

1. 绪论

"常" → 单一自变量求微分. 阶数 - 最高阶微分的阶数.

一般形式 1) 微分(全通)方程.

2) 微分方程 (自变量, 未知函数及其导数)

$$F(t, \psi(t), \psi'(t)) = 0.$$

$$F(t, \psi(t), \psi'(t), \dots, \psi^{(n)}(t)) = 0 \quad \text{or} \quad F(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0.$$

线性. $\frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x) y = f(x) \quad a_1(x), \dots, a_n(x), f(x) \text{ 为已知函数.}$

$$F(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0. \rightarrow y = \psi(x) \text{ 显式解.}$$

$$\Phi(x, y) = 0. \text{ 隐式解.}$$

通解: $y = \psi(x, c_1, \dots, c_n)$

定解: 当 $x = x_0$ 时, $y = y_0, \frac{dy}{dx} = y_0^{(1)}, \dots, \frac{d^{n-1} y}{dx^{n-1}} = y_0^{(n-1)}$ 左侧为给定常数 (初值条件).

积分曲线: 一阶微分方程 $\frac{dy}{dx} = f(x, y)$ 解 $y = \psi(x)$. 积分曲线

通解 $y = \psi(x, c)$ 一族曲线.

特解 $\psi(x_0) = y_0$ 过 (x_0, y_0) 曲线.

曲线上某点斜率 $f(x, y) \Leftrightarrow$ 是积分曲线.

方向场: $f(x, y)$ 过各点小线段斜率. 等倾斜线.

解出最高阶导数的形式: $z^{(n)} = g(t, z, z^{(1)}, \dots, z^{(n-1)}) \Rightarrow$

$$\begin{cases} \frac{dz}{dt} = z' \\ \vdots \\ \frac{d z^{(n-1)}}{dt} = g(t, z, \dots, z^{(n-1)}) \end{cases}$$

向量形式

$$\frac{d\vec{y}}{dt} = \vec{f}(t, \vec{y}), \vec{y} \in \mathbb{R}^n$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} z \\ z' \\ \vdots \\ z^{(n-1)} \end{bmatrix}$$

$$\vec{f}(t, \vec{y}) = \begin{bmatrix} f_1(t, y_1, \dots, y_n) \\ \vdots \\ f_n(t, y_1, \dots, y_n) \end{bmatrix}$$

Sept 19th.

$$2.1.2. (1) \frac{dy}{dx} = \frac{1}{(x+y)^2}$$

$$\Rightarrow x+y=t, \quad y=t-x.$$

$$\Leftrightarrow \frac{dt - dx}{dx} = \frac{1}{t^2}$$

$$\Leftrightarrow t^2 dt = (t^2 + 1) dx$$

$$(1 - \frac{1}{t^2+1}) dt = dx.$$

$$t - \arctan t = x + C$$

$$y = \arctan(x+y) + C.$$

$$(4) \frac{dy}{dx} = (x+1)^2 + (4y+1)^2 + 8xy + 1. \\ = (x+1)^2 + 8y(x+1) + 16y^2 + 2.$$

$$\frac{dy}{dx} = (x+1+4y)^2 + 2.$$

$$x+1+4y=t, \quad y = \frac{t-x-1}{4}$$

$$\frac{dt - dx}{dx} = 4(t^2 + 2)$$

$$\Rightarrow dt = (4t^2 + 9) dx$$

$$\Rightarrow \frac{1}{6} \frac{d(\frac{2}{3}t)}{\frac{4t^2+1}{9}} = dx$$

$$\frac{1}{6} \arctan \frac{2}{3}(x+4y+1) = x+C$$

$$(2) \frac{dy}{dx} = \frac{2x-y+1}{x-2y+1}$$

$$\left\{ \begin{array}{l} 2x-y+1=0 \\ x-2y+1=0 \end{array} \right. \Rightarrow \left(-\frac{1}{3}, \frac{1}{3} \right).$$

$$\begin{cases} x_1 = x + \frac{1}{3} \\ y_1 = y - \frac{1}{3} \\ \frac{dx_1}{dy_1} = \frac{2x_1 - y_1}{x_1 - 2y_1} \end{cases}$$

