Equations of Plane in Space. Some extra problems

Assume that some plane α in the space $\mathbb{E}$ is chosen and fixed. In order to study various equations determining this plane we choose some coordinate system O, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in the space $\mathbb{E}$. Locus of plane α may be described by the radius vectors of points contained in it.

We derived and started investigate some forms of the equation of this plane.

Vectorial parametric equation:

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a} + \tau\boldsymbol{b}, \tag{1}$$

where $\boldsymbol{r}_0$ is radius vector of initial point, $\boldsymbol{a}$ and $\boldsymbol{b}$ are direction vectors of the plane.

This equation yields coordinate parametric form:

$$\begin{cases} x = x_0 + a_x t + b_x \tau \\ y = y_0 + a_y t + b_y \tau \\ x = z_0 + a_z t + b_z \tau, \end{cases} \tag{2}$$

where x_0, y_0, z_0, a_i and b_i are components of $\boldsymbol{r}_0$, $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Vectorial normal equation:

$$(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = 0, \text{ or} \tag{3}$$

$$\boldsymbol{r} \cdot \boldsymbol{n} = D, \tag{4}$$

where $\boldsymbol{n}$ is normal vector, and constant D is result of dot product of this normal vector and radius vector of initial point.

Expression normal vector as cross product of direction vectors yield vectorial form of canonical equation of plane:

$$(\boldsymbol{r} - \boldsymbol{r_0}, \boldsymbol{a}, \boldsymbol{b}) = 0 \tag{5}$$

or

$$(\boldsymbol{r}, \boldsymbol{a}, \boldsymbol{b}) = D \tag{6}$$

Writing of the dot product in coordinate form yields general equation of the plane:

$$Ax + By + C + D = 0, \tag{7}$$

where A, $B$and C are covariant coordinates of normal vector.

Solving of the mixed product in canonical equation produces coordinate canonical equation:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = 0 \tag{8}$$

or

$$\begin{vmatrix} x & y & z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = D' \tag{9}$$

Accurate analysis of the problem on constructing the equation of plane passing through three points yields two forms for such equation:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{10}$$

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_1 - x0 & y_1 - y_0 & z_1 - z_0 \\ x_2 - x0 & y_2 - y_0 & z_2 - z_0 \end{vmatrix} = 0 \tag{11}$$

While these tree points are just intercepts with coordinate axes $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$, this equation reduces to

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{12}$$

Normalization of the general equation of plane with $\pm\sqrt{A^2 + B^2 + C^2}$ yields normal form of the plane equation

$$x \cos\alpha + y \cos\beta + x \cos\gamma - p = 0, \tag{13}$$

Letting direct measure of the distance from the plane to any point.

Finally, proper beam of planes (family of planes crossing by a line) has expression

$$p(A_1 x + B_1 y + C_1 z + D_1) + s(A_2 x + B_2 y + C_2 z + D_2) = 0.$$

And proper bundle of planes (family of planes crossing in a single point) has expression

$$p(A_1 x + B_1 y + C_1 z + D_1) + q(A_2 x + B_2 y + C_2 z + D_2) + s(A_3 x + B_3 y + C_3 z + D_3) = 0.$$

Planes involved into these equation must be not parallel or coincide. Hence, for the bundle following condition must be fulfilled:

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0.$$

Now let us take a look on some problems involving equations of planes

1 Problems corner

In problems above we deal with right orthonormal bases.

Problem 1

Find the point of intersection of the planes:

$$\begin{aligned} x + 2y - z - 6 = 0 \\ 2x - y + 3z + 13 = 0 \\ 3x - 2y + 3z + 16 = 0 \end{aligned}$$

Solution

Here we have three linear equations. The solution of these simultaneous equations determines the coordinates of the point of intersection of the three planes.

Determinant of the matrix of this system:

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & -2 & 3 \end{vmatrix} = 10$$

System has single honest solution.

$$\Delta_1 = \begin{vmatrix} 6 & 2 & -1 \\ -13 & -1 & 3 \\ -16 & -2 & 3 \end{vmatrix} = -10$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & -1 \\ 2 & -13 & 3 \\ 3 & -16 & 3 \end{vmatrix} = 20$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & -13 \\ 3 & -2 & -16 \end{vmatrix} = -30$$

$$\begin{aligned} x &= \frac{\Delta_1}{\Delta} = \frac{-10}{10} = -1 \\ y &= \frac{\Delta_2}{\Delta} = \frac{20}{10} = 2 \\ z &= \frac{\Delta_3}{\Delta} = \frac{-30}{10} = -3 \end{aligned}$$

Hence, intersection point is $(-1, 2, -3)$

Problem 2

Find the equation of the plane passing through the line of intersection $3x + y - 5z + 7 = 0$ and $x - 2y + 42 - 3 = 0$ and through the point $(-3, 2, -4)$.

