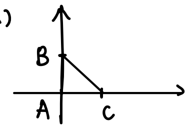
MA. HW20

1. Calculate a curvilinear integral of the first kind over a flat curve Γ :

(a) $\int_\Gamma (x+y)ds$, $\Gamma-$ border of a triangle with vertices $(0;0)$; $(1;0)$ and $(0;1)$;

(b) $\int_\Gamma \frac{ds}{\sqrt{x^2+y^2+4}}$, Γ is a segment with the ends $(0;0)$ and $(1;2)$.

(a)



AB : $\begin{cases} x=0 \\ y=y \end{cases}$ BC : $\begin{cases} x=x \\ y=1-x \end{cases}$ AC : $\begin{cases} x=x \\ y=0 \end{cases}$

$$\int_\Gamma (x+y)ds = \int_{\Gamma_{AB}+\Gamma_{BC}+\Gamma_{AC}} (x+y)ds = \int_0^1 y\cdot\sqrt{0^2+1^2}\,dy + \int_0^1 1\cdot\sqrt{1+(-1)^2}\,dx + \int_0^1 x\sqrt{1^2}\,dx$$
$$= \frac{1}{2} + \sqrt{2} + \frac{1}{2} = 1+\sqrt{2}.$$

(b). $\begin{cases} x=x \\ y=2x. \end{cases}$ $0\le x\le 1$

$$\int \frac{1}{\sqrt{5x^2+4}}\cdot\sqrt{5}\,dx = \int \frac{1}{\sqrt{x^2+(\frac{5}{4})}}\,dx. \quad \frac{x=\frac{\sqrt5}{2}\tan t}{dx=\frac{\sqrt5}{2}\sec^2 t} \quad \int \frac{\frac{\sqrt5}{2}\sec^2 t}{\frac{\sqrt5}{2}\sec t}\,dt = \ln|\sec t+\tan t| + C. = \ln\left|\frac{\sqrt{x^2+\frac{5}{4}}}{\frac{\sqrt5}{2}} + \frac{x}{\frac{\sqrt5}{2}}\right| + c$$

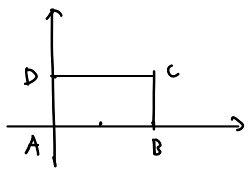
$\frac{2}{\sqrt5}$ $\frac{4}{5}$. $\frac{4}{5}$

$$\int_\Gamma f(x,y)\,ds = \cancel{\ln}\sqrt{5}$$

$\ln\frac{3+\sqrt5}{2}$

$$\frac{\frac{3}{\sqrt5}}{\frac{2}{\sqrt5}} \quad \frac{2}{\sqrt5}$$

2. Calculate the curvilinear integral $\int_\Gamma xyds$ if Γ — border of a rectangle with vertices $(0;0),(4;0),(4;2),\ (0;2)$.



$$\Gamma = \Gamma_{AB} + \Gamma_{BC} + \Gamma_{CD} + \Gamma_{AD}.$$

$$\int_\Gamma xy\,ds = \int_0^4 0\cdot dx + \int_0^2 4y\,dy + \int_0^4 2x\,dx + \int_0^2 0\cdot dy$$
$$= 2y^2\Big|_0^2 + x^2\Big|_0^4 = 24.$$

4. $\int_\Gamma \left(x^2+y^2\right)^n ds, \Gamma$ - circle $x^2+y^2=a^2$.

$\begin{cases} x = a\cos t \\ y = a\sin t. \end{cases}$

$$\int_0^{2\pi} (a^2)^n\cdot a\,dt = 2\pi a^{2n+1}$$

(* second kind ? but no direction)

5. $\int_\Gamma f(x,y)ds, \Gamma$ - circle $x^2+y^2=ax$, if $f(x;y)=\sqrt{x^2+y^2}$.

$a>0$. $\begin{cases} x = \frac{a}{2}+\frac{a}{2}\cos t \\ y = \frac{a}{2}\sin t. \end{cases}$

$$\int_0^{2\pi} \sqrt{(\tfrac{a}{2}+\tfrac{a}{2}\cos t)^2+(\tfrac{a}{2}\sin t)^2}\cdot\frac{a}{2}\,dt. = \frac{a^2}{2\sqrt2}\int_0^{2\pi}\sqrt{1+\cos t}\,dt. = \frac{a^2}{\sqrt2}\int_0^{\pi}\sqrt2\cos\frac{t}{2}\,dt = 2a^2\sin\frac{t}{2}\Big|_0^{\pi} = 2a^2$$

$a<0$. $\begin{cases} x = \frac{a}{2}-\frac{a}{2}\cos t \\ y = -\frac{a}{2}\cos t \end{cases}$ $\frac{a^2}{2\sqrt2}\int_0^{2\pi}\sqrt{1-\cos t}\,dt = \frac{a^2}{2}\int_0^{2\pi}\left|\sin\frac{t}{2}\right|dt = -a^2\cos\frac{t}{2}\Big|_0^{\pi} = 2a^2$

$$\int_\Gamma f(x,y)\,ds = 2a^2$$

6. $\int_\Gamma f(x;y)ds, \Gamma$ - the right lobe of the lemniscate given in polar coordinates by the equation $r^2 = a^2\cos 2\varphi$ if: $f(x;y) = x\sqrt{x^2-y^2}$.

