Complex analysis.

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1 Analytic continuation

1.1 Direct analytic continuation. Analytic continuation. Analytic function

Definition 1.1. *Let $f_1 \in H(D_1)$, $f_2 \in H(D_2)$, Δ is connected component of $D_1 \cap D_2$. If $f_1(z) = f_2(z)$ for every $z \in \Delta$ then function f_2 is called direct analytic continuation of function f_1 from domain D_1 to domain D_2 along the domain Δ.*

If the analytic continuation exists then it is unique by the uniqueness theorem applied to the domain D_2. If Δ^* is another connected component of $D_1 \cap D_2$ it may happen that $f_1|_{\Delta^*} \neq f_2|_{\Delta^*}$ so analytic continuations along Δ and Δ^* may be not equal.

Example 1.1. *The most simple example of analytic continuation is continuation of holomorphic function by continuity to the removable singular point. For example, a function $f(z) = \frac{\sin z}{z}$ is defined and holomorphic in $\mathbb{C}\backslash\{0\}$, and 0 is a removable singular point. Holomorphic continuation of function f to 0 is defined by Laurent's series at this point (whi in this case coincides with Taylor's series)*

$$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots.$$

Another example is a continuation of holomorphic functions that are defined by integrals with parameter.

Lemma 1.2. *Suppose that $D \subset \mathbb{C}$ is a domain and*

$$\varphi = \varphi(t,z) : [a,b] \times D \to \mathbb{C}$$

is a continuous function that is holomorphic in $z \in D$ for every fixed parameter $t \in [a,b]$. Consider a function f, defined by the integral

$$f(z) = \int_a^b \varphi(t,z)dt.$$

Then f is holomorphic in D.

Proof. By uniform continuity of function f on sets $[a,b] \times K$, where K is a compact subset of D, we deduce the continuity of f in D. Hence, by Morera theorem, it is enough to prove that for every triangle Δ compactly supported in D we have

$$\int_{\partial\Delta} f(z)dz = 0.$$

This follows from Fubini's theorem and Cauchy theorem in the following way:

$$\int_{\partial\Delta} f(z)dz = \int_{\partial\Delta}\int_a^b \varphi(t,z)dtdz$$
$$= \int_a^b \int_{\partial\Delta} \varphi(t,z)dzdt = \int_a^b 0dt = 0.$$

$\square$

1.2 Analytic continuation of $\Gamma-$function.

By the definition, $\Gamma-$function is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$$

where

$$t^{z-1} := e^{(z-1)\ln t} \quad \text{for } t > 0, \quad \operatorname{Re} z > 0.$$

To prove convergence of this integral at zero and on infinity we decompose it into a sum of two integrals

$$I_1(z) = \int_0^1 e^{-t} t^{z-1} dt \quad \text{and} \quad I_2(z) = \int_1^\infty e^{-t} t^{z-1} dt.$$

The integral $I_2(z)$ converges for every complex number z since the integral

$$\int_1^\infty \left| e^{-t} t^{z-1} \right| dt = \int_1^\infty e^{-t} t^{\operatorname{Re} z - 1} dt$$

is convergent for every $z \in \mathbb{C}$. Moreover, for $|z| \leqslant R$

$$\left| e^{-t} t^{z-1} \right| \leqslant e^{-t} t^{R-1},$$

Hence, the integral $I_2(z)$ is a uniform limit on compacts in $\mathbb{C}$ of functions

$$f_n(z) := \int_1^n e^{-t} t^{z-1} dt$$

as $n \to \infty$. Indedd,

$$|I_2(z) - f_n(z)| = \left| \int_n^\infty e^{-t} t^{z-1} dt \right| \leqslant \int_n^\infty e^{-t} t^{R-1} dt \to 0$$

as $n \to \infty$. Since functions f_n are entire (holomorphic in $\mathbb{C}$) then by Weierstrass theorem on series of holomorphic functions function I_2 is holomorphic in $\mathbb{C}$.

Analogously, the integral

$$I_1(z) = \lim_{n\to\infty} \int_{1/n}^1 e^{-t} t^{z-1} dt$$

is convergent and defines a holomorphic function for every $\operatorname{Re} z > 0$ since the integral

$$\int_0^1 e^{-t} t^{\alpha-1} dt$$

is convergent for every $\alpha > 0$. Consequently, the function $\Gamma(z)$ is defined and holomorphic for every $\operatorname{Re} z > 0$.

We will show now that Γ-function defined above has meromorphic continuation to the whole complex plane $\mathbb{C}$.

Notice that for every $z \in \mathbb{C}$ and $t \in [0,1]$ the function $e^{-t}t^z$ can be expressed as a series

$$e^{-t}t^{z-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{z+n-1}.$$

This series for $\operatorname{Re} z > 1$ has the following properties

1. All terms of a series are continuous in $t \in [0,1]$ and
$$\int_0^1 t^{z+n-1} dt = \frac{1}{z+n};$$

2. The series is uniformly convergent on $[0,1]$.

Hence,

$$I_1(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} \text{ for } \operatorname{Re} z > 1.$$

Notice that a series in the right-hand side is converges for every $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Moreover, if we substract terms with indexes $n =$

$0, 1, \dots, N-1$ then the remainder series

$$f_N(z) := \sum_{n=N}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n}$$

will uniformly converge on compacts of half-plane

$$D_N := \{z \in \mathbb{C} : \operatorname{Re} z > -N\}$$

Consequently, function f_N is holomorphic in D_N and $\Gamma-$function in the half-plane $\{\operatorname{Re} z > 1\}$ is expressed by the formula

$$\Gamma(z) = \sum_{n=0}^{N-1} \frac{(-1)^n}{n!} \frac{1}{z+n} + f_N(z) + I_2(z). \tag{1}$$

Moreover, by uniqueness theorem, this formula is true everywhere, where both sides of equality are holomorphic, that is for $\operatorname{Re} z > 0$.

Notice that the right-hand side of the equality (1) defines a function that is holomorphic in $D_N \backslash \{0, -1, \dots, -(N-1)\}$, coinsides with Γ in $\{\operatorname{Re} z > 0\}$, and concequently defines direct analytic continuation of Γ from domain $\{\operatorname{Re} z > 0\}$ to domain $D_N \backslash \{0, -1, \dots, -(N-1)\}$. By uniqueness theorem this continuation is unique and, since, N is arbitrary we see that

$$\Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^{\infty} e^{-t} t^{z-1} dt, \quad z \in \mathbb{C}\{0, -1, -2, \dots\}.$$

The obtained formula defines analytic continuation of $\Gamma(z)$ from domain $\{\operatorname{Re} z > 0\}$ to domain $\mathbb{C} \backslash \{0, -1, -2, \dots\}$. Points $z = -n, \quad n = 0, 1, 2, \dots$ are poles of the first order with residues equal to $\frac{(-1)^n}{n!}$. In other words the primary part of Laurent's series of $\Gamma(z)$ in punctured neighbourhood of point $z = -n$ is equal to $\frac{(-1)^n}{n!}\frac{1}{z+n}$. This defines a meromorphic continuation of function $\Gamma(z)$ to the function meromorphic in $\mathbb{C}$ with simple poles at points $z = 0, -1, -2, \ldots$.

1.3 Analytic continuation of logarithm

We begin from a disk $U = \{|z-1| < 1\}$ centered at $z = 1$ of radius 1 in which the logarithm can be defined by the series

$$\ln z \equiv f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n.$$

This series converges uniformly on compacts in U and, consequently, allows term-by-term differentiation

$$f'(z) = \sum_{n=1}^{\infty}(1-z)^{n-1} = \frac{1}{1-(1-z)} = \frac{1}{z}$$

for $z \in U$. Consequently, by Newton-Leibniz formula

$$f(z) = \int_1^z \frac{d\zeta}{\zeta} \quad \text{for} \quad z \in U,$$

where the integral is by the segment $[1, z]$. Defining

$$f_0(z) = \int_1^z \frac{d\zeta}{\zeta} \quad \text{for} \quad z \in \mathbb{C}\backslash(-\infty, 0]$$

(the integral is taking by segment $[1, z]$), we see that function f_0 is holomorphic in a slit domain $D_0 := \mathbb{C}\backslash(-\infty, 0)$ and $f_0 \equiv f$ in disk U.

