Surface integrals

BACKGROUND INFORMATION

1. **Surface integral of the first kind.**

Let the surface S be given parametrically:

$$x = x(u; v), \quad y = y(u; v), \quad z = z(u; v), \quad (u; v) \in \bar{D} \qquad (1)$$

moreover, the functions $x(u; v), y(u; v), z(u; v)$ are differentiable in the measurable domain D. Let the function $f(x; y; z)$ be given on this surface.

The surface integral of the first kind $\iint_S f(x; y; z)dS$ from the function $f(x; y; z)$ over the surface S can be defined as follows:

$$\iint_S f(x; y; z)dS = \iint_D f(x(u; v); y(u; v); z(u; v))\sqrt{EG - F^2}dudv \qquad (2)$$

where

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2, \quad G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

If the integral function on the right side of equality (2) is continuous in D (in particular, if the function f is continuous on S, and the functions (1) are continuously differentiable in $\bar{D}$, then the integral $\iint_S f(x; y; z)dS$ obviously exists.

The surface integral can also be defined as the limit of the corresponding integral sums (see, for example, [3] or [4]).

If the surface S is given by the equation

$$z = z(x; y), \quad (x; y) \in \bar{D} \qquad (3)$$

where $z(x; y)$ is a function differentiable in D, then equality (2) takes the form

$$\iint_S f(x; y; z)dS = \iint_D f(x; y; z(x; y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}dxdy \qquad (4)$$

Often the surface S cannot be given in the form (3) or (1), but it can be divided into parts S_i so that each of the parts allows representation in the desired form. In such cases, the integral over the surface S is understood as the sum of integrals over its parts:

$$\iint_S f\,dS = \sum_{i=1}^n \iint_{S_i} f\,dS_i \qquad (5)$$

If $f(x; y; z)$ is the density of the mass distributed over the surface of S, then the integrals $(2), (4)$ give the mass of the entire surface.

The potential at the point M_0 of a simple layer distributed with density $\mu(x; y; z)$ on the surface of S is called the integral

$$V(x_0; y_0; z_0) = \iint_S \frac{\mu(x; y; z)}{r}\,dS$$

where r is the distance between the point $M(x; y; z)$ of the surface S and the point $M_0(x_0; y_0; z_0)$.

2. **Surface integrals of the second kind*).**

Let the surface S be given parametrically:

$$x = x(u; v), \quad y = y(u; v), \quad z = z(u; v), \quad (u; v) \in \bar{D} \qquad (6)$$

the functions $x(u; v), y(u; v), z(u; v)$ are continuously differentiable in D, and the rank of the matrix

$$\left\| \begin{array}{ccc} x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{array} \right\|$$

is equal to 2. At each point $(u; v)$ of such surface, there are two oppositely directed unit normal vectors, each of which is a continuous function of the point $(u; v)$ of the surface S. The choice of one of them is called the orientation of the surface. If the surface S is the boundary of a bounded region, then it is said that it can be oriented by external or internal (with respect to this region) normals. The surface S, oriented by an external normal, is called its outer side, and the oriented inner normal is called its inner side.

For an oriented surface S, a surface integral of the second kind is determined.

Let $\cos \alpha, \cos \alpha, \cos \gamma$ be the guiding cosines of the normal

$$\left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{array} \right|$$

to the surface (1). Let the surface S be oriented by the unit normal vector $(\cos \alpha; \cos \beta; \cos \gamma)$, and let the functions $P(x; y; z), Q(x; y; z), R(x; y; z)$. The surface integral of the second kind

$$\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy \qquad (6)$$

is defined through a surface integral of the first kind by the formula

2

$$\iint_S Pdydz + Qdzdx + Rdxdy = \iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS \qquad (7)$$

If the surface S is oriented in the opposite way, i.e. by the normal $(-\cos\alpha; -\cos\beta; -\cos\gamma)$, then only the sign of the surface integral changes.

For integral (6), the following formula holds:

$$\iint_S Pdydz + Qdzdx + Rdxdy = \iint_D \begin{vmatrix} P & Q & R \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{vmatrix} dudv \qquad (8)$$

In the special case $P = 0, Q = 0$, formula (8) has the form

$$\iint_S Rdxdy = \iint_D R(x(u;v); y(u;v); z(u;v))\frac{\partial(x,y)}{\partial(u,v)}dudv \qquad (9)$$

Formulas for integrals are written in the same way

$$\iint_S Pdydz, \quad \iint_S Qdzdx$$

If the surface S is given explicitly, then formula (9) is simplified.