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi. \end{cases} \qquad \varphi \in [-\tfrac{\pi}{4}, \tfrac{\pi}{4}]$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r\cos\varphi \cdot r\sqrt{\cos 2\varphi} \cdot \sqrt{r^2 + a^2 \frac{\sin^2 2\varphi}{\cos 2\varphi}}\, d\varphi$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\cos\varphi \cdot \sqrt{\cos 2\varphi} \cdot a \cdot \frac{1}{\sqrt{\cos 2\varphi}}\, d\varphi = a^3\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\varphi \cos\varphi \, d\varphi.$$

$$= a^3 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1-2\sin^2\varphi)\cos\varphi\, d\varphi = \sqrt{2}a^3 - 2a^3\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\varphi\, d(\sin\varphi). = \frac{2\sqrt{2}}{3}a^3.$$

7. $\int_\Gamma \left(x^{4/3} + y^{4/3}\right) ds, \Gamma$ - astroid $x^{2/3} + y^{2/3} = a^{2/3}$.

$$\begin{cases} x = a\cos^3\varphi \\ y = a\sin^3\varphi \end{cases} \qquad \varphi \in [0, 2\pi).$$

$$\int_0^{2\pi} a^{\frac{4}{3}}(\cos^4\varphi + \sin^4\varphi) \cdot 3a\sqrt{\cos^4\varphi\sin^2\varphi + \cos^2\varphi\sin^4\varphi}\, d\varphi.$$

$$= 12a^{\frac{7}{3}} \int_0^{\frac{\pi}{2}} (\cos^4\varphi + \sin^4\varphi)\cos\varphi\sin\varphi\, d\varphi = 12a^{\frac{7}{3}}\left[\int_0^{\frac{\pi}{2}} -\cos^5\varphi\, d(\cos\varphi) + \int_0^{\frac{\pi}{2}} \sin^5\varphi\, d(\sin\varphi)\right] = 4a^{\frac{7}{3}}$$

8. $\int_\Gamma f(x;y)ds, \Gamma-$ the arch of the cycloid $x = a(t-\sin t)$, $y = a(1-\cos t)$, $0 \le t \le 2\pi$ if: $f(x;y) = y$; (b) $f(x;y) = y^2$.

$$\sqrt{[x'(t)]^2 + [y'(t)]^2} = |a| \cdot \sqrt{(1-\cos t)^2 + (\sin t)^2} = \left|2a \cdot \sin\tfrac{t}{2}\right|$$

(a) $\int_J f_1(x,y)ds = 2|a|\cdot a\int_0^{2\pi}(1-\cos t)\cdot \sin\frac{t}{2}\, dt = 4a|a|\int_0^{2\pi} \sin^3\frac{t}{2}\, dt. = \frac{32}{3}a|a|$

(b) $\int_J f_2(x,y)ds = 2|a|^3\int_0^{2\pi}(1-\cos t)^2\sin\frac{t}{2}\, dt = 16|a|^3\int_0^{\pi}\sin^5\frac{t}{2}\, d(\frac{t}{2}) = \frac{128}{15}|a|^3$

$= 32a^3\int_0^{\frac{\pi}{2}} \sin^5\theta\, d\theta = \frac{4\times 2}{5\times 3} \times 32a^3 = \frac{256}{15}a^3$

9. $\int_\Gamma \sqrt{2y^2+z^2}ds, \Gamma$ - circle $x^2+y^2+z^2 = a^2, x = y$. $(a>0)$

$$\int_J \sqrt{2y^2+z^2}\, ds = \int_J \sqrt{x^2+y^2+z^2}\, ds = a\int_J ds.$$

Find J's length J is the circle. center at $(0,0,0)$. has radius a

$$\int_J \sqrt{2y^2+z^2}\, ds = a \cdot 2\pi a = 2\pi a^2$$

10. $\int_\Gamma (x+y)ds, \Gamma$ is a quarter of the circle $x^2+y^2+z^2=a^2, y=x$, located in the I octant.

$$2x^2+z^2=a^2 \Rightarrow \begin{cases} x=y=\frac{a}{\sqrt{2}}\cos t \\ z=a\sin t. \end{cases} \quad t\in[0,\tfrac{\pi}{2}]$$

$$\int_0^{\frac{\pi}{2}} \sqrt{2}a\cos t \cdot a\,dt = \sqrt{2}a^2\int_0^{\frac{\pi}{2}} \cos t\,dt = \sqrt{2}a^2.$$

11. (a) $\int_\Gamma xdy - ydx, \Gamma-$ curve $y=x^3, 0 \leqslant x \leqslant 2$;
(b) $\int_\Gamma 2xydx - x^2dy, \Gamma-$ the arc of the parabola $y=\sqrt{\frac{x}{2}}, 0 \leqslant x \leqslant 2$.

