Differential Geometry

Part II.

Differential Geometry of Smooth Surfaces

2025

Chapter 4

Concept of the Surface and Contact Applied to Surfaces

4.1 Concept of the Surface

Elementary Surface

Definition (Elementary Region). A plane region is considered an **elementary region** if it is the image of an open circle (i.e. the interior of a circle) under an arbitrary topological mapping.

Briefly, this is expressed as follows: an elementary region is a region that is homeomorphic to a circle.

Hence, an elementary region satisfies certain conditions:

- *Connected*: It consists of a single "piece."
- *Simple Shape*: It has smooth and regular boundaries (e.g., a polygon or a circle).
- *Finite Area*: It has a finite extent and does not extend infinitely.
- *Open Region*: It does not include its boundary (in contrast to closed regions).

Definition (Elementary Surface). A set Φ of points in space will be called an **elementary surface** if it is the image of an elementary region in a plane under an arbitrary topological mapping of this region to space.

Example. Consider the **unit circle** C in the uv-plane, defined as:

$$C = \{(u, v) \in \mathbb{R}^2 \mid u^2 + v^2 < 1\}.$$

Let $f : C \to \mathbb{E}$ be a topological mapping given by:

$$f(u, v) = (u, v, \sqrt{1 - u^2 - v^2}).$$

The image of f is the **upper hemisphere** of the unit sphere in 3D space. This upper hemisphere is an **elementary surface** because:

1. C is an **elementary region** in the plane (homeomorphic to an open circle).

2. f is a topological mapping from C into space.

Suppose Φ is an elementary surface and G is an elementary plane region whose image under a topological mapping f is the surface Φ. Let u and v be the Cartesian coordinates of an arbitrary point belonging to the region G, and let x, y, z be the coordinates of the corresponding point on the surface. The coordinates x, y, z of the point on the surface are functions of the coordinates of the point in the region G:

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v).$$

Definition (Equations of the Surface)**.** This system of equations, which determines the mapping f of the region G into space, is called the **equations** of the surface in the *parametric form.*

Simple Suface

Definition (Simple Surface)**.** A set Φ of points in space will be called a **simple surface** if this set is connected and every point X in Φ has a neighbourhood G such that the subset of Φ which lies in G is an *elementary surface.*

Remark. A simple surface is a connected set where every point has a neighbourhood that is an elementary surface. Since an elementary surface already satisfies this condition globally (every point has a neighbourhood that is itself an elementary surface), it trivially satisfies the definition of a simple surface. But the set of elementary surfaces forms only a part of the set of all simple surfaces. For example, the sphere is a simple, but not elementary, surface.

Proposition. *If a closed set of points is deleted from any simple surface in such a way that the remaining part is still open and connected, then the remaining part will also be a simple surface.*

Definition (Complete Simple Surface)**.** A simple surface is said to be **complete** if the limit point of any convergent sequence of points on the surface is also a point on the surface.

Example. A sphere and a paraboloid are complete surfaces, but a spherical segment without the circumference bounding it is not.

Definition (Closed Simple Surface)**.** If a simple complete surface is finite, then it is said to be **closed**.

Example. Besides spheres, the surface of a torus, obtained by revolving a circumference about a straight line lying in the plane of the circumference and not intersecting it, is, for example, also a closed surface.

Definition (Neighborhood of a Point on a Simple Surface)**.** A neighbourhood of a point X on a simple surface is the common part of the surface Φ and some neighbourhood of the point X in space.

Example. Consider a sphere Φ in space. Let X be a point on the sphere. A neighbourhood of X in space could be an open ball centred at X. The intersection of this ball with the sphere is a neighbourhood of X on the sphere, which is a "disk" on the sphere centred at X.

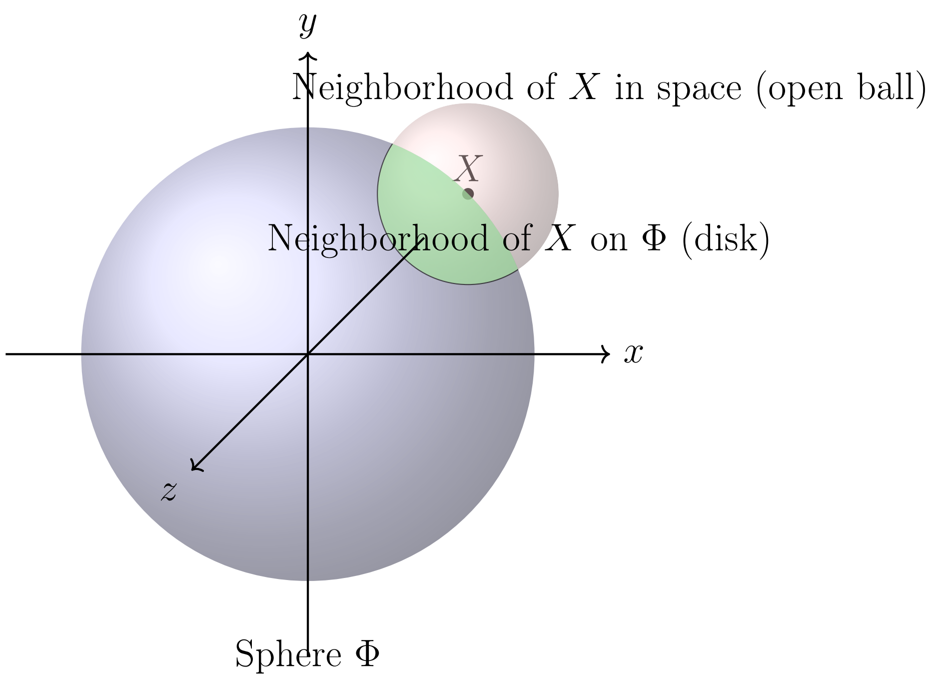


Figure 4.1. Neighbourhood of a Point on a Simple Surface

In agreement with the definition, each point of the simple surface has a neighbourhood which is an elementary surface. In the sequel, in speaking of a neighbourhood of a point on a surface, we shall have in mind such an elementary neighbourhood.

General Surface

Definition (General Surface)**.** A set of points Φ in space is called a **general surface** if there exists a simple surface S and a mapping $f : S \to \mathbb{E}$ such that:

1. f is **continuous**,
2. f is **locally one-to-one** (i.e., for every point $p \in S$, there exists a neighborhood U of p such that f is injective on U), and
3. $\Phi = f(S)$.

Definition. We say that the mapping f_1 of a simple surface Φ_1 and the mapping f_2 of a simple surface Φ_2 **define the same general surface** Φ if there exists a homeomorphism $h : \Phi_1 \to \Phi_2$ such that

$$f_1(p) = f_2(h(p)) \quad \text{for all } p \in \Phi_1.$$

Example. Consider a sphere Φ_1 parameterised by spherical coordinates and a sphere Φ_2 parameterised by Cartesian coordinates.

Under appropriate mappings f_1 and f_1, the images of corresponding points coincide on the same sphere Φ.

Suppose the general surface Φ is the image under a topological into space of a simple surface $\overline{\Phi}$.

Definition (Convergence on a General Surface)**.** We shall say that a sequence of points $f(X_n)$ on the surface Φ **converges** to the point $f(X)$ if the sequence of points X_n on the simple surface $\overline{\Phi}$ converges to the point X.

Definition (Neighborhood of a Point on a General Surface)**.** A neighbourhood of the point $f(X)$ on the surface Φ is the image of an arbitrary neighbourhood of the point X on the surface $\overline{\Phi}$ under the mapping f.

Although the convergence of sequences of points on a general surface Φ and the neighbourhoods of points on Φ are defined as the images of convergent sequences and neighborhoods on a simple surface, starting with some definite mapping, these concepts do not depend on the particular character of the mapping f in the sense that starting with another mapping f_1 of another simple surface which defines the same general surface Φ, we arrive at the same convergent sequences and the same neighborhoods of points on the surface Φ.

If a simple surface, in particular an elementary surface, is considered as a general surface, then the concept of convergence of points on it is equivalent to the concept of geometric convergence, and the concept of neighbourhood is equivalent to the concept of geometric neighbourhood introduced for simple surfaces.

4.2 Regular surface

Definition (Regular Surface)**.** A surface is said to be **regular** (k-times differentiable) if each of the points on this surface has a neighbourhood, permitting a regular parametrisation, i.e. allowing one to write the equations in the parametric form

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v),$$

where f_1, f_2, f_3 are regular (k-times continuously differentiable) functions, defined in an elementary region G of the u, v-plane.

For $k = 1$, the surface is said to be **smooth**.

Definition (Analytic surface)**.** A surface is said to be **analytic** if it allows an analytic parametrisation (the functions f_1, f_2, f_3 are analytic) in a sufficiently small neighbourhood of each of its points.

Definition (Regular and Singular Points)**.** A point P on a regular surface will be called a **regular point** if the surface permits a regular parametrisation in a neighbourhood of this point

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

satisfying the condition that the rank of the matrix

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

evaluated at the point P equals two.

In the contrary case, a point of the surface is called a **singular point**.

Definition (Singular Curve)**.** A curve on a surface, all points of which are singular points of the surface, is called a **singular curve**.

In the sequel, if the contrary is not expressly stated, we shall assume that all points on the surface considered are regular points.

Theorem 4.2.1. *Let G be an open subset of the u, v-plane, and let $x(u, v)$, $y(u, v)$, $z(u, v)$ be C^1-functions defined on G. If the Jacobian matrix*

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

has rank 2 everywhere in G, then the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

define a surface Φ in space. Moreover, the mapping $\mathbf{r} : G \to \Phi$ given by $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ is continuous and locally one-to-one.

Proof. We prove that the mapping $\mathbf{r} : G \to \mathbb{E}$ given by

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

is locally one-to-one under the given conditions.

Assume, for contradiction, that the mapping is not locally one-to-one. Then there exists a point $(u_0, v_0) \in G$ such that in every neighborhood of (u_0, v_0), there are distinct points (u_1, v_1) and (u_2, v_2) with $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$.

By the mean value theorem, we have:

$$x(u_1, v_1) - x(u_2, v_2) = (u_1 - u_2)x_u(u^*, v^*) + (v_1 - v_2)x_v(u^*, v^*),$$

$$y(u_1, v_1) - y(u_2, v_2) = (u_1 - u_2)y_u(u^*, v^*) + (v_1 - v_2)y_v(u^*, v^*),$$

$$z(u_1, v_1) - z(u_2, v_2) = (u_1 - u_2)z_u(u^*, v^*) + (v_1 - v_2)z_v(u^*, v^*),$$

where (u^*, v^*) is some point on the line segment between (u_1, v_1) and (u_2, v_2). Since $\mathbf{r}(u_1, v_1) = \mathbf{r}(u_2, v_2)$, the left-hand sides of these equations are zero. Writing this in matrix form, we have:

$$\begin{pmatrix} x_u(u^*, v^*) & x_v(u^*, v^*) \\ y_u(u^*, v^*) & y_v(u^*, v^*) \\ z_u(u^*, v^*) & z_v(u^*, v^*) \end{pmatrix} \begin{pmatrix} u_1 - u_2 \\ v_1 - v_2 \end{pmatrix} = \mathbf{0}.$$

Since $(u_1, v_1) \neq (u_2, v_2)$, the vector $(u_1 - u_2, v_1 - v_2)$ is nonzero. This implies that the Jacobian matrix has rank less than two at (u^*, v^*).

By continuity, the Jacobian matrix must also have rank less than two at (u_0, v_0), contradicting the hypothesis that it has rank two everywhere in G.

Thus, the mapping $\mathbf{r}$ is locally one-to-one.

□

Remark. With a proper choice of x, y, z-coordinates axes, some simple surfaces permit a parametrisation for the entire surface of the form

$$x = u, \quad y = v, \quad z = f(u, v),$$

where $f(u, v)$ is a function defined in a region G of the u, v-plane. The equations of this surface can be written in the equivalent form $z = f(x, y)$.

Such a parametrisation of the surface differs from others by its greater graphicalness. The correspondence between points on the surface and points of the region in the x, y-plane is realised by a projection by straight lines, parallel to the z-axis.

Theorem 4.2.2 (Implicit Definition of a Surface)**.** *Let $\varphi(x, y, z)$ be a C^1-function defined in an open subset of $\mathbb{R}^3$, and let M be the set of points (x, y, z) satisfying $\varphi(x, y, z) = 0$. Suppose $(x_0, y_0, z_0) \in M$ is a point where the gradient $\nabla\varphi = (\varphi_x, \varphi_y, \varphi_z)$, $(\nabla\varphi)^2 = \varphi_x^2 + \varphi_y^2 + \varphi_z^2$ is nonzero. Then there exists a neighborhood of (x_0, y_0, z_0) in which M is a regular elementary surface.*

Proof. The condition

$$\varphi_x^2 + \varphi_y^2 + \varphi_z^2 \neq 0$$

implies that at least one of the partial derivatives φ_x, φ_y, φ_z is nonzero at (x_0, y_0, z_0). Without loss of generality, assume $\varphi_z(x_0, y_0, z_0) \neq 0$.

By the implicit function theorem, there exists a neighborhood U of (x_0, y_0) and a smooth function $f(x, y)$ defined in U such that $z = f(x, y)$ and $\varphi(x, y, f(x, y)) = 0$ for all $(x, y) \in U$.

The set M in this neighborhood is then given by the graph of $f(x, y)$, i.e.,

$$M = \{(x, y, z) \mid (x, y) \in U, z = f(x, y)\}.$$

Since $f(x, y)$ is smooth, M is a smooth surface locally parameterized by (x, y). Moreover, the gradient condition ensures that M is nonsingular at (x_0, y_0, z_0).

Thus, M is a regular elementary surface in the neighbourhood of (x_0, y_0, z_0). $\square$

4.3 Parametrisation of a Surface

A regular surface permits an infinite number of parametrisations in a neighbourhood of each of its points.

Lemma 4.3.1. *Suppose*

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

is a regular parametrization of a surface in a neighborhood of the point $Q(u_0, v_0)$, meaning that the tangent vectors $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$ are linearly independent at (u_0, v_0).

Let $\varphi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ be smooth functions satisfying the conditions

$$\begin{cases} u_0 = \varphi(\alpha_0, \beta_0) \\ v_0 = \psi(\alpha_0, \beta_0), \end{cases} \qquad \begin{vmatrix} \varphi_\alpha & \varphi_\beta \\ \psi_\alpha & \psi_\beta \end{vmatrix} \neq 0$$

at the point (α_0, β_0). Then the equations

$$x = x(\varphi, \psi), \quad y = y(\varphi, \psi), \quad z = z(\varphi, \psi)$$

define a new regular parametrisation of the surface in a neighbourhood of (α_0, β_0).

This follows from the fact that the mapping $(\alpha, \beta) \mapsto (u, v)$ *given by*

$$u = \varphi(\alpha, \beta), \quad v = \psi(\alpha, \beta)$$

is a local diffeomorphism due to the nonzero Jacobian determinant. Consequently, the composition $(x(\varphi, \psi), y(\varphi, \psi), z(\varphi, \psi))$ *is smooth, and its tangent vectors remain linearly independent, ensuring that it is a regular parametrisation.*

In the investigation of regular surfaces, it is convenient to use special parametrisations. We shall consider the special parametrisation which is most frequently used.

Proof. **Part 1:** Suppose $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ is a regular parametrization of a surface in a neighborhood of $Q(u_0, v_0)$. Since the Jacobian matrix

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

has rank 2, we can assume without loss of generality that the determinant

$$\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \neq 0$$

at Q. By the implicit function theorem, the system

$$x = x(u, v), \quad y = y(u, v)$$

can be solved for u and v as smooth functions of x and y, i.e., $u = \varphi(x, y)$ and $v = \psi(x, y)$, in a neighborhood of (x_0, y_0). Substituting these into $z = z(u, v)$, we obtain $z = \overline{z}(x, y) = z(\varphi(x, y), \psi(x, y))$. Since φ and ψ are smooth and the original parametrization is regular, $\overline{z}(x, y)$ is smooth, and the tangent vectors

$$\frac{\partial}{\partial x}(x, y, \overline{z}(x, y)) \quad \text{and} \quad \frac{\partial}{\partial y}(x, y, \overline{z}(x, y))$$

are linearly independent. Thus, $(x, y, \overline{z}(x, y))$ is a regular parametrization of the surface.

Part 2: Suppose $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ is a regular parametrization of a surface in a neighborhood of $Q(u_0, v_0)$. Consider two differential equations

$$A_1(u, v)du + B_1(u, v)dv = 0, \quad A_2(u, v)du + B_2(u, v)dv = 0$$

in a neighborhood of (u_0, v_0), where the determinant

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0.$$

If $\varphi(u,v) = \overline{u}$ and $\psi(u,v) = \overline{v}$ are solutions of these equations with $\varphi_u^2 + \varphi_v^2 \neq 0$ and $\psi_u^2 + \psi_v^2 \neq 0$ at (u_0, v_0), then the mapping $(\overline{u}, \overline{v}) \mapsto (u, v)$ is invertible in a neighborhood of (u_0, v_0) because

$$\begin{vmatrix} \varphi_u & \varphi_v \\ \psi_u & \psi_v \end{vmatrix} \neq 0.$$

This ensures that the surface can be reparametrized as

$$x = \overline{x}(\overline{u}, \overline{v}), \quad y = \overline{y}(\overline{u}, \overline{v}), \quad z = \overline{z}(\overline{u}, \overline{v}),$$

where $\overline{x}, \overline{y}, \overline{z}$ are smooth functions. The regularity of the original parametrisation implies that the new parametrisation is also regular. □

4.4 Singular Points on Regular Surface

Suppose $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$ is a regular parametrisation of a regular surface.

Suppose the rank of the matrix

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

is less than two at the point $Q(u_0, v_0)$ on this surface.

We shall make use of the vector equation of the surface, $\boldsymbol{r} = \boldsymbol{r}(u,v)$. Then the condition that the rank of the above matrix be less than two is expressed by the fact that the vector product $\boldsymbol{r}_u \times \boldsymbol{r}_v = 0$.

Lemma 4.4.1. *Suppose $P(u,v)$ is a point on the surface near $Q(u_0, v_0)$. The necessary condition for the point $Q(u_0, v_0)$ to be singular is that the cross product of the tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ vanishes at Q, i.e., $\boldsymbol{r}_u \times \boldsymbol{r}_v = 0$. If Q is singular, the unit normal vector*

$$\boldsymbol{\xi}(u,v) = \frac{\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)}{|\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)|}$$

is undefined at Q, and the limit of $\boldsymbol{\xi}(u,v)$ as $P \to Q$ does not exist.

Proof. Indeed, if Q is a regular point, there is an arbitrary parameterisation in a neighbourhood of Q $\boldsymbol{r}(\alpha, \beta)$ granting $\boldsymbol{r}_\alpha \times \boldsymbol{r}_\beta \neq 0$ at Q, thus expression

$$\boldsymbol{\xi}(u,v) = \frac{\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)}{|\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)|} = \boldsymbol{\xi}(\alpha,\beta) = \frac{\boldsymbol{r}_\alpha(\alpha,\beta) \times \boldsymbol{r}_\beta(\alpha,\beta)}{|\boldsymbol{r}_\alpha(\alpha,\beta) \times \boldsymbol{r}_\beta(\alpha,\beta)|}$$

tends to a definenite leimit as $P \to Q$. □

Remark. The assertion that $\boldsymbol{\xi}(u,v)$ has no definite limit as $P \to Q$ is the evidence that Q is singular is not valid.

Lemma 4.4.2. *Suppose $P(u,v)$ is a point on the surface near $Q(u_0,v_0)$. The point Q is a regular point if the tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are linearly independent at Q, i.e., $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$. The unit normal vector at Q is given by*

$$\boldsymbol{\xi}(u,v) = \frac{\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)}{|\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)|}.$$

The unit normal vector $\boldsymbol{\xi}(u,v)$ is well-defined at Q if Q is a regular point and the vector function $\boldsymbol{\xi}(u,v)$ tends to a definite limit as $P \to Q$.

Proof. The point Q is a regular point on the surface if the tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are linearly independent at Q, i.e., $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$. The unit normal vector at Q is given by

$$\boldsymbol{\xi}(u,v) = \frac{\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)}{|\boldsymbol{r}_u(u,v) \times \boldsymbol{r}_v(u,v)|}.$$

For a smooth surface, the unit normal vector $\boldsymbol{\xi}(u,v)$ depends continuously on the position of the point P. Consequently, if Q is a regular point, $\boldsymbol{\xi}(u,v)$ is well-defined at Q and varies continuously in a neighbourhood of Q.

The regularity of Q is determined solely by the non-vanishing of $\boldsymbol{r}_u \times \boldsymbol{r}_v$ at Q. The continuity of $\boldsymbol{\xi}(u,v)$ is a property of the smoothness of the surface but does not define regularity.

□

Types of Singular Points

1. **Fold Singularities (Whitney Umbrella)**:

   - **Description**: At a fold singularity, the surface behaves like $z = x^2$ near the origin. The tangent plane is well-defined along certain directions but not others.

- **Example**: Consider the surface given by $z = x^2$. The origin $(0, 0, 0)$ is a fold singularity because the tangent plane is vertical along the y-axis but changes smoothly along the x-axis.

2. **Cusp Singularities**:

   - **Description**: A cusp singularity occurs when the surface resembles $z = x^3 + y^2$ near the origin. The tangent plane is not well-defined at the cusp point.
   - **Example**: The surface $z = x^3 + y^2$ has a cusp singularity at $(0, 0, 0)$ because the tangent plane oscillates wildly as x approaches zero.

3. **Double Points (Self-Intersections)**:

   - **Description**: At a double point, the surface intersects itself. Locally, the surface appears as two intersecting sheets.
   - **Example**: Consider the surface defined by $z^2 = x^2 + y^2$. The origin $(0, 0, 0)$ is a double point because the surface intersects itself along the z-axis.

4. **Conical Points**:

   - **Description**: A conical singularity is a point where the surface resembles a cone. The tangent plane does not exist at the apex of the cone.
   - **Example**: The surface $z = \sqrt{x^2 + y^2}$ has a conical singularity at the origin $(0, 0, 0)$. The tangent plane is undefined at this point.

5. **Boundary Points**:

   - **Description**: Boundary points are singularities that occur at the edges of the surface, where the surface abruptly ends.
   - **Example**: The surface defined by $z = \sqrt{1 - x^2 - y^2}$ has boundary points where $x^2 + y^2 = 1$. The tangent plane is undefined at these points.

Types of Singular Curves

1. **Cuspidal Curves**:

- **Description**: A cuspidal curve is a curve on the surface where the tangent plane is not well-defined. Each point on the curve is a cusp singularity.
- **Example**: The curve $z = x^3$ on the surface $z = x^3 + y^2$ is a cuspidal curve. Every point on this curve is a cusp singularity.

2. **Fold Curves**:

- **Description**: A fold curve is a curve on the surface where the surface folds over itself. The tangent plane is well-defined along certain directions but not others.
- **Example**: The curve $z = x^2$ on the surface $z = x^2 + y^3$ is a fold curve. The tangent plane is vertical along the y-axis but changes smoothly along the x-axis.

3. **Intersection Curves**:

- **Description**: An intersection curve is a curve where two sheets of the surface intersect. Each point on the curve is a double point.
- **Example**: The curve $z = 0$ on the surface $z^2 = x^2 + y^2$ is an intersection curve. The two sheets of the surface intersect along this curve.

Examples

1. **Whitney Umbrella**:

- The Whitney Umbrella can be parametrised explicitly using two parameters u and v. The parametric equations are:

$$\boldsymbol{r}(u, v) = (u, v^2, uv)$$

  Here:

  - u and v are parameters,
  - $x = u$,
  - $y = v^2$,
  - $z = uv$.

  It acts like $z = x^2$ near the origin, but not at the origin itself

- **Analysis**:

– The tangent vectors are:

$$\boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = (1, 0, v)$$

$$\boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = (0, 2v, u)$$

– The normal vector is the cross product:

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = (-2v^2, -u, 2v)$$

– The normal vector vanishes when:

$$(-2v^2, -u, 2v) = (0, 0, 0)$$

This occurs when $v = 0$ and $u = 0$, which corresponds to the point $(0, 0, 0)$. This is the singular point of the Whitney Umbrella.

- **Surface Equation**: $z = x^2$.
- **Analysis**: The tangent vectors are $\boldsymbol{r}_u = (1, 0, 2x)$ and $\boldsymbol{r}_v = (0, 1, 0)$. The cross product $\boldsymbol{r}_u \times \boldsymbol{r}_v = (-2x, 0, 1)$ vanishes at $x = 0$, indicating a fold singularity at $(0, 0, 0)$.

2. **Cuspidal Parabola**:
   - **Surface Equation**: $z = x^3 + y^2$.
   - **Analysis**: The tangent vectors are $\boldsymbol{r}_u = (1, 0, 3x^2)$ and $\boldsymbol{r}_v = (0, 1, 2y)$. The cross product $\boldsymbol{r}_u \times \boldsymbol{r}_v = (-3x^2, -2y, 1)$ vanishes at $(0, 0, 0)$, indicating a cusp singularity at the origin.

3. **Intersection of Two Planes**:
   - **Surface Equation**: $z^2 = x^2 + y^2$.
   - **Analysis**: The tangent vectors are $\boldsymbol{r}_u = (1, 0, x/\sqrt{x^2 + y^2})$ and $\boldsymbol{r}_v = (0, 1, y/\sqrt{x^2 + y^2})$. The cross product $\boldsymbol{r}_u \times \boldsymbol{r}_v = (-y/\sqrt{x^2 + y^2}, x/\sqrt{x^2 + y^2}, 1)$ vanishes at $(0, 0, 0)$, indicating a double point at the origin.

Remarks and Further Observations

- Singular points can often be studied by examining the behaviour of the surface in a small neighbourhood around the point. For example, near a conical singularity, the surface resembles a cone.

- Techniques from singularity theory, such as blow-ups and resolution of singularities, can be used to analyze the local structure of the surface near singular points.

4.5 Classifying Singular Points on Implicit Surfaces

A surface defined by the equation $\varphi(x, y, z) = 0$ may have **singular points** where the usual regularity conditions fail. These points are of particular interest because they represent locations where the surface behaves differently from its smooth, regular neighbourhoods. Here, we systematically analyse these points and classify them based on their geometric and algebraic properties.

Necessary Condition for Singularity

A point $Q(x_0, y_0, z_0)$ is **potentially singular** if the gradient of φ vanishes at Q:

$$\nabla\varphi(Q) = 0 \quad \text{(i.e., } \varphi_x = \varphi_y = \varphi_z = 0 \text{ at } Q\text{)}.$$

This condition is **necessary but not sufficient** for singularity. To illustrate:

- **Isolated Point**:
  - Example: $\varphi = x^2 + y^2 + z^2$.
  - At the origin $(0, 0, 0)$, $\nabla\varphi = (2x, 2y, 2z) = (0, 0, 0)$.
  - Here, the equation $\varphi = 0$ has only the trivial solution $(0, 0, 0)$, which is an isolated point rather than a surface.
- **Singular Surface**:
  - Example: $\varphi = z^2 - x^3$.
  - At the origin $(0, 0, 0)$, $\nabla\varphi = (-3x^2, 0, 2z) = (0, 0, 0)$.
  - This defines a **cuspidal surface**, where the surface has a sharp point (a cusp) at the origin.

Second-Order Analysis

To classify singular points more precisely, we examine the *Taylor expansion* of φ near the point Q:

$$\varphi(x,y,z) \approx \frac{1}{2}\Big[\varphi_{xx}(x-x_0)^2 + \varphi_{yy}(y-y_0)^2 + \varphi_{zz}(z-z_0)^2 + \\ + 2\varphi_{xy}(x-x_0)(y-y_0) + 2\varphi_{xz}(x-x_0)(z-z_0) + 2\varphi_{yz}(y-y_0)(z-z_0)\Big].$$

This can be written compactly as:

$$\varphi(x,y,z) \approx \sum_{i,j=1}^{3} a_{ij}\xi_i\xi_j,$$

where $\xi_1 = x - x_0$, $\xi_2 = y - y_0$, $\xi_3 = z - z_0$, and $a_{ij} = \frac{\partial^2\varphi}{\partial\xi_i\partial\xi_j}(Q)$ are the entries of the **Hessian matrix**.

The behavior of $\varphi = 0$ near Q depends on the nature of *quadratic form* $\sum a_{ij}\xi_i\xi_j$:

Singular points on implicit surfaces can be classified based on the **quadratic form** derived from the Hessian matrix of the defining function φ. Below are the main types:

1. Isolated Singular Points

- **Description**: The quadratic form is **definite** (all eigenvalues of the Hessian matrix have the same sign, either all positive or all negative).
- **Geometry**: The equation $\varphi = 0$ has only the trivial solution $\xi_i = 0$, meaning the surface degenerates to a single isolated point.
- **Example**:

$$\varphi = x^2 + y^2 + z^2.$$

  - The Hessian matrix:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

    has all eigenvalues equal to 2 (positive definite).
  - At the origin, $\varphi = 0$ has only the solution $(0,0,0)$, which is an isolated point.

2. Conical or Cuspidal Singular Points

- **Description**: The quadratic form is **indefinite** (the Hessian matrix has both positive and negative eigenvalues).

- **Geometry**: Solutions to $\varphi = 0$ form a cone, cusp, or similar singular surface. The singular point is non-isolated, and the surface self-intersects or has a sharp feature.

- **Example**:
$$\varphi = z^2 - x^2 - y^2.$$
  - The Hessian matrix:
$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$
has eigenvalues $-2, -2, 2$ (indefinite).
  - At the origin, $\varphi = 0$ defines a **double cone**, which self-intersects at the origin.

3. Degenerate Singular Points

- **Description**: The quadratic form is **degenerate** (the Hessian matrix has at least one zero eigenvalue).

- **Geometry**: The behavior of the surface depends on higher-order terms in the Taylor expansion. The surface may have a flat direction or degenerate into a line, plane, or more complex structure.

- **Example**:
$$\varphi = x^4 + y^4 - z^2.$$
  - The Hessian matrix:
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$
has eigenvalues $0, 0, -2$. The surface near the origin is flatter in the x- and y-directions due to the zero eigenvalues but still forms a singular point.

Remark on Zero Eigenvalues of the Hessian Matrix

The presence of **zero eigenvalues** in the Hessian matrix indicates that the quadratic form alone is insufficient to fully describe the behaviour of the surface near the singular point. In such cases:

- **Higher-Order Analysis**: Higher-order terms in the Taylor expansion of φ must be considered to determine the geometry of the surface near the singular point.
  - Example: For $\varphi = x^4 - y^4 - z^2$, the Hessian matrix at the origin has zero eigenvalues, but the higher-order terms (x^4, y^4) reveal the structure of the surface.
- **Flat Directions**: Zero eigenvalues correspond to **flat directions** in the quadratic form, meaning the *surface has no curvature* (see next chapters for clarification of the term, the curvature of the surface) along certain axes at the singular point.
  - Example: For $\varphi = x^4 + y^4 - z^2$, the surface is flat in the x- and y-directions but has curvature in the z-direction.
- **Higher-Dimensional Singularities**: In some cases, the zero eigenvalues indicate that the singular point is part of a **higher-dimensional singular set** (e.g., a singular curve or plane).
  - Example: For $\varphi = x^3 - y^3 + z^2$, the Hessian matrix has zero eigenvalues, and the singular points lie along the z-axis.

Summary of Singular Point Classification

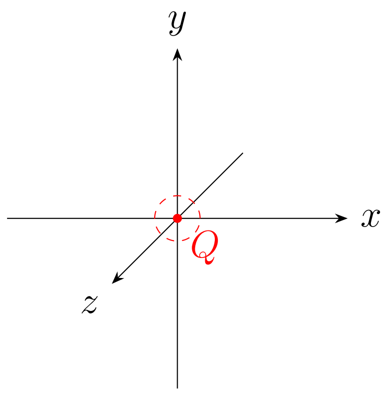
Type	**Quadratic Form**	**Eigenvalues**	**Geometry**
Isolated Point	Definite (all λ same sign)	All $\lambda > 0$ or $\lambda < 0$	Degenerates to a single point
Conical/ Cuspidal Point	Indefinite (mixed signs)	Some $\lambda > 0$, some $\lambda < 0$	Forms a cone, cusp, or self-intersection
Degenerate Point	Degenerate (zero eigen-values)	At least one $\lambda = 0$	Requires higher-order analysis

Implications and Practical Classification

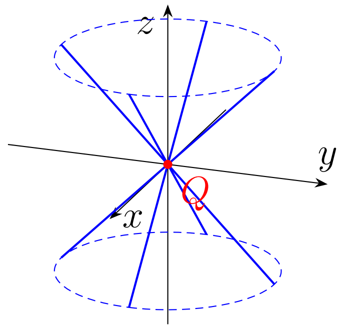
- **Regular Points**:
  - If $\nabla\varphi \neq 0$ at Q, the surface is **regular** near Q (by the Implicit Function Theorem).
  - Example: $\varphi = x + y + z$ has $\nabla\varphi = (1, 1, 1)$, so there are no singular points.
- **Singular Points**:
  - If $\nabla\varphi = 0$, the quadratic form determines the type of singularity:
    * **Isolated Point**: All eigenvalues of the Hessian matrix have the same sign.
    * **Conical or Cuspidal Singular Points**: Eigenvalues of the Hessian matrix have mixed signs (e.g., cones, self-intersections).
    * **Degenerate Singular Points**: At least one eigenvalue is zero (e.g. flat regions, higher-dimensional singular structures).

Additional Examples

- **Elliptic Cone**:
  - $\varphi = z^2 - 2x^2 - 3y^2$.
  - Hessian matrix:
$$\begin{pmatrix} -4 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$
  - Eigenvalues: $-4, -6, 2$ (indefinite).
  - The origin is a **singular point** defining an elliptic cone.
- **Hyperbolic Paraboloid**:
  - $\varphi = x^2 - y^2 - z$.
  - Hessian matrix:
$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$
  - Eigenvalues: $2, -2, 0$ (indefinite).
  - The origin is a **singular point**, and the surface is a hyperbolic paraboloid.



Isolated Singular Point
$\varphi = x^2 + y^2 + z^2$



Conical Singularity
$\varphi = z^2 - x^2 - y^2$

Figure 4.2. Types of singular points: (left) isolated point where no surface exists, (right) conical singularity where the surface self-intersects.

4.6 Tangent Plane to a Surface

Suppose Φ is a surface, and P is a point on it. Let α be any plane passing through the point P.

We take a point Q on the surface Φ and denote its distance from the point P and from the plane α by d respectively h.

Definition. We call the plane α the tangent plane to surface Φ at the point P if the ratio

$$\lim_{Q \to P} \frac{h}{d} = 0.$$

.

Theorem 4.6.1. *Let Φ be a smooth surface parametrized by $\boldsymbol{r}(u, v)$, where $\boldsymbol{r}$ is a continuously differentiable (C^1) vector function. At each point $P(u, v) \in S$, the tangent plane α exists and is uniquely spanned by the partial derivative vectors*

$$\boldsymbol{r}_u(u, v) = \frac{\partial \boldsymbol{r}}{\partial u} \quad \text{and} \quad \boldsymbol{r}_v(u, v) = \frac{\partial \boldsymbol{r}}{\partial v}.$$

That is, the tangent plane α is the set of all vectors of the form

$$\boldsymbol{v} = a\boldsymbol{r}_u(u, v) + b\boldsymbol{r}_v(u, v),$$

where $a, b \in \mathbb{R}$, provided $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are linearly independent. This holds for all regular points on the surface Φ, where the parametrisation is smooth and non-degenerate.

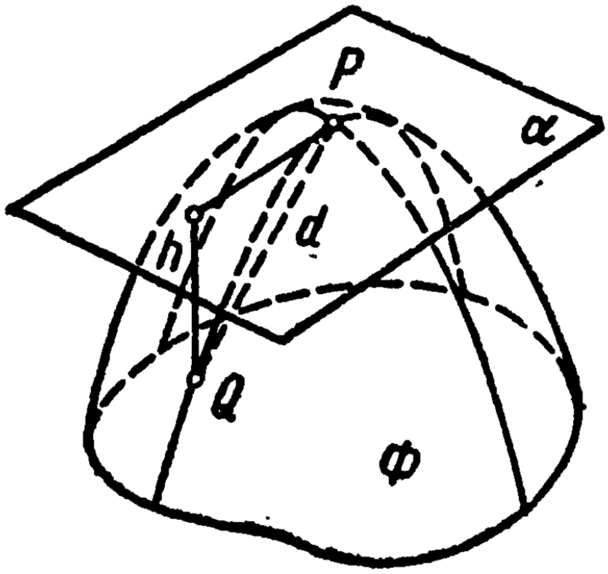


Figure 4.3. Tangent plane

Proof. 1. **Definition of the Tangent Plane:** The tangent plane α at a point $P \in S$ is defined as the plane such that for any point $Q \in S$ approaching P, the ratio of the distance h from Q to α and the distance d from Q to P tends to zero:

$$\lim_{Q \to P} \frac{h}{d} = 0.$$

2. **Parametrization and Taylor Expansion:** Let $P = \boldsymbol{r}(u, v)$ and $Q = \boldsymbol{r}(u+\Delta u, v+\Delta v)$. Using the Taylor expansion of $\boldsymbol{r}$ around (u, v), we have:

$$\boldsymbol{r}(u+\Delta u, v+\Delta v) = \boldsymbol{r}(u, v)+\boldsymbol{r}_u(u, v)\Delta u+\boldsymbol{r}_v(u, v)\Delta v+\text{higher-order terms.}$$

Here, $\boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u}$ and $\boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v}$ are the partial derivatives of $\boldsymbol{r}$.

3. **Distance from Q to α:** The tangent plane α is spanned by $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$. The distance h from Q to α is the perpendicular distance from Q to the plane. Using the linear approximation, we have:

$$h = \left\|\boldsymbol{r}(u + \Delta u, v + \Delta v) - \boldsymbol{r}(u, v) - \boldsymbol{r}_u(u, v)\Delta u - \boldsymbol{r}_v(u, v)\Delta v\right\|.$$

By the Taylor expansion, this simplifies to:

$$h = \|\text{higher-order terms}\|.$$

4. **Distance from Q to P:** The distance d from Q to P is:

$$d = \left\|\boldsymbol{r}(u + \Delta u, v + \Delta v) - \boldsymbol{r}(u, v)\right\|.$$

Using the linear approximation, this becomes:

$$d \approx \|\boldsymbol{r}_u(u,v)\Delta u + \boldsymbol{r}_v(u,v)\Delta v\| .$$

5. **Limit of the Ratio $\frac{h}{d}$:** As $Q \to P$, $\Delta u \to 0$ and $\Delta v \to 0$. The higher-order terms in h become negligible compared to the linear terms in d. Therefore:

$$\lim_{Q\to P} \frac{h}{d} = \lim_{(\Delta u,\Delta v)\to(0,0)} \frac{\|\text{higher-order terms}\|}{\|\boldsymbol{r}_u(u,v)\Delta u + \boldsymbol{r}_v(u,v)\Delta v\|} = 0.$$

This shows that the plane spanned by $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ satisfies the definition of the tangent plane.

6. **Uniqueness of the Tangent Plane:** Any other plane that satisfies the condition $\lim_{Q\to P} \frac{h}{d} = 0$ must coincide with the plane spanned by $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$. Therefore, the tangent plane is unique.

7. **Regular Point:** At a regular point $P(u,v)$, $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are linearly independent. This ensures that they span a 2-dimensional plane in $\mathbb{E}$.

8. **Tangent Vectors:** The vectors $\boldsymbol{r}_u(u,v)$ and $\boldsymbol{r}_v(u,v)$ are tangent to the surface Φ at $P(u,v)$. This is because:

   - $\boldsymbol{r}_u(u,v)$ represents the tangent vector to the u-curve on S (where v is held constant).
   - $\boldsymbol{r}_v(u,v)$ represents the tangent vector to the v-curve on S (where u is held constant).

9. **Spanning the Tangent Plane:** The tangent plane α at $P(u,v)$ is the set of all vectors of the form:

$$\boldsymbol{v} = a\boldsymbol{r}_u(u,v) + b\boldsymbol{r}_v(u,v), \quad a,b \in \mathbb{R}.$$

This is because any vector in the tangent plane must be a linear combination of the tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$.

10. **Conclusion:** At every regular point $P(u,v)$, the tangent plane α exists, is unique, and is spanned by the vectors $\boldsymbol{r}_u(u,v)$ and $\boldsymbol{r}_v(u,v)$.

$\square$

4.7 Equations of Tangent Plane

It is not difficult to write down the various forms of equations of the tangent plane once we know its direction.

The tangent plane to a surface at a given point is the plane that best approximates the surface near that point. It is defined using the gradient of the surface function and provides a linear approximation to the surface at the point of tangency.

Tangent Plane to an Explicit Surface $z = f(x, y)$

For a surface defined by $z = f(x, y)$, the equation of the tangent plane at the point $(a, b, f(a, b))$ is given by:

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b),$$

where $f_x(a, b)$ and $f_y(a, b)$ are the partial derivatives of f with respect to x and y, evaluated at (a, b).

Tangent Plane to an Implicit Surface $F(x, y, z) = 0$

For a surface defined implicitly by $F(x, y, z) = 0$, the tangent plane at the point (a, b, c) is given by:

$$\nabla F(a, b, c) \cdot (x - a, y - b, z - c) = 0,$$

where $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ is the gradient of F at (a, b, c).

4.8 Distance from a Point to a Surface

Definition. Suppose Φ is a surface and Q is any point in space. The distance of the point Q from the surface is the greatest lower bound of the distances of points on the surface from the point Q. If the point Q lies on the surface, its distance from the surface equals zero.

