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2.3.1. APPLICATION OF THE LAPLACE TRANSFORM TO SOLVING FIRST-ORDER LINEAR EQUATIONS

An unknown function satisfying a first-order linear partial differential equation and given conditions can be found using a one-time or two-time Laplace transform, depending on the type of conditions.

In the first case, the transformation is applied to a partial differential equation for one of the independent variables, assuming that the other remains unchanged.

The result is an operator equation with respect to the image, which is an ordinary differential equation with a parameter.

After integrating the operator equation from the image found from it, the original is found as a solution to the original equation.

In the second case, the Laplace transform is applied sequentially, resulting in an equation from which a two-fold image of the desired function is found.

Using inverse transformations, the original function is restored.

The solution of the partial differential equation found using the two-fold Laplace transform does not depend on the sequence in which the forward and reverse transformations were applied.

Example 1

In the region $x > 0$, $y > 0$, using the Laplace transform to find a solution to the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x + y ,$$

satisfying the conditions: $u(0, y) = u(x,0) = 1$.

<u>Solution:</u>

We apply the Laplace transform with respect to the variable x to the given equation , assuming $u(x,y) \leftrightarrow U(p,y)$. Since

$$\frac{\partial u}{\partial x} \leftrightarrow pU(p,y) - u(0,y) = pU(p,y) - 1 ,$$

$$\frac{\partial u}{\partial y} \leftrightarrow \frac{\partial U(p,y)}{\partial y} ,$$

$$u(x,0) = 1 \leftrightarrow U(p,0) = \frac{1}{p} ,$$

the specified transformation gives the operator equation:

$$pU(p,y) - 1 + \frac{\partial U(p,y)}{\partial y} = \frac{1}{p^2} + \frac{y}{p} ,$$

to which the condition

$$U(p,0) = \frac{1}{p}$$

should be added.

Thus, a single Laplace transform with respect to the variable x gives the problem

$$\begin{cases} pU(p,y)+\dfrac{\partial U(p,y)}{\partial y}=\dfrac{1}{p^2}+\dfrac{y}{p}+1, \quad y>0, \\ U(p,0)=\dfrac{1}{p}. \end{cases} \tag{*}$$

The resulting equation can be considered as an ordinary first - order differential equation with constant coefficients for the function U, with an independent variable y and a parameter p. Let's solve the Cauchy problem (*) in two ways.

First, by solving a differential equation, it is possible to construct its general solution:

$$U(p,y)=Ce^{-py}+\frac{y}{p^2}+\frac{1}{p},$$

and select a solution that satisfies the given initial condition:

$$U(p,y)=\frac{y}{p^2}+\frac{1}{p}.$$

It is easy to build a corresponding original for the found image:

$$u(x,y)=yx+1.$$

The second method involves solving the problem (*) using the Laplace transform with respect to the variable y.

Assuming $U(p,y)\leftrightarrow V(p,q)$, we construct the operator equation

$$qV(p,q)-\frac{1}{p}+pV=\frac{1}{p^2q}+\frac{1}{pq^2}+\frac{1}{q},$$

from where we find

$$V(p,q)=\frac{1}{p^2q^2}+\frac{1}{pq}.$$

By performing inverse transformations, we find a solution to the problem formulated in the example condition.

Both the first method, the one—time Laplace transform, and the second, the two-time Laplace transform, give the same result.

2.4. CLASSIFICATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

A linear partial differential equation of the second order is called the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f(x)\,,\ x \in D\,, \qquad (2.3)$$

where the coefficients are real functions of the point x in the region D:

$$a_{ij} = a_{ij}(x)\,,\ b_i = b_i(x)\,,\ c = c(x)\,.$$

Equation (2.3) corresponds to the characteristic form:

$$Q(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j\,,$$

which is quadratic.

At each fixed point $x \in D$, using a non-special affine transformation of variables:

$$\lambda_i = \lambda_i(\mu_1, \mu_2, \ldots, \mu_n)\,,\ i = 1, 2, \ldots, n$$

the quadratic form Q can be reduced to the canonical form

$$\tilde{Q}(\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i=1}^{n} \alpha_i \mu_i^2\,, \qquad (2.4)$$

where $\alpha_i \in \{-1, 0, 1\}$.

The canonical form of the quadratic form determines the type of equation (2.3).

The linear equation (2.3) will be called *elliptical* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, all $\alpha_i \neq 0$ and one sign.

Equation (2.3) will be called *hyperbolic* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, all $\alpha_i \neq 0$, but not all of the same sign.

Equation (2.3) will be called *parabolic* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, at least one of the coefficients $\alpha_k = 0$.

Example 1

Determine the type of equation for $u = u(x, y)$:

$$u_{xx} - 4u_{xy} + 8u_{yy} + u_x - 6u + y = 0\ .$$

Solution:

The given equation corresponds to the quadratic form

$$Q(\lambda_1, \lambda_2) = \lambda_1^2 - 4\lambda_1\lambda_2 + 8\lambda_2^2,$$

which we bring to the canonical form by sequentially highlighting the complete squares:

$$Q(\lambda_1, \lambda_2) = \lambda_1^2 - 4\lambda_1\lambda_2 + 8\lambda_2^2 = \lambda_1^2 - 4\lambda_1\lambda_2 + 4\lambda_2^2 + 4\lambda_2^2 =$$

$$= (\lambda_1 - 2\lambda_2)^2 + (2\lambda_2)^2 = \mu_1^2 + \mu_2^2 = \tilde{Q}(\mu_1, \mu_2)\,.$$

Since both coefficients in the canonical form of a quadratic form have the same sign, the given equation has an elliptical type in the entire domain of setting the variables x, y.

Example 2

Determine the type of equation for $u = u(x, y, z)$:

$$u_{xx} - 4u_{yy} + 2u_{xz} + 4u_{yz} + 2u_x - u_y = xyz^2.$$

<u>Solution:</u>

The given equation corresponds to the quadratic form

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 - 4\lambda_2^2 + 2\lambda_1\lambda_3 + 4\lambda_2\lambda_3.$$

Let's bring it to a canonical form, sequentially highlighting the full squares:

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + 2\lambda_1\lambda_3 + \lambda_3^2 - \lambda_3^2 - 4\lambda_2^2 + 4\lambda_2\lambda_3 =$$
$$= (\lambda_1 + \lambda_3)^2 - (2\lambda_2 - \lambda_3)^2 = \mu_1^2 - \mu_2^2 = \tilde{Q}(\mu_1, \mu_2, \mu_3).$$

Since one of the coefficients in the canonical form of the quadratic form is 0 (for μ_3^2), the given equation has a parabolic type in the entire domain of setting the variables x, y, z.

2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

Consider a second-order equation, linear with respect to the higher derivatives, for an unknown function $u(x,y)$ of two independent variables x and y:

$$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+F(x,y,u,u_x,u_y)=0, \qquad (2.5)$$

where the real functions $a(x,y), b(x,y), c(x,y)$ are defined in the domain D.

2.5.1. REPLACING INDEPENDENT VARIABLES

Let's introduce independent variables

$$\xi=\xi(x,y), \;\; \eta=\eta(x,y), \qquad (2.6)$$

where ξ, η are twice continuously differentiable functions in the domain D.

We require that the Jacobian of the transformation be nonzero:

$$\frac{D(\xi,\eta)}{D(x,y)}\neq 0 .$$

Let's try to choose the transformation (2.6) in such a way that equation (2.5) has the simplest form in the new variables. We transform equation (2.5) to new variables, assuming

$$U(\xi,\eta)=u\big(x(\xi,\eta),y(\xi,\eta)\big).$$

Then we get

$$u_x = U_\xi \xi_x + U_\eta \eta_x ,$$

$$u_y = U_\xi \xi_y + U_\eta \eta_y ,$$

$$u_{xx} = U_{\xi\xi} \xi_x^2 + 2U_{\xi\eta} \xi_x \eta_x + U_{\eta\eta} \eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx} ,$$

$$u_{yy} = U_{\xi\xi} \xi_y^2 + 2U_{\xi\eta} \xi_y \eta_y + U_{\eta\eta} \eta_y^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy} ,$$

$$u_{xy} = U_{\xi\xi} \xi_x \xi_y + U_{\xi\eta} \left(\xi_x \eta_y + \xi_y \eta_x \right) + U_{\eta\eta} \eta_x \eta_y + U_\xi \xi_{xy} + U_\eta \eta_{xy} .$$

In the new variables, equation (2.5) will take the form

$$\bar{a} U_{\xi\xi} + 2\bar{b} U_{\xi\eta} + \bar{c} U_{\eta\eta} + \bar{F} = 0 , \qquad (2.7)$$

where

$$\bar{a} = a\xi_x^2 + 2b\xi_x \xi_y + c\xi_y^2 , \qquad (2.8)$$

$$\bar{c} = a\eta_x^2 + 2b\eta_x \eta_y + c\eta_y^2 ,$$

$$\bar{b} = a\xi_x \eta_x + 2b(\xi_x \eta_y + \xi_y \eta_x) + c\xi_y \eta_y ,$$

$\bar{F} = \bar{F}\left(\xi, \eta, U, U_\xi, U_\eta\right) -$ a function that does not depend on the higher derivatives.

Definition 1. Equation (2.5) has at the point (x,y):

– *hyperbolic* type if $b^2 - ac > 0$ at point (x,y),

– *elliptical* type if $b^2 - ac < 0$ at point (x,y),

– *parabolic* type if $b^2 - ac = 0$ at point (x,y).

If the type of equation is preserved at all points of the domain D, then the equation is called an equation of this type in the entire domain D.

If an equation belongs to different types at different points in the domain, then it is called a *mixed type* equation in the domain D.

2.5.2.THE EQUATION OF CHARACTERISTICS

Now let's figure out how to introduce new variables x and h so that equation (2.5) takes the simplest form.

<u>Assumption.</u> Equation (2.5) belongs to a certain type in the entire domain D and $a(x,y)$ and $c(x,y)$ not equal to zero at the same time.

We assume that $a(x,y) \neq 0$.

It can be seen from the relation (2.8) that in order for $\bar{a} = 0$, it is necessary as a function of $\xi(x,y)$ to take the solution of the equation:

$$az_x^2 + 2bz_x z_y + cz_y^2 = 0. \tag{2.9}$$

Definition 2. Equation (2.9) is called the characteristic equation for equation (2.5).

<u>Lemma.</u> Let the function $z(x,y)$ be continuously differentiable in the domain D and such that $z_y \neq 0$. In order for the family of curves $z(x,y) = C$ to represent the characteristics of equation (2.5), it is necessary

and sufficient that the expression $z(x,y)=C$ be the general integral of the ordinary differential equation

$$a(x,y)(dy)^2-2b(x,y)dxdy+c(x,y)(dx)^2=0. \qquad (2.10)$$

Definition 3.

Equation (2.10) is called the *equation of characteristics* for equation (2.5).

Assuming $\xi=\varphi(x,y)$, where $\varphi(x,y)=C$ is the integral of equation (2.10), we nullify the coefficient at $U_{\xi\xi}$ in equation (2.7).

If $\psi(x,y)=C$ is another integral of equation (2.10), independent of $\varphi(x,y)$, then assuming $\eta=\psi(x,y)$, we also zero the coefficient at $U_{\eta\eta}$.

Equation (2.10) splits into two equations:

$$\frac{dy}{dx}=\frac{b+\sqrt{b^2-ac}}{a}, \qquad (2.11)$$

$$\frac{dy}{dx}=\frac{b-\sqrt{b^2-ac}}{a}. \qquad (2.12)$$

Definition 4. Solutions of equations (2.11), (2.12) are called *characteristics* for equation (2.5).

2.5.3.CANONICAL FORMS OF EQUATIONS

Consider the region D, at all points of which equation (2.5) has the same type.

1. For an equation of hyperbolic type $b^2 - ac > 0$, the right-hand sides of equations (2.11) and (2.12) are valid and different.

Their general integrals, $\varphi(x,y) = C_1$ and $\psi(x,y) = C_2$, define families of characteristics that do not touch each other.

Choosing $\xi = \varphi(x,y), \eta = \psi(x, y)$, we get $\bar{a} = 0$, $\bar{c} = 0$.

Therefore, equation (2.7), after division by $\bar{b} \neq 0$, takes the form

$$U_{\xi\eta} = \bar{F}(\xi, \eta, U, U_\xi, U_\eta). \qquad (2.13)$$

Definition 5. The form of equation (2.13) is called *the first canonical form* of the hyperbolic type equation.

Another canonical form is often used, which can be obtained by replacing:

$$\alpha = \frac{1}{2}(\xi - \eta),\ \beta = \frac{1}{2}(\xi + \eta).$$

In this case, the equation takes the form

$$U_{\alpha\alpha} - U_{\beta\beta} = \bar{F}_1(\xi, \eta, U, U_\xi, U_\eta).$$

2. Let the equation (2.5) be of elliptical type in the domain D, that is, $b^2 - ac < 0$.

Then the equations of characteristics (2.11) and (2.12) with real coefficients a,b,c have complex conjugate right-hand sides. All the characteristics will be complex.

Assuming that the coefficients a,b,c are defined in the complex domain, and making a formal substitution:

$$\xi = \xi(x,y), \ \eta = \xi^*(x,y),$$

where $\xi(x,y) = C_1$ and $\xi^*(x,y) = C_2$ - complex conjugate integrals (2.11) and (2.12), we obtain the equation

$$U_{\xi\eta} = \bar{F}_2(\xi,\eta,U,U_\xi,U_\eta) \qquad (2.14)$$

in the complex domain.

If we make another replacement:

$$\alpha = \frac{1}{2}(\xi+\eta) = \operatorname{Re}\xi, \ \beta = -\frac{i}{2}(\xi-\eta) = \operatorname{Im}\xi,$$

then equation (2.14) will take the form

$$U_{\alpha\alpha} + U_{\beta\beta} = \bar{F}_3(\xi,\eta,U,U_\xi,U_\eta) \qquad (2.15)$$

already in the real domain.

Definition 6. The form (2.15) of the transformed equation (2.5) is the canonical form of an elliptic equation.

3. Finally, let us consider a parabolic equation in the region D:

$$b^2 - ac = 0 .$$

In this case, there is only one equation of characteristics

$$\frac{dy}{dx} = \frac{b}{a} .$$

Let $\xi(x,y) = C$ be its integral. Let's take an arbitrary twice differentiable function $\eta(x,y)$ such that the condition

$$\frac{D(\xi,\eta)}{D(x,y)} \neq 0$$

is satisfied.

Then, when replacing $\xi = \xi(x,y), \eta = \eta(x,y)$, equation (2.7) takes the form

$$U_{\eta\eta} = \bar{F}_4(\xi,\eta,U,U_\xi,U_\eta) . \qquad (2.16)$$

Definition 7.

