Indefinite integral

Elena Lebedeva

Saint Petersburg State University

Notation: $\langle a,\ b\rangle \in \{(a,\ b)\,,\ [a,\ b]\,,\ (a,\ b]\,,\ [a,\ b)\}\,.$

Definition (A primitive)

Let $f, F : \langle a, b\rangle \to \mathbb{R}$. A function F is **a primitive** *(or* **inverse derivative, antiderivative***) of a function f on $\langle a, b\rangle$ if*

$$F'(x) = f(x), \quad x \in \langle a,\ b\rangle.$$

Theorem (The set of all primitives)

If F is a primitive of f on $\langle a,\ b\rangle$, then the set of all primitives is

$$\{F + C \,|\, C \in \mathbb{R}\}\,.$$

Proof. Let G be a primitive of f on $\langle a,\ b\rangle$. Then $(F-G)'(x) = f(x) - f(x) = 0$. By Lagrange's theorem it follows that $F - G$ is constant. $\square$

Remark. If we replace $\langle a, b\rangle$ to a more complicated set, say, $\langle a, b\rangle \sqcup \langle c, d\rangle$, then the constants on each component may differ from each other.

Example. $f(x) = \frac{1}{1+x^2}$, $F(x) = \arctan(x)$, $G(x) = \operatorname{arccot}(\frac{1}{x})$.

$F'(x) = f(x)$, $x \in \mathbb{R}$, $\qquad G'(x) = -\frac{1}{1+(1/x)^2}\left(-\frac{1}{x^2}\right) = f(x)$, $x \in \mathbb{R} \setminus \{0\}$.

$$G(x) = \begin{cases} F(x), & x > 0, \\ F(x) + \pi, & x < 0 \end{cases}$$

Definition (A indefinite integral)

The set of all primitives of f *on* $\langle a, b\rangle$ *is called* **the indefinite integral** *of* f *on* $\langle a, b\rangle$*. It is denoted by* $\int f(x)\,\mathrm{d}x$ *or* $\int f$*, where the sign* $\int$ *is called* **the indefinite integral** *sign,* f *is called* **the integrand***, and* $f(x)\,\mathrm{d}x$ *is called* **a differential form***. The operation of finding a primitive has the name "indefinite integration".*

$$\int f(x)\,\mathrm{d}x := \{F + C \,|\, C \in \mathbb{R}\}$$

Which functions have primitives?

- Later, we will prove that **any continuous function has a primitive**.
- If the function has points of discontinuity on $\langle a, b\rangle$ of the first kind, then it has no primitive (It follows from the Darboux theorem: the function $F'(x)$ assumes on $[a, b]$ all the values between $F'(a)$ and $F'(b)$).
- An example of a function which has a point of discontinuity of the second kind and it has a primitive.

$$F(x) = \begin{cases} x^2 \sin(1/x^3), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

$$f(x) = F'(x) = \begin{cases} 2x \sin(1/x^3) - 3/x^2 \cos(1/x^3), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

How to find an indefinite integral?

Table of standard integrals

1. $\int 0 \, \mathrm{d}x = C.$
2. $\int x^a \, \mathrm{d}x = \frac{x^{a+1}}{a+1} + C, \;\; a \neq -1.$
3. $\int \frac{\mathrm{d}x}{x} = \log|x| + C, \; x \neq 0.$
4. $\int a^x \, \mathrm{d}x = \frac{a^x}{\log a} + C, \;\; a > 1, \;\; a \neq 1.$
5. $\int \sin x \, \mathrm{d}x = -\cos x + C, \;\; \int \cos x \, dx = \sin x + C.$
6. $\int \frac{\mathrm{d}x}{\cos^2 x} = \tan x + C, \;\; x \neq \frac{\pi}{2} + \pi k, \;\; k \in \mathbb{Z}.$
7. $\int \frac{\mathrm{d}x}{\sin^2 x} = -\cot x + C, \;\; x \neq \pi k, \;\; k \in \mathbb{Z}.$
8. $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x + C = -\arccos x + C.$
9. $\int \frac{\mathrm{d}x}{1+x^2} = \mathrm{arctg}\, x + C = -\mathrm{arcctg}\, x + C.$
10. $\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm 1}} = \log\left|x + \sqrt{x^2 \pm 1}\right| + C.$
11. $\int \frac{\mathrm{d}x}{1-x^2} = \log\left|\frac{1+x}{1-x}\right| + C.$

Theorem (Arithmetical properties of indefinite integrals)

Assume that functions $f, g : \langle a, b\rangle \to \mathbb{R}$ have primitives, $\alpha \in \mathbb{R}$. Then

1. *(additivity) $f + g$ has a primitive and*
$$\int (f+g) = \int f + \int g;$$
2. *(homogeneity) αf has a primitive and for $\alpha \neq 0$*
$$\int \alpha f = \alpha \int f.$$

Recall that
$$A + B = \{x + y \,|\, x \in A,\ y \in B\},$$
$$\alpha A = \{\alpha x \,|\, x \in A\}, \quad x + B = \{x + y \,|\, y \in B\}.$$

Theorem (Change of variables in an indefinite integral)

Suppose $f : \langle a,\ b\rangle \to \mathbb{R}$, $\varphi : \langle c,\ d\rangle \to \langle a,\ b\rangle$, *$F$ is a primitive of f on* $\langle a,\ b\rangle$, *φ is differentiable on* $\langle c,\ d\rangle$. *Then*

$$\int f(\varphi(t))\varphi'(t)\,\mathrm{d}t = F(\varphi(t)) + C.$$

The proof follows from the rule of differentiation of a composite function (chain rule). Indeed,

$$(F(\varphi(t)) + C)' = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t). \quad \square$$

Here is a convenient way to apply the Theorem.

$$\int f(\varphi(t))\varphi'(t)\,\mathrm{d}t = \int f(\varphi(t))\,\mathrm{d}\varphi(t) = [x = \varphi(t)]$$

$$= \int f(x)\,\mathrm{d}x = F(x) + C = F(\varphi(t)) + C.$$

Example. $\displaystyle\int \tan(t)\,\mathrm{d}t = \int \frac{\sin t\,\mathrm{d}t}{\cos t} = \int \frac{-(\cos t)'\,\mathrm{d}t}{\cos t} = -\int \frac{\mathrm{d}(\cos t)}{\cos t}$

$$= [x = \cos t] = -\int \frac{\mathrm{d}x}{x} = -\log|x| + C = -\log|\cos t| + C.$$

Example. $\displaystyle\int \frac{e^t\,\mathrm{d}t}{e^{2t}+1} = \int \frac{\mathrm{d}(e^t)}{(e^t)^2+1} = \arctan e^t + C$

One more example. $I(x)=\displaystyle\int\frac{x^2+1}{x^4+1}\,\mathrm{d}x.$

Let $x\neq 0$.

$$I=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,\mathrm{d}x=\int\frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}=\frac{1}{\sqrt{2}}\,\mathrm{arctan}\,\frac{x-\frac{1}{x}}{\sqrt{2}}+\begin{cases}C_1, & x<0,\\ C_2, & x>0.\end{cases}$$