Lemma 4.8.1. *Suppose Φ is a smooth surface defined by the equation $F(x, y, z) = 0$. Let $O(x_0, y_0, z_0)$ be a point on Φ, and assume that the partial derivatives of F do not vanish simultaneously at O, i.e., $\nabla F(x_0, y_0, z_0) \neq 0$.*

Let $Q(x, y, z)$ be a point in space near O but not lying on the surface Φ. Denote the minimum distance from Q to the surface Φ as h.

If Q is sufficiently close to O, then the value of $F(x, y, z)$ at point Q is proportional to the distance h. Specifically, the ratio $\frac{F(x,y,z)}{h}$ approaches a finite, non-zero limit as Q approaches O from the outside of the surface Φ.

That is,

$$\lim_{Q \to O} \frac{F(x, y, z)}{h} = C,$$

C is a finite, non-zero constant.

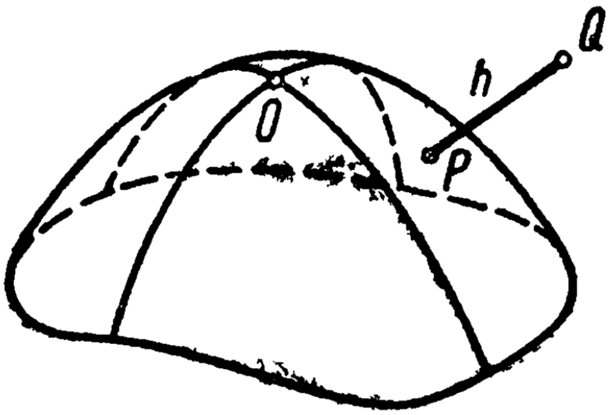


Figure 4.4. Distance from point to surface

Proof. Let Φ be a smooth surface defined by the equation $F(x, y, z) = 0$, and let $O(x_0, y_0, z_0)$ be a point on Φ. Assume that the gradient of F at O is non-zero, i.e.,

$$\nabla F(x_0, y_0, z_0) = (F_x(x_0, y_0, z_0),\ F_y(x_0, y_0, z_0),\ F_z(x_0, y_0, z_0)) \neq \mathbf{0}.$$

This ensures that the surface Φ is regular (non-singular) at O.

Let $Q(x, y, z)$ be a point in space near O but not lying on the surface Φ. Denote the minimum distance from Q to the surface Φ as h. We aim to show that

$$\lim_{Q \to O} \frac{F(x, y, z)}{h} = \|\nabla F(x_0, y_0, z_0)\|,$$

where $\|\nabla F(x_0, y_0, z_0)\|$ is a finite, non-zero constant.

Since F is smooth, we can approximate F near O using its first-order Taylor expansion:

$$F(x, y, z) = F(x_0, y_0, z_0) + F_x(x_0, y_0, z_0)(x - x_0) +$$

$$F_y(x_0, y_0, z_0)(y - y_0) +$$
$$F_z(x_0, y_0, z_0)(z - z_0) + \mathcal{O}(r^2),$$

where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$ is the distance from Q to O, and $\mathcal{O}(r^2)$ represents higher-order terms that vanish faster than r as $r \to 0$.

Since O lies on Φ, we have $F(x_0, y_0, z_0) = 0$. Thus, the expansion simplifies to:

$$F(x, y, z) = F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) + \mathcal{O}(r^2).$$

The distance h from Q to Φ is given by the projection of the vector $\boldsymbol{Q} - \boldsymbol{O} = (x - x_0, y - y_0, z - z_0)$ onto the direction of the gradient $\nabla F(x_0, y_0, z_0)$. For Q sufficiently close to O, we have:

$$h \approx \frac{|\nabla F(x_0, y_0, z_0) \cdot (\boldsymbol{Q} - \boldsymbol{O})|}{\|\nabla F(x_0, y_0, z_0)\|},$$

where $\cdot$ denotes the dot product, and $\|\nabla F(x_0, y_0, z_0)\|$ is the magnitude of the gradient.

From the Taylor expansion, we have:

$$F(x, y, z) \approx \nabla F(x_0, y_0, z_0) \cdot (\boldsymbol{Q} - \boldsymbol{O}).$$

From the approximation of h, we have:

$$\nabla F(x_0, y_0, z_0) \cdot (\boldsymbol{Q} - \boldsymbol{O}) \approx h \cdot \|\nabla F(x_0, y_0, z_0)\|.$$

Thus,

$$F(x, y, z) \approx h \cdot \|\nabla F(x_0, y_0, z_0)\|.$$

As $Q \to O$, the higher-order terms $\mathcal{O}(r^2)$ become negligible, and the approximation becomes exact. Therefore,

$$\lim_{Q \to O} \frac{F(x, y, z)}{h} = \|\nabla F(x_0, y_0, z_0)\|.$$

Since $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, this limit is a finite, non-zero constant.

$\square$

4.9 Contact of Surface and Curve

Definition. Let Φ be a smooth surface, and let γ be a curve that intersects Φ at a point O. For any point Q on γ, let h denote the shortest distance from Q to

Φ, and let d denote the arc length along γ from O to Q. We say that the curve γ has an **order of contact** n with the surface Φ at the point O if the following limit holds:

$$\lim_{Q \to O} \frac{h}{d^n} = 0.$$

Theorem 4.9.1. *Let S be an elementary regular surface and γ a regular curve, both passing through a common point O. Suppose:*

1. *The surface S is locally described by $F(x, y, z) = 0$ near O, where F is smooth and $\nabla F \neq 0$ at O.*

2. *The curve γ is parametrized near O by a regular smooth map:*

$$\gamma(t) = \big(x(t), y(t), z(t)\big), \quad \gamma(t_0) = O.$$

*Then, the curve γ has **contact of order** n with the surface S at O if and only if the composite function $F(t) = F\big(x(t), y(t), z(t)\big)$ satisfies:*

$$F(t_0) = 0, \quad \left.\frac{d^k F}{dt^k}\right|_{t=t_0} = 0 \quad \text{for} \quad k = 1, \ldots, n.$$

Proof. Let $t = t_0$ correspond to the point O. Since γ is regular, $r'(t_0) \neq 0$, and thus $|r(t) - r(t_0)| \sim |t - t_0|$ as $t \to t_0$.

By the lemma, for $Q = \gamma(t)$ near O, $|F(Q)|$ is proportional to the distance from Q to S. Therefore:

$$\frac{F(x(t), y(t), z(t))}{|r(t) - r(t_0)|^n} \to 0 \quad \text{as} \quad t \to t_0$$

if and only if $F(t) = F(x(t), y(t), z(t))$ vanishes to order n at t_0.

By Taylor's theorem, this occurs precisely when:

$$F(t_0) = F'(t_0) = \cdots = F^{(n)}(t_0) = 0,$$

which is the definition of n-th order contact. $\square$

Example

Let the surface S be the paraboloid $z = x^2 + y^2$ ($F(x, y, z) = z - x^2 - y^2 = 0$) and the curve $\gamma(t) = (t, t, t^2 + t^3)$.

At $t = 0$ (point O):

$$F(t) = t^3 - t^2, \quad F(0) = F'(0) = 0, \quad F''(0) = -2 \neq 0.$$

Thus, γ has first-order but not second-order contact with S at O.

Let us match with the definition of contact.

For $\gamma(t) = (t, t, t^2 + t^3)$ and $S : z = x^2 + y^2$:

- Distance $h \approx t^2$, arc length $d \approx \sqrt{2}|t|$.
- First-order limit: $\lim_{t \to 0} \frac{h}{d} = 0$ (order-1 contact).
- Second-order limit: $\lim_{t \to 0} \frac{h}{d^2} = \frac{1}{2} \neq 0$ (not order-2).

Example. Osculating Sphere

Definition. The **osculating sphere** to a curve $\boldsymbol{r}(s)$ (parametrised by arc length) at a point P is the sphere that has third-order contact with the curve at P. This means the sphere and curve agree in position, first derivative (tangent), second derivative (curvature), and third derivative at P.

Given:

- Curve $\boldsymbol{r}(s)$ with Frenet-Serret frame $\{\boldsymbol{\tau}, \boldsymbol{n}, \boldsymbol{b}\}$
- Curvature $k_1(s)$ and torsion $k_2(s)$
- Sphere equation: $(\boldsymbol{r} - \boldsymbol{a})^2 = R^2$

First Derivative Condition Differentiating the sphere equation once:

$$\frac{d}{ds}\left[(\boldsymbol{r} - \boldsymbol{a})^2\right] = 2(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{\tau} = 0$$

$$\boxed{(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{\tau} = 0}$$

This shows the centre $\boldsymbol{a}$ lies in the normal plane.

Second Derivative Condition Differentiating a second time:

$$\frac{d^2}{ds^2}\left[(\boldsymbol{r} - \boldsymbol{a})^2\right] = 2(\boldsymbol{r} - \boldsymbol{a}) \cdot k_1\boldsymbol{n} + 2\boldsymbol{\tau} \cdot \boldsymbol{\tau} = 0$$

$$2(\boldsymbol{r} - \boldsymbol{a}) \cdot k_1\boldsymbol{n} + 2 = 0 \implies \boxed{(\boldsymbol{r} - \boldsymbol{a}) \cdot \boldsymbol{n} = -\frac{1}{k_1}}$$

Third Derivative Condition Differentiating a third time:

$$\frac{d^3}{ds^3}\left[(\boldsymbol{r}-\boldsymbol{a})^2\right] = 2(\boldsymbol{r}-\boldsymbol{a})\cdot\boldsymbol{r}''' + 6\boldsymbol{r}'\cdot\boldsymbol{r}'' = 0$$

Using the Frenet-Serret formula for $\boldsymbol{r}'''$:

$$\boldsymbol{r}''' = -k_1^2\boldsymbol{\tau} + k_1'\boldsymbol{n} - k_1k_2\boldsymbol{b}$$

Substituting:

$$2(\boldsymbol{r}-\boldsymbol{a})\cdot(-k_1^2\boldsymbol{\tau} + k_1'\boldsymbol{n} - k_1k_2\boldsymbol{b}) + 6k_1\boldsymbol{\tau}\cdot\boldsymbol{n} = 0$$

Simplifying (since $\boldsymbol{\tau}\cdot\boldsymbol{n} = 0$ and $(\boldsymbol{r}-\boldsymbol{a})\cdot\boldsymbol{\tau} = 0$):

$$2(\boldsymbol{r}-\boldsymbol{a})\cdot(k_1'\boldsymbol{n} - k_1k_2\boldsymbol{b}) = 0$$

Using the second condition $(\boldsymbol{r}-\boldsymbol{a})\cdot\boldsymbol{n} = -1/k_1$:

$$2\left(k_1'\left(-\frac{1}{k_1}\right) - k_1k_2[(\boldsymbol{r}-\boldsymbol{a})\cdot\boldsymbol{b}]\right) = 0$$

$$\boxed{(\boldsymbol{r}-\boldsymbol{a})\cdot\boldsymbol{b} = -\frac{k_1'}{k_1^2k_2}}$$

Final Expressions The center $\boldsymbol{a}$ can be expressed as:

$$\boldsymbol{a} = \boldsymbol{r} + \frac{1}{k_1}\boldsymbol{n} + \frac{k_1'}{k_1^2k_2}\boldsymbol{b}$$

The radius R is:

$$R = |\boldsymbol{r}-\boldsymbol{a}| = \sqrt{\left(\frac{1}{k_1}\right)^2 + \left(\frac{k_1'}{k_1^2k_2}\right)^2}$$

$$\boxed{\begin{array}{l} \boldsymbol{a} = \boldsymbol{r} + \dfrac{1}{k_1}\boldsymbol{n} + \dfrac{k_1'}{k_1^2k_2}\boldsymbol{b} \\ R = \sqrt{\dfrac{1}{k_1^2} + \left(\dfrac{k_1'}{k_1^2k_2}\right)^2} \end{array}}$$

4.10 Osculating Paraboloid

Let S be a regular (twice continuously differentiable) surface, and let P be a point on S. Consider a paraboloid U containing P with its axis parallel to the surface normal at P. For any point Q on S, denote:

- $d =$ distance from Q to P,
- $h =$ perpendicular distance from Q to the paraboloid U.

Definition (Osculating Paraboloid)**.** The paraboloid U is called the *osculating paraboloid* of S at P if the ratio

$$\lim_{Q \to P} \frac{h}{d^2} = 0.$$

Degenerate cases (parabolic cylinder or plane) are included when the paraboloid loses its ellipticity or hyperbolicity.

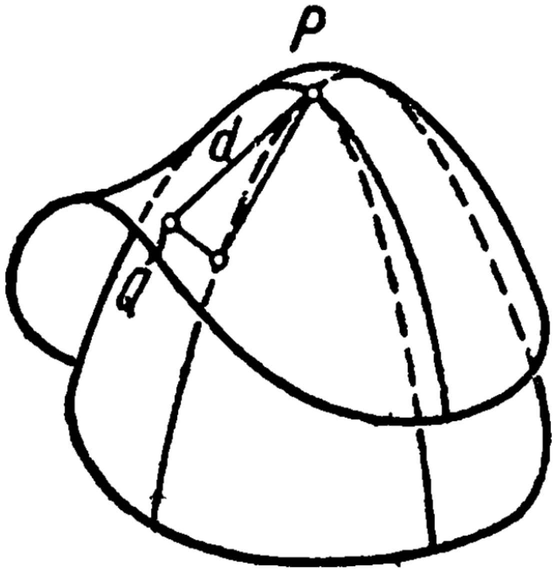


Figure 4.5. Osculating parabaloid

Theorem 4.10.1 (Existence and Uniqueness of the Osculating Paraboloid)**.** *At every point P of a regular surface S, there exists a unique osculating paraboloid U, which may degenerate into a parabolic cylinder or a plane. The coefficients of U are determined by the second fundamental form of S at P.*

Remark. The osculating paraboloid provides a second-order approximation of S near P, analogous to the osculating circle for curves.

Proof. To simplify the geometric setup, we place our point of interest at the origin with choosing the proper direction of the tangent plane.

Local Coordinate Setup Let Φ be a regular surface of class C^2 and $P \in \Phi$. Without loss of generality:

- Place P at the origin $(0, 0, 0)$.
- Align the tangent plane at P with the xy-plane.
- Orient the normal vector along the positive z-axis.

By the Implicit Function Theorem, near P the surface can be expressed as:

$$z = f(x, y), \quad \text{where } f(0, 0) = 0 \text{ and } \nabla f(0, 0) = (0, 0).$$

Second-Order Taylor Expansion Since $f \in C^2$, we have the Taylor approximation:

$$f(x, y) = \frac{1}{2}\left(Ax^2 + 2Bxy + Cy^2\right) + o(x^2 + y^2), \quad \text{as } (x, y) \to (0, 0)$$

where:

$$A = \frac{\partial^2 f}{\partial x^2}(0, 0), \quad B = \frac{\partial^2 f}{\partial x \partial y}(0, 0), \quad C = \frac{\partial^2 f}{\partial y^2}(0, 0).$$

Defining the Osculating Paraboloid Consider a general paraboloid U with axis parallel to z:

$$U : z = \frac{1}{2}\left(ax^2 + 2bxy + cy^2\right).$$

The vertical distance h from Φ to U is:

$$h = \left| f(x, y) - \frac{1}{2}\left(ax^2 + 2bxy + cy^2\right) \right|.$$

Imposing the Osculation Condition The condition $\frac{h}{d^2} \to 0$ as $Q \to P$, where $d = \sqrt{x^2 + y^2 + f(x, y)^2}$, requires:

$$\frac{\left|\frac{1}{2}\left((A - a)x^2 + 2(B - b)xy + (C - c)y^2\right) + o(x^2 + y^2)\right|}{x^2 + y^2} \to 0.$$

This implies the coefficients must match:

$$a = A, \quad b = B, \quad c = C.$$

Thus, the unique osculating paraboloid is:

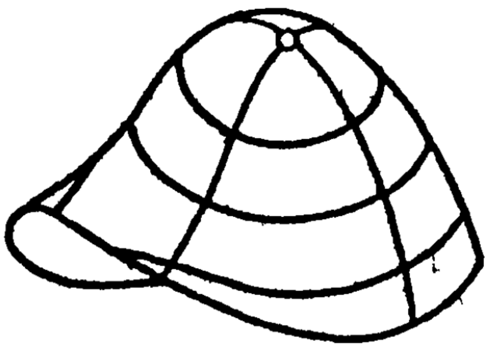
$$U : z = \frac{1}{2}\left(Ax^2 + 2Bxy + Cy^2\right).$$

Degenerate Cases

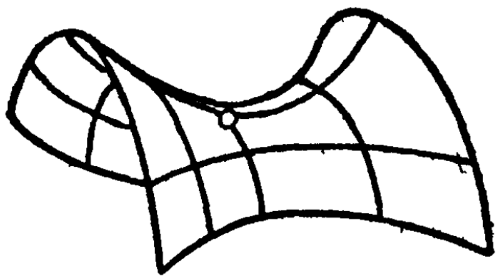
- **Plane**: If $A = B = C = 0$, U degenerates to the tangent plane $z = 0$.
- **Parabolic Cylinder**: If $AC - B^2 = 0$ but not all coefficients vanish, U becomes a parabolic cylinder.

Conclusion The osculating paraboloid exists uniquely and is determined by the second-order Taylor coefficients of f at P. The condition $h/d^2 \to 0$ enforces exact matching of the quadratic terms, with degeneracies occurring when the quadratic form is singular or vanishes. $\square$

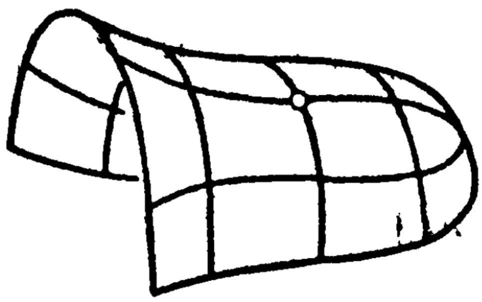
4.11 Classification of Points on a Surface



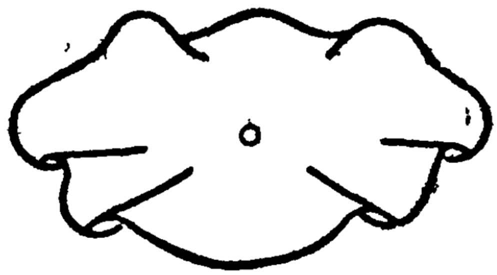
(a) Elliptic point



(b) Hyperbolic point



(c) Parabolic point



(d) Umbilical point

Figure 4.6. Classification of Points on a Surface.

The existence and uniqueness of the osculating paraboloid at every point of a regular surface permit us to make the following classification of points on a surface.

1. A point on a surface is called an **elliptic point** if the osculating paraboloid at this point is an *elliptic paraboloid*.

2. A point on a surface is called a **hyperbolic point** if the osculating paraboloid at this point is a *hyperbolic paraboloid*.

3. A point on a surface is called a **parabolic point** if the osculating paraboloid at this point degenerates into a *parabolic cylinder*.

4. A point on a surface is called an **umbilical point** if the osculating paraboloid at this point degenerates into a *plane* (i.e. the tangent plane to the surface at this point).

4.12 Envelope of a Family of Surfaces

An envelope of a family of surfaces is a surface that is tangent to each member of the family at some point, and whose tangent plane coincides with the tangent plane of the family member at that point.

Just as envelopes of curves are tangent to each curve in a family, envelopes of surfaces are tangent to each surface in a family. The key difference lies in the dimension: surfaces require analysing tangent planes rather than tangent lines.

Definition. Suppose $\{S\}$ is a family of smooth surfaces depending on one or two parameters. A surface F is called the envelope of the family if the following conditions are satisfied:

1. for every point $P \in F$ one can find a surface γ_P in the family $\{S\}$ which is tangent to F at the point P;

2. every surface γ in the family $\{S\}$ is tangent to F;

3. no surface in the family has a region in common with F.

Example. A smooth surface which does not contain pieces of a plane is the envelope of its tangent planes. The family of tangent planes may be either a one-parameter (cylinder) or a two-parameter (e.g. sphere) family.

In geometry and its applications to various fields of science, the problem frequently arises of finding the envelope for a given family.

Example. In optics or acoustics, a wavefront is defined as the locus of points where waves arrive at the same instant. For example, consider a point source emitting light or sound in a medium. The wavefronts are spherical surfaces centred at the source.

If the source moves, the wavefronts change positions. Analysing the envelope of these spherical wavefronts can reveal regions of constructive or destructive interference (e.g., the bright or dark regions in optics).

In the study of caustics (bright patterns formed by reflection or refraction), the envelope of reflected or refracted wavefronts creates intricate patterns that are visually observable in everyday life.

To simplify the discussion, we shall make some auxiliary assumptions concerning the nature of enveloping of the surfaces of the family by the surface F. Namely, we shall assume that for every point P of the envelope, we can specify a region G_P of variation of the parameters of the family, satisfying the following conditions:

1. For each point Q of the surface F, near P, only one surface of the family can be found having parameters belonging to G_P.
2. If $\boldsymbol{r}(u, v)$ is any smooth parametrization of the surface F and $a(u, v)$, $b(u, v)$ (which reduces to only $a(u, v)$ in the case of a one parameter family) are parameters of the surface, tangent to F at the point (u, v), then $a(u, v)$ and $b(u, v)$ are smooth functions of u and v.

Theorem 4.12.1 (Envelope of a One-Parameter Family of Surfaces). *Let $\{F_a\}$ be a family of smooth surfaces parameterised by a real parameter a, defined implicitly by the equation*

$$\varphi(x, y, z, a) = 0,$$

where φ is a smooth function of x, y, z, and a. If a smooth surface F is the envelope of this family, then for every point (x, y, z) on F, there exists a value of the parameter a such that the following two conditions are satisfied:

1. *The point (x, y, z) lies on the surface F_a:*
$$\varphi(x, y, z, a) = 0.$$
2. *The surface F_a is tangent to the envelope F at (x, y, z), which is equivalent to:*
$$\frac{\partial \varphi}{\partial a}(x, y, z, a) = 0.$$

The envelope F is thus the locus of points where the family of surfaces $\{F_a\}$ is tangent to a common surface.

Proof. Let F be the envelope of the family $\{F_a\}$. By definition, F is tangent to each surface F_a at some point. We parameterize F as $\boldsymbol{r}(u,v) = (x(u,v), y(u,v), z(u,v))$, and for each (u,v), there exists a parameter $a(u,v)$ such that F is tangent to $F_{a(u,v)}$ at $\boldsymbol{r}(u,v)$.

Since $\boldsymbol{r}(u,v)$ lies on $F_{a(u,v)}$, we have the identity:

$$\varphi(x(u,v), y(u,v), z(u,v), a(u,v)) \equiv 0.$$

Differentiating this identity with respect to u and v, we obtain:

$$\varphi_x x_u + \varphi_y y_u + \varphi_z z_u + \varphi_a a_u = 0, \quad (1)$$

$$\varphi_x x_v + \varphi_y y_v + \varphi_z z_v + \varphi_a a_v = 0. \quad (2)$$

Tangency Condition: The surfaces F and $F_{a(u,v)}$ are tangent at $\boldsymbol{r}(u,v)$, meaning their normal vectors are parallel. The normal vector of $F_{a(u,v)}$ is $\nabla\varphi = (\varphi_x, \varphi_y, \varphi_z)$, while the tangent plane of F is spanned by $\boldsymbol{r}_u = (x_u, y_u, z_u)$ and $\boldsymbol{r}_v = (x_v, y_v, z_v)$. Since $\nabla\varphi$ is orthogonal to the tangent plane of F, we have:

$$\varphi_x x_u + \varphi_y y_u + \varphi_z z_u = 0, \quad (3)$$

$$\varphi_x x_v + \varphi_y y_v + \varphi_z z_v = 0. \quad (4)$$

Substituting (3) and (4) into (1) and (2), we deduce:

$$\varphi_a a_u = 0, \quad \varphi_a a_v = 0.$$

Conclusion : Assume for contradiction that $\varphi_a \neq 0$ at some point $P = \boldsymbol{r}(u_0, v_0)$. By continuity, $\varphi_a \neq 0$ in a neighborhood of P, which implies $a_u = a_v = 0$ in that neighborhood. Thus, a is constant, meaning F coincides with a single surface F_a locally. This contradicts the definition of an envelope, which must be tangent to a *family* of distinct surfaces.

Therefore, $\varphi_a = 0$ at P. Since P was arbitrary, this holds for all points on F, proving the theorem. □

Example. Consider the family of spheres $\{F_a\}$ centered at $(a, 0, 0)$ on the x-axis, with a fixed radius R. The implicit equation of each sphere in the family is:

$$\varphi(x, y, z, a) = (x-a)^2 + y^2 + z^2 - R^2 = 0.$$

The first condition ensures that the point (x, y, z) lies on the surface F_a:

$$(x-a)^2 + y^2 + z^2 - R^2 = 0.$$

The second condition ensures that the surface F_a is tangent to the envelope F at (x, y, z). Compute the partial derivative of φ with respect to a:

$$\frac{\partial \varphi}{\partial a} = -2(x - a).$$

Setting this equal to zero gives:

$$-2(x - a) = 0 \quad \Rightarrow \quad x = a.$$

From the previous step, we have $x = a$. Substitute this into the first condition:

$$(x - x)^2 + y^2 + z^2 - R^2 = 0 \quad \Rightarrow \quad y^2 + z^2 = R^2.$$

The equation $y^2 + z^2 = R^2$ represents a **cylinder** of radius R centered along the x-axis. This cylinder is the envelope of the family of spheres.

Geometric Interpretation

- Each sphere F_a is centered at $(a, 0, 0)$ with radius R.
- As the center a varies, the spheres overlap along the cylinder $y^2 + z^2 = R^2$.
- The envelope is the **cylinder** because it is the surface where the family of spheres is tangent to a common surface.

Imagine stacking many spheres of radius R along the x-axis. The common region where these spheres "touch" forms a cylindrical surface. This cylinder is the envelope of the family of spheres.

Theorem 4.12.2 (Envelope of a Two-Parameter Family of Surfaces)**.** *Let $\{G_{a,b}\}$ be a two-parameter family of smooth surfaces defined by the equation*

$$\varphi(x, y, z, a, b) = 0,$$

*where a and b are parameters. The **envelope** G of this family is the locus of points (x, y, z) that satisfy the following conditions:*

1. *The point (x, y, z) lies on the surface $G_{a,b}$:*

$$\varphi(x, y, z, a, b) = 0.$$

2. *The surface $G_{a,b}$ is **tangent** to the envelope G at (x, y, z), which is equivalent to:*

$$\frac{\partial \varphi}{\partial a}(x, y, z, a, b) = 0 \quad \text{and} \quad \frac{\partial \varphi}{\partial b}(x, y, z, a, b) = 0.$$

Proof. Let $(x(u,v), y(u,v), z(u,v))$ be a local parametrization of the envelope G, where u and v are parameters. By definition, each point on G must lie on some surface $G_{a,b}$ from the family, so there exist functions $\alpha(u,v)$ and $\beta(u,v)$ such that:

$$\varphi(x(u,v), y(u,v), z(u,v), \alpha(u,v), \beta(u,v)) = 0. \tag{4.1}$$

Differentiating (4.1) concerning u and v (using the chain rule) and noting that the envelope is tangent to $G_{\alpha,\beta}$, the terms involving x_u, y_u, z_u (and x_v, y_v, z_v) vanish due to tangency. This yields:

$$\begin{cases} \varphi_\alpha \alpha_u + \varphi_\beta \beta_u = 0, \\ \varphi_\alpha \alpha_v + \varphi_\beta \beta_v = 0. \end{cases} \tag{4.2}$$

Assume, for contradiction, that at least one of φ_α or φ_β is non-zero at a point P. Without loss of generality, suppose $\varphi_\alpha \neq 0$ in a neighborhood of P. Then, equations (4.2) form a homogeneous linear system in φ_α and φ_β. For this system to have a non-trivial solution, the determinant of the coefficient matrix must vanish:

$$\begin{vmatrix} \alpha_u & \alpha_v \\ \beta_u & \beta_v \end{vmatrix} = \alpha_u \beta_v - \alpha_v \beta_u = 0.$$

This implies that α and β are functionally dependent, meaning there exists a relation $F(\alpha, \beta) = 0$ in a neighborhood of P. However, this contradicts the assumption that the family $\{G_{a,b}\}$ **essentially depends on two independent parameters** a and b.

Thus, the only possibility is that $\varphi_\alpha = \varphi_\beta = 0$ at P. Since P was arbitrary, this holds for all points on the envelope, completing the proof. □

Example. Consider the family of spheres defined by

$$\varphi(x, y, z, a, b) = (x-a)^2 + (y-b)^2 + z^2 - R^2 = 0,$$

where R is a fixed radius. The envelope G is obtained by solving:

1. $\varphi(x, y, z, a, b) = 0$,
2. $\frac{\partial \varphi}{\partial a} = -2(x-a) = 0$,
3. $\frac{\partial \varphi}{\partial b} = -2(y-b) = 0$.

Solving $\frac{\partial \varphi}{\partial a} = 0$ and $\frac{\partial \varphi}{\partial b} = 0$ gives:

$$x = a, \quad y = b.$$

Substituting $x = a$ and $y = b$ into the original equation $\varphi(x, y, z, a, b) = 0$ yields:

$$(x - x)^2 + (y - y)^2 + z^2 - R^2 = 0,$$

which simplifies to:

$$z^2 = R^2.$$

Thus, the envelope of the family of spheres consists of the two parallel planes:

$$z = R \quad \text{and} \quad z = -R.$$

4.13 Envelope of a Family of Planes

Suppose F is the envelope of a one-parameter family of planes $\{\Pi_a\}$. We shall explain the structure of the surface F.

Theorem 4.13.1. *Let $\{\Pi_a\}$ be a one-parameter family of planes, where each plane is defined by the equation*

$$\Pi_a : \phi(x, y, z, a) = 0,$$

where ϕ is a smooth function of the variables x, y, z, and the parameter a. The ***envelope*** *of this family is the surface F defined by the following conditions:*

1. *For each point (x, y, z) on F, there exists a value of the parameter a such that:*
$$\phi(x, y, z, a) = 0.$$

2. *The plane Π_a is tangent to the envelope F at the point (x, y, z), which is equivalent to:*
$$\frac{\partial \phi}{\partial a}(x, y, z, a) = 0.$$

The envelope F is, in general, a ***developable surface****, which can be one of the following:*

- *A* ***cylindrical surface****, if the family of planes is parallel to a fixed direction.*

- *A* ***conical surface****, if the family of planes passes through a fixed point (the vertex of the cone).*

- *A* ***tangent surface to a space curve****, if the family of planes is tangent to a curve in space.*

Proof. We analyse the envelope using the vector equation of the planes:

$$\boldsymbol{r} \cdot \boldsymbol{b}(a) + \alpha(a) = 0,$$

where $\boldsymbol{b}(a)$ is the unit normal vector and $\alpha(a)$ is a scalar function.

Key Assumptions

- $\boldsymbol{b}(a)$ is *not* constant (otherwise, the planes are parallel and no envelope exists).
- $\boldsymbol{b}'(a) \neq \boldsymbol{0}$ (ensures the family is non-degenerate).

The envelope F satisfies the system:

$$\boldsymbol{r} \cdot \boldsymbol{b} + \alpha = 0, \quad \boldsymbol{r} \cdot \boldsymbol{b}' + \alpha' = 0. \tag{4.3}$$

Intuition: The envelope is the set of points where neighboring planes Π_a and Π_{a+da} intersect. This forms a *ruled surface* (a union of straight lines).

Case Analysis We classify the envelope based on the intersection of three planes:

$$\boldsymbol{r} \cdot \boldsymbol{b} + \alpha = 0, \quad \boldsymbol{r} \cdot \boldsymbol{b}' + \alpha' = 0, \quad \boldsymbol{r} \cdot \boldsymbol{b}'' + \alpha'' = 0.$$

Case 1: Cylindrical Surface

Condition: The three planes have no common intersection (parallel to a fixed line).

Let $\boldsymbol{n}$ be a unit vector parallel to all planes. Then:

$$\boldsymbol{b} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{b}' \cdot \boldsymbol{n} = 0, \quad \boldsymbol{b}'' \cdot \boldsymbol{n} = 0.$$

Differentiating the first two gives $\boldsymbol{b} \cdot \boldsymbol{n}' = 0$ and $\boldsymbol{b}' \cdot \boldsymbol{n}' = 0$. Since $\boldsymbol{n}' \cdot \boldsymbol{n} = 0$, we conclude $\boldsymbol{n}' = \boldsymbol{0}$ (i.e., $\boldsymbol{n}$ is constant).

The cross product $\boldsymbol{b} \times \boldsymbol{b}'$ is nonzero and parallel to $\boldsymbol{n}$, so its direction is fixed. The envelope is a **cylinder** with rulings parallel to $\boldsymbol{n}$:

$$\boldsymbol{r} = -\alpha \boldsymbol{b} - \frac{\alpha'}{|\boldsymbol{b}'|^2} \boldsymbol{b}' + \nu(\boldsymbol{b} \times \boldsymbol{b}'), \quad \nu \in \mathbb{R}.$$

Non-degeneracy: The Jacobian condition $\boldsymbol{r}_a \times \boldsymbol{r}_\nu \neq \boldsymbol{0}$ holds everywhere, confirming F is a regular surface.

Case 2: Conical Surface

Condition: The three planes intersect at a fixed point S for all a.

Taking S as the origin, we get $\alpha = \alpha' = 0$, simplifying the envelope to:

$$\boldsymbol{r} = \nu(\boldsymbol{b} \times \boldsymbol{b}').$$

This is a **cone** with vertex S and rulings along $\boldsymbol{b} \times \boldsymbol{b}'$.

Case 3: Tangent Surface to a Curve

Condition: The three planes intersect at a point $S(a)$ that varies with a.

The position vector $\tilde{\boldsymbol{r}}(a)$ of $S(a)$ satisfies:

$$\tilde{\boldsymbol{r}} \cdot \boldsymbol{b} + \alpha = 0, \quad \tilde{\boldsymbol{r}} \cdot \boldsymbol{b}' + \alpha' = 0, \quad \tilde{\boldsymbol{r}} \cdot \boldsymbol{b}'' + \alpha'' = 0.$$

Differentiating the first two identities shows $\tilde{\boldsymbol{r}}' \parallel \boldsymbol{b} \times \boldsymbol{b}'$. Thus, the envelope is the **tangent surface** to the curve $\tilde{\boldsymbol{r}}(a)$:

$$\boldsymbol{r} = \tilde{\boldsymbol{r}}(a) + \nu(\boldsymbol{b} \times \boldsymbol{b}').$$

Conclusion The envelope F is always a developable surface, with its type determined by the relative positions of the planes Π_a. The three cases exhaust all possibilities, completing the proof. □

Example. **Cylindrical Envelope**

Consider the family of planes parallel to the z-axis:

$$\Pi_a : x \cos a + y \sin a = 1.$$

The envelope of this family is a cylinder of radius 1 centred along the z-axis.

Conical Envelope

Consider the family of planes passing through the origin:

$$\Pi_a : x \cos a + y \sin a + z \tan a = 0.$$

The envelope of this family is a cone with its vertex at the origin.

Tangent Surface

Consider the family of planes tangent to the helix:

$$\Pi_a : x \cos a + y \sin a + za = 0.$$

The envelope of this family is the tangent surface to the helix.

4.14 Problems Corner

Problem 1

Find the equation of the surface formed by all half-lines that **originate** at the point (a, b, c) and pass through at least one point on the parabola given by:

$$y^2 = 2px, \quad z = 0.$$

Express your answer as an implicit equation in x, y, and z.

Solution

Parametrize the Parabola The parabola in the xy-plane can be parametrised as:

$$\boldsymbol{r}(t) = \left(\frac{t^2}{2p}, t, 0\right), \quad t \in \mathbb{R}$$

Equation of Half-Lines The half-lines from (a, b, c) to points on the parabola are:

$$\boldsymbol{R}(s, t) = (a, b, c) + s\left(\frac{t^2}{2p} - a, t - b, -c\right), \quad s \geq 0$$

Step 3: Express Coordinates

$$\begin{aligned} x &= a + s\left(\frac{t^2}{2p} - a\right) \\ y &= b + s(t - b) \\ z &= c - sc \end{aligned}$$

Eliminate Parameters From $z = c(1 - s)$, we get $s = 1 - \frac{z}{c}$. Substituting into y:

$$t = b + \frac{c(y - b)}{c - z}$$

Substituting into x gives the implicit equation:

$$\boxed{(cy - bz)^2 = 2pc(c - z)(x - a) + 2pa(c - z)^2}$$

Problem 2

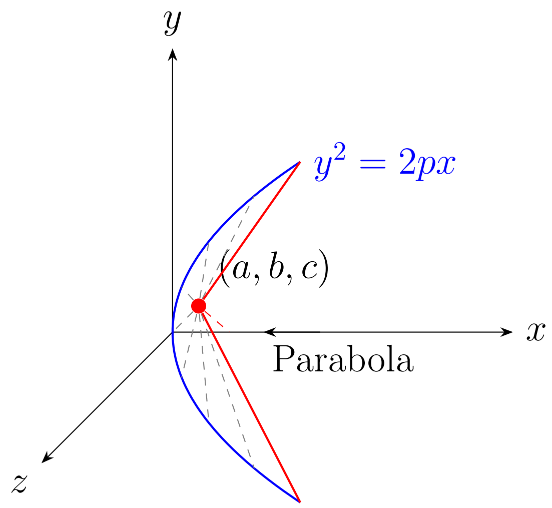


Figure 4.7. Surface formed by half-lines from (a, b, c) to the parabola $y^2 = 2px$ in the $z = 0$ plane. The red lines indicate the boundary of the resulting conical surface.

Investigate whether the point $Q(0, 0, 0)$ on the surface defined by

$$x = u^3, \quad y = v^3, \quad z = (u^6 + v^6)^{1/3}$$

is singular. Describe the local geometric shape of the surface near Q.

Solution

Parametric Surface Definition The surface is given parametrically by:

$$\boldsymbol{r}(u, v) = \left(u^3, v^3, (u^6 + v^6)^{1/3}\right).$$

At $(u, v) = (0, 0)$, we have $\boldsymbol{r}(0, 0) = (0, 0, 0) = Q$.

Singularity Check A point is singular if the tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are linearly dependent.

$$\boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = \left(3u^2, 0, \frac{2u^5}{(u^6 + v^6)^{2/3}}\right),$$
$$\boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = \left(0, 3v^2, \frac{2v^5}{(u^6 + v^6)^{2/3}}\right).$$

At $(u, v) = (0, 0)$, both tangent vectors become $\mathbf{0}$, and their cross product vanishes:

$$\boldsymbol{r}_u \times \boldsymbol{r}_v = \mathbf{0}.$$

Thus, Q is a **singular point**.

Local Geometric Analysis To understand the surface near Q, we examine its behaviour as $(u, v) \to (0, 0)$.

Implicit Form: Eliminating parameters u, v via $u = x^{1/3}$ and $v = y^{1/3}$ gives:

$$z = (x^2 + y^2)^{1/3}.$$

The implicit equation is:

$$\phi(x, y, z) = z^3 - x^2 - y^2 = 0.$$

Gradient Analysis: The gradient at Q:

$$\nabla\phi = (-2x, -2y, 3z^2)\big|_{(0,0,0)} = (0, 0, 0),$$

confirming singularity.

Hessian Analysis: The Hessian matrix at Q:

$$H = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

has eigenvalues $-2, -2, 0$, indicating a **degenerate singularity**.

Geometric Interpretation Near Q, the surface $z^3 = x^2 + y^2$ behaves like a **cuspoidal surface**:

- Sharper than a cone ($z^2 = x^2 + y^2$)
- "Pinched" appearance at the origin
- Flatter cross-sections compared to a standard cone

1. The point $Q(0, 0, 0)$ is **singular** (vanishing tangent vectors).
2. Locally, the surface resembles a **cusp** defined by $z^3 = x^2 + y^2$.
3. The singularity is **non-isolated** and **degenerate** (Hessian has zero eigenvalue).

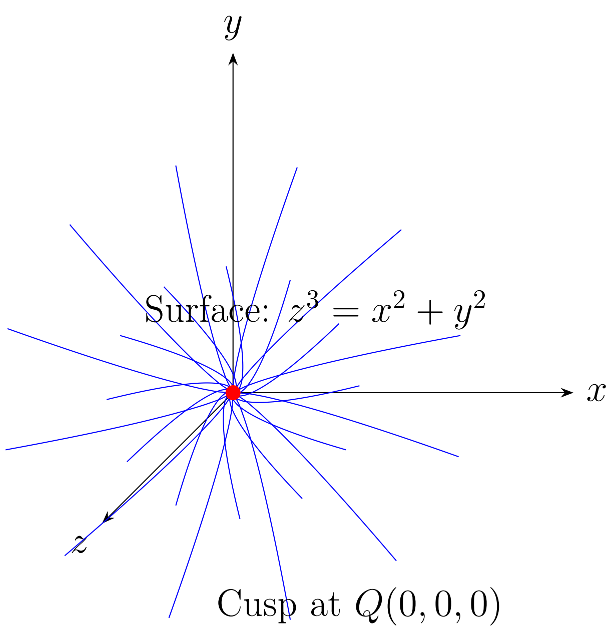


Figure 4.8. Visualisation of the cuspoidal surface near the singular point Q. The red dot marks the singularity.

Problem 3

Investigate whether the point $Q(0, 0, 0)$ on the surface defined by

$$x = u^2 - v^2, \quad y = v2uv, \quad z = u^5$$

is singular. Describe the local geometric shape of the surface near Q.