Solution

We are looking for a plane form the proper beam shaped with $3x + y - 5z + 7 = 0$ and $x - 2y + 4z - 3 = 0$.

Substituting coordinates of point into the equations of given lines, we yield:

$$3 \cdot (-3) + 2 + -5 \cdot (-4) + 7 = 20$$
$$-3 - 2 \cdot 2 + 4 \cdot (-4) - 3 = -26.$$

Hence, this point is not part of both that planes.

Therefore, general expression for the equation of this plane must be written:

$$p(3x + y - 5z + 7) + q(x - 2y + 4z - 3) = 0;$$

While both p and q are not zeros, we will look for their ratio $k = \frac{q}{p}$ instead:

$$(3x + y - 5z + 7) + k(x - 2y + 4z - 3) = 0$$

Substitution of calculated quantities of first and second expressions yield:

$$20 - 26k = 0$$
$$k = \frac{10}{13}.$$

Equation of the plane is

$$(3x + y - 5z + 7) + \frac{10}{13}(x - 2y + 4z - 3) = 0$$

Multiply it by 13 and simplify:

$$49x - 7y - 25z + 61 = 0.$$

Problem 3

Find the equations of the planes which bisect the dihedral angles between the planes

$$6z - 6y + 7z + 21 = 0$$

and

$$2x + 3y - 6z - 12 = 0.$$

Solution

Let (x_1, y_1, z_1) be any point on the bisecting plane. Then the distances of it from the two planes must be equal in magnitude.

Normalization ratio for the first plane is $-\sqrt{36+36+49} = -\sqrt{21} = -11$. Minus because $D = 21$ and is positive.

Normalization ratio for the first plane is $\sqrt{2+9+36} = -\sqrt{21} = 7$. Plus because $D = -12$ and is negative

Equivalence of distances:

$$\frac{6z - 6y + 7z + 21}{-11} = \pm\frac{2x + 3y - 6z - 12}{7}$$

Sign $\pm$ means that we are looking for both bisectors.

Simplifying and separating signs, we obtain two planes:

$$64x - 9y - 17z + 15 = 0;$$
$$20x - 75y + 115z + 279 = 0.$$

Problem 4

Find the equation of the plane through the points $A(1, -2, 2)$, $B(-3, 1, -2)$ and perpendicular to the plane $2x + y - 2 + 6 = 0$.

Solution

While plane in question is perpendicular with $2x + y - 2 + 6 = 0$, normal vector of $2x + y - 2 + 6 = 0$ is collinear with plane in question.

Hence, one of direction vector of this plane has coordinates $(2, 1, -2)$.

As a second direction vector we take $\overrightarrow{AB}$ with coordinates $(-4, 3, -4)$.

We took A s initial point.

Coordinates of these two vectors are not proportional, hence we can start writing canonical equation:

$$\begin{vmatrix} (x-1) & (y+2) & (z-2) \\ 2 & 1 & -1 \\ -4 & 3 & -4 \end{vmatrix} =$$

Calculating this determinant yields:

$$10(z-2) + 12(y+2) - x + 1 = 0$$
$$x - 12y - 10z - 5 = 0$$

Problem 5

Find the equations of the planes parallel to $2x - 3y - 6z - 14 = 0$ and distant 5 units from the origin.

Solution
Improper beam containing given plane may be expressed as

$$2x - 3y - 6z - k = 0$$

Normalization ratio of this family of planes is $\sqrt{4+9+36} = \sqrt{49} = 7$.
Distance of 5 units from any has now expression:

$$\pm 5 = \frac{2x - 3y - 6z - k}{7}$$

Substituting coordinates of the origin $(0, 0, 0)$ into it we yield:

$$\pm 5 = \frac{2 \cdot 0 - 3 \cdot 0 - 6 \cdot 0 - k}{7},$$

Thus, $k = \pm 35$
Planes are

$$2x - 3y - 6z + 35 = 0$$
$$2x - 3y - 6z - 35 = 0$$

Problem 6
Show that the planes

$$\Pi_1 : \; 7x + 4y - 4z + 30 = 0,$$
$$\Pi_2 : \; 36x - 51y + 12z + 17 = 0,$$
$$\Pi_3 : \; 14x + 8y - 8z - 12 = 0,$$
$$\Pi_4 : \; 12x - 17y + 4z - 3 = 0$$

form four faces of a rectangular parallelepiped.

