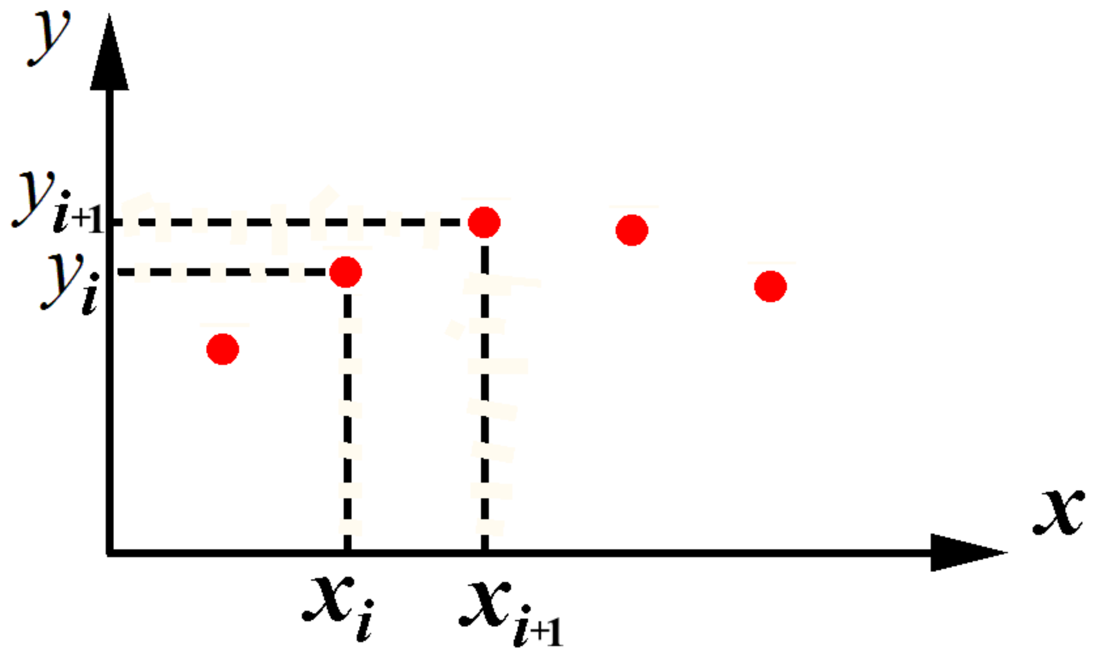
Chapter 5. Interpolation by polynomials

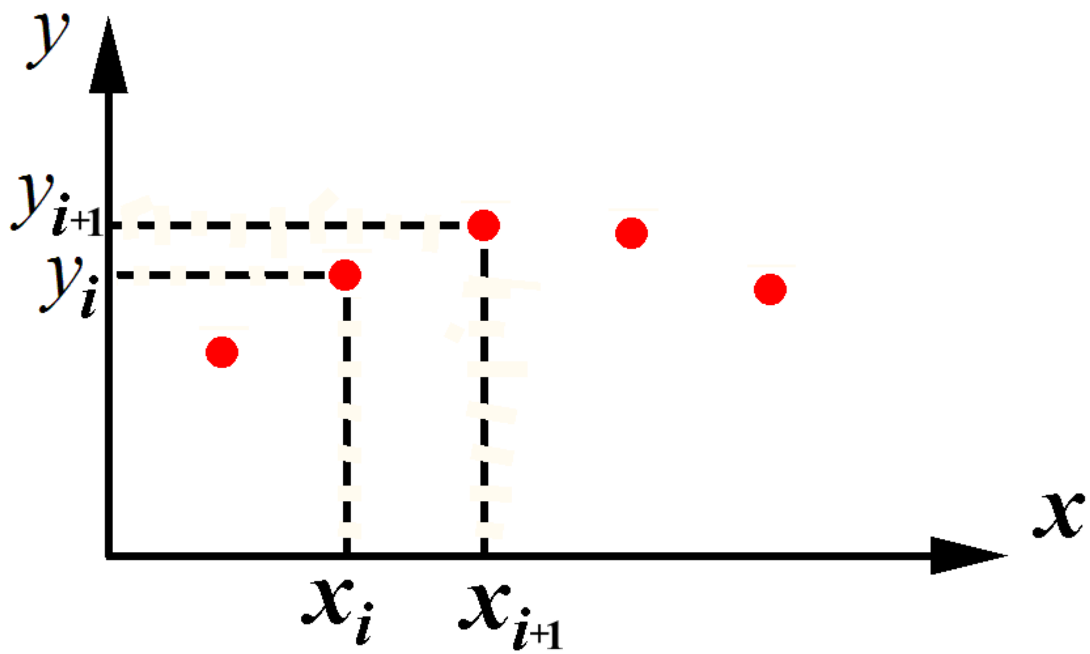
Suppose that values of a function $f(x)$ are given at finite number of points/nodes $x_0, x_1, x_2, \ldots, x_n$ (the function is given by table/array): $y_0, y_1, y_2, \ldots, y_n$.
What value can be adopted as approximate value of f at a point x lying between nodes?