Solution

Verify Q lies on the surface At $(u, v) = (0, 0)$:

$$x = 0^2 - 0^2 = 0, \quad y = 2 \cdot 0 \cdot 0 = 0, \quad z = 0^5 = 0.$$

Thus, $Q(0, 0, 0)$ corresponds to the parameter values $(0, 0)$.

Check for singularity Compute the tangent vectors:

$$\mathbf{r}_u = \begin{pmatrix} 2u \\ 2v \\ 5u^4 \end{pmatrix}, \quad \mathbf{r}_v = \begin{pmatrix} -2v \\ 2u \\ 0 \end{pmatrix}$$

At $(u, v) = (0, 0)$:

$$\mathbf{r}_u = \mathbf{0}, \quad \mathbf{r}_v = \mathbf{0}$$

The Jacobian matrix has rank 0:

$$J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Since both tangent vectors vanish, Q is a **singular point**.

Local Geometry Analysis For small u, v, retain dominant terms:

$$x \approx u^2 - v^2, \quad y \approx 2uv, \quad z \approx u^5$$

- For $v = 0$: $z \approx x^{5/2}$ (high-order cusp)
- For $u = 0$: $x = -v^2, y = z = 0$ (parabola in x)

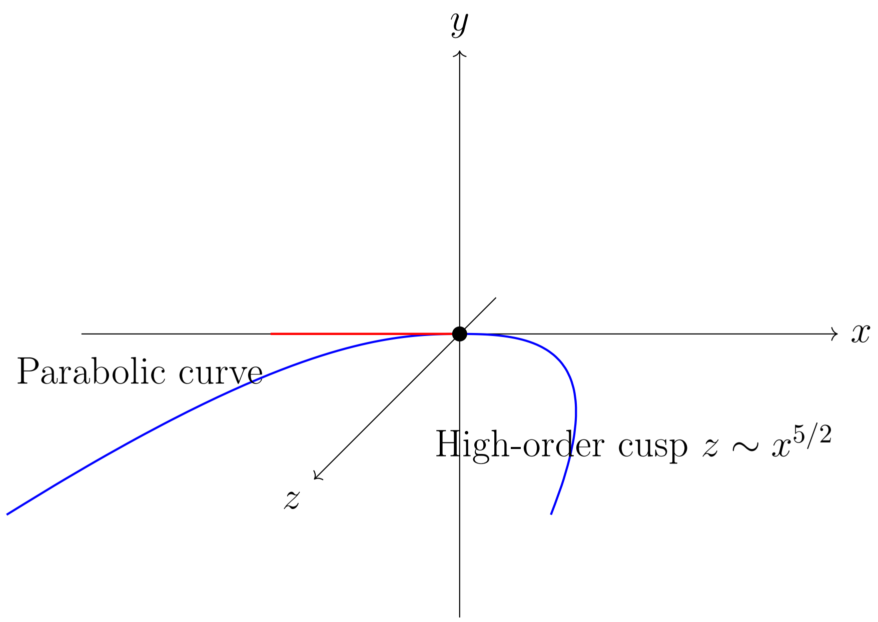


Figure 4.9. Local geometry near the singular point $Q(0, 0, 0)$. The surface has both a high-order cusp and a flat parabolic curve.

Conclusion

1. **Singularity**: $Q(0, 0, 0)$ is singular (vanishing tangent vectors).
2. **Local Geometry**:
   - Along $v = 0$: High-order cusp $z \sim x^{5/2}$

- Along $u = 0$: Flat parabolic curve $x = -v^2$

3. The singularity is **non-isolated** - it extends as a curve when $u = 0$.

Problem 4
Justify that all points on the surface defined by

$$x = u, \quad y = v^2, \quad z = v^3,$$

where $v = 0$ (i.e., the curve $(u, 0, 0)$ for $u \in \mathbb{R}$), are singular. Describe the local geometric shape of the surface near these points.

Solution

Parametric Surface Definition The surface is given by:

$$\boldsymbol{r}(u, v) = \left(u, v^2, v^3\right), \quad u \in \mathbb{R}, v \in \mathbb{R}.$$

The points where $v = 0$ form the curve $\boldsymbol{r}(u, 0) = (u, 0, 0)$.

Singularity Verification A point is singular if the tangent vectors are linearly dependent.

$$\text{Tangent vectors:} \quad \boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u} = (1, 0, 0),$$
$$\boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v} = (0, 2v, 3v^2).$$

At $v = 0$:

$$\boldsymbol{r}_v = (0, 0, 0) \implies \boldsymbol{r}_u \times \boldsymbol{r}_v = \boldsymbol{0}.$$

Thus, all points $(u, 0, 0)$ are **singular**. The surface is regular for $v \neq 0$.

Step 3: Local Geometry Analysis For fixed u, the yz-profile is:

$$y = v^2, \quad z = v^3 \implies z^2 = y^3.$$

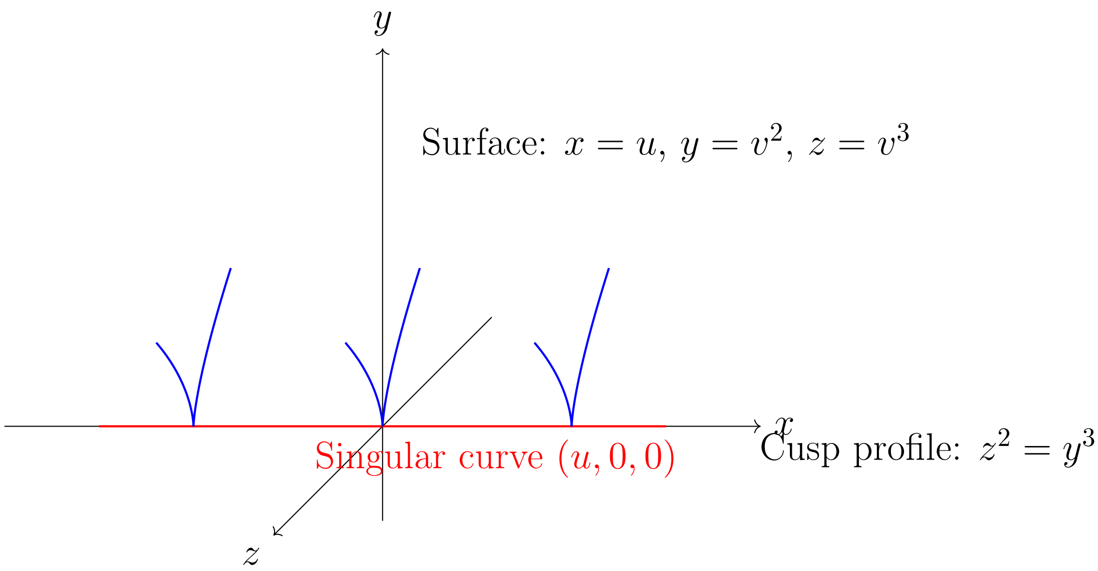


Figure 4.10. The surface has a line of cusp singularities along the x-axis. Each transverse slice shows a semi-cubical parabola.

Key Observations:

- **Singularity**: The entire x-axis is singular because $\boldsymbol{r}_v = \mathbf{0}$ when $v = 0$.
- **Cusp Behaviour**: Near singular points, the surface pinches to a cusp described by $z^2 = y^3$.
- **Regularity**: For $v \neq 0$, the surface is smooth with $\boldsymbol{r}_u \times \boldsymbol{r}_v = (0, -3v^2, 2v) \neq \mathbf{0}$.

Mathematical Justification

Implicit Form Eliminating v gives:

$$z^2 = y^3.$$

The gradient of $\phi(y, z) = z^2 - y^3$:

$$\nabla\phi = (-3y^2, 2z) \implies \text{Singular when } \nabla\phi = (0, 0).$$

This occurs only at $(y, z) = (0, 0)$, confirming the singularity along $(u, 0, 0)$.

Power Series Expansion Near $v = 0$:

$$y = v^2, \quad z = v^3 \implies z \approx \operatorname{sgn}(v) y^{3/2}.$$

The odd-order term in z creates the cusp asymmetry.

Conclusion

1. The curve $(u, 0, 0)$ is singular due to vanishing $\boldsymbol{r}_v$.
2. Locally, the surface forms a **line of cusps** with $z^2 = y^3$ profiles.
3. The cusp is sharper than a standard cone ($z^2 = y^2$) but smoother than a needle-like singularity.

Problem 5
Consider the surface defined by $\phi(x, y, z) = z^2 - x^3 - y^3 = 0$.

1. Classify the singular point at the origin $(0, 0, 0)$ by analyzing the gradient and Hessian of ϕ.
2. Describe the local geometric shape of the surface near the origin using a lowest-order approximation.

Solution

Verify Singularity at Origin First, check if $(0, 0, 0)$ is a singular point by evaluating the gradient:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (-3x^2, -3y^2, 2z).$$

At the origin:

$$\nabla\phi(0, 0, 0) = (0, 0, 0).$$

Since the gradient vanishes, $(0, 0, 0)$ is indeed a **singular point**.

Classify the Singularity To classify the singularity, analyse the Hessian matrix H of second derivatives:

$$H = \begin{pmatrix} \frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial x\partial z} \\ \frac{\partial^2\phi}{\partial y\partial x} & \frac{\partial^2\phi}{\partial y^2} & \frac{\partial^2\phi}{\partial y\partial z} \\ \frac{\partial^2\phi}{\partial z\partial x} & \frac{\partial^2\phi}{\partial z^2} & \frac{\partial^2\phi}{\partial z^2} \end{pmatrix} = \begin{pmatrix} -6x & 0 & 0 \\ 0 & -6y & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

At the origin:

$$H(0, 0, 0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

- **Rank**: The Hessian has rank 1 (only one non-zero eigenvalue).
- **Eigenvalues**: $0, 0, 2$.

This indicates a **degenerate singularity** where the surface flattens in the xy-plane but has quadratic behaviour in z.

Lowest-Order Approximation Near the origin, approximate ϕ by its lowest-order terms:

$$z^2 \approx x^3 + y^3.$$

This reveals:

- For $x, y \geq 0$: The surface behaves like $z = \pm(x^{3/2} + y^{3/2})$.
- Along $x = 0$: $z^2 = y^3$ (semi-cubical parabola).
- Along $y = 0$: $z^2 = x^3$ (semi-cubical parabola).

Step 4: Geometric Interpretation The surface near the origin resembles:

- A **double cusp** where two sheets meet sharply along the x- and y-axes.
- Cross-sections show **cuspidal edges** (see Figure 1).
- The singularity is **non-isolated** - it extends as a cuspidal edge when x or y is fixed at zero.

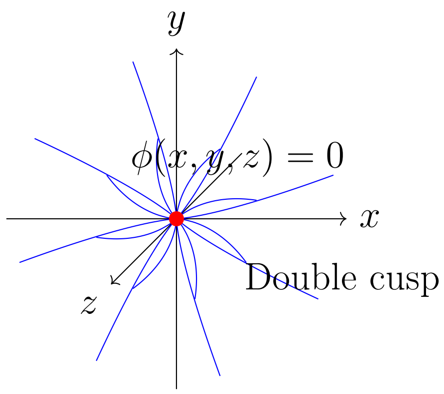


Figure 4.11. Local geometry of $z^2 = x^3 + y^3$ near the origin. The surface has two cuspidal sheets meeting at the singular point.

Conclusion

1. The origin is a **degenerate singular point** (Hessian rank 1).
2. Locally, the surface forms a **double cusp** described by $z^2 \approx x^3 + y^3$.
3. The geometry combines features of two semi-cubical parabolas orthogonal to each other.

Problem 6
Find the equation of the tangent plane to the surface $z = x^2 + y^2$ at the point $(1, 2, 5)$.

Solution
1. Compute the partial derivatives:

$$f_x(x, y) = 2x \quad \text{and} \quad f_y(x, y) = 2y.$$

2. Evaluate the partial derivatives at $(1, 2)$:

$$f_x(1, 2) = 2(1) = 2 \quad \text{and} \quad f_y(1, 2) = 2(2) = 4.$$

3. Substitute into the tangent plane equation:

$$z = 5 + 2(x - 1) + 4(y - 2).$$

4. Simplify:

$$z = 2x + 4y - 5.$$

Thus, the equation of the tangent plane is $z = 2x + 4y - 5$.

Problem 7
Find the equation of the tangent plane to the surface $x^2 + y^2 - z = 0$ at the point $(2, 3, 13)$.

Solution
1. Compute the partial derivatives:

$$F_x(x, y, z) = 2x, \quad F_y(x, y, z) = 2y, \quad F_z(x, y, z) = -1.$$

2. Evaluate the partial derivatives at $(2, 3, 13)$:

$$F_x(2, 3, 13) = 4, \quad F_y(2, 3, 13) = 6, \quad F_z(2, 3, 13) = -1.$$

3. Substitute into the tangent plane equation:

$$4(x-2)+6(y-3)-1(z-13)=0.$$

4. Simplify:

$$4x+6y-z-13=0.$$

Thus, the equation of the tangent plane is $4x+6y-z-13=0$.

Problem 8

Find the equation of the osculating paraboloid to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

at the point $(0,0,c)$. Assume $a,b,c>0$.

Solution

Rewrite the Ellipsoid Equation

The ellipsoid equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Solve for z near the point $(0,0,c)$:

$$z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}.$$

Employ Taylor Expansion

Approximate z using a second-order Taylor expansion around $(x,y)=(0,0)$:

$$z\approx c\left(1-\frac{x^2}{2a^2}-\frac{y^2}{2b^2}\right).$$

This simplifies to:

$$z=c-\frac{cx^2}{2a^2}-\frac{cy^2}{2b^2}.$$

Identify the Osculating Paraboloid

The equation obtained above is the equation of an elliptic paraboloid. Thus, the osculating paraboloid to the ellipsoid at $(0,0,c)$ is:

$$z=c-\frac{cx^2}{2a^2}-\frac{cy^2}{2b^2}.$$

Final Answer:

$$\boxed{z = c - \frac{cx^2}{2a^2} - \frac{cy^2}{2b^2}}$$

Problem 9

Find the position of the centre and the radius of the osculating sphere of the helix

$$x = a\cos t, \quad y = a\sin t, \quad z = bt$$

at point $(a, 0, 0)$. Assume $a, b > 0$.

Solution

Parametrize the Helix

The helix is given parametrically by:

$$\boldsymbol{r}(t) = (a\cos t, a\sin t, bt).$$

At $t = 0$, the point on the helix is $(a, 0, 0)$.

Solution

The centre of the osculating sphere is given by:

$$\boldsymbol{a} = \boldsymbol{r}(0) + \frac{1}{k_1}\boldsymbol{n} + \frac{k_1'}{k_1^2 k_2}\boldsymbol{b}.$$

The radius of the osculating sphere is:

$$R = \sqrt{\frac{1}{k_1^2} + \left(\frac{k_1'}{k_1^2 k_2}\right)^2}.$$

The derivatives of the position vector are:

$$\boldsymbol{r}'(t) = (-a\sin t, a\cos t, b).$$

At $t = 0$:

$$\boldsymbol{r}'(0) = (0, a, b).$$

$$\boldsymbol{r}''(t) = (-a\cos t, -a\sin t, 0).$$

At $t = 0$:

$$\boldsymbol{r}''(0) = (-a, 0, 0).$$

Curvature k_1:

$$k_1 = \frac{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}.$$

At $t = 0$:

$\boldsymbol{r}'(0) \times \boldsymbol{r}''(0) = (0, a, b) \times (-a, 0, 0) = (0 \cdot 0 - b \cdot 0, b \cdot (-a) - 0 \cdot 0, 0 \cdot 0 - a \cdot (-a)) = (0, -ab, a^2)$

The magnitude is:

$$\|\boldsymbol{r}'(0) \times \boldsymbol{r}''(0)\| = \sqrt{0^2 + (-ab)^2 + a^4} = a\sqrt{a^2 + b^2}$$

The curvature is:

$$k_1 = \frac{a\sqrt{a^2 + b^2}}{(a^2 + b^2)^{3/2}} = \frac{a}{a^2 + b^2}.$$

Since k_1 is constant, $k_1' = 0$ **the center simplifies to**:

$$\boldsymbol{a} = \boldsymbol{r}(0) + \frac{1}{k_1}\boldsymbol{n},$$

the radius simplifies to:

$$R = \frac{1}{k_1} = \frac{a^2 + b^2}{a}.$$

Hence, only the normal vector $\boldsymbol{n}$ remains an unknown value.
The principal normal vector $\boldsymbol{n}$ is:

$$\boldsymbol{n} = \frac{\boldsymbol{\tau}'(0)}{\|\boldsymbol{\tau}'(0)\|}$$

First, compute $\boldsymbol{\tau}'(t)$:

$$\boldsymbol{\tau}'(t) = \frac{\mathbf{r}''(t)}{\sqrt{a^2 + b^2}}$$

Thus, for this parameterisation, we can write

$$\boldsymbol{n} = \frac{\boldsymbol{r}''}{|\boldsymbol{r}''|}.$$

At $t = 0$:

$$\boldsymbol{n}(0) = \frac{\boldsymbol{r}''(0)}{|\boldsymbol{r}''(0)|} = \frac{(-a, 0, 0)}{a} = (-1, 0, 0).$$

The center is:

$$\boldsymbol{a} = \boldsymbol{r}(0) + \frac{1}{k_1}\boldsymbol{n} = (a, 0, 0) + \frac{1}{k_1}(-1, 0, 0) = (a, 0, 0) + \frac{a^2 + b^2}{a}(-1, 0, 0) = \left(-\frac{b^2}{a}, 0, 0\right).$$

Problem 10

Show that every tangent plane to the surface

$$z = xf\left(\frac{y}{x}\right)$$

passes through the origin.

Solution

To show that all tangent planes to the surface $z = xf\left(\frac{y}{x}\right)$ pass through the origin, we proceed as follows:

Compute the partial derivatives of z The surface is defined by:

$$z = xf\left(\frac{y}{x}\right).$$

Let $u = \frac{y}{x}$. Then, the partial derivatives of z with respect to x and y are:

1. **Partial derivative with respect to x:**

$$\frac{\partial z}{\partial x} = f\left(\frac{y}{x}\right) + x \cdot f'\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right).$$

2. **Partial derivative with respect to y:**

$$\frac{\partial z}{\partial y} = x \cdot f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = f'\left(\frac{y}{x}\right).$$

Write the equation of the tangent plane The equation of the tangent plane to the surface at a point (x_0, y_0, z_0) is:

$$z - z_0 = \frac{\partial z}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial z}{\partial y}(x_0, y_0)(y - y_0).$$

Substituting the partial derivatives and $z_0 = x_0 f\left(\frac{y_0}{x_0}\right)$, we get:

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left(f\left(\frac{y_0}{x_0}\right) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right)(x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0).$$

Simplify the tangent plane equation Expand the right-hand side:

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = f\left(\frac{y_0}{x_0}\right)(x - x_0) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)(x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0).$$

Simplify the equation:

$$z = x_0 f\left(\frac{y_0}{x_0}\right) + f\left(\frac{y_0}{x_0}\right)(x - x_0) - \frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)(x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0).$$

Show that the tangent plane passes through the origin To show that the tangent plane passes through the origin, substitute $(x, y, z) = (0, 0, 0)$ into the tangent plane equation:

$$0 = x_0 f\left(\frac{y_0}{x_0}\right) + f\left(\frac{y_0}{x_0}\right)(0 - x_0) - \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right)(0 - x_0) + f'\left(\frac{y_0}{x_0}\right)(0 - y_0).$$

Simplify the equation:

$$0 = x_0 f\left(\frac{y_0}{x_0}\right) - x_0 f\left(\frac{y_0}{x_0}\right) + y_0 f'\left(\frac{y_0}{x_0}\right) - y_0 f'\left(\frac{y_0}{x_0}\right).$$

This simplifies to:

$$0 = 0.$$

Thus, the tangent plane equation is satisfied at the origin.

Conclusion All tangent planes to the surface $z = xf\left(\frac{y}{x}\right)$ pass through the origin.

Problem 11
Show that the surfaces defined by

$$\begin{aligned} x^2 + y^2 + z^2 &= ax \\ x^2 + y^2 + z^2 &= by \\ x^2 + y^2 + z^2 &= cz \end{aligned}$$

intersect orthogonally at their common points. Assume $a, b, c > 0$.

Solution
To show that the surfaces intersect orthogonally, we must verify that their gradients are pairwise orthogonal at their common points. Let:
1. $F_1(x, y, z) = x^2 + y^2 + z^2 - ax$,
2. $F_2(x, y, z) = x^2 + y^2 + z^2 - by$,
3. $F_3(x, y, z) = x^2 + y^2 + z^2 - cz$.

We compute the gradients of F_1, F_2, and F_3 and show that their dot products are zero at the points of intersection.

Compute the Gradients 1. Gradient of F_1:

$$\nabla F_1 = \left(\frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial y}, \frac{\partial F_1}{\partial z}\right) = (2x - a, 2y, 2z).$$

2. Gradient of F_2:

$$\nabla F_2 = \left(\frac{\partial F_2}{\partial x}, \frac{\partial F_2}{\partial y}, \frac{\partial F_2}{\partial z} \right) = (2x, 2y - b, 2z).$$

3. Gradient of F_3:

$$\nabla F_3 = \left(\frac{\partial F_3}{\partial x}, \frac{\partial F_3}{\partial y}, \frac{\partial F_3}{\partial z} \right) = (2x, 2y, 2z - c).$$

Find the Points of Intersection The points of intersection satisfy all three equations:

$$x^2 + y^2 + z^2 = ax, \quad x^2 + y^2 + z^2 = by, \quad x^2 + y^2 + z^2 = cz.$$

From these, we get:

$$ax = by = cz = k,$$

where k is a constant. Without loss of generality, assume $k \neq 0$. Then:

$$x = \frac{k}{a}, \quad y = \frac{k}{b}, \quad z = \frac{k}{c}.$$

Substituting into $x^2 + y^2 + z^2 = k$:

$$\left(\frac{k}{a}\right)^2 + \left(\frac{k}{b}\right)^2 + \left(\frac{k}{c}\right)^2 = k.$$

Simplify:

$$k^2 \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) = k \quad \Rightarrow \quad k = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}.$$

Thus, the points of intersection are:

$$\left(\frac{k}{a}, \frac{k}{b}, \frac{k}{c} \right), \quad \text{where } k = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}.$$

Verify Orthogonality of Gradients At the points of intersection, the gradients are:

1. $\nabla F_1 = \left(2 \cdot \frac{k}{a} - a, 2 \cdot \frac{k}{b}, 2 \cdot \frac{k}{c}\right)$, 2. $\nabla F_2 = \left(2 \cdot \frac{k}{a}, 2 \cdot \frac{k}{b} - b, 2 \cdot \frac{k}{c}\right)$, 3. $\nabla F_3 = \left(2 \cdot \frac{k}{a}, 2 \cdot \frac{k}{b}, 2 \cdot \frac{k}{c} - c\right)$.

Now, check the dot products of the gradients:

1. Dot product of ∇F_1 and ∇F_2:

$$\nabla F_1 \cdot \nabla F_2 = \left(2 \cdot \frac{k}{a} - a\right)\left(2 \cdot \frac{k}{a}\right) + \left(2 \cdot \frac{k}{b}\right)\left(2 \cdot \frac{k}{b} - b\right) + \left(2 \cdot \frac{k}{c}\right)\left(2 \cdot \frac{k}{c}\right).$$

Simplify:

$$\nabla F_1 \cdot \nabla F_2 = 0.$$

2. Dot product of ∇F_1 and ∇F_3: Similarly:

$$\nabla F_1 \cdot \nabla F_3 = 0.$$

3. Dot product of ∇F_2 and ∇F_3: Similarly:

$$\nabla F_2 \cdot \nabla F_3 = 0.$$

Conclusion Since the dot products of the gradients are zero at the points of intersection, the surfaces intersect orthogonally.

Problem 12

Prove that the family of surfaces defined by $f(x, y, z) = a$, where f is a regular function of x, y, z and a is a real parameter, does not have an envelope unless f explicitly depends on a.

Solution

Understand the Family of Surfaces The given family of surfaces is defined by:

$$f(x, y, z) = a,$$

where:

- f is a regular (smooth) function of x, y, z,
- a is a real parameter.

Each value of a corresponds to a distinct surface in the family. For example, if $f(x, y, z) = x^2 + y^2 + z^2$, the family consists of concentric spheres of radius $\sqrt{a}$ centered at the origin.

Recall the Definition of an Envelope An **envelope** of a family of surfaces satisfies two conditions:

1. It is tangent to each surface in the family at some point.
2. Its tangent plane at each point coincides with the tangent plane of the family surface at that point.

Mathematically, the envelope is found by solving the system of equations:

$$\phi(x, y, z, a) = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial a} = 0,$$

where $\phi(x, y, z, a)$ is the equation defining the family of surfaces.

Apply the Envelope Conditions For the given family of surfaces $f(x, y, z) = a$, rewrite the equation as:

$$\phi(x, y, z, a) = f(x, y, z) - a = 0.$$

The two conditions for the envelope are:

1. $\phi(x, y, z, a) = f(x, y, z) - a = 0$,
2. $\frac{\partial \phi}{\partial a} = \frac{\partial}{\partial a}(f(x, y, z) - a) = -1 = 0$.

Analyse the Second Condition The second condition yields:

$$\frac{\partial \phi}{\partial a} = -1 = 0.$$

This is a contradiction because $-1 \neq 0$.

Interpret the Result The contradiction implies that there is no solution to the system of equations $\phi(x, y, z, a) = 0$ and $\frac{\partial \phi}{\partial a} = 0$. Therefore, the family of surfaces $f(x, y, z) = a$ does not have an envelope.

Special Case (f Explicitly Depends on a) If f explicitly depends on a, i.e., $f(x, y, z, a)$, the partial derivative $\frac{\partial \phi}{\partial a}$ becomes:

$$\frac{\partial \phi}{\partial a} = \frac{\partial f}{\partial a} - 1.$$

The second condition $\frac{\partial \phi}{\partial a} = 0$ becomes:

$$\frac{\partial f}{\partial a} - 1 = 0 \implies \frac{\partial f}{\partial a} = 1.$$

In this case, an envelope may exist if $\frac{\partial f}{\partial a} = 1$.

Conclusion The family of surfaces $f(x, y, z) = a$ does not have an envelope unless f explicitly depends on a and satisfies the additional condition $\frac{\partial f}{\partial a} = 1$.

Final Answer The family of surfaces defined by $f(x, y, z) = a$, where f is a regular function of x, y, z and a is a real parameter, does not have an envelope unless f explicitly depends on a and satisfies $\frac{\partial f}{\partial a} = 1$. Otherwise, no envelope exists for this family of surfaces.

Problem 13

Prove that if the normals to a surface pass through a common point, then this surface is either a sphere or a region on a sphere.

Hint: Show that the surface satisfies the equation of a sphere centred at the common point.

Solution

Understand the Given Condition Let S be a surface, and suppose all the normals to S pass through a common point $P_0 = (x_0, y_0, z_0)$.

This means that for any point $P = (x, y, z)$ on the surface S, the normal vector $\boldsymbol{n}(P)$ at P lies along the line connecting P and P_0.

Recall the Equation of the Normal Line The normal vector $\boldsymbol{n}(P)$ at a point P on the surface S is perpendicular to the tangent plane at P. If the normal passes through P_0, then the vector $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ is parallel to $\boldsymbol{n}(P)$.

Thus, we can write:

$$\overrightarrow{P_0P} = \lambda \boldsymbol{n}(P),$$

where λ is a scalar.

Relate the Normal Vector to the Gradient If the surface S is defined implicitly by $F(x, y, z) = 0$, the normal vector $\boldsymbol{n}(P)$ is given by the gradient of F:

$$\boldsymbol{n}(P) = \nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right).$$

Since $\overrightarrow{P_0P}$ is parallel to $\boldsymbol{n}(P)$, we have:

$$(x - x_0, y - y_0, z - z_0) = \lambda \nabla F.$$

This gives the system of equations:

$$x - x_0 = \lambda \frac{\partial F}{\partial x}, \quad y - y_0 = \lambda \frac{\partial F}{\partial y}, \quad z - z_0 = \lambda \frac{\partial F}{\partial z}.$$

Eliminate the Scalar λ From the three equations above, we can eliminate λ by taking ratios:

$$\frac{x - x_0}{\frac{\partial F}{\partial x}} = \frac{y - y_0}{\frac{\partial F}{\partial y}} = \frac{z - z_0}{\frac{\partial F}{\partial z}}.$$

Show That S Satisfies the Equation of a Sphere Assume the surface S is defined implicitly by $F(x, y, z) = 0$. To prove that S is a sphere, we show that $F(x, y, z)$ can be written in the form:

$$F(x, y, z) = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 - R^2 = 0,$$

where R is the radius of the sphere centered at P_0.

Consider the gradient of F:

$$\nabla F = (2(x - x_0), 2(y - y_0), 2(z - z_0)).$$

From Step 3, $\overrightarrow{P_0P} = \lambda \nabla F$, so:

$$(x - x_0, y - y_0, z - z_0) = \lambda (2(x - x_0), 2(y - y_0), 2(z - z_0)).$$

This implies:

$$\lambda = \frac{1}{2}.$$

Substituting $\lambda = \frac{1}{2}$ into $\overrightarrow{P_0P} = \lambda \nabla F$ gives:

$$(x - x_0, y - y_0, z - z_0) = \frac{1}{2} (2(x - x_0), 2(y - y_0), 2(z - z_0)),$$

which is consistent.

Thus, the surface S satisfies the equation:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2,$$

where R is the distance from P_0 to any point P on S.

Conclusion If all normals to the surface S pass through a common point P_0, then S satisfies the equation of a sphere centred at P_0. Therefore, S is either a sphere or a region on a sphere.

Final Answer If the normals to a surface pass through a common point P_0, then the surface satisfies the equation of a sphere centred at P_0. Hence, the surface is either a sphere or a region on a sphere.

Problem 14

Find the envelope of the family of planes that cut off a tetrahedron of constant volume V from the positive octant $x, y, z > 0$.

Solution

Consider a plane given by the equation:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

where $a, b, c > 0$. This plane intersects the coordinate axes at the point $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

Volume of the Tetrahedron The volume of the tetrahedron formed by this plane and the coordinate plane is:

$$V = \frac{1}{6} \cdot a \cdot b \cdot c.$$

Since the volume is constant, we have:

$$a \cdot b \cdot c = 6V = \text{const}.$$

Equation of the Envelope To find the envelope, eliminate the parameters a, b, c from the system of equations. Using the method of Lagrange multipliers, consider the function:

$$F(x, y, z, a, b, c) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 + \lambda(a \cdot b \cdot c - 6V).$$

The partial derivatives with respect to a, b, c yield:

$$-\frac{x}{a^2} + \lambda bc = 0 \quad \Rightarrow \quad \lambda = \frac{x}{a^2 bc},$$

$$-\frac{y}{b^2} + \lambda ac = 0 \quad \Rightarrow \quad \lambda = \frac{y}{ab^2 c},$$

$$-\frac{z}{c^2} + \lambda ab = 0 \quad \Rightarrow \quad \lambda = \frac{z}{abc^2}.$$

Equating the expressions for λ, we obtain:

$$\frac{x}{a^2bc} = \frac{y}{ab^2c} = \frac{z}{abc^2}.$$

From this:

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k.$$

Substituting into the plane equation:

$$k + k + k = 1 \quad \Rightarrow \quad k = \frac{1}{3}.$$

Thus:

$$a = 3x, \quad b = 3y, \quad c = 3z.$$

Substituting into the constant volume condition:

$$(3x) \cdot (3y) \cdot (3z) = 6V \quad \Rightarrow \quad 27xyz = 6V \quad \Rightarrow \quad xyz = \frac{2V}{9}.$$

Answer:

$$\boxed{xyz = \frac{2V}{9}}$$

Chapter 5

The First Quadratic Form of a Surface

Suppose Φ is a regular surface, $\boldsymbol{r} = \boldsymbol{r}(u, v)$ is any regular parametrisation of Φ, and $\boldsymbol{n}$ is the unit normal vector to the surface at the point (u, v).

There are **three quadratic forms** which are related to the surface:

$$d\boldsymbol{r}^2, \quad -d\boldsymbol{r} \cdot d\boldsymbol{n}, \quad d\boldsymbol{n}^2.$$

The first quadratic form $I = d\boldsymbol{r}^2$ is positive definite since it assumes only nonnegative values and vanishes only when $du = dv = 0$:

$$d\boldsymbol{r}^2 = 0 \text{ if } d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv = 0$$

. While $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$, this is possible only for

$$du = dv = 0.$$

We shall use the notation

$$\boldsymbol{r}_u^2 = E, \quad \boldsymbol{r}_u \cdot \boldsymbol{r}_v = F, \quad \boldsymbol{r}_v^2 = G$$

for the coefficients of the first quadratic form of the surface. Thus,

$$I = dr^2 = (\boldsymbol{r}_u du + \boldsymbol{r}_v dv)^2 = \boldsymbol{r}_u^2 du^2 + 2\boldsymbol{r}_u \cdot \boldsymbol{r}_v du dv + \boldsymbol{r}_v^2 dv^2 = E du^2 + 2F du dv + G dv^2.$$

We also call this first quadratic form *the first fundamental form* of the surface.

5.1 Length of a Curve on a Surface

Definition. Let Φ be a simple surface and γ a curve. We say γ *lies on* Φ if every point of γ is also a point of Φ.

Let P_0 be a common point of γ and Φ. Suppose:

- The surface Φ is parametrized near P_0 as $\boldsymbol{r}_\Phi = \boldsymbol{r}_\Phi(u, v)$, with P_0 corresponding to parameters (u_0, v_0).
- The curve γ is parametrized near P_0 as $\boldsymbol{r}_\gamma = \boldsymbol{r}_\gamma(t)$, with P_0 corresponding to $t = t_0$.

For $|t - t_0| < \delta$ (where δ is sufficiently small), each point $P(t)$ of γ lies in the parametrized neighborhood of P_0 on Φ. Thus, there exist unique functions $u(t)$ and $v(t)$ such that:

$$\boldsymbol{r}_\gamma(t) = \boldsymbol{r}_\Phi(u(t), v(t)).$$

Definition. The equations $u = u(t)$ and $v = v(t)$ are called the *equations of the curve γ on the surface Φ*.

Theorem 5.1.1 (Regular Curves on Regular Surfaces)**.** *Let Φ be a regular surface with parametrization $\boldsymbol{r}_\Phi(u, v)$ and γ a regular curve on Φ with parametrization $\boldsymbol{r}_\gamma(t)$. Suppose in a neighbourhood of $P \in \Psi$:*

- *$\boldsymbol{r}_\Phi$ is **regular**: $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq 0$ for all (u, v) in a neighborhood of P.*
- *$\boldsymbol{r}_\gamma$ is **regular**: $\boldsymbol{r}_\gamma'^2(t) \neq 0$ for all t near t_0, where $\boldsymbol{r}_\gamma(t_0) = P$.*

If γ lies on Φ, then locally $\boldsymbol{r}_\gamma(t) = \boldsymbol{r}_\Phi(u(t), v(t))$, and the parametric equations $u = u(t)$, $v = v(t)$ satisfy:

$$u'(t)^2 + v'(t)^2 \neq 0.$$

Remark. This ensures the **regularity of the reparametrization**: the curve $(u(t), v(t))$ in the parameter domain is non-singular.

Proof. 1. **Regular Case**:
Since γ lies on Φ, we have $\mathbf{r}_\gamma(t) = \mathbf{r}_\Phi(u(t), v(t))$. Differentiating concerning t:

$$\mathbf{r}_\gamma'(t) = \mathbf{r}_u \cdot u'(t) + \mathbf{r}_v \cdot v'(t).$$

By regularity of γ, $\mathbf{r}_\gamma'(t) \neq 0$, so $u'(t)$ and $v'(t)$ cannot both vanish. Thus:

$$u'(t)^2 + v'(t)^2 \neq 0.$$

The implicit function theorem guarantees the existence of $u(t)$ and $v(t)$ because the Jacobian matrix $[\mathbf{r}_u \, \mathbf{r}_v]$ has full rank (due to $\mathbf{r}_u \times \mathbf{r}_v \neq 0$).

2. **General Case (Remark)**:
For a general surface $\Phi = \varphi(\bar{\Phi})$, where φ is a continuous bijection and $\bar{\Phi}$ is a simple surface, the curve γ is the image of a curve $\bar{\gamma}$ on $\bar{\Phi}$. The regularity of γ and Φ ensures the existence of $u(t), v(t)$ as above. $\square$

Let Φ be a regular surface with regular parametrisation $\boldsymbol{r} = \boldsymbol{r}(u, v)$, and let γ be a regular curve on Φ given by $u = u(t)$, $v = v(t)$. The arc length of γ between points $P_0 = \boldsymbol{r}(u(t_0), v(t_0))$ and $P = \boldsymbol{r}(u(t), v(t))$ is:

$$s(t_0, t) = \int\limits_{t_0}^{t} \left\| \frac{d\boldsymbol{r}}{dt} \right\| dt = \int\limits_{t_0}^{t} \sqrt{E\, u'(t)^2 + 2F\, u'(t)v'(t) + G\, v'(t)^2}\, dt, =$$

$$= \int\limits_{\gamma(P_0,P)} |d\boldsymbol{r}(u, v)| dt = \int\limits_{\gamma(P_0,P)} \sqrt{I}.$$

where $E = \boldsymbol{r}_u^2$, $F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v$, and $G = \boldsymbol{r}_v^2$ are coefficients of the first fundamental form I.

We see that to measure the lengths of curves on a surface, it is sufficient to know the first quadratic form of the surface.

In this connection, we say that the first quadratic form defines **a metric on the surface**.

Definition. $\sqrt{I}$ is called the *line element*.

Remark. The first quadratic form does not define the surface uniquely. It is easy to introduce examples of various surfaces which have the same quadratic forms for corresponding parametrisations. But, generally speaking, for two surfaces taken arbitrarily, there does not exist a parametrisation for which the first quadratic forms of the surfaces coincide.

5.2 Angle Between Curves on a Surface

Definition. A **direction** on a regular surface Φ parametrised by $\boldsymbol{r}(u, v)$ is determined by a tangent vector in the tangent plane at a point $P \in \Phi$. Given a parameter change (du, dv), the corresponding direction is represented by:

$$d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv,$$

where $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are the partial derivatives of $\boldsymbol{r}(u, v)$.

Remark. The notation $(du : dv)$ denotes the *ratio* of parameter changes (invariant under scaling), but the vector $d\boldsymbol{r}$ explicitly defines the direction.

Definition. The **angle** θ between two directions $(du_1 : dv_1)$ and $(du_2 : dv_2)$ at P is the angle between their tangent vectors:

$$d\boldsymbol{r}_1 = \boldsymbol{r}_u du_1 + \boldsymbol{r}_v dv_1, \quad d\boldsymbol{r}_2 = \boldsymbol{r}_u du_2 + \boldsymbol{r}_v dv_2.$$

Using the first fundamental form,

$$
\begin{aligned}
&d\boldsymbol{r}_1 \cdot d\boldsymbol{r}_2 = |d\boldsymbol{r}_1||d\boldsymbol{r}_2| \cos\theta, \\
&d\boldsymbol{r}^2 = Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2 = I(d_1), \\
&d\boldsymbol{r}_2^2 = Edu_2^2 + 2Fdu_2dv_2 + Gdv_2^2 = I(d_2), \\
&d\boldsymbol{r} \cdot \delta\boldsymbol{r} = Edudu_2 + F(dudv_2 + dvdu_2) + Gdvdv_2 = I(d_1, d_2).
\end{aligned}
$$

The angle is:

$$
\cos\theta = \frac{Edu_1du_2 + F(du_1dv_2 + du_2dv_1) + Gdv_1dv_2}{\sqrt{Edu_1^2 + 2Fdu_1dv_1 + Gdv_1^2}\sqrt{Edu_2^2 + 2Fdu_2dv_2 + Gdv_2^2}} = \\
= \frac{I(d_1, d_2}{\sqrt{I(d_1)I(d_2)}}.
$$

Definition (Direction of a Curve on a Surface)**.** Let Φ be a regular surface parametrized by $\boldsymbol{r}(u, v)$, and let γ be a curve on Φ.

- **Geometric Definition**: The curve γ has *direction* $(du : dv)$ at a point $P = \boldsymbol{r}(u, v)$ if its tangent vector at P is parallel to:

$$d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv.$$

  Here, $(du : dv)$ represents the ratio of components in the basis $\{\boldsymbol{r}_u, \boldsymbol{r}_v\}$.

- **Parametric Definition**: If γ is parametrized as $\boldsymbol{r}(t) = \boldsymbol{r}(u(t), v(t))$, its direction at P is $(u'(t) : v'(t))$, since:

$$\boldsymbol{r}'(t) = \boldsymbol{r}_u u'(t) + \boldsymbol{r}_v v'(t).$$

  This matches the geometric definition with $du = u'(t)dt$, $dv = v'(t)dt$.

