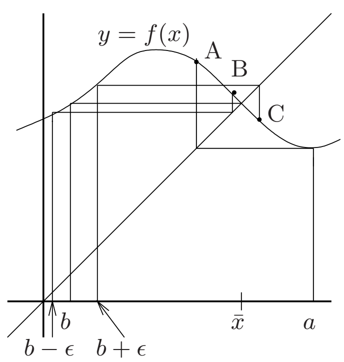
Lecture 7

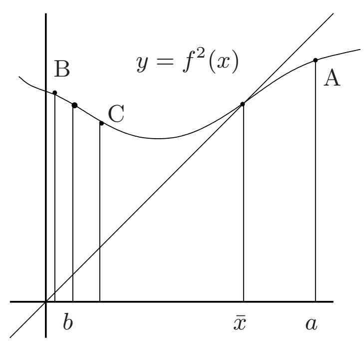
Dynamics of the logistic map $f_a(x) = ax(1-x)$ on [0,1]

The Appearance of "Two-Cycles" (cont'd)

Explaining the bifurcation from a fixed point to a two-cycle

Given a function $f(x)$, we can obtain an idea of what the graph of its iterate $g(x) = f^2(x)$ looks like from the graph of f: For every point x, we find $y = f(x)$, then travel horizontally to the line $y = x$ and then input $f(x)$ into f to obtain $f^2(x)$. This is illustrated in the figure below. Clearly, fixed points of f, $\bar{x} = f(\bar{x})$, are fixed points of $g(x)$. The point $x = a$ gets mapped, after two applications of f, to $g(a) = f^2(a)$. The point $x = b$ has been chosen so that $f(b) = \bar{x}$, i.e. b is a preperiodic point of $\bar{x}$. A point just to the right of $x = b$, say $x = b + \epsilon$, will be mapped to a value $f(b+\epsilon) > f(b) = \bar{x}$. However, from the graph, $f(f(b+\epsilon)) < \bar{x}$. The reader is encouraged to examine other x values in this example.

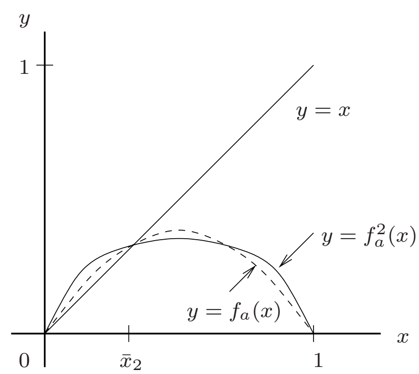




Let us now return to the logistic map $f_a(x) = ax(1-a)$ which we shall analyze with the help of the graphs of f_a and various iterates plotted on Pages 82 and 83 below. For $0 < a < 2$, the graph of $f_a^2(x)$ is not fundamentally different from the graph of $f_a(x)$. When $0 < a \leq 1$, $f_a(x) < x$, so that $f_a^2(x) < f_a(x)$, implying that the graph of f_a^2 is obtained by somewhat "squashing down" the graph of of f_a.

For $1 < a < 2$, the fixed point $\bar{x}_2(a) = \frac{a-1}{a}$ appears in the interval $\left(0, \frac{1}{2}\right)$. It will, of course, be a fixed point of f_a^2 as well. Since $f_a(x)$ is increasing on $(0, \bar{x}_2(a))$ and $f_a(x) > x$, it follows that $f_a^2(x) > f_a(x)$ for $x \in (0, \bar{x}_2(a))$. Thus, the graph of f_a^2 is obtained by "pulling up" the graph of f_a between $x = 0$ and $x = \bar{x}_2(a)$.

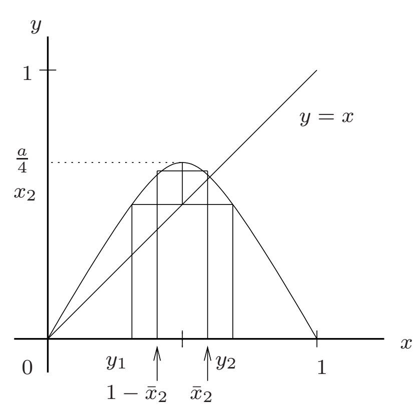
For $x \in \left(\bar{x}_2(a), \frac{1}{2}\right)$, $f_a(x) < x$ so that $f_a^2(x) < f_a(x)$, i.e. we once again "push down" on the graph of f_a to obtain the graph of f_a^2. All of the above behaviour is mirrored on the other half of the interval, i.e. $x \in \left(\frac{1}{2}, 1\right)$ since $f_a(x) = f_a(1-x)$. The net result is shown schematically below.



When $a = 2$, the fixed point $\bar{x}_2 = \frac{1}{2}$ coincides with the maximum value of $f_2(x)$ on $[0, 1]$. At this point, the graph of f_a^2 is quite flat at $x = \frac{1}{2}$.

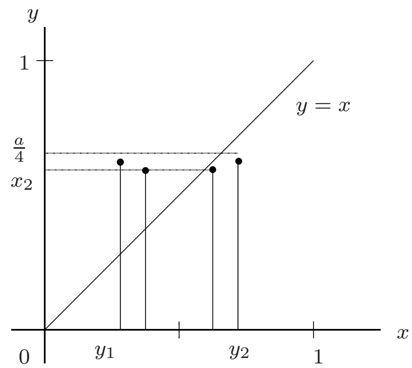
When $a > 2$, $\bar{x}_2 > \frac{1}{2}$ so that $f_a\left(\frac{1}{2}\right) > f_a(\bar{x}_2) = \bar{x}_2$. In other words, there is a piece of the graph of $f_a(x)$ that lies above the fixed point value $\bar{x}_2$. More precisely, $f_a(x) > \bar{x}_2$ for all $x \in (1 - \bar{x}_2, \bar{x}_2)$: see diagram below. There are some serious consequences for the graph of $f_a^2(x)$.

Firstly, note that $f_a(1 - \bar{x}_2) = f_a(\bar{x}_2) = \bar{x}_2$. It follows that $f_a^2(1 - \bar{x}_2) = f_a^2(\bar{x}_2) = \bar{x}_2$. Therefore, we know two points on the graph of $f_a^2(x)$. (We also know that $f_a^2(0) = f_a^2(1) = 0$.)



Graph of $f(x)$ for a slightly greater than 2.

Now, we know that $f_a\left(\frac{1}{2}\right) = \frac{a}{4}$, the absolute maximum value of $f_a(x)$ on $[0, 1]$. We ask the question: For what value(s) of x does $f_a(x) = \frac{a}{4}$? Answer: The values of x such that $f_a(x) = \frac{1}{2}$, i.e. the "preimages" of $x = \frac{1}{2}$. There are two such points, y_1 and y_2, which we have shown on the above figure. So far, we have:

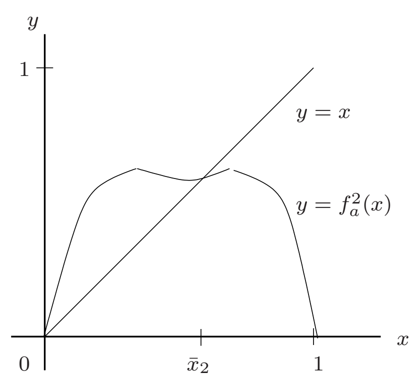


It now remains to fill in the graph, somehow joining these points. A few observations, using graphical methods, will help:

1) $f_a\left(\frac{1}{2}\right) = \frac{a}{4}$ and $f_a\left(\frac{a}{4}\right) < \bar{x}_2 \Longrightarrow f_a^2\left(\frac{1}{2}\right) < \bar{x}_2$. Moreover, if we move slightly away from $x = \frac{1}{2}$, $f_a^2\left(\frac{1}{2} \pm \epsilon\right) > f_a^2\left(\frac{1}{2}\right)$, implying that $x = \frac{1}{2}$ is a local minimum of $f_a^2(x)$.

2) Consider y_1, where $f_a(y_1) = \frac{1}{2}$. Since $x = \frac{1}{2}$ is a local maximum of $f_a(x)$, moving slightly away from y_1, i.e. $x = y_1 \pm \epsilon$ will cause us to move slightly away from the max value $\frac{1}{2}$. Hence, $f_a^2(y_1 \pm \epsilon) < f_a^2\left(\frac{1}{2}\right) = \frac{a}{4}$. Thus, $x = y_1$ and $x = y_2$ are local maxima of $f_a^2(x)$.

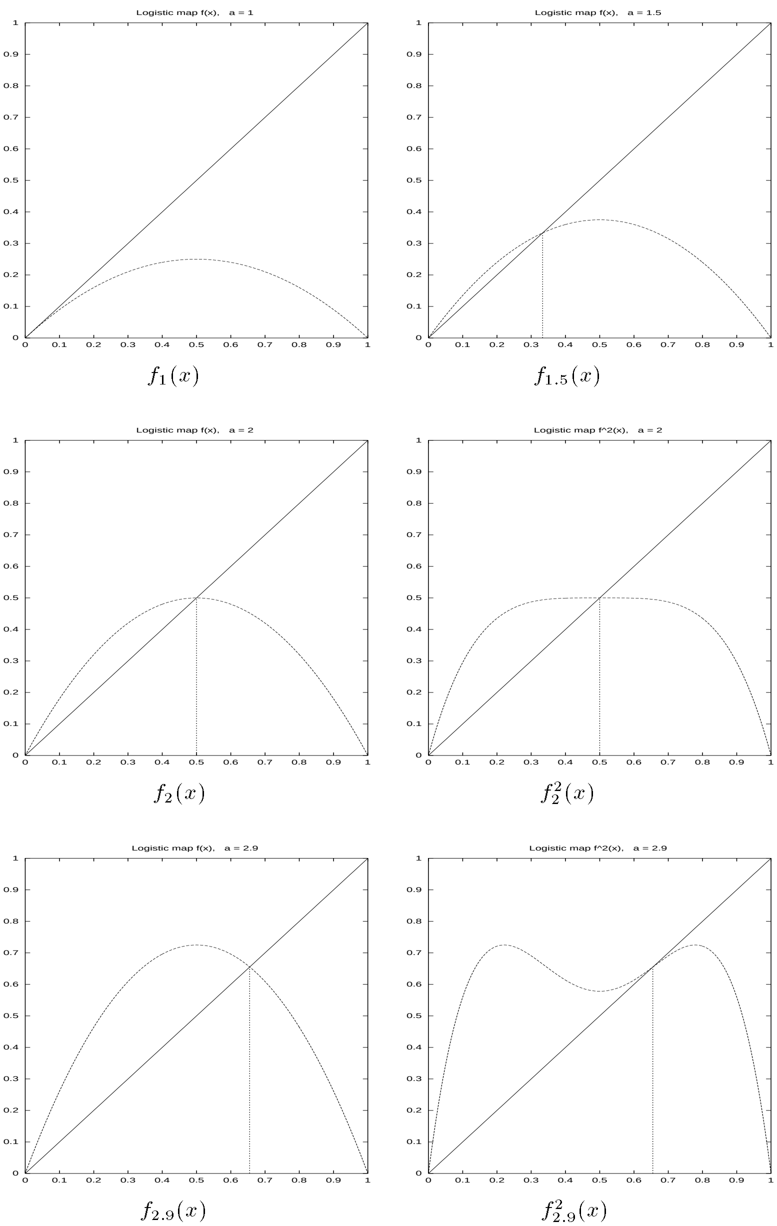
We now have enough information to fill in the details to obtain a rough idea of the graph of $f_a^2(x)$:



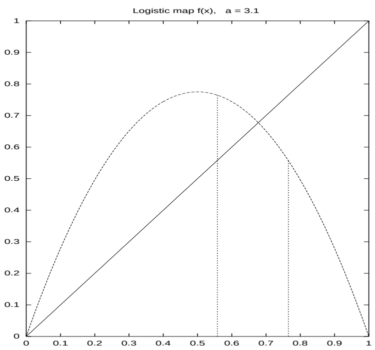
Graph of $f^2(x)$ for a slightly greater than 2.