Recall that limits of function f_0 on the upper and lower edge of a slit $(-\infty, 0]$ do not coincide. This implies that domain of function f_0 can not be extended without breaking the analyticity (and even continuity) of f_0.

2 Weierstrass theory of analytic continuation

Definition 2.1. *Suppose $f \in \mathcal{A}(D)$, $g \in \mathcal{A}(E)$. If there exists a sequence of domains D_0, $D_1, \ldots, D_n$ and functions $f_k \in \mathcal{A}(D_k)$ such that $D_0 = D$, $f_0 = f$, $D_n = E$, $f_n = g$ and for every $k \in [1 : n]$ f_k is a direct analytic continuation of f_{k-1} from D_{k-1} to D_k. Then function g is analytic continuation of function f along a chain of domains D_0, $D_1, \ldots, D_n$.*

Remark 2.2. *In this definition we don't mention connected components of overlaps $D_{k-1} \cap D_k$ while they have influence on the result. By induction analytic continuation along the fixed chain of domains and connected components of overlaps is unique. Also in the definition of the analytic continuation we may assume without loss of generality that intermediate domains are open disks.*

2.1 Elements and analytic continuation

Definition 2.3. *Element is a pair $F = (U, f)$ that consists of a disk $U = \{|z-a| < R\}$ centered at a and the function f that is holomorphic in this disk. A point a is called a center of an element and R its radius. Element F is called canonical if U coincides with disk of convergence of Taylor's series of function f with center at a.*

Example of a canonical element is a pair of a disk $U = \{|z-1| <\}$ centered at 1 and function $f(z) = \ln z$ defined by its Taylor's series

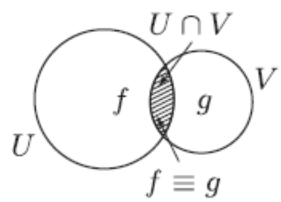
$$\ln z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n}.$$

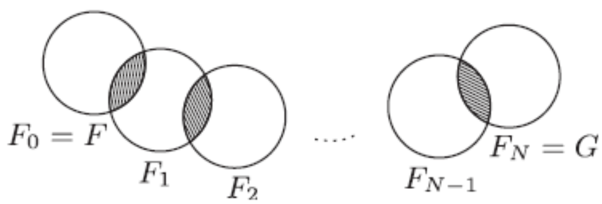
Definition 2.4. *Elements $F = (U, f)$ and $G = (V, g)$ are direct analytic continuations (shortly, DAC) of each other if*

$$U \cap V \neq \varnothing, \quad \text{and} \quad f(z) = g(z),\ z \in U \cap V.$$

An element G is analytic continuation of an element F along the chain $F_0 = F,\ F_1, \ldots, F_{N-1},\ F_N = G$ if

$$F_{n+1} \text{ is DAC of } F_n \text{ for every } n = 0, 1, \ldots, N-1.$$





2.2 Properties of direct analytic continuation.

Property 1. Weierstrass property. If $G = (V, g)$ is DAC of an element $F = (U, f)$ and center b of a disk V belongs to U then Taylor's series of function g is obtained by Taylor's decomposition of function f at point b, that is

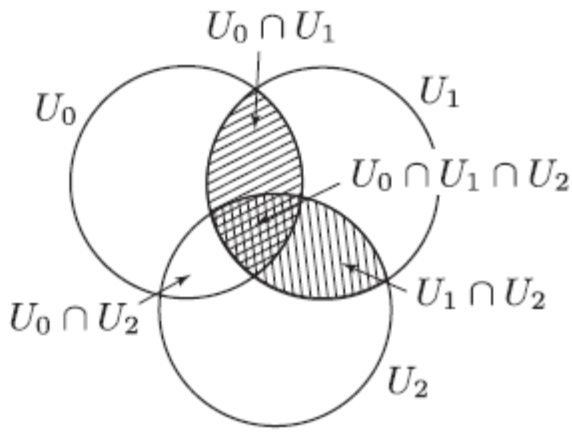
$$g(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!}(z-b)^n$$

for every $z \in V$. Inversely, for every point $b \in U$ if the function g is defined by this formula and V is the disk of convergence of this series then (V, g) is DAC (U, f).

Property 2. Triangle property. Suppose that element $F_1 = (U_1, f_1)$ is DAC of an element $F_0 = (U_0, f_0)$ and element $F_2 = (U_2, f_2)$ is DAC of an element F_1. If

$$U_0 \cap U_1 \cap U_2 \neq \varnothing$$

then F_2 is DAC of F_0.



Proof. First property follows from theorem on Taylor's decomposition of a holomorphic function. To prove second property notice that on the subset set $U_0 \cap U_1 \cap U_2$ of a set $U_0 \cap U_2$ we have $f_2 \equiv f_1 \equiv f_0$. By uniqueness theorem it follows that

$$f_2 \equiv f_0 \quad \text{on } U_0 \cap U_2,$$

and F_2 is DAC F_0. $\square$

2.3 Analytic continuation of canonic elements along a path.

Definition 2.5. *A family of canonical elements*

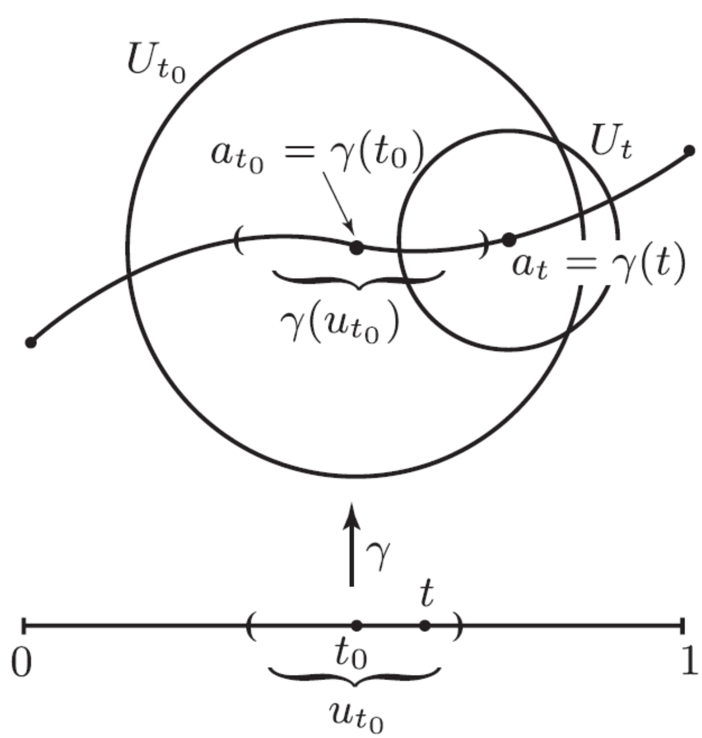
$$F_t = (U_t, f_t), \quad t \in I = [0,1],$$

is analytic continuation of a canonical element F_0 along a path $\gamma : I \to \mathbb{C}$ if

1. *center a_t of an element F_t coincides with $\gamma(t)$ and radius $R(t)$ is strictly positive for every $t \in I$;*

2. *for every $t_0 \in I$ there exists a neighborhood $u_{t_0} \subset I$ of a point t_0 such that for every $t \in u_{t_0}$*

$$\gamma(t) \in U_{t_0} \quad \textit{and} \quad F_t \textit{ is DAC of } F_{t_0}.$$