Definition (Angle Between Two Curves on a Surface)**.** Let γ and $\overline{\gamma}$ be two regular curves on a surface Φ parametrized by $\boldsymbol{r}(u, v)$, intersecting at a point $P = \boldsymbol{r}(u_0, v_0)$. The **angle** between γ and $\overline{\gamma}$ at P is the angle θ between their tangent vectors in the tangent plane $T_P\Phi$:

- If γ and $\overline{\gamma}$ are parametrized as $\boldsymbol{r}(u(t), v(t))$ and $\boldsymbol{r}(\overline{u}(\tau), \overline{v}(\tau))$, their tangent vectors at P are:

$$\boldsymbol{T}_1 = \boldsymbol{r}_u u'(t) + \boldsymbol{r}_v v'(t), \quad \boldsymbol{T}_2 = \boldsymbol{r}_u \overline{u}'(\tau) + \boldsymbol{r}_v \overline{v}'(\tau).$$

- The angle is computed via the first fundamental form ($E = \boldsymbol{r}_u \cdot \boldsymbol{r}_u$, $F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v$, $G = \boldsymbol{r}_v \cdot \boldsymbol{r}_v$):

$$\cos\theta = \frac{Eu'\overline{u}' + F(u'\overline{v}' + v'\overline{u}') + Gv'\overline{v}'}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}\sqrt{E\overline{u}'^2 + 2F\overline{u}'\overline{v}' + G\overline{v}'^2}}.$$

The angle θ is independent of the parametrizations of γ, $\overline{\gamma}$, and Φ.

5.3 Coordinate Curves

Definition (Coordinate Curves)**.** Let Φ be a **regular surface** parametrized by $\boldsymbol{r}(u, v)$, where (u, v) belong to an open domain in $\mathbb{R}^2$.

1. The u-**curve** (or v-*constant curve*) is obtained by fixing $v = v_0$ and varying u:
$$\boldsymbol{r}(u, v_0), \quad \text{with tangent vector} \quad \boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u}.$$
2. The v-**curve** (or u-*constant curve*) is obtained by fixing $u = u_0$ and varying v:
$$\boldsymbol{r}(u_0, v), \quad \text{with tangent vector} \quad \boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v}.$$

Let us list properties of coordinate curves

- The **coordinate net** is the grid formed by the u- and v-curves.
- The tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ span the tangent plane at each point.
- The curves are **regular** (no self-intersections or cusps) if $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq \boldsymbol{0}$ (i.e., the parametrization is regular).
- The coordinate net is **orthogonal** if $\boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0$.

Example (Spherical Coordinates). For spherical coordinates $\boldsymbol{r}(\theta, \phi)$, the θ-curves are longitudes, and the ϕ-curves are latitudes. These are orthogonal ($F = 0$).

The last property has its expression as the following theorem

Theorem 5.3.1 (Orthogonality of Coordinate Curves)**.** *Let Φ be a regular surface parametrized by $\boldsymbol{r}(u, v)$ with first fundamental form coefficients E, F, and G. The following conditions are equivalent:*

1. *The coordinate curves are orthogonal (intersect at right angles)*

2. The mixed coefficient F of the first fundamental form vanishes identically

3. The tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are orthogonal at every point

4. The first fundamental form is diagonal: $\mathrm{I} = E\,du^2 + G\,dv^2$

Symbolically, the orthogonality condition may be expressed as:

$$F(u,v) = \boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0 \quad \forall (u,v) \in \mathcal{D}$$

where $\mathcal{D}$ is the parameter domain.

Proof. We prove the equivalences systematically.
1. Coordinate curves orthogonal $\iff$ $F = 0$:

- ($\Rightarrow$) The u-curves ($v = v_0$) have tangent $\boldsymbol{r}_u$, while v-curves ($u = u_0$) have tangent $\boldsymbol{r}_v$. Their orthogonality implies:

$$\boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0 \quad \forall (u,v)$$

  By definition, $F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v$, so $F = 0$.

- ($\Leftarrow$) If $F = 0$, then $\boldsymbol{r}_u \cdot \boldsymbol{r}_v = 0$, proving the coordinate curves are orthogonal.

2. $F = 0$ $\iff$ First fundamental form is diagonal:
The first fundamental form is:

$$\mathrm{I} = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

When $F = 0$, it becomes diagonal ($\mathrm{I} = E\,du^2 + G\,dv^2$). Conversely, diagonality requires $F = 0$.
3. $F = 0$ $\iff$ $\boldsymbol{r}_u \perp \boldsymbol{r}_v$:
This is immediate from $F = \boldsymbol{r}_u \cdot \boldsymbol{r}_v$.
4. Geometric interpretation:
At any point $\boldsymbol{r}(u_0, v_0)$:

- u-curve tangent: $\boldsymbol{r}_u$

- v-curve tangent: $\boldsymbol{r}_v$

- Angle θ between them satisfies:

$$\cos\theta = \frac{F}{\sqrt{EG}}$$

  Thus $\theta = \pi/2$ if $F = 0$.

Conclusion: All four statements are equivalent, with $F = 0$ being the key algebraic condition. $\square$

The next statement gives us an approach to establish a curvilinear coordinate net on an arbitrary smooth surface

Theorem 5.3.2 (Existence of Orthogonal Parametrization)**.** *Let Φ be a regular surface of class C^k $(k \geq 1)$, and let $P_0 = \boldsymbol{r}(u_0, v_0)$ be an arbitrary point on Φ. Given any smooth family of curves $\mathcal{F}_1$ defined near P_0 by $\varphi(u, v) =$ constant with $\nabla\varphi \neq 0$ at P_0, there exists:*

1. *A neighborhood $U \subset \Phi$ of P_0*
2. *A regular C^k reparametrization $\boldsymbol{r}^*(s, t)$ of $\boldsymbol{r}|_U$*

such that:

- *The curves $\varphi =$ constant become s-curves (i.e., $t =$ constant)*
- *The orthogonal trajectories become t-curves (i.e., $s =$ constant)*
- *The new parametrization satisfies $F^* = 0$ (orthogonal coordinates)*

Moreover, the first fundamental form in these coordinates becomes:

$$\mathrm{I}^* = E^*\,ds^2 + G^*\,dt^2$$

Proof. The proof is constructive:

1. Solve (5.1) to find orthogonal trajectories
2. Compute integrating factor μ from (5.3)
3. The new coordinates are given by (5.4) and (5.5)

Differential Equation Setup The tangent direction of $\mathcal{F}_1$ is $(\varphi_v : -\varphi_u)$. The orthogonal trajectories must satisfy:

$$\begin{aligned} E\varphi_v du + F(\varphi_v dv - \varphi_u du) - G\varphi_u dv &= 0 \\ (E\varphi_v - F\varphi_u)du + (F\varphi_v - G\varphi_u)dv &= 0 \end{aligned} \tag{5.1}$$

Non-Degeneracy Condition At P_0, we verify:

$$E\varphi_v^2 - 2F\varphi_v\varphi_u + G\varphi_u^2 = \|\nabla\varphi\|_{\mathrm{I}}^2 > 0 \tag{5.2}$$

since I is positive definite and $\nabla\varphi \neq 0$. Thus (5.1) is non-degenerate near P_0.

Integrating Factor Construction There exists a C^1 integrating factor $\mu(u,v) \neq 0$ such that:

$$\mu\left[(E\varphi_v - F\varphi_u)du + (F\varphi_v - G\varphi_u)dv\right] = d\psi \tag{5.3}$$

This follows from:

Lemma 5.3.3 (Integrating Factor)**.** *For a 1-form $\omega = Pdu + Qdv$ with $P, Q \in C^1$ and $P^2 + Q^2 \neq 0$, there locally exists μ making $\mu\omega$ exact.*

Coordinate Transformation Define new coordinates:

$$s = \varphi(u,v) \tag{5.4}$$

$$t = \psi(u,v) \tag{5.5}$$

The Jacobian is non-singular because:

$$\frac{\partial(s,t)}{\partial(u,v)} = \varphi_u\psi_v - \varphi_v\psi_u = E\varphi_v^2 - 2F\varphi_v\varphi_u + G\varphi_u^2 > 0 \tag{5.6}$$

Orthogonality Verification In (s,t)-coordinates:

- t-curves correspond to $\varphi =$ constant (original $\mathcal{F}_1$)
- s-curves satisfy (5.1) and are orthogonal to $\mathcal{F}_1$
- The first fundamental form becomes:

$$\mathrm{I}^* = E^* ds^2 + G^* dt^2 \quad \text{with} \quad F^* = \langle \boldsymbol{r}_s^*, \boldsymbol{r}_t^* \rangle = 0 \tag{5.7}$$

Conclusion The map $(u,v) \mapsto (s,t)$ gives the required orthogonal parametrisation in a neighbourhood of P_0. □

Corollary 5.3.4. *In any orthogonal parametrisation, the coordinate curves are lines of curvature if and only if both $E_t^* = 0$ and $G_s^* = 0$ identically.*

5.4 Area of a Region on the Surface

Let F be a smooth surface, and let G be a region on F bounded by a finite number of piecewise smooth curves. We decompose G into small subregions g using piecewise smooth curves. For each subregion g, choose a point P in g and project g onto the tangent plane at P. If g is sufficiently small, this projection is one-to-one, resulting in a region $\overline{g}$ in the tangent plane bounded by piecewise smooth curves. Let $A(\overline{g})$ denote the area of $\overline{g}$.

Definition (Area)**.** The **area of the region** G on the surface F is defined as the limit of the sum of the areas $A(\overline{g})$, where the limit is taken as the size of the subregions g approaches zero. Mathematically, this is expressed as:

$$A(G) = \lim_{\operatorname{diam}(g)\to 0} \sum A(\overline{g}).$$

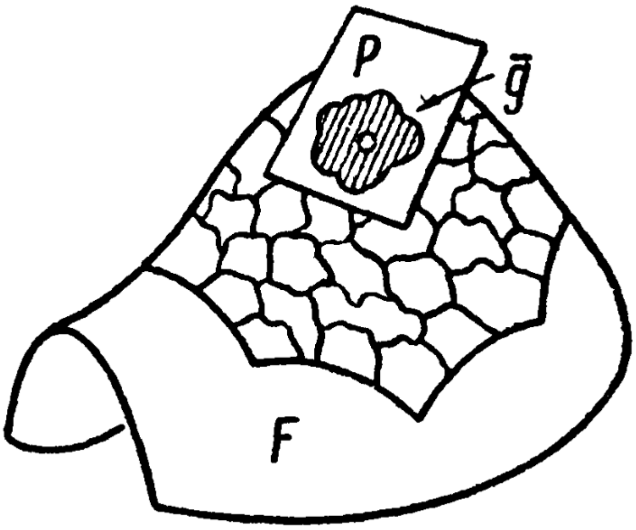


Figure 5.1. Definition of **area**

For simplicity, assume a smooth parametrization $\boldsymbol{r} = \boldsymbol{r}(u, v)$ exists for the surface F. The region G on F corresponds to a region $\tilde{G}$ in the u-v plane, bounded by piecewise smooth curves. The decomposition of $\tilde{G}$ into subregions $\tilde{g}$ corresponds to a decomposition of G into subregions g.

Theorem 5.4.1 (Surface Area of a Parametrized Surface)**.** *Let*

$$\boldsymbol{r}(u, v) = (x(u, v), y(u, v), z(u, v))$$

be a smooth parametrization of a surface F, where $\boldsymbol{r}_u = \frac{\partial \boldsymbol{r}}{\partial u}$ and $\boldsymbol{r}_v = \frac{\partial \boldsymbol{r}}{\partial v}$ are the partial derivatives of $\boldsymbol{r}$. Assume that $\boldsymbol{r}_u \times \boldsymbol{r}_v \neq \boldsymbol{0}$ (i.e., the surface is regular). Let G be a region on the surface F, and let $\tilde{G}$ be the corresponding region in the u-v plane. Then, the area of G is given by:

$$A(G) = \iint_{\tilde{G}} |\boldsymbol{r}_u \times \boldsymbol{r}_v| \, du \, dv.$$

Proof. 1. **Setup**
The parametrization $\boldsymbol{r}(u, v)$ maps the region $\tilde{G}$ in the u-v plane to the region G on the surface F. The partial derivatives $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are tangent vectors to the

surface at each point, and their cross product $\boldsymbol{r}_u \times \boldsymbol{r}_v$ is a normal vector to the surface.

2. **Mapping and Transformation**

For a small subregion $\tilde{g} \subset \tilde{G}$, the corresponding region $g \subset G$ is approximately a parallelogram in the tangent plane. The area of this parallelogram is given by $|\boldsymbol{r}_u \times \boldsymbol{r}_v| \, \Delta u \, \Delta v$.

3. **Limit Argument**

The area of G is obtained by summing the areas of these parallelograms and taking the limit as the size of $\tilde{g}$ approaches zero. By the uniform continuity of $\boldsymbol{r}_u \times \boldsymbol{r}_v$, this limit converges to the integral:

$$A(G) = \iint_{\tilde{G}} |\boldsymbol{r}_u \times \boldsymbol{r}_v| \, du \, dv.$$

4. **Additivity of Surface Area**

If G is divided into subregions G_1 and G_2, the corresponding regions in the u-v plane are $\tilde{G}_1$ and $\tilde{G}_2$. By the additivity of the integral, we have:

$$A(G) = A(G_1) + A(G_2).$$

□

Remark. The surface area is defined by only its first quadratic form. In fact,

$$|\boldsymbol{r}_u \times \boldsymbol{r}_v|^2 = r_u^2 r_v^2 - (\boldsymbol{r}_u \cdot \boldsymbol{r}_v)^2 = EG - F^2.$$

It follows that

$$A = \iint \sqrt{EG - F^2} dudv.$$

5.5 Conformal Mapping

Definition (Conformal Mapping)**.** Let Φ_1 and Φ_2 be regular surfaces in space. A mapping $f : \Phi_1 \to \Phi_2$ is said to be **conformal** if:

1. f is **bijective** (one-to-one and onto),
2. f is **smooth** (differentiable with a smooth inverse),
3. f **preserves angles** between curves: for any two curves γ_1 and γ_2 on Φ_1 intersecting at a point p, the angle between their tangent vectors at p is equal to the angle between the tangent vectors of the corresponding curves $f(\gamma_1)$ and $f(\gamma_2)$ at $f(p)$.

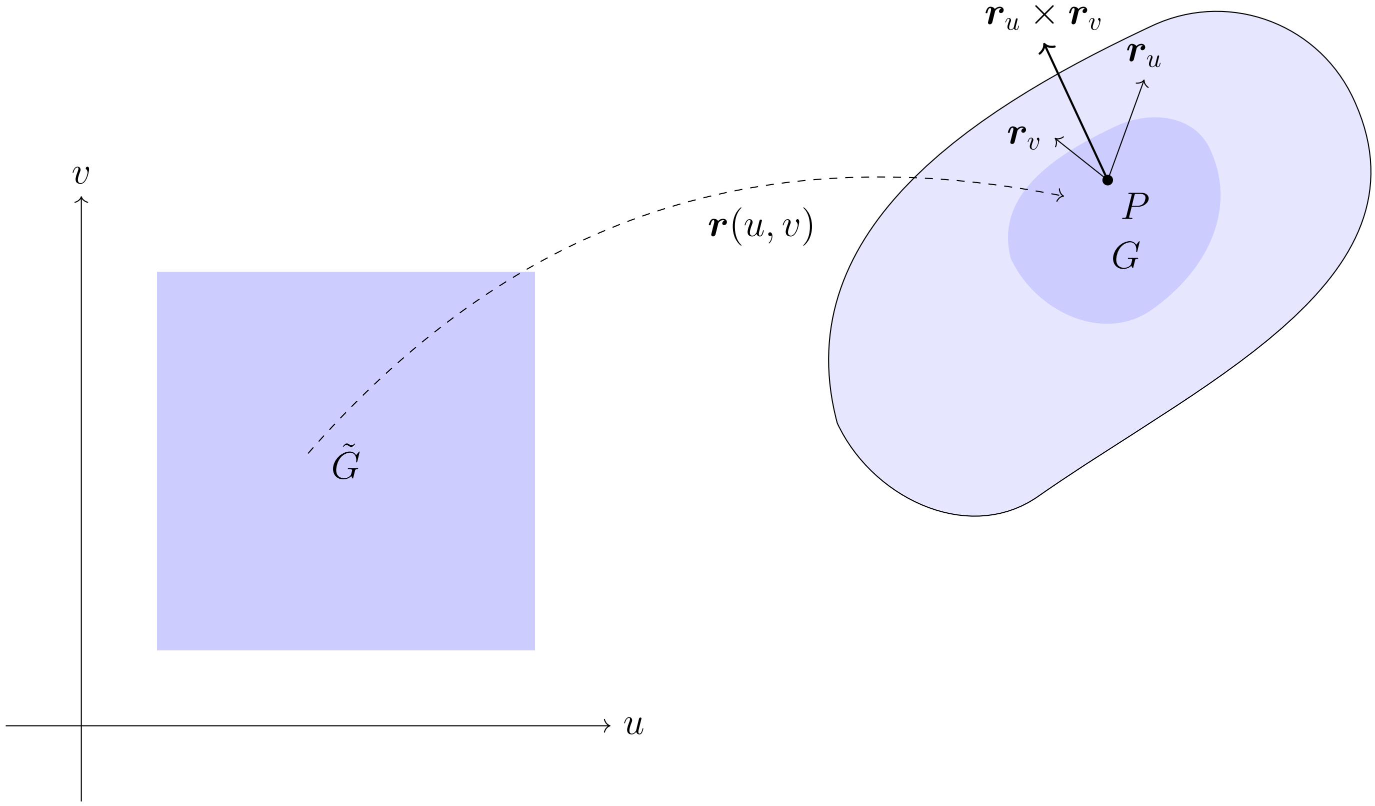


Figure 5.2. Illustration of the surface area theorem. The region $\tilde{G}$ in the u-v plane is mapped to the region G on the surface F. The tangent vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ at a point P on F define the normal vector $\boldsymbol{r}_u \times \boldsymbol{r}_v$.

Theorem 5.5.1. *Suppose Φ_1 and Φ_2 are regular surfaces, and let $P_1 \in \Phi_1$ and $P_2 \in \Phi_2$ be points on these surfaces. Let $\boldsymbol{r}_1(u,v)$ and $\boldsymbol{r}_2(u,v)$ be regular parametrizations of Φ_1 and Φ_2 in neighborhoods of P_1 and P_2, respectively, with P_1 and P_2 corresponding to the parameter values $u = u_0$, $v = v_0$. Suppose the coefficients of the first fundamental forms of Φ_1 and Φ_2, in these parametrizations, satisfy*

$$\frac{E_1}{E_2} = \frac{F_1}{F_2} = \frac{G_1}{G_2} = \lambda,$$

where $\lambda > 0$ is a constant. Then the mapping $f : \Phi_1 \to \Phi_2$, defined by $f(\boldsymbol{r}_1(u,v)) = \boldsymbol{r}_2(u,v)$, is conformal in a neighborhood of P_1.

Proof. Let γ_1 and γ_1' be two curves on Φ_1 intersecting at P_1, parametrized by $(u(t), v(t))$ and $(u'(t), v'(t))$, respectively. Their corresponding curves on Φ_2, denoted γ_2 and γ_2', are defined by the same parameterizations $(u(t), v(t))$ and $(u'(t), v'(t))$.

The angle θ_1 between γ_1 and γ_1' at P_1 is given by:

$$\cos\theta_1 = \frac{E_1\dot{u}\dot{u}' + F_1(\dot{u}\dot{v}' + \dot{v}\dot{u}') + G_1\dot{v}\dot{v}'}{\sqrt{E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2} \cdot \sqrt{E_1\dot{u}'^2 + 2F_1\dot{u}'\dot{v}' + G_1\dot{v}'^2}}.$$

Similarly, the angle θ_2 between γ_2 and γ_2' at P_2 is:

$$\cos\theta_2 = \frac{E_2\dot{u}\dot{u}' + F_2(\dot{u}\dot{v}' + \dot{v}\dot{u}') + G_2\dot{v}\dot{v}'}{\sqrt{E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2} \cdot \sqrt{E_2\dot{u}'^2 + 2F_2\dot{u}'\dot{v}' + G_2\dot{v}'^2}}.$$

Since $E_1 = \lambda E_2$, $F_1 = \lambda F_2$, and $G_1 = \lambda G_2$, the terms in $\cos\theta_1$ and $\cos\theta_2$ are proportional by λ. Thus, $\theta_1 = \theta_2$, and the mapping f preserves angles. Hence, f is conformal.

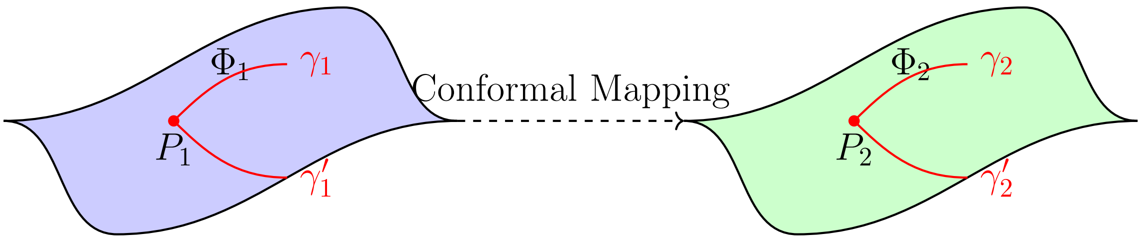


Figure 5.3. Conformal mapping of a neighbourhood of an arbitrary point

□

Remark. We use the notation $\dot{u}(t)$ to denote the derivative u by t in this proof.

Theorem 5.5.2. *Suppose Φ_1 and Φ_2 are regular surfaces and that P_1, P_2 are arbitrary points on these surfaces.*

Then there exists a conformal mapping of some neighbourhood of the point P_1 on the surface Φ_1 onto some neighbourhood of the point P_2 on the surface Φ_2.

Remark. The proof of this theorem is based on the possibility of parametrising a regular surface in a neighbourhood of an arbitrary point in such a way that its first quadratic form assumes the form

$$I = \lambda(u, v)(du^2 + dv^2)$$

with the parametrisation.

We shall not carry out the proof of this assertion; we shall only point out that the surfaces Φ_1 and Φ_2 are parametrized in neighborhoods of the points P_1 and P_2 respectively in such a way that a conformal mapping of a neighborhood of the point P_1 on the surface Φ_1 onto a neighborhood of the point P_2 on the surface Φ_2 is obtained by identifying points with the same coordinates.

5.6 Stereographic Projection

Definition. Let $\mathbb{S}^2$ be a sphere of radius R centered at the point $(0, 0, R)$ in three-dimensional Euclidean space. The **stereographic projection** of $\mathbb{S}^2$ onto its equatorial plane (the xy-plane) is a mapping defined as follows:

1. **Projection Point**: Let $N = (0, 0, 2R)$ denote the **north pole** of the sphere, and let $S = (0, 0, 0)$ denote the **south pole**.

2. **Mapping**: For any point $P \neq N$ on the sphere $\mathbb{S}^2$, the **stereographic projection** of P, denoted P', is obtained by projecting P from the north pole N onto the equatorial plane. Geometrically, P projects onto the plane along the straight line connecting N to P, and P' is the intersection of this line with the xy-plane (Figure 5.4).

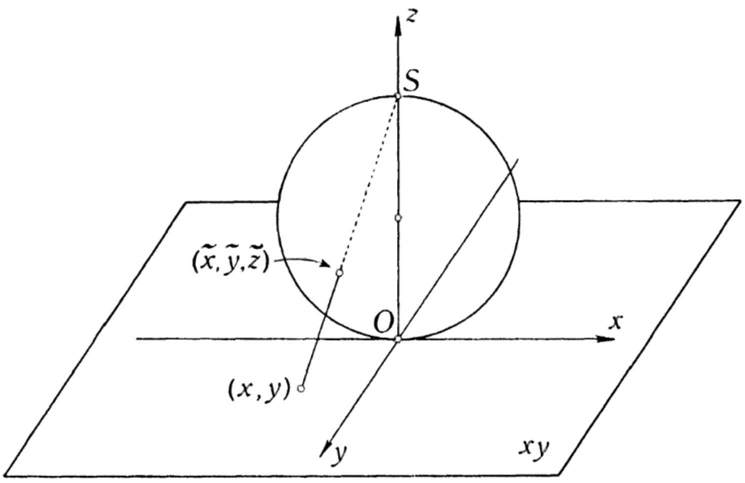


Figure 5.4

3. **Formulas**: If the coordinates of P on the sphere are $(\tilde{x}, \tilde{y}, \tilde{z})$, then the coordinates of its projection P' on the plane xy-plane are $(x, y, 0)$, where:

$$x = \frac{2R\tilde{x}}{2R - \tilde{z}}, \quad y = \frac{2R\tilde{y}}{2R - \tilde{z}}.$$

   Conversely, for any point $(x, y, 0)$, its corresponding point P on the sphere $\mathbb{S}^2$ has coordinates:

$$\tilde{x} = \frac{4R^2 x}{x^2 + y^2 + 4R^2}, \quad \tilde{y} = \frac{4R^2 y}{x^2 + y^2 + 4R^2}, \quad \tilde{z} = \frac{2R(x^2 + y^2)}{x^2 + y^2 + 4R^2}.$$

The stereographic projection is essentially a way to "flatten" the sphere onto a plane. It captures the geometry of the sphere in such a way that:

- The equator maps to the unit circle on the plane.
- The southern hemisphere maps to the interior of the circle, while the northern hemisphere maps to the exterior.

Theorem 5.6.1. *The stereographic projection of a sphere $\mathbb{S}^2$ of radius R centered at $(0,0,R)$ onto the xy-plane is a conformal mapping. Specifically:*

- *It preserves angles between intersecting curves on the sphere; that is, the angle between two curves at a point P on the sphere is equal to the angle between their projected images at P' on the plane.*
- *It is a bijective mapping between points on the sphere (excluding the north pole) and points on the plane.*
- *The first quadratic form of the plane is a scalar multiple of the first quadratic form of the sphere, ensuring conformality.*

Proof. We establish the relationship between the coordinates $(\tilde{x},\tilde{y},\tilde{z})$ of a point on the sphere and the coordinates (x,y) of its image in the plane.

1. **Projection Geometry**:
The projection is defined by the line connecting the north pole $N=(0,0,2R)$ to the point $P=(\tilde{x},\tilde{y},\tilde{z})$ on the sphere. By similar triangles, we have:

$$\frac{\tilde{x}}{x}=\frac{\tilde{y}}{y}=\frac{\tilde{z}-2R}{-2R}.$$

2. **Sphere Equation**: The point P lies on the sphere centered at $(0,0,R)$, so it satisfies:

$$\tilde{x}^2+\tilde{y}^2+(\tilde{z}-R)^2=R^2.$$

Substituting $\tilde{z}=2R-\frac{2R\tilde{z}}{\tilde{z}}$, the equation simplifies to:

$$\tilde{x}^2+\tilde{y}^2+(\tilde{z}-2R)\tilde{z}=0.$$

3. **Solving for $\tilde{x},\tilde{y},\tilde{z}$**: Combining the projection relationship and the sphere equation, we solve for $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ in terms of x and y:

$$\tilde{x}=\frac{4R^2x}{x^2+y^2+4R^2},\quad \tilde{y}=\frac{4R^2y}{x^2+y^2+4R^2},\quad \tilde{z}=\frac{2R(x^2+y^2)}{x^2+y^2+4R^2}.$$

4. **First Quadratic Forms**: The first quadratic form of the plane is:

$$ds^2_{\text{plane}}=dx^2+dy^2.$$

For the sphere, we compute the differentials $d\tilde{x}$, $d\tilde{y}$, and $d\tilde{z}$ and express the first quadratic form as:

$$ds^2_{\text{sphere}} = d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2 = \frac{16R^4(dx^2 + dy^2)}{(x^2 + y^2 + 4R^2)^2}.$$

5. **Conformality**: Since ds^2_{sphere} is a scalar multiple of ds^2_{plane}, the stereographic projection preserves angles between intersecting curves. Thus, it is a conformal mapping. □

Remark. The stereographic projection is particularly important in complex analysis, where the sphere $\mathbb{S}^2$ is identified with the **Riemann sphere**. Here, the projection serves as a correspondence between points in the plane $\mathbb{C}$ and points on the sphere, with the north pole representing the "point at infinity."

5.7 Isometric Surfaces

Definition (Isometric Surfaces)**.** Two surfaces Φ_1 and Φ_2 are said to be **isometric** if there exists a bijective (one-to-one and onto) map $f : \Phi_1 \to \Phi_2$ that preserves the length of all curves on the surface. Specifically, for any curve $\gamma : [a, b] \to \Phi_1$, the length of γ on Φ_1 is equal to the length of $f \circ \gamma$ on Φ_2.

- **Bijective Mapping:** The map f must be one-to-one and onto, ensuring a perfect correspondence between the two surfaces.
- **Length Preservation:** The mapping must preserve the intrinsic geometry, meaning the length of all corresponding curves is identical. This implies that the first fundamental form (metric tensor) of the two surfaces is the same under the mapping.

Isometric surfaces have the same **intrinsic geometry**, even if their **extrinsic geometry** (how they are embedded in space) might differ. For example, a flat plane and a cylinder are locally isometric, even though their shapes in space are different.

Lemma 5.7.1 (Isometric Mapping via Identical First Fundamental Forms)**.** *Let Φ_1 and Φ_2 be regular surfaces and that P_1 and P_2 are points on these surfaces; let $\boldsymbol{r} = \boldsymbol{r}_1(u, v)$, $\boldsymbol{r} = \boldsymbol{r}_2(u, v)$ be regular parametrizations of the surfaces in neighborhoods of the points P_1 and P_2.*

Suppose the first quadratic forms of the surfaces, corresponding to these parametrisations, are identical. Then a mapping of a neighbourhood of the point P_1 on the *surface Φ_1 onto a neighbourhood of the point P_2 on the surface Φ_2 in which points with the same coordinates u, v are set into correspondence, is isometric.*

Proof. If the curve γ_1 on the surface Φ_1 is defined by the equations $u = u(t)$, $v = v(t)$, then the curve on the surface Φ_2 which corresponds to it is defined by the same equations. Using the formula for arc length, we obtain the same length. $\square$

Remark. Identical surfaces are, obviously, isometric. The converse is not true in general.

Example. The rectangular region

$$0 < x < \frac{\pi}{2}, \quad 0 < y < 1$$

in the xOy-plane is *isometric* to the region on the cylinder

$$x^2 + y^2 = 1,$$

defined by the conditions

$$0 < z < 1, \quad x > 0, \quad y > 0.$$

The region on the cylinder indicated permits the parametrisation

$$x = \cos u, \quad y = \sin u, \quad z = v, \quad 0 < u < \pi/2, \quad 0 < v < 1.$$

A linear element on the cylinder, corresponding to such a parametrisation, is

$$du^2 + dv^2$$

.

Lemma 5.7.2 (Parametrization Under Isometry)**.** *Let Φ_1 and Φ_2 be regular isometric surfaces, and let $P_1 \in \Phi_1$ be an arbitrary point. Suppose $\mathbf{r}_1(u, v)$ is a regular parametrization of Φ_1 in a neighborhood of P_1.*

Then there exists a regular parametrization $\mathbf{r}_2(u, v)$ of Φ_2 in a neighborhood of the corresponding point $P_2 \in \Phi_2$ (under the isometry) such that:

1. *The points on Φ_1 and Φ_2 under the isometry have the same coordinates (u, v).*

2. *The first fundamental forms of Φ_1 and Φ_2 corresponding to these parametrizations are identical.*

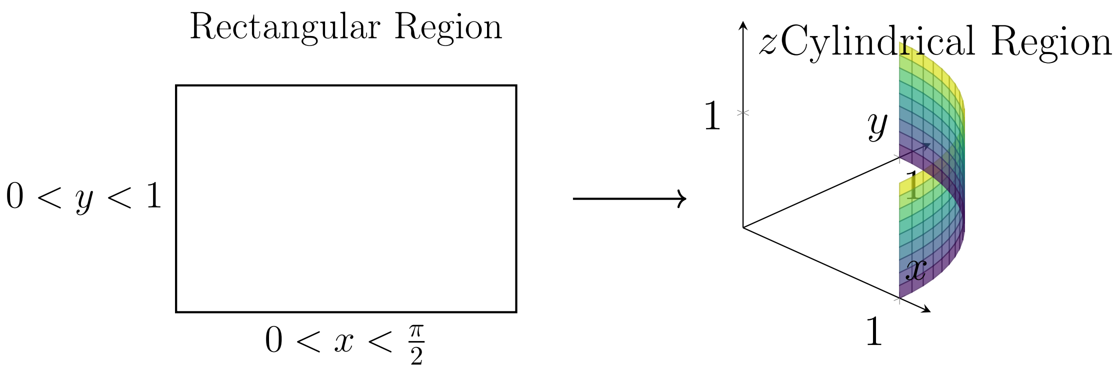


Figure 5.5. Illustration of the isometric mapping between the rectangular region and the cylindrical region.

Proof. Since Φ_1 and Φ_2 are isometric, there exists a diffeomorphism $\phi : \Phi_1 \to \Phi_2$ that preserves arc length. Let $\mathbf{r}_1(u, v)$ be a regular parametrization of Φ_1 near P_1, and let $P_2 = \phi(P_1)$. Define the parametrization $\mathbf{r}_2(u, v)$ of Φ_2 near P_2 by $\mathbf{r}_2(u, v) = \phi(\mathbf{r}_1(u, v))$.

The mapping ϕ ensures that points with coordinates (u, v) on Φ_1 correspond to points with the same coordinates (u, v) on Φ_2. Since ϕ is a diffeomorphism, $\mathbf{r}_2(u, v)$ is a regular parametrization.

Let γ_1 be a curve on Φ_1 given by $u = u(t)$, $v = v(t)$. Under the isometry, the corresponding curve γ_2 on Φ_2 is given by the same equations. Since ϕ preserves arc length, we have:

$$\int_{t_0}^{t} \sqrt{E_1 u'^2 + 2F_1 u'v' + G_1 v'^2}\, dt = \int_{t_0}^{t} \sqrt{E_2 u'^2 + 2F_2 u'v' + G_2 v'^2}\, dt.$$

As this holds for all t and arbitrary u', v', it follows that $E_1 = E_2$, $F_1 = F_2$, and $G_1 = G_2$. Thus, the first fundamental forms of Φ_1 and Φ_2 are identical under these parametrizations. □

Theorem 5.7.3 (Isometric Mapping via First Fundamental Form)**.** *A necessary and sufficient condition for a neighborhood of a point P_1 on a regular surface Φ_1 to be mapped isometrically onto a neighborhood of a point P_2 on a regular surface Φ_2 is that there exist regular parametrizations of these neighborhoods such that the first fundamental forms of the surfaces, corresponding to these parametrizations, are identical. Specifically, the coefficients E, F, G of the first fundamental forms must satisfy:*

$$E_1 = E_2, \quad F_1 = F_2, \quad G_1 = G_2.$$

Proof. The theorem is a combination of the previous two lemmas (5.7.1, 5.7.2). □

Corollary 5.7.4. *Let Φ_1 and Φ_2 be isometric surfaces. Then:*

1. *Angles between corresponding curves on Φ_1 and Φ_2 are equal.*

2. *Corresponding regions on Φ_1 and Φ_2 have identical areas.*

This follows because angles and areas are determined by the first fundamental form, which is identical for isometric surfaces.

5.8 Remark on Bending of Surfaces

We shall say that a surface is defined "in small" by its first quadratic form if for any sufficiently small neighbourhood ω of the point P on an analytic surface, there exist surfaces isometric to ω and not coinciding with it.

A surface is defined uniquely "in the large" by the first quadratic form, if any regular surface Φ' which is isometric to Φ is congruent to Φ. For example, an arbitrary regular closed convex surface is uniquely defined by the first quadratic form.

Definition (Bending of a surface)**.** A **bending** of a surface is a continuous deformation of it under which lengths of curves on the surface remain invariant.

The bending of a surface can be illustrated graphically by bending a sheet of paper.

Since lengths of curves remain invariant under bending of a surface, and consequently at any given moment of bending, the surface is isometric to the initial surface, the first quadratic form, for the corresponding parametrisation, remains invariant under bending.

Proposition. *The surface is always bendable "in the small".*

At every point of an analytic surface which is not an umbilical point, there exists a neighbourhood permitting a continuous bending.

Proposition. *There exist surfaces "in the large" which do not permit continuous bending. For example, all closed convex surfaces are of this sort.*

5.9 Problems Corner

Problem 1

Find the first quadratic form for the surface of revolution

$$x = f(u)\cos v, \quad y = f(u)\sin v, \quad z = g(u).$$

Show that a surface of revolution can be parametrised in such a way that its first quadratic form will have the form

$$I = du^2 + G(u)dv^2$$

Solution

1. **Parametric Surface**: The surface of revolution is given by

$$\mathbf{r}(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big).$$

2. **Compute Partial Derivatives**:

$$\mathbf{r}_u = \big(f'(u)\cos v, f'(u)\sin v, g'(u)\big), \quad \mathbf{r}_v = \big(-f(u)\sin v, f(u)\cos v, 0\big).$$

3. **Coefficients of the First Fundamental Form**: The first fundamental form is given by

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v.$$

Substituting:

$$E = (f'(u))^2 + (g'(u))^2, \quad F = 0, \quad G = f(u)^2.$$

4. **First Quadratic Form**:

$$I = \big((f'(u))^2 + (g'(u))^2\big)\,du^2 + f(u)^2\,dv^2.$$

5. **Special Parametrization**: For the first quadratic form to have the form $I = du^2 + G(u)dv^2$, choose a parametrization such that $(f'(u))^2 + (g'(u))^2 = 1$. This can always be achieved by reparametrizing u using arc length.

Problem 2

Find the arc length of a curve defined by the equation $u = v$ on the surface with

$$I = du^2 + \sinh^2 u dv^2$$

as its first quadratic form.

Solution

1. **Curve Parametrization**: The curve is given by $u = v$. Let v be the parameter, so $u(v) = v$.

2. **First Fundamental Form**: Substitute $u = v$ and $du = dv$ into the first quadratic form:

$$I = du^2 + \sinh^2 u\, dv^2 = dv^2 + \sinh^2 v\, dv^2 = (1 + \sinh^2 v)\, dv^2.$$

Using the identity $\cosh^2 v - \sinh^2 v = 1$, we get $1 + \sinh^2 v = \cosh^2 v$.

3. **Arc Length**: The arc length s is given by:

$$s = \int \sqrt{I}\, dv = \int \sqrt{\cosh^2 v}\, dv = \int \cosh v\, dv.$$

Integrating:

$$s = \sinh v + C.$$

For a specific interval, evaluate at the limits.

The arc length of the curve is

$$s = \sinh v + C.$$

For a specific interval $v \in [a, b]$, the length is $\sinh b - \sinh a$.

Problem 3

Show that the coordinate net u, v on the helicoid

$$x = au\cos v, \quad y = au\sin v, \quad z = bv$$

is orthogonal

Solution

1. **Parametric Surface**: The helicoid is given by

$$\mathbf{r}(u, v) = \big(au\cos v, au\sin v, bv\big).$$

2. **Compute Partial Derivatives**:

$$\mathbf{r}_u = \big(a\cos v, a\sin v, 0\big), \quad \mathbf{r}_v = \big(-au\sin v, au\cos v, b\big).$$

3. **Dot Product**: The coordinate net is orthogonal if $\mathbf{r}_u \cdot \mathbf{r}_v = 0$:

$$\mathbf{r}_u \cdot \mathbf{r}_v = (a\cos v)(-au\sin v) + (a\sin v)(au\cos v) + (0)(b) =$$
$$= -a^2u\cos v\sin v + a^2u\cos v\sin v = 0.$$

The coordinate net u, v on the helicoid is orthogonal.

Problem 4

Find curves on a sphere which intersect the meridians of the sphere at a constant angle (i.e. the loxodromes).

Solution

1. **Spherical Coordinates**: A sphere of radius R can be parameterized using spherical coordinates:

$$x = R\sin\theta\cos\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\theta,$$

where θ is the polar angle and ϕ is the azimuthal angle.

2. **Loxodrome Condition**: A loxodrome is a curve that intersects the meridians (lines of constant ϕ) at a constant angle α. The tangent vector of the loxodrome must satisfy the condition:

$$\frac{d\theta}{d\phi} = \cot\alpha\csc\theta.$$

3. **Differential Equation**: The differential equation for the loxodrome is:

$$\frac{d\theta}{d\phi} = \cot\alpha\csc\theta.$$

This can be solved by separation of variables:

$$\int \sin\theta\, d\theta = \cot\alpha \int d\phi.$$

Integrating both sides:

$$-\cos\theta = \cot\alpha\phi + C,$$

where C is the constant of integration.

4. **Solution**: The equation of the loxodrome is:

$$\cos\theta = -\cot\alpha\phi + C.$$

This describes a curve on the sphere that intersects the meridians at a constant angle α.

Problem 5

Find the area of the quadrilateral on the helicoid

$$x = au\cos v, \quad y = au\sin v, \quad z = bv$$

bounded by the curves

$$u = 0, \quad u = \frac{b}{a}, \quad v = 0, \quad v = 1.$$

Solution
1. **Parametric Surface**: The helicoid is given by:

$$\mathbf{r}(u, v) = (au \cos v, au \sin v, bv).$$

2. **First Fundamental Form**: Compute the partial derivatives:

$$\mathbf{r}_u = (a \cos v, a \sin v, 0), \quad \mathbf{r}_v = (-au \sin v, au \cos v, b).$$

The coefficients of the first fundamental form are:

$$E = a^2, \quad F = 0, \quad G = a^2u^2 + b^2.$$

The first fundamental form is:

$$I = a^2du^2 + (a^2u^2 + b^2)dv^2.$$

3. **Area Calculation**: The area A of the region bounded by $u = 0$, $u = \frac{b}{a}$, $v = 0$, and $v = 1$ is given by:

$$A = \int_{v=0}^{1} \int_{u=0}^{\frac{b}{a}} \sqrt{EG - F^2}\, du\, dv.$$

Substituting the values of E, F, and G:

$$A = \int_{v=0}^{1} \int_{u=0}^{\frac{b}{a}} \sqrt{a^2(a^2u^2 + b^2)}\, du\, dv = a \int_{v=0}^{1} \int_{u=0}^{\frac{b}{a}} \sqrt{a^2u^2 + b^2}\, du\, dv.$$

The inner integral is:

$$\int_{u=0}^{\frac{b}{a}} \sqrt{a^2u^2 + b^2}\, du = \frac{b^2}{2a} \left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right) \right).$$

The outer integral is:

$$A = a \cdot \frac{b^2}{2a} \left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right) \right) \cdot 1 = \frac{b^2}{2} \left(\sqrt{2} + \ln\left(1 + \sqrt{2}\right) \right).$$

Problem 6

Show that if a surface permits a parametrisation for which the coefficients of the first quadratic form do not depend on u and v, then this surface is locally isometric to a plane.