If $x_i < x < x_{i+1}$, then $f(x) = ?$ approximately ?

Term "interpolation" means approximation between nodes



An evident approach is to plot a polynomial $P(x)$ that passes through the given points (x_i , y_i), and then to assume that $f(x) \approx P(x)$.



(We already used a linear function $P_1(x)$ to obtain trapezoids formula, as well as a parabola $P_2(x)$ to obtain Simpson's formula).

Is it possible to plot a single polynomial that passes through all $n+1$ points (x_i , y_i) in the plane (x, y) ?

A general form of n^{th} degree polynomial is:

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

It involves $n+1$ coefficient a_i .

We can impose the condition $P_n(x_i) = y_i$, $i = 0, 1, \ldots n$

$$a_0 + a_1 x_i + a_2 x_i^2 + \ldots + a_n x_i^n = y_i$$

and obtain the system of $n+1$ algebraic equations with respect to a_i . The system can be solved with methods described in Chapter 2.

Unfortunately, this way of getting $P_n(x)$ needs a great deal of computations.

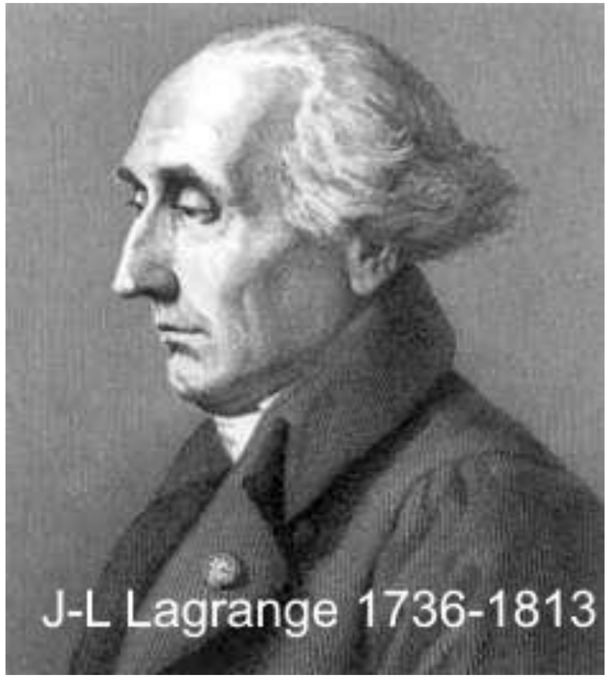
Lagrange suggested a form of the required polynomial which does not need solving any algebraic equations.

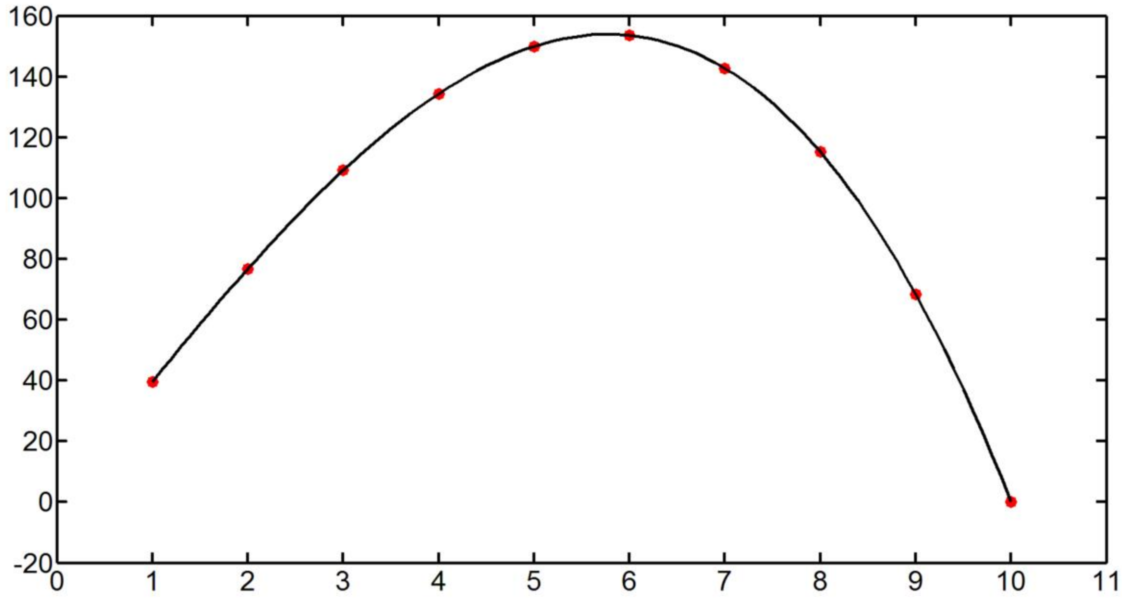
Lagrange's polynomial:

$$P(x) = y_0 \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} +$$

$$+\, y_1 \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} +$$

$$\ldots + y_n \frac{(x-x_0)(x-x_2)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\ldots(x_n-x_{n-1})}$$





