# Equations of Straight Line on Plane. Equations in Arbitrary Skew-Angular Basis

## 1 Equations of line

Assume that some plane $\alpha$ in the space $\mathbb{E}$ is chosen and fixed.

On this plane we choose arbitrary coordinate system: origin $O$, and a pair of unit vectors.

Let us enumerate and discuss some forms of equations of line we derived.

Vectorial parametric equation of this line is

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a}. \tag{1}$$

Here $\boldsymbol{r}_0$ is radius vector of arbitrary initial point $A \in a$, and $\boldsymbol{a} \parallel a$ is non-zero direction vector. $t$ is parameter.

Vectorial normal equation of this line is

$$\boldsymbol{r} \cdot \boldsymbol{n} = D. \tag{2}$$

Here $\boldsymbol{n} \perp a$ is non-zero normal vector, and $D = \boldsymbol{r}_0 \cdot \boldsymbol{n}$, and $\boldsymbol{r}_0$ is radius vector of arbitrary initial point $A \in a$.

Coordinate parametric equations of the line are

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t. \end{cases} \tag{3}$$

Here $a_x$ and $a_y$ are coordinates of direction vector and may not be zeros in the same time.

Canonical equation of the line is:

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} \tag{4}$$

It also has form for line passing through pint $B(x_1, y_1)$:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0}, \tag{5}$$

and form for two intercept points:

$$\frac{x}{a} + \frac{y}{b} = 1 \tag{6}$$

Here $(a, 0)$, and $(0, b)$ are intercept points.

While we derived these equations, we did not ever make any assumptions about the nature and features of the basis.

But while we derived slope-point, slope-intercept, general and normal forms we relied on features of rectangular bases through the expression for slope of through expression for dot product in Cartesian coordinates.

But while we take canonical equation, and rearrange terms, we yield:

$$\frac{x-x_0}{a_x}=\frac{y-y_0}{a_y}$$
$$a_y(x-x_0)=a_x(y-y_0)$$
$$a_yx-a_xy+(a_xy_0-a_yx_0)=0,$$

Which resembles form of general equation $Ax+By+C=0$.

Let us reveal meaning of the terms of general equation of line in arbitrary skew-angular basis. To do it we need first introduce **dual basis** and **covariant coordinates**.

## 2 Dual basis (对偶基)

Suppose that we introduced on plane or in space arbitrary skew-angular coordinate system with basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ on plane or $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in space.

**Definition.** We say that basis $\{\boldsymbol{e}^j\}$ is **dual** for arbitrary skew angular basis $\{\boldsymbol{e}_i\}$ if fulfilled condition

$$\boldsymbol{e}_i\boldsymbol{e}^j=\delta_i^j$$

$i,j=1,2$ on plane, or $i,j=1,2,3$ in space

## 3 Dual basis on plane

Suppose expansion of $\boldsymbol{e}^j$, $j=1,2$ by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is

$$\boldsymbol{e}^1=e_1^1\boldsymbol{e}_1+e_2^1\boldsymbol{e}_2$$
$$\boldsymbol{e}^2=e_1^2\boldsymbol{e}_1+e_2^2\boldsymbol{e}_2$$

Real numbers $e_m^n$ and are coordinates for basis vector of dual basis in original basis.

Each expansion yields two equations fulfilling condition $\boldsymbol{e}_i\boldsymbol{e}^j=\delta_i^j$.

For first dual basis vector $\boldsymbol{e}^1$ these conditions are:

$$\begin{cases}\boldsymbol{e}^1\cdot\boldsymbol{e}_1=e_1^1\boldsymbol{e}_1\cdot\boldsymbol{e}_1+e_2^1\boldsymbol{e}_2\cdot\boldsymbol{e}_1=1\\ \boldsymbol{e}^1\cdot\boldsymbol{e}_2=e_1^1\boldsymbol{e}_1\cdot\boldsymbol{e}_2+e_2^1\boldsymbol{e}_2\cdot\boldsymbol{e}_2=0\end{cases}$$
$$\begin{cases}e_1^1g_{11}+e_2^1g_{12}=1\\ e_1^1g_{21}+e_2^1g_{22}=0\end{cases}$$
$$G\begin{pmatrix}e_1^1\\ e_2^1\end{pmatrix}=\begin{pmatrix}1\\ 0\end{pmatrix}.$$

Numerical parameters $g_ij$ are components of Gram matrix of original basis, and $G$ is Gram matrix itself.

