No. 李开言 2022111221

Date · 2 · 28

Higher Algebra

2.28

Exercise 2.3. $v_1=(1,1,1,-1)$ $v_2=(0,-1,-1,1)$ $v_3=(1,1,0,-1)\in\mathbb{R}^4$

a. $v=(0,1,0,1)$ we need to show whether ~~can~~ v can be written as a combination of v_1,v_2,v_3 or not.

$$xv_1+yv_2+zv_3=v. \Rightarrow \begin{cases} x+z=0 \\ x-y+z=1 \\ x-y=0 \\ -x+y-z=1 \end{cases}$$

no solution.

v doesn't belong to $\text{span}(v_1,v_2,v_3)$

b. $v=(1,0,1,0)$ similar procedure as a.

$$\begin{cases} x+z=1 \\ x-y+z=0 \\ x-y=0 \\ -x+y-z=0 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=1 \\ z=-1 \end{cases}$$

thus. v belongs to $\text{span}(v_1,v_2,v_3)$.

Exercise 2.4. $\mathbb{R}[t]_2=\{f\in\mathbb{R}[t] \mid \deg f\le 2\}$. to show $\{f_1,f_2,f_3\}$ is a span, they must can represent any at^2+bt+c when. $a,b,c\in\mathbb{R}$.

a. $xf_1+yf_2=at^2+bt+c$ (a,b,c are arbitrary).

$$\begin{cases} x-y=a. \\ x=b. \\ x+y=c. \end{cases} \Rightarrow b+(b-a)=c.$$

a,b,c must subject to the equation $2b=a+c$. can't be arbitrary.

thus. $f_1.f_2$ is not span

b. $xf_1+yf_2+zf_3=at^2+bt+c$.

$$\begin{cases} x-y+z=a \\ x+z=b. \\ x+y=c \end{cases} \Rightarrow \begin{cases} x=a+c-b \\ y=b-a \\ z=2b-a-c \end{cases}$$

for any. $a.b.c$ we can always find $x.y.z$ satisfied the condition.

c. $xf_1+yf_2+zf_3=at^2+bt+c$

$$\begin{cases} x-y+2z=a \\ x+z=b \\ x+y=c \end{cases} \Rightarrow \frac{c+a}{2}=b,$$

$a.b.c$ can't be arbitrary. i.e. not span.

Exercise 2.5.

$\forall v\in V$, since. $\{v_1,v_2,v_3\}$ is a spanning set. Then there exists $a_1,a_2,a_3\in\mathbb{R}$.

$v=a_1v_1+a_2v_2+a_3v_3$.

let $v=x(v_1+v_2)+y(v_2-v_3)+z(v_1+v_2+v_3)$

$$\Rightarrow \begin{cases} x+z=a_1 \\ x+y+z=a_2 \\ z-y=a_3 \end{cases} \Rightarrow \begin{cases} x=2a_1-a_2-a_3 \\ y=a_2-a_1 \\ z=a_2+a_3-a_1 \end{cases}$$

Also, we have. $2a_1-a_2-a_3\in\mathbb{R}$, a_2-a_1, $a_2+a_3-a_1\in\mathbb{R}$

Thus. $\{v_1+v_2, v_2-v_3, v_1+v_2+v_3\}$ is also a spanning set of V.

i.e. $v=(2a_1-a_2-a_3)(v_1+v_2)+(a_2-a_1)(v_2-v_3)+(a_2+a_3-a_1)(v_1+v_2+v_3)$.

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exercise	2.3	2.4	2.5	2.6	2.7	2.8
score	2	2	2	2	2	2

Exercise 2.6.

1° $\mathrm{Span}(v_1, \dots v_n) + \mathrm{Span}(v_1', \dots v_m') \subset \mathrm{Span}(v_1, \dots v_n, v_1' \dots v_m')$.

$\forall\ v \in \mathrm{span}(v_1, \dots v_n)\quad v' \in \mathrm{span}(v_1', \dots v_m')$

$v + v' = a_1v_1 + \cdots + a_nv_n + a_1'v_1' + \cdots + a_m'v_m'$ since $a_1, \dots a_n, a_1' \dots a_m' \in \mathbb{R}$.

thus. $v + v' \in \mathrm{span}(v_1, \dots v_n, v_1' \dots v_m')$.

Since $v. v'$ are arbitrary, the "$\subset$" holds.

2° "$\supset$" the inverse inclusion

$\forall\ u + u' \in \mathrm{Span}(v_1 \dots v_n, v_1' \dots v_m')$.

$u + u' = b_1v_1 + \cdots b_nv_n + b_1'v_1' + \cdots + b_m'v_m'$ $b_1, \dots b_n, b_1' \dots b_m' \in \mathbb{R}$.

i.e. $u = \sum_{i=1}^{n} b_iv_i \qquad u' = \sum_{j=1}^{m} b_j'v_j'$ #

thus $u \in \mathrm{Span}(v_1 \dots v_n)$ $u' \in \mathrm{span}(v_1' \dots v_m')$. then, "$\supset$" holds.

Exercise 2.7.

proof: ① $U_1 \cup U_2$ is a subspace $\Rightarrow U_1 \subset U_2$ or $U_2 \subset U_1$

$\exists\ u_1 \in U_1$ and $u_2 \in U_2$. $u_1, u_2 \in U_1 \cup U_2$.

By the definition of subspace. $u_1 + u_2 \in U_1 \cup U_2$

Assume that. $U_1 \not\subset U_2$. and $U_2 \not\subset U_1$.

then $u_1 \notin U_2$. $u_1 + u_2 \notin U_2$ (otherwise. $u_1 + u_2 + (-u_2) \in U_2$. i.e. $u_1 \in U_2$)

$u_2 \notin U_1$. $u_1 + u_2 \notin U_1$ (similarly)

thus. $u_1 + u_2 \notin U_1 \cup U_2$

② $U_1 \subset U_2$ or $U_2 \subset U_1 \Rightarrow U_1 \cup U_2$ is a subspace

without loss of generality. we assume. $U_1 \subset U_2$.

$U_1 \cup U_2 = U_2$ is a subspace.

Exercise 2.8

proof:

$\forall\ v = (x, y, z) \in F^3$. $v = (a, a, 0) + (0, c, d)$. $\begin{cases} a = x \\ a + c = y \\ d = z \end{cases} \Rightarrow \begin{cases} a = x \\ c = y - x \\ d = z. \end{cases}$ the linear system has unique solution

thus. $V = U_1 \oplus U_3$

$\forall\ v = x', y', z' \in F^3$. $v = (b, 0, 0) + (0, c, d)$ $\begin{cases} b = x' \\ c = y' \\ d = z' \end{cases}$ the linear system has unique solution

thus. $V = U_2 \oplus U_3$.

Higher Algebra.

3.2. (Week 2. Thu.)

Exercise 2.9.

proof: by the equation $nx=0 \Rightarrow x=0$.

we can claim that. $U_1 \cap U_2 = \{0\}$.

Now we need to show. U_1, U_2 are subspace:

$\forall u_1 = (x_1, \dots x_n)$ s.t. $\sum_{i=1}^{n} x_i = 0$.

$u_1' = (x_1', \dots x_n')$ s.t. $\sum_{j=1}^{n} x_j = 0$. $u_1, u \in U_1$.

$u_1 + u_1' = (x_1 + x_1', \dots x_n + x_n')$

$\sum_{i=1}^{n} x_i + x_i' = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i' = 0 + 0 = 0$.

$\forall a \in F$. $au_1 \overset{\text{def.}}{=} (ax_1, \dots ax_n)$.

$\sum_{i=1}^{n} ax_i = a \cdot \sum_{i=1}^{n} x_i = a \cdot 0 = 0$ Thus. U_1 is a subspace of U_0

$\forall u_2 = (x, \dots, x)$ $u_2' = (x', \dots x') \in U_2$.

$u_2 + u_2' = (x + x', \dots x + x')$.

since $\begin{cases} x' = x' \\ x = x \end{cases} \Rightarrow \{ x + x' = x + x'$. $u_2 + u_2' \in U_2$

$\forall a \in F$. $au_2 = (ax, \dots ax)$. $\begin{cases} x = x \\ a \in F \end{cases} \Rightarrow ax = ax$. $au_2 \in U_2$.

Thus. U_2 is a subspace.

Then we show $V = U_1 + U_2$.

1° $V \subset U_1 + U_2$.

$\forall v \in V$. $v = (v_1, \dots v_n). = \left(\frac{\sum_{i=1}^{n} v_i}{n}, \dots, \frac{\sum_{i=1}^{n} v_i}{n} \right) + \left(v_1 - \frac{\sum_{i=1}^{n} v_i}{n}, \dots, v_n - \frac{\sum_{i=1}^{n} v_i}{n} \right)$

Clearly, the former one $\in U_2$. the latter one $\in U_1$.

2°. $U_1 + U_2 \subset V$. Trivially by proposition 2.5.

Since $V = U_1 + U_2$, $U_1 \cap U_2 = \{0\}$. by proposition 2.6.

We have $U_1 \oplus U_2 = V$

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Exercise 2.10

proof: First we need to show. U_1 and U_2 are subspace of V.

$\forall f_1(t), f_2(t) \in U_1$. i.e. $f_1(t) = f_2(-t)$, $f_1(t) = f_2(-t)$ $t \in \mathbb{R}$.

addition. $(f_1+f_2)(t) = f_1(t) + f_2(t) = f_1(-t) + f_2(-t) = (f_1+f_2)(-t)$. $\Rightarrow f_1+f_2 \in U_1$

scalar multiplication: $\forall a \in F$. $(af_1)(t) = a \cdot f_1(t) = a f_1(-t) = (af_1)(-t)$. $\Rightarrow af_1 \in U_1$.

Thus. U_1 is a subspace.

$\forall f_3(t), f_4(t) \in U_2$. i.e. $t \in \mathbb{R}$. s.t. $f_3(-t) = -f_3(t)$ $f_4(-t) = -f_4(t)$

addition: $(f_3+f_4)(-t) = f_3(-t) + f_4(-t) = -(f_3(t) + f_4(t)) = -(f_3+f_4)(t)$. $\Rightarrow f_3+f_4 \in U_2$

scalar multiplication. $(af_3)(-t) = a(f_3(-t)) = a(-f_3(t)) = -af_3(t) = -(af_3)(t)$.

$\Rightarrow af_3 \in U_2$

Thus. U_2 is a subspace.

Then we need to prove $V = U_1 + U_2$.

1°. $V \subset U_1 + U_2$.

$\forall f \in V$. i.e. $f \in \mathbb{R}[x]$ and $\deg f \le n$.

we let $f = \sum_{i\ge 1}^{n} a_i x^i = \sum_{k=0}^{[\frac{n}{2}]} a_k x^{2k} + \sum_{k=1}^{[\frac{n}{2}]+\frac{1-(-1)^n}{2}} a_k x^{2k-1}$

clearly, the former ~~plo~~ polynomial $\in U_1$. the latter one $\in U_2$.

$\therefore f \in U_1 + U_2$. since f is arbitrary. $V \subset U_1 + U_2$

2° the inverse inclusion is trivial by Pro 2.5.

Finally we need to show ~~U_1~~ $U_1 \cap U_2 = \{0\}$.

$\forall f \in U_1 \cap U_2$: i.e. $(f \in U_1) \wedge (f \in U_2)$.

i.e. $\begin{cases} f(t) = f(-t) \\ f(-t) = -f(t). \end{cases}$ $\Rightarrow f(t) = -f(t)$. $\Rightarrow f(t) = 0$ $(t \in \mathbb{R})$. (since t is arbitrary then $f=0$).

i.e. $U_1 \cap U_2 = \{0\}$

By Pro 2.6, we have $U_1 \cap U_2 = \{0\}$ and $V = U_1 + U_2$. we can claim. $V = U_1 \oplus U_2$

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Exercise 2.11

proof: I. First shows. $U_1 . U_2 . U_3$ are subspace of V

(1) $\forall u_1 = \left\{ \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix} \mid x_1 \in F \right\}$ $\forall u_1' = \left\{ \begin{pmatrix} 0 & x_1' \\ -x_1' & 0 \end{pmatrix} \mid x_1' \in F \right\} \in U_1$.

$$u_1 + u_1' = \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & x_1' \\ -x_1' & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1+x_1' \\ -x_1-x_1' & 0 \end{pmatrix} = \begin{pmatrix} 0 & x_1+x_1' \\ -(x_1+x_1') & 0 \end{pmatrix} \in U_1.$$

$\forall a \in F$ $au_1 = a\begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax_1 \\ a\cdot(-x_1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & ax_1 \\ -(ax_1) & 0 \end{pmatrix} \in U_1$. Thus. U_1 is a subspace of V.

(2) $\forall u_2 = \left\{ \begin{pmatrix} a & b \\ c & c \end{pmatrix} \mid a+b+c=0 \right\}$. $U_2' = \left\{ \begin{pmatrix} a' & b' \\ c' & c' \end{pmatrix} \mid a'+b'+c'=0 \right\} \in U_2$.

$$u_2 + u_2' = \begin{pmatrix} a & b \\ c & c \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & c' \end{pmatrix} = \begin{pmatrix} a+b & b+b' \\ c+c' & c+c' \end{pmatrix}$$

$(a+a') + (b+b') + (c+c') = (a+b+c) + (a'+b'+c') = 0+0=0$. i.e. $u_2 + u_2' \in U_2$.

$\forall t \in F$. $tu_2 = t\begin{pmatrix} a & b \\ c & c \end{pmatrix} = \begin{pmatrix} at & bt \\ ct & ct \end{pmatrix}$ $at+bt+ct = (a+b+c)t = 0\cdot t = 0$. i.e. $tu_2 \in U_2$.

Thus. U_2 is a subspace of V.

(3) $\forall u_3 = \left\{ \begin{pmatrix} y & y \\ y & 0 \end{pmatrix} \mid y \in F \right\}$ $u_3' = \left\{ \begin{pmatrix} y' & y' \\ y' & 0 \end{pmatrix} \mid y' \in F \right\}$. $u_3 . u_3' \in U_3$.

$u_3 + u_3' = \begin{pmatrix} y & y \\ y & 0 \end{pmatrix} + \begin{pmatrix} y' & y' \\ y' & 0 \end{pmatrix} = \begin{pmatrix} y+y' & y+y' \\ y+y' & 0 \end{pmatrix}$. since $y+y' \in F$. $u_3 + u_3' \in U_3$.

$\forall a \in F$, $au_3 = a\begin{pmatrix} y & y \\ y & 0 \end{pmatrix} = \begin{pmatrix} ay & ay \\ ay & 0 \end{pmatrix}$ $ay \in F$. $au_3 \in U_3$. Thus U_3 is a subspace of V.

2. Then shows $V = U_1 + U_2 + U_3$.

for any $v = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ s.t. $x_1, x_2, x_3, x_4 \in F$. $v \in V$.

we can always find.
$$v = \begin{pmatrix} 0 & \frac{x_1+x_2+3x_4-2x_3}{3} \\ -\frac{x_1+x_2+3x_4-2x_3}{3} & 0 \end{pmatrix} + \begin{pmatrix} \frac{2x_1-x_2-x_3}{3} & \frac{x_2+x_3-2x_1-3x_4}{3} \\ x_4 & x_4 \end{pmatrix} + \begin{pmatrix} \frac{x_1+x_2+x_3}{3} & \frac{x_1+x_2+x_3}{3} \\ \frac{x_1+x_2+x_3}{3} & 0 \end{pmatrix}. \quad (1)$$

such that. the first matrix of RHS $\in U_1$. the second one $\in U_2$, the third one $\in U_3$.

what's more. the linear system

$$\begin{cases} a+y = x_1 \\ x+b+y = x_2 \\ -x+c+y = x_3 \\ c = x_4 \\ a+b+c = 0. \end{cases}$$

has only unique solution as

$$\begin{cases} x = \frac{x_1+x_2+3x_4-2x_3}{3} \\ y = \frac{x_1+x_2+x_3}{3} \\ a = \frac{2x_1-x_2-x_3}{3} \\ b = \frac{x_2+x_3-2x_1-3x_4}{3} \\ c = x_4. \end{cases}$$

That's. the sum (1) is unique. since v is arbitrary. i.e. $V = U_1 \oplus U_2 \oplus U_3$ □

KOKUYO

Exercise 3.1.

a. let $xv_1 + yv_2 + zv_3 = 0$.

we obtain $\begin{cases} x+z=0 \\ y-z=0 \\ -x+y=0 \end{cases}$ $\Rightarrow$ $x=y=z=0$. i.e. the LS has only zero solution.

Thus. v_1, v_2, v_3 are linearly independent. □

b. let $xv_1 + yv_2 + zv_3 = 0$

we obtain $\begin{cases} x-y=0 \\ x-z=0 \\ y-z=0. \end{cases}$ $\Rightarrow$ $\begin{cases} x=z \\ y=z \\ z \text{ is free.} \end{cases}$ the LS has infinite many solution. That's we can find not all zero $x_1, x_2, x_3 \in \mathbb{R}$ such as $\begin{cases} x=1 \\ y=1 \\ z=1. \end{cases}$ s.t. $xv_1 + yv_2 + zv_3 = 0$.

Thus, v_1, v_2, v_3 are linearly dependent. □

Exercise 3.2.

proof: let $x_1\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} + x_2\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} + x_3\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} + x_4\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0$ $x_1, x_2, x_3, x_4 \in \mathbb{R}$.

we need to show that the equation holds if and only if $x_1 = x_2 = x_3 = x_4 = 0$.

$\begin{cases} -x_1 + x_2 - x_3 + x_4 = 0 \\ x_1 - x_2 = 0 \\ -x_1 + x_3 + x_4 = 0 \\ -x_2 + x_3 = 0 \end{cases}$ the LS has unique solution: $\Rightarrow$ $x_1 = x_2 = x_3 = x_4 = 0$.

Thus. there four matrices are linearly independent. □

Exercise 3.3.

proof: Assume v_1+v_2, v_2-v_3, $v_1+v_2+v_3$ are linearly dependent. that is, there exist $a_1, a_2, a_3 \in F$. a_1, a_2, a_3 are not all zero. s.t. $a_1(v_1+v_2) + a_2(v_2-v_3) + a_3(v_1+v_2+v_3) = 0$

i.e. $a_1v_1 + a_1v_2 + a_2v_2 - a_2v_3 + a_3v_1 + a_3v_2 + a_3v_3 = 0$. (by right distributivity).