The continuous function $f(x)=\dfrac{x^2+1}{x^4+1}$ has the primitive $I(x)$ on any $[a,b]\subset\mathbb{R}$. The primitive is continuous. Therefore,

$$\lim_{x\to+0}I(x)=\lim_{x\to-0}I(x)\ \Rightarrow\ \frac{\pi}{2\sqrt{2}}+C_1=-\frac{\pi}{2\sqrt{2}}+C_2$$

$$\Rightarrow\ C_1=-\frac{\pi}{2\sqrt{2}}+C,\ \ C_2=\frac{\pi}{2\sqrt{2}}+C.\ \text{Set } I(0)=C,\ \text{ then}$$

$$I(x)=\frac{1}{\sqrt{2}}\,\mathrm{arctan}\,\frac{x^2-1}{x\sqrt{2}}+\frac{\pi}{2\sqrt{2}}\mathrm{sign}x+C.$$

Theorem (Integration by parts in an indefinite integral)

Suppose f, g are differentiable on $\langle a, b\rangle$, $f'g$ has a primitive. Then fg' has a primitive and

$$\int fg' = fg - \int f'g.$$

Proof. The derivative of the product is $(fg)' = f'g + fg'$.
So, the function $fg' = (fg)' - f'g$ has a primitive as a difference.
Applying arithmetical properties of indefinite integrals we get

$$\int fg' = \int \left((fg)' - f'g\right) = \int (fg)' - \int f'g = fg - \int f'g. \quad \square$$

One may proceed as follows.

$$\int f(x)g'(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}(g(x)) = f(x)g(x) - \int g(x)\,\mathrm{d}(f(x))$$

$$= f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x.$$

Example.

$$\int x^2 \sin x\,\mathrm{d}x = \int x^2(-\cos x)'\,\mathrm{d}x = \int x^2\,\mathrm{d}(-cosx)$$

$$= -x^2\cos x - \int(-\cos x)\,\mathrm{d}(x^2) = -x^2\cos x + 2\int x\cos x\,\mathrm{d}x$$

$$= -x^2\cos x + 2\int x(\sin x)'\,\mathrm{d}x = -x^2\cos x + 2\int x\,\mathrm{d}(\sin x)$$

$$= -x^2\cos x + 2x\sin x - 2\int \sin x\,\mathrm{d}x = -x^2\cos x + 2x\sin x + 2\cos x + C.$$

Example.

$$I = \int e^{ax} \cos bx \,\mathrm{d}x = \frac{1}{a} \int \cos bx \,\mathrm{d}(e^{ax}) = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a} \int e^{ax} \sin bx \,\mathrm{d}x$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} \int \sin bx \,\mathrm{d}(e^{ax})$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} \int e^{ax} \cos bx \,\mathrm{d}x$$

$$= \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I$$

$$\Rightarrow \quad I = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C.$$

Remark. A primitive of an elementary function might not be an elementary function.

Example.

- $\mathrm{Ei}(x) = \displaystyle\int \frac{\mathrm{e}^x}{x}\,\mathrm{d}x, \quad \lim_{x\to-\infty} \mathrm{Ei}(x) = 0$ (the exponential integral);
- $\mathrm{Si}(x) = \displaystyle\int \frac{\sin x}{x}\,\mathrm{d}x, \quad \lim_{x\to 0} \mathrm{Si}(x) = 0$ (the sine integral);
- $\mathrm{Ci}(x) = \displaystyle\int \frac{\cos x}{x}\,\mathrm{d}x, \quad \lim_{x\to+\infty} \mathrm{Ci}(x) = 0$ (the cosine integral);
- $\left.\begin{array}{ll} \mathrm{S}(x) = \int \sin x^2\,\mathrm{d}x, & \lim_{x\to 0} \mathrm{S}(x) = 0 \\ \mathrm{C}(x) = \int \cos x^2\,\mathrm{d}x, & \lim_{x\to 0} \mathrm{C}(x) = 0 \end{array}\right\}$ (the Fresnel integrals);
- $\Phi(x) = \displaystyle\int \mathrm{e}^{-x^2}\,\mathrm{d}x, \quad \lim_{x\to-\infty} \Phi(x) = 0$ (the Euler-Poisson integral);

Classes of functions whose primitives are elementary

Notation: $R(x) = \frac{P(x)}{Q(x)}$ is a rational function in x, $R(u,\ v) = \frac{P(u,v)}{Q(u,v)}$ is a rational function in u and v.

1. $\displaystyle\int R(x)\,\mathrm{d}x.$
2. $\displaystyle\int R(\cos x,\ \sin x)\,\mathrm{d}x.$
3. $\displaystyle\int R\left(x,\ \sqrt[n]{\frac{ax+b}{cx+d}}\right)\,\mathrm{d}x.$
4. $\displaystyle\int R(x,\ \sqrt{ax^2+bx+c})\,\mathrm{d}x.$
5. $\displaystyle\int x^m(a+bx^n)^p\,\mathrm{d}x$, where $m,\ n,\ p \in \mathbb{Q}$, and $p \in \mathbb{Z}$, or $\frac{m+1}{p} \in \mathbb{Z}$, or $p + \frac{m+1}{p} \in \mathbb{Z}$.

Integration of rational functions

Theorem

Let P, Q be two polynomials, $Q(x) = C\prod_{i=1}^{n}(x-a_i)^{k_i} \cdot \prod_{i=1}^{m}(x^2+p_ix+q_i)^{l_i}$, $a_i, p_i, q_i, C \in \mathbb{R}$, $n, m, k_i, l_i \in \mathbb{N}$, $p_i^2 - 4q < 0$. *Then*

$$\frac{P(x)}{Q(x)} = P_0(x) + \sum_{i=1}^{n}\sum_{t=1}^{k_i}\frac{A_{i,t}}{(x-a_i)^t} + \sum_{i=1}^{m}\sum_{s=1}^{l_i}\frac{M_{i,s}x+N_{i,s}}{(x^2+p_ix+q_i)^s},$$

where P_0 is a polynomial, $deg(P_0) = deg(P) - deg(Q)$. The fractions in the RHS are called **partial fractions**.

Remark. To find undetermined coefficients $A_{i,t}$, $M_{i,s}x$, $N_{i,s}$ we put all the terms on the RHS over a common denominator, then equating the coefficients of the resulting numerator to the corresponding coefficients of P.

