Real Analysis 2024. Homework 1.

1. Prove that any open subset of a real line can be expressed as at most countable union of disjoint open intervals.

   Proof. Any open set in $\mathbb{R}$ can be considered as the union of connected components, which are open intervals. These components do not intersect are disjoint and each contains a rational number, consequently, any open set in $\mathbb{R}$ has no more than countable number of components. $\square$

2. Prove that Borel σ-algebra in $\mathbb{R}$ can be generated by the family of open rays $\{(-\infty, a) : a \in \mathbb{R}\}$.

   Proof. It is enough to notice that for $-\infty < a < b$ the open interval can be obtained from set of open rays by countable number of set-operations

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right) = \bigcup_{n=1}^{\infty} \left((-\infty, b) \setminus \left(-\infty, a + \frac{1}{n}\right]\right).$$

   And by the previous consideration any open set is at most countable union of disjoint intervals. Hence, $\sigma-$algebra generated by the family open rays contains open sets, and consequently Borel $\sigma-$algebra. $\square$

3. Let $(X, \mathcal{A})$ be measurable space, X' be a set, $\mathcal{A}'_{\min} = \{\varnothing, X'\}$ be a minimal σ-algebra on X'. Prove that every map $f : X \to X'$ is measurable.

   Proof. Let $E \in \mathcal{A}'_{\min}$. Then we have only two cases

$$f^{-1}(E) = \begin{cases} \varnothing, & E = \varnothing; \\ X, & E = X'. \end{cases}$$

   Since $\varnothing, X$ belong to any $\sigma-$algebra on X this means measurability of map f. $\square$

Real Analysis 2024. Homework 2.

1. Let (f_n) be a sequence of real-valued measurable functions on X. Then the set
$$E = \{x \in X \mid f_n(x) \text{ converges to some limit } \}$$
is measurable. (Hint: use Cauchy criterion).

Proof. By translating the Cauchy criterion in terms of set operations, one can write
$$E = \bigcap_{k \geqslant 1} \bigcup_{N \geqslant 1} \{x || f_n(x) - f_m(x) | < 1/k\}$$
and the result follows because $|f_n - f_m|$ is measurable. $\square$

2. Assume that μ is a measure on $(X, \mathcal{A})$. Prove that
$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$$
for every $E, F \in \mathcal{A}$.

Proof. Notice that
$$E = (E \backslash (E \cap F)) \cup (E \cap F); \quad F = (F \backslash (E \cap F)) \cup (E \cap F);$$
$$E \cup F = (E \backslash (E \cap F)) \cup (E \cap F) \cup (E \cap F).$$
Sets in the unions above are disjoint and measurable. Hence,
$$\mu E = \mu(E \backslash (E \cap F)) + \mu(E \cap F);$$
$$\mu F = \mu(F \backslash (E \cap F)) + \mu(E \cap F);$$
$$\mu(E \cup F) = \mu(E \backslash (E \cap F)) + \mu(E \cap F) + \mu(E \cap F).$$
and
$$\mu E + \mu F = \mu(E \backslash (E \cap F)) + \mu(F \backslash (E \cap F)) + 2\mu(E \cap F) = \mu(E \cup F) + \mu(E \cap F).$$
$\square$

3. Prove that Borel $\sigma-$algebra $\mathcal{B}_{\mathbb{R}^2}$ is the same as product $\sigma-$algebra $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

   Proof. First of all notice that the product of two Borel sets is always Borel, consequently, $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\mathbb{R}^2}$.

   Assume that $G \in \mathcal{B}_{\mathbb{R}^2}$. We will

   $\square$

4. Prove that a Cantor's set is a Borel set (see MA(3)).

   Proof. By the construction, Cantor's set is obtained as the intersection of a nested sequence of closed sets. Consequently, it is a set of type F_σ and is a Borel set. $\square$

5. Consider a sequence of measures μ_i on a $\sigma-$algebra $\mathcal{A}$ and a sequence of nonnegative numbers $\alpha_i \in \mathcal{A}$. For a set $E \in \mathcal{A}$ let

$$\mu(E) = \sum_{i=1}^{\infty} \alpha_i \mu_i(E).$$

   Prove that μ is a measure on $\mathcal{A}$.

   Proof. First, $\mu(\varnothing) = 0$ since $\mu_i(\varnothing) = 0$. To prove countable additivity consider a sequence of disjoint sets E_k and $E = \bigcup\limits_{k=1}^{\infty} E_k$. Hence,

$$\mu(E) = \sum_{i=1}^{\infty} \alpha_i \mu_i(E) = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_i \mu_i(E_k) \right) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_i \mu_i(E_k) \right) = \sum_{k=1}^{\infty} \mu(E_k). \quad (1)$$

   $\square$

   The change of the order of summation in the proof can be explained in the following lemma.

Lemma 0.1. *Assume that* $a_{k,i} \geq 0$, $k, i \in \mathbb{N}$. *Then*

$$S_1 = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} a_{k,i} = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{k,i} = S_2.$$

Proof. Assume first that

$$S_1 = +\infty.$$

Then for every $M > 0$ there exists $N > 0$ such that

$$\sum_{k=1}^{N} \sum_{i=1}^{\infty} a_{k,i} > M$$

Then

$$S_2 = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{k,i} \geq \sum_{i=1}^{\infty} \sum_{k=1}^{N} a_{k,i} = \sum_{k=1}^{N} \sum_{i=1}^{\infty} a_{k,i} > M.$$

Since $M > 0$ is arbitrary this implies that $S_2 = +\infty = S_1$. (And finitenss of one of the sums implies finitness of the other).

Assume that $S_1 < +\infty$. Then for every $\varepsilon > 0$ there exists $N > 0$ such that

$$S_1 - \varepsilon \leq \sum_{k=1}^{N} \sum_{i=1}^{\infty} a_{k,i} = \sum_{i=1}^{\infty} \sum_{k=1}^{N} a_{k,i} \leq S_1.$$

Since $\sum_{k=1}^{N} a_{k,i} \leq \sum_{k=1}^{\infty} a_{k,i}$ then

$$S_1 - \varepsilon \leq \sum_{i=1}^{\infty} \sum_{k=1}^{N} a_{k,i} \leq \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{k,i} = S_2.$$

Since $\varepsilon > 0$ is arbitrary this implies that $S_1 \leq S_2$ and, by symmetry, $S_2 \leq S_1$ and, hence, $S_1 = S_2$. $\square$

Real Analysis 2024. Homework 3.

