RA HW10

Represent the function $f(x)$ by the Fourier integral:

1. $f(x) = \begin{cases} 1, & \text{if } \quad |x| < \tau \\ 0, & \text{if } \quad |x| > \tau \end{cases}$

2. $f(x) = \begin{cases} e^{-\alpha x} \sin \omega x, & \text{if } \quad x > 0, \\ 0, & \text{if } \quad x < 0; \end{cases} \quad \alpha > 0;$

1. $f(x) = \frac{1}{\pi} \int_0^{+\infty} dy \int_{-\infty}^{+\infty} f(t) \cos y(x-t)\, dt = \frac{1}{\pi} \int_0^{+\infty} dy \int_{-\tau}^{\tau} \cos y(x-t)\, dt. = \frac{2}{\pi} \int_0^{+\infty} \frac{\cos xy \sin \tau y}{y} dy \quad (x \neq \tau)$

2. $f(x) = \frac{1}{\pi} \int_0^{+\infty} dy \int_{-\infty}^{+\infty} f(t) \cos y(x-t)\, dt = \frac{1}{\pi} \int_0^{+\infty} dy \int_0^{+\infty} e^{-\alpha t} \sin \omega t \cos y(x-t)\, dt. \quad (x \neq 0)$

3. Represent the Fourier integral function $f(x)$ by continuing it in an odd way on the interval $(-\infty; 0)$ if: $f(x) = \begin{cases} \sin x, & \text{if } \quad 0 \leqslant x \leqslant \pi \\ 0, & \text{if } \quad x > \pi; \end{cases}$

$$b(y) = \frac{2}{\pi} \int_0^{+\infty} f(t) \sin yt\, dt = \frac{2}{\pi} \int_0^{\pi} \sin t \sin yt\, dt = \frac{1}{\pi} \int_0^{\pi} \cos(y-1)t - \cos(y+1)t\, dt =$$

$$= \begin{cases} \frac{\sin(y-1)\pi}{(y-1)\pi} - \frac{\sin(y+1)\pi}{(y+1)\pi} & y \neq \pm 1 \\ 1 & y = 1 \\ -1 & y = -1 \end{cases} \qquad f(x) = \int_0^{+\infty} b(y) \sin xy\, dy$$

for $x \in (-\infty, 0)$. $f(x) = -f(-x) = -\int_0^{+\infty} b(y) \sin(-xy)\, dy = \int_0^{+\infty} b(y) \sin xy\, dy.$

$= + \int \sin y\pi \sin xy \cdot \frac{1}{\pi} \cdot \left[\frac{1}{y+1} - \frac{1}{y-1}\right] = \frac{2}{\pi} \int \frac{\sin \pi y}{1-y^2} \cos xy\, dy.$

4. Represent the Fourier integral function $f(x)$ by continuing it in an even way on the interval $(-\infty; 0)$ if: $f(x) = e^{-\alpha x}, x \geqslant 0, \alpha > 0;$

$$a(y) = \frac{2}{\pi} \int_0^{+\infty} f(t) \cos yt\, dt = \frac{2}{\pi} \int_0^{+\infty} e^{-\alpha t} \cos yt\, dt =$$

$$\int e^{-\alpha t} \cos yt\, dt = \frac{e^{-\alpha t} \sin yt}{y} + \frac{\alpha}{y} \int e^{-\alpha t} \sin yt\, dt.$$

$$\int e^{-\alpha t} \sin yt\, dt = -\frac{e^{-\alpha t} \cos yt}{y} - \frac{\alpha}{y} \int e^{-\alpha t} \cos yt\, dt.$$

$$\int e^{-\alpha t} \cos yt\, dt = \frac{e^{-\alpha t} \left(\frac{\sin yt}{y} - \frac{\alpha}{y^2} \cos yt\right)}{1 + \frac{\alpha^2}{y^2}} \qquad \text{thus } a(y) = \frac{2}{\pi} \cdot \frac{\alpha}{y^2+\alpha^2}$$

$f(x) = \frac{2\alpha}{\pi} \int_0^{+\infty} \frac{\cos xy}{y^2+\alpha^2} dy.$ $\quad x \in (-\infty, 0).$ $f(x) = f(-x) = \frac{2\alpha}{\pi} \int_0^{+\infty} \frac{\cos xy}{y^2+\alpha^2} dy.$

Find the Fourier transform of the function $f(x)$

5. $f(x) = \begin{cases} 1, & \text{if} \quad |x| \leqslant 1, \\ 0, & \text{if} \quad |x| > 1; \end{cases}$

