Mathematical analysis 3. Homework 1.

1. Does the function $d(x,y) = \arctan|x-y|$ define the metric on $\mathbb{R}$?

   Solution. First two axioms of the metric are obviously satisfied. To check triangle inequality notice that
   $$|x-y| \le |x-z| + |z-y|$$
   and it is enough to prove that
   $$\arctan(t+s) \le \arctan t + \arctan s, \quad t,s \ge 0. \tag{1}$$
   Fix $t \ge 0$ and consider differentiable function
   $$f(s) = \arctan(t+s) - (\arctan t + \arctan s), \quad s \ge 0.$$
   Then
   $$f'(s) = \frac{1}{1+(t+s)^2} - \frac{1}{1+s^2} < 0$$
   and, consequently, f is decreasing and $f(s) \le f(0) = 0$ for $s \ge 0$. This proves estimate (1) and the triangle inequality.

2. Does the function $d(x,y) = \sqrt{|x-y|}$ define the metric on $\mathbb{R}$?

   Solution. First two axioms of the metric are obviously satisfied. To check triangle inequality notice that
   $$|x-y| \le |x-z| + |z-y| \le |x-z| + |z-y| + 2\sqrt{|x-z|\,|z-y|} \le (\sqrt{|x-z|} + \sqrt{|z-y|})^2.$$
   Consequently, the triangle inequality holds,
   $$d(x,y) = \sqrt{|x-y|} \le \sqrt{|x-z|} + \sqrt{|z-y|} = d(x,z) + d(z,y).$$

   Remark. Considering the first two problems one can notice that these results may be generalized in the following way. Assume that (X,ρ) is a metric space and function $\varphi : [0,+\infty) \to [0,+\infty)$ is differentiable and strictly increasing, $\varphi(0) = 0$. Then $d_1(x,y) = \varphi(\rho(x,y))$ is the metric.

3. Let (X,d) be a metric space. Let $Y = 2^X$ be a set of all subsets of X. Does the function
   $$\rho(E,F) = \inf\{d(x,y) : x \in E,\ y \in F\}$$
   define a metric on Y.

   Solution. Assume that X contains at least 2 points $a, b \in X$, $a \ne b$. Then $\rho(\{a\},\{a,b\}) = 0$ while $\{a\} \ne \{a,b\}$ and ρ is not a metric.

4. Prove that a sphere
$$S = S(a,r) = \{x \in \mathbb{R} : \rho(a,x) = r\}$$
is a closed set.

Solution. Assume that $x_n \in S$ and $x_n \to x \in X$. Then
$$r - \rho(x_n, x) = \rho(a, x_n) - \rho(x_n, x) \le \rho(a,x) \le \rho(a, x_n) + \rho(x_n, x) = r + \rho(x_n, x)$$
and $\rho(a,x) = \rho(a,x_n) = r$ since $\rho(x_n, x) \to 0$ as $n \to \infty$.

5. Let H be the set of all real sequences $x = (x_1, x_2, x_3, \ldots)$ such that $|x_n| \le 1$ for all $n \in \mathbb{N}$. For $x, y \in H$ let
$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n| .$$
Prove that d is a metric on H.

Solution. First of all notice that
$$d(x,y) \le 2 \sum_{n=1}^{\infty} 2^{-n} < \infty$$
and the definition of d is correct. Now, we will check axioms of metric.

If $d(x,y) = 0$ then $|x_n - y_n| = 0$ for every n then $x = y$.

Symmetry is obvious. To prove triangle inequality, notice that for every $x, y, z \in H$
$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n| \le \sum_{n=1}^{\infty} 2^{-n}(|x_n - z_n| + |z_n - y_n|) =$$
$$\sum_{n=1}^{\infty} 2^{-n} |x_n - z_n| + \sum_{n=1}^{\infty} 2^{-n} |z_n - y_n| = d(x,z) + d(z,y).$$

Remark. Considering the first two problems one can notice that these results may be generalized in the following way. Assume that (X, ρ) is a metric space and function $\varphi : [0, +\infty) \to [0, +\infty)$ is differentiable and strictly increasing, $\varphi(0) = 0$. Then $d_1(x,y) = \varphi(\rho(x,y))$ is the metric.

Mathematical analysis 3. Homework 2.

1. Let (X, d) be a metric space, $a \in X$, $r > 0$.

   (a) Prove that the sphere $S = \{x \in X : d(a, x) = r\}$ is the closed set.

   (b) Let $D \subset X$. Prove that $\partial D = \mathrm{cl}(D) \setminus \mathrm{int}(D)$ and that ∂D is closed.

   (c) Explain why $\varnothing$ and X are open and closed sets.

   Solution. (a) Was proved in the previous HW.

   (b) We will provide the proof by two inclusions. Let $x \in \partial D$. Then for every V_x neighborhood of x intersections $V_x \cap D$ and $V_x \cap (X \setminus D)$ are not empty. Since $V_x \cap D \neq \varnothing$ for every neighborhood of x then either $x \in D$ or x is the limit point of D. In both cases $x \in \mathrm{cl}\, D$. Since $V_x \cap (X \setminus D) \neq \varnothing$ then $V_x \not\subset D$ and $x \notin \mathrm{int}\, D$ since V_x is arbitrary. Hence, $x \in \mathrm{cl}\, D \setminus \mathrm{int}\, D$ and $\partial D \subset \mathrm{cl}\, D \setminus \mathrm{int}\, D$.

   Let $x \in \mathrm{cl}\, D \setminus \mathrm{int}\, D$ and V_x be a neighborhood of x. Since $x \in \mathrm{cl}\, D$ then $V_x \cap D \neq \varnothing$. Since $x \notin \mathrm{int}\, D$ then $V_x \cap (X \setminus D) \neq \varnothing$. Consequently, $x \in \partial D$.

   (c) Let $x \in X$ then $x \in B(x, 1) \subset X$ and x is inner point of X. Consequently, X is open. Let $x_n \in X$ be convergent. By definition this means that the limit is a point of X and this implies closeness of X.

   Since $\varnothing = X \setminus X$ then by previous $\varnothing$ is open and closed as the complement of closed and open set X.

2. Prove that $X = \mathbb{Q}$ with metric $d(x, y) = |x - y|$ is not complete. What is the reason for $\mathbb{R}$ with the same metric to be complete (refer to and formulate the theorem from the 1st semester).

   Solution. First, notice that there is no such $x \in \mathbb{Q}$ that $x^2 = 2$. Let

   $$s_n = \max\{s \in \mathbb{N} : s^2 \leq 2 \cdot 10^{2n}\} \text{ and } x_n = \frac{s_n}{10^n}.$$

   Then, x_n is a Cauchy sequence since

   $$|x_n - x_m| \leq \frac{2}{10^{\min(n,m)}}$$

   and $x_n^2 \to 2$ since $2 \cdot 10^{2n} - 1 < s_n^2 \leq 2 \cdot 10^{2n}$. Consequently, if $x = \lim x_n$ then $x^2 = 2$ and there is no such x in $\mathbb{Q}$.

   The reason for completeness of $\mathbb{R}$ with respect to $d(x, y) = |x - y|$ is (Bolzano-)Cauchy criterion. A real sequence $\{x_n\}$ of converges if and only if for every $\varepsilon > 0$ there exists $N > 0$ such that $|x_n - x_m| < \varepsilon$ for every $n, m \geq N$.

3. Prove that a discrete space is always complete.

Solution. Recall that discrete metric on a set X is a function

$$d(x,y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y. \end{cases}$$

Assume that $\{x_n\}$ is a Cauchy sequence with respect to X. Then there exist $N > 0$ such that $d(x_,x_m) < \frac{1}{2}$ when $m, n \geq N$. But the inequality $d(x,y) < \frac{1}{2}$ for discrete metric implies that $d(x,y) = 0$ and, consequently, $x = y$. Hence $x_n = x_N$ when $n \geq N$ and $x_n \to x_N$ as $n \to \infty$. So, every Cauchy sequence in discrete metric is constant starting from some number and, consequently, convergent. Hence (X,d) is complete.

Mathematical analysis 3. Homework 3.

1. Let (X, d) be a metric space.

   (a) Prove that any open set can be expressed as a union of some family of balls.

   (b) Prove that a set K is compact if and only if for any cover of K by open balls there exists finite subcover.

   Solution. **(a)** Let G be open, then for every $x \in G$ there exist a neighbourhood V_x (that is an open ball) that is contained in G. Consequently,

$$G = \bigcup_{x \in G} V_x.$$

   (b) The necessety is obvious since open balls are open sets. Assume that for any cover of a set K by open balls there exists a finite subcover and let $K \subset \bigcup\limits_{\alpha \in A} G_\alpha$, where $\{G_\alpha\}_{\alpha \in A}$ is a family of open sets. Consider the following cover of K by open balls

$$K \subset \bigcup_{\alpha \in A} G_\alpha = K \subset \bigcup_{\alpha \in A} \bigcup_{x \in G_\alpha} V_x^\alpha,$$

   where V_x^α is a neighborhood of x that is contained in G_α. Then, by assumption, there exists a finite subcover and, consequently, $\alpha_1, \ldots, \alpha_n \in A$ such that

$$K \subset \bigcup_{k=1}^{n} G_{\alpha_k}.$$

2. Let $x^k = (x_1^k, x_2^k, \ldots, x_n^k)$ be a sequence of points in $\mathbb{R}^n$. Prove that $\{x^k\}$ is convergent in Euclidean metric if and only if sequences of coordinates $\{x_j^k\}_{k=1}^\infty$ are convergent, $1 \le j \le n$, in $\mathbb{R}$.

   Solution. Assume that $x^k \to a = (a_1, \ldots, a_n)$ then for every $1 \le j \le n$

$$\left|x_j^k - a_j\right| \le \rho(x^k, a) \to 0, \quad k \to \infty,$$

   and $x_j^k \to a_j$ as $k \to \infty$.

   Assume now that $x_j^k \to a_j$ as $k \to \infty$ for every $1 \le j \le n$. Let $a = (a_1, \ldots, a_n)$. Then

$$\rho(x^k, a)^2 = \left|x_1^k - a_1\right|^2 + \ldots + \left|x_n^k - a_n\right|^2 \to 0, \quad k \to \infty.$$

3. Prove that $\mathbb{R}^n$ is separable.

   Solution. To prove this it is enough to notice that $\mathbb{Q}^n$ is countable and dense in $\mathbb{R}^n$.

4. Consider a space of polynomials $\mathcal{P} = \left\{ p(x) = \sum\limits_{k=1}^{n} a_k x^k : k \in \mathbb{N},\ a_k \in \mathbb{R} \right\}$ on $[a, b]$ with uniform metric

$$d(p, q) = \max_{x \in [a,b]} |p(x) - q(x)|, \quad p, q \in \mathcal{P}.$$

Prove that this space is separable. Check the completeness of this space.

Solution. Consider a set

$$\mathcal{Q} = \left\{ p(x) = \sum_{k=1}^{n} a_k x^k : k \in \mathbb{N},\ a_k \in \mathbb{Q} \right\}.$$

Then $\mathcal{Q}$ is countable and dense in $\mathcal{P}$.

Consider a sequence of Taylor polynomials of function e^x

$$p_n(x) = \sum_{k=1}^{n} \frac{x^k}{k!}.$$

Then p_n is Cauchy sequence (since $p_n \rightrightarrows e^x$ on $[a, b]$) while it has no limit in $\mathcal{P}$.

5. Find the limit of a sequence $\{x^n\}$ if

   (a) $x^n = \left(\frac{1}{n}, \frac{n+1}{n}\right)$ in $\mathbb{R}^2$;

   (b) $x^n = \left(\left(1 + \frac{2}{n}\right)^{n/2}, e^{-n}\right)$ in $\mathbb{R}^2$;

   (c) $f_n(x) = \arctan(nx)$ in $C[1, 2]$.

Solution. Limit of a sequence in Euclidean space is vector of limits of coordinates. This allows to solve (a) and (b)

   (a) $\lim x^n = (0, 1)$.

   (b) $\lim x^n = (e, 0)$.

