# Notes. Review CA.

1. "diff" means $\mathbb{C}$-diff.

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

$$\boxed{\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \end{cases}}$$

2. directional function

$$\lim_{\substack{\Delta z \to 0 \\ \arg \Delta z = \theta}} \frac{\Delta f}{\Delta z} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar z}(z_0)\, e^{-2i\theta} =: f'_\theta(z_0).$$

Lemma 2.7. $f$ is $\mathbb{R}$-diff, $f'_\theta(z_0) = f'(z_0)$ for every $\theta \in \mathbb{R}$ $\iff$ $\frac{\partial f}{\partial \bar z}(z_0) = 0$.

3. conformal mappings.

"at some point $z_0$": $\mathbb{C}$-diff at $z_0$ and $f'(z_0) \neq 0$.

"conformal onto" 针对域(集合). 需要 $df$ 是一一映射.

判定: $f$ is conformal if $f$ - $\mathbb{R}$-diff. and $df(z_0)$. linear map of $\mathbb{R}^2$ (rotation/scaling). and bijective.

4. Harmonic function.

$u: \mathbb{R}^2 \to \mathbb{R}$. $\quad \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ 任何在区域 D 内的解析函数, 实部和虚部均为调和函数.

$f = u + iv$ satisfy the C-R. condition. then $v$. is conjugate harmonic function of $u$.

5. Geometric meaning of complex derivative.

$\arg f'(z_0)$. $|f'(z_0)|$ 仅与 $z_0$ 有关. 与曲线方向无关 旋转角. 伸缩率不变.

$w = f(z)$ 将 $z_0$ 无穷小圆 → $w_0$ 无穷小圆. 半径之比 $f'(z_0)$

6. Antiderivative.

holomorphic func. local a.d. 一定存在 ($\forall U \subset D$).

global a.d. in D. 不一定存在.

global a.d. along path 一定存在 (path 要求 -- 保证了 $F$ 单值)

N-L formula. $\int_\gamma f\,dz = \Phi(\beta) - \Phi(\alpha)$. $\Phi$ be a.d. of $f$ along $\gamma$.

7. Homotopy. $\gamma_1, \gamma_2$. (起始点相同. "圈数相同").

Contractible closed. homotopic to constant path.

8. Taylor and Laurent Series.

Taylor 存在性. $f \in H(D)$. $U_R(a) = \{z \in \mathbb{C} : |z - a| < R\} \subset D$. → 只要满足这个, 就收敛.

$$C_n := \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)\,d\zeta}{(\zeta - a)^{n+1}}, \quad n = 0, 1, 2, 3, \ldots \qquad 0 < r < R.$$

$$f(z) = \sum_{n=0}^{\infty} C_n (z - a)^n. \quad |z - a| < R$$

Cauchy-Hadamart's formula. $\sum_{n=0}^{\infty} b_n (z-a)^n$ radius of disk of conv. $R = \left(\overline{\lim} \sqrt[n]{|b_n|}\right)^{-1}$ $\overline{\lim}\, x_n = -\underline{\lim}\,(-x_n)$

$\Downarrow$

uni. conv. $K \subset U_R(a)$. div. $z \in \mathbb{C} \setminus \overline{U_R(a)}$. $\overline{\lim}\, x_n = \dfrac{1}{\underline{\lim} \frac{1}{x_n}}$.

coefficients. $c_n = \dfrac{f^{(n)}(a)}{n!}$ $f^{(m)}(z) = \sum_{n=m}^{\infty} \dfrac{n!}{(n-m)!} b_n z^{n-m}$

Laurent 存在性. $f \in H(V)$. $V = \{z \in \mathbb{C} : r < |z-a| < R\}$. $0 \le r < R \le +\infty$

$$c_n := \frac{1}{2\pi i} \int_{|\zeta - a| = \rho} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta \text{ for all } n \in \mathbb{Z} \text{ and } r < \rho < R.$$

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n. \quad r < |z-a| < R$$

$$R = \left(\overline{\lim_{n\to\infty}} |c_n|^{1/n}\right)^{-1} \qquad r := \overline{\lim_{n\to\infty}} |c_{-n}|^{1/n}.$$

$|z-a| < r$. main part $\sum_{n=-\infty}^{-1} c_n (z-a)^n$ div.