We solve this system of equations using Cramer's rule:

$$\Delta = \det G \quad \Delta_1 = \begin{vmatrix} 1 & g_{12} \\ 0 & g_{22} \end{vmatrix} = g_{22} \quad \Delta_1 = \begin{vmatrix} g_{11} & 1 \\ g_{12} & 0 \end{vmatrix} = -g_{12}$$

$$e_1^1 = \frac{g_{22}}{\det G} \quad e_2^1 = -\frac{g_{12}}{\det G}$$

For second dual basis vector $\boldsymbol{e}^2$ these conditions are:

$$\begin{cases} \boldsymbol{e}^2 \cdot \boldsymbol{e}_1 = e_1^2 \boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + e_2^2 \boldsymbol{e}_2 \cdot \boldsymbol{e}_1 = 0 \\ \boldsymbol{e}^2 \cdot \boldsymbol{e}_2 = e_1^2 \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 + e_2^2 \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1 \end{cases}$$

$$\begin{cases} e_1^2 g_{11} + e_2^2 g_{12} = 0 \\ e_1^2 g_{21} + e_2^2 g_{22} = 1 \end{cases}$$

$$G \begin{pmatrix} e_1^2 \\ e_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And we also solve it with Cramer's rule:

$$\Delta = \det G \quad \Delta_1 = \begin{vmatrix} 0 & g_{12} \\ 1 & g_{22} \end{vmatrix} = -g_{12} \quad \Delta_1 = \begin{vmatrix} g_{11} & 0 \\ g_{12} & 1 \end{vmatrix} = g_{11}$$

$$e_1^2 = -\frac{g_{12}}{\det G} \quad e_2^2 = \frac{g_{11}}{\det G}$$

Hence, coordinates of the dual basis vectors in the original basis are:

$$\boldsymbol{e}^1 = \begin{pmatrix} \dfrac{g_{22}}{\det G} \\ -\dfrac{g_{12}}{\det G} \end{pmatrix} \quad \boldsymbol{e}^2 = \begin{pmatrix} -\dfrac{g_{12}}{\det G} \\ \dfrac{g_{11}}{\det G} \end{pmatrix}$$

Hence, we can express components of Gram matrix for dual basis:

$$\begin{aligned} g^{11} = \boldsymbol{e}^1 \cdot \boldsymbol{e}^1 &= \frac{g_{22}}{\det G}\frac{g_{22}}{\det G} g_{11} - 2\frac{g_{22}}{\det G}\frac{g_{12}}{\det G} g_{12} + \frac{g_{12}}{\det G}\frac{g_{12}}{\det G} g_{22} = \\ &= \frac{g_{22}}{(\det G)^2}(g_{11}g_{22} - (g_{12})^2) = \frac{g_{22}}{(\det G)^2}\det G = \frac{g_{22}}{\det G} \end{aligned}$$

$$\begin{aligned} g^{22} = \boldsymbol{e}^2 \cdot \boldsymbol{e}^2 &= \frac{g_{12}}{\det G}\frac{g_{12}}{\det G} g_{11} - 2\frac{g_{12}}{\det G}\frac{g_{11}}{\det G} g_{12} + \frac{g_{11}}{\det G}\frac{g_{11}}{\det G} g_{22} = \\ &= \frac{g_{21}}{(\det G)^2}(g_{11}g_{22} - (g_{12})^2) = \frac{g_{11}}{(\det G)^2}\det G = \frac{g_{11}}{\det G} \end{aligned}$$

$$
\begin{aligned}
g^{12} = \boldsymbol{e}^1 \cdot \boldsymbol{e}^2 = -\frac{g_{22}}{\det G}\frac{g_{12}}{\det G}g_{11} + \frac{g_{22}}{\det G}\frac{g_{11}}{\det G}g_{12} + \\
+\frac{g_{12}}{\det G}\frac{g_{12}}{\det G}g_{12} - \frac{g_{12}}{\det G}\frac{g_{11}}{\det G}g_{22} = \\
\frac{g_{12}}{(\det G)^2}(-g_{22}g_{11} + g_{22}g_{11} + (g_{12})^2 - g_{11}g_{22}) = \\
-\frac{g_{12}}{(\det G)^2}(g_{11}g_{22} - (g_{12})^2) = -\frac{g_{12}}{(\det G)^2}\det G = -\frac{g_{12}}{\det G}
\end{aligned}
$$

Gram matrix for dual basis has expression int terms of elements of original Gram matrix

$$
\begin{pmatrix} \dfrac{g_{22}}{\det G} & -\dfrac{g_{12}}{\det G} \\ -\dfrac{g_{12}}{\det G} & \dfrac{g_{11}}{\det G} \end{pmatrix} = \frac{1}{\det G}\begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix}
$$

From general course od algebra we recap: for any non-degenerate matrix $2 \times 2$ $A$ ins inversion yields

$$
A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
$$

Hence, **Gram matrix of dual basis is inverse Gram matrix of original basis**.

For this Gram matrix of dual basis utilized notation:

$$
\begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix}
$$

正交的.

Since Gram matrix for orthonormal basis is unity matrix, orthonormal basis is dual with itself.

What is procedure to transform coordinates of arbitrary vector to dual basis?

Before answering this question we need to introduce some general definitions.

# 4 Transformation of basis vectors

Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$ be two honest skew-angular bases, "old" and "new" one.