   (c) $f_n(x) = \arctan(nx)$ in $C[1, 2]$.
   Solution. Notice, that

$$\lim_{n \to +\infty} f_n(x) = \pi/2$$

   and

$$d(f_n, \pi/2) = \max_{x \in [1,2]} |\pi/2 - \arctan nx| = |\pi/2 - \arctan 2n| \to 0,\ n \to +\infty.$$

Mathematical analysis 3. Homework 4.

1. Consider a space ℓ^∞ of bounded sequences. Prove that it is a Banach space with respect to the norm (check axioms of the norm and completeness)

$$\|x\|_\infty = \sup |x_k|.$$

Solution. Positive definiteness and homogeneity are obvious. To check triangle inequality notice that

$$|x_k| + |y_k| \le \sup |x_k| + \sup |y_k|.$$

Hence,

$$\|x+y\|_\infty = \sup |x_k + y_k| \le \sup |x_k| + \sup |y_k| \le \|x\|_\infty + \|y\|_\infty.$$

Now, we will check completeness of ℓ^∞. Assume that $\{x^n\} \subset \ell^\infty$ is a Cauchy sequence. Since

$$|x_k^n - x_k^m| \le \|x^n - x^m\|_\infty$$

then the sequence of coordinates $\{x_k^n\}_{n=1}^\infty$ is Cauchy for every $k \in \mathbb{N}$ and has limit $a_k = \lim\limits_{n\to+\infty} x_k^n$. Let $a = \{a_k\}$. We need to prove that $a \in \ell^\infty$ and $\|x^n - a\|_\infty \to 0$.

Since $\{x^n\} \subset \ell^\infty$ is a Cauchy sequence then it is bounded, that is, there exists $M > 0$ such that for every $n \in \mathbb{N}$

$$\sup_{k\in\mathbb{N}} |x_k^n| \le M.$$

Consequently, $|x_k^n| \le M$ for every $k, n \in \mathbb{N}$. Hence $|a_k| \le M$ for every k and $a \in \ell^\infty$.

Let $\varepsilon > 0$ then there exists $N \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} |x_k^n - x_k^m| < \varepsilon$$

for every $n, m \ge N$ and, consequently,

$$|x_k^n - x_k^m| < \varepsilon$$

for every $n, m \ge N$ and $k \in \mathbb{N}$. Now letting $m \to \infty$ we see that

$$|x_k^n - a_k| \le \varepsilon$$

for every $n \ge N$ and $k \in \mathbb{N}$. Hence, $\|x^n - a\| \le \varepsilon$ for every $n \ge N$.

2. Prove that ℓ^∞ is not separable.

Solution. Consider a set of sequences of 0 and 1

$$C = \{\{x_n\} : x_n = 0 \text{ or } x_n = 1\}.$$

Then C is uncountable and distance between every two different points is greater or equal than one. Assume that $Q \subset \ell^\infty$ is everywhere dense. Then for every $x \in C$ the intersection $Q \cap B(x, 1/2)$ is not empty. This implies that Q is uncountable since balls $B(x, 1/2)$ do not intersect each other.

3. Consider a space ℓ^p of sequences such that

$$\|x\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} < \infty.$$

Prove that $(\ell^p, \|\|_p)$ is a Banach space.

Solution. Positive definiteness and homogeneity are obvious. The triangle inequality is provided by Minkowski inequality

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}.$$

Completeness is proved analogously to completeness of ℓ^∞.

4. Consider the metric $\rho(x, y) = \frac{|x-y|}{1+|x-y|}$ on $\mathbb{R}$. Prove that $\rho(0, x)$ is not a norm.

Solution. Notice that $\rho(0, \alpha x) = \frac{|\alpha||x|}{1+|\alpha||x|} \neq |\alpha| \rho(0, x)$ if $\alpha \neq 0$ and $x \neq 0$. Thus, $\rho(0, x)$ is not positive homogeneous.

5. Let $f, g \in C[a, b]$. Prove that the set $E = \{h \in C[a, b] : f \leq h \leq g\}$ is closed with respect to the uniform metric.

Solution. Assume that $f_n \in E$ and $f_n \to f$ in $C[a, b]$ then $f_n \rightrightarrows f$ and if $h(x) \leq f_n(x) \leq g(x)$ for every $x \in [a, b]$ and $n \in \mathbb{N}$ then $h(x) \leq f(x) \leq g(x)$ and $f \in E$. Consequently, E is closed.

6. Provide an example of a nested sequence of open subsets G_n of $\mathbb{R}$, $G_{n+1} \subset G_n$, such that $\bigcap_{k=1}^{\infty} G_k = \varnothing$.

Solution. $G_n = (0, 1/n)$.

7. Let (X, d) be a complete metric space, $\{G_n\}$ a sequence of open subsets. Prove that if $\operatorname{cl} G_{n+1} \subset G_n$ for every n then $\bigcap_{k=1}^{\infty} G_k \neq \varnothing$.

Solution. This is not true. Consider $G_k = (k, +\infty)$.

Mathematical analysis 3. Homework 5.

1. Let (X, ρ_X) and (Y, ρ_Y) be two metric spaces and $f : D \subset X \to Y$, a be a limit point of D. Assume that Y is complete. Prove that the function f has a limit at a if and only if

$$\forall \varepsilon > 0\ \exists V_a\ \forall u, v \in \dot{V}_a \cap D\ \rho_Y(f(u), f(v)) < \varepsilon. \tag{1}$$

Solution. Assume that $\{x_m\} \subset D \setminus \{a\}$ and $x_n \to a$. Let $\varepsilon > 0$ and choose V_a by assumption (1). Since $x_n \to a$ then there exists $N > 0$ such that $x_n \in V_a$ for every $n \geq N$. Consequently,

$$\rho_Y(f(x_n), f(x_m)) < \varepsilon, \quad n, m \geq N,$$

and the sequence $y_n = f(x_n)$ is a Cauchy sequence in a complete space (Y, ρ_Y). Hence, there exists $A = \lim\limits_{n\to+\infty} f(x_n)$. Since the sequence $\{x_n\}$ is arbitrary then by Heine definition of a limit

$$A = \lim_{x\to a} f(x).$$

2. Explain why is the completeness of Y necessary in this theorem. **Solution.** Let $X = \mathbb{R}$, $Y = \mathbb{R} \setminus \{0\}$ with Euclidean metric $d(x, y) = |x - y|$. Consider a function

$$f : X \to Y$$

such that $f(x) = x$, $x \neq 0$, and $f(0) = 1$. Then f has no limit at 0 while it satisfies Cauchy condition (1).

3. Consider $C[a, b]$ with the uniform norm $\|f\| = \max\limits_{x\in[a,b]} |f(x)|$. Prove that the operator $A : C[a, b] \to C[a, b]$ defined by

$$Af(x) = \int_a^x f$$

is bounded and find its norm.

Solution. First, estimate

$$\|Af\| = \max_{x\in[a,b]} \left| \int_a^x f \right| \leq \max_{x\in[a,b]} \int_a^x |f| \leq \int_a^b |f| \leq (b - a) \max |f| = (b - a) \|f\|.$$

Consequently, A is bounded and $\|A\| \leq b - a$.

Let $f \equiv 1$. Then

$$\|Af\| = \max_{x\in[a,b]} \left| \int_a^x \right| = \max_{x\in[a,b]} (x - a) = b - a = (b - a) \|1\|.$$

Consequently, $\|A\| = b - a$.

4. Let $A \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^1)$. Then matrix (A) is a row vector $(a_1, \ldots, a_n)$. Prove that $\|A\|^2 = \sum\limits_{k=1}^{n} a_k^2$.

   Solution. By cauchy Bunyakowski-Schwartz inequality (or by the theorem on estimate of the norm of linear operator in Euclidean spaces)

$$\|Ax\| \leq \left(\sum_{k=1}^{n} a_k^2 \right)^{1/2} \|x\|$$

   and if $x = (a_1, \ldots, a_n)$ then the equality holds. This finishes the proof.

5. Consider linear operator $A : \mathbb{R}^n \to \mathbb{R}^m$. Find the estimate for $\|A\|$ if both $\mathbb{R}^n$ and $\mathbb{R}^m$ are considered with norm $\|\|_\infty$.

   Solution. Consider,

$$\|Ax\|_\infty = \max_{1 \leq k \leq m} \left| \sum_{j=1}^{n} a_{kj} x_j \right| \leq \max_{1 \leq k \leq m} \sum_{j=1}^{n} |a_{kj}| \max |x_j| = \|x\|_\infty \max_{1 \leq k \leq m} \sum_{j=1}^{n} |a_{kj}| \,.$$

   Consequently,

$$\|A\| \leq \max_{1 \leq k \leq m} \sum_{j=1}^{n} |a_{kj}| \,.$$

Mathematical analysis 3. Homework 6.

1. Explain why the function f is differentiable and find its partial derivatives

   (a) $f(x,y) = x^2 + xy + y^2$;

   (b) $g(x,y,z) = z + \sin(xy^2)$;

   (c) $h(x,y,z) = \sqrt{x^2+y^2+z^2}$. Is this function differentiable at $(0,0,0)$?

   Solution. Functions f, g and function h (if $(x,y,z) \neq 0$) are differentiable as compositions of differentiable functions.

   (a) $f'_x = 2x + y$; $\quad f'_y = 2y + x$.

   (b) $g'_x = y^2\cos(xy^2)$; $\quad g'_y = 2xy\cos xy^2$; $\quad g'_z = 1$.

   (c) $h'_x = \frac{x}{\sqrt{x^2+y^2+z^2}}$; $\quad h'_y = \frac{y}{\sqrt{x^2+y^2+z^2}}$; $\quad h'_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$ if $x^2+y^2+z^2 \neq 0$.
   The partial derivative

   $$h'_x(0,0,0) = \lim_{t\to 0}\frac{\sqrt{t^2+0^2+0^2}}{t} = \lim_{t\to 0}\frac{|t|}{t}$$

   doesn't exist.

2. Consider the function

   $$f(x,y) = \begin{cases} 1, & y = \sin x,\ x \neq 0; \\ 0, & y \neq \sin x \text{ or } y = x = 0. \end{cases}$$

   Calculate partial derivatives of f at $(0,0)$. Is this function differentiable at $(0,0)$?

   Solution. Considering the graph of $y = \sin x$ we see that

   $$f'_x(0,0) = \lim_{t\to 0}\frac{f(t,0) - f(0,0)}{t} = 0;$$

   $$f'_y(0,0) = \lim_{t\to 0}\frac{f(0,t) - f(0,0)}{t} = 0;$$

   while f is not continuous since

   $$\lim_{x\to 0} f(x,\sin x) = 1 \neq f(0,0).$$

3. Prove that function $f(x,y) = \sin(x+y)$ is Lipschitz, i.e. there exist $M > 0$ such that

   $$|f(x_1,y_1) - f(x_2,y_2)| \leq M\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}.$$

   Solution. The domain of f is $\mathbb{R}^2$. Moreover, f is differentiable on $\mathbb{R}^2$ and

   $$\operatorname{grad} f = (\cos(x+y), \sin(x+y))$$

with

$$\|\mathrm{grad}\, f\| \le \sqrt{2}.$$

Consequently, by corollary of Lagrange's theorem for mappings

$$|f(x_1, y_1) - f(x_2, y_2)| \le \sqrt{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

4. Find the domain and Jacobi matrix of the function

$$f(x, y) = \begin{pmatrix} xy; \\ \ln(1 + x + y). \end{pmatrix}$$

Solution. Function f is defined and differentiable on

$$D = \{(x, y) : x + y > -1\}.$$

Rows of the Jacobi matrix of vector-valued function are gradients of coordinate functions and

$$(f') = \begin{pmatrix} y & x \\ \frac{1}{1+x+y} & \frac{1}{1+x+y} \end{pmatrix}$$

Mathematical analysis 3. Homework 7.

Find the limit $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$ and iterated limits $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y)$, $\lim\limits_{y\to b}\lim\limits_{x\to a} f(x,y)$ or prove that some of these limits do not exist.

1. $f_1(x,y) = \frac{y\sin(x)+x\sin y}{|x|+|y|}$, $a=b=0$;
2. $f_2(x,y) = \frac{\ln(1+x\sin(x+y))}{|x|+y^2}$, $a=b=0$;
3. $f_3(x,y) = \sin\frac{\pi x}{2x+y}$, $a=b=+\infty$;
4. $f_4(x,y) = \frac{x^y}{1+x^y}$, $a=+\infty$, $b=0+$.