6. $f(x) = \begin{cases} e^{ix}, & \text{if} \quad x \in [0;\pi], \\ 0, & \text{if} \quad x \notin [0;\pi]; \end{cases}$

7. $f(x) = e^{-x^2/2}\cos\alpha x;$

5. $$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} P.V. \int_{-\infty}^{+\infty} f(t)\, e^{-iyt}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iyt}\, dt = \frac{1}{\sqrt{2\pi}} \frac{e^{-iyt}}{-iy} \Big|_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-iy} - e^{iy}}{-iy} = \sqrt{\frac{2}{\pi}} \cdot \frac{\sin y}{y}$$

6. $$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} P.V. \int_{-\infty}^{+\infty} f(t)\, e^{-iyt}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} e^{it} \cdot e^{-iyt}\, dt = \frac{1}{\sqrt{2\pi}} \int_0^{\pi} e^{it(1-y)}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{i\pi(1-y)} - 1}{i(1-y)}$$

7. $$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} P.V. \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2}} \cos\alpha t\, e^{-iyt}\, dt. = \sqrt{\frac{2}{\pi}}\, P.V. \int_0^{+\infty} e^{-\frac{t^2}{2}} \cos\alpha t \cos yt\, dt$$

$$= \sqrt{\frac{1}{2\pi}} \int_0^{+\infty} e^{-\frac{t^2}{2}} \left[\cos(y+\alpha)t + \cos(y-\alpha)t\right] dt$$

Let $g(y) = \int_0^{+\infty} e^{-\frac{t^2}{2}} \cos(y+\alpha)t\, dt.$ by Leibniz rule

$$g'(y) = -\int_0^{+\infty} t \cdot e^{-\frac{t^2}{2}} \sin(y+\alpha)t\, dt. = 2 e^{-\frac{t^2}{2}} \sin(y+\alpha)t \Big|_0^{+\infty} - 2(y+\alpha) \int_0^{\infty} e^{-\frac{t^2}{2}} \cos(y+\alpha)t\, dt = 2(y+\alpha)\, g(y)$$

$$\frac{d g(y)}{dy} = 2(y+\alpha)\, g(y) \qquad g(y) = C \cdot e^{y^2+2\alpha y}$$

$$g(-\alpha) = \int_0^{+\infty} e^{-\frac{t^2}{2}}\, dt = \sqrt{\frac{\pi}{2}} \qquad \text{thus } C = e^{\alpha^2}\sqrt{\frac{\pi}{2}}.$$

$$\frac{e^{-\frac{(y+\alpha)^2}{2}} + e^{\frac{(y+\alpha)^2}{2}}}{2}$$

similarly, $\int_0^{+\infty} e^{-\frac{t^2}{2}} \cos(y-\alpha)t\, dt = e^{\alpha^2}\sqrt{\frac{\pi}{2}}\, e^{y^2-2\alpha y}$

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{\frac{\pi}{2}} \cdot \left[e^{y^2-2\alpha y+\alpha^2} + e^{y^2+2\alpha y+\alpha^2}\right] = \frac{1}{2}\left[e^{(y-\alpha)^2} + e^{(y+\alpha)^2}\right]$$

8. Let $\hat{f}(y) = F[f(x)]$. To prove that: $F\left[e^{i\alpha x} f(x)\right] = \hat{f}(y-\alpha), \quad \alpha \in R;$

$$F\left[e^{i\alpha x} f(x)\right] = \frac{1}{\sqrt{2\pi}} P.V. \int_{-\infty}^{+\infty} e^{i\alpha t} f(t) \cdot e^{-iyt}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} P.V. \int_{-\infty}^{+\infty} f(t)\, e^{-i(y-\alpha)t}\, dt. = \hat{f}(y-\alpha).$$

Real Analysis 2024. Homework 11.