So, integrating of $P(x)/Q(x)$ reduces to integrating the individual terms

$$1.\ \frac{1}{(x-a)^t}, \qquad 2.\ \frac{Mx+N}{(x^2+px+q)^s}, \qquad 3.\ P_0(x).$$

$$1.\ \int \frac{\mathrm{d}x}{(x-a)^t} = \begin{cases} \log|x-a|+c, & t=1, \\ \frac{-1}{(t-1)(x-a)^{t-1}}+c, & t\geq 2. \end{cases}$$

$$2.\ \int \frac{Mx+N}{(x^2+px+q)^s}\,\mathrm{d}x = \frac{M}{2}\int \frac{(2x+p)\,\mathrm{d}x}{(x^2+px+q)^s}$$

$$+\left(N-\frac{Mp}{2}\right)\int \frac{\mathrm{d}x}{(x^2+px+q)^s}.$$

$$\int \frac{(2x+p)dx}{(x^2+px+q)^s} = \int \frac{d(x^2+px+q)}{(x^2+px+q)^s} = \begin{cases} \log(x^2+px+q)+c, & s=1, \\ \frac{-1}{(s-1)(x^2+px+q)^{s-1}}+c, & s\geq 2. \end{cases}$$

To calculate the second integral in the RHS we select a full square in the denominator

$$\int \frac{\mathrm{d}x}{(x^2+px+q)^s} = \int \frac{\mathrm{d}x}{((x+p/2)^2+q-p^2/4)^s}$$

$$= \frac{1}{(q-p^2/4)^{s-1/2}} \int \frac{\mathrm{d}\frac{x+p/2}{\sqrt{q-p^2/4}}}{\left(\left(\frac{x+p/2}{\sqrt{q-p^2/4}}\right)^2+1\right)^s} = \left[t = \frac{x+p/2}{\sqrt{q-p^2/4}}\right]$$

$$= \frac{1}{(q-p^2/4)^{s-1/2}} \int \frac{\mathrm{d}t}{(t^2+1)^s}$$

Integrating by parts, we get

$$I_s = \int \frac{\mathrm{d}t}{(t^2+1)^s} = \frac{t}{(t^2+1)^s} + 2s \int \frac{t^2\,\mathrm{d}t}{(t^2+1)^{s+1}}$$

$$= \frac{t}{(t^2+1)^s} + 2s\left(\int \frac{\mathrm{d}t}{(t^2+1)^s} - \int \frac{\mathrm{d}t}{(t^2+1)^{s+1}}\right) = \frac{t}{(t^2+1)^s} + 2s(I_s - I_{s+1})$$

Therefore, we obtain the recursive formula for I_s:

$$I_{s+1} = \frac{t}{2s(t^2+1)^s} + \frac{2s-1}{2s} I_s, \quad s \in \mathbb{N}, \qquad I_1 = \int \frac{dt}{t^2+1} = \arctan t + c.$$

Example. $I = \int \frac{3x^2 - x + 2}{(x^2+1)^2(x-1)}\,\mathrm{d}x$. We represent the integrand in the form

$$\frac{3x^2 - x + 2}{(x^2+1)^2(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Equating the nominators

$$3x^2 - x + 2 = A(x^2+1)^2 + (Bx+C)(x-1)(x^2+1) + (Dx+E)(x-1).$$

Equating the coefficients, $\begin{cases} A + B = 0, \\ -B + C = 0, \\ 2A - C + D + B = 3, \\ C - B + E - D = -1, \\ A - C - E = 2, \end{cases} \Leftrightarrow \begin{cases} A = 1, \\ B = -1, \\ C = -1, \\ D = 1, \\ E = 0. \end{cases}$

$$I = \int \frac{\mathrm{d}x}{x-1} - \int \frac{x+1}{x^2+1}\,\mathrm{d}x + \int \frac{x\,\mathrm{d}x}{(x^2+1)^2} = \log|x-1| - \frac{1}{2}\int \frac{\mathrm{d}(x^2+1)}{x^2+1}$$

$$- \int \frac{\mathrm{d}x}{x^2+1} + \frac{1}{2}\int \frac{\mathrm{d}(x^2+1)}{(x^2+1)^2}$$

$$= \log|x-1| - \frac{1}{2}\log|x^2+1| - \arctan x - \frac{1}{2(x^2+1)} + c.$$

Example. If the denominator has only real roots, there is another way to find undetermined coefficients. $I = \int \frac{x\,\mathrm{d}x}{x^3 - 3x + 2}$.

$$\frac{x}{x^3 - 3x + 2} = \frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)} + \frac{C}{x+2}$$

$$\frac{x}{x+2} = A + B(x-1) + C(x+2)(x-1)^2 \Rightarrow A = \frac{x}{x+2}\Big|_{x=1} = \frac{1}{3}$$

$$B = \left(\frac{x}{x+2}\right)'\Big|_{x=1} = \frac{2}{(x+2)^2}\Big|_{x=1} = \frac{2}{9}$$

$$\frac{x}{(x-1)^2} = \frac{A(x+2)}{(x-1)^2} + \frac{B(x+2)}{(x-1)} + C \Rightarrow C = \frac{x}{(x-1)^2}\Big|_{x=-2} = -\frac{2}{9}$$

$$I = \frac{1}{3}\int \frac{\mathrm{d}x}{(x-1)^2} + \frac{2}{9}\int \frac{\mathrm{d}x}{x-1} - \frac{2}{9}\int \frac{\mathrm{d}x}{x+2} = -\frac{1}{3(x-1)} + \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| + c.$$

The Ostrogradsky method of integration

Suppose P, Q are polynomials and $deg(P) < deg(Q)$, then

$$\int \frac{P}{Q} = \frac{P_1}{Q_1} + \int \frac{P_2}{Q_2},$$

where Q_1 is the greatest common divisor of Q and its derivative Q', and $Q_2 := Q/Q_1$, P_1/Q_1 and P_2/Q_2 are proper fractions. Undetermined coefficients of polynomials P_1 and P_2 are calculated by differentiating the above integral identity called the **Ostrogradsky formula**. Thus, if

$$Q(x) = C \prod_{i=1}^{n} (x - a_i)^{k_i} \cdot \prod_{i=1}^{m} (x^2 + p_i x + q_i)^{l_i}$$

then

$$Q_1(x) = C \prod_{i=1}^{n} (x - a_i)^{k_i - 1} \cdot \prod_{i=1}^{m} (x^2 + p_i x + q_i)^{l_i - 1},$$

$$Q_2(x) = \prod_{i=1}^{n} (x - a_i) \cdot \prod_{i=1}^{m} (x^2 + p_i x + q_i).$$

Example. $I = \int \frac{\mathrm{d}x}{(x^3+1)^2}$.