**Definition.** The procedure of changing an old basis for a new one can be understood as a transition from an old basis to a new basis or, in other words, as a **direct transition**

**Definition.** Changing a new basis for an old one is understood as an **inverse transition**

Suppose basis vectors of new basis have their expressions as unit vectors of old one:

$$
\begin{aligned}
\boldsymbol{e}'_1 = S_1^1\boldsymbol{e}_1 + S_1^2\boldsymbol{e}_2 + S_1^3\boldsymbol{e}_3 \\
\boldsymbol{e}'_2 = S_2^1\boldsymbol{e}_1 + S_2^2\boldsymbol{e}_2 + S_2^3\boldsymbol{e}_3 \\
\boldsymbol{e}'_3 = S_3^1\boldsymbol{e}_1 + S_3^2\boldsymbol{e}_2 + S_3^3\boldsymbol{e}_3
\end{aligned}
$$

Coefficients $S^i_j$ here are coordinates of vectors of new basis with respect to old one

Let us write these coordinates as columns:

$$\boldsymbol{e}'_1 = \begin{pmatrix} S^1_1 \\ S^2_1 \\ S^3_1 \end{pmatrix} \quad \boldsymbol{e}'_2 = \begin{pmatrix} S^1_2 \\ S^2_2 \\ S^3_2 \end{pmatrix} \quad \boldsymbol{e}'_3 = \begin{pmatrix} S^1_3 \\ S^2_3 \\ S^3_3 \end{pmatrix},$$

and glue by that columns **direct transition matrix**:

$$S = \begin{pmatrix} S^1_1 & S^1_2 & S^1_3 \\ S^2_1 & S^2_2 & S^2_3 \\ S^3_1 & S^3_2 & S^3_3 \end{pmatrix}$$

*Remark.* If elements of a double index array are enumerated by indices on different levels, then in composing a matrix of these elements the upper index is used as a row number, while the lower index is used a column number

列举.

*Remark.* If elements of a double index array are enumerated by indices on the same level, then in composing a matrix of these elements the first index is used as a row number, while the second index is used a column number

Now we rewrite our relation between basis vectors with summation:

$$\boldsymbol{e}'_j = \sum_{i=1}^{3} S^i_j \boldsymbol{e}_i,$$

or write matrix expression:

$$\begin{pmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{pmatrix} \begin{pmatrix} S^1_1 & S^1_2 & S^1_3 \\ S^2_1 & S^2_2 & S^2_3 \\ S^3_1 & S^3_2 & S^3_3 \end{pmatrix}$$

We call these expressions **direct transition formulas**

*Remark.* If elements of a single index array are enumerated by lower indices, then in matrix presentation they are written in a row, i. e. they constitute a matrix whose height is equal to unity

*Remark.* If elements of a single index array are enumerated by upper indices, then in matrix presentation they are written in a column, i. e. they constitute a matrix whose width is equal to unity

Now let's consider the inverse transition from the new basis $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$ to the old basis e $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$.

In the inverse transition procedure the vectors of an old basis are expanded in a new basis:

$$\begin{aligned} \boldsymbol{e}_1 &= T^1_1 \boldsymbol{e}'_1 + T^2_1 \boldsymbol{e}'_2 + T^3_1 \boldsymbol{e}'_3 \\ \boldsymbol{e}_2 &= T^1_2 \boldsymbol{e}'_1 + T^2_2 \boldsymbol{e}'_2 + T^3_2 \boldsymbol{e}'_3 \\ \boldsymbol{e}_3 &= T^1_3 \boldsymbol{e}'_1 + T^2_3 \boldsymbol{e}'_2 + T^3_3 \boldsymbol{e}'_3 \end{aligned}$$

There is also representation as columns:

$$\boldsymbol{e}_1 = \begin{pmatrix} T_1^1 \\ T_1^2 \\ T_1^3 \end{pmatrix} \quad \boldsymbol{e}_2 = \begin{pmatrix} T_2^1 \\ T_2^2 \\ T_2^3 \end{pmatrix} \quad \boldsymbol{e}_3 = \begin{pmatrix} T_3^1 \\ T_3^2 \\ T_3^3 \end{pmatrix},$$

and representation as **inverse transition matrix**

$$T = \begin{pmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 \end{pmatrix}$$

Corresponding transformation expressions are

$$\boldsymbol{e}_j = \sum_{i=1}^{3} T_j^i \boldsymbol{e}'_i,$$

or in matrix form:

$$\begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{pmatrix} \begin{pmatrix} T_1^1 & T_2^1 & T_3^1 \\ T_1^2 & T_2^2 & T_3^2 \\ T_1^3 & T_2^3 & T_3^3 \end{pmatrix}$$

We call these expressions **inverse transition formulas**
Suppose on a plane we have basis "old" basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and "new" one $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$.
**Direct transition** has a form of

$$\begin{pmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} \begin{pmatrix} S_1^1 & S_2^1 \\ S_1^2 & S_2^2 \end{pmatrix}.$$

**Inverse transition** has a form of

$$\begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 \end{pmatrix} \begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix}$$