Solution.

1. $|f_1(x,y)| \le \frac{2|xy|}{|x|+|y|} \le |x|+|y| \to 0$ if $(x,y)\to(0,0)$.
2. First,

$$\lim_{x\to 0}\lim_{y\to 0}\frac{\ln(1+x\sin(x+y))}{|x|+y^2} = \lim_{x\to 0}\frac{\ln(1+x\sin x)}{|x|} = 0;$$

$$\lim_{y\to 0}\lim_{x\to 0}\frac{\ln(1+x\sin(x+y))}{|x|+y^2} = \lim_{x\to 0}\frac{\ln(1+0)}{y^2} = 0;$$

Consider the limit

$$\lim_{\substack{x\to 0\\ y\to 0}} f_2(x,y) = \lim_{\substack{x\to 0\\ y\to 0}}\frac{x\sin(x+y)}{|x|+y^2} = \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2+xy}{|x|+y^2}$$

we notice that

$$\left|\frac{x^2+xy}{|x|+y^2}\right| \le \frac{|x|}{|x|+y^2}(|x|+|y|) \le |x|+|y|$$

Hence,

$$\lim_{\substack{x\to 0\\ y\to 0}} f_2(x,y) = \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2+xy}{|x|+y^2} = 0.$$

3. First,

$$\lim_{x\to+\infty}\lim_{y\to+\infty}\sin\frac{\pi x}{2x+y}=\lim_{x\to+\infty}0=0;$$

$$\lim_{y\to+\infty}\lim_{x\to+\infty}\sin\frac{\pi x}{2x+y}=\lim_{y\to+\infty}\sin\frac{\pi}{2}=1;$$

The limit

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}}\sin\frac{\pi x}{2x+y}$$

doesn't exist since

$$\lim f(n,n)=\sin\frac{\pi}{3}\neq\lim f(n,2n)=\sin\frac{\pi}{4}.$$

4. First,

$$\lim_{x\to+\infty}\lim_{y\to 0+}\frac{x^y}{1+x^y}=\lim_{x\to+\infty}\frac{1}{2}=\frac{1}{2};$$

and

$$\lim_{y\to 0+}\lim_{x\to+\infty}\frac{x^y}{1+x^y}=\lim_{y\to+\infty}1=1;$$

Let $q>1$. Then

$$\lim_{n\to+\infty}f(q^n,1/n)=\frac{q}{1+q}$$

and the limit

$$\lim_{\substack{x\to+\infty\\ y\to 0+}}\frac{x^y}{1+x^y}$$

doesn't exist.

Mathematical analysis 3. Homework 8.

1. Calculate $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$ where

$$f(x,y) = \begin{cases} xy(x^2-y^2)/(x^2+y^2), & x^2+y^2 \neq 0; \\ 0, & x=y=0. \end{cases}$$

Solution. Then

$$f'_x(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$$

and for $(x,y) \neq (0,0)$

$$f'_x(x,y) = y\frac{x^2-y^2}{x^2+y^2} + 2x^2y\frac{x^2+y^2-(x^2-y^2)}{(x^2+y^2)^2} = y\frac{x^4-y^4+4x^2y^2}{(x^2+y^2)^2}.$$

Consequently,

$$f''_{xy}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} \frac{-y^5}{y^5} = -1.$$

Analogously, $f'_y(0,0) = 0$ and for $(x,y) \neq (0,0)$

$$f'_y(x,y) = x\frac{x^4-y^4+4x^2y^2}{(x^2+y^2)^2}.$$

Thus $f''_{yx}(0,0) = 1 \neq f''_{xy}(0,0) = -1$.

2. Calculate $\frac{\partial^3 f}{\partial x \partial y \partial z}$ if

$$f = \sqrt{xy^3z^5}.$$

Solution.

$$f'_x = \frac{1}{2}\sqrt{\frac{y^3z^5}{x}}; \quad f''_{xy} = \frac{3}{4}\sqrt{\frac{yz^5}{x}}; \quad f'''_{xyz} = \frac{15}{8}\sqrt{\frac{yz^3}{x}};$$

3. Find second differential d^2u at $(0,0)$ if

$$u(x,y) = y \operatorname{arctg} \frac{x}{1+2y}.$$

Solution 1.

$$u'_x = \frac{y}{1+2y} \frac{1}{1+(x/(1+2y))^2};$$

$$u'_y = \operatorname{arctg} \frac{x}{1+2y} - \frac{2xy}{(1+2y)^2} \frac{1}{1+(x/(1+2y))^2};$$

$$u''_{xx} = \frac{y}{1+2y} \left(\frac{1}{1+(x/(1+2y))^2} \right)'_x \Bigg|_{x=y=0} = 0;$$

$$u''_{xy} = \frac{1}{1+2y} \left(\frac{1}{1+(x/(1+2y))^2} \right) + \frac{y}{1+2y} \left(\frac{1}{1+(x/(1+2y))^2} \right)'_y \Bigg|_{x=y=0} = 2;$$

$$u''_{yy} = \frac{-2x}{(1+2y)^2} \operatorname{arctg} \frac{x}{1+2y} - \frac{2x}{(1+2y)^2} \frac{1}{1+(x/(1+2y))^2}$$
$$- \frac{2xy}{(1+2y)^2} \left(\frac{1}{1+(x/(1+2y))^2} \right)'_y \Bigg|_{x=y=0} = 0;$$

$$d^2u = 2dxdy.$$

Solution 2. Consider Taylor's decomposition of f at $(0,0)$ with residue $o(\rho^2)$, where $\rho = \sqrt{x^2+y^2}$.

$$u(x,y) = y \operatorname{arctg} \frac{x}{1+2y} = \frac{xy}{1+2y} + yo\left(\frac{x^2}{(1+2y)^2} \right) =$$
$$xy(1+o(1)) + o(\rho^2) = xy + o(\rho^2),$$

since $\arctan t = t + o(t^2)$ as $t \to 0$. Hence,

$$d^2u = 2dxdy.$$

4. Find second differential d^2u, where

$$u(x,y) = f(x^2+y^2, x^2-y^2, 2xy)$$

and $f \in C^2$.

Solution.

$$\begin{aligned} u'_x = & \ 2xf'_1 + 2xf'_2 + 2yf'_3; \\ u'_y = & \ 2yf'_1 - 2yf'_2 + 2xf'_3; \end{aligned}$$

$$u''_{xx} = 2(f'_1 + f'_2) + 4x^2 f''_{11} + 4x^2 f''_{22} + 4y^2 f''_{33} + 8x^2 f''_{12} + 8xyf''_{13} + 8xyf''_{23};$$

$$u''_{xy} = 2f'_3 + 2y(2xf''_{11} + 2xf''_{12} + 2yf''_{13}) - 2y(2xf''_{12} + 2xf''_{22} + 2yf''_{23}) +$$
$$2x(2xf''_{13} + 2xf''_{23} + 2yf''_{33}) =$$
$$2f'_3 + 4xy(f''_{11} - f''_{22} + f''_{33}) + 4(x^2 + y^2)f''_{13} + 4(x^2 - y^2)f''_{23};$$

$$u''_{yy} = 2(f'_1 - f'_2) + 2y(2yf''_{11} - 2yf''_{12} + 2xf''_{13}) - 2y(2yf''_{12} - 2yf''_{22} + 2xf''_{23}) +$$
$$2x(2yf''_{13} - 2yf''_{23} + 2xf''_{33}) =$$
$$2(f'_1 - f'_2) + 4y^2 f''_{11} + 4y^2 f''_{22} + 4x^2 f''_{33} - 8y^2 f''_{12} + 8xyf''_{13} - 8xyf''_{23};$$

5. Prove that $\Delta u = 0$ if

$$u(x, y) = e^x(x \cos y - y \sin y).$$

Solution.

$$u''_{xx} = (e^x)''_{xx}(x \cos y - y \sin y) + 2(e^x)'_x(x \cos y - y \sin y)'_x + e^x(x \cos y - y \sin y)''_{xx} =$$
$$e^x(x \cos y - y \sin y + 2 \cos y);$$

$$u''_{yy} = e^x(-x \cos y + y \sin y - 2 \cos y);$$
$$\Delta u = u''_{xx} + u''_{yy} = 0.$$

6*. Find a function $\varphi(t)$ if the function $f = \varphi\left(\frac{x^2+y^2}{x}\right)$ satisfies Laplace equation $\Delta f = 0$.

Solution.

$$f'_x = \varphi'\left(\frac{x^2+y^2}{x}\right)\left(1-\frac{y^2}{x^2}\right);$$

$$f'_y = \varphi'\left(\frac{x^2+y^2}{x}\right)\frac{2y}{x};$$

$$f''_{xx} = \varphi''\left(\frac{x^2+y^2}{x}\right)\left(1-\frac{y^2}{x^2}\right)^2 + \varphi'\left(\frac{x^2+y^2}{x}\right)\frac{2y^2}{x^3};$$

$$f''_{yy} = \varphi''\left(\frac{x^2+y^2}{x}\right)\frac{4y^2}{x^2} + \varphi'\left(\frac{x^2+y^2}{x}\right)\frac{2}{x};$$

$$\Delta f = \varphi''\left(\frac{x^2+y^2}{x}\right)\left(\frac{x^2+y^2}{x^2}\right)^2 + 2\varphi'\left(\frac{x^2+y^2}{x}\right)\frac{y^2+x^2}{x^3} = 0.$$

Let $t = \frac{x^2+y^2}{x}$. Then

$$\varphi''(t)\frac{t^2}{x^2} + 2\varphi'(t)\frac{t}{x^2} = 0$$

and

$$(\ln\varphi'(t))' = \frac{\varphi''(t)}{\varphi'(t)} = -\frac{2}{t} = (-2\ln t)';$$

$$\ln|\varphi'(t)| = c - 2\ln|t|; \quad \varphi'(t) = \frac{a}{t^2}; \quad \varphi(t) = \frac{a}{t} + b, \quad a, b \in \mathbb{R}.$$

Mathematical analysis 3. Homework 9.

1. Find extremal points of function f

   (a) $f = \frac{x+y}{xy} - xy$;

   (b) $f = x^3/3 + 3x^2e^y - e^{-y^2}$;

   (c) $f = \sin x \sin y \sin(x+y)$, $x, y \in (0, \pi)$;

   (d) $f = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$.

2. Prove that function $f = (y^2 - x)(y^2 - 2x)$ has minimum on every line passing through a point $(0,0)$ but has no minimum at $(0,0)$ as a function of two variables.

Solutions.

1a. First, notice that $xy \neq 0$ and find stationary points from the system

$$f'_x = -1/x^2 - y = 0;$$
$$f'_y = -1/y^2 - x = 0.$$

Hence, $x^2y = xy^2$ and $x = y = -1$. To check sufficient condition consider

$$f''_{xx} = 2/x^3 = -2;\ f''_{xy} = -1;\ f''_{yy} = -2. \tag{1}$$

Consequently,

$$d^2f = -2(dx^2 + dxdy + dy^2) = -(dx + dy)^2 - dx^2 - dy^2$$

is negative definite and f has at $(-1,-1)$ strict minimum.

1b. First we find stationary points from the system

$$f'_x = x^2 + 6xe^y = 0;$$
$$f'_y = 3x^2e^y + 2ye^{-y^2} = 0.$$

Case 1. Let $x = 0$. Then $2ye^{-y^2} = 0$ and $y = 0$. Then

$$d^2f = 6dx^2 + 2dy^2$$

is positive definite and u has minimum at $(0,0)$, $u(0,0)=-1$.

Case 2. Let $x \neq 0$. Then $x=-6e^y$ and

$$108e^{3y}+2ye^{-y^2}=0.$$

Noticing that

$$54e^{y^2+3y}+y>54e^{-9/4}+y>5+y$$

we see that $y<-5$ and $y^2+3y>10$. Let

$$g(y)=54e^{y^2}+y.$$

Then

$$g'(y)=54(2y+3)e^{y^2+3y}+1<0,\ \ y<-5,$$

and

$$g(y)>g(-5)>0,\ \ y<-5.$$

Consequently, g has no zeros and in this case the system has no solutions.