Solution

1. **First Fundamental Form**: The first fundamental form of a surface is given by

$$I = E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

where E, F, and G are coefficients that depend on the parametrisation of the surface.

2. **Coefficients Independent of u and v**: If E, F, and G do not depend on u and v, they are constants. Hence, the first fundamental form becomes

$$I = E_0\,du^2 + 2F_0\,du\,dv + G_0\,dv^2,$$

where E_0, F_0, and G_0 are constants.

3. **Comparison to the Plane**: The first fundamental form of a plane is

$$I = du^2 + dv^2.$$

This is a special case of the first fundamental form with $E_0 = 1$, $F_0 = 0$, and $G_0 = 1$.

4. **Local Isometry**: Two surfaces are locally isometric if there exists a mapping between them that preserves the first fundamental form. If the first fundamental form of a surface has constant coefficients, it can be transformed into the first fundamental form of a plane by a suitable coordinate transformation.

5. **Conclusion**: If a surface has a parametrisation for which the coefficients E, F, and G of the first fundamental form are constant, then the surface is locally isometric to a plane.

Problem 7

Prove that it is impossible to map a sphere locally onto a plane.

Solution

1. **First Fundamental Form of a Sphere**: The first fundamental form of a sphere of radius R in spherical coordinates (θ, ϕ) is

$$I = ds^2 = R^2\,d\theta^2 + R^2 \sin^2\theta\,d\phi^2.$$

Here, $E = R^2$, $F = 0$, and $G = R^2 \sin^2\theta$.

2. **First Fundamental Form of a Plane**: The first fundamental form of a plane in Cartesian coordinates (u, v) is

$$I = ds^2 = du^2 + dv^2.$$

Here, $E = 1$, $F = 0$, and $G = 1$.

3. **Local Isometry Condition**: For a local isometry to exist, there must be a coordinate transformation $(\theta, \phi) \to (u, v)$ such that

$$R^2\,d\theta^2 + R^2 \sin^2\theta\,d\phi^2 = du^2 + dv^2.$$

4. **Contradiction in Scaling**: The coefficients of the sphere's first fundamental form (R^2 and $R^2 \sin^2\theta$) depend on R and θ, while the plane's first fundamental form has constant coefficients (1, 0, and 1). No coordinate transformation can eliminate this dependence.

5. **Conclusion**: Since the first fundamental form of the sphere cannot be made identical to that of the plane, it is impossible to map the sphere locally onto the plane in a distance-preserving (isometric) way.

Definition. A mapping of one surface onto another is said to be equiareal if regions which correspond under this mapping have the same areas.

Problem 8

Prove that if a mapping of one surface onto another is conformal and equiareal, then it is isometric.

Solution

1. **Definitions**:

- A mapping is **conformal** if it preserves angles.
- A mapping is **equiareal** if it preserves area.
- A mapping is **isometric** if it preserves distances.

2. **Conformal Mapping**: For a mapping to be conformal, the first fundamental form of the first surface must be proportional to the first fundamental form of the second surface. That is, there exists a positive scalar function $\lambda(u, v)$ such that:

$$E_1 = \lambda E_2, \quad F_1 = \lambda F_2, \quad G_1 = \lambda G_2,$$

where E_1, F_1, G_1 are the coefficients of the first fundamental form of the first surface, and E_2, F_2, G_2 are the coefficients of the first fundamental form of the second surface.

3. **Equiareal Mapping**: For a mapping to be equiareal, the area of any region on the first surface must be equal to the area of its image on the second surface. This implies that the determinant of the first fundamental form of the first surface is equal to the determinant of the first fundamental form of the second surface:

$$\sqrt{E_1G_1 - F_1^2} = \sqrt{E_2G_2 - F_2^2}.$$

4. **Combining Conformal and Equiareal Conditions**: From the conformal condition, we have:

$$E_1 = \lambda E_2, \quad F_1 = \lambda F_2, \quad G_1 = \lambda G_2.$$

Substituting these into the equiareal condition:

$$\sqrt{(\lambda E_2)(\lambda G_2) - (\lambda F_2)^2} = \sqrt{E_2G_2 - F_2^2}.$$

Simplifying:

$$\lambda\sqrt{E_2G_2 - F_2^2} = \sqrt{E_2G_2 - F_2^2}.$$

This implies that $\lambda = 1$.

5. **Isometric Mapping**: If $\lambda = 1$, then the first fundamental forms of the two surfaces are identical:

$$E_1 = E_2, \quad F_1 = F_2, \quad G_1 = G_2.$$

This means that the mapping preserves distances, and hence it is isometric.

6. **Conclusion**: If a mapping is both conformal and equiareal, it must be isometric because the conditions of conformality and equiareality together imply that the first fundamental forms of the two surfaces are identical, which is the definition of an isometric mapping.

Chapter 6

The Second Quadratic Form of a Surface

Let Φ be a regular surface, $\boldsymbol{r} = \boldsymbol{r}(u, v)$ a regular parametrization of Φ, and $\boldsymbol{n}(u, v)$ the unit normal vector to the surface at the point $P(u, v)$. The **second quadratic form** (or **second fundamental form**) of the surface is a quadratic form defined as

$$II = -d\boldsymbol{r} \cdot d\boldsymbol{n},$$

which can be expressed in terms of the parameters u and v as

$$II = L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

where the coefficients L, M, and N are given by

$$L = -\boldsymbol{r}_u \cdot \boldsymbol{n}_u, \quad M = -\frac{1}{2}(\boldsymbol{r}_u \cdot \boldsymbol{n}_v + \boldsymbol{r}_v \cdot \boldsymbol{n}_u), \quad N = -\boldsymbol{r}_v \cdot \boldsymbol{n}_v.$$

6.1 Coefficients of the Second Quadratic Form

Since $d\boldsymbol{r} \cdot \boldsymbol{n} = 0$ and differentiating this equation yields

$$d(d\boldsymbol{r} \cdot \boldsymbol{n}) = d^2\boldsymbol{r} \cdot \boldsymbol{n} + d\boldsymbol{r} \cdot d\boldsymbol{n} = 0,$$

it follows that

$$II = d^2\boldsymbol{r} \cdot \boldsymbol{n}.$$

Expanding this expression, we obtain

$$II = (\boldsymbol{r}_{uu} \cdot \boldsymbol{n})\,du^2 + 2(\boldsymbol{r}_{uv} \cdot \boldsymbol{n})\,du\,dv + (\boldsymbol{r}_{vv} \cdot \boldsymbol{n})\,dv^2.$$

Thus, the coefficients can also be written as

$$L = \boldsymbol{r}_{uu} \cdot \boldsymbol{n}, \quad M = \boldsymbol{r}_{uv} \cdot \boldsymbol{n}, \quad N = \boldsymbol{r}_{vv} \cdot \boldsymbol{n}.$$

Explicit Formulas Using Determinants

The unit normal vector $\boldsymbol{n}$ is given by

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|},$$

where $|\boldsymbol{r}_u \times \boldsymbol{r}_v| = \sqrt{EG - F^2}$ and E, F, G are the coefficients of the first fundamental form. Using the scalar triple product, the coefficients L, M, and N can be expressed as determinants:

$$L = \frac{(\boldsymbol{r}_{uu}\, \boldsymbol{r}_u\, \boldsymbol{r}_v)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{\begin{vmatrix} x_{uu} & y_{uu} & z_{uu} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}}{\sqrt{EG - F^2}},$$

$$M = \frac{(\boldsymbol{r}_{uv}\, \boldsymbol{r}_u\, \boldsymbol{r}_v)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{\begin{vmatrix} x_{uv} & y_{uv} & z_{uv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}}{\sqrt{EG - F^2}},$$

$$N = \frac{(\boldsymbol{r}_{vv}\, \boldsymbol{r}_u\, \boldsymbol{r}_v)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|} = \frac{\begin{vmatrix} x_{vv} & y_{vv} & z_{vv} \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix}}{\sqrt{EG - F^2}}.$$

Example. For a surface defined explicitly by $z = z(x, y)$, the coefficients of the second fundamental form are given by

$$L = \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}}, \quad M = \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}}, \quad N = \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}}.$$

Second Quadratic Firm and Osculating Paraboloid

The equation of the osculating paraboloid, which approximates the surface near a point up to second order, is given by

$$z = \frac{1}{2}(Lx^2 + 2Mxy + Ny^2).$$

The shape of this paraboloid determines the local behaviour of the surface.

The second fundamental form measures the rate of change of the normal vector as one moves along the surface, providing information about the local curvature. For example:

- If $LN - M^2 > 0$, the point is **elliptic**.
- If $LN - M^2 < 0$, the point is **hyperbolic**.
- If $LN - M^2 = 0$, the point is **parabolic** or **umbilic**.

6.2 Curvature of a Curve Lying on a Surface

Let Φ be a regular surface, and let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ be a regular parametrization of Φ. Let γ be a regular curve on the surface that passes through the point $P(u, v)$ and has the direction $(du : dv)$ at this point. Let $\boldsymbol{r} = \boldsymbol{r}(s)$ be the natural parametrization of the curve γ.

The vector $\boldsymbol{r}''$ is directed along the principal normal to the curve, and its magnitude equals the curvature of the curve. From this, it follows that

$$\boldsymbol{r}'' \cdot \boldsymbol{n} = k \cos\theta,$$

where k is the curve's curvature, and θ is the angle between the principal normal to the curve and the normal to the surface.

Expanding the dot product, we obtain

$$\boldsymbol{r}'' \cdot \boldsymbol{n} = (\boldsymbol{r}_{uu} \cdot \boldsymbol{n})u'^2 + 2(\boldsymbol{r}_{uv} \cdot \boldsymbol{n})u'v' + (\boldsymbol{r}_{vv} \cdot \boldsymbol{n})v'^2.$$

Therefore,

$$k \cos\theta = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2} = \frac{II}{I},$$

where L, M, N are the coefficients of the second fundamental form, and E, F, G are the coefficients of the first fundamental form.

The right-hand side of this equation depends only on the direction of the curve at the point $P(u, v)$. Thus,

$$k \cos\theta = k_0 = \text{constant}$$

at the point $P(u, v)$ for all curves γ that pass through this point and have the same direction at P (i.e., the same tangent).

Definition (Normal Curvature)**.** The quantity k_0 is called the **normal curvature** of the surface in the given direction $(du : dv)$. Within a sign, it is equal to the curvature of the curve obtained by intersecting the surface with a plane perpendicular to the tangent plane and having the direction $(du : dv)$.

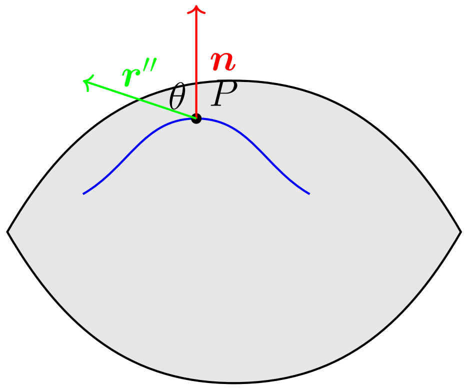


Figure 6.1. Angle between normal to surface and principal normal to curve lying on surface

Theorem 6.2.1. *The normal curvature of a surface Φ at a point $P(u,v)$ in the direction $(du : dv)$ is equal to the normal curvature of the osculating paraboloid to Φ at P in the same direction.*

Proof. The osculating paraboloid is a second-order approximation of the surface Φ at the point P. It can be parametrised as

$$\boldsymbol{r}(u,v) = (u-u_0)\boldsymbol{r}_u+(v-v_0)\boldsymbol{r}_v+\frac{1}{2}\left[L(u-u_0)^2+2M(u-u_0)(v-v_0)+N(v-v_0)^2\right]\boldsymbol{n},$$

where $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$ are the partial derivatives of the surface parametrization at P, $\boldsymbol{n}$ is the unit normal vector to the surface at P, and L, M, N are the coefficients of the second fundamental form of Φ at P.

To show that the normal curvatures coincide, we observe that:

1. The first fundamental form of the osculating paraboloid at P coincides with that of Φ because the paraboloid is tangent to the surface at P.

2. The second fundamental form of the osculating paraboloid at P also coincides with that of Φ because the paraboloid approximates the surface up to second order.

Since the normal curvature k_0 in a given direction is determined by the ratio of the second fundamental form to the first fundamental form, it follows that the normal curvature of Φ at P in the direction $(du : dv)$ is equal to the normal curvature of the osculating paraboloid at P in the same direction. □

Definition (Indicatrix of Curvature (Dupin Indicatrix))**.** Let $P(u,v)$ be a point on a surface, and let k_0 be the normal curvature of the surface at P in a direction $(du : dv)$. The **indicatrix of curvature** is the geometric locus of points obtained by drawing, from P in every direction $(du : dv)$, a segment of length $|1/k_0|^{1/2}$.

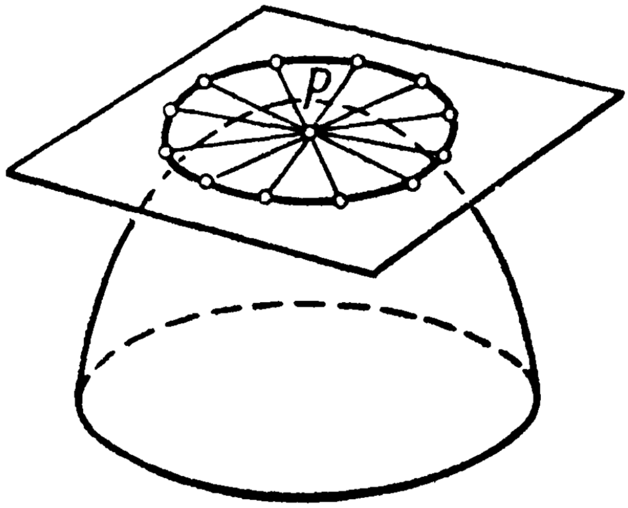


Figure 6.2. Indicatrix of Curvature

Theorem 6.2.2. *At a point P on a surface, the indicatrix of curvature is:*

- *An ellipse if P is an elliptic point ($LN - M^2 > 0$),*
- *A pair of conjugate hyperbolas if P is a hyperbolic point ($LN - M^2 < 0$),*
- *A pair of parallel straight lines if P is a parabolic point ($LN - M^2 = 0$).*

Proof. Let P be a point on the surface, and let the tangent plane to the surface at P be equipped with Cartesian coordinates. Take P as the origin, and let the coordinate axes align with the directions of the basis vectors $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$. Let (x, y) be the coordinates of a point on the indicatrix of curvature corresponding to the direction $(du : dv)$.

By definition, the vector from P to the point on the indicatrix is given by

$$x\boldsymbol{r}_u + y\boldsymbol{r}_v = \left|\frac{1}{k_0}\right|^{\frac{1}{2}} \cdot \frac{\boldsymbol{r}_u du + \boldsymbol{r}_v dv}{|\boldsymbol{r}_u du + \boldsymbol{r}_v dv|}.$$

Squaring both sides of this equation and noting that $x : y = du : dv$, we obtain

$$Ex^2 + 2Fxy + Gy^2 = \frac{Edu^2 + 2Fdudv + Gdv^2}{|Ldu^2 + 2Mdudv + Ndv^2|},$$

where E, F, G are the coefficients of the first fundamental form, and L, M, N are the coefficients of the second fundamental form. Since $x : y = du : dv$, we can rewrite this as

$$Ex^2 + 2Fxy + Gy^2 = \frac{Ex^2 + 2Fxy + Gy^2}{|Lx^2 + 2Mxy + Ny^2|}.$$

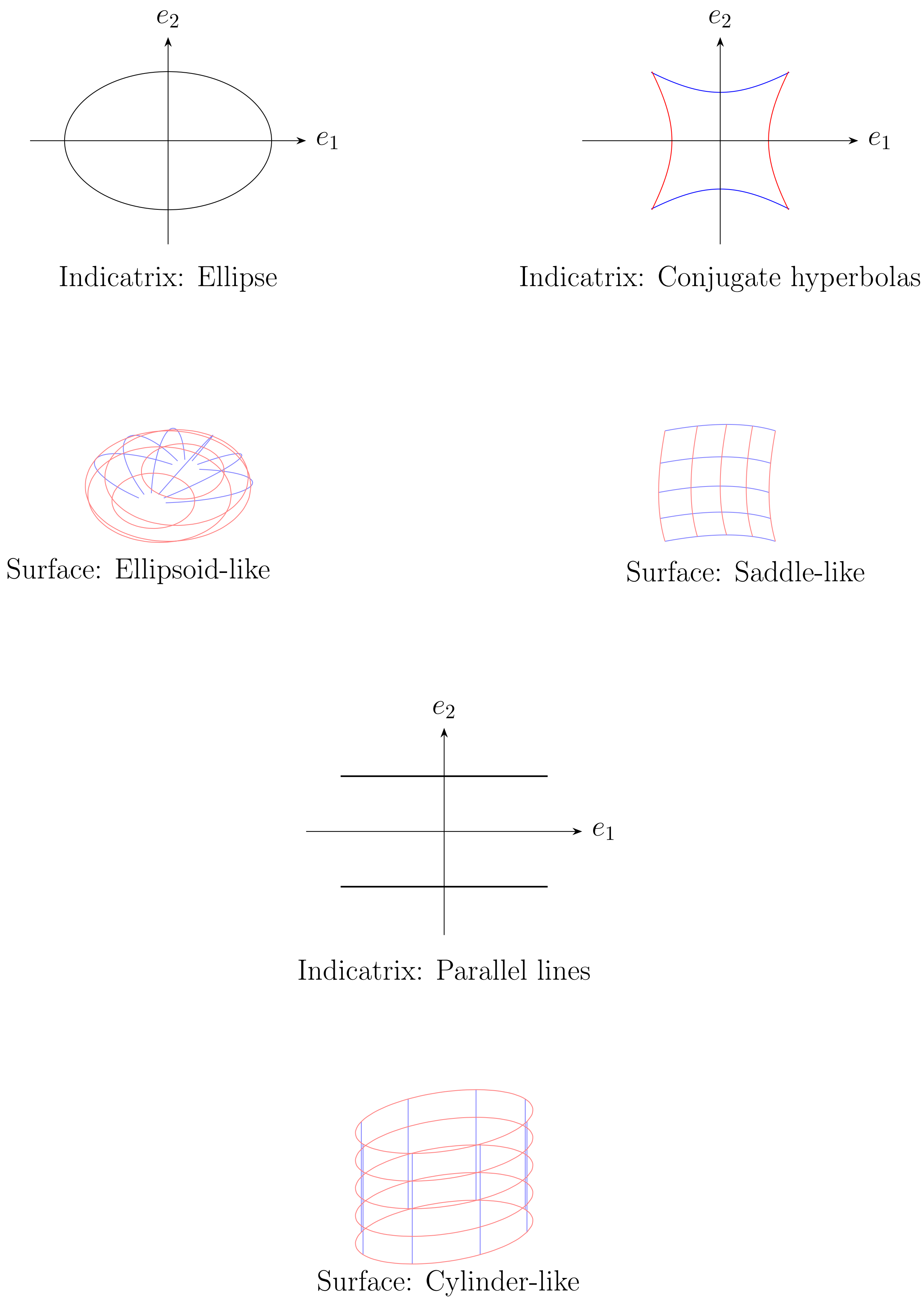


Figure 6.3. Indicatrcies of curvature and corresponding local surfaces

Dividing both sides by $Ex^2 + 2Fxy + Gy^2$ (which is nonzero for non-degenerate directions), we obtain

$$|Lx^2 + 2Mxy + Ny^2| = 1.$$

This is the equation of the indicatrix of curvature. The shape of the indicatrix depends on the sign of $LN - M^2$:

- If $LN - M^2 > 0$, the equation represents an ellipse.
- If $LN - M^2 < 0$, the equation represents a pair of conjugate hyperbolas.
- If $LN - M^2 = 0$, the equation represents a pair of parallel straight lines.

□

Corollary 6.2.3. *Let Φ be a surface, and let $P(u, v)$ be a point on Φ. Let U be the osculating paraboloid to Φ at P. Then, the indicatrix of curvature of Φ at P coincides with the indicatrix of curvature of U at P.*

6.3 Asymptotic Behaviour of Surfaces

Definition (Asymptotic Direction)**.** A direction $(du : dv)$ on a regular surface at the point $P(u, v)$ is called an **asymptotic direction** if the normal curvature of the surface in this direction vanishes. Mathematically, a direction $(du : dv)$ is asymptotic if and only if the following condition holds:

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0,$$

where L, M, and N are the coefficients of the second fundamental form of the surface.

Remark. Let us observe various types of points on the surface.

- At an **elliptic point**, there are no asymptotic directions because the second fundamental form is definite (either positive or negative).
- At a **hyperbolic point**, there are two distinct asymptotic directions because the second, fundamental form is indefinite.
- At a **parabolic point**, there is exactly one asymptotic direction because the second fundamental form is semi-definite.
- At an **umbilical point**, every direction is asymptotic because the second fundamental form is proportional to the first fundamental form.

Definition (Asymptotic Curve)**.** A curve on a surface is called an **asymptotic curve** if its tangent direction at every point is an asymptotic direction. The differential equation of the asymptotic curves is given by:

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0.$$

Example. If a straight line lies on a surface, it is an asymptotic curve because its tangent direction satisfies the condition for being an asymptotic direction.

Theorem 6.3.1. *If γ is an asymptotic curve on a surface S, then the osculating plane of γ is tangent to the surface at every point on γ.*

Proof. Let γ be an asymptotic curve on the surface S, and let P be any point on γ. Let $\boldsymbol{r}(s)$ be the arc-length parametrization of γ, where $\boldsymbol{r}(0) = P$.

The osculating plane of γ at P is spanned by the tangent vector $\boldsymbol{\tau} = \boldsymbol{r}'(0)$ and the principal normal vector $\boldsymbol{n}_c = \boldsymbol{r}''(0)/\|\boldsymbol{r}''(0)\|$.

Since γ is an asymptotic curve, its normal curvature vanishes. This implies that the acceleration vector $\boldsymbol{r}''(0)$ lies in the tangent plane of S at P:

$$\boldsymbol{r}''(0) \cdot \boldsymbol{n} = 0,$$

where $\boldsymbol{n}$ is the unit normal vector to the surface S at P.

Since both $\boldsymbol{\tau}$ and $\boldsymbol{r}''(0)$ (and thus $\boldsymbol{n}_c$) lie in the tangent plane of S at P, the osculating plane (spanned by $\boldsymbol{\tau}$ and $\boldsymbol{n}_c$) must coincide with the tangent plane of S at P.

□

Theorem 6.3.2. *The coordinate curves u = constant and v = constant are asymptotic if and only if the coefficients L and N in the second fundamental form vanish. Such a parametrisation, where the coordinate curves are asymptotic, is always possible in the neighbourhood of a hyperbolic point on the surface.*

Proof. **Part 1**: Coordinate curves are asymptotic if and only if $L = 0$ and $N = 0$.

Suppose u = constant and v = constant are asymptotic curves. For the curve u = constant, the tangent direction is $(du : dv) = (0 : 1)$. For this direction, the condition for being asymptotic is:

$$L(0)^2 + 2M(0)(1) + N(1)^2 = N = 0.$$

Similarly, for the curve v = constant, the tangent direction is $(du : dv) = (1 : 0)$. For this direction, the condition for being asymptotic is:

$$L(1)^2 + 2M(1)(0) + N(0)^2 = L = 0.$$

Thus, $L = 0$ and $N = 0$ are necessary and sufficient conditions for the coordinate curves to be asymptotic.

Part 2: Existence of such a parametrisation at a hyperbolic point.

At a hyperbolic point, $K = LN - M^2$ is negative, and the second fundamental form is indefinite. This implies that there exist two distinct asymptotic directions at the point.

Let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ be a parametrization of the surface. By the properties of hyperbolic points, we can choose new coordinates (u', v') such that the asymptotic directions align with the coordinate curves $u' =$ constant and $v' =$ constant. In this new parametrisation, the second fundamental form becomes:

$$L' du'^2 + 2M' du' dv' + N' dv'^2,$$

and by Part 1, $L' = 0$ and $N' = 0$ for the coordinate curves to be asymptotic.

Such a parametrisation is always possible in the neighbourhood of a hyperbolic point because the asymptotic directions are well-defined and vary smoothly in the neighbourhood.

□

Definition (Conjugate Directions)**.** Two directions $(du : dv)$ and $(\delta u : \delta v)$ at a point P on a regular surface Φ are said to be **conjugate directions** if the straight lines g' and g'' passing through P in these directions are polar conjugate concerning the osculating paraboloid of the surface at P.

Remark. The osculating paraboloid at P is given by:

$$z = \frac{1}{2}(Lx^2 + 2Mxy + Ny^2),$$

where L, M, and N are the coefficients of the second fundamental form. The condition for polar conjugacy is:

$$L\, du\, \delta u + M(du\, \delta v + dv\, \delta u) + N\, dv\, \delta v = 0.$$

This is both necessary and sufficient for the directions to be conjugate.

Definition (Conjugate Net)**.** A **conjugate net** on a surface is formed by two families of curves γ'_α and γ''_β such that:

1. At every point on the surface, exactly one curve from each family passes through it.

2. The tangent directions of the curves from different families are conjugate at every point.

Remark. If the coordinate net $u =$ constant and $v =$ constant forms a conjugate net, then the coefficient M of the second fundamental form vanishes.

Let (du, dv) be an asymptotic direction. By definition, the normal curvature in this direction is zero, which implies:

$$L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0.$$

This is exactly the condition for the direction to be self-conjugate. Therefore, every asymptotic direction is self-conjugate.

Theorem 6.3.3 (Existence of Conjugate Parametrization)**.** *In the neighbourhood of any point P that is not an umbilical point, a surface can be parametrised such that the coordinate curves form a conjugate net. One family of curves in this net can be chosen arbitrarily, provided that the curves in this family are not asymptotic.*

Proof. Let P be a non-umbilical point on the surface Φ. Since P is not an umbilical point, the principal curvatures at P are distinct, and the principal directions are well-defined and orthogonal.

Let γ_α be a family of curves on the surface passing through P such that the curves γ_α are not asymptotic. This means that the tangent vectors to γ_α are not asymptotic directions at P.

At each point in the neighbourhood of P, let $(du : dv)$ denote the direction of the chosen family of curves γ_α. We seek another family of curves γ_β such that their tangent directions $(\delta u : \delta v)$ are conjugate to $(du : dv)$. The condition for conjugacy is given by:

$$L\,du\,\delta u + M(du\,\delta v + dv\,\delta u) + N\,dv\,\delta v = 0,$$

where L, M, and N are the coefficients of the second fundamental form.

Since the curves γ_α are not asymptotic, the direction $(du : dv)$ satisfies $L\,du^2 + 2M\,du\,dv + N\,dv^2 \neq 0$. This ensures that the conjugacy condition is non-degenerate, and there exists a unique direction $(\delta u : \delta v)$ conjugate to $(du : dv)$.

Choose a new parametrisation of the surface in the neighbourhood of P such that:

- The curve $u =$ constant corresponds to the family γ_α.
- The curve $v =$ constant corresponds to the conjugate family γ_β.

In this parametrisation, the conjugacy condition simplifies to:

$$L\,du\,\delta u + M(du\,\delta v + dv\,\delta u) + N\,dv\,\delta v = 0.$$

For the coordinate direction $(du : 0)$ and $(0 : \delta v)$, this reduces to:

$$M\, du\, \delta v = 0.$$

Since du and δv are arbitrary, it follows that $M = 0$. Therefore, the second fundamental form in this parametrisation becomes:

$$L\, du^2 + N\, dv^2.$$

This shows that the coordinate curves form a conjugate net, and the theorem is proved. □

6.4 Principal Directions on a Surface

Definition (Principal Direction)**.** The direction $(du : dv)$ on a surface is called a **principal direction** if the normal curvature of the surface in this direction attains an extremal value.

Geometrically, this direction aligns with the axes of the *indicatrix of curvature.*
In general, there are two principal directions at each point on a surface. These directions are:

1. **Orthogonal**: They satisfy the orthogonality condition.
2. **Conjugate**: They satisfy the conjugacy condition.

Mathematically, these conditions are expressed as:

$$\begin{aligned} I(d,\delta) &= E\, du\, \delta u + F(du\, \delta v + dv\, \delta u) + G\, dv\, \delta v = 0 \quad \text{(orthogonality condition)},\\ II(d,\delta) &= L\, du\, \delta u + M(du\, \delta v + dv\, \delta u) + N\, dv\, \delta v = 0 \quad \text{(conjugacy condition)}. \end{aligned}$$

By eliminating δu and δv from these equations, we obtain the **necessary and sufficient condition** for a direction $(du : dv)$ to be a principal direction:

$$\begin{vmatrix} E\, dv + F\, dv & F\, du + G\, dv \\ L\, du + M\, dv & M\, du + N\, dv \end{vmatrix} = 0.$$

This condition can also be written in a more symmetric form:

$$\begin{vmatrix} dv^2 & -du\, dv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0.$$

Special Cases

Principal directions are not defined in two cases:

1. **Umbilical Point**: Here, $L = M = N = 0$, and any direction is a principal direction (the normal curvature vanishes in all directions).

2. **Spherical Point**: At such a point, the indicatrix of curvature is a circle, meaning the coefficients of the first fundamental form are proportional to those of the second fundamental form. As with umbilical points, any direction is a principal direction.

Definition (Principal Curvature)**.** The normal curvatures corresponding to the principal directions are called the **principal curvatures**.

These represent the maximum and minimum normal curvatures at a given point on the surface.

Theorem 6.4.1 (Rodrigues's Theorem)**.** *If the direction* (d) *is a principal direction, then* $d\boldsymbol{n} = -k\, d\boldsymbol{r}$*, where* k *is the normal curvature of the surface in this direction. Conversely, if* $d\boldsymbol{n} = \lambda\, d\boldsymbol{r}$ *in the direction* (d)*, then* (d) *is a principal direction.*

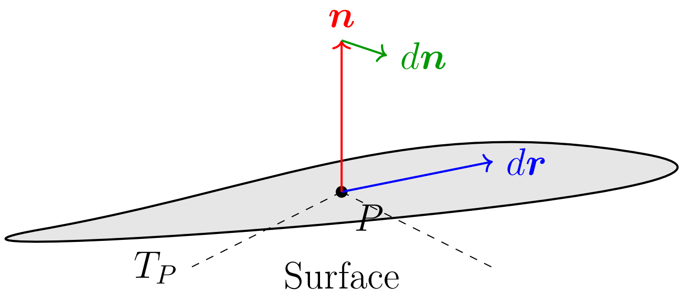


Figure 6.4. Illustration of $d\boldsymbol{r}$ and $d\boldsymbol{n}$ on a Surface.

The Figure 6.4 depicts a point P on a smooth surface. The blue arrow represents the differential

$$d\boldsymbol{r} = \boldsymbol{r}_u du + \boldsymbol{r}_v dv,$$

a small displacement in the tangent plane T_P. The red arrow represents the unit normal vector

$$d\boldsymbol{n} = \boldsymbol{n}_u du + \boldsymbol{n}_v dv$$

at point P, perpendicular to the tangent plane. The green arrow illustrates the differential $d\boldsymbol{n}$, showing how the normal vector changes as one moves in the direction of $d\boldsymbol{r}$. Rodrigues's Theorem connects $d\boldsymbol{r}$ and $d\boldsymbol{n}$ through the normal curvature of the surface.

Proof. We prove the direct and converse parts of the assertion.

Direct Part Suppose (d) is a principal direction. The vector $d\boldsymbol{n}$, being perpendicular to the normal vector $\boldsymbol{n}$, can be expressed as a linear combination of the tangent vectors $d\boldsymbol{r}$ and $\delta\boldsymbol{r}$, where $\delta\boldsymbol{r}$ is another principal direction perpendicular to $d\boldsymbol{r}$. Thus, we can write:

$$\boldsymbol{n} = \lambda\, d\boldsymbol{r} + \mu\, \delta\boldsymbol{r}.$$

Taking the scalar product of this equation with $d\boldsymbol{r}$ and using the fact that $d\boldsymbol{n}\cdot\delta\boldsymbol{r} = 0$ (due to the conjugacy of the directions (d) and (δ)) and $d\boldsymbol{r}\cdot\delta\boldsymbol{r} = 0$ (due to the orthogonality of these directions), we obtain:

$$\mu\, \delta r^2 = 0 \quad \Longrightarrow \quad \mu = 0.$$

Thus, the expression simplifies to:

$$d\boldsymbol{n} = \lambda\, d\boldsymbol{r}.$$

Taking the scalar product of this equation with $d\boldsymbol{r}$, we get:

$$d\boldsymbol{r}\cdot d\boldsymbol{n} = \lambda\, dr^2 \quad \Longrightarrow \quad \lambda = -k,$$

where k is the normal curvature. This completes the first part of the theorem.

Converse Part Suppose the direction (d) satisfies $d\boldsymbol{n} = \lambda\, d\boldsymbol{r}$. Let (δ) be a direction perpendicular to (d). Taking the scalar product of the equation $d\boldsymbol{n} = \lambda\, d\boldsymbol{r}$ with $\delta\boldsymbol{r}$, we get:

$$d\boldsymbol{n}\cdot\delta\boldsymbol{r} = 0.$$

This implies that the directions (d) and (δ) are conjugate. Since they are also orthogonal, they must be principal directions. This completes the proof.

□

6.5 Lines of Curvature

Definition (Line of Curvature)**.** A curve on a surface is called a **line of curvature** if its direction at every point aligns with a principal direction of the surface.

The **differential equation for lines of curvature** can be derived using the coefficients of the first and second fundamental forms of the surface. If the surface is parametrized by (u, v), the lines of curvature satisfy the condition:

$$(EM - FL)\, du^2 + (EN - GL)\, du\, dv + (FN - GM)\, dv^2 = 0,$$

or

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ E & F & G \\ L & M & N \end{vmatrix} = 0$$

where:

- E, F, G are the coefficients of the first fundamental form,
- L, M, N are the coefficients of the second fundamental form.

If the surface is parametrized in such a way that the coordinate curves $u =$ constant and $v =$ constant correspond to the lines of curvature, then the mixed term F of the first fundamental form and the mixed term M of the second fundamental form vanish. Specifically:

$$F = 0 \quad \text{and} \quad M = 0.$$

This simplification makes it easier to analyse the surface geometrically.

If the point P on the surface is neither spherical nor umbilical (i.e., not a point where the principal curvatures are equal), the surface can indeed be parametrised locally in a way that the coordinate curves align with the lines of curvature.

Theorem 6.5.1. *If two surfaces intersect along some curve γ under a constant angle and if this curve is a line of curvature on one of the surfaces, then it will also be the line of curvature on the other surface.*

Proof. Let γ be a curve of intersection between two surfaces S_1 and S_2, and let γ be a line of curvature on S_1. Let $\boldsymbol{r}$ be the position vector of points on γ, and let $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ be the unit normal vectors to S_1 and S_2, respectively.

Since γ is a line of curvature on S_1, the differential of the normal vector $d\boldsymbol{n}_1$ along γ satisfies:

$$d\boldsymbol{n}_1 = \lambda_1 d\boldsymbol{r},$$

where λ_1 is a scalar function describing the rate of change of $\boldsymbol{n}_1$ in the direction of γ.

For the surface S_2, the differential of the normal vector $d\boldsymbol{n}_2$ can be written as:

$$d\boldsymbol{n}_2 = \lambda_2 d\boldsymbol{r} + \mu\boldsymbol{n}_1 + \nu\boldsymbol{n}_2,$$

where λ_2 is a scalar function describing the rate of change of $\boldsymbol{n}_2$ in the direction of γ, and μ and ν are scalar functions.

Taking the scalar product of $d\boldsymbol{n}_2$ with $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, we obtain:

$$\boldsymbol{n}_1 \cdot d\boldsymbol{n}_2 = \mu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_1) + \nu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2),$$

$$\boldsymbol{n}_2 \cdot d\boldsymbol{n}_2 = \mu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) + \nu(\boldsymbol{n}_2 \cdot \boldsymbol{n}_2).$$

Since $\boldsymbol{n}_2$ is a unit normal vector, $\boldsymbol{n}_2 \cdot d\boldsymbol{n}_2 = 0$. Similarly, $\boldsymbol{n}_1 \cdot d\boldsymbol{n}_2$ can be rewritten using the product rule:

$$\boldsymbol{n}_1 \cdot d\boldsymbol{n}_2 = d(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) - \boldsymbol{n}_2 \cdot d\boldsymbol{n}_1 = 0 - \boldsymbol{n}_2 \cdot (\lambda_1 d\boldsymbol{r}) = -\lambda_1(\boldsymbol{n}_2 \cdot d\boldsymbol{r}).$$

Substituting these results into the earlier equations, we get:

$$\mu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_1) + \nu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) = -\lambda_1(\boldsymbol{n}_2 \cdot d\boldsymbol{r}),$$

$$\mu(\boldsymbol{n}_1 \cdot \boldsymbol{n}_2) + \nu(\boldsymbol{n}_2 \cdot \boldsymbol{n}_2) = 0.$$

If the surfaces are not tangent along γ, the determinant of the system is nonzero, and the only solution is $\mu = \nu = 0$. This implies:

$$d\boldsymbol{n}_2 = \lambda_2 d\boldsymbol{r},$$

which shows that γ is a line of curvature on S_2.

If the surfaces are tangent along γ, consider a new surface S_3 that intersects S_1 at a constant angle. Since γ is a line of curvature on S_1, it will also be a line of curvature on S_3. Because S_3 intersects S_2 at the same angle, γ must also be a line of curvature on S_2. □

Corollary 6.5.2. *If a sphere (or a plane) intersects any surface at a constant angle, then the intersection curve is a line of curvature.*

Proof. On a sphere (or on a plane), every curve is a line of curvature. By the theorem, the intersection curve must also be a line of curvature on the other surface. □

6.6 Mean and Gaussian Curvatures of a Surface

Let us prove two statements first.

Theorem 6.6.1 (Euler's Formula for Normal Curvature)**.** *Given a surface and a point on it, the normal curvature k_θ in any direction can be determined using the principal curvatures k_1 and k_2 of the surface and the angle θ between the chosen direction and a principal direction. The relationship is given by* ***Euler's formula****:*

$$k_\theta = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Proof. Let S be a surface and O a point on S. At O, we introduce a Cartesian coordinate system (x, y, z) such that:

- The tangent plane to S at O coincides with the x, y-plane.
- The normal vector to S at O aligns with the z-axis.
- The x- and y-axes align with the **principal directions** of S at O.

Let $z = z(x, y)$ be the local equation of S in a neighbourhood of O. At O, since the tangent plane is the x, y-plane, we have:

$$z_x = 0 \quad \text{and} \quad z_y = 0.$$

The first fundamental form I and the second fundamental form II at O are given by:

$$I = dx^2 + dy^2, \quad II = r\,dx^2 + 2s\,dx\,dy + t\,dy^2.$$

Here, r, s, and t are coefficients derived from the second derivatives of $z(x, y)$.

Since the x- and y-axes align with the principal directions at O, the directions $(dx : 0)$ and $(0 : dy)$ are conjugate. This implies that the mixed term in II vanishes, i.e., $s = 0$. Thus, the second fundamental form simplifies to:

$$II = r\,dx^2 + t\,dy^2.$$

The normal curvature k in a direction $(dx : dy)$ is given by the ratio of the second fundamental form to the first fundamental form:

$$k = \frac{r\,dx^2 + t\,dy^2}{dx^2 + dy^2}.$$

Evaluating k in the principal directions:

- Along the x-axis ($dy = 0$), the normal curvature is $k_1 = r$.
- Along the y-axis ($dx = 0$), the normal curvature is $k_2 = t$.

Euler's Formula:

Let θ be the angle between an arbitrary direction $(dx : dy)$ and the x-axis. Expressing dx and dy in terms of θ:

$$dx = \cos\theta, \quad dy = \sin\theta.$$

Substituting these into the expression for k:

$$k_\theta = \frac{r\cos^2\theta + t\sin^2\theta}{\cos^2\theta + \sin^2\theta}.$$

Since $\cos^2\theta + \sin^2\theta = 1$, we obtain Euler's formula:

$$k_\theta = k_1\cos^2\theta + k_2\sin^2\theta.$$

□

Euler's formula shows how the normal curvature k_θ varies with the direction θ relative to the principal directions. The principal curvatures k_1 and k_2 represent the maximum and minimum normal curvatures at the point O, and k_θ interpolates between these values based on the angle θ.

Theorem 6.6.2. *The principal curvatures k_1 and k_2 are the roots of the quadratic equation*

$$\begin{vmatrix} L-kE, & M-kF \\ M-kF, & N-kG \end{vmatrix} = 0, \tag{6.1}$$

where E, F, G and L, M, N are the coefficients of the first and second fundamental forms, respectively.