1. Find Fourier series for the function $f(x) = x\sin x$ on $[-\pi, \pi]$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[-x\cos x\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos x\,dx\right] = 1$$

$n \neq 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx.$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}x\cos(n+1)x\Big|_{-\pi}^{\pi} + \frac{1}{n+1}\int_{-\pi}^{\pi}\cos(n+1)\,dx + \frac{1}{n-1}x\cos(n-1)x\Big|_{-\pi}^{\pi} - \frac{1}{n-1}\int_{-\pi}^{\pi}\cos(n-1)\,dx.\right]$$

$$= \frac{1}{2\pi}\cdot\left[2\pi\frac{(-1)^n}{n+1} - 2\pi\frac{(-1)^n}{n-1}\right] = (-1)^n\frac{2}{1-n^2}$$

$n=1.$ $$a_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\sin 2x = \frac{1}{2\pi}\left[-\frac{1}{2}x\cos 2x\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\cos 2x\,dx.\right] = -\frac{1}{2}$$

since $f(-x) = -x\sin(-x) = x\sin x = f(x)$. even function.

$$f(x) \sim \frac{1}{2} + \sum_{n=2}^{\infty}(-1)^n\frac{2}{1-n^2}\cos nx.$$

$$1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^n\cos nx}{1-n^2}$$

2. Find cos and sin series for the function $f(x) = x - x^2/2$, $0 \le x \le 1$.

$l = \frac{1}{2}$

$$a_0 = \int_0^1 f(x)\,dx = \frac{x^2}{2} - \frac{x^3}{6}\Big|_0^1 = \frac{1}{3}.$$

$$a_n = 2\int_0^1 f(x)\cos nx = 2\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\Big|_0^1 + \frac{1}{n^3}\sin nx - \frac{1}{n^2}x\cos nx - \frac{1}{2n}x^2\sin nx\Big|_0^1\right] = \frac{\sin n}{n} + \frac{2\sin n}{n^3} - \frac{2}{n^2}$$

$$b_n = 2\int_0^1 f(x)\sin nx = 2\left[-\frac{1}{n}x\cos nx + \frac{1}{n^2}\sin nx\Big|_0^1 + \frac{1}{2n}x^2\cos nx - \frac{1}{n^2}x\sin nx - \frac{1}{n^3}\cos nx\Big|_0^1\right]$$

$$= -\frac{\cos n}{n.} - \frac{2\cos n}{n^3} + \frac{2}{n^3}$$

$$f \sim \frac{1}{3} + \sum_{n=1}^{+\infty}\left(\frac{\sin n}{n} + \frac{2\sin n}{n^3} - \frac{2}{n^2}\right)\cos 2n\pi x + \left(-\frac{\cos n}{n} - \frac{2\cos n}{n^3} + \frac{2}{n^3}\right)\sin 2n\pi x$$

3. Prove that

$$\sin x\ln\left(2\cos\frac{x}{2}\right) = \frac{1}{4}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx, \quad -\pi < x < \pi$$

$$RHS = \operatorname{Im}\left(\frac{z}{4} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}z^n\right).$$

$$\frac{z}{4} + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}z^n = \frac{1}{2}\left[\frac{z}{2} + \sum_{n=2}^{\infty}(-1)^n z^n\left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right] = \frac{1}{2}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^n}{n+1} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{n+1}}{n}\right].$$

$$= \frac{1}{2}\left[\frac{z - \ln(z+1)}{z} + z\ln(z+1)\right] = \frac{1}{2} + \frac{1}{2}\left(z - \frac{1}{z}\right)\ln(z+1)$$

$$\ln(e^{ix}+1) = \ln(1+\cos x + i\sin x) = \ln|2\cos\tfrac{x}{2}|\,e^{i\frac{x}{2}} = i\frac{x}{2} + \ln|2\cos\tfrac{x}{2}|$$

$$RHS = \operatorname{Im}\left[\frac{1}{2}\left(z - \frac{1}{z}\right)\ln(z+1)\right] = \operatorname{Im}\left[i\sin x\left(i\frac{x}{2} + \ln|2\cos\tfrac{x}{2}|\right)\right] = \sin x\ln\left(2\cos\tfrac{x}{2}\right) = LHS \qquad x\in(-\pi,\pi).\ \cos\tfrac{x}{2} > 0.$$

4. Calculate the sum of the trigonometric series

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin nx}{n(n+1)}.$$

$\triangle$: $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\sin nx}{n(n+1)} = \text{Im} \sum_{n=1}^{\infty}(-1)^n \frac{e^{inx}}{n(n+1)}$

$$\sum_{n=1}^{\infty}(-1)^n \frac{z^n}{n(n+1)} = \sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n} - \sum_{n=1}^{\infty}\frac{(-1)^n z^n}{n+1} = -\sum_{n=0}^{\infty}\frac{(-1)^n z^{n+1}}{n+1} - \frac{1}{z}\sum_{n=1}^{\infty}\frac{(-1)^n z^{n+1}}{n+1}$$