1c. First we find stationary points from the system

$$\begin{aligned} f'_x &= \cos x\sin y\sin(x+y)+\sin x\sin y\cos(x+y)=\sin y\sin(2x+y)=0;\\ f'_y &= \sin x\cos y\sin(x+y)+\sin x\sin y\cos(x+y)=\sin x\sin(x+2y)=0.\end{aligned}$$

Consequently, since $x,y\in(0,\pi)$ then

$$2x+y=x+2y=\pi \text{ or } 2x+y=x+2y=2\pi$$

and $x_1=y_1=\pi/3$ or $x_2=y_2=2\pi/3$.

To check the sufficient condition we calculate d^2f. For (x_k,y_k)

$$\begin{aligned} f''_{xx} &= 2\sin y\cos(2x+y)=(-1)^k\sqrt{3};\\ f''_{xy} &= \cos y\sin(2x+y)+\sin y\cos(2x+y)=(-1)^k\sqrt{3};\\ f''_{yy} &= 2\sin x\cos(x+2y)=(-1)^k\sqrt{3}.\end{aligned}$$

Hence,

$$d^2f=(-1)^k\sqrt{3}(dx^2+2dxdy+dy^2)=(-1)^k\sqrt{3}(dx+dy)^2.$$

Notice that f is continuous on $[0,\pi]^2$ and is equal to 0 on the boundary, has positive and negative values in $(0,\pi)^2$. Consequently, it obtains its maximal and minimal values in stationary points in $(0,\pi)$ and

- $x = y = \pi/3$ is a point of maximum, $u(\pi/3, \pi/3) = \frac{3\sqrt{3}}{8}$;
- $x = y = 2\pi/3$ is a point of maximum, $u(2\pi/3, 2\pi/3) = -\frac{3\sqrt{3}}{8}$;

1d. First we find stationary points from the system

$$\begin{aligned} f'_x &= 1 - \frac{y^2}{4x^2} = 0; \\ f'_y &= \frac{y}{2x} - \frac{z^2}{y^2} = 0; \\ f'_z &= \frac{2z}{y} - \frac{2}{z^2} = 0. \end{aligned}$$

Consequently, $y = 2x$, $z^2 = y^2$; $z^3 = y$. This system has two solutions

$$(x_1, y_1, z_1) = (1/2, 1, 1); \quad (x_2, y_2, z_2) = (-1/2, -1, -1).$$

To check the sufficient condition we calculate $d^2 f$. For (x_k, y_k, z_k)

$$f''_{xx} = \frac{y^2}{2x^3} = (-1)^{k-1}4; \quad f''_{yy} = \frac{1}{2x} + \frac{2z^2}{y^3} = (-1)^{k-1}3; \ f''_{zz} = \frac{2}{y} + \frac{4}{z^3} = (-1)^{k-1}6;$$
$$f''_{xy} = \frac{y}{2x^2} = (-1)^{k-1}2; \quad f''_{xz} = 0; \quad f''_{yz} = -\frac{2z}{y^2} = (-1)^{k-1}2;$$

Case 1. At $(1/2, 1, 1)$ matrix of quadratic form $d^2 f$ is equal to

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 6 \end{pmatrix}$$

Consequently, $\Delta_1 = 4 > 0$; $\Delta_2 = 8 > 0$; $\Delta_3 = 4(18 - 4) - 2(12 - 0) = 32 > 0$. Hence, $d^2 L$ is positive definite and f has strict local minimum at $(1/2, 1, 1)$ with $f(1/2, 1, 1) = 1/2 + 1/2 + 1 + 2 = 4$.
Case 2. At $(-1/2, -1, -1)$ matrix of quadratic form $d^2 f$ is equal to

$$\begin{pmatrix} -4 & -2 & 0 \\ -2 & -3 & -2 \\ -0 & -2 & -6 \end{pmatrix}$$

Consequently, $\Delta_1 = -4$; $\Delta_2 = 8$; $\Delta_3 = -32 < 0$.

Hence, d^2L is negative definite and f has strict local maximum at $(1/2, 1, 1)$ with $f(-1/2, -1, -1) = -4$.

2. Prove that function $f = (y^2 - x)(y^2 - 2x)$ has minimum on every line passing through the point $(0, 0)$ but has no minimum at $(0, 0)$ as a function of two variables.

Let $y = kx$. $k \in \mathbb{R}$. Then

$$f(x, kx) = (k^2x^2 - x)(k^2x^2 - 2x) = x^2(kx - 1)(kx - 2) > 0 = f(0, 0), \ 0 < |x| < 1/|k|.$$

Also $f(0, y) = y^2 > 0 = f(0, 0)$ for every $y \neq 0$. Consequently, f has minimum at $(0, 0)$ along any line passing this point. At the same time

$$f\left(\frac{3}{2}y^2, y\right) = -y^2/4 < f(0, 0)$$

and f has no minimum at $(0, 0)$.

Mathematical analysis 3. Homework 10.

1. Find Taylor's decomposition of function f at (x_0, y_0) with residue $o(\rho^2)$, where $\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$.

   (a) $f = \frac{\cos x}{\cos y}$, $(x_0, y_0) = (0,0)$;

   (b) $f = x^y$, $(x_0, y_0) = (1,1)$;

   Solution. 1a.

$$f = \frac{\cos x}{\cos y} = \frac{1 - \frac{x^2}{2} + o(x^2)}{1 - \frac{y^2}{2} + o(y^2)} = \left(1 - \frac{x^2}{2} + o(x^2)\right)\left(1 + \frac{y^2}{2} + o(y^2)\right) =$$
$$1 - \frac{x^2}{2} + \frac{y^2}{2} + o(\rho^2).$$

   1b.

$$f = x^y = e^{y \ln x} = e^{\ln x} e^{(y-1)\ln(1+(x-1))} = x e^{(y-1)((x-1)+o(x-1))} =$$
$$(1+(x-1))e^{(x-1)(y-1)+o(\rho^2)} = (1+(x-1))(1+(x-1)(y-1)+o(\rho^2)) =$$
$$1 + (x-1) + (x-1)(y-1) + o(\rho^2).$$

2. Find Taylor's decomposition of the function

$$f = \ln(xy + z^2)$$

   at $(x_0, y_0, z_0) = (0,0,1)$ with residue $o(\rho^2)$, where

$$\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}.$$

   Solution.

$$f = \ln(xy + z^2) = (1 + 2(z-1) + (z-1)^2 + xy) =$$
$$2(z-1) + (z-1)^2 + xy - \frac{1}{2}(2(z-1) + (z-1)^2 + xy)^2 + o(\rho^2) =$$
$$2(z-1) + (z-1)^2 + xy - 2(z-1)^2 + o(\rho^2) = 2(z-1) - (z-1)^2 + xy + o(\rho^2).$$

3. Find partial derivative of the first and second order of the implicit function u defined by the equation

$$u = x + \operatorname{arctg}\frac{y}{u-x}.$$

Solution. Let

$$F(x, y; u) = u - x - \operatorname{arctg}\frac{y}{u-x}.$$

Then

$$\begin{aligned}
F'_u &= \quad +1 + \tfrac{y}{(u-x)^2}\tfrac{1}{1+y^2/(u-x)^2} = 1 + \tfrac{y}{(u-x)^2+y^2} = \tfrac{y+y^2+(u-x)^2}{(u-x)^2+y^2};\\
F'_x &= -1 - \tfrac{y}{(u-x)^2}\tfrac{1}{1+y^2/(u-x)^2} = -1 - \tfrac{y}{(u-x)^2+y^2} = -\tfrac{y+y^2+(u-x)^2}{(u-x)^2+y^2};\\
F'_y &= \qquad -\tfrac{1}{u-x}\tfrac{1}{1+y^2/(u-x)^2} = \tfrac{x-u}{(u-x)^2+y^2}.
\end{aligned}$$

Consequently,

$$u'_x = -\frac{F'_x}{F'_u} = 1; \quad u'_y = -\frac{F'_y}{F'_u} = \frac{u-x}{y+y^2+(u-x)^2}.$$

Hence, $u''_{xx} = u''_{xy} = 0$ and

$$\begin{aligned}
u''_{yy} = \frac{u'_y(y+y^2+(u-x)^2) - (u-x)(1+2y+2(u-x)u'_y)}{(y+y^2+(u-x)^2)^2} = \\
\frac{(u-x) - (u-x)(1+2y+2(u-x)u'_y)}{(y+y^2+(u-x)^2)^2} = \\
\frac{2(x-u)(y+(u-x)u'_y)}{(y+y^2+(u-x)^2)^2} = \frac{2(x-u)(y(y(y+1)+(u-x)^2)+(u-x)^2)}{(y+y^2+(u-x)^2)^3} = \\
\frac{2(x-u)(y^2(y+1)+(y+1)(u-x)^2)}{(y+y^2+(u-x)^2)^3} = \frac{2(x-u)(y+1)(y^2+(u-x)^2)}{(y+y^2+(u-x)^2)^3}.
\end{aligned}$$

4. Find Taylor's decomposition of the implicit function u at $(x_0, y_0) = (1, 1)$ that is defined by the equation

$$u^3 - 2xu + y = 0, \; u(1,1) = 1;$$

with residue $o(\rho^2)$, where $\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$.

Solution. Let

$$F(x,y;u) = u^3 - 2xu + y.$$

Then

$$F'_u = 3u^2 - 2x \neq 0 \text{ if } x = y = 1,\ u(1,1) = 1;$$

$$F'_x = -2u; \quad F'_y = 1.$$

Hence,

$$u'_x = \left.\frac{2u}{3u^2 - 2x}\right|_{x=y=u=1} = 2; \quad u'_y = -\left.\frac{1}{3u^2 - 2x}\right|_{x=y=u=1} = -1;$$

$$\begin{aligned} u''_{xx} &= \left.\tfrac{2u'_x(3u^2-2x)-2u(6uu'_x-2)}{(3u^2-2x)^2}\right|_{x=y=u=1} = -16 \\ u''_{xy} &= \left.\tfrac{6uu'_x-2}{(3u^2-2x)^2}\right|_{x=y=u=1} = 10 \\ u''_{yy} &= \left.\tfrac{6uu'_y}{(3u^2-2x)^2}\right|_{x=y=u=1} = -6. \end{aligned}$$

Consequently,

$$u(x,y) = 1+2(x-1)-(y-1)-8(x-1)^2+10(x-1)(y-1)-3(y-1)^2+o(\rho^2),\ \rho \to 0.$$

Mathematical analysis 3. Homework 11.

1. Find extremal points of the implicit functions $u(x, y)$ defined by the equation

$$2x^2 + 2y^2 + u^2 + 8yu - u + 8 = 0.$$

Solution. Let

$$F(x, y, u) = 2x^2 + 2y^2 + u^2 + 8yu - u + 8.$$

Then condition of the theorem on the implicit function is

$$F'_u = 2u + 8y - 1 \neq 0$$

and stationary points can be found from conditions

$$u'_x = -\frac{F'_x}{F'_u} = -\frac{4x}{2u + 8y - 1} = 0;$$
$$u'_y = -\frac{F'_y}{F'_u} = -\frac{4y + 8u}{2u + 8y - 1} = 0.$$

Consequently, $x = 0$ and $y = -2u$. Applying the equation of the implicit function we see that

$$8u^2 + u^2 - 16u^2 - u + 8 = -7u^2 - u + 8 = 0;$$

Hence we have two solutions $x = 0$, $y = -2$, $u(0, -2) = 1$, and $x = 0$, $y = 16/7$, $u(0, 16/7) = -8/7$.

Case 1. $x = 0$, $y = -2$; $u(0, -2) = 1$. Then $F'_u = -15 \neq 0$ and

$$u''_{xx} = -\frac{4}{2u+8y-1} + \frac{8xu'_x}{(2u+8y-1)^2} = 4/15;$$
$$u''_{xy} = \frac{4x(2u'_y+8)}{(2u+8y-1)^2} = 0.$$
$$u''_{yy} = -\frac{4+8u'_y}{2u+8y-1} + \frac{(4y+8u)(2u'_y+8)}{(2u+8y-1)^2} = 4/3.$$

Consequently, $d^2u = 4/15dx^2 + 4/3dy^2$ is positive definite and u has local minimum at $(0, -2)$.