$$\int \frac{\mathrm{d}x}{(x^3+1)^2}\,\mathrm{d}x = \frac{Ax^2+Bx+C}{x^3+1} + D\int \frac{\mathrm{d}x}{x+1} + \int \frac{Ex+F}{x^2-x+1}\,\mathrm{d}x.$$

$$\frac{1}{(x^3+1)^2} = \left(\frac{Ax^2+Bx+C}{x^3+1}\right)' + \frac{D}{x+1} + \frac{Ex+F}{x^2-x+1}.$$

$$1 = -Ax^4-2Bx^3-3Cx^2+2Ax+B+D(x^5-x^4+x^3+x^2-x+1)+(Ex+F)(x^4+x^3+x+1)$$

$$\begin{cases} D+E=0, \\ -A-D+E+F=0, \\ -2B+D+F=0, \\ -3C+D+E=0, \\ 2A-D+E+F=0, \\ B+D+F=1, \end{cases} \Leftrightarrow \begin{cases} A=0, \\ B=\frac{1}{3}, \\ C=0, \\ D=\frac{2}{9}, \\ E=-\frac{2}{9}, \\ F=\frac{4}{9}. \end{cases}$$

$$I = \frac{x}{3(x^3+1)} + \frac{2}{9}\log|x+1| - \frac{2}{9}\int \frac{x-2}{x^2-x+1}\,\mathrm{d}x.$$

$$
\begin{aligned}
2\int \frac{x-2}{x^2-x+1}\,\mathrm{d}x &= \int \frac{2x-1}{x^2-x+1}\,\mathrm{d}x - 3\int \frac{\mathrm{d}x}{x^2-x+1} \\
&= \log\left|x^2-x+1\right| - 3\int \frac{\mathrm{d}x}{(x-1/2)^2+3/4} \\
&= \log\left|x^2-x+1\right| - 4\int \frac{\mathrm{d}x}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} \\
&= \log\left|x^2-x+1\right| - 2\sqrt{3}\int \frac{\mathrm{d}\left(\frac{2x-1}{\sqrt{3}}\right)}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} \\
&= \log\left|x^2-x+1\right| - 2\sqrt{3}\arctan\frac{2x-1}{\sqrt{3}} + c.
\end{aligned}
$$

$$
I = \frac{x}{3(x^3+1)} + \frac{1}{9}\log\frac{(x+1)^2}{x^2-x+1} + \frac{2}{3\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + c.
$$

Example. For what condition $I = \int \frac{\alpha x^2 + 2\beta x + \gamma}{(ax^2+2bx+c)^2}\,\mathrm{d}x$ is rational?

The case $ax^2 + 2bx + c = a(x - x_1)^2$ is excluded. It is included into the homework.

$$I = \frac{Ax+B}{ax^2+2bx+c} + \int \frac{Cx+D}{ax^2+2bx+c}\,\mathrm{d}x$$

$$I \text{ is rational} \Leftrightarrow C = D = 0$$

$$\frac{\alpha x^2 + 2\beta x + \gamma}{(ax^2+2bx+c)^2} = \frac{A(ax^2+2bx+c) - (Ax+B)(2ax+2bx)}{(ax^2+2bx+c)^2}$$

$$\begin{cases} -Aa = \alpha, \\ -Ba = \beta, \\ Ac - 2Bb = \gamma. \end{cases} \Leftrightarrow \begin{cases} A = -\frac{\alpha}{a}, \\ B = -\frac{\beta}{a}, \\ -\frac{\alpha c}{a} + 2\frac{\beta b}{a} = \gamma. \end{cases} \qquad a\gamma + \alpha c = 2\beta b.$$

Thus, any primitive of a rational function is a linear combination of a rational function, arctan**, and** log .

Integrals of the form $\int R(\cos x, \sin x)\,\mathrm{d}x$

1. Change of variable $t = \tan(x/2)$, $x \neq \pi + 2\pi n$, $n \in \mathbb{Z}$.

$$\cos x = \frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}, \qquad \sin x = \frac{2\tan(x/2)}{1+\tan^2(x/2)},$$

$$\mathrm{d}t = (\tan(x/2))'\,\mathrm{d}x = \frac{\mathrm{d}x}{2\cos^2(x/2)},$$

$$\mathrm{d}x = 2\cos^2(x/2)\,\mathrm{d}t = \frac{2\,\mathrm{d}t}{1+\tan^2(x/2)}.$$

$$\int R(\cos x, \sin x)\,\mathrm{d}x = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)\frac{2}{t^2+1}\,\mathrm{d}t.$$

Example. $I = \displaystyle\int \frac{\mathrm{d}x}{2\sin x - \cos x + 5}, \quad 2\pi n - \pi < x < 2\pi n + \pi, \quad n \in \mathbb{Z}.$

$$I = \int \frac{\frac{2}{1+t^2}\,\mathrm{d}t}{2\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 5} = \int \frac{\mathrm{d}t}{3t^2 + 2t + 2}$$

$$= \frac{1}{\sqrt{5}} \arctan \frac{3t+1}{\sqrt{5}} + c_n = \frac{1}{\sqrt{5}} \arctan \frac{3tan(x/2)+1}{\sqrt{5}} + c_n.$$

By the continuity, $I(2\pi n + \pi - 0) = I(2\pi n + \pi + 0)$,

$$\frac{\pi}{2\sqrt{5}} + c_n = \frac{-\pi}{2\sqrt{5}} + c_{n+1} \Rightarrow c_n = \frac{\pi n}{\sqrt{5}} + c.$$

$$2\pi n - \pi < x < 2\pi n + \pi \Rightarrow \frac{x+\pi}{2\pi} - 1 < n < \frac{x+\pi}{2\pi} \Rightarrow n = \left\lfloor \frac{x+\pi}{2\pi} \right\rfloor$$

$$I = \frac{1}{\sqrt{5}} \arctan \frac{3tan(x/2)+1}{\sqrt{5}} + \frac{\pi}{\sqrt{5}} \left\lfloor \frac{x+\pi}{2\pi} \right\rfloor + c.$$

2. If $R(u,v) = R(-u,v)$, then there exists a rational function R_1 such that
$$R(u,v) = R_1(u^2,v). \tag{1}$$
If
$$R(-u,v) = -R(u,v),$$
then there exists a rational function R_2 such that
$$R(u,v) = R_2(u^2,v)u.$$
It is sufficient to apply (1) to the function $R(u,v)/u$.
$$\int R(\cos x, \sin x)\,\mathrm{d}x = \int R_2(\cos^2 x, \sin x)\cos x\,\mathrm{d}x$$
$$= \int R_2(1-\sin^2 x, \sin x)\,\mathrm{d}\,(\sin x)\,.$$
So, the substitution $t = \sin x$ rationalizes the integral.
If
$$R(u,-v) = -R(u,v),$$
then analogously the substitution $t = \cos x$ rationalizes the integral.

Example. $\displaystyle\int \frac{\cos x \, \mathrm{d}x}{\cos^4 x + \sin^4 x + 2\sin^2 x + 1}$

$$= \int \frac{\mathrm{d}\sin x}{(1-\sin^2 x)^2 + \sin^4 x + 2\sin^2 x + 1} = [t = \sin x] = \frac{1}{2}\int \frac{\mathrm{d}t}{t^4+1}$$

$$= \frac{1}{4\sqrt{2}}\int \frac{t+\sqrt{2}}{t^2+t\sqrt{2}+1}\,\mathrm{d}t - \frac{1}{4\sqrt{2}}\int \frac{t-\sqrt{2}}{t^2-t\sqrt{2}+1}\,\mathrm{d}t$$

$$= \frac{1}{4\sqrt{2}}\int \frac{t+\sqrt{2}/2}{t^2+t\sqrt{2}+1}\,\mathrm{d}t + \frac{1}{8}\int \frac{1}{(t+\sqrt{2}/2)^2+1/2}\,\mathrm{d}t$$

$$-\frac{1}{4\sqrt{2}}\int \frac{t-\sqrt{2}/2}{t^2-t\sqrt{2}+1}\,\mathrm{d}t + \frac{1}{8}\int \frac{1}{(t-\sqrt{2}/2)^2+1/2}\,\mathrm{d}t$$

$$= \frac{1}{8\sqrt{2}}\log\frac{\sin^2 x + \sqrt{2}\sin x + 1}{\sin^2 x - \sqrt{2}\sin x + 1}$$

$$+\frac{1}{4\sqrt{2}}\left(\arctan\left(\sqrt{2}\sin x + 1\right) + \arctan\left(\sqrt{2}\sin x - 1\right)\right) + c.$$