**Theorem 4.1.** *The direct transition matrix $S$ and the inverse transition matrix $T$ are inverse to each other. $S = T^{-1}$, $T = S^{-1}$, $ST = TS = Id$*

*Proof.* There are summation formulas connecting bases vectors ($n = 2$ on plane, and $n = 3$ in space):

$$\boldsymbol{e}'_i = \sum_{k=1}^{n} S_i^k \boldsymbol{e}_k \quad \boldsymbol{e}_j = \sum_{i=1}^{n} T_j^i \boldsymbol{e}'_i$$

Combining of these formulas yields (we're expressing $\boldsymbol{e}_j$ within its own basis):

$$\boldsymbol{e}_j = \sum_{i=1}^{n} T_j^i \left( \sum_{k=1}^{n} S_i^k \boldsymbol{e}_k \right) = \sum_{k=1}^{n} \left( \sum_{i=1}^{n} T_j^i S_i^k \right) \boldsymbol{e}_k = \sum_{k=1}^{n} \delta_j^k \boldsymbol{e}_k$$

Thus, we expressed element of linearly independent system through itself. This together with uniqueness of vector expansion yields

$$\delta^i_j = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases},$$

and

$$\sum_{i=1}^{n} T^i_j S^k_i = \delta^k_j$$

This yields $ST = Id$. □ □

This statement together with the fact that *determinant of matrix product is product of determinants* yields

$$\det(ST) = \det S \cdot \det T = \det Id = 1$$

退化/非退化矩阵

Recap that a matrix with zero determinant is called **degenerate**. If the determinant of a matrix is nonzero, such a matrix is called **non-degenerate**

For any two bases in the space or on a plane the corresponding transition matrices $S$ and $T$ are non-degenerate and the product of their determinants is equal to the unity.

Each non-degenerate $3 \times 3$ matrix $S$ is a transition matrix relating some basis in the space with some other basis in this space.

Each non-degenerate $2 \times 2$ matrix $S$ is a transition matrix relating some basis on plane with some other basis in this plane.

# 5 Transformation of coordinates of arbitrary vector

We will write all formulas for space case. Planar case just overlooks third coordinate.

Suppose vector $\boldsymbol{x}$ is expressed with coordinates in "old" basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$,

$$\boldsymbol{x} \mapsto \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

and we are looking for its expression in coordinates of "new" basis $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$

$$\boldsymbol{x} \mapsto \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

Both expressions are explicit and unique, so we can explicitly express one with another.

$$\boldsymbol{x} = \sum_{j=1}^{3} x^j \boldsymbol{e}_j = \sum_{j=1}^{3} x^j \left( \sum_{i=1}^{3} T^i_j \boldsymbol{e}'_i \right) =$$

$$= \sum_{j=1}^{3} \sum_{i=1}^{3} x^j T^i_j \boldsymbol{e}'_i = \sum_{i=1}^{3} \left( \sum_{j=1}^{3} T^i_j x^j \right) \boldsymbol{e}'_i$$

Thus, we expressed our vector $\boldsymbol{x}$ as expansion on the basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$. Coordinates of $\boldsymbol{x}$ in this basis are:

$$x'^i = \sum_{j=1}^{3} T^i_j x^j$$

**Definition.** These formulas expressing the coordinates of an arbitrary vector $\boldsymbol{x}$ in a new basis through its coordinates in an old basis are called the **direct transformation formulas**.

**Definition.** The formulas expressing the coordinates of an arbitrary vector $\boldsymbol{x}$ in an old basis through its coordinates in a new basis are called the **inverse transformation formulas**

These formulas have analogous derivation and expression:

$$x^i = \sum_{j=1}^{3} S^i_j x'^j$$

Matrix representation for these transformations is:

$$\begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = T \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$
$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = S \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$$

On the plane we overlook third coordinate:

$$x'^i = \sum_{j=1}^{2} T^i_j x^j$$

$$x^i = \sum_{j=1}^{2} S^i_j x'^j$$

$$\begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix} = T \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = S \begin{pmatrix} x'^1 \\ x'^2 \end{pmatrix}$$

# 6 Transition to dual basis on plane

Vectors of dual basis on plane have expression with original one:

$$\boldsymbol{e}^1 = \frac{1}{\det G}(\boldsymbol{e}_1 g_{22} + \boldsymbol{e}_2(-g_{12}))$$

$$\boldsymbol{e}^2 = \frac{1}{\det G}(\boldsymbol{e}_1(-g_{12}) + \boldsymbol{e}_2 g_{22})$$

Hence, *direct transition matrix to dual basis* is matrix

$$S = \begin{pmatrix} \frac{g_{22}}{\det G} & -\frac{g_{12}}{\det G} \\ -\frac{g_{12}}{\det G} & \frac{g_{11}}{\det G} \end{pmatrix} = G^{-1},$$

The inverse Gram matrix of original basis.

Thus, *inverse transition matrix from dual basis* is matrix

$$T = G,$$

the Gram matrix of original basis.

