\in\mathbb{N},\ \exists m,n>N$. s.t. $|x_m-x_n|\geq\varepsilon$)

Def 8. $x_n < x_{n+1}$ increasing
$x_n \leq x_{n+1}$ nondecreasing (for all $n\in\mathbb{N}$). monotonically 单调地.
...

Def 9. Bounded above. $\exists M\in\mathbb{R}$. s.t. $x_n < M$. for all $n\in\mathbb{N}$.

Monotone Bound Criterion (Weierstrass)

Thm 5 (Weierstrass). (单调有界准则). In order for a nondecreasing sequence to have a limit it's necessary and sufficient that it be bound above

Proof: "$\Leftarrow$" By the least upper bound principle. there exists $\alpha=\sup_{n\in\mathbb{N}}\{x_n\}$

$\forall\varepsilon>0$ by the definition of α. there exists x_N such that $\alpha-\varepsilon < x_N$

when $n>N$, $\alpha-\varepsilon < x_N \leq x_n \leq \alpha < \varepsilon+\alpha$, i.e. $|x_N-\alpha|<\varepsilon$.

e.g.1 $\lim_{n\to\infty}\frac{n}{q^n}=0$ ($\lim_{n\to\infty}\frac{n^k}{q^n}=0 \Rightarrow (\lim_{n\to\infty}\frac{n}{q^{\frac{n}{k}}})^k$) $x_{n+1}=\frac{n+1}{n}x_n$.

Proof: (increasing) $x_{n+1}=\frac{n+1}{nq}x_n$. (* technique: 递推式两边同时取极限. 不一定有用 推出 $A=A$!. 对幂指型也要注意. 不可直接出现 A^n 的形式 ($A\geq1$ 大小不定))

e.g.2 $\boxed{\lim_{n\to\infty}\sqrt[n]{n}=1}$. (corollary: $\frac{n}{q^n}<1$ q 代入 $1+\varepsilon$)

Proof: ① for any given $\varepsilon>0$. $\exists N\in\mathbb{N}$. $1\leq n<(1+\varepsilon)^n$ for all $n\in\mathbb{N}$.

Then $1-\varepsilon<1\leq\sqrt[n]{n}<1+\varepsilon$.

② $1-\varepsilon<1\leq\sqrt[n]{n}=\sqrt[n]{\sqrt{n}\cdot\sqrt{n}\cdot\underbrace{1\cdot1\cdot1\cdot1\cdots}_{n-2个}} \leq \frac{2\sqrt{n}+n-2}{n} = \frac{2}{\sqrt{n}}+1-\frac{2}{n} < \frac{2}{\sqrt{n}}+1 < 1+\varepsilon$.

e.g.3 $\boxed{\lim_{n\to\infty}\sqrt[n]{a}=1.\ (a>0)}$

Proof: $a\geq1$. For any $\varepsilon>0$. $\exists N\in\mathbb{N}$. $1\leq a<n<(1+\varepsilon)^n$ for all $n>N$.

$0<a<1$ $\lim_{n\to\infty}\sqrt[n]{a} = \lim\frac{1}{\sqrt[n]{\frac{1}{a}}} = \frac{1}{\lim_{n\to\infty}\sqrt[n]{\frac{1}{a}}} = 1$.

e.g.4 $\lim_{n\to\infty}\frac{q^n}{n!}=0$. q is any real number, $n\in\mathbb{N}$. *property: $\lim_{n\to\infty}x_n=0 \Leftrightarrow \lim_{n\to\infty}|x_n|=0$.

$q=0$. trivial

$|q|<1$. $\lim_{n\to\infty}q^n\cdot\lim\frac{1}{n!}=0\cdot0=0$.

$|q|\geq1$ $\lim_{n\to\infty}\frac{|q|^n}{n!}=0 \Leftrightarrow \lim_{n\to\infty}\frac{q^n}{n!}=0$.

$\Rightarrow \frac{x_{n+1}}{x_n}=\left|\frac{q}{n+1}\right|<1$. 两边取limit

Bernoulli's inequality $(1+\alpha)^n\geq1+n\alpha$. for $n\in\mathbb{N}$. $\alpha>-1$ (induction)

$e := \lim_{n\to\infty}(1+\frac{1}{n})^n$

△1 $y_n = (1+\frac{1}{n})^{n+1}$ is decreasing

$$\frac{y_{n-1}}{y_n} = \frac{(1-\frac{1}{n-1})^n}{(1+\frac{1}{n})^{n+1}} = \frac{n^{2n}}{(n^2-1)^n}\cdot\frac{n}{n+1} = (1+\underbrace{\frac{1}{n^2-1}}_{\alpha})^n\frac{n}{n+1} \geq (1+\frac{n}{n^2-1})\frac{n}{n+1} > (1+\frac{1}{n})\frac{n}{n+1} = 1$$

$\{y_n\}$ is convergent. (lower bound 0).

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty}(1+\frac{1}{n})^{n+1}\cdot(1+\frac{1}{n})^{-1} = \lim_{n\to\infty}(1+\frac{1}{n})^{n+1}.$$

△2. $(1+\frac{1}{n})^n = C_n^0 + C_n^1\cdot\frac{1}{n} + C_n^2\cdot(\frac{1}{n})^2 + \cdots + C_n^n\cdot(\frac{1}{n})^n$

$= 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n}).$

$(1+\frac{1}{n+1})^{n+1} = 1+1+\frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1}) + \cdots \frac{1}{n!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1}) + (\frac{1}{n+1})^{n+1}$

$(1+\frac{1}{n+1})^{n+1} > (1+\frac{1}{n})^n$

upper bound: $(1+\frac{1}{n})^n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

$< 1+1+\frac{1}{2!}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}$

$< 3.$

Subsequences and Partial Limits of a Sequence.

<u>Def 11</u> If $x_1, x_2, \dots x_n, \dots$ is a sequence and $n_1 < n_2 < \dots < n_k < \dots$ an increasing sequence of natural numbers, then the sequence $x_{n_1}, x_{n_2} \dots x_{n_k} \dots$ is called a subsequence of $\{x_n\}$
$k \leq n_k$

Lemma 1. Every bounded sequence of real numbers contains a convergent subsequence. 有界数列必有收敛子列. 单调数列, 存在收敛子列, 则收敛.
(Bolzano - Weierstrass). 先证极限点 √. (致密性定理)

the set of value of sequence → finite constant. / → infinite ⟶ limit point α ($|x_{n_1}-\alpha|<1$, $|x_{n_k}-x|<\frac{1}{k}$. $|x_{n_{k+1}}-x|<\frac{1}{k+1}$ ...)

<u>Def 12</u> We shall write $x_n \to +\infty$ and say that the sequence $\{x_n\}$ tends to positive infinity if for each number c there exists $N\in\mathbb{N}$ such that $x_n > c$ for all $n > N$

$(x_n\to\infty) := \forall c\in\mathbb{R}\ \exists N\in\mathbb{N}.\ \forall n\in N\ (c<|x_n|)$

unbounded $\overset{\Leftarrow\sqrt{}}{\not\Rightarrow}$ infinity e.g. $n^{(-1)^n}$

Lemma 2. From each sequence of real numbers one can extract either a convergent subsequence or a subsequence that ~~that~~ tends to infinity.

consider the sequence $i_n = \inf_{k\geq n} x_k$.

<u>Def 13</u>. The number $l = \lim_{n\to\infty}\inf_{k\geq n} x_k$ is called the inferior limit of the sequence $\{x_k\}$

denote. $\underline{\lim}_{k\to\infty} x_k$. or $\liminf_{k\to\infty} x_k$. (Can be $\pm\infty$). when $\liminf_{n\to\infty} x_k = -\infty$. not bounded below.

Coro: $\overline{\lim}\, x_n = -\underline{\lim}(-x_n)$

$\overline{\lim}\, x_n = \frac{1}{\underline{\lim}\frac{1}{x_n}}$.

<u>Def 14</u> $\overline{\lim}_{k\to\infty} x_k := \lim_{n\to\infty}\sup_{k\geq n} x_k$.

e.g. $\underline{\lim}_{k\to\infty} x_k = 0$ $\quad \overline{\lim}_{k\to\infty} x_k = +\infty$ when $x_k = k^{(-1)^k}$ $(k\in\mathbb{N})$

e.g. $\underline{\lim}_{k\to\infty} x_k = +\infty$ $\quad \overline{\lim}_{k\to\infty} x_k = +\infty$ when $x_k = k$ $(k\in\mathbb{N})$

e.g. $x_k = \frac{(-1)^k}{k}$, $k\in\mathbb{N}$.

$$\underline{\lim}_{k\to\infty}\frac{(-1)^k}{k} = \lim_{n\to\infty}\left|\begin{matrix}-\frac{1}{n}, \text{ if } n=2m+1\\ -\frac{1}{n+1}, \text{ if } n=2m\end{matrix}\right| = 0 \qquad \overline{\lim}_{k\to\infty}\frac{(-1)^k}{k} = \lim_{n\to\infty}\left|\begin{matrix}\frac{1}{n}, \text{ if } n=2m\\ \frac{1}{n+1}, \text{ if } n=2m+1\end{matrix}\right| = 0.$$

Def 15. A number / $-\infty$ / $+\infty$ is called partial limit of a sequence, if the sequence contains a subsequence converging to that number

Pro 1. The inferior and superior limits of a bounded sequence are respectively the smallest and largest partial of the sequence ↳ 不含 $\pm\infty$.

Coro* If $\lim_{n\to\infty} x_n = A$, and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then $\lim_{k\to\infty} x_{n_k} = A$. (since $\lim_{k\to\infty} n_k = +\infty$)

Coro 3. A sequence has a limit or tends to +/- infinity if and only if its inferior and superior limits are the same.

"⇒" ① $s = \lim_{k\to\infty} x_{n_k} = A$. similarly $i = A$.

② Assume $i < s$. (x_n 对应的是 i_n)

Let $\varepsilon = \frac{s-i}{3}$, $\forall N \in \mathbb{N}$, $\exists (N_1 > N) \wedge (N_1 \in \mathbb{N})$, $|i_n - i| < \frac{\varepsilon}{6}$ when $n > N_1$.

s.t. $i_{n_1} \le x_{n_1} < i_{n_1} + \frac{\varepsilon}{6}$. $|x_{n_1} - i_{n_1}| < \frac{\varepsilon}{6}$.

$|x_{n_1} - i| = |x_{n_1} - i_{n_1} + i_{n_1} - i| \le |x_{n_1} - i_{n_1}| + |i_{n_1} - i| \le \frac{\varepsilon}{3}$

similarly, we can find x_{n_2} where $n_2 > N$ s.t. $s - \frac{\varepsilon}{3} < x_{n_2}$.

$|x_{n_2} - x_{n_1}| \ge \frac{\varepsilon}{3}$. contradicts ~~wh~~ with the sequence is convergent.

Coro 4. A sequence converges ⇔ every subsequence of it converges

odd and even subsequences are convergent.

(推广 mod 3,4,... 覆盖所有项即可)

3.1.4 Elementary Facts about Series.

Def 16. $\sum_{n=1}^{\infty} a_n$. a series or an infinite series.

(can be number / divergent)

Def 17. The elements of the sequence $\{a_n\}$, when regarded as elements of the series, are called the terms of the series.

Def 18. The sum ~~sum~~ $S_n = \sum_{k=1}^{n} a_k$. is called the partial sum.

Def 19. the sequence $\{S_n\}$, converges ⇔ series is convergent.

Def 20. The $\lim_{n\to\infty} S_n = s$ of the sequence of partial sums of the series if it exists, is called the sum of the series. $\sum_{n=1}^{\infty} a_n = s$. (可数无穷多项取值)

The Cauchy convergence criterion for a series.

The series $a_1+\cdots+a_n+\cdots$ converges $\Leftrightarrow$ for every $\varepsilon>0$ and $p\in\mathbb{N}$ $(p\in\mathbb{N})\vee(p=0)$, there exists $N\in\mathbb{N}$, such that the inequalities $n<N$, $|a_n+a_{n+1}+\cdots+a_{n+p}|<\varepsilon$

coro7. A necessary condition for convergence of the series $a_1+\cdots+a_n+\cdots$ is that the terms tend to zero as $n\to\infty$, that is necessary that $\lim\limits_{n\to\infty}a_n=0$. (必要条件一般项趋于0)

e.g. 20. geometric series. $1+q+q^2+\cdots+q^n+\cdots$ $\quad S_n=\frac{1-q^n}{1-q}$ $(q\neq1)$.

e.g. 21 harmonic series. $1+\frac{1}{2}+\cdots+\frac{1}{n}+\cdots$ (取 $x_{2n}-x_n$, By cauchy… divergent. 阶不可能快)
(each term from the second on is the harmonic mean of the two terms on either side of it).

* 无穷项相加的重排、结合(括号)运算需慎重(发散→收敛???). 绝对收敛才可 √. absolutely convergent.

Def21 The series $\sum\limits_{n=1}^{\infty}a_n$ is absolutely convergent if $\Leftarrow$ the series $\sum\limits_{n=1}^{\infty}|a_n|$ converges

e.g. $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\cdots$ partial sums are $\frac{1}{n}$ or 0. convergent, but not absolutely.
convergence, not absolutely convergence. $\Rightarrow$ conditional convergence.

Thm7. Criterion for convergence of series of nonnegative terms
A series $\{a_n\}$ whose terms are nonnegative converges if and only if the sequence of partial sums bounded above. (部分和数列有上限,正项级数收敛).

Thm 8. Comparison theorem.
$\sum\limits_{n=1}^{\infty}a_n$, $\sum\limits_{n=1}^{\infty}b_n$, be two series with nonnegative ← 前提☆. $\exists N\in\mathbb{N}$, s.t. $a_n\le b_n$ for all $n>N$.
$\sum\limits_{n=1}^{\infty}b_n$ converges $\Rightarrow\sum\limits_{n=1}^{\infty}a_n$ converges. $\sum\limits_{n=1}^{\infty}b_n$ diverges $\Leftarrow\sum\limits_{n=1}^{\infty}a_n$ diverges.
(Fact: $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ ⊙. Fourier series).

coro8 The Weierstrass M-test for absolute convergence.
Let $\sum\limits_{n=1}^{\infty}a_n$ and $\sum\limits_{n=1}^{\infty}b_n$ be series. Suppose $\exists N\in\mathbb{N}$, s.t. $|a_n|\le b_n$. for all $n>N$.
Then a sufficient condition for absolute convergence of the series $\sum\limits_{n=1}^{\infty}a_n$ is that the series $\sum\limits_{n=1}^{\infty}b_n$ converge.

~~数样~~ 一般地,先考查绝对收敛

coro9 Cauchy's test. (基于自身) 根值判别法
Let $\sum\limits_{n=1}^{\infty}a_n$. be a given series and $\alpha=\overline{\lim\limits_{n\to\infty}}\sqrt[n]{|a_n|}$
a) $\alpha<1$. $\sum\limits_{n=1}^{\infty}a_n$ converges absolutely. b) $\alpha>1$ $\sum\limits_{n=1}^{\infty}a_n$ diverges ⇓ 本身也发散.
c) $\alpha=1$ both serieses are existed.
$\sum\limits_{n=1}^{\infty}\frac{1}{n}$ diverges $\quad\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Proof: we can find $N\in\mathbb{N}$.
s.t. $(\sqrt[n]{|a_n|}<q)\wedge(q=\frac{1+\alpha}{2})$ for all $n>N$
$\sqrt[k]{|a_k|}\le\sup\limits_{k\ge n}\sqrt[k]{|a_k|}<q$
i.e. $|a_n|<q^n$, when $n>N$ → 实质也借助了 $b_n=q^n$
$q<1$. $\sum\limits_{n=1}^{\infty}q^n$ converges

e.g. (幂级数 "收敛半径") $\to x\in(r-,r+)$ 收敛. power series.

$*\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n}$ alternative series

$\sum_{n=1}^{\infty}(2+(-1)^n)^n x^n$ we find $\alpha=\overline{\lim_{n\to\infty}}|2+(-1)^n|\cdot|x|=3|x|$. $|x|<\frac{1}{3}$ converges $|x|>\frac{1}{3}$ diverges.

$x=\pm\frac{1}{3}$. we consider $a_{2k}=(2+(-1)^{2k})^{2k}\cdot x^{2k}=3^{2k}(\frac{1}{3})^{2k}=1$. (考察一般项, 不趋于0, 一定 diverge).

Coro10. (d'Alembert's test 比值判别法).

Suppose the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\alpha$. exists (上极限也可 ✓) for the series $\sum_{n=1}^{\infty}a_n$.

$\alpha>1$ diverges $0<\alpha<1$ converges $\alpha=1$ both possible.

(In some series, we find the same α by Cauchy's and D'Alembert's test)

$*\sum_{n=1}^{\infty}\frac{1}{n^p}$ p-series (一般认为 $p>0$) $p\le 1$ diverge $(\frac{1}{n}\le\frac{1}{n^p})$ $p>1$ converges.

Proof: $\sum_{n=1}^{\infty}\frac{1}{n^p}=1+(\frac{1}{2^p})+(\frac{1}{3^p}+\frac{1}{4^p})+(\frac{1}{5^p}+\cdots+\frac{1}{8^p})+(\frac{1}{9^p}+\cdots+\frac{1}{16^p})+(\frac{1}{17^p}+\cdots+\frac{1}{32^p})+\cdots$

$<1+\frac{1}{2^p}+\frac{1}{2^{p-1}}+\frac{1}{4^{p-1}}+\frac{1}{8^{p-1}}+\cdots$

$=1+\frac{1}{2^p}+\sum_{n=1}^{\infty}(\frac{1}{2^{p-1}})^n$

比较判别法: 一般有效, 可为C敛散性一致.

Proposition 2. (Cauchy). If $a_1\ge a_2\ge\cdots\ge 0$. the series $\sum_{n=1}^{\infty}a_n$ converges if and only if the series $\sum_{k=0}^{\infty}2^k a_{2^k}=a_1+2a_2+4a_4+8a_8+\cdots$ converges (证明结果是具有相同

Consider e as a sum of a series.

$e_n=(1+\frac{1}{n})^n$. $S_n=1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots$

$(1+\frac{1}{n})^n=1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots+\frac{1}{k!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{k-1}{n})+\cdots\frac{1}{n!}(1-\frac{1}{n})\cdots(1-\frac{n-1}{n})<S_n$

For any given $k\in\mathbb{N}$,

$1+1+\frac{1}{2!}(1-\frac{1}{n})+\cdots\frac{1}{k!}(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})<e_n$.

When $n\to\infty$, LHS $\to S_k$. RHS $\to e$. since k is arbitrary. $S_n\le e$ (保存性)

estimate the difference $e-S_n$.

$e-S_n=\frac{1}{(n+1)!}\left[1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\cdots\right]<\frac{1}{(n+1)!}\left[1+\frac{1}{(n+2)}+\frac{1}{(n+2)^2}+\cdots\right.$

$=\frac{1}{(n+1)!}\frac{1}{1-\frac{1}{n+2}}=\frac{n+2}{n!(n+1)^2}<\frac{1}{n!n}$

estimate equality $e=S_n+\frac{\theta_n}{n!n}$ where $0<\theta_n<1$ ← "θ"在拉格朗日定理中再次出现

if e is rational number $e=\frac{p}{q}$. where $p,q\in\mathbb{N}$. $q!e$ must be an integer.

$q!e=q!(S_q+\frac{\theta_q}{q!q})=q!+\frac{q!}{1!}+\frac{q!}{2!}+\cdots+\frac{q!}{q!}+\frac{\theta_q}{q!}$ ← $\frac{\theta_q}{q!}$ can't be integer.

Theorem (Leibniz series).

$\sum_{n=1}^{\infty}(-1)^n U_n$. s.t. $U_n>0$. U_n is decreasing. $\lim_{n\to\infty}U_n=0$.

The Limit of a Function.

"ε-δ" $x \to x_0 / x_0^+ / x_0^-$ "ε-X" $x \to \pm\infty / \infty / -\infty$.

去心邻域 $\mathring{U}_\delta(x_0)$ (Deleted neighborhood).

$\forall \varepsilon > 0.\ \exists \delta > 0\ \forall x \in E\ (0 < |x-a| < \delta \Rightarrow |f(x) - A| < \varepsilon). \iff \lim_{E \ni x \to a} f(x) = A.$

x 来自 E. 否则默认趋于最大定义域

Def2. A deleted neighborhood of a point is a neighborhood of the point from which itself has been removed.

$(\lim_{E \ni x \to a} f(x) = A) := \forall V_{\mathbb{R}}^{\varepsilon}(A)\ \exists \mathring{U}_E^{\delta}(a)\ (f(\mathring{U}_E^{\delta}(a)) \subset V_{\mathbb{R}}^{\varepsilon}(A))$

(自变量邻域). 点 a 在 E 中的去心 δ 邻域.

$(\lim_{E \ni x \to a} f(x) = A) := \forall V_{\mathbb{R}}(A)\ \exists \mathring{U}_E(a)\ (f\mathring{U}_E(a)) \subset V_{\mathbb{R}}(A))$

☆☆☆ 证在 $x=0$ 无极限. 找子极限. 子极限本身不行. 其他数也不行.

e.g. signum x: $\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases}$ $\lim_{x \to 0} |\operatorname{sgn} x| = 1.$

有极限 $\Rightarrow$ 加绝对值有极限. 极限为0.等价

Pro1 (Heine's Theorem) 数列与函数极限的联系. 可导 > 连续 > 有极限.

The relation $\lim_{E \ni x \to a} f(x) = A$ holds $\iff$ for every sequence of the point $x_n \in E \setminus a$. converging to a. the sequence $\{f(x_n)\}$ converges to A.

Proof: "$\Rightarrow$" $\forall \varepsilon > 0$, we want $|f(x_n) - A| < \varepsilon$.

since $\lim_{x \to a} f(x) = A$, we can find $\delta > 0$, s.t. $|f(x) - A| < \varepsilon$. when $0 < |x - a| < \delta$

since $\lim_{n \to \infty} x_n = a$, there exists $N \in \mathbb{N}$, s.t. $|x_n - a| < \delta$ when $n > N$.

Hence, $|f(x_n) - A| < \varepsilon$.

"$\Leftarrow$" Assume that $\lim_{x \to a} f(x) \neq A$. then there exists $\varepsilon_0 > 0$

s.t. $\forall \frac{1}{n} > 0\ (n \in \mathbb{N})$. there exists $0 < |x_n - a| < \frac{1}{n}$, $|f(x_n) - A| \geq \varepsilon_0$.

Since $\lim_{n \to \infty} x_n = a$. by the condition, we have $\lim_{n \to \infty} f(x_n) = A$.

which contradicts with $|f(x_n) - A| \geq \varepsilon_0$, $n \in \mathbb{N}$.

Thus the assumption is false. i.e. $\lim_{n \to \infty} f(x) = A$.

General Property

① ultimately constant. (与极限过程有关) ultimately bounded $(\exists \lim_{E \ni x \to a} f(x)) \Rightarrow (f: E \to \mathbb{R}$ is ultimately bounded) bounded (range of function is bounded).

$|f(x)| < C$. $f(x) < C$. $C < f(x)$.

e.g. $f(x) = \sin\frac{1}{x} + x\cos\frac{1}{x}$. ultimately bounded as $x \to 0$.

② * $\lim_{E \ni x \to a} f(x) = A$ is unique.

③ Arithmetic Operations (limit and arithmetic change the order)

④ infinitesimal $\lim_{x \to a} f(x) = 0$. $(\lim_{x \to a} f(x) = A) \iff (f(x) = A + \alpha(x) \wedge \lim_{x \to a} \alpha(x) = 0)$.

⑤ preserve the order. ($\lim < \Rightarrow f(x) <$ $f(x) \leq \Rightarrow \lim \leq$ $f(x) < \Rightarrow \lim \leq$.

⑥ sequeezee theorem

e.g. $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$.

几何. $\cos^2 x<\frac{\sin x}{x}<1$.

$\lim\limits_{x\to 0}(1-\sin^2 x) = 1-(\lim\limits_{x\to 0}\sin x)(\lim\limits_{x\to 0}\sin x)=1-0=1$

proof. $\lim\limits_{x\to x_0}\cos x=\cos x_0$.

$\forall\varepsilon>0$. $|\cos x-\cos x_0| = 2\cdot\left|\sin\frac{x+x_0}{2}\right|\cdot\left|\sin\frac{x-x_0}{2}\right|$

$|x-x_0|\to 0$, $\left|\sin\frac{x-x_0}{2}\right|\to 0$.

3.2.4 Existence of the Limit of a Function.

Def. oscillation (振荡中的振幅) of a function $f:\mathbb{X}\to\mathbb{R}$ on set $E\subset\mathbb{X}$.

$$\omega(f;E):=\sup_{x_1,x_2\in E}|f(x_1)-f(x_2)|$$

The Cauchy criterion (for the existence of the limit of a function.

A function $f:\mathbb{X}\to\mathbb{R}$ has a limit $\Leftrightarrow$ for every $\varepsilon>0$ there exists $\delta>0$.

s.t. for any $x,x'\in\mathring{U}_\delta(x_0)$, $|f(x)-f(x')|<\varepsilon$. (i.e. $\omega(f.\mathring{U}_\delta(x_0))<\varepsilon$.)

趋于 trans to.

"$\Leftarrow$" Let $\varepsilon_n=\frac{1}{n}$, $\exists\delta_n>0$, s.t $\omega(f,\mathring{U}_{\delta_n}(x_0))<\frac{1}{n}$

We can choose $x_n\in\mathring{U}_{\delta_n}(x_0)$, we obtain sequence $\{x_n\}$, $\{f(x_n)\}$

$\forall\varepsilon>0$, Let $N=[\frac{1}{\varepsilon}]$, $|f(x_n)-f(x_m)|<\varepsilon$, when $n,m\in N$ ($<\frac{1}{N}<\varepsilon$)

the sequence $\{f(x_n)\}$ is Cauchy's. Let $\lim\limits_{n\to\infty}f(x_n)=A$.

$\forall\varepsilon>0$, $\exists N_1$, s.t. $|f(x_n)-A|<\frac{\varepsilon}{2}$. when $n>N_1$

$\exists N_2$. s.t. $|f(x)-f(x_n)|<\frac{\varepsilon}{2}$, when $x,x_n\in\mathring{U}_{\delta_{N_2}}(x_0)$ ($\frac{1}{N_2}<\frac{\varepsilon}{2}$). $\frac{1}{N}\downarrow$

set $N=\max\{N_1,N_2\}$. $|f(x)-A|<\varepsilon$ (when $n>N$, $x\in\mathring{U}_{\delta_N}(x_0)$ (i.e. $0<|x-x_0|<\delta_N$)

(用区间套公理：所有δ邻域的上确界、下确界形成两个集合. 此处闭区间套不言. 区间数可能不可数).
并且该上、下确界区间只趋于0. (唯一).

The Limit Composite Function.

外函数在u_0连续即可去。

$\lim\limits_{u\to u_0}f(u)=A$, $u=g(x)$, $\lim\limits_{x\to x_0}g(x)=u_0$, $\exists\delta_0>0$ such that $g(x)\neq u_0$. when $x\in\mathring{U}_{\delta_0}(x_0)$.

Then $\lim\limits_{x\to x_0}f(g(x))=A$. (内函数极限值是外函数极限点)

$\lim\limits_{x\to\infty}(1+\frac{1}{x})^x=e$. $\left(1+\frac{1}{[x]+1}\right)^{[x]}<\left(1+\frac{1}{x}\right)^x<\left(1+\frac{1}{[x]}\right)^{[x]+1}$

$\Leftrightarrow\lim\limits_{t\to 0}(1+t)^{\frac{1}{t}}=e$ (一的无穷次方未定式).

The Limit of a Monotonic Function

Thm. Suppose $f: X \to \mathbb{R}$ s.t. f is nondecreasing on interval $I \subset X$. Then $\lim_{x\to x_0^-} f(x)$. $\lim_{x\to x_0^+} f(x)$ exist.
$x_0 \in I$. (此处界是 $f(x_0)$. 证明时表示出确界. 确界与极限建立关系).
(单调. 保证单侧极限).

Comparison of the Asymptotic Behavior of Functions. (无穷大. 小. 相对比较).

渐近的.

$\pi(x) \to$ number of primes not larger than x. $\pi(x) = \frac{x}{\ln x} + o(\frac{x}{\ln x})$ [asymptotic law]
$\pi(x) \to +\infty$.. grows approximately like $\frac{x}{\ln x}$.

Def. The function f is said to be infinitesimal compared with the function g over the $\mathring{U}_\delta(x_0)$. we write $f = o(g)$. if the relation $f(x) = \alpha(x) g(x)$ holds ultimately over the $\mathring{U}_\delta(x_0)$. where $\alpha(x)$ is a function that is infinitesimal over $\mathring{U}_\delta(x_0)$.
($\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$. $\Leftrightarrow$ $f(x) = o(g(x))$ reads "f is little-oh of g).

① $f = o(g)$. f is an infinitesimal of higher order than g over $\mathring{U}_\delta(x_0)$ $\Leftrightarrow \lim_{x\to \square} \frac{f(x)}{g(x)} = 0$.
"表达到 $+\infty / 0$ 效果更快的是高阶".
(若 $f(x)$. $g(x)$ 均为无穷大. $g(x)$ 高阶)

② $f(x) = O(g(x))$, $\lim_{x\to\square} \frac{f(x)}{g(x)} = C \neq 0$. ($\frac{f(x)}{g(x)}$ is ultimately bounded, the $\lim_{x\to\square}\frac{f(x)}{g(x)}$ can be not existed!)
(f is big-oh of g over). $\neq$ f, g of same order

the relation $f(x) = \beta(x) g(x)$ holds ultimately over $\mathring{U}_\delta(x_0)$, where $\beta(x)$ is ultimately bounded over $\mathring{U}_\delta(x_0)$.

e.g. verify $(\frac{1}{x} + \sin x) x = O(x)$. as $x \to \infty$
$\frac{(\frac{1}{x}+\sin x)x}{x} = \frac{1}{x} + \sin x$, $\frac{1}{x}$ is ultimately bounded; $\sin x$ is bounded.
hence. $\frac{1}{x} + \sin x$ is ultimately bounded.

Def. The functions f and g are of the same order over $\mathring{U}_\delta(x_0)$, we write $f \asymp g$ over $\mathring{U}_\delta(x_0)$. if $f = O(g)$ and $f = O(g)$ simultaneously
$f \asymp g$
(为了防止 $\beta(x)$ 的 bound 中有0. 双向成立).
$\Leftrightarrow$ over the $\mathring{U}_\delta(x_0)$. exists $c_1 > 0$. and $c_2 > 0$.

Def 25. the relation $f(x) = \gamma(x) g(x)$ holds ultimately over $\mathring{U}_\delta(x_0)$ where $\lim_{x\to x_0} \gamma(x) = 1$. we say that the function f behaves asymptotically like g over $\mathring{U}_\delta(x_0)$, or, more briefly, that f is equivalent to g over $\mathring{U}_\delta(x_0)$. $f \sim g$ $(x \to \square)$

近似的. 渐近的.

e.g. $\lim_{x\to 0} \frac{\sin x}{x} = 1$. $\Leftrightarrow$ $\sin x = x + o(x)$

错位 ×

* 离散版 积分第一中值定理.
$f(x)$ 在 (a,b) 连续. $g(x)$ 在 (a,b) 内有定义, $g(x) > 0$. 证. 对于 (a,b) 内任意 n 个点 $x_1, x_2, \dots, x_n$, 至少存在一点 $\xi \in (a,b)$
不变号即可
使得 $\sum_{i=1}^n f(x_i) g(x_i) = f(\xi) \sum_{i=1}^n g(x_i)$

证: w.l.o.g. $x_1 \le x_2 < \dots < x_n$. 在 $[x_1, x_n] \subset (a,b)$ 上. 连续函数 $f(x)$ has ~~max~~ $f(x)_{max} = M$. $f(x)_{min} = m$.

$m \le f(x_i) \le M$. $\quad m\, g(x_i) \le f(x_i) g(x_i) \le M \cdot g(x_i)$

$$\Rightarrow m \cdot \sum_{i=1}^n g(x_i) \le \sum_{i=1}^n f(x_i) g(x_i) \le M \cdot \sum_{i=1}^n g(x_i)$$

$$m \le \frac{\sum_{i=1}^n f(x_i) g(x_i)}{\sum_{i=1}^n g(x_i)} \le M$$ 介值定理 $\exists\, \xi \in [x_1, x_n]$ $f(\xi) = \frac{\sum_{i=1}^n f(x_i) g(x_i)}{\sum_{i=1}^n g(x_i)}$

e.g. $(1+x)^{\alpha}-1 \sim \alpha x$ (when $x \to 0$).

proof: $\lim\limits_{x\to 0} \dfrac{(1+x)^{\alpha}-1}{\alpha x} \xlongequal{(1+x)^{\alpha}-1=t} \lim\limits_{t\to 0} \dfrac{t}{\ln(t+1)} = 1.$

$(\alpha \ln(1+x) = \ln(t+1))$.

$\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \ln(1+x)^{\frac{1}{x}}$

$= \lim\limits_{x\to 0} \ln e = 1.$

<u>Pro.3.</u> If $f \sim \tilde{f}$ over $(\mathring{U}_{\delta}(x_0))$, then $\lim\limits_{x\to x_0} f(x)g(x) = \lim\limits_{x\to x_0} \tilde{f}(x)g(x)$, provided one of these limits exists. (only in multiplication (division)).

$(\sqrt{x^2+x}-x \sim x$ as $x \to +\infty)$.

<u>Pro4.</u> (1) $o(f) + o(f) = o(f)$

(2) $o(f)$ is also $O(f)$

(3) $o(f) + O(f) = O(f)$

(4) $O(f) + O(f) = O(f)$

(5) if $g(x) \neq 0$ then $\dfrac{o(f(x))}{g(x)} = o\left(\dfrac{f(x)}{g(x)}\right)$ and $\dfrac{O(f(x))}{g(x)} = O\left(\dfrac{f(x)}{g(x)}\right)$.

对无穷小：取低(阶)不取高.

$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \cdots$ for $x \in \mathbb{R}$

$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots + \frac{(-1)^k}{(2k)!}x^{2k} + \cdots$ for $x \in \mathbb{R}$.

$\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^k}{(2k+1)!}x^{2k+1} + \cdots$ for $x \in \mathbb{R}$.

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots + \frac{(-1)^{n-1}}{n}x^n + \cdots$ for $|x|<1$

$(1+x)^{\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots$ for $|x|<1$

* $\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^n + \cdots$ ($|u|<0$. 等比求和).

§4. Continuous Functions.

Def. For any $\varepsilon > 0$, there exists $\delta > 0$, when $|x - x_0| < \delta \wedge x \in E$. s.t. $|f(x) - f(x_0)| < \varepsilon$. (=0 is allowed).

$\Leftrightarrow \lim\limits_{x\to x_0} f(x) = f(x_0) \Leftrightarrow \omega(f;a) = \lim\limits_{\delta\to +0} \omega(f; U_E^{\delta}(a))$ $(U_{\delta}(a))$.

Def: Oscillation $\omega(f; U_E^{\delta}(a)) = \sup\limits_{x_1, x_2 \in U_{\delta}(a)} |f(x_1) - f(x_2)|$

Def: each point continuous $\to$ set continuous.

For closed interval $[a,b]$. a right continuous. / left continuous.

e.g. $|\sin x - \sin x_0| = \left|2\cos\frac{x+x_0}{2}\sin\frac{x-x_0}{2}\right| \le 2\left|\sin\frac{x-x_0}{2}\right| \le 2\left|\frac{x-x_0}{2}\right| \le |x-x_0| < \varepsilon$. (Let $\delta = \varepsilon$)

e.g. $f(x) = a^x$

proof: $\forall x_0 \in \mathbb{R}$. $\forall \varepsilon > 0$.

$|a^x - a^{x_0}| < \varepsilon \Leftrightarrow a^{x_0}|a^{x-x_0} - 1| < \varepsilon$.

$\Leftrightarrow \ln(1 - a^{-x_0}\varepsilon) < |x - x_0| \ln a < \ln(1 + a^{-x_0}\varepsilon)$ 分 $a<1$, $a>1$, $a=1$. 取 $\delta = \min\{\cdots\}$.

* 基本初等函数在其定义域内连续.

* arctan. range: $(-\frac{\pi}{2}, \frac{\pi}{2})$. can't represent any angle. $\pm\pi$ adjust. * $|x-x_0| < |x_0| \Rightarrow$ 同号.

$(\arctan x_1 - \arctan x_2 = \arctan \frac{x_1 - x_2}{1+x_1x_2})$

4.1.2. Discontinuity.

The points of discontinuity

1. First kind. (left/right limit both exists).

① removable discontinuity a point of discontinuity $a \in E$ of the function $f: E \to \mathbb{R}$ is s.t. there exists a continuous function $\tilde{f}: E \to \mathbb{R}$ s.t. $f|_{E \setminus a} = \tilde{f}|_{E \setminus a}$.

② jump discontinuity. left limit ≠ right limit.

2. Second kind.

Def. If $a \in E$ is a point of discontinuity of the function $f: E \to \mathbb{R}$ and at least one of the two limits $\lim_{x \to a^-} f(x)$ $(a-0)$, $\lim_{x \to a^+} f(x)$ $(a+0)$ does not exist, then a is called a discontinuity of second kind.

e.g. Dirichlet function $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0. & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

(discontinuous at every point)

Riemann function $R(x) = \begin{cases} \frac{1}{n} & x = \frac{m}{n} \in \mathbb{Q}. \text{ where } \frac{m}{n} \text{ is in lowest terms}, n \in \mathbb{N}. \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ (既约分数)

Especially. if $x \in \mathbb{Z}$. $R(x) = 0$. has period, $T=1$.

For any point $a \in \mathbb{R}$. any neiborhood $U(a)$ of it, and any number $N \in \mathbb{N}$, the neiborhood $U(a)$ contains only a finite number of rational numbers $\frac{m}{n}$. (with $n < N$) 限制分母. 有限个.

By shrinking (收缩) the neighborhood, one can then assume the denominators of all rational numbers in the neighborhood (except possibly for the point a itself if $a \in \mathbb{Q}$) are larger than N. 找邻域. 排除有限个. 其余分母大于N

Thus any point $x \in \mathring{U}(a)$. we have $|R(x)| < 1/N$

We have thereby shown that $\lim_{x \to a} R(x) = 0$. at any point $a \in \mathbb{R} \setminus \mathbb{Q}$.

irrational points, integer points ⇒ continuous.

other rational points ⇒ removable discontinuity.

4.2. Properties of Continuous Functions.

4.2.1 Local.

Thm I $f: E \to \mathbb{R}$. is continuous at point a.

1°. f is bounded. in some neighborhood $U_E(a)$ of a.

