

Mar 15th

1. (max 6 points) From a deck of 52 cards, 6 cards are drawn.

What is the probability that among the drawn cards there will be

a) exactly three red cards, exactly one spade, and at least 2 queens;

b) exactly 4 red cards, exactly one heart, and exactly 2 aces;

c) exactly 2 black cards, exactly 2 diamonds, exactly 2 kings, and 1 ten.

a). $|N| = C_{52}^6$

$|A|$: 1) $\boxed{Q} \times 2$: 2 black

1 spade 1 red

1 club 1 red

2 red

$\boxed{\text{other}}$: 3 red + 1 club.

2 red + 2 club.

2 red + 1 spade + 1 club

1 red + 1 spade + 2 club.

$$C_{24}^3 \cdot C_{12}^1$$

$$C_{24}^2 \cdot C_{12}^2 \cdot C_2^1$$

$$C_{24}^2 \cdot C_{12}^1 \cdot C_{12}^1 \cdot C_2^1$$

$$C_{24}^1 \cdot C_{12}^1 \cdot C_{12}^2$$

2) $\boxed{Q} \times 3$: without heart

diamond

spade

club

$\boxed{\text{other}}$: 2 red. 1 club.

2 red 1 club

1 red 1 spade. 1 club

1 red 2 club.

$$C_{24}^2 \cdot C_{12}^1$$

$$C_{24}^2 \cdot C_{12}^1$$

$$C_{24}^1 \cdot C_{12}^1 \cdot C_{12}^1$$

$$C_{24}^1 \cdot C_{12}^2$$

3) $\boxed{Q} \times 4$

$\boxed{\text{other}}$ 1 red 1 club

$$C_{24}^1 \cdot C_{12}^1$$

$$P(A) = \frac{C_{24}^3 \cdot C_{12}^1 + C_{24}^2 \cdot C_{12}^2 \cdot C_2^1 + C_{24}^2 \cdot C_{12}^1 \cdot C_{12}^1 \cdot C_2^1 + C_{24}^1 \cdot C_{12}^1 \cdot C_{12}^2 + C_{24}^2 \cdot C_{12}^1 \cdot C_2^1 + C_{24}^1 \cdot C_{12}^1 \cdot C_{12}^1 + C_{24}^1 \cdot C_{12}^2 + C_{24}^1 \cdot C_{12}^1}{C_{52}^6}$$

$$= \frac{171168}{20358520} \approx 0.84\%$$

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b). $|B|$ 1) \boxed{A} . 2 black.

1 black. 1 ♠

1 black 1 ♠

2 red.

2) $\boxed{\text{other}}$ 1 ♠ 3 ♠.

1 black 3 ♠

1 black 1 ♠ 2 ♠

2 black 2 ♠

$$C_{12}^1 \cdot C_{12}^3$$

$$C_2^1 \cdot C_{24}^1 \cdot C_{12}^3$$

$$C_2^1 \cdot C_{24}^1 \cdot C_{12}^1 \cdot C_{12}^2$$

$$C_{24}^2 \cdot C_{12}^2$$

$$P(B) = \frac{C_{12}^1 \cdot C_{12}^3 + C_2^1 \cdot C_{24}^1 \cdot C_{12}^3 + C_2^1 \cdot C_{24}^1 \cdot C_{12}^1 \cdot C_{12}^2 + C_{24}^2 \cdot C_{12}^2}{C_{52}^6} = \frac{2640 + 10560 + 38016 + 1816}{20358520} = 0.34\%$$

c). 2 black 2 ♠ 2 ♠.

(1) 2 \boxed{K} 2 black.

(2) 1 $\boxed{10}$

$\left\{ \begin{array}{l} \spadesuit \\ \heartsuit \end{array} \right.$

(3) $\boxed{\text{other}}$

1 ♠ 2 ♠

1 ♠ 2 ♠

$$C_{11}^1 \cdot C_{11}^2$$

$$C_{11}^1 \cdot C_{11}^2$$

2 red.

$\left\{ \begin{array}{l} \spadesuit \text{ or } \heartsuit \\ \heartsuit \text{ or } \spadesuit \end{array} \right.$

1 black 1 ♠ 1 ♠.