The form (2.16) of the transformed equation (2.5) is the *canonical form* of a parabolic equation.

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2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

We are considering $u(x, y)$ - unknown function of two independent variables.

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + F\left(x,y,u,u_x,u_y\right) = 0 \quad \textbf{(*)}$$

where $a(x,y),\ b(x,y),\ c(x,y)$ − are defined in the domain D.

$b^2 - ac > 0$ - *hyperbolic type*

$b^2 - ac = 0$ - *parabolic type*

$b^2 - ac < 0$ - *elliptical type*

Depending on what type of equation, you can find such a coordinate transformation:

It is possible to move from coordinates (x,y) to coordinates (ξ,η):

$$(x,y) \to (\xi,\eta).$$

You can find such a transformation and make it.

$$\xi = \varphi(x,y)$$

$$\eta = \psi(x,y)$$

Then the equation (*) can be written much more simply (depending on the type of equation).

And depending on the equation, it will be clear how to solve it.

The coefficients $a(x,y),\ b(x,y),\ c(x,y)$ – may depend on $x,\ y$, so if they are constant, then the equation will always have the same type for all $x,\ y$.

If the coefficients are variable, if they depend on $x,\ y$, then it may happen that the equation in one part of the plane has a hyperbolic type, and in another part of the plane has an elliptical type.

And then in each separate part it is necessary to solve it separately.

It will be necessary to find its canonical form separately for each part.

To bring the equation to a canonical form, it is <u>necessary to determine its type.</u>

Usually, the problem asks you to determine the type of equation.

And then we write <u>the characteristic equation:</u>

$$a(x,y)dy^2 - 2b(x,y)dx\,dy + c(x,y)dx^2 = 0$$

This is an ordinary second-order differential equation.

It can be solved as a quadratic equation with respect to dy.

If we solve such a quadratic equation:

$$dy = \frac{b \pm \sqrt{b^2 - ac}}{a}dx$$

Convert, multiply by a :

$$a\,dy-\left(b\pm\sqrt{b^2-ac}\right)dx=0$$

And then we see that the solution will depend on what type of equation it is.

If the **equation is hyperbolic**, then there is a positive number under the square root, everything is fine, we have two solutions.

There will be two equations:

$$a\,dy-\left(b\pm\sqrt{b^2-ac}\right)dx=0 \quad \Rightarrow \begin{cases}\varphi(x,y)=C_1\\ \psi(x,y)=C_2\end{cases}$$

The fact is that if we now use these two independent integrals (these are the functions φ and ψ that we found) in order to replace the variable:

$$\xi=\varphi(x,y)$$

$$\eta=\psi(x,y)$$

Then, when substituting these variables into the equation (*), the equation becomes much simpler.

In the case of a **parabolic type** equation, under the square root of 0, therefore, two equations will not work. Let's look at a specific example.

In the case of an **elliptic type,** there will be a negative number under the square root, respectively, there will be complex numbers. Let's show you a specific example.

You can always find the functions $\varphi,\ \psi$, and make a replacement that will bring the original equation to a much simpler form.

And after that, you can write down the *canonical form* of this equation.

To create a new function from ξ and η, we should theoretically replace it ($u(x,y)$) with a some function $U(\xi,\eta)$:

$$u(x,y) \rightarrow U(\xi,\eta)=U(\varphi(x,y),\psi(x,y))=u(x,y)$$

In theory, we should replace one function $u(x,y)$ with some other function $U(\xi,\eta)$.

Example 1

To bring the following differential equation to a canonical form:

$$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}+6\frac{\partial u}{\partial y}=0$$

Solution:

The type of equation:

$$a=1,\ \ b=1,\ c=-3$$

$$b^2-ac=1+3=4\ \ >0$$

This equation has a hyperbolic type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$dy^2-2dydx-3dx^2=0$$

We solve this equation as a quadratic one.

$$a\,dy-\left(b\pm\sqrt{b^2-ac}\right)dx=0$$

$$dy-\left(1\pm\sqrt{4}\right)dx=0$$

$$\begin{cases} dy-3dx=0 \\ dy+dx=0 \end{cases}$$

It is good that we have an equation with constant coefficients: a, b, c do not depend on x, y.

$$\begin{cases} y-3x=C_1 \\ y+x=C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi=y-3x \\ \eta=y+x \end{cases}$$

When moving to these variables, the equation (*) is greatly simplified.

$$u(x,y)\to U(\xi,\eta)$$

The derivative of the whole function:

$$u_x=U_\xi\xi_x+U_\eta\eta_x$$
$$u_y=U_\xi\xi_y+U_\eta\eta_y$$

Take the second derivatives of $u_{x,}$ u_y.

$$u_x=U_\xi\cdot(\text{-}3)+U_\eta\cdot 1$$

$$u_y=U_\xi\cdot 1+U_\eta\cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx}=-3\left(U_{\xi\xi}\cdot\xi_x+U_{\xi\eta}\cdot\eta_x\right)+1\left(U_{\eta\xi}\cdot\xi_x+U_{\eta\eta}\cdot\eta_x\right)=$$

$$=-3\left(U_{\xi\xi}\cdot(-3)+U_{\xi\eta}\cdot 1\right)+1\left(U_{\eta\xi}\cdot(-3)+U_{\eta\eta}\cdot 1\right)=$$

$$=9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}$$

$$u_{xy}=-3\left(U_{\xi\xi}\cdot\xi_y+U_{\xi\eta}\cdot\eta_y\right)+1\left(U_{\eta\xi}\cdot\xi_y+U_{\eta\eta}\cdot\eta_y\right)=$$

$$= -3\left(U_{\xi\xi}\cdot 1 + U_{\xi\eta}\cdot 1\right) + 1\left(U_{\eta\xi}\cdot 1 + U_{\eta\eta}\cdot 1\right) =$$

$$= -3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = 1\left(U_{\xi\xi}\cdot\xi_y + U_{\xi\eta}\cdot\eta_y\right) + 1\left(U_{\eta\xi}\cdot\xi_y + U_{\eta\eta}\cdot\eta_y\right) =$$

$$= 1\cdot U_{\xi\xi} + U_{\xi\eta}\cdot 1 + U_{\eta\xi}\cdot 1 + U_{\eta\eta}\cdot 1 =$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Substituting everything into our equation:

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} + 2\left(-3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - 3\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) +$$

$$+ 2\left(U_{\xi}\cdot(-3) + U_{\eta}\right) + 6\left(U_{\xi} + U_{\eta}\right) = 0$$

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} - 6U_{\xi\xi} - 4U_{\xi\eta} + 2U_{\eta\eta} - 3U_{\xi\xi} - 6U_{\xi\eta} - 3U_{\eta\eta} -$$

$$-6U_{\xi} + 2U_{\eta} + 6U_{\xi} + 6U_{\eta} = 0$$

$$-16U_{\xi\eta} + 8U_{\eta} = 0$$

$$-U_{\xi\eta} = \frac{1}{2}U_{\eta}$$

This expression is already the canonical form of a hyperbolic equation. This is the answer to this task.

In principle, in general, the **canonical form of a hyperbolic equation** will be as follows:

$$U_{\xi\eta} = \Phi\left(\xi, \eta, U, U_{\xi}, U_{\eta}\right).$$

When we moved on to the new variables, there should be a single derivative of this order (mixed) in the left part, with a coefficient equal to one; in the right part - some kind of function Φ, which contains independent variables $\xi, \eta, U, U_{\xi}, U_{\eta}$.

Example 2

$$\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial x \partial y} + 13\frac{\partial^2 u}{\partial y^2} = 0$$

Solution:

The type of equation:

$$a = 1, \;\; b = \text{-}3, \;\; c = 13$$

$$b^2 - ac = 9 - 13 = -4 \;\; < 0$$

This equation has an elliptical type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$1 \cdot dy^2 + 6dxdy + 13dx^2 = 0$$

$$dy - \left(-3 \pm \sqrt{-4}\right)dx = 0$$

$$dy-\left(-3\pm 2i\right)dx=0$$

Here is a complex number.

There are two equations here, but they merge into each other during the operation of complex conjugation.

Recall that such a complex conjugation is:

A complex number $z=a+ib$

A complex conjugate number $\overline{z}=a-ib$

Here, one equation can be obtained from another by complex conjugation. Therefore, it does not make much sense to consider the second equation. We have a real and imaginary part of this equation.

Let's make one equation:

$$dy-\left(-3+2i\right)dx=0$$

We will integrate:

$$y-\left(-3+2i\right)x=A=C_1+iC_2$$

the real part of the equation $y+3x=C_1$

the imaginary part of the equation $-2x=C_2$

$$\begin{cases} y+3x=C_1 \\ -2x=C_2 \end{cases}$$

These are the functions that we can use in this case, as a substitute for variables.

$$\begin{cases} \xi = y+3x \\ \eta = -2x \end{cases}$$

Replace $u(x,y) \to U(\xi,\eta)$.

$$u_x = U_\xi \cdot \xi_x + U_\eta \cdot \eta_x = U_\xi \cdot 3 + U_\eta \cdot (-2)$$

$$u_y = U_\xi \cdot \xi_y + U_\eta \cdot \eta_y = U_\xi \cdot 1 + U_\eta \cdot 0$$

$$u_x = 3U_\xi - 2U_\eta$$

$$u_y = U_\xi$$

$$u_{xx} = 3\left(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x\right) - 2\left(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x\right) =$$

$$= 3\left(U_{\xi\xi} \cdot 3 + U_{\xi\eta} \cdot (-2)\right) - 2\left(U_{\eta\xi} \cdot 3 + U_{\eta\eta} \cdot (-2)\right) =$$

$$= 9U_{\xi\xi} - 12U_{\xi\eta} + 4U_{\eta\eta}$$

$$u_{xy} = 3\left(U_{\xi\xi}\xi_y + U_{\xi\eta}\eta_y\right) - 2\left(U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y\right) =$$

$$= 3\left(U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0\right) - 2\left(U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 0\right) =$$

$$= 3U_{\xi\xi} - 2U_{\eta\xi}$$

$$u_{yy} = U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0 = U_{\xi\xi}$$

We substitute all derivatives:

$$9U_{\xi\xi} - 12U_{\xi\eta} + 4U_{\eta\eta} - 18U_{\xi\xi} + 12U_{\xi\eta} + 13U_{\xi\xi} = 0$$

$$4U_{\xi\xi} + 4U_{\eta\eta} = 0$$

$$U_{\xi\xi} + U_{\eta\eta} = 0$$

This is the canonical form of the elliptical type.

General view (canonical view) elliptic equations:

$$U_{\xi\xi} + U_{\eta\eta} = \Phi\left(\xi, \eta, U, U_{\xi}, U_{\eta}\right)$$

with a coefficient equal to one (on the left side). If you get something else, you should look for an error.

Example 3

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 1\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + cu = 0$$

Solution:

The type of equation:

$$a = 1, \ \ b = \text{-}1, \ \ c = 1$$

$$b^2 - ac = (-1)^2 - 1 = 0$$

This equation has a parabolic type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$dy^2 + 2dxdy + dx^2 = 0$$

$$(dy + dx)^2 = 0$$

$$dy + dx = 0$$

$$y + x = C_1$$

That is, $\xi = y + x$.

Where do we get η from?

In the case of parabolic type equations, the function η can generally be taken any (linearly independent, which is written for ξ):

$$\eta = x.$$

Usually take x, the constant will not be linearly independent.

That is,

$$\begin{cases} \xi = y + x \\ \eta = x \end{cases}$$

If you doubt that these functions are linearly independent (or if ξ is a composite function), then we remember about replacing variables, and turn to topics about integrals if we do some kind of transformation (replacing variables from $(x,y) \to (\xi,\eta)$).

This transformation will be non-degenerate, so the functions will be linearly independent.

The Jacobian of the transition (determinant) will not be equal to 0.

We can make such a determinant, calculate, the transformation is non-degenerate, therefore we can choose the function as $\eta = x$.

$$(x,y) \to (\xi,\eta)$$

The transformation is non-degenerate and the Jacobian of the transition is non-zero.

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

The transformation is non-degenerate.

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi \cdot 1 + U_\eta \cdot 1;$$

$$u_y = U_\xi;$$

$$u_{xx} = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 1 + U_{\eta\xi} \cdot 1 + U_{\eta\eta} =$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{xy} = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0 + U_{\eta\xi} \cdot 1 =$$

$$= U_{\xi\xi} + U_{\eta\xi}$$

$$u_{yy} = U_{\xi\xi}$$

All derivatives have been found.

Let's substitute it into the original equation:

$$U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta} - 2U_{\xi\xi} - 2U_{\eta\xi} + U_{\xi\xi} + \alpha U_\xi + \alpha U_\eta + \beta U_\xi + cU = 0$$

$$U_{\eta\eta} = -(\alpha + \beta) U_\xi - \alpha U_\eta - cU$$

A parabolic type equation has been reduced to a canonical form.

In general, the canonical form of a parabolic equation is:

$$U_{\eta\eta} = \Phi\left(\xi, \eta, U, U_\xi, U_\eta\right).$$

Example 4 (EQUATION WITH VARIABLE COEFFICIENTS)

$$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} - 2x \frac{\partial u}{\partial x} = 0$$

Solution:

The type of equation:

$$a = y^2, \ \ b = 0, \ c = -x^2$$

$$b^2 - ac = x^2 y^2 \ \ > 0, \qquad x \neq 0, \ y \neq 0$$

The hyperbolic type is everywhere on the entire plane (except the axes).