$$= -\ln(z+1) - \frac{\ln(z+1)-z}{z} = -\frac{\ln(z+1)}{z} - \ln(z+1) + 1$$

$\ln(e^{ix}+1) = \ln(1+\cos x + i\sin x) = \ln|2\cos\frac{x}{2}|\, e^{i\varphi(x)}$, where. $\varphi(x) = \arcsin\frac{\sin x}{\sqrt{2+2\cos x}} = \arcsin\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos\frac{x}{2}} = \arcsin\sin\frac{x}{2} = \frac{x}{2}$

$$= \ln|2\cos\tfrac{x}{2}| + \tfrac{x}{2}i$$

$\ln(e^{ix}+1)\left(-\frac{1}{z}-1\right).$

$$\sum_{n=1}^{\infty}(-1)\frac{\sin nx}{n(n+1)} = \text{Im}\left[-\frac{\ln(e^{ix}+1)}{e^{ix}} - \ln(e^{ix}+1)\right] = \text{Im}\left[(1+\cos x - i\sin x)(\ln|2\cos\tfrac{x}{2}| + i\tfrac{x}{2})\right]$$

$$= \sin x \ln|2\cos\tfrac{x}{2}| - \tfrac{x}{2}(\cos x + 1).$$

5. Calculate the sum of the trigonometric series

$$\frac{\cos 3x}{1\cdot 3\cdot 5} - \frac{\cos 5x}{3\cdot 5\cdot 7} + \frac{\cos 7x}{5\cdot 7\cdot 9} + \dots .$$

the original series $\displaystyle = \sum_{n=1}^{\infty}\frac{\cos(2n+1)x\,(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)} = \text{Re}\sum_{n=1}^{\infty}\frac{z^{2n+1}(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)}$

$$\sum_{n=1}^{\infty}\frac{z^{2n+1}(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)} = \frac{1}{8}\sum_{n=1}^{\infty}\frac{z^{2n+1}(-1)^{n+1}}{2n-1} + 2\frac{z^{2n+1}(-1)^{n}}{2n+1} + \frac{z^{2n+1}(-1)^{n+1}}{2n+3}$$

arctan z arctan z. arctan z

$$= \frac{1}{8}\left[\frac{z^2}{\tan z} + 2\left[\frac{1}{\tan z} - z\right] + \frac{\frac{1}{\tan z} - z + \frac{z^3}{3}}{z^2}\right] = \frac{1}{8}\left[\frac{z^2+2+\frac{1}{z^2}}{\tan z} - \frac{1}{z} - \frac{5}{3}z\right]$$

$$= \frac{1}{8}\, z^2\arctan z + 2\left[\arctan z - z\right] + \frac{\arctan z - z + \frac{z^3}{3}}{z^2}$$

$\arctan z\,(z+\frac{1}{z})^2$

$= \frac{1}{2i}\ln\frac{i-z}{i+z}\,(z+\frac{1}{z})^2$

$= \frac{1}{2i}\left(\ln\left|\frac{\sin(\frac{x}{2}-\frac{\pi}{4})}{\sin(\frac{x}{2}+\frac{\pi}{4})}\right| - \frac{\pi}{2}i\right)(4\cos^2 x).$

$= +\pi\cos^2 x$

$$\Rightarrow \text{Re}\sum_{n=1}^{\infty}\frac{z^{2n+1}(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)} = -\frac{\pi}{8}\cos^2 x - \frac{\cos x}{3}$$

$\ln(i+e^{ix}) = \ln|2\sin(\frac{x}{2}+\frac{\pi}{4})| + (\frac{x}{2}+\frac{\pi}{4})i.$

$\ln(i-e^{ix}) = \ln|2\sin(\frac{x}{2}-\frac{\pi}{4})| + (\frac{x}{2}-\frac{\pi}{4})i.$

$\ln|\cos x + i(1+\sin x)|.)$

$$\arctan\frac{1+\sin x}{\cos x} = \frac{(\sin\frac{x}{2}+\cos\frac{x}{2})^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \frac{\sin(\frac{x}{2}+\frac{\pi}{4})}{\sin(\frac{x}{2}+\frac{3\pi}{4})} = \arctan\tan(\tfrac{x}{2}+\tfrac{\pi}{4}) = \tfrac{x}{2}+\tfrac{\pi}{4}$$

6. Express function $f(x)$ by the Fourier integral

$\sin(x+\frac{\pi}{2}) = \cos x.$ $\sin x = \cos(x-\frac{\pi}{2})$

$\sin(x+\frac{\pi}{2}) = \cos x$

$$f(x) = \begin{cases} \sin x, & |x| \le \pi; \\ 0, & |x| > \pi. \end{cases}$$

f - odd function. $\mathcal{F}_s[f](y) = 2\int_0^{\pi} \sin x \sin(2\pi y x)\, dx$

1) $y \neq \frac{1}{2\pi}$ $= \int_0^{\pi} \cos(2\pi y-1)x - \cos(2\pi y+1)x\, dx = \left[\frac{\sin(2\pi y-1)\pi}{2\pi y-1} - \frac{\sin(2\pi y+1)\pi}{2\pi y+1}\right]$.