Let, for example, the surface S is given by the equation

$$z = z(x;y), \quad (x;y) \in \bar{D} \qquad (10)$$

where $z(x;y)$ is a function continuously differentiable in $\bar{D}$. Then

$$\iint_S Rdxdy = \pm \iint_D R(x;y;z(x;y))dxdy \qquad (11)$$

where D is the projection of the surface S onto the plane $z = 0$.

Before the double integral in formula (11), a plus sign is taken if the surface S is oriented by normals forming an acute angle with the z axis, and a minus sign if the surface S is oriented by normals forming an obtuse angle with the z axis. In the first case, it is said that the integral is taken on the upper side of the surface, in the second - on its lower side.

If the surface S is not representable in the form of (10) or (1), but it can be divided into a finite number of parts, each of which is representable in this form, then the surface integral of the second kind over the surface S is understood as the sum of integrals over its parts.

3

EXAMPLES WITH SOLUTIONS

Example 1. Calculate the integral $\iint_S \frac{dS}{\sqrt{x^2+y^2+z^2}}$ if S is a part of a cylindrical surface

$$x = r\cos u, \quad y = r\sin u, \quad z = v; \quad 0 \leqslant u \leqslant 2\pi, \quad 0 \leqslant v \leqslant H.$$

Δ In this case, the formula (2) is applicable, and $E = r^2, G = 1, F = 0$. Therefore

$$\iint_S \frac{dS}{\sqrt{x^2+y^2+z^2}} = \int_0^{2\pi} \int_0^H \frac{r\,du\,dv}{\sqrt{r^2+v^2}} = 2\pi r \int_0^H \frac{dv}{\sqrt{r^2+v^2}} = 2\pi r \ln\frac{H+\sqrt{r^2+H^2}}{r}.$$

Example 2. Calculate the integral $I = \iint_S z^2 dS$, where S is the complete surface of the cone $\sqrt{x^2+y^2} \leqslant z \leqslant 2$.

Δ Let S_1 be the lateral surface of the cone, S_2 be its base; then

$$I = \iint_{S_1} z^2 dS_1 + \iint_{S_2} z^2 dS_2$$

We apply the formula (4) to the first integral. On the side surface of the cone

$$z = \sqrt{x^2+y^2}$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}, \quad \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{2}$$

Therefore,

$$\iint_{S_1} z^2 dS_1 = \iint_{x^2+y^2 \leqslant 4} \left(x^2+y^2\right)\sqrt{2}\,dxdy = \sqrt{2}\int_0^{2\pi}\int_0^2 r^3 dr d\varphi = 8\sqrt{2}\pi.$$

Based on the cone $z = 2$, so the second integral is equal to the fourfold area of the base of the cone $4\pi 2^2$. So, $I = 8\pi(2+\sqrt{2})$.

Example 3. Calculate the integral $\iint_S z\,dxdy$, where S is the lower side of the part of the conical surface $z^2 = x^2 + y^2, 0 < z \leqslant H$.

Δ The surface S is oriented by normals that make up an obtuse angle with the axis z. According to formula (11), taking the minus sign in it, we reduce the integral to a double, which we calculate by going to the polar coordinates:

$$\iint_S z\,dxdy = -\iint_{x^2+y^2 \leqslant H^2} \sqrt{x^2+y^2}\,dxdy = -\int_0^{2\pi} d\varphi \int_0^H r^2 dr = -\frac{2}{3}\pi H^3.$$

4

Example 4. Calculate integrals: a) $\iint_S z^2 dxdy$; b) $\iint_S zdxdy$; where S is a hemisphere $x^2 + y^2 + z^2 = R^2, y \geqslant 0$, oriented by an external normal.

$\blacktriangle$a) Let's split the surface S into parts S_1 and S_2, located respectively above and below the plane $z = 0$. Then

$$\iint_S z^2 dxdy = \iint_{S_1} z^2 dxdy + \iint_{S_2} z^2 dxdy$$

Surfaces S_1 and S_2 have the same projection D on the plane $z = 0$. According to formula (11) we obtain

$$\iint_{S_1} z^2 dxdy = \iint_D \left(R^2 - x^2 - y^2\right) dxdy$$

since the external normal to the surface S_1 forms an acute angle with the z axis;

$$\iint_{S_1} z^2 dxdy = -\iint_D \left(R^2 - x^2 - y^2\right) dxdy$$

since the external normal to the surface S_2 forms an obtuse angle with the axis z. Therefore,

$$\iint_S z^2 dxdy = 0$$

b) As in the case of a), splitting the surface S into parts S_1 and S_2 and applying formula (11), we get

$$\iint_{S_1} zdxdy = \iint_D \sqrt{R^2 - x^2 - y^2} dxdy$$

$$\iint_{S_2} zdxdy = -\iint_D \left(-\sqrt{R^2 - x^2 - y^2}\right) dxdy$$

Therefore,

$$\iint_S zdxdy = 2\iint_D \sqrt{R^2 - x^2 - y^2} dxdy = 2 \cdot \frac{\pi}{3} R^3 = \frac{2\pi}{3} R^3,$$

since the last integral is equal to the volume of the fourth part of the ball of radius R.