Lemma 2.6. *If $\{F_t : t \in I\}$ and $\{\tilde{F}_t : t \in I\}$ are two analytic continuations of canonic element $F_0 = \tilde{F}_0$ along a path γ then $F_1 = \tilde{F}_1$.*

Proof. Consider a set

$$E := \{t \in I : F_t = \tilde{F}_t\}.$$

This set is

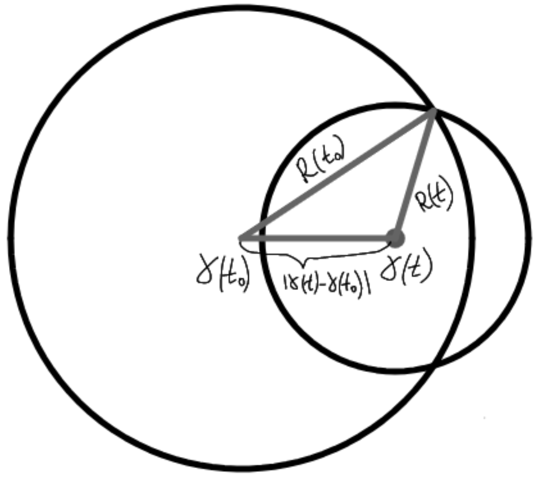
1. not empty since $0 \in E$;
2. open since if $t_0 \in E$ then
$$u_{t_0} \cap \tilde{u}_{t_0} \subset E.$$
3. closed. Indeed, if $t_0 \in I$ is a limit point of a set E then in intersection $u_{t_0} \cap \tilde{u}_{t_0}$ there exists a point $t_1 \in E$. Hence, canonical elements F_{t_0} and $\tilde{F}_{t_0}$ are DAC of an element $F_{t_1} = \tilde{F}_{t_1}$, have the same center $\gamma(t_0)$ and, consequently, coincide.

Consequently, $I = E$ since I is connected. □

Lemma 2.7. *Let $R(t)$ be the radius of an element F_t Then either $R(t) = +\infty$ or $R : I \to \mathbb{R}$ is continuous function.*

Proof. If $R(t_0) = +\infty$ for some $t_0 \in I$ then $R(t) = +\infty$ for every $t \in u_{t_0}$ and analogously to previous lemma we can prove that a set $\{t : R(t) = +\infty\}$ coincides with a segment.

Assume that $R(t_0) < \infty$ for every $t_0 \in I$. Then for $t \in u_{t_0}$ the intersection $\partial U_t \cap \partial U_{t_0} \neq \varnothing$.



Let $a \in \partial U_t \cap \partial U_{t_0}$. Then in a triangle with vertexes at points $\gamma(t)$, a, $\gamma(t_0)$ the following inequality is satisfied

$$|R(t) - R(t_0)| \le |\gamma(t) - \gamma(t_0)| \,.$$

Hence, the continuity of γ implies continuity of $R(t)$. $\square$

Notice that we have used the following estimate for the radius of convergence:

Lemma 2.8. *Suppose that $f \in H(D)$ and $a \in D$, $R = \operatorname{dist}(a, \ \partial D)$, $U = \{z \in \mathbb{C} : |z - a| < R$. Then the Taylor's series of function f with center at a converges to f in U_0.*

Proof. Let $r \in (0, R_0)$ and $U = \{z \in \mathbb{C} : |z - a| < r\} \subset D$. Then

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta) d\zeta}{\zeta - z},$$

where

$$\frac{f(\zeta)}{(\zeta - a) - (z - a)} = \frac{f(\zeta)}{\zeta - a} \cdot \frac{1}{1 - \frac{z-a}{\zeta-a}} = \sum_{n=0}^{\infty} \frac{(z-a)^n f(\zeta)}{(\zeta - a)^{n+1}}, \ \ z \in U.$$

The series for a fixed z converges uniformly in $\zeta \in \partial U$, consequently,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, z \in U, \tag{2}$$

where

$$a_n = \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta = \frac{f^{(n)}(a)}{n!}.$$

Since $r \in (0, R)$ is arbitrary the equality (2) holds in U_0. $\square$

Theorem 2.9 (Ostrowski-Hadamard Gap Theorem)**.** *Let $s > 1$, let $\{n_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that $n_{k+1}/n_k \geq s$ for all k, and let $\{a_k\}_{k=1}^{\infty}$ be a sequence in $\mathbb{C}$. If*

$$f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$$

and the radius of convergence of the series is equal to 1 then f has no analytic continuation outside of $|z| < 1$.

Definition 2.10. *Let $f \in H(D)$. A point a is a regular point of function*

Example 2.1. *The series*

$$f(z) = \sum_{k=1}^{\infty} z^{2^k}$$

defines the function that is analytic in the unit disk $|z| < 1$ and has no analytic continuation outside of this disk.

2.4 Equivalence of analytic continuation along a chain and along a path.

Theorem 2.11. *(1) Suppose that a family of canonical elements $\{F_t : t \in I\}$ defines an analytic continuation of element F_0 along a path γ. Then there exists a finite sequence of points*

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

such that the element F_1 coincides with analytic continuation of element F_0 along a chain

$$F_0 = F_{t_0}, \quad F_{t_1}, \quad \dots, \quad F_{t_n} = F_1.$$

(2) Inveresely, if a canonical element G is an analytic continuation of a canonical element F by some chain of canonical elements

$$F = F_0, \quad F_1, \quad \dots, \quad F_n = G.$$

Denote by $\gamma : I \to \mathbb{C}$ a polygonal curve that connects centers of element $F_0, F_1, \dots, F_n$. Then there exists a family of canonical elements F_t, $t \in I$, that perform analytic continuation of F_0 along path γ such that

$$F_{t=1} = F_n.$$

Proof. **(1)** By Lemma 2.7 there exists $\varepsilon > 0$ such that $R(t) \geq \varepsilon$ for every $t \in I$. Hence, by uniform continuity of $\gamma(t)$ there exists $\delta > 0$ such that

$$|s - t| < \delta \quad \Longrightarrow \quad |\gamma(s) - \gamma(t)| < \varepsilon.$$

From open cover of $I = [0,1]$ by intervals

$$I_t := u_t \cap (t - \delta/2, t + \delta/2)$$

we can choose a finite subcover $I_{t_1}, \ldots, I_{t_{n-1}}$, where $t_1 < t_2 < \cdots < t_{n-1}$ and let $t_0 = 0$, $t_n = 1$. Then

$$|\gamma(t_j) - \gamma(t_{j-1})| < \varepsilon, \quad j = 1, \ldots, n,$$

(by definition of δ) and element F_{t_j} is DAC of element $F_{t_{j-1}}$ (by definition u_t for $t = t_{j-1}, t_j$).
(2) It is enough to prove the statemnt for $n = 1$ (The general case will the immideately follow by induction). Hence, assume that $F_1 = G$ is DAC of an element F_0 and a polygonal curve $\gamma : I \to \mathbb{C}$ is a segment that connects centers $\gamma(0)$ and $\gamma(1)$ of canonical elements F_0, F_1. Let $F_k = (U_k, f_k)$ for $k = 0, 1$ and for every $t \in [0,1]$ the canonical element $F_t = (U_t, f_t)$ with center at $\gamma(t)$ by Taylor's series expansion of f_0 or f_1 centered at $\gamma(t)$. The independence of f_t of choice of f_0 or f_1 in for $\gamma(t) \in U_0 \cap U_1$ follows from triangle property.