2) $y = \frac{1}{2\pi}$ $\mathcal{F}_s[f](\frac{1}{2\pi}) = \pi$

$$J(f)(x) = P.V. \int_{-\infty}^{+\infty} \hat{f}(y) \sin(2\pi x y)\, dy$$

$$= 2\int_0^{\pi} \left[\frac{\sin(2\pi y-1)\pi}{2\pi y-1} - \frac{\sin(2\pi y+1)\pi}{2\pi y+1}\right] \sin(2\pi x y)\, dy.$$

7. Prove that

$$\mathcal{F}[f(x)\cos(2\pi\alpha x)] = \frac{\mathcal{F}[f](y-\alpha) + \mathcal{F}[f](y+\alpha)}{2}, \quad \alpha \in \mathbb{R}.$$

Pf: $\mathcal{F}(f(x)\cos(2\pi\alpha x)) = \int_{-\infty}^{+\infty} f(t)\cos(2\pi\alpha t)\cdot e^{-2\pi i y t}\, dt = \int_{-\infty}^{+\infty} f(t) \frac{e^{2\pi\alpha t i} + e^{-2\pi\alpha t i}}{2} e^{-2\pi i y t}\, dt$

$= \frac{1}{2}\int_{-\infty}^{+\infty} f(t)\left[e^{-2\pi(y-\alpha)t i} + e^{-2\pi(y+\alpha)t i}\right] dt = \frac{1}{2}\left[\mathcal{F}[f](y-\alpha) + \mathcal{F}[f](y+\alpha)\right]$

8. Prove that Fourier transform of function $\frac{1}{1+x^{12}}$ is C^{10}–smooth.

Pf: by the property of smoothness. it suffices to check that

$g(t) = t^{10} \cdot \frac{1}{1+t^{12}}$ is integrable on $\mathbb{R}$.

$$\int_{-\infty}^{+\infty} \frac{t^{10}}{1+t^{12}}\, dt = 2\left[\int_1^{+\infty} \frac{1}{\frac{1}{t^{10}}+t^2}\, dt + \int_0^1 \frac{t^{10}}{1+t^{12}}\, dt\right]$$

$$\int_1^{+\infty} \frac{1}{\frac{1}{t^{10}}+t^2}\, dt \le \int_0^{+\infty} \frac{dt}{1+t^2} = \arctan t \Big|_0^{+\infty} = \frac{\pi}{2} \qquad \int_0^1 \frac{t^{10}}{1+t^{12}}\, dt \le \int_0^1 \frac{dt}{1+t^{12}} \le \int_0^1 dt = 1.$$

thus. $\int_{-\infty}^{+\infty} g(t)\, dt < +\infty$, $\hat{f} \in C^{(r)}(\mathbb{R})$ for every $r \in [1:10]$

9. Calculate Fourier transform of the function

$$f(x) = \begin{cases} 2 - x^2, & |x| \le 1; \\ 1, & 1 < |x| < 2; \\ 0, & |x| \ge 2. \end{cases}$$

$$\hat{f}(y) = 2\int_0^1 (2-t^2)\cos(2\pi t y)\,dt + \int_1^2 e^{-2\pi i y t}\,dt.$$

$$= 4\int_0^1 \cos(2\pi t y)\,dt - 2\int_0^1 t^2\cos(2\pi t y)\,dt + \int_1^2 e^{-2\pi i y t}\,dt$$

$$= \frac{4}{2\pi y}\sin(2\pi y t)\Big|_0^1 + \frac{1}{-2\pi i y}e^{-2\pi i y t}\Big|_1^2 - \frac{t^2}{\pi y}\sin(2\pi t y) - \frac{1}{\pi^2 y^2}t\cos(2\pi t y) + \frac{1}{2\pi^3 y^3}\sin(2\pi t y)\Big|_0^1$$

$$= \frac{1}{\pi y}\sin(2\pi y) + \frac{e^{-2\pi y i} - e^{-4\pi y i}}{2\pi i y} - \frac{\cos(2\pi y)}{\pi^2 y^2} + \frac{\sin(2\pi y)}{2\pi^3 y^3}$$

Real Analysis 2024. Homework 12.