Let
$$R(-u,-v)=R(u,v), \tag{2}$$
then
$$R(u,v)=R\left(u,\frac{v}{u}u\right)=R_3\left(u,\frac{v}{u}\right)\underbrace{=}_{(2)}R_3\left(-u,\frac{v}{u}\right)\underbrace{=}_{(1)}R_4\left(u^2,\frac{v}{u}\right).$$
$$\int R(\cos x,\sin x)\,\mathrm{d}x=\int R_4\left(\cos^2 x,\frac{\sin x}{\cos x}\right)\,\mathrm{d}x$$
$$=\int R_4\left(\frac{1}{1+\tan^2 x},\tan x\right)\,\mathrm{d}x=\left[t=\tan x,\ \ \mathrm{d}t=\frac{\mathrm{d}x}{\cos^2 x},\ \ \mathrm{d}x=\frac{\mathrm{d}t}{1+t^2}\right]$$
$$=\int R_5(t)\frac{\mathrm{d}t}{1+t^2}$$
So, the substitution $t=\tan x$ rationalizes the integral.

Example. $\displaystyle\int\frac{\sin x\,\mathrm{d}x}{\cos^2 x(\sin x+\cos x)}=\int\frac{\tan x\,\mathrm{d}x}{\cos^2 x(\tan x+1)}$
$$=\int\frac{\tan x\,\mathrm{d}\tan x}{\tan x+1}=\tan x-\log|\tan x+1|+c.$$

The substitutions $t = \cos x$, $t = \sin x$, $t = \tan x$ are sufficient to rationalize **any** integral $\int R(\cos x, \sin x)\,\mathrm{d}x$.

$$R(u,v) = \frac{R(u,v) - R(-u,v)}{2} + \frac{R(-u,v) - R(-u,-v)}{2}$$

$$+\frac{R(-u,-v) + R(u,v)}{2} =: R_{01}(u,v) + R_{02}(u,v) + R_{03}(u,v).$$

$$R_{01}(-u,v) = -R_{01}(u,v), \quad R_{02}(u,-v) = -R_{02}(u,v),$$

$$R_{03}(-u,-v) = R_{03}(u,v).$$

$$R(\cos x, \sin x) = \underbrace{R_{01}(\cos x, \sin x)}_{t=\sin x} + \underbrace{R_{02}(\cos x, \sin x)}_{t=\cos x} + \underbrace{R_{03}(\cos x, \sin x)}_{t=\tan x}.$$

Example. Let us prove that

$$I = \int \frac{a_1 \sin x + b_1 \cos x}{a \sin x + b \cos x}\,\mathrm{d}x = Ax + B \log|a \sin x + b \cos x| + C.$$

$$\exists \quad A,\ B \quad a_1 \sin x + b_1 \cos x = A\,(a \sin x + b \cos x) + B\,(a \cos x - b \sin x)$$

$$\begin{cases} Aa - Bb = a_1 \\ Ab + Ba = b_1 \end{cases} \Leftrightarrow \begin{cases} A = \frac{a_1 a + b_1 b}{a^2 + b^2} \\ B = \frac{a b_1 - a_1 b}{a^2 + b^2} \end{cases}$$

$$I = A \int \frac{a \sin x + b \cos x}{a \sin x + b \cos x}\,\mathrm{d}x + B \int \frac{\mathrm{d}(a \sin x + b \cos x)}{a \sin x + b \cos x}\,\mathrm{d}x.$$

Example. $I = \int \cos x \cos 3x \cos 2x \,\mathrm{d}x = \frac{1}{2} \int (\cos 4x + \cos 2x) \cos 2x \,\mathrm{d}x$

$$= \frac{1}{4} \int (\cos 6x + \cos 2x + \cos 4x + 1)\ \mathrm{d}x$$

$$= \frac{1}{4} \left(x + \frac{1}{6} \sin 6x + \frac{1}{4} \sin 4x + \frac{1}{2} \sin 2x \right) + C.$$

Example.

$$I = \int \frac{\mathrm{d}x}{\sin(x+a)\sin(x+b)} = \frac{1}{\sin(a-b)} \int \frac{\sin\left((x+a)-(x+b)\right)}{\sin(x+a)\sin(x+b)} \,\mathrm{d}x$$

$$= \frac{1}{\sin(a-b)} \left(\int \frac{\cos(x+b)}{\sin(x+b)} \,\mathrm{d}x - \int \frac{\cos(x+a)}{\sin(x+a)} \,\mathrm{d}x \right)$$

$$= \frac{1}{\sin(a-b)} \log \left| \frac{\sin(x+b)}{\sin(x+a)} \right| + C. \quad a \neq b.$$

Example. $n \in \mathbb{Z}_+$

$$I_n = \int \sin^n x \, \mathrm{d}x = -\int \sin^{n-1} x \, \mathrm{d}(\cos x) = -\cos x \sin^{n-1} x$$

$$+(n-1) \int \sin^{n-2} x \cos^2 x \, \mathrm{d}x = -\cos x \sin^{n-1} x$$

$$+(n-1) \int \sin^{n-2} x (1 - \sin^2 x) \, \mathrm{d}x = -\cos x \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n,$$

$$I_n = \frac{1}{n} \left((n-1) I_{n-2} - \cos x \sin^{n-1} x \right).$$

$$I_0 = \int \mathrm{d}x = x + C, \quad I_1 = \int \sin x \, \mathrm{d}x = -\cos x + C.$$

Integrals of the form $\int R\left(x, \sqrt[m]{\dfrac{\alpha x+\beta}{\gamma x+\delta}}\right) \mathrm{d}x$

Suppose $t^m = \dfrac{\alpha x+\beta}{\gamma x+\delta}$, then $x = \dfrac{\delta t^m - \beta}{\alpha - \gamma t^m}$, $\quad \mathrm{d}x = \dfrac{\alpha\delta - \beta\gamma}{(\alpha - \gamma t^m)^2} m t^{m-1}\, \mathrm{d}t.$

$$\int R\left(x, \sqrt[m]{\frac{\alpha x+\beta}{\gamma x+\delta}}\right) \mathrm{d}x = \int R\left(\frac{\delta t^m - \beta}{\alpha - \gamma t^m}, t\right) \frac{\alpha\delta - \beta\gamma}{(\alpha - \gamma t^m)^2} m t^{m-1}\, \mathrm{d}t.$$