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$y^2 dy^2 - x^2 dx^2 = 0$$

$$(ydy)^2 = (xdx)^2$$

$$ydy = \pm xdx$$

$$2ydy = \pm 2xdx$$

We will integrate:

$$\begin{cases} y^2 = x^2 + C_1 \\ y^2 = -x^2 + C_2 \end{cases}$$

$$\begin{cases} y^2 - x^2 = C_1 \\ y^2 + x^2 = C_2 \end{cases}$$

Substitution:

$$\begin{cases} \xi = y^2 - x^2 \\ \eta = y^2 + x^2 \end{cases}$$

We substitute all this, and we get the canonical form:

$$\xi_x = -2x$$
$$\xi_y = 2y$$
$$\eta_x = 2x$$
$$\eta_y = 2y$$

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi(-2x) + U_\eta(2x) = 2x(-U_\xi + U_\eta)$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y = U_\xi(2y) + U_\eta(2y) = 2y(U_\xi + U_\eta)$$
$$u_{xx} = \left[2x \cdot (-U_\xi + U_\eta)\right]'_x = 2(-U_\xi + U_\eta) + 2x\left[-U_\xi + U_\eta\right]'_x =$$
$$= 2(-U_\xi + U_\eta) + 2x(-U_{\xi\xi}\xi_x - U_{\xi\eta}\eta_x + U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x) =$$
$$= 2(-U_\xi + U_\eta) + 2x(-U_{\xi\xi}(-2x) - U_{\xi\eta}(2x) + U_{\eta\xi}(-2x) + U_{\eta\eta}(2x)) =$$
$$= 2(-U_\xi + U_\eta) + 4x^2(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta})$$

We used the formula: $(u\upsilon)' = u'\upsilon + \upsilon' u$

$\left[-U_\xi + U_\eta\right]'_x$ - and here we take it as a derivative of a complicated function

u_{xy} we don't need it, it's not in the equation.

$$u_{yy} = 2\left(U_{\xi} + U_{\eta}\right) + 2y\left(U_{\xi\xi}\xi_y + U_{\xi\eta}\eta_y + U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y\right) =$$
$$= 2\left(U_{\xi} + U_{\eta}\right) + 2y\left(U_{\xi\xi}(2y) + U_{\xi\eta}(2y) + U_{\eta\xi}(2y) + U_{\eta\eta}(2y)\right) =$$
$$= 2\left(U_{\xi} + U_{\eta}\right) + 4y^2\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right)$$

We substitute it into the equation:

$$2y^2\left(-U_{\xi} + U_{\eta}\right) + 4x^2y^2\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) -$$
$$-2x^2\left(U_{\xi} + U_{\eta}\right) - 4x^2y^2\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) -$$
$$-4x^2\left(-U_{\xi} + U_{\eta}\right) = 0$$

$$-16x^2y^2U_{\xi\eta} + \left(-2y^2 + 2x^2\right)U_{\xi} + \left(2y^2 - 6x^2\right)U_{\eta} = 0$$

Let's express x^2 and y^2:

$$\begin{cases} \xi = y^2 - x^2 & (1) \\ \eta = y^2 + x^2 & (2) \end{cases}$$

let's add two equations:

$$y^2 = \frac{\xi + \eta}{2}$$

Subtract from (2) equation (1) equation:

$$x^2 = \frac{\eta - \xi}{2}$$

$$-16\frac{\eta^2-\xi^2}{4}U_{\xi\eta}+\left(-\xi-\eta+\eta-\xi\right)U_{\xi}+\left(\xi+\eta-3\eta+3\xi\right)U_{\eta}=0$$

$$-4\left(\eta^2-\xi^2\right)U_{\xi\eta}-2\xi U_{\xi}+\left(4\xi-2\eta\right)U_{\eta}=0$$

$$U_{\xi\eta}=\frac{1}{4\left(\eta^2-\xi^2\right)}\left(-2\xi U_{\xi}+\left(4\xi-2\eta\right)U_{\eta}\right)$$

11.10.24

2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

Algorithm

1) Find $b^2 - ac$, determine the type of equation.
2) We find the first integrals of the characteristic equations:

in the case when $a \neq 0$: $\dfrac{dy}{dx} = \dfrac{b \pm \sqrt{b^2 - ac}}{a}$,

in the case when $c \neq 0$: $\dfrac{dx}{dy} = \dfrac{b \pm \sqrt{b^2 - ac}}{c}$

3) The first integrals have the form:

in the case of hyperbolic type: $\varphi(x,y) = C, \ \psi(x,y) = C;$

in the case of elliptic type: $\alpha(x,y) \pm i\beta(x,y) = C$,

in the case of parabolic type: $\delta(x,y) = C$.

4) We replace variables:

in the case of hyperbolic type: $\begin{cases} \xi = \varphi(x,y); \\ \eta = \psi(x,y). \end{cases}$

in the case of elliptical type: $\begin{cases} \xi = \alpha(x,y); \\ \eta = \beta(x,y). \end{cases}$

in the case of parabolic type: $\begin{cases} \xi = \delta(x,y); \\ \eta = \varepsilon(x,y). \end{cases}$

where $\varepsilon(x,y)$ is any function of C^1 such that $\begin{vmatrix} \delta_x & \delta_y \\ \varepsilon_x & \varepsilon_y \end{vmatrix} \neq 0$.

The result of the replacement will be the canonical form of the equation.

Example 1

Bring it to a canonical form

$$u_{xx} + 2u_{xy} - 3u_{yy} + u_x + u_y = 0$$

Solution:

The type of equation:

$a = 1,\ \ b = 1,\ c = \text{-}3$

$b^2 - ac = 1 + 3 = 4\ \ > 0$

This equation has a hyperbolic type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$dy^2 - 2dydx - 3dx^2 = 0$

We solve this equation as a quadratic one.

$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$

$dy - \left(1 \pm \sqrt{4}\right)dx = 0$

$$\begin{cases} dy - 3dx = 0 \\ dy + dx = 0 \end{cases}$$

It is good that we have an equation with constant coefficients: a, b, c do not depend on x, y.

$$\begin{cases} y - 3x = C_1 \\ y + x = C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi = y - 3x \\ \eta = y + x \end{cases}$$

When moving to these variables, the equation (*) is greatly simplified.

$$u(x,y) \to U(\xi,\eta)$$

The derivative of the whole function:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

Take the second derivatives of $u_{x,}$ u_y.

$$u_x = U_\xi \cdot (\text{-}3) + U_\eta \cdot 1$$

$$u_y = U_\xi \cdot 1 + U_\eta \cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx} = -3\left(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x\right) + 1\left(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x\right) =$$

$$= -3\left(U_{\xi\xi} \cdot (-3) + U_{\xi\eta} \cdot 1\right) + 1\left(U_{\eta\xi} \cdot (-3) + U_{\eta\eta} \cdot 1\right) =$$

$$= 9U_{\xi\xi} - 6U_{\eta\xi} + U_{\eta\eta}$$

$$u_{xy} = -3\left(U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y\right) + 1\left(U_{\eta\xi} \cdot \xi_y + U_{\eta\eta} \cdot \eta_y\right) =$$

$$= -3\left(U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 1\right) + 1\left(U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 1\right) =$$

$$= -3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = 1\left(U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y\right) + 1\left(U_{\eta\xi} \cdot \xi_y + U_{\eta\eta} \cdot \eta_y\right) =$$

$$= 1 \cdot U_{\xi\xi} + U_{\xi\eta} \cdot 1 + U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 1 =$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Substituting everything into our equation:

$$u_{xx} + 2u_{xy} - 3u_{yy} + u_x + u_y = 0$$

$$9U_{\xi\xi} - 6U_{\eta\xi} + U_{\eta\eta} + 2\left(-3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) -$$
$$-3\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) + U_{\xi} \cdot (-3) + U_{\eta} \cdot 1 + U_{\xi} \cdot 1 + U_{\eta} \cdot 1 = 0$$

$$9U_{\xi\xi} - 6U_{\eta\xi} + U_{\eta\eta} - 6U_{\xi\xi} - 4U_{\xi\eta} + 2U_{\eta\eta} -$$
$$-3U_{\xi\xi} - 6U_{\xi\eta} - 3U_{\eta\eta} - 3U_{\xi} + U_{\eta} + U_{\xi} + U_{\eta} = 0$$

$$-16U_{\xi\eta} + 2U_{\eta} - 2U_{\xi} = 0$$

$$U_{\xi\eta} = \frac{2U_{\xi} - 2U_{\eta}}{-16}$$

Example 2 EQUATION WITH VARIABLE COEFFICIENTS

To bring to a canonical form in each area where the type is preserved, the equation

$$yu_{xx} + u_{yy} = 0$$

<u>Solution:</u>

Step 1. The type of equation:

$a = y, \;\; b = 0, \; c = 1$

$\Delta = b^2 - ac = 0 - y \cdot 1 = -y$

Therefore,

1) in the half-plane $y < 0$, $\Delta > 0$ $\Rightarrow$ means hyperbolic type,

2) in the half-plane $y > 0$, $\Delta < 0$ $\Rightarrow$ means elliptical type,

3) on the straight $y = 0$ discriminant $\Delta = 0$ $\Rightarrow$ means parabolic type.

Step 2.

Next, we need to bring it to a canonical form.

We write the <u>characteristic equation:</u>

$y dy^2 + dx^2 = 0$

We solve this equation as a quadratic one.

Since $c = 1 \neq 0$, the characteristic equations have the form:

$\frac{dx}{dy} = \frac{b \pm \sqrt{\Delta}}{c}$, that is, $\frac{dx}{dy} = \pm\sqrt{-y}$

This is an equation with separable variables. We solve them:

1. in the half-plane $y < 0$

$$dx = \pm\sqrt{-y}dy \qquad \Rightarrow \qquad x + c = \mp\frac{2}{3}(-y)^{\frac{3}{2}}.$$

Therefore, the first integrals have the form:

$$\boxed{\varphi(x,\, y) = x + \tfrac{2}{3}(-y)^{\frac{3}{2}}} = c, \qquad \boxed{\psi(x,\, y) = x - \tfrac{2}{3}(-y)^{\frac{3}{2}}} = c$$

2. in the half-plane $y > 0$

$$dx = \pm i\sqrt{y}dy \qquad \Rightarrow \qquad x + c = \pm i\frac{2}{3}y^{\frac{3}{2}}.$$

Therefore, the first integrals have the form:

$$\alpha(x,\, y) \pm i\beta(x,\, y) = c,$$

where

$$\boxed{\alpha(x,\,y) = x}, \qquad \boxed{\beta(x,\,y) = \tfrac{2}{3}\,y^{\frac{3}{2}}}$$

3. on the straight $y = 0$

$$dx = 0 \cdot dy \qquad \Rightarrow \qquad x = c$$

Therefore, the first integral (the only linearly independent one) has the form:

$$\boxed{\delta(x,\,y) = x.}$$

Step 3.

Replacing variables.

According to the algorithm, it is necessary to carry out a replacement.

1. in the half-plane $y < 0$

$$\begin{cases} \xi = x + \frac{2}{3}(-y)^{\frac{3}{2}}; \\ \eta = x - \frac{2}{3}(-y)^{\frac{3}{2}}. \end{cases}$$

$$\xi_x = 1,$$
$$\xi_y = -\sqrt{-y},$$
$$\eta_x = 1,$$
$$\eta_y = \sqrt{-y}$$

Then by entering the function $U(\xi,\eta)$, we get:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

Take the second derivatives of $u_{x,}\ u_y$.

$$u_x = U_\xi \cdot 1 + U_\eta \cdot 1 = U_\xi + U_\eta$$

$$u_y = U_\xi \cdot \left(-\sqrt{-y}\right) + U_\eta \cdot \left(\sqrt{-y}\right) = \sqrt{-y}\left(-U_\xi + U_\eta\right)$$

From these derivatives we take the second derivatives:

$$u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = -y\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - \frac{1}{2\sqrt{-y}}\left(-U_\xi + U_\eta\right)$$

Substituting the found derivatives into the original equation, we get:

$$yu_{xx} + u_{yy} = 0$$

$$y\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) - y\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - \frac{1}{2\sqrt{-y}}\left(-U_\xi + U_\eta\right) =$$

$$= y\left[4U_{\xi\eta} - \frac{1}{2(-y)^{\frac{3}{2}}}\left(-U_\xi + U_\eta\right)\right] = 0$$

Dividing by $4y$ and expressing $2(-y)^{\frac{3}{2}} = \frac{3}{2}(\xi - \eta)$, we get the canonical form:

$$U_{\xi\eta} - \frac{1}{6(\xi - \eta)}\left(-U_\xi + U_\eta\right) = 0$$

2. in the half-plane $y > 0$

$$\begin{cases} \xi = x; \\ \eta = \frac{2}{3} y^{\frac{3}{2}}. \end{cases}$$

$$\xi_x = 1,$$
$$\xi_y = 0,$$
$$\eta_x = 0,$$
$$\eta_y = \sqrt{y}$$

Then by entering the function $U(\xi, \eta)$, we get:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

$$u_x = U_\xi + U_\eta \cdot 0 = U_\xi$$
$$u_y = U_\xi \cdot 0 + U_\eta \sqrt{y} = U_\eta \sqrt{y}$$

$$u_{xx} = U_{\xi\xi}$$
$$u_{yy} = U_{\eta\eta} y + \frac{1}{2\sqrt{y}} U_\eta$$

Substituting the found derivatives into the original equation, we get:

$$y u_{xx} + u_{yy} = 0$$

$$yU_{\xi\xi}+U_{\eta\eta}y+\frac{1}{2\sqrt{y}}U_{\eta}=y\left(U_{\xi\xi}+U_{\eta\eta}\right)+\frac{1}{2\sqrt{y}}U_{\eta}=$$

$$=y\left(U_{\xi\xi}+U_{\eta\eta}+\frac{1}{2y^{\frac{3}{2}}}U_{\eta}\right)=\left[2y^{\frac{3}{2}}=3\eta\right]=$$

$$=y\left(U_{\xi\xi}+U_{\eta\eta}+\frac{1}{3\eta}U_{\eta}\right)=0$$

Dividing by y, we get the canonical form:

$$U_{\xi\xi}+U_{\eta\eta}+\frac{1}{3\eta}U_{\eta}=0$$

3. on the straight $y=0$

$$\begin{cases} \xi = x; \\ \eta = y. \end{cases}$$

We need to arbitrarily choose $\eta(x,y)$ so that the functions ξ,η form a linearly independent pair.

$\xi_x=1,$
$\xi_y=0,$
$\eta_x=0,$
$\eta_y=1$

Then by entering the function $U(\xi,\eta)$, we get:

$u_x=U_{\xi}\xi_x+U_{\eta}\eta_x$
$u_y=U_{\xi}\xi_y+U_{\eta}\eta_y$

$$u_x = U_\xi$$
$$u_y = U_\eta$$

$$u_{xx} = U_{\xi\xi}$$
$$u_{yy} = U_{\eta\eta}$$

Substituting the found derivatives into the original equation for $y=0$, we get:

$$u_{yy} = U_{\eta\eta} = 0$$

So, the canonical form of the original equation on the line $y=0$:

$U_{\eta\eta} = 0$ or, what is the same, $u_{yy} = 0$.

<u>Answer:</u>

$$\begin{cases} U_{\xi\eta} - \dfrac{1}{6(\xi-\eta)}\left(-U_\xi + U_\eta\right) = 0 & \textit{in the area } y<0, \textit{ hyperbolic type} \\ U_{\xi\xi} + U_{\eta\eta} + \dfrac{1}{3\eta} U_\eta = 0 & \textit{in the area } y>0, \textit{ elliptical type} \\ U_{\eta\eta} = 0 & \textit{in the area } y=0, \textit{ parabolic type.} \end{cases}$$

REDUCTION TO THE CANONICAL FORM OF PARTIAL DIFFERENTIAL EQUATIONS OF THE 2ND ORDER WITH CONSTANT COEFFICIENTS

In this section, we will consider second-order partial differential equations with constant coefficients and n independent variables:

$$\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + f(x_1,\, \dots,\, x_n;\; u,\, u_{x_1},\, \dots,\, u_{x_n}) = 0, \tag{1}$$

$a_{ij} = const \in \mathbb{R},\ i, j = \overline{1,n}$.