Solution
First, we notice, that Π_1 and Π_3 have proportional A, B, and C:

$$\frac{7}{14} = \frac{4}{8} = \frac{-4}{-8} \left(= \frac{1}{2}\right),$$

as well Π_2 and Π_4:

$$\frac{36}{12} = \frac{-51}{-17} = \frac{12}{4} \, (= 3) \, .$$

Further, the first and second planes are perpendicular since

$$7 \cdot 36 + 4 \cdot (-51) - 4 \cdot 12 = 252 - 204 - 48 = 0.$$

Problem 7
Determine the locus of the equation $x^2 + y^2 - 2xy - 4z^2 = 0$.

Solution
We isolate full squared terms in the left side of equation:

$$x^2 - 2xy^2 - 4z^2 = (x - y)^2 - 4z^2 = (x - y - 2z)(x - y + 2z).$$

Letting this transformed term to be zero yields two equations:

$$(x - y - 2z)(x - y + 2z) = 0$$

$$\begin{cases} x - y - 2z = 0; \\ x - y + 2z = 0 \end{cases}$$

Problem 8
Find on the axis z point equidistant to the planes

$$\begin{aligned} &x + 4y - 3z - 2 = 0 \\ &5x + z + 8 = 0 \end{aligned}$$

Basis is right orthonormal.

Solution
We transform both equations into normal form first.
Normalization ratio for first equation $\sqrt{1 + 16 + 9} = \sqrt{26}$, taken with plus, as $D = -2$.
Normalization ratio for first equation $-\sqrt{25 + 1} = -\sqrt{26}$, taken with plus, as $D = 8$.
We look point with coordinates $(0, 0, c)$ which is equidistant from these planes:

$$\frac{0 + 4 \cdot 0 - 3c - 2}{\sqrt{26}} = -\frac{5 \cdot 0 + c + 8}{\sqrt{26}}$$

$$\begin{aligned} -3c - 2 &= -c - 8 \\ -2c &= -6 \\ c &= 3 \end{aligned}$$

This point is $(0, 0, 3)$.

Problem 9
Given three points $A(6, 1, -1)$, $B(0, 5, 4)$, $C(5, 2, 0)$.

Write equation of a plane, if distances from that plane to points are $d_A = -1$, $d_B = 3$, $d_C = 0$.

Solution

Let

$$ax + by + cx - p = 0$$

be normal equation of the line (we hide cosines into these lowercase letters). While are direction cosines of a line, condition

$$a^2 + b^2 + c^2 = 1$$

fulfilled

Given conditions have algebraic expression:

$$\begin{cases} 6a + b - c - p = -1 \\ 5b + 4c - p = 3 \\ 5a + 2b - p = 0 \\ a^2 + b^2 + c^2 = 1 \end{cases}$$

We express a, b, and $c$with p first solving the system, and then find p form the fourth equation.

$$\begin{cases} 6a + b - c = p - 1 \\ 5b + 4c = p + 3 \\ 5a + 2b = p \end{cases}$$

$$\Delta = \begin{vmatrix} 6 & 1 & -1 \\ 0 & 5 & 4 \\ 5 & 2 & 0 \end{vmatrix} = -3$$

$$\Delta_1 = \begin{vmatrix} p-1 & 1 & -1 \\ p+3 & 5 & 4 \\ p & 2 & 0 \end{vmatrix} = -2(p+3) + 9p - 8(p-1) = 2 - p$$

$$\Delta_2 = \begin{vmatrix} 6 & p-1 & -1 \\ 0 & p+3 & 4 \\ 5 & p & 0 \end{vmatrix} = 5(p+3) - 24p + 20(p-1) = p - 5$$

$$\Delta_3 = \begin{vmatrix} 6 & 1 & p-1 \\ 0 & 5 & p+3 \\ 5 & 2 & p \end{vmatrix} = 6(5p-2(p+3)) + 5(p+3) - 25(p-1) = 4-2p$$

$$a = \frac{2-p}{-3} \quad b = \frac{p-5}{-3} \quad c = \frac{4-2p}{-3}$$

Condition of the direction cosines is:

$$\left(\frac{2-p}{-3}\right)^2 + \left(\frac{p-5}{-3}\right)^2 + \left(\frac{4-2p}{-3}\right)^2 = 1$$

$$(2-p)^2 + (p-5)^2 + (4-2p)^2 = 9$$

$$\begin{aligned} &6p^2 - 30p + 45 = 9 \\ &6p^2 - 30p + 36 = 0 \\ &p^2 - 5p + 6 = 0 \end{aligned}$$

Solutions of this equation are $p = 2$ and $p = 3$.
Hence,

$$p = 2, \quad a = 0, \quad b = 1, \quad c = 0,$$

$$y - 2 = 0.$$

$$p = 3, \quad a = \frac{1}{3}, \quad b = \frac{2}{3}, \quad c = \frac{2}{3},$$

$$x + 2y + 2z - 9 = 0$$