2° If $f(a) \neq 0$. in some neighborhood. $U_E(a)$ all the values of the function have the same sign as $f(a)$

3° If $g: U_E(a) \to \mathbb{R}$ is defined in some neighborhood of a and is continuous at a

then a) $(f+g) := f+g$

b) $(f \cdot g) := f \cdot g$

c) $(\frac{f}{g})(x) := \frac{f(x)}{g(x)}$ (provide $g(a) \neq 0$)

are defined in some neighborhood. of a and. continuous at a.

e.g. proof. $\lim_{x \to a} (f+g)(x) = \lim_{x \to a} (f(x) + g(x))$

$= \lim_{x \to a} (f(x)) + \lim_{x \to a} (g(x))$

$= f(a) + g(a)$

$= (f+g)(a)$

4°. If $g: Y \to \mathbb{R}$. is continuous at a point $b \in Y$ and f is such that $f: E \to Y$, $f(a) = b$, and f is continuous at. then the composite $(g \circ f)$ is defined on E and continuous at a (内函数不连续也可

coro 1 The composition of a finite number of continuous functions is continuous at each point of its domain of definition. (Follows by induction from assertion 4° of Thm 1)

4.2.2 Global Properties of Continuous Functions.

The Bolzano - Cauchy intermediate-value theorem. (零点存在定理) 中值 介值定理

If a function that is continuous on a closed interval assumes values with different signs at the endpoints of the interval, then there is a point in the interval where it assumes the value 0.

proof: Let us divide the interval $[a,b]$ into a half. (二分法)

i) $\frac{a+b}{2}=0$. the theorem holds ii) It must assume opposite values at the endpoints of one of the two subintervals. In that interval we proceed as we did with the original interval, that is, we bisect it and continue the process.

In the first case. we hit a point $c\in[a,b]$ where $f(c)=0$

In the second case, we obtain a sequence $\{I_n\}$ of nested closed interval whose lengths tend to zero and at whose endpoint f assumes values with opposite signs there exist $c\in[a,b]$ (By 区间套) common to all intervals. two sequence of endpoint $\{x_n'\}$ and $\{x_n''\}$, $f(x_n')<0$, $f(x_n'')>0$ while $\lim\limits_{n\to\infty} x_n' = \lim\limits_{n\to\infty} x_n'' = c$ $\lim\limits_{n\to\infty} f(x_n') = f(c) \leq 0$. $\lim\limits_{n\to\infty} (f(x_n'')) = f(c) \geq 0$. $\Rightarrow f(c)=0$.

Corollary. If the function φ is continuous on an open interval and assumes values $\varphi(a)=A$ and $\varphi(b)=B$ at point a and b, then for any number C between A and B, there is a point c between a and b at which $\varphi(c)=C$ $\boxed{\text{介值性}}$.

proof: the function $f(x)=\varphi(x)-C$ is defined and continuous

$f(a)\cdot f(b)=(A-C)(B-C)<0$. By ... theorem ..

The Weierstrass maximum-value theorem (最值定理).

A function that is continuous on a closed interval is bounded on that interval. Moreover there is a point in the interval where the function assumes its maximum value and a point where it assumes its minimal value.

proof: (First prove the boundedness)

closed interval $E=[a,b]$.

any point $x\in E$. find $U(x)$, s.t. function is bounded on the set $U_E(x)=U(x)\cap E$. local properties.

extract a finite system $U(x_1)\dots U(x_n)$ of open intervals that together cover the $[a,b]$

$U(x)$. bounded. $m_k\leq f(x)\leq M_k$. $\min\{m_1\dots m_k\}\leq f(x)\leq \{M_1,\dots M_N\}\Rightarrow f(x)$ is bounded.

Let $M=\sup\limits_{x\in E} f(x)$. assume M is not the maximum. $y=\frac{1}{M-f(x)}$. (continuous) & not bounded). contradict.

Def: Compact set. from every covering of E by open intervals one can extract a finite subcovering. (有限开覆盖).

在欧氏空间中. 即有界闭集.

Def: Uniformly continuous (更强. δ 统一. 不随 x_0 变化). 点越来越近, 差小不下去

For every $\varepsilon > 0$, there exists $\delta > 0$ s.t. $|f(x_1) - f(x_2)| < \varepsilon$ for all points $x_1, x_2 \in E$ s.t. $|x_1 - x_2| < \delta$.

区分: $f: E \to \mathbb{R}$ is continuous on $E := (\forall a \in E\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in E\ (|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon)$

在E上点点连续. pointwisely continuous.

e.g. $f: E \to \mathbb{R}$. is unbounded in every neiborhood of a fixed point $x_0 \in E$.

then it is not uniformly continuous

prove not uniformly continuity. ① find unbounded point. (every neighborhood of a fixed point.)

② find pairs of points. distance. $\to 0$.

difference of value $>$ a number

x^2 找 $\sqrt{x+1}$ 与 $\sqrt{x}$.

$\sin(x^2)$ 找 $\sqrt{\frac{\pi}{2}(n+1)}$ 与 $\sqrt{\frac{\pi}{2}n}$

The Cantor-Heine theorem on uniform continuity.

A function that is continuous on a closed interval is uniformly continuous on that interval

proof: $f: E \to \mathbb{R}$. $E = [a,b]$ and $f \in C(E)$

Since f is continuous at any point. $x \in E$. it follows that. for every give $\varepsilon > 0$. we can find a δ-neighbophood $U^{\delta}(x)$ of x s.t. $\omega(f; U_E^{\delta}(x)) < \varepsilon$. on the set $U_E^{\delta}(x)$.

We suppose $V(x) = U^{\frac{1}{2}\delta(x)}(x)$. For $[a,b]$ we can find a finite cover $V(x_1), V(x_2), \dots V(x_n)$

Let $\delta = \min\{\frac{1}{2}\delta(x_1), \dots \frac{1}{2}\delta(x_n)\}$. We need $|f'(x) - f''(x'')| < \varepsilon$ for any $|x' - x''| < \varepsilon$.

Since the system of $V(x_n)$ covers E. there exists an interval $V(x_i)$ of this system that contains x'. s.t. $|x' - x_i| < \frac{1}{2}\delta(x_i)$

$|x'' - x_i| \le |x' - x''| + |x' - x_i| < \delta + \frac{1}{2}\delta(x_i) \le \frac{1}{2}\delta(x_i) \times 2 = \delta(x_i)$

consequently $x', x'' \in U_E^{\delta(x_i)}(x_i)$, and $|f(x') - f(x'')| \le \omega(f; U_E^{\delta(x_i)}(x_i)) < \varepsilon$.

e.g. For $f \in C(\mathbb{R})$. $\lim\limits_{x \to \infty} f(x) = A$. $\Rightarrow$ uniformly continuous. on $\mathbb{R}$.

proof: 1°. Let X. $\forall |x_1|, |x_2| > X$. $|f(x) - A| < \frac{\varepsilon}{2}$ $\quad |f(x_1) - f(x_2)| \le |f(x_1) - A| + |f(x_2) - A|$

2°. $x \in [-X, X]$. $\delta_1 > 0$. (closed interval ...).

3°. continuous at $x = X$. $\exists \delta_2 > 0$. $\omega(f; U_{\delta(X)}(X)) < \varepsilon$. (local properties)

$x = -X$. $\exists \delta_3 > 0$. $\omega(f; U_{\delta(-X)}(X)) < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ ...

Proposition 1. A continuous mapping $f: E \to \mathbb{R}$ of a closed interval $E = [a,b]$ into $\mathbb{R}$ is injective $\Leftrightarrow$ f is strictly monotonic on $[a,b]$ (闭区间连续函数. 单调 $\Leftrightarrow$ 单射.

proof: if not. $x_1 < x_2 < x_3$. $f(x_2) < f(x_1) < f(x_3)$

$\exists \xi \in [x_2, x_3]$. $f(\xi) = f(x_1)$

Proposition 2. Each strictly monotonic $f: X \to \mathbb{R}$ defined on a numerial set $X \subset \mathbb{R}$ has an inverse $f^{-1}: Y \to \mathbb{R}$ defined on set $Y = f(X)$ of values of f and has the same kind of monotonicity on Y that f has on X.

proof: $\forall y_1, y_2 \in Y$ and $y_1 < y_2$

$f^{-1}(y_1) < f^{-1}(y_2) \Leftrightarrow f(f^{-1}(y_1)) < f(f^{-1}(y_2)) \Leftrightarrow y_1 < y_2$.

Proposition 3. The discontinuities of a function $f: E \to \mathbb{R}$ that is monotonic on the set $E \subset \mathbb{R}$. can be only discontinuities of first kind.
(满足函数单调有限准则. 有左右极限).

Coro 1. If a is a point of discontinuity of a monotonic function $f: E \to \mathbb{R}$, then at least one of the limits $\lim_{x\to a^-} f(x) = f(a^-)$, $\lim_{x\to a^+} f(x) = f(a^+)$ exists, and strict inequality holds in at least one of inequalities. $f(a^-) \le f(a) \le f(a^+)$ (nondecreasing) $f(a^-) \ge f(a) \ge f(a^+)$ (nonincreasing). 至少有一侧的严格不等号成立.
The function assumes no values in the open interval defined by the strict inequality. Open intervals of this kind determined by different points of discontinuity have no point in common

Coro 2: The set of point of discontinuity of a monotonic function is at most countable.
discontinuity point $\Rightarrow$ open interval $\to$ a rational number in the open interval
(at most countable)

Proposition 4. A criterion for continuity of a monotonic function 单调函数连续 $\Leftrightarrow f:[a,b] \to [f(a),f(b)]$ 或 $[f(b),f(a)]$
A monotonic function $f: E \to \mathbb{R}$ defined on a closed interval $E=[a,b]$ is continuous if and only if its set of values $f(E)$ is closed interval with endpoints $f(a)$ and $f(b)$
proof: "$\Rightarrow$:" 介值 + 单调.
monotonicity implies all values on $[a,b]$. lie between $f(a)$ and $f(b)$
continuity implies must assume all values intermediate between $f(a)$ and $f(b)$
values of f on $[a,b]$ is indeed the closed interval with the endpoint $f(a)$ and $f(b)$
converse: "$\Leftarrow$" 有跳跃间断. 值域不连续
Assume f is discontinuous at some point $c \in [a,b]$.
by corollay 1. one of the open intervals $(f(c^-), f(c))$ and $(f(c), f(c^+))$ is defined and nonempty and contains no value of f.
and nonempty and contains no value of f.
But. since f is monotonic, that interval is contained in $[f(a), f(b)]$.
the $(f(c^-), f(c))$ or $(f(c), f(c^+))$ cannot be contained in the range of $[f(a), f(b)]$

Thm 5. The inverse function theorem
A function $f: X \to \mathbb{R}$ that is strictly monotonic on a set $X \subset \mathbb{R}$ has inverse $f^{-1}: Y \to \mathbb{R}$ defined on the set $Y = f(X)$ of values of f. The function $f^{-1}: Y \to \mathbb{R}$ is monotonic and has the same type of monotonicity on Y that f has on X. (与 proposition 2 同. X 是子集不一定是区间)
If in addition X is a closed interval $[a,b]$ and f is continuous on X, then the set $Y = f(X)$ is the closed interval with endpoint $f(a)$ and $f(b)$ and the function $f^{-1}: Y \to \mathbb{R}$ is continuous on it. (反函数连续性).
proof: $f^{-1}(Y) = X = [a,b]$. monotonic. $\Rightarrow f^{-1}$ continuous (by pro 4).

e.g. $\underset{[0,\pi]}{x} = \arccos \underset{[-1,1]}{y}$. $\underset{[-\frac{\pi}{2},\frac{\pi}{2}]}{x} = \arcsin \underset{[-1,1]}{y}$

Chapter 5 Differential Calculus

§1. Differentiable Functions (可微函数).

e.g. simultaneous velocity (瞬时速度): $V(t) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ ($\Delta t \to 0^+$ / $\Delta t \to 0^-$ v).

slope of tangent line (切线斜率): $k = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

tangent line 切线.
normal line 法线

Def. 0: A function $f: E \to \mathbb{R}$ defined on set $E \subset \mathbb{R}$ is differentiable at a point $a \in E$, that is a limit point of E if there exist a linear function $A(x-a)$ of the increment $(x-a)$ (增量) of the argument such that $f(x)-f(a)$ can be represented as

$$\underbrace{f(x)-f(a)}_{\Rightarrow \Delta y} = A \cdot \underbrace{(x-a)}_{\Rightarrow \Delta x} + \underbrace{o(x-a)}_{\text{近似、线性}} \quad \text{as } x \to a,\ x \in E.$$

$A(x-a)$ is called differential (of the function f at a)

Def. 1. The number $f'(a) = \lim_{x \to a} \frac{f(x)-f'(a)}{x-a}$ is called derivative of the function f at a.

(derivable $\Leftrightarrow$ differentiable)

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} \Leftrightarrow \frac{f(x)-f(a)}{x-a} = f'(a) \cdot \alpha(x) \quad (\alpha(x) \to 0)$$

$$\Leftrightarrow f(x)-f(a) = f'(a)(x-a) + \underbrace{o(x-a)}_{\text{加减时切保留}} \quad (x \to a,\ x \in E)$$

Thus, differentiability of a function at a point is equivalent to the existence of its derivative at the same point.

- the differetial provides the best linear approximation to the increment of the function in a neighborhood of the same point (微分提供线性逼近).

Def 2: $\Delta x(h) := (x+h) - x = h$. increment of argument

$\Delta f(x;h) := f(x+h) - f(x)$ increment of function 视作二元函数.

Def 3. The function $h \mapsto A(x)h$ which is linear in h, is called the differential of $f: E \to \mathbb{R}$ at the point $x \in E$, denoted $df(x)$ or $Df(x)$. (微分视作函数, 映射角度)

Thus. $df(x)(h) = A(x)h$. ($dy = f'(x)\Delta x$.) 与Δx也有关

↑其实与h无关, 只与x有关.

$\Delta f(x;h) - df(x)(h) = \alpha(x;h)$

e.g. $f(x) = x$. $dx(h) = h$ (恒等映射) (the differential of an independent variable equals its increment).

$df(x)(h) = f'(x)dx(h) \Leftrightarrow df(x) = f'(x)\,dx \Rightarrow f'(x) = \frac{df(x)(h)}{dx(h)}$ (微商).

the ratio of functions $df(x)$ and dx is constant and equals $f'(x)$

Geometric

Written $f(x)$ as a power function of x_0. (任何函数, 再成polynomial的逼近, 与原函数 $a_1 \sim n$ 阶导在 x_0 处相等)

$$f(x) = C_0 + C_1(x-x_0) + \cdots + C_n(x-x_n)^n + o((x-x_0)^n),\ \text{as } x \to x_0$$

$$C_0 = \lim_{x \to x_0} f(x)$$

$$C_1 = \lim_{x \to x_0} \frac{f(x)-C_0}{x-x_0} \quad \cdots \quad C_n = \lim_{x \to x_0} \frac{f(x) - [C_0 + (x-x_0)C_1 + \cdots + C_{n-1}(x-x_0)^{n-1}]}{(x-x_0)^n}$$

especially $f(x) = C_0 + C_1(x-x_0) + o(x-x_0)$. tangent linear.

Def 5. If $f: E \to \mathbb{R}$ and $g: E \to \mathbb{R}$ are continuous at $x_0 \in E$, that is a limit point E and $f(x) - g(x) = o((x-x_0)^n)$. ($x \to x_0$) we say f and g have nth order contact at x_0 "阶" 接触.
(contact of order at least n).

For $n=1$. f and g are tangent to each other at x_0.

$\boxed{\text{e.g.}}$ instantaneous velocity and acceleration of a point mass

vector $\mathbf{r}(t) = (x(t), y(t))$.

velocity $\vec{v}(t) = \dot{r}(t) = (\dot{x}(t), \dot{y}(t))$

acceleration $\mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{r}(t) = (\ddot{x}(t), \ddot{y}(t))$

$\vec{r}(t) = (r\cos(\omega t+\alpha), r\sin(\omega t+\alpha))$

inner product (内积). $\langle \vec{v}(t), \vec{r}(t)\rangle = -r^2\omega\sin\omega t\cos\omega t + r^2\omega\cos\omega t\sin\omega t = 0$.

$\vec{a}(t) = -\omega^2\vec{r}(t)$. (与位置向量平行, "向心").

$\boxed{\text{Counter - e.g.}}$ $f(x) = |x|$. continuous. / can not derivate

$f'(0-) = -1$. $f'(0+) = +1$.

$\boxed{\text{e.g.}}$ $e^{x+h} - e^x = e^x h + o(h)$ as $h \to 0$. $\Leftrightarrow \frac{de^x}{dx} = e^x$

without Taylor $\lim_{h\to 0}\frac{e^h - e^0}{h-0} = 1$

proof: $e^{x+h} - e^x = e^x(e^h - 1) = e^x(h + o(h)) = e^x h + o(h)$

more general : $a^{x+h} - a^x = a^x(a^h - 1) = a^x(e^{h\ln a} - 1) = a^x(h\ln a + o(h\ln a))$

$= a^x(\ln a)h + o(h)$ $(h\to 0)$

$\therefore \frac{da^x}{dx} = a^x\ln a$;

$\boxed{\text{e.g.}}$ $\ln|x+h| - \ln|x| = \frac{1}{x}h + o(h)$ $(h\to 0,\ x\neq 0)$. $\Leftrightarrow d\ln|x| = \frac{1}{x}dx$.

proof: $\ln\left|\frac{x+h}{x}\right| = \ln\left|1+\frac{h}{x}\right|. = \ln(1+\frac{h}{x})$ $(h\to 0)$

$\lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0}\frac{\ln(1+h)}{h} = 1. \Rightarrow \lim_{\frac{h}{x}\to 0}\frac{\ln(1+\frac{h}{x})}{\frac{h}{x}} = 1$ ...

more general. $\log_a|x+h| - \log_a|x| = \frac{\ln|x+h|}{\ln a} - \frac{\ln|x|}{\ln a} = \cdots = \frac{1}{x\ln a}h + o(h)$ $(h\to 0.\ x\neq 0)$.

5.2 Basic Rule of Differentiation.

1. $(f+g)'(x) = (f'+g')(x)$. ①

$(f\cdot g)'(x) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$ ②

$(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$ ③

proof②: $\lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$

$= \lim_{h\to 0} f(x+h)\cdot\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\cdot g(x)$

$f(x)$连续

$= f(x)\cdot g'(x) + f'(x)g(x)$

proof② 2: (differential)

$f(x+h) - f(x) = f'(x)\cdot h + o(h)$

$g(x+h) - g(x) = g'(x)\cdot h + o(h)$ $(h\to 0)$

$f(x+h)g(x+h) - f(x)g(x)$

$= [f'(x)g(x) + g'(x)f(x)]\cdot h + o(h)$

(Remark. $o(h) = f'(x)g'(x)h^2 + o(h)\cdots$).

proof③. $\Leftrightarrow [\frac{1}{g(x)}]' = -\frac{g'(x)}{g^2(x)}$

$\frac{1}{g(x+h)} - \frac{1}{g(x)} + \frac{g'(x)}{g^2(x)} = o(h)$

LHS $= \frac{-g'(x)h + o(h)}{[g(x)+g'(x)+o(h)]g(x)} + \frac{g'(x)}{g^2(x)}h$

$\lim_{h\to 0}\frac{\cdots}{h} = \frac{-g'(x)}{g^2(x)}$

$= \frac{-g'(x)h + o(h)}{g^2(x) + O(h)} + \frac{g'(x)h}{g^2(x)}$

$\therefore = -\frac{g'(x)}{g^2(x)}\cdot h + \underbrace{o(h)}_{o(h)}\cdot h$.

$= -\frac{g'(x)h}{g^2(x)} + \frac{g'(x)h}{g^2(x)} + o(h)$

coro1 The derivative of a linear combination of differentiable functions equals the same linear combination of the derivatives of these functions.

$$(c_1 f + c_2 g)'(x) = (c_1 f)'(x) + (c_2 g)'(x) = c_1 f'(x) + c_2 g'(x)$$

$$(c_1 f_1 + \cdots + c_n f_n)'(x) = c_1 f_1'(x) + \cdots + c_n f_n'(x)$$

coro2. $(f_1 \cdots f_n)'(x) = \sum_{i=1}^{n} f_1(x) f_2(x) \cdots f_i'(x) \cdots f_n(x)$

2. Composite Function (Chain Rule)

$y = g(u)$ $u = f(x)$.

proof: $g(u_0+h) - g(u_0) = g'(u_0)h + \alpha h$. $(\alpha \to 0, h \to 0.\ \alpha|_{h=0} = 0)$.

$f(x_0+l) - f(x_0) = f'(x_0) \cdot l + \beta \cdot l$. $(\beta \to 0,\ l \to 0)\ \beta|_{l=0} = 0)$

$$g[f(x_0+l)] - g[f(x_0)] = g'(u_0)[\underbrace{f(x_0+l) - f(x_0)}_{h}] + \alpha[\underbrace{f(x_0+l) - f(x_0)}_{h}]$$

$$= g'(u_0)[f'(x_0) \cdot l + \beta l] + \alpha[f'(x_0) l + \beta l]$$

$$= g'(u_0) f'(x_0) l + o(l)$$

$$(f_n \circ \cdots \circ f_1)'(x) = f_n'(y_{n-1}) f_{n-1}'(y_{n-2}) \cdots f_1'(x) \quad 读 \leftarrow \ 求导 \rightarrow.$$

absolute error $f(x+h) - f(x)$

relative error $\dfrac{|f'(x) h|}{|f(x)|} = \dfrac{|df(x) h|}{|f(x)|}$

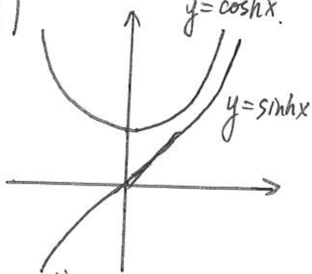
3. Inverse Function. $(f^{-1})'(y_0) = (f'(x_0))^{-1}$ $(f'(x_0) \neq 0)$

proof: f and f^{-1} are mutually inverse. $f(x) - f(x_0)$ and $f^{-1}(y) - f^{-1}(y_0)$ where $y = f(x)$. both nonzero if $x \neq x_0$. (f is continuous at x_0. f^{-1} is continous at y_0.)

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{x \to x_0} \frac{1}{f'(x_0)} \quad (y \to y_0 \Rightarrow x \to x_0).$$

e.g. $\arcsin' y = \dfrac{1}{\sin' x} = \dfrac{1}{\cos x} = \dfrac{1}{\sqrt{1-y^2}}$ (original $y = \sin x$ $|y| < 1$; inverse: $\arcsin y = x$ $|x| < \frac{\pi}{2}$)

e.g. $\arctan' y = \dfrac{1}{\sec^2 x} = \dfrac{1}{1 + \tan^2 x} = \dfrac{1}{1+y^2}$



e.g. The hyperbolic (双曲函数). and inverse hyperbolic functions.

$\sinh x = \frac{1}{2}(e^x - e^{-x})$ 双曲正弦. 又作 $\operatorname{sh} x$. 奇

$\sinh^2 x - \cosh^2 x = 1$.

$\cosh x = \frac{1}{2}(e^x + e^{-x})$ 双曲余弦. 又作 $\operatorname{ch} x$ 偶.

For $\frac{1}{2}(e^x - e^{-x}) = y$ we have $e^x = y + \sqrt{1+y^2}$ ($y - \sqrt{1+y^2}$ must be negative)

i.e. $x = \ln(y + \sqrt{1+y^2})$.

thus, $\operatorname{arcsinh} y = \ln(y + \sqrt{1+y^2})$, $y \in \mathbb{R}$.

(cosh x is even function).

$\operatorname{arcosh}_- y = \ln(y - \sqrt{y^2-1})$ $(x < 0)$

$\operatorname{arcosh}_+ y = \ln(y + \sqrt{y^2-1})$ $(x \geq 0)$

similarly $\tanh x = \dfrac{\sinh x}{\cosh x}$. $\operatorname{arctanh} y = \dfrac{1}{2} \ln \dfrac{1+y}{1-y}$

$\tanh' x = \dfrac{1}{\cosh^2 x}$ $\coth' x = -\dfrac{1}{\sinh^2 x}$

we find.

$\sinh' x = \cosh x$

$\cosh' x = \sinh x$

$\operatorname{arcsinh}' y = \dfrac{1}{\sinh' x} = \dfrac{1}{\cosh x} = \dfrac{1}{\sqrt{1 + \sinh^2 x}} = \dfrac{1}{1+y_2}$

$\operatorname{arcosh}_-' y = \dfrac{1}{\sinh x} = -\dfrac{1}{\sqrt{y^2-1}}$, $y > 1$.

$\operatorname{arcosh}_+' y = \dfrac{1}{\sinh x} = \dfrac{1}{\sqrt{y^2-1}}$, $y > 1$

4. Parametric function (参数函数) Very Simple Implicit Function (最简单的隐函数)

$$\begin{cases} x=\varphi(t) \\ y=\psi(t) \end{cases}$$ ($x(t)$ has inverse function $t=\varphi^{-1}(x)$).

$y=\psi(t)$, $t=\varphi^{-1}(x)$

$$y_x' = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\psi'(t)}{\varphi'(t)} = \boxed{\psi'(t)\cdot\frac{1}{\varphi'(t)}}$$ ← 对应求导数的倒数.

$$y''_{xx} = (y_x')' = \frac{(y_x')'_t}{x'_t} = \frac{\frac{y''_{tt}x'_t - y'_t x''_{tt}}{(x'_t)^2}}{x'_t} = \frac{x'_t y''_{tt} - x''_{tt} y'_t}{(x'_t)^3}$$

5. Higher-Order Derivatives.

the second derivative function $f''(x)$. $\frac{d^2 f(x)}{dx^2} = \frac{d\left(\frac{df(x)}{dx}\right)}{dx}$.

Def. the derivative of order n. $f^{(n)}(x) = \frac{d^n f(x)}{dx^n}$

by convention $f^{(0)}(x) := f(x)$

△ 高阶一点.. 低阶一片.

continuous derivative up to order n inclusive : $C^{(n)}(E, \mathbb{R})$, $C^n(E)$.

e.g.

$f(x) = a^x$ $\qquad f^{(n)}(x) = a^x \ln^n a$

$f(x) = \sin x$ $\qquad f^{(n)}(x) = \sin(x + \frac{n\pi}{2})$

$f(x) = \cos x$ $\qquad f^{(n)}(x) = \cos(x + \frac{n\pi}{2})$.

$f(x) = x^\alpha$ $\qquad f^{(n)}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)x^{\alpha-n}$

$f(x) = \log_a|x|$ $\qquad f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{\ln a} x^{-n}$

Leibniz's formula.

$$(uv)^{(n)} = \sum_{m=0}^{n} \underbrace{\binom{n}{m}}_{C_n^m} u^{(n-m)} v^{(m)}$$

proof by induction.

Assume $n=k$. $(uv)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} u^{(i)} v^{(k-i)}$.

$$(uv)^{k+1} = [(uv)^{(k)}]' = [\sum_{i=0}^{k} C_k^i u^{(i)} v^{(k-i)}] = \sum_{i=0}^{k} C_k^i (u^{(i+1)} v^{(k-i)} + u^{(i)} v^{(k+1-i)}).$$

$$= \sum_{i=0}^{k} C_k^i u^{(i+1)} v^{(k-i)} + \sum_{i=0}^{k} C_k^i u^{(i)} v^{(k+1-i)}$$

$$= \sum_{j\geq 1}^{k+1} C_k^{j-1} u^{(j)} v^{(k+1-j)} + \sum_{i=0}^{k} C_k^i u^{(i)} v^{(k+1)-i}$$

$$= \sum_{i=1}^{k} (C_k^{i-1} + C_k^i) \cdot u^{(i)} \cdot v^{(k+1-j)} + C_{k+1}^0 u^{(0)} \cdot v^{(k+1)} + C_{k+1}^{k+1} u^{(k+1)} v^{(0)}$$

$$= \sum_{i=0}^{k+1} C_{k+1}^i u^{(i)} v^{(k+1-j)}$$

e.g. Polynomial. $P_n(x) = \sum_{i=0}^{n} c_i x^i$ $\qquad P_n^{(k)}(x) = 0$. for $k > n$.

$P_n(x) = P_n^{(0)}(0) + \frac{1}{1!} P_n^{(1)}(0) x + \frac{1}{2!} P_n^{(2)}(0) x^2 + \cdots + \frac{1}{n!} P_n^{(n)}(0) x^n$ (Maclaurin's series).

5.3 Basic Theorems of Differential Calculus

Def 1. A point $x_0 \in E \subset \mathbb{R}$ is called a local maximum (resp. local minimum) and the value of a function $f: E \to \mathbb{R}$ at that point a local maximum value (resp. ...) if there $\boxed{\text{exists}}$ $U_E(x_0)$ s.t. $x \in U_E(x_0) \Rightarrow f(x) \le f(x_0)$. (resp. ...)

Def 2. strict local maximum. $\mathring{U}_E(x_0)$. $f(x) < f(x_0)$

复数. 注意端点也可能是.

Def 3. local maxima/minima are called local extrema (极值点)
local extreme value (极值)

Def 4. An extremum $x_0 \in E$. if x_0 is a limit point of both set $E_- = \{x \in E \mid x < x_0\}$ and $E_+ = \{x \in E \mid x > x_0\}$

$\Leftrightarrow$ interior extremum

Fermat Lemma. If a function $f: E \to \mathbb{R}$ is differentable at an interior extremum $x_0 \in E$. then $f'(x_0) = 0$ (necessary condition

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \le 0 \qquad f'_-(x_0) \ge 0.$$

Since $f'_+(x_0) = f'_-(x_0)$

Rolle's Theorem If a function $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$. and differentiable on the open interval (a,b). $f(a) = f(b)$, there exists $\xi \in (a,b)$ s.t. $f'(\xi) = 0$.

proof: Since $f(x)$ is continuous on $[a,b]$, $f(x)$ must have maximum value M and minimum value m. on $[a,b]$

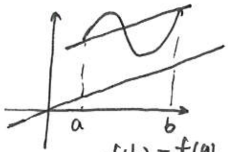
1° If $M = m = C$ then $f(x) \equiv C$. $\forall x \in (a,b)$, $f'(x) \equiv 0$.

2° If $M > m$. Since $f(a) = f(b)$. at least one of the extreme value M or m can be reached within (a,b). w.l.g. suppose $\exists \xi \in (a,b)$ s.t. $f(\xi) = M$.
By Fermat's Lemma. $f'(\xi) = 0$

Lagrange's finite-increment theorem. (有限增量).
If a function $f: [a,b] \to \mathbb{R}$ is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b). there exists a point $\xi \in (a,b)$. s.t. $f(b) - f(a) = f'(\xi)(b-a)$

proof: (method A)

Let $F(x) = f(x) - \dfrac{f(b) - f(a)}{b - a} x$. s.t. $F(a) = F(b)$



$y = f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)$ 割线.

Coro 1. Criterion for monotonicity of a function.
If the derivative of a function is nonnegative (resp. positive) at every point of an open interval, then the function is nondecreasing (resp. increasing) on the interval.

proof: $\forall x_1, x_2 \in I$. s.t. $x_1 < x_2$. then by Lagrange's Theorem, $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$
where $\xi \in (x_1, x_2)$. Since $f'(x) > 0$. $(x \in I)$. $f(x_2) - f(x_1) > 0$.
i.e. $f(x_2) > f(x_1)$. $f(x)$ is increasing on I.

Coro 2. Criterion for a function to be constant.
A function that is continuous on a closed interval $[a,b]$ is constant on it if and only if its derivative equals <u>zero</u> at every point of the interval (a,b)

"$\Rightarrow$" obviously

"$\Leftarrow$" $\forall x_1, x_2 \in [a,b]$. By Lagrange's Theorem, there exists ξ between x_1, x_2 s.t. $f(x_1) - f(x_2) = f'(\xi)(x_1 - x_2)$. Since $f'(x) \equiv 0$ in (a,b) it follows that $f(x_1) = f(x_2)$.

Cauchy's finite-increment theorem.

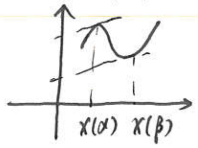
Let $x=x(t)$ and $y=y(t)$. be functions that are continuous on a closed interval $[\alpha,\beta]$. and differentiable on open interval. (α,β).
Then there exists a point $\tau\in[\alpha,\beta]$. such that.

$$x'(\tau)(y(\beta)-y(\alpha)) = y'(\tau)(x(\beta)-x(\alpha))$$

If in addition $x'(t)\neq 0$. for each $t\in(\alpha,\beta)$. then $x(\alpha)\neq x(\beta)$ and we have the equality.

$$\frac{y(\beta)-y(\alpha)}{x(\beta)-x(\alpha)} = \frac{y'(\tau)}{x'(\tau)}$$

geometric: $\begin{cases} x=x(t) \\ y=y(t) \end{cases}$



proof: Let. $F(t) = x(t)[y(\beta)-y(\alpha)] - y(t)[x(\beta)-x(\alpha)]$

$F(\alpha) = F(\beta)$

5.3.3. Taylor's Formular.

the more derivatives of two functions coincide (including $f^{(0)}$) at a point. the better these functions approximate each other in a neighborhood of a point.

$P_n(x) = P_n(x_0;x) = C_0 + C_1(x-x_0) + \cdots + C_n(x-x_0)^n$ (approximations of function in the neighborhood).

the algebraic polynomial can be represented as

$$P_n(x) = P_n(x_0) + \frac{P_n'(x_0)}{1!}(x-x_0) + \cdots + \frac{P_n^{(n)}(x_0)}{n!}(x-x_0)^n \quad (1)$$

that is $C_k = \frac{P_n^{(k)}(x_0)}{k!}$ $(k=0,1,\dots n)$

If f having derivatives up to order n inclusive at point x_0. we can write $P_n(x)$. who has the same derivatives up to order n. at x_0.

(1) is called the Taylor polynomial of order n of $f(x)$ at x_0.

Then we consider the remainder $r_n(x_0;x) = f(x) - P_n(x_0;x)$

n+1阶导数存在未必连续

Thm2. function f. continuous on the closed interval with end-points x and x_0. along with its first n derivatives. and it has a derivative of order $n+1$ at the inferior points of this interval. Then for any function φ that is continuous on this closed interval has a nonzero derivative at its interior points, there exists a point ξ between x_0 and x such that.

$$r_n(x_0;x) = \frac{\varphi(x)-\varphi(x_0)}{\varphi'(\xi)\,n!} f^{(n+1)}(\xi)(x-\xi)^n$$

proof: Consider the auxiliary function. $F(t) = f(x) - P_n(t;x)$. $\rightarrow$ of argument t.! (变量是t) on $[x,x_0]/[x_0,x]$.

$$F'(t) = -\left[f'(t) - \cancel{\frac{f'(t)}{1!}} + \cancel{\frac{f''(t)}{1!}(x-t)} - \cancel{\frac{f''(t)}{2!}2(x-t)} + \cdots + \cancel{\frac{f^{(n)}}{(n-1)!}(x-t)^{n-1}} - \frac{f^{(n)}}{n!}n(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right]$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

By Cauchy's ... $\dfrac{F(x)-F(x_0)}{\varphi(x)-\varphi(x_0)} = \dfrac{F'(\xi)}{\varphi'(\xi)}$

Coro 1. (Cauchy's formula for the remainder term) $[\varphi(t) = x-t]$. $\varphi(x)=0$ $\varphi'(t)=-1$. $\varphi(x_0) = x-x_0$

$$r_n(x_0;x) = \frac{1}{n!}f^{(n+1)}(\xi)(x-\xi)^n(x-x_0).$$

Coro 2. (The Lagrange form of the remainder). $[\varphi(t) = (x-t)^{n+1}]$

$$r_n(x_0;x) = \frac{1}{(n+1)!}f^{(n+1)}(\xi)(x-x_0)^{n+1}$$

$x_0=0$. called. MacLaurin's formula.

e.g. 1. $e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + r_n(0;x)$.

$\sum_{n=1}^{\infty} \frac{|x|^n}{n!}$ $\rho = \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = 0$

必有 $\frac{|x|^{n+1}}{(n+1)!} \to 0$.

$r_n(0;x) = \frac{1}{(n+1)!} e^{\xi} x^{n+1}$ $(|\xi| < |x|)$.

$|r_n(0;x)| < \frac{|x|^{n+1}}{(n+1)!} e^{|x|}$ for any given $x \in \mathbb{R}$. if $n \to \infty$. $\frac{|x|^{n+1}}{(n+1)!} \to 0$.

$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

similarly. $a^x = 1 + \frac{\ln a}{1!} x + \frac{\ln^2 a}{2!} x^2 + \cdots + \frac{\ln^n a}{n!} x^n + \cdots$

e.g. 2. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots + \frac{(-1)^n}{(2n+1)!} x^{2n+1} + \cdots$

$r_n(0;x) = \frac{1}{(n+1)!} \sin\left(\xi + \frac{\pi}{2}(n+1)\right) x^{n+1}$

e.g. 3. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots + (-1)^n \frac{1}{2n!} x^{2n} + R_{2n+1}$

$\therefore R_{2n+1} = \frac{\cos[(2n+2)\frac{\pi}{2} + \xi]}{(2n+2)!} x^{2n+2}$

$r_n(0;x) = \frac{1}{(n+1)!} \cos\left(\xi + \frac{\pi}{2}(n+1)\right) x^{n+1}$

e.g. 4. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + \frac{(-1)^{n-1}}{n} x^n + r_n(0;x)$. $x \in (-1,1]$

Lagrange: $r_n(0;x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) x^{n+1} = \frac{(-1)^n}{(1+\xi)^{n+1} \cdot (n+1)} x^{n+1}$ $\left|\frac{x}{1+\xi}\right|$ 难比较与1的大小

Cauchy: $r_n(0;x) = \frac{1}{n!} f^{(n+1)}(x-\xi)^n \cdot x = (-1)^n \frac{x}{1+\xi} \left(\frac{x-\xi}{1+\xi}\right)^n$.

$|x|<1$. $\left|\frac{x-\xi}{1+\xi}\right| = \frac{|x|-|\xi|}{|1+\xi|} \leq \frac{|x|-|\xi|}{1-|\xi|} = 1 - \frac{1-|x|}{1-|\xi|} \leq 1 - \frac{1-|x|}{1-|0|} = |x|$ $n \to +\infty$. $r_n(0;x) \to 0$.

$|x| \geq 1$: its general term does not tend to zero.

e.g. 5. $(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + r_n(0;x)$ $x \in (-1,1)$. $|\rho| = 1$ (不知 con/di).