Direct transformation formula to covariant coordinates has form:

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = G \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Inverse transformation formula from covariant coordinates has form:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = G^{-1} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

# 7 Dot product in covariant coordinates

Suppose vector $\boldsymbol{a}$ has expression with covariant coordinates $(a^1, a^2)$, and vector $\boldsymbol{b}$ has expression with original coordinates.

We are looking for the dot product of these two vectors.

Suppose corresponding Gram matrix is

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{2}2 \end{pmatrix}$$

As matrix of direct transition to dual basis is $G^{-1}$, we restore ordinary coordinates of $\boldsymbol{a}$ formula:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = G^{-1} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \frac{1}{\det G} \begin{pmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{pmatrix} \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \frac{1}{\det G} \begin{pmatrix} a^1 g_{22} - a^2 g_{12} \\ a^2 g_{11} - a^1 g_{12} \end{pmatrix}$$

Let us write now the dot product itself:

$$\begin{aligned}
\boldsymbol{a} \cdot \boldsymbol{b} &= a_1 b_1 g_{11} + a_1 b_2 g_{12} + a_2 b_1 g_{12} + a_2 b_2 g_{22} = \\
&= \frac{1}{\det G}[(a^1 g_{22} - a^2 g_{12}) b_1 g_{11} + (a^1 g_{22} - a^2 g_{12}) b_2 g_{12} + \\
&\quad +(a^2 g_{11} - a^1 g_{12}) b_1 g_{12} + (a^2 g_{11} - a^1 g_{12}) b_2 g_{22}] = \\
&= \frac{1}{\det G}[a^1 b_1 g_{11} g_{22} - a^2 b_1 g_{11} g_{12} + a^1 b_2 g_{12} g_{22} - a^2 b_2 g_{12} g_{12} + \\
&\quad +a^2 b_1 g_{11} g_{12} - a^1 b_1 g_{12} g_{12} + a^2 b_2 g_{11} g_{22} - a^1 b_2 g_{12} g_{22}] = \\
&= \frac{1}{\det G}[a^1 b_1 (g_{11} g_{22} - g_{12} g_{12}) + a^2 b_2 (-g_{12} g_{12} + g_{11} g_{22}) + \\
&+a^1 b_2 (g_{12} g_{22} - g_{12} g_{22}) + a^1 b_2 (-g_{11} g_{12} + g_{11} g_{12})] = \\
&= \frac{1}{\det G}[\det G (a^1 b_1 + a^2 b_2)] = a^1 b_1 + a^2 b_2
\end{aligned}$$

**Definition.** For any pair of vectors on plane $\boldsymbol{a}$ and $\boldsymbol{b}$, where one expressed with covariant coordinates $(a^1, a^2)$, and second expressed with ordinary coordinates $(b_1, b_2)$ with respect to arbitrary skew-angular basis, there is formula for dot product of these two vectors:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^1 b_1 + a^2 b_2$$

# 8 General equation of the line in arbitrary skew-angular basis

Suppose line in question expressed with vectorial normal equation (2):

$$\boldsymbol{r} \cdot \boldsymbol{n} = D. \tag{7}$$

Let coordinates of radius vector in arbitrary skew-angular basis be $\begin{pmatrix} x \\ y \end{pmatrix}$, and covariant coordinates of normal vector be $\begin{pmatrix} x^n \\ y^n \end{pmatrix}$.

Dot product yields us:

$$xx^n + yy^n = D$$

Condition that normal vector is not-zero automatically restrict $x^n$ and $y^n$ be zeros in the same time.

Reordering of terms yields

$$xx^n + yy^n - D = 0$$

or

$$Ax + By + C = 0 \tag{8}$$

Here $A = x^n$, $B = x^n$, $C = -D$.

We call equation in form (8) the **general equation of the line** in skew-angular basis.

Terms $A$ and $B$ in this equation have meaning of the covariant coordinates of normal vector.

Comparing this with expressions for canonical equation:

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y}$$
$$a_y(x - x_0) = a_x(y - y_0)$$
$$a_y x - a_x y + (a_x y_0 - a_y x_0) = 0,$$

yields us that coordinates of one of direction vectors are $(B, -A)$.

# 9 Generalization for term slope

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Suppose that vector of our skew-angular basis are unit ones.

Line is expressed with general equation:

$$Ax + By + C = 0$$

Direction vector of the line has coordinates proportional with $(B, -A)$.

Letting $B \neq 0$, we express the $y$ coordinate explicitly:

$$y = -\frac{A}{B}x - C$$

Let one of direction vector have coordinates $(p, q)$, angle the line shapes with first coordinate be $\varphi$, and angle between axes be $\omega$ (see sketch).

This direction vector resembles diagonal of the parallelogram $OPDQ$, $\overrightarrow{OP}$ is direction vector, $OP = p$, $OQ = q$.