$|z-a| > R$. regular part. $\sum_{n=1}^{\infty} c_n (z-a)^n$ div.

# Holomorphity.

说明复积分值与路径无关.

Cauchy-Goursat thm. $f \in H(D)$. $\forall \Delta \subset D$. $\int_{\partial\Delta} f(z)\,dz = 0$.

Cauchy inequality. $|C_n| \le \frac{M(r)}{r^n}$. $M(r) := \max_{|z-a|=r} |f(z)|$ (both. Taylor/ Laurent series holds).

Liuville theorem. $f \in H(\mathbb{C})$. $|f| \le M$. then $f$ is constant.

(1) 三种等价定义. (在 a 处) ① $f$ - $\mathbb{C}$-diff in some $U_a$.

② $f$ - can be decomposed into a power series. center at $a$. conv. in some $U_a$.

③ $f \in C(U_a)$ $\int_{\partial\Delta} f\,dz = 0$. for any $\Delta \subsetneq U_a$. (Morer thm).

(2) $f \in H(D)$. $f$ has all order of derivatives in $D$.

(3) 在零处的分解. $f(z) = (z-a)^n g(z)$ $f$ is holomorphic at $a$. $f(a)=0$. $f$ is not identically zero in any $U_a$

(order of zero). $g$ is holomorphic at $a$. and $g(z) \neq 0$. in some $U_a$.

$n = \min\{m \geq 1.\ C_m \neq 0\} = \min\{m \ge 1 : f^{(m)}(a) \neq 0\}$. multiplicity of zero. of $f$ at $a$.

(4) Uniqueness. $f, g \in H(D)$. $E = \{z \in D : f = g\}$. has a limit point in $D$. Then $f \equiv g$ in $D$. (此定理可把很多 $\mathbb{R}$ 上的运算试推广到 $\mathbb{C}$ 上)

(5) Maximum modulus principle. $f \in H(D)$ If $\exists p \in D$. s.t. $|f(z)| \le |f(p)|$ for every $z \in D$. then $f$ is constant.

给定域 (开.连通集). 非常值函数的模无最大值; $f \in H(D) \cap C(\bar{D})$ $|f|$ 在 $\partial D$ 上取得最大值.

$f \in H(D)$. not const. not vanish. 无最小值

(6). integer functions. $f \in H(\mathbb{C})$.

**Definition.** A function that is holomorphic in the entire complex plane $\mathbb{C}$ is called an integer.

整函数仅在 $z=\infty$ 有奇点且必为孤立奇点

整函数 regular part. $\sum_{n=0}^{\infty} C_{-n} z^{-n}$ 有限. by Liouville. - 定是常数.

$\begin{cases} \text{fixable.} \iff f \equiv \text{const.} \\ \text{pole} \iff f = \sum_{n=1}^{N} C_n z^n \quad (\text{polynomial}) \\ \text{essential} \iff \text{infinite many } C_n \neq 0.\ n \geq 0. \end{cases}$ 整超越函数 (收敛半径 $R = \infty$)

(7). meromorphic function (亚纯函数)

def. $\mathbb{C}$ 上除极点外再无其他奇点.

thm. $f$ is meromorphic. has fixable / pole on $z=\infty$. $f$ is rational ( $Q(z) = \frac{P_1(z)}{P_2(z)}$ )

1. 复平面有关计算

2. 可微性证明等式.（C-R方程）

$f(x,y)=u(x,y)+i\,v(x,y)$　　$f: \mathbb{C}\to\mathbb{C}$.

(1) 可微 ① $\mathbb{R}$-diff. u.v. 可微 $\begin{cases} u(x,y)-u(x_0,y_0) = A_1(x-x_0)+B_1(y-y_0)+\bar{o}(|\Delta z|) \\ v(x,y)-v(x_0,y_0) = A_2(x-x_0)+B_2(y-y_0)+\bar{o}(|\Delta z|) \end{cases}$　　$\Delta z = d((x,y),(x_0,y_0))\to 0$.