Note that $|f_a^{2\prime}(\bar{x}_2)| < 1$ for $1 < a < 3$.

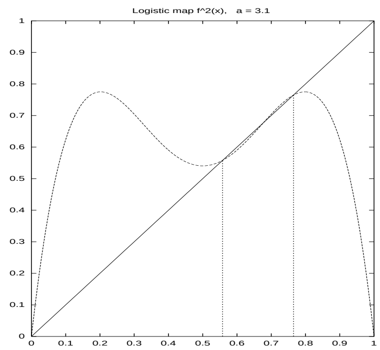
As a increases from 2 to 3, the minimum of $f_a^2(x)$ at $x = \frac{1}{2}$ moves further downward and the maxima $f_a(y_1) = f_a(y_2) = \frac{a}{4}$ move upward. The result: the slope of $f_a^2(\bar{x}_2)$ approaches the value 1, as can be seen in the case $a = 2.9$ on the next page.



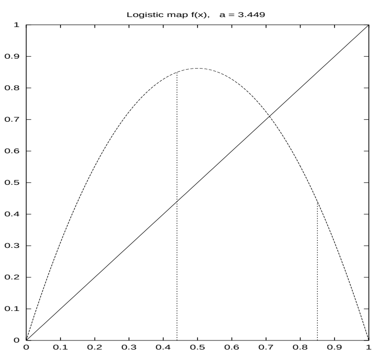
Graphs of $f_a(x)$ and $f_a^2(x)$, showing fixed points.



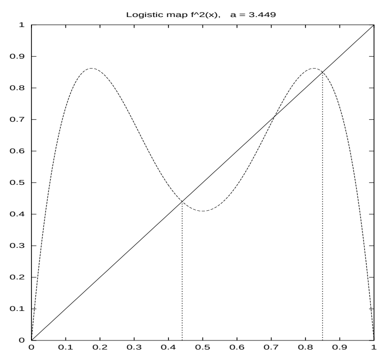
$f_{3.1}(x)$



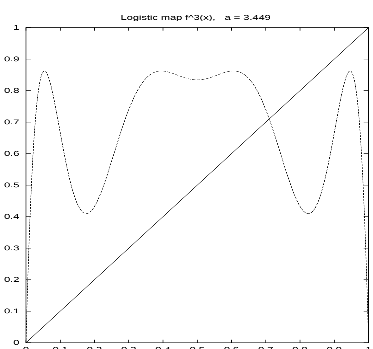
$f^2_{3.1}(x)$



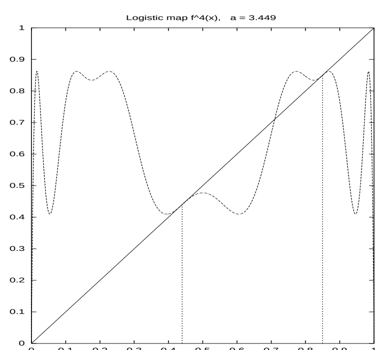
$f_{3.449}(x)$



$f^2_{3.449}(x)$



$f^3_{3.449}(x)$



$f^4_{3.449}(x)$

Graphs of $f_a(x)$ and various iterates, showing two-cycles.

A $a = 3$, where the fixed point of $f_3(x)$ is $\bar{x}_2 = \frac{2}{3}$, we have that $f_3'(\bar{x}_2) = -1$, implying that $(f_3^2)'(\bar{x}_2) = 1$. Why is this? Because $f_2(\bar{x}_2) = \bar{x}_2$ implies that

$$f_3^2(\bar{x}_2) = \bar{x}_2 \quad \Longrightarrow \quad (f_3^2)'(\bar{x}_2) = f_3'(f_3(\bar{x}_2))f_3'(\bar{x}_2) = f_3'(\bar{x}_2)f_3'(\bar{x}_2) = (-1)(-1) = 1\,. \tag{1}$$

The fixed point $\bar{x}_2(a) = \frac{2}{3}$ is neutral. For $a = 3+\epsilon$, where $0 < \epsilon \ll 1$, $f_a'(\bar{x}_2) > 1$ so that $(f_a^2)'(\bar{x}_2) > 1$. The result, as can be seen in the case $a = 3.1$, is that the graph of $f_a^2(x)$ must intersect the line $y = x$ at two additional points $p_1 < \bar{x}_2$ and $p_2 > \bar{x}_2$. In other words, a two-cycle (p_1, p_2) appears for $a > 3$: $f_a(p_1) = p_2$, $f_a(p_2) = p_1$. As we computed earlier, this two-cycle is attractive for $3 < a < 1 + \sqrt{6} \cong 3.449$.

In summary, the behaviour of the graph of $f_a^2(x)$ for values of a running from $2 < a < 3$ to $a = 3$ and on to $a = 1 + \sqrt{6}$ explains the appearance of a two-cycle (p_1, p_2) that is attractive for $a \in (3, 1 + \sqrt{6})$. For these a-values, *most* iteration sequences $x_{n+1} = f_a(x_n)$ will approach these attractive two-cycles. The word "most" is emphasized, since **not all** sequences will approach the two-cycle. For example, if $x_0 = 0$, then $x_n = 0$. As well, if $x_0 = 1$, then $x_n = 0$ for $n \geq 1$. The reader is encouraged to find other sequences that do not approach the two-cycle (p_1, p_2).

Dynamics of the iteration procedure $x_{n+1} = ax(1-x)$ for $a > 1 + \sqrt{6}$

A natural question to ask is, "What happens for $a \geq 1+\sqrt{6}$?" At $a = 1+\sqrt{6}$, the two-cycle is neutral but it still attracts neighbouring iterates. What has happened is that the values of the derivatives $(f^2)'(p_1)$ and $(f^2)'(p_2)$ have changed in value from $+1$ at $p_1 = p_2 = \frac{2}{3}$ when $a = 3$ to -1 at p_1 and p_2 when $a = 1+\sqrt{6}$. Remarkably, it appears as if **$f_a^2(x)$ is behaving at p_1 and p_2 as $a \to 1+\sqrt{6}$ in the same way that $f_a(x)$ behaves at $\bar{x}_2$ as $a \to 3$.** A look at the graph of $h(x) = f_a^2(f_a^2(x)) = f_a^4(x)$ shows that this is, indeed, the case. At $x = p_1$ and $x = p_2$ we now have scaled-down copies of the same "curling up" behaviour that f_a^2 exhibited at $a = 3$. The result is a birth of two new points at each of p_1 and p_2, implying the creation of a four-cycle of points (q_1, q_2, q_3, q_4) such that $f(q_1) = q_2$, $f(q_2) = q_3$, $f(q_3) = f(q_4)$, $f(q_4) = q_1$. Numerical experiments have shown that this four-cycle is stable until $a \cong 3.544$. The process repeats itself and at $a \cong 3.544$, an eight-cycle is born.

On the next two pages are presented plots of iteration sequences $x_{n+1} = ax_n(1-x_n)$, for several values of a to illustrate the period-doubling phenomenon. In all cases the starting point of the iteration sequence was $x_0 = \frac{1}{3}$. The values of a are:

i) $a = 3.2, \quad x_n \to$ 2-cycle,

ii) $a = 3.5$, $x_n \to$ 4-cycle,

iii) $a = 3.55$, $x_n \to$ 8-cycle,

iv) $a = 3.565$, $x_n \to$ 16-cycle.

Bifurcations and the "period-doubling route to Chaos"

Let us now quickly summarize what we have found so far for the logistic map $f_a(x) = ax(1-x)$ and the associated iteration procedure $x_{n+1} = f_a(x_a)$.

1) For $0 < a < 1$, the fixed point $\bar{x}_1 = 0$ is attractive or stable, and all iterates $x_n \to \bar{x}_1$ as $n \to \infty$.

2) At $a = 1$, $\bar{x}_1 = 0$ is neutral. Nevertheless, all iterates $x_n \to 0$ as $n \to \infty$.

3) For $1 < a < 3$, the fixed point $\bar{x}_2(a) = \frac{a-1}{a}$ is attractive, with $x_n \to \bar{x}_2(a)$ as $n \to \infty$.

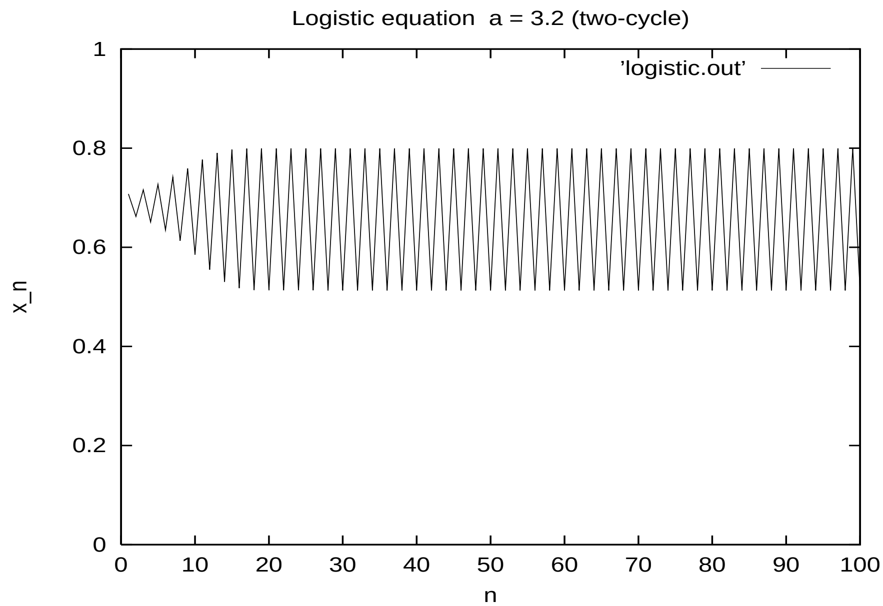
4) At $a = 3$, $\bar{x}_2 = \frac{2}{3}$ is neutral. Nevertheless, all iterates $x_n \to \bar{x}_2$ as $n \to \infty$.