$\Rightarrow (a_1+a_3)v_1 + (a_1+a_2+a_3)v_2 + (a_3-a_2)v_3 = 0$. (by commutative and associate law).

Since v_1, v_2, v_3 are linearly independent. we obtain

$\begin{cases} a_1+a_3=0 \\ a_1+a_2+a_3=0 \\ a_3-a_2=0 \end{cases}$ $\Rightarrow$ the LS has unique solution $a_1=a_2=a_3=0$. which contradicts with v_1+v_2, v_2-v_3, $v_1+v_2+v_3$ are linearly dependent. Thus, they are linearly independent □.

Exercise 3.4.

proof: For any set $(v_1, \dots v_n)$. $n \in \mathbb{N}$. without loss of generality. we assume $v_j = v_i$ $(1 \le i, j \le n, i \ne j)$

then we can let. $a_1 = \cdots = a_{j-1} = a_{j+1} = \cdots = a_{i-1} = a_{i+1} = \cdots = a_n = 0$. $a_i = 1$ $a_j = -1$.

Since $v_i = v_j$. $\sum_{k=1}^{n} a_k v_k = 0$. Thus, the set of vectors is linearly dependent □

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Exercise 3.10.

Proof: First we need to show ~~F^n~~ $g_1 \dots g_n$ are l.i.

$a_1 g_1 + a_2 g_2 \cdots + a_n g_n = 0$.

$$\text{i th}\begin{cases} a_2 + a_3 + \cdots + a_n = 0 & ① \\ a_1 + a_3 + \cdots a_n = 0 & ② \\ \vdots \\ a_1 + \cdots + a_{i-1} + a_{i+1} + a_n = 0 \\ \vdots \\ a_1 + \cdots + a_{n-1} = 0. & ⓝ \end{cases}$$

① − ⓝ $a_1 = a_n$.
① − (n−1) $a_1 = a_{n-1}$
⋮
① − ② $a_1 = a_2$

Thus $a_1 = a_2 = \cdots = a_n = 0$. has only zero solution.
i.e. $g_1 \dots g_n$ are l.i.

Then we need to show for any $g \in F^n$. $g = (x_1, x_2, \dots, x_n)$. $x_i \in \mathbb{R}$. $1 \le i \le n$.

$\exists!\ a_1, a_2 \dots a_n \in \mathbb{R}$. s.t. $a_1 g_1 + a_2 g_2 + \cdots + a_n g_n = (x_1, x_2 \dots x_n)$

$$\begin{cases} a_2 + \cdots + a_n = x_1 & ① \\ a_1 + a_3 + \cdots + a_n = x_2 & ② \\ \vdots \\ a_1 + a_{i-1} + a_{i+1} + \cdots + a_n = x_i & ⓘ \\ \vdots \\ a_1 + \cdots + a_{n-1} = x_n & ⓝ \end{cases}$$

for any equation ⓘ. we add a_i ~~the~~ to both side.
then we obtain.
$a_1 + x_1 = a_2 + x_2 = \cdots = a_i + x_i = \cdots = a_n + x_n$ (*)

by the chain of equation (*). we can make substitution.

$(a_1 + x_1 - x_2) + (a_1 + x_1 - x_3) + \cdots + (a_1 + x_1 - x_n) = x_1$

$$a_1 = \frac{\sum_{i=1}^{n} x_i - (n-1)x_1}{n-1} \qquad \text{similarly,} \quad a_i = \frac{\sum_{i=1}^{n} x_i - (n-1)x_i}{n-1}$$

by this operation, we obtain unique set of $a_1, a_2 \dots a_n$ from given $x_1, x_2 \dots x_n$.

Thus, any $g \in F^n$ can be represent as a linear combination of $g_1 \dots g_n$ uniquely.
i.e. $g_1 \dots g_n$ form a basis of F^n.

Exercise 3.11.

Let $v > 0$. v is linear independent with itself. trivially

for any vector $u \in$ exotic space. $u = u(v^{-1} \oplus v) \neq uv^{-1} \oplus uv = uv^{-1}uv = u^2$

$$\underset{\text{Axiom 8}}{u} = u \cdot \underset{\text{Axiom 3.}}{1} = u \cdot (v^{-1} v) \underset{\text{def.}}{=} u(v^{-1} + v) = u \cdot (-1) \cdot v + uv \underset{\text{Axiom 6. def. Axiom 7}}{=} -uv + uv \quad u \in \mathbb{R}.$$

thus. u can be written as a linear combination of v.

v is a ~~p~~ basis. of exotic space.

All 2

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3.3. (Week 2. Fri)

Exercise 3.5.

proof: we can find a vector set (v_1, v_2, v_3). $v_1 = (1,0,0)$. $v_2 = (0,1,0)$. $v_3 = (0,0,1)$.

$\forall v = (x, y, z) \in \mathbb{R}^3$. we can find $x, y, z \in \mathbb{R}$.

s.t. $x v_1 + y v_2 + z v_3 = v$. that is. (v_1, v_2, v_3) is a spanning set of $\mathbb{R}^3$.

Assume there exist $v_1', v_2', v_3', v_4' \in \mathbb{R}^3$. s.t. (v_1', v_2', v_3', v_4') are l.i.

which means the number of vectors in a l.i. set bigger than the number of vectors in a spanning set. that is contradictary with Theorem 3.4.

Thus, Any four vectors in $\mathbb{R}^3$ are l.d. □

Exercise 3.6.

proof: First we let. $(1,0,0) = v_1$, $(1,1,0) = v_2$, $(0,1,0) = v_3$, $(0,0,1) = v_4$. $(1,1,1) = v_5$

Since we can find $v_1 + 0v_2 + v_3 + v_4 + (-1) v_5 = \vec{0}$. v_1, v_2, v_3, v_4, v_5 are l.d.

Now by lemma 3.2. since ~~$v_5 \in$~~ $v_5 = v_1 + 0 \cdot v_2 + v_3 + v_4$, then we can exclude v_5. from the ~~span~~ set and ~~v_5~~ $\operatorname{span}(v_1, v_2, v_3, v_4) = \operatorname{span}(v_1, v_2, v_3, v_4, v_5)$

Repeat the process, since $v_3 = (-1) v_1 + v_2$. $v_3 \in \operatorname{span}(v_1, v_2)$.

similarly. we can exclude the v_3 from the set and spanning space unchanged.

Let ~~$x v_1$~~ $x v_1 + y v_2 + z v_4 = 0$

$$\begin{cases} x + y = 0 \\ y = 0 \\ z = 0 \end{cases} \Rightarrow x = y = z = 0.$$

Thus the set (v_1, v_2, v_4) are l.i. □

Exercise 3.7.

proof: $\forall A = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in V$. s.t. $A^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = A$. i.e. $c = b$.

$a, b, c, d \in \mathbb{R}$.

we can find a set $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$.

For any $A \in V$. $\exists$ $a, c, d \in \mathbb{R}$.

s.t. $a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ □

Exercise 3.8

proof: ① First we let $v_1 = (1,-1,0,0)$, $v_1 \in U$. and $\operatorname{span}(v_1) \subset U$

② Second we can find $v_2 = (1,0,-1,0)$. $v_2 \notin \operatorname{span}(v_1)$. (Obviously, $v_2 \neq a v_1$. $a \in \mathbb{R}$).

s.t. $v_2 \in U$ and $\operatorname{span}(v_1, v_2) \subset U$.

③ Then we let $v_3 = (1,0,0,-1)$. since the equation $v_3 = a v_1 + b v_2$ has no solution, $v_3 \notin \operatorname{span}(v_1, v_2)$. s.t. $v_3 \in U$ and $\operatorname{span}(v_1, v_2, v_3) \subset U$.

④ Finally we let $v_4 \in U$. suppose $v_4 = (a_1, a_2, a_3, a_4)$. $a_1 + a_2 + a_3 + a_4 = 0$.

Let $v_4 = x v_1 + y v_2 + z v_3$

$$\begin{cases} x + y + z = a_1 \\ -x = a_2 \\ -y = a_3 \\ -z = a_4 \end{cases} \Rightarrow \begin{cases} x = -a_2 \\ y = -a_3 \\ z = -a_4 \end{cases}$$

the LS has a solution that is. $v_4 \in \operatorname{span}(v_1, v_2, v_3)$

Since v_4 can be arbitrary from U. then $U \subset \operatorname{span}(v_1, v_2, v_3)$

by ③. $\operatorname{span}(v_1, v_2, v_3) \subset U$. Now we have $U = \operatorname{span}(v_1, v_2, v_3)$. □

Exercise 3.9 (correct the question)

$\forall f = at^3 + bt^2 + ct^0 + dt \in \mathbb{R}[t]_3$, and $a, b, c, d \in \mathbb{R}$.

First to show that f can be uniquely written as a linear combination of f_1, f_2, f_3, f_4

let $x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 = f$

$$\begin{cases} x_1 - x_2 = d \\ x_1 - x_4 = c \\ -x_2 + x_3 = b \\ x_2 + x_3 + x_4 = a \end{cases} \Rightarrow \begin{cases} x_1 = \frac{a+c-b+2d}{3} \\ x_2 = \frac{a+c-b-d}{3} \\ x_3 = \frac{a+2b+c-d}{3} \\ x_4 = \frac{a-b-2c+2d}{3} \end{cases}$$

the LS has unique solution.

For any $f \in \mathbb{R}[t]_3$. we can find proper x_1, x_2, x_3, x_4. satisfy the condition.

Then show f_1, f_2, f_3, f_4 are l.i.

$x_1' f_1 + x_2' f_2 + x_3' f_3 + x_4' f_4 = 0$

$$\begin{cases} x_1 - x_2 = 0 \\ x_1 - x_4 = 0 \\ -x_2 + x_3 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases} \Rightarrow x_1 = x_2 = x_3 = x_4 = 0$$

the LS has only trivial solution.

Thus. f_1, f_2, f_3, f_4 form a basis of $\mathbb{R}[t]_3$

3.7 (Week 3 Tue.)

Exercise 3.12.

If $(f,a)\in \mathbb{R}[t]_3\times\mathbb{R}$, then $f=a_0+a_1t+a_2t^2+a_3t^3$ and $f\in V$ iff

$$\begin{cases} a_0+a_1+a_2+a_3=a \\ a_0-a_1+a_2-a_3=0 \end{cases}$$

the solution of this system is $\begin{cases} a_2=\frac{a}{2}-a_0 \\ a_3=\frac{a}{2}-a_1 \end{cases}$

$$\begin{aligned} f&=a_0+a_1t+(\tfrac{a}{2}-a_0)t^2+(\tfrac{a}{2}-a_1)t^3 \\ &=a(\tfrac{t^3}{2}+\tfrac{t^2}{2})+a_1(-t^3+t)+a_0(-t^2+1) \end{aligned}$$

then $(f,a)=a(\frac{t^3}{2}+\frac{t^2}{2},1)+a_1(-t^3+t,0)+a_0(-t^2+1,0)$. ①

~~thus~~ for any given $(f,a)\in V$, when can find $a_1,a_2\in\mathbb{R}$ such that the equation ① holds. thus $(\frac{t^3}{2}+\frac{t^2}{2},1)$. $(-t^3+t,0)$ $(-t^2+1,0)$ is a basis of V

Exercise 3.13.

Let $(A,B)\in V$. $A=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4\end{pmatrix}$ $B=\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4\end{pmatrix}$ $(a_i,b_i\in\mathbb{R},\ 1\le i\le 4)$

then we have $\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4\end{pmatrix}\begin{pmatrix} 1 \\ 1\end{pmatrix}=\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4\end{pmatrix}\begin{pmatrix} 0 \\ 1\end{pmatrix} \iff \begin{pmatrix} a_1+a_2 \\ a_3+a_4\end{pmatrix}=\begin{pmatrix} b_2 \\ b_4\end{pmatrix}$

$\begin{pmatrix} a_1 & a_3 \\ a_2 & a_4\end{pmatrix}\begin{pmatrix} -1 \\ 0\end{pmatrix}=\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4\end{pmatrix}\begin{pmatrix} 1 \\ -1\end{pmatrix} \iff \begin{pmatrix} -a_1 \\ -a_2\end{pmatrix}=\begin{pmatrix} b_1-b_2 \\ b_3-b_4\end{pmatrix}$

i.e. $\begin{cases} b_1=a_2 \\ b_2=a_1+a_2 \\ b_3=a_3+a_4-a_2 \\ b_4=a_3+a_4 \end{cases}$ $(A,B)=\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4\end{pmatrix}\times\begin{pmatrix} a_2 & a_1+a_2 \\ a_3+a_4-a_2 & a_3+a_4\end{pmatrix}=a_1\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\times\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}+a_2\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\times\begin{pmatrix} 1 & 1 \\ -1 & 0\end{pmatrix}$

$+a_3\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\times\begin{pmatrix} 0 & 0 \\ 1 & 1\end{pmatrix}+a_4\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\times\begin{pmatrix} 0 & 0 \\ 1 & 1\end{pmatrix}$

For any $A.B\in V$, we can find $a_1,a_2,a_3,a_4\in\mathbb{R}$. Let the equation above holds.

thus. $\begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}\times\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$. $\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\times\begin{pmatrix} 1 & 1 \\ -1 & 0\end{pmatrix}$. $\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\times\begin{pmatrix} 0 & 0 \\ 1 & 1\end{pmatrix}$. $\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}\times\begin{pmatrix} 0 & 0 \\ 0 & 1\end{pmatrix}$ form a basis of V

Exercise 3.14.

Consider the matrix, let's reduce it to the row echelon form:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & -1 & -1 \\ 2 & 2 & 0 & 0 & -1 \\ 1 & 1 & 5 & 5 & 2 \\ 1 & -1 & -1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & -2 & -2 & -1 \\ 0 & 0 & 4 & 4 & 2 \\ 0 & -2 & -2 & -1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

clearly, the procedure don't change the span

Thus. for any $v \in \operatorname{span}(v_1, v_2, v_3, v_4, v_5)$. we can find proper $x, y, z \in \mathbb{R}$.

such that $v = x(1,0,0,\frac{1}{2},0) + y(0,1,0,-\frac{1}{2},-\frac{1}{2}) + z(0,0,1,1,\frac{1}{2})$.

Exercise 3.15.

Solution. ~~we~~ first we consider matrix $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. let's reduce it the row echelon form.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 1 & -2 & -1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & -3 \end{pmatrix}$$

Now we let $v_4 = (0,0,0,1,0)$.

Let $v_1' = (1,0,0,1,1)$. $v_2' = (0,1,0,0,-1)$. $v_3' = (0,0,1,-2,-3)$.

obviously. $v_4 = xv_1' + yv_2' + zv_3'$ iff $x = y = z = 0$. Thus. $v_4 \in \mathbb{R}^5 \setminus \operatorname{span}(v_1, v_2, v_3)$

Then we repeat the process. consider $\begin{pmatrix} v_1' \\ v_2' \\ v_3' \\ v_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} v_1'' \\ v_2'' \\ v_3'' \\ v_4'' \end{pmatrix}$

we let $v_5 = (0,0,0,0,1)$. similarly we can proof $v_5 \in \mathbb{R}^5 \setminus \operatorname{span}(v_1, v_2, v_3, v_4)$

finally we consider $\begin{pmatrix} v_1'' \\ v_2'' \\ v_3'' \\ v_4'' \\ v_5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} v_1^3 \\ v_2^3 \\ v_3^3 \\ v_4^3 \\ v_5^3 \end{pmatrix}$

since we only use the elementary row operation, then $\operatorname{span}(v_1, v_2, v_3, v_4, v_5) = \operatorname{span}(v_1^3, v_2^3, v_3^3, v_4^3, v_5^3)$

For any $v = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5$,

we have $v = a_1 v_1^3 + a_2 v_2^3 + a_3 v_3^3 + a_4 v_4^3 + a_5 v_5^3$. i.e. $v \in \operatorname{span}(v_1^3, v_2^3, v_3^3, v_4^3, v_5^3) = \operatorname{span}(v_1, v_2, v_3, v_4, v_5)$

that is, v_1, v_2, v_3, v_4, v_5 form a basis of $\mathbb{R}^5$

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3.10 (Week 3 Thu. & Fri.)

Exercise 3.16.

Let $B = \{v_1, v_2, v_3, \dots, v_n\}$. $v_1 \dots v_n \in V$

proof: 1 ⇒ 2: Since B is a basis of V.

then for any $v \in V$, $\exists a_i \in \mathbb{R}$. $i \in [1, n]$. s.t. $v = a_1 v_1 + \cdots + a_n v_n$ and a_i not all equal to 0.

that is, any $v \in V$. the set $(v, v_1, \dots v_n)$ is l.d. □

2 ⇒ 1 Since B is a maximal l.i. set.

that is, any $v \in V$. set $(v, v_1, \dots v_n)$ is l.d. i.e. $\exists a, a_1 \dots a_n \in \mathbb{R}$.

s.t. $av + a_1 v_1 + \cdots + a_n v_n = 0$. and $a, a_1 \dots a_n$ not all 0.

we exclaim that $a \neq 0$ (if so, we have $a_1 v_1 + \cdots + a_n v_n$. by l.i of B. we have $a_i = 0$).

thus $v = -\frac{a_1}{a} v_1 - \frac{a_2}{a} v_2 - \cdots - \frac{a_n}{a} v_n$. i.e. $v \in span(v_1 \dots v_n)$

Since v is arbitrary and. $v_1 \dots v_n$ are l.i ⇒ $v_1 \dots v_n$ form a basis of V. □

1 ⇒ 3 Assume the reverse, if we can extract v_j from B and.

$V = span(v_1 \dots v_n) = span(v_1 \dots v_{j-1}, v_{j+1} \dots v_n)$ that the number of spanning set is $n-1$.

Since B is a basis. B is a l.i. so we have the number of vectors in a l.i set larger than the number of vector in a spanning set. which contradicts with Thm 3.4. □

3 ⇒ 1 We need to show. B is l.i.