2 black. 1 ♠ or 1 ♠.

$$C_2^1 \cdot C_{22}^1 \cdot C_{11}^1 \cdot C_{11}^1$$

$$C_2^1 \cdot C_{22}^2 \cdot C_{11}^1$$

1 black 1 ♠

$\left\{ \begin{array}{l} \text{black.} \\ \heartsuit \\ \spadesuit \end{array} \right.$

1 ♠ 2 ♠

1 black 2 ♠

1 black 1 ♠ 1 ♠

$$C_2^1 \cdot C_{12}^1 \cdot C_{11}^1 \cdot C_{11}^2$$

$$C_2^1 \cdot C_{22}^1 \cdot C_{11}^2$$

$$C_2^1 \cdot C_{22}^1 \cdot C_{11}^1 \cdot C_{11}^1$$

1 black 1 ♠

- same as 1 black 1 ♠.

$$P(B) = \frac{C_{11}^1 \cdot C_{11}^2 \times 2 + C_{22}^1 \cdot C_{11}^1 \cdot C_{11}^1 \times 2 + C_2^1 \cdot C_{22}^2 \cdot C_{11}^1 + C_2^1 \cdot C_2^1 [C_2^1 \cdot C_{11}^1 \cdot C_{11}^2 + C_{22}^1 \cdot C_{11}^2 + C_{22}^1 \cdot C_{11}^1 \cdot C_{11}^1]}{C_{52}^6}$$

$$= \frac{31944}{C_{52}^6} \approx 0.16\%$$

2. (max 4 points) Let there be 14 identical candies. These candies are randomly distributed among 5 different bags.

a) What is the probability that no bag will be empty?

b) What is the probability that no bag will be empty if some candies could have been eaten

0000... 00000 14 candy. bags be labeled 1-5.

↑ insert 4. bars. between them. we obtain 5 groups. → put in 5 bags.

$$|A| = \hat{C}_{15}^4 = C_{18}^4$$

a). $A = C_{13}^4$ (no repetition of places can be chosen for bars, the right and left end also not allowed).

$$P(A) = \frac{C_{13}^4}{C_{18}^4} = \frac{13 \times 12 \times 11 \times 10}{18 \times 17 \times 16 \times 15} = \frac{143}{612}$$

b). Assume n candies be eaten.

$$P(B) = \begin{cases} \frac{C_{13-n}^4}{C_{18-n}^4} & n \leq 9. \\ 0 & n \geq 10. \end{cases}$$

3. (max 6 points)

Two numbers (x, y) are chosen at random, the first from the interval $[-1, 7]$, the second from $[-3, 3]$.

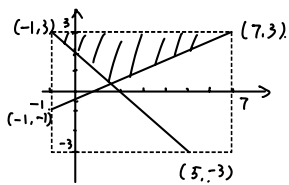
a) What is the probability $P(\{x+y > 2\} \cap \{x-2y < 1\})$?

b) What is the probability $P(\{xy^2 < 10\} | \{x > y\})$?

c) Which of the following events $y > 0$, $xy > 1$, $2y - 3x < 5$ are independent, and are they jointly independent?

$$S = b \times 8 = 48.$$

a)

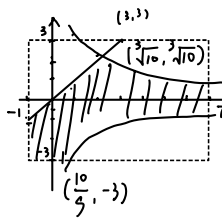


$$\begin{cases} x+y-2=0 \\ x-2y-1=0 \end{cases} \Rightarrow \begin{cases} x=\frac{5}{3} \\ y=\frac{1}{3} \end{cases}$$

$$|A| = S_A = \frac{1}{2} \cdot 8 \cdot (3 - \frac{1}{3}) = \frac{32}{3}$$

$$P(A) = \frac{|A|}{|M|} = \frac{2}{3}$$

b)



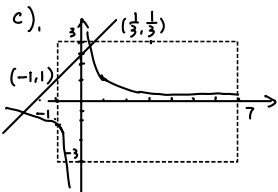
$$\begin{cases} y=\sqrt[3]{\frac{10}{x}} \\ y=x \end{cases} \Rightarrow x=\sqrt[3]{10}$$