Lagrange: $r_n(0;x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) x^{n+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+\xi)^{\alpha-(n+1)}}{(n+1)!} x^{n+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!} (1+\xi)^\alpha \cdot \left(\frac{x}{1+\xi}\right)^{n+1}$

Cauchy: $r_n(0;x) = \frac{1}{n!} f^{(n+1)}(x-\xi)^n \cdot x = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!} (1+\xi)^{\alpha-n-1} (x-\xi)^n x = \left(\frac{x-\xi}{1+\xi}\right)^n \cdot (1+\xi)^{\alpha-1} \cdot x$

若 $|x|<1$

$|r_n(0;x)| \leq \left|\alpha\left(1-\frac{\alpha}{1}\right) \cdots \left(1-\frac{\alpha}{n}\right)\right| \cdot |x|^{n+1} \cdot (1+\xi)^{\alpha-1}$. 此处 $(1+\xi)^{\alpha-1}$ 有界 $|-\xi$ 与 n 有关). 用一次得一个 ξ

① $\alpha-1>0$. $<2^{\alpha-1}$. ② $\alpha-1<0$ $1<(1+\xi)^{\alpha-1}<(1+x)^{\alpha-1}$

When $n+1$. the RHS. is multiplied by $\left|\left(1-\frac{\alpha}{(n+1)}x\right)\right|$. since $|x|<1$. we shall have $\left|\left(1-\frac{\alpha}{(n+1)}\right)x\right| < q < 1$. (independently of the value of α provide $|x|<q<1$ and n is sufficiently large.)

即使. $1-\frac{\alpha}{n+1} > 1$. 也有 $\left|\left(1-\frac{\alpha}{n+1}\right)x\right| < 1$.

Def: Taylor series. ($f(x)$ has derivatives of all orders $n \in \mathbb{N}$ at a point x_0).

$f(x_0) + \frac{1}{1!} f'(x_0)(x-x_0) + \cdots + \frac{1}{n!} f^{(n)}(x-x_0)^n + \cdots$

Remark. ① not be thought that if the Taylor series of an infinitely differentiable function converges in some neighborhood of x_0 泰勒级数不一定在 x_0 某个邻域内收敛.

since for given any sequence $c_0, c_1 \ldots c_n \ldots$ we can construct $f(x)$. s.t $f^{(n)}(x) = c_n$ $(n \in \mathbb{N})$

② not be thought. series converge $\Rightarrow$ it necessarily converges to function that generated it. 泰勒级数不一定收敛到产生它的函数.

(只对 analytic function 成立).

e.g. $f(x) = \begin{cases} e^{-\frac{1}{x^2}}. & x \neq 0 \\ 0, & x = 0. \end{cases}$ $f^{(i)} \equiv 0$. the series $\equiv 0$.

* 导函数极限定理. $f(x)$ 在 $[x_0, x_0+\delta)$ 上连续，在 $(x_0, x_0+\delta)$ 内可导. 且 $\lim_{x\to x_0^+} f'(x)$ 存在. 则 $\lim_{x\to x_0^+} f'(x) = f'_+(x_0)$
(在 $x=x_0$ 右可导)

proof: $f'_+(x_0) = \lim_{x\to x_0^+} \frac{f(x)-f(x_0^+)}{x-x_0} = \lim_{x\to x_0^+} f'(\xi)$ $(x_0 < \xi < x_0)$. when $x\to x_0^+$, $\xi \to x_0^+$

hence $f'_+(x_0) = \lim_{x\to x_0^+} f'(x)$.

Coro 1. If $f(x)$ is derivable on $[a,b]$. then $f'(x)$ doesn't have the points of discontinuity of second kind.

proof: Assume $\exists x_0 \dots$

We have $f'_+(x_0) = \lim_{x\to x_0^+} f'(x)$. $f'_-(x_0) = \lim_{x\to x_0^-} f'(x)$.

since $f(x)$ is derivable on x_0. $f'_-(x_0) = f'(x_0) = f'_+(x_0)$

i.e. $\lim_{x\to x_0^+} f'(x) = f'(x_0) = \lim_{x\to x_0^-} f'(x)$. i.e. $f'(x)$ is continuous on x_0.

Peano Remainder. $P_n(x_0, x) = C_0 + C_1(x-x_0) + \dots + C_n(x-x_0)^n$

the polynomial is unique. (得到多项式. 一定是泰勒多项式).

Lemma. If a function $\varphi: E \to \mathbb{R}$. defined on a closed interval E with endpoint x_0. is such that it has derivatives up to order n (不超过n) inclusive at x_0 and $\varphi(x_0) = \varphi'(x_0) = \dots \varphi^{(n)}(x_0) = 0$.
then $\varphi(x) = o((x-x_0)^n)$ as $x\to x_0$, $x\in E$.

proof by induction:

1° $n=1$. $\varphi(x) = \varphi(x_0) + \varphi'(x_0)(x-x_0) + o(x-x_0)$. $\varphi(x_0) = \varphi'(x_0) = 0$.

2° Assume $n = k-1 \geq 1$ holds

When $n=k$. s.t. $\varphi(x_0) = \varphi'(x_0) = \dots = \varphi^{(k)}(x_0) = 0$.

then. $\varphi'(x_0) = (\varphi')'(x_0) = \dots = (\varphi')^{(k-1)}(x_0) = 0$ (其实是一种转述).

by assumption $\varphi'(x_0) = o((x-x_0)^{k-1})$.

By Lagrange's Theorem $\varphi(x) = \varphi(x) - \varphi(x_0) = \varphi'(\xi)(x-x_0) = \alpha(\xi)(\xi - x_0)^{k-1}(x-x_0)$

since ξ lies between x and x_0. $\alpha(\xi) \to 0$ as $\xi \to x_0$.

$|\varphi(x)| \leq |\alpha(\xi)(x-x_0)^n|$ i.e. $\varphi(x) = o(|x-x_0|^k)$. (as $x\to x_0$)

The local Taylor formula: $f(x) = P_n(x_0; x) + o((x-x_0)^n)$.

when $\frac{f^{(n+1)}(\xi)}{(n+1)!}$ is bounded. then the Lagrange's Remainder $\to$ Peano's Remainder.

$f(x) = C_0 + C_1 x + \dots + C_n x^n + o(x^n)$ as $x \to 0$

$f'(x) = C_0' + C_1' x + \dots + C_n' x^n + o(x^{n+1})$ $C_{k-1}' = k C_k$.

$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots + \frac{(-1)^n}{2n+1} x^{2n+1} + O(x^{2n+3})$

$\arcsin x = x + \frac{1}{3!}x^3 + \frac{(3!!)^2}{5!} x^5 + \dots + \frac{[(2n-1)!!]^2}{(2n+1)!} x^{2n+1} + O(x^{2n+3})$.

5.4. The study of Function Using the Methods of Differential Calculus.

Proposition 1. f is differentiable on (a,b).

$f'(x)>0 \Rightarrow \underline{f \text{ is increasing} \Rightarrow f'(x)\geq 0}$ → proof: $f'_+(x)=\lim\limits_{\Delta x\to 0^+}\frac{f(x+\Delta x)-f(x)}{\Delta x}\geq 0$ (保号性否).

$f'(x)\geq 0 \Leftrightarrow$ nondecreasing $\Rightarrow f'(x)\geq 0$.

$f'(x)\equiv 0 \Leftrightarrow f\equiv \text{const}$.

Proposition 2. (Necessary conditions for an interior extremum)

In order for a point x_0 to be an extremum of a function $f: U(x_0)\to\mathbb{R}$ defined on $U(x_0)$, a necessary condition is that one of the following two conditions holds: either the function is not differentiable at x_0 or $f'(x_0)=0$

Proposition 3. (Sufficient conditions for an extremum in terms of the first derivative)

$f: U(x_0)\to\mathbb{R}$. differentiable $\mathring{U}(x_0)$. continuous at x_0. (itself).

Let $\mathring{U}^-(x_0)=\{x\in U(x_0)\mid x<x_0\}$. $\mathring{U}^+(x_0)$

a) $(\forall x\in \mathring{U}^-(x_0)<0\ \ f'(x)<0)\wedge(\forall x\in\mathring{U}^+(x_0)\ f'(x)<0)\Rightarrow f$ has no extremum at x_0.

b) $(\forall x\in\mathring{U}^-(x_0)<0\ \ f'(x)<0)\wedge(\forall x\in\mathring{U}^+(x_0)\ f'(x)>0)\Rightarrow x_0$ is a strict local minimum of f.

(sign change).

Not necessary. e.g. $f(x)=\begin{cases}2x^2+x^2\sin\frac{1}{x} & \text{for } x\neq 0.\\ x & \text{for } x=0.\end{cases}$

$x=0$ is a strict local minimum $f'(x)=4x+2x\sin\frac{1}{x}-\cos\frac{1}{x}$ not constant sign in any $\mathring{U}$

Proposition 4. (Sufficient conditions for an extremum in terms of higher-order derivative)

$f: U(x_0)\to\mathbb{R}$. defined on a neighborhood $U(x_0)$ of x_0 has derivative of order up to n. inclusive at x_0 $(n\geq 1)$

If $f'(x_0)=\cdots=f^{(n-1)}(x_0)=0$. and $f^{(n)}(x_0)\neq 0$. then there is no extremum at x_0 if n is odd.

if n is even, then point x_0 is local extremum (in fact a strict local minimum if $f^{(n)}(x_0)>0$. and a strict local maximum if $f^{(n)}(x_0)<0$).

proof: $f(x)=f(x_0)+\frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n+o((x-x_0)^n)$

$$\Rightarrow \frac{f(x)-f(x_0)}{(x-x_0)^n}=\frac{1}{n!}f^{(n)}(x_0)+\frac{o((x-x_0)^n)}{(x-x_0)^n}$$ (无穷小之和. 符号难判断).

(取极限). 1° n is even. $f(x)-f(x_0)$ has the same sign with $f^{(n)}(x_0)$.

2° n is odd

(Remark: When $n=2$. due to the sign-preservering property. $\lim\limits_{x\to x_0}\frac{f'(x)}{x-x_0}$).

e.g.7. For $x>0$. $f(x)=x^\alpha-\alpha x+\alpha-1 = \alpha(x^{\alpha-1}-1)$

若 $\alpha\in(0,1)$. $f(x)\leq 0$

$(\alpha<0)\wedge(\alpha>1)$ $f(x)\geq 0$. (Remark. x replaced by $x+1$. Bernoulli's inequality

a. Young's inequalities.

If $a>0$ and $b>0$. and the number p and q s.t. $p\neq 0,1$. $q\neq 0,1$. and $\frac{1}{p}+\frac{1}{q}=1$. then

$a^{\frac{1}{p}}b^{\frac{1}{q}}\leq\frac{1}{p}a+\frac{1}{q}b$. if $p>1$.

$a^{\frac{1}{p}}b^{\frac{1}{q}}\geq\frac{1}{p}a+\frac{1}{q}b$, if $p<1$ (equality holds when $a=b$.

(let $x=\frac{a}{b}$. $\alpha=\frac{1}{p}$ $(\frac{1}{q}=1-\frac{1}{p})$

b. Hölder's Inequalities.

($x_i \geq 0$. $y_i \geq 0$. $i=1,\dots n$, and $\frac{1}{p}+\frac{1}{q}=1$).

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}} \quad \text{for } p>1.$$

$$\sum_{i=1}^{n} x_i y_i \geq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q\right)^{\frac{1}{q}}. \quad \text{for } p<1.\ p\neq 0.$$

($p=q=2$. Cauchy's. inequality). Equality is possible. $\Leftrightarrow$ the vectors ($x_1^p,\dots x_n^p$) and ($y_1^q,\dots y_n^q$). are proportional. (成比例).

proof: Let. $X=\sum_{i=1}^{n} x_i^p > 0$. $Y=\sum_{i=1}^{n} y_i^q > 0$. ($a=\frac{x_i^p}{X}$. $b=\frac{y_i^q}{Y}$. by Young inequality).

$$\frac{x_i y_i}{X^{\frac{1}{p}} Y^{\frac{1}{q}}} \leq \frac{1}{p}\frac{x_i^p}{X}+\frac{1}{q}\frac{y_i^q}{Y}$$

Summing over i from 1 to n. we obtain. $\frac{\sum_{i=1}^{n} x_i y_i}{X^{\frac{1}{p}} Y^{\frac{1}{q}}} \leq 1$.

c. Minkowski's inequalities. (Let $x_i \geq 0$. $y_i \geq 0$. $i=1,\dots n$.).

$$\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{1/p} \leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}}.$$ 【p=2. 欧氏空间中的三角不等式(三维坐标. 距离)】

proof:

$$\sum_{i=1}^{n}(x_i+y_i)^p = \sum_{i=1}^{n} x_i(x_i+y_i)^{p-1} + \sum_{i=1}^{n} y_i(x_i+y_i)^{p-1}. \quad \text{(Hölder's).}$$

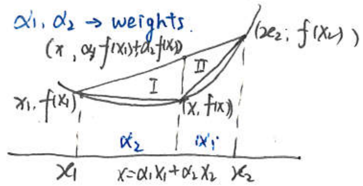
$$\leq \left(\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{q}} + \left(\sum_{i=1}^{n} y_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}(x_i+y_i)^p\right)^{\frac{1}{q}}$$

($1-\frac{1}{q}=\frac{1}{p}$).

Convex Function (下凸). 'convex downward. 一阶导↑. 二阶导>0

$f(\alpha_1 x_1+\alpha_2 x_2) \leq \alpha_1 f(x_1)+\alpha_2 f(x_2)$. holds for any point $x_1, x_2 \in (a,b)$. any number $\alpha_1 \geq 0$. $\alpha_2 \geq 0$ s.t. $\alpha_1+\alpha_2=1$. (If. the equality is strict whenever $x_1 \neq x_2$ and $\alpha_1\alpha_2 \neq 0$. the function is strictly convex on (a,b)).

Concave Fuction (上凸) convex upward.



$\alpha_1=\frac{x_2-x}{x_2-x_1}$ $\quad \alpha_2=\frac{x-x_1}{x_2-x_1}$. $\quad x=\alpha_1 x_1+\alpha_2 x_2$

$f(x) \leq \frac{x_2-x}{x_2-x_1}f(x_1)+\frac{x-x_1}{x_2-x_1}f(x_2)$.

$\Rightarrow \frac{f(x)-f(x_1)}{x-x_1} \leq \frac{f(x_2)-f(x)}{x_2-x}$ ($x_2-x_1=(x_2-x)+(x-x_1)$).

↘ 取 $\lim\limits_{x\to x_1}$. $\lim\limits_{x\to x_2}$ 分别得到2个半边

$\Rightarrow f'(x_1) \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq f'(x_2)$.

凸凹性 ⇔ 一阶导单调性

$\Rightarrow f'(x_1) < f'(\xi_1) = \frac{f(x)-f(x_1)}{x-x_1} < \frac{f(x_2)-f(x)}{x_2-x} = f'(\xi_2) < f'(x_2)$. ($x_1<\xi_1<x<\xi_2<x_2$).

Proposition 6. $f:(a,b)\to\mathbb{R}$ is differentiable.

strictly convex $\Leftrightarrow$ all points of the graph except the point of tangency lie strictly above the tangent line.

proof: "$\Rightarrow$" (necessity) Let $x_0 \in (a,b)$

$$y=f(x_0)+f'(x_0)(x-x_0).$$

so that. $f(x)-y(x)=f(x)-f(x_0)-f'(x_0)(x-x_0)=\left(f'(\xi)-f'(x_0)\right)(x-x_0)$ $\quad \xi$ between x, x_0.

f is convex. $f'(x)$ nondecreasing. $f'(\xi)-f'(x_0)$. $x-x_0$. 's signs are same. $f(x)-y(x)>0$. ($x\neq x_0$)

"$\Leftarrow$" (sufficiency)

the inequality holds $f(x)-y(x)=f(x)-f(x_0)-f'(x_0)(x-x_0)\geq 0$.

then $\frac{f(x)-f(x_0)}{x-x_0} \leq f'(x_0)$ ($x<x_0$). $\quad \frac{f(x)-f(x_0)}{x-x_0} \geq f'(x_0)$ ($x>x_0$).

for any triple of point $x_1. x. x_2$. ($x_1. x_2 \to x$. $x\to x_0$).

e.g. $e^x \geq x+1.$

$\ln x \leq x-1$ (proof by necessity of prop. 6.).

Def: the point of inflection (拐点).

$f: U(x_0) \to \mathbb{R}$. is defined and differentiable $U(x_0)$ of $x_0 \in \mathbb{R}$.

if the function is convex downward on the set $\mathring{U}^-(x_0)$. and convex. upward on $\mathring{U}^+(x_0)$.

① at the point $(x_0, f(x_0))$ the graph of the function passes from one side of the tangent line to the other.

② must have $f''(x_0) = 0$. ($f'(x)$ has a extremum).

③ if $f''(x)$ is defined. on $U(x_0)$. and has one sign on $\mathring{U}^-(x_0)$ and opposite sign. on $\mathring{U}^+(x_0)$ (sufficient for $f'(x)$ to be monotonic in $\mathring{U}^-(x_0)$ and $\mathring{U}^+(x_0)$ but with opposite monotonicity)

An expound of prop. 4.

$f'(x_0) = f^{(2)}(x_0) = \cdots f^{(n-1)}(x_0) = 0.\quad f^{(n)} \neq 0.$

$(x_0, f(x_0)) <$ point of inflection, n is odd. / point of extremum, n is even (必居其一. 只居其一).

$$f^{(2)}(x) = f^{(2)}(x_0) + f^{(3)}(x_0)(x-x_0) + \frac{1}{2!} f^{(4)}(x_0)(x-x_0)^2 + \cdots + \frac{1}{(n-2)!} f^{(n)}(x_0)(x-x_0)^{n-2} + o((x-x_0)^{n-2}).$$

$$\Leftrightarrow \frac{f^{(2)}(x)}{(x-x_0)^{n-2}} = \frac{1}{(n-2)!} f^{(n)}(x_0) + \frac{o((x-x_0)^{n-2})}{(x-x_0)^{n-2}}$$

($n-2$ has same parity with n).

n is odd. $f^{(2)}(x)$ has opposite sign.

Fact: $f(x)$ is continuous on interval I and has unique extremum x_0.

if x_0 is a local min (max) point of $f(x)$ then x_0 is a ~~ma~~ min (max) point of $f(x)$

注：最值在区间内点取得，必是极值(

the passing of a curve from one side of it's tangent line to the other at a point is a sufficient condition for the ~~point~~ inflection point.

$$f(x) = \begin{cases} 2x^3 + x^3 \sin \frac{1}{x^2}. & x \neq 0 \\ 0 & x = 0 \end{cases}$$ (凹凸性不断变化)

Jensen's inequality $f: (a,b) \to \mathbb{R}$ convex (downward). (延森不等式).

$x_1, x_2, x_3, \ldots x_n \in (a,b)$. $\alpha_1, \alpha_2 \ldots \alpha_n$ nonnegative. $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \cdots + \alpha_n f(x_n)$$

proof by induction.

show that if $n=m-1$ holds, it's also valid for $n=m$.

$\alpha_1, \ldots \alpha_n$. Then $\beta = \alpha_2 + \cdots + \alpha_n > 0$ and $\frac{\alpha_2}{\beta} + \cdots + \frac{\alpha_n}{\beta} = 1$.

Using the convexity of. the function

$$f(\alpha_1 x_1 + \cdots + \alpha_n x_n) = f\left(\alpha_1 x_1 + \beta\left(\frac{\alpha_2}{\beta} x_2 + \cdots + \frac{\alpha_n}{\beta} x_n\right)\right) \leq \alpha_1 f(x_1) + \beta f\left(\frac{\alpha_2}{\beta} x_2 + \cdots \frac{\alpha_n}{\beta} x_n\right).$$

$$\leq \alpha_1 f(x_1) + \beta\left(\frac{\alpha_2}{\beta} f(x_2) + \cdots + \frac{\alpha_n}{\beta} f(x_n)\right) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n).$$

($f(x) = x^p$. $x \geq 0$. $p > 1$. we obtain Hölder's inequality.

5.4.4. L'Hôpital's Rule.

L'Hôpital's Rule. Suppose the functions $f:(a,b)\to\mathbb{R}$ and $g:(a,b)\to\mathbb{R}$ are differentiable on the open interval (a,b) $(-\infty\leqslant a<b\leqslant+\infty)$ with $g'(x)\neq 0$ on (a,b) and

$$\frac{f'(x)}{g'(x)}\to A \text{ as } x\to a^+ \ (-\infty\leqslant A\leqslant+\infty).$$

→ 广义极限.

then $\frac{f(x)}{g(x)}\to A$ as $x\to a^+$.

in each of the following two cases.

1° $(f(x)\to 0)\wedge(g(x)\to 0)$. as $x\to a+$

or 2° $g(x)\to\infty$ as $x\to a+$ ($f(x)\not\to\infty$, $\lim=0$). ($x\to b-$ holds).

proof1: case 1°. let. $f(a):=\lim\limits_{x\to a}f(x)=0$. $g(a):=\lim\limits_{x\to a}g(x)=0$.

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{f'(\xi)}{g'(\xi)}.$$

when $x\to a$ ($\xi\to a$. (ξ lies between a and x).

i.e. $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}.$

proof2°: case 1° $g'(x)\neq 0$. $g'(x)$'s sign is invarible. (By Darboux's Theorem).

i.e. $g(x)$ is strictly monotonic.

shrinking the interval (a,b) if necessary by shifting toward the endpoint a.

we can assume that $g(x)\neq 0$ on (a,b). ($\Rightarrow g(x)=0$ 至多一个点，可除去).

By Cauchy's theorem. for $x,y\in(a,b)$. $\exists\,\xi\in(a,b)$. $\frac{f(x)-f(y)}{g(x)-g(y)}=\frac{f'(\xi)}{g'(\xi)}$.

i.e. $\frac{f(x)}{g(x)}=\frac{f(y)}{g(x)}+\frac{f'(\xi)}{g'(\xi)}\left(1-\frac{g(y)}{g(x)}\right)$.

As $x\to a+$. we shall make y tend to $a+$ in such a way that $\frac{f(y)}{g(x)}\to 0$. and $\frac{g(y)}{g(x)}\to 0$.

(先固定一个，另一个无限趋近…).

e.g. proof $\lim\limits_{x\to+\infty}\frac{x^\alpha}{a^x}=0$. $(a>1.\ \alpha>0)$.

1° $\lim\limits_{x\to+\infty}\frac{x}{a^x}=\lim\limits_{x\to+\infty}\frac{1}{a^{x}\ln a}=0,$

2° $\lim\limits_{x\to+\infty}\frac{x^\alpha}{a^x}=\left(\lim\limits_{x\to+\infty}\frac{x}{a^{\frac{x}{\alpha}}}\right)^\alpha=\left(\lim\limits_{+\infty}\frac{x}{(a^{\frac{1}{\alpha}})^x}\right)^\alpha=0.$

O'Stolz Theorem. (discrete version of L'Hôpital's Rule).

1. ($\frac{\infty}{\infty}$). $\{x_n\}$ strictly increasing, and $\lim\limits_{n\to+\infty}x_n=+\infty$. 若 $\lim\limits_{n\to\infty}\frac{y_n-y_{n-1}}{x_n-x_{n-1}}=\begin{cases}a.\ (\text{scalar})\\+\infty\\-\infty\end{cases}$

then $\lim\limits_{n\to\infty}\frac{y_n}{x_n}=\begin{cases}a.\ (\text{scalar})\\+\infty\\-\infty\end{cases}$

2° ($\frac{0}{0}$). Let $\lim\limits_{n\to+\infty}y_n=0$. $n\to+\infty$. x_n strictly decreasing. trend to 0.

if $\lim\limits_{n\to\infty}\frac{y_n-y_{n-1}}{x_n-x_{n-1}}=\begin{cases}a\ (\text{scalar})\\+\infty\\-\infty\end{cases}$

proof: $\forall \varepsilon > 0.\ \exists N \in \mathbb{N}.$ when $n > N$, $a-\varepsilon < \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}} < a+\varepsilon.$

i.e. $(a+\varepsilon)(x_n - x_{n-1}) > y_n - y_{n-1} > (a-\varepsilon)(y_n - y_{n-1})$

$(a+\varepsilon)(x_{N+1} - x_N) > y_{N+1} - y_N < (a-\varepsilon)(x_{N+1} - x_N)$

$(a+\varepsilon)(x_{N+2} - x_{N+1}) > y_{N+2} - y_{N+1} > (a-\varepsilon)(x_{N+2} - x_{N+1})$

$\vdots$

$(a+\varepsilon)(x_n - x_{n-1}) > y_n - y_{n-1} > (a-\varepsilon)(x_n - x_{n-1})$

add up. $/x_n$. $\quad (a+\varepsilon)\Big(1 - \underbrace{\frac{x_N}{x_n}}_{\to 0}\Big) > \frac{y_n}{x_n} - \underbrace{\frac{y_N}{x_n}}_{\to 0} > (a-\varepsilon)\Big(1 - \frac{x_N}{x_n}\Big).$ (可以重新取ε).

$\therefore \lim\limits_{n\to\infty} \dfrac{y_n}{x_n} = \left(\lim\limits_{n\to\infty} \dfrac{\frac{1}{y_n}}{\frac{1}{x_n}}\right)^{-1}$

Def. (asymptote)

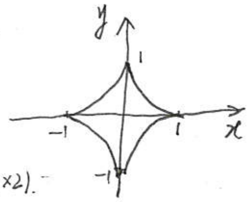
The line $b+kx$. the function $y=f(x)$ as $x \to -\infty$ (or $+\infty$) if $f(x) - (c_0 + c_1 x) = o(1)$ (oblique asymptote)

$x = a$ vertical asymptote (垂直)

$y = a$ horizontal asymptote (水平).

$k = \lim\limits_{\substack{x\to+\infty \\ -\infty}} \dfrac{f(x)}{x}$, $b = \lim\limits_{\substack{x\to+\infty \\ -\infty}} [f(x) - kx]$

e.g. $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases}$ $0 \le t \le 2\pi$ "星形线" $\quad y' = -\tan t.$



$x^{\frac{2}{3}} + y^{\frac{2}{3}} = \cos^2 t + \sin^2 t = 1.$ (对称性×2).

5.7 Primitives. (原函数).

Def. $F(x)$ is a primitive of a function $f(x)$ on an interval. F is differentiable satisfies the equation $F'(x) = f(x)$ / $dF(x) = f(x)dx$.

not "the" 不唯一.

Remark. the primitive can have different ~~apparan~~ appearances

$f(x) = \frac{1}{1+x^2}$. $\nearrow F(x) = \operatorname{arccot}\frac{1}{x}\ (x \neq 0).$ $\searrow F(x) = \arctan x.\ (x \in \mathbb{R})$

Fact: every ~~fact~~ function that is continuous on an interval has a primitive on that interval

Pro 1. $F_1(x)$, $F_2(x)$ are primitives of $f(x)$ on the same interval, then the difference $F_1(x) - F_2(x)$ is constant on that interval

($F_1(x)$. $F_2(x)$ are being compared on a connected interval) 原函数:在连续区间上考虑

indefinite integration (积分运算) $\int f(x)dx$. $\qquad \int$ ~~indefite~~ indefinite integral sign

f integrand (被积函数). $f(x)dx$ differential form (被积表达式).

if $F(x)$ is any particular primitive of $f(x)$ on the interval. $\int f(x)dx = F(x) + C$

any other primitive can be obtained from the particular primitive $F(x)$ by adding C.

$d\int f(x)dx = dF(x) = F'(x)dx = f(x)dx$ ①

对原函数全体求微分 $\quad$ 对一特定原函数求微分

$\int dF(x) = \int F'(x)dx = F(x) + C$ ②

① & ② shows a reciprocity between differentiation and integration

5.7.2. The Basic General Methods

a. $\int (\alpha u(x) + \beta v(x))\,dx = \alpha \int u(x)\,dx + \beta \int v(x)\,dx + c$

b. $\int (uv)'\,dx = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx + c$ integration by part.

光滑 → 导函数连续.

c. If $\int f(x)dx = F(x)+c$ on an interval I_x and $\varphi: I_t \to I_x$ is a $\underline{\text{smooth}}$ (continuously differentiable) mapping of the interval I_t into I_x, then

$$\int (f\circ\varphi)(t)\,\varphi'(t)\,dt = (F\circ\varphi)(t) + C$$

$$\underline{\underline{\varphi(t)=x}} \int f(x)\,dx = F(x) + C$$

$$\int f(\varphi(t))\varphi'(t)\,dt = \int f(\varphi(t))\,d\varphi(t) \underline{\underline{x=\varphi(t)}} \int f(x)\,dx$$

→ 第一换元积分(凑微分)

← 第二换元积分(凑√. $x=\varphi(t)$ 有反函)

proof: $\frac{d}{dt}\left(\int f[\varphi(t)]\,\varphi'(t)\,dt\right) = f[\varphi(t)]\,\varphi'(t).$

$$\frac{d\,[F(\varphi(t))]}{dt} = F'(\varphi(t))\cdot\varphi'(t) = f(\varphi(t))\cdot\varphi'(t).$$

(right-hand sides = ~~right~~ left-hand sides).

e.g. $\int \frac{1}{1-x^2}\,dx = \int (\frac{1}{1+x})(\frac{1}{1-x})\,dx = \frac{1}{2}\int (\frac{1}{1+x} + \frac{1}{1-x})\,dx.$

$$= \frac{1}{2}\left(\int \frac{1}{1-x}\,dx + \int \frac{1}{1+x}\,dx\right) = \frac{1}{2}\left(-\int \frac{1}{1-x}\,d(1-x) + \int \frac{1}{1+x}\,d(1+x)\right)$$ (凑微分法).

$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$$

Linearity of the Indefinite Integral.

the primitive of a linear combination of functions can be found as the same linear combination of the primitives of the function.

* some basic integration (第一换元积分法)

$$\int \tan x\,dx = -\int \frac{(-\sin x)}{\cos x}\,dx = -\int \frac{d(\cos x)}{\cos x} = -\ln|\cos x| + c.$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a^2}\int \frac{1}{1+(\frac{x}{a})^2}\,dx = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}\cdot\frac{1}{a}\cdot dx = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}\cdot d(\frac{x}{a}) = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \sin^2 x\cos^3 x\,dx = \int \sin^2 x\cos^2 x\,d\sin x = \int (1-\sin^2 x)\sin^2 x\,d\sin x) = \int \sin^2 x - \sin^4 x\,d\sin x.$$

$$= \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

(积化和差)

$$\int \cos 3x\cdot\cos 2x\,dx = \int \frac{\cos 5x + \cos x}{2}\,dx = \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + c$$

$$\int \sec x\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{d(\sin x)}{1-\sin^2 x} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| = \frac{1}{2}|\sec x + \tan x| + c.$$

$$\int \csc\,dx = -\int \frac{-\sin x}{\sin^2 x}\,dx = -\int \frac{\cancel{\cos x}}{1-\cos^2 x}\,d(\cos x) = -\frac{1}{2}\ln\left|\frac{1+\cos x}{1-\cos x}\right|\,d\cos x = -\ln|\csc x + \cot x| + C$$

$$d\cos^2 x = -\sin 2x\,dx.$$

$$\int \frac{dx}{x(x^3+3)} = \frac{1}{3}\int \left(\frac{1}{x} - \frac{x^2}{x^3+3}\right)dx = \frac{1}{3}\ln|x| - \frac{1}{9}\ln|x^3+3| + C$$

$$\int \frac{x+1}{\sqrt{3+4x-4x^2}}\,dx = -\frac{1}{8}\int \frac{d(3+4x-4x^2)}{\sqrt{3+4x-4x^2}} + \frac{3}{2}\int \frac{dx}{\sqrt{3+4x-4x^2}}$$

$$\int \frac{dx}{\sqrt{1+12x-9x^2}} = \int \frac{dx}{\sqrt{5-(3x-2)^2}} = \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{1-(\frac{3x-2}{\sqrt{5}})^2}}\,dx = \frac{1}{3}\int \frac{d(\frac{3x-2}{\sqrt{5}})}{\sqrt{1-(\frac{3x-2}{\sqrt{5}})^2}} = \frac{1}{3}\arcsin\frac{3x-2}{\sqrt{5}}$$

第二换元积分法（$x=\psi(t)$ 有 $\psi^{-1}(t)$，一般为了去$\sqrt{\ }$）.

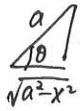
Let $x=a\sin t \ (t\in[-\frac{\pi}{2},\frac{\pi}{2}])$. $\sqrt{a^2-x^2}=a\cos t$. $dx=a\cos t\,dt$

$x=a\tan t \ (t\in(-\frac{\pi}{2},\frac{\pi}{2}))$. $\sqrt{a^2+x^2}=a\sec t$ $dx=a\sec^2 t\,dt$

$x=a\sec t \ (t\in[0,\frac{\pi}{2})$ or $t\in[\pi,\frac{3\pi}{2}))$ $\sqrt{x^2-a^2}=a\tan t$. $dx=a\sec t\tan t\,dt$.

e.g.1 $\int\sqrt{a^2-x^2}\,dx \xlongequal[\theta=\arcsin\frac{x}{a}]{x=a\sin\theta} \int a\cos\theta\cdot a\cos\theta \, d\theta=\frac{a^2}{2}\int(1+\cos2\theta)\,d\theta$

$=\frac{a^2}{2}(\theta+\frac{1}{2}\sin2\theta)+C=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}$



辅助三角形.

e.g.2. $\int\frac{1}{\sqrt{x^2+a^2}}\,dx \xlongequal{x=a\tan\theta} \int\frac{a\sec^2\theta\,d\theta}{a\sec\theta}=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C=\cdots$

Some Basic Equalities of Indefinite Integral.

$\int\tan x\,dx=-\ln|\cos x|+C$

$\int\cot x\,dx=\ln|\sin x|+C$

$\int\sec x\,dx=\ln|\sec x+\tan x|+C$

$\int\csc x\,dx=\ln|\csc x-\cot x|+C$

$\int\frac{1}{a^2+x^2}\,dx=\frac{1}{a}\arctan\frac{x}{a}+C$

$\int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

$\int\frac{1}{\sqrt{a^2-x^2}}\,dx=\arcsin\frac{x}{a}+C$

$\int\frac{1}{\sqrt{x^2\pm a^2}}\,dx=\ln|x+\sqrt{x^2\pm a^2}|+C$

$\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a}{2}\arcsin\frac{x}{a}+C$

$\int\sqrt{x^2\pm a^2}\,dx=\frac{x}{2}\sqrt{x^2\pm a^2}+\frac{a^2}{2}\ln|x+\sqrt{x^2\pm a^2}|+C$

e.g. $\int\frac{1}{(x^2+a^2)^{\frac{3}{2}}}\,dx \xlongequal{x=a\tan\theta} \int\frac{1}{a^3\sec^3\theta}\,a\cdot\sec^2\theta\,d\theta=\frac{1}{a^2}\int\cos\theta\,d\theta=\frac{1}{a^2}\sin\theta+C$

$\int\frac{\sqrt{a^2-x^2}}{x^4}\,dx \xlongequal{x=a\sin\theta} \int\frac{a\cos\theta}{a^4\sin^4\theta}\,a\cos\theta\,d\theta=\frac{1}{a^2}\int\cot^2\theta\cdot\csc^2\theta=-\frac{1}{a^2}\int\cot^2\theta\cdot d(\cot\theta)$

$\int\sqrt{5-4x-x^2}\,dx = \int\sqrt{9-(x+2)^2}\,dx \xlongequal{x+2=3\sin\theta} \int3\cos\theta\cdot3\cos\theta\,d\theta=\int\cos^2\theta\,d\theta.$

$\int\sqrt{x^2-a^2}\,dx \xlongequal{x=a\sec\theta} \int a\tan\theta\cdot a\sec\theta\cdot\tan\theta\,d\theta=a^2\int\sec\theta\cdot\tan^2\theta\,d\theta=a^2\int\underbrace{\sec\theta(\sec^2\theta-1)d\theta}_{分部}$

Integration by part.（分部积分法）.

$uv' = (uv)' - u'v.$

$\int uv'dx = uv - \int vu'dx$ or $\int u\,dv = uv - v\,du$ （注意把握谁升谁降）.

e.g.(5) ① $\int x\tan^2 x\,dx \cancel{= x\int \tan^2 x\,dx - \int \tan^2 x}$ $[\tan^2 x = \sec^2 x - 1]$

$= \int x\sec^2 x\,dx - \int x\,dx = x\tan x - \int \tan x\,dx - \frac{1}{2}x^2$

$= x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$

有ln. 除ln以外先做不定积分.

② $\int \underset{u}{\ln x}\,\underset{v}{dx} = x\ln x - \int x\cdot\frac{1}{x}dx = \overset{x(\ln x-1)}{\cancel{(x-1)\ln x}} + C$

③ $\int x\ln x\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2\frac{1}{x}dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$

其导数更易计算.

④ $\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx.$

$= x\arcsin x + \int \frac{1}{2\sqrt{1-x^2}}\,d(1-x^2)$

$= x\arcsin x + \sqrt{1-x^2}$

⑤ $\begin{cases} \int e^x\cos x\,dx = e^x\sin x + \int e^x\sin x\,dx. \\ \int e^x\sin x\,dx = -e^x\cos x + \int e^x\cos x\,dx \end{cases}$ 解方程法.

⑥ $\int \sec^3 x\,dx = \sec x\cdot\tan x - \int \sec x\cdot\tan^2 x\,dx$

$= \sec x\cdot\tan x - \int \sec^3 x\,dx + \int \sec x\,dx$

$\int \sec^3 x\,dx = \frac{\sec x\cdot\tan x + \ln|\sec x + \tan x|}{2} + C$

⑦ $\int \ln(1+\sqrt{1+x^2})\,dx = x\ln(1+\sqrt{1+x^2}) - \int \frac{x^2}{x^2+1+\sqrt{x^2+1}}dx$

$= x\ln(1+\sqrt{1+x^2}) - \int \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}}dx$

$= x\ln(1+\sqrt{1+x^2}) - x + \ln|x+\sqrt{1+x^2}| + C$

⑧（分段函数. 注意C的一致）.

e.g. $f(x) = \begin{cases} \ln x, & x \geq 1 \\ \frac{1}{2} - \frac{1}{x+1}\,dx, & x < 1 \end{cases}$

$\int f(x)\,d(x) = \begin{cases} x(\ln x - 1) + C_1 & x \geq 1 \\ \frac{1}{2}x - \arctan x + C_2, & x < 1 \end{cases}$ （在 $x=1$ 连续）.

$\therefore \lim_{x\to 1^+} = C_1 - 1 \qquad \lim_{x\to 1^-} = \frac{1}{2} - \frac{\pi}{4} + C_2$

§ 5.7.4. Integration of some kinds of functions.

1. Rational Function.

Def. $R(x) = \dfrac{P_n(X)}{Q_m(X)} = \dfrac{a_0 x^n + \cdots + a_n}{b_0 x^m + \cdots + b_m}$ ($P_n(X)$, $Q_m(x)$ are polynomials).

$n \geq m$. improper fraction. $m > n$. proper fraction.

For any improper fraction. $R(x)$ can be written as a sum of polynomial and a proper fraction.

$\boxed{\text{Thm}}$: For any reduced rational fraction $\frac{P(X)}{Q(X)}$ (proper), it can be represent as the sum of finite ~~any~~ many of reduced fraction, if $Q(X)$ can be factorized on real numbers. field. as.

$Q(X) = b_0(x-a)^\lambda \cdots (x^2+px+q)^\mu \cdots, \quad \lambda, \mu \in \mathbb{N}$.