Assume the reverse. By Lemma 3.2. there exists $v_j \in V$ $(2 \leq j \leq n)$.

such that $v_j \in Span(v_1 \dots v_{j-1})$ and. $Span(v_1 \dots v_n) = span(v_1 \dots v_{j-1}, v_{j+1} \dots v_n)$

which contradicts with B is minimal of spanning set. □

Exercise 3.17

proof: If $v_1 \dots v_n$ form a basis of V, then we are done. if not, we let $v_{n+1} \in V \setminus span(v_1 \dots v_n)$.

By coro. 3.3. the vectors $v_1 \dots v_{n+1}$ are also l.i. therefore. this procedure can be repeated. since V has a basis. that is V is finite-dimensional. Thm 3.4. implies that the process terminates in a finite number of step. let the number be $n-m$

thus. $v_1 \dots v_m$ form a basis. by the def. of basis, $v_{n+1} \dots v_m$ are l.i.

by Thm 3.9. $v_{n+1}, \dots v_m$ can be extended to a basis, let it be B. □

B is pre-determined.

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Exercise 3.18.

let. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = x_1 \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x_3 \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $(x_1, x_2, x_3, x_4 \in \mathbb{R})$

$$\begin{cases} x_2 = 1 \\ x_1 + x_4 = 1 \\ -x_3 + x_4 = 0 \\ -x_1 + x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 1. \\ x_3 = 0. \\ x_4 = 0 \end{cases}$$

then coordinate of A is $(1, 1, 0, 0)$

Exercise 3.19.

Since. (a_1, a_2, a_3) are coordinates of a vector v. relative to a basis.

$v = a_1 v_1 + a_2 v_2 + a_3 v_3$

we suppose the coordinates of $v_1 - v_3, -v_2, v_1 + v_2 + v_3$. be b_1, b_2, b_3.

$v = b_1(v_1 - v_3) + b_2(-v_2) + b_3(v_1 + v_2 + v_3) = a_1 v_1 + a_2 v_2 + a_3 v_3$

$$\begin{cases} b_1 + b_3 = a_1 \\ -b_2 + b_3 = a_2 \\ -b_1 + b_3 = a_3 \end{cases} \Rightarrow \begin{cases} b_1 = \frac{a_1 - a_3}{2} \\ b_2 = \frac{a_1 + a_3 - 2a_2}{2} \\ b_3 = \frac{a_1 + a_3}{2} \end{cases}$$

now we obtain the asked coordinates.

Exercise 4.1.

Let $f = \sum_{i=0}^{6} a_i x^i$ since $f(x) = f(-x)$.

i.e. $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + a_4 x^4 - a_5 x^5 + a_6 x^6$.

$\Rightarrow a_1 x + a_3 x^3 + a_5 x^5 = 0$. since the equation holds for any $x \in \mathbb{R}$. then $a_1 = a_3 = a_5 = 0$.

Thus, $V = \{ f = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 \mid f \in \mathbb{R}[x]_6 \}$

Now can find $f_1 = 1$. $f_2 = x^2$, $f_3 = x^4$ $f_4 = x^6$. clearly. f_1, f_2, f_3, f_4 are l.i.

For any $v = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 \in V$, we can find $a_0, a_2, a_4, a_6 \in \mathbb{R}$.

s.t. $v = a_0 f_1 + a_2 f_2 + a_4 f_3 + a_6 f_4$. that is $\{f_1, f_2, f_3, f_4\}$ spans V.

i.e. $\{f_1, f_2, f_3, f_4\}$ form a basis of V.

Thus $\dim V = 4$.

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Exercise 4.2.

proof: "$\Rightarrow$" (sufficient). let. $\operatorname{span}(v_1 \cdots v_n) = U$

since $(v_1 \dots v_n)$ are l.i. then $(v_1 \dots v_n)$ form a basis of U.

the number of vectors in basis is n. that is, $\dim \operatorname{span}(v_1 \dots v_n) = \dim U = n$

"$\Leftarrow$" (necessity). let $\operatorname{span}(v_1 \dots v_n) = U$

since $\dim U = n$. there exist a basis of U, $\{v_1', v_2' \dots v_n'\}$.

Assume $\{v_1 \dots v_n\}$ is l.d. by Lemma 3.2. $\exists\, v_j \in U$, s.t. $2 \le j \le n$. $v_j \in \operatorname{span}(v_1 \dots v_{j-1})$.

and $\operatorname{span}(v_1 \dots v_{j-1}, v_{j+1}, \dots v_n) = \operatorname{span}(v_1 \dots v_n) = U$.

that is the number of vectors in a spanning is $n-1$, while the number of l.i set. (the basis) is n, $n-1 < n$, which contradicts with Thm 3.4. Thus. $v_1 \dots v_n$ are l.i. $\square$.

Exercise 4.3. (in 3.9. $f_1 = t+1$. $f_2 = t^3 - t^2 - 1$. $f_3 = t^3 + t$. $f_4 = t^3 - t$).

by pro 4.4. we form $A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$ since $f_1' = t^3$ $f_2' = t^2$ $f_3' = t^0$ $f_4' = 0$. form a basis of $\mathbb{R}[t]_3$.

$$\begin{cases} f_1 = f_3' + f_4' & f_2 = f_1' - f_2' - f_4' \\ f_3 = f_1' + f_2' & f_4 = f_1' - f_3' \end{cases}$$

$$\det A = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{vmatrix}$$

since $\det A \ne 0$. A is invertible. Thus. f_1, f_2, f_3, f_4 form a basis of $\mathbb{R}[t]_3 = 3$.

Exercise 4.4.

Proof: let two ~~the~~ 3-dim subspace be U_1 and U_2. the 5-dim space be V.

by pro 2.5. $U_1 + U_2$ is a subspace of V. by the def of subspace.

we have $\dim V \ge \dim(U_1 + U_2)$. i.e. $\dim(U_1 + U_2) \le 5$.

by Thm 4.5. we have $\dim U_1 \cap U_2 = \dim U_1 + \dim U_2 - \dim(U_1 + U_2)$.

thus. $\dim U_1 \cap U_2 \ge 1$.

the dimension of the intersection of U_1 and U_2 is at least. 1. $\Rightarrow$ it's non-zero.

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No. 李舟言

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3.14 (Week 4. Tue.)

Exercise 4.5. Let $U_1 = \operatorname{span}(v_1, v_2)$. $U_2 = \operatorname{span}(u_1, u_2)$. $U_3 = \operatorname{span}(t_1, t_2, t_3)$

proof: ~~first~~ first we show $V = U_1 + U_2 + U_3$.

$\forall v \in V$. $v = (a_1, a_2, a_3, a_4, a_5)$

let. $v = x_1(1,1,-1,1,0) + x_2(1,1,0,-1,1) + x_3(1,-1,1,0,1) + x_4 (1,1,-1,0,1) + x_5(-1,0,1,1,1)$

$+ x_6(0,1,1,1,-1) + x_7(1,1,0,0,-2)$.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 - x_5 + x_7 = a_1 \\ x_1 + x_2 - x_3 + x_4 + x_6 + x_7 = a_2 \\ -x_1 + x_3 - x_4 + x_5 + x_6 = a_3 \\ x_1 - x_2 + x_5 + x_6 = a_4 \\ x_2 + x_3 + x_4 + x_5 - x_6 - 2x_7 = a_5 . \end{cases} \quad (1)$$

this LS must have free variable.

thus. the LS has non-trivial solution.

i.e. v can be expressed as L.C. of $(v_1, v_2, u_1, u_2, t_1, t_2, t_3)$ that is $V = U_1 + U_2 + U_3$.

We find that. $t_1 - t_2 + t_3 = 0$. i.e. $t_2 \in \operatorname{span}(t_1, t_3) = \operatorname{span}(t_1, t_2, t_3) = U_3$.

since t_1, t_3 are not linear, then t_1, t_3 are l.i. so t_1, t_3 form a basis of U_3.

similarly. we have v_1, v_2 form a basis of U_1. u_1, u_2 form a basis of U_2.

① Thus. the union of bases for U_i has 6 vectors. but $V = \mathbb{R}^5$. $\dim V = 5$.
which means the union can't be a basis ~~of~~ for $V \Rightarrow$ not satisfied the pro 4.6.4 so the sum is not directed. $(4\times \Rightarrow 1\times)$. $\to$ x is a cross. means not satisfied

② Also, we have $\dim U_1 + \dim U_2 + \dim U_3 = 2+2+2 = 6$. $\dim V = 5$. $6 \neq 5$.
~~ol~~ $\Rightarrow$ not satisfied with pro 4.6.5. $(5\times \Rightarrow 1\times)$.

③ Now we consider the LS. (1). Let $a_1 = a_2 = a_3 = a_4 = a_5 = 0$.

the coefficient matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 & 1 & 1 \\ -1 & 0 & 1 & -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 & -2 \end{bmatrix} \xrightarrow{\text{row echelon}} \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 1 & -5 & -6 \\ 0 & 0 & 0 & 0 & 2 & 1 & -1 \end{bmatrix}$ since $5 < 7$. the LS has free variable.

that is the LS. (1) has non-trival solution. $(3\times \Rightarrow 1\times)$

thus. $\exists\, u_1' \in U_1$, $u_2' \in U_2$, $u_3' \in U_3$ not all zero vector u_1', u_2', u_3' s.t. $u_1' + u_2' + u_3' = 0$.

④ Let's check the $U_3 \cap (U_1 + U_2)$

let. $v \in U_3 \cap (U_1 + U_2)$. suppose $v = y_1 t_1 + y_3 t_3 = z_1 v_1 + z_2 v_2 + z_3 u_1 + z_4 u_2$.

$$\begin{cases} -y_1 + y_3 = z_1 + z_2 + z_3 + z_4 \\ y_3 = z_1 + z_2 - z_3 + z_4 \\ y_1 = -z_1 + z_3 - z_4 . \\ y_3 = z_1 - z_2 . \\ y_1 - 2y_3 = z_2 + z_3 + z_4 . \end{cases} \qquad \begin{cases} y_1 = -2z_3 . \\ y_3 = z_2 + 2z_3 . \\ z_1 = 2z_2 + 2z_3 \\ z_4 = z_3 - 2z_2 \end{cases}$$

thus. by let arbitrary z_2, z_3.

we can find $v \neq 0$.

i.e. $U_3 \cap (U_1 + U_2)$

Thus. the sum not direct. $(2\times \Rightarrow 1\times)$

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Exercise 5.1.

a. Let $A_1 = (a_{ij})_{m\times n}$. $A_2 = (b_{ij})_{m\times n}$. $a \in F$.

$L(A_1) = A_1^T$ $L(A_2) = A_2^T$

$L(A_1 + A_2) = L[(a_{ij} + b_{ij})_{m\times n}] = (a_{ij}+b_{ij})_{m\times n}^T = (a_{ji} + b_{ji})_{n\times m} = (a_{ji})_{n\times m} + (b_{ji})_{n\times m}$

$= A_1^T + A_2^T = L(A_1) + L(A_2)$

$L(aA_1) = (a \cdot a_{ji})_{n\times m} = a \cdot (a_{ji})_{n\times m} = a \cdot L(A_1)$ Thus. L is l.t.

b. Let A_1. A_2 are both identity matrix of $n\times n$.

$L(A_1) = L(A_2) = 1^n = 1$.

$L(A_1 + A_2) = 2^n = 2^n$. since $2^n \neq 1$ $(n \neq 0)$. $L(A_1) + L(A_2) \neq L(A_1 + A_2)$

L is not linear transformation

c. Let $f_1 = \sum_{i=0}^{n} a_i t^i$. $f_2 = \sum_{i=0}^{n} b_i t^i$ $a \in F$. $f_1, f_2 \in F[t]_n$.

$L(f_1 + f_2) = L(\sum_{i=0}^{n}(a_i + b_i)t^i) = (a_0 + b_0, a_1 + b_1, \dots, a_n + b_n) = (a_0, a_1, \dots a_n) + (b_0, b_1, \dots b_n)$

$= L(f_1) + L(f_2)$

$L(af_1) = L(a\sum_{i=0}^{n} a_i t^i) = (a \cdot a_0, aa_1, \dots, a \cdot a_n) = a \cdot (a_0, a_1, \dots a_n) = a \cdot L(f_1)$.

Thus, L is l.t.

d. Let $L: (a,b) \to (-a, b)$. (a,b) are coordinates on the geometric plane.

$L((a_1, b_1) + (a_2, b_2)) = L(a_1 + a_2, b_1 + b_2) = (-(a_1 + a_2), b_1 + b_2) = (-a_1, b_1) + (-a_2, b_2)$

$= L(a_1, b_1) + L(a_2, b_2)$

$L(a(a_1, b_1)) = (-aa_1, ab_1) = a(-a_1, b_1) = aL(a_1, b_1)$.

thus. L is l.t.

e. Let $L: (a,b) \to (a', b')$ (a,b) be any coordinates on a geometric plane.

let (a_1, b_1), $(a_2, b_2) \in$ the plane.

$L((a_1, b_1) + (a_2, b_2)) = L(a_1 + a_2, b_1 + b_2) = (a_1' + a_2', b_1' + b_2') = (a_1', b_1') + (b_1', b_2')$

$= L(a_1, b_1) + L(a_2, b_2)$

$L(a(a_1, b_1)) = (aa_1', ab_1') = a(a_1', b_1') = aL(a_1, b_1)$

Thus. L is linear transformation

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Exercise 5.2.

Solution: let $v_3 = x_1 v_1 + x_2 v_2$.

$$\begin{cases} x_1 + 2x_2 = 1 \\ -x_1 + x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{3} \\ x_2 = \frac{2}{3} \end{cases}$$

since $L \in \mathcal{L}(F^2, W)$. L is l.t.

$L(v_3) = L(-\frac{1}{3}v_1 + \frac{2}{3}v_2) = -\frac{1}{3}L(v_1) + \frac{2}{3}L(v_2) = -\frac{1}{3}w_1 + \frac{2}{3}w_2$.

Exercise 5.3.

proof: Since L. L' are linear transformation.

then $\forall\ v \in V$, $\exists\ a_1 \dots a_n \in F$. s.t. $v = a_1 v_1 + \cdots + a_n v_n$.

$L(v) = L(a_1 v_1 + \cdots + a_n v_n) = a_1 L(v_1) + a_2 L(v_2) + \cdots + a_n L(v_n)$

$= a_1 L'(v_1) + a_2 L'(v_2) + \cdots + a_n L'(v_n) = L'(a_1 v_1 + \cdots + a_n v_n) = L'(v)$.

since v is arbitrary from V. $L = L'$.

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3.16 (Week 4. Thu.).

Exercise 5.4.

Proof: Let $L, L', L'' \in \mathcal{L}(V, W)$ $a, b \in F$ ~~$\forall v \in$~~ v is arbitrary vector in V.

1) addition associative. $((L+L')+L'')(v) = (L+(L'+L''))(v)$.

$((L+L')+L'')(v) = (L+L')(v) + L''(v)$ (By linearity of l.t.).

$= L(v) + L'(v) + L''(v)$ (similarly as above).

$= L(v) + (L'(v) + L''(v))$ (By associative law of addition. W is a space).

$= L(v) + (L'+L'')(v)$ (again by linearity of l.t.)

$= (L + (L'+L''))(v)$ (by linearity of l.t.)

2) commutative. of addition. $(L+L')(v) = (L'+L)(v)$

$(L+L')(v) = L(v) + L'(v)$ (By linearity of l.t.)

$= L'(v) + L(v)$ (By commutative law in space W).

$= (L'+L)(v)$ (By linearity of l.t.)

3). left distributivity. $((a+b)L)(v) = (aL+bL)v$.

$((a+b)L)(v) = (a+b) \cdot L(v)$ (By scalar multiplication of l.t.)

$= a \cdot L(v) + b \cdot L(v)$ (left distributivity of space W).

$= (aL+bL)(v)$ (By linearity of l.t.).

4). $((a \cdot b) \cdot L)(v) = (a \cdot (b \cdot L))(v)$

$((a \cdot b) \cdot L)(v) = (a \cdot b) \cdot L(v)$ ① By scalar multiplication of l.t.

$= a \cdot (b \cdot L(v))$ ② By associative of multiplication ~~of~~ in Space W.

$= a \cdot (b \cdot L)(v)$ ③ simillar as step ①

$= (a \cdot (b \cdot L))(v)$. ④ similar as step ①

5). $(1 \cdot L)(v) = L(v)$. for any $v \in V$.

$(1 \cdot L)(v) = 1 \cdot L(v)$. (By scalar multiplication of l.t.)

$= L(v)$. (By axiom VIII in vector space W.).

the multiplication identity.

Check properties of product

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Exercise 5.5.

Solution: find the kernal. let. $L(\bar{X})=0$. $\Rightarrow$ $\bar{X}=\bar{X}^T$.

$$Ker(L) = \{ \bar{X} \in M_n(F) \mid \bar{X} = \bar{X}^T \}$$

$$Im(L) = \{ \bar{X} - \bar{X}^T \mid \bar{X} \in M_n(F) \}$$

1) the $Ker(L)$ contains all symmetric matrix of $M_n(F)$.

$\dim Ker(L) = \frac{n(n+1)}{2}$, since we can form a basis of $Ker(L)$

s.t. $A_{ij} = \begin{bmatrix} 0 & \cdots & \vdots & \cdots \\ \cdots & \cdots & 1 & \cdots \\ & & \vdots & \ddots \\ & & & 0 \end{bmatrix}$ ith row with $a_{ij}=a_{ji}=1$. and other entries are 0. $(1\le i\le j\le n)$

j th column.

we have $\frac{n(n+1)}{2}$ numbers of vector satisfied that condition.

2). let $\bar{X}=(a_{ij})_{n\times n}$. $\bar{X}^T=(a_{ji})_{n\times n}$.

$$\bar{X}-\bar{X}^T = \begin{bmatrix} 0 & a_{12}-a_{21} & \cdots & a_{1n}-a_{n1} \\ a_{21}-a_{12} & 0 & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1}-a_{1n} & \cdots & \cdots & 0 \end{bmatrix}$$

we find that the diagonal entries are 0. the i th row j th column 's entry and j th row i th column 's entry are inverse to each other.

thus we form $A_{ij} = \begin{bmatrix} 0 & -1 & \cdots & 0 \\ 1 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix}$ s.t. $a_{ij}=1$ $a_{ji}=-1$. and $1\le i<j\le n$

we have $\frac{n(n-1)}{2}$ numbers of vector in a basis of $Im(L)$.

