Real Analysis 2024. Samples.

1. Suppose $f, g \in L^p(X,\mu)$. Prove that $f+g \in L^p(X,\mu)$.
2. Suppose $f \in L^p(X,\mu)$, $g \in L^\infty$. Prove that $fg \in L^p(X,\mu)$.
3. Is it true that every a.e. convergenct sequence of functions contains subsequence that is convergent in measure? Explain the answer.
4. Is it true that $L^1(x,\mu) \subset L^p(X,\mu)$ if $p>1$? Is it true that $L^p(x,\mu) \subset L^1(X,\mu)$ if $p>1$?
5. Let $(X,\mathcal{A})$ be a measurable space. The identify map
$$\mathrm{id} : (X,\mathcal{A}) \to (X,\mathcal{A})$$
is map such that $\mathrm{id}(x) = x$. Is this map measurable?
6. Let $(X,\mathcal{A})$ be a measurable space, $E \subset X$. Under which condition the characteristic function χ_E is measurable?
7. Provide the definition of $\sigma-$algebra generated by a family of subsets.
8. Provide the definition of the integral with respect to a measure (for simple functions, nonnegative measurable functions, arbitrary measurable functions). What is the condition for existence of the integral?
9. Choose the correct statements. Explain the choice.
   (a) If $\int_E f(x)d\mu = 0$ then $f=0$ a.e.
   (b) If $\int_E |f(x)|\, d\mu = 0$ then $f=0$ a.e.
   (c) If f is finite a.e. then $\int_E f(x)d\mu$ is finite.
10. How is the norm in space $L^\infty(X,\mu)$ defined?
11. Find condition on p, q under which the integral
$$\int_{\mathbb{R}^2} \frac{dxdydz}{(x^2+y^2+z^2)^p(1+x^2+y^2+z^2)^q}$$
is finite.
12. Find conditions on $p, q \in \mathbb{R}$ under which
$$I = \iint_{|x|+|y|\leq 1} \frac{dxdy}{|x|^p+|y|^q} < \infty.$$
13. Calculate Fourier series of function $f(x) = \cosh x$ on $[-\pi,\pi]$.
14. Calculate Fourier series of function $f(x) = \sin^6 x \cos^4 x$ on $[-\pi,\pi]$.
15. Calculate Fourier series of function $x \sin x$ on $[-\pi,\pi]$.
16. Calculate cos and sin Fourier series of function
$$f(x) = \begin{cases} \pi/2 - x, & 0 < x < \pi/2; \\ 0, & \pi/2 \leq x < \pi. \end{cases}$$

17. * Calculate the sum $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2n-1)x}{n}$.

18. * Calculate the sum $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{\cos(2n-1)x}{(2n-1)2n}$.

19. Calculate the sum $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{(2n-1)!}$.

20. Calculate the sum $\sum\limits_{n=2}^{\infty}(-1)^{n}\frac{\cos nx}{n^2-1}$.

21. Express the following functions by Fourier integral
    - $f(x) = e^{-x^2}$;
    - $f(x) = xe^{-x^2}$;
    - $f(x) = \begin{cases} 2-3x, & 0 \leqslant x \leqslant 2/3 \\ 0, & x > 2/3 \end{cases}$ continuing f in odd way to $\mathbb{R}$.

22. Calculate Fourier transform of the following functions:
    - $f(x) = \begin{cases} \cos x, & x \in [0,\pi], \\ 0, & x \notin [0,\pi]; \end{cases}$
    - $f(x) = e^{-x^2/2}\cos \alpha x$;
    - $f(x) = \left(x^2 e^{-|x|}\right)'$.

23. Use Bessel inequality to prove that if Fourier coefficients of function $f \in C[-\pi,\pi]$ are equal to 0 then this function is identically 0.

24. Prove that Fourier transform of function $f(x) = \frac{\arctan x}{1+x^9}$ is C^7-smooth.

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