Definition 1

The characteristic quadratic form of equation (1) is the expression:

$$Q(\lambda_1,\, \dots,\, \lambda_n) = \sum_{i,j=1}^{n} a_{ij}\, \lambda_i \lambda_j. \tag{2}$$

The normal form of the quadratic form (2) is its form:

$$\tilde{Q}(\mu_1,\, \dots,\, \mu_n) = \sum_{k=1}^{n} \beta_k\, \mu_k^2, \qquad \beta_k \in \{-1,\, 0,\, 1\}. \tag{3}$$

The canonical form of equation (1) is the form in which its characteristic quadratic form takes the normal (or canonical) form:

$$\sum_{k=1}^{n} \beta_k\, u_{x_k x_k} + g(x_1, \ldots, x_n;\ u, u_{x_1}, \ldots, u_{x_n}) = 0. \qquad \textbf{(4)}$$

Definition 2

Equation (1) refers to

1) **hyperbolic type**, if the coefficients β_k are different from zero and not all of the same sign;

2) **elliptical type**, if all coefficients β_k are nonzero and all of the same sign;

3) **parabolic type,** if at least one of the coefficients β_k is zero.

<u>Algorithm</u>

1) We reduce the characteristic quadratic form to the canonical (normal) form (3) (by the method of selecting complete squares). We write out the transformation matrix that performs this process:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \mu_n \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \vdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \vdots & \alpha_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \vdots & \alpha_{nn} \end{pmatrix}}_{A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \lambda_n \end{pmatrix}, \qquad \det A \neq 0.$$

2) We find the matrix Γ of the substitution of variables according to the law:

$$\Gamma = \left(A^T\right)^{-1}.$$

3) We replace variables:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma_{11} & \gamma_{12} & \vdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \vdots & \gamma_{2n} \\ \dots & \dots & \ddots & \dots \\ \gamma_{n1} & \gamma_{n2} & \vdots & \gamma_{nn} \end{pmatrix}}_{\Gamma} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

The result of the replacement will be the canonical form (4) of equation (1).

Example 3

To bring the equation to a canonical form:

$$u_{xx} + 2u_{xy} + 5u_{yy} - 32u = 0.$$

<u>Solution:</u>

Step 1

The characteristic quadratic form of this equation has the form

$$Q(\lambda_1,\ \lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + 5\lambda_2^2.$$

Let's bring it to the canonical form:

$$Q(\lambda_1,\ \lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + 5\lambda_2^2 = (\lambda_1 + \lambda_2)^2 + (2\lambda_2)^2 = \mu_1^2 + \mu_2^2,$$

where

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_2; \\ \mu_2 = 2\lambda_2 \end{cases}$$

that is

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}}_{A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Step 2

Let's find the matrix of substitution of variables Γ:

$$\Gamma = \left(A^T\right)^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

Step 3

We replace the variables:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

that is,

$$\begin{cases} \xi = x; \\ \eta = \frac{1}{2}(-x+y). \end{cases}$$

To put new variables in the original equation, let's put

$$U(\xi,\eta) = u(x,y)$$

and find $u_x, u_y, u_{xx}, u_{xy}, u_{yy}$ as derivatives of a complicated function $U(\xi(x,y),\eta(x,y))$:

$$u_x = U_\xi - \frac{1}{2}U_\eta$$

$$u_y = \frac{1}{2}U_\eta$$

$$u_{xx} = U_{\xi\xi} - U_{\xi\eta} + \frac{1}{4}U_{\eta\eta}$$

$$u_{xy} = \frac{1}{2}U_{\xi\eta} - \frac{1}{4}U_{\eta\eta}$$

$$u_{yy} = \frac{1}{4}U_{\eta\eta}$$

Substituting the found derivatives into the left side of the original equation and giving similar ones, we get:

$$u_{xx} + 2u_{xy} + 5u_{yy} - 32u = \left(U_{\xi\xi} - U_{\xi\eta} + \frac{1}{4}U_{\eta\eta}\right) + 2\left(\frac{1}{2}U_{\xi\eta} - \frac{1}{4}U_{\eta\eta}\right) +$$

$$+5\left(\frac{1}{4}U_{\eta\eta}\right) - 32U = U_{\xi\xi} + U_{\eta\eta} - 32U$$

Answer:

the equation has an elliptical type,

$$U_{\xi\xi} + U_{\eta\eta} - 32U = 0, \text{ where } \xi = x,\ \ \eta = \frac{1}{2}(-x + y).$$

HOMEWORK 7 (The deadline is October 14, 2024)

Version 1 $u_{xx} - yu_{yy} = 0$.	**Version 2** $xu_{xx} - 2\sqrt{xy}u_{xy} + yu_{yy} + \frac{1}{2}u_y = 0$.
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Guan Haochen	Liu Yudong
Liu Tianxing	Wang Youshen
Ma Yueyang	Lu Qibo
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Li Sicheng	Xia Xinglin
Li Kaiyan	Huang Yifan
Yang Guowei	Shen Xingye
Kong Xiangning	Wang Haojun
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Li Xinyi	Wang Leihan
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Zhang Hongbo	
Yan Shukun	

14.10.2024

2.6. BASIC EQUATIONS OF MATHEMATICAL PHYSICS

14.10.2024

2.6. BASIC EQUATIONS OF MATHEMATICAL PHYSICS

The subject of the theory of equations of mathematical physics is the study of differential, integral and functional equations describing natural phenomena. The construction of a mathematical model of the process begins with the establishment of values that are decisive for the process under study. Further, using physical laws (principles) expressing the relationship between these quantities, an equation (system of equations) in partial derivatives is constructed and additional conditions (initial and boundary) to the equation (system) are drawn up.

We will mainly study second-order partial differential equations with one unknown function, in particular the wave equation, the heat equation and the Laplace equation, commonly called the classical equations of mathematical physics.

2.6.1. THE OSCILLATION EQUATION

Many problems of mechanics (vibrations of strings, rods, membranes and three-dimensional volumes) and physics (electromagnetic oscillation) lead to an oscillation equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \operatorname{div}(k\operatorname{grad}u) - qu + F(x,t),$$

where the unknown function $u = u(x,t)$ depends on n $(n = 1,2,3)$ spatial variables $x = (x_1, x_2, ..., x_n)$ and time t, coefficients ρ, k, q are determined by the properties of the medium, $F(x,t)$ is the density of the external disturbance:

$$\operatorname{div}(k \operatorname{grad} u)=\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(k \frac{\partial u}{\partial x_i}\right).$$

Consider a stretched string fixed at the ends. By string we mean a thin thread that does not exert any resistance to changing its shape, unrelated to changing its length. The tension force T_0 acting on the string is assumed to be significant, so the effect of gravity can be ignored.

Let the string be directed along the x axis in the equilibrium position.

We will consider only the *transverse vibrations* of the string, assuming that the movement occurs in the same plane and that all points of the string move perpendicular to the x axis.

Let's denote by $u(x,t)$ the displacement of the string points at time t from the equilibrium position.

Considering further only *small vibrations* of the string, we will assume that the displacement $u(x,t)$, as well as the derivative $\frac{\partial u}{\partial x}$, are so small that their squares and products can be neglected compared to the quantities themselves.

For each fixed value of t, the graph of the function $u(x,t)$ obviously gives the shape of the string at this point in time (Fig. 1).

Denote by $F(x,t)$ the density of external forces acting on the string at point x at time t and directed perpendicular to the x axis.

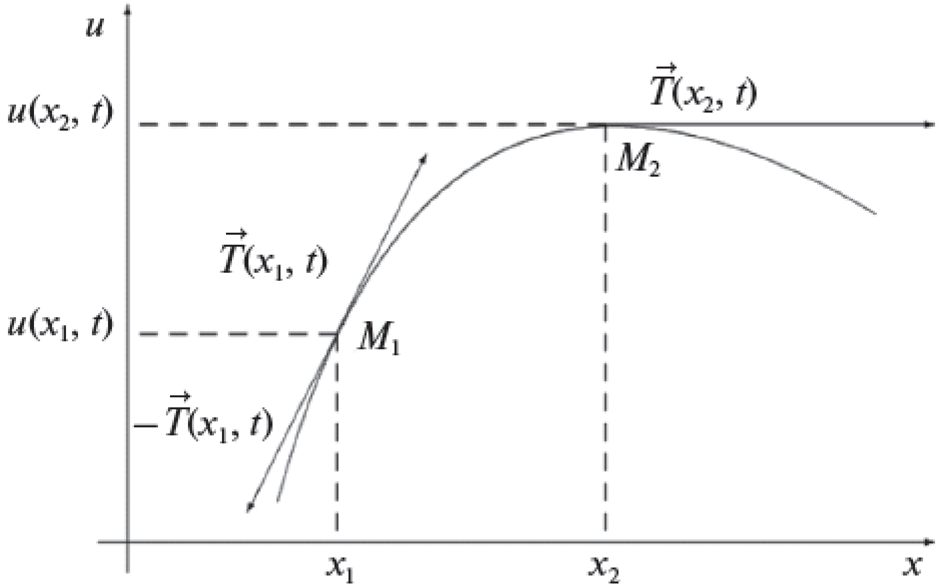

Fig. 1. Instantaneous profile of the string section (x_1, x_2) at time t

Let $\rho(x)$ be the linear density of the string, then the function $u(x,t)$ satisfies the differential *equation of string vibrations*:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + F(x,t).$$

If $\rho(x) = \rho = \text{const}$, that is, in the case of a homogeneous string, the equation is usually written as

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t),\ a = \sqrt{\frac{T_0}{\rho}},\ f(x,t) = \frac{F(x,t)}{\rho}.$$

This equation will be called the *one-dimensional wave equation.*

If there is no external force, then we have: $F(x,t) = 0$ and get the *equation of free vibrations of the string*

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$$

The equation of small transverse vibrations of the membrane A =0 is similar:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + F(x,t).$$

If the density ρ is constant, then the membrane oscillation equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + f(x,t), \quad a = \sqrt{\frac{T_0}{\rho}}, \quad f(x,t) = \frac{F(x,t)}{\rho}.$$

The last equation will be called the *two-dimensional wave equation*.

Three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) + f(x,t)$$

describes the processes of sound propagation in a homogeneous medium and electromagnetic waves in a homogeneous nonconducting medium. This equation is satisfied by the density of the gas, its pressure and velocity potential, as well as the components of the electric and magnetic field strengths and the corresponding potentials.

We will write the wave equations using the single formula

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u + f,$$

where Δ is the Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

2.6.2. THE EQUATION OF THERMAL CONDUCTIVITY (HEAT EQUATION)

The processes of heat propagation or particle diffusion in the medium are described by the heat equation

$$\rho\frac{\partial u}{\partial t}=\operatorname{div}(k\operatorname{grad}u)-qu+F(x,t).$$

Let's derive the equation of heat propagation.

Denote by $u(x,t)$ the temperature of the medium at point $x=(x_1,x_2,x_3)$ at time t, and by $\rho(x)$, $c(x)$ and $k(x)$, respectively, its density, specific density and thermal conductivity coefficient at point x.

Let $F(x,t)$ be the intensity of heat sources at point x at time t.

Let's calculate the heat balance in an arbitrary volume V over a period of time $(t,t+\Delta t)$. Denote by S the boundary of V and let $\vec{n}-$ be the external normal to it.

According to Fourier's law, through the surface S, an amount of heat

$$Q_1=\iint_S k(x)\frac{\partial u}{\partial n}dS\Delta t=\Delta t\iint_S (k(x)\operatorname{grad}u,\vec{n})dS,$$

enters the volume V, equal, by virtue of the Gauss's-Ostrogradsky's formula (theorem):

$$Q_1=\iiint_V \operatorname{div}(k(x)\operatorname{grad}u)dx\Delta t.$$

Due to the thermal sources in volume V the amount of heat

$$Q_2=\iiint_V F(x,t)dx\Delta t.$$

Since the temperature in volume V has increased by

$$u(x,t+\Delta t)-u(x,t)\approx\frac{\partial u}{\partial t}\Delta t\ ,$$

over a period of time $(t,t+\Delta t)$, it is necessary to expend the amount of heat

$$Q_3=\iiint\limits_V c(x)\rho(x)\frac{\partial u}{\partial t}dx\Delta t\ .$$

On the other hand, $Q_3=Q_1+Q_2$ and therefore

$$\iiint\limits_V\left[\operatorname{div}(k(x)\operatorname{grad}u)+F-c(x)\rho(x)\frac{\partial u}{\partial t}\right]dx\Delta t=0\ ,$$

from where, due to the arbitrariness of the volume V, we obtain the equation of heat propagation

$$c(x)\rho(x)\frac{\partial u}{\partial t}=\operatorname{div}(k(x)\operatorname{grad}u)+F(x,t)\ . \qquad (2.17)$$

If the medium is homogeneous, that is, $c(x)$, $\rho(x)$ and $k(x)$ are constants, then equation (2.17) takes the form

$$\frac{\partial u}{\partial t}=a^2\Delta u+f\ , \qquad (2.18)$$

where

$$a^2=\frac{k}{c\rho}\ ,\ f=\frac{F}{c\rho}\ .$$

Equation (2.18) is called the heat equation or the *diffusion equation.*

2.6.3. THE STATIONARY EQUATION

For stationary processes $F(x,t)=F(x), u(x,t)=u(x)$, and the equations of oscillations and heat take the form

$$-\mathrm{div}(k\,\mathrm{grad}\,u)+qu=F(x). \qquad (2.19)$$

For $k=const, q=0$, equation (2.19) is called the Poisson equation:

$$\Delta u=-f\ ,\ f=\frac{F}{k}. \qquad (2.20)$$

For $f=0$, the equation (2.20) is called the Laplace equation:

$$\Delta u=0.$$

Let us consider the potential flow of fluid without sources, namely: let inside a certain volume V with a boundary S, which has a stationary flow of an incompressible fluid (density $\rho=\mathrm{const}$), characterized by a velocity $\vec{v}(x_1,x_2,x_3)$. If the fluid flow is not vortex ($\mathrm{rot}\,\vec{v}=0$), then the velocity $\vec{v}$ is a potential vector, that is,

$$\vec{v}=\mathrm{grad}\,u, \qquad (2.21)$$

where u is a scalar function called the *velocity potential*.

If there are no sources, then

$$\mathrm{div}\,\vec{v}=0. \qquad (2.22)$$

Now from formulas (2.21) and (2.22) we get

$$\mathrm{div}\,\mathrm{grad}\,u=0,$$

or

$$\Delta u=0,$$

that is, the velocity potential satisfies the Laplace equation.

