Equations of mathematical physics

Content

Introduction

The course includes the main sections:

— Fundamentals of operational calculus;

— Classification of partial differential equations;

— Hyperbolic equations;

— Parabolic equations;

— Elliptical equations.

1. FUNDAMENTALS OF OPERATIONAL CALCULUS

1.1. The concepts of the original and the Laplace image. Properties of the Laplace transform

Definition 1. *An original function* is any complex-valued function $f(t)$ of a valid argument t that satisfies the conditions:

1) $f(t)$ is Riemann integrable on any finite interval of the t axis (locally integrable);

2) $f(t)=0$ for all $t<0$;

3) $M>0$ and $\alpha>0$ are constants for which

$$|f(t)| \le Me^{\alpha t}. \tag{1.1}$$

The lower edge α_0 of all numbers α for which the inequality (1.1) is valid is called the *growth index* of the function $f(t)$.

The first condition in definition 1 is sometimes formulated as follows: on any finite interval of the t axis, the function $f(t)$ is continuous, except, perhaps, a finite number of discontinuity points of the first kind.

The simplest original function is the Heaviside function:

$$\theta(t) = \begin{cases} 0, & t<0, \\ 1, & t\ge 1. \end{cases}$$

Obviously, for any function $\varphi(t)$ it is true:

$$\varphi(t)\theta(t) = \begin{cases} 0, & t<0, \\ \varphi(t), & t\ge 1. \end{cases}$$

If, for $t \geq 0$, the function $\varphi(t)$ satisfies conditions 1 and 3 of definition 1, then the function $\varphi(t)\theta(t)$ is the original. In the future, to shorten the record, we will, as a rule, write $\varphi(t)$ instead of $\varphi(t)\theta(t)$, assuming that the functions we are considering are continued by zero for negative values of the argument t.

Definition 2. *The image of the function* $f(t)$ *according to Laplace* is called the function $F(p)$ of the complex variable $p = s + i\sigma$, defined by the equality

$$F(p) = \int_0^{+\infty} f(t)e^{-pt}dt \,. \qquad (1.2)$$

Theorem 1 (on the analyticity of the image). For any original $f(t)$, its image $F(p)$ is defined and is an analytical function of the variable p in the half—plane $Re\, p > \alpha_0$, where α_0 is the growth index of the function $f(t)$, while the equality is valid:

$$\lim_{\mathrm{Re}\, p \to +\infty} |F(p)| = 0 \,.$$

Theorem 2 (uniqueness). The Laplace image $F(p)$ is unique in the sense that two functions $f_1(t)$ and $f_2(t)$ having the same images coincide at all points of continuity at $t > 0$.

There are several ways to record the correspondence between the original and the image:

$$f(t) \leftrightarrow F(p)\,,\ f(t) \doteq F(p)\,,\ L\{f(t)\} = F(p)\,.$$

Example 1.

Using the definition, find the image of the function $f(t) = sin3t$.

Solution:

For the function $f(t) = sin3t$, we have $\alpha_0 = 0$. Therefore, the image $F(p)$ will be defined and analytically in the half-plane $Re\,p > 0$. Let us apply formula (1.2) to a given function, using the rule of integration in parts and the restriction on the set of values of the variable p, which ensures the convergence of the integral, when performing transformations:

$$F(p) = \int_0^{+\infty} e^{-pt}\sin 3t dt = -\frac{1}{p}e^{-pt}\sin 3t\Big|_0^{+\infty} + \frac{3}{p}\int_0^{+\infty} e^{-pt}\cos 3t dt =$$

$$= \frac{3}{p}\left(-\frac{1}{p}e^{-pt}\cos 3t\Big|_0^{+\infty} - \frac{3}{p}\int_0^{+\infty} e^{-pt}\sin 3t dt\right) = \frac{3}{p^2} - \frac{9}{p^2}F(p)\,.$$