Case 1. $x = 0$, $y = 16/7$; $u(0, -2) = -8/7$. Then $F'_u = -16/7 + 128/7 - 1 = 105/7 = 15 \neq 0$ and

$$u''_{xx} = -\frac{4}{2u+8y-1} + \frac{8xu'_x}{(2u+8y-1)^2} = -4/15;$$
$$u''_{xy} = \frac{4x(2u'_y+8)}{(2u+8y-1)^2} = 0.$$
$$u''_{yy} = -\frac{4+8u'_y}{2u+8y-1} + \frac{(4y+8u)(2u'_y+8)}{(2u+8y-1)^2} = -4/3.$$

Consequently, $d^2u = -4/15dx^2 - 4/3dy^2$ is negative definite and function u has local maximum at $(0, 16/7)$.

Answer. 1. $x = 0$, $y = -2$ is a point of local minimum for implicit function u with value $u(0, -2) = 1$.

2. $x = 0$, $y = 16/7$ is a point of local minimum for implicit function u with value $u(0, 16/7) = 8/7$.

2. Find conditional extremum using Lagrange's function

   (a) $u = xyz$, $\varphi = x^2 + y^2 + z^2 = 3$;

   (b) $u = \ln(xy)$, $\varphi = x^3 + xy + y^3 = 0$;

   (c) $u = x + 2y$, $\varphi = x^2 - 8y^2 = 8$

Solution.

2a. Notice that

$$\operatorname{grad}(x^2 + y^2 + z^2) = (2x, 2y, 2z) \neq (0, 0), \quad x^2 + y^2 + z^2 \neq 0.$$

Consider Lagrange's function

$$L = xyz + \lambda(x^2 + y^2 + z^2).$$

First, we find stationary points of L

$$\begin{aligned} L'_x &= yz + 2\lambda x = 0; \\ L'_y &= xz + 2\lambda y = 0; \\ L'_z &= xy + 2\lambda z = 0; \\ x^2 + y^2 + z^2 &= 3. \end{aligned}$$

Hence,

$$\lambda x^2 = \lambda y^2 = \lambda z^2.$$

If $\lambda = 0$ then two of coordinates (x, y, z) are 0 and one is $\pm\sqrt{3}$. In this case function u is 0 at this point and has negative and positive values in any neighborhood of this point. Consequently, in this case u has no extremum at these points.

Assume that $\lambda \neq 0$. Then $x^2 = y^2 = z^2 = 1$. Then $x, y, z \in \{-1, 1\}$.

Let $(x, y, z) = (1, 1, 1)$. Then $\lambda = -1/2$ and

$$\begin{aligned} d^2L = 2\lambda(dx^2 + dy^2 + dz^2) + 2zdxdy + 2ydxdz + 2xdydz = \\ -(dx^2 + dy^2 + dz^2) + 2dxdy + 2dxdz + 2dydz \end{aligned}$$

and we need to investigate this differential with assumption that

$$d(x^2 + y^2 + z^2) = 2xdx + 2ydy + 2zdz = 2(dx + dy + dz).$$

With this assumption $dz = -dx - dy$ and

$$d^2L = -(dx^2 + dy^2 + (dx + dy)^2) + 2dxdy - 2dx(dx + dy) - 2dy(dx + dy) = -4(dx^2 + dy^2)$$

is negative definite. Hence, $(1, 1, 1)$ is the point of local maximum.

To investigate other points notice that

$$u((-1)^i x, (-1)^k x, (-1)^j x) = (-1)^{i+j+k} u(x, y, z).$$

Consequently, $(-1,-1,1)$, $(-1,1,-1)$, $(1,-1,-1)$ are points of local maximum. while $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, $(-1,-1,-1)$ are points of local minimum

2b. Notice that

$$\operatorname{grad}(x^3 + xy + y^3) = (3x^2 + y, x + 3y^2) \neq (0,0)$$

since the domain of $\ln(xy$ is defined by condition $xy > 0$. Let

$$L = \ln(xy) + \lambda(x^3 + xy + y^3).$$

First, we find stationary points of L.

$$\begin{aligned} L'_x &= \frac{1}{x} + \lambda(3x^2 + y) = 0; \\ L'_y &= \frac{1}{y} + \lambda(3y^2 + x) = 0; \\ & x^3 + xy + y^3 = 3. \end{aligned}$$

Hence,

$$\begin{aligned} 1 + \lambda(3x^3 + xy) = 0; \\ 1 + \lambda(3y^3 + xy) = 0; \\ x^3 + xy + y^3 = 0. \end{aligned}$$

Hence, $2x^3 + xy = 3y^3 + xy$ and $x = y$. Applying the connection condition we see that

$$2x^3 + x^2 = 0,$$

and $x = y = -1/2$, $\lambda = -1/(3x^3 + xy) = 8$. Then $\operatorname{grad}\varphi \neq (0,0)$ and we can apply Lagrange's method.

Now we will check the sufficient condition, definiteness of the second differential d^2L with condition $d\varphi = 0$.

$$\begin{aligned} L''_{xx} = -\frac{1}{x^2} + 6\lambda x = -4 - 24 = -20; \\ L''_{yy} = -\frac{1}{y^2} + 6\lambda y = -20; \\ L''_{xy} = \lambda = 8; \\ d(x^3 + xy + y^3) = (3x^2 + y)dx + (x + 3y^2)dy = \frac{dx + dy}{4}; \end{aligned}$$

Hence, $dx = -dy$ and

$$d^2L = -20dx^2 + 16dxdy - 20dy^2 = -56dx^2$$

is negative definite and u has local maximum at $(-1/2, -1/2)$, $u(-1/2, -1/2) = -2\ln 2$.

2c. Let

$$L = x + 2y + \lambda(x^2 - 8y^2);$$

First, we find stationary points of L.

$$\begin{aligned} L'_x &= 1 + 2\lambda x = 0; \\ L'_y &= 2 - 16\lambda y = 0; \\ x^2 - 8y^2 &= 8. \end{aligned}$$

Hence, $\lambda \neq 0$ and $x = -4y$. Applying the connection condition $x^2 - 8y^2 = 8$ we get two solutions $(x, y) = (-4, 1)$, $\lambda = 1/8$; and $(x, y) = (4, -1)$, $\lambda = -1/8$;

Now, we will check the sufficient condition. Notice that the connection condition implies restriction

$$d(x^2 - 8y^2) = 2xdx - 16ydy = 2y\left(\frac{x}{y}dx - 8dy\right) = -8y(dx - 2dy).$$

Consequently, $dx = 2dy$ and

$$d^2L = 2\lambda(dx^2 - 8dy^2) = -8\lambda dy^2.$$

Hence, at $(x, y) = (-4, 1)$, $\lambda = 1/8$, d^2L is positive definite and u has local minimum, $u(-4, 1) = -2$; at $(x, y) = (4, -1)$, $\lambda = -1/8$, d^2L is negative definite and u has local maximum, $u(4, -1) = 2$.

3. Investigate the function u for conditional extremum and find out if the Lagrange's method can be applied to this problem

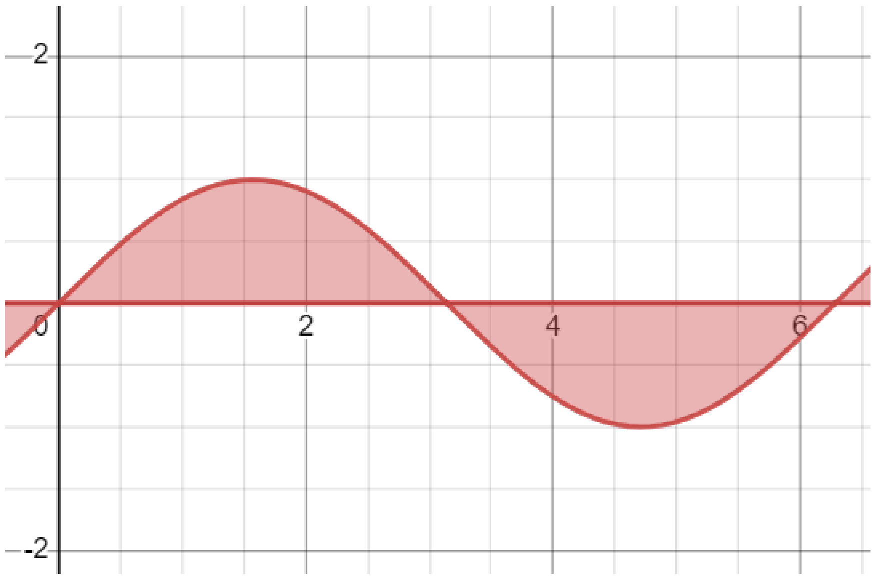
$$u = x^4 + y^4, \quad (x-1)^3 - y^2 = 0$$

Mathematical analysis 3. Homework 12.

1. Change the order of integration in the double integral

$$I = \int_0^{2\pi} dx \int_0^{\sin x} f(x, y) dy.$$

Solution. The domain of integration is bounded by the graph of $\sin x$ and $y = 0$, $0 \leq x \leq 2\pi$,



Let $y \in (-1, 1) \setminus \{0\}$. Then the equation $y = \sin x$ has two solutions on $(0, 2\pi)$, then

$$x_1 = \arcsin y,\ x_2 = \pi - \arcsin y,\ 0 < y < 1;$$

$$x_1 = \pi - \arcsin y,\ x_2 = 2\pi + \arcsin y,\ -1 < y < 0.$$

Hence,

$$I = \int_{-1}^{0} dy \int\limits_{\pi - \arcsin y}^{2\pi + \arcsin y} f(x, y) dx + \int_0^1 dy \int\limits_{\arcsin y}^{\pi - \arcsin y} f(x, y) dx.$$

2. Calculate

$$I = \iint_{\Omega} \left(x^2 + y^2\right) dx dy,$$

where Ω is a parallelogram bounded by $y = x, y = x + a, y = a$ and $y = 3a$, $(a > 0)$.

Solution.

$$I = \int_a^{3a} dy \int_{y-a}^{y} (x^2 + y^2) dx = \int_a^{3a} (x^3/3 + y^2 x)\Big|_{y-a}^{y} dy =$$

$$\int_a^{3a} \left(\frac{y^3 - (y-a)^3}{3} + ay^2 \right) dy = \left(\frac{y^4 - (y-a)^4}{12} + ay^3/3 \right)\Big|_a^{3a} = 14a^4.$$

3. Convert Cartesian coordinates to polar coordinates in the integral

$$I = \int_0^2 dx \int_x^{x\sqrt{3}} f\left(\sqrt{x^2+y^2}\right) dy.$$

Solution. Consider the polar change

$$x = r\cos\varphi,\ y = r\sin\varphi,\ \varphi \in [-\pi, \pi),\ r > 0.$$

Domain of integration is defined by inequalities

$$0 < x < 2, \quad x < y < x\sqrt{3}.$$

Then $x, y > 0$, $\varphi \in (0, \pi/2)$,

$$r\cos\varphi < r\sin\varphi < \sqrt{3}r\cos\varphi$$

and $1 < \tan\varphi < \sqrt{3}$. Hence,

$$\frac{\pi}{4} < \varphi < \frac{\pi}{3}, \quad 0 < r < \frac{2}{\cos\varphi},$$

and

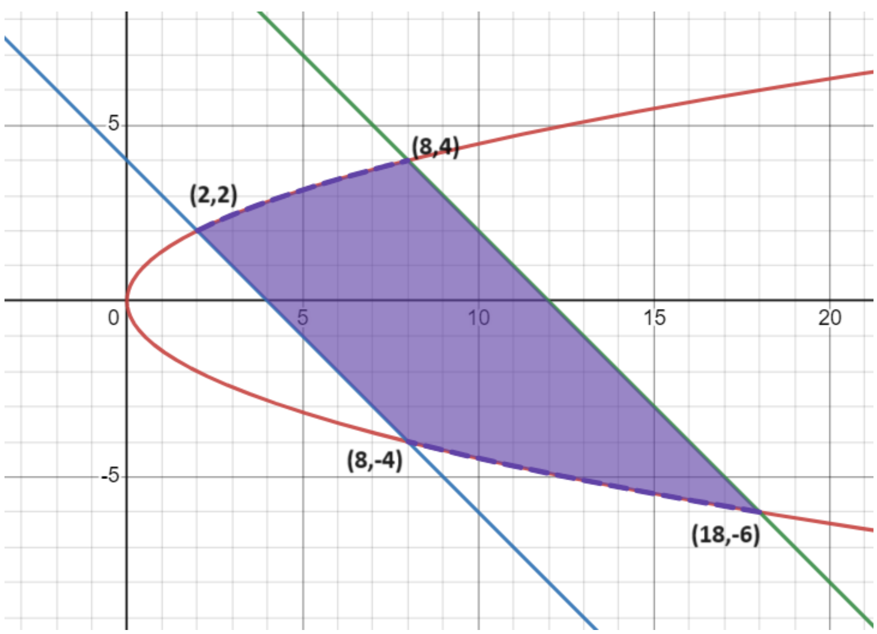
$$I = \int_{\pi/4}^{\pi/3} d\varphi \int_0^{2/\cos\varphi} f(r^2) r dr.$$

4. Calculate

$$I = \int_\Omega (x+y) dxdy,$$

where Ω is a set bounded by curves $y^2 = 2x$, $x + y = 4$, $x + y = 12$.