2.7. FORMULATION OF BASIC BOUNDARY VALUE PROBLEMS FOR A SECOND-ORDER DIFFERENTIAL EQUATION

2.7.1. CLASSIFICATION OF BOUNDARY VALUE PROBLEMS

As shown, the linear equation of the second order

$$\rho\frac{\partial^2 u}{\partial t^2}=\operatorname{div}(k\operatorname{grad}u)-qu+F(x,t) \qquad (2.23)$$

describes the processes of vibrations, equation

$$\rho\frac{\partial u}{\partial t}=\operatorname{div}(k\operatorname{grad}u)-qu+F(x,t) \qquad (2.24)$$

describes the processes of diffusion, and equation

$$-\operatorname{div}(k\operatorname{grad}u)+qu=F(x) \qquad (2.25)$$

describes stationary processes.

Let $G\subset R^n$ be the area where the process takes place and S be its boundary. Thus, G is the domain of setting equation (2.25). The domain of setting equations (2.23) and (2.24) will be considered cylinder $\Omega_T=G\times(0,T)$ height T and base G. Its boundary consists of the lateral surface $S\times(0,T)$ and two bases: the lower $\bar{G}\times\{0\}$ and the upper $\bar{G}\times\{T\}$ (Fig. 2).

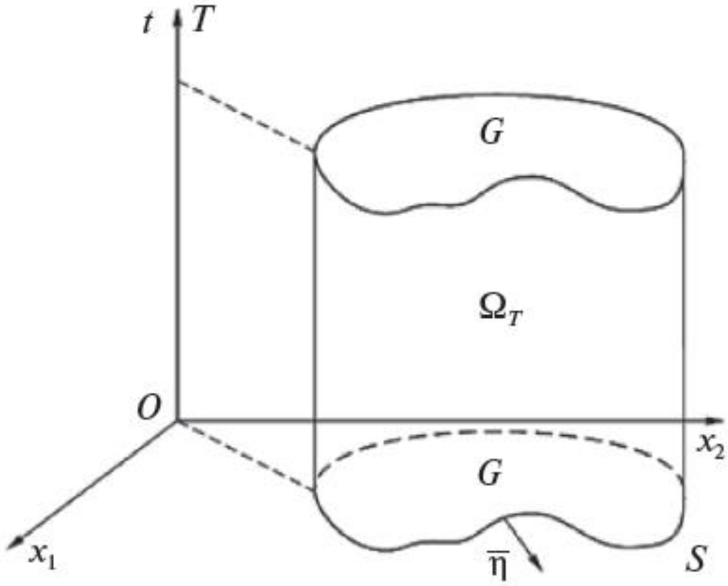

Fig. 2. The field of setting the equations of oscillations and diffusion

We will assume that the coefficients ρ, k, q of equations (2.23)– (2.25) do not depend on time t; further, in accordance with their physical meaning, we will assume that $\rho(x) > 0,\ k(x) > 0,\ q(x) \geq 0, x \in \overline{G}$.

Under these assumptions, the oscillation equation (2.23) is of the hyperbolic type, the diffusion (heat equation) equation (2.24) is of the parabolic type, and the stationary equation (2.25) is of the elliptical type.

Further, in order to fully describe the physical process, it is necessary, in addition to the equation describing this process, to specify the initial state of this process (initial conditions) and the regime at the boundary of the region in which the process occurs (boundary conditions).
There are three types of problems for differential equations.

1) The Cauchy problem for hyperbolic and parabolic equations: initial conditions are specifying, the region G coincides with the entire space R^n, there are no boundary conditions.
2) Boundary value problem for elliptic type equations: boundary conditions are specifying at the boundary S, the initial conditions, of course, are absent.
3) A mixed problem for hyperbolic and parabolic equations: both initial and boundary conditions are specifying, $G \neq R^n$.

Let us describe in more detail each of the listed boundary value problems for the equations (2.23)– (2.25) under consideration.

2.7.2.THE CAUCHY PROBLEM

For the oscillation equation (2.23), the Cauchy problem is posed as follows: find a function $u(x,t)$ of class $C^2(t>0)\cap C^1(t\geq 0)$ satisfying equation (2.23) in the half-space $t>0$ and the initial conditions at $t=0$:

$$u\big|_{t=0} = u_0(x),\ \frac{\partial u}{\partial t}\bigg|_{t=0} = u_1(x). \tag{2.26}$$

At the same time, it is necessary:

$$F \in C(t>0),\ u_0 \in C^1\left(R^n\right),\ u_1 \in C\left(R^n\right).$$

For the thermal conductivity equation (heat equation) (2.24), the Cauchy problem is posed as follows: find a function $u(x,t)$ of class $C^2(t>0)\cap C^1(t\geq 0)$ satisfying equation (2.24) in the half-space $t>0$ and the initial conditions at $t=0$:

$$u\big|_{t=0} = u_0(x). \qquad (2.27)$$

At the same time, it is necessary that

$$F \in C(t>0),\ u_0 \in C\left(R^n\right).$$

The above statement of the Cauchy problem admits the following generalization. Let the differential equations of the 2nd order be given:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_{i0}\frac{\partial^2 u}{\partial x_i \partial t} + \Phi\left(x,t,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n},\frac{\partial u}{\partial t}\right), \quad (2.28)$$

piecewise smooth surface Σ: $t=\sigma(x)$ and functions u_0 and u_1 on Σ. The Cauchy problem for equation (2.28) consists in finding, in some part of the domain $t>\sigma(x)$ adjacent to the surface Σ, a solution $u(x,t)$ satisfying the boundary conditions on Σ

$$u\big|_{\Sigma} = u_0,\ \left.\frac{\partial u}{\partial n}\right|_{\Sigma} = u_1,$$

where $\vec{n}$ is the normal to Σ directed towards increasing t.

2.7.3. BOUNDARY VALUE PROBLEM FOR ELLIPTIC TYPE EQUATIONS. A MIXED TASK

The boundary value problem for equation (2.25) consists in finding a function $u(x)$ of class $C^2(G) \cap C^1(\bar{G})$ satisfying in the domain G equation (2.25) and a boundary condition on S of the form

$$\alpha u + \beta \left. \frac{\partial u}{\partial n} \right|_S = \nu \, , \quad (2.29)$$

where $\alpha, \beta, \nu -$ are given continuous functions on S, and

$$\alpha \geq 0 \, , \; \beta \geq 0 \, , \; \alpha + \beta > 0 \, .$$

The following types of boundary conditions are distinguished (2.29).

Boundary condition of the first kind $(\alpha = 1, \beta = 0)$:

$$u|_S = u_0 \, .$$

The boundary condition of the second kind $(\alpha = 0, \beta = 1)$:

$$\left. \frac{\partial u}{\partial n} \right|_S = u_1 \, .$$

Boundary condition of the third kind $(\alpha \geq 0, \beta = 1)$:

$$\alpha u + \left. \frac{\partial u}{\partial n} \right|_S = u_2 \, .$$

The corresponding boundary value problems are called problems of *the I, II and III kind.* For the Laplace and Poisson equations, the boundary value problem of the first kind:

$$\Delta u = -f \ , \ u\big|_S = u_0$$

is called the *Dirichlet problem*; the boundary value problem of the second kind:

$$\Delta u = -f \ , \ \frac{\partial u}{\partial n}\bigg|_S = u_1$$

is called the *Neumann problem.*

For the oscillation equation (2.23), the mixed problem is posed as follows: find a function $u(x,t)$ of class $C^2(\Omega_\infty)\cap C^1(\bar{\Omega}_\infty)$ satisfying equation (2.23) in the cylinder Ω_∞, the initial conditions (2.26) at $t=0$ and the boundary condition (2.29) at $x \in S,\ t \geq 0$.

Similarly, for the diffusion equation (2.24), the mixed problem is posed as follows: find a function $u(x,t)$ of class $C^2(\Omega_\infty)\cap C^1(\bar{\Omega}_\infty)$ satisfying equation (2.24) in the cylinder Ω_∞, the initial condition (2.27) at $t=0$ and the boundary condition (2.29) at $x \in S,\ t \geq 0$.

2.7.4. THE CORRECTNESS OF THE FORMULATION OF MATHEMATICAL PHYSICS PROBLEMS

Since the problems of mathematical physics describe real physical processes, the mathematical formulation of these problems must meet the following requirements:

1) the solution exists in some class of M1 functions;

2) the solution is the only one in a certain class of M2 functions;

3) the solution continuously depends on the data of the problem (initial and boundary data, free term, coefficients of the equation, and so on).

The continuous dependence of the solution u on the data of the problem $\tilde{u}$ means the following: let the sequence $\tilde{u}_k, k = 1,2,...$, in some sense tends to $\tilde{u}$ and $\tilde{u}_k, k = 1,2,...,\ u$ are the corresponding solutions to the problem; then $u_k \to u,\ k \to \infty$ in the sense of convergence, appropriately chosen.

The requirement of continuous dependence of the solution is due to the fact that the data of a physical problem, as a rule, are determined from experiment approximately, and therefore one must be sure that the solution of the problem will not significantly depend on measurement errors.

A problem satisfying the listed requirements 1)-3) is called *correctly posed*, and the corresponding set of functions $M_1 \cap M_2$ is a *correctness class.*

Consider the following system of differential equations with N unknown functions $u_1, u_2, ..., u_N$:

$$\frac{\partial^{k_i} u_i}{\partial t^{k_k}} = \Phi_i\left(x, t, u_1, u_2, \ldots, u_N, \ldots, \frac{\partial^{\alpha_0+\alpha_1+\ldots+\alpha_n} u_j}{\partial t^{\alpha_0}\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}}, \ldots\right), \qquad (2.30)$$

where $i = 1,2,\ldots,N$.

Here, the right-hand sides [illegible] not contain derivatives of order higher than k_i and derivatives wi[illegible]ct to t of order higher than $k_i - 1$, that is

$$\alpha_0 + \alpha_1 + \ldots + \alpha_n \le k_i,\ \alpha_0 \le k_i - 1.$$

For system (2.30), we set the following Cauchy problem: to find a solution $u_1, u_2, \ldots, u_N$ of this system satisfying the initial conditions at $t = t_0$:

$$\left.\frac{\partial^k u_i}{\partial t^k}\right|_{t=t_0} = \varphi_{ik}(x),\ k = 0,1,\ldots,k_i - 1,\ i = 1,2,\ldots,N, \qquad (2.31)$$

where $\varphi_{ik}(x)$ are the given functions in some domain $G \subset R^n$.

16.10.24

3. HYPERBOLIC EQUATIONS

3.1. THE STRING OSCILLATION EQUATION AND ITS SOLUTION BY THE D'ALEMBERT METHOD

The study of methods for solving boundary value problems for hyperbolic equations begins with the Cauchy problem for the equation of free vibrations of a string:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 , \tag{3.1}$$

$$\begin{cases} u(x,0) = \varphi(x), \\ \dfrac{\partial u(x,0)}{\partial t} = \psi(x). \end{cases} \tag{3.2}$$

3.1.1. D'ALEMBERT FORMULA

We transform equation (3.1) to a canonical form containing a mixed derivative. The equation of characteristics

$$\left[\frac{dx}{dt}\right]^2 - a^2 = 0$$

splits into two equations:

$$\frac{dx}{dt} - a = 0 , \ \frac{dx}{dt} + a = 0 ,$$

the integrals of which are

$$x - at = C_1 , \ x + at = C_2 .$$

Now, assuming

$$\xi = x + at\ ,\ \ \eta = x - at\ ,$$

equation (3.1) is transformed to the form

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0\,. \qquad (3.3)$$

The general solution of equation (3.3) is determined by the formula

$$u = f_1(\xi) + f_2(\eta)\,,$$

where $f_1(\xi)$ and $f_2(\eta)$ are arbitrary functions. Returning to the variables x, t, we get

$$u = f_1(x + at) + f_2(x - at)\,. \qquad (3.4)$$

The resulting solution depends on two arbitrary functions f_1 and f_2. It is called the *D'Alembert solution*.

Next, substituting formula (3.4) into (3.2), we will have

$$f_1(x) + f_2(x) = \varphi(x)\,, \qquad (3.5)$$

$$af_1'(x) - af_2'(x) = \psi(x)\,, \qquad (3.6)$$

from where, integrating the second equality (3.6), we get

$$f_1(x) - f_2(x) = \frac{1}{a}\int_{x_0}^{x}\psi(y)dy + C\,, \qquad (3.7)$$

where x_0 and C are constants. From the formulas (3.5) and (3.7) we find

$$f_1(x) = \frac{1}{2}\left[\varphi(x) + \frac{1}{a}\int_{x_0}^{x}\psi(y)dy + C\right],$$

$$f_2(x) = \frac{1}{2}\left[\varphi(x) - \frac{1}{a}\int_{x_0}^{x}\psi(y)dy - C\right].$$

At the same time, taking into account the formula (3.4), we have

$$u(x,t) = \frac{1}{2}\left[\varphi(x+at) + \frac{1}{a}\int_{x_0}^{x+at}\psi(y)dy + C + \varphi(x-at) - \frac{1}{a}\int_{x_0}^{x-at}\psi(y)dy - C\right]$$

and finally we get the formula

$$u(x,t) = \frac{\varphi(x+at)+\varphi(x-at)}{2} + \frac{1}{2a}\int_{x-at}^{x+at}\psi(y)dy\,. \qquad (3.8)$$

The formula (3.8) is called the *D'Alembert's formula.*

It is not difficult to verify that formula (3.8) satisfies equation (3.1) and initial conditions (3.2) given that $\varphi(x) \in C^2(R)$ and $\psi(x) \in C^1(R)$. Thus, the described method proves both the uniqueness and the existence of a solution to the problem.

Example 1

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

if

$$u\big|_{t=0} = x^2,\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0\,.$$

Solution:

We know that $u_{tt} - a^2 u_{xx} = 0$.

Then $a = 1$.

We know that $\begin{cases} u(x;0) = \varphi(x) \\ \dfrac{\partial u}{\partial t}(x;0) = \psi(x) \end{cases}$

Since $u(x;0) = x^2$, then $\varphi(x) = x^2$.

Since $u'_t(x;0) = 0$, then $\psi(x) = 0$.

$$f_1 = \frac{1}{2}\left[x^2 + \frac{1}{1}\int_{x_0}^{x} 0dy + C\right] = \frac{1}{2}\left[x^2 + C\right]$$

$$f_2 = \frac{1}{2}\left[x^2 - C\right]$$

We know that $u = f_1(x - at) + f_2(x + at)$ and $a = 1$:

$$u = f_1(x - t) + f_2(x + t) = \frac{1}{2}\left[(x - t)^2 + C\right] + \frac{1}{2}\left[(x + t)^2 - C\right] =$$

$$= \frac{1}{2}\left[x^2 - 2xt + t^2 + x^2 + 2xt + t^2\right] = x^2 + t^2$$

We have $u = x^2 + t^2$.

