

Topics for the Final Test

It is necessary to know the definitions of the listed concepts and the exact formulations of the theorems and lemmas. Proofs and algorithmic steps are not required.

Part I: Optimization Theory

1. Unconstrained Optimization Concepts (Definitions)

- You should know the definitions of: The unconstrained optimization problem; local / strict local / global minimum; stationary point; descent direction; gradient and Hessian; positive (semi)definiteness.

Relevant Theorems:

- Theorem 1 (Taylor's Theorem).
- Theorem 2 (Necessary Condition for a Local Minimum).
- Theorem 3 (Second-Order Necessary Condition).
- Theorem 4 (Second-Order Sufficient Condition).
- Theorem 5 (Global Minimum for Convex Functions).

2. Line-Search Concepts (Definitions)

- You should know the definitions of: Descent Direction; Sufficient Decrease (Armijo) condition; Curvature condition; Wolfe and Strong Wolfe conditions; Goldstein conditions; step length (α).

Relevant Results:

- Lemma 1 (Existence of Step Lengths).
- Theorem 6 (Global Convergence of Line Search Methods).

3. Newton and Quasi-Newton Methods (Statements)

- Know: Newton step definition; local quadratic convergence statement; the Hessian modification idea; Quasi-Newton secant condition and positive-definite updates (BFGS/SR1) — their definitions and theorem statements.

Relevant Theorems/Lemmas:

- Theorem 8 (Local Convergence of Newton's Method).
- Theorem 9, Theorem 10, Theorem 11 (Quasi-Newton / Superlinear Convergence Statements).

4. Trust-Region Methods

- Know: The quadratic trust-region model; trust-region radius strategy; the Cauchy point (closed-form); dogleg path idea; the hard case description.

Relevant Results:

- Lemma 2, Lemma 3 (Dogleg Properties) and Lemma 4 (Cauchy Decrease).
- Theorem 12 (Characterization of Trust-Region Solution / Moré–Sorensen Type Result).
- Theorem 14–15 (Global Convergence Statements for Trust-Region Algorithms).

5. Conjugate Gradient and Conjugate Directions

- Know: Definition of A -conjugacy; Krylov subspaces; finite termination property; orthogonality of residuals; spectral convergence bounds; preconditioning concept.

Relevant Theorems:

- Theorem 16 (Convergence of Conjugate Direction Method)
- Theorem 17 (Expanding Subspace Minimization).
- Theorem 18 (Orthogonality and Krylov Subspaces).
- Theorem 19 (Finite-Termination Property).

6. Constrained Optimization — Core Definitions

- You should know: problem statement; feasible point and feasible set; active set; feasible sequence; tangent cone; set of linearized feasible directions; active/inactive constraints; LICQ and MFCQ (statements as in the notes).

7. Lagrangian, KKT, and First-Order Optimality

- Know: The Lagrangian function; KKT conditions (primal feasibility, stationarity, dual feasibility, complementary slackness); role of constraint qualifications for necessity/sufficiency.

Relevant Items in the Notes:

- Lemma 5 (Tangent Cone and First-Order Feasible Directions).
- Lemma 6 (Farkas) — the theorem of the alternative used to derive multipliers.
- Theorem 21 (First-Order Necessary Conditions).
- Theorem 22.

8. Second-Order Optimality for Constrained Problems

- Know: Critical cone definition; second-order necessary and sufficient conditions (formulated with the Lagrangian Hessian on the critical cone); normal cone definition.

Relevant Theorems:

- Theorem 23 (Second-Order Necessary Conditions).
- Theorem 24 (Second-Order Sufficient Conditions).
- Theorem 25 (Geometric Necessary Condition).

9. Duality Theory (Nonlinear and LP)

- Know: Dual function; weak duality; duality for convex quadratic programming; Wolfe dual (construction and interpretation); First-Order Conditions for the Dual Problem.

Relevant Theorems:

- Theorem 26 (Concavity of the Dual Objective).
- Theorem 27 (Weak Duality).
- Theorem 28 (Solutions of the Dual Problem)
- Theorem 31 (Strong Duality / Conditions).
- Theorem 30 (Wolfe Dual Formulation).

10. Linear Programming and the Simplex Method (Concepts)

- Know: Standard form of LP; vertices and basic feasible points; statement of the Fundamental Theorem of Linear Programming; primal–dual relationships.

Relevant Results:

- Theorem 32 (Fundamental Theorem of Linear Programming).
- Theorem 33 (Vertices and Basic Feasible Points).
- Theorem 34 (Termination Properties of the Simplex Method).

11. Quadratic Programming (QP)

- Know: Definition of a quadratic program; equality-constrained QP; KKT system for QP; sufficiency conditions for convex QP (positive semidefinite Hessian); relation to KKT matrix inertia and solution uniqueness.