$$\begin{aligned} S &= \int_{-1}^{\frac{10}{7}} (x+3) dx + \int_{\frac{10}{7}}^{\sqrt[3]{10}} (x+\sqrt[3]{\frac{10}{x}}) dx + \int_{\sqrt[3]{10}}^7 2\sqrt[3]{\frac{10}{x}} dx \\ &= \frac{x^2}{2} + 3x \Big|_{-1}^{\frac{10}{7}} + \frac{x^2}{2} + 2\sqrt[3]{10} \cdot x^{\frac{1}{2}} \Big|_{\frac{10}{7}}^{\sqrt[3]{10}} + 4\sqrt[3]{10} \cdot x^{\frac{1}{2}} \Big|_{\sqrt[3]{10}}^7 \\ &= \frac{50}{162} - \frac{1}{2} + 3 \cdot \frac{19}{9} + \frac{\sqrt[3]{100}}{2} - \frac{50}{162} + 2 \cdot 10^{\frac{1}{2}} \cdot 10^{\frac{1}{6}} - 2 \cdot 10^{\frac{1}{2}} \cdot 10^{\frac{1}{6}} \cdot \frac{1}{3} + 4\sqrt[3]{10} \cdot \sqrt{7} - 4\sqrt[3]{10} \cdot 10^{\frac{1}{6}} \\ &= -\frac{1}{2} + \frac{19}{3} - \frac{20}{3} + \frac{10^{\frac{2}{3}}}{2} + 2 \cdot 10^{\frac{2}{3}} - 4 \cdot 10^{\frac{2}{3}} + 4\sqrt[3]{10} \cdot \sqrt{7} = -\frac{5}{6} - \frac{3}{2} \cdot 10^{\frac{2}{3}} + 4\sqrt[3]{10} \cdot \sqrt{7} \end{aligned}$$

$$S(x > y) = 48 - \frac{1}{2} \cdot 4^2 = 40.$$

$$P(A|B) = \frac{-\frac{5}{6} - \frac{3}{2} \cdot 10^{\frac{2}{3}} + 4\sqrt[3]{10} \cdot \sqrt{7}}{40}$$

c)



$$C = \{y > 0\}, D = \{xy > 1\}, E = \{2y - 3x < 5\}.$$

$$\begin{cases} xy=1 \\ 2y-3x=5 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{3} \\ y=3 \end{cases}$$

$$|D| = \int_{\frac{1}{3}}^7 \frac{1}{x} dx + \int_{-\frac{1}{3}}^{\frac{1}{3}} \frac{1}{x} dx = 24 - \ln 21 - \ln 3$$

$$|C| = 24 \quad |E| = \frac{4}{3}$$

$$|D \cap E| = \int_{\frac{1}{3}}^7 3 - \frac{1}{x} dx = 20 - \ln 21.$$

$$|C \cap D| = \int_{\frac{1}{3}}^7 \frac{1}{x} dx = \ln 21$$

$$|C \cap E| = S_C - S_D = 21 - \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3}$$

we have $\begin{cases} P(D) \cdot P(E) \neq P(D \cap E) \\ P(C) \cdot P(D) \neq P(C \cap D) \\ P(C) \cdot P(E) \neq P(C \cap E) \end{cases}$
pairs are not mutually independent.
they are not jointly independent

4. (max 8 points)

- a) If $P(A) = 0.7$, $P(B) = 0.5$, show that $P(A|B) \geq 0.3$. Is it possible to improve this estimation?
 b) Prove that $P(C|D) \geq 1 - \frac{P(\bar{C})}{P(D)}$.
 c) Let $P(A) = p$, $P(B) = 1 - \epsilon$, for small values of ϵ . Find upper and lower bounds for $P(A|B)$.
 d) If A, B are independent and $P(A) = 0.6$, $P(AB) = 0.3$, find $P(A \setminus B | A \cup B)$.

$$a). P(A|B) = \frac{P(AB)}{P(B)} \geq \frac{P(A) + P(B) - 1}{P(B)} = \frac{0.7}{0.5} = 0.4 > 0.3.$$

the inequality holds since $P(A) + P(B) - P(AB) = P(A \cup B) \leq 1$.

$$b). P(C|D) = \frac{P(CD)}{P(D)} \geq \frac{P(C) + P(D) - 1}{P(D)} = \frac{P(D) - P(\bar{C})}{P(D)} = 1 - \frac{P(\bar{C})}{P(D)}.$$

c). by a) and b). $\max(0, P(A) + P(B) - 1) \leq P(AB) \leq \min(P(A), P(B))$.