3). the dimension of $M_n(F)$ is n^2. (form a basis $\{A_{ij}\}$ with $a_{ij}=1$. and other $=0$.).

thus. $\dim \ker L + \dim Im = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2 = \dim V$.

Excise 5.6.

Solution. Define $L \in \mathcal{L}(M_2(\mathbb{R}))$. by $L(X) = XA - BX$.

the $Ker(L) = \{ \bar{X} \in M_2(\mathbb{R}) \mid XA = BX \}$

Suppose $\bar{X} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $XA = \begin{pmatrix} a_{12} & a_{11} \\ a_{22} & a_{21} \end{pmatrix}$ $BX = \begin{pmatrix} a_{11} & a_{12} \\ a_{11}-a_{21} & a_{12}-a_{22} \end{pmatrix}$

$$\begin{cases} a_{11} = a_{12} \\ a_{22} = a_{11}-a_{21} \\ a_{21} = a_{12}-a_{22} \end{cases} \Rightarrow \begin{cases} a_{12} = a_{11} \\ a_{22} = a_{11}-a_{21}. \\ a_{11}.\ a_{21} \text{ is free.} \end{cases}$$

Thus. $\dim Ker(L) = 2$.

$\dim U = \dim Im(L) = \dim M_2(\mathbb{R}) - Ker(L) = 2^2 - 2 = 2$. (to be continue ...).

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Exercise 6.5

Solution. $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ thus. $[v]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$$L(v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \Rightarrow \text{thus } [L(v)]_{B'} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$L\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$L\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$L\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 2\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$L\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[L]_{BB'} = \begin{bmatrix} 2 & 2 & 0 & 1 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$[L]_{BB'}[v]_B = \underset{3\times 4}{\begin{bmatrix} 2 & 2 & 0 & 1 \\ -2 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}} = [1 \ \ -1 \ \ 0]^T \overset{\text{in } W.}{=} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

~~the~~

the thm. 6.4 's condition satisfied.

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Week 6. Tue.

Exercise 7.4.

Proof: 1). additivity. $\forall L, L' \in K \subset \mathcal{L}(V,W)$

$(L+L')(v) = L(v) + L'(v)$. (by linearity of space).

$= 0 + 0 = 0$. thus. $L+L' \in K$.

2) scalar multiplication

$\forall a \in F, (aL)(v) = a \cdot L(v)$ (by multiplication of space)

$= a \cdot 0 = 0$. thus $(aL) \in K$.

Thus. K is a subspace.

Let $T(L) = L(v)$. s.t $T \in \mathcal{L}(\mathcal{L}(V,W), W)$.

the $\ker(T) = \{ L \in \mathcal{L}(V,W) \mid T(L) = 0 \} = K$

$Im(T) = \{ T(L) \mid L \in \mathcal{L}(V,W) \} = \{ L(v) \mid L \in \mathcal{L}(V,W) \} = W$.

remains to prove. $\forall w \in W$. $\exists L \in \mathcal{L}(V,W)$. s.t. $L(v) = w$.

(Only tools is Thm 5.1)

why ? .

we need to extend the given v. to a basis.

By theorem 5.6. $\dim \mathcal{L}(V,W) = \dim \ker(T) + \dim Im(T)$.

thus $K = \dim \mathcal{L}(V,W) - \dim Im(T) = \dim V \cdot \dim W - \dim W = m(n-1)$. $\square$

Exercise 8.1.

Solution: $\forall f \in \mathbb{R}[t]_3$. $f = \sum_{i=0}^{3} a_i t^i$

$L(f) = \lambda f = f(2t) \Rightarrow (\lambda-8)a_3t^3 + (\lambda-4)a_2t^2 + (\lambda-2)a_1t + (\lambda-1)a_0 = 0$.

$\lambda \neq 1, 2, 4, 8 \Rightarrow a_0 = a_1 = a_2 = a_3 = 0$

$\Rightarrow f = 0$ impossible.

thus. we have

eigenvalues	8	4	2	1
eigenvector	a_3t^3	a_2t^2	a_1t	any constant

a_3, a_2, a_1 are arbitrary real numbers

non-zero.

Exercise 8.2

"$\Rightarrow$" if 0 is an eigenvalue. then $\exists$ non-zero $v \in V$. s.t. $L(v) = 0$.

then $v \in \ker(L)$. thus $\ker(L) \neq \{0\}$.

"$\Leftarrow$" if $\ker(L) \neq \{0\}$. $\exists v \in V$ $v \neq 0$. s.t. $v \in \ker(L)$.

that is $L(v) = 0 = 0 \cdot v$. thus 0 is an eigenvalue.

Exercise 8.3.

Proof 1): "$\Rightarrow$" since $\lambda \in F$ is an eigenvalue $\exists v \in V$. $v \neq 0$. s.t. $L(v) = \lambda v$.

since L is invertible. $L^{-1}(\lambda v) = v = \frac{1}{\lambda} \cdot (\lambda v)$. $\frac{1}{\lambda}$ is an eigenvalue. λv is an eigenvector, respectively. (T.B.C)

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(brought forward).

"$\Leftarrow$" λ^{-1} is an eigenvalue of L^{-1}. that is, $\exists v' \in V$. $v' \neq 0$.

s.t. $L^{-1}(v') = \lambda^{-1} v'$.

since L is invertible.

$L(\lambda^{-1} v') = v' = (\lambda \cdot \lambda^{-1}) \cdot v' = \lambda \cdot (\lambda^{-1} v')$

where λ is an eigenvalue and $\lambda^{-1} v'$ is an eigenvector, respectively.

L^{-1} is a l.t.

$L^{-1}(\lambda v) = v \Leftrightarrow \lambda L^{-1}(v) = v$

$\Leftrightarrow L^{-1}(v) = \lambda^{-1} v$.

2). "$\Rightarrow$" $L(v) = \lambda v.^{(1)}$ since L is invertible.

then $L^{-1}(\lambda v) = v. = (\lambda^{-1} \cdot \lambda) . v = \lambda^{-1} \cdot (\lambda v).^{(2)}$ thus. λ^{-1} is eigenvalue. λv is eigenvector by equation (1). we know that if we change v by $\lambda^{-1} v$. the eigenvalue will not change apply it to the equation (2). $L^{-1}(v) = \lambda^{-1} v$.

"$\Leftarrow$" $L^{-1}(v) = \lambda^{-1} v.^{(1')}$ since L is invertible

$L(\lambda^{-1} v) = v. = \lambda(\lambda^{-1} v).^{(2')}$ λ is a eigenvalue. $\lambda^{-1} v$ is eigenvector.

also by equation (2'). we know that. if we change v by λv. the eigenvalue keeps same.

similarly. we have $L(v) = \lambda v$. $\square$.

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Exercise 7.4'. $v. v' \in V$ are linear independent $K = \{L \in \mathcal{L}(V, W) \mid L(v) = L(v') = 0\}$

find the dimension of K.

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3.30 (Week 6. Thu).

Exercise 8.4.

(3) proof: since u is an eigenvector of $L \in \mathcal{L}(V)$. $\exists \lambda \in F$. s.t. $L(u) = \lambda u$.

$[L]_B [u_B]_B = [L(u)]_B = [\lambda u]_B = \lambda [u]_B$.

Let $[L]_B = A$. $[u]_B = X$. we have $AX = \lambda X$. λ is a eigenvalue of A.

(4) proof: Since ~~X~~ $\exists \lambda \in F$. s.t. $AX = \lambda X$. suppose $X = [u]_B$. $A = [L]_B$

$[L(u)]_B = [L]_B [u]_B = \lambda [u]_B = [\lambda u]_B$

that is $L(u) = \lambda u$. i.e. u is an eigenvector of L.

Exercise 8.5.

Proof: Firstly we need to ~~sh~~ show U is L-invariant.

$\forall u \in U$, assume that $u = a_1(v_1+v_2) + a_2(v_3+v_4)$. $a_1, a_2 \in \mathbb{F}$

$$L(u) = L(a_1(v_1+v_2) + a_2(v_3+v_4))$$
$$= a_1(L(v_1) + L(v_2)) + a_2(L(v_3) + L(v_4))$$
$$= a_1(v_1 - 2v_2 + v_3 - v_1 + 2v_2 + v_4) + a_2(2v_1 + v_2 - v_3 + v_4 + v_2 + v_3 - v_4).$$
$$= a_1(v_3+v_4) + 2a_2(v_1+v_2).$$
$$= 2a_2(v_1+v_2) + a_1(v_3+v_4). \in U$$

. thus. U is L-invariant.

We can simply claim that v_1+v_2, v_3+v_4 forms a basis of U

since them are l.i. and $U = \mathrm{span}(v_1+v_2, v_3+v_4)$.

thus. $\dim U = 2$.

the ~~m~~ rank of matrix $= 2$.

By Thm 6.9. We can find $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ satisfy the condition

$[L|_U]_B = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

Yes, but please find the corresponding basis.

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4.4. (Week 7 Tue.)

Exercise 8.6.

i) $L(f_1) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = -\frac{1}{2}g_1 - \frac{5}{2}g_2 - \frac{1}{2}g_3.$

$L(f_2) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} = (-2)g_2 + (-1)g_3$

$L(f_3) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = -\frac{1}{2}(g_1 + g_2 - g_3) = -\frac{1}{2}g_1 - \frac{1}{2}g_2 + \frac{1}{2}g_3$

$[L]_{F,G} = \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{5}{2} & -2 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$

ii) $L(f_1) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = 5g_1' + 2g_2' + g_3'$

$L(f_2) = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix} = 3g_1' + 2g_2' + g_3' \quad \Rightarrow [L]_{F,G'} = \begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

$L(f_3) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = 2g_1'$

iii). $g_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2g_1' + g_2' + 2g_3'$

$g_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -3g_1' - g_2' - g_3. \quad \Rightarrow M_{G'\to G} = \begin{bmatrix} 2 & -3 & 3 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$

$g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = 3g_1' + 0 + g_3'$

we need to show.

iv). $[L]_{F,G'} = M^{-1}_{G\to G'}[L]_{F,G} \Leftrightarrow [L]_{F,G'} = M_{G'\to G}[L]_{F,G}$

$RHS = \begin{bmatrix} 2 & -3 & 3 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{5}{2} & -2 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 5 & 3 & 2 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} = LHS. \quad \square$

v). $\omega = 4g_1' + g_2' + 2g_3'$

$\omega = \frac{1}{2}g_1 - \frac{1}{2}g_2 + \frac{1}{2}g_3.$

the coordinates related to basis G is $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} := [V]_G$

related to basis G' is $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix} := [V]_{G'}$

(vi). By thm 6.6. we have $[V]_{G'} = M_{G'\to G}[V]_G$

$RHS = \begin{bmatrix} 2 & -3 & 3 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} = LHS \quad \square$

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Exercise 9.1. $f(t) = a_1(\sin t - \frac{\sqrt{2}}{2}) + a_2(\cos t - \frac{\sqrt{2}}{2}) + a_3(\sin 2t - 1) + a_4 \cos 2t$.
let $\{\sin t - \frac{\sqrt{2}}{2},\ \cos t - \frac{\sqrt{2}}{2},\ \sin 2t - 1,\ \cos 2t\}$ forms a basis.

Solution: $f(\frac{\pi}{4}) = 0. \Rightarrow a_0 + \frac{\sqrt{2}}{2} a_1 + \frac{\sqrt{2}}{2} a_2 + a_3 = 0. \Rightarrow a_3 = -a_0 - \frac{\sqrt{2}}{2}(a_1 + a_2)$

1) Firstly, we can find that $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ form a basis. let ~~them~~ it be B.

? what does each component mean?

2) $L(1) = 1 = 1 \cdot 1$. $\quad L(\sin t) = \cos t = 1 \cdot \cos t$. $\quad 1, \sin t, \cos t \notin V$.

$L(\cos t) = \sin t = 1 \cdot \sin t \qquad L(\cos 2t) = -\cos 2t$

$A = [L]_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad \chi_A(t) = |tE_n - A| = \begin{vmatrix} t-1 & 0 & 0 & 0 \\ 0 & t & -1 & 0 \\ 0 & -1 & t & 0 \\ 0 & 0 & 0 & t+1 \end{vmatrix} = (t^2-1)^2$

3) find the roots of $\chi_A(t)$. them are $t=1$ and $t=-1$

4) i) For $\lambda_1 = 1$. $\quad \lambda E_n - A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow \begin{cases} x_2 - x_3 = 0 \\ -x_3 + x_2 = 0 \\ 2x_4 = 0 \end{cases}$

then we have $v_1 \in V$. s.t. $v_1 = \alpha + \beta \sin t + \beta \cos t - \sqrt{2}\beta \sin 2t$. (where α, β are parameters)
v_1 is an eigenvectors of L associated with $\lambda_1 = 1$.

ii) For $\lambda_2 = -1$ $\quad \lambda E_4 - A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

we have $v_2 \in V$. s.t. $v_2 = \beta'(\sin t - \cos t) + \alpha' \cos 2t$ (where α', β' are parameters)
v_2 is an eigenvectors of L associated with $\lambda_2 = -1$

Exercise 9.2. $\quad V = \mathbb{C}^2 = \{[c_1, c_2]^T \mid c_1, c_2 \in \mathbb{C}\}$. $\quad V = M_{2,1}(\mathbb{C})$

Solution First we form $v_1 = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, v_2 = \begin{pmatrix}0 & 0\\i & 0\end{pmatrix}, v_3 = \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, v_4 = \begin{pmatrix}0 & 0\\0 & i\end{pmatrix}$ as a basis of V. let it be B.

$v_1 = (1, 0) \quad v_2 = (i, 0) \quad v_3 = (0, 1) \quad v_4 = (0, i)$

$L(v_1) = \begin{pmatrix}1 & 0\\i & 0\end{pmatrix} = v_1 + v_2$. $\quad L(v_2) = \begin{pmatrix}1 & 0\\i & 0\end{pmatrix} = v_1 + v_2$.

$L(v_3) = \begin{pmatrix}0 & 1\\0 & i\end{pmatrix} = v_3 + v_4$. $\quad L(v_4) = \begin{pmatrix}0 & 1\\0 & i\end{pmatrix} = v_3 + v_4$. $\quad [L]_B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

$A = [L]_B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad \chi_A(t) = |tE_n - A| = \begin{bmatrix} t-1 & -1 & 0 & 0 \\ -1 & t-1 & 0 & 0 \\ 0 & 0 & t-1 & -1 \\ 0 & 0 & -1 & t-1 \end{bmatrix} = t^2(t-2)^2$ ✓

the roots of $\chi_A(t)$ are $t=0$ and $t=2$. ✓

$$\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y = z \\ x = -w \end{cases} \quad X = \begin{pmatrix} a \\ b \\ b \\ -a \end{pmatrix}$$

1) For $\lambda_1 = 0$. $|-A| = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$ $\quad \underline{X} = \begin{bmatrix} \alpha \\ -\alpha \\ \beta \\ -\beta \end{bmatrix}$ (where α, β are parameters)

$(a+bi, b-ai)$ $\overset{\shortparallel}{}$ $(z, -iz)^T$.

$v_1' = (\alpha - \alpha i, \beta - \beta i) \in \mathbb{C}^2$ is an eigenvector associated with $\lambda = 0$. X

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{matrix} y = -z. \\ x = w. \end{matrix}$$

2) For $\lambda_2 = 2$. $|2E_n - A| = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ $\Rightarrow \underline{X} = \begin{bmatrix} \alpha' \\ \alpha' \\ \beta' \\ \beta' \end{bmatrix}$

$(a+bi, -b+ai)$ $\overset{\shortparallel}{}$ $(z, iz)^T$.

$v_2' = \begin{pmatrix} \alpha' & \beta' \\ \alpha' i & \beta' i \end{pmatrix}$ is an eigenvector associated with $\lambda = 2$.

$= (\alpha' + \alpha' i, \beta' + \beta' i) \in \mathbb{C}^2$ X

Exercise 9.3.

Proof: "$\Rightarrow$" $\lambda \in F$ is an eigenvalue of A. i.e. $\exists \underline{X} \in M_{n,1}(F)$. $AX = \lambda X$.

i.e. $\begin{pmatrix} B & * \\ 0 & C \end{pmatrix} \begin{pmatrix} \underline{X_1} \\ \underline{X_2} \end{pmatrix} = \lambda \underline{X}$ s.t. $\begin{pmatrix} \underline{X_1} \\ \underline{X_2} \end{pmatrix} = \underline{X}$. $\underline{X_1} \in M_{m \times 1}(F)$ $\underline{X_2} \in M_{n-m \times 1}(F)$.

what if $\underline{X_2} = \underline{0}$?

$\begin{pmatrix} B\underline{X_1} + *\underline{X_2} \\ C\underline{X_2} \end{pmatrix} = \begin{pmatrix} \lambda \underline{X_1} \\ \lambda \underline{X_2} \end{pmatrix} \Rightarrow C\underline{X_2} = \lambda \underline{X_2} \Rightarrow \lambda$ is a eigenvalue of C.

if $\underline{X_2} = 0$. $B\underline{X_1} = \lambda \underline{X_1}$. since. $\underline{X_2}, \underline{X_1}$ can't be zero simultaneously since $\underline{X} \neq 0$.

λ is a eigenvalue of B.

"$\Leftarrow$" w.l.o.g. let λ is an eigenvalue of B.

i.e. $\exists \underline{X} \in M_{m \times 1}(F) \setminus \{0\}$ i.e. $B\underline{X} = \lambda \underline{X}$

now we consider the product $\begin{pmatrix} B & * \\ 0 & C \end{pmatrix} \begin{pmatrix} \underline{X}_{m \times 1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n \times 1} = \begin{pmatrix} B\underline{X} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \underline{X} \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} \underline{X} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

thus. $\exists \underline{X}' = \begin{pmatrix} \underline{X} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in M_{n \times 1}(F)$. s.t. $\lambda \underline{X} = A\underline{X}$.

i.e. λ is an eigenvalue of A.

$|\chi_A(\lambda) = \begin{vmatrix} \lambda E_n - B & * \\ 0 & \lambda E_{n-m} - C \end{vmatrix} = |\lambda E_m - B||\lambda E_{n-m} - C|$

No. 李卄言

Date 2022.11.24

Week 7. (Fri.) 4.7.