To check that obtained family $\{F_t, t \in I\}$ is analytic continuation along γ it remains to define neighborhoods u_{t_0}. We choose u_{t_0} such that $\gamma(u_{t_0})$ is subset of U_0 or U_1. Then for every $t \in u_{t_0}$ element F_t is DAC F_{t_0}. □

Theorem 2.12 (Construction of analytic continuation along path)**.** *Suppose $D \subset \mathbb{C}$ is domain, $f \in H(D)$, G_0 and G_1 are two subdomains of D and f has antiderivatives in G_1 and G_2. Consider a path that* *connects $z_0 \in G_0$ and $z_1 \in G_1$ and let*

$$\Phi_0(z) = \int_{z_0}^{z} f(\xi)d\xi, \ z \in G_0; \tag{3}$$

$$\Phi_1(z) = \int_{\gamma} f(\xi)d\xi + \int_{z_1}^{z} f(\xi)d\xi, \ z \in G_1. \tag{4}$$

Then Φ_1 is analytic continuation of Φ_0 along path γ.

Remark 2.13. *Integrals $\int_{z_0}^{z} f(\xi)d\xi$ and $\int_{z_1}^{z} f(\xi)d\xi$ in (3) and (4) can be considered along any path in G_0 and G_1 respectively since f has antiderivatives both in G_0 and G_1. Functions Φ_0 and Φ_1 are antiderivatives of f in G_0 and G_1 respectively.*

Proof. Let $\gamma : I = [0,1] \to D$ be a path such that $\gamma(0) = z_0$ and $\gamma(1) = z_1$. We will prove that for every $\tau \in I$ there exists a neighborhood $U_{\gamma(\tau)} \subset D$, function $\Phi_\tau \in H(U_{\gamma(\tau)})$ and neighborhood u_τ in I with the following properties

1. $\gamma(u_\tau) \subset U_{\gamma(\tau)}$;

2. $\Phi_t = \Phi_\tau$ in $U_{\gamma(\tau)} \cap U_{\gamma(t)}$ for every $t \in u_\tau$.

For $\tau = 0$ we choose $U_{z_0} \subset G_0$ and for $\tau = 1$ we choose $U_{z_1} \subset G_1$. For $\tau \in (0,1)$ we choose a disk $U_{\gamma(t)} \subset D$ and let

$$\Phi_\tau(z) = \int_{\gamma|_{[0,\tau]}} f + \int_{\gamma(\tau)}^{z} f, \quad z \in U_{\gamma(\tau)}.$$

This definition is correct since in a disk $U_{\gamma(\tau)}$ the integral doesn't depend on path.

The neighborhood u_τ is chosen by continuity of γ to satisfy the first property.

Let's check the second property. Let $t \in u_\tau$ and $z \in U_{\gamma(\tau)} \cap U_{\gamma(t)}$. If $t > \tau$ then

$$\Phi_\tau(z) = \int_{\gamma|_{[0,\tau]}} f + \int_{\gamma|_{[\tau,t]}} f + \int_{\gamma(t)}^{z} f = \Phi_t(z),$$

where the last integral is considered by a segment that connects $\gamma(t)$ and z. If $t < \tau$ then the second integral should be considered along the path $(\gamma|_{[\tau,t]})^-$. □

2.5 Monodromy theorem

Theorem 2.14. *Suppose γ_0, γ_1 are two paths with common endpoints*

$$\gamma_0(0) = \gamma_1(0) = a, \quad \gamma_0(1) = \gamma_1(1) = b,$$

homotopic to each other in $\mathbb{C}$. Let $\Gamma : I \times I \to \mathbb{C}$ be a homotopy of these two paths

$$\gamma(s,0) = a, \quad \gamma(s,1) = b \quad \text{for every } s \in I,$$

paths

$$\gamma_s : I \to \mathbb{C}, \quad \gamma_s(t) := \Gamma(s,t),$$

perform deformation of path γ_0 to path γ_1. Suppose that canonical element $F = (U, f)$ centered at a has analytic continuation $\{F_{s,t} : t \in I\}$ along every path γ_s, $s \in I$. Then results of continuations F along γ_0 and γ_1 coincide

$$F_{01} = F_{11}.$$

Proof. Let $s_0 \in I$. By Lemma 2.7 there exists $\varepsilon > 0$ such that radius $R(s_0, t)$ of an element $F_{s_0,t}$ satisfies the estimate +

$$R(s_0, t) > \varepsilon \quad \text{for every } t \in I.$$

By the uniform continuity of γ there exists a neighborhood $v = v_{s_0} \subset I$ of a point s_0 such that for every $s \in v$

$$\max_{t \in I} |\gamma(s_0, t) - \gamma(s, t)| < \frac{\varepsilon}{4}.$$

We will prove that

(1) for every $s \in v$ the result of analytic continuation of F along γ_s coincides with the result of analytic continuation of F along γ_{s_0};

To check this we will construct for $s \in v$, $t \in I$ a new family of canonical elements $\widetilde{F}_{st}$ that is defined in the following way. An element $\widetilde{F}_{st} := \left(\widetilde{U}_{st}, \widetilde{f}_{st}\right)$ will have center at a point $\gamma(s, t)$ and is defined by Taylor series $\tilde{f}_{st}$ of a function $f_{s_0 t}$ at point $\gamma(s, t)$.

It is enough to check the following.

(2) For every $s \in v$ a family of elements $\left\{\widetilde{F}_{st} : t \in I\right\}$ performs analytic continuation of $\widetilde{F}_{s0} = F$ along γ_s. Indeed, from this assertion, the result F_{s1} of analytic continuation of F along γ_s coincides with $\widetilde{F}_{s1} := F_{s_0 1}$, which proves assertion **(1)**.

To prove **(2)** We need to construct for every $t_0 \in I$ a neighborhood $\tilde{u}_{t_0}$. Let $\delta > 0$ be such that

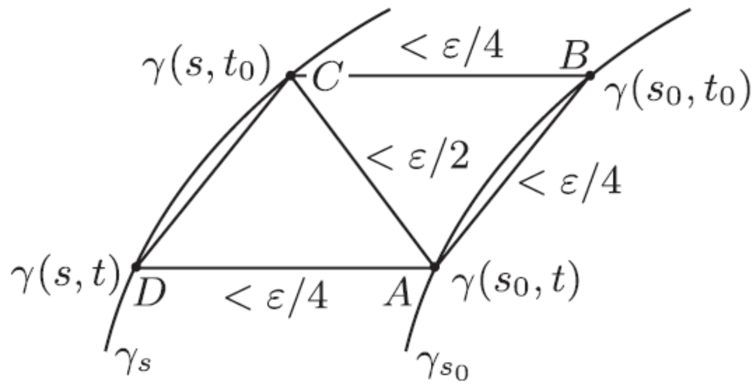
$$|\gamma(s_0, t_1) - \gamma(s_0, t_2)| < \frac{\varepsilon}{4}$$

for $|t_1 - t_2| < \delta$ and prove that neighborhoods

$$\widetilde{u}_{t_0} := u_{t_0} \cap \left(t_0 - \frac{\delta}{2}, t_0 + \frac{\delta}{2}\right)$$

satisfy the property from the definition of analytic continuation along the curve.

Indeed, a point $\gamma(s_0, t)$ belongs to disks $U_{s_0t_0}$, U_{s_0t}, $\widetilde{U}_{st_0}$, $\widetilde{U}_{st}$.



Consequently, applying the triangle property of direct analytic continuation we prove that (triangle ABC)

- $\widetilde{F}_{st_0}$ is DAC of $F_{s_0t_0}$ (by definition of $\widetilde{F}$);
- $F_{s_0t_0}$ is DAC F_{s_0t} (by definition u_{t_0}).

Consequently, again by triangle property applied to triangle ACD we see that

- $\widetilde{F}_{st_0}$ is DAC F_{s_0t};
- F_{s_0t} is DAC of $\widetilde{F}_{st}$ (by definition of $\widetilde{F}$.

