Differential Geometry. Home Assignments

Problem 1

Find curvature, torsion, and Frenet frame for the curves expressed with equations

$$1) x = 1 - \sin t, \quad y = 1 - \cos t, \quad z = 4 \sin \frac{t}{2}$$

$$2) x = t - \sin t, \quad y = 1 - \cos t, \quad z = 4 \sin \frac{t}{2}$$

for point $t = 0$.

Problem 2

Find a singular curve of the surface

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u + \ln \tan \frac{u}{2}$$

Problem 3

Justify that normal lines (i.e. lines perpendicular to the tangent plane and emanating from the tangency point) of the surface

$$x = f(u) \cos v, \quad y = f(u) \sin v, \quad z = g(u)$$

intersect the Oz axis.

HW3. Week 13th.

Problem 1

Find curvature, torsion, and Frenet frame for the curves expressed with equations

$$1) x = 1 - \sin t, \quad y = 1 - \cos t, \quad z = 4\sin\frac{t}{2}$$

$$2) x = t - \sin t, \quad y = 1 - \cos t, \quad z = 4\sin\frac{t}{2}$$

for point $t = 0$.

$\vec{r}(t) = (x(t), y(t), z(t))$

1).

$$\vec{r}'(t) = (-\cos t, \sin t, 2\cos\tfrac{t}{2}) \qquad \vec{r}'(0) = (-1, 0, 2)$$

$$\vec{r}''(t) = (\sin t, \cos t, -\sin\tfrac{t}{2}) \xrightarrow{t=0} \vec{r}''(0) = (0, 1, 0)$$

$$\vec{r}'''(t) = (\cos t, -\sin t, -\tfrac{1}{2}\cos\tfrac{t}{2}) \qquad \vec{r}'''(0) = (1, 0, -\tfrac{1}{2})$$

$$k_1 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\begin{vmatrix} i & j & k \\ -1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix}}{|\sqrt{5}|^3} = \frac{|(-2, 0, -1)|}{5\sqrt{5}} = \frac{1}{5}.$$

$$k_2 = \frac{(\vec{r}', \vec{r}'', \vec{r}''')}{|\vec{r}'(t) \times \vec{r}''(t)|} = \frac{\begin{vmatrix} -1 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} \end{vmatrix}}{5} = \frac{\frac{1}{2} + (-1)\cdot 2.}{5} = -\frac{3}{10}.$$

$$\vec{\tau} = (-\tfrac{1}{\sqrt{5}}, 0, \tfrac{2}{\sqrt{5}}) \qquad \vec{n} = (0, 1, 0) \qquad \vec{b} = \vec{\tau} \times \vec{n} = \begin{vmatrix} i & j & k \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{vmatrix} = (-\tfrac{2}{\sqrt{5}}, 0, -\tfrac{1}{\sqrt{5}})$$

2)

$$\vec{r}'(t) = (1-\cos t, \sin t, 2\cos\tfrac{t}{2}) \qquad \vec{r}'(0) = (0, 0, 2)$$

$$\vec{r}''(t) = (\sin t, \cos t, -\sin\tfrac{t}{2}) \xrightarrow{t=0.} \vec{r}''(0) = (0, 1, 0)$$

$$\vec{r}'''(t) = (\cos t, -\sin t, -\tfrac{1}{2}\cos\tfrac{t}{2}) \qquad \vec{r}'''(0) = (1, 0, -\tfrac{1}{2})$$

$$k_1 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{\begin{vmatrix} i & j & k \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix}}{2^3} = \frac{|(-2, 0, 0)|}{8} = \frac{1}{4}.$$

$$k_2 = \frac{(\vec{r}', \vec{r}'', \vec{r}''')}{|\vec{r}'(t) \times \vec{r}''(t)|^2} = \frac{\begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{2} \end{vmatrix}}{4} = -\frac{1}{2}$$

$$\vec{\tau} = (0, 0, 1). \qquad \vec{n} = (0, 1, 0) \qquad \vec{b} = \vec{\tau} \times \vec{n} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = (-1, 0, 0)$$

Problem 2
Find a singular curve of the surface

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u + \ln \tan \frac{u}{2}$$

Sol: denote $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.

$\vec{r}_u = (\cos u \cos v, \cos u \sin v, -\sin u + \frac{1}{\tan\frac{u}{2}} \cdot \frac{1}{\cos^2\frac{u}{2}} \cdot \frac{1}{2}) = (\cos u \cos v, \cos u \sin v, -\sin u + \frac{1}{\sin u})$.

$\vec{r}_v = (-\sin u \sin v, \sin u \cos v, 0)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos u \cos v & \cos u \sin v & -\sin u + \frac{1}{\sin u} \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} = (\sin^2 u \cos v - \cos v, \sin^2 u \sin v - \sin v, \sin u \cos u).$$

denote $\vec{r}_u \times \vec{r}_v = \vec{0} \Rightarrow \{(u,v) \mid u = \frac{\pi}{2} + 2k\pi, v \in \mathbb{R}\}$.

thus. the curve is $\begin{cases} x = \cos v \\ y = \sin v \\ z = 0 \end{cases}$ .unit circle in xOy plane.

On this curve. we have $\operatorname{rank}\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix} < 2$. each point is singular.

Problem 3
Justify that normal lines (i.e. lines perpendicular to the tangent plane and emanating from the tangency point) of the surface

$$x = f(u) \cos v, \quad y = f(u) \sin v, \quad z = g(u)$$

intersect the Oz axis.

Sol: denote the $\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$.

the tangent vector. $\vec{r}_u = (f'(u) \cos v, f'(u) \sin v, g'(u))$

$\vec{r}_v = (-\sin v f(u), \cos v f(u), 0)$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ f'(u) \cos v & f'(u) \sin v & g'(u) \\ -\sin v f(u) & \cos v f(u) & 0 \end{vmatrix} = (-\cos v f(u) g'(u), -\sin v f(u) g'(u), f'(u) f(u))$$

emanate from $\vec{r}(u,v)$. : $\vec{l}(s) = \vec{r}(u,v) + s\,\vec{n}(u,v)$.

$= (\cos v f(1 - sg'), \sin v f(1 - sg'), g + s \cdot f'f)$.

intersects the Oz axis : $\begin{cases} \cos v f(1 - sg') = 0 \\ \sin v f(1 - sg') = 0. \end{cases} \Rightarrow s = \frac{1}{g'(u)}$.

if $g'(u) \equiv 0$. $g(u) = \text{const}$.

$l \parallel Oz$ or tangency point on Oz.

thus. normal line : $\vec{l}(s) = (\cos v f(1 - sg'), \sin v f(1 - sg'), g + s \cdot f'f)$.

intersects the Oz axis at : $(0, 0, g(u) + \frac{f(u) f'(u)}{g'(u)})$