1. Consider a function $f : [0,1] \to [0,1]$ defined as following:

$$f(x) = \begin{cases} x + 1/2, & 0 \le x < 1/2; \\ x - 1/2, & 1/2 \le x \le 1. \end{cases}$$

   Find the image of Lebesgue measure with respect to this transform.

   Solution. Let $E \subset [0,1]$ be measurable. Then

$$f^{-1}(E) = A_1 \cup A_2,$$

   where

$$A_1 = \{x + 1/2 : x \in E \cap [0, 1/2]\}$$

   and

$$A_2 = \{x - 1/2 : x \in E \cap [1/2, 1)\}.$$

   Since $\mu(A_1) = \mu(E \cap [0, 1/2])$ and $\mu(A_2) = \mu(E \cap [1/2, 1))$. Consequently,

$$f_*(\mu)(E) = \mu(f^{-1}(E)) = \mu(A_1) + \mu(A_2) = \mu(E \cap [0, 1/2]) + \mu(E \cap [1/2, 1)) = \mu(E)$$

   and Lebesgue measure is preserved under map f.

2. Consider a measure defined as linear combination of Dirac measures

$$\mu = \delta_0 + \delta_1 + 2\delta_2$$

   on $\mathcal{A}_{max}$ on $\mathbb{R}$. Consider a function $f(x) = x^2$. Prove that it is measurable map with respect to $\sigma-$algebra $\mathcal{A}_{max}$. Find $f_*(\mu)$.

Solution.

$$f_*(\mu)(E) = \delta_0(f^{-1}(E)) + \delta_1(f^{-1}(E)) + 2\delta_2(f^{-1}(E)) =$$

$$\begin{cases} 0, & 0,1,2 \notin f^{-1}(E); \\ 1, & 0 \in f^{-1}(E),\ 1,2 \notin f^{-1}(E); \\ 1, & 1 \in f^{-1}(E),\ 0,2 \notin f^{-1}(E); \\ 2, & 0,1 \in f^{-1}(E),\ 2 \notin f^{-1}(E); \\ 2, & 2 \in f^{-1}(E),\ 0,1 \notin f^{-1}(E); \\ 4, & 0,1,2 \in f^{-1}(E) \end{cases} =$$

$$\begin{cases} 0, & 0,1,4 \notin E; \\ 1, & 0 \in E,\ 1,4 \notin E; \\ 1, & 1 \in E,\ 0,4 \notin E; \\ 2, & 0,1 \in E,\ 4 \notin E; \\ 2, & 4 \in E,\ 0,1 \notin E; \\ 4, & 0,1,4 \in E \end{cases}$$

Consequently,

$$f_*(\mu) = \delta_0 + \delta_1 + 2\delta_4.$$

3. Calculate the integral of function $f(x) = x^3$ on $[0, +\infty)$ with respect measure

$$\mu = \frac{\delta_1 + 2\delta_2 + 3\delta_3}{6}.$$

Solution.

$$\int_{[0,+\infty)} f(x)d\mu = \frac{1}{6}(f(1) + 2f(2) + 3f(3)) = \frac{1}{6}(1 + 16 + 81) = \frac{49}{3}.$$

4. Explain why the counting measure on $\mathbb{R}$ is not $\sigma-$finite.

Solution. The counting measure ν of a set $A \subset \mathbb{R}$ is finite if and only if A is finite. Consequently, if

$$E = \bigcup_{k=1}^{\infty} A_k, \quad \nu(A_k) < \infty,$$

then E is at most countable, while $\mathbb{R}$ is uncountable.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let E_j, $1 \le j \le k$, be measurable sets. Prove that

$$\sum_{i=1}^{k} \mu E_i \leqslant \mu \left(\bigcup_{i=1}^{k} E_i \right) + \sum_{i<j} \mu \left(E_i \cap E_j \right).$$

Proof. First, by additivity of a measure

$$\mu E_1 + \mu E_2 = \mu E_1 + \mu \left(E_2 \backslash E_1 \right) + \mu \left(E_1 \cap E_2 \right) = \mu \left(E_1 \cup E_2 \right) + \mu \left(E_1 \cap E_2 \right).$$

We will prove that the identity by induction over k. Let

$$\sum_{i=1}^{k} \mu E_i \leqslant \mu \left(\bigcup_{i=1}^{k} E_i \right) + \sum_{1 \le i<j \le k} m \left(E_i \cap E_j \right).$$

Then

$$\sum_{i=1}^{k+1} \mu E_i = \sum_{i=1}^{k} \mu E_i + \mu E_{k+1} \leqslant \mu \left(\bigcup_{i=1}^{k} E_i \right) + \sum_{1<i<j<k} \mu \left(E_i \cap E_j \right) + \mu E_{k+1} =$$

$$= \mu \left(\bigcup_{i=1}^{k+1} E_i \right) + \mu \left(\left(\bigcup_{i=1}^{k} E_i \right) \cap E_{k+1} \right) + \sum_{1<i<j \leqslant k} \mu \left(E_i \cap E_j \right) \leqslant$$

$$\leqslant \mu \left(\bigcup_{i=1}^{k+1} E_i \right) + \sum_{1 \leqslant i<k} \mu \left(E_i \cap E_{k+1} \right) + \sum_{1<i<j<k} \mu \left(E_i \cap E_j \right) =$$

$$= \mu \left(\bigcup_{i=1}^{k+1} E_i \right) + \sum_{1<i<j \leqslant k+1} \mu \left(E_i \cap E_j \right).$$

$\square$

Real Analysis 2024. Homework 4.

1. Consider a counting measure on $\mathbb{N}$. Describe spaces of mesurable and of integrable functions.

   Any function $f : \mathbb{N} \to \mathbb{C}$ is a sequence of complex numbers $c_k = f(k)$. Any such function is measurable with respect to counting measure since the counting measure ν is defined on maximal $\sigma-$algebra $\mathcal{A}_{\max} = 2^{\mathbb{N}}$. The integral

$$\int_{\mathbb{N}} |f|\, d\nu = \sum_{k=1}^{\infty} |c_k|$$

   is finite iff a series $\sum\limits_{k=1}^{\infty} c_k$ is absolutely convergent. This space is called

$$\ell^1 = \{\{c_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} |c_k| < \infty\}.$$

2. Assume that $f_n \in L(E)$ is increasing sequence, $f_n \to f$ pointwise and $f \in L(E)$. Prove that $f_n \to f$ in $L(E)$.