Got equality

$$F(p) = \frac{3}{p^2} - \frac{9}{p^2}F(p)\,.$$

From here we find

$$F(p) = \frac{3}{p^2+9}\,.$$

Thus, the following correspondence is valid:

$$\sin 3t \leftrightarrow \frac{3}{p^2+9}, \ \mathrm{Re}\, p > 0.$$

Properties of the Laplace transform

1. **Linearity.** If $f(t) \leftrightarrow F(p)$, $g(t) \leftrightarrow G(p)$, then for any complex λ and μ it is performed

$$\lambda f(t) + \mu g(t) \leftrightarrow \lambda F(p) + \mu G(p), \ \mathrm{Re}\, p > \max(\alpha_0, \beta_0),$$

here and further, α_0, β_0 are the growth indicators of the function $f(t), g(t)$, respectively.

2. **Similarity.** If $f(t) \leftrightarrow F(p)$, then for $\forall \alpha > 0$ it is true

$$f(at) \leftrightarrow \frac{1}{a} F\left(\frac{p}{a}\right), \ \mathrm{Re}\, p > a\alpha_0.$$

3. **Differentiation of the original.** If $f(t), f'(t), ..., f^{(n)}(t)$ are originals and $f(t) \leftrightarrow F(p)$ for $Re\, p > \alpha_0$, then

$$f^{(n)}(t) \leftrightarrow p^n F(p) - p^{n-1} f(+0) - p^{n-2} f'(+0) - \ldots - p f^{(n-2)}(+0) - f^{(n-1)}(+0),$$

where

$$f^{(k)}(+0) = \lim_{t \to +0} f^{(k)}(t), \ k = 0, 1, \ldots, n-1.$$

4. **Image differentiation.** If $f(t) \leftrightarrow F(p)$, then

$$F^{(n)}(p) \leftrightarrow (-t)^n f(t), \ \mathrm{Re}\, p > \alpha_0.$$

5. **Integration of the original.** If $f(t) \leftrightarrow F(p)$, then

$$\int_0^t f(\tau)d\tau \leftrightarrow \frac{F(p)}{p}, \ \operatorname{Re} p > \alpha_0 .$$

6. **Image integration.** If $f(t) \leftrightarrow F(p)$ and $\frac{f(t)}{t}$ are the original, then

$$\int_p^\infty F(\xi)d\xi \leftrightarrow \frac{f(t)}{t}, \ \operatorname{Re} p > \alpha_0 .$$

7. **The delay property.** If $f(t) \leftrightarrow F(p)$ and $f(t)=0$ for $t<\tau$, where $\tau>0$, then

$$f(t-\tau) \leftrightarrow e^{-\tau p} F(p), \ \operatorname{Re} p > \alpha_0 .$$

Remark. The following formulation of the delay property is possible: if $f(t) \leftrightarrow F(p)$, then for any $\tau>0$ there is

$$f(t-\tau)\theta(t-\tau) \leftrightarrow e^{-\tau p} F(p), \ \operatorname{Re} p > \alpha_0 .$$

8. **The displacement property.** If $f(t) \leftrightarrow F(p)$, then for any complex λ

$$e^{\lambda t} f(t) \leftrightarrow F(p-\lambda), \ \operatorname{Re} p > \alpha_0 + \operatorname{Re}\lambda .$$

9. **The image of the convolution.** *The convolution of functions* f *and* g is a function that is denoted by $f \cdot g$ and is defined by equality

$$(f \cdot g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau .$$

The convolution of functions has the property of symmetry, that is,

$$(f \cdot g)(t) = (g \cdot f)(t) .$$

If $f(t) \leftrightarrow F(p)$ and $g(t) \leftrightarrow G(p)$, then

$$(f \cdot g)(t) \leftrightarrow F(p)G(p), \ \mathrm{Re}\, p > \max(\alpha_0, \beta_0).$$

Here is a table of originals and images of some elementary functions:

Original $f(t)$	Image $F(p)$	Original $f(t)$	Image $F(p)$
1	$\frac{1}{p}$	$sh\, at$	$\frac{a}{p^2 - a^2}$
e^{-at}	$\frac{1}{p+a}$	$ch\, at$	$\frac{p}{p^2 - a^2}$
t	$\frac{1}{p^2}$	$e^{-at} \cos \omega t$	$\frac{p+a}{(p+a)^2 + \omega^2}$
$\sin at$	$\frac{a}{p^2 + a^2}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(p+a)^2 + \omega^2}$
$\cos at$	$\frac{p}{p^2 + a^2}$	$e^{at}\, sh\, \omega t$	$\frac{\omega}{(p-a)^2 - \omega^2}$
$t^n, n \in Z$	$\frac{n!}{p^{n+1}}$	$e^{at}\, ch\, \omega t$	$\frac{p-a}{(p-a)^2 - \omega^2}$
$t^n e^{at}$	$\frac{n!}{(p-a)^{n+1}}$	$t\, sh\, \omega t$	$\frac{2\omega p}{(p^2 - \omega^2)^2}$
$t \sin \omega t$	$\frac{2p\omega}{(p^2 + \omega^2)^2}$	$t\, ch\, \omega t$	$\frac{p^2 + \omega^2}{(p^2 - \omega^2)^2}$
$t \cos \omega t$	$\frac{p^2 - \omega^2}{(p^2 + \omega^2)^2}$	$e^{at}\, t \sin \omega t$	$\frac{2\omega(p-a)}{((p-a)^2 + \omega^2)^2}$
$e^{at}\, t \cos \omega t$	$\frac{(p-a)^2 - \omega^2}{((p-a)^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	$\frac{1}{(p^2 + \omega^2)^2}$
$\frac{1}{2\omega^3}(\omega t\, ch\, \omega t - sh\, \omega t)$	$\frac{1}{(p^2 - \omega^2)^2}$	$\sin(\omega t \pm \varphi)$	$\frac{\omega \cos \varphi \pm p \sin \varphi}{p^2 + \omega^2}$
$\cos(\omega t \pm \varphi)$	$\frac{p \cos \varphi \mp \omega \sin \varphi}{p^2 + \omega^2}$		

Example 2.

Using the properties of the Laplace transform and the table of the main originals and images, find images of the following functions:

1) $f(t)=e^{-4t}\sin 3t\cos 2t$;
2) $f(t)=e^{(t-2)}\sin(t-2)$;
3) $f(t)=t^2e^{3t}$;
4) $f(t)=\dfrac{\sin^2 t}{t}$.

<u>Solution:</u>

1) Let's transform the expression for the function $f(t)$ as follows:

$$f(t)=e^{-4t}\sin 3t\cos 2t=\frac{1}{2}e^{-4t}\left(\sin 5t+\sin t\right)=\frac{1}{2}e^{-4t}\sin 5t+\frac{1}{2}e^{-4t}\sin t .$$

Since $sin t\leftrightarrow\frac{1}{p^2+1}$ and $sin 5t\leftrightarrow\frac{5}{p^2+25}$, then, using the properties of linearity and displacement, for the image of the function $f(t)$ we will have

$$F(p)=\frac{1}{2}\left(\frac{5}{(p+4)^2+25}+\frac{1}{(p+4)^2+1}\right).$$

2) Since $sin t\leftrightarrow\frac{1}{p^2+1}, e^t\, sin t\leftrightarrow\frac{1}{(p-1)^2+1}$, then, using the delay property, we will have

$$f(t)=e^{(t-2)}\sin(t-2)\leftrightarrow F(p)=\frac{e^{-2p}}{(p-1)^2+1}.$$

3) Since $t^2\leftrightarrow\frac{2}{p^3}$, then by the displacement property we have

$$f(t)=t^2e^{3t} \leftrightarrow F(p)=\frac{2}{(p-3)^3}.$$

For comparison, we present a method for constructing an image of function $f(t)=t^2e^{3t}$ using the image differentiation property:

$$e^{3t} \leftrightarrow \frac{1}{p-3};\ te^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{p-3}\right)=\frac{1}{(p-3)^2};$$

$$t^2e^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{(p-3)^2}\right)=\frac{2}{(p-3)^3}.$$

We got the same result.