Hence, $\widetilde{F}_{st}$ is DAC of $\widetilde{F}_{st_0}$ for every $t \in \tilde{u}_{t_0}$. This proves assertion (2), and, consequently, assertion (1).

This proves that G_s is locally constant with respect to $s \in [0, 1]$, and, hence, G_s doesn't depend on $s \in [0, 1]$. $\square$

Corollary 2.14.1 (Monodromy theorem)**.** *Let $D \subset \mathbb{C}$ be a simply connected domain, $F = (U, f)$ is a canonical element with center at point $a \in D$. Suppose that this element has analytic continuation along any path γ in D with starting point at a. Then for every $b \in D$ analytic continuation F along any path γ that begins at a and ends at b provides the same result $G = (V, g)$. This means that analytic continuation of F all possible paths define a function that is holomorphic in D. In the neighborhood V of a point b this function is defined by the Taylor's series g of an element $G = (V, g)$ obtained by analytic continuation of an element $F = (U, f)$ along arbitrary path that starts at a and ends at b. This function defines analytic continuation of function f to domain D.*

3 Analytic functions

Definition 3.1. *Let $D \subset \mathbb{C}$ be a domain and $F_0 = (U_0, f_0)$ be a canonical element with center at point $a \in D$ and such that $U_0 \subset D$ that has analytic continuation along any path γ in D with starting point at a. A set $\mathcal{F}$* "analytic . contin." 解析素. *of all canonical elements that can be obtained by continuation F_0 of all such paths is called an analytic function in domain D generated by element F_0.*

Example 3.1. *The simplest example of an analytic function in domain D is a set $\mathcal{F}$ of all canonical elements $F_a = (U_a, f_a)$, $a \in D$, of some holomorphic function $f \in H(D)$. Here, $f_a(z)$ is a sum of Taylor's function f with center at $a \in D$ and U_a is a disk of convergence of f. In future we will identify this analytic function $\mathcal{F}$ with usual holomorphic function f and say that $\mathcal{F}$ is single valued in D.*

Monodromy theorem implies that every analytic function in simply connected domain is single valued.

Definition 3.2. *Suppose that $\mathcal{F}$ is analytic function in domain D and $D_1 \subset D$ is subdomain. If there exists a canonical element $F_1 = (U_1, f_1) \in \mathcal{F}$ continuations of which along all possible paths $\gamma \subset D_1$ defines a holomorphic function $g \in H(D_1)$ then analytic function $\mathcal{F}$ allows single-valued (holomorphic) branch in domain D_1 and pair (D_1, g) is called a holomorphic branch (analytic element) of analytic function $\mathcal{F}$ in domain D_1.*

Notice that a class of sets that we call analytic functions doesn't change if we consider analytic function $\mathcal{F}$ as a set of all its analytic branches (not necessary canonical). Herewith the definition of analytic function $\mathcal{F}$ as a set of all branches (analytic elements) of any of its (analytic) element along a chain.

Definition 3.3. *Analytic elements (D_1, f_1) and (D_2, f_2) are called equivalent at point $a \in D_1 \cap D_2$ if $f_1 \equiv f_2$ in some neighborhood of a. Classes of equivalence are called germs at point a.*

A set of all germs at point a generates a ring denoted by $\mathcal{O}_a$. If the germ of function f at point a to denote by $\{f\}_a$ then operation on the germs are defined as following

$$\{f\}_a + \{g\}_a := \{f + g\}_a, \quad \{f\}_a\{g\}_a := \{fg\}_a.$$

Analytic continuation of a germ φ_0 along path $\gamma : I \to \mathbb{C}$ is a family of germs $\varphi_t \in \mathcal{O}_{\gamma(t)}$ such that for every $t_0 \in I$ there exists a neighborhood

$u = u_{t_0} \subset I$ of a point t_0, a domain $D_{t_0} \subset \mathbb{C}$ and a function $f \in H(D_{t_0})$ such that

$$\gamma(u) \subset D_{t_0} \text{ and } \{f\}_{\gamma(t)} = \varphi_t \text{ for every } t \in u.$$

Summarizing definitions of this paragraph we see that analytic function can be considered in three ways

(1) As a family of canonical elements obtained by analytic continuation of some initial element.

(2) A set of branches obtained by analytic continuation of some initial element (along a chain or a path).

(3) A set of germs obtained by analytic continuation (along paths)

Definition 3.4. *Analytic function $\mathcal{F}$ on D is called complete if no germ of $\mathcal{F}$ has analytic continuation outside of D.*

Definition 3.5. *Values of an analytic function $\mathcal{F}$ at point z are values of all its elements (or branches, or germs) defined at z. A set of all values of $\mathcal{F}$ at point z is denoted by $\mathcal{F}(z)$, that is*

$$\mathcal{F}(z) = \{f(z) : (U, f) \in \mathcal{F},\ z \in U\}.$$

Notice that $\mathcal{F}$ is single valued if and only if for every $z \in D$ a set $\mathcal{F}(z)$ consists of one value.

Example 3.2 (Analytic function $\sqrt{z}$.)**.** *We define the initial element f_0 of this function as*

$$f_0(z) = \sqrt{|z|}e^{i \arg z/2}, \quad -\pi < \arg z < \pi.$$

The function f_0 defined by this formula is holomorphic in a slit plane $\mathbb{C}\backslash(-\infty, 0]$.

From equality $f_0(z)^2 = z$ we see that

$$f_0'(z) = \frac{1}{2f_0(z)}.$$

Consider Taylor series of function f_0 with center at point $z = 1$. This series converges in a disk $U_0 = \{|z-1| < 1\}$ which coincides with disk of convergence since $f'(z) \to \infty$ as $z \to 0$.

Lemma 3.6. *Canonical element (U_0, f_0) has analytic continuation along any path $\gamma \subset \mathbb{C}\backslash\{0\}$ with starting point at $z = 1$ and doesn't have analytic continuation along any path $\gamma \subset \mathbb{C}$ that passes through 0.*

Proof. Let $\gamma : I \to \mathbb{C}\backslash\{0\}$ with $\gamma(0) = 1$. Then

$$\gamma(t) = |\gamma(t)|\, e^{i\theta(t)},$$

where $\theta : I \to \mathbb{R}$ is some continuous function with $\theta(0) = 0$. Let

$$U_t := \{z \in \mathbb{C} : |z - \gamma(t)| < |\gamma(t)|\}$$

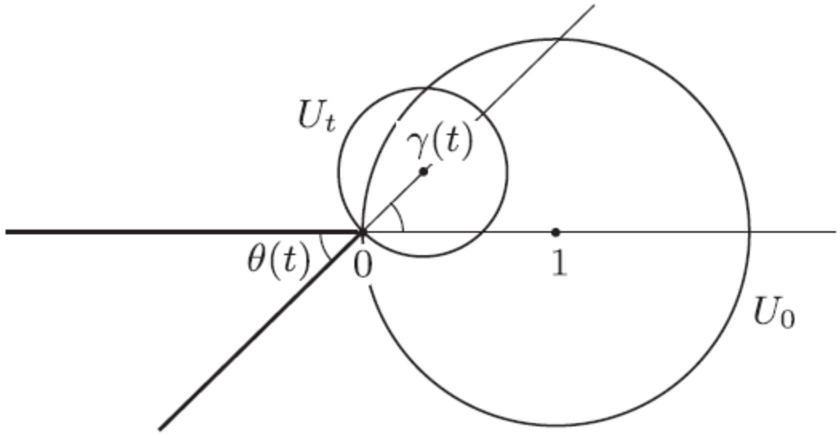
and define a family of elements $f_t \in H(U_t)$ that are analytic continuations of an element (U_0, f_0) along a path γ by the formula

$$f_t(z) = \sqrt{|z|}e^{i \arg z/2}, \quad -\pi + \theta(t) < \arg z < \pi + \theta(t).$$

Function f_t defined by this formula is holomorphic in a slit plane $\mathbb{C}\backslash\mathbb{R}_{\theta(t)}$, where

$$\mathbb{R}_\theta = e^{i\theta}\mathbb{R}_- = \{xe^{i\theta} : x \in (-\infty, 0]\}$$

is a ray from the origin by angle $\pi + \theta$. We choose a neighborhood u_{t_0} so that $\gamma(u_{t_0}) \subset U_{t_0}$ by continuity of $\theta(t)$.