Then $\dfrac{P(X)}{Q(X)} = \dfrac{A_1}{(x-a)^\lambda} + \dfrac{A_2}{(x-a)^{\lambda-1}} + \cdots + \dfrac{A_\lambda}{(x-a)} + \cdots + \dfrac{M_1 x + N_1}{(x^2+px+q)^\mu} + \cdots + \dfrac{M_\mu x + N_\mu}{x^2+px+q}$

(every part of the equation is called partial fraction.) unique factorization

e.g. $\displaystyle\int \frac{x^2+1}{x(x-1)^2}\,dx = \int \left(\frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{x-1}\right) dx.$

i.e. $x^2+1 = A(x-1)^2 + Bx + Cx(x-1)$.

when $x=0$, $A=1$. when $x=1$, $B=2$. $\Rightarrow C=0$. (find some special value).

① $\cos\theta + 1 = 2\cos^2\frac{\theta}{2}$.

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

② 设倍次. 可为高次根式解号

$(1+\sqrt[3]{x})\sqrt{x}$ 令 $x = t^6 \ (t>0)$

③ $$\cos x = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})} \qquad \sin x = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}.$$

④ $$\sum_{k=1}^{N} \sin k\alpha = \frac{\sin\frac{N\alpha}{2}\sin\frac{(N+1)\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

⑤ $$\sin^3 k = \frac{3\sin k - \sin 3k}{4}.$$

⑥ $$\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(n+\frac{1}{2})x}{2\sin\frac{x}{2}}$$

Indefinite Integral

Notation $\langle a,b\rangle \in \{(a,b), [a,b], (a,b], [a,b)\}$

Def. Primitive / Inverse derivative / antiderivative.

$F: \langle a,b\rangle \to \mathbb{R}$. $F'(x) = f(x)$. $x \in \langle a,b\rangle$.

Remark: $\{F(x)+C\}$. if $\langle a,b\rangle$ is more complicated. $\langle a,b\rangle \cup \langle c,d\rangle$. constant of each component can be differ. from each other

Def. Indefinite integral

The set of all primitive of f on $\langle a,b\rangle$.

$\int$ ~ sign $f(x)$ integrand $f(x)\,dx$ differential form — additivity. homogeneity

finding primitive → indefinite integration.

Fact: (1) Any continuous function has primitive

(2). f has point of discontinuity of first kind. no primitive (By Darboux Thm. $F(x)$ assumes on $[a,b]$ all the values between $F'(a)$ and $F'(b)$).

(3) f has point of discon- second kind maybe have a primitive.

$$F(x) = \begin{cases} x^2\sin\left(\frac{1}{x^3}\right). & x \neq 0 \\ & x = 0 \end{cases} \quad \varphi'(x) = f(x) \text{ can have primitives}$$

Thm. $f: \langle a,b\rangle \to \mathbb{R}$. $\varphi: \langle c,d\rangle \to \langle a,b\rangle$. φ is differentiable.

$$\int f(\varphi(t))\,\varphi'(t)\,dt = F(\varphi(t)) + C$$

proof: $(F(\varphi(t)) + C)'$ (Chain Rule).

Remark: Always. let the primitive continuous.

$$I(x) = \int \frac{x^2+1}{x^4+1}\,dx. \quad \text{let } x \neq 0. \quad I = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx = \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})+2} = \frac{1}{\sqrt{2}}\,\mathrm{arc}\,\frac{x-\frac{1}{x}}{\sqrt{2}} + \begin{cases} C_1 & x>0 \\ C_2 & x<0. \end{cases}$$

find a relation between C_1 and C_2 $\quad \lim_{x\to 0+} = \lim_{x\to 0-}$.

$$I(x) = \frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}\,x} + \frac{\pi}{2\sqrt{2}}\,\mathrm{sign}\,x + C.$$

Thm (Integration by part).

f, g differentiable. f has primitive. $\int fg' = fg - \int f'g$

Remark. A primitive of an elementary function might not elementary function.

$Ei(x) = \int \frac{e^x}{x}\,dx$ (the exponential integral) $\lim_{x\to-\infty} Ei(x) = 0$

$Si(x) = \int \frac{\sin x}{x}\,dx$ (the sine ~) $\lim_{x\to 0} Si(x) = 0$

$Ci(x) = \int \frac{\cos x}{x}\,dx$ the cosine ~) $\lim_{x\to+\infty} Ci(x) = 0$

$$\left.\begin{aligned} S(x) &= \int \sin^2 x\,dx & \lim_{x\to 0} S(x) = 0 \\ C(x) &= \int \cos^2 x\,dx & \lim_{x\to 0} C(x) = 0 \end{aligned}\right\} \text{(the Fresnel integrals)}$$

$\phi(x) = \int e^{-x^2}\,dx$ $\quad \lim_{x\to-\infty} \phi(x) = 0$ (the Euler-Poisson integral).

Classes of functions whose primitive are elementary.

$R(x)=\frac{P(x)}{Q(x)}$. $R(u,v)=\frac{P(u,v)}{Q(u,v)}$ (Rational).

① $\int R(x)dx$. ② $\int R(\sin x,\cos x)dx$ ③ $\int R(x,\sqrt[n]{\frac{ax+b}{cx+d}})dx$ ④ $R(x,\sqrt{ax^2+bx+c})dx$

⑤ $\int x^m(a+bx^n)^p dx$. where $m.n.p\in\mathbb{Q}$. $p\in\mathbb{Z}$ or $\frac{m+1}{p}\in\mathbb{Z}$ or $p+\frac{m+1}{p}\in\mathbb{Z}$.

Integration of Ration Function

Thm. P.Q. be two ploynomial

$Q(x)=C\prod_{i=1}^{n}(x-a_i)^{k_i}\cdot\prod_{i=1}^{m}(x^2+p_ix+q_i)^{l_i}$

$a_i.p_i.q_i.C\in\mathbb{R}$. $m,n,k_i,l_i\in\mathbb{N}$. $p_i^2-4q_i<0$.

Then $\frac{P(x)}{Q(x)}=p_0(x)+\sum_{i=1}^{n}\sum_{t=1}^{k_i}\frac{A_{i,t}}{(x-a_i)^t}+\sum_{i=1}^{m}\sum_{s=1}^{l_i}\frac{M_{i,s}x+N_{i,s}}{(x^2+p_ix+q_i)^s}$.

p_0 is polynomial. $\deg(P_0)=\deg(P)-\deg(Q)$

$$\int\frac{dx}{(x^2+px+q)^s}=\int\frac{dx}{((x+\frac{p}{2})^2+q-\frac{p^2}{4})^s}=\frac{1}{(q-\frac{p^2}{4})^{s-\frac{1}{2}}}\int\frac{d\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}}{\left((\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}})^2+1\right)^s}\xlongequal{t=\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}}\frac{1}{(q-\frac{p^2}{4})^{s-\frac{1}{2}}}\int\frac{dt}{(t^2+1)^s}.$$

let. $I_s=\int\frac{dt}{(t^2+1)^s}=\frac{t}{(t^2+1)^s}+2s\int\frac{t^2dt}{(t^2+1)^{s+1}}=\frac{t}{(t^2+1)^s}+2s\Big(\underbrace{\int\frac{dt}{(t^2+1)^s}}_{I_s}-\underbrace{\int\frac{dt}{(t^2+1)^{s+1}}}_{I_{s+1}}\Big)$

$I_{s+1}=\frac{t}{2s(t^2+1)^s}+\frac{2s-1}{2s}I_s$. $I_1=\arctan t+C$.

*. If the denominantor has only real roots.

e.g. $I=\int\frac{x}{x^3-3x+2}dx=\int\frac{x}{(x-1)^2(x+2)}dx.=\int(\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{x+2})dx$.

$\frac{x}{x+2}=A+B(x-1)+C(x+2)(x-1)\Rightarrow A=\frac{x}{x+2}\Big|_{x=1}=\frac{1}{3}$.

$B=\left(\frac{x}{x+2}\right)'\Big|_{x=1}=\frac{2}{(x+2)^2}\Big|_{x=1}=\frac{2}{9}$.

$\frac{x}{(x-1)^2}=\frac{A(x+2)}{(x-1)^2}+\frac{B(x+2)}{x-1}+C$ $C=\frac{x}{(x-1)^2}\Big|_{x=-2}=-\frac{2}{9}$.

The Ostrogradsky method of integration (重根. 重因式).

Suppose P.Q are polynomial and $\deg(P)<\deg(Q)$.

实质上降次. $\int\frac{P_2}{Q_2}$ 仍需用

then $\int\frac{P}{Q}=\frac{P_1}{Q_1}+\int\frac{P_2}{Q_2}$.

更待定系数法

Q_1 is g.c.d of Q and its detrivative Q'

($P_1.P_2$用待定系数法找).

$Q_2:=\frac{Q}{Q_1}$, $\frac{P_1}{Q_1}$ and $\frac{P_2}{Q_2}$ are proper fraction.

Undetermined coefficients of polynomials P_1 and P_2 are calculated by differentiating the above integral identity called Ostrogradsky formula.

$$Q(x) = C\prod_{i=1}^{n}(x-a_i)^{k_i}\prod_{i=1}^{m}(x^2+p_ix+q_i)^{l_i}$$

the $Q_1(x) = C\prod_{i=1}^{n}(x-a_i)^{k_i-1}\cdot\prod_{i=1}^{m}(x^2+p_ix+q)^{l_i-1}$ (重因式乘积, 各降一次 (与 Q' 取 g.c.d)).

$$Q_2(x) = \prod_{i=1}^{n}(x-a_i)\prod_{i=1}^{m}(x^2+p_ix+q_i)$$

e.g. $I = \int\frac{dx}{(x^3+1)}$

$$\int\frac{dx}{(x^3+1)^2} = \frac{Ax^2+Bx+C}{x^3+1} + D\int\frac{dx}{x+1} + \int\frac{Ex+F}{x^2-x+1}dx$$

$$\frac{1}{(x^3+1)^2} = \left(\frac{Ax^2+Bx+C}{x^3+1}\right)' + \frac{D}{x+1} + \frac{Ex+F}{x^2-x+1}$$

e.g. what condition $I = \int\frac{\alpha x^2+2\beta x+\gamma}{(ax^2+2bx+c)^2}dx$ is rational? $\left(R(x) = \frac{P(x)}{Q(x)}\right)$.

$Q_1 = Q_2 = ax^2+2bx+c$. $I = \frac{Ax+B}{ax^2+2bx+c} + \int\frac{Cx+D}{ax^2+2bx+c}dx$.

① $ax^2+2bx+c = a(x-x_1)(x-x_2)$

not rational (will have log).

② $ax^2+2bx+c$ $b^2-ac<0$. not.

③ $ax^2+2bx+c = a(x-x_1)^2$ $\int\frac{Cx+D}{a(x-x_1)^2}dx$. s.t. $C=D=0$. (not sufficient and necessary condition)

the original function $= \frac{\alpha x^2+2\beta x+\gamma}{(ax^2+2bx+c)^2} = \frac{A(ax^2+2bx+c)-(Ax+B)(2ax+2bx)}{(ax^2+2bx+c)^2}$.

$$\begin{cases}-Aa=\alpha.\\-Ba=\beta.\\Ac-2Bb=\gamma\end{cases} \Rightarrow \begin{cases}A=-\frac{\alpha}{a}.\\B=-\frac{\beta}{a}.\end{cases} \qquad a\gamma+\alpha c = 2\beta b$$

$$-\frac{\alpha c}{a}+2\frac{\beta b}{a}=\gamma.$$

Integrals of the form $\int R(\cos x, \sin x)\,dx$

$R(u,v) = \frac{P(u,v)}{Q(u,v)}$ $\quad P(u,v) = \sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}u^iv^j$.

let $t = \tan\frac{x}{2}$. $x \neq \pi+2\pi n$. $n\in\mathbb{Z}$ (注意有分段函数, 需用连续性找 C_i 之间的关系?).

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} \qquad \sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$dt = \left(\tan\frac{x}{2}\right)'dx = \frac{dx}{2\cos^2(\frac{x}{2})} \qquad dx = 2\cos^2\left(\frac{x}{2}\right)dt = \frac{2\,dt}{1+\tan^2\frac{x}{2}} = \frac{2\,dt}{1+t^2}$$

1.

$$\int R(\cos x, \sin x)\,dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{t^2+1}dt.$$

2. If $R(u,v) = R(-u,v)$ then there exists a rational function R_1.

such that. $\underline{R(u,v) = R_1(u^2,v)}$. (1)

$R(u,v) = \sum_{i,j} a_{ij} u^i v^j$.

$R(u,v) = R(u,v)$.

$R(u,v) = R(-u,v) = \sum_{i,j} a_{ij} u^{2i} v^j$ (把 $a_{ij}v^j$ 看成系数).

If $R(-u,v) = -R(u,v)$.

$\underline{R(u,v) = R_2(u^2,v)u.}$

(It's sufficient to apply (1) to the function $R(u,v)/u$.)

e.g. $\int R(\cos x, \sin x)dx = \int R_2(\cos^2 x, \sin x)\cos x\, dx$

$= \int R_2(1-\sin^2 x, \sin x)\, d(\sin x)$ $\quad t = \sin x$.

If $R(u,-v) = -R(u,v)$. similarly. let $t = \cos x$ ($-\cos x$ maybe).

The suitations above are determined by the order of u, v. (odevity)

If $R(-u,-v) = R(u,v)$ (2) $R(u,v) = R(u, \frac{v}{u}u) = R_3(u, \frac{v}{u})$.
also rational

e.g. $R(u,v) = u(u+v) = u(u + \frac{v}{u}\cdot u)$

$R_3(u,v) = u(u+vu)$

把所有 u, v 换成 $-u, -v$.

$R_3(u, \frac{v}{u}) = R_3(-u, \frac{-v}{-u}) = R_3(-u, \frac{v}{u}) = R_4(u^2, \frac{v}{u})$.

$\int R(\cos x, \sin x)dx = \int R_4(\cos^2 x, \frac{\sin x}{\cos x})dx = \int R_4(\frac{1}{1+\tan^2 x}, \tan x)$.

$\xlongequal{t=\tan x}$ $(dt = \frac{dx}{\cos^2 x}$ $\quad dx = \frac{dt}{1+t^2})$. $= \int R_5(t)\frac{dt}{1+t^2}$

the substitution $t = \tan x$ rationalizes the integrals.

General the substitution $t = \cos x$, $t = \sin x$, $t = \tan x$.

$R(u,v) = \frac{R(u,v) - R(-u,v)}{2} + \frac{R(-u,v) - R(-u,-v)}{2} + \frac{R(-u,-v) - R(u,v)}{2}$

$:= R_{01}(u,v) + R_{02}(u,v) + R_{03}(u,v)$.

($R_{01}(-u,v) = -R_{01}(u,v)$ $\quad R_{02}(u,-v) = -R_{02}(u,v)$ $\quad R_{03}(-u,-v) = R_{03}(u,v)$)

$R(\cos x, \sin x) = R_{01}(\cos x, \sin x) + R_{02}(\cos x, \sin x) + R_{03}(\cos x, \sin x)$

$I = \int \frac{a_1\sin x + b_1\cos x}{a\sin x + b\cos x}dx = Ax + B\ln|a\sin x + b\cos x| + C$

Integrals of the form $\int R\left(x, \sqrt[m]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right) dx$.

Suppose $t^m = \frac{\alpha x+\beta}{\gamma x+\delta}$, then $x = \frac{\delta t^m - \beta}{\alpha - \gamma t^m}$ $\quad dx = \frac{\alpha\delta - \beta\gamma}{(\alpha - \gamma t^m)^2} m \cdot t^{m-1} dt$.

the original $= \int R\left(\frac{\delta t^m - \beta}{\alpha - \gamma t^m}, t\right) \frac{\alpha\delta - \beta\gamma}{(\alpha - \gamma t^m)^2} m \cdot t^{m-1} dt$

e.g. $\int \sqrt[4]{\left(\frac{x+2}{x+1}\right)^3} \cdot \frac{dx}{(x+2)^2}$

let $\frac{x+2}{x-1} = t^4$. $\quad x \pm 2 = \frac{3+t^4}{t^4-1}$ (分有错误√). $\quad dx = \frac{-12t^3 dt}{(t^4-1)^2}$

Integrals of the form $\int R(x, \sqrt{ax^2+bx+c})$.

We denote $Y := ax^2+bx+c$ $\quad y := \sqrt{Y}$ replacing $y^2 = Y$,

we get $R(x,y) = \frac{P_1(x) + P_2(x)y}{P_3(x) + P_4(x)y} = \frac{(P_1(x) + P_2(x)y)(P_3(x) - P_4(x)y)}{(P_3(x) + P_4(x)y)(P_3(x) - P_4(x)y)} = R_1(x) + R_2(x)y$
$= R_1(x) + R_3(x)\frac{1}{y}$.

we select the quotient of a rational function $R_3(x)$, a polynomial $T(x)$.

$R_3 = T(x) + \frac{Q(x)}{S(x)}$. $\quad$ where $\deg Q < \deg S$.

we decompose a rational function Q/S into a sum of partial fractions.

So it is sufficient to integrate the following three types of function

A. $\frac{P(x)}{\sqrt{ax^2+bx+c}}$, P is a polynomial. $\quad = Q(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$

B. $\frac{1}{(x-x_0)^k \sqrt{ax^2+bx+c}}$, $k \in \mathbb{N}$. $\quad t = \frac{1}{x-x_0}$ (change of variable)

C. $\frac{Ax+B}{(x^2+px+q)^m \sqrt{ax^2+bx+c}}$, $A, B, p, q \in \mathbb{R}$, $m \in \mathbb{N}$, $p^2 - 4q < 0$.

Case 1. $ax^2+bx+c = a(x^2+px+q)$

let $a>0$. $\quad = \frac{1}{\sqrt{a}} \int \frac{Ax+B}{(x^2+px+q)^{m+\frac{1}{2}}} dx$

$Ax+B = \frac{A}{2}(2x+p) + B - \frac{Ap}{2}$.

Then. $\int \frac{Ax+B}{(x^2+px+q)^{m+\frac{1}{2}}} dx = \frac{A}{2} \int \frac{d(x^2+px+q)}{(x^2+px+q)^{m+\frac{1}{2}}} + \int \frac{(B - \frac{Ap}{2})dx}{(x^2+px+q)^{m+\frac{1}{2}}}$

let $t = (\sqrt{x^2+px+q})'$

$d(t\sqrt{x^2+px+q}) = d(x + \frac{p}{2})$.

$\sqrt{x^2+px+q}\, dt + d\sqrt{x^2+px+q} \cdot t = dx$

$\Downarrow$

$dx \cdot (\sqrt{x^2+px+q})'$

$\therefore \frac{dx}{\sqrt{x^2+px+q}} = \frac{dt}{1-t^2}$

Case 2. $ax^2+bx+c \neq a(x^2+px+q)$.

If $p \neq \frac{b}{a}$ $\quad x = \frac{\alpha t+\beta}{t+1}$

If $p = \frac{b}{a}$ and $q \neq \frac{c}{a}$. $\quad x = t - \frac{p}{2}$

$= \int \frac{(Mt+N)dt}{(t^2+\lambda)^m \sqrt{\delta t^2+r}} = \int \frac{Mt\,dt}{(t^2+\lambda)^m \sqrt{\delta t^2+r}} + \int \frac{N\,dt}{(t^2+\lambda)^m \sqrt{\delta t^2+r}}$

$\downarrow$ $\quad u = \sqrt{\delta t^2+r}$

$\downarrow$ $\quad v = (\sqrt{\delta t^2+r})' \Rightarrow \frac{dt}{\sqrt{\delta t^2+r}} = \frac{dv}{\delta - v^2}$

Trigonometric method.

(make a suitable linear substitution. then the integral is reduced to one of the cases:

$\int R(t, \sqrt{t^2+1})\, dt$ $\quad t = \tan x. (t = \sinh x)$

$\int R(t, \sqrt{t^2-1})\, dt$ $\quad t = \frac{1}{\cos x}\ (t = \cosh x)$

$\int R(t, \sqrt{1-t^2})\, dt$ $\quad t = \sin x / \cos x.\ (t = \tanh x)$

Integration of differential binomial $\int x^m (a+bx^n)^p dx$.

1. $p \in \mathbb{Z}$, $m = \frac{m_1}{m_2}$, $n = \frac{n_1}{n_2}$, then $t = x^{\frac{1}{k}}$. (k is the least common multiple of $m_2. n_2$) 最小公倍数.

2. $\int x^m (a+bx^n)^p dx \underline{\underline{t=x^n}} \frac{1}{n} \int (a+bt)^p t^{\frac{m+1}{n}-1} dt = \frac{1}{n} \int (a+bt)^p t^q dt.$

$(\frac{m+1}{n})$. $q \in \mathbb{Z}$ $\quad t = (a+bt)^{\frac{1}{p_2}} = (a+bx^n)^{\frac{1}{p_2}}$ $\quad p = \frac{p_1}{p_2}$

3. $p+q \in \mathbb{Z}$. $\quad \int (a+bt)^p t^q dt = \int (\frac{a+bt}{t})^p t^{p+q} dt.$

$$t = (\frac{a+bt}{t})^{\frac{1}{p_2}} = (ax^{-n} + b)^{\frac{1}{p_2}}.$$

In all other cases. the integral of a differential binomial cannot be reduced to elementary function (P.L. Chebyshev).

Definite Integral

Def. Tagged ~~pat~~ partition of an interval.

Let $[a,b]$ be a closed interval $-\infty < a < b < +\infty$. A set of points $T = \{x_k\}_{k=0}^{n}$ such that $a = x_0 < x_1 < \cdots x_n = b$ is called a partition of an interval $[a,b]$.

Intervals $[x_k, x_{k+1}]$ are called intervals of the partition.

We use notation $\Delta x_k = |x_{k+1} - x_k|$ for the length of ~~an~~ interval $[x_k, x_{k+1}]$.

Then the value $\lambda = \lambda_T = \max\limits_{0 \le k \le n-1} \Delta x_k$ is a mesh (网) $\to \lambda_T$ of the partition T

a pair (T, ξ) is a tagged partition (标记的分割) of an interval $[a,b]$. if T is a partition of an interval $[a,b]$, and $\xi = \{\xi_k\}_{k=0}^{n-1}$ is a set of tags s.t. $\xi_k \in [x_k, x_{k+1}]$.

Def. Riemann sum.

Let $f: [a,b] \to \mathbb{R}$. A Riemann sum of a function f with respecti to a tagged partition (T, ξ) is defined as

$$\sigma = \sigma(f, T, \xi) = \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k.$$

f is Riemann-integrable (可积的) on $[a,b]$ if there exist a number $I \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. for every (T, ξ) with mesh $\lambda_T < \delta$ ($\forall T, \forall \xi$),

we have $\quad |\sigma(f, T, \xi) - I| < \varepsilon.$

I is called Riemann-integral. $I := \int_a^b f(x)\,dx$.

denote a set of all Riemann-integrable functions on a segment $[a,b]$ as $R[a,b]$

$$\int_a^b f(x)\,dx = \lim_{\lambda(T) \to 0} \sum_{k=0}^{n-1} f(\xi_k) \Delta x_k.$$

$$\lim_{B} \sigma(T, \xi) = I \iff \forall V(I)\ \exists (T, \xi) \in B.\ \sigma(T, \xi) \subset V(I) \to (I-\varepsilon, I+\varepsilon))$$

(A base B in the set (T, ξ), the element B_d, $d > 0$, of the base B consists of all tagged partitioned (T, ξ) for which $\lambda(T) < d$

- $B_d \neq \emptyset$. • If $d_1, d_2 > 0$, and $d = \min\{d_1, d_2\}$, then $B_{d_1} \cap B_{d_2} = B_d \in B$

we denote the base B by $\lim\limits_{\lambda(T) \to 0}$).

Thm. (A necessary condition for integrability

If $f \in R[a,b]$, then f is bound on $[a,b]$

proof: Assume the converse.

Consider the partition $T = \{x_k\}_{k=0}^{n}$

The function f is not bounded on some $[x_r, x_{r+1}]$. Fix tag $\xi \in [x_k, x_{k+1}]$ for $k \neq r$.

$\sigma(f, T, \xi) = f(\xi_r)\Delta x_r + \underbrace{\sum_{k \neq r} f(\xi_k)\Delta x_k}_{:= \alpha}$. $\quad |\sigma(f, T, \xi)| \geq |f(\xi_r)| \Delta x_r - \alpha$.

For any $A > 0$. Since f is not bounded on ~~som~~ $[x_r, x_{r+1}]$.

we can always find $|f(\xi_r)| > \dfrac{A + |\alpha|}{\Delta x_r}$ ($\forall A > 0$, $\exists \xi_r > 0$).

$|\sigma(f, T, \xi)| \geq |f(\xi_r)| \Delta x_r - |\alpha| > A$.

Then $\sigma(f, T, \xi)$ is not bounded.

Remark. the theorem is not sufficient.

counter-e.g. Dirichlet function $D(x) = \begin{cases} 1. & x \in Q \\ 0. & x \notin Q \end{cases}$

If $\xi_k \in Q$. $\sigma(f_D. T. \xi) = \sum_{k=0}^{n} f(\xi_k)\Delta x_k = b-a$.

$\xi \notin Q$ $\sigma(f_D. T. \xi) = \sum_{k=0}^{n} f(\xi_k)\Delta x_k = 0$

$\sigma(f_D, T. \xi)$ depends on $\xi \Rightarrow f_D \notin R[a,b]$.

Upper and lower Darboux sums

Def. $f: [a,b) \to \mathbb{R}$. $T = \{x_k\}_{k=0}^{n}$ be a partition of $[a,b]$

$$M_k := \sup_{x \in [x_k, x_{k+1}]} f(x) \qquad m_k := \inf_{x \in [x_k, x_{k+1}]} f(x). \qquad (k=0, \dots n-1)$$

the upper and lower Darboux sums of f with respect to a partition T.

$$S = S_T(f) := \sum_{k=0}^{n-1} M_k \Delta x_k. \qquad s = s_T(f) := \sum_{k=0}^{n-1} m_k \Delta x_k.$$

Remark: The upper and lower Darboux might not be Riemann. sums. (上确界不一定取得到).

Properties. 1. $S_T(f) = \sup_{\xi} \sigma(f. T. \xi)$. $s_T(f) = \inf_{\xi} \sigma(f. T, \xi)$.

proof: $s_T(f) = \inf_{\xi} \sigma(f. T. \xi)$.

$f(\xi_k) \geq m_k$. for $k=0, \dots, n-1$. $\xi_k \in [x_k, x_{k+1}]$

$\Rightarrow \sigma(f. T. \xi) = \sum_{k=0}^{n-1} f(\xi_k)\Delta k \geq \sum_{k=0}^{n-1} m_k \Delta k = s_T(f)$. (a bound below ✓).

Let's prove $\forall \varepsilon. \exists \xi^\circ$ $\sigma(f. T. \xi^\circ) < s_T(f) + \varepsilon$ (f is bounded below).

by the def. of inf. $\forall k=0, \dots n-1$. $\exists \xi_k^\circ$. $f(\xi_k^\circ) > m_k + \frac{\varepsilon}{b-a}$.

then. $\sigma(f. T. \xi) < \sum_{k=0}^{n-1}(m_k + \frac{\varepsilon}{b-a})\Delta x_k = \sum_{k=0}^{n-1} m_k \Delta k + \frac{\varepsilon}{b-a}\sum_{k=0}^{n-1}\Delta k = \sum_{b-a}^{n-1} m_k \Delta k + \varepsilon$.

2. The upper sum does not increase and the lower sum does not decrease. when new points are added to partition.

proof: Consider $S_T(f)$. $T = \{x_k\}_{k=0}^{n}$ new point $c \in (x_r + x_{r+1})$ new partition T.

$$S_T(f) = \sum_{k=0}^{r-1} M_k \Delta x_k + M_r(x_{r+1} - x_r) + \sum_{k=r+1}^{n-1} M_k \Delta x_k.$$

$$S_T(f) = \sum_{k=0}^{r-1} M_k \Delta x_k + M'(c - x_r) + M''(x_{r+1} - c) + \sum_{k=r+1}^{n-1} M_k \Delta x_k$$

($M' = \sup_{x \in [x_r, c)} f(x)$. $M'' := \sup_{x \in [c, r+1)} f(x)$ $M' \leq M_r$ $M'' \leq M_r$.

$M'(c - x_r) + M''(x_{r+1} - c) \leq M_r(x_{r+1} - x_r)$ $S_T(f) \leq S_T(f)$. □

3. $\forall T_1, T_2$. $s_{T_1} \leq S_{T_2}$.

proof: Let T_1, T_2. be two partitions. Denote $T := T_1 \cup T_2$ (过渡).

Then $s_{T_2}(f) \leq s_T(f) \leq S_T(f) \leq S_{T_1}(f)$.

Darboux Integrals.

Def. upper: $I^* := \inf_{\mathbb{T}} S_{\mathbb{T}}(f)$. Lower: $I_* := \sup_{\mathbb{T}} s_{\mathbb{T}}(f)$

Theorem: (Criterion for integrability)

f is bounded on $[a,b]$. Then the following conditions are equivalent.

1. $f \in \mathcal{R}[a,b]$
2. $\forall \varepsilon\ \exists \mathbb{T}\ (S_{\mathbb{T}}(f) - s_{\mathbb{T}}(f) < \varepsilon)$
3. $\forall \varepsilon\ \exists \delta = \delta(\varepsilon).\ \forall \mathbb{T}\ (\lambda(\mathbb{T}) < \delta.\ S_{\mathbb{T}}(f) - s_{\mathbb{T}}(f) < \varepsilon.)$.

若已知可积，可用：
$\int_a^b f = \sup_{\mathbb{T}} s_{\mathbb{T}}(f) = \inf_{\mathbb{T}} S_{\mathbb{T}}(f)$. 求积分值.

proof: $1. \Rightarrow 2.$

$$I = \int_a^b f \Leftrightarrow \forall \varepsilon > 0.\ \exists \delta > 0.\ \forall (\mathbb{T}, \xi).\ (\lambda(\mathbb{T}) < \delta \to I - \frac{\varepsilon}{3} < \sigma < I + \frac{\varepsilon}{3})$$

$$S_{\mathbb{T}}(f) = \sup_{\xi} \sigma(f, \mathbb{T}, \xi). \Rightarrow S_{\mathbb{T}}(f) \leq I - \frac{\varepsilon}{3}.$$

$$s_{\mathbb{T}}(f) = \inf_{\xi} \sigma(f, \mathbb{T}, \xi) \Rightarrow s_{\mathbb{T}}(f) \geq I - \frac{\varepsilon}{3} \qquad 0 \leq S_{\mathbb{T}}(f) - s_{\mathbb{T}}(f) \leq \frac{2\varepsilon}{3} < \varepsilon.$$

$2. \Rightarrow 3.$ (用现有分割对新分割进行范围限制). 与中文教材中 $\lim_{\lambda \to 0} \bar{S}(P) = \inf\{\bar{S}(P) \mid \bar{S}(P) \in \bar{S}\}$ "达布引理"一回事 代δ.

Fix $\varepsilon > 0$. and a partition $\mathbb{T} = \{x_k^*\}_{k=0}^n$: $S_{\mathbb{T}}(f) - s_{\mathbb{T}}(f) < \frac{\varepsilon}{2}$. (subject to. 2.)

We need to find $\delta = \delta(\varepsilon)$. s.t. for any $T = \{x_k\}_{k=0}^N$ $\lambda(T) < \delta$. $S_T(f) - s_T(f) < \varepsilon$ holds

We choose $\delta = \frac{\varepsilon}{8nK}$. $K := \sup|f([a,b])|$ ($2K \geq M - m$

$$S_T(f) - s_T(f) = \sum^a (M-m)\Delta + \sum^b (M-m)\Delta$$

$\sum^a$: intervals of T contains at least one point x_k^*. others are $\sum^b$

the amount of terms in $\sum^a$ is $\leq 2n$ (可能在区间端点，有2个区间包含它).

$$\sum^a (M-m)\Delta \leq 2n \cdot \frac{\varepsilon}{8nK} \cdot 2K < \frac{\varepsilon}{2}.$$

$\sum^b = \sum_{j=0}^{n-1} \sum^j$. $\sum^j$ corresponds to the intervals $[x_k, x_{k+1}]$ of T.
→ 不含 x_k^* 的，放大到对应的 (x_j^*, x_{j+1}^*)
如有多个，则放大为一个.

$[x_k, x_{k+1}] \subset (x_j^*, x_{j+1}^*)$.

$$\sum_{j=0}^{n-1} \sum^j (M-m)\Delta \leq \sum_{j=0}^{n-1} (M_j - m_j) \sum^j \Delta \leq \sum_{j=0}^{n-1} (M_j - m_j)\Delta_j < \frac{\varepsilon}{2}$$

$3. \Rightarrow 1.$ $I^* := \inf_{\mathbb{T}} S_{\mathbb{T}}(f) \leq S_{\mathbb{T}}(f)$ and $I_* := \sup_{\mathbb{T}} s_{\mathbb{T}}(f) \geq s_{\mathbb{T}}(f)$. $I^* \geq I_*$.

$$0 \leq I^* - I_* \leq S_{\mathbb{T}}(f) - s_{\mathbb{T}}(f) \to 0 \text{ as } \lambda_{\mathbb{T}} \to . \Rightarrow I^* = I_* = I_0 \text{ (by 3.)}.$$

$$\begin{array}{l} s_{\mathbb{T}}(f) \leq I_0 \leq S_{\mathbb{T}}(f) \\ s_{\mathbb{T}}(f) \leq \sigma \leq S_{\mathbb{T}}(f) \end{array} \Rightarrow |\sigma - I_0| \leq S_{\mathbb{T}}(f) - s_{\mathbb{T}}(f) \to \text{ as } \lambda_{\mathbb{T}} \to 0 \Rightarrow I_0 = \int_a^b f$$

e.g. Riemann function $f_R(x) = \begin{cases} \frac{1}{q}. & x = \frac{p}{q}. \\ 0, & x \notin \mathbb{Q} \text{ or } x \backslash 0. \end{cases}$ $\qquad f_R \in \mathcal{R}[0,1]$

$s_{\mathbb{T}}(f_R) = 0$.

Let us prove. $S_{\mathbb{T}}(f_R) \to 0$ as $\lambda_{\mathbb{T}} \to 0$.

Fix $\varepsilon > 0$. find $N \in \mathbb{N}$: $\frac{1}{N} < \frac{\varepsilon}{2}$ The amount of ration number $\frac{p}{q} \in [0,1]$. $q \leq N$ is finite. denote as C_N similarly, the interval contains them no large than $2C_N$

Let $\delta = \frac{\varepsilon}{4C_N}$, $\mathbb{T}: \lambda_{\mathbb{T}} < \delta$.

$$S_{\mathbb{T}}(f_R) = \sum_{M_k \geq \frac{1}{N}} M_k \Delta x_k + \sum_{M_k < \frac{1}{N}} M_k \Delta x_k \leq 2C_N \cdot \overset{M_k < 1}{1} \cdot \delta + \frac{1}{N}\sum \Delta x_k < \varepsilon$$

Def. Oscillation.

$f: D \subset \mathbb{R} \to \mathbb{R}$. The oscillation of the function f on the set D is defined as.

$$\omega(f, D) = \sup_{x, y \in D} |f(x) - f(y)|$$

Remark: $\omega(f, D) = \sup_{x \in D} f(x) - \inf_{x \in D} f(x)$.

Corollary: $f \in R[a,b] \Leftrightarrow \forall \varepsilon \exists \mathcal{T} = \{x_k\}_{k=0}^{n}$. $\sum_{k=0}^{n-1} \omega_k(f) \Delta x_k < \varepsilon$.

where $\omega_k(f) := \omega(f, [x_k, x_{k+1}]) = M_k - m_k$.

Thm. Integrability of a restriction.

Suppose $f \in R[a,b]$. and $[c,d] \subset [a,b]$. then $f \in R[c,d]$

proof: find new partition. $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2$. be $[c,d]$. $[a,c]$. $[d,b]$

$\mathcal{T}$: $a = x_0 < x_1 \cdots < x_\mu = c < \cdots < x_\nu = d < \cdots < x_n$

$$S_{\mathcal{T}_0}(f) - s_{\mathcal{T}_0}(f) = \sum_{k=\mu}^{\nu-1} \omega_k(f) \Delta x_k \le \sum_{k=0}^{n-1} \omega_k(f) \Delta x_k = S_{\mathcal{T}}(f) - s_{\mathcal{T}}(f) < \varepsilon$$

Thm. Additivity of the integral w.r.t. the interval

$$\int_a^c f = \int_a^b f + \int_b^c f$$

$S_{\mathcal{T}}(f) - s_{\mathcal{T}}(f) = S_{\mathcal{T}_1}(f) - s_{\mathcal{T}_1}(f) + S_{\mathcal{T}_2}(f) - s_{\mathcal{T}_2}(f) < \varepsilon. \Rightarrow f \in R[a,b]$. (2⇒1.)

Let. $\mathcal{T}_1^k$, $\mathcal{T}_2^k$ be sequences of partitions of $[a,c]$. $[c,b]$ $\lambda_{\mathcal{T}_1^k} \to 0$. $\lambda_{\mathcal{T}_2^k} \to 0$. (when $k \to \infty$).

$\mathcal{T}^k := \mathcal{T}_1^k \cup \mathcal{T}_2^k$, $\lambda_{\mathcal{T}^k} \to 0$ when $k \to \infty$. $S_{\mathcal{T}^k}(f) = S_{\mathcal{T}_1^k}(f) + S_{\mathcal{T}_2^k}(f)$ when $k > 0$. $\int_a^b f = \int_a^c f + \int_c^b f$.

(3⇒1.)

Thm Integrability of continuous functions.

If $f \in C[a,b]$. then $f \in R[a,b]$

proof: the f is uniformly continuous on $[a,b]$

$\forall \varepsilon \exists \delta$. $\forall c, d \in [a,b]$. $(|c-d| < \delta, |f(c) - f(d)| < \frac{\varepsilon}{b-a})$

If $\mathcal{T} = \{x_k\}_{k=0}^{n}$. $\lambda_{\mathcal{T}} < \delta$. then $\omega_k(f) = \sup_{c,d \in [x_k, x_{k+1}]} |f(c) - f(d)| < \frac{\varepsilon}{b-a}$.

$$\Rightarrow \sum_{k=0}^{n-1} \omega_k(f) \Delta x_k < \frac{\varepsilon}{b-a} \sum_{k=0}^{n-1} \Delta x_k = \varepsilon.$$

Thm Integrability of monotone functions.