4. <u>HOMEWORK (using the image integration property)</u>

For a function defined as follows:

$$f(t)=\begin{cases} 0, & t<t_1, \\ f_1(t), & t_1 \le t < t_2, \\ f_2(t), & t_2 \le t < t_3, \\ \dots & \\ f_{n-1}(t), & t_{n-1} \le t < t_n, \\ f_n(t), & t \ge t_n, \end{cases}$$

using the Heaviside function, you can write an analytical form that is convenient to use when constructing the corresponding image.

It is easy to verify that for a function $g_k(t)$ equal to

$$g_k(t)=\begin{cases} 0, & t<t_k, \\ f_k(t), & t_k \le t < t_{k+1}, \\ 0, & t \ge t_{k+1}, \end{cases}$$

the following representation is valid using the Heaviside function:

$$g_k(t) = f_k(t)\theta(t - t_k) - f_k(t)\theta(t - t_{k+1}) . \quad (1.3)$$

And for the function

$$g_n(t) = \begin{cases} 0, & t < t_n, \\ f_n(t), & t \geq t_n, \end{cases}$$

there is

$$g_n(t) = f_n(t)\theta(t - t_n) . \quad (1.4)$$

Assuming that k varies from 1 to $n-1$, the function $f(t)$ can be considered as the sum of the functions $g_k(t)$ and $g_n(t)$:

$$f(t) = \sum_{k=1}^{n-1} g_k(t) + g_n(t) .$$

And then, using expressions (1.3) and (1.4), we get

$$f(t) = f_1(t)\theta(t - t_1) - \sum_{k=2}^{n} \left(f_k(t) - f_{k-1}(t) \right) \theta(t - t_k) . \quad (1.5)$$

Example 3.

Build an image for the function $f(t)$:

$$f(t) = \begin{cases} 0, & t < a, \\ \varphi(t), & a \leq t < b, \\ 0, & t \geq b. \end{cases}$$

<u>Solution:</u>

Let's write an expression for the function $f(t)$ using the Heaviside function:

$$f(t) = \varphi(t)\theta(t-a) - \varphi(t)\theta(t-b).$$

Since

$$\varphi(t) = \varphi(t-a+a) \text{ и } \varphi(t) = \varphi(t-b+b),$$

then, having found the images for the functions $\varphi(t+a)$ and $\varphi(t+b)$,

$$\varphi(t+a) \leftrightarrow \Phi_1(p), \ \varphi(t+b) \leftrightarrow \Phi_2(p),$$

we construct an image for the function $f(t)$, taking into account the lag property

$$f(t) \leftrightarrow F(p) = \Phi_1(p)e^{-ap} - \Phi_2(p)e^{-bp}.$$

Example 4.

Find the image $F(p)$ of the function $f(t)$:

$$f(t) = \begin{cases} 0, & t \in (-\infty, 0), \\ 1, & t \in (0, a), \\ \dfrac{2a-t}{a}, & t \in (a, 3a), \\ \dfrac{t-4a}{a}, & t \in [3a, \infty). \end{cases}$$

<u>Solution:</u>

Let's find an image of the function $f(t)$, having previously written an expression for it using the Heaviside function $\theta(t)$. To do this, use the formula (1.5). Since for a given function

$t_1 = 0$, $t_2 = a$, $t_3 = 3a$ and

$$f_1(t)=1\,,\ f_2(t)=\frac{2a-t}{a}\,,\ f_3(t)=\frac{t-4a}{a}\,,$$

then we will have

$$f(t)=\theta(t)-\left(\frac{2a-t}{a}-1\right)\theta(t-a)+\left(\frac{t-4a}{a}-\frac{2a-t}{a}\right)\theta(t-3a)=$$

$$=\theta(t)-\frac{t-a}{a}\theta(t-a)+\frac{2(t-3a)}{a}\theta(t-3a)\,.$$

Applying the properties of linearity and delay to the constructed expression, we find the desired image $F(p)$:

$$F(p)=\frac{1}{p}-\frac{1}{ap^2}e^{-ap}+\frac{2}{ap^2}e^{-3ap}\,.$$