OR:

$$u(x,t) = \frac{\varphi(x + at) + \varphi(x - at)}{2},$$

where $a = 1,\ \varphi(x) = x^2$.

$$u(x,t)=\frac{(x+t)^2+(x-t)^2}{2}=\frac{x^2+2xt+t^2+x^2-2xt+t^2}{2}=$$

$$=\frac{2\left(x^2+t^2\right)}{2}=x^2+t^2$$

Let's check three conditions:

1) $\dfrac{\partial^2 u}{\partial t^2}=\dfrac{\partial^2 u}{\partial x^2}$

$u_x=2x$

$u_{xx}=2$

$u_t=2t$

$u_{tt}=2$

2) $u\big|_{t=0}=x^2$

$u=x^2+0=x^2$

3) $\left.\dfrac{\partial u}{\partial t}\right|_{t=0}=0$

$u_t=2t$

$u_t(0)=2\cdot 0=0$

Example 2

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2}=4\frac{\partial^2 u}{\partial x^2},$$

if

$$u\big|_{t=0}=0\,,\ \frac{\partial u}{\partial t}\bigg|_{t=0}=x\,.$$

<u>Solution:</u>

We know that $u_{tt}-a^2u_{xx}=0$.

Then $a=2$.

We know that $\begin{cases} u(x;0)=\varphi(x) \\ \dfrac{\partial u}{\partial t}(x;0)=\psi(x) \end{cases}$

Since $u(x;0)=0$, then $\varphi(x)=0$.

Since $u_t'(x;0)=x$, then $\psi(x)=x$.

We know that

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int\limits_{x-at}^{x+at}\psi(y)dy$$

and $a=2$:

$$u(x,t)=\frac{1}{4}\int\limits_{x-2t}^{x+2t}ydy=\frac{1}{8}y^2\Big|_{x-2t}^{x+2t}=\frac{1}{8}\Big[(x+2t)^2-(x-2t)^2\Big]=$$

$$=\frac{1}{8}\Big[x^2+4xt+4t^2-x^2+4xt-4t^2\Big]=\frac{1}{8}[8xt]=xt$$

We have $u(x,t)=xt$.

<u>Let's check three conditions:</u>

1) $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$

$u_x = t$
$u_{xx} = 0$
$u_t = x$
$u_{tt} = 0$

2) $u\big|_{t=0} = 0$

$u = x \cdot 0 = 0$

3) $\left.\dfrac{\partial u}{\partial t}\right|_{t=0} = x$

$u_t = x$

3.1.2. THE INHOMOGENEOUS EQUATION

Consider the Cauchy problem for an inhomogeneous oscillation equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),\ x \in R,\ t > 0, \qquad (3.9)$$

$$u(x,0) = \varphi(x),\ \frac{\partial u(x,0)}{\partial t} = \psi(x),\ x \in R. \qquad (3.10)$$

It is easy to verify that the solution of the problem (3.9), (3.10) $u = u(x,t)$ is represented in the form

$$u = \upsilon + \omega, \qquad (3.11)$$

where υ is the solution of the Cauchy problem (3.1), (3.2), and ω is the solution of the following problem:

$$\begin{cases} \dfrac{\partial^2 \omega}{\partial t^2} = a^2 \dfrac{\partial^2 \omega}{\partial x^2} + f(x,t), & x \in R, \quad t > 0, \\ \omega(x,0) = 0, \quad \dfrac{\partial \omega(x,0)}{\partial t} = 0, & x \in R. \end{cases} \qquad (3.12)$$

Let $W(x,t;\tau)$ be the solution of the auxiliary Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 W}{\partial t^2} = a^2 \dfrac{\partial^2 W}{\partial x^2}, & x \in R, \quad t > \tau, \\ W(x,t;\tau)\big|_{t=\tau} = 0, \quad \dfrac{\partial W(x,0)}{\partial t}\Big|_{t=\tau} = f(x,\tau). \end{cases} \qquad (3.13)$$

We show that the solution of the $\omega(x,t)$ problem (3.12) is determined by the formula

$$\omega(x,t) = \int_0^t W(x,t;\tau)\,d\tau\,, \qquad (3.14)$$

where $W(x,t;\tau)-$ is the solution to the problem (3.13).

Really,

$$\omega(x,0) = 0\,,\quad \frac{\partial \omega(x,t)}{\partial t} = W(x,t;t) + \int_0^t \frac{\partial W(x,t;\tau)}{\partial t} d\tau\,,$$

therefore, $\dfrac{\partial \omega(x,0)}{\partial t} = 0$ by virtue of the initial condition (3.13).

And finally:

$$\frac{\partial^2 \omega}{\partial t^2} - a^2 \frac{\partial^2 \omega}{\partial x^2} = \frac{\partial W(x,t;\tau)}{\partial t}\Big|_{t=\tau} + \int_0^t \frac{\partial^2 W(x,t;\tau)}{\partial t^2} - a^2 \frac{\partial^2 W(x,t;\tau)}{\partial x^2} d\tau = f(x,t)\,.$$

The solution of the problem (3.13) is determined by the D'Alembert's formula:

$$W(x,t;\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi\,. \qquad (3.15)$$

Now, using the formulas (3.8), (3.11), (3.14) and (3.15), we find that the solution of the initial problem (3.9), (3.10) is given by the formula

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)dy+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau\,.$$

Example 3

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+x\sin t\,,$$

if

$$u\big|_{t=0}=\sin x\,,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=\cos x\,,\ x\in R\,.$$

<u>Solution:</u>

$u_{tt}-a\,u_{xx}=f\left(x;t\right)$

$a=1$

$f\left(x;t\right)=x\cdot\sin t$

$u\big|_{t=0}=\sin x : u\left(x;0\right)=\sin x,$ so $\varphi\left(x\right)=\sin x$

$u'_t\big|_{t=0}=\cos x : u'_t\left(x;0\right)=\cos x,$ so $\psi\left(x\right)=\cos x$

Since $u\left(x;t\right)=\upsilon\left(x;t\right)+\omega\left(x;t\right)$, we will find $\upsilon\left(x;t\right)$ and $\omega\left(x;t\right)$.

Let $\upsilon(x;t)=\upsilon_1+\upsilon_2$.

$$\upsilon_1(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}$$

$$\upsilon_1=\frac{1}{2}\left(\varphi(x+t)+\varphi(x-t)\right)=\frac{1}{2}\left(\sin(x+t)+\sin(x-t)\right)$$

$$\upsilon_2=\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\,dy=\frac{1}{2}\int_{x-t}^{x+t}\cos y\,dy=\frac{1}{2}\sin y\Big|_{x-t}^{x+t}=$$

$$=\frac{1}{2}\left(sin(x+t)-sin(x-t)\right)$$

$$\upsilon(x;t)=\frac{1}{2}\left(\sin(x+t)+\sin(x-t)\right)+\frac{1}{2}\left(sin(x+t)-sin(x-t)\right)=$$
$$=\sin(x+t)$$

We have $\upsilon(x;t)=\sin(x+t)$.

Now let's find $\omega(x;t)$.

Since $f(x;t)=x\cdot\sin t$, then

$$\omega=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi;\tau)\,d\xi\,d\tau=\frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}\xi\cdot\sin\tau\,d\xi\,d\tau=\frac{1}{2}\int_0^t\sin\tau\,d\tau\,\frac{\xi^2}{2}\Big|_{x-t+\tau}^{x+t-\tau}=$$

$$=\frac{1}{4}\int_0^t\sin\tau\left[(x+t-\tau)^2-(x-t+\tau)^2\right]d\tau=$$

$$=\frac{1}{4}\int_0^t\sin\tau\left[(x+t-\tau+x-t+\tau)(x+t-\tau-x+t-\tau)\right]d\tau=$$

$$=x\int_0^t\sin\tau\cdot(t-\tau)\,d\tau=x(t-\sin t)$$

We get

$$u = \sin(x+t) + xt - x\sin t$$

3.1.3. THE CONTINUATION METHOD

The first boundary value problem

The first boundary value problem for the oscillation equation on a half-line with a homogeneous boundary condition is set as follows: find a solution to the oscillation equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \; x > 0, \; t > 0, \qquad (3.16)$$

satisfying the boundary condition

$$u(0,t) = 0, \; t > 0 \qquad (3.17)$$

and the initial conditions

$$u(x,0) = \varphi(x), \; \frac{\partial u(x,0)}{\partial t} = \psi(x), \; x \geq 0. \qquad (3.18)$$

Let's add the conjugation conditions

$$\varphi(0) = 0, \; \psi(0) = 0$$

to ensure the continuity of the functions $u(x,t)$ and $\frac{\partial u(x,t)}{\partial t}$ at zero.

Let's define the functions $\varphi(x)$ and $\psi(x)$ in an odd way on the entire line by specifying new functions Φ and Ψ:

$$\Phi(x)=\begin{cases}\varphi(x), & x\geq 0,\\ -\varphi(-x), & x<0,\end{cases}$$

$$\Psi(x)=\begin{cases}\psi(x), & x\geq 0,\\ -\psi(-x), & x<0.\end{cases}$$

Consider a modified Cauchy problem:

$$\begin{cases}\dfrac{\partial^2 U}{\partial t^2}=a^2\dfrac{\partial^2 U}{\partial x^2}, & -\infty<x<\infty,\quad t>0,\\ U(x,0)=\Phi(x), & \dfrac{\partial U(x,0)}{\partial t}=\Psi(x).\end{cases}$$

In this case, to find $U(x,t)$, we can apply the D'Alembert formula:

$$U(x,t)=\frac{\Phi(x+at)+\Phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(y)dy\,.$$

Let's take the function $U(x,t)$ as the function we need $u(x,t)$ for $x\geq 0, t\geq 0$. Obviously, conditions (3.16) and (3.18) for $x\geq 0, t\geq 0$ are fulfilled immediately — this follows from the definition of the functions $\Phi(x)$ and $\Psi(x)$. The fulfillment of condition (3.17) follows from the following transformations:

$$u(0,t)=U(0,t)=\frac{\Phi(at)+\Phi(-at)}{2}+\frac{1}{2a}\int_{-at}^{at}\Psi(y)dy\,.$$

Due to the odd number of functions $\Phi(x)$ and $\Psi(x)$, the first and second terms vanish, which gives the fulfillment of condition (3.17).

We express $\Phi(x)$ and $\Psi(x)$ through the original functions $\varphi(x)$ and $\psi(x)$, respectively:

$$x \geq at\ , \begin{cases} \Phi(x+at) = \varphi(x+at), \\ \Phi(x-at) = \varphi(x-at), \\ \Psi(y) = \psi(y), \quad y \in [x-at, x+at]; \end{cases}$$

$$x < at\ , \begin{cases} \Phi(x+at) = \varphi(x+at), \\ \Phi(x-at) = -\varphi(at-x). \end{cases}$$

Now let's write down an auxiliary formula for solving the first boundary value problem:

at $x < at$,

$$\int_{x-at}^{x+at} \Psi(y)dy = \int_{x-at}^{0} \Psi(y)dy + \int_{0}^{x+at} \Psi(y)dy =$$

$$= \int_{x-at}^{0} -\psi(-y)dy + \int_{0}^{x+at} \psi(y)dy =$$

{let's put $-y = y$ }

$$= \int_{at-x}^{0} \psi(y)dy + \int_{0}^{x+at} \psi(y)dy = \int_{at-x}^{x+at} \psi(y)dy\,.$$

Then the general formula will be as follows:

$$u(x,t) = \begin{cases} \dfrac{\varphi(x+at)+\varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(y)dy, \quad x \geq at, \\ \dfrac{\varphi(x+at)-\varphi(at-x)}{2} + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} \psi(y)dy, \quad x < at. \end{cases}$$

The second boundary value problem

The second boundary value problem for the equation of oscillations on a half-line with a homogeneous boundary condition is set as follows: find a solution to the equation of oscillations

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \ x > 0, \ t > 0, \tag{3.19}$$

satisfying the boundary condition

$$\frac{\partial u(0,t)}{\partial x} = 0, \ t \ge 0, \tag{3.20}$$

and the initial conditions:

$$u(x,0) = \varphi(x), \ \frac{\partial u(x,0)}{\partial t} = \psi(x), \ x \ge 0. \tag{3.21}$$

We will act in the same way as in the previous case, however, only an even continuation will suit us here:

$$\Phi(x) = \begin{cases} \varphi(x), & x \ge 0, \\ \varphi(-x), & x < 0, \end{cases}$$

$$\Psi(x) = \begin{cases} \psi(x), & x \ge 0, \\ \psi(-x), & x < 0. \end{cases}$$

The new Cauchy problem and the solution for it according to the D'Alembert formula will look the same as in the previous case:

$$U(x,t) = \frac{\Phi(x+at) + \Phi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(y)dy.$$

Similarly, let the function $u(x,t)=U(x,t)$ for $x>0,\ t>0$. Then the fulfillment of conditions (3.19) and (3.21) is again obvious. Let's check the condition (3.20). Differentiating the D'Alembert formula and using the fact that the derivative of an even function is odd, we get

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial U(0,t)}{\partial x}=\frac{\Phi'(at)+\Phi'(-at)}{2}+\frac{1}{2a}\left[\Psi(at)-\Psi(-at)\right].$$

From the odd $\Phi'(t)$ and the parity $\Psi(t)$, it can be seen that both terms are equal to zero.

The general formula is obtained similarly:

$$u(x,t)=\begin{cases}\dfrac{\varphi(x+at)+\varphi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(y)dy, & x\geq at,\\ \dfrac{\varphi(x+at)+\varphi(at-x)}{2}+\dfrac{1}{2a}\left[\displaystyle\int_{0}^{at-x}\psi(y)dy+\int_{0}^{x+at}\psi(y)dy\right], & x<at.\end{cases}$$

17.10.24

3.2.THE EQUATION OF STRING VIBRATION AND ITS SOLUTION BY THE METHOD OF SEPARATION OF VARIABLES (FOURIER METHOD)

The method of separation of variables, or the Fourier method, is one of the most common methods for solving partial differential equations. We will present this method for the problem of vibrations of a string fixed at the ends.

3.2.1. THE EQUATION OF FREE VIBRATIONS OF THE STRING

The following mixed problem is considered.

Task 1.

