Complex Analysis 2024. Homework 13.

1. Consider primary branch of logarithm in $U = \{z \in \mathbb{C} : |z-1| < 1\}$ defined by the series

$$\ln z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n.$$

Let $a \in U$. Prove that Taylor series centered at point a converges on the disk of radius $r = |a|$.

Proof. First notice that

$$(\ln z)' = \sum_{n=1}^{\infty}(-1)^{n-1}(z-1)^{n-1} = \frac{1}{1+(z-1)} = \frac{1}{z},$$

and

$$\frac{1}{z} = \frac{1}{a+(z-a)} = \frac{1}{a}\frac{1}{1+\frac{z-a}{a}} = \frac{1}{a}\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{z-a}{a}\right)^{n-1}$$

for $z \in U$ such that $\left|\frac{z-a}{a}\right| < 1$.

Consequently, for $z \in U$ such that $\left|\frac{z-a}{a}\right| < 1$ we have

$$\ln z = \ln a \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{z-a}{a}\right)^n$$

while the series in the RHS converges for $|z-a| < |a|$. □

2. Prove that the relation "to be analytic continuation" is equivalence relation on a family of all canonical elements.

Proof. Reflexivity and symmetry are obvious.

To prove transitivity suppose that element g is analytic continuation of element F along chain

$$F = F_0, \dots, F_n = G$$

and element H is analytic continuation of element G along chain

$$F_n = G, \ldots, F_{n+m} = H.$$

Hence element H is analytic continuation of element G along chain

$$F = F_0, \ldots, F_{n+m} = H.$$

$\square$

3. *Prove that on the boundary of disk of convergence of a power series the sum
$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$
has a singular point.

Definition 0.1. *Let $a \in \mathbb{C}$, $R \in (0, +\infty)$, $U = \{z \in \mathbb{C} : |z - z_0| < R\}$, $F = (U, f)$ be a canonical element*

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k, \quad |z - z_0| < R.$$

A point $z_1 \in \overline{U}$ is called a regular point of element (U, f) if there exists DAC of F with center at z_1. Otherwise z_1 is a simgular point.

Proof. Assume the converse. Then every point $a \in \partial U$ has neighborhood V_a and function $f_a \in H(V_a)$ such that $f_a = f$ on $U \cap V_a$. This defines an open cover of a circle ∂U (that is a compact set) and we can cover it by finite number of such neighborhoods:

$$\partial U \subset \bigcup_{n=1}^{N} V_{a_n}.$$

Moreover, there exist $\varepsilon > 0$ such that

$$\widetilde{U} = \{z \in \mathbb{C} : |z| < R + \varepsilon\} \subset \bigcup_{n=1}^{N} V_{a_n}.$$

Consider a function

$$g(z) = f(z), z \in U; \quad g(z) = f_{a_n}(z), \quad z \in V_{a_n} \cap \widetilde{U}.$$

This definition is correct since $f_{a_n}(z) = f_{a_m}(z)$ on the intersection $z \in V_{a_n} \cap V_{a_m}$ by the uniqueness theorem ($f_{a_n}(z) = f_{a_m}(z)$, $z \in V_{a_n} \cap V_{a_m} \cap U$.) Consequently $g \in H(\widetilde{U})$ and its Taylor's series coinsides with Taylor series for f (at 0) and converges on the $\widetilde{U}$ which contradicts the assumption that U is the disk of convergence. $\square$