Proof. Let S be a surface, and let E, F, G and L, M, N be the coefficients of the first and second fundamental forms, respectively. The principal curvatures k_1 and k_2 are the maximum and minimum values of the normal curvature, which is given by the ratio of the second and first fundamental forms:

$$\frac{II}{I} = \frac{L\,dx^2 + 2M\,dx\,dy + N\,dy^2}{E\,dx^2 + 2F\,dx\,dy + G\,dy^2}.$$

Without loss of generality, assume $k_1 \geq k_2$. Then, k_1 is the maximum and k_2 is the minimum of the ratio $\frac{II}{I}$. Let $\overline{\xi}$ and $\overline{\eta}$ be the values of dx and dy that maximize this ratio. At these values, we have:

$$II - k_1 I \leq 0,$$

with equality holding for $dx = \overline{\xi}$ and $dy = \overline{\eta}$.

Taking the partial derivatives of $II - k_1 I$ with respect to dx and dy, we obtain the following system of equations:

$$\frac{\partial}{\partial dx}(II - k_1 I) = 0 \quad \Rightarrow \quad L\overline{\xi} + M\overline{\eta} - k_1(E\overline{\xi} + F\overline{\eta}) = 0,$$

$$\frac{\partial}{\partial dy}(II - k_1 I) = 0 \quad \Rightarrow \quad M\overline{\xi} + N\overline{\eta} - k_1(F\overline{\xi} + G\overline{\eta}) = 0.$$

Eliminating $\overline{\xi}$ and $\overline{\eta}$ from the above system, we obtain the determinant equation:

$$\begin{vmatrix} L-k_1E, & M-k_1F \\ M-k_1F, & N-k_1G \end{vmatrix} = 0.$$

The same reasoning applies to k_2, yielding the same quadratic equation. Thus, both principal curvatures k_1 and k_2 are roots of (6.1). □

Definition (Mean Curvature)**.** The **mean curvature** H of a surface is defined as half the sum of the principal curvatures k_1 and k_2:

$$H = \frac{1}{2}(k_1 + k_2).$$

The mean curvature has the following properties:

- **Half-Sum of Perpendicular Normal Curvatures:** If k_θ and $k_{\theta+\pi/2}$ are the normal curvatures of a surface in two mutually perpendicular directions, then the mean curvature H is half their sum:

  $$H = \frac{1}{2}(k_\theta + k_{\theta+\pi/2}).$$

  This follows directly from Euler's formula, which relates the normal curvature k_θ to the principal curvatures:

  $$k_\theta = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

- **Mean Value of Normal Curvatures:** The mean curvature H is also the average of the normal curvatures k_θ over all directions at a given point on the surface:

  $$H = \frac{1}{2\pi}\int_0^{2\pi} k_\theta \, d\theta.$$

  To verify this, substitute Euler's formula into the integral and evaluate:

  $$\begin{aligned} H &= \frac{1}{2\pi}\int_0^{2\pi} \big(k_1\cos^2\theta + k_2\sin^2\theta\big)\, d\theta \\ &= \frac{1}{2\pi}\left(k_1\int_0^{2\pi}\cos^2\theta\, d\theta + k_2\int_0^{2\pi}\sin^2\theta\, d\theta\right) \\ &= \frac{1}{2\pi}\left(k_1\cdot\pi + k_2\cdot\pi\right) \\ &= \frac{1}{2}(k_1 + k_2). \end{aligned}$$

  Thus, the mean curvature H is indeed the average of the normal curvatures over all directions.

Definition (Gaussian Curvature)**.** The **Gaussian curvature** K of a surface is defined as the product of the principal curvatures k_1 and k_2:

$$K = k_1 k_2.$$

The mean curvature H and Gaussian curvature K can be expressed in terms of the coefficients of the first and second fundamental forms (E, F, G and L, M, N, respectively):

$$H = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2},$$
$$K = \frac{LN - M^2}{EG - F^2}.$$

These expressions follow from the fact that k_1 and k_2 are the roots of the quadratic equation:

$$\begin{vmatrix} L - kE, & M - kF \\ M - kF, & N - kG \end{vmatrix} = 0.$$

The sign of the Gaussian curvature K classifies points on the surface:

- $K > 0$: Elliptic point,
- $K < 0$: Hyperbolic point,
- $K = 0$: Parabolic or umbilical point.

A point P on a surface is an umbilical point if and only if $K = 0$ and $H = 0$ at P.

Definition (Spherical Image (Gauss Map))**.** Let M be a set of points on a surface. Marking off unit normal vectors to the surface at each point in M defines a set M' on the unit sphere. This set M' is called the **spherical image** of M.

Theorem 6.6.3 (Gauss's Theorem)**.** *Let O be a point on a surface, and let G be a region on the surface containing O. As G shrinks to O, the ratio of the area of the spherical image of G to the area of G tends to the absolute value of the Gaussian curvature at O:*

$$\frac{A(G')}{A(G)} \to |K(O)|.$$

Proof. Parametrise the surface in a neighbourhood of O such that the coordinate curves align with the principal directions at O. Let $\boldsymbol{r}(u, v)$ represent the surface, and let $\boldsymbol{n}(u, v)$ be the unit normal vector. The spherical image G' of G is parametrized by:

$$\tilde{\boldsymbol{r}} = \boldsymbol{n}(u, v).$$

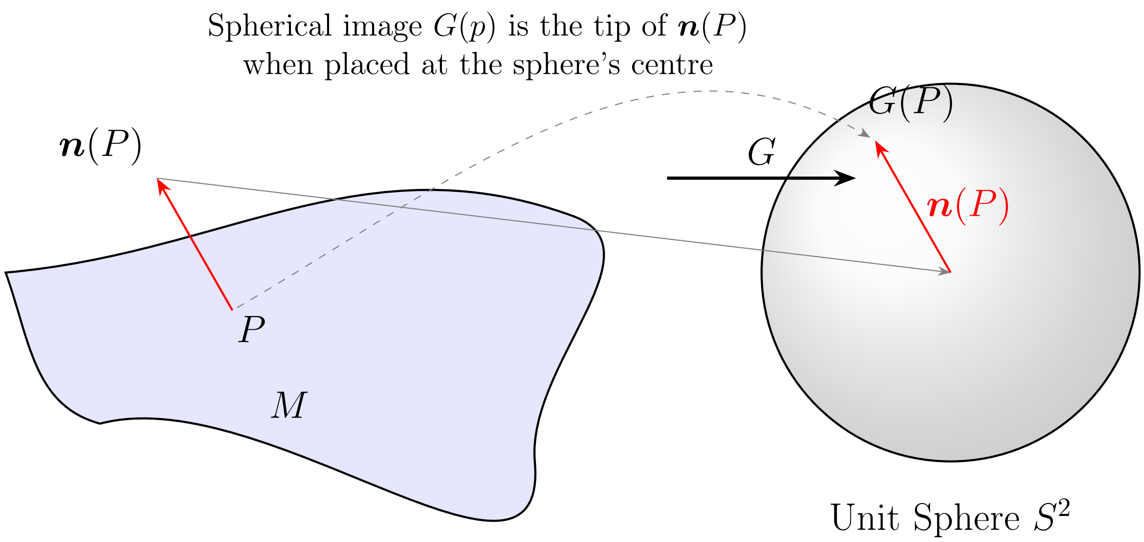


Figure 6.5. Spherical Image

The area of G' is given by:

$$A(G') = \iint_G |\boldsymbol{n}_u \times \boldsymbol{n}_v| \, du \, dv.$$

Since $\boldsymbol{n}_u = -k_1 \boldsymbol{r}_u$ and $\boldsymbol{n}_v = -k_2 \boldsymbol{r}_v$ at O, we have:

$$|\boldsymbol{n}_u \times \boldsymbol{n}_v| = |k_1 k_2||\boldsymbol{r}_u \times \boldsymbol{r}_v|.$$

As G shrinks to O, the ratio becomes:

$$\frac{A(G')}{A(G)} \to \frac{|\boldsymbol{n}_u \times \boldsymbol{n}_v|(O)}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|(O)} = |k_1 k_2| = |K(O)|.$$

□

6.7 Particular Case. Ruled Surfaces

Definition (Elementary Ruled Surface). A surface Φ is called an **elementary ruled surface** if, for every point P on the surface, there exists a straight line (generator) such that an open interval containing P lies entirely on the surface, but the endpoints of this interval do not belong to the surface.

Example. Suppose $\boldsymbol{a}(u)$ and $\boldsymbol{b}(u)$ are two vector functions defined in a neighborhood of $u = u_0$, satisfying $\boldsymbol{b}(u_0) \neq 0$ and $\boldsymbol{b}(u_0) \times \boldsymbol{a}'(u_0) \neq 0$. The vector equation

$$\boldsymbol{r} = \boldsymbol{a}(u) + v\boldsymbol{b}(u), \quad |u - u_0| < \varepsilon, \quad |v| < \varepsilon,$$

defines an elementary ruled surface for sufficiently small ε.

Explanation: At $u = u_0$ and $v = 0$, we have $\boldsymbol{r}_u \times \boldsymbol{r}_v = \boldsymbol{a}'(u_0) \times \boldsymbol{b}(u_0) \neq 0$, ensuring the surface is smooth and regular in this neighborhood. For any point (u', v') on the surface, the straight line $\boldsymbol{r} = \boldsymbol{a}(u') + t\boldsymbol{b}(u')$ lies entirely on the surface for $|t| < \varepsilon$, and its endpoints do not belong to the surface. Thus, the surface is an elementary ruled surface.

Definition (General Ruled Surface)**.** A surface is called a **general ruled surface** if every point on the surface has a neighbourhood that is an elementary ruled surface.

Definition (Rectilinear Generators)**.** Straight-line segments on a ruled surface are called **rectilinear generators**. Along these generators, the normal curvature vanishes, making them asymptotic curves. As a result, ruled surfaces cannot have elliptic points, and their Gaussian curvature is always non-positive.

Parametrisation of Ruled Surfaces

Theorem 6.7.1 (Parametrization of Ruled Surfaces)**.** *A ruled surface can be locally parametrised in the form*

$$\boldsymbol{r} = \boldsymbol{a}(u) + v\boldsymbol{b}(u)$$

in a neighbourhood of every point, provided one of the following conditions holds:

1. *The point is hyperbolic.*
2. *All points in a neighbourhood of the point are parabolic.*
3. *All points in a neighbourhood of the point are umbilical.*

Proof. We check all three parts of the proof separately.

Hyperbolic Point: At least one family of asymptotic curves in a neighbourhood of the point P consists of straight lines. If $\boldsymbol{r} = \boldsymbol{a}(u)$ is the equation of the asymptote γ and $\boldsymbol{b}(u)$ is a unit vector in a second asymptotic direction, then the surface can be defined in a neighborhood of the point P using the equation

$$\boldsymbol{r} = \boldsymbol{a}(u) + v\boldsymbol{b}(u).$$

2. **Parabolic Point**: The rectilinear generators are lines of curvature. Only one rectilinear generator passes through each point Q near P. We draw the curve γ, $\boldsymbol{r} = \boldsymbol{a}(u)$, through the point P on the surface such that its direction at P does not coincide with the direction of the generator. The unit vector $\boldsymbol{b}(u)$ on the generator is a regular function of u. The surface can be defined in a neighbourhood of P using the equation

$$\boldsymbol{r} = \boldsymbol{a}(u) + v\boldsymbol{b}(u).$$

3. **Umbilical Point**: Any direction is a principal direction at an umbilic point, and the normal curvature in any direction equals zero. By Rodrigues's theorem, $dn = 0$ in a neighbourhood of P. Consequently, $\boldsymbol{n} = \boldsymbol{n}_0 =$ constant. Since $\boldsymbol{n} \cdot d\boldsymbol{r} = 0$, we have $\boldsymbol{n}_0 \cdot (\boldsymbol{r} - \boldsymbol{r}_0) = 0$. Thus, a sufficiently small neighbourhood of P lies on a plane. Let $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ be any independent constant vectors in this plane. Then, in a neighbourhood of P, the surface can be defined using the equation

$$\boldsymbol{r} = \boldsymbol{a}_0 u + \boldsymbol{b}_0 v.$$

□

Developable Surfaces

Definition (Developable Surface)**.** A surface Φ is called a **developable surface** if it is locally isometric to a plane, meaning every point has a neighbourhood that can be flattened onto a plane without distortion. A necessary and sufficient condition for developability is that the Gaussian curvature K vanishes everywhere.

Structure of Developable Surfaces

1. **Case 1: Mean Curvature** $H = 0$: If the mean curvature $H = 0$ in a neighbourhood of the point P, the principal curvatures vanish at every point near P. Consequently, every point near P is an umbilical point, and the neighbourhood of P is a planar region.

2. **Case 2: Mean Curvature** $H \neq 0$: If $H \neq 0$ in a neighbourhood of P, introduce a coordinate net consisting of lines of curvature. Let the u-curves (i.e., $v =$ constant) be the lines of curvature along which the normal curvature vanishes. By Rodrigues's theorem, $\boldsymbol{n}_u = 0$, so the normals to the surface along the u-curves are parallel. The u-curves lie in a plane, and since $(\boldsymbol{n}_v)_u = (\boldsymbol{n}_u)_v = 0$, the normals to the u-curves are parallel. This implies that the u-curves are straight lines.

Conclusion: In both cases, a developable surface is a ruled surface where the tangent plane remains unchanged along the rectilinear generators.

6.8 Particutar Case. Surfaces of Revolution

Definition (Surface of Revolution)**.** A surface F is said to be a **surface of revolution** if it is generated by rotating a plane curve (called the **generating curve**) about a fixed axis. The curves of intersection of the surface with planes passing through the axis of rotation are called **meridians**, and the curves of intersection with planes perpendicular to the axis are called **parallels**.

Parametric Equations of a Surface of Revolution

We derive the equation of the surface of revolution generated by rotating the curve

$$x = \varphi(u), \quad z = \psi(u)$$

in the xz-plane about the z-axis. The point $(\varphi(u), 0, \psi(u))$ on the curve γ is transformed, upon rotation through an angle v, into the point

$$(\varphi(u)\cos v, \varphi(u)\sin v, \psi(u)).$$

Thus, the parametric equations of the surface of revolution are

$$x = \varphi(u)\cos v, \quad y = \varphi(u)\sin v, \quad z = \psi(u).$$

Here, the curves $v =$ constant are **meridians**, and $u =$ constant are **parallels**.

The First Quadratic Form

The coefficients of the first quadratic form are computed as

$$\begin{aligned}
E &= (\varphi' \cos v)^2 + (\varphi' \sin v)^2 + \psi'^2 = \varphi'^2 + \psi'^2, \\
F &= (\varphi' \cos v)(-\varphi \sin v) + (\varphi' \sin v)(\varphi \cos v) = 0, \\
G &= (-\varphi \sin v)^2 + (\varphi \cos v)^2 = \varphi^2.
\end{aligned}$$

Thus, the first quadratic form is

$$ds^2 = (\varphi'^2 + \psi'^2)du^2 + \varphi^2 dv^2.$$

The meridians and parallels form an orthogonal net ($F = 0$).

The Second Quadratic Form

The coefficients of the second quadratic form are computed as

$$L = \frac{\begin{vmatrix} \varphi'' cosv, & \varphi'' sinv, & \psi'' \\ \varphi' cosv, & \varphi' sinv, & \psi' \\ -\varphi sinv, & \varphi cosv, & 0 \end{vmatrix}}{EG - F^2} = \frac{\varphi(\psi''\varphi' - \psi'\varphi'')}{\varphi^2(\varphi'^2 + \psi'^2)},$$

$$M = \frac{\begin{vmatrix} -\varphi' sinv, & \varphi' cosv, & 0 \\ \varphi' cosv, & \varphi' sinv, & \psi' \\ -\varphi sinv, & \varphi cosv, & 0 \end{vmatrix}}{EG - F^2} = 0,$$

$$N = \frac{\begin{vmatrix} -\varphi cosv, & -\varphi sinv, & 0 \\ \varphi' cosv, & \varphi' sinv, & \psi' \\ -\varphi sinv, & \varphi cosv, & 0 \end{vmatrix}}{EG - F^2} = \frac{\psi'}{\varphi'^2 + \psi'^2}.$$

Thus, the second quadratic form is

$$II = \frac{\varphi(\psi''\varphi' - \psi'\varphi'')}{\varphi^2(\varphi'^2 + \psi'^2)} du^2 + \frac{\psi'}{\varphi'^2 + \psi'^2} dv^2.$$

Since $M = 0$, the parallels and meridians form a conjugate net. Combining this with the orthogonality of the net ($F = 0$), we conclude that the parallels and meridians are *lines of curvature.*

Principal Curvatures

Let k_1 be the curvature of a meridian and k_2 the curvature of a parallel. Let θ be the angle formed by the tangent to the meridian with the axis of revolution. Since the meridian plane intersects the surface orthogonally, the normal curvature of the surface in the direction of the meridian equals k_1. By Meusnier's theorem, the normal curvature in the direction of the parallels is $k_2 \cos\theta$. If d is the length of the segment of the normal to the surface extending to the axis, then $k_2 \cos\theta = \frac{1}{d}$.

Example: Surface of Revolution with Constant Negative Gaussian Curvature

Consider a surface of revolution with constant negative Gaussian curvature K. Let the z-axis be the axis of revolution, and let the generating curve in the xz-plane be given by $x = x(z)$. The normal curvature of the surface in the direction of the meridian is

$$k_1 = \frac{x''}{(1 + x'^2)^{3/2}},$$

and the normal curvature in the direction of the parallels is

$$k_2 = -\frac{1}{x(1 + x'^2)^{1/2}}.$$

Thus, the Gaussian curvature is

$$K = -\frac{x''}{x(1 + x'^2)^2}.$$

Multiplying by xx' and integrating, we obtain

$$Kx^2 + c = \frac{1}{1 + x'^2},$$

where c is an integration constant. Setting $c = 1$, we simplify the equation and solve for x and z in terms of a parameter θ. This leads to the parametric equations of the generating curve (the **tractrix**):

$$x = \frac{1}{\sqrt{-K}} \sin\theta, \quad z = \frac{1}{\sqrt{-K}} \left(\cos\theta + \ln\tan\frac{\theta}{2} \right).$$

Its distinguishing property is the fact that the segment of the tangent from the point of tangency to the z-axis is constant.

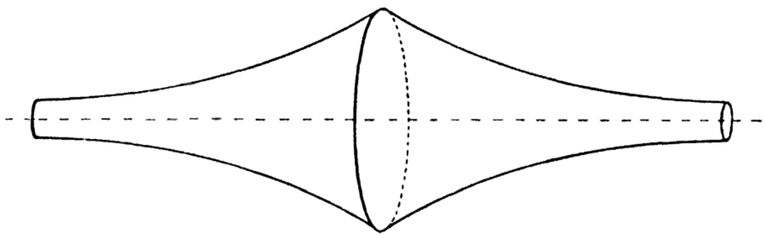


Figure 6.6. Pseudosphere.

Definition (Pseudosphere)**.** The surface obtained by rotating the tractrix about the z-axis is called a **pseudosphere**. Its equations are

$$x = \frac{1}{\sqrt{-K}} \sin\theta\cos\varphi, \quad y = \frac{1}{\sqrt{-K}} \sin\theta\sin\varphi, \quad z = \frac{1}{\sqrt{-K}} \left(\cos\theta + \ln\tan\frac{\theta}{2} \right).$$

6.9 Problems Corner

Problem 1
Determine the asymptotic curves of the catenoid

$$x = \cosh u \cos v, \quad y = \cosh u \sin v, \quad z = u$$

Solution
The catenoid is parametrised by:

$$\boldsymbol{r}(u, v) = (\cosh u \cos v, \cosh u \sin v, u)$$

1: Compute the first fundamental form coefficients The first fundamental form coefficients are:

$$E = \langle \boldsymbol{r}_u, \boldsymbol{r}_u \rangle, \quad F = \langle \boldsymbol{r}_u, \boldsymbol{r}_v \rangle, \quad G = \langle \boldsymbol{r}_v, \boldsymbol{r}_v \rangle$$

Compute the partial derivatives:

$$\boldsymbol{r}_u = (\sinh u \cos v, \sinh u \sin v, 1), \quad \boldsymbol{r}_v = (-\cosh u \sin v, \cosh u \cos v, 0)$$

Compute the coefficients:

$$E = \sinh^2 u \cos^2 v + \sinh^2 u \sin^2 v + 1 = \cosh^2 u$$

$$F = \sinh u \cos v(-\cosh u \sin v) + \sinh u \sin v(\cosh u \cos v) = 0$$
$$G = \cosh^2 u \sin^2 v + \cosh^2 u \cos^2 v = \cosh^2 u$$

2: Compute the second fundamental form coefficients The second fundamental form coefficients are:

$$L = \langle \boldsymbol{r}_{uu}, N \rangle, \quad M = \langle \boldsymbol{r}_{uv}, N \rangle, \quad N = \langle \boldsymbol{r}_{vv}, N \rangle$$

Compute the normal vector N:

$$N = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|} =$$
$$= \frac{(-\cosh u \cos v, -\cosh u \sin v, \sinh u \cosh u)}{\cosh^2 u} =$$
$$= \left(-\frac{\cos v}{\cosh u}, -\frac{\sin v}{\cosh u}, \frac{\sinh u}{\cosh u} \right)$$

Compute the second derivatives:

$$\begin{aligned}
\boldsymbol{r}_{uu} &= (\cosh u \cos v, \cosh u \sin v, 0), \\
\boldsymbol{r}_{uv} &= (-\sinh u \sin v, \sinh u \cos v, 0), \\
\boldsymbol{r}_{vv} &= (-\cosh u \cos v, -\cosh u \sin v, 0)
\end{aligned}$$

Compute the coefficients:

$$L = \langle \boldsymbol{r}_{uu}, \boldsymbol{n} \rangle = -1, \quad M = \langle \boldsymbol{r}_{uv}, \boldsymbol{n} \rangle = 0, \quad N = \langle \boldsymbol{r}_{vv}, \boldsymbol{n} \rangle = 1$$

3: Determine the asymptotic curves The asymptotic curves satisfy:

$$Ldu^2 + 2Mdudv + Ndv^2 = 0 \implies -du^2 + dv^2 = 0 \implies du = \pm dv$$

Thus, the asymptotic curves are:

$$u = \pm v + C$$

Problem 2

Show that on the helicoid, one family of asymptotic curves consists of straight lines and that the other consists of helices.

Solution

The helicoid is parametrised by:

$$\boldsymbol{r}(u, v) = (u \cos v, u \sin v, v)$$

1: Compute the first fundamental form coefficients Compute the partial derivatives:

$$\boldsymbol{r}_u = (\cos v, \sin v, 0), \quad \boldsymbol{r}_v = (-u \sin v, u \cos v, 1)$$

Compute the coefficients:

$$E = \cos^2 v + \sin^2 v = 1,$$

$$\begin{aligned}
F &= \cos v(-u \sin v) + \sin v(u \cos v) = 0 \\
G &= u^2 \sin^2 v + u^2 \cos^2 v + 1 = u^2 + 1
\end{aligned}$$

2: Compute the second fundamental form coefficients Compute the normal vector $\boldsymbol{n}$:

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|} = \frac{(\sin v, -\cos v, u)}{\sqrt{1+u^2}}$$

Compute the second derivatives:

$$\boldsymbol{r}_{uu} = (0,0,0), \quad \boldsymbol{r}_{uv} = (-\sin v, \cos v, 0), \quad \boldsymbol{r}_{vv} = (-u\cos v, -u\sin v, 0)$$

Compute the coefficients:

$$L = \langle \boldsymbol{r}_{uu}, \boldsymbol{n}\rangle = 0, \quad M = \langle \boldsymbol{r}_{uv}, \boldsymbol{n}\rangle = -\frac{1}{\sqrt{1+u^2}}, \quad N = \langle \boldsymbol{r}_{vv}, \boldsymbol{n}\rangle = 0$$

3: Determine the asymptotic curves The asymptotic curves satisfy:

$$Ldu^2 + 2Mdudv + Ndv^2 = 0 \implies 2Mdudv = 0 \implies du = 0 \text{ or } dv = 0$$

Thus, the asymptotic curves are:

$$u = \text{constant} \quad \text{(straight lines)}, \quad v = \text{constant} \quad \text{(helices)}$$

On the helicoid, one family of asymptotic curves consists of straight lines (where $u = $ constant), and the other family consists of helices (where $v =$ constant).

Problem 3

Determine the principal curvatures of the paraboloid

$$z = a(x^2 + y^2)$$

at the origin.

Solution

1: Parametrise the Surface Parametrise the surface as:

$$\boldsymbol{r}(x,y) = (x, y, a(x^2+y^2))$$

2: Compute the First Fundamental Form The first fundamental form is given by:

$$E = \langle \boldsymbol{r}_x, \boldsymbol{r}_x\rangle, \quad F = \langle \boldsymbol{r}_x, \boldsymbol{r}_y\rangle, \quad G = \langle \boldsymbol{r}_y, \boldsymbol{r}_y\rangle$$

Compute the partial derivatives:

$$\boldsymbol{r}_x = (1, 0, 2ax), \quad \boldsymbol{r}_y = (0, 1, 2ay)$$

Compute the coefficients:

$$E = 1 + (2ax)^2, \quad F = (1)(0) + (0)(1) + (2ax)(2ay) = 0, \quad G = 1 + (2ay)^2$$

At the origin $(0, 0)$:

$$E = 1, \quad F = 0, \quad G = 1$$

3: Compute the Second Fundamental Form The second fundamental form is given by:

$$L = \langle \boldsymbol{r}_{xx}, N \rangle, \quad M = \langle \boldsymbol{r}_{xy}, N \rangle, \quad N = \langle \boldsymbol{r}_{yy}, N \rangle$$

First, compute the normal vector N:

$$N = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\|\boldsymbol{r}_x \times \boldsymbol{r}_y\|}$$

Compute the cross product:

$$\boldsymbol{r}_x \times \boldsymbol{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2ax \\ 0 & 1 & 2ay \end{vmatrix} = (-2ax, -2ay, 1)$$

Compute the magnitude:

$$\|\boldsymbol{r}_x \times \boldsymbol{r}_y\| = \sqrt{(-2ax)^2 + (-2ay)^2 + 1} = \sqrt{4a^2(x^2 + y^2) + 1}$$

At the origin $(0, 0)$:

$$N = (0, 0, 1)$$

Compute the second derivatives:

$$\boldsymbol{r}_{xx} = (0, 0, 2a), \quad \boldsymbol{r}_{xy} = (0, 0, 0), \quad \boldsymbol{r}_{yy} = (0, 0, 2a)$$

Compute the coefficients:

$$L = \langle \boldsymbol{r}_{xx}, N \rangle = 2a, \quad M = \langle \boldsymbol{r}_{xy}, N \rangle = 0, \quad N = \langle \boldsymbol{r}_{yy}, N \rangle = 2a$$

Step 4: Compute the Principal Curvatures The principal curvatures are the eigenvalues of the matrix of the second quadratic form (the shape operator) S, which is given by:

$$S = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2a \end{pmatrix}$$

The eigenvalues of S are $\kappa_1 = 2a$ and $\kappa_2 = 2a$.

Final Answer: The principal curvatures of the paraboloid $z = a(x^2 + y^2)$ at the origin are $\kappa_1 = 2a$ and $\kappa_2 = 2a$.

Problem 4
Find the lines of curvature of the hyperboloid $z = axy$.

Solution

1: Parametrise the Surface Parametrise the surface as:

$$\boldsymbol{r}(x, y) = (x, y, axy)$$

2: Compute the First Fundamental Form The first fundamental form is given by:

$$E = \langle \boldsymbol{r}_x, \boldsymbol{r}_x \rangle, \quad F = \langle \boldsymbol{r}_x, \boldsymbol{r}_y \rangle, \quad G = \langle \boldsymbol{r}_y, \boldsymbol{r}_y \rangle$$

Compute the partial derivatives:

$$\boldsymbol{r}_x = (1, 0, ay), \quad \boldsymbol{r}_y = (0, 1, ax)$$

Compute the coefficients:

$$E = 1 + (ay)^2, \quad F = (1)(0) + (0)(1) + (ay)(ax) = a^2xy, \quad G = 1 + (ax)^2$$

3: Compute the Second Fundamental Form The second fundamental form is given by:

$$L = \langle \boldsymbol{r}_{xx}, N \rangle, \quad M = \langle \boldsymbol{r}_{xy}, N \rangle, \quad N = \langle \boldsymbol{r}_{yy}, N \rangle$$

First, compute the normal vector N:

$$N = \frac{\boldsymbol{r}_x \times \boldsymbol{r}_y}{\|\boldsymbol{r}_x \times \boldsymbol{r}_y\|}$$

Compute the cross product:

$$\boldsymbol{r}_x \times \boldsymbol{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & ay \\ 0 & 1 & ax \end{vmatrix} = (-ay, -ax, 1)$$

Compute the magnitude:

$$\|\boldsymbol{r}_x \times \boldsymbol{r}_y\| = \sqrt{(-ay)^2 + (-ax)^2 + 1} = \sqrt{a^2(x^2 + y^2) + 1}$$

Compute the normal vector:

$$N = \frac{(-ay, -ax, 1)}{\sqrt{a^2(x^2 + y^2) + 1}}$$

Compute the second derivatives:

$$\boldsymbol{r}_{xx} = (0, 0, 0), \quad \boldsymbol{r}_{xy} = (0, 0, a), \quad \boldsymbol{r}_{yy} = (0, 0, 0)$$

Compute the coefficients:

$$L = \langle \boldsymbol{r}_{xx}, N \rangle = 0, \quad M = \langle \boldsymbol{r}_{xy}, N \rangle = \frac{a}{\sqrt{a^2(x^2 + y^2) + 1}}, \quad N = \langle \boldsymbol{r}_{yy}, N \rangle = 0$$

4: Determine the Principal Curvatures The principal curvatures κ_1 and κ_2 are the roots of the quadratic equation:

$$\begin{vmatrix} L - \kappa E, & M - \kappa F \\ M - \kappa F, & N - \kappa G \end{vmatrix} = 0.$$

Substituting the coefficients $L = 0$, $M = \frac{a}{\sqrt{a^2(x^2+y^2)+1}}$, $N = 0$, $E = 1 + a^2y^2$, $F = a^2xy$, and $G = 1 + a^2x^2$, we get: <

$$\begin{vmatrix} 0 - \kappa(1 + a^2y^2), & \frac{a}{\sqrt{a^2(x^2+y^2)+1}} - \kappa(a^2xy) \\ \frac{a}{\sqrt{a^2(x^2+y^2)+1}} - \kappa(a^2xy), & 0 - \kappa(1 + a^2x^2) \end{vmatrix} = 0.$$

Expanding the determinant:

$$\left(-\kappa(1 + a^2y^2)\right)\left(-\kappa(1 + a^2x^2)\right) - \left(\frac{a}{\sqrt{a^2(x^2 + y^2) + 1}} - \kappa(a^2xy)\right)^2 = 0.$$

Simplifying:

$$\kappa^2(1 + a^2y^2)(1 + a^2x^2) = \left(\frac{a}{\sqrt{a^2(x^2 + y^2) + 1}} - \kappa(a^2xy)\right)^2.$$

Collecting terms and simplifying, we obtain the principal curvatures:

$$\kappa_1 = \frac{a}{\sqrt{a^2(x^2 + y^2) + 1}}, \quad \kappa_2 = -\frac{a}{\sqrt{a^2(x^2 + y^2) + 1}}.$$

5: Differential Equation for Lines of Curvature

The equation for the lines of curvature is:

$$(EM - FL)\,du^2 + (EN - GL)\,du\,dv + (FN - GM)\,dv^2 = 0.$$

Substitute the values of E, F, G, L, M, N:

$$\left[(1 + a^2y^2)\left(\frac{a}{\sqrt{1 + a^2(x^2 + y^2)}}\right) - (a^2xy)(0)\right] dx^2$$

$$+ \left[(1 + a^2y^2)(0) - (1 + a^2x^2)(0)\right] dx\,dy$$

$$+ \left[(a^2xy)(0) - (1 + a^2x^2)\left(\frac{a}{\sqrt{1 + a^2(x^2 + y^2)}}\right)\right] dy^2 = 0.$$

Simplify:

$$\frac{a(1 + a^2y^2)}{\sqrt{1 + a^2(x^2 + y^2)}}dx^2 - \frac{a(1 + a^2x^2)}{\sqrt{1 + a^2(x^2 + y^2)}}dy^2 = 0.$$

Multiply through by $\sqrt{1 + a^2(x^2 + y^2)}$:

$$a(1 + a^2y^2)\,dx^2 - a(1 + a^2x^2)\,dy^2 = 0.$$

Divide through by a:

$$(1 + a^2y^2)\,dx^2 = (1 + a^2x^2)\,dy^2.$$

Rewrite as:

$$\frac{dx^2}{1 + a^2x^2} = \frac{dy^2}{1 + a^2y^2}.$$

6: Determine Lines of Curvature

Take the square root of both sides and integrate:

$$\int \frac{dx}{\sqrt{1 + a^2x^2}} = \pm \int \frac{dy}{\sqrt{1 + a^2y^2}} + C,$$

where C is a constant of integration. Using the standard integral:

$$\int \frac{dt}{\sqrt{1 + a^2t^2}} = \frac{1}{a}\ln\left(at + \sqrt{1 + a^2t^2}\right),$$

we obtain:

$$\frac{1}{a}\ln\left(ax+\sqrt{1+a^2x^2}\right)=\pm\frac{1}{a}\ln\left(ay+\sqrt{1+a^2y^2}\right)+C.$$

Multiply through by a:

$$\ln\left(ax+\sqrt{1+a^2x^2}\right)=\pm\ln\left(ay+\sqrt{1+a^2y^2}\right)+C_1,$$

where $C_1=aC$.

Final Answer: The principal curvatures of the hyperboloid $z=axy$ are:

$$\kappa_1=\frac{a}{\sqrt{a^2(x^2+y^2)+1}},\quad \kappa_2=-\frac{a}{\sqrt{a^2(x^2+y^2)+1}}.$$

The lines of curvature are given by:

$$\ln\left(ay+\sqrt{1+a^2y^2}\right)\pm\ln\left(ax+\sqrt{1+a^2x^2}\right)=\text{constant}.$$

Problem 5

Find the mean and Gaussian curvatures of the hyperboloid $z=axy$ at the origin.

Solution

1: Parametrise the Surface Parametrise the surface as:

$$\boldsymbol{r}(x,y)=(x,y,axy)$$

2: Compute the First Fundamental Form at the Origin At the origin $(0,0)$:

$$E=1,\quad F=0,\quad G=1$$

3: Compute the Second Fundamental Form at the Origin At the origin $(0,0)$:

$$L=0,\quad M=a,\quad N=0$$

4: Compute the Mean and Gaussian Curvatures The mean curvature H is given by:

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{(1)(0) - 2(0)(a) + (1)(0)}{2((1)(1) - 0^2)} = 0$$

The Gaussian curvature K is given by:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{(0)(0) - a^2}{(1)(1) - 0^2} = -a^2$$

Final Answer: The mean curvature of the hyperboloid $z = axy$ at the origin is $H = 0$, and the Gaussian curvature is $K = -a^2$.

Chapter 7

The Fundamental Equations of the Theory of Surfaces

7.1 The Gauss Formula

Gaussian Curvature

The Gaussian curvature K of a surface is given by:

$$K = \frac{LN - M^2}{EG - F^2},$$

where:

- E, F, G are the coefficients of the first fundamental form,
- L, M, N are the coefficients of the second fundamental form.

The coefficients L, M, N can be expressed in terms of the surface parameterization $\boldsymbol{r}(u, v)$ and the unit normal vector $\boldsymbol{n}$:

$$L = \boldsymbol{r}_{uu} \cdot \boldsymbol{n}, \quad M = \boldsymbol{r}_{uv} \cdot \boldsymbol{n}, \quad N = \boldsymbol{r}_{vv} \cdot \boldsymbol{n}.$$

Gaussian Curvature in Terms of the First Fundamental Form

The Gaussian curvature K can also be expressed solely in terms of the coefficients of the first fundamental form and their derivatives.

Substituting everywhere the expressions for the coefficients of the second quadratic form

$$L = \frac{(\boldsymbol{r}_{uu}\boldsymbol{r}_u\boldsymbol{r}_v)}{\sqrt{EG - F^2}}, \quad M = \frac{(\boldsymbol{r}_{uv}\boldsymbol{r}_u\boldsymbol{r}_v)}{\sqrt{EG - F^2}}, \quad N = \frac{(\boldsymbol{r}_{vv}\boldsymbol{r}_u\boldsymbol{r}_v)}{\sqrt{EG - F^2}}$$

we have

$$K = \frac{1}{(EG-F^2)^2}\{(\boldsymbol{r}_{uu}\boldsymbol{r}_u\boldsymbol{r}_v)(\boldsymbol{r}_{vv}\boldsymbol{r}_u\boldsymbol{r}_v) - (\boldsymbol{r}_{uv}\boldsymbol{r}_u\boldsymbol{r}_v)^2\}.$$

$$K = \frac{1}{(EG-F^2)^2}\left\{\begin{vmatrix} \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_{vv}, & \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_u, & \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_v \\ \boldsymbol{r}_u\cdot\boldsymbol{r}_{vv}, & E, & F \\ \boldsymbol{r}_v\cdot\boldsymbol{r}_{vv}, & F, & G \end{vmatrix} - \begin{vmatrix} \boldsymbol{r}_{uv}^2, & \boldsymbol{r}_{uv}\cdot\boldsymbol{r}_u, & \boldsymbol{r}_{uv}\cdot\boldsymbol{r}_v \\ \boldsymbol{r}_u\cdot\boldsymbol{r}_{uv}, & E, & F \\ \boldsymbol{r}_v\cdot\boldsymbol{r}_{uv}, & F, & G \end{vmatrix}\right\} =$$

$$= \frac{1}{(EG-F^2)^2}\left\{\begin{vmatrix} \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_{vv} - \boldsymbol{r}_{uv}^2, & \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_u, & \boldsymbol{r}_{uu}\cdot\boldsymbol{r}_v \\ \boldsymbol{r}_u\cdot\boldsymbol{r}_{vv}, & E, & F \\ \boldsymbol{r}_v\cdot\boldsymbol{r}_{vv}, & F, & G \end{vmatrix} - \begin{vmatrix} 0, & \boldsymbol{r}_{uv}\cdot\boldsymbol{r}_u, & \boldsymbol{r}_{uv}\cdot\boldsymbol{r}_v \\ \boldsymbol{r}_v\cdot\boldsymbol{r}_{uv}, & E, & F, \\ \boldsymbol{r}_v\cdot\boldsymbol{r}_{uv}, & F, & G \end{vmatrix}\right\}$$

Differentiating the expressions $\boldsymbol{r}_u^2 = E$, $\boldsymbol{r}_u\cdot\boldsymbol{r}_v = F$, $\boldsymbol{r}_v^2 = G$ with respect to u and v, we obtain

$$\begin{aligned}
\boldsymbol{r}_{uu}\cdot\boldsymbol{r}_u &= \frac{1}{2}E_u, \\
\boldsymbol{r}_{uv}\cdot\boldsymbol{r}_u &= \frac{1}{2}E_v, \\
\boldsymbol{r}_{vv}\cdot\boldsymbol{r}_v &= \frac{1}{2}G_v, \\
\boldsymbol{r}_{uv}\cdot\boldsymbol{r}_v &= \frac{1}{2}G_u, \\
\boldsymbol{r}_{uu}\cdot\boldsymbol{r}_v &= F_u - \frac{1}{2}E_v, \\
\boldsymbol{r}_{vv}\cdot\boldsymbol{r}_u &= F_v - \frac{1}{2}G_u.
\end{aligned}$$

If we now differentiate the fifth equation concerning v, the fourth with respect to u, and then subtract the resultant equations termwise, we obtain

$$\boldsymbol{r}_{uu}\cdot\boldsymbol{r}_{vv} - \boldsymbol{r}_{uv}^2 = -\frac{1}{2}G_{uu} + F_{uv} - \frac{1}{2}E_{vv}.$$

Finally, we get

$$K = \frac{1}{(EG-F^2)^2}\left\{\begin{vmatrix} (-\frac{1}{2}G_{uu} + F_{uv} - \frac{1}{2}E_{vv}), & \frac{1}{2}E_u, & (F_u - \frac{1}{2}E_v) \\ (F_v - \frac{1}{2}G_u), & E, & F \\ \frac{1}{2}G_v, & F, & G \end{vmatrix} - \right.$$
$$\left. - \begin{vmatrix} 0, & \frac{1}{2}E_v, & \frac{1}{2}G_u \\ \frac{1}{2}E_v, & E, & F \\ \frac{1}{2}G_u, & F, & G \end{vmatrix}\right\}. \tag{7.1}$$

Finally, if $F = 0$ (i.e. coordinate curves are orthogonal), the Gaussian curvature is given by:

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial v}\right)\right). \tag{7.2}$$

Corollaries

Corollary 7.1.1 (Gaussian Curvature of Isometric Surfaces)**.** *Isometric surfaces have the same Gaussian curvature at corresponding points. Gaussian curvature is invariant under bending.*

Corollary 7.1.2 (Gaussian Curvature of Developable Surfaces)**.** *Developable surfaces are locally isometric to a plane. Hence, their Gaussian curvature vanishes everywhere.*

Corollary 7.1.3 (Dependence of Fundamental Forms)**.** *The first and second fundamental forms of a surface are not independent.*

Example: Special Parametrisation

If the first fundamental form is given by:

$$I = du^2 + G dv^2,$$

then the Gaussian curvature simplifies to:

$$K = -\frac{1}{\sqrt{G}}\left(\sqrt{G}\right)_{uu}.$$

7.2 Derived Formulas for Surfaces (Analogous to Frenet Formulas)

The derived formulas for a surface are analogous to the Frenet formulas for curves. These formulas describe the derivatives of the tangent vectors $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, and the normal vector $\boldsymbol{n}$ in terms of the surface's first and second fundamental forms.