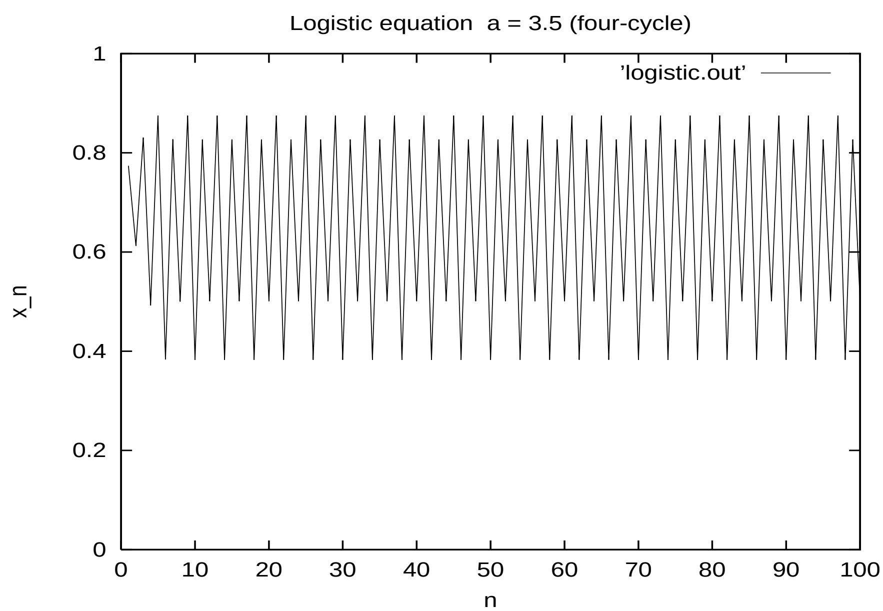
5) For $3 < a < 1 + \sqrt{6}$, the two-cycle $(p_1(a), p_2(a))$ computed in the previous section is attractive.

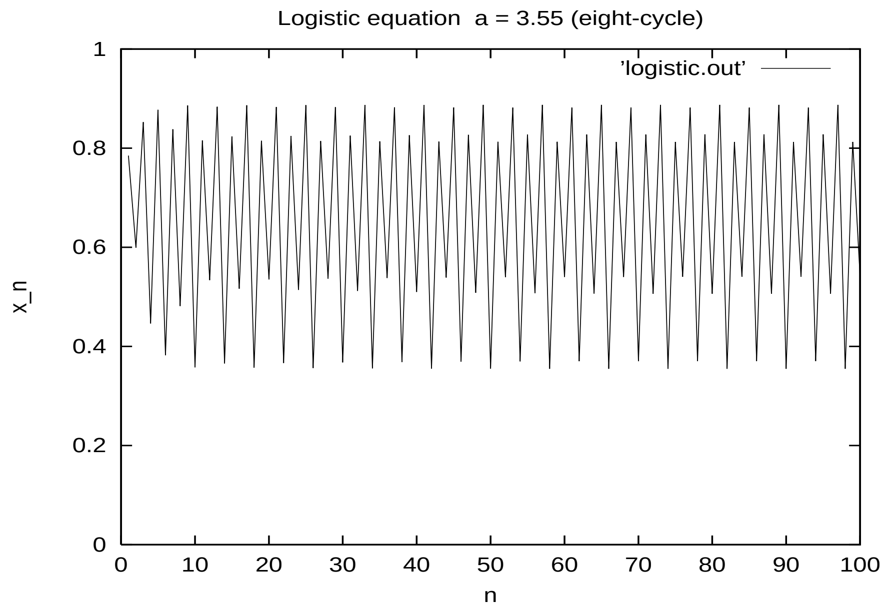
The a-values $0, 1, 3$ and $1 + \sqrt{6}$ represent "bifurcation points" of the discrete dynamical system $x_{n+1} = f_a(x_n)$. "Bifurcations" are changes in the dynamical behaviour of a system as one or more parameters are varied. These changes are normally due to the changes in stability properties of fixed points or periodic orbits, e.g. from attractive/stable to repulsive/unstable. Such changes in stability may be accompanied by the appearance of new periodic orbits which may "take over" in the attraction of iterates x_n.

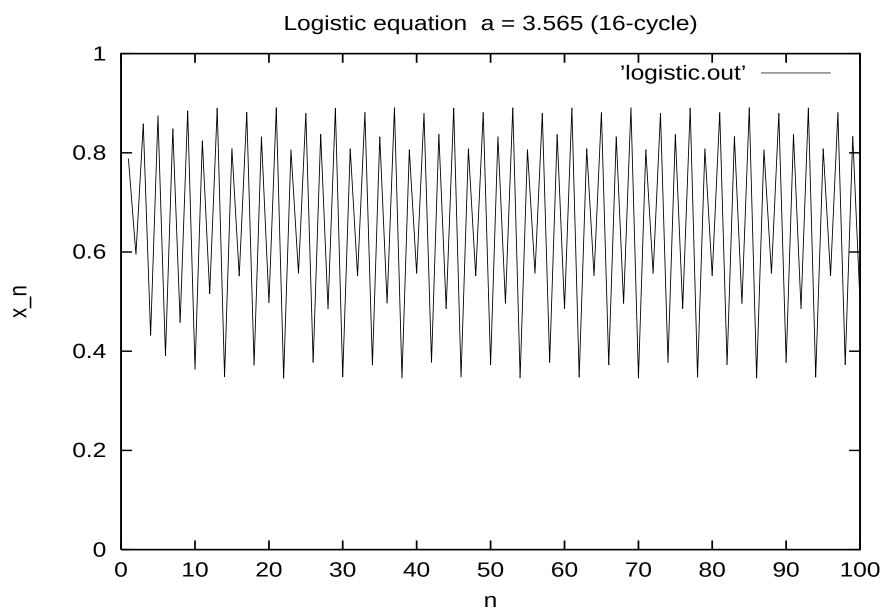
Let us examine the bifurcations that take place with the logistic map $f_a(x)$ as the parameter "a" is allowed to vary from $a = 0$. We can think of "a" as a turnable parameter – once we lock in on a desired a-value, we than "turn on" the iteration process $x_{n+1} = f_a(x_n)$ and observe the long-term behaviour of the iterates x_n.

For $0 < a < 1$, the fixed point $\bar{x}_2(a) = \frac{a-1}{a} < 0$ and plays no role in the dynamics of the iteration process on $[0, 1]$. The fixed point $\bar{x}_1 = 0$ is **attractive or stable** and attracts all iteration sequences $\{x_n\}$ as $n \to \infty$. At $a = 1$, $\bar{x}_2(1) = 0$ "collides" with $\bar{x}_1 = 0$. The single fixed point $\bar{x} = 0$ is **neutral**. As "a" increases past 1, the fixed point $\bar{x}_2 = \frac{a-1}{a}$ is now **attractive** and $\bar{x}_1 = 0$ is **repulsive or unstable**. It is as if the "collision" caused the two fixed points to exchange their


Logistic equation a = 3.2 (two-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100


Logistic equation a = 3.5 (four-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100


Logistic equation a = 3.55 (eight-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100


Logistic equation a = 3.565 (16-cycle)
'logistic.out'
x_n
n
0
0.2
0.4
0.6
0.8
1
0
10
20
30
40
50
60
70
80
90
100

stability characteristics. The point $a = 1$ is a **bifurcation point**. (In technical jargon, it is a "transverse bifurcation point".) At $a = 3$, the fixed point $x_2 = \frac{a-1}{a}$ ceases to be attractive. As a increases past 3, an **attractive two-cycle**, (p_1, p_2), is created; $\bar{x}_2(a) = \frac{a-1}{a}$ is now **repulsive**. The point $a = 3$ is a **period-doubling bifurcation point** (since we go from a period one orbit – a fixed point – to a period two orbit – a two-cycle). This phenomenon is also called a **pitchfork bifurcation** because of the curves traced out by the fixed point $\bar{x}_2(a)$ and the two-cycle, cf. Page 93. At $a = 1+\sqrt{6}$, the two-cycle ceases to be stable and bifurcates (period-doubling again) into a four-cycle.

What is now remarkable is that this period-doubling behaviour continues: from 2- to 4-cycle, 4- to 8-cycle, 8- to 16-cycle,etc. A great deal of theoretical and computational work has been done for this apparently simple iteration process. On the next page, the **asymptotic behaviour of iterates** $x_{n+1} = f_a(x_n)$ is plotted as a function of a. For a fixed value of a, say $a = K$, the intersection of the line $a = K$ and the points will give the points in $[0, 1]$ to which the iterates $\{x_n\}$ are attracted. This is a completion of the figure drawn in the previous lecture, with the exception that the repulsive periodic points are omitted. The figure on the right is a blow-up of the region $3.4 \leq a \leq 4$ to give the reader an idea of the complexity - and beauty - of this process.