Exercise 9.4.

by. Hamilton-Cayley. thm.

we have. $-C_0E_n = C_1A + C_2A^2 + C_3A^3$.

$$-C_0A^{-1} = C_1E_n + C_2A + C_3A^2.$$

$$A = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & -1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 3 & -2 & -2 \\ -3 & 1 & 1 \\ -3 & 3 & 0 \end{pmatrix} . \quad A^3 = \begin{pmatrix} -9 & 4 & 4 \\ 6 & -5 & -2 \\ 6 & -6 & -3 \end{pmatrix}$$

we have
$$\begin{cases} C_0 - C_1 + 3C_2 - 9C_3 = 0. \\ C_0 + C_2 - 5C_3 = 0. \\ C_0 - C_1 - 3C_3 = 0 \\ 2C_1 - 2C_2 + 4C_3 = 0. \end{cases} \Rightarrow \begin{cases} C_0 = 3C_3 \\ C_1 = 0 \\ C_2 = 2C_3 \\ C_3 \text{ is free} \end{cases}$$

$$\Rightarrow \quad -3A^{-1} = 2A + A^2.$$

$$A^{-1} = -\frac{2}{3}A + -\frac{1}{3}A^2 = \begin{pmatrix} \frac{2}{3} & -\frac{4}{3} & 0 \\ -\frac{2}{3} & 0 & \frac{2}{3} \\ -\frac{4}{3} & -\frac{2}{3} & \frac{2}{3} \end{pmatrix} + \begin{pmatrix} -1 & \frac{2}{3} & \frac{2}{3} \\ 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \end{pmatrix}. \square$$

Exercise 9.5.

We need to show that. $\chi_A(A) = 0$. let. $A = [L]_B$

let $B = \left\{ v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ form a basis of V.

$$L(v_1) = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}. \quad L(v_2) = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} \quad L(v_3) = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \quad L(v_4) = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}$$

$$[L]_B = \begin{bmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{bmatrix} \qquad \chi_A(t) = |tE_n - A| = [(t-a)(t-d) - bc]^2 = [t^2 - (a+d)t + (ad-bc)]^2$$
$$= t^4 - 2(a+d)t^3 + ((a+d)^2 + 2ad - 2bc) - 2(a+d)(ad-bc)t + (ad-bc)^2.$$

we need to show $C_0E_n + C_1A + C_2A^2 + C_3A^3 + C_4A^4 = 0$.

let $C = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$. $A = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$ $A^2 = \begin{pmatrix} C^2 & 0 \\ 0 & C^2 \end{pmatrix}$ ...

$$LHS = \begin{pmatrix} (ad-bc)^2 & & & 0 \\ & (ad-bc)^2 & & \\ & & (ad-bc)^2 & \\ 0 & & & (ad-bc)^2 \end{pmatrix} + 2(a+d)(ad-bc)\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} + [(a+d)^2 + 2ad - 2bc]\begin{pmatrix} C^2 & 0 \\ 0 & C^2 \end{pmatrix}$$

$$-2(a+d)(ad-bc)\begin{pmatrix} C^3 & 0 \\ & C^3 \end{pmatrix} + \begin{pmatrix} C^4 & 0 \\ 0 & C^4 \end{pmatrix} = 0_{4\times4} = RHS \quad (t.b.c)$$

Since $\chi_L = \chi_{[L]_B}$. $[\chi_L(L)]_B = [\chi_{[L]_B}(L)]_B = \chi_{[L]_B}([L]_B) = 0$. by lemma 9.9. $\chi_L(L) = 0$.

KOKUYO

(brought forward).

In fact, since $\chi_A(t) = \left(t^2-(a+d)t-(ad-bc)\right)^2 = 0 \iff t^2-(a+d)t+(ab-bc)=0$.

we just need to check $C_0'E_n + C_1'A + C_2'A^2 = 0$. where $C_0 = (ad-bc)$, $C_1 = -(a+d)$, $C_2 = 1$.

that is,
$$LHS = \begin{pmatrix} ad-bc & & 0 \\ & \ddots & \\ 0 & & ad-bc \end{pmatrix} + \begin{pmatrix} -a(a+d) & -c(a+d) & 0 \\ -b(a+d) & -d(a+d) & \\ 0 & & -(a+d)C \end{pmatrix} + \begin{pmatrix} a^2+bc & ac+cd & \\ ab+bd & bc+d^2 & \\ & & C^2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & & \\ & 0 & 0 & 0 \\ & & 0 & 0 \end{pmatrix} = 0_{4\times 4} = RHS \quad \square.$$

Exercise 10.1.

a. L has unique eigenvalue 0 associated with eigenvector any nonzero constant.
$\dim V = n+1$. $\dim V_0(L) = 1$. , by Thm 10.1. L is not diagonalizable. if $n \neq 0$

b. $\forall\, v = \sum_{i=1}^{n} a_i t^i \in V$. $L(v) = \sum_{i=1}^{n} i a_i t^i$

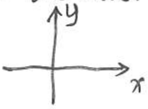
thus we have

eigenvalue	n	$n-1$	$\cdots$	1	0
associated eigenvector	$a_n t^n$	$a_{n-1}t^{n-1}$	$\cdots$	$a_1 t$	a_0

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are arbitrary non-zero number.

thus $\dim V_n(L) + \dim V_{n-1}(L) + \cdots + \dim V_0(L) = n+1 = \dim V$. By thm 10.1.
the L is diagonalizable $\square$.

c. we have $\lambda_1 = 1$. associated ~~with~~ eigenvector, is any vector on x-axis.



$\lambda_2 = -1$ associated eigenvector is any vector on y-axis.
we can extract $\vec{e_1} = \begin{pmatrix}1\\0\end{pmatrix} \in V_{\lambda_1}(L)$. $\vec{e_2} = \begin{pmatrix}0\\1\end{pmatrix} \in V_{\lambda_2}(L)$.
for any $v \in$ the plane. $\exists\, a_1, a_2 \in \mathbb{R}$. $v = a_1\vec{e_1} + a_2\vec{e_2}$
$\{e_1, e_2\}$ form a basis of the plane. by thm 10.1, reflection is diagonalizable $\square$.

Exercise 10.2.

Find the eigenvalue(s) of L. $\dim V = 4$.

$$\chi_A(t) = |tE_n - A| = \begin{vmatrix} t & 1 & -2 & 1 \\ 2 & t-1 & 4 & -2 \\ 2 & -2 & t+5 & -2 \\ 1 & -1 & 2 & t \end{vmatrix} = (t+1)^4.$$ L has unique eigenvalue $\lambda = -1$.

$\exists\, \bar{X}$. $A\bar{X} = -\bar{X}$. $=(A+E_4)\bar{X} = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -2 & 2 & -4 & 2 \\ -2 & 2 & -4 & 2 \\ -1 & 1 & -2 & 1 \end{pmatrix}\bar{X} = 0$. let $\bar{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$

only one restriction: $x_1 - x_2 + 2x_3 - x_4 = 0$. the $\dim V_{-1}(L) = 3 < \dim V = 4$.

L is not diagonalizable.

No. 李开言

Date 2022111221

4.11 (Week 8. Tue.)

Exercise 10.3.

Solution: Let $L \in \mathcal{L}(M_{3,1}(F))$ $L(X) = AX$.

Let $\chi_A(t) = |tE_n - A| = \begin{vmatrix} t+7 & 4 & 4 \\ -8 & t-5 & -4 \\ -4 & -2 & t-3 \end{vmatrix} = (t-1)^2(t+1)$.

i). $\lambda = 1$. $\begin{pmatrix} 8 & 4 & 4 \\ -8 & -4 & -4 \\ -4 & -2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow 2x+y+z=0$. $X = \begin{pmatrix} -\frac{a+b}{2} \\ a \\ b \end{pmatrix}$.

ii) $\lambda = -1$ $\begin{pmatrix} 6 & 4 & 4 \\ -8 & -6 & -4 \\ -4 & -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+y=0 \\ x+2z=0 \end{cases}$ $X = \begin{pmatrix} a \\ -a \\ -\frac{a}{2} \end{pmatrix}$

$\dim V_1(L) + \dim V_{-1}(L) = \dim L$. L is diagonalizable.

$A = [L]_F$. where F is standard basis. A is diagonalizable.

we learn that $C = \left\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -\frac{1}{2} \end{pmatrix} \right\}$ forms a basis of L.

and such that $[L]_C$ is diagonal. $[L]_C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} := D$.

thus. we denote $U = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \end{pmatrix}$ which has C as its columns.

that is what we want.

Exercise 10.4.

Proof: If an operator $L \in \mathcal{L}(V)$ is nilpotent. then $\chi_N(t) = t^n$.

that is, $\lambda = 0$ is its only eigenvalue.

we have $AX = 0$ $(A = [L]_B$. B is some basis of $V)$.

the L is diagonalizable if and only if. dimension of eigenspace associated to 0.

equals $\dim V$. We have the dimension of eigenspace $= \dim V - \operatorname{rank}(A)$.

thus. $\dim V_0(L) = \dim V$ if and only if $\operatorname{rank}(A) = 0$.

that means $A = 0$. i.e. L is zero operator.

(t, b, c)

No.

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Exercise 10.5

Proof: "$\Rightarrow$" Since N is nilpotent, there exist a basis C of V.

s.t. $[N]_C$ is strictly upper triangular.

if $B=C$. then we are done

if $B \neq C$. by thm 6.7. we have. $[N]_B = M_{B\to C} [N]_C M_{C\to B}$ where $M_{B\to C}$ is invertible

since. $[N]_C^n = 0$. (by calculation, every time we multiple $[N]_C$, turns the term parallel to the diagonal to 0.

then $[N]_B^n = (M_{B\to C} [N]_C M_{B\to C}^{-1})(M_{B\to C} [N]_C M_{B\to C}^{-1}) \cdots (M_{B\to C} [N]_C M_{B\to C}^{-1})$

$= M_{B\to C} [N]_C^n M_{B\to C}^{-1} = 0.$

by def. $[L]_B^n = 0$ shows $[N]_B$ is nilpotent.

"$\Leftarrow$" obviously. $[N]_B^j = 0$ for some $j \in \mathbb{N}$ (by definition)

for any $v \in V$. suppose its coordinates w.r.t B is $[v]_B$.

then $[N^j(v)]_B = [N]_B^j [v]_B = 0.$

i.e. $N^j(v) = 0.$

~~B~~ By Pro 10.4. N is nilpotent

2 2 2

Campus

No. 李开言

Date 20221122

Week 8 Thu. Fri.

Exercise 10.6.

Solution: Consider $L \in \mathcal{L}(M_{3,1}(F))$. $L(X) = AX$.

Suppose there exist a basis C in $M_{3,1}(F)$. s.t. $Q = [L]_C = \begin{bmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix}$

~~thus we have~~ let $C = \{c_1, c_2, c_3\}$.

we have $L(c_1) = \begin{pmatrix} 3 & 1 & 3 \\ -4 & -2 & -4 \\ -1 & 0 & -1 \end{pmatrix} \begin{bmatrix} c_1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$L(c_2) = \begin{pmatrix} 3 & 1 & 3 \\ -4 & -2 & -4 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_2 \end{pmatrix} = x c_1$ let $x = 1$. $\Rightarrow c_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

$L(c_3) = \begin{pmatrix} 3 & 1 & 3 \\ -4 & -2 & -4 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} c_3 \end{pmatrix} = y c_1 + z c_2$. let $y = z = 1$. $\Rightarrow c_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Since we have $A = [L]_F$. F is the standard basis of $M_{n,1}(F)$

Clearly L is nilpotent iff A is nilpotent

we can denote $U = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ (has C as its column).

Please check c_1, c_2, c_3 are linerly independent.

Verify: $A = UQU^{-1} \iff AU = UQ$.

$AU = \begin{bmatrix} 3 & 1 & 3 \\ -4 & -2 & -4 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$UQ = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

$AU = UQ$.

thus. the equality holds.

Exercise 11.1.

1°. $V(\lambda, L) \subset V(\lambda^{-1}, L^{-1})$

$\forall v \in V(\lambda, L)$. $v \neq 0$. $\exists m \geq 0$. $(L - \lambda id_V)^m(v) \neq 0$. $(L - \lambda id_V)^{m+1}(v) = 0$

we denote. $(L - \lambda id_V)^m(v) = u$. $\Rightarrow$ ~~$\Leftarrow$~~ $v = (L - \lambda id_V)^{-m}(u)$

$\Rightarrow (L - \lambda id_V)(u) = 0$. $\Rightarrow L(u) = \lambda u \Rightarrow L^{-1}(\lambda u) = u$.

since L^{-1} is also subspace. $\lambda L^{-1}(u) = u \Rightarrow L^{-1}(u) = \lambda^{-1} u$. $\Rightarrow (L^{-1} - \lambda^{-1} id_V)(u) = 0$.

u is an eigenvector of L^{-1} associated with λ^{-1}

$\therefore (L - \lambda id_V)^{m+1} \underset{\text{premultiple } L^{-(m+1)}}{=} (L^{-1}L - L^{-1}\lambda id_V)^{m+1} = (id_V - \lambda L^{-1})^{m+1} = \lambda^{m+1}(\lambda^{-1} id_V - L^{-1})^{m+1}$

~~$= 0$~~

$\overset{?}{=} 0$ why?

No.

Date

N is missing here

why?

we have $(\lambda^{-1}id_V - L^{-1})^{m+1} = 0$ i.e. $(L^{-1}-\lambda^{-1}id_V)^{m+1}(v) = 0$.

thus $v \in V(\lambda^{-1}, L^{-1})$. since v is arbitrary from $V(\lambda, L)$. thus $V(\lambda,L) \subset V(\lambda^{-1},L^{-1})$

2° $V(\lambda^{-1},L^{-1}) \subset V(\lambda, L)$

$\forall v' \in V(\lambda^{-1},L^{-1})$. i.e. for some $m \in \mathbb{N}$. $(L^{-1}-\lambda^{-1}id_V)^m(v) = 0$.

$(L^{-1}-\lambda^{-1}id_V)^m \xlongequal{\text{multiple }(\lambda L)^m} (\lambda id_V - L)^m = 0$

i.e. $(L-\lambda id_V)^m = 0$. i.e. $v \in V(\lambda, L)$. i.e. $V(\lambda^{-1},L^{-1}) \subset V(\lambda, L)$

thus we have $V(\lambda^{-1},L^{-1}) = V(\lambda, L)$

Exercise 11.2.

Solution: i) let F be standard basis of V. $A = [L]_F = \begin{bmatrix} 1&1&1&1&1&1 \\ 0&1&0&2&1&0 \\ 0&0&1&0&1&2 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}$

$\chi_{[L]_F}(t) = |tE_n - [L]_F| = (t-1)^6$

V has unique eigenvalue $\lambda = 1$.

we have the equality $(A-E_6)X = 0$. $X = \begin{pmatrix} a \\ b \\ -b \\ c \\ -2c \\ c \end{pmatrix}$ where a.b.c are parameters

let $A - \lambda E_6 = B$. $B^2 = \begin{pmatrix} 0&0&0&2&2&2 \\ & & 0 & & & \end{pmatrix}$ $B^3 = 0$

thus we have for any $v \in V$. $(L-\lambda id_V)^3(v) = 0$.

In conclusion. we have eigenvalue $\lambda = 1$.

eigenspace $V_{\lambda=1}(L) = \{a + bt - bs + ct^2 - 2cts + cs^2 / a, b, c \in \mathbb{R}\}$.

generalized eigenspace $V(1, L) = V$.

ii) since $V(1,L) = V$. let $B = \{1, t, s, t^2, ts, s^2\}$.

$[L]_B|_{V(\lambda,L)} = [L|_{V(\lambda,L)}]_B = \begin{bmatrix} 1&1&1&1&1&1 \\ 0&1&0&2&1&0 \\ 0&0&1&0&1&2 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}$

iii) $[L|_{V(\lambda,L)}]_B - \lambda E_n = \begin{bmatrix} 0&1&1&1&1&1 \\ 0&0&0&2&1&0 \\ 0&0&0&0&1&2 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}$ $([L|_{V(\lambda,L)}]_B - \lambda E_n)^2 = \begin{pmatrix} 0 & \begin{matrix} 2&2&2 \\ 0&0&0 \end{matrix} \\ 0 & 0 \end{pmatrix}$

$([L|_{V(\lambda,L)}]_B - \lambda E_n)^3 = 0$ thus $(L|_{V(\lambda,L)} - \lambda id_{V(\lambda,L)})^3 = 0$

i.e. $L|_{V(\lambda,L)} - \lambda id_{V(\lambda,L)}$ is nilpotent.

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Exercise 12.1

Solution: let $B=\{1, t, s, t^2, st, s\}$. B forms a basis of V.

$$[L]_B=\begin{bmatrix} 1&1&1&1&1&1\\ &1&0&2&1&0\\ &&1&0&1&2\\ &&&1&0&0\\ &0&&&1&0\\ &&&&&1 \end{bmatrix}$$

$\chi_L(t)=(t-1)^6$. L has unique eigenvalue 1. and eigenvectors $x=\begin{pmatrix}a\\b\\-b\\c\\-2c\\c\end{pmatrix}$

put $C=[L]_B-E_4=\begin{bmatrix} 0&1&1&1&1&1\\ &0&0&2&1&0\\ &&0&0&1&2\\ &&&0&0&0\\ &0&&&0&0\\ &&&&&0 \end{bmatrix}$

To find independent non-trivial zero linear combinations of the rows of C.

we solve $C^Tz=0$

$\Rightarrow$ we have $z=\begin{pmatrix}z_1\\z_2\\ \vdots\\z_6\end{pmatrix}=z_4\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}+z_5\begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix}+z_6\begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$.

Extend C, we form $\hat{C}=\begin{bmatrix} & & C & & & \\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}$

Then solve $\hat{C}\bar{X}_1=0$. we have $\bar{X}_1=\begin{pmatrix}x_1\\x_2\\x_3\\ \vdots\\x_6\end{pmatrix}=x_1\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}$

$C\bar{X}=0$ we have $\bar{X}=x_1\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}+x_4\begin{pmatrix}0\\0\\0\\1\\-2\\1\end{pmatrix}$

the extension is $(0\ 0\ 0\ 1\ -2\ 1)^T$

① Let $\hat{C}\bar{X}_2=\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}$ $\Rightarrow$ $\bar{X}_2=x_1\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}$ which is solvable.