Example. $\displaystyle \int \frac{\mathrm{d}x}{\sqrt[4]{(x-1)^3(x+2)^5}} = \int \sqrt[4]{\left(\frac{x+2}{x-1}\right)^3} \frac{\mathrm{d}x}{(x+2)^2}$ 为什么提取(x+2)

$$= \left[\frac{x+2}{x-1} = t^4,\ x = \frac{t^4+2}{t^4-1},\ x+2 = \frac{3t^4}{t^4-1},\ \mathrm{d}x = \frac{-12t^3\, \mathrm{d}t}{(t^4-1)^2}\right]$$

$$= \int t^3 \frac{(t^4-1)^2}{9t^8} \frac{-12t^3}{(t^4-1)^2} \mathrm{d}t = -\frac{4}{3}\int \frac{\mathrm{d}t}{t^2} = \frac{4}{3t} + c = \frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + c.$$

Integrals of the form $\int R\left(x, \sqrt{ax^2+bx+c}\right)\,\mathrm{d}x$

We denote $Y := ax^2 + bx + c$, $y := \sqrt{Y}$. Replacing $y^2 = Y$, we get

$$R(x,y) = \frac{P_1(x)+P_2(x)y}{P_3(x)+P_4(x)y} = \frac{(P_1(x)+P_2(x)y)(P_3(x)-P_4(x)y)}{(P_3(x)+P_4(x)y)(P_3(x)-P_4(x)y)}$$

$$= R_1(x) + R_2(x)y = R_1(x) + R_3(x)\frac{1}{y}.$$

We select the quotient of a rational function $R_3(x)$, a polynomial $T(x)$,

$$R_3(x) = T(x) + \frac{Q(x)}{S(x)},$$

where $degQ < degS$, and Q/S is a proper rational function. We decompose a rational function Q/S into a sum of partial fractions. So, it is sufficient to integrate the following three types of functions.

A. $\frac{P(x)}{\sqrt{ax^2+bx+c}}$, P is a polynomial;

B. $\frac{1}{(x-x_0)^k\sqrt{ax^2+bx+c}}$, $k \in \mathbb{N}$;

C. $\frac{Ax+B}{(x^2+px+q)^m\sqrt{ax^2+bx+c}}$, $A, B, p, q \in \mathbb{R}$, $m \in \mathbb{N}$, $p^2 - 4q < 0$.

Type A. There exists a polynomial $Q(x)$ with $deg(Q) < deg(P)$ and a constant λ such that

$$\int \frac{P(x)}{\sqrt{ax^2+bx+c}}dx = Q(x)\sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

Example.

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}\,\mathrm{d}x = (Ax^2+Bx+C)\sqrt{1+2x-x^2} + \lambda \int \frac{\mathrm{d}x}{\sqrt{2-(x-1)^2}}.$$

$$\frac{x^3}{\sqrt{1+2x-x^2}} = \left((Ax^2+Bx+C)\sqrt{1+2x-x^2}\right)' + \frac{\lambda}{\sqrt{2-(x-1)^2}}.$$

$$x^3 = (2Ax+B)(1+2x-x^2) + (Ax^2+Bx+C)(1-x) + \lambda.$$

$$\begin{cases} -3A = 1, \\ 5A - 2B = 0, \\ 2A + 3B - C = 0, \\ B + C + \lambda = 0, \end{cases} \Leftrightarrow \begin{cases} A = -1/3, \\ B = -5/6, \\ C = -19/6, \\ \lambda = 4. \end{cases}$$

$$I = -\frac{2x^2+5x+19}{6}\sqrt{1+2x-x^2} + 4\arcsin\frac{x-1}{\sqrt{2}} + c.$$

Type B. $\displaystyle\int \frac{\mathrm{d}x}{(x-x_0)^k\sqrt{ax^2+bx+c}}$, $k \in \mathbb{N}$, can be reduced to an integral of type A by the change of variable $t = \frac{1}{x-x_0}$.

Example. $$I = \int \frac{\mathrm{d}x}{x^3\sqrt{x^2+1}} = \left[x = \frac{1}{t},\ \ \mathrm{d}x = -\frac{\mathrm{d}t}{t^2},\ \ t > 0\right]$$

$$= -\int \frac{t^3\,\mathrm{d}t}{t^2\sqrt{1+1/t^2}} = -\int \frac{t^2\,\mathrm{d}t}{\sqrt{t^2+1}} = -\int \frac{t^2+1-1}{\sqrt{t^2+1}}\,\mathrm{d}t$$

$$= -\underbrace{\int \sqrt{t^2+1}\,\mathrm{d}t}_{=:J} + \int \frac{\mathrm{d}t}{\sqrt{t^2+1}} = -J + \log\left|t+\sqrt{t^2+1}\right|.$$

$$J = \int \sqrt{t^2+1}\,\mathrm{d}t = t\sqrt{t^2+1} - \int \frac{t^2\,\mathrm{d}t}{\sqrt{t^2+1}} = t\sqrt{t^2+1} + I.$$

$$\begin{cases} I + J = \log\left|t+\sqrt{t^2+1}\right|, \\ -I + J = t\sqrt{t^2+1}. \end{cases} \qquad \begin{aligned} J &= \frac{1}{2}\left(\frac{\sqrt{x^2+1}}{x^2} + \log\left|\frac{x+\sqrt{x^2+1}}{x}\right|\right) + c, \\ I &= \frac{1}{2}\left(-\frac{\sqrt{x^2+1}}{x^2} + \log\left|\frac{x+\sqrt{x^2+1}}{x}\right|\right) + c. \end{aligned}$$

Type C. Consider an integral $\displaystyle\int \frac{Ax+B}{(x^2+px+q)^m\sqrt{ax^2+bx+c}}\,\mathrm{d}x$, where $A, B, p, q \in \mathbb{R}$, $m \in \mathbb{N}$, $p^2-4q<0$. There are two cases

1. $ax^2+bx+c = a(x^2+px+q)$,
2. $ax^2+bx+c \neq a(x^2+px+q)$.

Case 1. By $p^2-4q<0$, it follows that $a>0$. We get

$$\int \frac{Ax+B}{(x^2+px+q)^m\sqrt{ax^2+bx+c}}\,\mathrm{d}x = \frac{1}{\sqrt{a}}\int \frac{Ax+B}{(x^2+px+q)^{m+\frac{1}{2}}}\,\mathrm{d}x.$$

Let $Ax+B = \frac{A}{2}(2x+p)+B-\frac{Ap}{2}$. Then

$$\int \frac{Ax+B}{(x^2+px+q)^{m+\frac{1}{2}}}\,\mathrm{d}x = \frac{A}{2}\int \frac{d(x^2+px+q)}{(x^2+px+q)^{m+\frac{1}{2}}} + \int \frac{\left(B-\frac{Ap}{2}\right)\mathrm{d}x}{(x^2+px+q)^{m+\frac{1}{2}}}.$$

The first integral is from the list of basic integrals

$$\int \frac{\mathrm{d}(x^2+px+q)}{(x^2+px+q)^{m+\frac{1}{2}}} = \left(-m+\frac{1}{2}\right)^{-1}(x^2+px+q)^{-m+\frac{1}{2}}+c.$$