Below the figures are listed numerical values of the bifurcation points a_k at which 2^{k-1}-cycles give birth to 2^k-cycles. Early numerical experiments suggested that the bifurcation points a_k approach a limiting value "a_∞" as $k \to \infty$. In fact, an examination of the first differences $\Delta_k = a_{k+1} - a_k$ and ratios of first differences $R_k = \Delta_{k+1}/\Delta_k$ (some values are shown in the table) suggested that the a_k were approaching a_∞ geometrically: If

$$\frac{\Delta_{k+1}}{\Delta_k} = \frac{a_{k+1} - a_k}{a_k - a_{k-1}} \to r < 1 \quad \text{as} \quad k \to \infty \tag{2}$$

then

$$a_{k+n} - a_{k+n-1} \simeq (a_{k+1} - a_k) r^n. \tag{3}$$

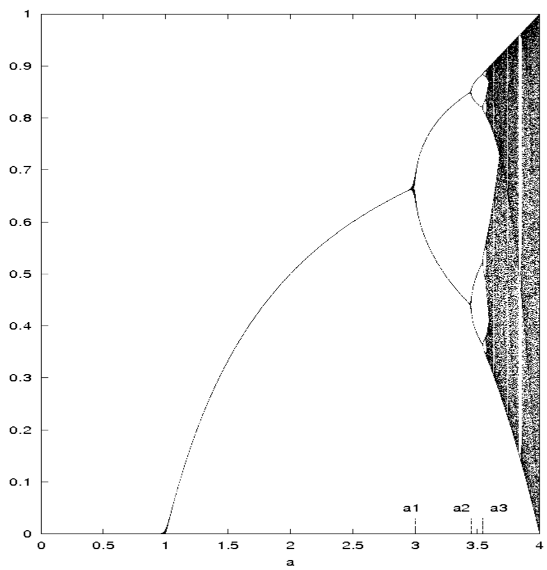
The ratio $\delta = \frac{1}{r} \cong 4.6692$ is referred to as "Feigenbaum's constant". It is now known that

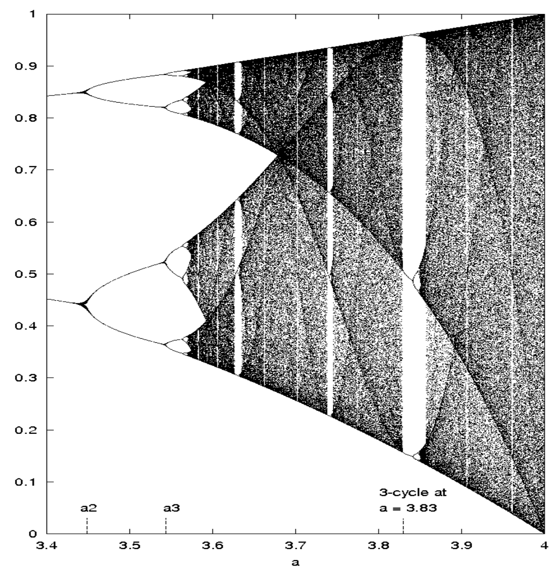
$$\lim_{k\to\infty} a_k = a_\infty \cong 3.5699456. \tag{4}$$

A natural question is, "What happens at $a = a_\infty$?" The answer is that the iterates x_n are attracted to a "fractal" subset of $[0, 1]$. Very briefly, such a fractal subset is "full of holes" in the sense that it has no intervals of the form $[a_k, b_k]$. Nevertheless, it still has a lot of points – an uncountable infinity of points, to be precise. (A set such as $S = \left\{\frac{1}{n} \,|\, n \geq 1\right\}$ has a countable infinity of points since

Asymptotic Behaviour of Iteration Sequences

$$x_{n+1} = ax_n(1 - x_n) \textbf{ vs. } a$$



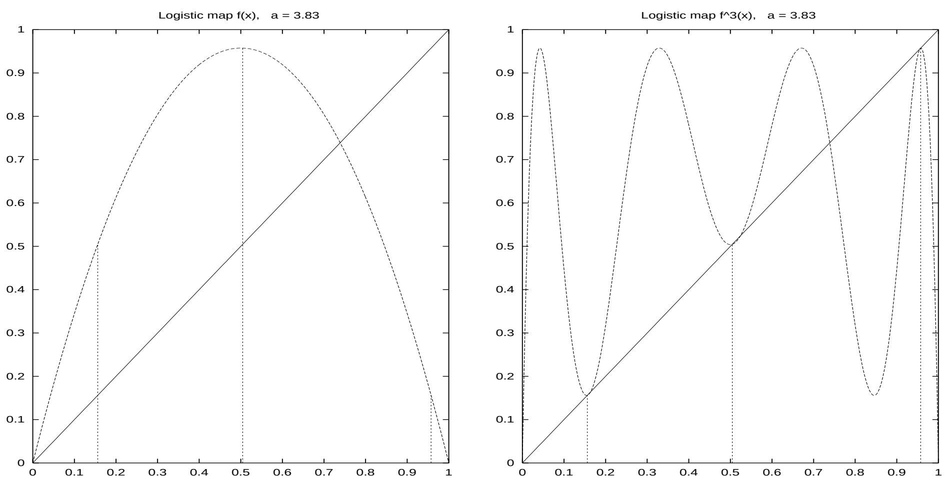


Finite-Difference Analysis of Bifurcation Points a_k

k	a_k	$\Delta_k = a_{k+1} - a_k$	$r_k = \Delta_k/\Delta_{k+1}$
1	3		
2	3.449499	0.449499	4.752
3	3.544090	0.094591	4.656
4	3.564407	0.020317	4.668
5	3.568759	0.004352	4.449
6	3.569692	0.000932	4.669
7	3.569891	0.000200	4.669
8	3.569934	0.000043	

they can all be put into one-to-one correspondence with the integers. The real line, and even the real interval $[0,1]$, are uncountable – points in these sets cannot be put into one-to-one correspondence with the integers. We'll discuss this in more detail later in the course.) Thus, in an "intuitive" sense, a fractal set is a very "thick" set. We shall return to this idea in the next section.

Before closing we simply mention that some rather exotic dynamical behaviour is demonstrated by iteration sequences $x_{n+1} = f_a(x_n)$ for $a > a_\infty \cong 3.5699456$. For example, at $a \cong 3.83$, there is a quite prominent three-cycle that eventually undergoes period-doubling bifurcations to 6, then 12, then 24, etc. cycles, up to a "$3 \cdot 2^\infty$" cycle. At various a-values, 5-cycles appear, only to undergo the same type of period doubling, i.e. $5 \to 10 \to 20 \to 40$ etc. A graph of $f_a(x)$ for $a = 3.83$ along with the 3-cycle is shown below.



Graphs of $f_a(x)$ and $f_a^2(x)$ for $a = 3.83$, showing the three-cycle

As written earlier, volumes have been written on this dynamical system and related ones. In fact, some rather remarkable results show that there is a kind of "universality" property shared by one parameter families of mappings $g_c(x)$ of an interval, say $[0,1]$ into itself. For $c_{\min} \le c \le c_{\max}$,

1) $g_c(0) = g_c(1) = 0$

2) $g_c(x)$ has a single (global) maximum at an interior point $p \in (0,1)$

3) $g_c'(x) > 0$ for $x \in [0,p)$; $g_c'(x) < 0$ for $x \in (p,1]$; $g_c'(p) = 0$, $g''(p)$ exists.

These maps are known as "unimodal maps". As the parameter c is increased, the maps exhibit period-doubling bifurcations, of 2^{k-1}- to 2^k- cycles for $k = 1, 2, 3, \ldots$. An example of such a family of non-quadratic unimodal maps is

$$g_c(x) = c \sin \pi x, \qquad x \in [0, 1],$$

for $c \in [0, 1]$. The reader may wish to explore the asymptotic behaviour of iteration sequences as the parameter c is varied from 0 to 1, in particular, locating the point c_1 where an attractive fixed point becomes neutral, giving rise to an attractive two-cycle. For such unimodal maps, the convergence of the bifurcation points $c_k \to c_\infty$ is geometric.

Some important comments about the period-doubling scheme $2 \to 4 \to 8 \to 16 \to \cdots$ cycles

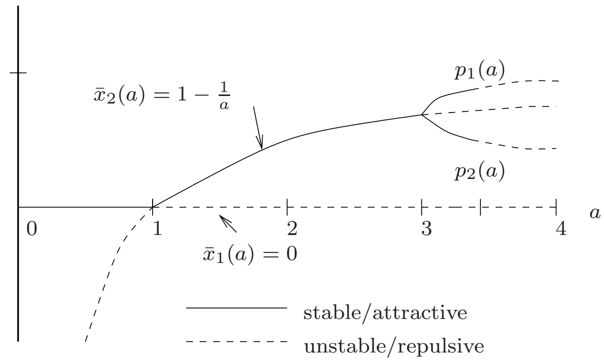
In the previous sections, we examined the "period-doubling" scheme for the logistic iteration process

$$x_{n+1} = a x_n (1 - x_n)\,, \tag{5}$$

as the parameter a is increased. For the range of parameter values $1 \leq a \leq 3$ the asymptotic behaviour of the iterates $\{x_n\}$ in Eq. (5) is simple – they approach the fixed point $\bar{x}_2(a) = 1 - \dfrac{1}{a}$ which was shown to be **attractive**. At $a = 3$, the fixed point $\bar{x}_2 = \dfrac{2}{3}$ is indifferent, but iterates still approach it in the limit.

For parameter values $3 < a < 1 + \sqrt{6}$, the iterates approach a two-cycle $(p_1(a), p_2(a))$ – this two-cycle is **attractive** over this range of parameters. But it is important to remember that the fixed point $\bar{x}_2(a) = 1 - \dfrac{1}{a}$ still exists for this range of parameter values. However, it is now **repulsive**, so it won't attract iterates. Numerically, even if you think you are exactly on this fixed point, you would probably be "blown off" and attracted to the two-cycle, unless you are working with an infinite number of decimal digit accuracy. (Even if the fixed point is a rational number, and you work with rational arithmetic, you would have problems with irrational values of a.)

The existence of the repulsive fixed point $\bar{x}_2(a)$ for $a > 3$ was indicted in a bifurcation diagram in the previous lecture - it is reproduced below. Unstable/repulsive fixed points are shown with dotted curves.



At $a = 1 + \sqrt{6}$, the two-cycle ceases to be attractive, giving rise to an **attractive four-cycle** for values of a slightly greater than $1 + \sqrt{6}$. For these values of a, the two-cycle is still present, but **repulsive**. This kind of "forking" or **bifurcation** will take place at all the period-doubling parameter values a_k. The result is that at any value of $a < a_\infty$, for which, say, there is an attractive 2^N-cycle, there are

$$1 + 2 + \cdots + 2^{N-1} = 2^N - 1 \tag{6}$$

repulsive cycle points in the interval [0,1]. And they lie between the 2^N points of the attractive 2^N-cycle!

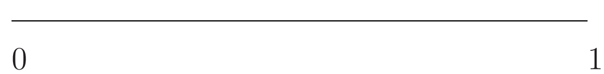
Behaviour at $a_\infty = 3.5699456$

We now return to the question of the behaviour of the iteration scheme $x_{n+1} = f_a(x_n)$ at the limit point $a_\infty = \lim_{k\to\infty} a_k$, where the a_k are bifurcation points at which 2^{k-1}-cycles give rise to 2^k-cycles. The limiting operation $\lim_{k\to\infty} a_k = a_\infty$ is accompanied by a procedure involving a "limit" of sets of points – here, the sets S_k, $k = 1, 2, 3, \ldots$, are composed of points that make up the periodic 2^k-cycle. Note that each set S_k is composed of a finite number of points. The "limiting" operation $\lim_{k\to\infty} S_k$ will involve a "closure" of subsequences of points that have limits. As a result, not only will the limiting set "S_∞" have an infinite number of points but S_∞ will have an **uncountable infinity** of points. In other words, S_∞ will have as many points as there are real numbers in the interval $[0, 1]$! We shall show how such a strange "fractal" set can be produced below.