$$\text{since } \epsilon \text{ is small. } p - \epsilon \leq P(AB) \leq p. \Rightarrow \frac{p - \epsilon}{1 - \epsilon} \leq P(A|B) \leq \frac{p}{1 - \epsilon}.$$

$$d). P(B) = \frac{P(AB)}{P(A)} = 0.5 \text{ since } A, B \text{ independent.}$$

$$P(\overline{A|B} | A \cup B) = \frac{P(\overline{A|B} \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cup B) - P(A \cap B)}{P(A) + P(B) - P(AB)} = \frac{0.8 - [P(A \cap (A \cup B)) - P(AB \cap (A \cup B))]}{0.8} = \frac{0.8 - (P(A) - P(AB))}{0.8} = \frac{5}{8}$$

5. (max 6 points)

A bag contains 45 candies, including chocolate, nut, mint, and wafer candies, which are distributed in the ratio 5:3:4:3. Children take turns picking one candy at a time.

- a) What is the probability that the third candy taken is either mint or nut?
 b) What is the probability that the first two candies are the same type, given that the third candy was chocolate?
 c) What is the probability that the fourth candy taken is a wafer, given that exactly one wafer candy was taken among the first three?

a). Consider the first two candy.

denote $H_1 = \{\text{no nut/mint}\}$ $H_2 = \{1 \text{ nut} + 1 \text{ wafer/cho}\}$ $H_3 = \{1 \text{ mint} + 1 \text{ wafer/cho}\}$ $H_4 = \{2 \text{ nut}\}$ $H_5 = \{2 \text{ mint}\}$ $H_6 = \{1 \text{ nut} / 1 \text{ mint}\}$

$$P(A) = \sum P(H_i) P(A|H_i) = \frac{C_{21}^1}{C_{45}^1} \cdot \frac{C_{24}^2}{C_{45}^2} + \frac{C_{20}^1}{C_{43}^1} \cdot \left[\frac{C_9^1 \cdot C_{24}^1}{C_{45}^2} + \frac{C_{12}^1 \cdot C_{34}^1}{C_{45}^2} \right] + \frac{C_{19}^1}{C_{43}^1} \left[\frac{C_9^2}{C_{45}^2} + \frac{C_{12}^2}{C_{45}^2} + \frac{C_9^1 \cdot C_{12}^1}{C_{45}^2} \right]$$

$$= \frac{21 \times 12 \times 23 + 20 \times 9 \times 24 + 20 \times 12 \times 24 + 19 \times 9 \times 4 + 19 \times 11 \times 6 + 19 \times 12 \times 9}{43 \times 22 \times 45} = \frac{21 \times 12 \times 23 + 20 \times 21 \times 24 + 19 \times 21 \times 10}{43 \times 22 \times 45}$$

$$= \frac{946 \times 21}{43 \times 22 \times 45} = \frac{21}{45} = \frac{7}{15} \quad \text{the result will be the same as the situation that if we did not consider the 1st and 2nd Children. and directly compute } P(\text{Mint/nut}) = \frac{9+12}{45} = \frac{7}{15}.$$

b) A: 3rd is cho B: first two with same type

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{C_{15}^2 \cdot C_{13}^1 + C_9^2 \cdot C_{15}^1 + C_{12}^2 \cdot C_{15}^1 + C_9^2 \cdot C_{15}^1}{C_{45}^2 \cdot 43}}{15/45} = 3 \cdot \frac{15 \cdot 7 \cdot 13 + 9 \cdot 4 \cdot 15 + 6 \cdot 11 \cdot 15 + 9 \cdot 4 \cdot 15}{43 \cdot 22 \cdot 45} = \frac{72166 + 91}{43 \cdot 22} = \frac{229}{906} \approx 24.2\%$$

$$c). P(B|A) = \frac{9-1}{45-3} = \frac{8}{42} = \frac{4}{21}$$

6*. (max 2 points)

A random 9-digit number is chosen. What is the probability that the sum of its digits is greater than 74?

\Rightarrow I understand it means "strict"

Sol: the maximum sum of 9-digit is 81.

it suffices to put 0-6 "-1" on 9-digits (no risk for the 1st digit to be 0)

$$P = \frac{\sum_{n=0}^6 \tilde{C}_9^n}{9 \times 10^8} = \frac{1 + C_9^1 + C_{10}^2 + C_{11}^3 + C_{12}^4 + C_{13}^5 + C_{14}^6}{9 \times 10^8} = \frac{5005}{9 \times 10^8} \approx 5.56 \times 10^{-4} \%$$