Solution. First, we consider the sketch of the domain of integration



This allows us to calculate the integral in the following way:

$$I = \int_2^8 dx \int_{4-x}^{\sqrt{2x}} (x+y)dy + \int_8^{18} dx \int_{-\sqrt{2x}}^{12-x} (x+y)dy =$$
$$\int_2^8 \left(x\big(\sqrt{2x} - (4-x)\big) + \frac{2x-(4-x)^2}{2} \right) dx +$$
$$\int_8^{18} \left(x\big((12-x) + \sqrt{2x}\big) + \frac{(12-x)^2 - 2x}{2} \right) dx = \frac{826}{5} + \frac{5678}{15} = \frac{8156}{15}$$

Solution 2. Consider the change $t = x + y$. Then Jacobian is equal to $J = 1$. And Ω in new coordinates is defined by inequalities

$$4 < t < 12, \quad y^2 < 2(t-y)$$

The last inequality is equivalent to $(y+1)^2 < 2t + 1$ and

$$-1 - \sqrt{2t+1} < y < -1 + \sqrt{2t+1}.$$

Hence,

$$I = \int\limits_4^{12} dt \int\limits_{-1-\sqrt{2t+1}}^{-1+\sqrt{2t+1}} t dy = 2\int\limits_4^{12} t\sqrt{2t+1}dt = \frac{2}{15}(1+2t)^{3/2}(-1+3t)\Big|_4^{12} = \frac{8156}{15}.$$

5. Calculate the integral considering the polar change

$$I = \iint_\Omega \sqrt{a^2 - x^2 - y^2}dxdy,$$

where the set Ω bounded by the loop of the lemniscate

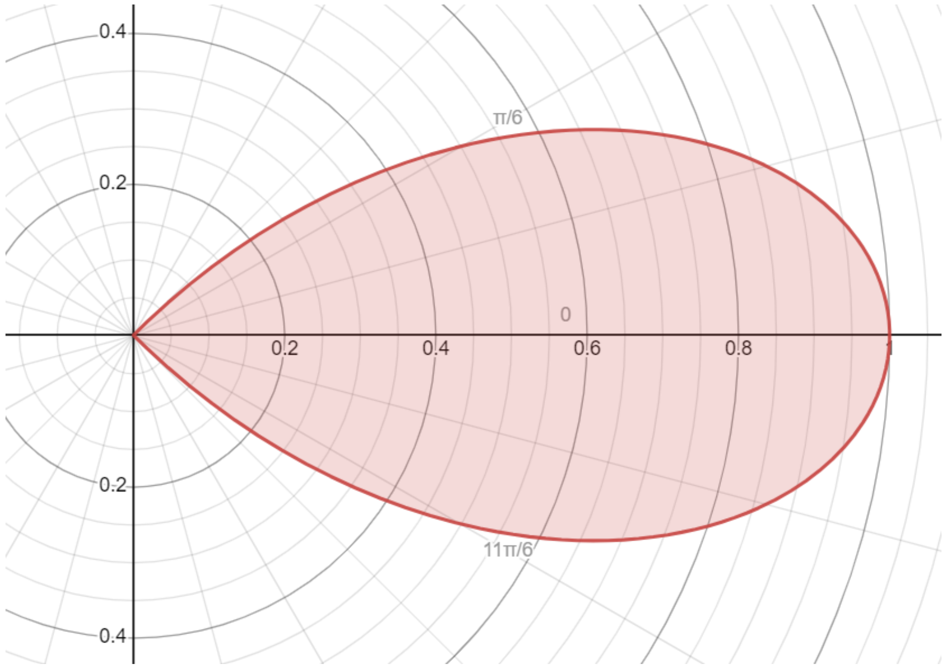
$$\left(x^2+y^2\right)^2 = a^2\left(x^2 - y^2\right), \quad x \geqslant 0.$$

Solution. Consider the polar change

$$x = ar\cos\varphi,\ y = ar\sin\varphi,\ \varphi \in [-\pi/2, \pi/2)\,,\ r > 0.$$

In these coordinate lemniscate is expressed in the following form

$$r^2 = \cos^2\varphi - \sin^2\varphi = \cos 2\varphi,\ \varphi \in (-\pi/4, \pi/4)\,.$$



The Jacobian is equal to $J = a^2 r$ and

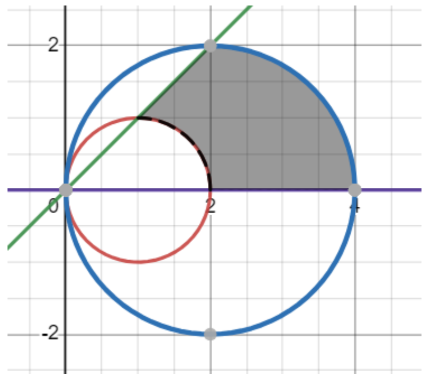
$$I = a^4 \int_{-\pi/4}^{\pi/4} d\varphi \int_0^{\sqrt{\cos 2\varphi}} r\sqrt{1-r^2}dr = \frac{2a^4}{3}\int_0^{\pi/4} -(1-r^2)^{3/2}\Big|_0^{\sqrt{\cos 2\varphi}} d\varphi =$$
$$\frac{2a^4}{3}\int_0^{\pi/4}(1-(1-\cos(2\varphi))^{3/2})d\varphi = \frac{2a^4}{3}\int_0^{\pi/4}(1-2\sqrt{2}\sin^3\varphi)d\varphi = \frac{a^4}{18}\left(20-16\sqrt{2}+3\pi\right)$$

Mathematical analysis 3. Homework 13.

1. Considering polar coordinates calculate the area of the set Ω bounded by the curves

$$x^2 + y^2 = 2x, \quad x^2 + y^2 = 4x, \quad y = x, \quad y = 0.$$

Solution. First, we sketch the set



Consider the polar change

$$x = r\cos\varphi,\ y = r\sin\varphi,\ \varphi \in [-\pi, \pi)\,,\ r > 0.$$

The curves from the statement of the theorem are

$$r = 2\cos\varphi, \quad r = 4\cos\varphi, \quad \varphi = \frac{\pi}{4}, \quad \varphi = 0.$$

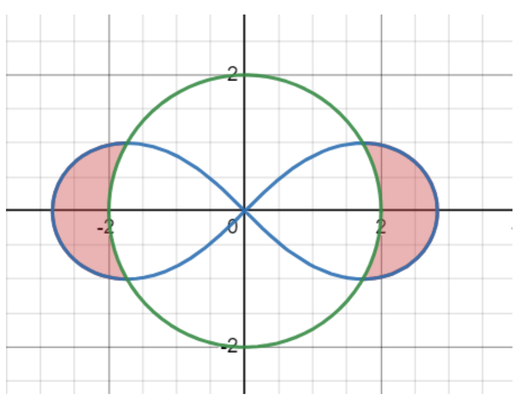
Since Jacobian of the change is $J = r$

$$\text{Area}(\Omega) = \int_0^{\pi/4} d\varphi \int_{2\cos\varphi}^{4\cos\varphi} r dr = 6\int_0^{\pi/4} \cos^2\varphi d\varphi = \frac{6 + 3\pi}{4}.$$

2. Calculate the area of the set Ω bounded by a curve

$$(x^2 + y^2)^2 = 8(x^2 - y^2), \quad x^2 + y^2 \geq 4.$$

Solution. First, we sketch the set



Consider the polar change

$$x = 2r\cos\varphi,\ y = 2r\sin\varphi,\ \varphi \in [-\pi, \pi)\,,\ r > 0.$$

The curves from the statement of the theorem are

$$r^2 = 2\cos(2\varphi), \quad r = 1.$$

Hence, $\varphi \in (-\pi/2, -5\pi/6) \cup (-\pi/6, \pi/6) \cup (5\pi/6, \pi/2)$. Since the Jacobian of the change is $J = 4r$ and the figure is symmetric with respect to $x = 0$ and $y = 0$ then

$$\text{Area}(\Omega) = 16\int_0^{\pi/6} d\varphi \int_1^{2\cos 2\varphi} r dr = 8\int_0^{\pi/6} (1 - 4\cos^2 2\varphi) d\varphi = \frac{4\pi}{3} + 2\sqrt{3}.$$

3. Calculate the area of the set Ω bounded by curves

$$x^2 = y, \quad x^2 = 4y, \quad x^3 = y^2, \quad x^3 = 2y^2.$$

Solution. Consider new coordinates

$$u = \frac{x^2}{y}, \quad v = \frac{x^3}{y^2}.$$

Expressing old varialbes via new ones we see that

$$x = \frac{u^2}{v}, \quad y = \frac{x^2}{u} = \frac{u^3}{v^2}.$$

Hence,

$$|J| = \left|\det\begin{pmatrix} x'_u & x'_v \\ y'_u & y'_v \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ \frac{3u^2}{v^2} & -\frac{2u^3}{v^3} \end{pmatrix}\right| = \frac{u^4}{v^4}$$

and

$$\Omega' = \{(u, v) : 1 < u < 4, \quad 1 < v < 2\}\,.$$

Consequently, the area of Ω is equal to

$$S(\Omega) = \iint\limits_{\Omega} dxdy = \int\limits_1^4 u^4 du \int\limits_1^2 \frac{dv}{v^4} = \frac{1}{15}(4^4 - 1)(1 - 2^{-3}) = \frac{119}{8}.$$

4. Let $a, b > 0$. Calculate the area of the set bounded by a curve

$$\sqrt[4]{\frac{x}{a}} + \sqrt[4]{\frac{y}{b}} = 1;\ x, y > 0.$$

Solution. Consider new coordinates

$$x = ar\cos^8\varphi,\ \ y = br\sin^8\varphi,\ \ \varphi \in (0, \pi/2)\,,\ \ r > 0.$$

Then the curve in new coordinates is defined by equation $r = 1$ and Jacobian is equal to

$$J = 8abr\sin^7\varepsilon\cos^7\varphi$$

Hence

$$\text{Area} = 8ab\int_0^1 rdt\int_0^{\pi/2}\sin^7\varepsilon\cos^7\varphi d\varphi = \frac{ab}{70}.$$

Mathematical analysis 3. Homework 14.

