Recall that two polynomials

$$f(x)=a_n x^n+\cdots+a_1x+a_0$$

and

$$g(x)=b_m x^m+\cdots+b_1(x)+b_0$$

are equal if and only if the coefficients of each power of x are equal.

Thus as polynomials with coefficients in $\mathbb{F}$ ($=\mathbb{R}$) the polynomial

$$f(x)=a_3x^3+a_2x^2+a_1x+a_0$$

is equal to

$$g(x)=x^3+1$$

if and only if $a_0=a_3=1$ and $a_1=a_2=0$.

On the other hand, two functions $f(x)$ and $g(x)$ defined on the ~~set~~ field $\mathbb{F}$ are equal if and only if for all a in $\mathbb{F}$, the numbers $f(a)=g(a)$.

Any polynomial over the field $\mathbb{F}$ defines a function on $\mathbb{F}$, as we have seen. Thus two polynomials that are equal as polynomials are equal as functions. ~~Conversely~~ Conversely, is it possible for two polynomials to be ~~different~~ different as polynomials but be equal as functions? This phenomenon cannot happen if $\mathbb{F}$ is an infinite field, such as the real numbers.

Thm 6 Let $\mathbb{F}$ be a field of numbers and let $f(x)$ and $g(x)$ be two polynomials over $\mathbb{F}$ of degree not exceeding n. If there exist $n+1$ distinct numbers $\alpha_1,\cdots,\alpha_{n+1}\in\mathbb{F}$ such that

$$f(\alpha_i)=g(\alpha_i),\quad 1\le i\le n+1,$$

then $f(x)=g(x)$ as polynomials.

Different polynomials define different functions over infinite fields.

Proof: Set $h(x)=f(x)-g(x)$. We have ~~h(x)~~ $\deg h(x)\le n$ or $h(x)=0$. By the assumption, $h(\alpha_i)=f(\alpha_i)-g(\alpha_i)=0$ for $1\le i\le n+1$. So that $h(x)$ has $n+1$ roots. Hence, ~~If h(x)≠0~~ $h(x)$ must be the zero polynomial by D'Alembert's Theorem, and $f(x)=g(x)$ as polynomials. □

§8. Factorization for Real / Complex polynomials

$\boxed{\text{Theorem 1. (Fundamental Theorem of Algebra)}}$ Every polynomial $f(x)$ in $\mathbb{C}[x]$ of degree ≥ 1 has a root in $\mathbb{C}$.

$\boxed{\text{Remark 1.}}$ The only irreducible polynomials in $\mathbb{C}[x]$ are of degree one.

$\boxed{\text{Theorem 2}}$. Every polynomial $f(x)$ in $\mathbb{C}[x]$ of degree ≥ 1 factors into a product of ~~polynomial~~ polynomials of degree 1.

$\boxed{\text{Remark 2}}$. A polynomial $f(x)$ in $\mathbb{C}[x]$ of degree ≥ 1 has a normalized factorization:

$$f(x) = a(x-\alpha_1)^{l_1}(x-\alpha_2)^{l_2}\cdots(x-\alpha_s)^{l_s},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_s$ are distinct roots of $f(x)$, $l_1, l_2, \ldots, l_s$ are positive integers, and $l_1 + l_2 + \cdots + l_s = \deg(f)$. Hence, a polynomial $f(x)$ in $\mathbb{C}[x]$ of degree n has exactly n roots in $\mathbb{C}$.

$\boxed{\text{proposition 1}}$. Let $f(x)$ be a polynomial with real coefficients. ~~over $\mathbb{R}$~~ If $z = a+bi$ is a root of $f(x)$, where a and b are real numbers, then $\bar{z} = a-bi$ is also a root of $f(x)$.

proof. Write $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_n, a_{n-1}, \ldots, a_0$ are real numbers.