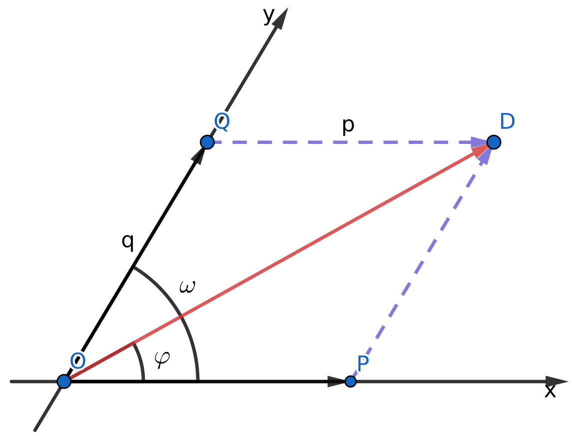


Figure 1: Sketch of the direction vector with coordinates $(p, q)$ in skew-angular basis with unit basis vectors

In the triangle $\triangle OQD$ $\angle O = \omega - \varphi$, $\angle Q = \pi - \omega$, $\angle D = \varphi$, $QD = p$.

Application of sines theorem yields:

$$\frac{QD}{\sin\angle O} = \frac{OQ}{\sin\angle D}$$
$$\frac{p}{\sin(\omega-\varphi)} = \frac{q}{\sin\varphi}$$
$$\frac{q}{p} = \frac{\sin\varphi}{\sin(\omega-\varphi)} = -\frac{A}{B}$$

**Definition.** General meaning of the coefficient $m$ in equation of line

$$y = mx + b$$

expressed in arbitrary skew-angular basis with unit basis vectors is ratio:

$$m = \frac{\sin\varphi}{\sin(\omega-\varphi)}.$$

Here $\varphi$ is angle shaped by line with ~~positive $Ox$~~ direction, and $\omega$ is the angle from first to second basis vectors.

Now we can yield evidences for parallelism and perpendicularity for two lines with coefficients $m$ and $m'$.

Parallelism of lines may be expressed as equality $m = m'$.

To derive evidence of perpendicularity, let us express perpendicularity of two direction vectors by letting their dot product to be zero.

Gram matrix in our case is $\begin{pmatrix} 1 & \cos\omega \\ \cos\omega & 1 \end{pmatrix}$.

Dot product

$$a_x a'_x + (a_x a'_y + a'_x a_y)\cos\omega + a_x a'_x = 0$$

For non-zero $B$s, we are free to divide it with $a_x a'_x$:

$$1 + (m + m')\cos\omega + mm' = 0$$

General expression for the tangent of the angle between two lines in this case is

$$\tan\alpha = \frac{(m' - m)\sin\omega}{1 + (m + m')\cos\omega + mm'}$$

This clarification of the meaning of $m$ in skew-angular basis preserves form of *slope-point* and *slope-intercept* equations

**Problem 1**
Suppose basis is formed with unit vectors shaping angle $2\pi/3$. Express equation of the line passing through the point $(-1, -3)$, and shaping angle $\pi/3$ with $Ox$

**Solution**
Coefficient $m$ for this line in specified basis is

$$m = \frac{\sin\dfrac{\pi}{3}}{\sin\left(\dfrac{2\pi}{3} - \dfrac{\pi}{3}\right)} = 1$$

Thus, equation of the line is:

$$\begin{aligned} x + 1 &= y + 3 \\ y &= x - 2 \end{aligned}$$

**Problem 2**
Suppose basis is formed with unit vectors shaping angle $2\pi/3$.
Straight line in this basis has equation

$$y = 3x - 4$$

Write the equation of perpendicular line passing through the point (5,0).

**Solution**
Angle coefficient of this line is $m = 3$.
Evidence of perpendicularity:

$$\begin{aligned} 1 + (m + m')\cos\omega + mm' &= 0 \\ 1 - \frac{1}{2}(m' + 3) + 3m' &= 0 \\ 2 - m' - 3 + 6m' &= 0 \\ m' &= \frac{1}{5} \end{aligned}$$

Equation of the line:

$$y = \frac{1}{5}x - 1$$

# 10 Changing a coordinate system. General definitions

Let us now return to the question of coordinates transformation and see how can we apply investigation of straight lines to this problem.

For practical applications it may be important to move the origin, rigidly rotate basis, flip basis vector, or maybe change angles between basis vector.

We start brief discussion on this pic with some general definition.

To explain equations of lines with respect to arbitrary basis we introduced transition between bases. Now we generalize these terms for coordinate systems and take into account displacement of the origin.

Basis itself carries no information exact location of the object. We need to add origin point to overlook it.

Let $O$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $O'$, $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$ be two Cartesian coordinate systems in the space. They consist of the bases $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ and $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, $\boldsymbol{e}'_3$ complemented with origins $O$ and $O'$.

Direct transposition from the first basis to second supported with direct transition matrix $S$. Inverse transition is supported with matrix $T$.