   Proof. Let $g_n = f - f_n \geq 0$. Then g_n is decreasing sequence and by monotone convergence theorem

$$\int_E |f - f_n|\, d\mu = \int_E g_n d\mu \to 0, \quad n \to \infty.$$

   □

3. Let f be $\mu-$measurable on E and denote $E_t = E(|f| > t)$. Prove that

$$\mu E_t \leq \frac{1}{t^p} \int_E |f|^p\, d\mu.$$

   Proof. Note that

$$E_t = E(|f|^p > t^p).$$

   Hence, by Chebyshev's inequality applied to $|f|^p$ we have

$$\mu E_t \leq \frac{1}{t^p} \int_E |f|^p\, d\mu.$$

   □

4. Prove that a measure μ is $\sigma-$finite if and only if there exists a positive integrable function ($f > 0$ on X and $\int_X f d\mu < +\infty$).

Proof. Assume first that μ is $\sigma-$finite. Then

$$X = \bigcup_{k=1}^{\infty} X_k, \quad \mu X_k < \infty.$$

Let

$$E_n = \left(\bigcup_{k=1}^{n} X_k\right) \setminus \left(\bigcup_{k=1}^{n-1} X_k\right), \ n > 1; \ E_1 = X_1.$$

Then E_n is measurable, $X = \bigcup\limits_{n=1}^{\infty} E_n$, and $c_n = \mu(E_n) < \infty$. Let

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \chi_{E_n} > 0.$$

Since $E_m \cap E_n$, $n \neq m$, then by monotone convergence theorem

$$\int_X f d\mu = \sum_{n=1}^{\infty} \frac{1}{2^n c_n} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Assume now that there exists function $f > 0$ such that $\int_X f d\mu < \infty$. Consider a set $X_k = X(f > 1/k)$. Then, by Chebyshev's inequality

$$\mu(X_k) \leq k \int_X |f| < \infty.$$

And since $f(x) > 0$ for every $x \in X$ we see that

$$X = \bigcup_{k=1}^{\infty} X_k, \quad \mu X_k < \infty.$$

$\square$

5. Consider $f_n(x) = \frac{1}{n} \left(\frac{\sin nx}{x}\right)^2$. Prove that

   (a) $f \in L(0, \pi)$;

(b) $f_n(x) \to 0,\ n \to \infty$ for every $x \in (0, \pi)$;

(c) There is no such function $g \in L(0, \pi)$ such that $f_n(x) \le g(x)$ for every $x \in (0, \pi)$ and every $n \in \mathbb{N}$.

Proof. (a)

$$\int_0^{\pi} |f_n(x)|\, dx = \frac{1}{n} \int_0^{\pi} \left(\frac{\sin nx}{x}\right)^2 dx = [x = t/n] =$$
$$\int_0^{n\pi} \left(\frac{\sin x}{x}\right)^2 dx < \int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx < +\infty.$$

(b) $f_n(x) \le \frac{1}{nx^2} \to 0$, $n \to \infty$, $x \in (0, \pi)$.

(c) Assume the converse. In this case, by Lebesgue thm on dominated convergence, we must have $\int f_n dx \to 0$, $n \to \infty$. At the same time

$$\int f_n dx \to \int_0^{+\infty} \left(\frac{\sin x}{x}\right)^2 dx,\ n \to \infty.$$

$\square$

Real Analysis 2024. Homework 5.

1. Let $p \in X$ and δ_p be Dirac measure. Prove that

$$\int_X f d\delta_p = f(p)$$

for every function $f : X \to \mathbb{R}$.

Proof. First, consider the case when $f = \chi_E$. Then, by the definition of the integral

$$\int_X f d\delta_p = 1 \cdot \delta_p(E) + 0 \cdot \delta_p(X \setminus E) = \begin{cases} 0, & p \notin E; \\ 1, & p \in E \end{cases} = \chi_E(p) = f(p).$$

Then, by linearity (or by definition) this statement is true for simple functions. Let $f \geq 0$ be a measurable (actually, any) function $f : X \to \mathbb{R}$. If ϕ is simple and $\phi \leq f$ then $\phi(p) \leq f(p)$ and

$$\int_X \phi d\delta_p = \phi(p) \leq f(p).$$

So

$$\int_X f d\delta_p = \sup\left\{\int_X \phi d\delta_p : \phi \text{ is simple and } \phi \leq f\right\} \leq f(p).$$

Although let $\phi_0(p) = f(p)\chi_{\{p\}}$. Then

$$\int_X \phi_0 d\delta_p = f(p)$$

and

$$\int_X f d\delta_p = \sup\left\{\int_X \phi d\delta_p : \phi \text{ is simple and } \phi \leq f\right\} = f(p).$$

□

2. Let $n \in \mathbb{N}$, $0 \leq k \leq 2^n - 1$. Consider an interval

$$\Delta_{k,n} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)$$

and let $f_{n,k}(x) = \chi_{\Delta_{k,n}}$ be a characteristic function of this interval. This defines a countable family of functions that can be considered as a sequence. For example,

$$g_1 = f_{1,0},\ g_2 = f_{1,1},\ g_3 = f_{2,0},\ g_4 = f_{2,1},\ g_5 = f_{2,2},\ g_6 = f_{2,3},\ g_7 = f_{3,0}, \ldots.$$

Let $m_n = 1 + 2 + \ldots + 2^{n-1} = 2^n - 1$ and for $m_n \leq j < m_{n+1}$ we let $k = j - m_n$. Then the above sequnce is defined as follows:

$$g_j = f_{n,j-m_n}, \quad m_n \leq j < m_{n+1}.$$

(a) Prove that a sequence g_j converges in $L^1[0,1]$.

(b) Indicate a subsequence that converges to the limit function a.e.

Proof. (a) $\int_0^1 g_j dx = \int_0^1 f_{n,j-m_n} dx = \frac{1}{2^n} \to 0$.