1. Calculate the volume of the set Ω bounded by the following surfaces

$$z = x^2 + y^2,\ x^2 + y^2 = x,\ x^2 + y^2 = 2x,\ z = 0.$$

Solution. Consider cylindric coordinates

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \quad r > 0,\ \varphi \in [-\pi, \pi),\ z \in \mathbb{R}.$$

Then $|J| = r$ and surfaces are expressed as

$$z = r^2,\ r = \cos\varphi,\ r = 2\cos\varphi,\ z = 0.$$

Hence,

$$\mathrm{Vol}(\Omega) = \int_{-\pi/2}^{\pi/2} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r^3 dr = \frac{15}{4}\int_{-\pi/2}^{\pi/2} \cos^4\varphi d\varphi = \frac{15}{64}(8 + 3\pi)$$

2. Let $a, b, c > 0$. Calculate the volume of the set bounded by the following surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a} + \frac{y}{b}, \quad z = 0.$$

Consider cylindric coordinates

$$x = ar\cos\varphi, \quad y = br\sin\varphi, \quad z = z, \quad r > 0,\ \varphi \in [-3\pi/4, 5\pi/4),\ z \in \mathbb{R}.$$

Then $|J| = abr$ and surfaces are expressed as

$$z = cr^2, \quad r = \cos\varphi + \sin\varphi = \sqrt{2}\cos\left(\varphi + \frac{\pi}{4}\right), \quad z = 0.$$

Let $\theta = \varphi + \frac{\pi}{4}$ then $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ since $r > 0$. Hence,

$$\mathrm{Vol}(\Omega) = abc\int_{-\pi/2}^{\pi/2} d\theta \int_{0}^{\sqrt{2}\cos\theta} r^3 dr = abc\int_{-\pi/2}^{\pi/2} \cos^4\theta d\theta = \frac{3\pi abc}{8}.$$

3. Consider all rearrangements of order of integration in

$$I = \int\limits_{-1}^{1} dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_{\sqrt{x^2+y^2}}^{1} f(x, y, z) dz$$

Solution. First, rearrange the inner integrals. The system of inequalities

$$-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}; \quad \sqrt{x^2 + y^2} \le z \le 1.$$

is equivalent to

$$|x| \le z \le 1; \quad |y| \le \sqrt{z^2 - x^2}.$$

Hence

$$I = \int_{-1}^{1} dx \int_{|y|}^{1} dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)dy. \tag{1}$$

To change order of integration by x and y notice that the system of inequalities

$$|x| \le 1; \quad |y| \le \sqrt{1-x^2}.$$

is equivalent to

$$|y| \le 1; \quad |x| \le \sqrt{1-y^2}.$$

Hence

$$I = \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx \int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)dz$$

The order of x and z is changed analogously to the first case

$$I = \int_{-1}^{1} dy \int_{|y|}^{1} dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-y^2}} f(x,y,z)dx.$$

Finally to change order of integration by z and by x in (**??**) we notice that

$$|x| \le z \le 1; \ |x| \le 1$$

if and only if

$$0 \le z \le 1; \ -z \le x \le z.$$

Hence

$$I = \int_{0}^{1} dz \int_{-z}^{z} dx \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f(x,y,z)dy.$$

Analogously,

$$I = \int_{0}^{1} dz \int_{-z}^{z} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f(x,y,z)dx.$$

4. Calculate

$$I = \iiint_{\Omega} \sqrt{x^2 + y^2} dxdydz,$$

where Ω is bounded by the surfaces

$$x^2 + y^2 = z^2, \quad z = 1.$$

Solution. Consider cylindric coordinates

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \quad r > 0,\ \varphi \in [-\pi, \pi),\ z \in \mathbb{R}.$$

Then $|J| = r$ and surfaces are expressed as

$$r = |z|, \quad z = 1.$$

Hence,

$$I = \int_{-\pi}^{\pi} d\varphi \int_0^1 dz \int_0^z r^2 dr = \frac{2\pi}{3} \int_0^1 z^3 dz = \frac{\pi}{6}.$$

Mathematical analysis 3. Homework 15.

1. Calculate $F'(t)$ if
$$F(t) = \iiint\limits_{x^2+y^2+z^2\le t^2} f(x^2+y^2+z^2)dxdydz,$$
where f is a differentiable function.

Solution. Consider spherical coordinates Let $r>0, \varphi\in[0,2\pi), \psi\in[-\pi/2,\pi/2)$ and
$$x = r\cos\varphi\cos\psi, \quad y = r\sin\varphi\cos\psi, \quad z = r\sin\psi.$$
Then $J = r^2\cos\psi$ and
$$F(t) = 2\pi\int_{-\pi/2}^{\pi/2}\cos\psi d\psi\int_0^t r^2 f(r^2)dr = 4\pi\int_0^t r^2 f(r^2)dr$$
and $F'(t) = 4\pi t^2 f(t^2)$.

2. Let $a>0$. Calculate the volume of a set bounded by surfaces
$$az = x^2+y^2, \quad z = \sqrt{x^2+y^2}.$$

Solution. If
$$x^2+y^2 \le az \le a\sqrt{x^2+y^2}$$
then $x^2+y^2\le a^2$. Consider cylindric coordinates
$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \quad r>0,\ \varphi\in[-\pi,\pi),\ z\in\mathbb{R}.$$
Then $J = r$ and
$$\mathrm{Vol}(\Omega) = 2\pi\int_0^a rdr\int_{r^2/a}^r dz = 2\pi\int_0^a\left(r^2-\frac{r^3}{a}\right) = \frac{2\pi a^4}{12}.$$

3. Calculate the volume of a set bounded by the surface
$$x^2+y^2+z^4 = 1.$$

Solution. Consider cylindric coordinates
$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \quad r>0,\ \varphi\in[-\pi,\pi),\ z\in\mathbb{R}.$$
Then $|J| = r$ and $|z|\le(1-r^2)^{1/4}$. Consequently,
$$\mathrm{Vol}(\Omega) = 4\pi\int_0^1 r(1-r^2)^{1/4}dr = \frac{16\pi}{5}.$$

4. Calculate

$$I = \iiint_\Omega \sqrt{x^2+y^2+z^2}dxdydz,$$

where $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leqslant z\}$.

Solution. Consider spherical coordinates Let $r > 0, \varphi \in [0, 2\pi), \psi \in [-\pi/2, \pi/2)$ and

$$x = r\cos\varphi\cos\psi, \quad y = r\sin\varphi\cos\psi, \quad z = r\sin\psi.$$

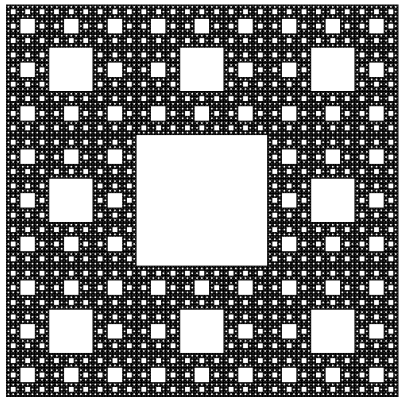
Then $J = r^2\cos\psi$ and Ω is defined by the following inequality

$$r \leq \sin\psi, \ \psi \in (0, \pi/2).$$

Hence

$$I = \int_0^{\frac{\pi}{2}} \cos\psi d\psi \int_0^{2\pi} d\varphi \int_0^{\sin\psi} r^3 dr = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^4\psi\cos\psi d\psi = \frac{\pi}{10}\sin^5\psi\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{10}$$

1. Consider a square $[0,1]^2$. . The square is cut into 9 congruent subsquares in a 3-by-3 grid, and the central subsquare is removed. The same procedure is then applied recursively to the remaining 8 subsquares, ad infinitum. The obtained set S is called Sierpinski carpet. Calculate the Lebesgue measure of S.



Solution. At the $n-$th step we remove 8^{n-1} square of area 3^{-2n}. Hence, by countable additivity of the measure

$$m_2(S) = 1 - \sum_{n=1}^{\infty} \frac{8^{n-1}}{9^n} = 1 - \frac{1}{9}\frac{1}{1-8/9} = 0.$$

2. Can the unbounded Lebesgue measurable subset of $\mathbb{R}$ have finite positive Lebesgue measure? **Solution.** Yes. Consider a set

$$E = \bigcup_{n=1}^{\infty}[n, n+1/2^n].$$

This set is unbounded and has measure $m(E) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} = 1.$

3. Assume that E_1 and E_2 are $\mu-$measurable. Prove that $E_1 \cup E_2$ are measurable and

$$\mu(E_1 \cup E_2) = \mu E_1 + \mu E_2 - \mu(E_1 \cap E_2).$$

Solution. Notice that

$$E_j = (E_j \setminus (E_1 \cap E_2)) \cup E_1 \cap E_2$$

Hence,

$$\mu(E_j \setminus (E_1 \cap E_2)) = \mu(E_j) - \mu(E_1 \cap E_2).$$

Also

$$E_1 \cup E_2 = (E_1 \setminus (E_1 \cap E_2)) \cup E_1 \cap E_2 \cup (E_2 \setminus (E_1 \cap E_2)).$$

Consequently,

$$\mu(E_1 \cup E_2) = \mu(E_1 \setminus (E_1 \cap E_2)) + \mu(E_1 \cap E_2) + \mu(E_2 \setminus (E_1 \cap E_2)) = \mu E_1 + \mu E_2 - \mu(E_1 \cap E_2).$$

4. Assume that $A_k \subset [0,1]$ are Lebesgue measurable sets such that

$$mA_1 + mA_2 + \ldots + mA_n > n - 1.$$

Prove that $m\left(\bigcap_{k=1}^{n} A_k\right) > 0$.

Solution.

$$m\left(\bigcap_{k=1}^{n} A_k\right) = 1 - m\left([0,1] \setminus \bigcap_{k=1}^{n} A_k\right) = 1 - m\left(\bigcup_{k=1}^{n} ([0,1] \setminus A_k)\right) \geq$$

$$1 - \left(\sum_{k=1}^{n} m\left([0,1] \setminus A_k\right)\right) = 1 - \left(n - \sum_{k=1}^{n} mA_k\right) > 0.$$

5. Assume that $E \subset \mathbb{R}$ is measurable and has positive measure. Prove that there exist $x, y \in E$ such that $x - y \in \mathbb{Q}$.

Solution. Assume the converse. Since $\mu E > 0$ then there exists $A > 0$ such that $\mu(E \cap [-A, A]) = \delta > 0$. Without loss of generality assume that $E \subset [-A, A]$. Let $E_q = \{x + q : x \in E\}$. Then $\mu E_q = \mu E$ and for every $q_1, q_2 \in \mathbb{Q}$

$$E_{q_1} \cap E_{q_2} = \varnothing, \quad q_1 \neq q_2.$$

Since $E \subset [-A, A]$ then $E_q \subset [-A, A+1]$ if $q \in [0,1]$. Hence,

$$\bigcup_{q \in \mathbb{Q} \cap [0,1]} E_q \subset [-A, A+1]$$

and

$$\mu \bigcup_{q \in \mathbb{Q} \cap [0,1]} E_q < 2A + 1.$$

At the same time sets E_q are mutually disjoint and have the same positive measure. Consequently,

$$\mu \bigcup_{q \in \mathbb{Q} \cap [0,1]} E_q = \sum_{q \in \mathbb{Q} \cap [0,1]} \mu E_q = +\infty,$$

which leads to a contradiction.

Mathematical analysis 3. Homework 18.

1. Assume that $f : [a, b] \to \mathbb{R}$ is differentiable. Prove that f' is a (Lebesgue) measurable function.

   Solution. Consider a sequence

$$f_n(x) = \begin{cases} n(f(x+1/n) - f(x)), & x \in [a, b-1/n]; \\ 0, & x \in [b-1/n, b]. \end{cases}$$

   Then f_n is measurable and $\lim\limits_{n\to\infty} f_n(x) \to f'(x)$, $x \in [a, b)$. Consequently, f' is measurable on $[a, b)$ as a pointwise limit of measurable functions (and, consequently, on $[a, b]$).

2. Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Prove that f is measurable iff $f^{-1}(\{-\infty\})$, $f^{-1}(\{+\infty\})$ are measurable sets and f is measurable on Y.

   Solution. If f is measurable then Y, $f^{-1}(\{-\infty\})$, $f^{-1}(\{+\infty\})$ are measurable sets as intersections of Lebesgue sets of function f.

   Assume that $f^{-1}(\{-\infty\})$, $f^{-1}(\{+\infty\})$ are measurable sets and f is measurable on Y. Then Lebesgue sets of f are measurable since

$$E(f > a) = E(f|_Y > a) \cup f^{-1}(\{+\infty\});\ E(f \ge a) = E(f|_Y \ge a) \cup f^{-1}(\{+\infty\})$$

$$E(f < a) = E(f|_Y < a) \cup f^{-1}(\{-\infty\});\ E(f \le a) = E(f|_Y \le a) \cup f^{-1}(\{-\infty\}).$$

3. Let $\{f_n\}$ is a sequence of measurable functions on X. Prove that

$$E = \{x : \lim f_n(x) \text{ exists}\}$$

   is a measurable set.