(a)
$$\int_0^2 (3x^3 - x^3)\,dx = 2\int_0^2 x^3\,dx = \frac{x^4}{2}\Big|_0^2 = 8$$

(b).
$$\int_0^2 2x\cdot\sqrt{\frac{x}{2}}\,dx - x^2\cdot\frac{\frac{1}{2}}{2\sqrt{\frac{x}{2}}}\,dx = \frac{3\sqrt{2}}{4}\int_0^2 x^{\frac{3}{2}}\,dx = \frac{12}{5}.$$

12. (a) $\int_\Gamma (xy-y^2)\,dx + xdy, \Gamma-$ curve $y=2\sqrt{x}, 0 \leqslant x \leqslant 1$;
(b) $\int_\Gamma (x^2-2xy)\,dx + (y^2-2xy)\,dy, \Gamma-$ the arc of the parabola $y=x^2$, $-1 \leqslant x \leqslant 1$

(a)
$$\int_0^1 (2x\sqrt{x} - 4x)\,dx + x\cdot\frac{\cancel{2}\ 1}{\sqrt{x}}\,dx. = \int_0^1 2x^{\frac{3}{2}} - 4x + \cancel{2}x^{\frac{1}{2}}\,dx.$$

$$= \frac{4}{5}\cdot x^{\frac{5}{2}} - 2x^2 + \frac{\cancel{4}\ 2}{3}\cdot x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{15} \quad -\frac{8}{15}$$

(b).
$$\int_{-1}^1 (x^2-2x^3)\,dx + (x^4-2x^3)\cdot 2x\,dx$$

$$= \int_{-1}^1 (2x^5 - 4x^4 - 2x^3 + x^2)\,dx = \frac{x^6}{3} - \frac{4}{5}x^5 - \frac{x^4}{2} + \frac{x^3}{3}\Big|_{-1}^1 = -\frac{14}{15}$$

13. (a) $\int_\Gamma xy^2dx$, $\Gamma-$ the arc of the circle $x=\cos t$, $y=\sin t$, $0 \leqslant t \leqslant \pi/2$
(b) $\int_\Gamma ydx - xdy$, $\Gamma-$ ellipse $x=a\cos t$, $y=b\sin t$, $0 \leqslant t \leqslant 2\pi$;

(a).
$$\int_0^{\frac{\pi}{2}} \cos t\sin^2 t\,(-\sin t)\,dt = -\int_0^{\frac{\pi}{2}} \sin^3 t\,d(\sin t) = -\frac{1}{4}$$

(b).
$$ab\int_0^{2\pi} \sin t\cdot(-\sin t)\,dt - \cos t\cdot\cos t\,dt = -2\pi ab.$$

14. $\int_\Gamma y dx + z dy + x dz$, $\Gamma-$ the spiral of the helix $x = a\cos t$, $y = a\sin t$, $z = bt$, $0 \leqslant t \leqslant 2\pi$.

$$\int_\Gamma y\,dx = \int_0^{2\pi} a\sin t \cdot (-a\sin t)\,dt$$

$$= -a^2 \int_0^{2\pi} \sin^2 t\,dt = -a^2 \int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt = -a^2\pi.$$

$$\int_\Gamma z\,dy = \int_0^{2\pi} bt \cdot (a\cos t)\,dt = ab\left[t\sin t \Big|_0^{2\pi} - \int_0^{2\pi} \sin t\,dt\right] = 0$$

$$\int_\Gamma x\,dz = \int_0^{2\pi} a\cos t \cdot b\,dt. = ab\int_0^{2\pi} \cos t\,dt = 0.$$

$$I = -a\pi^2$$

15. $\int_\Gamma \left(y^2 - z^2\right) dx + 2yzdy - x^2dz$, Γ - curve $x = t$, $y = t^2$, $z = t^3$, $0 \leqslant t \leqslant 1$.

$$\int_0^1 (t^{\cancel{2}\,4} - t^{\cancel{3}\,6})\,dt + 2t^5 \cdot 2t\,dt - t^2 \cdot 3t^2\,dt$$

$$= \int_0^1 4t^6 - 3t^4 - t^3 + t^2\,dt = \frac{4}{7}t^7 - \frac{3}{5}t^5 - \frac{1}{4}t^4 + \frac{1}{3}t^3 \Big|_0^1 = \frac{23}{420}$$

$$3t^6 - 2t^4 \qquad \frac{3}{7}t^6 - \frac{2}{5}t^5 \Big|_0^1 = \frac{1}{35}$$