(b) First, note that for every $x \in [0,1]$ and for every $J \in \mathbb{N}$ we can find $j_1 > J$ such that $g_{j_1}(x) = 1$. and $j_2 > J$ such that $g_{j_2}(x) = 0$. Hence, g_j doesn' converge for every $x \in [0,1]$. At the same time

$$g_{m_n} = f_{n,0} = \chi_{[0,1/2^n]} \to 0, \quad x \neq 0.$$

□

3. Let $\mu(X) < +\infty$, $f_n \in L^1(X,\mu)$, and $f_n \rightrightarrows f$ on X. Prove that $f \in L^1(X,\mu)$ and $\|f_n - f\|_1 \xrightarrow[n\to\infty]{} 0$.

Proof. By monotonicity of the integral we have

$$\|f_n - f\|_1 \leq \mu(X) \sup_{x\in X} |f_n(x) - f(x)| \to 0,\ n \to +\infty.$$

□

4. Provide an example of a set and a measure such that $\mu(X) = +\infty$, $f_n, f \in L^1(X,\mu))$, and $f_n \rightrightarrows f$ on X but $\|f_n - f\|_1 \not\to 0$.

Let $X = [0,\infty)$, μ be Lebesgue measure,

$$f_n = \frac{1}{n}\chi_{[n,2n)}.$$

Then $|f_n| < 1/n$ so $f_n \rightrightarrows 0$ on X. At the same time

$$\|f_n\|_1 = 1 \not\to 0.$$

Real Analysis 2024. Homework 6.

1. Consider a space $(\mathbb{R}^2, \mathcal{B}_2, \mu_2)$ and a function $f(x,y) = \frac{1}{|x|+|y|}$ prove that $f \in L^1([-1,1]^2)$.

Proof. Consider

$$E_k = \left\{(x,y) \in [-1,1]^2 : \frac{1}{k+1} < |x|\,, |y| \leq \frac{1}{k}\right\}.$$

Then

$$\int_{E_k} \frac{dxdy}{|x|+|y|} \leq 2(k+1)\mu_2(E_k) \leq \frac{8(k+1)}{k^2(k+1)^2} \leq \frac{8}{k^3}.$$

Hence,

$$\int_{[-1,1]^2} \frac{dxdy}{|x|+|y|} \leq \sum_{k=1}^{\infty} \frac{1}{k^3} < +\infty.$$

$\square$

2. Consider $\mathbb{N}$ with counting measure. Describe space $L^p(\mathbb{N})$.

$$L^p(\mathbb{N}) = \left\{ \{c_k\}_{k=1}^{\infty} : \left\|\{c_k\}\right\|_p = \left(\sum_{k=1}^{\infty} |c_k|^p < \infty\right)^{1/p} \right\}.$$

3. Prove Hölder's inequality for simple functions using Hölder's inequality for finite sums. Apply theorem on monotone sequence to obtain Hölder's inequality for nonnegative measurable functions.

Proof. Suppose $\frac{1}{p} + \frac{1}{q} = 1$. Let Let

$$f = \sum_{k=1}^{n} a_k \chi_{A_k}, \quad g = \sum_{j=1}^{m} b_j \chi_{B_j},$$

where

$$X = \bigcup_{k=1}^{n} A_k = \bigcup_{j=1}^{m} B_j;$$

$$A_{k_1} \cap A_{k_2} = \varnothing,\ k_1 \neq k_2;\ \ B_{j_1} \cap B_{j_2} = \varnothing,\ j_1 \neq j_2.$$

Consequently,

$$fg = \sum_{\substack{1 \le k \le n \\ 1 \le j \le m}} a_k b_j \chi_{A_k \cap B_j}$$

and

$$\int_X |fg| \, d\mu = \sum_{\substack{1 \le k \le n \\ 1 \le j \le m}} |a_k b_j| \, \mu(A_k \cap B_j) = \sum_{\substack{1 \le k \le n \\ 1 \le j \le m}} |a_k| \, \mu(A_k \cap B_j)^{1/p} \, |b_j| \, \mu(A_k \cap B_j)^{1/q} \le$$

$$\left(\sum_{\substack{1 \le k \le n \\ 1 \le j \le m}} |a_k|^p \, \mu(A_k \cap B_j) \right)^{1/p} \left(\sum_{\substack{1 \le k \le n \\ 1 \le j \le m}} |b_j|^q \, \mu(A_k \cap B_j) \right)^{1/q} =$$

$$\left(\sum_{1 \le k \le n} |a_k|^p \, \mu(A_k) \right)^{1/p} \left(\sum_{1 \le j \le m} |b_j|^q \, \mu(B_j) \right)^{1/q} = \left(\int_X |f|^p \, d\mu \right)^{1/p} \left(\int_X |g|^q \, d\mu \right)^{1/q}.$$

If $f \in L^p(X, \mu)$ and $g \in L^q(X, \mu)$ consider increasing sequences of simple functions φ_n, ψ_n such that

$$\varphi_n(x) \le \varphi_{n+1}(x), \;\; f(x) = \lim_{n \to +\infty} \varphi_n(x), \quad x \in X;$$

$$\psi_n(x) \le \psi_{n+1}(x), \;\; g(x) = \lim_{n \to +\infty} \psi_n(x), \quad x \in X.$$

Then

$$\varphi_n(x)\psi_n(x) \le \varphi_{n+1}(x)\psi_{n+1}(x), \;\; fg(x) = \lim_{n \to +\infty} \varphi_n(x)\psi_n(x), \quad x \in X;$$

and by the monotone convergence theorem we have

$$\int_X |fg| \, d\mu = \lim_{n \to +\infty} \int_X \varphi_n \psi_n d\mu \le$$

$$\lim_{n \to +\infty} \left(\int_X |\varphi_n|^p \, d\mu \right)^{1/p} \left(\int_X |\psi_n|^q \, d\mu \right)^{1/q} =$$

$$\left(\int_X |f|^p \, d\mu \right)^{1/p} \left(\int_X |g|^q \, d\mu \right)^{1/q}.$$

□

4. Prove the General Lebesgue Dominated Convergence Theorem. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g_n and dominates $\{f_n\}$ on E in the sense that

$$|f_n| \leq g_n \text{ a.e. on } E \text{ for all } n.$$

Prove that if

$$\lim_{n\to\infty} \int_E g_n = \int_E g < \infty,$$

then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. The following assertions hold simultaneously a.e.

(a) $|f_n(x)| \leqslant g_n(x)$ for every $n \in \mathbb{N}$;

(b) $f_n(x) \to f(x)$.