Let $\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}=\bar{X}_j^{(2)}$. $\hat{C}\bar{X}=\bar{X}_j^{(2)}$ is not solvable.

Let $C\bar{X}=\begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}$ $\Rightarrow$ $\bar{X}=x_1\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}+x_4\begin{pmatrix}0\\0\\0\\1\\-2\\1\end{pmatrix}+\begin{pmatrix}0\\0\\0\\0\\0\\ \frac{1}{2}\end{pmatrix}$

Hence let $\begin{pmatrix}0\\0\\0\\0\\0\\ \frac{1}{2}\end{pmatrix}$ be a basis vector

③ let $\hat{C}\underline{X} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ which is inconsistent.

hence, let. $C\underline{X} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$. $\Rightarrow \underline{X} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ Jordan basis $J = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$.

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 12.2.

$$\chi_A(t) = \begin{vmatrix} t-1 & 0 & -1 & -2 \\ 0 & t-1 & 0 & -1 \\ 0 & 0 & t-1 & -1 \\ 0 & 0 & 0 & t-2 \end{vmatrix} = (t-1)^3(t-2)$$

$C_1 = A - E_4 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad C_2 = A - 2E_4 = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

similarly as 12.1. We need to solve ① $C_1^T z = 0$

$z = z_3 \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z_4 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow$ extend to $\hat{C}_1 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

② $C_2^T z = 0$. $\Rightarrow z = z_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \hat{C}_2 = \begin{bmatrix} C \\ 0\ 0\ 0\ 1 \end{bmatrix}$

Now to $\lambda = 1$. Let $\hat{C}_1 \underline{X} = 0$. $\Rightarrow \underline{X_1} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

To $C_1 \underline{X}$. $\underline{X} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ let. $\underline{X}_1^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad Y_1^{(1)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

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To $\lambda=2$. Let $\hat{C}_2\bar{X}=0$. $\Rightarrow \bar{X}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ has only trivial solution.

then let $C_2\bar{X}=0$. $\bar{X}=x_4\begin{pmatrix}3\\1\\1\\1\end{pmatrix}$.

Let $\bar{X}_2^{(1)}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ $\quad Y_2^{(1)}=\begin{bmatrix}3\\1\\1\\1\end{bmatrix}$

Now to $\lambda=1$. $\hat{C}_1\bar{X}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $\Rightarrow$ inconsistent.

$C_1\bar{X}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\Rightarrow \bar{X}=x_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$

$\Rightarrow$ let $Y_1^{(2)}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$ $\quad Y_2^{(2)}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

to $\lambda=2$. ~~let~~ we have $Y_2^{(1)}=\begin{pmatrix}3\\1\\1\\1\end{pmatrix}$.

$\Rightarrow$ In conclusion. $J=\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\cancel{1}\,2\end{bmatrix}$

Exercise 12.3

$A=\begin{bmatrix}0&-1&1&0\\0&-2&1&1\\-1&0&-1&1\\1&-1&1&-1\end{bmatrix}$

S: Consider $L\in\mathcal{L}(M_{n,1}(F))$. $L(X)=AX$.

Let $\mathcal{F}$ be standard basis. thus $[L]_{\mathcal{F}}=A$.

$\chi_L(t)=\begin{vmatrix}t&1&-1&0\\0&t+2&-1&-1\\1&0&t+1&-1\\-1&1&-1&t+1\end{vmatrix}=(t+1)^4$ unique eigenvalue $\lambda=-1$.

let $C=[L]_{\mathcal{F}}+E_4=\begin{bmatrix}1&-1&1&0\\0&-1&1&1\\-1&0&0&1\\1&-1&1&0\end{bmatrix}$

then solve the L.S. $C^T Z = 0$. $\begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & -1 & 0 & -1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = 0.$

$$\Rightarrow Z = z_3 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + z_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

extend to $\hat{C} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$

we have no distinct row between C and $\hat{C}$. their basis are same.

① let $\hat{C}\underline{X} = 0$. $\Rightarrow \underline{X} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

② let $\hat{C}\underline{X} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ $\Rightarrow$ inconsistent.

let $C\underline{X} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $\Rightarrow \underline{X} = x_3 \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$

let $Y_1^{(2)} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$

③ let $\hat{C}\underline{X} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ inconsistent

then $C\underline{X} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow \underline{X} = x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$

let $Y_2^{(2)} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$

In conclusion. $J = \begin{bmatrix} -1 & 1 & 0 & 0 \\ & -1 & 0 & 0 \\ 0 & & -1 & 1 \\ & & & -1 \end{bmatrix}$

Jordan basis $C = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\}.$

denote $U = M_{F \to C}$.

$U = \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (has C in its columns).

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Exercise 12.4.

Proof: Consider $L \in \mathcal{L}(M_{n,1}(\mathbb{C}))$, $L(X) = AX$.

By thm 12.4. there exists a basis $\mathcal{B}$ s.t. $[L]_{\mathcal{B}} = J$. (J is Jordan matrix

then denote $\mathcal{B}'$ ~~w~~ w.r.t to $[L]_{\mathcal{B}'} = J'$ is a subdiagonal alternative form.

Thus, by thm 6.8..

we can find an invertible $U \in M_n(\mathbb{C})$ s.t. $J = UJ'U^{-1}$

let F be standard basis of L. $[L]_F = A$.

there exist an invertible matrix $U' \in M_n(\mathbb{C})$, s.t. $J = U'AU'^{-1}$

thus we have $A = (U'^{-1}U)J'(U^{-1}U')$

$(U'^{-1}U)(U^{-1}U') = E_n$, which is inverse to each other. $\square$.

$A \downarrow J \downarrow J'$ why not?

Exercise 12.5.

$$\chi_A(t) = \begin{vmatrix} t+1 & 0 & 1 \\ -1 & t+2 & 0 \\ 1 & 1 & t \end{vmatrix} = (t+1)^3. \quad \text{unique eigenvalue } \lambda = -1.$$

put $C = A + E_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$. solve $C\underline{X_1} = 0$. $\underline{X_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t_1$

$$\Rightarrow C\underline{X_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \underline{X_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t_2 + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\Rightarrow C\underline{X_3} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow \underline{X_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} t_3 + \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

thus we have the transition matrix $U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix}$ $U^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{pmatrix}$

Let $A = [L]_{\mathcal{B}}$ $\quad A' = J = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$$A^{101} = U[A']^{101}U^{-1} =$$

$$A'^{101} = \begin{pmatrix} (-1)^{101} & \binom{101}{1} & -\binom{101}{2} \\ & -1 & \binom{101}{1} \\ & & -1 \end{pmatrix} = \begin{pmatrix} -1 & 101 & -5050 \\ 0 & -1 & 101 \\ & & -1 \end{pmatrix}.$$

$$A^{101} = \begin{pmatrix} 5049 & -4848 & 4949 \\ 5151 & -4948 & 5050 \\ 101 & -99 & 100 \end{pmatrix}$$

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Exercise 12.6.

$$B^2 = \begin{pmatrix} 5 & -6 & 2 \\ 4 & -5 & 2 \\ 4 & -6 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 4 & -6 & 2 \\ 4 & -6 & 2 \\ 4 & -6 & 2 \end{pmatrix}.$$

by lemma 12.8. $\exists S \in M_3(F)$. s.t. $S^2 = E_3 + \begin{pmatrix} 4 & -6 & 2 \\ 4 & -6 & 2 \\ 4 & -6 & 2 \end{pmatrix}$.

$$S = E_3 + \sum_{j=1}^{2} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-j+1)}{1\cdot 2 \cdots j} Q^j = E_3 + \frac{1}{2}Q + 0\cdot Q^2 = \begin{pmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 2 & -3 & 2 \end{pmatrix}.$$

Exercise 12.7.

Let B be a basis of V. $[L]_B$ is invertible. since L is invertible

By Pro 12.9. $\exists A \in M_n(F)$. $A^2 = [L]_B$. A is invertible.

since L is invertible. $\exists T \in \mathcal{L}(V)$. $[T]_{B'} = A$ (B' is some basis of V).

thus we have $T^2 = L$.

and since A is invertible. T is invertible.

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Exercise 13.1

Solution for $f(X,Y,Z) = 2X^2+4XY+3XZ+Y^2+2YZ-Z^2$

$$= -\left(Z-Y-\tfrac{3}{2}X\right)^2 + 2\left(Y+\tfrac{7}{4}X\right)^2 - \tfrac{15}{8}X^2$$

Let $X' = X \quad Y' = Y+\frac{7}{4}X \quad Z' = -\frac{3}{2}X - Y + Z$.

thus we have $\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{4} & 1 & 0 \\ -\frac{3}{2} & -1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$

denote $\begin{pmatrix} 1 & 0 & 0 \\ \frac{7}{4} & 1 & 0 \\ -\frac{3}{2} & -1 & 1 \end{pmatrix} = L$ we have $A = L^T A' L$. where $A' = \begin{pmatrix} -\frac{15}{8} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$A' = L^T A L$

Exercise 13.2.

~~by~~ $G_B = C G_{B'} C^T$ ✗ $G_{B'} = C^T G_B C$

proof: by def. of Gram matrix. $[V]_B$, $[V]_{B'}$ be the coordinates column

$q(v) = [V]_B^T G_B [V]_B = [V]_{B'}^T G_{B'} [V]_{B'}$

$[V]_{B'} = [V]_B^{(T)?} M_{B\to B'} = [V]_B^T C$

Thm. 6.6 $[V]_B = M_{B\to B'} [V]_{B'}$

$[V]_{B'}^T = ([V]_B^T C)^T = C^T [V]_B$

$G_{B'} = M_{B\to B'}^T G_B M_{B\to B'}$

$\Rightarrow \quad [V]_B^T G_B [V]_B = [V]_B^T C G_{B'} C^T [V]_B$

$\Rightarrow \quad G_B = C G_{B'} C^T$

Exercise 13.3.

Proof: "$\Rightarrow$" $h(x,y) = h(y,x)$. Let B be standard basis $\{v_1, \dots, v_n\}$ $v_i = (0 \dots \overset{i\text{th position}}{i} \dots 0)$

$h(x,y) = \sum_{i,j=1}^{n} a_{ij} h(v_i, v_j) \qquad h(y,x) = \sum_{i,j=1}^{n} a_{ji} h(v_j, v_i)$

since. $h(v_i,v_j) = h(v_j,v_i)$. then we have $a_{ji} = a_{ij}$. i.e. $A^T = A$.

"$\Leftarrow$" $A^T = A$.

$h(x,y) = x^T A y = (x^T A y)^T$ (the size of matrix is 1×1)

$= \cancel{XA}\ y^T A^T \cancel{X} = y^T A X = h(y,x)$.

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Ex 13.4. (1) proof: let $B = \{e_1, e_2\}$ be standard basis of $\mathbb{R}^2$.

"$\Rightarrow$" $q(xe_1 + ye_2) = (x, y)\, G\, (x, y)^T = x^2 - y^2$.

thus $G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Let $L\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$. $A = [L]_B$. $A^T G A = G$.

let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \begin{cases} ab - cd = 0 \\ a^2 - c^2 = 1 \\ b^2 - d^2 = -1 \end{cases}$

$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$. 并

if $x^2 - y^2 = 1$, $(ax+by)^2 - (cx+dy)^2 = (a^2x^2 - c^2x^2) + (b^2y^2 - d^2y^2) = x^2 - y^2 = 1$.

"$\Leftarrow$" similarly. we let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$(ax+by)^2 - (cx+dy)^2 = x^2 - y^2 = 1$. $\xrightarrow{\text{more comments}}$ We also have $\begin{cases} ab - cd = 0 \\ a^2 - c^2 = 1 \\ b^2 - d^2 = -1 \end{cases}$

thus we have $A^T G A = G$. by Pro 13.5. L is orthogonal w.r.t q.

(2) "$\Rightarrow$" by prove of (1).

we denote $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and have. $\begin{cases} ab - cd = 0 \\ a^2 - c^2 = 1 \\ b^2 - d^2 = -1 \end{cases}$

$\det A = ad - bc = 1$. $\Rightarrow \begin{cases} ab - cd = 0 & ① \\ ad - bc = 1 & ② \\ a^2 - c^2 = 1 & ③ \\ b^2 - d^2 = -1 & ④ \end{cases}$

by ① $ab = cd$: if $d = 0$. b^2 has no real root.

if $c = 0$ $\begin{cases} a = 1 \\ b = 0 \\ d = 1 \end{cases}$

if $d, c \neq 0$. let $\frac{a}{d} = \frac{b}{c} = k$. substitute in ②. $(d^2 - b^2)k = 1$. $\Rightarrow k = 1$ by ④.

$\Rightarrow a = d,\ b = c$ $a^2 = b^2 + 1$.

let $b = \sinh t$ $a = \pm\cosh t$. thus we have $A = \pm\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$. t is parameter

$b = -\sinh t$ $a = \pm\cosh t$.

"$\Leftarrow$". $A = \pm\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ we have the $\begin{cases} ab - cd = 0 \\ a^2 - c^2 = 1 \\ b^2 - d^2 = -1 \end{cases} \Rightarrow A^T G A = G$ holds.

by pro 13.5. L is orthogonal w.r.t q.

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Ex 13.5 (1)

proof: 1° Let $\mathcal{B}$ be standard basis ($v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. $v_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $v_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$).

$q(\underline{X}) = q(av_1 + bv_2 + cv_3 + dv_4) = (a\ b\ c\ d)\, A\, (a\ b\ c\ d)^T = ad - bc.$

where $A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$ A is symmetric. thus q is quadratic

2° let $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$. $C^{-1} = \det C^{-1} \begin{pmatrix} c_{22} & -c_{21} \\ -c_{12} & c_{11} \end{pmatrix}$

$\gamma(v_1) = \det C^{-1} \begin{pmatrix} c_{11}c_{22} & -c_{11}c_{21} \\ c_{21}c_{22} & -c_{21}^2 \end{pmatrix}$ $\gamma_3(v_3) = \det C \begin{pmatrix} c_{12}c_{22} & -c_{12}c_{21} \\ c_{22}^2 & -c_{21}c_{22} \end{pmatrix}$

$\gamma(v_2) = \det C \begin{pmatrix} -c_{11}c_{12} & c_{11}^2 \\ -c_{12}c_{21} & c_{11}c_{21} \end{pmatrix}$ $\gamma_4(v_4) = \det C \begin{pmatrix} -c_{12}^2 & c_{11}c_{12} \\ -c_{12}c_{22} & c_{11}c_{22} \end{pmatrix}$

$[\gamma_C]_{\mathcal{B}} = \det C \begin{bmatrix} c_{11}c_{22} & -c_{11}c_{12} & c_{12}c_{22} & -c_{12}^2 \\ -c_{11}c_{21} & c_{11}^2 & -c_{12}c_{21} & c_{11}c_{12} \\ c_{21}c_{22} & -c_{12}c_{21} & c_{22}^2 & -c_{12}c_{22} \\ -c_{21}^2 & c_{11}c_{21} & -c_{21}c_{22} & c_{11}c_{22} \end{bmatrix}$

since we have $[\gamma_C]_{\mathcal{B}}\, A\, [\gamma_C]_{\mathcal{B}}^T = A$. by Pro 13.5. γ_C is orthogonal w.r.t q

13.5 (2) proof:

similar as 1° form the standard basis. $\mathcal{B}$.

by ~~q~~ $\varphi(X) = a^2 + 2bc + d^2$ $A' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ A' is symmetric.

~~since~~ thus φ is quadratic form. on V.

the $[\gamma_C]_{\mathcal{B}}$ is completely same as the matrix $[\gamma_C]_{\mathcal{B}}$ in 1.

Check. $[\gamma_C]_{\mathcal{B}} A' [\gamma_C]_{\mathcal{B}}^T = A'$ holds as well. there are simpler ways

then by Pro 13.5. we have γ_C is orthogonal w.r.t φ.

Exercise 13.6.

(1) proof: "$\subset$".

$\forall v \in A^{\perp}$. we need to check. ~~$\forall \in$~~ $v \in (\mathrm{span}(A))^{\perp}$

it is sufficient to check. $h(v,u)=0$ for all $u \in \mathrm{span}(A)$

if $u \in A$. trivial

if $u \in \mathrm{span}(A) \setminus A$. $\exists u_1 \in A$, s.t. $u = a u_1$. $h(v,u) = h(v, au_1) = a\,h(v,u_1) = 0$.

thus. $v \in (\mathrm{span}(A))^{\perp}$

"$\supset$"

$\forall v \in (\mathrm{span}(A))^{\perp}$ it is sufficient to check. $h(v,u)=0$ for all $u \in A$.

since $A \subset \mathrm{span}(A)$. all $u \in A \subset \mathrm{span}(A)$. ~~then v~~

thus $v \in A^{\perp}$ $\square$.

(2) i/ additivity

$\forall v_1, v_2 \in A^{\perp}$.

$h(v_1+v_2, u) = h(v_1; u) + h(v_2; u) = 0+0=0. \Rightarrow v_1+v_2 \in A^{\perp}$

ii/ scalar multiplicatio. $\forall v \in A^{\perp}$, $a \in F$.

$h(av, u) = a \cdot h(v,u) = a \cdot 0 = 0 \quad \Rightarrow av \in A^{\perp}$.

thus $A^{\perp}$ is a subspace of V.

Exercise 13.7.

S: since $(U_1 \cap U_2)^{\perp} = U_1^{\perp} + U_2^{\perp}$

① find $U_1^{\perp}$ $\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 + x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{1}{3}x_3 + \frac{2}{3}x_4 \\ x_2 = -\frac{2}{3}x_3 - \frac{1}{3}x_4 \end{cases}$ $\bar{X} = x_3 \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$

$U_1^{\perp} = \mathrm{span}\left(\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \right)$

② $U_2^{\perp}$ $\begin{cases} 2x_1 - x_2 + x_4 = 0 \\ x_1 - x_2 + 3x_3 + 7x_4 = 0 \end{cases} \Rightarrow U_2^{\perp} = \mathrm{span}\left(\begin{pmatrix} 3 \\ 6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 13 \\ 0 \\ 1 \end{pmatrix} \right)$.

since $\begin{pmatrix} 6 \\ 13 \\ 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \\ 1 \\ 0 \end{pmatrix} - 2\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}$ $U_1^{\perp} + U_2^{\perp} = \mathrm{span}\left(\begin{pmatrix} 3 \\ 6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} \right)$

to $U_1 \cap U_2$. all vector in it should orthogonal with $\forall u \in U_1^{\perp} + U_2^{\perp}$.

let $\forall u' \in U_1 \cap U_2$. $u' = (x_1', x_2', x_3', x_4')^T$.