Suppose that $\gamma : I \to \mathbb{C}$ is such that $\gamma(t_0) = 0$ for some $t_0 \in I$. Then for every continuation $\{F_t : t \in I\}$ along γ we have

$$f'_{t_0}(z) = \frac{1}{2f_{t_0}(z)} \to \infty, \quad z \to \gamma(t_0) = 0$$

since the equality $f'_t(z) = \frac{1}{2f_t(z)}$ remains true for every t by uniqueness theorem. This proves that continuation along γ is impossible. $\square$

Analytic function on $\mathbb{C}\backslash\{0\}$ generated by a set of analytic continuations of initial element (U_0, f_0) along all possible paths in $\mathbb{C}\backslash\{0\}$is denoted by $\sqrt{z}$. This function is double-valued in the following sense: for every $z \in \mathbb{C}\backslash\{0\}$ there exist exactly two canonical elements centered at z.

3.1 Analytic function $\mathrm{Ln}\, z$.

The initial element (U_0, f_0) of this function is a disk $U_0 = \{z : |z-1| < 1\}$ of radius 1 with center at $z = 1$ and Taylor series f_0 defined by

$$f_0(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n = \ln|z| + i \arg z, \quad -\pi < \arg z < \pi.$$

This element allows continuation along any path $\gamma \subset \mathbb{C}\backslash\{0\}$ that starts from $z = 1$ either by the formula

$$f_t(z) = \ln|z| + i \arg z, \quad -\pi + \theta(t) < \arg z < \pi + \theta(t)$$

or by the integral

$$f_t(z) = \int_{\gamma|_{[0,t]}} \frac{d\zeta}{\zeta} + \int_{\gamma(t)}^{z} \frac{d\zeta}{\zeta},$$

where the second integral is considered with respect to the segment that connects $\gamma(t)$ and z in the disk

$$U_t = \{z \in \mathbb{C} : |z - \gamma(t)| < |\gamma(t)|\,.$$

As result we obtain the function that has infinite (countable) number of elements at every point of $\mathbb{C}\backslash\{0\}$.

3.1.1 Properties of $\operatorname{Ln} z$.

L1. If (G, f) is a holomorphic branch of logarithm then $f'(z) = \frac{1}{z}$ for every $z \in G$. In particular, $z \notin G$.

Proof. For initial element (U_0, f_0) the equality is true

$$f_0'(z) = \frac{1}{z}.$$

If (G, f) is analytic continuation of (U_0, f_0) then (G, f') is analytic continuation (U_0, f_0') along the same chain. By the uniqueness of analytic continuation $f'(z) = \frac{1}{z}$ for every $z \in G\backslash\{0\}$. Finally $0 \notin G$ since f' has no analytic continuation at 0. □

L2. Let G be domain in $\mathbb{C}$. Then the followin assertions are equivalent.

1. There exists analytic branch of logarithm in G.

2. There exists antiderivative of function $1/z$ in G.

L3. Domain of $\operatorname{Ln} z$ is $\mathbb{C}\setminus\{0\}$.
L4. If (G, f) is a holomorphic branch of logarithm then $e^{f(z)}$ for every $z \in G$.

Proof. On $(0, 1)$ the equality $e^{f_0(z)} = z$ is true by the definition of real logarithm. Hence, by the uniqueness theorem it is true in U_0. If (G, f) is analytic continuation of (U_0, f_0) then (G, e^f) is analytic continuation of $(U_0, e^{f_0}) = (U_0, z)$ along the same sequence of disks. Hence, by the uniqueness theorem $e^{f(z)} = z$. □

L5. If (G, f) is a holomorphic branch of logarithm, $k \in \mathbb{Z}$, then $(G, f + 2\pi ki)$ is also holomorphic branch of logarithm and there is no other branches in G.

Proof. Let $k \in \mathbb{Z}\setminus\{0\}$. For $r > 0$ by γ_r^k we denote a path defined by a circle of radius r centered at 0 traversed k times counterclockwise if $k \in \mathbb{N}$ and $-k$ time clockwise if $-k \in \mathbb{N}$. Then

$$\int_{\gamma_r^k} \frac{d\xi}{\xi} = 2k\pi i.$$

If f is holomorphic branch of logarithm, $z_1 \in G$. Then $f'(z) = \frac{1}{z}$ and

$$f(z) = C + \int_{z_1}^{z} \frac{d\xi}{\xi}$$

for some constant C. Consequently, for $r = |z_1|$ we have

$$g(z) = f(z) + 2\pi ki = C + \int_{z_1}^{z} \frac{d\xi}{\xi} + \int_{\gamma_r^k} \frac{d\xi}{\xi}$$

and by Theorem on analytic continuation along path g is analytic continuation of (G, f) and, hence, of (U_0, f_0).

To prove that there is no other holomorphic branches suppose that g is a branch in G. Then $g = f + A$ for some $A \in \mathbb{C}$ by **L1** as antiderivative of $1/z$. Hence,

$$z = e^{g(z)} = e^{f(z)} e^A = z e^A$$

and $A = 2k\pi i$ for some $k \in \mathbb{Z}$. $\square$

L6. $\operatorname{Ln} z = \{w \in \mathbb{C} : e^w = z\},\ z \neq 0.$
L7. $\operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z,\ z \neq 0$. In particular, for principal branches of logarithm

$$\ln z = \ln |z| + i \arg z.$$

3.2 Operations on analytic functions.

Suppose that $\mathcal{F}$ and $\mathcal{G}$ are two analytic functions in domain $D \subset C$ and $F = (U, f)$, $G = (V, g)$ are two canonical elements. Then canonical elements centered at a of functions f', $f + g$, fg allow analytic continuations along any path in D. The result of these continuations is denoted by $\mathcal{F}'$, $\mathcal{F} + \mathcal{G}$, $\mathcal{F}\mathcal{G}$. Notice that constructed sets do not necessary define the unique analytic continuation. It may happen that each of the sets $\mathcal{F}'$, $\mathcal{F} + \mathcal{G}$, $\mathcal{F}\mathcal{G}$ decomposes into some number of **different (!)** analytic functions.

Example 3.3. • *The sum*

$$\sqrt{z} + \sqrt{z}$$

consists of two analytic functions $2\sqrt{z}$ *and* 0.

- *The sum*

$$\operatorname{Ln} z + i \operatorname{Ln} z$$

consists of countable number of different analytic functions.

- $(\operatorname{Ln} z)' = 1/z$ *defines a unique single-valued function.*

To define composition $\mathcal{G} \circ \mathcal{F}$ assume that $\mathcal{F}$ is analytic function in domain $D \subset \mathbb{C}$ which all values belong to domain D_1 on which function $\mathcal{G}$ is defined. Consider canonical elements $F = (U, f)$ and (V, g) of these functions centered at $a \in D$ and $f(a) \in f(D)$ respectively. Hence, canonical element with center a with function $g \circ f$ allows continuation along any path in D. The set of all such continuations is denoted by $\mathcal{G} \circ \mathcal{F}$ and may consist of one ore several analytic functions.