Considering the limit in the first one we see that $|f(x)| \leqslant g(x)$ for a.e. $x \in E$. Consequently, $f_n, f \in L(E, \mu)$ and by the Chebyshev inequality functions $f_n, f, g_n g$ are finite a.e. Noticing that $g_n + f_n \geqslant 0$ a.e. and applying Fatou's theorem we obtain

$$\begin{aligned}\int_E g + \int_E f = \int_E (g+f) \leqslant \underline{\lim} \int_E (g_n + f_n) = \\ = \underline{\lim} \left(\int_E g_n + \int_E f_n \right) = \int_E g + \underline{\lim} \int_E f_n.\end{aligned}$$

Hence

$$\int_E f \leqslant \underline{\lim} \int_E f_n \tag{1}$$

Analogously,

$$\begin{aligned}\int_E g - \int_E f = \int_E (g-f) \leqslant \underline{\lim} \int_E (g_n - f_n) = \\ = \int_E g + \underline{\lim} \left(- \int_E f_n \right) = \int_E \Phi - \overline{\lim} \int_E f_n.\end{aligned}$$

Consequently,

$$\overline{\lim} \int_E f_n \leqslant \int_E f. \tag{2}$$

Considering estimates (1) and (2) we see that

$$\int_E f \leqslant \underline{\lim} \int_E f_n \leqslant \overline{\lim} \int_E f_n \leqslant \int_E f.$$

This implies the existence of the limit $\lim \int_E f_n$ and that this limit is equal to the integral $\int_E f$. $\square$

5. Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ if and only if $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: Use the General Lebesgue Dominated Convergence Theorem.)

   Proof. Notice that

   $$\left| \int_E |f| - \int_E |f_n| \right| \leq \int_E ||f| - |f_n|| \leq \int_E |f - f_n|$$

   which proves that if $\int_E |f - f_n| \to 0$ then $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$.

   Assume that $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. Let $g_n = |f_n| + |f|$. Then

   $$|f_n - f| \leq |f_n| + |f|,$$

   while

   $$\int_E (|f_n| + |f|) \to 2 \int_E |f|.$$

   Consequently, by Lebesgue dominated convergence theorem

   $$\int_E |f_n - f| \to 0$$

   since $|f_n - f| \to 0$ a.e. on E. $\square$

6. For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. Is it true that f is integrable over $[1, \infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges? Is

it true that f is integrable over $[1,\infty)$ if and only if the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

Solution. Both are not true. Consider a function

$$f(x) = \begin{cases} 1, & x \in [n, n+1/2),\ n \in \mathbb{N}; \\ -1, & x \in [n+1/2, n+1),\ n \in \mathbb{N}. \end{cases}$$

Then $a_n = 0$. The series converges absolutely and while

$$\int\limits_0^{+\infty} |f|\, dx = +\infty.$$

Real Analysis 2024. Homework 7.

1. For f in $L^1[a,b]$, define $\|f\| = \int_a^b x^2|f(x)|dx$. Show that this is a norm on $L^1[a,b]$.

   Proof. **(1)** $\|f\| \geq 0$ and if $\|f\| = 0$ then $x^2\,|f(x)| = 0$ for a.e. $x \in [a,b]$ and, hence, $|f(x)| = 0$ for a.e. $x \in [a,b]$.

   (2)
$$\|\alpha f\| = \int_a^b x^2|\alpha f(x)|dx = |\alpha| \int_a^b x^2|f(x)|dx = |\alpha|\,\|f\|$$

   (3)
$$\|f+g\| = \int_a^b x^2|f(x)+g(x)|dx \leq \int_a^b x^2|f(x)|dx + \int_a^b x^2|g(x)|dx = \|f\|+\|g\|$$

   □

2. Prove that $L^p[0,1] \neq L^q[0,1]$ if $p \neq q$.

   Proof. Suppose $p > q$. Let $f(x) = \frac{1}{x^{1/p}}$. Then $f \notin L^p([0,1])$ and $f \in L^q([0,1])$.

   □

3. Assume that $\mu(E) < \infty$. Prove that
$$\|f\|_\infty = \lim_{p\to+\infty} \|f\|_p$$
   for any measurable on E function f.

   Proof. Assume first that $\|f\|_\infty < +\infty$. Since
$$\|f\|_p \leq \mu(E)^{1/p}\,\|f\|_\infty$$
   then
$$\lim_{p\to+\infty} \|f\|_p \leq \|f\|_\infty\,. \tag{1}$$
   Let $\varepsilon > 0$ then $|f| > \|f\|_\infty - \varepsilon$ on a set F of a positive measure. In this case,
$$\|f\|_p \geq \mu(F)^{1/p}(\|f\|_\infty - \varepsilon)$$

and letting $p \to +\infty$ we see that

$$\lim_{p\to+\infty} \|f\|_p \geq \|f\|_\infty - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and by estimate (1) we see that

$$\lim_{p\to+\infty} \|f\|_p = \|f\|_\infty .$$

Assume now that $\|f\|_\infty = +\infty$. Then for every $M > 0$ $|f| > M$ on a set of a positive measure. Analogously to previous consideration we see that

$$\lim_{p\to+\infty} \|f\|_p \geq M.$$

Since $M > 0$ is arbitrary we obtain

$$\lim_{p\to+\infty} \|f\|_p = +\infty = \|f\|_\infty .$$

□

4. Suppose $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $f \in L^p(E)$. Show that $f = 0$ a.e. if and only if

$$\int_E f \cdot g = 0 \text{ for all } g \in L^q(E).$$

Proof. If $f = 0$ a.e. then $fg = 0$ a.e. and

$$\int_E f \cdot g = 0.$$

Assume now that $f \in L^p$

$$\int_E f \cdot g = 0 \text{ for all } g \in L^q(E).$$

Let $f(x) = |f|\, e^{i\theta(x)}$ and $g(x) = e^{-i\theta(x)}\, |f|^{p-1}$. Then

$$\|g\|_q^q = \int_E |g|^q = \int_E |f|^{q(p-1)} \int_E |f|^p = \|f\|_p^p < \infty$$

and $g \in L^q(E)$. At the same time

$$0 = \int_E f \cdot g = \int_E |f|^p$$

and $f = 0$ a.e.

□

5. (The L^p Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions that converges pointwise a.e. on E to f. For $1 \leq p < \infty$, suppose there is a function g in $L^p(E)$ such that for all $n, |f_n| \leq g$ a.e. on E. Prove that $f_n \to f$ in $L^p(E)$.