$\begin{cases} 3x_1' + 6x_2' + x_3' = 0 \\ \frac{1}{3}x_1' - \frac{2}{3}x_2' + x_3' = 0 \\ \frac{2}{3}x_1' - \frac{1}{3}x_2' + x_4' = 0 \end{cases} \Rightarrow \begin{cases} x_1' = -\frac{5}{4}x_4' \\ x_2' = \frac{1}{2}x_4' \\ x_3' = \frac{3}{4}x_4' \end{cases}$ $U_1 \cap U_2 = \mathrm{span}\left(\begin{pmatrix} -\frac{5}{4} \\ \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} \right)$

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No. 李昇言 Date 2022/11/22

Week 12 Tue (Ex 13.7 ~~is~~ was handed last time).

Exercise 14.1.

(1) Proof: let $f(X) = \sum_{i=0}^{3} a_i v_i$ $\quad g(X) = \sum_{i=0}^{3} b_i v_i$

$h(f(X), g(X)) = \int_{-1}^{1} f(x) g(x)\, dx = \int_{-1}^{1} g(x) f(x) = h(g(X), f(X))$.

Thus h is symmetric

$\forall \alpha \in F.\ \forall f_1, f_2 \in \mathbb{R}[X]^3.\ \forall g_1, g_2 \in \mathbb{R}[X]^3$

$h(\alpha f_1 + f_2, g_1) = \int_{-1}^{1} (\alpha f_1 + f_2) g_1 = \int_{-1}^{1} \alpha f_1 g_1 + f_2 g_1 = \alpha \int_{-1}^{1} f_1 g_1 + \int_{-1}^{1} f_2 g_1 = \alpha h(f_1, g_1) + h(f_2, g_1)$

$h(f_1, \alpha g_1 + g_2) = \int_{-1}^{1} f_1 (\alpha g_1 + g_2) = \alpha \int_{-1}^{1} f_1 g_1 + \int_{-1}^{1} f_1 g_2 = \alpha h(f_1, g_1) + h(f_1, g_2)$

Thus h is bilinear

(2) by. Thm 14.2. We have $e_n = v_n - \sum_{k=1}^{n-1} \frac{h(v_n, e_k)}{h(e_k, e_k)}$.

let $B_0 = \{e_0, e_1, e_2, e_3\}$

$e_0 = v_0 = 1$

$e_1 = v_1 - \frac{h(v_1, e_0)}{h(e_0, e_0)} = X - \frac{0}{2} = X.$

$e_2 = v_2 - \frac{h(v_2, e_1)}{h(e_1, e_1)} e_1 - \frac{h(v_2, e_0)}{h(e_0, e_0)} e_0 = X^2 - \frac{\frac{2}{3}}{2} = X^2 - \frac{1}{3}.$

$e_3 = v_3 - \frac{h(v_3, e_2)}{h(e_2, e_2)} e_2 - \frac{h(v_3, e_1)}{h(e_1, e_1)} e_1 - \frac{h(v_3, e_0)}{h(e_0, e_0)} e_0 = X^3 - \frac{3}{5} X.$

Thus we find $B_0 = \{1, X, X^2 - \frac{1}{3}, X^3 - \frac{3}{5} X\}$.

Exercise 14.2.

$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$. denote rows of A by. $v_1 = (1\ 1\ 1)$ $\quad v_2 = (-1\ 1\ 0)$ $\quad v_3 = (-1\ 1\ 1)$.

Consider h be the dot product. Now apply the orthogonalization prosess

put $e_1 = v_1$

$e_2 = v_2 - \frac{h(v_2, e_1)}{h(e_1, e_1)} e_1 = v_2 - 0 e_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}^T = v_2$

$e_3 = v_3 - \frac{h(v_3, e_2)}{h(e_2, e_2)} e_2 - \frac{h(v_3, e_1)}{h(e_1, e_1)} e_1 = v_3 - e_2 - \frac{1}{3} e_1 = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}^T = v_3 - v_2 - \frac{1}{3} v_1$

thus we have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{3} & -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$

that is $\begin{pmatrix}1&0&0\\0&1&0\\-\frac{1}{3}&-1&1\end{pmatrix}A=\begin{pmatrix}1&1&1\\-1&\phi&0\\-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix}$.

$$A=\begin{pmatrix}1&0&0\\0&1&0\\\frac{1}{3}&1&1\end{pmatrix}\begin{pmatrix}1&1&1\\-1&1&0\\-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix}$$

In order to make the second factor to be orthogonal matrix we need to divide the rows by their Length.

$$A=\begin{pmatrix}1&0&0\\0&1&0\\\frac{1}{3}&1&1\end{pmatrix}\begin{pmatrix}\sqrt{3}&0&0\\0&\sqrt{2}&0\\0&0&\sqrt{\frac{2}{3}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{3}}&0&0\\0&\frac{1}{\sqrt{2}}&0\\0&0&\frac{\sqrt{3}}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1&1&1\\-1&1&0\\-\frac{1}{3}&-\frac{1}{3}&\frac{2}{3}\end{pmatrix}$$

$$=\underbrace{\begin{pmatrix}\sqrt{3}&0&0\\0&\sqrt{2}&0\\\sqrt{\frac{2}{3}}&\sqrt{2}&\sqrt{\frac{2}{3}}\end{pmatrix}}_{L'}\underbrace{\begin{pmatrix}\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}&\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}&0\\-\sqrt{\frac{1}{6}}&-\sqrt{\frac{1}{6}}&\sqrt{\frac{2}{3}}\end{pmatrix}}_{Q'}$$

Method II $A_k=$ Gram matrix $h|_{span(e_1\dots e_k)}$.

$B=\{e_1\dots e_n\}$ is standard basis in $\mathbb{R}^n$.

$h(x^T,y)=x^TAy=(x_1\dots x_k)A_k\begin{pmatrix}y_1\\\vdots\\y_k\end{pmatrix}$

let an orthogonal basis $B'=\{e_1'\dots e_n'\}$. let. $C=M_{B\to B'}$.

$A'=C^TAC$. new Gram matrix

$\Rightarrow A_k'=C_k^TA_kC_k$ (checked)

Denote $A_k=(C_k^T)^{-1}A_k'C_k^{-1}$

$\downarrow$ diagonal

Exercise 14.3.

Pf: 1°: Assume the converse. $\exists k$, $|A_k|=0$

we part the L.D. $L=\begin{pmatrix}L_k&0\\B&L'\end{pmatrix}$ $D=\begin{pmatrix}D_k&0\\0&D'\end{pmatrix}$.

$$LDL^T=\begin{pmatrix}L_kD_k&0\\BD_k&L'D'\end{pmatrix}\begin{pmatrix}L_k^T&B^T\\0&L'^T\end{pmatrix}=\begin{pmatrix}L_kD_kL_k^T&\dots\\\dots&\dots\end{pmatrix}.$$

Thus we have $A_k=L_kD_kL_k^T$

$|A_k|=|L_k|\,|D_k|\,|L_k^T|$.

L_k. L_k^T is triangular. the determinant equal to the product of diagonal entries ①

$|L_k|\neq0$. $|D_k|\neq0$. $|L_k^T|\neq0$. $\Rightarrow|A_k|\neq0$. contradicts. thus all $|A_k|\neq0$.

2° In $F=\mathbb{R}$. Since L has units on diagonal, By statement ① $|L_k|=1^k=1$, $|L_k^T|=1$.

thus we have $|A_k|=|L_k|\,|D_k|\,|L_k^T|=|D_k|$

D_k is diagonal since D is diagonal. $|A_k|$ is equal to the product of k diagonal elements in D $\square$.

Week 12. Fri.

$$\frac{\frac{2}{7}-\frac{2}{15}}{\frac{2}{5}-\frac{2}{9}} = \frac{\frac{8}{105}}{\frac{4}{45}}$$

Exercise 14.4.

Pf: P is defined as a linear operator. the linearity is trivial.

1). $\forall v \in V$. $P(v) = P(u+w) = u = u+0$.

$P^2(v) = P(u+0) = u$. thus $P = P^2$.

2). "$\subset$". trivially by the definition of P.

"$\supset$". $\forall u \in U$. put. $P(u) = P(u+0) = u$. thus $u \in \operatorname{Im} P$.

3). Since $V = U \oplus U^\perp$ by Pro 14.6.

$U = \operatorname{Im} P$. $\dim V = \dim \operatorname{Im} P + \dim \ker P$. $\Rightarrow$ thus. $\dim \ker P = \dim U^\perp$.

"$\subset$" $\forall v \in \ker P$. $P(v) = 0 \Rightarrow P(0+v) = 0$.

$v = 0+v$ is the unique represation where $0 \in U$. $v \in W$.

thus we have $\ker P = U^\perp$

Exercise 14.5. $h(f,g) = \int_{-1}^{1} fg$.

Solution: First we need a orthogonal basis of $U = \{e_1, e_2, e_3\}$.

Denote $\mathcal{B} = \{v_1 = 1, v_2 = X, v_3 = X^2\}$. a basis of U.

Let $e_1 = v_1 = 1$.

$$e_2 = v_2 - \frac{h(v_2, e_1)}{h(e_1, e_1)} e_1 = X$$

$$e_3 = v_3 - \frac{h(v_3, e_2)}{h(e_2, e_2)} e_2 - \frac{h(v_3, e_1)}{h(e_1, e_1)} e_1 = X^2 - \frac{1}{3}.$$

Thus we have $\{1, X, X^2 - \frac{1}{3}\}$ is an orthogonal basis of U.

$$\operatorname{Proj}_U(1) = \sum_{i=1}^{3} \frac{h(1, e_i)}{h(e_i, e_i)} e_i = 1$$

$$\operatorname{Proj}_U(X) = \sum_{i=1}^{3} \frac{h(X, e_i)}{h(e_i, e_i)} e_i = X$$

$$\operatorname{Proj}_U(X^2) = \sum_{i=1}^{3} \frac{h(X^2, e_i)}{h(e_i, e_i)} e_i = \frac{1}{3} + X^2 - \frac{1}{3} = X^2.$$

$$\operatorname{Proj}_U(X^3) = \sum_{i=1}^{3} \frac{h(X^3, e_i)}{h(e_i, e_i)} e_i = \frac{3}{5}X$$

$\rightarrow$ actually. in 14.1. $X^3 = \overset{U^\perp}{\overset{\cap}{x^3 - \frac{3}{5}x}} + \overset{U}{\overset{\cap}{\frac{3}{5}x}}$

$$\operatorname{Proj}_U(X^4) = \sum_{i=1}^{3} \frac{h(X^4, e_i)}{h(e_i, e_i)} e_i = \frac{1}{5} + \frac{6}{7}\left(X^2 - \frac{1}{3}\right) = -\frac{3}{35} + \frac{6}{7}X^2.$$

$$[\operatorname{Proj}_U] = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{3}{35} \\ 0 & 1 & 0 & \frac{3}{5} & 0 \\ 0 & 0 & 1 & 0 & \frac{6}{7} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Exercise 14.6

Pf: Let $\{y_{k+1} \dots y_k\}$ is a basis of $U^\perp$. Then $\{y_1 \dots y_n\}$ forms a basis of $\mathbb{R}^n$.

By Thm 6.1. It's sufficient to check. $Y(Y^TY)^{-1}Y^T \cdot y_i = \mathrm{Proj}_U(y_i)$ $i=1,\dots n$. (1)

$Y(Y^TY)^{-1}Y^TY = Y[(Y^TY)^{-1}Y^TY] = Y$

Thus the equation (1) holds for $i=1,\dots,k$

$Y(Y^TY)^{-1}Y^T y_j = Y Y^T Y^{-1}(Y^T y_j)$

for $j>i$. $Y^T y_j = \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} [y_j] = 0$. since $h(y_i, y_j) = 0$ $1 \le i \le k$. $k+1 \le j \le n$.

$y_i \in U$. $y_j \in U^\perp$. □

Exercise 14.7.

Pf: 1. Let $V = \mathbb{R}^3$ equipped with dot product h.

Let $v_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

thus. by Pro 14.12. $|h(v_1, v_2)| \le ||v_1|| \cdot ||v_2||$.

$\Rightarrow$ $|h(v_1, v_2)|^2 \le ||v_1||^2 \cdot ||v_2||^2$.

$\Rightarrow$ $\left((a\ b\ c)\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)^2 \le (\sqrt{a^2+b^2+c^2})^2 \cdot (\sqrt{x^2+y^2+z^2})^2$

i.e. $(ax+by+cz)^2 \le (a^2+b^2+c^2)(x^2+y^2+z^2)$.

2. Let $V = \mathbb{R}^4$ equipped with dot product h.

Let $v_1 = \begin{pmatrix} a+b \\ b+c \\ a \\ c \end{pmatrix}$ $v_2 = \begin{pmatrix} x+y \\ y+z \\ x \\ z \end{pmatrix} \in \mathbb{R}^4$.

by Pro 14.12. $|h(v_1, v_2)| \le ||v_1|| \cdot ||v_2||$

$\Rightarrow$ $h^2(v_1, v_2) \le ||v_1||^2 \cdot ||v_2||^2$

$\Rightarrow$ $(2ax+2by+2cz+ay+bx+bz+cy)^2 \le (2a^2+2b^2+2c^2+2ab+2bc)(2x^2+2y^2+2z^2+2xy+2yz)$

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Ex14.8 (Ex.14.7 is handed on Tue.)

1. we have
$$\begin{pmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{pmatrix} = \begin{pmatrix} -1\cdot a+b \\ 0\cdot a+b \\ 1\cdot a+b \\ 2\cdot a+b \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}.$$

we denote $A=\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$. $x=\begin{pmatrix} a \\ b \end{pmatrix}$. $b=\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$.

we need to find the least square solution of. $Ax=b$

By pro. 14.15. $x_0=(A^TA)^{-1}A^Tb = \frac{1}{20}\begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix}$

$$= \frac{1}{20}\begin{pmatrix} -6 & -2 & 2 & 6 \\ 8 & 6 & 4 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 16 \\ 6 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 0.3 \end{pmatrix}.$$

we have $f(x) = 0.8x+0.3$.

2、设
$$\begin{pmatrix} f(-1) \\ f(0) \\ f(1) \\ f(2) \end{pmatrix} = \begin{pmatrix} 1\cdot c-b+a \\ 0\cdot c+0\cdot b+a \\ 1\cdot c+1\cdot b+a \\ 4\cdot c+2\cdot b+a \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} c \\ b \\ a \end{pmatrix} \qquad b=\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

denote : $\overset{\shortparallel}{A}$ $\overset{\shortparallel}{x}$.

$$x_0=(A^TA)^{-1}A^Tb = \begin{pmatrix} \frac14 & -\frac14 & -\frac14 \\ -\frac14 & \frac{9}{20} & \frac{3}{20} \\ -\frac14 & \frac{3}{20} & \frac{11}{20} \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 5 & -5 & -5 & 5 \\ -11 & 3 & 7 & 1 \\ 3 & 11 & 9 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{20}\begin{pmatrix} 20 \\ -12 \\ -14 \end{pmatrix} = \begin{pmatrix} 1 \\ -0.6 \\ -0.7 \end{pmatrix}$$

$f(x) = -0.7-0.6x+x^2$

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Ex 14.9.

By proposition 14.18. we have the form of unique solution

$$f(x) = \sum_{i=1}^{n} f(c_i) \prod_{k \neq i} \frac{x - c_k}{c_i - c_k}$$

$$= \frac{1}{24}(x-1)(x-2)(x-3)(x-4) - \frac{1}{6}x(x-2)(x-3)(x-4) + \frac{1}{4}x(x-1)(x-3)(x-4) - \frac{1}{6}x(x-1)(x-2)(x-4)$$
$$+ \frac{1}{24}x(x-1)(x-2)(x-3)$$

$$= \frac{1}{12}(x-1)(x-2)^2(x-3) - \frac{1}{3}x(x-2)^2(x-4) + \frac{1}{4}x(x-1)(x-3)(x-4)$$

$$f(x) - 1 = \frac{(x-0)(x-1)(x-2)(x-3)}{(4-0)(4-1)(4-2)(4-3)}$$

x	0	1	2	3	4
$f(x)-1$	0	0	0	0	1

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Ex 15.1.

let. v, v' be arbitrary vector s.t. $v \in V$. $v' \in V'$ take arbitrary bases B. B' related to V and V' respectively.

$(L(v), v')_{V'} = (v, L^*(v'))_V$

is equivalent to $[L(v)]_{B'}^T G' \overline{[v']_{B'}} = [v]_B^T G \overline{[L^*(v')]_B}$

$[L(v)]_{B'} = [A \cdot v]_{B'} = A[v]_B \qquad [L^*(v')]_B = A^*[v']_{B'}$.

Thus we have $[v]_B^T A^T G' \overline{[v']_{B'}} = [v]_B^T G \overline{A^*} \overline{[v']_{B'}}$.

$\Rightarrow A^T G' = G \overline{A^*}$ which is what we want.

Ex 15.2.

$f''(x) - 2xf'(x) = e^{x^2} \cdot (e^{-x^2} \cdot f'(x))'$.

1. Pf: Actually we need $\int_{-\infty}^{+\infty} e^{-x^2}[f''(x) - 2xf'(x)]\, g(x)\, dx = \int_{-\infty}^{+\infty} e^{-x^2} f(x)\, [g''(x) - 2x^2 g(x))\, dx$.

(Since the space is real, we omit the conjugation).

$LHS = \int_{-\infty}^{+\infty} [e^{-x^2} f'(x)]'\, g(x)\, dx$

$RHS = \int_{-\infty}^{+\infty} [e^{-x^2} g'(x)]'\, f(x)\, dx$. Thus we obtain the same expression.

they are not the same yet.