A monotone function is integrable.

proof: Suppose f is decreasing on $[a,b]$

1°. $f(a) = f(b)$ $f = c$. $\sigma(f, \mathcal{T}, \xi) = c(b-a)$

2°. $f(a) \neq f(b)$. then we fix ε. choose $\delta = \frac{\varepsilon}{f(a) - f(b)}$ consider $\mathcal{T} = \{x_k\}_{k=0}^{n}$. $\lambda_{\mathcal{T}} < \delta$.

$$\omega_k(f) = \sum_{k=0}^{n-1} \omega_k(f) \Delta x_k = \sum_{k=0}^{n-1} (f(x_k) - f(x_{k+1})) \Delta x_k < \frac{\varepsilon}{f(a) - f(b)} \underbrace{\sum_{k=0}^{n-1} (f(x_k) - f(x_{k+1}))}_{f(a) - f(b)} = \varepsilon \ \square.$$

Lemma. The integrability and value of the integral do not change if we change values of an integrable function at a finite number of points.

Let $f \in R[a,b]$. $\tilde{f}: [a,b] \to \mathbb{R}$., $\{x \mid f(x) \neq \tilde{f}(x)\} = \{x_1, \dots x_m\}$

$f(x)$ is bounded. $|f(x)| \le K$ then $|\tilde{f}(x)| \le \max\{|\tilde{f}(x_1)|, \cdots |\tilde{f}(x_m)|, K\} := \tilde{K}$

$$|\sigma(f, \mathcal{T}, \xi) - \sigma(\tilde{f}, \mathcal{T}, \xi)| = \left| \sum_{k: f(\xi_k) \neq \tilde{f}(\xi_k)} (f(\xi_k) - \tilde{f}(\xi_k)) \Delta x_k \right| \le 2m(K + \tilde{K}) \lambda_{\mathcal{T}} \to 0 \ (\text{as } \lambda_{\mathcal{T}} \to 0)$$

(看区间数)

Remark: The function f might not be defined at a finite number of points of $[a,b]$. nevertheless, f might be Riemann-integrable on $[a,b]$

Def. Piecewise continuous function

A function $f:[a,b]\to\mathbb{R}$ is called piecewise continuous if the set of points of discontinuity is either empty or finite and all discontinuities are of a first kind (jump discontinuities).

Thm. A piecewise continuous is integrable.

proof: Let $\{c_1 \dots c_m\}$ be the points of discontinuity of f on $[a,b]$.

Denote $c_0 := a$, $c_{m+1} := b$. The function is continuous on (c_k, c_{k+1}) $(k=0,\dots n)$

$f(c_k \pm 0)$ are finite. there are at most two points. where f differs from a continuous function on $[c_k, c_{k+1}]$.

By Lemma. $f\in\mathcal{R}[c_k, c_{k+1}]$. By additivity of the integral. w.r.t. the interval of integration $f\in\mathcal{R}[a,b]$

e.g. f be monotonic on $[0,1]$. Let us prove.

$$\left|\int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right|^{①} < O\left(\frac{1}{n}\right)^{④}$$

$$\begin{cases} s_T(f)\le \int_0^1 f \le S_T(f) \\ s_T(f)\le \sigma \le S_T(f)\end{cases} \Rightarrow \left|\int_0^1 f-\sigma\right|^{②} \le S_T(f)-s_T(f)^{③} \quad T=\left\{\frac{k}{n}\right\}_{k=0}^{n}$$

$$S_T(f)-s_T(f) = \frac{1}{n}\,|f(1)-f(0)| = O\left(\frac{1}{n}\right). \qquad [① < ② < ③ = ④]$$

Def. Set of measure zero. (countable set $\overset{\Rightarrow}{\not\Leftarrow}$ measure zero).

set $E\subset\mathbb{R}$ has measure zero if for any ε. $\exists$ a covering of set E by at most countable system $\{(a_n, b_n)\}_n$ of intervals. such that $\sum_n |b_n - a_n| < \varepsilon$.

Any at most countable set has measure zero

Thm. (Lebesque's criterion for Riemann integrability).

$f\in\mathcal{R}[a,b] \Leftrightarrow f$ is bounded on $[a,b]$. and the points of discountinuity of f form a set of measure zero.

e.g. Riemann. function f_R. is bounded. the set of points of discontinuity of f_R is $\mathbb{Q}\setminus\{0\}$.

The Cantor set. (an example. integrable. the set of points of discontinuity is not countable.

properties: 1. card C = card $[0,1]$

2. C has measure zero.

construction: $F_1=[0,1] \xrightarrow{\text{remove }(\frac{1}{3},\frac{2}{3})} F_2=[0,\frac{1}{3}]\cup[\frac{2}{3},0] \xrightarrow[(\frac{1}{9},\frac{2}{9}),(\frac{7}{9},\frac{8}{9})]{\text{remove}} F_3=[0,\frac{1}{9}]\cup[\frac{2}{9},\frac{1}{3}]\cup[\frac{2}{3},\frac{7}{9}]\cup[\frac{8}{9},1]$ (middle interval (open))

each step remove open middle third from each closed interval (designed on the previous step).

As a result. we have sets $(F_n)_{n=1}^{\infty}$

$F:=\bigcap_{k=1}^{\infty} F_k$ is called the Cantor set.

1. F is not countable. Let $x\in[0,1]$. has ternary (三进制) representation $x=0.x_1x_2\dots x_k$. $x_k\in\{0,1,2\}$

Then $F=\{x=0.x_1x_2\dots : x_k\in\{0,2\}\}$

Using the binary representation for a real number, we conclude that F is equipollent to $[0,1]$.

2. F is of measure zero

The sum of the lengths of all open interval:

$$\frac{1}{3}+2\cdot\frac{1}{9}+4\cdot\frac{1}{27}+\dots = \frac{1}{3}\left(1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\dots\right) = \frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}} = 1$$

Consider the function $f=\mathbb{1}_F$. It is bounded and the set of points of discontinuity of is F

Thm. Integrability and arithmetic operation

If $f, g \in R[a,b]$. $\alpha \in \mathbb{R}$. then αf, $f+g$, $|f|$, $fg \in R[a,b]$

In addition if $\inf g([a,b]) > 0$, then $\frac{f}{g} \in R[a,b]$

Proof: $|f(x)| \le C_1$ $|g(x)| \le C_2$.

Let $x_1, x_2 \in E \subset [a,b]$. ($E$ is arbitrary)

$|f(x_1)g(x_1) - f(x_2)g(x_2)| \le |f(x_1) - f(x_2)|\,|g(x_1)| + |g(x_1) - g(x_2)|\,|f(x_2)| \le C_2|f(x_1) - f(x_2)| + C_1|g(x_1) - g(x_2)|$

$\le C_2\,\omega(f, E) + C_1\,\omega(g, E)$. $\Rightarrow$ $\omega(fg, E) \le C_2\,\omega(f, E) + C_1\,\omega(g, E)$

For any $\mathcal{T} = \{x_k\}_{k=0}^{n}$ we get $\omega_k(fg) \le C_2\,\omega_k(f) + C_1\,\omega_k(g)$

$$\Rightarrow \sum_{k=0}^{n-1} \omega_k(fg)\,\Delta x_k \le C_2 \sum_{k=0}^{n-1} \omega_k(f)\,\Delta x_k + C_1 \sum_{k=0}^{n-1} \omega_k(g)\,\Delta x_k$$

The case of f. $\big||f(x_1)| - |f(x_2)|\big| \le |f(x_1) - f(x_2)| \le \omega(f, E) \Rightarrow \omega(|f|, E) \le \omega(f, E)$

The case of $1/g$. $m := \inf g([a,b])$

$$\left|\frac{1}{g(x_1)} - \frac{1}{g(x_2)}\right| = \left|\frac{g(x_1) - g(x_2)}{g(x_1)g(x_2)}\right| \le \frac{|g(x_1) - g(x_2)|}{m^2} \le \frac{1}{m^2}\,\omega(g, E) \Rightarrow \omega\left(\frac{1}{g}, E\right) \le \frac{1}{m^2}\,\omega(g, E)$$

If $a > b$ and $f \in R[a,b]$. then $\int_a^b f := -\int_b^a f$. $\int_a^a f = 0$

Properties of Riemann integral.

1. Linearity. If $f, g \in R[a,b]$. $\alpha, \beta \in \mathbb{R}$. then

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

2. monoticity of integral. If $a < b$ $f, g \in R[a,b]$. $f(x) \le g(x)$ for $x \in [a,b]$. then $\int_a^b f \le \int_a^b g$

proof: Let $(\mathcal{T}^n, \xi^n)$ be a sequence of tagged partition $\mathcal{T}^n = \{x_k^n\}_{k=0}^{N_n}$ $\xi^n = (\xi_k^n)_{k=0}^{N_n - 1}$ $\lambda(\mathcal{T}^n) \to 0$.

then $\sum_{k=0}^{N_n-1} f(\xi_k^n)\,\Delta x_k^n \le \sum_{k=0}^{N_n-1} g(\xi_k^n)\,\Delta x_k^n$

Coro 1. If $m \le f(x) \le M$. $x \in [a,b]$. $m(b-a) \le \int_a^b f \le M(b-a)$

Coro 2. If $f(x) \ge 0$. $x \in [a,b]$. $\int_a^b f \ge 0$.

即保证了有无限个点 $f \ne 0$ $(f > 0)$

3. If $f \in R[a,b]$. $f \ge 0$. $\exists x_0 \in [a,b]$. ($f(x_0) > 0$, f is continuous at x_0) then $\int_a^b f > 0$.

proof: f is continuous. at $x_0 \Rightarrow$ for $\varepsilon = \frac{f(x_0)}{2}$ $\exists \delta$ $\forall x \in (x_0 - \delta, x_0 + \delta)$.

$|f(x) - f(x_0)| < \frac{f(x_0)}{2} \Rightarrow f(x) > \frac{f(x_0)}{2}$.

We denote $[c,d] := [a,b] \cap [x_0 - \delta, x_0 + \delta]$.

$$\int_a^b f = \Big(\underbrace{\int_a^c}_{\ge 0} + \int_c^d + \underbrace{\int_d^b}_{\ge 0}\Big) f \ge \int_c^d f \ge \int_c^d \frac{f(x_0)}{2} = \frac{f(x_0)}{2}(d-c) > 0.$$

4. $\left|\int_a^b f\right| \le \left|\int_a^b |f|\right|$

proof: $a<b$.

$-|f| \le f \le |f| \Rightarrow -\int_a^b |f| \le \int_a^b f \le \int_a^b |f| \Rightarrow \left|\int_a^b f\right| \le \int_a^b |f|$

$b<a$. $\quad \int_a^b f = -\int_b^a f$

$$\left|\int_a^b f\right| = \left|\int_b^a f\right| \le \int_b^a |f| = \left|\int_a^b |f|\right|$$

Thm. (The first mean value theorem)

If $f, g \in R[a,b]$. $g \ge 0$ (or $g \le 0$) on $[a,b]$. and $m \le f \le M$. then.

$$\exists\, \mu \in [m, M] \qquad \int_a^b fg = \mu \int_a^b g$$

proof: w.l.g. let $g \ge 0$.

$m \le f \le M. \Rightarrow mg \le fg \le Mg \Rightarrow m\int_a^b g \le \int_a^b fg \le M\int_a^b g$ (monotonicity of integral)

If $\int_a^b g = 0$ then $\int_a^b fg = 0$ and any μ is appropriate.

If $\int_a^b g > 0$. then $m \le \boxed{\dfrac{\int_a^b fg}{\int_a^b g}} \le M$ $\rightarrow \mu$

coro1 If $f \in C[a,b]$, $g \in R[a,b]$, $g \ge 0$ (or $g \le 0$) on $[a,b]$.

then $c \in [a,b]$, $\int_a^b fg = f(c)\int_a^b g$ (介值性)

coro2. If $f \in R[a,b]$. $m \le f \le M$. then $\exists \mu \in [m,M]$. $\int_a^b f = \mu(b-a)$ (μ is call integral arithmetic mean)

coro2.1 If $f \in C[a,b]$. then $\exists c \in [a,b]$ $\int_a^b f = f(c)(b-a)$

Thm. (Integral with variable upper limit).

Let $f \in R[a,b]$, $x \in [a,b]$. $\phi(x) := \int_a^x f$ then (function $\phi(x)$ is called an integral with variable upper limit)

1) $\phi \in C[a,b]$

2) If f is continuous at $x_0 \in [a,b]$. then ϕ is differentiable at x_0 and $\phi'(x_0) = f(x_0)$

proof: $\exists M$. $|f| \le M$. let x_0, $x_0+\Delta x \in [a,b]$

1. $|\phi(x_0+\Delta x) - \phi(x_0)| = \left|\int_a^{x_0+x} f - \int_a^{x_0} f\right| = \left|\int_{x_0}^{x_0+\Delta x} f\right| \le \left|\int_x^{x_0+\Delta x} |f|\right|$

$$= \begin{cases} \int_{x_0}^{x_0+\Delta x} |f| & \Delta x \ge 0 \\ \int_{x_0+\Delta x}^{x_0} |f| & \Delta x \le 0 \end{cases} \le \begin{cases} \int_{x_0}^{x_0+\Delta x} M, & \Delta x \ge 0 \\ \int_{x_0+\Delta x}^{x_0} M, & \Delta x \le 0 \end{cases} = M|\Delta x| \to 0. \quad (\text{as } \Delta x \to 0).$$

2. $$\frac{\phi(x_0+\Delta x) - \phi(x_0)}{\Delta x_0} - f(x_0) = \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} f(t)\,dt - \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} f(x_0)\,dt.$$

$$= \frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x} (f(t) - f(x_0))$$

f is continuous at $x_0 \Rightarrow \forall \varepsilon\ \exists \delta\ \forall t \in (x_0-\delta, x_0+\delta)$ $|f(t) - f(x_0)| < \varepsilon$.

Choose Δx: $x_0+\Delta x \in (x_0-\delta, x_0+\delta)$ then $\left|\frac{1}{\Delta x}\int_{x_0}^{x_0+\Delta x}(f(t)-f(x_0))dt\right| \le \frac{1}{|\Delta x|}\left|\int_{x_0}^{x_0+\Delta x}|f(t)-f(x_0)|dt\right|$

$$= \begin{cases} \frac{1}{|\Delta x|}\int_{x_0}^{x_0+\Delta x}|f(t)-f(x_0)|dt. & \Delta x \ge 0 \\ \frac{1}{|\Delta x|}\int_{x_0+\Delta x}^{x_0}|f(t)-f(x_0)|dt & \Delta x \le 0 \end{cases} \le \frac{1}{|\Delta x|}\varepsilon|\Delta x| = \varepsilon.$$

Remark: If $f \in C[a,b]$. then $\forall x \in [a,b]$. $\phi'(x) = f(x)$

Any continuous function has a primitive $\phi(x)=\int_a^x f$.

An arbitrary primitive F of f is $F(x)=\phi(x)+C$.

For $x=a$ we get $F(a)=\int_a^a f+C=C$. that is $\int_a^x f=F(x)-F(a)$

Theorem (Newton - Leibniz formula).

If $f\in R[a,b]$, F is a primitive of f on $[a,b]$. then.

$$\int_a^b f=F(b)-F(a)=:F(x)\Big|_a^b.$$

proof: Let $\tau^n=\{x_k^n\}_{k=0}^{N_n}$ be a sequence of partition. $\lambda(\tau^n)\to 0$. as $n\to\infty$.

then $F(b)-F(a)=\sum\limits_{k=0}^{N_n-1}(F(x_{k+1}^n)-F(x_k^n))=\sum\limits_{k=0}^{N_n-1}F'(\xi_k^n)\Delta x_k^n=\sum\limits_{k=0}^{N_n-1}f(\xi_k^n)\Delta x_k^n\to\int_a^b f.$ (as $n\to\infty$)

$[F(b)-F(x_{N_n-1}^n)]+[F(x_{N_n-1}^n)-\ldots]+\cdots+[F(x_1)-F(a)]$

e.g. prove the inequality $\frac{4}{9}(e-1)<\int_0^1\frac{e^x\,dx}{(x+1)(2-x)}<\frac{1}{2}(e-1)$

$f(x)=\frac{e^x}{(x+1)(2-x)}$ do not has an elementary primitive $f(x)\in[0,1]$ has a primitive.

$\frac{4}{9}\int_0^1 e^x\,dx<\int_0^1\frac{e^x\,dx}{(x+1)(2-x)}<\frac{1}{2}\int_0^1(e-1)$. $\Leftarrow$ $g(x)=\frac{1}{(x+1)(2-x)}$ $\max\limits_{x\in[0,1]}g(x)=g(1)=g(0)=\frac{1}{2}$, $\min\limits_{x\in[0,1]}g(x)=g(\frac{1}{2})=\frac{4}{9}$.

注意. continuous 是前提 (不是有限不连续).

Thm. If $f\in R[a,b]$. F is continuous on $[a,b]$. and F is a primitive of f on $[a,b]$ except a finite number of point. then $\int_a^b f=F(b)-F(a)=:F(x)\Big|_a^b$.

proof: Let $c_1\ldots c_m$ be all points where $F'(x)\neq f(x)$. We denote $c_0:=a$. $c_{m+1}:=b$.

Then $\int_{c_k}^{c_{k+1}}f=\lim\limits_{\varepsilon\to 0}\int_{c_k+\varepsilon}^{c_{k+1}-\varepsilon}f=\lim\limits_{\varepsilon\to 0}(F(c_{k+1}-\varepsilon)-F(c_k+\varepsilon)).=F(c_{k+1})-F(c_k)$

By additivity. $\int_a^b f=\sum\limits_{k=0}^m\int_{c_k}^{c_{k+1}}f=\sum\limits_{k=0}^m F(c_{k+1})-F(c_k)=F(b)-F(a)$

e.g. $\int_{-1}^1\operatorname{sign}t\,dt=|t|\Big|_{-1}^1=0$. $(F(t)=|t|)$.

Remark: ① $F\in C[a,b]$. is important For $f(x)=0$. $F(x)=\operatorname{sign}x$.

$$0=\int_{-1}^1 f\neq F\Big|_{-1}^1=2.$$

注由导数恢复原函数

② The fundamental theorem of integral calculus is result on a restoring a function via its derivative: If F is differentiable on $[a,b]$. and $F'\in R[a,b]$. then $\int_a^x F'+F(a)=F(x)$ for any $x\in[a,b]$.

Thm (Integration by parts in the Riemann integral).

If f,g are differentiable on $[a,b]$ and $f',g'\in R[a,b]$. then $\int_a^b f'g=fg\Big|_a^b-\int_a^b fg'$

proof: f,g are differentiable on $[a,b]\Rightarrow f,g$ are continuous on $[a,b]\Rightarrow f,g\in R[a,b]$.

since $f',g'\in R[a,b]$. $\Rightarrow f'g,\ fg'\in R[a,b]$. $\Rightarrow f'g+fg'\in R[a,b]$

$$\int_a^b f'g+\int_a^b fg'=\int_a^b(f'g+fg')=fg\Big|_a^b.$$

Thm. (Change of Variable in Riemann integral)

If $f \in C[a,b]$. $\varphi [\alpha,\beta] \to [a,b]$. φ is differentiable on $[a,b]$. $\varphi' \in R[\alpha,\beta]$.

then $\int_\alpha^\beta (f\circ\varphi)\varphi' = \int_{\varphi(\alpha)}^{\varphi(\beta)} f$ $\rightarrow \Delta$ maybe $\varphi(\beta) < \varphi(\alpha)$.

proof: $f\circ\varphi \in C[\alpha,\beta] \Rightarrow f\circ\varphi \in R[\alpha,\beta] \xrightarrow{\varphi' \in R[\alpha,\beta]} (f\circ\varphi)\varphi' \in R[\alpha,\beta]$

Let F be a primitive of f on $[a,b]$.

then $[F\circ\varphi]'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\cdot\varphi'(t)$

By the fundamental theorem of integral calculus.

$$\int_\alpha^\beta (f\circ\varphi)\varphi' = F\circ\varphi\Big|_\alpha^\beta = F\Big|_{\varphi(\alpha)}^{\varphi(\beta)} = \int_{\varphi(\alpha)}^{\varphi(\beta)} f. \ \square$$

e.g1. Legendre polynomial $P_n(x) = \frac{1}{2^n n!}\frac{d^n[(x^2-1)^n]}{dx^n}$.

$\int_{-1}^1 Q_m(x)P_n(x)\,dx = 0$. for any polynomial Q of order $m<n$.

integration by part: $\int_{-1}^1 Q_m(x)\frac{d^n[(x^2-1)^n]}{dx^n}dx = Q_m(x)\frac{d^{n-1}[(x^2-1)^n]}{dx^{n-1}}\Big|_{-1}^1 - \int_{-1}^1 Q_m'(x)\frac{d^{n-1}[(x^2-1)^n]}{dx^{n-1}}dx$

repeat m times $= (-1)^m \int_{-1}^1 Q_m^{(m)}(x)\frac{d^{n-m}[(x^2-1)^n]}{dx^{n-m}}dx = (-1)^m Q_m^{(m)} x \cdot \frac{d^{n-m-1}[(x^2-1)^n]}{dx^{n-m-1}}\Big|_{-1}^1 = 0$.

e.g2. Let $f: \mathbb{R} \to \mathbb{R}$. be a continuous periodic (周期的) function. T be a period of f.

Prove $\int_a^{a+T} f = \int_0^T f$. a is an arbitrary real number.

$$\int_a^{a+T} = \int_a^T f + \int_T^{a+T} f = \int_a^T f + \int_T^{a+T} f(x-T)dx \overset{t=x-T}{=\!=} \int_a^T f + \int_0^a f(t)\,dt$$

Δ $\int_T^{a+T} f(x-T)\,dx = \int_T^a f(t)\,dt$.

* Euler formula. $\sin kx = \frac{1}{2i}(e^{ikx} - e^{-ikx})$. $k=1\ldots n$.

$e^{ix} = \cos x + i\sin x$. (regard as def.)

$z = e^{ix}$ $\bar{z} = e^{-ix}$

$z = a+bi$
$= \mathrm{Re}\, z + \mathrm{Im}\, z$.
(Real part / imaginary part)

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$

$\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$\sin x = \mathrm{Im}\, e^{ix}$. $\sin x + \cdots + \sin nx = \mathrm{Im}(e^{ix} + \cdots + e^{inx}) = \mathrm{Im}\left(e^{ix}\cdot\frac{1-e^{inx}}{1-e^{ix}}\right)$

$$= \mathrm{Im}\left(\frac{(e^{ix} - e^{i(n+1)x})(1-e^{-ix})}{(1-e^{ix})(1-e^{-ix})}\right) = \frac{\sin x - \sin(n+1)x + \sin nx}{2-2\cos x}$$

reps $\mathrm{Im} \Rightarrow \mathrm{Re}$. We obtain. $\cos x + \cdots + \cos nx = \frac{\cos x - \cos(n+1)x + \cos nx - 1}{2-2\cos x}$

$$\lim_{x\to\pi-} \frac{\sin nx}{\sin x} = \lim_{t\to 0+}\frac{\sin n(\pi-t)}{\sin(\pi-t)} = (-1)^{n-1}n.$$

e.g. 3. $I = \int_0^\pi \frac{\sin nx}{\sin x}dx$.

第k项和第n+1-k项

$$f(x) = \frac{e^{inx}-e^{-inx}}{e^{ix}-e^{-ix}} = \sum_{k=0}^{n-1} e^{i((n-1)-2k)x} = \sum_{k=1}^{n} e^{i(n+1-2k)x} = \begin{cases} 2(\cos(n-1)x + \cos(n-3)x + \cdots + \cos x & n \text{ is even} \\ 2(\cos(n-1)x + \cos(n-3)x + \cdots + \cos 2x + 1 & n \text{ is odd}\end{cases}$$

$\int_0^\pi \cos(n-k)x\,dx = \frac{\sin(n-k)x}{n-k}\Big|_0^\pi = 0$, $k=1,\ldots,n-1$. we finally have $I = \begin{cases} 0. & n \text{ is even} \\ \pi. & n \text{ is odd}\end{cases}$

e.g. 4. $\int_0^x e^{t^2}dt \sim \frac{e^{x^2}}{2x^2}$ (as $x\to+\infty$).

1) $\lim\limits_{x\to+\infty} \frac{2x^2\int_0^x e^{t^2}dt}{e^{x^2}} = \lim\limits_{x\to+\infty} \frac{2\int_0^x e^{t^2}dt + 2x\cdot e^{x^2}}{2xe^{x^2}} = \lim\limits_{x\to+\infty}\left(\frac{e^{x^2}}{e^{x^2}+2x^2e^{x^2}}+1\right) = 1.$ if $F(x)=\int_a^{\varphi(x)} f(t)dt$. $F'(x)=\varphi'(x)f(\varphi(x))$

using L'Hopital's rule twice. $[\,F(x)=\int_0^x f(t)dt$. $dt\in C(\mathbb{R})$. $F'(x)=f(x)\,]$ $F(0)$

Lemma. (Summation by parts or Abel transformation)

$\sum\limits_{k=1}^{n} a_k b_k = A_n b_n + \sum\limits_{k=1}^{n-1} A_k(b_k-b_{k+1})$ where $A_k := \sum\limits_{i=1}^{k} a_i$ $(A_0=0)$

常令 $a_k = \Delta x/\Delta f$.

* If $m\le A_k\le M$. $b_i\ge 0$. $b_i\ge b_{i+1}$. then $mb_1 \le \sum\limits_{k=1}^{n} a_k b_k \le Mb_1$ (b_i non-negative. decreasing)

Lemma. If. $f\in R[a,b]$. $g\ge 0$. g is nonincreasing on $[a,b]$

then $\exists \xi\in[a,b]$, $\int_a^b fg = g(a)\int_a^\xi f$ $\quad J=\{x_k\}_{k=0}^n$. $L:=\sup f([a,b])$

proof: $\int_a^b fg = \sum\limits_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} fg = \sum\limits_{k=0}^{n-1} g(x_k)\int_{x_k}^{x_{k+1}} f(x)dx + \sum\limits_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} f(x)(g(x)-g(x_k))dx =: S_1+S_2.$

$|S_2| \le \sum\limits_{k=0}^{n-1}\int_{x_k}^{x_{k+1}} |f(x)||g(x)-g(x_k)|dx \le L\sum\limits_{k=0}^{n-1}\omega_k(g)\cdot\int_{x_k}^{x_{k+1}} 1 \le L\cdot\sum\limits_{k=0}^{n-1}\omega_k(g)\cdot\Delta x_k.$

$(\lambda_T\to 0)$ $S_1\to\int_a^b fg$. (by $|S_2|\to 0$)

$S_1 = \sum\limits_{k=0}^{n-1} g(x_k)\int_{x_k}^{x_{k+1}} f(x)dx = \sum\limits_{k=0}^{n-1} g(x_k)(F(x_{k+1})-F(x_k)) = \sum\limits_{k=1}^{n} g(x_{k-1})(F(x_k)-F(x_{k-1}))$

where $F(x):=\int_a^x f$. we denote. $a_k := F(x_k)-F(x_{k-1})$ $\quad b_k := g(x_{k-1})$

Then $A_n = \sum\limits_{k=1}^{n} a_k = F(x_n)-\overset{0}{\overset{\uparrow}{F(x_0)}}$ $\quad F\in C[a,b]$. $\quad m\le F(x_n)\le M$.

by Lemma, $mg(a) \le \sum\limits_{k=1}^{n} g(x_{k-1})(F(x_k)-F(x_{k-1})) \le Mg(a) \Rightarrow mg(a)\le\int_a^b fg\le Mg(a)$

1) If $g(a)=0$. any ξ. 2) If $g(a)\neq 0$. $m\le\frac{\int_a^b fg}{g(a)}\le M$. By the Bolzano intermediate value theorem...

Thm (The second mean value theorem for the Riemann integral)

If $f\in R[a,b]$. g is monotonic. then $\exists\xi\in[a,b]$. $\int_a^b fg = g(a)\int_a^\xi f + g(b)\int_\xi^b f$

Proof: Let g be nondecreasing on $[a,b]$. Then $g_1(x) := g(b)-g(x)$ is nonnegative and increasing on $[a,b]$. By last lemma. $\exists\xi\in[a,b]$.

$\int_a^b fg_1 = g_1(a)\int_a^\xi f \Leftrightarrow g(b)\int_a^b f - \int_a^b fg = (g(b)-g(a))\int_a^\xi f$

$\Leftrightarrow g(b)\left(\int_a^b f - \int_a^\xi\right) + g(a)\int_a^\xi f = \int_a^b fg.$ (If $g(x)$ is nonincreasing. $g_1(x) := g(x)-g(b)$)

e.g. $I=\int_0^{2\pi}\frac{\sin x}{x}dx$ (sine function). define the sign

$I=\int_\pi^{2\pi}\frac{\sin x}{x}dx + \int_0^\pi\frac{\sin x}{x}dx \overset{x=t+\pi}{=\!=\!=} \int_0^\pi\frac{\sin x}{x}dx - \int_0^\pi\frac{\sin t}{t+\pi}dt = \pi\int_0^\pi\frac{\sin x}{x(x+\pi)}$

$\overset{\text{the first mean theorem}}{=\!=\!=\!=\!=} \pi\cdot\frac{\sin\xi}{\xi}\int_0^\pi\frac{1}{x+\pi}dx.$ $(\exists\xi\in[0,\pi])$ $= \pi\cdot\frac{\sin\xi}{\xi}\ln 2 > 0$

$\frac{1}{x+\pi}$. $\frac{\sin x}{x}\in C[0,1]$.

the first (second) mean theorem — to estimate. the integral e.g. estimate. $I=\int_{100}^{200}\sin\pi x^2\,dx$. ($g=\frac{1}{\sqrt{t}}$ $f=\sin t$. second mean value thm.

$$I \overset{t=\pi x^2}{=\!=} \frac{1}{2\sqrt{\pi}}\int_{100^2\pi}^{200^2\pi}\frac{\sin t}{\sqrt{t}}\,dt = \frac{1}{2\sqrt{\pi}}\left(\frac{1}{100\sqrt{\pi}}\int_{100^2\pi}^{\xi}\sin t\,dt+\frac{1}{200\sqrt{\pi}}\int_{\xi}^{200^2\pi}\sin t\,dt\right)=\frac{1-\cos\xi}{400\pi}$$

$100^2\pi<\xi<200^2\pi$. $0<I<\frac{1}{200\pi}$ ($\sin x$ 一个周期积分$=0$).

Thm (Taylor's formula with the remainder in integral form)

若 $f\in C^{n+1}(a,b)$ $n\in\mathbb{N}$. $x, x_0\in(a,b]$.

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+\frac{1}{n!}\int_{x_0}^{x}f^{(n+1)}(t)(x-t)^n\,dt.$$

proof by induction. skip. $(n-1)$ holds.

$$\frac{1}{(n-1)!}\int_{x_0}^{x}f^{n}(t)(x-1)^{n-1}dt = \frac{-1}{n!}\int_{x_0}^{x}f^{n}(t)\,d(x-t)^n$$

$$=-\frac{1}{n!}\left(f^{(n)}(t)(x-t)^n\Big|_{t=x_0}^{t=x}-\int_{x_0}^{x}f^{(n+1)}(t)(x-t)^n dt\right)$$

$$=\frac{1}{n!}\left(f^{(n)}(x_0)(x-x_0)^n+\int_{x_0}^{x}f^{(n+1)}(t)(x-t)^n dt\right).$$

Remark: $\frac{1}{n!}\int_{x_0}^{x}f^{(n+1)}(t)(x-t)^n dt=\frac{f^{(n+1)}(c)}{n!}\int_{x_0}^{x}(x-t)^n dt=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$

(与拉格朗日型余项不同. 此处需 $n+1$ 阶导连续 拉: n 阶连续 $n+1$ 可导)

* notation: $f\in C^k[a,b]$. f is k-smooth. ($k=1$. f is smooth)

e.g. Wallis formula: $\pi=\lim_{n\to\infty}\frac{1}{n}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2$

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1}x\,dx<\int_0^{\frac{\pi}{2}}\sin^{2n}x\,dx<\int_0^{\frac{\pi}{2}}\sin^{2n-1}x\,dx.$$

$$\frac{(2n)!!}{(2n+1)!!}<\frac{(2n-1)!!}{(2n)!!}\cdot\frac{\pi}{2}<\frac{(2n-2)!!}{(2n-1)!!}$$

(apply previous.

$I=\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\begin{cases}\frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}, & n=2k\\ \frac{(2k)!!}{(2k+1)!!} & n=2k+1\end{cases}$

$I_n=\frac{n-1}{n}I_{n-2}$

$$\Rightarrow\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}<\frac{\pi}{2}<\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\cdot\frac{1}{2n}\rightarrow \text{denote by } 2x_n$$

$\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}}\cos^n x\,dx$

$\pi<x_n<\frac{2n+1}{2n}\pi\Rightarrow x_n\to\pi\ (n\to\infty)$

Wallis Formula.

★ $\lim_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}=\frac{\pi}{2}$.

Thm. Hölder's inequality for integral.

$f, g\in C[a,b]$. $\frac{1}{p}+\frac{1}{q}=1$. ($p, q$ conjugate exponents).

$$\left|\int_a^b fg\right|\le\left(\int_a^b|f|^p\right)^{\frac{1}{p}}\left(\int_a^b|g|^q\right)^{\frac{1}{q}}$$

$\sum_{i=1}^{n}x_iy_i\le\left(\sum_{i=1}^{n}x_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n}y_i\right)^{\frac{1}{q}}$ ($p>1$. $x_i\ge0$. $y_i\ge0$)

$x_k=a+\frac{k(b-a)}{n}$. $a_k=f(x_k)(\Delta x_k)^{\frac{1}{p}}$ $b_k=g(x_k)(\Delta x_k)^{\frac{1}{q}}$. $\Rightarrow a_kb_k=f(x_k)g(x_k)\Delta x_k$.

$$\left|\sum_{k=0}^{n-1}f(x_k)g(x_k)\Delta x_k\right|\le\left(\sum_{k=0}^{n-1}|f(x_k)|^p\Delta x_k\right)^{\frac{1}{p}}\left(\sum_{k=0}^{n-1}|g(x_k)|^q\Delta x_k\right)^{\frac{1}{q}}$$

Coro. ($p=2$. $q=2$). Cauchy's inequality for integrals

$$\left|\int_a^b fg\right|\le\sqrt{\int_a^b f^2}\cdot\sqrt{\int_a^b g^2}$$

Thm. (Minkowski's inequality for integrals)

suppose $f, g \in C[a,b]$. $p \geq 1$. then

$$\left(\int_a^b |f+g|^p\right)^{\frac{1}{p}} \leq \left(\int_a^b |f|^p\right)^{\frac{1}{p}} + \left(\int_a^b |g|^p\right)^{\frac{1}{p}}$$

Thm. Chebyshev's inequality for integrals.

Suppose f increase, g decreases on $[a,b]$. Then.

$$\frac{1}{b-a}\int_a^b fg \leq \left(\frac{1}{b-a}\int_a^b f\right)\cdot\left(\frac{1}{b-a}\int_a^b g\right)$$

proof: Let $A = \frac{1}{b-a}\int_a^b f$. $E = \{x \in [a,b] : f(x) \leq A\}$. $E \neq \phi$

(otherwise. $f > A$ on $[a,b]$. we obtain $A > A$).

Let $c = \sup E$. Then $A - f \geq 0$. $g \geq g(c)$. on $[a,c)$ and $A - f \leq 0$. $g \leq g(c)$. on $(c,b]$.

Then $\int_a^b (A-f)g = \int_a^c (A-f)g + \int_c^b (A-f)g \geq g(c)\int_a^b (A-f) = 0$.

$\int_a^b (A-f)g \geq 0. \Rightarrow A\int_a^b g - \int_a^b fg \geq 0$

Coro. (Chebyshev's inequality for sums)

Let $n \in \mathbb{N}$. $a, b \in \mathbb{R}^n$. $a_1 \leq \dots \leq a_n$. $b_1 \geq \dots \geq b_n$.

Then $\frac{1}{n}\sum_{k=1}^n a_k b_k \leq \left(\frac{1}{n}\sum_{k=1}^n a_k\right)\left(\frac{1}{n}\sum_{k=1}^n b_k\right)$

(apply ... by function $f, g : [0,1] \to \mathbb{R}$. values a_k and b_k on $(\frac{k-1}{n}, \frac{k}{n})$.

$\left(\int_a^b = \sum_{k=1}^n \int_{\frac{k-1}{n}}^{\frac{k}{n}} f = \frac{1}{n}\sum_{k=1}^n a_k.\right)$

Def. (Average of a function)

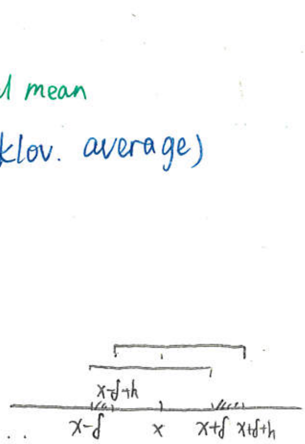
Let $f \in [a,b]$, for any $[a,b] \subset \mathbb{R}$. F_δ arithmatical mean

$F_\delta(x) = \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)dt$. $\to$ the average of f (the Steklov. average)

Properties. 1). $F_\delta \in C(\mathbb{R})$ (smooth)

2). $f \in C^k(\mathbb{R})$. $F_\delta(x) \in C^{k+1}(\mathbb{R})$

3). If $f \in C(\mathbb{R})$. $\lim_{\delta \to 0^+} F(\delta)(x) = f(x)$.



proof (1). Let $|f(x)| \leq M$. and $\forall h$ s.t $|h| < \delta$.

$|F_\delta(x+h) - F_\delta(x)| = \frac{1}{2\delta}\left|\int_{x+\delta}^{x+\delta+h} f(t)dt + \int_{x-\delta+h}^{x-\delta} f(t)dt\right| \leq \frac{1}{2\delta}(M|h| + M|h|) = \frac{M}{\delta}|h|$

(2) $[f(\varphi(x))]' \quad \frac{d\int_a^{\varphi(x)} f(t)dt}{dx} = \frac{d\int_a^{\varphi} f(t)dt}{d\varphi}\cdot\frac{d\varphi}{dx} = f(\varphi(x))\cdot\varphi'(x)$. $(\varphi(x) := \int_a^x f)$.

$F_\delta(x) = \frac{1}{2\delta}\int_a^{x+\delta} f(t) - \frac{1}{2\delta}\int_a^{x-\delta} f(t)dt$. $\quad F_\delta'(x) = \frac{f(x+\delta) - f(x-\delta)}{2\delta}$ $\left(\begin{array}{l}\varphi(x) = x+\varphi \\ \varphi(x) = x-\varphi\end{array}\right)$

(3). $F_\delta(x) \overset{t=x+u}{=\!=} \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x+u)du$.

$\exists \mathcal{J}$, $F_\delta(x) = \frac{1}{2\delta}f(x+\mathcal{J})\cdot 2\delta = f(x+\mathcal{J})$. $|\mathcal{J}| \leq \delta$.

Improper Integral

– a generalization of Riemann integral in two directions. – unbounded domain
unbounded function

Def. local integrability.

f is local integrable on $D \subset \mathbb{R}$. and write $f \in R_{loc}(D)$. if $f \in R[c,d]$. for every $[c,d] \subset D$.

e.g. $e^x \in R_{loc}(\mathbb{R})$. $\frac{1}{x} \in R_{loc}(\mathbb{R}\setminus 0)$.