(形式)定义: $\frac{\partial f}{\partial z}=\frac{1}{2}\left(\frac{\partial f}{\partial x}-i\frac{\partial f}{\partial y}\right)$　　$\frac{\partial f}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial f}{\partial x}+i\frac{\partial f}{\partial y}\right)$　　$df(z_0)=\frac{\partial f}{\partial z}(z_0)\,dz+\frac{\partial f}{\partial \bar{z}}(z_0)\,d\bar{z}$.

② $\mathbb{C}$-diff　$\mathbb{R}$-diff $+\ \frac{\partial f}{\partial \bar{z}}(z_0)=0$.　即: $\begin{cases} u'_x = v'_y \\ u'_y = -v'_x \end{cases}$　(C-R方程).

(2) polar change.　$z=re^{i\varphi}$　$\bar{z}=re^{-i\varphi}$.　$\Rightarrow$　$\frac{\partial r}{\partial \bar{z}}=\frac{e^{i\varphi}}{2}$　$\frac{\partial \varphi}{\partial \bar{z}}=\frac{ie^{i\varphi}}{2r}$

$$\frac{\partial}{\partial \bar{z}}=\frac{\partial r}{\partial \bar{z}}\cdot\frac{\partial}{\partial r}+\frac{\partial \varphi}{\partial \bar{z}}\cdot\frac{\partial}{\partial \varphi}=\frac{e^{i\varphi}}{2}\left(\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial \varphi}\right).$$

C-R. equation: $\begin{cases} \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \varphi} \\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \varphi}. \end{cases}$

(3). f-diff. $\Leftrightarrow$ $f'(z_0)=\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$.

3. 复数域上初等函数的性质

$\cos z=\frac{e^{iz}+e^{-iz}}{2}$　　$\sin z=\frac{e^{iz}-e^{-iz}}{2i}$.

$e^z=e^{x+iy}=e^x(\cos y+i\sin y)$.

sin, cos not bounded in $\mathbb{C}$.

4. 刻画z平面上的函数　(path / equation)

说明转n圈.

## 5. 多值函数

$\mathrm{Ln}\, z = \ln|z| + i \arg z + 2\pi k i,$

$w^n = z \Rightarrow w = \sqrt[n]{r}\, e^{i\frac{\varphi+2\pi k}{n}} \quad k \in [0:n-1]$

## 6. 计算复积分

(1) $\displaystyle\int_\gamma (z-a)^n dz = \begin{cases} 0, & n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i, & n = -1 \end{cases} \qquad \gamma(t) = a + re^{it} \quad t \in [0, 2\pi]$

(2) Cauchy integral formula 前提. $D \subset \mathbb{C}$. $D$-open. $f \in H(D)$. $\gamma$ oriented boundary of $G$. $\overline{G} \subset D$.

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} d\xi. \quad \forall z \in G.$$

## 7. 计算洛朗级数

"Expand $f(z) = \cdots$ in Laurent series valid for domain $r < |z-a| < R$."

1. 先整理. $f(z)$. 所有项有 "$z-a$" 或 $\frac{1}{z-a}$ 的形式.

2. 根据 $r$, $R$ 作满足收敛半径约束的 Taylor 展开. ( $|z-a| < R$. $\frac{1}{|z-a|} < \frac{1}{r}$ ).

## 8. 全纯函数奇点和类别

$\exists \{z_n\}$ $z_n = \frac{1}{\pi n}$. $\mathrm{ctg}\frac{1}{z_n} \xrightarrow{\text{every } n} \infty$

<u>def.</u> $f \in H(\dot{V}_a)$ isolated singular point. (一旦确定 isolated. 则必为三类之一. 也有 unisolated. 如 $\mathrm{ctg}\frac{1}{z}$. 在 $z=0$.)

可以用于证明 main part 有限.

fixable. $\lim\limits_{z \to 0} f(z) = A \in \mathbb{C}$. $\Leftrightarrow$ $C_n = 0$ at $n < 0$ $\Leftrightarrow$ $f(z)$. bounded in some $\dot{V}_a'(\varepsilon')$.

pole $\lim\limits_{z \to a} f(z) = \infty$ $\Leftrightarrow$ $f(z) = \sum\limits_{n=-N}^{\infty} C_n (z-a)^n$. $C_{-n} \neq 0$. $n \in [1:N]$. ($N$ - order of pole).

$\Leftrightarrow$ $g(z) := \frac{1}{f(z)}$ is holomorphic at $a$. and $g(a) = 0$ ($N$-order of zero)

essential no limit.