$(L(f), g)_V = (f, L(g))_V$, L is self-adjoint.

$(L(f), g) = -\int_{-\infty}^{\infty} f'(x)\, (e^{-x^2} g'(x) - 2xe^{-x^2} g(x))\, dx + \int_{-\infty}^{\infty} f(x)(2xe^{-x^2} g'(x) + (2-2x)e^{-x^2} g(x))$

$= -\int e^{-x^2} f'(x)\, g'(x) + \int (2xe^{-x^2} f(x) g(x))'\, dx$

$= -\int e^{-x^2} f'(x)\, g'(x) = (f, L(g))$.

2. Solution: Let the standard basis $\{1, x, x^2, x^3, x^4\}$

$L(1) = 0 \qquad L(x) = -2x. \qquad L(x^2) = 2 - 4x^2 \qquad L(x^3) = 6x - 6x^3 \qquad L(x^4) = 12x^2 - 8x^4$

$$A = [L]_B = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 6 & 0 \\ 0 & 0 & -4 & 0 & 12 \\ 0 & 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 & -8 \end{bmatrix} \qquad |\lambda E_n - A| = \begin{vmatrix} \lambda & 0 & -2 & 0 & 0 \\ 0 & \lambda+2 & 0 & -6 & 0 \\ 0 & 0 & \lambda+4 & 0 & -12 \\ 0 & 0 & 0 & \lambda+6 & 0 \\ 0 & 0 & 0 & 0 & \lambda+8 \end{vmatrix} = \lambda(\lambda+2)(\lambda+4)(\lambda+6)(\lambda+8)$$

Let $\lambda X = AX$. $\Leftrightarrow (A - \lambda E_5)X = 0$

$\lambda_1 = 0 \Rightarrow X = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \lambda_4 = -6 \Rightarrow X = \alpha_4 \begin{bmatrix} 0 \\ 3 \\ 0 \\ -2 \\ 0 \end{bmatrix}$

$\lambda_2 = -2 \Rightarrow X = \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \lambda_5 = -8 \Rightarrow X = \alpha_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \begin{matrix} 1 \\ 0 \\ -4 \\ 0 \\ \frac{4}{3} \end{matrix}$

$\lambda_3 = -4 \Rightarrow X = \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \; -\frac{1}{2}$ (where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are parameters.)

to make it clear,

eigenvalue	0	-2	-4	-6	~~-8~~
corresponding eigenvector	α_1	$\alpha_2 x$	$\alpha_3 x^2$	$\alpha_4 x^3$	~~$\alpha_5 x^4$~~

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3. By Pro 15.5, the eigenvectors belongs to the different eigenvalues of self adjoint operator are orthogonal to each other. thus, 5 vectors in 2) are exactly the orthogonal basis. the 5 vectors in the standard basis is correspond to the 5 different eigenvalues and belongs to different eigenspace.

So they are orthogonal to each other.

Thus, the standard basis is also an orthogonal basis.

Ex. 15.3.

Solution: denote $A = G_{h,\mathcal{E}} = \begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix}$. denote an operator $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbb{R}^3$.

We need to find an orthonormal basis, i.e. find orthogonal matrix Q. s.t. $Q^{-1}AQ$ is diag.

First find eigen value and vectors.

$\chi_A(t) = |\lambda E_3 - A| = \begin{vmatrix} t-1 & 2 & -2 \\ 2 & t+2 & -4 \\ -2 & -4 & t+2 \end{vmatrix} = (t-2)^2(t+7)$

$\lambda_1 = 2$. $(A-\lambda E_3)X = \begin{pmatrix} -1 & -2 & 2 \\ -2 & -4 & 4 \\ 2 & 4 & -4 \end{pmatrix} X = \alpha\begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} := u_2$. ($\alpha, \beta$) are parameter.

$\lambda_2 = -7$ $(A-\lambda E_3)X = \begin{pmatrix} 8 & -2 & 2 \\ -2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix} X = \alpha\begin{pmatrix} \frac{1}{2} \\ 1 \\ -1 \end{pmatrix} := u_1$

since $\left(\begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) \neq 0$ we use the orthogonalization process. $u_3 = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} - \frac{1}{4}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$

thus we obtain $\tilde{Q} = \begin{bmatrix} \frac{1}{2} & 0 & 1 \\ 1 & 1 & -\frac{1}{4} \\ -1 & 1 & \frac{1}{4} \end{bmatrix}$ have orthogonal columns.

$\Rightarrow$ divide by length $\Rightarrow$ $Q = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}$

since $h\left(\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}, \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\right) = (x'\ y'\ z')\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = (x\ y\ z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = h\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$

$\Rightarrow (x\ y\ z)\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}^T\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x\ y\ z)A\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

at the begining, we denote $Q^{-1}AQ = D$ i.e. $QDQ^{-1} = A$.

thus $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = Q^T = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2\sqrt{2}}{3} & -\frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{pmatrix}$ (comparing two equality above)

Denote $A = L - \lambda Id$. $(L-\lambda Id)(u) = 0$.

$((L-\lambda Id)(u), (L-\lambda Id)(u)) = (u, (L^* - \bar{\lambda} Id)(L-\lambda Id)(u))$.

$\Downarrow$

commutative

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1.4

Ex16.1.

Proof: Let $L(X) = AX$, A is the matrix w.r.t. the operator.

since L and L^* adjoint, we have $L^*(X) = \bar{A}^T X$

Denote $U = V_\lambda(L)$. By lemma16.4. U is L-invariant and L^*-invariant.

In the subspace U we consider the unitary scalar product in orthogonal basis.

$(L(u), u) = (u, L^*(u)) \Leftrightarrow \lambda u^T \bar{u} = u^T A^T \bar{u} \Leftrightarrow u^T(\lambda E - A^T)\bar{u} = 0$.

since u is non-zero. $(u,u) > 0$ because of the positive definite. $\lambda E - A^T = 0$. ×

Thus we have $\lambda E = A^T$. $\Leftrightarrow$ $\bar{\lambda} E = \bar{A}^T$ (seems a bit strange here but I can't find better ways ^^)

$\forall u \in U$. $L^*(u) = \bar{A}^T u = \bar{\lambda} u$. $\therefore \bar{\lambda}$ is ~~an~~ eigenvalue.

Ex16.2.

(1). Pf. Since $\vec{x} \in \mathbb{C}^n$ is an eigencolumn.

$A\vec{x} = \lambda \vec{x} \Leftrightarrow A\vec{u} + iA\vec{v} = (\vec{u} + i\vec{v})(a+ib) = (a\vec{u} - b\vec{v}) + i(a\vec{v} + b\vec{u})$

By comparing the both side, we have $A\vec{u} = a\vec{u} - b\vec{v}$ $A\vec{v} = a\vec{v} + b\vec{u}$

$\forall y \in span(u,v)$. let $y = a_1\vec{u} + a_2\vec{v}$ $(a_1, a_2 \in \mathbb{R})$

$Ay = A(a_1\vec{u} + a_2\vec{v}) = a_1 A\vec{u} + a_2 A\vec{v} = a_1(a\vec{u} - b\vec{v}) + a_2(a\vec{v} + b\vec{u})$

$= (a_1 a + a_2 a)\vec{v} + (a_2 b - a_1 b)\vec{u}$

since $a_1 a + a_2 a$, $a_2 b - a_1 b \in \mathbb{R}$. then $Ay \in Span(v,u)$

Since y is arbitrary, we have the invariance.

(2). Denote $U = span(u,v)$. $L(x) = Ax$.

By (1). We have U is L-invariant.

By Ex16.1. We have $x = u + iv$ is a eigenvector of L^* w.r.t. $\bar{\lambda} = a - bi$.

$\bar{A}^T \vec{x} = \bar{\lambda}\vec{x} \Leftrightarrow \bar{A}^T\vec{u} = a\vec{u} + b\vec{v}$ $\bar{A}^T\vec{v} = a\vec{v} - b\vec{u}$

① For $(L(u), v) = (u, L^*(v)) \Leftrightarrow (au - bv, v) = (u, av - bu)$

$\Leftrightarrow a(u,v) - b(v,v) = a(u,v) - b(u,u)$. $\Leftrightarrow b[(u,u) - (v,v)] = 0$.

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Since the $\lambda = a+bi$ is non-zero. $b \neq 0$.

that is $(u,u) - (v,v) = 0$. hence we have $\|v\|^2 = \|u\|$

② For $(L(u), u) = (u, L^*(u)) \Leftrightarrow (au - bv, u) = (u, au + bv)$.

$\Leftrightarrow a(u,u) - b(v,u) = a(u,u) + b(u,v)$

since $u, v \in \mathbb{R}^n$. $\Rightarrow b(v,u) = 0. \Rightarrow b \neq 0$, $(v,u) = 0$. i.e. $v \perp u$. $\square$.

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Ex 26.3.

1). Pf: $A = \begin{pmatrix} 12 & 3 & -4 \\ -3 & 12 & 0 \\ 4 & 0 & 12 \end{pmatrix}$ $\bar{A}^T = \begin{pmatrix} 12 & -3 & 4 \\ 3 & 12 & 0 \\ -4 & 0 & 12 \end{pmatrix}$.

$A\bar{A}^T = \begin{pmatrix} 144 & 0 & 0 \\ 0 & 153 & -12 \\ 0 & -12 & 160 \end{pmatrix}$ $\bar{A}^T A = \begin{pmatrix} 144 & 0 & 0 \\ 0 & 153 & -12 \\ 0 & -12 & 160 \end{pmatrix}$.

$LL^*(x) = L(\bar{A}x) = A\bar{A}^T x = \bar{A}^T A x = \bar{A}^T(L(x)) = L^*L(x)$. For any $x \in V$.

2). First compute the eigenvalue:

$|\lambda E_3 - A| = \begin{vmatrix} \lambda-12 & -3 & 4 \\ 3 & \lambda-12 & 0 \\ -4 & 0 & \lambda-12 \end{vmatrix} = (\lambda-12)(\lambda^2-24\lambda+169)$.

$\Rightarrow \lambda = 12$ and $\lambda = 12 \pm 5i$ thus we have the canonical form $A' = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 5 \\ 0 & -5 & 12 \end{pmatrix}$.

1°) $\lambda = 12$. $(A - \lambda E_n)X = 0$. $\Rightarrow$ $X = \alpha \begin{pmatrix} 0 \\ 1 \\ \frac{3}{4} \end{pmatrix} := u_3$.

2°) Denote $U = V_{12}(L)$, which is A and $\bar{A}^T$ - invariant. (since A is normal).

$\dim U^\perp = \dim \mathbb{R}^3 - \dim U = 2$.

take $u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\tilde{u}_2 = \begin{pmatrix} 0 \\ 1 \\ -\frac{4}{3} \end{pmatrix} \in U^\perp$. since $u_1, \tilde{u}_2$ are l.i. and $\dim U^\perp = 2$.

$\{u_1, u_2\}$ forms a basis.

Need not apply the orthogonalization process since $(u_1, u_2) = 0$ implies $u_1 \perp u_2$.

Thus we obtain $\tilde{Q} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ \frac{3}{4} & 0 & -\frac{4}{3} \end{pmatrix}$. which has orthogonal columns consists of eigenvectors of A.

divide each column by its length, we obtain $Q = \begin{pmatrix} 0 & 0 & 1 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ \frac{3}{5} & -\frac{4}{5} & 0 \end{pmatrix}$

which consists of orthonormal basiss of $\mathbb{R}^3$.

Ex. 26.4.

Denote $L: x \mapsto Ax$ on $\mathbb{C}^n$ s.t. $A^T = -A$, L is normal. (easy to check)

By thm. Thm 26.6. There exists an orthogonal basis B s.t. $[L]_B$ has block form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for imaginary eigenvalue.

By Ex 26.3. $\exists$ Q is orthogonal s.t. $[L]_B = Q^T A Q$.

we have $[L]_B^T = (Q^T A Q)^T = Q^T A^T Q = Q^T(-A)Q = -Q^T A Q = -[L]_B$

which implies $[L]_B$ is also skew-symmetric.

thus in every block $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, we have $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^T = -\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

i.e. $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} -a & b \\ -b & -a \end{pmatrix}$. $\Rightarrow$ which implies $a = -a$, i.e. $a = 0$.

Thus. all complex eigenvalues has zero real part.

Ex 16.5.

Pf: $\forall U \subset V$. $V = U \perp U^{\perp}$. By Pro 14.6.

$\forall v \in V$. v can be unique decomposed orthoganally. $v = u + w$. where $u \in U$, $w \in U^{\perp}$

$S_U^2 = S_U(S_U(v)) = S_U(w-u) = w - (-u) = w + u = v = Id_V$.

By. Pro 16.16. S_U is self-adjoint.

We can claim that for any self-adjoint operator s.t. $L^2 = Id$, It always exists a reflection S_U correspond to L for some U. i.e. $L = U$.

Ex 16.6.

$x + 2y - 3z = 0$. $\Rightarrow$ the corresponding subspace $U = \{ v \in V \mid v = \alpha \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$, α, β are parameters$\}$.

thus $u_3 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$ form a basis of U. $\dim U^{\perp} = \dim V - \dim U = 1$.

Find a $u_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ s.t. $u_1 \perp u_2$ $u_1 \perp u_3$ $u_1 \in U^{\perp}$. forms a basis of $U^{\perp}$

thus. $\forall v \in V$, $v = a_1 u_1 + a_2 u_2 + a_3 u_3$.

$L(v) = -a_1 u_1 + a_2 u_2 + a_3 u_3$. Let $\mathcal{F} = \{e_1, e_2, e_3\}$ be standard basis

$L(e_1) = L(\frac{1}{14} u_1 - \frac{2}{14} u_2 + \frac{3}{14} u_3) = -\frac{1}{14} u_1 - \frac{2}{14} u_2 + \frac{3}{14} u_3 = (\frac{12}{14}, -\frac{4}{14}, \frac{6}{14})^T$

$L(e_2) = L(\frac{1}{7} u_1 + \frac{5}{7} u_2 + \frac{3}{7} u_3) = -\frac{1}{7} u_1 + \frac{5}{7} u_2 + \frac{3}{7} u_3 = (-\frac{2}{7}, \frac{3}{7}, \frac{6}{7})^T$

$L(e_3) = L(-\frac{3}{14} u_1 + \frac{6}{14} u_2 + \frac{5}{14} u_3) = \frac{3}{14} u_1 + \frac{6}{14} u_2 + \frac{5}{14} u_3 = (\frac{6}{14}, \frac{12}{14}, -\frac{4}{14})^T$

$$[L]_{\mathcal{B}} = \begin{bmatrix} \frac{6}{7} & -\frac{2}{7} & \frac{3}{7} \\ -\frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & \frac{6}{7} & -\frac{2}{7} \end{bmatrix}$$

Ex 17.1. $f: V \times V \to \mathbb{C}$. $f: (x, v) \mapsto (x, L_f(v))$.

$\forall x, v \in V$. $(x, L_f(v)) = f(x, v)$

On the other hand. $(v, L_f(x)) = f(v, x)$.

f is arbitrary

In the complex inner space, f is unitary scalar product. $f(x,v) = f(\overline{v,x}) = f(v,x)$

Thus we have $(x, L_f(v)) = (v, L_f(x))$. $v, x \in V$. is arbitrary. L_f is self-adjoint.

Ex. 17.2.

Solution: denote $A = \begin{pmatrix} -4 & -6 \\ 3 & -8 \end{pmatrix}$.

consider a linear operator $L: \mathbb{R}^2 \to \mathbb{R}^2$. $L(X) = AX$.

$\chi_{L^*L}(t) = |tE - A^TA| = \begin{pmatrix} t-25 & 0 \\ 0 & t-100 \end{pmatrix} = (t-25)(t-100)$

$\lambda_1 = 25 \quad \lambda_2 = 100$.

By Pro 17.5. For any $v \in \mathbb{R}^2$. $v = v_1 + v_2$. $v_i \in V_{\lambda_i}(L)$.

$T(v) = \sum_{i=1}^{2} \sqrt{\lambda_i}\, v_i$. s.t. $T^2 = L^*L$ and T is uniquely positive operator.

We can find that, the matrix of operator T is $\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} := S$

then consider $Q_1 = AS^{-1}$ for left polar decomposition.

$Q_1 = \begin{pmatrix} -4 & -6 \\ 3 & -8 \end{pmatrix} \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{10} \end{pmatrix} = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix}$.

① $A = \begin{pmatrix} -\frac{4}{5} & -\frac{3}{5} \\ \frac{3}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$

then consider

$\chi_{LL^*} = |tE - AA^T| = \begin{vmatrix} t-52 & -36 \\ -36 & t-73 \end{vmatrix} = (t-25)(t-100)$.

LL^* has the same eigenvalue as L^*L.

however, we may face trouble when finding matrix of T' to satisfy $T'^2 = L^*L$

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Ex. 18.1.

Pf: By thm. 18.2. We can denote two vector space $U = \mathbb{R}^m$. $V = \mathbb{R}^n$.
s.t. $\dim V = n$ $\dim U = m$. consider a linear transformation $L: V \to U$.
$L(X) = AX$. $(x \in V)$.

In thm 18.2, we have proved that. $[L]_{B \to B'} = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$ has singular values of ~~A~~ on its diagonal.

By Thm. 6.7. its sufficient to check the transition matrix of standard basis to the orthonormal basis is orthogonal.

Let $B_\varepsilon = \{\varepsilon_1, \dots \varepsilon_n\}$ be standard basis of V. $B_e = \{e_1, \dots e_n\}$ be orthonormal basis of V.

~~A~~ the transition matrix $M_{B_\varepsilon \to B_e}$ consists of the coordinates columns of $e_1, \dots e_n$ w.r.t. B_ε.

thus $M^T M = \begin{bmatrix} (e_1, e_1) & \cdots & (e_1, e_n) \\ (e_2, e_1) & & \vdots \\ \vdots & & \\ (e_n, e_1) & & (e_n, e_n) \end{bmatrix}$. since $\{e_1 \dots e_n\}$ are orthonormal.

$(e_i, e_i) = 1$ $(e_j, e_i) = 0$ $(j \neq i)$, i.e. $M^T M = E_n$. $\Rightarrow M$ is orthogonal $\square$.