Def. Improper Integral.

(1) $\int_a^{+\infty} f := \lim\limits_{B\to+\infty} \int_a^B f$ $\quad (f \in R_{loc}([a,+\infty)))$

$\int_{-\infty}^{b} f := \lim\limits_{A\to-\infty} \int_A^b f$ $\quad (f \in R_{loc}[(-\infty, b])$

$\int_{-\infty}^{+\infty} f := \lim\limits_{A\to-\infty} \int_A^c f + \lim\limits_{B\to+\infty} \int_c^B f$ $\quad (f \in R_{loc}(\mathbb{R}))$. (do not depend on the choice of c)
– sometimes. write as $\int_{\mathbb{R}} f$.

(2). $f \in R_{loc}[a,b)$.
$\int_a^b f := \lim\limits_{\varepsilon\to0^+} \int_a^{b-\varepsilon} f$.

$f \in R_{loc}[a,c)\cup(c,d]$.
$\int_a^b f := \lim\limits_{\varepsilon_1\to0^+} \int_a^{c-\varepsilon_1} f + \lim\limits_{\varepsilon_2\to0^+} \int_{c+\varepsilon_2}^{b} f$.

Def. Let $-\infty \le a < c_1 \cdots < c_m < b \le \infty$ $f \in R_{loc}(a,c_1)\cup\cdots\cup(c_m,b)$.

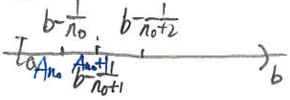
$\int_a^b f := \int_a^{d_1} f + \int_{d_1}^{c_1} f + \int_{c_1}^{d_2} f + \cdots + \int_{c_m}^{d_{m+1}} f + \int_{d_{m+1}}^{b} f$. where $a<d_1<c_1<\cdots<c_m<d_{m+1}$

the limit finite – convergent infinite – divergent.
for integrable function. Riemann integral coincides improper integral

Def. Singular point f is not bounded on any neighbourhood of p or $p=\pm\infty$

△ Improper function over $[a,b]$. is a generalization of Riemann integral to the locally integrable unbounded functions and only to them.
i.e. $f \in R_{loc}[a,b)$ f is bounded on $[a,b)$. $\Rightarrow f \in R[a,b]$.

proof: Use Lebesgue Criterion (measure zero).



Choose. n_0. s.t. $\frac{1}{n_0} < |b-a|$ $\quad I_n = [b-\frac{1}{n}, b-\frac{1}{n+1}]$.

$f \in R(I_n) \xRightarrow{\text{Leb.Cri.}}$ the set $A_{n_0} \cdots A_{n_0+n}$. is of measure zero.

* find a finite cover. let it be $\{A_{N_0} \dots A_{N_0+N}\}$. the sum of the set is of measure zero.

有限覆盖

$\exists (a_k^n, b_k^n).\ k \in \mathbb{N}$. $A_n \subset \bigcup_k (a_k^n, b_k^n)$. $\sum_k (b_k^n - a_k^n) < \frac{\varepsilon}{2^n}$ $\Rightarrow A \subset \bigcup_{k,n} (a_k^n, b_k^n)$ $\sum_{k,n} (b_k^n - a_k^n) < \sum_n \frac{\varepsilon}{2^n} < \varepsilon$.

(remind $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$. conv $\alpha>1$ div. $\alpha\le1$ p-series)

it's sufficient to consider "right" singular points only.

Let $b \in \mathbb{R}$ or $b=+\infty$. $f \in R_{loc}[a,b)$

$$\int_a^b f := \begin{cases} \lim\limits_{\varepsilon\to0^+} \int_a^{b-\varepsilon} f, & b \in \mathbb{R}. \\ \lim\limits_{B\to+\infty} \int_a^B f, & b=+\infty \end{cases}$$

Simple Properties

1). Additivity.

If. $\int_b^a f$ converges. $c \in (a,b)$. $\int_c^b$ converges. and $\int_b^a f = \int_c^a f + \int_b^c f$

Conversely. if $c \in (a,b)$. $\int_c^b f$ converges. then $\int_a^b f$ converges

(let $A \in (c,b)$. and $A \to b-$ pass the limit.).

Def. Improper integral $\int_A^b f$ is called the reminder of the integral $\int_a^b f$

(integral and its reminder div. conv. simultaneously.

2) If. $\int_a^b f$ converges. then $\int_A^b f \xrightarrow[A \to b-]{} 0$.

that is. $\int_A^b f = \int_a^b f - \int_a^A f \to 0$.

3) Linearity. $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ ($\int_a^b f$. $\int_g^b$ converges).

4) Monotonisity. if $\int_a^b f$. $\int_a^b g$ exists in $\bar{\mathbb{R}}$. $f \le g$ on $[a,b]$. $\int_a^b f \le \int_a^b g$. *$\bar{\mathbb{R}}$ ⇒ 任意实数. 包括 $\pm\infty$

Remark. $\int_a^b f$. $\int_a^b g$ conv. not imply. their product conv.

<u>Thm. Integration by parts.</u>

f, g. differentiable. on $[a,b)$. $b \in \bar{\mathbb{R}}$. $f', g' \in R_{loc}[a,b)$. $\int_a^b f' g = fg \big|_a^b - \int_a^b f g'$

(proof. let $B \in [a,b)$. pass the limit)

<u>Thm. Change variables.</u>

$f \in C[a,b)$. $g: [c,d) \to [a,b)$. g is differentiable on $[c,d)$. $g' \in R_{loc}[c,d)$.

$\exists \lim\limits_{t \to d-} g(t) = g(d-) \in \bar{\mathbb{R}}$. then $\int_a^d (f \circ g) g' = \int_{g(c)}^{g(d-)} f$

proof: $t \in [c,d)$. $x \in [a,b)$. denote $\phi(t) := \int_c^t (f \circ g) \circ g'$ $F(x) := \int_{g(c)}^x f$

by .. of proper integral $\phi(t) = F(g(t))$. then pass the limit.

Lemma Suppose $f \in R_{loc}[a,b)$. $f \ge 0$. then $\int_a^b f$ is conv. iff $F(x) := \int_a^x f$. $F(x)$ is bounded from above.

proof: $\forall x_1, x_2 \in [a,b)$. w.l.g. $x_1 < x_2$ $F(x_2) - F(x_1) = \int_{x_1}^{x_2} f \ge 0$. F is nondecreasing

$\Rightarrow \lim\limits_{\cancel{x} \to b^-} \int_a^b f = \lim\limits_{x \to b^-} F(x)$ is ~~de~~finite iff F is bounded from above

<u>Thm</u>. A comparsion test for convergece of improper integral

$f, g \in R_{loc}[a,b)$. $f, g \ge 0$ $f(x) = O(g(x))$ $x \to b-$ 1. $\int_a^b g$ conv. then $\int_a^b f$ conv.

2. $\int_a^b f$ div. then $\int_a^b g$ conv.

$f(x) = O(g(x)) \Rightarrow x \to b-$

$\exists k \, \exists C \, \forall x \in [c,b)$ $f(x) \le k g(x)$: (def. $o \underset{\not\leftarrow}{\Rightarrow} O$)

"O" $\frac{f(x)}{g(x)}$ ultimate bounded. $\lim \frac{f(x)}{g(x)}$ may~~be~~ be not exist !

Corollay (Limit Comparison test).

$f, g \in R_{loc}[a,b)$. $f, g \geq 0$ $\lim\limits_{x \to b-} \frac{f(x)}{g(x)} = k \in [0, +\infty]$

1) if $k \in (0, +\infty)$. $\int_a^b f$ and $\int_a^b g$ have same conv. or div.

2). if $k = 0$. $\int_a^b g$ conv. $\to$ $\int_a^b f$ conv.

3) if $k = +\infty$. $\int_a^b f$ conv. $\to$ $\int_a^b g$ conv.

e.g. $\int_2^{+\infty} \frac{dx}{x^\alpha \ln^\beta x}$ converges iff $\alpha > 1$. and β is arbitrary or $\alpha = 1$ and $\beta = 1$.

Thm (Cauchy Criterion)

Suppose $f \in R_{loc}[a,b)$, then $\int_a^b f$ converges iff

$\forall \varepsilon\ \exists b_0 \in [a,b)\ \forall b_1, b_2 \in [b_0, b)\ \left|\int_{b_1}^{b_2} f\right| < \varepsilon$.

$F(x) := \int_a^x f$ $\quad \left|\int_{b_1}^{b_2} f\right| = |F(b_2) - F(b_1)|$ use Cauchy criterion in function

Def. (Absolute convergent integral) 绝对收敛.

If the integral $\int_a^b |f|$ converges. then we say that $\int_a^b f$ is absolutely convergent.

Thm. If $\int_a^b f$ is absolutely convergent. then it is convergent.

Proof 1° Apply Cauchy Criterion and property $\left|\int_a^b |f|\right| \geq \left|\int_a^b f\right|$. (in Riemann integral).

2° Consider $f_+ := \max\{f, 0\}$. $f_- := \max\{-f, 0\}$.

They are called <u>a positive and a negative parts</u> of the function f.

respectively. $f = f_+ - f_-$ $\quad |f| = f_+ + f_-$. $f_+, f_- \leq |f|$.

If $\int_a^b |f|$ conv. $\int_a^b f_+$. $\int_a^b f_-$ conv. (by the comparison). $\int_a^b f = \int_a^b f_+ - \int_a^b f_-$

Def. Conditionally convergent integral. (条件收敛).

If an improper integral converges but not absolutely.

Thm. Let $f \in R_{loc}[a,b)$. g be monotonic on $[a,b)$. If one of the following pairs of condition holds.

1. Abel's Test. $\int_a^b f$ conv. & g is bounded on $[a,b)$ (In fact. we do not need monotonicity of g in Abel's test. $|\int_a^b f| < \frac{\varepsilon}{M}$. $|g| < M$).

2. Dirichlet's Test. $F(x) := \int_a^x f$ is bounded on $[a,b)$. & $\lim\limits_{x \to b-} g(x) = 0$. the sign of f should be constant (要用第一中值. 是条件) $\left|\int_a^b fg\right| = \left|M \int_a^b f\right| \leq M \cdot \frac{\varepsilon}{M}$.

then $\int_a^b fg$ conv.

proof Cauchy's Criterion. $\forall \varepsilon\ \exists b_0.\ \forall b_1, b_2 \in [b_0, b)$. $\left|\int_{b_1}^{b_2} fg\right| < \varepsilon$ (we find the needed. b_0 later).

w.l.g. $b_1 < b_2$. by second mean... $\left|\int_{b_1}^{b_2} fg\right| = \left|g(b_1)\int_{b_1}^{c} f + g(b_2)\int_c^{b_2} f\right| =: A$. $c \in (b_1, b_2)$

1. g is bounded. $|g(x)| < M$. $\quad \int_a^b f$ conv. $\Rightarrow \forall \varepsilon\ \underline{\exists b_0 \in [a,b)}\ \forall b_1, b_2, c \in [b_0, b)$

$$\left|\int_{b_1}^{c} f\right| < \frac{\varepsilon}{2M}. \quad \left|\int_c^{b_2} f\right| < \frac{\varepsilon}{2M}$$

2. $F(x)$ is bounded. $|F(x)| < K$.

$\left|\int_{b_1}^{c} f\right| = |F(c) - F(b_1)| \leq |F(c)| + |F(b_1)| \leq 2K$.

since $\lim_{x\to b-} g = 0$. $\exists\, b_0 \in [a,b)$. $\forall\, b_1, b_2 \in [b_0, b)$. $|g(b_1)|, |g(b_2)| < \frac{\varepsilon}{4K}$.

Fourier Transformation (傅里叶变换) (补充).

$e^{ix} = \cos x + i\sin x$

$e^{-ix} = \cos x - i\sin x$

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-i\omega x} dx = \int_{\mathbb{R}} f(x)\cos\omega x - i\int_{\mathbb{R}} f(x)\sin\omega x\, dx.$$

The Dirichlet integral $\int_0^{+\infty} \frac{\sin x}{x} dx$.

The Fresnel integral $\int_0^{+\infty} \sin(x^2)dx$. $\int_0^{+\infty} \cos(x^2)dx$

E.g. $\int_a^{+\infty} g(x)\sin x\, dx$. $\int_a^{+\infty} g(x)\cos x\, dx$. g is monotonic and nonnegative on $[a,+\infty)$. $a \in \mathbb{R}$.

1. $\int_a^{+\infty} g$ conv. the function abs. conv.
2. $\int_a^{+\infty} g$ div. $\begin{cases} \lim_{x\to\infty} g(x) = 0 & \text{cond. conv.} \\ \lim_{x\to\infty} g(x) = c > 0. & \text{div.} \end{cases}$

technique: improper integral from unbounded function to infinite integral. (无界函数→无界区间)

$f(x) \in R_{loc}[a,b)$

$$\int_a^b f(x)dx \xlongequal{t=\frac{b-a}{b-x}} (b-a)\int_1^{+\infty} f\left(b-\frac{b-a}{t}\right)\frac{dt}{t^2}.$$

Def. The principal value of Improper integral "same speed"

Let $f, g \in R_{loc}[a,c) \cup (c,b]$.

the quantity $p.v.\int_a^b f := \lim_{\varepsilon\to 0+}\left(\int_a^{c-\varepsilon} f + \int_{c+\varepsilon}^b f\right)$. (if the limit exists).

$f \in R_{loc}(-\infty,+\infty)$ $p.v.\int_{-\infty}^{+\infty} f := \lim_{A\to+\infty}\int_{-A}^{A} f$

Application of integration

Def. Additive interval function.

$\forall \alpha, \beta, \gamma \in [a,b]$. $I(\alpha,\gamma) = I(\alpha,\beta) + I(\beta,\gamma)$ ((α,β),… are ordered pair of point).

Then function $I(\alpha,\beta)$ is additive (oriented) interval function. (defined on intervals contained in $[a,b]$)

1) $\alpha=\beta=\gamma$. $I(\alpha,\alpha)=0$. 2) $\alpha=\gamma$ $I(\alpha,\beta)+I(\beta,\alpha)=0$.

Thm (The density of an additive interval function).

If. $\exists f \in R[a,b]$. $\forall \alpha,\beta$ $a \le \alpha \le \beta \le b$. $I(\alpha,\beta)$ be an additive interval function. &.

$$\inf_{x\in[\alpha,\beta]} f(x)(\beta-\alpha) \le I(\alpha,\beta) \le \sup_{x\in[\alpha,\beta]} f(x)(\beta-\alpha).$$

then $I(a,b) = \int_a^b f$ The function f is called the density of $I(\alpha,\beta)$.

proof: regard the LHS. RHS as $s_J(f)$. $S_J(f)$. pass to the limit $\lambda(J)\to 0$.

e.g. The area of curvilinear sector (曲边扇形).

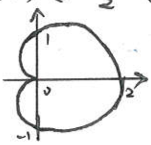
Let (r,t) be polar coordinates.

$T(c,d) = \{(r,t) : t\in[c,d],\ r\in[0,r(t)]\}$. $S(c,d)$ is the area of $T([c,d])$.

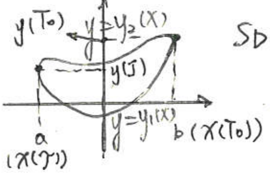
$S(a,b) = S(a,c) + S(c,b)$

$m := \inf_{t\in[c,d]} r(t)$ $M := \sup_{t\in[c,d]} r(t)$ $\frac{m^2}{2}(d-c) \le S(c,d) \le \frac{M^2}{2}(d-c)$. $S(c,d) = \frac{1}{2}\int_c^d r^2$

$r(t) = \cos t + 1$ $t\in[0,2\pi]$. cardioid (心脏形曲线)



General Method for boundary curve.



$S_D = \int_a^b (y_2(x) - y_1(x))\,dx$ ①

参数方程是整条  人为从最远点断开.

J: $\begin{cases} x = x(t) \\ y = y(t). \end{cases}$ $t\in[T_0,T_1]$. $x(T_0) = x(T_1)$. $y(T_0) = y(T_1)$.

Let $J \in [T_0,T_1]$ $x(t)$ is strictly monotonic. differentiable. on $[T_0,J]$. $[J,T_1]$.

Change variable in ①. $S_D = -\int_{T_0}^{J} y(t)\,x'(t)\,dt - \int_J^{T_1} y(t)\,x'(t)\,dt = -\int_{T_0}^{T_1} y(t)\,x'(t)\,dt$.

We have $x=x(t)$. $t\in[T_0,J]$; $x=x(t)$ $t\in[J,T_1]$.

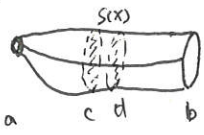
Changing x and y their place. $S_D = \int_{T_0}^{T_1} x(t)\,y'(t)\,dt$. $y_2(x(t)) = y(t)$, $t\in[T_0,J]$. $y_1(x(t)) = y(t)$ $t\in[J,T_1]$.

$$S_D = \frac{1}{2}\int_{T_0}^{T_1} (x(t)\,y'(t) - y(t)\,x'(t))\,dx. \quad \text{(Joining two equation)}$$

Calculate the volume (of solids).

Let $T \subset \mathbb{R}^3$. be a solid. $T(x) := \{(y,z)\in\mathbb{R}^2 ; (x,y,z)\in T\}$ (cross section in x). 截面

$S(x)$ be area of $T(x)$. ($S(x)$ is continous on $[a,b]$.



If T is obtain by revolving the curvilinear trapezoid corresponding to the function $y=f(x)$. $S(x) = \pi f^2(x)$ $V(c,d) = \pi\int_c^d f^2$

Def. (Path.)

A path in $\mathbb{R}^d$ is a mapping $\gamma: [a,b] \to \mathbb{R}^d$. $\gamma: t \mapsto (\gamma_1(t), \dots, \gamma_d(t))$.

all function γ_k are continuous on $[a,b]$. $\gamma(a)$, $\gamma(b)$. initial / terminal point

the path is closed if $\gamma(a) = \gamma(b)$.

If. $\gamma(t_1) = \gamma(t_2)$ implies $t_1 = t_2$ or $t_1, t_2 \in \{a, b\}$. path is simple $\Rightarrow$ 同一函数 → 单射
不同函数 → 无交点

$\gamma_k \in C^r[a,b]$. path is r-smooth ($r=1$., smooth).

If $\exists$ partition $\mathcal{T} = \{t_k\}_{k=0}^{n}$ of $[a,b]$, and restrictions $\gamma_{[t_k, t_{k+1}]}$. $k=0, \dots, n-1$, are smooth.

the path is piecewise smooth.

The image $\gamma([a,b])$. is called the support of path.

Def. Equivalent paths. (Support same)

Path $\gamma: [a,b] \to \mathbb{R}^d$ $\gamma^*: [c,d] \to \mathbb{R}^d$ are equivalent if there exists a strictly increasing onto (or surjective) function $\theta: [a,b] \to [c,d]$. such that $\gamma = \gamma^* \circ \theta$.

The function θ is called an admissible change of parameter. t is a parameter.

Def. curve.

The equivalence class of equivalent paths is called a curve

An element of the class is called a parametrization of a curve.

curve is smooth if $\exists$ a smooth parametrization.

Def. (The length of the path, rectifiable path → 可求长度的.)

let partition $\mathcal{T} = \{t_k\}_{k=0}^{n}$. of $[a,b]$. $\gamma: [a,b] \to \mathbb{R}^d$ be a path.

We connect $\gamma(t_k)$. $\gamma(t_{k+1})$ by line segments. to create a ~~pl~~ polygonal path.

$P_{\mathcal{T}}$ length of polygonal path. $S_\gamma := \sup_{\mathcal{T}} P_{\mathcal{T}}$ is the length of path γ.

If S_γ is finite. γ is rectifiable.

Lemma: The length of equivalent paths are equal

Proof: let $\theta: [a,b] \to [c,d]$. be admissible change of parameter for γ. γ^*

let partition $\mathcal{T} = \{t_k\}_{k=0}^{n}$ $\mathcal{T}^* = \{\theta(t_k)\}_{k=0}^{n}$ denote $|x_1 - x_2|$ the distance of point x_1. x_2

$$P_{\mathcal{T}} = \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| = \sum_{k=0}^{n-1} |\gamma^*(\theta(t_{k+1})) - \gamma^*(\theta(t_k))| = P_{\mathcal{T}^*}$$

$P_{\mathcal{T}} = P_{\mathcal{T}^*} \le \underbrace{S_{\gamma^*}}_{\sup_{\mathcal{T}}} \Rightarrow S_\gamma \le S_{\gamma^*}$ ($P_{\mathcal{T}}$ is arbitrary) $P_{\mathcal{T}^*} = P_{\mathcal{T}} \le \underbrace{S_\gamma}_{\sup_{\mathcal{T}^*}} \Rightarrow S_{\gamma^*} \le S_\gamma$. ($P_{\mathcal{T}^*}$ is arbitrary)

the lemma shows. path does not depend on parametrization.

Lemma (length additivity).

Suppose $\gamma:[a,b]\to\mathbb{R}^d$, $c\in(a,b)$. $\gamma^1:=\gamma|_{[a,c]}$ $\gamma^2:=\gamma|_{[c,b]}$. then $S_{\gamma_1}+S_{\gamma_2}=S_\gamma$.

proof: "$\le$" Let $\mathcal{T}_1, \mathcal{T}_2$ be partition of $[a,c]$, $[c,b]$.

Let $p_{\mathcal{T}_1}, p_{\mathcal{T}_2}$ be the lengths of corresponding polygonal paths.

$p_{\mathcal{T}_1}+p_{\mathcal{T}_2}=p_{\mathcal{T}}\le S_\gamma$ ($\because \mathcal{T}_1, \mathcal{T}_2$ is arbitrary. but $\mathcal{T}:=\mathcal{T}_1\cup\mathcal{T}_2$ is depend on $\mathcal{T}_1,\mathcal{T}_2$)

$\Rightarrow S_{\gamma_1}+S_{\gamma_2}\le S_\gamma$.

"$\ge$" let $\mathcal{T}$ be partition.

1° $c\in\mathcal{T}$. $\forall p_{\mathcal{T}}=p_{\mathcal{T}_1}+p_{\mathcal{T}_2}\le \sup p_{\mathcal{T}_1}+\sup p_{\mathcal{T}_2}=S_{\gamma_1}+S_{\gamma_2}$. $\Rightarrow$ $S_\gamma\le S_{\gamma_1}+S_{\gamma_2}$

2° $c\notin\mathcal{T}$. let $\mathcal{T}^*=\mathcal{T}\cup\{c\}$. let $\mathcal{T}=\{t_k\}_{k=0}^n$. $c\in(t_r,t_{r+1})$.

$$p_{\mathcal{T}}=\sum_{k=0}^{r-1}|\gamma(t_{k+1})-\gamma(t_k)|+|\gamma(t_{r+1})-\gamma(t_r)|+\sum_{k=r+1}^{n-1}|\gamma(t_{k+1})-\gamma(t_k)|\le p_{\mathcal{T}^*}=p_{\mathcal{T}_1}+p_{\mathcal{T}_2}.$$

$$|\gamma(t_{r+1})-\gamma(t_r)|\le|\gamma(c)-\gamma(t_r)|+|\gamma(t_{r+1})-\gamma(c)|$$

$\therefore \forall p_{\mathcal{T}}\le S_{\gamma_1}+S_{\gamma_2}$. $\Rightarrow S_\gamma\le S_{\gamma_1}+S_{\gamma_2}$.

Thm. (The length of a smooth part).

If $\gamma:[a,b]\to\mathbb{R}^d$. $\gamma_j\in C^1[a,b]$. $j=1,2\ldots d$. then γ is rectifiable and

$$S_\gamma=\int_a^b|\gamma'(t)|dt=\int_a^b\Big(\sum_{j=1}^d|\gamma_j'(t)|^2\Big)^{\frac12}dt.$$

$\int ds$
$$ds=\sqrt{d^2x+d^2y}=\sqrt{(x'(t)dt)^2+(y'(t)dt)^2}=\sqrt{(x'(t))^2+y'(t)^2}\,dt.$$

proof: 1. $p_{\mathcal{T}}=\sum_{k=0}^{n-1}|\gamma(t_{k+1})-\gamma(t_k)|=\sum_{k=0}^{n-1}\Big(\sum_{j=1}^d|\gamma_j(t_{k+1})-\gamma_j(t_k)|\Big)^{\frac12}$ (两点距离公式.)

by lagrange formula
$$=\sum_{k=0}^{n-1}\Big(\sum_{j=1}^d|\gamma_j'(t_k^*)|^2\Big)^{\frac12}\Delta t.\quad t_k^*\in(t_k,t_{k+1})$$

denote. $M_{j,[a,b]}:=\sup_{t\in[a,b]}|\gamma_j'(t)|$ $m_{j,[a,b]}:=\inf_{t\in[a,b]}|\gamma_j'(t)|$. (用 M.m 代替 $|\gamma_j'(t_k^*)|$.

$$\Big(\sum_{j=1}^d m_{j[a,b]}^2\Big)^{\frac12}(b-a)\le S_\gamma\le\Big(\sum_{j=1}^d M_{j[a,b]}^2\Big)^{\frac12}(b-a).$$ (不取决于 $\mathcal{T}$.)

2. to show $S_\gamma=\int_a^b|\gamma'(t)|dt$. we need to show $S_\gamma(x)=\int_a^x|\gamma'(t)|dt\Leftarrow S'_\gamma(x)=|\gamma'(x)|$ ($\gamma'(x)$ is continous)

$S(t)$ be length of $\gamma|_{[a,t]}$. by additivity. $S(t+\Delta t)-S(t)=\Delta S(t)$.

by 1. $$\Big(\sum_{j=1}^d m_{j[t,t+\Delta t]}^2\Big)^{\frac12}\Delta t\le\Delta S(t)\le\Big(\sum_{j=1}^d M_{j[t,t+\Delta t]}^2\Big)^{\frac12}\Delta t.\qquad(1).$$

(连续函数介值性)

$\exists t_j^*, t_j^{**}\in[t,t+\Delta t]$. $m_{j[t,t+\Delta t]}=|\gamma_j'(t_j^*)|$ $M_{j[t,t+\Delta t]}=|\gamma_j'(t_j^{**})|$

Let $t_j^*=t_j^*(\Delta t)$. $t<t_j^*(\Delta t)<t+\Delta t$. $\Rightarrow\lim_{\Delta t\to0}t_j^*(\Delta t)=t$. ($t_j^*$ 是与Δt有关的函数).

$\Rightarrow\lim_{\Delta t\to0}\gamma_j'(t_j^*(\Delta t))=\gamma_j'(t)$. similarly $\lim_{\Delta t\to0}\gamma_j'(t_j^{**}(\Delta t))=\gamma_j'(t)$

pass the limit $\Delta t\to0$ in (1). $\Big(\sum_{j=1}^d|\gamma_j'(t)|dt\Big)^{\frac12}=S'(t)$.

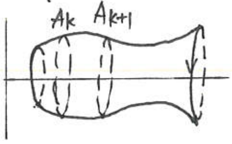
since $\sum_{j=1}^d|\gamma_j'(t)|dt$ is continuous $\xrightarrow{N-L}$ $S(x)=\int_a^x\Big(\sum_{j=1}^d|\gamma_j'(t)|dt\Big)^{\frac12}dt.$

Thm. (The area of a surface of revolution.

If. $x, y \in C'[\alpha,\beta]$. then the area of a surface of revolution is.

$$S = 2\pi \int_\alpha^\beta |y(t)| \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Surface of revolution.



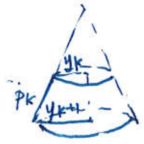
Let $(x(t), y(t))$. $t \in [\alpha,\beta]$

$\mathcal{T} = \{t_k\}_{k=0}^n$ be a partition of $[\alpha,\beta]$

$A_k = (x(t_k), y(t_k)) = (x_k, y_k)$

$p_k = \sqrt{(x_{k+1}-x_k)^2 + (y_{k+1}-y_k)^2}$ be a length of A_kA_{k+1}

S_k是两侧面扇形相减.



Rotating A_kA_{k+1} around x-axis we get a surface of a truncated cone.
the corresponding area is $S_k = \pi(y_k + y_{k+1})p_k$. If. $\lim_{\lambda(\mathcal{T})\to 0} \sum_{k=0}^{n-1} S_k$. exists.
the limit is called the area of a surface of revolution.

Def. Let $f: [a,b] \to \mathbb{R}$. (New class of function).
The quantity $\bigvee_a^b f = \sup_{\mathcal{T}} \sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)|$ is a <u>variation</u> of f on $[a,b]$

If $V_a^b f < +\infty$. f is the function of bounded variation on $[a,b]$. (1-dim rectifiable mapping).
The set of all functions of bounded variation on $[a,b]$ is denoted by $V[a,b]$

Properties. 1. additive. $a<c<b$. $\bigvee_a^b f = \bigvee_c^b f + \bigvee_a^c f$. (additivity of length)

2. If f is piecewise smooth on $[a,b]$. ($d=1$ in length of smooth part).

$$\bigvee_a^b f = \bigvee_a^b |f'|$$

3. Variation is monotone. If. $f: [a,b] \to \mathbb{R}$. $[\alpha,\beta] \subset [a,b]$. (proof by additivity)

$$\bigvee_\alpha^\beta f \le \bigvee_a^b f$$

Def of variation for a function defined on non-closed interval

If $f: \langle a,b\rangle \to \mathbb{R}$. $\bigvee_a^b f := \sup_{[\alpha,\beta]\subset\langle a,b\rangle} \bigvee_\alpha^\beta f$.

4. Let $(\gamma_1, \dots \gamma_m): [a,b] \to \mathbb{R}^m$. Then $S_\gamma < +\infty$ iff $\gamma_i \in V[a,b]$ for all $i = 1,\dots,m$

proof. "$\Rightarrow$". $S_\gamma = \sup_{\mathcal{T}} \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| = \sup_{\mathcal{T}} \sum_{k=0}^{n-1} \left(\sum_{j=1}^{d} (\gamma_j(t_{k+1}) - \gamma_j(t_k))^2\right)^{\frac{1}{2}} < \infty$

for any fixed $j \in \mathbb{N}$, $\sum_{k=0}^{n-1} |\gamma_j(t_{k+1}) - \gamma_j(t_k)| \le \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)|$

$\Rightarrow \sup \sum_{k=0}^{n-1} |\gamma_j(t_{k+1}) - \gamma_j(t_k)| \le \sup \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| < \infty$. $\gamma_j \in V[a,b]$.

"$\Leftarrow$" any $\gamma_i \in V[a,b]$, $\sum_{i=1}^{m} |\gamma_i(t_{k+1}) - \gamma_i(t_k)| < \infty$ (the sum of $\gamma_1 \dots \gamma_m \in V[a,b]$)

since $\sup_{\mathcal{T}} \sum_{k=0}^{n-1} \left(\sum_{j=1}^{m} (\gamma_j(t_{k+1}) - \gamma_j(t_k))^2\right)^{\frac{1}{2}} \le \sup_{\mathcal{T}} \sum_{k=0}^{n-1} \left(\sum_{i=1}^{m} |\gamma_i(t_{k+1}) - \gamma_i(t_k)|\right) < \infty$

$\hookrightarrow \sqrt{a_1^2 + \dots + a_n^2} < |a_1| + \dots + |a_n|$

thus $S_\gamma < \infty$.

properties 5. If f is monotone on $[a,b]$, then $f \in V[a,b]$ and

$$\overset{b}{\underset{a}{V}} f = |f(b) - f(a)|$$

proof: For any partition

$$\sum_{k=0}^{n-1} |f(x_{k+1}) - f(x_k)| = \left|\sum_{k=0}^{n-1} f(x_{k+1}) - f(x_k)\right| = |f(b) - f(a)| .$$

6. If $f \in V[a,b]$ the f is bounded on $[a,b]$

proof: $|f(x)| \leq \frac{1}{2}|f(a)| + \frac{1}{2}|f(b)| + \frac{1}{2}|f(x) - f(a)| + \frac{1}{2}|f(b) - f(x)|$ $\Rightarrow$ a given $\mathcal{T} = \{a, x, b\}$

$\leq \frac{1}{2}|f(a)| + \frac{1}{2}|f(b)| + \frac{1}{2}\overset{b}{\underset{a}{V}} f$ $\overset{b}{\underset{a}{V}} f$ is a supremum.

Thm. (Function of bounded variations and arithmetic operations)

Let $f, g \in V[a,b]$.

1. $f+g \in V[a,b]$
2. $fg \in V[a,b]$.
3. $\alpha f \in V[a,b]$ $(\alpha \in \mathbb{R})$
4. $|f| \in V[a,b]$.
5. if $\inf\limits_{x \in [a,b]} |g(x)| > 0$, then $\frac{f}{g} \in V[a,b]$.

Thm. (Criterion for a bounded variation)

Let $f: [a,b] \to \mathbb{R}$. $f \in V[a,b]$ iff f is represented as a difference of two increasing functions on $[a,b]$.

"$\Rightarrow$" we set $g(x) = \overset{x}{\underset{a}{V}} f$, $x \in [a,b]$ $h = g - f$. (remains to check. g, h is increasing)

$$g(x_2) - g(x_1) = \overset{x_2}{\underset{x_1}{V}} f \geq 0. \text{ (variation's additivity and non-negativity)}$$

$$h(x_2) - h(x_1) = \overset{x_2}{\underset{x_1}{V}} f - \underbrace{(f(x_2) - f(x_1))}_{\text{also a partition}} \geq 0$$

"$\Leftarrow$". last thm. Property 5.

properties 7. $V[a,b] \subset R[a,b]$

A monotonic function is integrable and a difference of integrable functions is integrable

8. The function of bounded variation can not have discontinuities of 2nd kind.

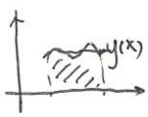
(follows from the criterion)

9. $V[a,b] \not\subset C[a,b]$. and $C[a,b] \not\subset V[a,b]$

proof: $\exists$ discontinous monotone function s.t $V[a,b] \not\subset C[a,b]$

a counter-example of $C[a,b]$ but not $V[a,b]$ $f(x) = \begin{cases} x\cos\frac{\pi}{x}, & x \in (0,1] \\ 0 & x = 0. \end{cases}$ [let partition $x_k = \frac{1}{k}$ $(k \in \mathbb{N})$]

volume of solid of revolution



rotate around x-axis. $V_x = \int_a^b \pi y^2(x)\, dx$ (截面)

rotate around y-axis $V_x = \int_a^b 2\pi |x| \cdot |y(x)|\, dx$ (切片)

Numerical Series

Def $\{a_k\}_{k=1}^{\infty}$ $\sum_{k=1}^{\infty} a_k$ numeric series. a_k. terms.

$S_n = \sum_{k=1}^{n} a_k$ partial sum $\quad \sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ (if the limit exists).

If sequence $\{S_n\}_{n=1}^{\infty}$ has limit S ($\mathbb{R}, \pm\infty$). we write $\sum_{k=1}^{\infty} a_k = S$.

conv. S is finite. div. S is infinite / series has no sum.

properties 1. $\sum_{k=1}^{\infty} a_k$ conv. $\to \sum_{k=m+1}^{\infty} a_k$ conv. $\quad \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{m} a_k + \boxed{\sum_{k=m+1}^{\infty} a_k} \to$ m-th remainder of $\sum_{k=1}^{\infty} a_k$.

2. $\sum_{k=1}^{\infty} a_k$ conv. $\to \sum_{k=m+1}^{\infty} a_k \to 0$ as $m \to 0$

3. linearity

$\boxed{4.}$ a sequence of $\mathbb{C}$ $\{z_k\}_{k=1}^{\infty}$. $\quad x_k = \operatorname{Re} z_k$. $y_k = \operatorname{Im} z_k$

$\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} x_k + i\sum_{k=1}^{\infty} y_k$. $\quad$ (z_k conv. $\to$ x_k. y_k conv. simultaneously)

5. monotonicity. $a_k, b_k \in \mathbb{R}$. $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k \in \overline{\mathbb{R}}^{\to\pm\infty}$ and $a_k \le b_k$ for all $k \in \mathbb{N}$.

$\sum_{k=1}^{\infty} a_k \le \sum_{k=1}^{\infty} b_k$

Necessary Condition: If $\sum_{k=1}^{\infty} a_k$ converges. then $a_n \to 0$ as $n \to \infty$ (一般项趋于0.是必要条件.

proof: $\sum_{k=1}^{\infty} a_k = s \quad a_n = S_n - S_{n-1} \Rightarrow$ pass the limit.

Thm (Cauchy's Criterion)

$\sum_{k=1}^{\infty} a_k$ converges $\Leftrightarrow \forall \varepsilon > 0. \exists N \in \mathbb{N}, \forall n > N \ \forall p \in \mathbb{N} \ \left|\sum_{k=n+1}^{n+p} a_k\right| < \varepsilon$.

Lemma. Let $a_k \ge 0$ for all $k \in \mathbb{N}$. $\sum_{k=1}^{\infty} a_k$ conv. $\Leftrightarrow \{S_n\}$. bounded from above.

Thm (Comparison test) Let $a_k, b_k \ge 0$. for all $k \ge 0$. $a_k = O(b_k)$ as $k \to \infty$ $\quad \exists K\ \forall k > K\ a_k \le K b_k$.

1. $\{b_k\}$ conv. $\to \{a_k\}$ conv.
2. $\{a_k\}$ div. $\to \{b_k\}$ div.

Limit form. $a_k \ge 0$. $b_k > 0$. exist $\lim_{k\to\infty} \frac{a_k}{b_k} = l \in [0, +\infty]$

1. $l \in [0, +\infty)$. $\{b_k\}$ conv. $\to \{a_k\}$ conv.
2. $l \in (0, +\infty]$. $\{a_k\}$ conv. $\to \{b_k\}$ conv.
3. $l \in (0, +\infty)$ series conv. / div. simultaneously.

Thm (Cauchy's Test). $a_k \ge 0$. $K = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$. $\quad K > 1$. div. $\quad K < 1$ conv.

Thm (d'Alembert's Test) $a_k > 0$. $\exists D = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \in [0, +\infty]$ $\quad D > 1$ div. $\quad D < 1$ conv.

Stirling's formula $\quad n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \quad n \to \infty$.

$\left(\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1. \text{ or } \lim_{n\to\infty} \frac{e^n n!}{n^n \sqrt{n}} = \sqrt{2\pi}. \right)$

Thm. (Cauchy's integral test).

Let f be monotone on $[1,+\infty)$. Then the series $\sum_{k=1}^{\infty} f(k)$ and $\int_1^{+\infty} f$

conv. or div. simultaneously.

proof: w.l.g. f decrease.

if some $f(x_0)<0$. them $\lim_{x\to+\infty} f(x) \le f(x_0) < 0$. integral dive to $-\infty$.

consider $f \ge 0$. $f(k+1) \le \int_k^{k+1} f \le f(k)$ (for $k\in\mathbb{N}$).