To express transition from one coordinate system to other we first employ the **origin displacement vectors**:

$$\boldsymbol{a} = \overrightarrow{OO'}, \quad \boldsymbol{a}' = \overrightarrow{O'O}$$
$$\boldsymbol{a} = -\boldsymbol{a}'$$

First means displacement the origin to point $O'$, second means displacement the origin to pint $O$.

Usual coordinate representation for $\boldsymbol{a}$ is by first basis, and usual coordinate representation for $\boldsymbol{a}'$ is by second basis:

$$\boldsymbol{a} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \quad \boldsymbol{a}' = \begin{pmatrix} a'^1 \\ a'^2 \\ a'^3 \end{pmatrix}$$

Let $X$ be arbitrary point. $\boldsymbol{r}_X = \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$ and $\boldsymbol{r}'_X = \begin{pmatrix} x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}$ are its radius vectors in both bases.

With terms of $\boldsymbol{a}$ and $\boldsymbol{a}'$ they are related as:

$$\boldsymbol{r}_X = \boldsymbol{a} + \boldsymbol{r}'_X \quad \boldsymbol{r}'_X = \boldsymbol{a}' + \boldsymbol{r}_X$$

Coordinates of $\boldsymbol{r}_X$ and $\boldsymbol{r}'_X$ may be successfully transformed with $S$ or $T$ matrix:

$$x^i = a^i + \sum_{j=1}^{3} S^i_j x'^j \quad x'^i = a'^i + \sum_{j=1}^{3} T^i_j x^j$$

These formulas are called the inverse and direct transformation formulas for the coordinates of a point under a change of a Cartesian coordinate system.

For the plane we need to overlook third coordinate.

# 11 Rotation of a rectangular coordinate system on a plane. The rotation matrix.

Let us introduce some assumptions. We suppose that both bases are orthonormal and the origins of coordinate system are coincided.

In this rigid system let us suppose that we obtained second basis from the first one only with rotation by angle $\varphi$

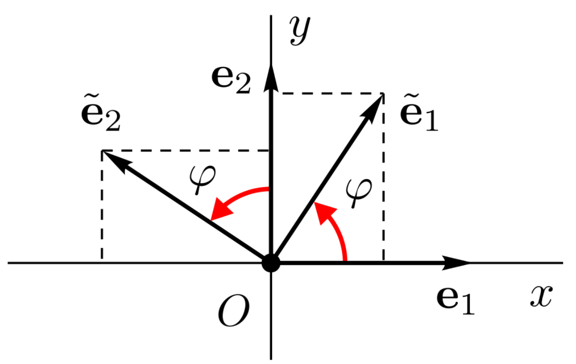


Figure 2: Rotation of the basis by angle $\varphi$

This case yields us representation for $S$ and $T$ matrices:

$$\boldsymbol{e}'_1 = \cos\varphi \cdot \boldsymbol{e}_1 + \sin\varphi \cdot \boldsymbol{e}_2 \boldsymbol{e}'_2 = -\sin\varphi \cdot \boldsymbol{e}_1 + \cos\varphi \cdot \boldsymbol{e}_2$$

$$S = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

To derive $T$ we inverse $S$:

$$T = \begin{pmatrix} \cos(-\varphi) & -\sin(-\varphi) \\ \sin(-\varphi) & \cos(-\varphi) \end{pmatrix}$$

These two matrices are called **rotation matrices by angle** $\varphi$. Direction of rotation is defined by the sign by $\varphi$.

Formulas to transform coordinates of arbitrary point in these bases are:

$$x'^1 = \cos(\varphi)x^1 + sin(\varphi)x^2,$$
$$x'^2 = -sin(\varphi)x^1 + \cos(\varphi)x^2,$$

$$x^1 = \cos(\varphi)x'^1 - \sin(\varphi)x'^2,$$
$$x^2 = \sin(\varphi)x'^1 + \cos(\varphi)x'^2.$$

# 12 Equations of coordinate axes. Transition, rotation and reflection

Normal equation of the line in orthonormal coordinates yields opportunity to easily take introduce account additional possibility to transform orthonormal basis on plane while it stays rectangular: change the order of vectors.

Suppose we introduced on a plane two coordinate systems with orthonormal bases. We will write coordinates of the first basis with clear letters: $(x, y)$, and coordinates in second basis with primes: $(x', y')$

Suppose axes of the second basis are expressed as normal lines equations in coordinates of first basis:

$$x\cos\varphi_1 + y\sin\varphi_1 - p_1 = 0$$
$$x\cos\varphi_2 + y\sin\varphi_2 - p_1 = 0,$$

$a_1 = \cos\varphi_1 \quad b_1 = \sin\varphi_1$

$a_2 = \cos\varphi_2 \quad b_2 = \sin\varphi_2$

or just

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

$\dfrac{a_1}{\sqrt{a_1^2 + b_1^2}}$

Let us take as a reference point $A_0(x_0, y_0)$ laying in the first quadrant with respect of both coordinate systems, thus:

$$a_1x_0 + b_1y_0 + c_1 > 0$$
$$a_2x_0 + b_2y_0 + c_2 > 0$$

Such point $A$ always exists. If not, we just change signs in one of equations.