   Solution. Consider a sequence $g_n = f_n\chi_E$. Then every g_n is measurable and $\{g_n\}$ converges pointwise on X. Let $g = \lim g_n$. Then

$$E = \{x : g(x) \neq 0\} \cup \{x : \lim f_n(x) = 0\}.$$

   The set $\{x : g(x) \neq 0\}$ is measurable since g is measurable function (as a limit of measurable functions). The set

$$E_0 = \{x : \lim f_n(x) = 0\} = \{x : \forall\varepsilon > 0\ \exists N \in \mathbb{N}\ :\ \forall n \ge N\ \ |f_n(x)| \le \varepsilon\} = \bigcap_{K=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{x : |f_n(x)| < \frac{1}{K}\right\}.$$

   is measurable since the set

$$\left\{x : |f_n(x)| < \frac{1}{K}\right\} = \left\{x : f_n(x) > -\frac{1}{K}\right\} \bigcap \left\{x : f_n(x) < \frac{1}{K}\right\}$$

   is measurable.

4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is monotone then f is measurable.

Solution. Assume that f is monotonic. Then $E(f > a) = f^{-1}(a, +\infty)$ is an interval and is, consequently, measurable.

5*. Assume that μ is a Borel measure. Prove that for any finite family of segments $I_1, \dots, I_n$ there exists a subfamily $\{J_1, \dots, J_m\}$ of mutually disjoint intervals such that

$$\sum_{k=1}^{m} \mu(J_k) \geq \frac{1}{2}\mu\left(\sum_{k=1}^{n} I_k\right).$$

Solution. First, we may assume that none of the intervals is contained in the union of others (otherwise we remove such intervals). Enumerate the intervals $I_k = (a_k, b_k)$ in ascending order of left endpoints

$$a_1 \leq a_2 \leq \ldots \leq a_n.$$

Then $b_k > b_{k+1}$ (otherwise $I_k \subset I_{k+1}$) and $a_{k+1} > b_{k-1}$ (otherwise $I_k \subset I_{k-1} \cup I_{k+1}$). Hence,

$$I_k \cap I_{k+2} = \varnothing, \ k \in \mathbb{N},$$

and intervals with even and of numbers form systems of mutually disjoint intervals. Also,

$$\mu\left(\sum_{k=1}^{n} I_k\right) \leq \sum_{k=1}^{n} \mu I_k \leq \sum_{k \text{ is even}} \mu I_k + \sum_{k \text{ is odd}} \mu I_k.$$

One of the sums in the right-hand side is greater or equal then half of the left-hand side. $\square$

Mathematical analysis 3. Homework 18.

1. Calculate Lebesgue integral $\int_0^1 f(x)dx$, where

$$f(x) = \begin{cases} x^2, & x \in \left(\frac{1}{3}, +\infty\right) \setminus \mathbb{Q}; \\ x^2, & x \in \left(-\infty, \frac{1}{3}\right) \setminus \mathbb{Q}; \\ 0, & x \in \mathbb{Q}. \end{cases}$$

Solution. Since $\mu\mathbb{Q}$. The integral coincides with the integral of the function

$$f_1(x) = \begin{cases} x^2, & x \in \left(\frac{1}{3}, +\infty\right); \\ x^2, & x \in \left(-\infty, \frac{1}{3}\right). \end{cases}$$

This function is Riemann integrable, and the Lebesgue integral coincides with Riemann integral:

$$\int\limits_0^1 f(x)dx = \int\limits_0^{1/3} x^2dx + \int\limits_{1/3}^1 x^3dx = \frac{1}{3^4} + \frac{1}{4}\left(1 - \frac{1}{3^4}\right) = \frac{7}{27}.$$

2. Construct a sequence of integrable functions $f_n : [0, 1] \to \mathbb{R}$ such that

- The limit $f(x) = \lim\limits_{n\to\infty} f_n(x)$ exists a.e.;
- $\lim\limits_{n\to\infty} \int\limits_0^1 f_n \to \int\limits_0^1 f$;
- There is no integrable function G such that $|f_n| \leq G$ a.e. for every $n \in \mathbb{N}$.

Solution. Consider a sequence

$$f_n = \begin{cases} 2^n/n, & x \in (2^{-n-1}, 2^{-n}); \\ 0, & x \notin (2^{-n-1}, 2^{-n}). \end{cases}$$

Then $f_n \to 0$ a.e. and $\int_0^1 f_n = \frac{1}{n} \to 0$.

Assume that $f_n \leq G$ then $\sum\limits_{n=1}^{\infty} f_n \leq G$ and

$$\int_0^1 G \geq \sum_{n=1}^{\infty} \int_0^1 f_n = \sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

3. Provide an example of function f such that the integral $\int_0^1 f$ exists as improper integral while f is not Lebesgue integrable.

Solution. Let
$$f(x) = \begin{cases} \frac{1}{x}, & \cos\frac{1}{x} x \in (0,1]; \\ 0, & x = 0. \end{cases}.$$
This function has improper integral on $[0,1]$. Indeed,

$$\int_0^{\to 1} \frac{1}{x}\cos\frac{1}{x}dx = \lim_{t\to+0}\int_t^1 \frac{1}{x}\cos\frac{1}{x}dx =$$
$$\lim_{t\to+0}\int_{\frac{1}{t}}^1 z\cdot\cos z\cdot\left(-\frac{1}{z^2}\right)dz == \lim_{t\to+0}\int\limits_1^{\frac{1}{t}}\frac{\cos z}{z}dz = \int\limits_1^{+\infty}\frac{\cos z}{z}dz,$$

and the improper integral $\int_1^{+\infty}\frac{\cos z}{z}dz$ converges by Dirichlet's test.

The Lebesgue integrability of f is equivalent to integrability of $|f|$. Consider the following estimate,
$$\frac{1}{x}\left|\cos\frac{1}{x}\right| \geq \frac{1}{x}\cos^2\frac{1}{x} = \frac{1}{2x} - \frac{1}{2x}\cos\frac{2}{x}.$$
The second term in the right hand side is integrable, while $1/x$ is not. Hence, f is not (Lebesgue) integrable.

4. Assume that the integral $\int_E fg dx$ exists and is finite for every $g \in L(E)$. Prove that f is bounded a.e.

Solution. Assume that for every $M > 0$ the set $E_M = E(|f| > M)$ has positive measure. WLOG assume that $f \geq 0$ and that μE is finite. Let $A_n = E_n \setminus E_{n+1}$, $n \in \mathbb{N}$. Then A_n is measurable and

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Tasks

2. Find the limit:

1. $\lim_{\alpha\to 0}\int_0^1 \sqrt{1+\alpha^2x^4}dx = 1$;

2. $\lim_{\alpha\to 0}\int_{-1}^1 \sqrt{x^2+\alpha^2}dx = 1$;

3. $\lim_{\alpha\to 1}\int_2^4 \frac{xdx}{1+x^2+\alpha^6} = (\ln 3)/2$;

4. $\lim_{\alpha\to 1}\int_0^1 x^2e^{\alpha x^3}dx = (e-1)/3$;

5. $\lim_{\alpha\to 0}\int_0^\pi x\cos(1+\alpha)xdx = -2$

13. Calculate $I'(\alpha)$, if:

1. $I(\alpha)=\int_0^1 \sin(\alpha x)dx$; $I'(\alpha)=\frac{\alpha\sin\alpha+\cos\alpha-1}{\alpha^2}$

2. $I(\alpha)=\int_1^3 \frac{\cos\left(\alpha x^3\right)}{x}dx$; $I'(\alpha)=\frac{\cos 27\alpha-\cos\alpha}{3\alpha}$

3. $I(\alpha)=\int_1^2 e^{\alpha x^2}\frac{dx}{x}$; $I'(\alpha)=\frac{e^{4\alpha}-e^{\alpha}}{2\alpha}$

4. $I(\alpha)=\int_2^3 \operatorname{ch}\left(\alpha^4x^2\right)\frac{dx}{x}$; $I'(\alpha)=\frac{2\left(\operatorname{ch}9\alpha^4-\operatorname{ch}4\alpha^4\right)}{\alpha}$.

14. Calculate $\Phi'(\alpha)$, If:

1. $\Phi(\alpha)=\int_0^\alpha \frac{\ln(1+\alpha x)}{x}dx$;

2. $\Phi(\alpha)=\int_\alpha^{2\alpha} \frac{\sin\alpha x}{x}dx$

3. $\Phi(\alpha)=\int_{\sin\alpha}^{\cos\alpha} e^{\alpha\sqrt{1-x^2}}dx$;

4. $\Phi(\alpha)=\int_{3\alpha}^{\alpha^2} e^{\alpha x^2}dx$;

5. $\Phi(\alpha)=\int_{\cos\alpha}^{\sin\alpha} e^{\alpha^4x^2}dx$;

6. $\Phi(\alpha)=\int_{e^{-\alpha}}^{e^\alpha} \ln\left(1+\alpha^2x^2\right)\frac{dx}{x}$;

7. $\Phi(\alpha)=\int_{\alpha e^{-\alpha}}^{\alpha e^\alpha} \ln\left(1+\alpha^2x^2\right)dx$

8. $\Phi(\alpha)=\int_{\operatorname{ch}\alpha}^{\operatorname{sh}\alpha} \ln\left(1+x^2+\alpha^2\right)dx$.

Answers

1. $\Phi'(\alpha)=\left(2\ln\left(1+\alpha^2\right)\right)/\alpha$;

2. $\Phi'(\alpha)=2\left(\sin 2\alpha^2-\sin\alpha^2\right)/\alpha$;

3. $\Phi'(\alpha)=\int_{\sin\alpha}^{\cos\alpha}\sqrt{1-x^2}e^{\alpha\sqrt{1-x^2}}dx-\sin\alpha\cdot e^{\alpha|\sin\alpha|}-\cos\alpha\cdot e^{\alpha|\cos\alpha|}$;

4. $\Phi'(\alpha) = \int_{3\alpha}^{\alpha} x^2 e^{\alpha x^2}\, dx + 2\alpha e^{\alpha^2} - 3e^{9\alpha^3}$;

5. $\Phi'(\alpha) = 4\alpha^3 \int_{\cos\alpha}^{\sin\alpha} x^2 e^{\alpha^4 x^2}\, dx + \cos\alpha \cdot e^{\alpha^4 \sin^2\alpha} + \sin\alpha \cdot e^{\alpha^4 \cos^2\alpha}$;

6. $\Phi'(\alpha) = \frac{1}{\alpha} \ln \frac{1+\alpha^2 e^{2\alpha}}{1+\alpha^2 e^{-2\alpha}} + \ln\left(1 + 2\alpha^2 \operatorname{ch} 2\alpha + \alpha^4\right)$;

7. $\Phi'(\alpha) = 4 \operatorname{sh}\alpha + \frac{2}{\alpha^2}\left(\operatorname{arctg}\left(\alpha^2 e^{-\alpha}\right) - \operatorname{arctg}\left(\alpha^2 e^{\alpha}\right)\right) + (\alpha+1)e^{\alpha} \ln\left(1 + \alpha^4 e^{2\alpha}\right) -$
$-(\alpha - 1)e^{-\alpha} \ln\left(1 + \alpha^4 e^{-2\alpha}\right)$;

8. $\Phi'(\alpha) = \frac{2\alpha}{\sqrt{1+\alpha^2}}\left(\operatorname{arctg} \frac{\operatorname{sh}\alpha}{\sqrt{1+\alpha^2}} - \operatorname{arctg} \frac{\operatorname{ch}\alpha}{\sqrt{1+\alpha^2}}\right) + \ + \operatorname{ch}\alpha \ln\left(\operatorname{ch}^2\alpha + \alpha^2\right) -$
$- \operatorname{sh}\alpha \ln\left(\operatorname{ch}^2\alpha + \alpha^2 + 1\right)$.