Fix $n\in\mathbb{N}$. $\sum_{k=1}^{n} f(k+1) \le \int_1^{n+1} f \le \sum_{k=1}^{n} f(k)$

pass the limit to ∞ $\sum_{k=2}^{\infty} f(k) \le \int_1^{+\infty} f \le \sum_{k=1}^{\infty} f(k)$

e.g. $\sum_{k=1}^{\infty} \frac{1}{k^p}$ conv. for $p>1$. and div for $p\le 1$. (compare with the integral $\int_1^{+\infty} \frac{dx}{x^\alpha}$ (p-series).

Def. Absolutely converge. $\sum_{k=1}^{\infty} a_k$ if series $\sum_{k=1}^{\infty} |a_k|$ converges

Lemma: If the series absolutely converges, then it converges.

($|\sum_{k=1}^{\infty} a_k| \le \sum_{k=1}^{\infty} |a_k|$ if the sum exists.

Thm. Let (a_k) be a sequence over $\mathbb{R}$ or $\mathbb{C}$. (b_k) be a monotone series.

Dirichlet's test: $A_n = \sum_{k=1}^{n} a_k$ is bouned. $b_n \to 0$ $(n\to\infty)$. $\sum_{k=1}^{\infty} a_k b_k$ converges.

Abel's test: If series $\sum_{k=1}^{\infty} a_k$ converges. $\{b_k\}$ is bounded. the series $\sum_{k=1}^{\infty} a_k b_k$ conv.

The series $\sum_{k=1}^{\infty} (-1)^{k-1} b_k$ or $\sum_{k=1}^{\infty} (-1)^k b_k$ where $b_k \ge 0$ for all k. := alternating series.

Thm (Leibniz's test).

Let $\{b_n\}$ be monotone. $b_n \to 0$. Then $\sum_{k=1}^{\infty} (-1)^{k-1} b_k$ converges.

(Dirichlet test implies it for $a_k = (-1)^{k-1}$)

Independent proof: let $\{b_n\}$ decrease. $b_n \ge 0$. Consider the sequence $\{S_{2m}\}$.

$S_{2m} - S_{2(m-1)} = b_{2m-1} - b_{2m} \ge 0$ (increase)

$S_{2m} = b_1 + (-b_2 + b_3) + \cdots + (-b_{2m-2} + b_{2m-1}) - b_{2m} \le b_1$ (bounded above)

So $\{S_{2m}\}$ converges to a limit S. Therefore, by $b_{2m+1} \to 0$ we get $S_{2m+1} = S_{2m} + b_{2m+1} \to S$.

thus $S_n \to S$.

Remark: $S_{2m} = (b_1 - b_2) + \cdots + (b_{2m-1} - b_{2m}) \ge 0$ $S_{2m} \le b_1$. $\Rightarrow$ $0 \le S \le b_1$.

The series satisfying Leibniz's test are Leibniz's series.

The reminder of the Leibniz's series. $0 \le (-1)^n (S - S_n) \le b_{n+1}$

(not exceed its first term ($\underline{(-1)^{n+1}} b_{n+1} + (-1)^{n+2} b_{n+2} + \cdots$). in modulus

the signs coincide with its first term). $\hookrightarrow$ since the decrease.

Thm. (Grouping terms). $A_j = \sum_{k=n_j+1}^{n_{j+1}} a_k$, $j \in \mathbb{Z}_+$ $\{n_j\}_{j\geq 0}^{\infty}$. $n_0=0$. strictly increasing integers sequence

1. If $\sum_{k=1}^{\infty} a_k = S$ ($S \in \overline{\mathbb{R}}$ or $\mathbb{C} \cup \{\infty\}$). then $\sum_{j\geq 0}^{\infty} A_j = S$.

2. If $\sum_{j\geq 0}^{\infty} A_j = S$ ($S \in \overline{\mathbb{R}}$ or $\mathbb{C} \cup \{\infty\}$). $a_n \to 0$. and there exists $L \in \mathbb{N}$ s.t each bracket contains no more than L terms. $\sum_{k\geq 1}^{\infty} a_k = S$.

3. If $a_k \in \mathbb{R}$, $\sum_{j\geq 0}^{\infty} A_j = S \in \overline{\mathbb{R}}$ and all terms in each group have the same sign.

then $\sum_{K\geq 1}^{\infty} a_K = S$. (non strict, 0 is also allowed.)

< We denote $S_n = \sum_{K=1}^{n} a_K$. $T_m = \sum_{j\geq 0}^{m} A_j$

proof: 1. $T_m = S_{n_{m+1}}$ that is $\{T_m\}$ is a consequence of $\{S_n\}$. Therefore if $S_n \to S$ then $T_m \to S$.

2. Let $\sum_{j\geq 1}^{\infty} A_j = S$ that is $S_{n_{m+1}} \to S$.

we need to show $S_n \to S$.

2. Fix $\varepsilon > 0$. find $M, K \in \mathbb{N}$ s.t. $|S_{n_m} - S| < \frac{\varepsilon}{2}$ ($\forall m > M$) $|a_k| < \frac{\varepsilon}{2L}$ ($\forall k > K$).

Denote $N = \max\{n_{M+1}, K\}$. For any $n > N$. $\forall m$. $n_m \leq n < n_{m+1}$. then $m > M$.

$$|S_n - S| \leq |S_n - S_{n_m}| + |S_{n_m} - S| \leq \sum_{k=n_m+1}^{n} a_k + |S_{n_m} - S| < L \cdot \frac{\varepsilon}{2L} + \frac{\varepsilon}{2} = \varepsilon.$$

3. Similarly. $\forall n > N$. $n_m \leq n < n_{m+1}$. then $m > M$.

sign same. $S_{n_m} \leq S_n \leq S_{n_{m+1}}$ (w.l.g. +).

$|S_n - S| \leq \max\{|S_{n_m} - S_n|, |S_{n_{m+1}} - S_n|\} < \varepsilon$.

Asymptotic Formula. $H_n = \log n + \gamma + \delta_n$. $\delta_n \to 0$ γ is Euler's constant.

By Cauchy's integral test.

Suppose f decrease on $[1, +\infty)$. $f \geq 0$.

denote $A_n := \sum_{k\geq 1}^{n} f(k) - \int_1^{n+1} f$

($A_{n+1} - A_n = f(n+1) - \int_{n+1}^{n+2} f \geq 0$ $\{A_n\}$ is increasing).

$0 \leq A_n \leq f(1)$. A_n is bounded. thus A_n has a limit. $\lim_{n\to\infty} A_n = \gamma \geq 0$ ($A_n = \gamma + \varepsilon_n$, $\varepsilon_n \to 0$)

e.g. For harmonic series. $H_n = \sum_{k\geq 1}^{n} \frac{1}{k} = \int_1^{n+1} \frac{dx}{x} + \gamma + \varepsilon_n = \ln(n+1) + \gamma + \varepsilon_n$.

$\Rightarrow H_n = \ln n + \gamma + \delta_n$

$$\gamma = \lim_{n\to\infty} \sum_{k\geq 1}^{n} \frac{1}{n} - \ln(n+1) = \lim_{n\to\infty} \sum_{k\geq 1}^{n} \frac{1}{k} - \sum_{k\geq 1}^{n} \ln\left(1+\frac{1}{k}\right) = \sum_{k\geq 1}^{\infty} \left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right).$$

$$\delta_n = H_n - \ln n - \gamma = \frac{1}{n} + \sum_{k\geq 1}^{n-1} \left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) - \sum_{k\geq 1}^{\infty} \left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right) = \frac{1}{n} - \sum_{k\geq n}^{\infty} \left(\frac{1}{k} - \ln\left(1+\frac{1}{k}\right)\right)$$

by estimation. the error $\delta_n \in (0, \frac{1}{2n})$.

Thm (Rearrangement of the absolutely convergent series). 项的重排.

Let $\sum\limits_{k=1}^{\infty} a_k$ be abs. conv. Let its sum be S. $\varphi: \mathbb{N} \to \mathbb{N}$ be a bijection.

Then the $\sum\limits_{k=1}^{\infty} a_{\varphi(k)}$ absolutely converges to S.

proof: ~~Let~~ $\sum\limits_{k=1}^{\infty} a_k \geq 0$ for all $k \in \mathbb{N}$ (w.l.g).

We denote $S_n = \sum\limits_{k=1}^{n} a_k$ $\quad T_n = \sum\limits_{k=1}^{n} a_{\varphi(k)}$.

For all n. we have $T_n \leq S_m \leq S$. (where $m = \max\{\varphi(1), \varphi(2), \dots \varphi(n)\}$).

So T_n conv. $T \leq S$. (apply the proof to φ^{-1}, $S \leq T$. Rearrange series $\leq$ initial series)

Lemma If the $\sum\limits_{k=1}^{\infty} a_k$. with real terms are conditionally convergent. then both

series $\sum\limits_{k=1}^{\infty} (a_k)_+$ and $\sum\limits_{k=1}^{\infty} (a_k)_-$ are div. ← 所有正项/所有负项.

proof: 1° both conv. → absolutely conv. $\sum a_k$.

2° conv. and div. → the difference of div and conv. conv. $\sum a_k$.

Thm. (B. Riemann. The rearrangement of the conditionally convergent series).

Let the series $\sum\limits_{k=1}^{\infty} a_k$ with real terms conditionally convergent. Then for any $S \in \overline{\mathbb{R}}$

there exists $\varphi(k)$ s.t. $\sum\limits_{k=1}^{\infty} a_{\varphi(k)} = S$. There exists the rearrangement $\varphi(k)$ s.t.

$\sum\limits_{k=1}^{\infty} a_{\varphi(k)}$ has no sum.

proof: Consider $S \in [0, +\infty)$. (symmetric).

$\{b_p\}$. $\{c_q\}$ be sequences of all nonnegative and negative terms of the series $b_p = a_{n_p}$ $c_q = a_{m_p}$.

By the Lemma $\sum\limits_{p=1}^{\infty} b_p$. $\sum\limits_{q=1}^{\infty} c_q$ div.

Let $p_0 = q_0 = 0$. Let p_1 be the least natural number s.t $\sum\limits_{p=1}^{p_1} b_p > S$

i.e. $\sum\limits_{p=1}^{p_1 - 1} b_p \leq S < \sum\limits_{p=1}^{p_1} b_p$

Let q_1 be the least natural number s.t. $\sum\limits_{q=1}^{q_1} c_q < S - \sum\limits_{p=1}^{p_1} b_p$

i.e. $\sum\limits_{p=1}^{p_1} b_p + \sum\limits_{q=1}^{q_1} c_q < S \leq \sum\limits_{p=1}^{p_1} b_p + \sum\limits_{q=1}^{q_1 - 1} c_q$.

$\sum\limits_{p=1}^{p_s} b_p + \sum\limits_{q=1}^{q_{s-1}} c_q > S > \sum\limits_{p=1}^{p_s - 1} b_p + \sum\limits_{q=1}^{q_{s-1}} c_q$.

By the divergence of the series.

We continue the procedure. $\sum\limits_{p=1}^{p_s} b_p > S - \sum\limits_{q=1}^{q_{s-1}} c_q$. ① $\quad \sum\limits_{p=1}^{q_s} c_q < S - \sum\limits_{p=1}^{p_s} b_p$. ②

the series $\underbrace{b_1 + \dots + b_{p_1}}_{B_1} + \underbrace{c_1 + \dots + c_{q_1}}_{C_1} + \dots + \underbrace{b_{p_{s-1}+1} + \dots + b_{p_s}}_{B_s} + \underbrace{c_{q_{s-1}+1} + \dots + c_q}_{C_s} + \dots$ is the rearrangment.

Denote the $T_{2n-1} = B_1 + C_1 + \dots + B_n$ $\quad T_{2n} = B_1 + C_1 + \dots + B_n + C_n$.

By ①. $T_{2s-1} - b_{p_s} \leq S < T_{2s-1}$. $\quad$ ② $T_{2s} < S < T_{2s} - c_{q_s}$. $\quad b_{p_s}$. $c_{q_s} \to 0$ since $\{a_n\}$ conv.

By the squeeze thm. $T_{2n-1} \to S$ $\quad T_{2n} \to S$. $\Rightarrow T_n \to S$

Product of the series.

For finite m. n. $\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{m} b_j\right) = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k b_j$.

Def. Let $\sum_{k=1}^{\infty} a_k$, $\sum_{j=1}^{\infty} b_j$ be numerical series. $\gamma = (\varphi, \psi): \mathbb{N} \to \mathbb{N}^2$ be a bijection. Then the series. $\sum_{l=1}^{\infty} a_{\varphi(l)} b_{\psi(l)}$ $\Rightarrow$ the product of series. (corresponding to the numeration γ)

Thm (O. Cauchy. The product of series)
If the series $\sum_{k=1}^{\infty} a_k$. $\sum_{j=1}^{\infty} b_j$ absolutely converge to the sums A and B, then for any numeration their product absolutely converges to AB.

Pf: Let $\gamma = (\varphi, \psi): \mathbb{N} \to \mathbb{N}^2$ be a bijection. We denote $\sum_{k=1}^{\infty} |a_k| = A^*$ $\sum_{j=1}^{\infty} |b_j| = B^*$

For all $v \in \mathbb{N}$. $\sum_{l=1}^{v} |a_{\varphi(l)} b_{\psi(l)}| \le \left(\sum_{k=1}^{n} |a_k|\right)\left(\sum_{j=1}^{m} |b_j|\right) \le A^* B^*$. $n = \max_{1 \le l \le v} \varphi(l)$ $m = \max_{1 \le l \le v} \psi(l)$.

So the partial sums of series. are bounded. from above. i.e. we have the abs. convergence.

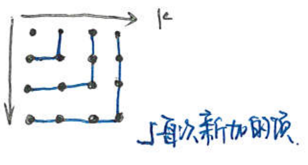
By the theorem of rearrangement. the abs. conv. series its sum does not depend on the arrangement.

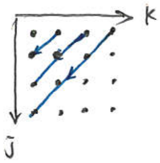
Lemma. If the series $\sum_{k=1}^{\infty} a_k$, $\sum_{j=1}^{\infty} b_j$ converge to the sums A and B. then their product "by squares" converges to AB. (do not need absolute conv. here).

$S_{n^2} = \sum_{k,j=1}^{n} a_k b_j = \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{n} b_j\right) \xrightarrow[n\to\infty]{} AB$.

We need. $S_n \to AB$.

Denote. $m_n := [\sqrt{n}]$. $S_n = S_{m_n^2} + \theta_n$. $\theta_n = a_{m_n+1} \underset{\downarrow 0}{B_J} + b_{m_n+1} \underset{\downarrow 0}{(A_K - A_M)}$. $S_n \to S_{m_n^2} \to AB$.





Def. The series. $\sum_{k=1}^{\infty} C_k$, where $C_k = \sum_{j=1}^{k} a_j b_{k+1-j}$ the product is called "by diagonal" or "Cauchy's product"

If. $\sum a_k \to A$. $\sum b_k \to B$. $C_k = \sum_{j=1}^{k} a_j a_{k+1-j} \to C$. $AB \stackrel{?}{=} C$.

Cauchy's product. of two div. might be conv. e.g. $a_k = \begin{cases} 1, & k=0 \\ 2^{k-1}, & k \in \mathbb{N} \end{cases}$ $b_j = \begin{cases} 1, & j=0 \\ -1, & j \in \mathbb{N} \end{cases}$

but $C_k = \underset{\substack{\Downarrow \\ a_k b_0}}{2^{k-1}} - 2^{k-2} - \dots 2^1 - 2^0 - \underset{\substack{\Downarrow \\ b_k a_0}}{1} = 0$.

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k \xrightarrow[n\to+\infty]{} \ln(1+x). \quad |x|<1.$$

$$\sum_{k=1}^{\infty} \sin \frac{kx}{k} = .$$

Functional Sequences and Series

1. Pointwise and Uniform Limit

Def. Pointwise Limit. (用数列逼近函数)

$\forall x \in E. \forall \varepsilon > 0 \ \exists N \in \mathbb{N}$. s.t. $\forall n \in N$ $|f_n(x) - f(x)| < \varepsilon$.
N depends on both ε and $x \in E$

$\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on the same set $f_n : E \to \mathbb{R}(\mathbb{C})$

Assume that for every $x \in E$ a (numerical) sequence $\{f_n(x)\}$ has a limit $f(x)$.

$f(x) = \lim_{n \to \infty} f_n(x)$ (define on E). $\Rightarrow$ a pointwise limit of a sequence $\{f_n\}_{n=1}^{\infty}$
(we write $f_n \to f$ on E).

Remark: A pointwise limit of continuous functions can be not continuous

e.g. $f_n(x) = x^n$. $x \in [0,1]$. $\quad f(x) = \begin{cases} 0, & x \in [0,1) \\ 1 & x = 1 \end{cases}$

Def. uniform limit. (函数逼近函数).

$f, f_n : E \to \mathbb{R}(\mathbb{C})$, $n \in \mathbb{N}$. A function is an uniform limit of a sequence $\{f_n\}$.
(a sequence $\{f_n\}$ converges uniformly to f on E). if

$\forall \varepsilon > 0. \exists N \in \mathbb{N}$ s.t. $\forall n > N \ \forall x \in E$. $|f_n(x) - f(x)| < \varepsilon$. (条件更强).
N only depends on ε.

In this case. $f_n \rightrightarrows f$ on E.

Lemma. Let $f, f_n : E \to \mathbb{R}$, $n \in \mathbb{N}$. The following assertion are equivalent: (to check the uniform convergence).

1. $f_n \rightrightarrows f$ on E.
2. $\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \forall n > N \ \sup_{x \in E} |f_n(x) - f(x)| < \varepsilon$.
3. $\lim_{n \to \infty} \sup_{x \in E} |f_n(x) - f(x)| = 0$.
4. $\exists \{\varepsilon_n\}$. nonnegative numbers such that $|f_n(x) - f(x)| \le \varepsilon_n$. $\forall x \in E$ and $\varepsilon_n \to 0$

proof: "3 ⇒ 4" it's enough to take $\varepsilon_n = \sup_{x \in E} |f_n(x) - f(x)|$.

"4 ⇒ 1" Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$, s.t. $\forall n \in N$, $\varepsilon_n < \varepsilon$.

consequencely. $\forall n > N$. $\forall x \in E$. $|f_n(x) - f(x)| < \varepsilon$.

[1 ⇒ 2. by def of sup.
(since n is arbitrary).
2 ⇒ 3. by def of lim.

* notation : $\operatorname{dist}(f, g) = \sup_{x \in E} |f(x) - g(x)|$

已知 $f_n(x)$ 找 $f(x)$. 主要目标是消 n. 多用一些极限性质 (然后证明是其收敛函数!).

Lemma. (Arithmetical properties).

1. $f_n \rightrightarrows f$. $g_n \rightrightarrows g$ on $\bar{X}$. $\alpha, \beta \in \mathbb{R}(\mathbb{C})$. Then $\alpha f_n + \beta g_n \rightrightarrows \alpha f + \beta g$
2. Let $f_n \rightrightarrows f$. g be bounded on $\bar{X}$. $f_n g \rightrightarrows f g$ on $\bar{X}$.

proof 1, 2. (use the supremum)

Thm. Bolzano - Cauchy criterion.

Let $f_n: E\to\mathbb{R}$. The uniform convergence of a sequence $\{f_n\}$ on E is equivalent to the following condition. $\forall\varepsilon>0\ \exists N\in\mathbb{N}$ s.t $\forall n,m>N,\ \forall x\in E.\ |f_n(x)-f_m(x)|<\varepsilon$

Pf: "$\Rightarrow$" $f_n\rightrightarrows f$ on E. $\forall\varepsilon>0.\ \exists N\in\mathbb{N}: \forall n>N\ \forall x\in E.\ |f_n(x)-f(x)|<\frac{\varepsilon}{2}$

$\forall n,m\in\mathbb{N}.\quad |f_n(x)-f_m(x)|\le|f_n(x)-f(x)|+|f(x)-f_m(x)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$

"$\Leftarrow$" (suffiency) $\{f_n(x)\}$ is a numerical Cauchy sequence.

$\Rightarrow\ \exists \lim\limits_{n\to\infty} f_n(x):=f(x).$

Fix $\varepsilon>0$. $\exists N>0.\ \forall n,m>N\ \forall x\in E.\ |f_n(x)-f_m(x)|<\frac{\varepsilon}{2}.$

Fix $n>N$ and let $m\to\infty$. $\Rightarrow |f_n(x)-f(x)|\le\frac{\varepsilon}{2}<\varepsilon.$

Thm. 1.7 (On permutations of limits). (极限的顺序变化).

Let E be a subset of $\mathbb{R}$ or $\mathbb{C}$, p be a limit point of E, and $f_n, f:\mathbb{R}\to\mathbb{R}$.

Assume: 1. $f_n\rightrightarrows f$ on E.

2. for every $n\in\mathbb{N}$ there exist a limit $A_n=\lim\limits_{x\to p} f_n(x)$.

Then. $\lim\limits_{n\to\infty}\lim\limits_{x\to p} f_n(x)=\lim\limits_{x\to p}\lim\limits_{n\to\infty} f_n(x).$

Pf: $\forall\varepsilon>0.\ \exists N\in\mathbb{N}.\ \forall n,m\in N.\ \forall x\in E.\ |f_n(x)-f_m(x)|<\varepsilon.$

$\Rightarrow \lim\limits_{x\to p}|f_n(x)-f_m(x)|\le\varepsilon. \Rightarrow |A_n-A_m|\le\varepsilon\Rightarrow\{A_n\}$ is a Cauchy sequence.

$\exists A=\lim\limits_{n\to\infty} A_n$ [the left part exists]

Now it is enough to show $A=\lim\limits_{x\to p} f(x)$.

Fix $\varepsilon>0$.

$\exists L>0.\ \forall n>L.\ \forall x\in E.\ |f_n(x)-f(x)|<\frac{\varepsilon}{3}.$ [$f_n\rightrightarrows f$]

$\exists K>0\ \forall k>K\quad |A_k-A|<\frac{\varepsilon}{3}.$ [$A=\lim\limits_{n\to\infty}A_n$]

Let $M=\max(L,K)+1$. $\exists\delta>0.\ 0<|x-p|<\delta.\ |f_M(x)-A_M|<\frac{\varepsilon}{3}.$ [$A_M=\lim\limits_{x\to p}f_M(x)$]

$|f(x)-A|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$

Coro 1.7.1 (Pointwise continuity of the uniform limit).

Let E be a subset of $\mathbb{R}$ or $\mathbb{C}$, $p\in E$. and $f_n, f:\mathbb{R}\to\mathbb{R}$ or $\mathbb{C}$. Assume that

1. $f_n\rightrightarrows f$ on E.
2. all functions f_n are continuous in p.

p is isolated point if $\exists V_p.\ V_p\cap E=\{p\}$.

Then the function f is continuous in p.

Pf. 1° p is isolated. trivial (continuous automatically)

2° p not isolated. condition in thm above satisfied. $A_n=f(p)$

Consequently $\lim\limits_{x\to p}f(x)=\lim\limits_{n\to\infty}A_n=f(p)$ f is continuous on p.

$\lim\limits_{x\to p}f(x)\overset{1.}{=}\lim\limits_{x\to p}\lim\limits_{n\to\infty}f_n(x)\overset{\text{Thm.}}{=}\lim\limits_{n\to\infty}\lim\limits_{x\to p}f_n(x)\overset{2.}{=}\lim\limits_{n\to\infty}A_n=f(p)$.

Coro[1.7.2] (Continuity of the uniform limit on a set.).

Let E be a subset of $\mathbb{R}$ ($\mathbb{C}$), $p \in E$. and $f_n, f: \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$).

Assume: 1. $f_n \rightrightarrows f$ on E

2. all functions f_n are continuous on E.

Then the function f is continuous on E. (Moreover, if every function f_n is uniform continuous then f is also uniform continuous).

Pf: $\forall \varepsilon > 0$ Since $f_n \rightrightarrows f$. $\exists L > 0$ s.t. $\forall n > L$. $x \in E$, $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$

Let $M = L+1$ then $\exists \delta > 0$. s.t. $|f_M(x) - f_M(y)| < \frac{\varepsilon}{3}$. $x, y \in E$. $|x-y| < \delta$.

(直接证 uniform. 一般时即取不同 δ)

Consequently, for every $x, y \in E$. s.t. $|x-y| < \delta$.

$$|f(x) - f(y)| \leq |f(x) - f_M(x)| + |f_M(x) - f_M(y)| + |f_M(y) - f(y)| < \varepsilon$$

Thm [1.8] Dini's theorem for a sequence.

Let K be compact subset of $\mathbb{R}$ or $\mathbb{C}$ and $f_n, f \in C(K)$. If f is a pointwise limit of a sequence $\{f_n\}$ and for every $x \in E$ a sequence $f_n(x)$ is increasing. Then $f_n \rightrightarrows f$ on E.

Pf: Let $\varepsilon > 0$. $g_n(x) = f(x) - f_n(x)$. $E_n = \{x \in [0,1] : g_n(x) < \varepsilon\}$.

$g_n(x) \to 0 \Rightarrow \exists N \in \mathbb{N}$ $\forall n > N$ $x \in E_n \Rightarrow [0,1] = \bigcup_{n=1}^{\infty} E_n \Rightarrow \exists$ finite subcover.

$\exists E_N \supset [0,1]$

Since g_n is continuous; then E_n is open

Also since g_n is monotonic. $E_{n+1} \supset E_n$.

$\Rightarrow \forall x \in [0,1]$. $g_N(x) < \varepsilon$. $\Rightarrow \forall n > N$. $g_n(x) < \varepsilon$. (i.e $|f(x) - f_n(x)| < \varepsilon$. $x \in K$)

Lemma 1.9. (Squeeze theorem (for Riemann integral))

Let $f: [a,b] \to \mathbb{R}$. Then f is Riemann integral $\Leftrightarrow \forall \varepsilon > 0$, $\exists g_1, g_2 \in R[a,b]$.

s.t. $g_1(x) \leq f(x) \leq g_2(x)$, $g_2(x) - g_1(x) < \varepsilon$. $x \in [a,b]$

Pf: Suppose $f \notin R[a,b]$: and let g_1 and g_2 satisfy $g_2 - g_1 \leq \frac{\varepsilon}{3(b-a)}$ and $g_1, g_2 \in R[a,b]$

$\forall \varepsilon > 0$. $\exists \delta > 0$. s.t. $\forall \mathcal{T} = \{x_k\}_{k=0}^{n}$ with $\max |x_{k+1} - x_k| < \delta$

we have $g_1, g_2 \in R[a,b]$.

then we have $S_{\mathcal{T}}(f) \leq S_{\mathcal{T}}(g_2) \leq \int_a^b g_2 + \frac{\varepsilon}{3}$.

$$s_{\mathcal{T}}(f) \geq s_{\mathcal{T}}(g_1) \geq \int_a^b g_1 - \frac{\varepsilon}{3}$$

$$S_{\mathcal{T}}(f) - s_{\mathcal{T}}(f) \leq \int_a^b (g_2 - g_1) + \frac{2\varepsilon}{3} < \frac{(b-a)\varepsilon}{3(b-a)} + \frac{2\varepsilon}{3} \varepsilon = \varepsilon.$$

then $f \in R[a,b]$ (We can always denote $f = g_1 = g_2$).

1.10

Thm. (Uniform Convergence and Integration)

Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$. If $\{f_n\}$ converges uniformly to $\{f\}$ on $[a,b]$ then $f \in R[a,b]$. and $\int_a^b f_n \to \int_a^b f$

proof: Let $\varepsilon > 0$. be arbitrary.

By uniformly convergence $\exists K$. s.t $\forall n > k$. $\forall x \in [a,b)$ $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$.

$\Rightarrow f_n(x) - \frac{\varepsilon}{b-a} < f(x) < f_n(x) + \frac{\varepsilon}{b-a}$.

Lemma 1.9. $f_n \pm \frac{\varepsilon}{b-a} \in R[a,b]$. $\quad f_n \in R[a,b]$.

Moreover. $\quad -\frac{\varepsilon}{b-a} < f_n(x) - f(x) < \frac{\varepsilon}{b-a}$

by monoticity. $\quad -\varepsilon < \int_a^b f_n - \int_a^b f < \varepsilon$ ...

Thm 1.11 (Integral of uniform limit, weaker).

Let $f_n \in C[a,b]$. and $f_n \rightrightarrows f$ on $[a,b]$ Then $\int_a^b f_n \to \int_a^b f$.

by Coro 1.7.2. $f \in C[a,b] \Rightarrow f \in R[a,b]$.

$\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$. $\forall n \in N$. $\forall x \in [a,b]$ $\quad |f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$.

$\left| \int_a^b f_n - \int_a^b f \right| \le \int_a^b |f_n - f| < \varepsilon$.

Δ $f_n \rightrightarrows f$, f_n is differentiable.

f not always differentiable $\quad f_n = \sqrt{\frac{nx^2+1}{n}}$. $\rightrightarrows f(x) = |x|$. $x \in [0,1]$. $f(x)$ not diff on 0

$f_n' \to f'$ not always at least pointwise conv. $\quad$ counter-e.g. $f_n(x) = \frac{\sin nx}{n}$ $\quad f_n' = \cos nx$.

Thm 1.12 (differentiation).

Let $f_n : [a,b] \to \mathbb{R}$. be a differentiable functions. Assume

- a sequence of derivative $\{f_n'\}$. uniformly converges on $[a,b]$. to some function φ.
- $\exists c \in [a,b]$. such that $f_n(c)$ converges.

Then. 1. a sequence $\{f_n\}$. uniformly converges on $[a,b]$ to some function f;

2. a function f is differentiable on $[a,b]$.

3. $f' = \varphi$. $\quad \left(\lim_{n\to\infty} f_n \right)' = \lim_{n\to\infty} f_n'$).

proof: Fix $p \in E := [a,b]$.

let $g_n(x) = \frac{f_n(x) - f_n(p)}{x - p}$ $\quad x \in E \setminus \{p\}$.

$|g_n(x) - g_m(x)| = \left| \frac{(f_n - f_m)(x) - (f_n - f_m)(p)}{x - p} \right| \overset{(\text{Lagrange})}{=} (f_n - f_m)'(\xi)$. (which is uniformly converges).

$\sup |g_n - g_m| \le \sup |f_n'(x) - f_m'(x)| < \varepsilon$.

thus $\{g_n\}$ is uniformly converges. on $[a,b] \setminus \{p\}$

(brought forward)

Let $p=c$. (只有这一步).

$$f_n(x) = \underbrace{\underbrace{(x-c)}_{\text{bounded}}\ \underbrace{g_n(x)}_{\text{u.c.}}}_{\text{u.c.}} + \underbrace{f_n(c)}_{\substack{\text{numerical converges.} \\ \Rightarrow \text{u.c.}}} \quad \Rightarrow f_n(x) \text{ is uniformly conv.}$$

$\Rightarrow g_n \rightrightarrows g = \dfrac{f(x)-f(p)}{x-p}$. $\quad g_n(x) \xrightarrow{x\to p} f_n'(p)$, $\quad g(x) \xrightarrow{x\to p} f'(p)$.

since $f_n \rightrightarrows \Rightarrow \forall x.\ g_n(x) \to \dfrac{f(x)-f(p)}{x-p} = g$ (p.w.c)

$\{g_n\}$ is uni. conv. the limit is unique

$\Rightarrow g_n \rightrightarrows g$.

$$f'(p) = \lim_{x\to p} g(x) = \lim_{x\to p}\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty}\lim_{x\to p} g_n(x) = \lim_{n\to\infty} f_n'(p) = \varphi(p) \text{ (p is arbitrary)}$$

§ 2. Functional series.

Def. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions defined on the same set $f_n: E\to \mathbb{R}(\mathbb{C})$

A functional series $\sum_{k=1}^{\infty} u_k(x)$ is a sequence of partial sums $S_n(x) = \sum_{k=1}^{n} u_k(x)$

Assume for every $x\in E$. this series converges.

This defines a pointwise sum $S(x) = \sum_{k=1}^{\infty} u_k(x) = \lim_{n\to\infty} S_n(x)$

the functional series converges uniformly on E. if a sequence of partial sums converges uniformly on E. $S_n \rightrightarrows S$.

1.) Any functional sequence $\{S_n\}_{n=1}^{\infty}$ can be considered as a series $\sum_{k=1}^{\infty} u_k(x)$ with terms

$u_1(x) = S_1 \quad u_k(x) = S_k(x) - S_{k-1}(x).\ k\geq 2.$

2.). $S_n \rightrightarrows S \iff S - S_n = \sum_{k=n+1}^{\infty} u_k(x) \rightrightarrows 0.$

Thm 2.2. (Cauchy criterion).

Let $u_n: E \to \mathbb{R}$.

$\sum_{k=1}^{n} u_n(x)$ uniformly conv. on $E \iff \forall \varepsilon>0\ \exists N\in\mathbb{N}.$ s.t $\forall n,m>N.\ \forall x\in E.\ |S_m(x)-S_n(x)| < \varepsilon.$

(w.l.g. $m>n$. $m=n+p$. $\quad \left|\sum_{k=n+1}^{n+p} u_k(x)\right| < \varepsilon$).

Coro 2.2.1. (Necessary condition for u.c. of series)

If a series $\sum_{k=1}^{n} u_n(x)$ u.c. on E then $u_n \rightrightarrows 0$. on E. $\quad \sup_{x\in E} |u_k(x)| \to 0.$

Thm 2.3. Let E be a subset of $\mathbb{R}(\mathbb{C})$. p be a limit point of E. $u_k: \mathbb{R}\to\mathbb{R}(\mathbb{C})$. Assume.

1. a series $\sum_{k=1}^{\infty} u_k$ uniformly converges to S on E.
2. for every $k\in\mathbb{N}$, $\exists\ u_k = \lim_{x\to p} a_k(x)$

Then the series converges to sum A. $\lim_{x\to p} S(x)$ exists and is equal to A.

$$\lim_{x\to p}\sum_{k=1}^{\infty} u_k(x) = \sum_{k=1}^{\infty}\lim_{x\to p} u_k(x)$$

(use the thm1.1 to proof).

Thm 2.4 $p \in E$. $u_k : \mathbb{R} \to \mathbb{R}$. Assume.

1. $\sum_{k=1}^{\infty} u_k \rightrightarrows S$.

2. all u_k are continuous in p.

$\Rightarrow$ Then the S is continuous in p.

(usage: show conv. on $(-1,1)$. just $\forall r \in (0,1)$. consider $(-r,r)$)

Thm 2.5 (Dini Thm).

Let $u_k \in C[0,1]$. $u_k \geq 0$. If $\sum_{k=1}^{\infty} u_k$ conv. to a continuous sum then it conv. uniformly on $[0,1]$

$[0,1]$ can be replaced by any compact set.

Thm 2.6. (Term-by-term Integration.)

Let $u_k \in R[a,b]$. Assume. a series $\sum_{k=1}^{\infty} u_k$ uniformly converges on $[a,b]$.

Then the sum is Riemann integrable and $\int_a^b \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \int_a^b u_k$

Thm 2.7 (Term-by-term differentiation)

Let $u_k : [a,b] \to \mathbb{R}$. differentiable. Assume.

- a series $\sum_{k=1}^{\infty} u_k'$ u.c on $[a,b]$
- $\exists c \in [a,b]$. s.t. $\sum_{k=1}^{\infty} u_k(c)$ converges.

Then: 1. $\sum_{k=1}^{\infty} u_k$ uniformly conv. on $[a,b]$

2. its sum is differentiable on $[a,b]$.

3. $\left(\sum_{k=1}^{\infty} u_k(x) \right)' = \sum_{k=1}^{\infty} u_k'(x)$

Thm 2.8 (Weierstrass M-test) (Sufficient condition)

函数→常数项级化.

Let $\{u_k\}$ be a sequence of real / complex valued functions on E and $\{M_k\}$ be a sequence of nonnegative numbers s.t. $|u_k(x)| \leq M_k$. $x \in E$. and $\sum_{k=1}^{n} M_k < +\infty$

Then the series $\sum_{k=1}^{n} u_k(x)$ converges uniformly. on E.

proof. $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$: $\forall n \in \mathbb{N}$. $\forall p \in \mathbb{N}$. $\left| \sum_{k=n+1}^{n+p} M_k \right| < \varepsilon$. numerical Cauchy criterion

for every $n > N$ and $p \in \mathbb{N}$

$$\sup_{x \in E} \left| \sum_{k=n+1}^{n+p} u_k(x) \right| \leq \sup_{x \in E} \sum_{k=n+1}^{n+p} |u_k(x)| \leq \sum_{k=n+1}^{n+p} M_k < \varepsilon.$$

Thm 2.9 (Dirichlet's test).

Let $\{u_k\}$ be a sequence of $\mathbb{R}(\mathbb{C})$ function on E. $\{v_k\}$ be function $E \to \mathbb{R}$. Assume.

- a sequence $U_n(x) = \sum_{k=1}^{n} u_k(x)$ is uniformly bounded, that is. $\exists M$. $\left| \sum_{k=1}^{n} u_k \right| \leq M$. $(\forall x \in E,\ n \in \mathbb{N})$
- For every $x \in E$ a sequence $\{v_n(x)\}$ is monotone. (关于n单调).
- $v_n \rightrightarrows 0$ on E

Then series $\sum^{\infty} u_k(X) v_k(X)$ converges uniformly on E

Lemma 2.10. (Abel's Lemma).

Let two numerical sequence. $\{a_k\}$. $\{b_k\}$. $A_n = \sum_{n=1}^{n} a_k$.

Then $\sum_{k=n+1}^{m} a_k b_k = A_m b_m - A_n b_{n+1} - \sum_{k=n+1}^{m-1} A_k (b_{k+1} - b_k) = (A_m - A_n) b_{n+1} - \sum_{k=n+1}^{m} (A_k - A_m)(b_{k+1} - b_k)$

proof: ① $\sum_{k=n+1}^{m} a_k b_k = \sum_{k=n+1}^{m} (A_k - A_{k-1}) b_k = \sum_{k=n+1}^{m} A_k b_k - \sum_{k=n+1}^{m} A_{k-1} b_k = \sum_{k=n+1}^{m} A_k b_k - \sum_{k=n}^{m-1} A_k b_{k+1}$. ← interchange index.

$= A_m b_m - A_n b_{n+1} - \sum_{k=n+1}^{m-1} A_k (b_{k+1} - b_k)$

② $\sum_{k=n+1}^{m} a_k b_k = \sum_{k=n+1}^{m} (A_k - A_m - (A_{k-1} - A_m)) b_k = \sum_{k=n+1}^{m} (A_k - A_m) b_k - \sum_{k=n}^{m-1} (A_k - A_m) b_{k+1}$

$= (A_{n+1} - A_m) b_{n+1} - \sum_{k=n}^{m-1} (A_k - A_m)(b_{k+1} - b_k)$

Use the Lemma to proof Thm 2.9.

