Equations of Mathematical Physics Homework 4

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1 September 9th

1.1 Find solutions of differential equation

$$x'' + x' + 6x = 3(cos3t - sin3t) \quad x(0) = 0, x'(0) = 3$$

Solution:
Denote that $x(t) \longleftrightarrow X(p) \quad f(t) = 3(cos3t - sin3t) \longleftrightarrow F(p)$, then we have:

$$F(p) = 3(\frac{p}{p^2+9} - \frac{3}{p^2+9}) = \frac{3(p-3)}{p^2+9}$$
$$A(p) = p^2 + p + 6 \quad B(p) = 3$$
$$X(p) = \frac{F(p)+B(p)}{A(p)} = \frac{3(p-3)+3(p^2+9)}{(p^2+9)(p^2+p+6)} = \frac{3}{p^2+9}$$

Thus, by the table of original and image of elementary functions, $x(t) = sin3t$.

2 September 11th

2.1 Find solutions of differential system

$$\begin{cases} x'(t) = -x(t) + y(t) + e^t \\ y'(t) = x(t) - y(t) + e^t \end{cases}$$
$$x(0) = y(0) = 1$$

Solution:
Denote that $x(t) \longleftrightarrow X(p) \quad y(t) \longleftrightarrow Y(p)$, then we have:

$$x'(t) \longleftrightarrow pX(p) - X(0) = pX(p) - 1$$
$$y'(t) \longleftrightarrow pY(p) - Y(0) = pY(p) - 1$$

Apply Laplace transform to both sides, we have:

$$\begin{cases} pX(p) - 1 + X(p) - Y(p) = \dfrac{1}{p-1} \\ pY(p) - 1 - X(p) + Y(p) = \dfrac{1}{p-1} \end{cases}$$

Rewrite these equations, we get:

$$\begin{cases} (p+1)X(p) - Y(p) = \dfrac{p}{p-1} \\ -X(p) + (p+1)Y(p) = \dfrac{1}{p-1} \end{cases}$$

Solve this linear system,

$$\Delta = \begin{vmatrix} p+1 & -1 \\ -1 & p+1 \end{vmatrix} = p^2 + 2p$$

$$\Delta_x = \begin{vmatrix} \frac{p}{p-1} & -1 \\ \frac{1}{p-1} & p+1 \end{vmatrix} = \frac{p^2+p+1}{p-1} \qquad \Delta_y = \begin{vmatrix} p+1 & \frac{p}{p-1} \\ -1 & \frac{1}{p-1} \end{vmatrix} = \frac{2p+1}{p-1}$$

$$X(p) = \frac{p^2+p+1}{p(p+2)(p-1)} \qquad Y(p) = \frac{2p+1}{p(p+2)(p-1)}$$

Then we restore originals from the images, we get the solutions:

$$x(t) = \sum \frac{p^2+p+1}{3p^2+2p-2} e^{pt} \Big|_{p=0,-2,1} = -\frac{1}{2} + e^t + \frac{3}{4} e^{-2t}$$

$$y(t) = \sum \frac{2p+1}{3p^2+2p-2} e^{pt} \Big|_{p=0,-2,1} = -\frac{1}{2} + e^t - \frac{3}{4} e^{-2t}$$