On the next page is plotted a typical iteration sequence $\{x_n\}$. Note that the "signal" x_n vs. n does not look periodic, yet it does not look far from being periodic. In fact, it can be shown that this motion is "chaotic" on the fractal attractor set S_∞. In an attempt to show the fractal nature of the set S_∞ to which the iterates x_n are attracted, the bottom figure on the next page is a "'histogram" plot of 10000 consecutive iterates x_n. The plot is useful to record both the location of iterates as well as the frequency at which they visit regions. 5000 bins representing the subintervals $I_k = [b_{k-1}, b_k)$, $k = 1, 2, \ldots 500$, where $b_k = k\Delta x$, $\Delta x = 1/5000$, were used. Initially, a bin-counting vector was set to zero, i.e. $\mathbf{ibin}(i) = 0$, $1 \leq i \leq 5000$. Picking an x_0 value (here $x_0 = 1/3$), the iteration process $x_{n+1} = f_a(x_n)$ was begun. For each $n \geq 1$, the location of x_n, i.e. the bin I_m in which x_n fell ($m = \mathrm{int}(x_n/\Delta x) + 1$) was determined and then the counter incremented by 1, i.e. $\mathbf{ibin}(m) := \mathbf{ibin}(m) + 1$.

The resolution of the histogram plot shown in the figure – 5000 bins – is still too course to reveal the fractal nature of the set S_∞ on which the x_n travel. Nevertheless, one may notice a kind of self-similar nesting of "gaps" in the set S_∞: The portion of S_∞ that lies in the region $0.47 < x < 0.56$ looks like a scaled down version of the entire set S_∞ lying in the region $0.33 < x < 0.89$.

In order to better understand the idea of a fractal set on the real line, we now introduce a classical example: the so-called "ternary Cantor set" on $[0, 1]$. The Cantor set is constructed by means of a "middle-thirds dissection" procedure. We begin with the unit interval $J_0 = [0, 1]$:

J_0:


Now remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from this set to produce the set $J_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$:

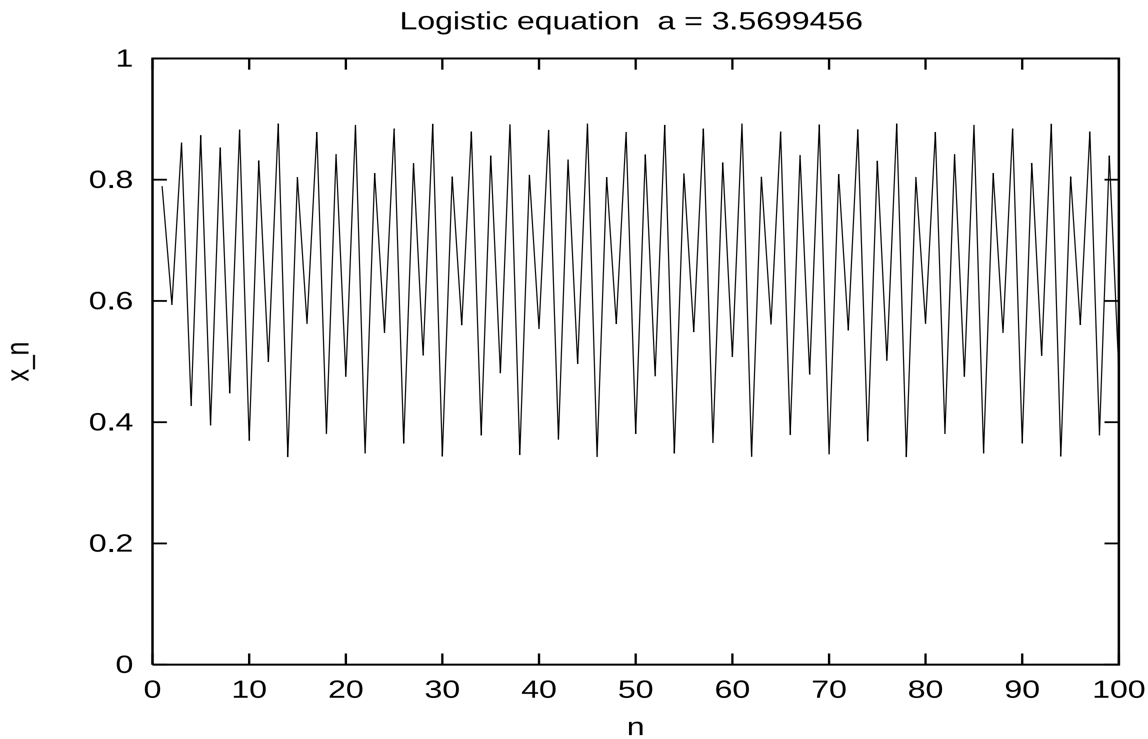
J_1:


Repeat the procedure, removing the "middle-thirds" open sets $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$ from J_1 to produce the set $J_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$:

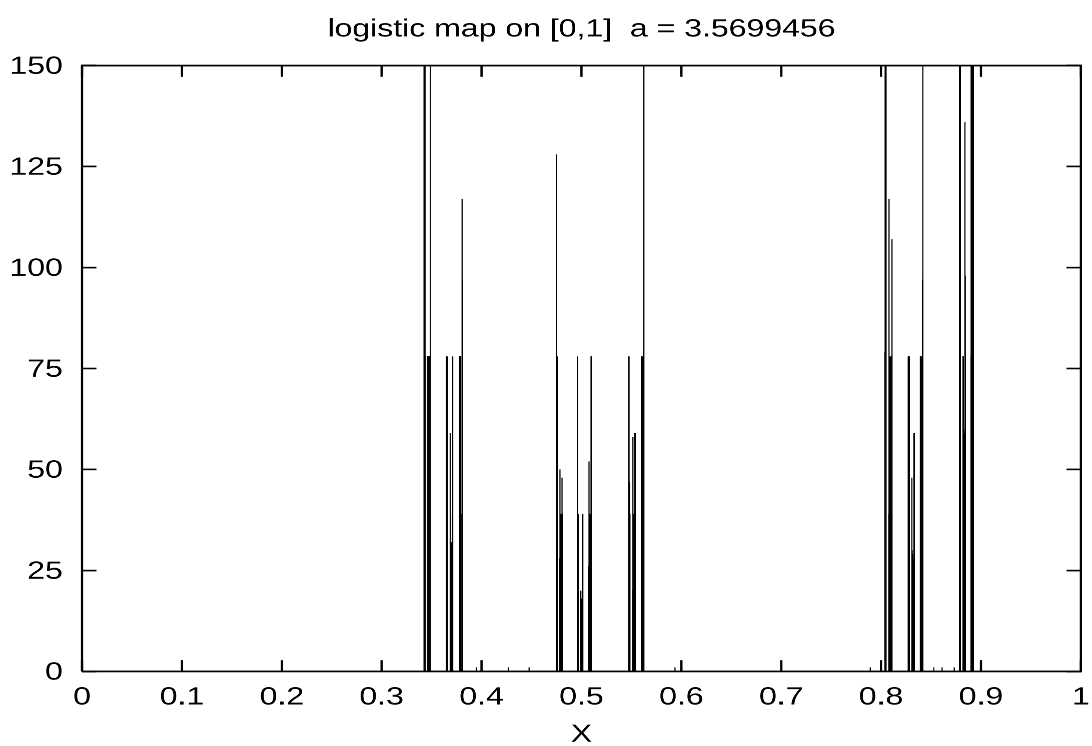
J_2:


Now continue this procedure to produce the sets $J_3, J_4, \ldots$. In the limit $k \to \infty$, the sets J_k converge (in a suitable fashion) to a limiting set, C, the ternary Cantor set on $[0, 1]$:

This set C is a fascinating set. It contains points that one may not have imagined to lie on this set, such as $x = \frac{1}{4}$. It also contains an uncountable infinity of points: The points in C can be put into a one-to-one correspondence with points on the real interval $[0, 1]$. This is quite surprising, especially



Plot of the first 100 iterates $x_{n+1} = f_a(x_n)$, $a = 3.5699456$



Histogram plot of 10000 iterates to show their distribution over [0,1] in light of the fact that we have removed virtually all of the interval $[0,1]$ in order to produce C. In fact, let us attempt to compute the "length" of the Cantor set C by examining the lengths of the sets J_n that were used to construct C:

$$\begin{array}{llll}
\text{Length of} & J_0 = [0,1] & \text{is} & L_0 = 1 \\
\text{Length of} & J_1 : 2 \times \frac{1}{3} & & L_1 = \frac{2}{3} \\
\text{Length of} & J_2 : 4 \times \frac{1}{9} & & L_2 = \frac{4}{9} \\
& \vdots & & \\
\text{Length of} & J_n : 2^n \cdot \frac{1}{3^n} & & L_n = \left(\frac{2}{3}\right)^n
\end{array}$$

Note that $\lim_{n\to\infty} L_n = 0$. In other words, the Cantor set has zero length! (We obtain the same result by computing the total length of intervals that have been removed from $[0,1]$ in order to construct C. This length is

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots = \frac{1}{3}\left[1 + \frac{2}{3} + \frac{4}{9} + \cdots\right] \tag{7}$$

$$= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{3}} \tag{8}$$

$$= 1. \tag{9}$$

Thus, length of $C = 1 - 1 = 0$.)

The Cantor set also exhibits an interesting property of "self-similarity". The subset $C_1 \subset C$ that lies on the interval $\left[0, \frac{1}{3}\right]$, is an exact scaled-down copy of C. If we were to magnify C_1 by a factor of 3 – by taking each point x_i of C_1 and moving it to $3x_1$ – then we would obtain C. Similarity, $C_2 \subset C$, the subset of C that lies on the interval $\left[\frac{2}{3}, 1\right]$, is an exact scaled-down copy of C. We may write

$$C = C_1 \cup C_2,$$

where C_1 and C_2 are shrunken copies of C. In other words, the Cantor set is a union of shrunken copies of itself. This is a feature exhibited by many "fractal" sets. Note that we have not, as of yet, formally defined the term "fractal" but shall return to this subject in Chapter 4.

Lecture 8

The Significance of Three-Cycles

`Relevant section of textbook by Gulick, Second Edition: 1.7`

As mentioned in the previous lecture, at around $a \cong 3.83$, the asymptotic behaviour of the iterates of the logistic function show that there is an attractive three-cycle. The textbook by Gulick mentions (Page 62) that this three-cycle exists in the interval $3.829 \leq a \leq 3.840$ As a increases from 3.840, the attractive three-cycle bifurcates into an attractive six-cycle, then a 12-cycle, etc.. In this section, we show that the existence of a three-cycle for a function $f(x)$ is a rather special situation since it implies the existence of other fixed points. There are two celebrated theorems, one by Li and Yorke (1975) and the other by Sharkovsky (1064) that prove that **when a continuous function $f(x)$ has a three-cycle, it has periodic points of all order $n \geq 1$.** This is indeed a remarkable result.

