Differential Geometry. Home Assignments

Assignment 4

Problem 1

Find the first fundamental form of the surface parametrised by

$$x = a \sin u \cos v, \quad y = a \sin u \sin v, \quad z = a \left(\ln \tan \frac{u}{2} + \cos u \right)$$

where a is a constant and $u \neq \frac{\pi}{2}$.

Problem 2

On the helicoid parametrised by

$$x = u \cos v, \quad y = u \sin v, \quad z = av$$

where aa is a constant, find the angle between the curves defined by

$$u + 2v = 0, \text{ and } 4u - v = 0.$$

Problem 3

The first fundamental form of a surface is given by

$$I = du^2 + (u^2 + a^2) dv^2$$

where a is a constant. Find the area of the region on this surface bounded by the curves

$$u = \pm av, \qquad v = 1.$$

HW4 Week 14

Problem 1
Find the first fundamental form of the surface parametrised by

$$x = a\sin u\cos v, \quad y = a\sin u\sin v, \quad z = a\left(\ln\tan\frac{u}{2} + \cos u\right)$$

where a is a constant and $u \neq \frac{\pi}{2}$.

Sol: $\vec{r}_u = \left(a\cos u\cos v,\ a\cos u\sin v,\ a\left(\frac{1}{\sin u} - \sin u\right)\right)$

$\vec{r}_v = (-a\sin u\sin v,\ a\sin u\cos v,\ 0).$

$\vec{r}_u^{\,2} = a^2\cos^2 u(\cos^2 v + \sin^2 v) + a^2\left(\frac{1}{\sin u} - \sin u\right)^2.$

$= a^2\cos^2 u + a^2\sin^2 u + \frac{a^2}{\sin^2 u} - 2a^2 = \frac{a^2}{\sin^2 u} - a^2 = a^2\cdot \mathrm{ctg}^2 u. \quad (u \neq \frac{\pi}{2})$

$\vec{r}_v^{\,2} = a^2\sin^2 u.$

$\vec{r}_v \cdot \vec{r}_u = 0.$

$I = a^2\mathrm{ctg}^2 u\,(du)^2 + a^2\sin^2 u\,(dv)^2 \quad (u \neq \frac{\pi}{2}).$

Problem 2
On the helicoid parametrised by

$$x = u\cos v, \quad y = u\sin v, \quad z = av$$

where aa is a constant, find the angle between the curves defined by

$$u + 2v = 0, \text{ and } 4u - v = 0.$$

Sol: parametrize these two curves.

$u+2v=0 \rightarrow \begin{cases} u(t) = t \\ v(t) = -\frac{1}{2}t \end{cases}$ $\quad 4u - v = 0 \rightarrow \begin{cases} \bar{u}(t) = t \\ \bar{v}(t) = 4t. \end{cases}$

$u'(t) = 1 \quad v'(t) = -\frac{1}{2}$
$\bar{u}'(t) = 1 \quad \bar{v}'(t) = 4.$

the intersection point is $(0,0)$.

the tangent vector of the surface at (u,v):

$\vec{r}_u = (\cos v,\ \sin v,\ 0) \qquad \vec{r}_v = (-u\sin v,\ u\cos v,\ a)$

$E = \vec{r}_u^{\,2} = 1 \qquad F = \vec{r}_u\cdot\vec{r}_v = 0 \qquad G = \vec{r}_v^{\,2} = u^2 + a^2$

$$\cos\theta = \frac{E\cdot u'\bar{u}' + F(u'\bar{v}' + \bar{u}'v') + Gv'\bar{v}'}{\sqrt{E\cdot u'^2 + 2Fu'v' + Gv'^2}\sqrt{E\cdot\bar{u}'^2 + 2F\bar{u}'\bar{v}' + G\bar{v}'}} = \frac{1 + (-2)\cdot(u^2+a^2)}{\sqrt{1+\frac{1}{4}(u^2+a^2)}\cdot\sqrt{1+16(u^2+a^2)}} = \frac{1-2a^2}{\sqrt{1+\frac{a^2}{4}}\cdot\sqrt{1+16a^2}}$$

$$\theta = \arccos\frac{1-2a^2}{\sqrt{1+\frac{a^2}{4}}\cdot\sqrt{1+16a^2}}$$

Problem 3

The first fundamental form of a surface is given by

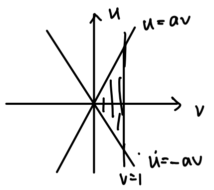
$$I = du^2 + (u^2 + a^2)dv^2$$

where a is a constant. Find the area of the region on this surface bounded by the curves

$$u = \pm av, \qquad v = 1.$$

Sol: by the expression of 1st fundamental form, $E=1$ $F=0$ $G=u^2+a^2$.

in $\tilde{G}$. (the u-v plane). the region is (w.l.g. $a>0$), a triangle :



$$A = \iint_{\tilde{G}} \sqrt{EG-F^2}\, dudv = \int_0^1 \left(\int_{-av}^{av} \sqrt{a^2+u^2}\, du \right) dv = 2\int_0^1 \left(\int_0^{av} \sqrt{a^2+u^2}\, du \right) dv.$$

$$\int_0^{av} \sqrt{a^2+u^2}\, du = \frac{1}{2}\left[u\sqrt{u^2+a^2} + a^2 \ln|u+\sqrt{u^2+a^2}|\right]\Big|_0^{av} = \frac{1}{2}a^2\left[v\sqrt{v^2+1} + \ln|av+a\sqrt{v^2+1}|\right] - \ln|a|$$

$$= \frac{a^2}{2}\left[v\sqrt{v^2+1} + \ln|v+\sqrt{v^2+1}|\right.$$

$$A = a^2\int_0^1 \left(v\sqrt{v^2+1} + \ln|v+\sqrt{v^2+1}|\right) dv.$$

$$\int_0^1 v\sqrt{v^2+1}\, dv = \frac{1}{2}\int_0^1 \sqrt{v^2+1}\, d(v^2+1) = \frac{1}{3}\cdot(v^2+1)^{\frac{3}{2}}\Big|_0^1 = \frac{1}{3}\left(2\sqrt{2}-1\right)$$

$$\int_0^1 \ln|v+\sqrt{v^2+1}|\, dv = v\cdot\ln|v+\sqrt{v^2+1}|\Big|_0^1 - \int_0^1 \frac{v}{\sqrt{v^2+1}}\, dv = \ln|\sqrt{2}+1| - \int_0^1 \frac{d(v^2+1)}{2\sqrt{v^2+1}} = \ln|\sqrt{2}+1| - \sqrt{v^2+1}\Big|_0^1$$

$$= \ln|\sqrt{2}+1| - \sqrt{2} + 1.$$

$$A = a^2\left(\ln(\sqrt{2}+1) + \frac{2-\sqrt{2}}{3} \right)$$