<u>Sokhotsky's thm.</u> $a \in \mathbb{C}$. essential. singular point. of $f$. $\forall A \in \overline{\mathbb{C}}$. $\exists \{z_n\} \to a$. s.t. $\lim\limits_{n \to \infty} f(z_n) = A$.

$a = \infty$. <u>def.</u> $f \in H(|z| > R)$ for some $R > 0$.

判别时考察函数 $g(z) := f(\frac{1}{z})$.

(1). fixable. $C_n = 0$ for all $n > 0$.

(2) pole. $\exists N > 0$ s.t. $C_N \neq 0$. but $C_n = 0$ when $n > N$.

(3) essential. $C_n \neq 0$ for an infinite many $n$.

在 $z = \infty$ 处 main / regular part 在形式上与有限点相反.

9. 算留数（前提是孤立奇点）

def. $f\in H(\dot{V}_a(\varepsilon))$ $\quad res_a f = \frac{1}{2\pi i}\int_{|z-a|=r} f(z)\,dz.$ $\quad 0<r<\varepsilon.$

(1) 计算给定函数的留数

① 根据定义. ② 根据 Laurent series. $res_a f = C_{-1}$ （对任意类留数均适用）.

③ 分类：

有限 a

simple pole. i) $C_{-1} = res_a f = \lim_{z\to a}(z-a)f(z)$

ii) $f(z) = \frac{\varphi(z)}{\psi(z)}$ $\quad \varphi.\psi\in H(U_a)$. and. $\varphi(a)\neq 0$. $\psi(a)=0$. but $\psi'(a)\neq 0$.

$res_a f = \frac{\varphi(a)}{\psi'(a)}.$

n-th pole $\quad res_a f = \frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}\{(z-a)^n f(z)\}}{dz^{n-1}}$

def. $f\in H(\{|z|>R_0\})$. $\quad res_\infty f = \frac{1}{2\pi i}\int_{\gamma_R^{-1}} f\,dz.$ $\quad res_\infty f = -C_{-1}.$ → $z=0$ 处

Relation. $f\in H(\mathbb{C}\setminus\{a_0, a_1 \dots a_N\})$. $\quad res_\infty f + \sum_{n=1}^{N} res_{a_n} f = 0.$

$\operatorname{Res}_{z=\infty} f(z) = -\operatorname{Res}_{t=0}\left[f\left(\frac{1}{t}\right)\frac{1}{t^2}\right]$

10. 解析函数奇点（支点）.

(1) 多值运算. 加法. 复合. ($\Phi\circ F$. $F$ is analytic in D. $F(D)\subset G$. $\Phi\in H(G)$. $\Phi\circ F$ 单值（最后一步单值）.

(2) 多值函数.

1. 对数函数

2. 幂函数.

转化：$z^\alpha = e^{\alpha \operatorname{Ln} z} = e^{\alpha(\ln z + 2k\pi i)} = e^{\alpha \ln z}\, e^{2k\pi i\alpha}.$

$\alpha\in\mathbb{Z}$. single-valued. $\alpha\geq 0$. fixable / defined $\alpha<0$. pole of order $|\alpha|$.

$\alpha\in\mathbb{Q}$. $\alpha = \frac{p}{q}$. $\quad q$ -values ($k=0,1,\dots,q-1$). branch point of order $q$ at $\infty/0$.

$\alpha\notin\mathbb{Q}$. Countable number of values. ( logarithmic branch point at $\infty/0$.

$\Phi(z) = z^{\frac{1}{n}}$. n-valued. analytic func. $\mathbb{C}\setminus\{0\}$. two branch. $z=0,\infty$. of order n.

$\operatorname{Ln} z$. $\quad$ two branch. $z=0,\infty$ of logarithmic

Lemma. $f$ has. zero at $a\in\bar{\mathbb{C}}$ of order 1. $\quad\sqrt{f(z)}$ branch point of order 2.

判断支点所数用 $F_0 = F_k$.

## 11. 积分计算.

(1) 算 $\oint_C f(z)dz$ ( c包围有奇点)  注意:对多值 $f(z)$. 轨迹 C 一般有辐角 $[-\pi,\pi]$

Cauchy thm. on residue.