The presentation in this and the next lecture will follow the treatment in Section 1.7, "Period-3 Points," of the textbook of Gulick (Page 57).

First of all, let us recall that we have seen examples of continuous functions with different cases of periodic points. For example:

1. The function $f(x) = x^2 + \dfrac{1}{2}$ has no periodic points, i.e., no fixed points, no two-cycles, etc.. (In most cases, we'll omit the graphs, leaving them to the reader to construct.)

2. The function $f(x) = x^2 + \dfrac{1}{4}$ has one fixed point, i.e., one period-1 point, and no other cycles or period-n points.

3. The function $f(x) = x^2$ has two fixed points or period-1 points and no cycles or period-n points, $n > 1$.

4. The function $f(x) = -x^{1/3}$ has one fixed point, $\bar{x} = 0$ and one two-cycle, namely, $(1, -1)$. Equivalently, it has two period-2 points, $p_1 = 1$ and $p_2 = -1$. (Exercise. Start by sketching the graph of $f(x)$.)

The above examples seem to indicate that the existence of fixed points and cycles is rather independent. But the following result, which we shall eventually prove, shows that the situation changes remarkably when f has a 3-cycle.

Theorem: If f is continuous and has a 3-cycle (or a period-3 orbit), then f has

1. a fixed point $\bar{x}$ (or a period-1 point) and
2. a two-cycle (or period-2 points).

Before we try to prove this result, it may be helpful to use a little graphical analysis to see why the above result holds. It's certainly not a proof, but it will give us some insight. If f has three period-3 points, $a < b < c$, then there are two possibilities:

1. $f(a) = b \quad f(b) = c \quad f(c) = a.$
2. $f(a) = c \quad f(c) = b \quad f(b) = a.$

On the next page are sketched some possible graphs representing these situations. (They were presented in the lecture.) Of course, the graphs of f have been assumed to be very simple – they could have many more oscillations between the period-3 points.

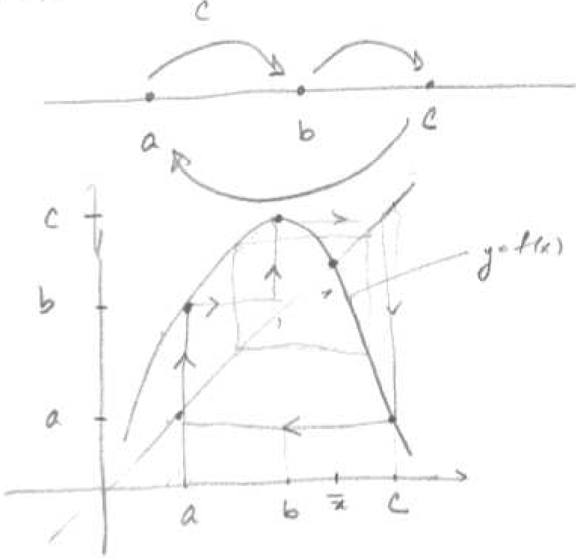
The top two graphs are those of $f(x)$ in each of the above situations. The graphs below them are sketches of the graph $f^2(x)$. To show the existence of a period-2 point, we'll have to show that the function $g(x) = f^2(x)$ has a fixed point. Note that a fixed point $\bar{x}$ of $f(x)$ is also a fixed point of $g(x) = f^2(x)$. That feature, of course, must be included in the graph of $g(x)$.

In each of the graphs of $f^2(x)$, we see the existence of fixed points that are not fixed points of $f(x)$. These points are period-2 points.

We now start our path toward a proof of the above theorem. We'll need some preliminary results. Two of these results should be familiar to you from Year 1 Calculus.

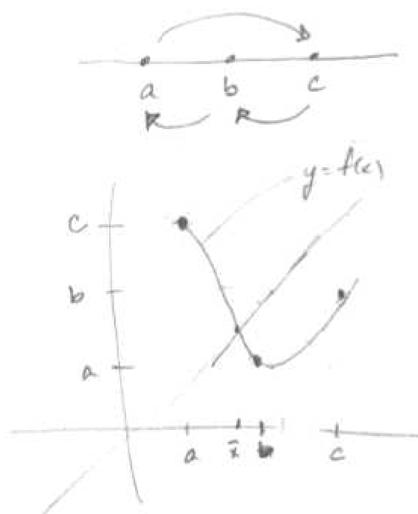
CASE 1

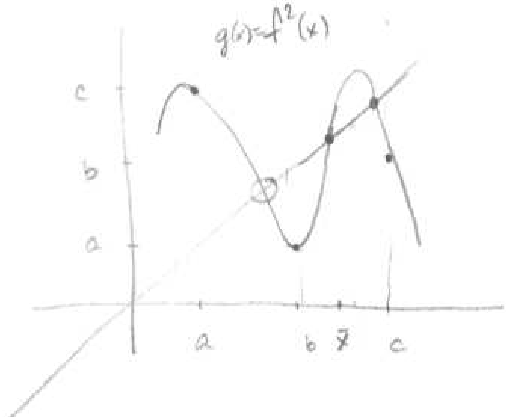
$f(a) = b \quad f(b) = c \quad f(c) = a$

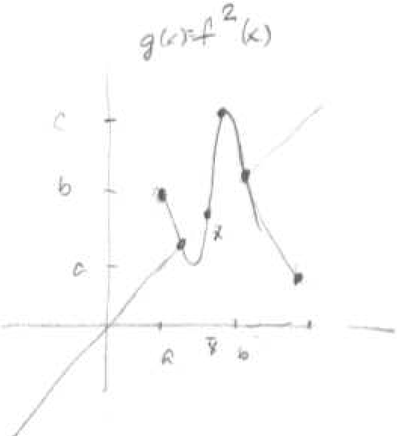


CASE 2

$f(a) = c \quad f(c) = b \quad f(b) = a$







$f^2(a) = f(f(a)) = f(b) = c$

$f^2(b) = f(f(b)) = f(c) = a$

$f^2(c) = f(f(c) = f(a) = b$

$f^2(a) = b$

$f^2(b) = (c)$

$f^2(c) = a$

Theorem 1 (Maximum-Minimum or Extreme-Value Theorem): Let f be continuous on $[a, b]$ Then f achieves maximum and minimum values on $[a, b]$. More precisely, There are numbers c and d in $[a, b]$ such that if x is any number in $[a, b]$, then

$$f(c) \leq f(x) \leq f(d)\,. \tag{10}$$

$f(c)$ and $f(d)$ are, respectively, the minimum and maximum values of $f(x)$ on $[a, b]$. We may also write

$$f(c) = \min_{a \leq x \leq b} f(x)\,, \quad f(d) = \max_{a \leq x \leq b} f(x)\,, \tag{11}$$

Theorem 2 (Intermediate Value Theorem): Once again, let f be continuous on $[a, b]$. Let p be any number between $f(a)$ and $f(b)$. Then there exists a number $c \in [a, b]$ such that $f(c) = p$.

Note: These theorems were most likely not proved in your first-year Calculus courses since the proofs require some rather advanced ideas from Analysis. We shall omit the proofs of these theorems in this course as well, referring the interested reader to an advanced book in Calculus or a book in Analysis.

We now proceed to prove some rather straightforward results.

Lemma 1: Let f be continuous on an interval J. Let $f(J)$ denote the set of all values of $f(x)$ for $x \in J$, i.e.,

$$f(J) = \{f(x)\,, \quad x \in J\,\}\,. \tag{12}$$

Then $f(J)$ is an interval.

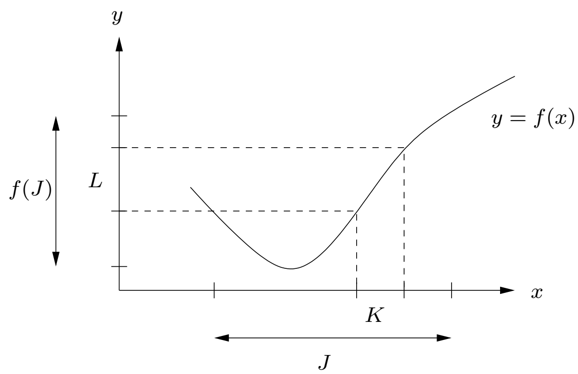
Proof: Assume the contrary, i.e., that $f(J)$ is not an interval. If $f(J)$ is not an interval, say $[c, d]$, then it would have to have "holes." Some possibilities,

1. It could be a set of nonintersecting intervals, i.e., $[a_1, b_1]$, $[a_2, b_2]$, $\cdots$ $[a_n, b_n]$.
2. It could be a set of points $\{y_1, y_2, y_n\}$.
3. A combination of 1 and 2.

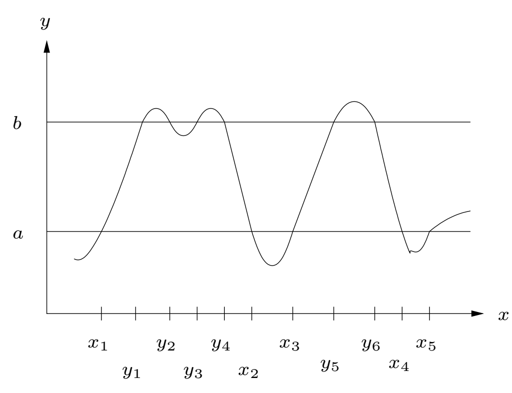
Whatever the case, if $f(J)$ were not an interval, then there would exist two numbers y and z in $f(J)$, with $y < z$, and a number p in (y, z) such that p is not in the set $f(J)$. This means that there is no $x \in J$ such that $f(x) = p$.

Since y and z are in $f(J)$, there is an $a \in J$ such that $f(a) = y$ and a $b \in J$ such that $f(b) = z$. From above, value p is an intermediate value, i.e., a value between $f(a)$ and $f(b)$. Therefore, there exists a $c \in J$ between a and b for which $f(c) = p$. But this then implies that $p \in f(J)$, which contradicts our earlier statement that $p \notin f(J)$. Therefore, our assumption that $f(J)$ is not an interval cannot be true. This implies that $f(J)$ is an interval, and the Lemma is proved.

Lemma 2: Let $f(x)$ be continuous on a closed interval J. Let $L = [a, b] \subseteq f(J)$. Then there is a closed interval K such that $K \subseteq J$ and $f(K) = L$. We sketch a possible situation below.