Let it be required to find a solution:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < l,\ t > 0,$$

satisfying the initial and boundary conditions:

$$u(x,0) = \varphi(x),$$

$$\frac{\partial u(x,0)}{\partial t} = \psi(x),$$

$$u(0,t) = 0,$$

$$u(l,t) = 0,\ t \ge 0.$$

We are looking for a solution to this problem in the form of a product:

$$u(x,t) = X(x)T(t),$$

substituting which into this equation, we have

$$X(x)T''(t) = a^2 X''(x)T(t).$$

Dividing both parts of this equation by $a^2\, X(x)T(t)$, we obtain

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}. \qquad (3.22)$$

The right side of equality (3.22) is a function of only variable x, and the left side is only t, so the right and left sides of equality (3.22), when changing their arguments. retain a constant value. It is convenient to denote this value by $-\lambda$, that is

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

$$X''(x) + \lambda X(x) = 0,$$

$$T''(t) + \lambda a^2 T(t) = 0.$$

The general solutions of these equations have the form

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

$$T(t) = C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t,$$

where $A,B,C,D-$ are arbitrary constants, and the function $u(x,t)$ is

$$u(x,t) = \left(A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\right)\left(C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t\right).$$

Constants A and B can be found using the boundary conditions of Task 1. Since

$$T(t) \not\equiv 0\,,$$

then

$$X(0)=0\,,\ X(l)=0\,.$$

$$X(0)=A=0\,,$$

$$X(l)=A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l=0\,,$$

that is,

$$A=0\,,$$

$$B\sin\sqrt{\lambda}l=0\,.$$

From where

$$\sqrt{\lambda}=\frac{k\pi}{l}\,,\ k=1,2,\dots\,.$$

So,

$$X(x)=B\sin\frac{k\pi}{l}x\,.$$

The values $\lambda=\dfrac{k^2\pi^2}{l^2}$ found are called *eigenvalues* for a given boundary value Task 1, and the functions $X\left(x\right)=B\sin\dfrac{k\pi}{l}x$ are called *eigenfunctions.*

With the values of λ found, we get

$$T(t)=C\cos\frac{ak\pi}{l}t+D\sin\frac{ak\pi}{l}t\,,$$

$$u_k(x,t) = \sin\frac{k\pi}{l}x\left(a_k\cos\frac{ak\pi}{l}t + b_k\sin\frac{ak\pi}{l}t\right),\ k = 1,2,\dots .$$

Since the equation is linear and homogeneous, the sum of the solutions is also a solution that can be represented as a series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{ak\pi}{l}t + b_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x .$$

In this case, the solution must satisfy the initial condition:

$$u(x,0) = \sum_{k=1}^{\infty} a_k\sin\frac{k\pi}{l}x = \varphi(x) .$$

If the function $\varphi(x)$ decomposes into a Fourier series in the interval $(0,l)$ in terms of sines, then

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x dx . \qquad (3.23)$$

From the initial condition

$$\frac{\partial u(x,0)}{\partial t} = \psi(x)$$

we have

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \sum_{k=1}^{\infty}\frac{ak\pi}{l}b_k\sin\frac{k\pi}{l}x = \psi(x) .$$

We determine the Fourier coefficients of this series:

$$\frac{ak\pi}{l}b_k = \frac{2}{l}\int_0^l \psi(x)\sin\frac{k\pi}{l}x dx ,$$

from where

$$b_k = \frac{2}{ak\pi}\int_0^l \psi(x)\sin\frac{k\pi}{l}xdx . \qquad (3.24)$$

Thus, the solution of the string oscillation equation can be represented as the sum of an infinite series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty}\left(a_k \cos\frac{ak\pi}{l}t + b_k \sin\frac{ak\pi}{l}t \right)\sin\frac{k\pi}{l}x , \qquad (3.25)$$

where a_k , b_k are determined by formulas (3.23) and (3.24).

Theorem. Let $\varphi(x) \in C^2([0,l])$, $\psi(x) \in C^1([0,l])$, in addition, $\varphi(x)$ has a third, and $\psi(x)$ has a second piecewise continuous derivative and the relations are fulfilled: $\varphi(0) = \varphi(l) = 0,\ \varphi''(0) = \varphi''(l) = 0,\ \psi(0) = \psi(l) = 0$. Then the sum of the series (3.25) with coefficients defined by formulas (3.23), (3.24) is the solution to Task 1.

Example 1

Find the deviation $u(x;t)$ from the equilibrium position of a homogeneous horizontal string fixed at the ends $x = 0$ and $x = l$, if at the initial moment the string had the shape $\frac{1}{8}\sin\frac{3\pi x}{l}$, and the initial velocities were absent.

Solution:

$$\begin{cases} u_{tt} = a^2 u_{xx} \qquad\qquad (*) \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \frac{1}{8}\sin\frac{3\pi x}{l} \\ u_t(x,0) = 0 \end{cases}$$

The method of separation of variables, the Fourier method:

we will look for a solution in the form

$$u(x,t) = T(t)\cdot X(x)$$

$$u_{tt}(x,t) = T''(t)\cdot X(x)$$

$$u_{xx}(x,t) = T(t)\cdot X''(x)$$

We substitute it into equation (*):

$$T''(t)X(x) = a^2 T(t)X''(x)$$

Divide by $a^2T(t)X(x)$, we get

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda\ .$$

Consider

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X(l) = 0 \end{cases}$$

We obtain an ordinary differential equation of the second order. This is the task of Sturm-Liouville theory.

We are solving this problem:

1) $\lambda = 0 \quad \Rightarrow \quad X''(x) = 0$

$$X(x) = C_1 x + C_2$$

Substituting into the boundary conditions, we get:

$$C_1 = 0$$
$$C_2 = 0$$

That is, for $\lambda = 0$, the only solution is: $X(x) \equiv 0$. This solution does not suit us.

2) $\lambda < 0 \quad \Rightarrow \quad X(x) \equiv 0$

3) $\lambda > 0 \quad \Rightarrow \quad X(x) = A\sin\left(\sqrt{\lambda}\, x\right) + B\cos\left(\sqrt{\lambda}\, x\right)$

Substitute the boundary conditions, we get:

$$X(0) = B = 0$$
$$X(l) = A\,sin\left(\sqrt{\lambda}\, l\right) = 0$$

There are two options:

If $A = 0$, then $X(x) \equiv 0$. It doesn't suit us.

If $A \neq 0$, then

$$sin\left(\sqrt{\lambda}\, l\right) = 0$$

$$\sqrt{\lambda}\, l = \pi n, \quad n \in \mathbb{Z}$$

$$\sqrt{\lambda} = \frac{\pi n}{l},$$

$$\lambda = \left(\frac{\pi n}{l}\right)^2$$

$$X(x) = A\sin\left(\frac{\pi n}{l}x\right)$$

The solution is definitely ambiguous. $A-$ can be any, $n-$ can be different numbers.

So

$$X_n(x) = A_n \sin\left(\frac{\pi n}{l}x\right)$$

We have found Sturm-Liouville's *eigenfunctions.*

$$T''(t) + a^2\lambda T(t) = 0$$

$$T''(t) + \left(\frac{a\pi n}{l}\right)^2 T(t) = 0$$

$$T(t) = C\, sin\left(\frac{a\pi n}{l}t\right) + D\, cos\left(\frac{a\pi n}{l}t\right)$$

$$T_n(t) = C_n \sin\left(\frac{a\pi n}{l}t\right) + D_n \cos\left(\frac{a\pi n}{l}t\right)$$

$$u_n(x,t) = \left(C_n \sin\left(\frac{a\pi n}{l}t\right) + D_n \cos\left(\frac{a\pi n}{l}t\right)\right) \cdot A_n \sin\left(\frac{\pi n}{l}x\right) =$$

$$= \left(a_n \sin\left(\frac{a\pi n}{l}t\right) + b_n \cos\left(\frac{a\pi n}{l}t\right)\right) \cdot \sin\left(\frac{\pi n}{l}x\right)$$

$$u_n(x,t) = \left(a_n \sin\left(\frac{a\pi n}{l}t\right) + b_n \cos\left(\frac{a\pi n}{l}t\right)\right) \cdot \sin\left(\frac{\pi n}{l}x\right)$$

We don't know these constants a_n and b_n, there are an infinite number of them.

Consider the initial conditions (initial amplitude and initial velocities).

$$u(x;0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{l}x\right) = \frac{1}{8}\sin\left(\frac{3\pi}{l}x\right)$$

$$u_t(x;0) = \sum_{n=1}^{\infty} a_n \cdot \frac{a\pi n}{l} \cdot 1 \cdot \sin\left(\frac{\pi n}{l}x\right) = 0$$

We get that $a_n = 0 \quad \forall n$.

$$b_1 \sin\left(\frac{\pi}{l}x\right) + b_2 \sin\left(\frac{2\pi}{l}x\right) + b_3 \sin\left(\frac{3\pi}{l}x\right) + b_4 \sin\left(\frac{4\pi}{l}x\right) + \dots = \frac{1}{8}\sin\left(\frac{3\pi}{l}x\right).$$

We get that $b_3 = \frac{1}{8}, \quad b_n = 0 \quad n \neq 3$.

<u>Answer:</u> $u(x,t) = \frac{1}{8}\cos\left(\frac{3a\pi}{l}t\right)\sin\left(\frac{3\pi}{l}x\right)$.

Example 2

Let the initial rejection of the string fixed at points $x=0$ and $x=l$ be zero, and the initial velocity

$$\frac{\partial u}{\partial t}=\begin{cases} v_0, & \left|x-\frac{l}{2}\right|<\frac{h}{2}, \\ 0, & \left|x-\frac{l}{2}\right|>\frac{h}{2}. \end{cases}$$

Determine the shape of the string for any time t.

<u>Solution:</u>

Here $\varphi(x)=0$, and $\psi(x)=v_0$ in the interval $\left(\frac{l-h}{2},\frac{l+h}{2}\right)$, and $\psi(x)=0$ outside this interval.

Therefore,

$$a_k=0,$$

$$b_k=\frac{2}{ak\pi}\int_{(l-h)/2}^{(l+h)/2} v_0 \sin\frac{k\pi}{l}x\,dx=-\frac{2v_0}{ak\pi}\frac{l}{k\pi}\cos\frac{k\pi}{l}x\Big|_{(l-h)/2}^{(l+h)/2}=$$

$$=\frac{2v_0 l}{ak^2\pi^2}\left[\cos\frac{k\pi(l-h)}{2l}-\cos\frac{k\pi(l+h)}{2l}\right]=\frac{4v_0 l}{ak^2\pi^2}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2l}.$$

Hence

$$u(x,t)=\frac{4v_0 l}{a\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2l}\sin\frac{ak\pi t}{l}\sin\frac{k\pi x}{l}.$$

Example 3

A string is given, fixed at the ends $x=0$ and $x=l$. Let's assume that at the initial moment the shape of the string has the form of a polyline OAB, shown in Fig. Find the shape of the string for any time t if there are no initial velocities.

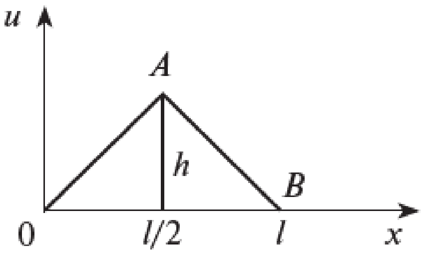

Fig. The shape of the string at the initial moment of time

Solution:

The angular coefficient of the straight line OA is equal to $\frac{h}{l/2}$, the equation of this straight line is $u=\frac{2h}{l}x$. The straight line AB cuts off the segments: l and $2h$ on the coordinate axes, which means that the equation of the straight line AB: $u=\frac{2h}{l}(l-x)$. So,

$$\varphi(x)=\begin{cases}\frac{2h}{l}x, & 0\le x\le\frac{l}{2}\\ \frac{2h}{l}(l-x), & \frac{l}{2}\le x\le l,\end{cases}$$

$$\psi(x)=0.$$

We find

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}dx =$$

$$= \frac{4h}{l^2}\int_0^{l/2} x\sin\frac{k\pi x}{l}dx + \frac{4h}{l^2}\int_{l/2}^{l}(l-x)\sin\frac{k\pi x}{l}dx\ ,$$

$$b_k = 0\ .$$

Integrating by parts, we get

$$a_k = -\frac{4h}{k\pi l}x\cos\frac{k\pi x}{l}\Bigg|_0^{l/2} + \frac{4h}{k\pi l}\int_0^{l/2}\cos\frac{k\pi x}{l}dx -$$

$$-\frac{4h}{k\pi l}(l-x)\cos\frac{k\pi x}{l}\Bigg|_{l/2}^{l} - \frac{4h}{k\pi l}\int_{l/2}^{l}\cos\frac{k\pi x}{l}dx =$$

$$= -\frac{2h}{k\pi}\cos\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\sin\frac{k\pi x}{l}\Bigg|_0^{l/2} + \frac{2h}{k\pi}\cos\frac{k\pi}{2} - \frac{4h}{k^2\pi^2}\sin\frac{k\pi x}{l}\Bigg|_{l/2}^{l} =$$

$$= \frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2} = \frac{8h}{k^2\pi^2}\sin\frac{k\pi}{2}\ .$$

Therefore,

$$u(x,t) = \frac{8h}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{2}\sin\frac{k\pi x}{l}\cos\frac{ak\pi t}{l}\ .$$

Example 4

The string fixed at the ends $x = 0$ and $x = l$ has the shape of a parabola

$$u(x,0) = \frac{4}{l^2}\,x(l-x)\ .$$

at the initial moment of time. Find the shape of the string at any given time if there are no initial velocities.

Solution:

Here

$$\varphi(x)=\frac{4}{l^2}x(l-x),$$

$$\psi(x)=0\,.$$

Therefore, we have

$$a_k=\frac{2}{l}\int_0^l\frac{4}{l^2}x(l-x)\sin\frac{k\pi x}{l}dx,$$

Applying the integration by parts method twice, we get

$$a_k=\frac{8}{l^3}\int_0^l(lx-x^2)\sin\frac{k\pi x}{l}dx=$$

$$=\frac{8}{l^3}\left(-\frac{l\left(lx-x^2\right)}{k\pi}\cos\frac{k\pi x}{l}\bigg|_0^l+\frac{l}{k\pi}\int_0^l\left(l-2x\right)\cos\frac{k\pi x}{l}dx\right)=$$

$$=\frac{8}{l^2k\pi}\int_0^l\left(l-2x\right)\cos\frac{k\pi x}{l}dx=$$

$$=\frac{8}{l^2k\pi}\left(\frac{l\left(l-2x\right)}{k\pi}\sin\frac{k\pi x}{l}\bigg|_0^l+\frac{2l}{k\pi}\int_0^l\sin\frac{k\pi x}{l}dx\right)=$$

$$=\frac{16}{lk^2\pi^2}\int_0^l\sin\frac{k\pi x}{l}dx=\frac{16}{lk^2\pi^2}\left(-\frac{l}{k\pi}\cos\frac{k\pi x}{l}\right)\bigg|_0^l=$$

$$=-\frac{16}{k^3\pi^3}\left(\cos k\pi-1\right)=\frac{16}{k^3\pi^3}\left(1-(-1)^k\right)=\begin{cases}\dfrac{32}{k^3\pi^3}, & k=2n+1,\\ 0, & k=2n.\end{cases}$$

Then the solution of the problem will take the following form:

$$u(x,t)=\frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{\left(2n+1\right)^3}\sin\frac{\left(2n+1\right)\pi x}{l}\cos\frac{a\left(2n+1\right)\pi t}{l}.$$

Task 1	$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = x^2, \quad \dfrac{\partial u(x,0)}{\partial t} = 1; \end{cases}$$
Task 2	$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = 1, \quad \dfrac{\partial u(x,0)}{\partial t} = 0, \\ u(0,t) = 0, \quad u\left(\dfrac{\pi}{2}, t\right) = 0; \end{cases}$$

Do it before October 20th. Homework is not accepted after October 20th.