Example 3.4. *The composition $\Phi \circ \mathcal{F}$ of function $\Phi(w) = w^2$ and $\mathcal{F} = \sqrt[4]{z}$ is unique analytic function $(\sqrt[4]{z})^2 = \sqrt{z}$ (that doesn't coincide with $\sqrt[4]{z}\sqrt[4]{z} = \{\sqrt{z}, i\sqrt{z}\}$) while the composition $\mathcal{F} \circ \Phi$ consists of two functions $\sqrt{w}$ and $i\sqrt{w}$.*

Remark 3.7. *Suppose that $\mathcal{F}$ is analytic function in $D \subset \mathbb{C}$, $\mathcal{F}(D) \subset G$ and $\Phi \in H(G)$. Then the composition $\Phi \circ \mathcal{F}$ defines unique analytic function in D.*

Proof. Indeed, if $\{f_\alpha : \alpha \in A\}$ is a set of all elements of $\mathcal{F}$ on some disk $U \subset D$ then all elements of composition $\Phi \circ \mathcal{F}$ on U are obtained by compositions $\Phi \circ f_\alpha$. Also if f_β is obtained by analytic continuation of f_α along some chain then $\Phi \circ f_\beta$ is obtained by analytic continuation from $\Phi \circ f_\alpha$ along the same chain. Hence, every element $\Phi \circ \mathcal{F}$ on U is obtained from any other element $\Phi \circ \mathcal{F}$ on U by analytic continuation along some path. Hence, $\mathcal{F}$ consists of the unique analytic function on D. $\square$

3.3 Power function z^α.

Definition 3.8. *Let $\alpha \in \mathbb{C}$. The function*

$$\Phi_\alpha(z) = e^{\alpha \operatorname{Ln} z}, \quad z \neq 0,$$

considered as the composition of analytic function $\operatorname{Ln} z$ and holomorphic function e^w, is called the power function.

The function

$$\varphi_\alpha(z) = e^{\alpha \ln z}, \quad z \in G_0 = \mathbb{C} \backslash (-\infty, 0],$$

is called the primary branch or the primary value of power function with degree α. Here, $\ln z$ denotes primary branch of $\ln z$.

By z^α we denote either the set of all values of the power function and particular values (depending on the context).

3.3.1 Properties of z^α.

1. Power function is the complete analytic function generated by the element (U, φ_α), where $U = \{z \in \mathbb{C} : |z-1| < 1\}$.

2. If (D, f) is a holomorphic branch of logarithm then $(G, e^{\alpha f})$ is a holomorphic branch of the power function.

   Proof. Indeed, if f is analytic continuation of the primary branch of logarithm $\ln z$ then $e^{\alpha f}$ is a analytic continuation of a primary branch of power function along the same chain. □

3. Since $z^\alpha = e^{\alpha \operatorname{Ln} z} = e^{\alpha(\ln z + 2k\pi i)} = e^{\alpha \ln z} e^{2k\pi i \alpha}$, $k \in \mathbb{Z}$, then there may be three cases

(a) If $\alpha \in \mathbb{Z}$ then $e^{2k\pi i\alpha} = 1$ and the function is single-valued. If $\alpha \geq 0$ then it is defined at 0.

(b) Suppose that $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$, $q \in \mathbb{N}$, $p \in \mathbb{Z}$ and the fraction is irreducible. Then z^α obtains exactly q different values, obtained, for example with the choice $k = 0, 1, \ldots, q-1$.

(c) If $\alpha \notin \mathbb{Q}$ then z^α obtains countable number of values.

4. For every branch we have $(z^\alpha)' = \frac{\alpha}{z} z^\alpha$.

Proof. Suppose that $\alpha \notin \mathbb{Z}$, $z \neq 0$. Then for every branch of $\ln z$

$$(z^\alpha)' = (e^{\alpha \ln z})' = e^{\alpha \ln z}(\alpha \ln z)' = \frac{\alpha}{z} z^\alpha.$$

□

5. For $\alpha \in \mathbb{C}$ let (D, f) be a holomorphic branch of a power function of degree α, $k \in \mathbb{Z}$. Then $\left(D, e^{2k\pi i\alpha} f\right)$ is also a holomorphic branch of the power function and there is no other branches in D.

Remark 3.9.

$$z^{\alpha+\beta} \neq z^\alpha z^\beta.$$

3.4 Isolated singular points of analytic function

Definition 3.10. *A point $a \in \mathbb{C}$ is isolated singular point of analytic function $\mathcal{F}$ if $\mathcal{F}$ is analytic function in some punctured neighborhood $\dot{V}_a$ of a, where*

$$\dot{V}_a = \{z \in \mathbb{C} : 0 < |z - a| < \varepsilon\}$$

if $a \in \mathbb{C}$ *or*

$$\dot{V}_\infty = \{z \in \mathbb{C} : |z| > \varepsilon^{-1}\}$$

for $a = \infty$.

Lemma 3.11. *Suppose that $\dot{V}_a$ is a punctured neighborhood of a point a. Then for every closed path $\gamma : I \to V$ with $\gamma(0) = \gamma(1) = z_0 \in \dot{V}_a$ there exists unique $n \in \mathbb{Z}$ such that γ is homotopic to γ_0^n, where*

$$\gamma_0(t) = a + (z_0 - a)\, e^{2\pi i t}, \quad 0 \leqslant t \leqslant 1,$$

is a circle of radius $|z_0 - a|$ centered at point a.

Proof. **Existence.** Consider a polar expression

$$z_0 - a = |z_0 - a|\, e^{i\varphi_0},$$

where $\varphi_0 = Arg(z_0 - a)$. Then

$$\gamma_0(t)^n = a + |z_0 - a|\, e^{i(\varphi_0 + 2\pi n t)}, \quad 0 \le t \le 1.$$

Path $\gamma(t)$ can be also written in a polar form:

$$\gamma(t) = a + |\gamma(t) - a|\, e^{i\varphi(t)}, \quad t \in I = [0, 1],$$

where $\varphi(t) = Arg(\gamma(t) - a)$ is a continuous real-valued function on a segment I with $\varphi(0) = \varphi_0$. Since $\gamma(0) = \gamma(1) = z_0$ then $\varphi(1) - \varphi(0) = 2\pi n$ for some $n \in \mathbb{Z}$. Define a map $\Gamma : I \times I \to V$ by the formula

$$\Gamma(s, t) = a + |z_0 - a|^s\, |\gamma(t) - a|^{1-s} e^{i\{(\varphi_0 + 2\pi n t)s + \varphi(t)(1-s)\}}.$$

The Γ is continuous, and

$$\Gamma(0, t) = a + |\gamma(t) - a| e^{i\varphi(t)} = \gamma(t)$$

$$\Gamma(1,t) = a + |z_0 - a|e^{i(\varphi_0 + 2\pi n t)} = a + |z_0 - a|e^{2\pi i n t} = \gamma_)^n(t).$$

Consequently, Γ is a homotopy of γ to γ_0^n.

Uniqueness. If $\gamma \sim \gamma_0^n \sim \gamma_0^m$ then

$$n = \frac{1}{2\pi i}\int_{\gamma_0^n} \frac{dz}{z-a} = \frac{1}{2\pi i}\int_{\gamma_0^m} \frac{dz}{z-a} = m.$$

□

Lemma 3.12. *Suppose that a is isolated singular point of $\mathcal{F}$ defined in some punctured neighborhood $\dot{V}_a$. If the result of the analytic continuation of some canonical element $F_0 \in \mathcal{F}$ (with center at $z_0 \in \dot{V}_a$) along circle γ_0 coincides with F_0 then $\mathcal{F}$ is single-valued on $\dot{V}_a$.*

Proof. By the previous lemma for any two paths γ_1 and γ_2 that begin at z_0 and end at z there exists $n \in \mathbb{Z}$ such that

$$\gamma_2 = \gamma_1 \cup \gamma_0^n.$$

Hence, by assumption of the lemma the result of analytic continuations along paths γ_1 and γ_2 coincide. Hence, $\mathcal{F}$ is single-valued. □

3.5 Classification of isolated singular points

Suppose that $\mathcal{F}$ analytic in punctured neighborhood $\dot{V}_a$ of a point a generated by the canonical element $F_0 \in \mathcal{F}$ with center at $z_0 \in \dot{V}_a$ For $n \in \mathbb{Z}$ we denote by F_n the result of analytic continuation along γ_0^n defined above.