Example 5. Calculate the integral $K = \iint_S \frac{dydz}{x} + \frac{dzdx}{y} + \frac{dxdy}{z}$, where S is part of an ellipsoid

$$x = a\cos u \cos v, \quad y = b\sin u \cos v, \quad z = c\sin v$$
$$u \in [\pi/4; \pi/3], \quad v \in [\pi/6; \pi/4]$$

oriented by an external normal.

Δ Note that the functions $1/x, 1/y, 1/z$ are positive, and the angles formed by the external normal with the coordinate axes are sharp, so $K > 0$. Let's use the formula (8). Since

$$x'_u = -a \sin u \cos v, \quad y'_u = b \cos u \cos v, \quad z'_u = 0$$

that

$$x'_v = -a \cos u \sin v, \quad y'_v = -b \sin u \sin v, \quad z'_v = c \cos v$$

$$\begin{vmatrix} \frac{1}{x} & \frac{1}{y} & \frac{1}{z} \\ x'_u & y'_u & z'_u \\ x'_v & y'_v & z'_v \end{vmatrix} = \begin{vmatrix} \frac{1}{a \cos u \cos v} & \frac{1}{b \sin u \cos v} & \frac{1}{c \sin v} \\ -a \sin u \cos v & b \cos u \cos v & 0 \\ -a \cos u \sin v & -b \sin u \sin v & c \cos v \end{vmatrix} = p \cos v$$

where

$$p = \frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a}.$$

Therefore, by the formula (8) we get

$$K = p \int_{\pi/4}^{\pi/3} du \int_{\pi/6}^{\pi/4} \cos v\, dv = p\frac{\pi}{12} \left(\frac{\sqrt{2}}{2} - \frac{1}{2} \right) = \frac{\pi(\sqrt{2}-1)}{24} \left(\frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} \right)$$

TASKS

Calculate integrals.

1. $\iint (x + y + z)dS$, where:

   (a) S - the part of the plane $x + 2y + 4z = 4$ allocated by the conditions $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$;

   (b) S - the part of the sphere $x^2 + y^2 + z^2 = 1$ allocated by the condition $z \geqslant 0$.

2. $\iint (x^2 + y^2) dS$, where:

   (a) S - sphere $x^2 + y^2 + z^2 = R^2$;

   (b) S - cone surface $\sqrt{x^2 + y^2} \leqslant z \leqslant 1$.

3. $\iint_S (x^2 + y^2 + z^2) dS$, where:

   (a) S − sphere $x^2 + y^2 + z^2 = R^2$;

   (b) S - cube surface $|x| \leqslant a, |y| \leqslant a, |z| \leqslant a$;

   (c) S - octahedron surface $|x| + |y| + |z| \leqslant a$;

   (d) S - full cylinder surface $x^2 + y^2 \leqslant r^2, 0 \leqslant z \leqslant H$.

4. $\iint_S \frac{dS}{(1+x+y)^2}, S$ - the surface of the tetrahedron $x+y+z \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

5. (a) $\iint_S xyz dS$; (b) $\iint_S |xy|z dS$; where S is the part of the paraboloid $z = x^2 + y^2$ allocated by the condition $z \leqslant 1$.

6. (a) $\iint_S (x^2 + y^2) dS$; (b) $\iint_S \sqrt{x^2 + y^2} dS$; where S is the part of the conic surface $z = \sqrt{x^2 + y^2}$ allocated by the condition $z \leqslant 1$.

7. (a) $\iint_S (xy + yz + zx) dS$; (b) $\iint_S (x^2 y^2 + y^2 z^2 + z^2 x^2) dS$; where S is a part of the conical surface $z = \sqrt{x^2 + y^2}$ located inside the cylinder $x^2 + y^2 = 2x$.

8. (a) $\iint_S f(x; y; z) dS$;

   (b) $\iint_S \frac{dS}{f(x;y;z)}$;

   (c) $\iint_S (x^2 + y^2 + z^2)^{-3/2} \frac{dS}{f(x;y;z)}$;

   where $f = \sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}, S$ - ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

9. $\iint_S (x^2 + y^2 + (z - a)^2)^{-n/2} dS, n \in N, S−$ sphere $x^2 + y^2 + z^2 = R^2$.