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Regarding the error of approximation $f(x) \approx P(x)$
at $x_i < x < x_{i+1}$ **:**

Theorem

If there exist derivatives $f^{n+1}(x)$ of function $f(x)$ up to the order $n+1$ on segment $[x_0, x_n]$, then

$$f(x) - P(x) = f^{n+1}(c) \cdot (x-x_0)(x-x_1)\ldots(x-x_n)/(n+1)!$$

where $x_0 \leq c \leq x_n$, ! is the factorial

(proof is omitted)

Example

Estimate the error of approximation of the function $y = \ln x$ by Lagrange's polynomial of degree 2. Use the points:

x_i	$y_i = \ln x$
2.0	0.69315
2.5	0.91629
3.0	1.09861

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x0= 1
x1= 2
x2= 3
x3= 5
      y0= 2
      y1= 2.9
      y2= 4.2
      y3= 6
plot(x0,y0,'o',x1,y1,'o',x2,y2,'o',x3,y3,'o')        // nodal points
      for i=1:101
      x= 1+ (i-1)*4*0.01
      y=y0* (x-x1)*(x-x2)*(x-x3)   / ((x0-x1)*(x0-x2)*(x0-x3) ) +...
         y1* (x-x0)*(x-x2)*(x-x3)   / ((x1-x0)*(x1-x2)*(x1-x3) ) + ...
         y2* (x-x0)*(x-x1)*(x-x3)   / ((x2-x0)*(x2-x1)*(x2-x3) ) +...
         y3* (x-x0)*(x-x1)*(x-x2)   / ((x3-x0)*(x3-x1)*(x3-x2))
         xp(i)=x
         yp(i)=y
      end
plot(xp,yp,'k','LineWidth',3)                          // polynomial
            ypp=interpln([x0 x1 x2 x3; y0 y1 y2 y3],xp)
            plot(xp,ypp,'r','LineWidth',3)
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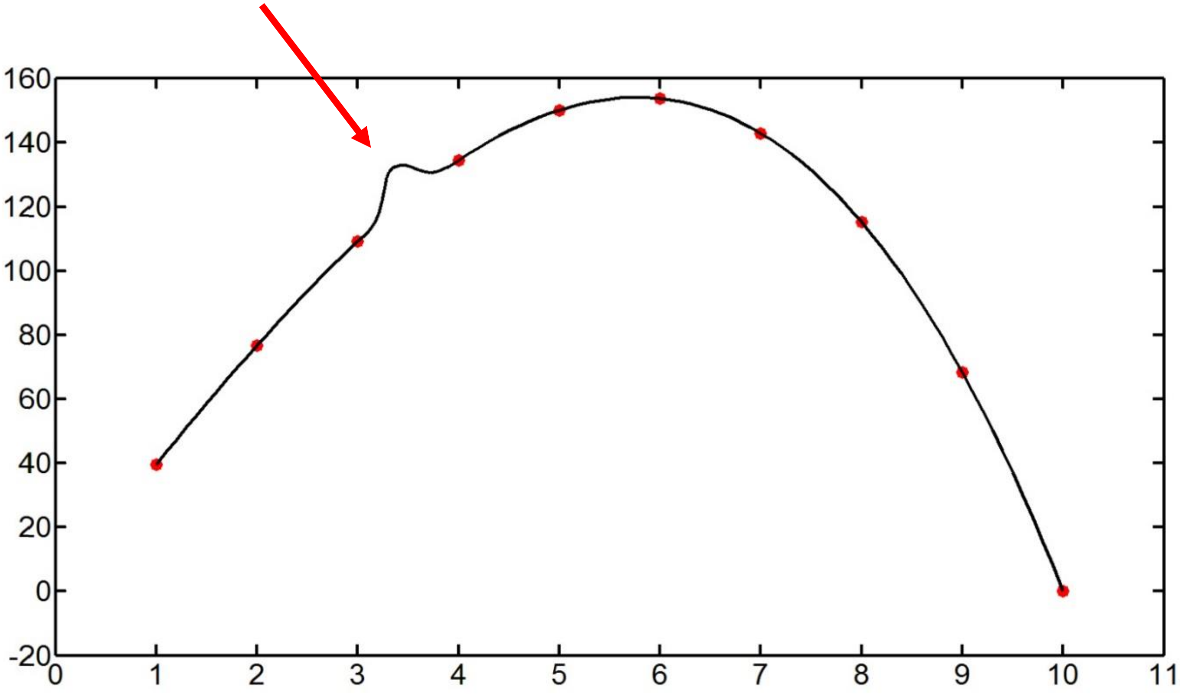
Sometimes, Lagrange polynomial exhibits an unusual behavior:



This may happen when degree *n* of the polynomial is large

This is a specific property of polynomials.

As a result, we get a bad approximation of function *f(x)*.

Even at *n*=4 and *n*=5 Lagrangian's polynomial happens to show inappropriate behavior:

Example. Let us approximate the function $f(x) = 1/(1+25x^2)$

$-1 \leq x \leq 1$