Proof. Since $|f_n| \leq g$ a.e. then $|f_n - f| \leq 2g$ a.e. and and $|f_n - f|^p \to 2^p |f|^p$ then by dominated convergence theorem

$$\int_E |f_n - f|^p \, d\mu \to \int_E |f - f|^p \, d\mu = 0.$$

and $f_n \to f$ in $L^p(E)$. $\square$

Real Analysis 2024. Homework 8.

1. Find condition on p under which the integral is finite

$$\iint_E \frac{dxdy}{(x^2+y^2)^p}, \qquad E = \{(x,y) : 0 < x < 1,\ 0 < y < x^2\}.$$

Solution.

$$I = \iint_E \frac{dxdy}{(x^2+y^2)^p} = \int_0^1 dx \int_0^{x^2} \frac{dy}{(x^2+y^2)^p} = [y = x^2 t] =$$

$$\int_0^1 dx \int_0^1 \frac{dy}{(x^2+t^2x^4)^p} = \int_0^1 \frac{dx}{x^{2p-2}} \int_0^1 \frac{dy}{(1+t^2x^2)^p}$$

Since

$$\frac{1}{2^p} \le \frac{1}{(1+t^2x^2)^p} \le 1, \quad 0 \le t, x \le 1,$$

then

$$\frac{1}{2^p} \int_0^1 \frac{dx}{x^{2p-2}} \le I \le \int_0^1 \frac{dx}{x^{2p-2}}$$

and I converges if and only if $p < 1/2$.

2. Find condition on p under which the integral is finite

$$\iint_{x+y>1} \frac{\sin x \sin y}{(x+y)^p} dxdy.$$

Caution: the integrability of f is equivalent to integrability of $|f|$!

Solution.

The integral converges if and only if

$$I = \iint_{x+y>1} \frac{|\sin x \sin y|}{(x+y)^p} dxdy$$

converges. Let $u = x$, $v = x + y$. The Jacobian of this change is equal to 1 and

$$I = \int_{v>1} \frac{|\sin u \sin(v-u)|}{v^p} dudv$$

The function

$$F(u) = \int_1^{+\infty} \frac{\sin(v-u)}{v^p} dv$$

is positive and 2π-periodic. Hence,

$$I = \int_{-\infty}^{+\infty} F(u)\,|\sin u|\,du = \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} F(u)\,|\sin u|\,du + \infty.$$

3. Find condition on p under which the integral is finite

$$I = \int_E \frac{dxdydz}{|x+y-z|^p}, \quad E = [-1,1]^3.$$

Solution. First, notice that the cube $[-1,1]^3$ is divided by a plane $z = x+y$ into two equal parts and

$$I = 2 \int\limits_{\substack{-1\le x,y\le 1,\\ x+y\le 1}} dxdy \int_{x+y}^{1} \frac{dz}{(z-x-y)^p}$$

The integral $\int_{x+y}^{1} \frac{dz}{(z-x-y)^p}$ converges if and only if $p<1$. In this case

$$\int_{x+y}^{1} \frac{dz}{(z-x-y)^p} = \frac{1}{1-p}\frac{1}{(1-x-y)^{p-1}}$$

and

$$I = \frac{2}{1-p} \int\limits_{\substack{-1\le x,y,\le 1\\ x+y\le 1}} (1-x-y)^{1-p} dxdy$$

This integral converges since $p < 1$ and function $(1-x-y)^{1-p}$ is continuous in $\{-1 \le x, y, \le 1; x+y \le 1\}$.

Fourier series

TASKS

1. Decompose the function into a Fourier series $f(x) = \text{sign } x, -\pi < x < \pi$, and using the resulting decomposition, find the sum of the Leibniz series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Answer: $signx = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{sin(2n-1)x}{2n-1}$, Leibniz series: $\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

Decompose the Fourier series of the function $f(x)$ on the specified interval, the length of the interval is the period (2-11).

2. $f(x) = \begin{cases} A, & 0 < x < l, \\ A/2, & x = l, \\ 0, & l < x < 2l, \end{cases}$ on the interval $(0.2l)$.

Answer: $\frac{A}{2} + \frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}sin\frac{2n-1}{l}\pi x$

3. $f(x) = |x|$ on the segment $[-1; 1]$.

Answer: $\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{cos\pi(2n-1)x}{(2n-1)^2}$

4. $f(x) = \begin{cases} ax, & -\pi < x < 0, \\ bx, & 0 \leqslant x < \pi, \end{cases}$ in the interval $(-\pi, \pi)$.

Answer: $\frac{(b-a)\pi}{4} - \sum_{n=1}^{\infty}((b-a)(1-(-1)^n)\frac{cosnx}{\pi n^2} + (-1)^n(a+b)\frac{sinnx}{n}$

5. $f(x) = \begin{cases} a, & -\pi/2 < x < \pi/2, \\ b, & \pi/2 \leqslant x < 3\pi/2, \end{cases}$ in the range $(-\pi/2, 3\pi/2)$.

Answer: $\frac{a+b}{2} + \frac{2(a-b)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}cos(2n-1)x$

6. $f(x) = x + \text{sign}\, x$ on the interval $(-\pi; \pi)$.

Answer: $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}(1+\pi)}{n}sinnx$

7. $f(x) = \pi^2 - x^2$ on the interval $(-\pi; \pi)$.

Answer: $\frac{2}{3}\pi^2 + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}cosnx$

8. $f(x) = x^3$ on the interval $(-\pi; \pi)$.

Answer: $\sum_{n=1}^{\infty}(-1)^n(\frac{12}{n^3} - \frac{2\pi^2}{n})sinnx$

9. $f(x) = e^{ax}, a \neq 0$, in the interval $(-\pi; \pi)$.

Answer: $\frac{2}{\pi}sh\alpha\pi(\frac{1}{2a} + \sum_{n=1}^{\infty}\frac{(-1)^n}{a^2+n^2}(acos(nx) - nsinnx))$

10. $f(x) = e^{2|x|}$ in the interval $(-\pi; \pi)$.

