Chapter 4. Trapezoids method for calculation of definite integrals

$$\int_a^b f(x)\,dx$$

When do you need to use a numerical method?

1st case: Function $f(x)$ **is given by a formula; however, the integral cannot be expressed in terms of elementary functions** $sin(x), cos(x), tan(x), exp(x),...$

For example,

$$\int_a^b e^{x \sin(\cos(\sin x))}\,dx$$

2nd case: values of function $f(x)$ are only given at finite number of points of the segment $[a,b]$:

$$y_0 , y_1 , y_2 , \ldots, y_n$$

$$\text{at} \qquad a=x_0 , x_1 , x_2 , \ldots, x_n=b$$

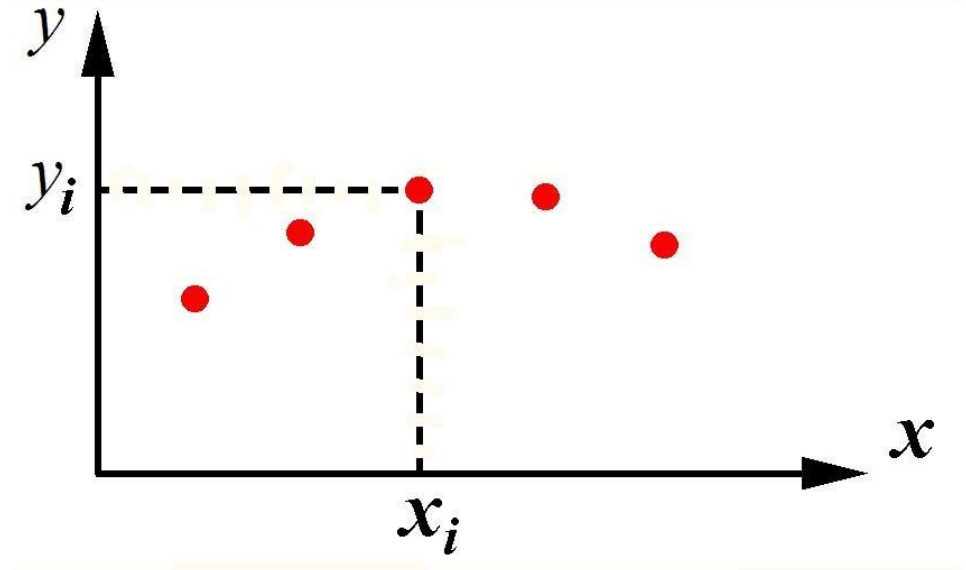
(for example, y_i are results of some experiments).

$$I=\int_{x_0}^{x_n} f(x)\, dx = ?$$

In case 1, when $f(x)$ is given by a formula, we can choose points x_i ourselves and calculate $y_i=f(x_i)$

Assume that $h_i = x_{i+1} - x_i$ $=\text{const} = h$

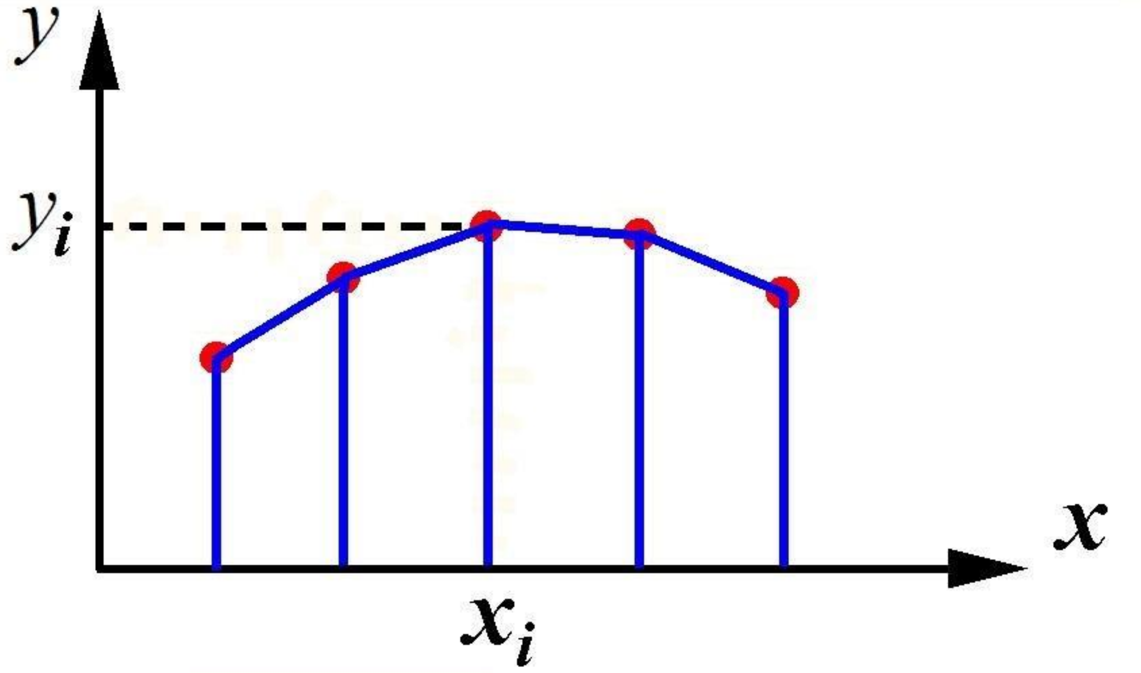
$$I = \int_{x_0}^{x_n} f(x)\, dx = \ ?$$