1. Let $f_n \underset{n\to\infty}{\longrightarrow} f$ in measure. Show that if $\mu(X) < +\infty$ and $g \in L^0(X)$, then $f_n g \underset{n\to\infty}{\longrightarrow} fg$ in measure. Is this true for an infinite measure?

2. Let $f_k^{(n)}(x) = \cos^{2k}(\pi n! x)(x \in \mathbb{R})$. Show that:

   (a) for every $x \in \mathbb{R}$, the limit $g_n(x) = \lim\limits_{k\to\infty} f_k^{(n)}(x)$ exists;

   (b) $g_n(x) \underset{n\to\infty}{\longrightarrow} \chi(x)$ everywhere on $\mathbb{R}$ (here $\chi = \chi_{\mathbb{Q}}$ is the Dirichlet function);

   (c) there is no sequence of continuous functions (and, in particular, no diagonal sequence $\{f_{k_n}^{(n)}\}$ that converges to the Dirichlet function pointwise on a non-degenerate interval.

3. Assume that the measure under consideration is $\sigma-$finite and a sequence of measurable functions f_k converges to zero almost everywhere. Show that $c_k f_k \to 0$ a.e. for some numerical sequence $c_k \to +\infty$. Hint. Assuming that the sequence $\{|f_k|\}$ is decreasing, apply the diagonal sequence theorem to the functions $f_k^{(n)} = nf_k$.

4. Suppose that $f_n \leq g_n \leq h_n$ a.e. on E, $f_n, h_n \underset{n\to\infty}{\longrightarrow} f$ in measure on E. Prove that $g_n \underset{n\to\infty}{\longrightarrow} f$.

5. Let $f : [0,1]^2 \to \mathbb{R}$ is continuous by the first argument (for arbitrary fixed second). Prove that if $f(x,y) \underset{y\to 0}{\longrightarrow} 0$ for a.e. $x \in [0,1]$ then for every $\varepsilon > 0$ there exists $e \subset [0,1]$ such that $\lambda(e) < \varepsilon$ and $f(x,y) \underset{y\to 0}{\longrightarrow} 0$ uniformly on $[0,1]\backslash e$. Hint: consider sets

$$G_n(\varepsilon) = \left\{ (x,y) \,\middle|\, 0 < x < 1, 0 < y < \frac{1}{n}, \left| f(x,y) \right| > \varepsilon \right\}$$

   and their projections on $0X-$axis.

Real Analysis 2024. Homework 12.

1. Let $f_n \underset{n\to\infty}{\longrightarrow} f$ in measure. Show that if $\mu(X) < +\infty$ and $g \in L^0(X)$, then $f_n g \underset{n\to\infty}{\longrightarrow} fg$ in measure. Is this true for an infinite measure?

Pf.: fix $\varepsilon>0$. $\forall \delta>0$. $\exists N$, s.t. $\mu(E(|f_n-f|>\varepsilon))<\frac{\delta}{2}$ holds for all $n>N$.

since $g\in L^0(X)$. and $\mu X<+\infty$ then $\exists M$. s.t. $\mu E(|g|>\frac{M}{\varepsilon})<\frac{\delta}{2}$

$E(|f_n-f|\cdot|g|>M) \subset E(|g|>\frac{M}{\varepsilon}) \cup E(|f_n-f|>\varepsilon)$

thus $\mu E(|f_ng-fg|>M)<\delta$ since δ is arbitrary, then $f_ng \xrightarrow[n\to\infty]{\mu} fg$.

For infinite measure. not true. counter-example:

$X=\mathbb{R}$. $f_n=\frac{1}{n}\chi_{[n,n+1)}$ $g(x)=x$. $f(x)=0$.

$\forall \varepsilon>0$. $\exists N=[\frac{1}{\varepsilon}]$. $\mu(E(|f_n-f|>\varepsilon))=0$ for all $n>N$. thus. $f_n \xrightarrow{\mu} f$.

$f_ng=\frac{x}{n}\chi_{[n,n+1)}$, $fg=0$. for $\varepsilon_0=\frac{1}{2}$. $E(|f_n-f|>\frac{1}{2})=1$ for all $n\in\mathbb{N}$. thus $f_ng \not\xrightarrow{\mu} fg$.