Proof: Let $x_1 < x_2 < x_3 < \cdots$ be points in J such that $f(x_i) = a$. The existence of at least one of these points is guaranteed by the fact that the point a belongs to $f(J)$. Similarly, let $y_1 < y_2 < \cdots$ be points in J such that $f(y_j) = b$. A possible situation is sketched below.



Now order these numbers in increasing value. For example, from the situation sketched in the above graph,

$$x_1 < y_1 < y_2 < y_3 < y_4 < x_2 < x_3 < y_5 < y_6 < x_4 \cdots$$

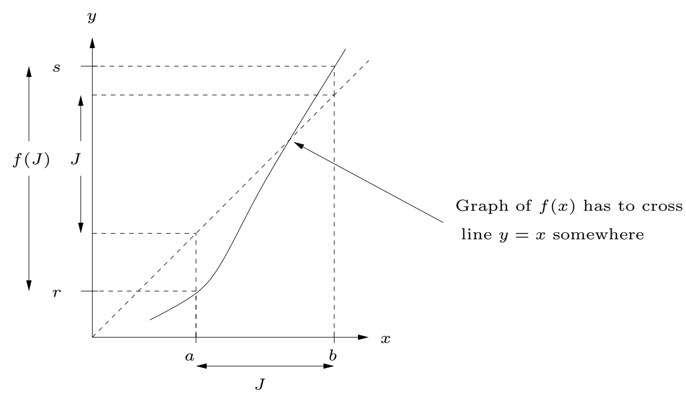
Now pick a consecutive pair

$$x_i < y_j \quad \text{or} \quad y_i < x_j$$

such that there are no other points x_k or y_l between them. From the graph, this implies that the graph of f travels from (x_i, a) to (y_j, b) or from (y_i, b) to (x_j, a) without crossing the lines $y = a$ or $y = b$ between these points. Graphically, we see that $[x_i, y_j]$ or $[y_i, x_j]$ is the interval K that satisfies the relation $f(K) = [a, b]$.

In the above graph, K could be any of the following intervals: $[x_1, y_1]$, $[y_4, x_2]$, $[x_3, y_5]$, $[y_6, x_4]$.

Lemma 3: Let f be continuous on an interval $J = [a, b]$ and $J \subset f(J)$. Then f has a fixed point in J. A possible situation is sketched below.



Graphically, we see that there is no way that the graph can extend from one side of J to the other without crossing the line $y = x$, implying the existence of a fixed point $\bar{x} = f(\bar{x})$. Of course, this is not a proof, but it is a nice picture to keep in mind.

Proof: By the Maximum-Minimum (Extreme Value) Theorem, f has a minimum value r and a maximum value s on J. This implies that the interval $f(J) = [r, s]$. Note that $r \neq s$: Otherwise, $f(J)$ consists of one point, which contradicts the original assumption that $J \subset f(J)$. Therefore there exist y and z in J such that

$$f(y) = r \quad \text{and} \quad f(z) = s\,.$$

Since $J \subset f(J)$, it follows that

$$r \leq y \leq s \quad \text{and} \quad r \leq z \leq s\,.$$

Perhaps the sketch above will show this more clearly.

Now define the following function on J:

$$g(x) = f(x) - x\,.$$

Since f is continuous on J and the function $h(x) = x$ is also continuous on J, it follows that $g(x)$ is continuous on J. Note that

$$g(y) = f(y) - y = r - y \leq 0 \quad \text{(we showed above that } r \leq y \text{)}$$

and

$$g(z) = f(z) - z = s - z \geq 0 \quad \text{(we showed above that } z \leq s \text{).}$$

There are three cases to consider:

1. $g(y) = 0$ which implies that $f(y) = y$, i.e., y is a fixed point of f.

2. $g(z) = 0$ which implies that $f(z) = z$, i.e., z is a fixed point of f.

3. $g(y) < 0$ and $g(z) > 0$, which implies that $g(x)$ changes sign as we move from y to z. We now employ the Intermediate Value Theorem, with the intermediate value 0 to conclude that there exists a point $c \in J$ such that

   $$g(c) = 0 \quad \Longrightarrow \quad f(c) - c = 0 \quad \Longrightarrow \quad f(c) = c\,.$$

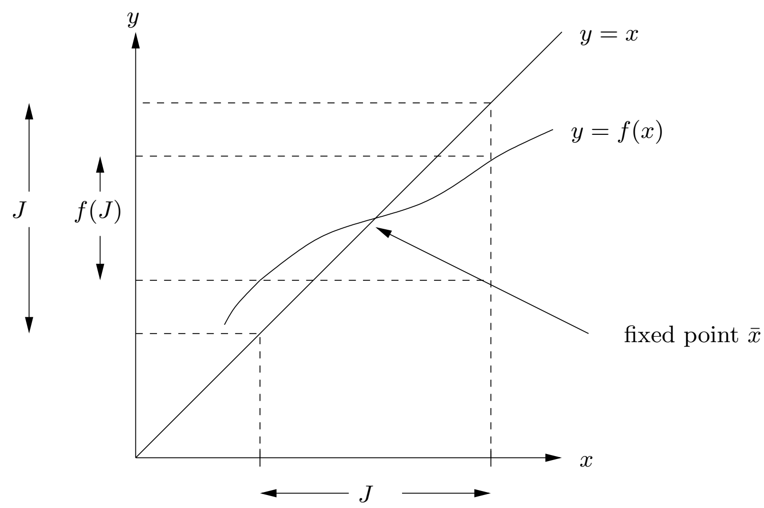
   Therefore, c is a fixed point of f.

Since all possible cases have been considered, f must have a fixed point and the theorem is proved.

Before going on to the next Lemma, let us investigate the consequences of some other possible relationships between a closed interval J and its image $f(J)$ under the action of a continuous map. For example, instead of the situation $J \subset f(J)$ considered above, what about the case

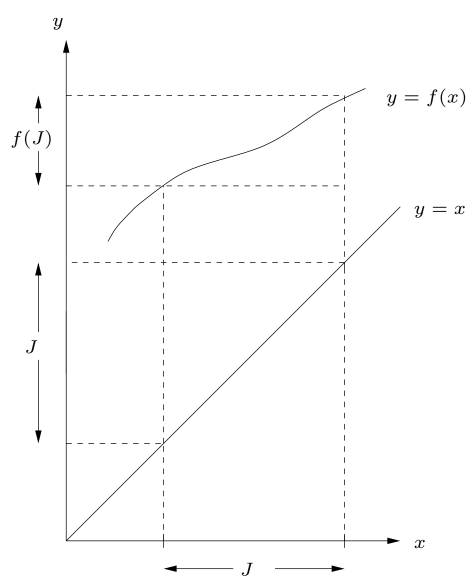
$$f(J) \subset J\,,$$

as sketched in the figure below. We see that, once again, there is no way that the graph can extend from one side of J to the other without crossing the line $y = x$, implying the existence of a fixed point $\bar{x} = f(\bar{x})$.



This is actually a special case of a more general theorem known as the “Schauder Fixed Point Theorem”, which is beyond the scope of this course. You may see it some day in an advanced course in analysis.

There is one more situation to consider - the one in which J and $f(J)$ do not intersect. A possible situation is sketched below.



We see that there is no possibility for the graph of $y = f(x)$ to intersect the line $y = x$, which suggests that no fixed point exists. In fact, this follows quite easy theoretically. If f has a fixed point $\bar{x}$, then $f(\bar{x}) = \bar{x}$ implies that $f(J)$ and J must have a common point, namely $\bar{x}$.

Lecture 9

The Signicance of Three-Cycles (cont'd)

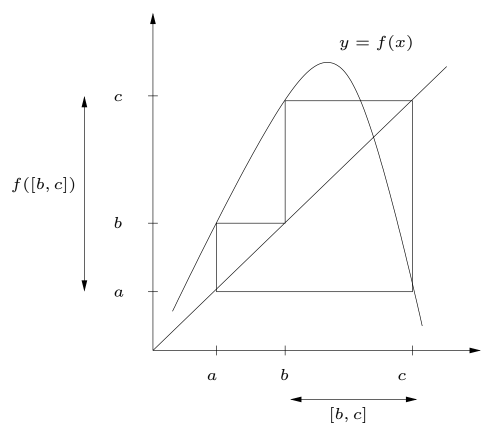
`Relevant section of textbook by Gulick, Second Edition: 1.7`

We continue with the discussion from the previous lecture.

Lemma 4: Let f be continuous and suppose that $f(a) = b$, $f(b) = c$ and $f(c) = a$ with, of course, a, b and c distinct. (In other words, f has a three-cycle (a, b, c).) Then f has both of the following:

1. a period-1 point, that is, a fixed point $\bar{x}$, i.e., $f(\bar{x}) = \bar{x}$,

2. a period-2 point p, i.e., $f^2(p) = p$. (This, implies, of course, that f has a two-cycle, i.e., (p, q), with $f(p) = q$ and $f(q) = p$, but the Lemma only establishes the existence of at least one period-2 point.)



Graphical illustration of a three-cycle: $f(a) = b$, $f(b) = c$, $f(c) = a$, $a < b < c$.

Proof: Without loss of generality, we shall assume that $a < b < c$, as sketched in the graph above. There is only one other case, $a < c < b$, and it can be proved in a method analogous to the one used for this case. (All other cases are equivalent to one of these two.)

Since

$$f(b) = c \quad \text{and} \quad f(c) = a\,,$$

It follows that

$$[a, c] \subset f([b, c])\,.$$

To see this: Note that a and c are two points that must be included in the set $f([b,c])$ which, from Lemma 1, must be an interval. As such, $[a,c]$ must be included in the set $f([b,c])$. It is possible, however, that there are points outside $[a,c]$ that also exist in $f([b,c])$.

Since

$$[b,c] \subseteq [a,c]$$

(technically $[b,c] \subset [a,c]$, but it doesn't hurt to keep the "equals" sign), it then follows that

$$[b,c] \subseteq f([b,c])\,.$$

We can now employ Lemma 3 to conclude that f has a fixed point $\bar{x} \in [b,c]$.