Trapezoids formula:

$$I = \int_{x_0}^{x_n} f(x)\, dx \approx h(y_0/2 + y_1 + y_2 + \ldots + y_{n-1} + y_n/2)$$

Derivation of the formula:



Suppose that $f(x)$ is well approximated on $[x_i, x_{i+1}]$ by the linear function

$$f(x) \approx y_i + (y_{i+1} - y_i)(x - x_i)/h$$

$$\int_{x_i}^{x_{i+1}} [\, y_i + (y_{i+1} - y_i)(x - x_i)/h \,]\, dx =$$

$$= y_i h + (y_{i+1} - y_i) \int (x - x_i)\, dx \,/\, h \;=$$

$$= y_i h + (y_{i+1} - y_i)\, [(x_{i+1} - x_i)^2 - (x_i - x_i)^2]/2h =$$

$$= y_i h + (y_{i+1} - y_i)\, h / 2 =$$

$$= (y_i + y_{i+1})\, h/2$$

Actually, this expression is evident, as an integral is known to be the area below the plot of function *f(x)* ; *in the case of linear function the area is halved sum of bases × height of trapezoid.*

$$\int_{x_0}^{x_1} f(x)\,dx \approx h\,(y_0 + y_1)\,/\,2$$

$$\int_{x_0}^{x_2} f(x)\,dx \approx h\,(y_0+y_1+y_1+y_2)/2 = h\,(y_0+2y_1+y_2)/2$$

$$\int_{x_0}^{x_3} f(x)\,dx \approx h\,(y_0+2y_1+2y_2+y_3)/2$$

Summation over all segments $[\,x_i\,,\,x_{i+1}]$, $i = 0, 1, 2, \ldots$ gives

$$\int_{x_0}^{x_n} f(x)\,dx \approx h\left(y_0/2 + y_1 + y_2 + \ldots + y_{n-1} + y_n/2 \right)$$

***Theorem* on the error of the trapezoid formula:**

$$\int_{x_0}^{x_n} f(x)\,dx = h\left(y_0/2 + y_1 + y_2 + \ldots + y_{n-1} + y_n/2\right) - \underbrace{- f''(c)\, h^2\, (x_n - x_0)/12}_{\text{Error}}$$

where c is some point in the interval $x_0 < x < x_n$

(Proof is omitted).

Sometimes $f''(c)$ **can be estimated clearly:**

$$f(x) = \sin x^2 \qquad \int_0^1 \sin x^2 \, dx$$

$$f'(x) = \qquad\qquad f''(x) =$$

$$|f''(x)| \le$$

Therefore

$$|f''(c)|\, h^2 (x_n - x_0) / 12 \le$$

Example: calculate the integral

$$\int_0^1 e^{x \sin(\cos(\sin x))}\, dx$$

Way 1: write a short Scilab code

```
n=100
h=1/n
x= 0: h :1
y= exp(x.*sin(cos(sin(x))))
Int=(y(1)+y(n+1))/2
for i=2: n
Int=Int+y(i)
end
Int=h*Int
disp(Int)  ;    plot(x,y)
```

Way 2. Scilab

```
x= 0 : 0.01 : 1

y=exp(x.*sin(cos(sin(x))))
Int=inttrap(x,y)
```

Another example

$$\int_{5}^{13} \sqrt{2x-1}\; dx$$

```
clear
x=5: 0.5 : 13
y=sqrt(2*x-1)
Int=inttrap(x,y)
32.663890
```

```
x=5: 0.1 : 13
y=sqrt(2*x-1)
Int=inttrap(x,y)
32.666556
```

In fact,

$$\int_{5}^{13} \sqrt{2x-1}\; dx$$

can be expressed in an analytic form:

$$= (1/3)\,(2x-1)^{3/2} \Big|_{x=5}^{x=13} =$$

$$= (1/3)\,(26-1)^{3/2} - (1/3)\,(10-1)^{3/2}$$

$$\text{Int*} = (1/3)(25^{3/2} - 9^{3/2}) =$$

Matlab:

```
Int= trapz(x,y)
```