Relevant Theorems:

- Theorem 35 (Global Solution for the Equality-Constrained QP).
- Theorem 36 (KKT Matrix and Inertia).
- Theorem 37 (Sufficiency for Convex QP).
- Theorem 39 (About Strictly Decreasing along p_k).

Part II: Linear Models and Design of Experiments

1. Descriptive Regression — General Setting

- Know: Regression purpose; dependent vs. independent variables; the model function $\eta(x_1, \dots, x_m)$; role of regression analysis as a descriptive modeling tool.

2. Parametric Regression and Least Squares Formulation

- Know: Idea of model fitting; residuals and error minimization; least squares criterion and its justification.

3. Least Squares Estimation (LSE)

- Know: Definition of the least squares estimator; system of normal equations; convexity of the least squares criterion.

Relevant Results:

- Lemma 1 (Normal Equations — Existence and Optimality Condition).

4. Classical Linear Regression Model

- Know: Definition of the classical linear regression model; assumptions of zero mean, uncorrelated and homoscedastic errors; definition of OLS.

5. Best Linear Unbiased Estimator (BLUE)

- Know: Definition and conditions for linear unbiased estimators; covariance dominance; conditions (a)–(c) for BLUE; variance minimization principle.

Relevant Results:

- Theorem 1 (Gauss–Markov Theorem).
- Lemma 2–Lemma 4 (Unbiasedness, Linearity, Variance Comparison).

6. Estimability of Linear Functions

- Know: Definition of an estimable function $\tau = T\theta$; equivalent conditions (Lemma 5).

Relevant Results:

- Lemma 5 (Equivalent Conditions of Estimability).

7. BLUE for a Linear Function of Parameters

- Know: Theorem 2 (BLUE for $\tau = T\theta$); expression for $\hat{\tau}$ and its covariance matrix.

8. OLS Estimation in the Singular Case

- Know: OLS solutions for singular $X^T X$; concept of the generalized inverse and Moore–Penrose pseudoinverse; Penrose conditions; representation of all OLS estimators via the generalized inverse.

Relevant Results:

- Theorem 3 (Generalized Inverse Solution).
- Lemma 6 (Condition for Generalized Inverse).
- Lemma 7 (Existence of Generalized Inverse).

9. Estimability of Linear Parametric Functions

- Know: Conditions of estimability of function $\tau = T\theta$; form of the BLUE for τ in the case of singular $X^T X$.

Relevant Results:

- Theorem 4 (BLUE for $\tau = T\theta$ in the General Linear Model).

10. Weighted and Generalized Least Squares (WLS / GLS)

- Know: Generalized linear regression model; definition and properties of the GLS estimator; covariance and optimality (BLUE in the generalized model).

Relevant Results:

- Properties of the GLS Estimator
- Definition of the GLS estimator and its covariance matrix.

11. Regression with Prior Information on Parameters

- Know: Four Typical Types of Prior Information; Model with Linear Constraints; Transformation and Reduced Model; Minimax Estimator; form of Bayesian estimator; the explicit forms of the Minimax Estimator.

Relevant Definitions:

- Definition (Minimax Estimator)
- Definition (Bayesian Estimator)
- Theorem 6 (Explicit Minimax Estimator under Set Constraints).

12. Design of Experiments — Basic Concepts

- Know: Model structure; standard assumptions (a)–(f); definition of discrete design ξ_N and approximate design ξ ; weights as probabilities; definition of a full design space.

13. Information Matrix and Variance of Estimates

- Know: Definition of the information matrix $M(\xi)$; relation to covariance of parameter estimates; criteria of optimality based on $M(\xi)$.

Relevant Definitions:

- Theorem 7 (Properties of Information Matrices).

14. Optimality Criteria in Design

- Know: D -, e_k -, E -, L - and G -optimality criteria; interpretation via $\det M(\xi)$, $\text{tr } M^{-1}(\xi)$, and $\lambda_{\min}(M(\xi))$.

15. Kiefer-Wolfowitz Equivalence Theorem

- Know: Statements of the Kiefer-Wolfowitz Equivalence Theorem and Lemma 11-14.

16. Equivalence Theorems and Extremal Polynomials

- Know: Equivalence theorem for E -optimality; equivalence theorem for L -optimality (non-singular and singular cases); classes Ξ_L and Ξ_L^* ; generalized extremal polynomial for singular designs.

Relevant Theorems:

- Equivalence Theorem for the E -Criterion

- Equivalence Theorem for L -Optimality
- Equivalence Theorem for L -Optimality (Singular Case)
- Theorem 8 (Extremal Polynomial in the Singular Case).

17. Analytical Solutions for D -Optimal Design

- Know: D -optimal designs for polynomial and trigonometric models.

Relevant Theorems:

- Theorem 9 (D -Optimal Designs for the Polynomial Model)
- Theorem 10 (D -Optimal Designs for the Trigonometric Model).