It now remains to show that f has a period-2 point. At some point, we'll have to examine the function $g(x) = f^2(x)$. In fact, from a look at the sketches of the graphs of $f(x)$ and $g(x)$ for Case 1 presented a few pages ago, we suspect that $g(x)$ has a fixed point in the interval $[a,b]$. In fact, we are tempted employ the same type of argument as we did above for $g(x)$ on $[a,b]$, i.e., since

$$g(a) = c \quad \text{and} \quad g(b) = a\,, \tag{13}$$

it follows that

$$[a,c] \subseteq g([a,b])\,. \tag{14}$$

But

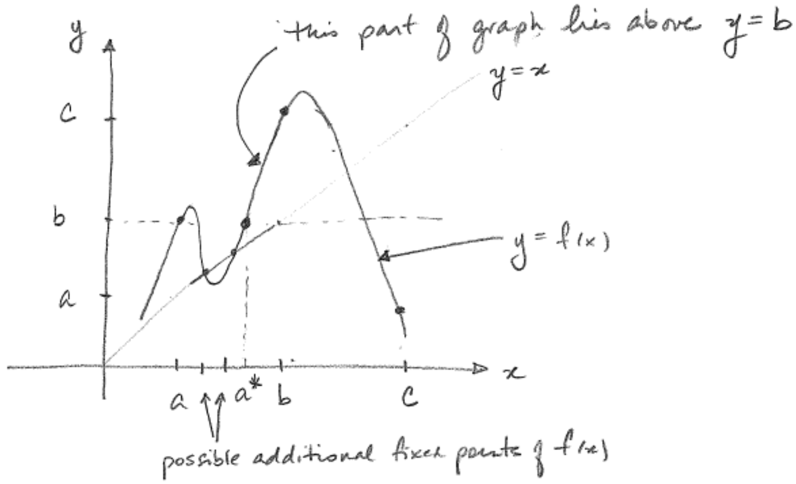
$$[a,b] \subset [a,c]\,, \tag{15}$$

which implies that

$$[a,b] \subset g([a,b])\,, \tag{16}$$

which, from Lemma 3, implies that $g(x) = f(f(x))$ has a fixed point in $[a,b]$.

The only problem with this argument is that we don't know if this fixed of $g(x)$ is a point of a two-cycle of $f(x)$ or a fixed point of $f(x)$. Indeed, the behaviour of $f(x)$ can be more complicated than the simple behaviour shown in the previous sketch. In the sketch below, we have modified the function $f(x)$ over the interval (a,b) to show that (i) there can be points other than a which f maps to b and (ii) f can have fixed points in (a,b). To show that f^2 has a fixed point which is not a fixed point of f will require a little more work, as we now show.



First of all, let a^* be the largest number such that

$$a \leq a^* < b \quad \text{and} \quad f(a^*) = b\,.$$

In the previous sketch, $a = a^*$. In the sketch immediately above, a^* is **not** a but a point closer to b. At the moment, it will not be clear why we need this point – that will come near the end of the proof.

We now have that

$$f(a^*) = b \quad \text{and} \quad f(b) = c\,.$$

Using the same reasoning as before, we can conclude that

$$[b, c] \subseteq f([a^*, b])\,.$$

Let us now examine what the function $g = f^2$ (i.e., $g(x) = f(f(x))$) does to the interval $[a^*, b]$. We examine what f^2 does to the endpoints:

$$f^2(a^*) = f(f(a^*)) = f(b) = c$$

and

$$f^2(b) = f(f(b)) = f(c) = a\,.$$

Once again using the same reasoning as before, we conclude that

$$[a, c] \subseteq f^2([a^*, b])\,.$$

But recall that $[a^*, b] \subset [a, c]$, which implies that

$$[a^*, b] \subseteq f^2([a^*, b])\,.$$

We can now use Lemma 3 to conclude that f^2 has a fixed point $p \in [a^*, b]$. There are two possibilities for p:

- p is a period-2 point of f, or

- p is a period-1 point of f, i.e., p is a fixed point of f.

We now show that the second possibility cannot be true. To do so, recall that

$$a^* < b < c \quad \text{and} \quad f(a^*) = b\,, \quad f(b) = c\,,$$

which implies that

$$f^2(a^*) = c\,.$$

Since $a^* \neq c$, a^* cannot be a fixed point of f^2, which means that $p \neq a^*$. We can then revise our earlier result that $p \in [a^*, b]$ to $p \in (a^*, b]$, i.e.,

$$a^* < p \leq b\,. \tag{17}$$

Likewise

$$f(b) = c \quad \text{and} \quad f(c) = a\,, \tag{18}$$

which implies that

$$f^2(b) = a\,. \tag{19}$$

Since $a \neq b$, it follows that neither a nor b can be fixed points of f^2, i.e., $p \neq a$ and $p \neq b$. The latter inequality is more important since we can revise the inequality in (17) to the following,

$$p \in (a^*, b) \quad \text{or} \quad a^* < p < b\,. \tag{20}$$

Now recall the definition of a^*: it is the point closest to, but less than, b such that $f(a^*) = b$. It therefore follows that

$$\text{if} \quad a^* < x < b \quad \text{then} \quad f(x) > b\,.$$

To see this: If $f(x) = b$ for some $a^* < x_1 < b$, then a^* isn't the largest value of $x < b$ for which $f(a) = b$. (x_1 is.) If $f(x) < b$ for some $a^* < x_1 < b$, then the graph of $f(x)$ would have to increase back to the value b at some $x_1 < x_2 < b$ in order to make its way to the value $c = f(b)$. But the existence of x_2 contradicts the original fact that a^* is the largest value of $x < b$ for which $f(a) = b$.

Now going back to the previous inequality, since $a^* < p < b$, it follows that

$$f(p) > b\,.$$

But $a^* < p < b$ also implies that $b > p$ so that we may add the following inequality to the above:

$$f(p) > b > p\,,$$

which implies that p cannot be a fixed point of f. We can therefore conclude that p is a period-2 point of f and the proof of the Lemma is finished.

We now use the results of the previous two lectures to prove the celebrated **Li-Yorke Theorem.** The material presented below closely follows the proof in Gulick's book (Section 1.7, p. 61, Theorem 1.19), but additional explanations are provided throughout the proof.

Theorem: (Li-Yorke) Suppose that f is continuous on the closed interval J, with $f(J) \subset J$. If f has a period-3 point, then f has points of all other periods, i.e., of period-n, $n = 1, 2, \cdots$.

Proof: As before, without loss of generality, we assume that $f(a) = b$, $f(b) = c$ and $f(c) = a$, with $a < b < c$. From Lemma 4 of the previous lecture, f has points of period 1 and 2. It remains to show that f has points of period n for $n > 3$.

Here is the main idea of the proof: It will be shown that for each $n > 3$, there is a point $p \in (b, c)$ such that

1. $f^k(p)$ lies in $[b, c]$ for $k = 1, 2, \cdots, n-2$.
2. $f^{[n-1]}(p)$ lies in (a, b).
3. $f^n(p) = p$ and lies in (b, c).

This proves that p is a period-n point.

The first step of the proof is to choose an $n > 3$ and keep this n fixed. Now let

$$J_0 = [b, c] .$$

Since $f(b) = c$ and $f(c) = a$, it follows that

$$[a, c] \subseteq f([b, c]) . \tag{21}$$

Since

$$J_0 = [b, c] \subseteq [a, c] \ ,$$

it follows that

$$[b, c] \subseteq f([b, c]) \quad \text{or} \quad J_0 \subseteq f(J_0) . \tag{22}$$

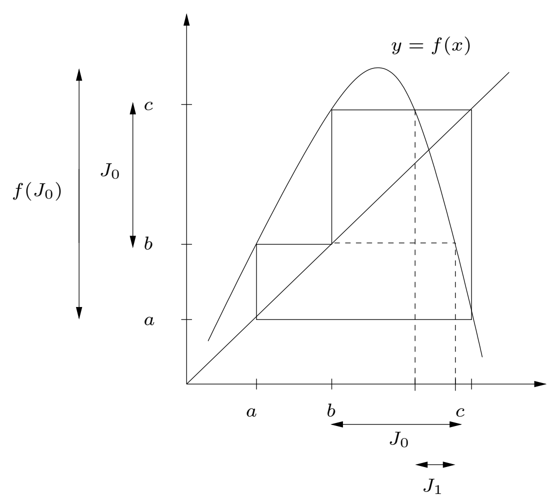
From Lemma 2 (previous lecture), it follows that there is a closed interval $J_1 \subseteq J_0$ such that

$$f(J_1) = J_0 .$$

This implies that

$$f^2(J_1) = f(f(J_1)) = f(J_0) .$$

Such a situation is sketched in the figure below.



But from (22), $J_0 \subseteq f(J_0)$ which implies that

$$J_0 \subseteq f^2(J_1) .$$

We use Lemma 2 once again – but now applied to the function f^2 – to conclude that there exists an interval $J_2 \subseteq J_1$ such that

$$f^2(J_2) = J_0 .$$

We repeat this idea one more time so that the reader gets the idea of the iterative process that is going to be performed. Consider

$$f^3(J_2) = f(f^2(J_2)) = f(J_0)\,.$$

Now use (22), i.e., $J_0 \subseteq f(J_0)$, to conclude that

$$J_0 \subseteq f^3(J_2)\,.$$

Once again, we use Lemma 2 to conclude that there exists an interval $J_3 \subseteq J_2$ such that

$$f^3(J_3) = J_0\,.$$

The reader should be able to see the pattern. We continue the process to obtain a nested sequence of closed intervals,

$$[b,c] = J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots \supseteq J_{n-2}\,, \tag{23}$$

which obey the properties

$$f^k(J_k) = J_0 = [b,c]\,, \quad \text{for} \quad k = 1, 2, \cdots, n-2\,. \tag{24}$$

Note that the procedure has been stopped at J_{n-2}.

Aside: The above construction will allow Step 1 in the list presented at the beginning of this proof to be accomplished. We now have to "move over" to the interval $[a,b]$.

The final step in the above recursive procedure produced subinterval J_{n-2} which satisfies

$$f^{[n-2]}(J_{n-2}) = J_0 = [b,c]\,.$$

Let us now apply the function f to both sides of the above equation:

$$f^{[n-1]}(J_{n-2}) = f(f^{[n-2]}(J_{n-2})) = f([b,c]) \supseteq [a,c] \supseteq [a,b]\,,$$

where second-to-last inclusion comes from Eq. (21). Once again invoking Lemma 2 – now applied to the function $f^{[n-1]}$ – we conclude that there is a closed interval $J_{n-1} \subseteq J_{n-2}$ such that

$$[a,b] = f^{[n-1]}(J_{n-1})\,. \tag{25}$$

From the above relation and the nested relations in (23), we have

$$f^n(J_{n-1}) = f(f^{[n-1]}(J_{n-1})) = f([a,b]) \supseteq [b,c] = J_0 \cdots \supseteq J_{n-2} \supseteq J_{n-1}\,. \tag{26}$$

In other words,

$$J_{n-1} \subseteq f^n(J_{n-1})\,. \tag{27}$$

We now use Lemma 3 of the previous lecture to conclude that there exists a point $p \in J_{n-1}$, and therefore in $[b,c]$, that is a fixed point of f^n.

Unfortunately, we ran out of time. The proof will have to be finished in the next lecture.