Answer: $\frac{e^{2\pi}-1}{2\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n e^{2\pi}-1}{n^2+4}cosnx$

11. $f(x) = \sin ax, a \in Z$ in the interval $(-\pi; \pi)$.
Answer: $\frac{2 sin\pi a}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n sinnx}{n^2 - a^2}, \quad -\pi < x < \pi$

Fourier integrals. Fourier Transform

TASKS

Represent the function $f(x)$ by the Fourier integral:

1. $f(x) = \begin{cases} 1, & \text{if} \quad |x| < \tau \\ 0, & \text{if} \quad |x| > \tau \end{cases}$

Answer: $f(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{sin\tau y}{y} cosxydy$

2. $f(x) = \begin{cases} e^{-\alpha x} \sin \omega x, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x < 0; \end{cases} \quad \alpha > 0;$

Answer: $f(x) = \frac{\omega}{\pi} \int_0^{+\infty} \frac{(\alpha^2+\omega^2-y^2)cosxy+2\alpha ysinxy}{(\alpha^2-\omega^2+y^2)^2+4\alpha^2\omega^2} dy$

3. Represent the Fourier integral function $f(x)$ by continuing it in an odd way on the interval $(-\infty; 0)$ if: $f(x) = \begin{cases} \sin x, & \text{if} \quad 0 \leqslant x \leqslant \pi \\ 0, & \text{if} \quad x > \pi; \end{cases}$

Answer: $f(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{sin\pi y}{1-y^2} sinxydy$

4. Represent the Fourier integral function $f(x)$ by continuing it in an even way on the interval $(-\infty; 0)$ if: $f(x) = e^{-\alpha x}, x \geqslant 0, \alpha > 0;$

Answer: $f(x) = \frac{2}{\pi} \int_0^{+\infty} \frac{siny}{y} cosxydy$

Find the Fourier transform of the function $f(x)$

5. $f(x) = \begin{cases} 1, & \text{if} \quad |x| \leqslant 1, \\ 0, & \text{if} \quad |x| > 1; \end{cases}$

Answer: $f(x) = \sqrt{\frac{2}{\pi}} \frac{siny}{y}$

6. $f(x) = \begin{cases} e^{ix}, & \text{if} \quad x \in [0; \pi], \\ 0, & \text{if} \quad x \notin [0; \pi]; \end{cases}$

Answer: $f(x) = \sqrt{\frac{2}{\pi}} \frac{sin\pi y}{1-y}$

7. $f(x) = e^{-x^2/2} \cos \alpha x;$

Answer: $f(x) = e^{-(y^2+\alpha^2)/2} ch\alpha y$

8. Let $\widehat{f}(y) = F[f(x)]$. To prove that: $F\left[e^{i\alpha x} f(x)\right] = \widehat{f}(y-\alpha), \quad \alpha \in R;$

Real Analysis 2024. Homework 11.

1. Find Fourier series for the function $f(x) = x\sin x$ on $[-\pi, \pi]$.

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty} \frac{(-1)^n \cos nx}{n^2-1}, \quad -\pi \leqslant x \leqslant \pi$$

2. Find cos and sin series for the function $f(x) = x - x^2/2$, $0 \le x \le 1$.

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\left(1+\frac{4}{\pi^2(2n-1)^2}\right)\frac{\sin(2n-1)\pi x}{2n-1} - \frac{\sin 2\pi n x}{2n}\right).$$

$$\frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos \pi n x;$$

3. Prove that

$$\sin x \ln\left(2\cos\frac{x}{2}\right) = \frac{1}{4}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx, \quad -\pi < x < \pi$$

Proof. Consider Fourier series for $\ln\left(2\cos\frac{x}{2}\right)$ multiply bi $\sin x$ and use formulas for product of *cos* and *sin* □

4. Calculate the sum of the trigonometric series

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n(n+1)}.$$

$$\frac{1}{2}x(1+\cos x) - \sin x \ln\left(2\cos\frac{x}{2}\right).$$

5. Calculate the sum of the trigonometric series

$$f(x) = \frac{\cos 3x}{1\cdot 3\cdot 5} - \frac{\cos 5x}{3\cdot 5\cdot 7} + \frac{\cos 7x}{5\cdot 7\cdot 9} + \ldots.$$

$$f(x) = \frac{\pi}{8}\cos^2 x - \frac{1}{3}\cos x$$

6. Express function $f(x)$ by the Fourier integral

$$f(x) = \begin{cases} \sin x, & |x| \le \pi; \\ 0, & |x| > \pi. \end{cases}$$

Fourier transform is equal to:

$$F[f] = -2i \frac{\sin(2\pi^2 y)}{1 - 4\pi^2 y^2}$$

and

$$f(x) = -2i \int_{-\infty}^{+\infty} \frac{\sin(2\pi^2 y) e^{2\pi i x y}}{1 - 4\pi^2 y^2} dy = 2 \int_{-\infty}^{+\infty} \frac{\sin(2\pi^2 y) \cos(2\pi x y)}{1 - 4\pi^2 y^2} dy.$$

7. Prove that

$$\mathcal{F}[f(x)\cos(2\pi\alpha x)] = \frac{\mathcal{F}[f](y-\alpha) + \mathcal{F}[f](y+\alpha)}{2}, \quad \alpha \in \mathbb{R}.$$

Proof.

$$\mathcal{F}[f(x)\cos(2\pi\alpha x)] = \int_{\mathbb{R}} f(x)\cos(2\pi\alpha x) e^{-2\pi i x y} dx = \int_{\mathbb{R}} f(x) \frac{e^{2\pi\alpha x i} + e^{2\pi\alpha x i}}{2} e^{-2\pi i x y} dx =$$
$$\frac{1}{2}\int_{\mathbb{R}} f(x) e^{-2\pi i x(y-\alpha)} dx + \frac{1}{2}\int_{\mathbb{R}} f(x) e^{-2\pi i x(y+\alpha)} dx = \frac{\mathcal{F}[f](y-\alpha) + \mathcal{F}[f](y+\alpha)}{2}.$$

□

8. Prove that Fourier transform of function $\frac{1}{1+x^{12}}$ is $C^{10}-$smooth.

Proof. This follows from the theorem on derivative of Fourier transform:

$$F[f]^{10}(y) = F[(-2\pi i x)^{10} f(x)](y)$$

if $\int_{\mathbb{R}} x^1 0 \, |f(x)| \, dx$ is finite. □

9. Calculate Fourier transform of the function

$$f(x) = \begin{cases} 2 - x^2, & |x| \le 1; \\ 1, & 1 < |x| < 2; \\ 0, & |x| \ge 2. \end{cases}$$

$$F[f] = 2 \frac{(2\pi y)^2 \sin(4\pi y) + 2\sin(2\pi y) - 4\pi y \cos(2\pi y)}{(2\pi y)^3}.$$

Real Analysis 2024. Homework 12. Solution.