To calculate the second integral

$$\int \frac{\mathrm{d}x}{(x^2+px+q)^{m+1/2}}$$

we apply the Abel change of variable

$$t = \left(\sqrt{x^2+px+q}\right)' = \frac{2x+p}{2\sqrt{x^2+px+q}}.$$

We notice that $t\sqrt{x^2+px+q} = x + p/2$ and

$$\sqrt{x^2+px+q}\,\mathrm{d}t + t\underbrace{\left(\sqrt{x^2+px+q}\right)'}_{=t}\mathrm{d}x = \mathrm{d}x, \quad \frac{\mathrm{d}x}{\sqrt{x^2+px+q}} = \frac{\mathrm{d}t}{1-t^2},$$

and

$$x^2+px+q = \frac{q-p^2/4}{1-t^2}.$$

Finally, we reduce the problem to the integration of a polynomial

$$\int \frac{\mathrm{d}x}{(x^2+px+q)^{m+\frac{1}{2}}} = \left(q - \frac{p^2}{4}\right)^{-m}\int (1-t^2)^{m-1}\,\mathrm{d}t.$$

Case 2. $ax^2 + bx + c \neq a(x^2 + px + q)$. If $p \neq \frac{b}{a}$, we apply the linear fractional change of variable $x = \frac{\alpha t + \beta}{t + 1}$ to the integral $\int \frac{(Ax + B)\,\mathrm{d}x}{(x^2 + px + q)^m \sqrt{ax^2 + bx + c}}$, where $\alpha, \beta \in \mathbb{R}$ are chosen so that the linear terms disappear simultaneously in new quadratic polynomials. If $p = \frac{b}{a}$ and $q \neq \frac{c}{a}$, we apply the linear change of variable $x = t - \frac{p}{2}$. The result of these changes is the integral

$$\int \frac{(Mt + N)\,\mathrm{d}t}{(t^2 + \lambda)^m \sqrt{\delta t^2 + r}} = \int \frac{Mt\,\mathrm{d}t}{(t^2 + \lambda)^m \sqrt{\delta t^2 + r}} + \int \frac{N\,\mathrm{d}t}{(t^2 + \lambda)^m \sqrt{\delta t^2 + r}}.$$

To calculate the first integral in the RHS we apply the change of variable $u = \sqrt{\delta t^2 + r}$, for the second integral the Abel change is used $v = \left(\sqrt{\delta t^2 + r}\right)'$.

$$\int \frac{t\,\mathrm{d}t}{(t^2+\lambda)^m\sqrt{\delta t^2+r}} = \Bigg[u=\sqrt{\delta t^2+r},\ t^2+\lambda=\frac{u^2+\lambda\delta-r}{\delta},$$

$$\mathrm{d}u=\frac{\delta t}{\sqrt{\delta t^2+r}}\,\mathrm{d}t\Bigg] = \delta^{m-1}\int\frac{\mathrm{d}u}{(u^2+\lambda\delta-r)^m}.$$

$$\int \frac{\mathrm{d}t}{(t^2+\lambda)^m\sqrt{\delta t^2+r}} = \Bigg[v=\left(\sqrt{\delta t^2+r}\right)'=\frac{\delta t}{\sqrt{\delta t^2+r}},\ v\sqrt{\delta t^2+r}=\delta t,$$

$$\sqrt{\delta t^2+r}\,\mathrm{d}v+v\underbrace{\left(\sqrt{\delta t^2+r}\right)'}_{=v}\mathrm{d}t=\delta\,\mathrm{d}t,\qquad \frac{\mathrm{d}t}{\sqrt{\delta t^2+r}}=\frac{\mathrm{d}v}{\delta-v^2}$$

$$v^2=\frac{\delta^2t^2}{\delta t^2+r},\ t^2=\frac{r}{\delta}\frac{v^2}{\delta-v^2},\ t^2+\lambda=\frac{(r-\lambda\delta)v^2+\lambda\delta^2}{\delta(\delta-v^2)}\Bigg]$$

$$=\delta^m\int\frac{(\delta-v^2)^{m-1}}{((r-\lambda\delta)v^2+\lambda\delta^2)^m}\,\mathrm{d}v.$$

Example. $I = \displaystyle\int \frac{11x-13}{(x^2-x+1)\sqrt{x^2+1}}\,\mathrm{d}x.$ Let $x = \dfrac{\alpha t+\beta}{t+1}$, then

$$x^2-x+1 = \frac{\alpha^2t^2+2\alpha\beta t+\beta^2-(\alpha t^2+\alpha t+\beta t+\beta)+t^2+2t+1}{(t+1)^2},$$

$$x^2+1 = \frac{\alpha^2t^2+2\alpha\beta t+\beta^2+t^2+2t+1}{(t+1)^2},$$

$$\begin{cases} 2\alpha\beta-\alpha-\beta+2=0, \\ 2\alpha\beta+2=0, \end{cases} \Rightarrow \begin{cases} \alpha=1, \\ \beta=-1. \end{cases} \qquad x = \frac{t-1}{t+1}$$

$$x^2-x+1 = \frac{t^2+3}{(t+1)^2}, \quad x^2+1 = \frac{2t^2+2}{(t+1)^2}, \quad 11x-13 = \frac{-2t-24}{t+1},$$

$$\mathrm{d}x = \frac{2\,\mathrm{d}t}{(t+1)^2}. \quad \Rightarrow \quad I = -2\sqrt{2}\int \frac{(t+12)\,\mathrm{d}t}{(t^2+3)\sqrt{t^2+1}}.$$

$$\int \frac{t\,\mathrm{d}t}{(t^2+3)\sqrt{t^2+1}} = \int \frac{\mathrm{d}\sqrt{t^2+1}}{t^2+3} = \int \frac{\mathrm{d}u}{u^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}}+c$$

$$\int \frac{\mathrm{d}t}{(t^2+3)\sqrt{t^2+1}} = \left[v = \left(\sqrt{t^2+1}\right)', \ \frac{\mathrm{d}v}{1-v^2} = \frac{\mathrm{d}t}{\sqrt{t^2+1}}, \right.$$

$$\left. t^2+3 = \frac{3-2v^2}{1-v^2} \right] = \int \frac{\mathrm{d}v}{3-2v^2} = \frac{1}{2\sqrt{6}} \log \left| \frac{\sqrt{3}+v\sqrt{2}}{\sqrt{3}-v\sqrt{2}} \right| + c$$

$$= \frac{1}{2\sqrt{6}} \log \left| \frac{\sqrt{3t^2+3}+\sqrt{2}t}{\sqrt{3t^2+3}-\sqrt{2}t} \right| + c$$

$$I = -2\arctan \frac{\sqrt{t^2+1}}{\sqrt{2}} - 4\sqrt{3}\log \left| \frac{\sqrt{3t^2+3}+\sqrt{2}t}{\sqrt{3t^2+3}-\sqrt{2}t} \right| + c$$

Trigonometric method for $\int R\left(x, \sqrt{ax^2+bx+c}\right)\,\mathrm{d}x$

We select a full square in the quadratic function ax^2+bx+c and make a suitable linear substitution, then the integral is reduced to one of the following cases

$$\int R\left(t, \sqrt{t^2+1}\right)\,\mathrm{d}t, \quad \int R\left(t, \sqrt{t^2-1}\right)\,\mathrm{d}t, \quad \int R\left(t, \sqrt{1-t^2}\right)\,\mathrm{d}t.$$

