~~Homework~~:

Exercise: For $f(x) = x^3 - 3x + 2$ and $g(x) = x^2 + x + 1$ in $\mathbb{Q}[x]$, find a greatest common divisor of $f(x)$ and $g(x)$, and find $u(x)$, $v(x)$ in $\mathbb{Q}[x]$ such that

$$d(x) = u(x) f(x) + v(x) g(x).$$

Solution: By Euclidean Algorithm, we have a sequence of equalities:

$$f(x) = (x-1)\, g(x) + (-3x+3)$$

$$g(x) = \left(-\tfrac{1}{3}x - \tfrac{2}{3}\right)(-3x+3) + 3$$

$$\cancel{3 = -3(-x+1)}$$

$$-3x+3 = 3(-x+1) + 0$$

Hence, 3 is a greatest common divisor of $f(x)$ and $g(x)$.

$$3 = g(x) - \left(-\tfrac{1}{3}x - \tfrac{2}{3}\right)(-3x+3)$$

$$= g(x) + \tfrac{1}{3}(x+2)\left[f(x) - (x-1)g(x)\right]$$

$$= g(x) - \tfrac{1}{3}(x+2)(x-1)g(x) + \tfrac{1}{3}(x+2)f(x)$$

$$= \left(1 - \tfrac{1}{3}(x+2)(x-1)\right)g(x) + \tfrac{1}{3}(x+2)f(x)$$

Taking $u(x) = \frac{1}{3}(x+2)$, $v(x) = 1 - \frac{1}{3}(x+2)(x-1) = -\frac{1}{3}x^2 - \frac{1}{3}x + \frac{5}{3}$, we obtain

$$3 = u(x) f(x) + v(x) g(x)$$