10. $\iint_S z^2 dS$, S is part of a conical surface $x = u \cos v \sin \alpha$, $y = u \sin v \sin \alpha$, $z = u \cos \alpha$, $\alpha =$const, $\alpha \in (0; \pi/2)$, allocated by the conditions $u \in [0; 1]$, $v \in [0; 2\pi]$.

11. $\iint_S z dS$, $S-$surface $x = u \cos v$, $y = u \sin v$, $z = v$, $u \in [0; 1]$, $v \in [0; 2\pi]$.

12. $\iint_S f(r) dS$, where $r = \sqrt{x^2 + y^2 + z^2}$, $f(r) = \begin{cases} 1 - r^2, & r \leqslant 1, \\ 0, & r \geqslant 1, \end{cases}$ S is the plane $x + y + z = a$.

13. $\iint_S f(r; z) dS$, where $r = \sqrt{x^2 + y^2}$, $f(r; z) = \begin{cases} r^2, & r \leqslant z, \\ 0, & r \geqslant z, \end{cases}$ $S-$sphere $x^2 + y^2 + z^2 = R^2$.

26. $\iint_S \left(x^2 + y^2\right) dxdy$, S is the underside of the circle $x^2 + y^2 \leqslant 4$, $z = 0$.

27. $\iint_S (2z - x) dydz + (x + 2z) dzdx + 3z dxdy$, S is the upper side of the triangle $x + 4$, $y + z = 4$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

28. (a) $\iint_S xz dxdy$;

    (b) $\iint_S yz dydz + zx dzdx + xy dxdy$;

    S inner side of the tetrahedron surface $x + y + z \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$.

29. $\iint_S f_1(x) dydz + f_2(y) dzdx + f_3(z) dxdy$, where f_1, f_2, f_3 are continuous functions, S is the outer side of the parallelepiped surface $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$.

30. (a) $\iint_S y dz dx$

    (b) $\iint_S x^2 dydz$;

    S is the outer side of the sphere $x^2 + y^2 + z^2 = R^2$.

31. (a) $\iint_S \left(x^5 + z\right) dydz$;

    (b) $\iint_S x^2 y^2 z dxdy$;

    S is the inner side of the hemisphere $x^2 + y^2 + z^2 = R^2$, $z \leqslant 0$.

32. $\iint_S x^2 dydz + z^2 dxdy$,

    S is the outer side of a part of the sphere $x^2 + y^2 + z^2 = R^2$, $x \leqslant 0$, $y \geqslant 0$.

33. $\iint_S x^2 dydz + y^2 dzdx + z^2 dxdy$, S is the outer side of the sphere $(x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$.

34. $\iint_S z^2 dxdy$, S is the inner side of the hemisphere $(x - a)^2 + (y - b)^2 + z^2 = R^2$, $z \geqslant 0$.

35. $\iint_S (x - 1)^3 dydz$, S is the outer side of the hemisphere $x^2 + y^2 + z^2 = 2x$, $z \leqslant 0$.

36. (a) $\iint_S dz dx$;

    (b) $\iint_S x dydz$;

(c) $\iint_S x^2 dydz$;

(d) $\iint_S \frac{dxdy}{z}$;

$S-$ the outer side of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

37. (a) $\iint_S yzdzdx$;

(b) $\iint_S x^3 dydz + y^3 dzdx$;

S is the outer side of the part of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $z \geqslant 0$.

38. $\iint_S (2x^2 + y^2 + z^2) \, dydz$, S is the outer side of the side surface of the cone $\sqrt{y^2 + z^2} \leqslant x \leqslant H$.

39. $\iint_S (y - z)dydz + (z - x)dzdx + (x - y)dxdy$, S is one of the sides of the surface $x^2 + y^2 = z^2$, $0 < z \leqslant H$.

40. $\iint_S yz^2 dxdz$, S is the inner side of a part of a cylindrical surface $x^2 + y^2 = r^2$, $y \leqslant 0$, $0 \leqslant z \leqslant r$.

41. $\iint_S yzdxdy + zxdydz + xydzdx$, S is the outer side of the cylinder part $x^2 + y^2 = r^2$, $x \leqslant 0$, $y \geqslant 0$, $0 \leqslant z \leqslant H$.

42. $\iint_S x^6 dydz + y^4 dzdx + z^2 dxdy$, S is the underside of a part of an elliptical paraboloid $z = x^2 + y^2$, $z \leqslant 1$.

43. $\iint_S xdydz + ydzdx + zdxdy$, S is the upper side of the hyperbolic paraboloid $z = x^2 - y^2$, $|y| \leqslant x \leqslant a$.