Derivatives of Tangent and Normal Vectors

Since the vectors $\boldsymbol{r}_u$, $\boldsymbol{r}_v$, and $\boldsymbol{n}$ form a basis, the derivatives of these vectors can be expressed as linear combinations of themselves. Specifically:

$$
\begin{aligned}
\boldsymbol{r}_{uu} &= \Gamma^1_{11}\boldsymbol{r}_u + \Gamma^2_{11}\boldsymbol{r}_v + \lambda_{11}\boldsymbol{n},\\
\boldsymbol{r}_{uv} &= \Gamma^1_{12}\boldsymbol{r}_u + \Gamma^2_{12}\boldsymbol{r}_v + \lambda_{12}\boldsymbol{n},\\
\boldsymbol{r}_{vv} &= \Gamma^1_{22}\boldsymbol{r}_u + \Gamma^2_{22}\boldsymbol{r}_v + \lambda_{22}\boldsymbol{n},\\
\boldsymbol{n}_u &= \alpha_{11}\boldsymbol{r}_u + \alpha_{12}\boldsymbol{r}_v + \alpha_{10}\boldsymbol{n},\\
\boldsymbol{n}_v &= \alpha_{21}\boldsymbol{r}_u + \alpha_{22}\boldsymbol{r}_v + \alpha_{20}\boldsymbol{n}.
\end{aligned}
\tag{7.3}
$$

Coefficients of the Normal Derivatives

The coefficients α_{10} and α_{20} vanish because the normal vector $\boldsymbol{n}$ has unit length:

$$\boldsymbol{n}_u \cdot \boldsymbol{n} = \frac{1}{2}(\boldsymbol{n}^2)_u = 0, \quad \boldsymbol{n}_v \cdot \boldsymbol{n} = \frac{1}{2}(\boldsymbol{n}^2)_v = 0.$$

The coefficients α_{11} and α_{12} are obtained by taking the dot product of $\boldsymbol{n}_u = \alpha_{11}\boldsymbol{r}_u + \alpha_{12}\boldsymbol{r}_v$ with $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$:

$$
\begin{aligned}
-L = \alpha_{11}E + \alpha_{12}F \quad &\Longrightarrow \quad \alpha_{11} = \frac{-LG + MF}{EG - F^2},\\
-M = \alpha_{11}F + \alpha_{12}G \quad &\Longrightarrow \quad \alpha_{12} = \frac{LF - ME}{EG - F^2}.
\end{aligned}
$$

Derivation of α_{21} and α_{22} is analogious.

The coefficients L, M, N are the components of the second fundamental form and are given by:

$$\lambda_{11} = L, \quad \lambda_{12} = M, \quad \lambda_{22} = N.$$

Christoffel Symbols

The Christoffel symbols of the second kind Γ^k_{ij} are obtained by taking the dot product of the equations for $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{uv}$, and $\boldsymbol{r}_{vv}$ with $\boldsymbol{r}_u$ and $\boldsymbol{r}_v$. This results in the following system of equations:

$$
\begin{cases}
\Gamma^1_{11}E + \Gamma^2_{11}F = \frac{1}{2}E_u,\\
\Gamma^1_{11}F + \Gamma^2_{11}G = F_u - \frac{1}{2}E_v,
\end{cases}
$$
$$
\begin{cases}
\Gamma^1_{12}E + \Gamma^2_{12}F = \frac{1}{2}E_v,\\
\Gamma^1_{22}F + \Gamma^2_{12}G = \frac{1}{2}G_u,
\end{cases}
$$
$$
\begin{cases}
\Gamma^1_{22}E + \Gamma^2_{22}F = F_v - \frac{1}{2}G_u,\\
\Gamma^1_{22}F + \Gamma^2_{22}G = \frac{1}{2}G_v.
\end{cases}
$$

Example: Special Case

If the first fundamental form is given by:

$$I = du^2 + G dv^2,$$

the Christoffel symbols simplify to:

$$\begin{aligned}
\Gamma^1_{11} &= 0, \quad \Gamma^2_{11} = 0, \\
\Gamma^1_{12} &= 0, \quad \Gamma^2_{12} = \frac{1}{2}\frac{G_u}{G}, \\
\Gamma^1_{22} &= -\frac{1}{2}G_u, \quad \Gamma^2_{22} = \frac{1}{2}\frac{G_v}{G}.
\end{aligned}$$

7.3 The Peterson-Codazzi Formulas

The Peterson-Codazzi formulas are fundamental relationships in the differential geometry of surfaces. They arise from the compatibility conditions for the second derivatives of the tangent and normal vectors. These formulas, along with the Gauss formula, provide a complete set of conditions for the existence of a surface with given first and second fundamental forms.

Compatibility Conditions

The starting point for deriving the Peterson-Codazzi formulas is the following set of obvious equalities:

$$\begin{aligned}
(\boldsymbol{r}_{uu})_v - (\boldsymbol{r}_{uv})_u &= 0, \\
(\boldsymbol{r}_{vv})_u - (\boldsymbol{r}_{uv})_v &= 0, \\
(\boldsymbol{n}_u)_v - (\boldsymbol{n}_v)_u &= 0.
\end{aligned}$$

These equalities express the symmetry of mixed partial derivatives.

Vector Equations

Substituting the derived expressions (7.3) for $\boldsymbol{r}_{uu}$, $\boldsymbol{r}_{uv}$, $\boldsymbol{r}_{vv}$, $\boldsymbol{n}_u$, and $\boldsymbol{n}_v$ into these equalities yields three vector equations of the form:

$$\begin{aligned}
A_1\boldsymbol{r}_u + B_1\boldsymbol{r}_v + C_1\boldsymbol{n} &= 0, \\
A_2\boldsymbol{r}_u + B_2\boldsymbol{r}_v + C_2\boldsymbol{n} &= 0, \\
A_3\boldsymbol{r}_u + B_3\boldsymbol{r}_v + C_3\boldsymbol{n} &= 0,
\end{aligned}$$

where $A_1, A_2, \ldots, C_3$ are expressions involving the coefficients of the first and second fundamental forms and their derivatives.

Scalar Equations

Taking the components of these vector equations, we obtain nine scalar equations:

$$
\begin{aligned}
A_1 &= 0, \quad B_1 = 0, \quad C_1 = 0,\\
A_2 &= 0, \quad B_2 = 0, \quad C_2 = 0,\\
A_3 &= 0, \quad B_3 = 0, \quad C_3 = 0.
\end{aligned}
$$

Of these nine relations, only three are independent. One is equivalent to the Gauss formula, while the other two are the Peterson-Codazzi formulas.

The Peterson-Codazzi Formulas

The Peterson-Codazzi formulas are given by:

$$
\begin{aligned}
&(EG - 2FF + GE)(L_v - M_u)\\
&- (EN - 2FM + GL)(E_v - F_u)\\
&+ \begin{vmatrix} E & E_u & L \\ F & F_u & M \\ G & G_u & N \end{vmatrix} = 0,
\end{aligned}
\tag{7.4}
$$

and

$$
\begin{aligned}
&(EG - 2FF + GE)(M_v - N_u)\\
&- (EN - 2FM + GL)(F_v - G_u)\\
&+ \begin{vmatrix} E & E_v & L \\ F & F_v & M \\ G & G_v & N \end{vmatrix} = 0.
\end{aligned}
\tag{7.5}
$$

These formulas ensure the compatibility of the first and second fundamental forms.

Let us simplify these formulas. We expand determinants first.

For the first equation:

$$
\begin{vmatrix} E & E_u & L \\ F & F_u & M \\ G & G_u & N \end{vmatrix} = E(F_uN - G_uM) - E_u(FN - GM) + L(FG_u - GF_u).
$$

For the second equation:

$$
\begin{vmatrix} E & E_v & L \\ F & F_v & M \\ G & G_v & N \end{vmatrix} = E(F_vN - G_vM) - E_v(FN - GM) + L(FG_v - GF_v).
$$

Substitute the expanded determinants into the original equations. After simplification, the equations reduce to:

$$(EG - 2F^2 + GE)(L_v - M_u) - (EN - 2FM + GL)(E_v - F_u) + \\ +E(F_u N - G_u M) - E_u(FN - GM) + L(FG_u - GF_u) = 0,$$

and

$$(EG - 2F^2 + GE)(M_v - N_u) - (EN - 2FM + GL)(F_v - G_u) + \\ +E(F_v N - G_v M) - E_v(FN - GM) + L(FG_v - GF_v) = 0.$$

The Christoffel symbols of the second kind are calculated as:

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \quad \Gamma^2_{11} = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)},$$

$$\Gamma^1_{12} = \frac{GE_v - FG_u}{2(EG - F^2)}, \quad \Gamma^2_{12} = \frac{EG_u - FE_v}{2(EG - F^2)},$$

$$\Gamma^1_{22} = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)}, \quad \Gamma^2_{22} = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$

Substitute the Christoffel symbols into the equations and simplify. After algebraic manipulation, the equations reduce to:

$$\frac{\partial L}{\partial v} - \frac{\partial M}{\partial u} = L\,\Gamma^1_{12} + M\,(\Gamma^2_{12} - \Gamma^1_{11}) - N\,\Gamma^2_{11}, \tag{7.6}$$

and

$$\frac{\partial M}{\partial v} - \frac{\partial N}{\partial u} = L\,\Gamma^1_{22} + M\,(\Gamma^2_{22} - \Gamma^1_{12}) - N\,\Gamma^2_{12}. \tag{7.7}$$

The Peterson-Codazzi equations, when simplified using Christoffel symbols, relate the derivatives of the coefficients of the second fundamental form (L, M, N) to the intrinsic geometry of the surface, encoded in the Christoffel symbols.

Historical Note

The Peterson-Codazzi formulas were first derived by Peterson and later independently rediscovered by Mainardi and Codazzi. They play a crucial role in the study of surfaces and their curvature properties.

7.4 Bonnet's Fundamental Theorem

Definition (Gauss-Peterson-Codazzi Conditions)**.** Let

$$E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

and

$$L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

be two arbitrary quadratic forms, with the first being positive definite. The following three conditions are called the **Gauss-Peterson-Codazzi conditions**:

1. The Gauss formula (7.1):

$$\frac{LN - M^2}{EG - F^2} = K(E, F, G). \tag{7.8}$$

2. The first Peterson-Codazzi formula (7.4):

$$\begin{aligned}&(EG - 2FF + GE)(L_v - M_u)\\ &- (EN - 2FM + GL)(E_v - F_u)\\ &+ \begin{vmatrix} E & E_u & L \\ F & F_u & M \\ G & G_u & N \end{vmatrix} = 0.\end{aligned} \tag{7.9}$$

3. The second Peterson-Codazzi formula (7.4):

$$\begin{aligned}&(EG - 2FF + GE)(M_v - N_u)\\ &- (EN - 2FM + GL)(F_v - G_u)\\ &+ \begin{vmatrix} E & E_v & L \\ F & F_v & M \\ G & G_v & N \end{vmatrix} = 0.\end{aligned} \tag{7.10}$$

These conditions ensure the compatibility of the first and second fundamental forms for the existence of a surface.

Theorem 7.4.1 (Bonnet's Theorem)**.** *Let*

$$E\,du^2 + 2F\,du\,dv + G\,dv^2,$$

and

$$L\,du^2 + 2M\,du\,dv + N\,dv^2,$$

be two arbitrary quadratic forms, with the first being positive definite. If the coefficients of these forms satisfy the Gauss-Peterson-Codazzi conditions defined in Equations (7.8), (7.9), *and* (7.10), *then there exists a surface, unique up to its position in space, for which these forms are the first and second fundamental forms, respectively.*

Proof. We first prove the existence of such a surface and then establish its uniqueness.

Existence

Consider the following system of differential equations for the vector functions $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}$:

$$\begin{aligned}
\boldsymbol{\xi}_u &= \Gamma_{11}^1 \boldsymbol{\xi} + \Gamma_{11}^2 \boldsymbol{\eta} + L\boldsymbol{\zeta}, \\
\boldsymbol{\xi}_v &= \Gamma_{12}^1 \boldsymbol{\xi} + \Gamma_{12}^2 \boldsymbol{\eta} + M\boldsymbol{\zeta}, \\
\boldsymbol{\eta}_u &= \Gamma_{12}^1 \boldsymbol{\xi} + \Gamma_{12}^2 \boldsymbol{\eta} + M\boldsymbol{\zeta}, \\
\boldsymbol{\eta}_v &= \Gamma_{22}^1 \boldsymbol{\xi} + \Gamma_{22}^2 \boldsymbol{\eta} + N\boldsymbol{\zeta}, \\
\boldsymbol{\zeta}_u &= \alpha_{11} \boldsymbol{\xi} + \alpha_{12} \boldsymbol{\eta}, \\
\boldsymbol{\zeta}_v &= \alpha_{21} \boldsymbol{\xi} + \alpha_{22} \boldsymbol{\eta}.
\end{aligned}$$

From the theory of differential equations, this system has a unique solution for given initial conditions if the integrability conditions are satisfied. The integrability conditions are:

$$\begin{aligned}
(\Gamma_{11}^1 \boldsymbol{\xi} + \Gamma_{11}^2 \boldsymbol{\eta} + L\boldsymbol{\zeta})_v - (\Gamma_{12}^1 \boldsymbol{\xi} + \Gamma_{12}^2 \boldsymbol{\eta} + M\boldsymbol{\zeta})_u &= 0, \\
(\Gamma_{12}^1 \boldsymbol{\xi} + \Gamma_{12}^2 \boldsymbol{\eta} + M\boldsymbol{\zeta})_v - (\Gamma_{22}^1 \boldsymbol{\xi} + \Gamma_{22}^2 \boldsymbol{\eta} + N\boldsymbol{\zeta})_u &= 0, \\
(\alpha_{11} \boldsymbol{\xi} + \alpha_{12} \boldsymbol{\eta})_v - (\alpha_{21} \boldsymbol{\xi} + \alpha_{22} \boldsymbol{\eta})_u &= 0.
\end{aligned}$$

Using the expressions for Γ_{ij}^k and α_{ij}, these conditions reduce to the Gauss-Peterson-Codazzi conditions, which are satisfied by assumption. Hence, the system of differential equations has a unique solution.

Let $\boldsymbol{\xi}_0, \boldsymbol{\eta}_0, \boldsymbol{\zeta}_0$ be initial vectors satisfying:

$$\begin{aligned}
&\boldsymbol{\xi}_0^2 = E(u_0, v_0), \quad \boldsymbol{\xi}_0 \cdot \boldsymbol{\eta}_0 = F(u_0, v_0), \quad \boldsymbol{\eta}_0^2 = G(u_0, v_0), \\
&\boldsymbol{\xi}_0 \cdot \boldsymbol{\zeta}_0 = 0, \quad \boldsymbol{\eta}_0 \cdot \boldsymbol{\zeta}_0 = 0, \quad \boldsymbol{\zeta}_0^2 = 1.
\end{aligned}$$

The solution $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}$ satisfies these initial conditions. Since $\boldsymbol{\xi}_v = \boldsymbol{\eta}_u$, there exists a vector function $\boldsymbol{r}(u, v)$ such that $\boldsymbol{r}_u = \boldsymbol{\xi}$ and $\boldsymbol{r}_v = \boldsymbol{\eta}$.

We now show that the surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$ has the given quadratic forms. From the solution of the differential system, we have:

$$\boldsymbol{\xi}^2 = E, \quad \boldsymbol{\eta}^2 = G, \quad \boldsymbol{\xi} \cdot \boldsymbol{\eta} = F,$$
$$\boldsymbol{\xi} \cdot \boldsymbol{\zeta} = 0, \quad \boldsymbol{\eta} \cdot \boldsymbol{\zeta} = 0, \quad \boldsymbol{\zeta}^2 = 1.$$

Thus, the surface has $E\,du^2 + 2F\,du\,dv + G\,dv^2$ as its first fundamental form. Since $\boldsymbol{\zeta}$ is a unit normal vector, the coefficients of the second fundamental form are:

$$L = \boldsymbol{\xi}_u \cdot \boldsymbol{\zeta}, \quad M = \boldsymbol{\xi}_v \cdot \boldsymbol{\zeta}, \quad N = \boldsymbol{\eta}_v \cdot \boldsymbol{\zeta}.$$

Hence, the surface also has $L\,du^2 + 2M\,du\,dv + N\,dv^2$ as its second fundamental form.

Uniqueness

Suppose Φ_1 and Φ_2 are two surfaces with the same fundamental forms. We associate corresponding points, directions, and normals at a fixed point (u_0, v_0). This is possible because the first fundamental forms coincide. Let $\boldsymbol{r}_1(u, v)$ and $\boldsymbol{r}_2(u, v)$ be the equations of the surfaces after this correspondence.

The system of differential equations for $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}$ is satisfied by:

$$\boldsymbol{\xi} = \boldsymbol{r}_{1u}, \quad \boldsymbol{\eta} = \boldsymbol{r}_{1v}, \quad \boldsymbol{\zeta} = \boldsymbol{n}_1,$$

and

$$\boldsymbol{\xi} = \boldsymbol{r}_{2u}, \quad \boldsymbol{\eta} = \boldsymbol{r}_{2v}, \quad \boldsymbol{\zeta} = \boldsymbol{n}_2.$$

Since both solutions coincide at (u_0, v_0), they are identical everywhere. Thus,

$$\boldsymbol{r}_{1u}(u, v) = \boldsymbol{r}_{2u}(u, v), \quad \boldsymbol{r}_{1v}(u, v) = \boldsymbol{r}_{2v}(u, v).$$

This implies $d\boldsymbol{r}_1(u, v) = d\boldsymbol{r}_2(u, v)$. Integrating, we find $\boldsymbol{r}_1(u, v) = \boldsymbol{r}_2(u, v) + \boldsymbol{c}$. Since $\boldsymbol{r}_1(u_0, v_0) = \boldsymbol{r}_2(u_0, v_0)$, we have $\boldsymbol{c} = 0$, and the surfaces are identical up to their position in space. $\square$

7.5 Problems Corner

Problem 1
Let the linear element of a surface be

$$ds^2 = \lambda(du^2 + dv^2),$$

where $\lambda = \lambda(u, v)$ is a positive function of the parameters u and v. Calculate the Gaussian curvature.

Solution

Given the linear element of the surface:

$$ds^2 = \lambda(u, v)(du^2 + dv^2),$$

where $\lambda(u, v)$ is a positive function, we calculate the Gaussian curvature K.

The first quadratic form coefficients are:

$$E = \lambda(u, v), \quad F = 0, \quad G = \lambda(u, v).$$

The Gaussian curvature K is given by:

$$K = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial u} \left(\frac{G_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial v} \left(\frac{E_v}{\sqrt{EG}} \right) \right).$$

Since $E = G = \lambda$ and $F = 0$, this simplifies to:

$$K = -\frac{1}{2\lambda} \left(\frac{\partial}{\partial u} \left(\frac{\lambda_u}{\lambda} \right) + \frac{\partial}{\partial v} \left(\frac{\lambda_v}{\lambda} \right) \right).$$

Compute the partial derivatives:

$$\frac{\partial}{\partial u} \left(\frac{\lambda_u}{\lambda} \right) = \frac{\lambda_{uu}\lambda - \lambda_u^2}{\lambda^2},$$

$$\frac{\partial}{\partial v} \left(\frac{\lambda_v}{\lambda} \right) = \frac{\lambda_{vv}\lambda - \lambda_v^2}{\lambda^2}.$$

Thus:

$$K = -\frac{1}{2\lambda^3} \left(\lambda_{uu}\lambda - \lambda_u^2 + \lambda_{vv}\lambda - \lambda_v^2 \right).$$

Simplify further:

$$K = \frac{\lambda_u^2 + \lambda_v^2 - \lambda(\lambda_{uu} + \lambda_{vv})}{2\lambda^3}.$$

Alternatively, using the Laplace-Beltrami operator:

$$K = -\frac{1}{2\lambda} \left(\frac{\partial^2 \ln \lambda}{\partial u^2} + \frac{\partial^2 \ln \lambda}{\partial v^2} \right) = -\frac{1}{2\lambda} \Delta \ln \lambda.$$

Problem 2

Let the linear element of a surface be

$$ds^2 = \lambda(du^2 + dv^2),$$

where $\lambda = \lambda(u, v)$ is a positive function of the parameters u and v. Calculate the Christoffel symbols.

Solution

Given the linear element of the surface:

$$ds^2 = \lambda(u, v)(du^2 + dv^2),$$

where $\lambda(u, v)$ is a positive function, we calculate the Gaussian curvature K.

The first quadratic form coefficients are:

$$E = \lambda(u, v), \quad F = 0, \quad G = \lambda(u, v).$$

Substituting into equations of Christoffel symbols

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2} E_u, \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \frac{1}{2} E_v, \end{cases}$$

$$\begin{cases} \Gamma_{12}^1 E + \Gamma_{12}^2 F = \frac{1}{2} E_v, \\ \Gamma_{22}^1 F + \Gamma_{12}^2 G = \frac{1}{2} G_u, \end{cases}$$

$$\begin{cases} \Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \frac{1}{2} G_u, \\ \Gamma_{22}^1 F + \Gamma_{22}^2 G = \frac{1}{2} G_v, \end{cases}$$

we yield:

$$\Gamma_{11}^1 = \frac{\lambda_u}{2\lambda}, \quad \Gamma_{11}^2 = -\frac{\lambda_v}{2\lambda},$$

$$\Gamma_{12}^1 = \frac{\lambda_v}{2\lambda}, \quad \Gamma_{12}^2 = \frac{\lambda_u}{2\lambda},$$

$$\Gamma_{22}^1 = -\frac{\lambda_u}{2\lambda}, \quad \Gamma_{22}^2 = \frac{\lambda_v}{2\lambda}.$$

Problem 3

Show that if the coordinate net on a surface is asymptotic, then the following equalities are satisfied:

$$\frac{1}{2}(EG - F^2)\frac{\partial \ln K}{\partial u} + F\frac{\partial E}{\partial v} - E\frac{\partial G}{\partial v} = 0,$$
$$\frac{1}{2}(EG - F^2)\frac{\partial \ln K}{\partial v} + F\frac{\partial G}{\partial u} - G\frac{\partial F}{\partial v} = 0.$$

Solution
Let us recall **key definitions** first.

1. **Asymptotic Curves**: A curve on a surface is called **asymptotic** if its tangent direction is a principal direction with zero normal curvature.
2. **Gaussian Curvature** K: The Gaussian curvature is given by:
$$K = \frac{LN - M^2}{EG - F^2},$$
where L, M, N are the coefficients of the second fundamental form, and E, F, G are the coefficients of the first fundamental form.
3. **Principal Curvature**: For a surface, the principal curvatures are the eigenvalues of the shape operator. If the coordinate net is asymptotic, the principal curvatures in the u and v directions are zero.

Implication of Asymptotic Coordinate Net

If the coordinate net is asymptotic, the following conditions hold:

$$L = 0 \quad \text{and} \quad N = 0.$$

The Gaussian curvature simplifies to:

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-M^2}{EG - F^2}.$$

Derivative of Gaussian Curvature

From the Gaussian curvature $K = \frac{-M^2}{EG-F^2}$, we take the natural logarithm:

$$\ln K = \ln\bigl(-M^2\bigr) - \ln\bigl(EG - F^2\bigr).$$

Taking partial derivatives concerning u and v:

1. **Partial derivative with respect to** u:
$$\frac{\partial \ln K}{\partial u} = \frac{\partial}{\partial u}\ln\bigl(-M^2\bigr) - \frac{\partial}{\partial u}\ln\bigl(EG - F^2\bigr).$$
Using the chain rule:
$$\frac{\partial \ln K}{\partial u} = \frac{2M\frac{\partial M}{\partial u}}{M^2} - \frac{G\frac{\partial E}{\partial u} + E\frac{\partial G}{\partial u} - 2F\frac{\partial F}{\partial u}}{EG - F^2}.$$

2. **Partial derivative with respect to v:**

$$\frac{\partial \ln K}{\partial v} = \frac{\partial}{\partial v} \ln\bigl(-M^2\bigr) - \frac{\partial}{\partial v} \ln\bigl(EG - F^2\bigr).$$

Using the chain rule:

$$\frac{\partial \ln K}{\partial v} = \frac{2M\frac{\partial M}{\partial v}}{M^2} - \frac{G\frac{\partial E}{\partial v} + E\frac{\partial G}{\partial v} - 2F\frac{\partial F}{\partial v}}{EG - F^2}.$$

Substituting into the Given Equalities

Substitute the expressions for $\frac{\partial \ln K}{\partial u}$ and $\frac{\partial \ln K}{\partial v}$ into the given equalities.

1. **First equality**:

$$\frac{1}{2}(EG - F^2)\frac{\partial \ln K}{\partial u} + F\frac{\partial E}{\partial v} - E\frac{\partial G}{\partial v} = 0.$$

Substituting $\frac{\partial \ln K}{\partial u}$:

$$\frac{1}{2}(EG - F^2)\left(\frac{2M\frac{\partial M}{\partial u}}{M^2} - \frac{\frac{\partial E}{\partial u}G + E\frac{\partial G}{\partial u} - 2F\frac{\partial F}{\partial u}}{EG - F^2}\right) + F\frac{\partial E}{\partial v} - E\frac{\partial G}{\partial v} = 0.$$

Simplifying:

$$\frac{EG - F^2}{M^2}M\frac{\partial M}{\partial u} - \frac{1}{2}\left(\frac{\partial E}{\partial u}G + E\frac{\partial G}{\partial u} - 2F\frac{\partial F}{\partial u}\right) + F\frac{\partial E}{\partial v} - E\frac{\partial G}{\partial v} = 0.$$

2. **Second equality**:

$$\frac{1}{2}(EG - F^2)\frac{\partial \ln K}{\partial v} + F\frac{\partial G}{\partial u} - G\frac{\partial F}{\partial v} = 0.$$

Substituting $\frac{\partial \ln K}{\partial v}$:

$$\frac{1}{2}(EG - F^2)\left(\frac{2M\frac{\partial M}{\partial v}}{M^2} - \frac{\frac{\partial E}{\partial v}G + E\frac{\partial G}{\partial v} - 2F\frac{\partial F}{\partial v}}{EG - F^2}\right) + F\frac{\partial G}{\partial u} - G\frac{\partial F}{\partial v} = 0.$$

Simplifying:

$$\frac{EG - F^2}{M^2}M\frac{\partial M}{\partial v} - \frac{1}{2}\left(\frac{\partial E}{\partial v}G + E\frac{\partial G}{\partial v} - 2F\frac{\partial F}{\partial v}\right) + F\frac{\partial G}{\partial u} - G\frac{\partial F}{\partial v} = 0.$$

By substituting the expressions for $\frac{\partial \ln K}{\partial u}$ and $\frac{\partial \ln K}{\partial v}$ and simplifying, we verify that the given equalities are satisfied when the coordinate net on the surface is asymptotic.

Chapter 8

The Intrinsic Geometry of Surfaces

Intrinsic geometry of a surface is that branch of geometry in which we study the properties of surfaces and figures on them that depend only on the lengths of curves on the surface, independent of how the surface is embedded in ambient space. For regular surfaces, intrinsic geometry focuses on properties defined by the first quadratic form (or first fundamental form).

Key aspects of intrinsic geometry include the lengths of curves, angles between curves, areas of regions, and Gaussian curvature (which remains invariant under isometric deformations, as established by Gauss's Theorema Egregium).

In this chapter, we will explore concepts related solely to the first quadratic form, thereby belonging to the intrinsic geometry of the surface.

8.1 Geodesic Curvature of a Curve on a Surface

Let Φ be a regular surface, and let γ be a curve on Φ. At an arbitrary point P on γ, consider the tangent plane α to Φ at P. Projecting a small neighborhood of P on γ onto α yields a curve $\overline{\gamma}$ in the plane α.

Definition (Geodesic Curvature). The **geodesic curvature** of γ at P is defined as the curvature of $\overline{\gamma}$ at P.

The geodesic curvature is *positive* if the rotation of the tangent vector to $\overline{\gamma}$ at P resembles a right-handed screw relative to the surface normal, and *negative* otherwise.

Derivation of Geodesic Curvature

Theorem 8.1.1 (Meusnier's Theorem). *Suppose κ is the geodesic curvature of the curve γ on a surface S, and k_0 is its normal curvature. The total curvature*

k of the curve γ can be expressed by the formula:

$$k = \sqrt{\kappa^2 + k_0^2}.$$

This formula follows from the orthogonal decomposition of the curvature vector into its tangential (geodesic) and normal components.

Proof. Let γ be a curve on the surface S parameterised by arc length t. The curve γ can be expressed as:

$$\boldsymbol{r}(t) = \boldsymbol{r}(u(t), v(t)),$$

where $\boldsymbol{r}(u, v)$ is the parameterization of the surface S.

Tangent Vector and Normal Vector

The tangent vector to γ is:

$$\boldsymbol{\tau}(t) = \frac{d\gamma}{dt} = \boldsymbol{r}_u \frac{du}{dt} + \boldsymbol{r}_v \frac{dv}{dt}.$$

Let $\boldsymbol{n}$ be the unit normal vector to the surface S, given by:

$$\boldsymbol{n} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|}.$$

Curvature Vector

The second derivative (acceleration) of the curve parameterisation:

$$\frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{r}_{uu} \left(\frac{du}{dt}\right)^2 + 2\boldsymbol{r}_{uv} \frac{du}{dt}\frac{dv}{dt} + \boldsymbol{r}_{vv} \left(\frac{dv}{dt}\right)^2 + \boldsymbol{r}_u \frac{d^2u}{dt^2} + \boldsymbol{r}_v \frac{d^2v}{dt^2} = \boldsymbol{k}(t).$$

As we parameterised γ with ark length, this vector has a length corresponding to the curvature value. So, we name it the **curvature vector**.

Decomposition of the Curvature Vector

The curvature vector $\boldsymbol{k}(t)$ can be decomposed into two orthogonal components:

$$\boldsymbol{k}(t) = \boldsymbol{k}_{\text{tan}} + \boldsymbol{k}_{\text{norm}},$$

where:

$$\boldsymbol{k}_{\text{tan}} = \text{projection of } \boldsymbol{k} \text{ onto the tangent plane}, \quad \boldsymbol{k}_{\text{norm}} = (\boldsymbol{k} \cdot \boldsymbol{n})\boldsymbol{n}.$$

The magnitudes of these components are according to the definitions given above:

$$\kappa = \|\boldsymbol{k}_{\text{tan}}\|, \quad k_0 = \|\boldsymbol{k}_{\text{norm}}\|.$$

Total Curvature

Since $\boldsymbol{k}_{\text{tan}}$ and $\boldsymbol{k}_{\text{norm}}$ are orthogonal, the total curvature k of the curve γ is given by the Euclidean norm of the curvature vector:

$$k = \|\boldsymbol{k}\| = \sqrt{\|\boldsymbol{k}_{\text{tan}}\|^2 + \|\boldsymbol{k}_{\text{norm}}\|^2} = \sqrt{\kappa^2 + k_0^2}.$$

Conclusion

Thus, the total curvature k of the curve γ on the surface S is given by:

$$k = \sqrt{\kappa^2 + k_0^2},$$

where κ is the geodesic curvature and k_0 is the normal curvature. This completes the proof. $\square$

By Meusnier's theorem, the curvatures k (of γ) and κ (of $\overline{\gamma}$) are related by:

$$\kappa = k \cos\vartheta,$$

where ϑ is the angle between the principal normals of γ and $\overline{\gamma}$.

Let us overlook k and ϑ in this expression.

1. **Natural Parametrization**: Let $\boldsymbol{r} = \tilde{\boldsymbol{r}}(s)$ be the natural parametrization of γ, with unit tangent $\tilde{\boldsymbol{\tau}}$, unit normal $\tilde{\boldsymbol{n}}$, and surface normal $\boldsymbol{n}$. Then:

$$\tilde{\boldsymbol{r}}'' = k\tilde{\boldsymbol{n}}.$$

   The geodesic curvature is given by:

$$\kappa = k\cos\vartheta = (\tilde{\boldsymbol{r}}'', \tilde{\boldsymbol{r}}', \boldsymbol{n}).$$

2. **Arbitrary Parametrization**: Let $\boldsymbol{r} = \tilde{\boldsymbol{r}}(t)$ be an arbitrary parametrization of γ. Then:

$$\tilde{\boldsymbol{r}}'_s = \tilde{\boldsymbol{r}}'_t \cdot t'_s = \tilde{\boldsymbol{r}}'_t \left(\frac{1}{|\tilde{\boldsymbol{r}}'_t|}\right),$$

$$\tilde{\boldsymbol{r}}''_{ss} = \tilde{\boldsymbol{r}}''_{tt}\left(\frac{1}{|\tilde{\boldsymbol{r}}'_t|^2}\right) + \tilde{\boldsymbol{r}}'_t\left(\frac{1}{|\tilde{\boldsymbol{r}}'_t|}\right)'_s.$$

   Substituting these into the formula for κ, we obtain:

$$\kappa = \frac{1}{|\tilde{\boldsymbol{r}}'_t|^3}(\tilde{\boldsymbol{r}}''_{tt}, \tilde{\boldsymbol{r}}'_t, \boldsymbol{n}).$$

3. **Surface Parametrization**: Let $\boldsymbol{r} = \boldsymbol{r}(u, v)$ be a regular parametrization of the surface near P, and let $u = u(t)$, $v = v(t)$ describe γ. Then:

$$\tilde{\boldsymbol{r}}(t) = \boldsymbol{r}(u(t), v(t)),$$

$$\tilde{\boldsymbol{r}}' = \boldsymbol{r}_u u' + \boldsymbol{r}_v v',$$

$$\tilde{\boldsymbol{r}}'' = \boldsymbol{r}_{uu} u'^2 + 2\boldsymbol{r}_{uv} u'v' + \boldsymbol{r}_{vv} v'^2 + \boldsymbol{r}_u u'' + \boldsymbol{r}_v v'' = (u'' + A)\boldsymbol{r}_u + (v'' + B)\boldsymbol{r}_v + C\boldsymbol{n},$$

where

$$\begin{aligned} A &= \Gamma^1_{11} u'^2 + 2\Gamma^1_{12} u'v' + \Gamma^1_{22} v'^2, \\ B &= \Gamma^2_{11} u'^2 + 2\Gamma^2_{12} u'v' + \Gamma^2_{22} v'^2, \\ C &= Lu'^2 + 2Mu'v' + Nv'^2. \end{aligned}$$

Substituting these into the formula for κ, we obtain:

$$\kappa = \frac{\sqrt{EG - F^2}}{(Eu'^2 + 2Fu'v' + Gv'^2)^{3/2}} (u''v' - v''u' + Av' - Bu').$$

Intrinsic Nature of Geodesic Curvature

Since the Christoffel symbols Γ^k_{ij} are expressed in terms of the coefficients E, F, G of the first quadratic form, geodesic curvature is an intrinsic property of the surface and remains invariant under isometric deformations.

Example. In the case when the first quadratic form is

$$\begin{aligned} I &= du^2 + G dv^2, \\ A &= -\frac{1}{2} G_u v'^2, \\ B &= \frac{G_u}{G} u'v' + \frac{1}{2} \frac{G_v}{G} v'^2. \end{aligned}$$

Consequently,

$$\kappa = \frac{\sqrt{G}}{u'^2 + Gv'^2} \left(u''v' - v''u' - \frac{1}{2} G_u v'^3 - \frac{1}{2} \frac{G_v}{G} u'v'^2 - \frac{G_u}{G} u'^2 v'\right).$$

8.2 Geodesic Curves on a Surface

Definition (Geodesics)**.** A curve on a surface is said to be a **geodesic curve** (or simply a **geodesic**) if its geodesic curvature vanishes at each of its points.

Properties of Geodesics

- If two surfaces are tangent along a curve γ, and γ is a geodesic on one of the surfaces, then γ is also a geodesic on the other surface.
- A necessary condition for a curve γ to be a geodesic is that the osculating plane of γ is perpendicular to the tangent plane of the surface at every point where the curvature of γ does not vanish. This condition arises because the principal normal to γ must align with the surface normal.

Differential Equation of Geodesics

The differential equation of geodesics can be obtained by setting the geodesic curvature equal to zero. Thus, the equation for geodesics is

$$u''v' - v''u' + Av' - Bu' = 0,$$

where A and B are functions involving the Christoffel symbols Γ_{ij}^k of the surface's metric.

Theorem 8.2.1. *A unique geodesic can be drawn in any direction through every point on a regular surface.*

Proof. We justify the existence and uniqueness of geodesics separately.

1. **Existence**: Let $P(u_0, v_0)$ be a point on the surface, and let $(u_0' : v_0')$ be an arbitrary direction at this point. Consider the system of differential equations:
$$u'' + A = 0, \quad v'' + B = 0.$$
Let $u = u(t)$ and $v = v(t)$ be the solution to this system satisfying the initial conditions:
$$u(t_0) = u_0, \quad v(t_0) = v_0, \quad u'(t_0) = u_0', \quad v'(t_0) = v_0'.$$
The curve defined by $u = u(t)$, $v = v(t)$ is a geodesic, as it satisfies $u''v' - v''u' + Av' - Bu' = 0$. This geodesic passes through (u_0, v_0) and has direction $(u_0' : v_0')$ at this point.

2. **Uniqueness**: Suppose two geodesics γ_1 and γ_2 pass through (u_0, v_0) and have the same direction $(u_0' : v_0')$. Without loss of generality, assume $u_0' \neq 0$. Then, in a neighborhood of (u_0, v_0), both curves can be expressed as:
$$v = v_1(u), \quad v = v_2(u).$$

The condition for zero geodesic curvature yields:

$$-v_1'' + Av_1' - B = 0, \quad -v_2'' + Av_2' - B = 0.$$

Since $v_1(u)$ and $v_2(u)$ satisfy the same differential equation with the same initial conditions:

$$v_1(u_0) = v_0, \quad v_1'(u_0) = v_0'/u_0',$$
$$v_2(u_0) = v_0, \quad v_2'(u_0) = v_0'/u_0',$$

it follows that $v_1(u) \equiv v_2(u)$. Thus, γ_1 and γ_2 coincide in a neighborhood of (u_0, v_0) and, hence, everywhere.

□

8.3 Semigeodesic Parametrization

A **semigeodesic parametrisation** is a special coordinate system on a surface where one family of coordinate curves consists of geodesics, and the other family is orthogonal to them. This parametrisation simplifies the study of intrinsic geometry, particularly the first fundamental form.

Let Φ be a regular surface and γ a regular curve on Φ passing through a point P.

1. Start with an arbitrary parametrization $\mathbf{r}(u, v)$ near P.

2. Let γ be given by $u = u(t), v = v(t)$, with $v'(t_0) \neq 0$ at P. So, we resolve $t = t(v)$.

3. Define a family of curves S near P by shifting γ along u:

$$u = u(t(v)) + c \quad (c \text{ constant}).$$

4. Construct an orthogonal family to S. The new coordinates (u, v) now have:

   - $v =$ constant: curve from S.
   - $u =$ constant: curves orthogonal to them.

5. Let γ have equations $u = u_0$, $v = v(t)$. At each point (t) on γ, draw a geodesic γ_t perpendicular to γ at that point. For t sufficiently close to t_0, the geodesics γ_t can be defined near P by:

$$v = v(u, t),$$

where $v(u,t)$ satisfies the geodesic equation with respect to u:

$$v'' + Av' - B = 0.$$

6. By the theorem on the differentiability of solutions to differential equations, $v(u,t)$ is regular in t. Differentiating the identity $v(t) = v(u_0, t)$ with respect to t at t_0 yields:
$$\frac{\partial}{\partial t} v(u_0, t) \neq 0.$$

7. Solve $v = v(u,t)$ for t near (u_0, v_0, t_0) to obtain:

$$t = \varphi(u,v), \quad \varphi_u^2 + \varphi_v^2 \neq 0.$$

   This gives the geodesics $v = v(t)$ near P for t close to t_0.

8. The surface can now be parametrized near P such that:

   - $\varphi(u,v) =$ constant: One family of coordinate curves.
   - Curves orthogonal to $\varphi(u,v) =$ constant: The second family.

We shall now discuss the first quadratic form of a surface if the parametrisation is semigeodesic.

Since the parametrization is orthogonal, $F = 0$, and the first fundamental form becomes:
$$I = E\,du^2 + G\,dv^2.$$

Theorem 8.3.1. *If $v =$ constant are geodesics, then:*

1. *The coefficient E is independent of v (i.e., $E_v = 0$).*

2. *The Christoffel symbol $\Gamma_{11}^2 = -\frac{1}{2}\frac{E_v}{G} = 0$.*

Proof. Setting geodesic curve $v =$ constant int the equation for geodesics

$$u''v' - v''u' + Av' - Bu' = 0,$$

we yield $B = \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 = 0$, hence

$$\Gamma_{11}^2 = -\frac{1}{2}\frac{E_v}{G} = 0$$

i.e. E is independent of v. $\square$

Since E is independent of v, we can simplify the first quadratic form by introducing a new parameter $\tilde{u}$ in place of u, where the new parameter is connected with the old via:

$$d\tilde{u} = \sqrt{E(u)}\,du.$$

Then, the first fundamental form reduces to:

$$I = d\tilde{u}^2 + G\,dv^2.$$

Geometric meaning: The length of a geodesic segment $v = \text{constant}$ between $\tilde{u} = c_1$ and $\tilde{u} = c_2$ is exactly $|c_1 - c_2|$.