Let $S_n = \sum_{k=1}^{n} u_k v_k$. $(m>n)$. $U_n := \sum_{n=1}^{n} u_k$

$|S_m - S_n| = |\sum_{k=n+1}^{m} u_k v_k| \le |U_m v_m| + |U_n v_{n+1}| + |\sum_{k=n+1}^{m-1} U_k (v_{k+1} - v_k)|$

since v_k is monotone. $v_{k+1} - v_k$ sign keeps same.

$|\sum_{k=n+1}^{m-1} U_k \, v_{k+1} - v_k| \le M |\sum_{k=n+1}^{m-1} (v_{k+1} - v_k)| = M (v_m - v_{n+1}) \le M(|v_m| + |v_{n+1}|)$.

$|S_m - S_n| \le M \cdot 4 \max(|v_m|, |v_{n+1}|)$. (Let $v_n < \frac{\varepsilon}{4M}$).

Coro 2.10.1 (Leibniz's test) alternating series

Let $\{v_n\}$ be a sequence of real-valued functions s.t. for every $x \in E$. a sequence $\{v_n(x)\}$ is monotone. $\sum_{k=1}^{\infty} (-1)^{k-1} v_k(x)$ conv uniformly $\Leftrightarrow v_n \rightrightarrows 0$ on E.

proof: "$\Rightarrow$" Let. $U_n = \sum_{k=1}^{n} (-1)^{k-1}$ $|U_n| \le 1$. apply Dirichlet test

"$\Leftarrow$" if $v_n \not\rightrightarrows 0$. necessary condition not satifies. the $\sum_{k=1}^{\infty} (-1)^{k-1} v_k(x)$ div.

Thm 2.11. (Abel's test).

Let E be compact. $\{u_n\} \in \mathbb{R}(\mathbb{C})$. $\{v_n\} \in \mathbb{R}$. on E.

Assume · $\sum_{n=1}^{\infty} u_n$ conv. uni. on E.

· For every $x \in E$. a sequence $\{v_n(x)\}$. is monotone.

· a sequence $\{v_n(x)\}$. is uniformly bounded.

$\Rightarrow \sum_{n=1}^{\infty} u_n(x) v_n(x)$. conv. uni. on E.

proof: Let $U_n = \sum_{n=1}^{n} u_n(x)$. (Let. $|U_n(x) - U_m(x)| < \frac{\varepsilon}{3M}$).

use the second equality of Lemma 2.10. to proof.

Investigation of series $\sum_{k=1}^{\infty} b_k \sin kx$, $\sum_{k=1}^{\infty} b_k \cos kx$, $\sum_{k=1}^{\infty} b_k e^{ikx}$

$e^{ix} = \cos x + i\sin x$.

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2} \qquad \sin nx = \frac{e^{inx} - e^{-inx}}{2}$$

Lemma 2.12 Assume that $\frac{x}{2\pi} \notin \mathbb{Z}$. Then

$$U_n(x) = \sum_{k=0}^{n} e^{ikx} = \frac{1-e^{i(n+1)x}}{1-e^{ix}}$$

$$V_n(x) = \sum_{k=1}^{n} \sin kx = \frac{\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}$$

$$W_n(x) = \sum_{k=0}^{n} \cos kx = \frac{\cos\frac{nx}{2}\sin\frac{(n+1)}{2}x}{\sin\frac{x}{2}}$$

若k=1需-1.

and $|U_n(x)|$, $|V_n(x)|$, $|W_n(x)| \leq \frac{1}{|\sin(\frac{x}{2})|}$ ⇐ 比较时$U_n(x)$中虚部直接不考虑.

$U_n(x) = W_n(x) + V_n(x)i$

$$\frac{1-e^{i(n+1)x}}{1-e^{ix}} = \frac{e^{i\frac{n+1}{2}x}}{e^{i\frac{x}{2}}} \cdot \frac{e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} = e^{i\frac{n}{2}x} \cdot \frac{\sin\frac{(n+1)}{2}x}{\sin\frac{x}{2}} = \left(\cos\frac{nx}{2} + \sin\frac{nx}{2}i\right) \cdot \frac{\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}$$

Thm 2.13. Let $[a,b] \subset (2\pi m, 2\pi(m+1))$ and b_k be monotone. $b_k \to 0$.

Then $\sum_{k=1}^{\infty} b_k \sin kx$, $\sum_{k=1}^{\infty} b_k \cos kx$, $\sum_{k=0}^{\infty} b_k e^{ikx}$ conv. uni. on $[a,b]$.

Follow the Dirichlet test.

since $\frac{x}{2} \in [\frac{a}{2}, \frac{b}{2}] \subset (\pi m, \pi(m+1))$. $\sin\frac{x}{2} \neq 0$. $\min|\sin\frac{x}{2}| = \min\{|\sin\frac{a}{2}|, |\sin\frac{b}{2}|\} = \max\frac{1}{|\sin\frac{x}{2}|}$.

Thm 2.14. Let $p \in (0,1]$. Then $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$, $\sum_{k=1}^{\infty} \frac{\cos kx}{k^p}$, $\sum_{k=1}^{\infty} \frac{e^{ikx}}{k^p}$ do not conv. uniformly on $(0, 2\pi)$

proof: Let $\underline{x_n = \frac{1}{2n}}$. $\sin kx_n \geq \sin\frac{1}{2}$ $(n \leq k \leq 2n)$.

Consider $\sum_{k=n+1}^{2n} \frac{\sin kx_n}{k^p} \geq n \cdot \sin\frac{1}{2} \cdot \frac{1}{(2n)^p} \geq \frac{n}{2n} \cdot \sin\frac{1}{2} = \frac{\sin\frac{1}{2}}{2} > 0$.

$\exists \varepsilon^* = \frac{\sin\frac{1}{2}}{2}$. $\forall N > \mathbb{N}$. $\exists n \geq N$. $\exists p \in \mathbb{N}$: $\underline{\exists x_N \in E}$. $|S_{n+p}(x_N) - S_n(x_N)| > \varepsilon^*$ (x可以与n有关.)

*证 pointwise conv. 时. fix x. 可用数项级数结论 (如 Cauchy/d'Alembert's test)

§ 3. Power Series.

Def. a function series $\sum_{n\geq 0}^{+\infty} a_n (x-x_0)^n$. a power series. a_n are coefficients of the ser.

$\{a_n\} \in \mathbb{R}$. $x_0 \in \mathbb{R}$. real. $\{a_n\} \in \mathbb{C}$. $x_0 \in \mathbb{C}$. complex.

Remark: change $z = x - x_0 \Rightarrow \sum_{n=0}^{+\infty} a_n z^n$.

For the compactness of reasoning we will further consider this form of power series.

Lemma 3.2. Assume the power series $\sum_{n\geq 0}^{+\infty} a_n z^n$ converges at z_0. Then it converges at every point z such that $|z| < |z_0|$. (复数. 比模长).

Proof: Assume a series $\sum_{n=0}^{+\infty} a_n z_0^2$ converges. Then terms of the series are uniformly bound

$\exists M$. s.t $|a_n z_0^n| < M$.

let. $0 \leq q = \left|\frac{z}{z_0}\right| < 1$. Then $|a_n z^n| = |a_0 z_0^n| \left|\frac{z}{z_0}\right|^n \leq M q^n$

Def. radius of convergence. $R \in [0, +\infty]$. if the series converges when $|z| < R$. diverges $|z| > R$. ($\hookrightarrow z \in \mathbb{C}$)

When $R = 0$. nowhere conv. $R = +\infty$ everywhere convergent.

By def. $R = \sup\{|x| : \text{series } \sum_{n=0}^{+\infty} c_n z_0^n \text{ converges at } x\}$.

Def. 1. partial limit. (of a real sequence $\{x_n\}_{n\geq 1}^{\infty}$). if there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to A$. as $k \to \infty$.

2. upper limit (maximal partial limit) every sequence has partial limit.

$\limsup\{x_n\} = \overline{\lim}\, x_n = \max\{A. \ A \text{ is a partial limit}\}$

3. Lower limit. (minimal partial limit)

$\liminf\{x_n\} = \underline{\lim}\, x_n = \min\{A : A \text{ is a partial limit}\}$

Lemma 3.5. Let $\{x_n\}$ be a real sequence. $\overline{\lim}\, x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k$. (or $\limsup x_k$)

$\underline{\lim}\, x_n = \lim_{n\to\infty} \inf_{k\geq n} x_k$ (or $\liminf x_k$)

Thm 3.6 (Cauchy-Adamar formula). Every power series has a radius convergence and.

$\frac{1}{R} = \overline{\lim} \sqrt[n]{|c_n|}$ (we assume $\frac{1}{+\infty} = 0$. $\frac{1}{0} = +\infty$). [use Cauchy's test]

Pf: $q = \overline{\lim} \sqrt[n]{|c_n| z^n} = |z| \overline{\lim} \sqrt[n]{|c_n|} = \frac{|z|}{R}$. ($=R$ 时的敛散性在此无法研究).

Lemma 3.7. Let $\{x_n\}$. $\{y_n\}$ be two real sequences. Assume that there exists a limit $\lim x_n \in (0, +\infty)$

Then $\overline{\lim}\, x_n y_n = \lim x_n \cdot \overline{\lim}\, y_n$

Pf: Let $x_{n_k} y_{n_k} \to C$. Then $x_{n_k} \to A$. and $y_{n_k} = \frac{x_{n_k} y_{n_k}}{x_{n_k}} \to \frac{C}{A}$. (partial limit of B). $\leq B$.

Let $y_{m_l} \to B$. $x_{m_l} y_{m_l} \to AB$. AB is partial limit $\leq C$.

Lemma 3.8. If there exists a finite or infinite $\lim\limits_{n\to+\infty}\left|\frac{c_n}{c_{n+1}}\right|$ then it is equal to the radius of convergence.

Pf: Let $R=\lim\limits_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$. $z\neq 0$. apply the d'Alembert's test. $D=\lim\limits_{n\to\infty}\frac{c_{n+1}z^{n+1}}{c_n z^n}=\frac{|z|}{R}$.

Def. disc of convergence. ("复平面上的盘"(因为是距离(模长)))

Assume that R is radius of convergence of power series $\sum z(w-w_0)^n$.

An open disc $B(w_0,R)=\{z\in\mathbb{C}: |w-w_0|<R\}$. a disc of convergence of series $\sum z(w-w_0)^n$

↑ "代表距离的一个数集". 用一个"半径"表示

(Assume $B(w_0,+\infty)=\mathbb{C}$. $B(w_0,0)=\emptyset$)

Def. interval of convergence.

R. radius of real power series. An open interval $(x_0-R,\ x_0+R)$ (of series $\sum a_n(x-x_0)^n$).

If $R\in(0,\infty)$. the interval of convergence of real power series is $\langle x_0-R,\ x_0+R\rangle$

4 cases can be possible.

Thm. 3.11. (Uniform convergence of a power series).

Let $R\in(0,+\infty]$. be the radius of convergence of a power series $\sum c_n z^n$

Then for every $r\in(0,R)$ this series converges uniformly on $\overline{B}(0,r):=\{z\in\mathbb{C}: |z|\le r\}$

disc closed

Proof: Let $|z|\le r<R$. Then $|c_k z^k|\le |c_k| r^k$,

the convergence implies (abs.). $\sum\limits_{k=0}^{\infty}|c_k|r^k<+\infty$. (the abs. we show in the proof of 3.10).

By M-test. the series conv. on closed disc $\overline{B}(0,r)$.

Coro 3.11.1. A sum of power series is continuous on the disc of convergence.

Pf: Let. $f(z)=\sum\limits_{k=0}^{\infty}c_k z^k$. $|z|<R$.

Let $z_1\in\mathbb{C}$ be s.t. $|z_1|<R$ and let $r\in(|z_1|,R)$. Thm 3.11 implies the uni. conv. in $\overline{B}(0,r)$.

Since the term of power series are continuous the sum f is continuous in $\overline{B}(0,r)$

i.e. f is continuous in z_1. z_1 is arbitrary.

Thm 3.12. (Abel's Thm). Let $R\in(0,+\infty]$ be the radius of convergence of real power series.

If the series $S(R)=\sum\limits_{n=0}^{+\infty}a_n R^n$ converges then it converges uniformly on $[0,R]$ and $S(R)=\lim\limits_{x\to R^-}S(x)$

Analogously. if. $S(-R)=\sum\limits_{n=0}^{+\infty}a_n(-R)^n$ converges then it converges uniformly on $[-R,0]$ and $S(-R)=\lim\limits_{x\to -R^+}S(x)$

Pf: Consider terms of power series as $a_k x^k=a_k R^k\left(\frac{x}{R}\right)^k$ numerical series $\sum\limits_{n=1}^{+\infty}a_n R^n$ conv. (uniformly).

$\left\{\left(\frac{x}{R}\right)^k\right\}$ is uniformly bounded on $[0,R]$ and decreasing.

$\Downarrow$ $\left|\frac{x}{R}\right|^k\le 1$.

By Abel's test. the series conv. uniformly. and it's sum is continuous on $[0,R]$. then $S(R)=\lim\limits_{x\to R^-}S(x)$

Thm 3.13. (On derivative and integral of a power series).

Let $R>0$. be a radius of convergence of real power series. $f(x)=\sum_{k=0}^{\infty} a_k x^k$

Then 1. a function f is smooth in the interval $(-R,R)$ and

$$f^{(m)}(x)=\sum_{k=m}^{+\infty}\frac{k!}{(k-m)!}a_k x^{k-m}$$

2. For every $x\in(-R,R)$. $\int_0^x f=\sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1}$

3. If a series $\sum_{k=0}^{\infty}\frac{a_k}{k+1}R^{k+1}$ conv. then the integral (may be improper) $\int_0^R f=\sum_{k=0}^{\infty}\frac{a_k}{k+1}R^{k+1}$

Pf: 1. by induction. $f'(x)=\sum_{k=1}^{\infty}kC_k x^{k-1}$ which is uniformly convergent in $[-r,r]$. $r<R$.
($C_k x^{k-1}$ uni. conv. $\lim\sqrt[k]{k}=1$. use $\overline{\lim}\sqrt[k]{kC_{k-1}}$

2. $\int_0^x f=\sum_{k=0}^{\infty}\int_0^x C_k x^k=\sum_{k=0}^{\infty}\frac{C_k}{k+1}x^{k+1}$ (Term-by-term integration)

3. $\int_0^R f=\lim_{x\to R-}F(x)-F(0)=\sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1}$

Actually. $\sum_{k=0}^{\infty}C_k(z-z_0)^k$ $\sum_{k=1}^{\infty}kC_k(z-z_0)^k$ $\sum_{k=0}^{\infty}\frac{C_k}{k+1}(z-z_0)^k$ equal.

Thm. 3.15. (Taylor Series)

Let $R\in(0,+\infty]$. $f(x)=\sum_{k=0}^{\infty}a_k(x-p)^k$. $|x-p|<R$. → Taylor series of f at point p.

Then a_k are defined uniquely $a_k=\frac{f^{(k)}(p)}{k!}$ → Taylor coefficients

Def. If $f\in C^{\infty}(p-R,p+R)$. or at p (at least) ($p=0$. McLaurin series),

Pf: by 3.13. $f^{(m)}(p)=\sum_{k\ge m}^{\infty}\frac{k!}{(k-m)!}a_k\cdot(x-p)^{k-m}\Big|_{x=p}=m!\,a_m$. (all terms $k\ge m+1$ vanish).

Remark: Partial sums of Taylor series are Taylor polynomials

$$f(x)=T_{n,p}(x)+O(x-p)^{n+1}$$

Taylor series may conv. or not conv. to the function f. (div. / conv., but not to f).

e.g. $f(x)=\begin{cases}e^{-\frac{1}{x^2}}, & x\neq 0\\ 0, & x=0.\end{cases}$ $f^{(k)}(x)=\begin{cases}P_{3k}(\frac{1}{x})\cdot e^{-\frac{1}{x^2}}, & x\neq 0\\ 0 & ,x=0\end{cases}$ (P_{3k} is a polynomial $\deg P_{3k}\le 3k$. $k=0$. $P_{3k}(\frac{1}{x})=1$).

$f^{(k+1)}(0)=\lim_{h\to 0}\frac{f^{(k)}(h)-f^{(k)}(0)}{h}=0$. (induction since $f^{(1)}=0$.

⇒ the Taylor series converges to 0 everywhere.

but $f(x)\neq\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k=0$. $(f^{(k)}(0)\to 0\ (x\to 0))$

Def. We say that a function f is analytic at a point p. if there exists a neighborhood of a point p in which this function can be decomposed as a power series. that is. $\exists\, r>0$. and numbers $a_1,a_2\ldots$ s.t. $f(x)=\sum_{k=0}^{\infty}a_n(x-p)^k$. $|x-p|<r$. $r>0$.

A function $f:(\alpha,\beta)\to\mathbb{R}$ is analytic (real-analytic). on (α,β) if analytic at every point of (α,β). the set of all real-analytic function on $(\alpha;\beta)$ is denoted by $\mathcal{A}(\alpha,\beta)$.

$\mathcal{A}(\alpha,\beta)\subsetneq C^{(\infty)}(\alpha,\beta)$.

Thm 3.18 (Sufficient condition for analyticity).

Let $f\in C^\infty(p-r,p+r)$ s.t. $\exists M>0$ s.t. $|f^{(k)}(x)|\le M$, $\forall k\in\mathbb{N}$, $x\in(p-r,p+r)$

$$\Rightarrow f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(p)}{k!}(x-p)^k,\ |x-p|<r.$$

Pf: $$|f(x)-T_{n,p}(x)|=\left|\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-p)^{n+1}\right|\le M\cdot\left|\frac{(x-p)^{n+1}}{(n+1)!}\right|\xrightarrow[n\to\infty]{}0$$

e.g. $\cos x$, $\sin x$, e^x, $\cosh x$, $\sinh x$. (since they have uniformly bounded derivatives on $(p-r,p+r)$).

e.g. $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ $\quad R=\lim\left|\frac{n+1}{n}\right|=1.$ $\quad f'(x)=\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}=\frac{1}{x+1},\ x\in(-1,1)$

$f(x)=f(0)+\int_0^x\frac{dt}{1+t}=\ln(1+x).$

Application:

1.) $e<\left(1+\frac{1}{n}\right)^{n+\frac12}<e^{1+\frac{1}{12n(n+1)}}$

Pf: $\ln(1+x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ $\qquad\Rightarrow\frac12\ln\left(\frac{1+x}{1-x}\right)=\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$

$\ln(1-x)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(-x)^n$

We fix $n\in\mathbb{N}$. Let $x=\frac{1}{2n+1}\Rightarrow\frac{1+x}{1-x}=\frac{1}{2n+1}\xrightarrow[\text{by }2n+1]{\text{multiply}}(n+\frac12)\ln(1+\frac1n)=1+\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2n+1)^{2k}}$

$\xrightarrow{\frac{1}{2k+1}\le\frac13}\le 1+\frac13\sum_{k=1}^{\infty}\frac{1}{(2n+1)^{2k}}\le 1+\frac13\frac{\frac{1}{(2n+1)^2}}{1-\frac{1}{(2n+1)^2}}=1+\frac{1}{12n(n+1)}$

2) **Stirling's Formula** Let $n\in\mathbb{N}$. $\exists\,\theta_n\in(0,1)$. $n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}}$

Pf: Let $a_n=\frac{n!\,e^n}{n^{n+\frac12}}$ $\qquad\frac{a_n}{a_{n+1}}=e^{-1}\left(1+\frac1n\right)^{n+\frac12}>1$

$\{a_n\}$ is strictly decreasing, positive, thus conv. $\lim_{n\to\infty}a_n=a$

Let $b_n=a_ne^{-\frac{1}{12n}}$ $\quad\frac{b_n}{b_{n+1}}<1$. $\{b_n\}$ increasing $b_n\le a_n$, $\lim_{n\to\infty}b_n=\lim_{n\to\infty}a_n\cdot\lim_{n\to\infty}e^{-\frac{1}{12n}}=a.$

$b_n=a_ne^{-\frac{1}{12n}}\le a\le a_ne^{-0\cdot\frac{1}{12n}}$. $\Rightarrow\exists\,\theta_n\in(0,1)$. $a_n=a\cdot e^{\frac{\theta_n}{12n}}$. $(n!\sim a\sqrt{n}\left(\frac{n}{2}\right)^n)$.

$N!!=\begin{cases}N\text{ odd} & \frac{(N+1)!}{(N+1)!!}\\ N\text{ even} & 2^{\frac N2}\left(\frac N2\right)!\end{cases}$

by Wallis formula

$$\pi=\left(\frac{(2n)!!}{(2n-1)!!}\right)^2=\frac1n\left(\frac{((2n)!!)^2}{(2n)!}\right)^2=\frac1n\cdot\left(\frac{2^{2n}(n!)^2}{(2n)!}\right)^2\sim\frac1n\left(\frac{a\sqrt n}{2}\right)^2=\frac{a^2}{2}.$$

3) **Binomial series.** Let $\alpha\in\mathbb{R}$. Then $(1+x)^\alpha=\sum_{k\ge0}\binom{\alpha}{k}x^k$. $x\in(-1,1)$. (particularly $\binom{\alpha}{0}=0$).

Pf: $\binom{\alpha}{k}=\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$ $(k\in\mathbb{N},\ \alpha\in\mathbb{R})$

$$R=\lim_{n\to\infty}\frac{\binom{\alpha}{n}}{\binom{\alpha}{n+1}}=\lim_{n\to\infty}\left(\frac{n+1}{n+\alpha}\right)=1.$$

原k=0是常数项1.

Let $S(x)=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k$. $\quad S'(x)=\sum_{k=1}^{\infty}k\binom{\alpha}{k}x^{k-1}=\sum_{k=0}^{\infty}(k+1)\binom{\alpha}{k+1}x^{k-1}=\sum_{k=0}^{\infty}(\alpha-k)\binom{\alpha}{k}x^k.\ |x|<1.$

$\Rightarrow\alpha S(x)-xS'(x)=S'(x)\Rightarrow\frac{S'(x)}{S(x)}=\frac{\alpha}{1+x}\Rightarrow(\ln S(x))'=\frac{\alpha}{1+x}$

$\int_0^x[\ln S(t)]'dt$

$\Rightarrow\ln S(x)=\ln S(0)+\alpha\ln(1+x)=\ln(1+x)^\alpha.\Rightarrow S(x)=(1+x)^\alpha.$

4). Taylor series for arcsin. ($\alpha=\frac12$ in 3)) $\Rightarrow(1-x^2)^{-\frac12}=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}$

$\Rightarrow\arcsin x=x+\sum^{\infty}\frac{((2n-1)!!)^2}{\cdots}x^{2n+1},\ |x|<1$

e.g. Sum of inverse square ($\sum_{n\geq 1}^{\infty} \frac{1}{n^2}$).

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!\,(2n+1)} x^{2n+1}$$

Let $\sin(t) = x$. by. Thm 3.13. $\frac{\pi^2}{8} = \int_0^{\frac{\pi}{2}} t\,dt = \int_0^{\frac{\pi}{2}} \arcsin(\sin t)\,dt = I_0 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!\,(2n+1)} I_{2n+1}$

where $I_{2n+1} = \int_0^{\frac{\pi}{2}} \sin^{2n+1} t\,dt = \frac{(2n)!!}{(2n+1)!!} = \frac{2^n n!}{(2n+1)!!}$

Thus we have $\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!(2n+1)} \cdot \frac{2^n \cdot n!}{(2n+1)!!} = 1 + \sum_{n\geq 1}^{\infty} \frac{1}{(2n+1)^2}$

$$S = \sum_{n=1}^{\infty} = \sum_{n\geq 0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n\geq 1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4} S \quad \Rightarrow S = 1 + \frac{1}{2^2} + \cdots = \frac{\pi^2}{6}$$

Hypergeometric series.

<u>Def.</u> A power series $\sum_{k\geq 0} a_k z^k$ is a hypergeometric series if $D_k = a_{k+1}/a_k = A(k)/B(k)$ for some polynomial A. B.

e.g. 1) $D_k = \frac{a+k}{(b+k)(1+k)}$. <u>Kummer's series</u> ${}_1F_1(a;b;x) = \sum_{x=1}^{\infty} \frac{(a)_k}{(b)_k (1)_k} x^k$ $b \geq 0$. $\to (1)_k = k!$

This series converges for every x. where $(q)_n = \begin{cases} 1 & k=0 \\ q(q+1)\ldots(q+n-1) & k>0. \end{cases}$

2) $D_k = \frac{(a+k)(b+k)}{(c+k)(1+k)}$ Gaussian function (or ordinary hypergeometric function)

分子$\to {}_2F_1 \leftarrow$分母. $= (a;b;c;x) = \sum_{k\geq 0}^{\infty} \frac{(a)_k (b)_k}{(1)_k (c)_k} z^k = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots$

$R = 1$. (the series also conv. for $x = \pm 1$).

This function can be considered as universal (since it includes various simple functions as special or limiting case)

general formula: ${}_nF_m = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_n)_k}{(b_1)_k \cdots (b_m)_k} \frac{x^k}{(1)_k}$

e.g. ${}_2F_1(-m;b;c:z) = \sum_{k=0}^{m} (-1)^k \binom{m}{n} \frac{(b)_n}{(c)_n} z^n$

${}_2F_1(1;1;2;-z) = \frac{\ln(1+z)}{z}$.

${}_2F_1(a;b;b;z) = (1-z)^{-a}$. $b \in \mathbb{R}$.

${}_2F_1(\frac{1}{2};\frac{1}{2};\frac{3}{2};z^2) = \frac{\arcsin z}{z}$.

${}_2F_1(1;b;b;z) = 1 + z + z^2 + z^3 + \ldots = \sum_{k=0}^{\infty} z^k$. $b \in \mathbb{R}$.

${}_1F_1(b;c;x) = \lim_{a\to+\infty} {}_2F_1(a;b;c;\frac{x}{a})$

Euler's hypergeometric differential equation. (A solution is hypergeometric function)

$$z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)z)\frac{dw}{dz} - ab\,w = 0.$$

Def. ordinary generating function. (of a sequence $\{a_n\}$). is a function $G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n$

(When the term generating function is used without qualification, it is usually taken to mean an ordinary generating function).

Methods: (to compute series by generating function).

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. consider $f'(x)$. (maybe we can find the uni. conv. function).

$$f(x) = f(0) + \int_0^x f'(t)\,dt. \qquad S = \lim_{x \to 1^-} f(x)$$

∘ Use power series to find solutions of differential equation.

For a given differential equation. we can let $y = \sum_{k=0}^{\infty} a_k z^k$

the equality shows the coefficient coincide. (some relation between the coefficient)

also we can use some given values (points) to compute basis coefficient (a_0. a_1. etc).

Complex number

Def. Let $\mathbb{C} = \mathbb{R}^2$ be a set of complex numbers and for $z = (a, b), w = (c, d) \in \mathbb{C}$

define: $z + w = (a+c, b+d)$

$zw = (ac - bd, ad + bc)$

imaginary unit: a complex number $i = (0, 1)$.

If $a, b \in \mathbb{R}$. $(a, b) = a + bi$.

Def. conjugate of z. $\bar{z} = a - bi$.

$a = Re(z)$. real part

$b = Im(z)$ imaginary part

$|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$ non-negative square root. of $z\bar{z}$

Remark. $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

$\bar{z} = z \iff z \in \mathbb{R}$.

$|z_1 z_2| = |z_1| \, |z_2|$.

$|z_1 \pm z_2| \le |z_1| + |z_2|$ $\quad ||z_1| - |z_2|| \le |z_1 - z_2|$

if $z \neq 0$. $\frac{1}{z} = z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$

Def. (disc). Let $p \in \mathbb{C}$. $r > 0$. a set $B(p, r) = \{z \in \mathbb{C} : |z - p| < r\}$

– a open disc of radius r with center at point p or a neighborhood of p.

A closed disc $\bar{B}(p, r) = \{z \in \mathbb{C} : |z - p| \le r\}$

Polar representation

Def. $z \in \mathbb{C} \setminus \{0\}$ the angle φ. measured from direction of vector 1. is called an argument of z.

(denote by. $\varphi = \arg z$. not unique. the set of all argument is $\operatorname{Arg} z$.

Sometimes we fix a semiopen interval of length 2π. usually $(-\pi, \pi]$. $[0, 2\pi)$.

$r = |z|$. $\varphi = \arg \varphi$. are coordinates (polar). of a point (x, y).

$z = r(\cos\varphi + i \sin\varphi)$. $(x = r\cos\varphi.\ y = r\sin\varphi)$. $\quad z = re^{i\varphi}$

$z^n = r^n(\cos n\varphi + i \sin n\varphi)$. $\quad z_1 z_2 = r_1 r_2 (\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2))$.

Complex sequences and function.

Def. Let $\{z_n\}$ be a sequence of complex numbers. Number z is a limit of a sequence z_n if $|z - z_n| \to 0$. as $n \to 0$. The sequence is convergent if it has a limit.

Thm 1.8. The convergence of complex sequence is equivalent to the convergence of real and imaginary parts and $\lim z_n = \lim Re z_n + i \lim Im z_n$.

Pf: "$\Rightarrow$" $z_n - z$. $|Re z_n - Re z| \le |z_n - z| \to 0$ $\quad |Im z_n - Im z| \le |z_n - z| \to 0$.

"$\Leftarrow$". $z_n = x_n + i y_n$. $x_n \to x$, $y_n \to y$.

$|z_n - (x + iy)| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0$. $z_n \to x + iy$.

Remark. for series. $\sum_{k=1}^{\infty} C_k$. $C_k \in \mathbb{C}$. conv. $\Leftrightarrow \sum_{k=1}^{\infty} Re\, C_k \cdot \sum_{k=1}^{\infty} Im\, C_k$ conv.

absolute. conv.

Remark: $f_k: E \to \mathbb{C}$. define pointwise. and uniform conv. of a functional sequence and of a functional series. as we did for real-valued functions

Bolzano - Cauchy. thm. Weierstrass-M test works.

Def. (neighborhood). Let $\varepsilon > 0$. An open disc $V_p = V_p(\varepsilon) = \{z \in \mathbb{C} : |z-p| < \varepsilon\}$ is called a neighborhood. and a set $\mathring{V}_p = \mathring{V}_p(\varepsilon) = V_p(\varepsilon) \setminus \{p\}$ is punctured neighborhood of $p \in \mathbb{C}$.

Def. Let $E \subset \mathbb{C}$. $p \subset \mathbb{C}$. Then.

1. limit point. (cluster, accumulation. point).
   p is a ~ of E. if every punctured. neighborhood $\mathring{V}_p$ of p the intersection has a common. point with E.
   $E \cap \mathring{V}_p = E \cap V_p \setminus \{p\} \neq \phi$
2. isolated point.
   p is a ~ of E. if $\exists V_p$ of p s.t. $E \cap V_p = \{p\}$

Lemma 1.11 (Equivalent condition).

1. p is a limit point
2. Every neiborhood of p has an infinite intersection with E.
3. There exists a sequence $\{z_n\}$ s.t. $z_n \in E$. $z_n \neq p$ and $z_n \to p$.

Def. (A limit of function f at point p).
Let. $D, G \subset \mathbb{C}$. $f: D \to G$. and let $p \in \mathbb{C}$. to be an accumulation point of D. A is called a limit of function f at point p if one of the following assertion holds:

Cauchy (ε-δ) def $\forall \varepsilon > 0\ \exists \delta > 0: \forall z \in D \setminus \{p\}.\ |z-p| < \delta \Rightarrow |f(z) - A| < \varepsilon$.

neighborhood. def. $\forall V_A\ \exists \mathring{V}_p: f(D \cap \mathring{V}_p) = f(D \cap V_p \setminus \{p\}) \subset V_A$.

Heine def. $\forall \{z_n\}: z_n \in D \setminus \{p\},\ z_n \to p \Rightarrow f(z_n) \to A$.

Thm 1.13. All three definitions of a limit of a function are equivalent.

Thm 1.14. Let $f, g: D \to G$ and $D, G \subset \mathbb{C}$, $p \in \mathbb{C}$. to be a limit point of D. Let $\lim_{z \to p} f(x) = A$. $\lim_{z \to p} g(x) = B$. Then. $\lim_{z \to p} (f+g)(x) = A+B$. $\lim_{z \to p} (fg)(x) = AB$. $\lim_{z \to p} |f(x)| = |A|$.

If $B \neq 0$ then $\lim_{z \to p} \frac{f(z)}{g(z)} = \frac{A}{B}$.

Def. 1.15 (Continuous).
Let $D, G \subset \mathbb{C}$. $f: D \to G$. A function f is called continuous at $z_0 \in D$. if (one of):

1. Either z_0 is an isolated or limit point of D. $f(z_0) = \lim_{z \to z_0} f(z)$
2. Weierstrass and Jordan (ε-δ). $\forall \varepsilon > 0.\ \exists \delta > 0\ \forall z \in D: |z - z_0| < \delta.\ |f(z) - f(z_0)| < \varepsilon$.
3. neiborhood. def. $\forall V_{f(z_0)}\ \exists V_{z_0}; f(V_{z_0}) \subset V_{f(z_0)}$.
4. Heine. $\forall \{z_n\}: z_n \in D.\ z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

Def. 1.16 (Discontinuity)
If f is not continuous in $z_0 \in D$. then one says that it has a discontinuity at $z_0 \in D$.

Def. 1.17 (inner point. interior).

A point $z_0 \in D$. is inner point of a set D. if there exist some neighborhood V_{z_0} of point z_0 such that $V_{z_0} \subset D$. The interior D° is a set of inner points of a set D.

Def. 1.18 (differentiable).

Let $D \subset \mathbb{C}$. $z_0 \in D^\circ$. A function $f: D \to \mathbb{C}$ is differentiable at point z_0 if exists $A \in \mathbb{C}$. s.t. $f(z) = f(z_0) + A(z - z_0) + o(z - z_0)$, $z \to z_0$

A is called a derivative of f at z_0 and denote $A = f'(z_0)$.

Remark: f is differentiable at $z_0 \iff \exists f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$

arithmetical and composition operator:

$(f \pm g)' = f' \pm g'$. $(fg)' = f'g + g'f$. $f(g(z))' = f'(g(z)) \cdot g'(z)$

Thm 1.19. (Derivate/Integral of power series.

Let $R > 0$ be a radius of convergence of real power series. $f(z) = \sum\limits_{k=0}^{\infty} c_k z^k$

Then $f^{(m)}(z) = \sum\limits_{k=m}^{\infty} \dfrac{k!}{(k-m)!} c_k z^{k-m}$

Pf: Let $r < R$. $z, w \in B(0, r)$. Then

$$\frac{f(w) - f(z)}{w - z} = \sum_{k=1}^{\infty} c_k \frac{w^k - z^k}{w - z} = \sum_{k=1}^{\infty} c_k (z^{k-1} + wz^{k-2} + \cdots + zw^{k-2} + w^{k-1}).$$ need to show this $f_k(z)$ uni. conv.

We have. $|c_k(z^{k-1} + \cdots + w^{k-1})| < |c_k| k \cdot r^{k-1}$, $\sum k|c_k| r^{k-1}$ conv. since $r < R$ (R is radius of $\sum k c_k x^k$)

$f_k(z)$ uni. conv. $\lim\limits_{w \to z} \dfrac{f(w) - f(z)}{w - z} = \sum\limits_{k=1}^{\infty} c_k \lim\limits_{w \to z} (\cdots) = \sum\limits_{k=1}^{\infty} k c_k z^{k-1}$.

Product: $f(z) = \sum c_k z^k$ $g(z) = \sum d_k z^k$.

$$(fg)(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} c_j d_{k-j}\right) z^k.$$

Def. $z \in \mathbb{C}$. we let. $e^z = \sum\limits_{k=0}^{\infty} \dfrac{z^k}{k!}$ $\sin z = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k}{(2k+1)!} z^{2k+1}$ $\cos z = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k}{(2k)!} z^{2k}$.

main properties: $(e^z)' = e^z$. $(\sin z)' = \cos z$ $(\cos z)' = -\sin z$.

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}. \left[\left(\sum_{k=0}^{\infty} \frac{z_1^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{z_2^k}{k!}\right) = \sum_{k=0}^{\infty} \frac{(z_1 + z_2)^k}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z_1^j z_2^{k-j}\right]$$

Euler's Formula: $e^{iz} = \cos z + i \sin z$.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2}.$$

if $z = \varphi \in \mathbb{R}$. $|e^{iz}| = 1$.

$\cos z = \sum\limits_{k=0}^{\infty} \dfrac{(-1)^k z^{2k}}{(2k)!} = \sum\limits_{k=0}^{\infty} \dfrac{(iz)^{2k}}{(2k)!}$. $i \sin z = \sum\limits_{k=0}^{\infty} \dfrac{(iz)^{2k+1}}{(2k+1)!}$. ★ $(-1)^k = (i)^{2k}$

$\cos z + i \sin z = \sum\limits_{k=0}^{\infty} \dfrac{(iz)^k}{k!} = e^{iz}$. Remark: $e^{i\pi} = -1$. $e^{\frac{i\pi}{2}} = i$. $e^{-\frac{i\pi}{2}} = -i$.

Trigonmetric (三角学的) identities.

$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

$\sin(z_1+z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$.

Function sin. cos. are not bounded in $\mathbb{C}$.

$\cos iy = \frac{e^{-y}+e^{y}}{2}$ $\rightarrow +\infty$. $y \rightarrow \pm\infty$.

$\sin iy = \frac{e^{-y}-e^{y}}{2}$ $\rightarrow \mp\infty$, $y \rightarrow \pm\infty$

Exponent $e^z \neq 0$. (if $e^z = 0 \Leftrightarrow e^{x+iy} = 0 \Leftrightarrow e^x(\cos y + i\sin y) = 0 \Leftrightarrow \cos y + i\sin y = 0$.
$\Leftrightarrow \sin y = \cos y = 0$. impossible).

Exponent e^z has periods equal to $2k\pi i$, $k \in \mathbb{Z} \setminus \{0\}$. and no other periods.

function sin. cos. have periods equal to $2k\pi$. and no other periods.

$e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi = 1$.

$e^{z+2k\pi i} = e^z \cdot e^{2k\pi i} = e^z$.

converse (other period T). $e^{z+T} = e^z$. $T = x+iy$. $|e^T| = |e^x||\cos y + i\sin y| = 1. \Rightarrow e^x = 1. \ x = 0$.

$e^{iy} = \cos y + i\sin y = 1 \Leftrightarrow y = 2k\pi i$.

$\ln(z) = \ln|z| + \mathrm{Arg}\, z = \ln|z| + \arg z + 2k\pi$.

$\forall z \neq 0$. $\exists w \in \mathbb{C}$. s.t. $e^w = z$. (the exponent obtains every value except 0).

solve the equation: $e^w = z$.

Let. $z = re^{i\varphi}$. $\varphi = \arg z \in [0, 2\pi)$. $w = x+iy$.

$e^{x+iy} = \boxed{e^x e^{iy} = re^{i\varphi}}$ $\rightarrow$ "equal". ~~If~~ we consider the figure on the disc. the direction must coincide. so. $\cos y = \cos\varphi$. (or simply consider the modulus).

$\Rightarrow x = \ln r$.