18.10.24

3. HYPERBOLIC EQUATIONS

3.2.2. THE INHOMOGENEOUS EQUATION

The following mixed problem is considered.

Task 2.

Let it be necessary to find a solution to the inhomogeneous equation of string vibrations:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),\ 0 < x < l,\ t > 0, \qquad (3.26)$$

satisfying the initial and boundary conditions:

$$u(x,0) = \varphi(x),\ \frac{\partial u(x,0)}{\partial t} = \psi(x), \qquad (3.27)$$

$$u(0,t) = 0,\ u(l,t) = 0,\ t \geq 0.$$

We will look for a solution to Task 2 in the form of a Fourier series expansion in x:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin \frac{k\pi}{l} x, \qquad (3.28)$$

considering t as a parameter.

Let's imagine the function $f(x,t)$ as a Fourier series:

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) \sin \frac{k\pi}{l} x,$$

$$f_k(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}x\,dx. \qquad (3.29)$$

Substituting series (3.28) and (3.29) into the original equation (3.26):

$$\sum_{k=1}^{\infty}\left[u_k''(t) + a^2\left(\frac{k\pi}{l}\right)^2 u_k(t) - f_k(t)\right]\sin\frac{k\pi}{l}x = 0,$$

we see that it will be satisfied if all the expansion coefficients are equal:

$$u_k''(t) + a^2\left(\frac{k\pi}{l}\right)^2 u_k(t) = f_k(t). \qquad (3.30)$$

To determine $u_k(t)$, we obtained an ordinary differential equation with constant coefficients.

Further, the initial conditions (3.27) give:

$$\varphi(x) = \sum_{k=1}^{\infty} u_k(0)\sin\frac{k\pi}{l}x,$$

$$\psi(x) = \sum_{k=1}^{\infty} u_k'(0)\sin\frac{k\pi}{l}x,$$

therefore,

$$u_k(0) = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,dx,$$

$$u_k'(0) = \frac{2}{l}\int_0^l \psi(x)\sin\frac{k\pi}{l}x\,dx, \qquad (3.31)$$

$$\varphi_k = u_k(0), \quad \psi_k = u_k'(0).$$

The conditions (3.31) completely determine the solution (3.30):

$$u_k(t) = \varphi_k \cos\frac{ak\pi}{l}t + \frac{l}{ak\pi}\psi_k \sin\frac{ak\pi}{l}t + \frac{l}{ak\pi}\int_0^t f_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)d\tau \,. \quad (3.32)$$

Thus, the desired solution to Task 2, according to formulas (3.28) and (3.32), will be written in the form

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\varphi_k \cos\frac{ak\pi}{l}t + \frac{l}{ak\pi}\psi_k \sin\frac{ak\pi}{l}t + \frac{l}{ak\pi}\int_0^t f_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)d\tau\right\}\times$$
$$\times \sin\frac{k\pi}{l}x,$$

where the values $\varphi_k, \psi_k, f_k(\tau)$ are calculated by (3.31) and (3.29), respectively.

Example 5.

Find a solution to the boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 2b, \; 0 < x < l, \; t > 0,$$

$$u(x,0) = 0, \; \frac{\partial u(x,0)}{\partial t} = 0,$$

$$u(0,t) = 0, \; u(l,t) = 0, \; t \geq 0.$$

<u>Solution:</u>

Here

$$\varphi(x) = 0, \; t, \; \psi(x) = 0, \; f(x,t) = 2b, \; a = 1.$$

Therefore,

$\varphi_k = 0$, $\psi_k = 0$.

$$f_k(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}x dx.$$

$$f_k(t) = \frac{2}{l}\int_0^l 2b\sin\frac{k\pi}{l}x dx = -\frac{4b}{l}\frac{l}{k\pi}\cos\frac{k\pi}{l}x\Big|_0^l =$$

$$= -\frac{4b}{k\pi}\left[(-1)^k - 1\right] = \begin{cases} 0, & k = 2n, \\ \dfrac{8b}{k\pi}, & k = 2n+1. \end{cases}$$

Further

$$\int_0^t f_{2n+1}(\tau)\sin\frac{(2n+1)\pi}{l}(t-\tau)d\tau = \frac{8b}{(2n+1)\pi}\int_0^t \sin\frac{(2n+1)\pi}{l}(t-\tau)d\tau =$$

$$= \frac{8b}{(2n+1)\pi}\frac{l}{(2n+1)\pi}\cos\frac{(2n+1)\pi}{l}(t-\tau)\Big|_0^t = \frac{8bl}{(2n+1)^2\pi^2}\left[1-\cos\frac{(2n+1)\pi}{l}t\right].$$

Hence

$$u(x,t) = \sum_{n=0}^{\infty}\frac{8bl^2}{(2n+1)^3\pi^3}\left[1-\cos\frac{(2n+1)\pi}{l}t\right]\sin\frac{(2n+1)\pi}{l}x.$$

4. PARABOLIC EQUATIONS

4.1. ONE-DIMENSIONAL EQUATION OF THERMAL CONDUCTIVITY. SETTING BOUNDARY VALUE PROBLEMS

The process of temperature distribution in a rod, thermally insulated from the sides and thin enough that at any given time the temperature at all points of the cross section can be considered a single one, can be described by the function $u(x,t)$, representing the temperature in the cross section x at time t. This function $u(x,t)$ is the solution of the equation:

$$c\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+F(x,t),$$

called *the heat equation.*

Here $\rho(x), c(x), k(x) -$ are, respectively, the density, specific heat capacity and thermal conductivity coefficient of the rod at point x, and $F(x,t) -$ is the intensity of heat sources at point x at time t.

To identify the only one solution to the heat equation, it is necessary to attach the initial and boundary conditions to the equation.

The initial condition, unlike a hyperbolic equation, consists only in setting the values of the function u at the initial moment of time t_0:

$$u(x,t_0)=\varphi(x).$$

The main types of boundary conditions are boundary value problems of the first, second and third types.

The first boundary value problem is set if the temperature at the end of the rod $x=0$ is maintained according to a certain law, for example:

$$u(0,t)=\mu(t)\ ,$$

where $\mu(t)$ is a given function of time.

The second boundary value problem is posed if the heat flow q is set at the end of the rod $x=l$, for example:

$$q(l,t)=-k\frac{\partial u(l,t)}{\partial x}\ ,$$

therefore, the boundary condition has the form

$$\frac{\partial u(l,t)}{\partial x}=\upsilon(t)=-\frac{1}{k}q(l,t)\ .$$

In particular, in the case of a thermally insulated end, there is no heat flow through it, that is, $\upsilon(t)=0$.

The third boundary value problem is formulated when heat exchange with the ambient occurs at the end of the rod $x=l$ according to *Newton's law:*

$$q(l,t)=H\left(u(l,t)-\theta(t)\right),$$

where $\theta(t)$ is the ambient temperature, H is the heat exchange coefficient, that is, the amount of heat that has passed through a single section of the rod per unit time with a temperature change of one degree.

The boundary condition has the form

$$\frac{\partial u(l,t)}{\partial x} = -\lambda\big(u(l,t) - \theta(t)\big),$$

where $\lambda = \dfrac{H}{k}$.

Some limiting cases are also considered. For example, if the process of thermal conductivity is studied in a very long rod. For a short period of time, the influence of the temperature regime set at the boundary in the central part of the rod has a very weak effect, and the temperature in this area is mainly determined only by the initial temperature distribution. In problems of this type, it is usually assumed that the rod has an infinite length. Thus, a problem with initial conditions (the Cauchy problem) is posed on the temperature distribution on an infinite line: to find a solution to the thermal conductivity equation in the region $-\infty < x < \infty$ and $t \geq t_0$ satisfying the condition

$$u(x,t_0) = \varphi(x), \; -\infty < x < \infty,$$

where $\varphi(x)$ is a given function.

Similarly, if the section of the rod whose temperature we are interested in is located near one end and far from the other, then in this case the temperature is practically determined by the temperature regime of the near end and the initial conditions. In problems of this type, it is usually assumed that the rod is semi-infinite, and the coordinate measured from the end varies within $0 \leq x < \infty$. Let us give as an example the formulation of the first boundary value problem for a semi—infinite rod: to find a solution to the thermal conductivity equation in the region $-0 < x < \infty$ and $t \geq t_0$ satisfying the conditions

$$u(x,t_0) = \varphi(x)\,,\ 0 < x < \infty\,,$$
$$u(0,t) = \mu(t)\,,\ t \geq t_0\,,$$

where $\varphi(x)$ and $\mu(t)$ are given functions.

4.2. A METHOD FOR SEPARATING VARIABLES FOR THE EQUATION OF THERMAL CONDUCTIVITY. INSTANT POINT SOURCE FUNCTION

4.2.1. HOMOGENEOUS BOUNDARY VALUE PROBLEM

The following first boundary value problem is considered.

Find a solution to a homogeneous equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}\,,\ 0 < x < l\,,\ 0 < t \leq T\,, \tag{4.1}$$

satisfying the initial condition

$$u(x,0) = \varphi(x)\,,\ 0 \leq x \leq l \tag{4.2}$$

and homogeneous boundary conditions

$$u(0,t) = 0\,,\ u(l,t) = 0\,,\ 0 \leq t \leq T\,. \tag{4.3}$$

We are looking for a solution to this problem in the form of a product

$$u(x,t) = X(x)T(t)\,,$$

substituting it into equation (4.1), we have

$$X(x)T'(t) = a^2 X''(x)T(t).$$

Dividing both parts of this equation by $a^2 X(x)T(t)$, we get

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}. \qquad (4.4)$$

The right side of equality (4.4) is a function of only the variable x, and the left side is only t, so the right and left sides of equality (4.4) retain a constant value when changing their arguments. It is convenient to denote this value by $-\lambda$, that is,

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

$$X''(x) + \lambda X(x) = 0,$$

$$T'(t) + \lambda a^2 T(t) = 0.$$

The general solutions of these equations have the form

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

$$T(t) = Ce^{-a^2\lambda t},$$

where $A, B, C -$ are arbitrary constants, and the function $u(x,t)$ is

$$u(x,t) = \left(A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\right)Ce^{-a^2\lambda t}.$$

Constants A and B can be found using the boundary conditions (4.3) of the problem. Since

$$T(t) \not\equiv 0,$$

then

$$X(0)=0, \ X(l)=0.$$

$$X(0)=A=0,$$

$$X(l)=A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l=0,$$

that is,

$$A=0 \text{ и } B\sin\sqrt{\lambda}l=0.$$

From where

$$\sqrt{\lambda}=\frac{k\pi}{l}, \ k=1,2,\dots.$$

So,

$$X(x)=B\sin\frac{k\pi}{l}x.$$

The values $\lambda=\frac{k^2\pi^2}{l^2}$ found are called *eigenvalues* for a given boundary value problem, and the functions $X(x)=B\sin\frac{k\pi}{l}x$ are called *eigenfunctions.*

When the values of λ are found, we get

$$T(t)=Ce^{-\frac{a^2k^2\pi^2}{l^2}t},$$

$$u_k(x,t)=a_k e^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi}{l}x,\ k=1,2,\dots .$$

Since equation (4.1) is linear and homogeneous, the sum of the solutions is also a solution that can be represented as a series:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}a_k e^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi}{l}x .$$

In this case, the solution must satisfy the initial condition (4.2):

$$u(x,0)=\sum_{k=1}^{\infty}a_k\sin\frac{k\pi}{l}x=\varphi(x) .$$

If the function $\varphi(x)$ decomposes into a Fourier series in the interval $(0,l)$ according to the sine, then

$$a_k=\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi .$$

Thus, the solution of the heat equation can be represented as the sum of an infinite series:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}\frac{2}{l}\int_0^l\varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi e^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi}{l}x . \qquad (4.5)$$

Theorem. Let $\varphi(x) \in C^1([0,l])$, $\varphi(0) = \varphi(l) = 0$. Then there is a unique solution to the problem (4.1)–(4.3), which is represented as an absolutely and uniformly converging series (4.5).

The solution (4.5) can be represented as

$$u(x,t) = \int_0^l G(x,\xi,t)\varphi(\xi)d\xi \ ,$$

where the function

$$G(x,\xi,t) = \frac{2}{l}\sum_{k=1}^{\infty} e^{-\frac{a^2k^2\pi^2}{l^2}t} \sin\frac{k\pi x}{l} \sin\frac{k\pi\xi}{l} \ ,$$

is introduced, called the *instantaneous point source function.*

The physical meaning of the function $G(x,\xi,t)$ is that, as a function of the argument x, it represents the temperature distribution in the rod $0 \le x \le l$ at time t, if at $t = 0$ the temperature was zero, and at this moment at point $x = \xi$ a certain amount of heat Q was instantly released, and at the ends of the rod is constantly maintained the temperature is zero.

Example 1

A thin homogeneous rod $0 \le x \le l$ is given , the side surface of which is thermally insulated. Find the temperature distribution $u(x,t)$ in the rod if the ends of the rod are maintained at zero temperature, and the initial temperature $u(x,0) = u_0 = const$.

Solution:

The problem is reduced to solving the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \ 0 < x < l, \ t > 0$$

under conditions

$u(x,0) = u_0 = \text{const}$,

$u(0,t) = u(l,t) = 0$.

Let's calculate:

$$\frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi = \frac{2}{l}\int_0^l u_0 \sin\frac{k\pi}{l}\xi d\xi = -\frac{2u_0}{k\pi}\cos\frac{k\pi}{l}\xi\Big|_0^l =$$

$$= -\frac{2u_0}{k\pi}\left((-1)^k - 1\right) = \begin{cases} \dfrac{4u_0}{k\pi}, & k = 2n+1, \\ 0, & k = 2n. \end{cases}$$

Then the solution will take the form (according to the formula (4.5)):

$$u(x,t) = \frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1} e^{-\frac{a^2(2n+1)^2\pi^2}{l^2}t} \sin\frac{(2n+1)\pi}{l}x.$$