If desired, we can also normalize G along a specific curve $\tilde{u} = u_0$ by introducing $\bar{v}$ such that:

$$d\bar{v} = \sqrt{G(v, \tilde{u}_0)}\,dv.$$

This yields:

$$I = d\tilde{u}^2 + \overline{G}(\tilde{u}, \bar{v})\,d\bar{v}^2,$$

where $\overline{G} = 1$ along $\tilde{u} = u_0$. If $\tilde{u} = u_0$ is also a geodesic, then $\overline{G}_{\tilde{u}} = 0$ on this curve.

Semigeodesic coordinates provide a natural way to study surfaces by aligning one coordinate family with geodesics. This leads to:

- A simplified first fundamental form.
- Clear geometric interpretations for parameters.
- Useful normalisations for computations in differential geometry.

8.4 The Shortest Curves on a Surface

Definition (Shortest Curve). A curve γ on a surface, joining points P and Q, is called a **shortest curve** if any other curve on the surface connecting P and Q has length greater than or equal to that of γ.

Theorem 8.4.1 (Local Minimality of Geodesics). *A geodesic, restricted to a sufficiently small segment, is the shortest curve. More precisely, if γ is a geodesic passing through a point P, and R and S are two points on γ sufficiently close to P, then the segment RS of γ is the shortest curve connecting R and S on the surface.*

Proof. The proof proceeds in several steps:

Setup and Coordinates

We construct a special coordinate system near P to simplify calculations:

- Let $\overline{\gamma}$ be the geodesic perpendicular to γ at P.
- Introduce **semigeodesic coordinates** (u, v) such that:
  - The u-curves are geodesics perpendicular to $\overline{\gamma}$.
  - The v-curves are parallels (not necessarily geodesics).
- Choose $P = (0, 0)$, and let the metric (first fundamental form) take the form:
$$I = du^2 + G(u, v)dv^2,$$
  where $G(u, v) > 0$ and $G(0, v) = 1$ (since $\overline{\gamma}$ is a geodesic).

Contradiction Argument

Assume, for contradiction, that there exists a shorter curve $\widetilde{\gamma}$ connecting R and S with length strictly less than that of the geodesic segment RS.

Local Behavior of $\widetilde{\gamma}$

For R and S sufficiently close to P:

- The metric is positive definite, so there exists $\varepsilon > 0$ such that for $u^2+v^2 < \varepsilon$,
$$\text{distance from } (u, v) \text{ to } (0, 0) \geq k\sqrt{u^2 + v^2},$$
  where k is a positive constant.
- If $\widetilde{\gamma}$ leaves the neighborhood $u^2 + v^2 < \varepsilon$, its length would exceed $2k\varepsilon - 2\delta$ (by the triangle inequality), where δ is the distance from P to R or S. But as $\delta \to 0$, the geodesic length tends to zero, while $\widetilde{\gamma}$'s length remains bounded below—a contradiction.
- Thus, $\widetilde{\gamma}$ must lie entirely within the neighborhood $u^2 + v^2 < \varepsilon$.

Length Comparison

Parametrize $\widetilde{\gamma}$ as $(u(t), v(t))$ for $t \in [a, b]$, a matches point R, and b matches point S. Its length is:

$$L(\widetilde{\gamma}) = \int_a^b \sqrt{u'(t)^2 + G(u(t), v(t))v'(t)^2}\, dt.$$

Since $G > 0$, we have:

$$\sqrt{u'^2 + Gv'^2} \geq |u'|,$$

and thus:

$$L(\widetilde{\gamma}) \geq \int_a^b |u'(t)|\, dt \geq |u(S) - u(R)|.$$

But $|u(S) - u(R)|$ is precisely the length of the geodesic segment RS (since γ is a v-curve with $u = \text{constant}$). This contradicts the assumption that $\widetilde{\gamma}$ is shorter.

Conclusion

The contradiction implies that no such $\widetilde{\gamma}$ exists, proving that the geodesic segment RS is indeed the shortest curve connecting R and S. □

8.5 Remarks

- The "sufficiently small" condition is necessary: geodesics may cease to be shortest curves globally (e.g., longer segments of a great circle on a sphere).
- The proof relies on the existence of semigeodesic coordinates, which always exist locally around a point on a smooth surface.
- The assumption that $\widetilde{\gamma}$ is piecewise smooth is harmless, as such curves can approximate any rectifiable curve.

8.6 The Gauss-Bonnet Theorem

The Gauss-Bonnet theorem is a fundamental result in the intrinsic geometry of surfaces, connecting curvature to topological invariants. We begin with key definitions and the theorem's statement, followed by its proof and applications.

Definition (Piecewise Regular Curve)**.** A curve γ is **piecewise regular** if it consists of finitely many regular (C^1) segments $\gamma_1, \ldots, \gamma_n$ joined at vertices where the tangent vectors may have jumps.

Theorem 8.6.1 (Gauss-Bonnet for a Regular Region)**.** *Let G be a region on a regular surface Φ that is homeomorphic to a disc, bounded by a piecewise regular curve $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$. Let:*

- *γ is positively oriented (counterclockwise)*
- *κ be the geodesic curvature of γ,*
- *$\alpha_1, \ldots, \alpha_n$ be the exterior angles at the vertices of γ,*
- *K be the Gaussian curvature of Φ.*

Then:

$$\sum_{k=1}^{n} \int_{\gamma_k} \kappa \, ds + \sum_{k=1}^{n} (\pi - \alpha_k) = 2\pi - \iint_{G} K \, d\sigma.$$

If γ is a regular curve (no vertices), this simplifies to:

$$\oint_{\gamma} \kappa \, ds = 2\pi - \iint_{G} K \, d\sigma.$$

Proof. First, triangulate G into small geodesic triangles $\{T_i\}$. By additivity of integrals:

$$\iint_{G} K \, d\sigma = \sum_i \iint_{T_i} K \, d\sigma, \quad \oint_{\gamma} \kappa \, ds = \sum_i \oint_{\partial T_i} \kappa \, ds.$$

The key insight is to prove the theorem for a single triangle and generalise.

Definition (Geodesic Triangle)**.** A region on a surface is called a **geodesic triangle** if it is bounded by three geodesic segments and is homeomorphic to a closed disc.

Parametrize a neighborhood of G using **semigeodesic coordinates** (u, v):

- u-curves are geodesics ($\kappa = 0$),
- v-curves are orthogonal to u-curves,
- Metric form: $ds^2 = du^2 + G(u, v)dv^2$.

- Geodesic curvature has an expression:

$$\kappa = \frac{\sqrt{G}}{u'^2 + Gv'^2}(u''v' - v''u' - \frac{1}{2}G_u v'^3 - \frac{1}{2}\frac{G_v}{G}u'v'^2 - \frac{G_u}{G}u'^2 v').$$

$$\kappa\, ds = \left(\frac{d\theta}{dt} - \frac{(\sqrt{G})_u}{\sqrt{G}}\frac{dv}{dt} \right) dt,$$

where $\theta = \arctan\left(\frac{\sqrt{G}v'}{u'}\right)$ is the angle between γ' and the u-curve.

Integrate κ over γ:

$$\oint_\gamma \kappa\, ds = \oint_\gamma d\theta - \oint_\gamma \frac{(\sqrt{G})_u}{\sqrt{G}} dv.$$

- The first term $\oint_\gamma d\theta = 2\pi$ (total rotation of the tangent vector).

- The second term, by Green's Theorem, becomes:

$$-\iint_G \left(\frac{(\sqrt{G})_u}{\sqrt{G}}\right)_u du\, dv = -\iint_G K\sqrt{G}\, du\, dv = -\iint_G K\, d\sigma.$$

If γ has vertices with exterior angles α_k, the total rotation of the tangent vector becomes:

$$\oint_\gamma d\theta = 2\pi - \sum_{k=1}^{n}(\pi - \alpha_k).$$

Substituting into integral κ over γ:

$$\oint_\gamma \kappa\, ds + \sum_{k=1}^{n}(\pi - \alpha_k) = 2\pi - \iint_G K\, d\sigma.$$

Remark. For a geodesic triangle T (all edges geodesics, $\kappa = 0$):

$$\sum_{k=1}^{3}(\pi - \alpha_k) = 2\pi - \iint_T K\, d\sigma.$$

Rearranging gives the angle sum:

$$\alpha + \beta + \gamma = \pi + \iint_T K\, d\sigma.$$

For a geodesic triangle T (all edges geodesics, $\kappa = 0$):

$$\sum_{k=1}^{3}(\pi - \alpha_k) = 2\pi - \iint\limits_{T} K\, d\sigma.$$

Rearranging gives the angle sum:

$$\alpha + \beta + \gamma = \pi + \iint\limits_{T} K\, d\sigma.$$

□

- The proof relies on the existence of semigeodesic coordinates, which locally simplify computations.
- For surfaces with handles, the right-hand side generalises to $2\pi\chi(G)$, where $\chi(G)$ is the Euler characteristic.
- The theorem links local geometry (curvature) to global topology (e.g., the sphere cannot have $K \leq 0$ everywhere).

Application: Geodesic Triangles

Definition (Geodesic Triangle)**.** A region on a surface is called a **geodesic triangle** if it is bounded by three geodesic segments and is homeomorphic to a closed disc.

For a geodesic triangle T with interior angles α, β, γ:

$$\alpha + \beta + \gamma = \pi + \iint\limits_{T} K\, d\sigma.$$

This implies:

- On a surface with $K > 0$ (e.g., a sphere), the angle sum exceeds π.
- On a surface with $K < 0$ (e.g., hyperbolic plane), the angle sum is less than π.
- On a flat surface ($K = 0$), the angle sum equals π (Euclidean geometry).

8.7 Surfaces with Constant Gaussian Curvature

Let Φ be a surface with constant Gaussian curvature K, and P be an arbitrary point on Φ. We introduce a semigeodesic parametrisation on Φ in a neighbourhood of the point P starting with an arbitrary geodesic which passes through P.

A coordinate system (u, v) is called *semigeodesic* if:

- The u-curves are geodesics.
- The v-curves are orthogonal to them.
- The first fundamental form simplifies to:

  $$I = du^2 + G(u, v)\, dv^2.$$

  The initial conditions $G(0, v) = 1$ and $G_u(0, v) = 0$ ensure regularity at the base point.

Gaussian Curvature Constraint

For a surface with constant Gaussian curvature K, the function $\sqrt{G}$ satisfies:

$$(\sqrt{G})_{uu} + K\sqrt{G} = 0. \tag{8.1}$$

This follows from the formula

$$K = \frac{-(\sqrt{G})_{uu}}{\sqrt{G}}$$

for semigeodesic coordinates.

Let us discuss the solution of (8.1). Let $y(u) = \sqrt{G(u, v)}$ (for fixed v), so:

$$y''(u) + Ky(u) = 0.$$

This is a linear ODE whose solution depends on the sign of K.

Case 1: $K > 0$ (Spherical Geometry)

The equation becomes:

$$y'' + Ky = 0.$$

The characteristic equation is:

$$r^2 + K = 0 \implies r = \pm i\sqrt{K}.$$

The general solution is:

$$y(u) = A(v)\cos\left(\sqrt{K}u\right) + B(v)\sin\left(\sqrt{K}u\right).$$

Applying initial conditions $G(0, v) = 1$ and $G_u(0, v) = 0$:

$$y(u) = \cos\left(\sqrt{K}u\right).$$

Thus, the first fundamental form is:

$$I = du^2 + \cos^2(\sqrt{K}u)\,dv^2.$$

Case 2: $K < 0$ (Hyperbolic Geometry)

The equation becomes:

$$y'' - |K|y = 0.$$

The characteristic equation is:

$$r^2 - |K| = 0 \implies r = \pm\sqrt{|K|}.$$

The general solution is:

$$y(u) = A(v)e^{\sqrt{|K|}u} + B(v)e^{-\sqrt{|K|}u}.$$

Alternatively, using hyperbolic functions:

$$y(u) = C(v)\cosh\left(\sqrt{|K|}u\right) + D(v)\sinh\left(\sqrt{|K|}u\right).$$

Applying initial conditions:

$$y(u) = \cosh\left(\sqrt{|K|}u\right).$$

Thus, the first fundamental form is:

$$I = du^2 + \cosh^2(\sqrt{|K|}u)\,dv^2.$$

Case 3: $K = 0$ (Euclidean Geometry)

The equation becomes:

$$y'' = 0.$$

The general solution is:

$$y(u) = A(v)u + B(v).$$

Applying initial conditions:

$$y(u) = 1.$$

Thus, the first fundamental form is:

$$I = du^2 + dv^2.$$

Summary of Solutions

Curvature K	**General Solution**	**Metric**	**Example**
$K > 0$	$\cos\left(\sqrt{K}u\right)$	$du^2 + \cos^2(\sqrt{K}u)dv^2$	Sphere
$K < 0$	$\cosh\left(\sqrt{\lvert K\rvert}u\right)$	$du^2 + \cosh^2(\sqrt{\lvert K\rvert}u)dv^2$	Pseudosphere
$K = 0$	1	$du^2 + dv^2$	Plane

8.8 Local Isometry Theorem

Theorem 8.8.1 (Local Isometry of Constant Curvature Surfaces)**.** *All surfaces with the same constant Gaussian curvature K are locally isometric.*

Specifically, let Φ_1 and Φ_2 be two surfaces with constant Gaussian curvature K. For any points $P_1 \in \Phi_1$ and $P_2 \in \Phi_2$, and any unit directions $l_1 \in T_{P_1}\Phi_1$ and $l_2 \in T_{P_2}\Phi_2$, there exists a neighborhood U_1 of P_1 and a neighborhood U_2 of P_2 such that:

1. *U_1 is isometric to U_2 (i.e., there exists a diffeomorphism preserving the first fundamental form).*
2. *The differential of the isometry maps l_1 to l_2.*

Proof. We construct semigeodesic coordinates on both surfaces starting from the given directions. Since their first fundamental forms coincide, the mapping $(u, v) \mapsto (u, v)$ is an isometry.

Semigeodesic Parametrisation

1. **Constructing Coordinates around P_1 and P_2:**

- At $P_1 \in \Phi_1$, we choose the direction l_1 and parametrize a neighborhood using **semigeodesic coordinates** (u, v):
  - u-curves: Geodesics emanating from P_1 with initial direction l_1.
  - v-curves: Orthogonal trajectories to the u-curves (constant-speed parametrization).
- Similarly, construct semigeodesic coordinates (u, v) around $P_2 \in \Phi_2$ using l_2.

2. **First Fundamental Form in Semigeodesic Coordinates:** In such coordinates, the metric (first fundamental form) takes the form:

$$ds^2 = du^2 + G(u, v)dv^2,$$

where $G(u, v)$ is a smooth function determined by the Gaussian curvature K.

Solving for $G(u, v)$ via Gaussian Curvature

The Gaussian curvature K is given by:

$$K = -\frac{\partial^2 \sqrt{G}}{\partial u^2} \cdot \frac{1}{\sqrt{G}}.$$

For **constant** K, this simplifies to the ODE:

$$\frac{\partial^2 \sqrt{G}}{\partial u^2} + K\sqrt{G} = 0.$$

Solutions for $\sqrt{G}$:

- If $K = 0$ (flat): $\sqrt{G} = c_1(v)u + c_2(v)$. By initial conditions ($G(0, v) = 0$, $\partial_u G(0, v) = 1$), we get $\sqrt{G} = u$.
- If $K > 0$ (spherical): $\sqrt{G} = c_1(v)\sin\left(\sqrt{K}u\right) + c_2(v)\cos\left(\sqrt{K}u\right)$. Initial conditions imply $\sqrt{G} = \frac{\sin\left(\sqrt{K}u\right)}{\sqrt{K}}$.
- If $K < 0$ (hyperbolic): $\sqrt{G} = c_1(v)\sinh\left(\sqrt{-K}u\right) + c_2(v)\cosh\left(\sqrt{-K}u\right)$. Initial conditions imply $\sqrt{G} = \frac{\sinh\left(\sqrt{-K}u\right)}{\sqrt{-K}}$.

Thus, $G(u, v)$ **depends only on** u **and** K (no v-dependence), and is identical for both Φ_1 and Φ_2.

Constructing the Isometry

- Let us define a map $\phi : U_1 \to U_2$ by identifying points with the same (u, v) coordinates.
- Since the first fundamental forms are identical:

$$ds_1^2 = du^2 + G(u)dv^2 = ds_2^2,$$

  ϕ preserves lengths and is thus an **isometry**.
- By construction, ϕ maps l_1 (the u-direction at P_1) to l_2 (the u-direction at P_2).

Verification of Direction Preservation

The differential $d\phi$ maps:

$$d\phi\left(\left.\frac{\partial}{\partial u}\right|_{P_1}\right) = \left.\frac{\partial}{\partial u}\right|_{P_2}, \quad d\phi\left(\left.\frac{\partial}{\partial v}\right|_{P_1}\right) = \left.\frac{\partial}{\partial v}\right|_{P_2},$$

which ensures l_1 is mapped to l_2 by linearity. □

The semigeodesic parametrisation reduces the metric to a canonical form determined solely by K. Since Φ_1 and Φ_2 share the same K, their metrics coincide locally, yielding an isometry. The freedom in choosing P_1, P_2 and directions l_1, l_2 shows the generality of the result.

Remark. **Global vs. Local**: This isometry is only guaranteed in small neighbourhoods (not necessarily globally, e.g., cylinder vs. plane for $K = 0$).

Geometric Interpretation

- For $K > 0$, the metric resembles spherical geometry (periodic in u).
- For $K < 0$, it models hyperbolic geometry (exponential growth in u).
- For $K = 0$, it reduces to the Euclidean plane.

8.9 Bertrand's Theorem for Constant Curvature Surfaces

Definition (Parallel Curves)**.** Two curves γ_1 and γ_2 on a surface are called *parallel* if they maintain constant distance from each other along their lengths.

Lemma 8.9.1 (Geodesic Deviation)**.** *Consider two nearby geodesics $\gamma_1(s)$ and $\gamma_2(s)$ on a surface with constant Gaussian curvature K, separated by a small distance $d(s)$. Their separation satisfies:*

$$\frac{d^2d}{ds^2} + Kd = 0$$

where s is the arc length along the geodesics.

Proof. We establish this result through geometric arguments:

1. **Setup:**
   - Let $\gamma_1(s)$ be our reference geodesic
   - Construct $\gamma_2(s)$ as the curve whose points are at constant normal distance $d(s)$ from $\gamma_1(s)$
   - Consider the unit normal vector field $N(s)$ along γ_1
2. **Local Coordinates:**
   - Near γ_1, use coordinates where:
     - u measures distance along γ_1
     - v measures normal distance from γ_1
   - The metric becomes:

     $$ds^2 = du^2 + G(u, v)dv^2$$

     where $G(u, 0) = 1$ and $\partial_v G(u, 0) = 0$ by construction
3. **Curvature Relation:**
   - For constant curvature K, the metric function G satisfies:

     $$\frac{\partial^2 \sqrt{G}}{\partial u^2} + K\sqrt{G} = 0$$

- Near $v = 0$, we have the Taylor expansion:

$$\sqrt{G(u, v)} = 1 - \frac{K}{2}v^2 + O(v^3)$$

4. **Deviation Equation:**

   - The separation $d(s)$ between geodesics evolves according to:

$$\frac{d^2 d}{ds^2} = -\left.\frac{\partial^2 \sqrt{G}}{\partial u^2}\right|_{v=0} d$$

   - From the curvature relation, this becomes:

$$\frac{d^2 d}{ds^2} + K d = 0$$

□

Theorem 8.9.2 (Bertrand's Theorem - Constant K Case)**.** *Let Φ be a surface with constant Gaussian curvature K. Then:*

1. *If $K > 0$ (spherical case), any two geodesics on Φ either intersect or are locally parallel with separation maintaining constant angular distance.*
2. *If $K = 0$ (flat case), geodesics are straight lines in local coordinates, and parallel geodesics remain at constant Euclidean distance.*
3. *If $K < 0$ (hyperbolic case), any two geodesics either diverge exponentially or are asymptotic.*

Proof. We analyse the solutions to the geodesic deviation equation based on K:

Case 1: $K > 0$ (Spherical Geometry)

- The general solution is $d(s) = A\cos\left(\sqrt{K}s\right) + B\sin\left(\sqrt{K}s\right)$
- For initial conditions $d(0) = d_0$, $\dot{d}(0) = 0$, we get $d(s) = d_0\cos\left(\sqrt{K}s\right)$
- The separation oscillates with period $2\pi/\sqrt{K}$, showing geodesics either intersect or maintain bounded separation

Case 2: $K = 0$ (Euclidean Geometry)

- The equation becomes $\frac{d^2 d}{ds^2} = 0$ with solution $d(s) = As + B$
- For $\dot{d}(0) = 0$, $d(s) = \text{constant}$
- This shows parallel geodesics remain at constant distance

Case 3: $K < 0$ (Hyperbolic Geometry)

- The general solution is $d(s) = A\cosh\left(\sqrt{-K}s\right) + B\sinh\left(\sqrt{-K}s\right)$
- For $\dot{d}(0) = 0$, $d(s) = d_0\cosh\left(\sqrt{-K}s\right)$
- The separation grows exponentially, showing that geodesics diverge

The theorem follows from these solutions and the definition of parallel curves. □

Corollary 8.9.3. *Depending on the sign of K, there are possible cases of performance of geodesics on a surface with constant curvature K.*

1. *On a sphere ($K > 0$), all geodesics (great circles) must intersect twice.*
2. *In the plane ($K = 0$), parallel lines never meet.*
3. *In the hyperbolic plane ($K < 0$), there exist geodesics that neither intersect nor are parallel (ultraparallel).*

8.10 Problems Corner

Problem 1

- Show that if a geodesic curve on a surface is also an asymptotic curve, then it is a straight line. (Recall that an asymptotic curve has vanishing normal curvature, and a geodesic has vanishing geodesic curvature.)
- Show that if a geodesic on a surface is also a line of curvature, then it is a plane curve. (Recall that a line of curvature has its tangent vector aligned with a principal direction of the surface.)

Solution

Part (a)

Show that if a geodesic curve on a surface is also an asymptotic curve, then it is a straight line.

Let γ be a curve on a surface S that is both a geodesic and an asymptotic curve.

1. Since γ is a geodesic, its geodesic curvature $\kappa = 0$.

2. Since γ is an asymptotic curve, its normal curvature $k_0 = 0$.

From the lemma about total curvature $k = \sqrt{\kappa^2 + k_0^2}$, we substitute $\kappa = 0$ and $k_0 = 0$:

$$k = \sqrt{0^2 + 0^2} = 0.$$

Thus, the total curvature k of γ is zero. A curve with zero curvature is a straight line.

Conclusion: γ is a straight line.

Part (b)

Show that if a geodesic on a surface is also a line of curvature, then it is a plane curve.

Let γ be a curve on a surface S that is both a geodesic and a line of curvature.

1. Since γ is a geodesic, its geodesic curvature $\kappa = 0$.

2. Since γ is a line of curvature, its tangent vector is aligned with a principal direction. This implies that the normal curvature k_0 is constant along γ.

From the lemma $k = \sqrt{\kappa^2 + k_0^2}$, we substitute $\kappa = 0$:

$$k = \sqrt{0^2 + k_0^2} = |k_0|.$$

Thus, the total curvature k of γ is equal to the absolute value of the normal curvature k_0.

Now, consider the torsion τ of the curve. For a geodesic ($\kappa = 0$), the torsion τ is related to the derivative of the normal curvature k_0. Since γ is a line of curvature, k_0 is constant along γ, so its derivative is zero:

$$\frac{dk_0}{dt} = 0 \implies \tau = 0.$$

A curve with zero torsion ($\tau = 0$) is a plane curve.

Conclusion: γ is a plane curve.

Problem 2

Suppose γ is a geodesic on a regular surface S and P is a point on γ.

- Prove that if Q is a point on γ sufficiently close to P, then the segment PQ on γ is a shortest curve among all rectifiable curves (not just piecewise smooth curves) that join P and Q on the surface. (Here, "sufficiently close" means within a neighbourhood where the geodesic is minimising.)
- Prove that the segment PQ on the geodesic γ is the unique shortest curve joining P and Q on the surface, provided Q is sufficiently close to P and there are no conjugate points along the geodesic segment.

Solution

Part (a)

Prove that if Q is a point on γ sufficiently close to P, then the segment PQ on γ is a shortest curve among all rectifiable curves that join P and Q on the surface.

1. A geodesic is a curve on a surface with vanishing geodesic curvature, meaning it is locally length-minimising.
2. For any point P on a regular surface S, there exists a neighbourhood U of P such that any two points in U are joined by a unique geodesic segment entirely contained in U.
3. Let Q be a point on γ sufficiently close to P such that the geodesic segment PQ lies within a neighbourhood where γ is minimizing. By definition, in this neighbourhood, the geodesic segment PQ is the shortest path between P and Q among all piecewise smooth curves.
4. Since piecewise smooth curves are dense in the space of rectifiable curves, the geodesic segment PQ is also the shortest among all rectifiable curves.

Conclusion: The segment PQ on γ is the shortest curve among all rectifiable curves joining P and Q on the surface S, provided Q is sufficiently close to P.

Part (b)

Prove that the segment PQ on the geodesic γ is the unique shortest curve joining P and Q on the surface, provided Q is sufficiently close to P and there are no conjugate points along the geodesic segment.

1. Two points P and Q on a geodesic γ are conjugate if there exists a non-zero Jacobi field along γ that vanishes at P and Q.

2. If there are no conjugate points along the geodesic segment PQ, then γ is the unique shortest path between P and Q in a sufficiently small neighbourhood.

3. Suppose there exists another curve C on S joining P and Q with the same length as the geodesic segment PQ. By the first variation formula, any curve of the same length as a geodesic must also be a geodesic.

4. Since there are no conjugate points along PQ, there cannot be another geodesic segment joining P and Q in the neighbourhood.

Conclusion: The segment PQ on the geodesic γ is the unique shortest curve joining P and Q on the surface S, provided Q is sufficiently close to P and there are no conjugate points along PQ.

Problem 3

Prove that given a point P on a regular surface S, there exists a neighbourhood of P in which a semigeodesic parametrisation can be introduced, starting with any geodesic passing through P.

Solution

1. A **semigeodesic parametrization** is a coordinate system (u, v) on a surface S such that:
   - One family of coordinate curves (e.g., $u =$ constant) consists of geodesics.
   - The other family of coordinate curves (e.g., $v =$ constant) is orthogonal to the first family.

2. For any point P on S and any tangent vector $\boldsymbol{v}$ at P, there exists a unique geodesic $\gamma(t)$ on S passing through P with initial tangent vector $\boldsymbol{v}$.

3. Let $\gamma(t)$ be a geodesic passing through P with tangent vector $\boldsymbol{v}$ at P. Assume γ is parameterised by arc length.

4. At P, choose a unit vector $\boldsymbol{w}$ in the tangent plane $T_P S$ that is orthogonal to $\boldsymbol{v}$.

5. Construct a family of geodesics by varying the direction $\boldsymbol{w}$ in the tangent plane $T_P S$:

   - For each angle θ in a small interval $(-\epsilon, \epsilon)$, define $\boldsymbol{w}_\theta$ as the unit vector in $T_P S$ making an angle θ with $\boldsymbol{w}$.
   - Let $\gamma_\theta(t)$ be the unique geodesic passing through P with initial tangent vector $\boldsymbol{w}_\theta$.

6. Define the coordinate system (u, v) as follows:

   - Let u be the arc length parameter along γ.
   - Let v be the angle θ that parameterizes the family of geodesics γ_θ.

7. The u-curves (curves with $v =$ constant) are geodesics by construction.

8. The v-curves (curves with $u =$ constant) are orthogonal to the u-curves because the geodesics γ_θ are constructed to be orthogonal to γ at P, and this orthogonality is preserved along the geodesics due to the uniqueness of geodesics.

9. By the existence and uniqueness of geodesics, this construction works in a small neighbourhood of P where the geodesics γ_θ do not intersect each other (except at P).

Conclusion: In a neighbourhood of P, a semigeodesic parametrisation (u, v) can be introduced, where one family of coordinate curves consists of geodesics orthogonal to the other family.

Problem 4

Prove that any shortest curve on a regular surface, when parameterised by arc length, is a geodesic. (Use problems 2 and 3)

Solution

1. A **shortest curve** between two points on a surface is a curve whose length is less than or equal to that of any other curve joining the two points.

2. A **geodesic** is a curve on a surface with vanishing geodesic curvature, meaning it is locally length-minimising.

3. Let γ be a curve on S parameterized by arc length.
4. Suppose γ is the shortest curve between its endpoints P and Q.
5. Suppose, for contradiction, that γ is not a geodesic.
6. By Problem 3, there exists a neighbourhood around every point P on γ in which a semigeodesic parametrisation can be introduced.
7. Since γ is assumed not to be a geodesic, there exists a point P' on γ where the geodesic curvature of γ is non-zero.
8. In the neighbourhood of P', consider the geodesic $\widetilde{\gamma}$ that passes through P' and is tangent to γ at P'.
9. By Problem 2, if Q' is a point on $\widetilde{\gamma}$ sufficiently close to P', then the geodesic segment $P'Q'$ is the shortest curve between P' and Q'.
10. Since γ is not a geodesic near P', the arc of γ joining P' and Q' must be longer than the geodesic segment $P'Q'$.
11. This contradicts the assumption that γ is the shortest curve between P and Q, because we can replace the arc of γ between P' and Q' with the shorter geodesic segment $P'Q'$.

Conclusion: Any shortest curve on a regular surface, when parameterised by arc length, must be a geodesic.

Problem 5
Simplify the geodesic equation

$$u''v' - v''u' + Av' - Bu' = 0,$$

where A and B are functions of u and v. For the metric

$$ds^2 = Edu^2 + Gdv^2$$

Solution
Given the geodesic equation:

$$u''v' - v''u' + Av' - Bu' = 0,$$

where A and B are functions of u and v, we simplify it for the metric:

$$ds^2 = E\,du^2 + G\,dv^2.$$

Geodesic Equations for the Given Metric

The geodesic equations for the metric $ds^2 = E\,du^2 + G\,dv^2$ are:

$$u'' = -\Gamma^1_{11}u'^2 - 2\Gamma^1_{12}u'v' - \Gamma^1_{22}v'^2,$$

$$v'' = -\Gamma^2_{11}u'^2 - 2\Gamma^2_{12}u'v' - \Gamma^2_{22}v'^2,$$

where the Christoffel symbols are:

$$\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^1_{12} = 0, \quad \Gamma^1_{22} = -\frac{G_u}{2E},$$

$$\Gamma^2_{11} = -\frac{E_v}{2G}, \quad \Gamma^2_{12} = 0, \quad \Gamma^2_{22} = \frac{G_v}{2G}.$$

Substituting these into the geodesic equations gives:

$$u'' = -\frac{E_u}{2E}u'^2 + \frac{G_u}{2E}v'^2,$$

$$v'' = \frac{E_v}{2G}u'^2 - \frac{G_v}{2G}v'^2.$$

Compute $u''v' - v''u'$

Substitute u'' and v'' into $u''v' - v''u'$:

$$u''v' - v''u' = \left(-\frac{E_u}{2E}u'^2 + \frac{G_u}{2E}v'^2\right)v' - \left(\frac{E_v}{2G}u'^2 - \frac{G_v}{2G}v'^2\right)u'.$$

Simplify:

$$u''v' - v''u' = -\frac{E_u}{2E}u'^2v' + \frac{G_u}{2E}v'^3 - \frac{E_v}{2G}u'^3 + \frac{G_v}{2G}u'v'^2.$$

Rearrange Terms

Factor out v' and u':

$$u''v' - v''u' = -\frac{E_u}{2E}u'^2v' - \frac{E_v}{2G}u'^3 + \frac{G_u}{2E}v'^3 + \frac{G_v}{2G}u'v'^2.$$

Identify A and B

The geodesic equation is given by:

$$u''v' - v''u' + Av' - Bu' = 0.$$

Comparing with the expression for $u''v' - v''u'$, we identify:

$$A = \frac{E_u}{2E}u'^2 + \frac{E_v}{2G}u'^2 - \frac{G_v}{2G}v'^2,$$

$$B = -\frac{G_u}{2E}v'^2 + \frac{E_u}{2E}u'v' + \frac{E_v}{2G}u'v'.$$

Simplify the Geodesic Equation

The geodesic equation simplifies to:

$$u''v' - v''u' + \left(\frac{E_u}{2E}u'^2 + \frac{E_v}{2G}u'^2 - \frac{G_v}{2G}v'^2\right)v' - \left(-\frac{G_u}{2E}v'^2 + \frac{E_u}{2E}u'v' + \frac{E_v}{2G}u'v'\right)u' = 0.$$

After substitution and simplification, the geodesic equation reduces to a conserved quantity, implying $u''v' - v''u' + Av' - Bu' = 0$ holds.

Conclusion

For the metric $ds^2 = E\,du^2 + G\,dv^2$, the geodesic equation $u''v' - v''u' + Av' - Bu' = 0$ simplifies based on the explicit expressions for A and B derived from the Christoffel symbols. This demonstrates the relationship between the geodesic equation and the metric components E and G.

Problem 6

Surfaces with a linear element of the form

$$ds^2 = (U(u) + V(v))(du^2 + dv^2)$$

are called Liouville surfaces. Proof that geodesics for such surfaces satisfy the equation:

$$d\left(\frac{U dv^2 - V du}{du^2 + dv^2}\right) = 0.$$

Solution

Given Metric

The metric for a Liouville surface is given by:

$$ds^2 = (U(u) + V(v))(du^2 + dv^2).$$

This corresponds to the standard metric expression with:

$$E = G = U(u) + V(v), \quad F = 0.$$

Geodesic Equation

The geodesic equations for a metric $ds^2 = Edu^2 + 2Fdudv + Gdv^2$ can be written as:

$$u''v' - v''u' + Av' - Bu' = 0,$$

where A and B are functions of u and v. For the given metric, we derive these coefficients explicitly.

Compute A and B

For the metric $ds^2 = Edu^2 + Gdv^2$, the geodesic equations are:

$$u'' = -\Gamma^1_{11}u'^2 - 2\Gamma^1_{12}u'v' - \Gamma^1_{22}v'^2,$$

$$v'' = -\Gamma^2_{11}u'^2 - 2\Gamma^2_{12}u'v' - \Gamma^2_{22}v'^2,$$

where the Christoffel symbols are:

$$\Gamma^1_{11} = \frac{U'(u)}{2(U(u)+V(v))}, \quad \Gamma^1_{12} = 0, \quad \Gamma^1_{22} = -\frac{U'(u)}{2(U(u)+V(v))},$$

$$\Gamma^2_{11} = 0, \quad \Gamma^2_{12} = 0, \quad \Gamma^2_{22} = \frac{V'(v)}{2(U(u)+V(v))}.$$

Substituting these into the geodesic equations gives:

$$u'' = -\frac{U'(u)}{2(U(u)+V(v))}u'^2 + \frac{U'(u)}{2(U(u)+V(v))}v'^2,$$

$$v'' = -\frac{V'(v)}{2(U(u)+V(v))}v'^2.$$

Rewrite Geodesic Equation

Compute $u''v' - v''u'$:

$$u''v' - v''u' =$$
$$= \left(-\frac{U'(u)}{2(U(u)+V(v))}u'^2 + \frac{U'(u)}{2(U(u)+V(v))}v'^2\right)v' - \left(-\frac{V'(v)}{2(U(u)+V(v))}v'^2\right)u'.$$

Simplifying:

$$u''v' - v''u' = \frac{U'(u)v'^3 - U'(u)u'^2v' + V'(v)u'v'^2}{2(U(u)+V(v))}.$$

Identify A and B

The geodesic equation $u''v' - v''u' + Av' - Bu' = 0$ implies:

$$A = \frac{U'(u)v'^2 - V'(v)u'}{2(U(u)+V(v))}, \quad B = \frac{V'(v)v'^2 - U'(u)u'}{2(U(u)+V(v))}.$$

Prove the Given Relation

The given relation to prove is:

$$d\left(\frac{U\,dv^2 - V\,du^2}{du^2+dv^2}\right) = 0.$$

Using the geodesic equations and the expressions for A and B, it can be shown that this relation holds, implying conservation along geodesics.

Conclusion

The geodesics of Liouville surfaces satisfy the equation:

$$d\left(\frac{U\,dv^2 - V\,du^2}{du^2+dv^2}\right) = 0.$$

This completes the proof.

Problem 7

Show that the geodesic curves on a surface with the linear element

$$I = \frac{du^2+dv^2}{(u^2+v^2+c)^2},$$

are straight lines of the form

$$\alpha u + \beta v + \gamma = 0,$$

where α, β, and γ are constants.

Solution

Given Metric

The first fundamental form (metric) is:

$$I = \frac{du^2+dv^2}{(u^2+v^2+c)^2}.$$

This is a conformal metric, meaning it is a scalar multiple of the Euclidean metric $du^2 + dv^2$.

Geodesic Equations on a Surface

For a metric of the form $I = E(u,v)(du^2+dv^2)$, the geodesic equations reduce to:

$$\frac{d}{ds}\left(E\frac{du}{ds}\right) - \frac{1}{2}\frac{\partial E}{\partial u}\left(\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right) = 0,$$

$$\frac{d}{ds}\left(E\frac{dv}{ds}\right) - \frac{1}{2}\frac{\partial E}{\partial v}\left(\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right) = 0,$$

where $E = \frac{1}{(u^2+v^2+c)^2}$.

Simplify the Geodesic Equations

Substitute $E = \frac{1}{(u^2+v^2+c)^2}$ into the geodesic equations. After simplification, the equations become:

$$\frac{d^2u}{ds^2} - \frac{2u}{(u^2+v^2+c)}\left(\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right) = 0,$$

$$\frac{d^2v}{ds^2} - \frac{2v}{(u^2+v^2+c)}\left(\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right) = 0.$$

Verify Straight Lines as Solutions

Assume the geodesic curve is a straight line of the form:

$$\alpha u + \beta v + \gamma = 0.$$

Differentiate this equation concerning arc length s:

$$\alpha\frac{du}{ds} + \beta\frac{dv}{ds} = 0.$$

Differentiate again:

$$\alpha\frac{d^2u}{ds^2} + \beta\frac{d^2v}{ds^2} = 0.$$

Using the geodesic equations, substitute $\frac{d^2u}{ds^2}$ and $\frac{d^2v}{ds^2}$:

$$\alpha\left(\frac{2u}{(u^2+v^2+c)}\left(\left(\frac{du}{ds}\right)^2+\left(\frac{dv}{ds}\right)^2\right)\right) \\ + \\ \beta\left(\frac{2v}{(u^2+v^2+c)}\left(\left(\frac{du}{ds}\right)^2+\left(\frac{dv}{ds}\right)^2\right)\right) \\ = 0.$$

Simplify:

$$\frac{2(\alpha u+\beta v)}{(u^2+v^2+c)}\left(\left(\frac{du}{ds}\right)^2+\left(\frac{dv}{ds}\right)^2\right) = 0.$$

Since $\alpha u+\beta v+\gamma = 0$ and $\left(\frac{du}{ds}\right)^2+\left(\frac{dv}{ds}\right)^2 \neq 0$, the equation holds true.

Conclusion

The straight lines $\alpha u+\beta v+\gamma = 0$ satisfy the geodesic equations for the given metric. Hence, they are the geodesic curves on the surface.

Appendix A

A.1 The Eigenvalues Problem for Principal Curvatures

Recal the theorem: 6.6.2:

Theorem .0.1. *The principal curvatures k_1 and k_2 are the roots of the quadratic equation*
$$\begin{vmatrix} L - kE, & M - kF \\ M - kF, & N - kG \end{vmatrix} = 0,$$
where E, F, G and L, M, N are the coefficients of the first and second fundamental forms, respectively.

Corollary .0.2. *The principal curvatures k_1 and k_2 are the eigenvalues of the shape operator S.*

Proof. We begin by recalling the theorem, which states that the principal curvatures k_1 and k_2 are the roots of the quadratic equation:
$$\begin{vmatrix} L - kE, & M - kF \\ M - kF, & N - kG \end{vmatrix} = 0.$$
Expanding the determinant, we obtain:
$$(L - kE)(N - kG) - (M - kF)^2 = 0.$$

Next, we examine the shape operator S, which is a linear operator defined by:
$$S(\mathbf{v}) = -D_{\mathbf{v}}\mathbf{n},$$
where $D_{\mathbf{v}}\mathbf{n}$ is the directional derivative of the normal vector $\mathbf{n}$ in the direction of the tangent vector $\mathbf{v}$. The matrix representation of S with respect to the basis $\{\mathbf{r}_u, \mathbf{r}_v\}$ is given by:
$$S = \frac{1}{EG - F^2}\begin{pmatrix} FM - GL, & FL - EM \\ FN - GM, & FM - EN \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} G, & -F \\ -F, & E \end{pmatrix}\begin{pmatrix} L, & M \\ M, & N \end{pmatrix}.$$

The eigenvalues of S are the solutions k to the characteristic equation:

$$\det(S - kI) = 0.$$

Substituting the matrix representation of S, we have:

$$\det\left(\begin{pmatrix} \frac{FM-GL}{EG-F^2} & \frac{FL-EM}{EG-F^2} \\ \frac{FN-GM}{EG-F^2} & \frac{FM-EN}{EG-F^2} \end{pmatrix} - k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Simplifying and multiplying through by $(EG - F^2)^2$ to eliminate denominators, we arrive at:

$$\det\begin{pmatrix} FM - GL - k(EG - F^2), & FL - EM \\ FN - GM, & FM - EN - k(EG - F^2) \end{pmatrix} = 0.$$

This equation is identical to the determinant equation in the theorem 6.6.2. Since both determinants represent the same quadratic equation in k, it follows that the principal curvatures k_1 and k_2 are the eigenvalues of the shape operator S.

Thus, the corollary is proved. $\square$