單值的

Definition 3.13. .

(1) *If $F_1 = F_0$ then analytic function $\mathcal{F}$ is single-valued and holomorphic in $\dot{V}_a$. In this case a is isolated singular point of single-valued character of $\mathcal{F}$.*

(2) *If $F_1 \neq F_0$ then a is branch point of $\mathcal{F}$. If $F_n = F_0$ for some $n \geq 2$ then is branch point of finite order of $\mathcal{F}$ and $m = \min\{n \geq 2 : F_n = F_0\}$ is the order of the branch point. In the converse case a is logarithmic branch point.*

Further classification of isolated singular points of single-valued character (to removable singular points, poles and essential singular points) is reduced to the classification of isolated singular points for holomorphic functions.

Lemma 3.14. *The definition given above doesn't depend on the choice of the initial canonical element.*

Proof. Consider another canonical element $\tilde{F}_0$ of analytic function $\mathcal{F}$ centered at $\tilde{z}_0 \in \dot{V}_a$. Suppose that $\tilde{\gamma}_0$ is a closed path

$$\gamma(t) = a + (\tilde{z}_0 - a)e^{2\pi i t}, \quad t \in I = [0, 1].$$

Denote by $\tilde{F}_n$ the result of analytic continuation of $\tilde{F}_0$ along $\tilde{\gamma}_0^n$.

Let $\lambda \subset \dot{V}_a$ be some path that connects $\tilde{z}_0$ with z such that $\tilde{F}_0$ is the analytic continuation of F_0 along λ. By λ_0 we denote the standart path that connects $\tilde{z}_0$ and z as following: first this path goes from $\tilde{z}_0$ to $|\tilde{z}_0|\, e^{i \arg z}$ counterclockwise and then by the segment of radius of $\dot{V}_a$ that connect this point with z. Then $\lambda \cup \lambda_0^{-1}$ is closed path, consequently,

$$\lambda \cup \lambda_0^{-1} \sim \gamma_0^m \quad \Rightarrow \quad \lambda \sim \gamma_0^m \cup \lambda_0.$$

Also

$$\tilde{\gamma}_0 \sim \lambda_0^{-1} \cup \gamma_0 \cup \lambda_0$$

and $\tilde{\gamma}_0 \sim \lambda_0^{-1} \cup \gamma_0 \cup \lambda_0$. Hence,

$$\widetilde{\gamma}_0^n \sim \lambda^{-1} \cup \gamma_0^n \cup \lambda, \quad n \in \mathbb{Z}.$$

Consequently, continuation of $\tilde{F}_n$ along λ coincides with F_n for every $n \in \mathbb{Z}$. Hence, conditions $F_n = F_0$ and $\tilde{F}_n = \tilde{F}_0$ are equivalent. $\square$

Definition 3.15. *Suppose that $\mathcal{F}$ is analytic in D and $\dot{V}_a \subset D$ is a punctured neighborhood of $a \in \mathbb{C}$. The restriction $\mathcal{F}\Big|_{\dot{V}_a}$ may consist of several analytic functions on V and each function may have it's own kind of singularity at a. Singular points of these branches are called singular points of function $\mathcal{F}$ above point a.*

Example 3.5. *Consider the analytic function defined by the formula*

$$\sqrt{1+\sqrt{z}} = \begin{cases} \varphi_0(1+\varphi_0) & + \\ \varphi_1(1+\varphi_0) & - \\ \varphi_0(1+\varphi_1) \\ \varphi_1(1+\varphi_1) \end{cases}$$

$\}$ 一个函数的两个分支

$$\mathcal{F}(z) = \frac{1}{\sqrt{1+\sqrt{z}-\sqrt{2}}}, \quad z \in \mathbb{C}\backslash\{0,1\}.$$

$\varphi_0 = \sqrt{z} : \mathbb{C} \backslash (-\infty, 0] \to \{\operatorname{Re} z > 0\}$

$\varphi_1 = \sqrt{z} : \mathbb{C} \backslash (-\infty, 0] \to \{\operatorname{Re} z < 0\}.$

$1+\sqrt{z} \in \{\operatorname{Re} z > 1\}$

Its restriction to the set

$e^{i\theta}$. $e^{i(\theta+2\pi)}$.

$$\dot{V}_1 = \{0 < |z-1| < 1/2\}$$

$$\mathcal{F}|_V = \left\{ \frac{1}{\varphi_0(1+\varphi_0(z))-\sqrt{2}}, \ \frac{1}{\varphi_1(1+\varphi_0(z))-\sqrt{2}}, \right.$$

consists of three different analytic functions, one of which has at $z = 1$

$\Downarrow$ $z=1$ pole $\quad$ $\Downarrow$ $z=1$ removable

branch point of order 2, another removable singularity, the third one the pole.

$$\frac{1}{\sqrt{1+\varphi_1(z)}-\sqrt{2}} \Rightarrow z=1 \text{ branch point.} \Big\}$$

So, we say that $\mathcal{F}$ has three singular points above $z = 1$ (branch point of order 2, removable singular point and the pole).

3.6 Examples of analytic functions and their singular points.

Example 3.6. *The root of order n $\Phi_{1/n}(z) = z^{1/n}$ is $n-$valued analytic function on $\mathbb{C}\backslash\{0\}$ that has two branch points $z = 0, \infty$ of order n. Similarly,* $\operatorname{Ln} z$ *has two logarithmic branch points $z = 0, \infty$.*

Example 3.7. *Analytic function* $\sqrt{\tan\frac{1}{z}}$ *has singular points of second order at* $z=\infty$ *and*

$$z=\frac{1}{\pi n} \text{ and } z=\frac{1}{\pi/2+\pi m}, \quad n,m\in\mathbb{Z}.$$

Point $z=0$ *is not isolated singular point.*

Lemma 3.16. *Suppose function* f *has at point* $a\in\overline{\mathbb{C}}$ *zero of order* 1. *Then analytic function* $\sqrt{f(z)}$ *has branch point of order* 2 *at* $z=a$.

Proof. Consider the case $a\in\mathbb{C}$. Then in some neighborhood U of a we have

$$f(z)=(z-a)g(z),\ g\in H(U),\ g(z)\neq 0, z\in U.$$

Hence, $g=h^2$ for some $h\in H(U)$ and

$$\sqrt{f(z)}=h(z)\sqrt{z-a}.$$

Consequently, $\sqrt{f}$ has at $z=a$ the same singularity as $\sqrt{z-a}$, that is the branch point of order 2. The case $a=\infty$ is reduced by the change $1/z$ to the case $z=0$. □

Example 3.8. *Let* $\alpha\in\mathbb{C}$. *The power function*

$$\Phi_\alpha(z)=z^\alpha=e^{\alpha\ln z}$$

is analytic function in $\mathbb{C}\backslash\{0\}$.

1. *If* $\alpha\in\mathbb{Z}$ *then* $\Phi_\alpha(z)$ *is single-valued function on* $\mathbb{C}\backslash\{0\}$ *that has at* $z=0$ *removable singularity for* $\alpha\geq 0$ *and pole of order* $|\alpha|$ *if* $\alpha<0$.

2. *If* $a\in\mathbb{Q}\backslash\mathbb{Z}$ *and* $a=p/q$ *is irreducible ration then* z^a *has branch point of order* q *at* ∞ *and* 0.