Then we apply the substitutions. For the first integral

$$t = \tan x, \quad \text{or} \quad t = \sinh x,$$

for the second one

$$t = \frac{1}{\cos x}, \quad \text{or} \quad t = \cosh x,$$

for the third one

$$t = \sin x, \quad \text{or} \quad t = \cos x, \quad \text{or} \quad t = \tanh x.$$

Example. $I = \int \frac{\mathrm{d}x}{(2x+1)^2\sqrt{4x^2+4x+5}} = [t = 2x+1]$

$$= \frac{1}{2}\int \frac{\mathrm{d}t}{t^2\sqrt{t^2+4}} = \left[t = 2\sinh u,\ \ 4\sinh^2 u + 4 = 4\cosh^2 u\right]$$

$$= \frac{1}{8}\int \frac{\cosh u\,\mathrm{d}u}{\sinh^2 u\cosh u} = \frac{1}{8}\int \frac{\mathrm{d}u}{\sinh^2 u} = -\frac{1}{8}\coth u + c$$

$$= -\frac{\sqrt{1+\sinh^2 u}}{8\sinh u} + c = -\frac{\sqrt{t^2+4}}{8t} + c = -\frac{\sqrt{4x^2+4x+5}}{8(2x+1)} + c.$$

Integration of the differential binomial $\int x^m(a+bx^n)^p\,\mathrm{d}x$

$a, b \in \mathbb{R}$, $m, n, p \in \mathbb{Q}$.

1. If $p \in \mathbb{Z}$, $m = \dfrac{m_1}{m_2}$, $n = \dfrac{n_1}{n_2}$, then we apply a substitution $t = x^{1/k}$, where k is the least common multiple of m_2 and n_2.

Example. $I = \displaystyle\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,\mathrm{d}x = \left[p = -2 \in \mathbb{Z},\ x = t^6,\ \mathrm{d}x = 6t^5\,\mathrm{d}t\right] =$

$$6\int \frac{t^8\,\mathrm{d}t}{(1+t^2)^2} = 6\int \left(t^4 - 2t^2 + 3 - \frac{3(t^2+1)+t^2}{(1+t^2)^2}\right)\mathrm{d}t = \frac{6}{5}t^5 - 4t^3$$

$$+18t - 18\int \frac{\mathrm{d}t}{1+t^2} - 6\int \frac{t^2\,\mathrm{d}t}{(1+t^2)^2} = \frac{6}{5}t^5 - 4t^3 + 18t - 18\arctan t - 6J.$$

$$J = \int \frac{t^2\,\mathrm{d}t}{(1+t^2)^2} = -\frac{1}{2}\int t\,\mathrm{d}\left(\frac{1}{1+t^2}\right) = -\frac{t}{2(1+t^2)} + \frac{1}{2}\arctan t + C,$$

$$I = \frac{6}{5}\sqrt[6]{x^5} - 4\sqrt{x} + 18\sqrt[6]{x} + \frac{3\sqrt[6]{x}}{1+\sqrt[3]{x}} + 21\arctan\sqrt[6]{x} + C.$$

Integration of the differential binomial $\int x^m(a+bx^n)^p\,\mathrm{d}x$

$a, b \in \mathbb{R}$, $m, n, p \in \mathbb{Q}$

2. $\displaystyle\int x^m(a+bx^n)^p\,\mathrm{d}x \underbrace{=}_{t=x^n} \frac{1}{n}\int (a+bt)^p t^{\frac{m+1}{n}-1}\,\mathrm{d}t = \frac{1}{n}\int (a+bt)^p t^q\,\mathrm{d}t.$

If $q \in \mathbb{Z}$, then the integral is rationalized via the substitution

$$u = (a+bt)^{1/p_2} = (a+bx^n)^{1/p_2}, \text{ where } p = \frac{p_1}{p_2}.$$

Example.

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,\mathrm{d}x = \int x^{-\frac{1}{2}}\left(1+x^{\frac{1}{4}}\right)^{\frac{1}{3}}\,\mathrm{d}x = \left[\frac{m+1}{n} = \frac{-\frac{1}{2}+1}{\frac{1}{4}} = 2\right.$$

$$\left.\Rightarrow t = \sqrt[3]{1+\sqrt[4]{x}},\ x = (t^3-1)^4,\ \mathrm{d}x = 12t^2(t^3-1)^3\,\mathrm{d}t\right]$$

$$= 12\int (t^6 - t^3)\,\mathrm{d}t = \frac{12}{7}t^7 - 3t^4 + c = \frac{12}{7}\sqrt[3]{\left(1+\sqrt[4]{x}\right)^7} - 3\sqrt[3]{\left(1+\sqrt[4]{x}\right)^4} + c.$$

Integration of the differential binomial $\int x^m(a+bx^n)^p\,\mathrm{d}x$

$a, b \in \mathbb{R}$, $m, n, p \in \mathbb{Q}$

3. $\displaystyle\int x^m(a+bx^n)^p\,\mathrm{d}x \underbrace{=}_{t=x^n} \frac{1}{n}\int (a+bt)^p t^{\frac{m+1}{n}-1}\,\mathrm{d}t$

$$= \frac{1}{n}\int (a+bt)^p t^q\,\mathrm{d}t = \frac{1}{n}\int \left(\frac{a+bt}{t}\right)^p t^{p+q}\,\mathrm{d}t.$$

If $p+q \in \mathbb{Z}$, then the integral is rationalized via the substitution

$$u = \left(\frac{a+bt}{t}\right)^{1/p_2} = (ax^{-n}+b)^{1/p_2}.$$

Example. $\displaystyle\int \sqrt[3]{3x-x^3}\,\mathrm{d}x = \int x^{\frac{1}{3}}(3-x^2)^{\frac{1}{3}}\,\mathrm{d}x = \Big[m = \frac{1}{3}, n = 2, p = \frac{1}{3},$

$$\frac{m+1}{n} + p = 1, 3x^{-2} - 1 = u^3\Big] = \int x\left(3x^{-2}-1\right)^{\frac{1}{3}}\,\mathrm{d}x = -\frac{9}{2}\int \frac{u^3\,\mathrm{d}u}{(u^3+1)^2}$$

$$= \frac{3}{2}\int u\,\mathrm{d}\left(\frac{1}{u^3+1}\right) = \frac{3u}{2(u^3+1)} - \frac{3}{2}\int \frac{\mathrm{d}u}{u^3+1}$$

$$= \frac{3u}{2(u^3+1)} - \frac{1}{4}\log\frac{(u+1)^2}{u^2-u+1} - \frac{\sqrt{3}}{2}\arctan\frac{2u-1}{\sqrt{3}} + C,$$

where $u = \dfrac{\sqrt[3]{3x - x^3}}{x}$.

In all other cases, the integral of a differential binomial cannot be reduced to elementary functions (P.L. Chebyshev, 1853).