D - open. connected set. simple boundary.  G - open. connected. $G \supset \bar{D}$.  $f \in H(G \setminus \{a_1, a_2 \dots a_n\})$.

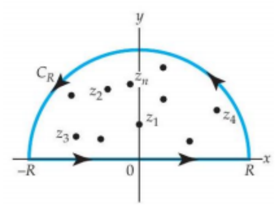
$$\int_{\partial D} f(z)\,dz = 2\pi i \sum_{j=1}^{\infty} \operatorname{res}_{a_j} f$$

(2) 三类特殊形式的积分. (构造闭路. 用留数)

① $\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$   F is rational   $\int_0^{2\pi} F(\cos\theta, \sin\theta) = \oint_C F\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz}$   $C: \{|z|=1\}$.

$dz = ie^{i\theta}d\theta$.  $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$  $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$  $\Rightarrow$  $d\theta = \frac{dz}{iz}$  $\cos\theta = \frac{1}{2}(z+z^{-1})$.  $\sin\theta = \frac{1}{2i}(z-z^{-1})$

② $\int_{-\infty}^{\infty} f(x)\,dx$   $f$ is rational.  $f = \frac{p}{q}$.  p. q no common factor (一般是算 P.V.)



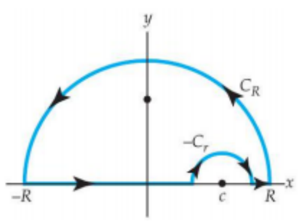

$$\text{P.V.}\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to+\infty}\int_{-R}^{R} f(x)\,dx. = \oint_C f(z)\,dz - \int_{C_R} f(z)\,dz.$$

$$\int_{C_R} f(z)\,dz \xrightarrow[R\to\infty]{} 0 \qquad \deg q \geq \deg p + 2.$$

③ $\int_{-\infty}^{\infty} f(x)\cos\alpha x\,dx$ .  $\int_{-\infty}^{\infty} f(x)\sin\alpha x$.  $f$ is rational.  $f = \frac{p}{q}$.  p. q no common factor

先算 $\int_C f(z)\cdot e^{i\alpha z}dz$. 分实部. 虚部.   $\int_{C_R} f(z)e^{i\alpha z}dz \xrightarrow[R\to\infty]{} 0$   $\deg q \geq \deg p + 1$.

△ 上述②③ 计算. 要满足 $f(z)$ 在 $[-R,R]$ 上无极点. 若有:




$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{C_r} + \int_r^R$$

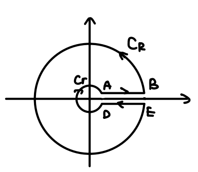
$\lim_{r\to 0}\int_{C_r} f(z)\,dz = \pi i \operatorname{Res}(f(z), c)$.  $C_r$: $z = c + re^{i\theta}$. $0 \leq \theta \leq \pi$

$$\text{P.V.}\int_{-\infty}^{+\infty} f(x)\,dx = \oint_C + \int_{C_r}.$$

(3) along a branch cut.

在做变换 $z = re^{i\theta}$. 多值函数可能表现出不同的值.

$\sqrt{z} = \sqrt{r}\,e^{i\frac{\theta}{2}}$   $\sqrt{z} = \sqrt{r}\,e^{i\frac{\theta+2\pi}{2}} = -\sqrt{r}\,e^{i\frac{\theta}{2}}$.




$$\int_0^{\infty}\frac{1}{\sqrt{x}(x+1)}\,dx$$

$$\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = \oint_C$$

12. Argument principle.

D – open, connect.   C – simple. closed. lying within D.

$f \in H(D \setminus N_p)$.   $N_p$ is pole inside C   $f(z) \neq 0$ on C   $N_0$ is zero inside C.

$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N_0 - N_p$.   注意是 $f(z)$ 的 0，别用 $\frac{f'(z)}{f(z)}$ 算！

13. Rouché's Thm. ( Location of zeros ).

D – open, connect.   C – simple. closed. lying within D.   $f, g \in H(D)$.

$|f(z) - g(z)| < |f(z)|$ holds for all $z$ on C.   $f, g$ have some number of zeros inside C.

一般取简单函数来作差. 然后取适合的C. 考察0的位置.