**Theorem 12.1.** *Any point $A(x, y)$ has coordinates with respect to the second coordinate system:*

$$x' = a_1x + b_1y + c_1$$
$$y' = a_2x + b_2y + c_2$$

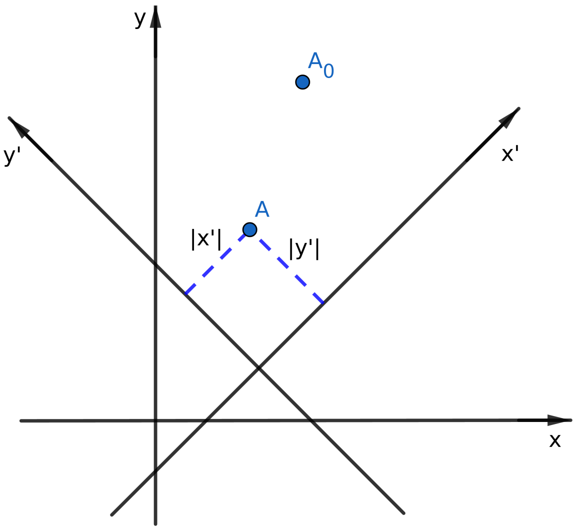


Figure 3: Relation between coordinate axes

*Proof.* These formulas numerically express distances from point $A$ to coordinate lines $Oy'$, and $Ox'$ respectively.

Sign of the distance depends on the relative disposition of point $A$ and origins. But for $A_0$ that distances are positive.

Thus, they are positive for all points with the same relative disposition.

Thus, changing type of disposition will explicitly change one of both signs. □

Returning to cosines and sines representation of $a_i$ and $b_i$, we rewrite condition of their perpendicularity as:

$$a_1a_2 + b_1b_2 = 0$$
$$\cos(\varphi_1 - \varphi_2) = 0$$

Thus, $\varphi_2 = \varphi_1 \pm \frac{\pi}{2} + 2\pi n$, $n$ is integer. Letting $\varphi_1$ be just $\varphi$ we yield:

$$x' = x\cos\varphi + y\sin\varphi + c_1 \tag{9}$$
$$y' = -x\sin\varphi + y\cos\varphi + c_2 \tag{10}$$

or

$$x' = x\cos\varphi + y\sin\varphi + c_1 \tag{11}$$
$$y' = x\sin\varphi - y\cos\varphi + c_2 \tag{12}$$

First formula moves coordinate system to the point $(c_1, c_2)$ and rotates it with angle $\varphi$. Such transformation we call the **motion**

Second formula additionally reflects figure.

**Problem 1**

Derive equation of the curve $x^2 + y^2 = a^2$ in the coordinate system expressed with lines:

$$x + y = 0$$
$$x - y = 0$$

**Solution**
First, we bring our equations into normal form: $k_1 = \sqrt{2}$, $k_2 = -\sqrt{2}$

$$x' = \frac{x + y}{\sqrt{2}}$$
$$y' = \frac{y - x}{\sqrt{2}}$$

Solving these equalities as system of equations with respect to $x$, $y$ yields:

$$x = \frac{x' - y'}{\sqrt{2}}$$
$$y = \frac{x' + y'}{\sqrt{2}}$$

Substituting and simplifying original equation:

$$\left(\frac{x' - y'}{\sqrt{2}}\right)^2 + \left(\frac{x' + y'}{\sqrt{2}}\right) + a^2 =$$
$$= x'^2 + y'^2 - 2x'y' + x'^2 + y'^2 + 2x'y'y + 2a^2 = 0$$

$$x'^2 + y'^2 + a^2 = 0$$

Original equation expressed a circle with radius $a$. Thus, it is natural that rotation of coordinates by $\pi/4$ saves the equation.

**Problem 2**
Discuss the locus of

$$y = \frac{ax + b}{cx + d}, \quad c \neq 0$$

**Solution**
First, we divide numerator and denominator by $c$:

$$y = \frac{px+q}{x+r}, \quad c \neq 0$$

Let new coordinate axes be lines $x - x_0 =, y - y_0 = 0$.
The equations are normal, new coordinates have expression:

$$x' = x - x_0$$
$$y' = y - y_0$$

$$x = x' + x_0$$
$$y = y' + y_0$$

Substituting and neutralizing denominator:

$$(x' + x_0 + r)(y' + y_0) = p(x' + x_0) + q$$

Simplify:

$$x'y' + (y_0 - p)x' + (x_0 + r)y' + (x_0 y_0 - p x_0 + r y_0 - q) = 0$$

Now we select $x_0$ and $y_0$ to sterilize linear terms with $x'$ and $y'$:

$$y_0 - p = 0, y_0 = p$$
$$x_0 + r = 0, x_0 = -r$$

Substituting and simplifying, we yield:

$$x'y' = q - pr.$$

This locus is hyperbola with asymptotes $x = -r$, and $y = p$.