If $z = a+bi$ is a root of $f(x)$, then

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0$$

Taking the conjugate of both sides of the last equation gives

$$\overline{a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0} = \bar{0}$$

Using a generalization of the properties $\overline{c+d} = \bar{c} + \bar{d}$ and $\overline{cd} = \bar{c}\,\bar{d}$ gives

$$\overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots + \overline{a_0} = \bar{0}.$$

Now use the property $\overline{c^n} = (\bar{c})^n$ and the fact that for any real number a, $\bar{a} = a$, to obtain

$$a_n(\bar{z})^n + a_{n-1}(\bar{z})^{n-1} + \dots + a_1 \bar{z} = 0$$

Hence $f(\bar{z}) = 0$ and $\bar{z} = a - bi$ is a root of $f(x)$. $\square$.

$\boxed{\text{Caution}}$: It is essential that the polynomial have only real coefficients. For instance, $f(x) = x - (1+i)$ has $1+i$ as a root, but the conjugate $1-i$ is not a root.

$\boxed{\text{Proposition 2}}$. If $f(x) = x^2 + bx + c$ is a real polynomial of degree 2, then $f(x)$ is irreducible over $\mathbb{R}$ if and only if $b^2 - 4c < 0$.

$\boxed{\text{Theorem 3}}$. No polynomials $f(x)$ in $\mathbb{R}[x]$ of degree > 2 are irreducible in $\mathbb{R}[x]$.

Proof. Let $f(x)$ in $\mathbb{R}[x]$ have degree > 2. We will show that $f(x)$ is not irreducible. We can assume that $f(x)$ has no real roots, by Root theorem.

Suppose α is a nonreal complex root of $f(x)$. Let

$$p(x) = (x - \alpha)(x - \bar{\alpha}),$$

where, if $\alpha = a + bi$, then $\bar{\alpha} = a - bi$ is the complex conjugate of α. Then

$$p(x) = x^2 - 2ax + a^2 + b^2$$

is in $\mathbb{R}[x]$ and $p(x)$ is irreducible in $\mathbb{R}[x]$ since its two roots are not real numbers. Dividing $f(x)$ by $p(x)$ in $\mathbb{R}[x]$ gives

$$f(x) = q(x)\,p(x) + r(x), \tag{1}$$

with $r(x) = 0$ or $\deg r(x) \leq 1$. Let $r(x) = r + sx$ be in $\mathbb{R}[x]$. Evaluate equation (1) at $x = \alpha$, we get $r(\alpha) = 0$, since α is a root of both $f(x)$ and $p(x)$. But then $r + s\alpha = 0$, and so unless $r = s = 0$, we conclude that α is real, a contradiction. Thus $r(x) = 0$, and $p(x) \mid f(x)$. Since $\deg p(x) = 2 < \deg f(x)$, $f(x)$ is not irreducible. $\square$

Theorem 4. Every polynomial in $\mathbb{R}[x]$ of degree ≥ 1 can be factored into a product of polynomials of degree 1 and irreducible polynomials of degree 2.

More precisely, a polynomials $f(x)$ in $\mathbb{R}[x]$ has a factorization of the form

$$f(x) = a(x-c_1)^{l_1} \cdots (x-c_s)^{l_s} (x^2+p_1x+q_1)^{k_1} \cdots (x^2+p_rx+q_r)^{k_r},$$

where $a, c_2, \cdots, c_s, p_1, \cdots, p_r, q_1, \cdots, q_r$ are real numbers, $l_1, \cdots, l_s, k_1, \cdots, k_r$ are positive integers, and $x^2+p_ix+q_i$ $(1 \leq i \leq r)$ are irreducible, that is, $p_i^2-4q_i<0$ for $i=1, \cdots, r$.

Excercises

1. Find a polynomial of least degree having only real coefficients and roots 3 and $2+i$.

2. Find all roots of $f(x) = x^4-7x^3+18x^2-22x+12$, given that $1-i$ is a root.

Solutions.

1. $f(x) = x^3-7x^2+17x-15 = (x-3)(x-(2+i))(x-(2-i))$

Of course, any nonzero multiple of $f(x)$ also satisfies the given conditions.

2. $f(x) = (x-(1-i))(x-(1+i))(x^2-5x+6) = (x-(1-i))(x-(1+i))(x-2)(x-3)$

The four roots of $f(x)$ are $1-i$, $1+i$, 2 and 3.

(Note that $(x-(1-i))(x-(1+i)) = x^2-2x+2$)