1. Let $f_n \underset{n\to\infty}{\longrightarrow} f$ in measure. Show that if $\mu(X) < +\infty$ and $g \in L^0(X)$, then $f_n g \underset{n\to\infty}{\longrightarrow} fg$ in measure. Is this true for an infinite measure?

   Proof. Suppose that $g \in L^0(X)$. This means that f is finite a.e. Since $\mu(X) < \infty$ then

   $$\lim_{M\to+\infty} \mu\left(\bigcap_{M>0} E(|g| > M)\right) = \mu\left(\bigcap_{M>0} E(|g| > M)\right) = 0$$

   (this is not true when $\mu(X) = +\infty$ consider, for example, function $g_0 : [1, +\infty)$ such that $g_0(x) = n, x \in [n, n+1)$.

   Let $\varepsilon > 0$ and choose $M > 0$ such that $E(|g| > M) < \varepsilon/2$.

   Let $\delta > 0$. Since $f_n \underset{n\to\infty}{\longrightarrow} f$ then there exists $N \in \mathbb{N}$ such that for every $n > N$ we have

   $$E\left(|f_n - f| > \delta/M\right) < \varepsilon/2.$$

   Finally

   $$E(|f_n g - fg| > \delta) \subset E(|g| > M) \bigcup (E(|g| > M) \cup E(|f_n - f| > \delta/M))$$

   and for $n > N$ we have

   $$\mu E(|f_n g - fg| > \delta) \leq \mu E(|g| > M) + \mu E(|f_n - f| > \delta/M) < \varepsilon.$$

   The statement is not true for $X = [1, +\infty)$, function g_0 mentioned above and sequence $f_n \equiv 1/n$ that converges to 0 uniformly. $\square$

2. Let $f_k^{(n)}(x) = \cos^{2k}(\pi n! x)(x \in \mathbb{R})$. Show that:

   (a) for every $x \in \mathbb{R}$, the limit $g_n(x) = \lim_{k\to\infty} f_k^{(n)}(x)$ exists;

   (b) $g_n(x) \underset{n\to\infty}{\longrightarrow} \chi(x)$ everywhere on $\mathbb{R}$ (here $\chi = \chi_{\mathbb{Q}}$ is the Dirichlet function);

   (c) there is no sequence of continuous functions (and, in particular, no diagonal sequence $\{f_{k_n}^{(n)}\}$ that converges to the Dirichlet function pointwise on a non-degenerate interval.

Proof. (a) If $n!x \notin \mathbb{N}$ then $|\cos(\pi n!x)| < 1$ and $g_n(x) = \lim_{k\to\infty} \cos^{2k}(\pi n!x) = 0$. Otherwise, $|\cos(\pi n!x)| = 1$ and $g_n(x) = \lim_{k\to\infty} \cos^{2k}(\pi n!x) = 1$.

(b) If $x \notin \mathbb{Q}$ then $n!x \notin \mathbb{N}$ for every $n \in \mathbb{N}$ and $g_n(x) = 0 = \chi(x)$. If $x = p/q \in \mathbb{Q}$ then $n!x \in \mathbb{N}$ for $n > q$ and $g_n(x) = 1 = \chi(x)$, $n > q$.

□

3. Assume that the measure under consideration is $\sigma-$finite and a sequence of measurable functions f_k converges to zero almost everywhere. Show that $c_k f_k \to 0$ a.e. for some numerical sequence $c_k \to +\infty$. Hint. Assuming that the sequence $\{|f_k|\}$ is decreasing, apply the diagonal sequence theorem to the functions $f_k^{(n)} = nf_k$.

Proof. **Case 1.** Suppose that the sequence $\{|f_k|\}$ is decreasing and let $f_k^{(n)} = nf_k$. Then for every $n \in \mathbb{N}$ $\lim_{k\to+\infty} f_k^{(n)} = 0$ a.e.

Hence, by the diagonal sequence theorem there exists a sequence k_n such that $\lim_{n\to+\infty} nf_{k_n} = 0$. a.e.

Let $c_k = n$, $k_n \le k_{n+1} - 1$. Then $c_k |f_k| \le nf_{k_n}$ for $k \ge k_n$. Hence $c_k f_k \to 0$ a.e.

Case 2. If $|f_n|$ is not decreasing let

$$g_n(x) = \sup_{k\ge n} |f_k(x)| .$$

Then g_n is decreasing and $g_n \to 0$ a.e. □

4. Suppose that $f_n \le g_n \le h_n$ a.e. on E, $f_n, h_n \underset{n\to\infty}{\longrightarrow} f$ in measure on E. Prove that $g_n \underset{n\to\infty}{\longrightarrow} f$.

Proof. Since

$$|g_n - f| \le \max(|h_n - f| , |f_n - f|)$$

then

$$\mu(|g_n - f| > \varepsilon) \le \mu(|f_n - f| > \varepsilon) + \mu(|h_n - f| > \varepsilon) \to 0,\ n \to +\infty,$$

for every $\varepsilon > 0$. □

5. Let $f : [0,1]^2 \to \mathbb{R}$ is continuous by the first argument (for arbitrary fixed second). Prove that if $f(x,y) \underset{y\to 0}{\longrightarrow} 0$ for a.e. $x \in [0,1]$ then for every $\varepsilon > 0$ there exists $e \subset [0,1]$ such that $\lambda(e) < \varepsilon$ and $f(x,y) \underset{y\to 0}{\longrightarrow} 0$ uniformly on $[0,1]\backslash e$. Hint: consider sets

$$G_n(\varepsilon) = \left\{(x,y) : \ 0 < x < 1, \ 0 < y < \frac{1}{n}, \ |f(x,y)| > \varepsilon\right\}$$

and their projections on $0X-$axis.

Proof. Consider a sequence $g_n(x) = \sup\limits_{y\in(0,1/n)} f(x,y)$. Then $g_n(x) \to 0$ a.e. on $[0,1]$ and, consequently, by Lebesgue theorem, in measure. Hence, applying Egorov theorem, there exists $e \subset [0,1]$ such that

$$g_n \rightrightarrows 0 \text{ on } [0,1]\backslash e,$$

which implies that $f(x,y) \underset{y\to 0}{\rightrightarrows} 0$ on $[0,1]\backslash e$

□