$$y_1 = tx_1$$

$$(x_1 - 2tx_1)dx_1 = (2x_1 - tx_1)(tdx_1 + x_1 dt)$$

$$\begin{aligned} x_1 dx_1 - 2tx_1 dx_1 &= 2tx_1 dx_1 + 2x_1^2 dt - t^2 x_1 dx_1 - x_1^2 dt \\ dx_1 &= 4t dx_1 + 2x_1 dt - t^2 dx_1 - tx_1 dt. \end{aligned}$$

$$(t^2 - 4t + 1) dx_1 = x_1 (2-t) dt.$$

$$\frac{dx_1}{x_1} = \frac{-\frac{1}{2}dt + 4t + 1}{t^2 - 4t + 1}$$

$$\ln|x_1| + \frac{1}{2}\ln|t^2 - 4t + 1| + C = 0$$

$$\ln|\frac{x+\frac{1}{3}}{x-\frac{1}{3}}| + \frac{1}{2}\ln|\frac{(y-\frac{1}{3})^2}{x+\frac{1}{3}}| - 4\left(\frac{y-\frac{1}{3}}{x+\frac{1}{3}}\right) + 1 + C = 0$$

$$\begin{cases} x = -\frac{1}{3} \\ y = \frac{1}{3} \end{cases}$$

$$(5) \frac{dy}{dx} = \frac{y^6 - 2x^2}{2xy^5 + x^2y^3}$$

$$y = t^m$$

$$(mt^{m-1}dt)(2x \cdot t^{5m} + x^2 t^{2m}) = (t^{6m} - 2x^2) dx$$

$$1+6m-1 = 2+3m-1 = 6m = 2. \Rightarrow m = \frac{1}{3}$$

$$\frac{2}{3}x \cdot t dt + \frac{1}{3}x^2 dt = t^2 dx - 2x^2 dx.$$

$$\text{Let } t = sx.$$

$$(\frac{2}{3}sx^2 + \frac{1}{3}s^2 x^3)(x ds + s dx) = (s^2 - 2)x^2 dx.$$

$$x=0 \text{ or } (\frac{2}{3}s + \frac{1}{3})(x ds + s dx) = (s^2 - 2) dx$$

$$(2s+1)x ds = (s-3)(s+2) dx.$$

$$\frac{(2s+1)ds}{(s-3)(s+2)} = \frac{dx}{x}$$

$$\ln|s^2 - 5 + b| + \frac{2}{5}\ln|\frac{s-3}{s+2}| = \ln|x|$$

$$\ln|\frac{y^6}{x^2} - \frac{y}{x} + b| + \frac{2}{5}\ln|\frac{y^2 - 3x}{y^2 + 2x}| = \ln|x| + C$$

$$(b) \frac{dy}{dx} = \frac{2x^3 + 3xy^2 + x}{3x^2 y + 2y^3 - y}$$

$$\Rightarrow \frac{dy^2}{dx^2} = \frac{2x^2 + 3y^2 + 1}{3x^2 + 2y^2 - 1}$$

$$\begin{cases} 2x^2 + 3y^2 + 1 = 0 \\ 3x^2 + 2y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} t = x^2 - 1 \\ s = y^2 + 1 \end{cases}$$

$$\frac{ds}{dt} = \frac{2t+3s}{3t+2s}$$

$$s = kt.$$

$$t(3+2k) + (tdk + kdt) = t(2+3k) dt \quad t>0 \Rightarrow \text{no soln.}$$

$$(3+2k)t + dk + (3+2k) \cdot kdt = (2+3k)dt$$

$$(3+2k)t dk + 2(k^2 - 1) dt = 0$$

$$-\frac{(3+2k)dk}{2(k^2-1)} = \frac{dt}{t}$$

$$-\frac{1}{2} \cdot \frac{d(k^2-1)+3}{k^2-1} dk = \frac{dt}{t}$$

$$2\ln|t| + \ln|k^2-1| + \frac{3}{2}\ln|\frac{k-1}{k+1}| + \tilde{C} = 0$$

$$2\ln|x^2-1| + \ln|\frac{y^2-1}{x^2-1}-1| + \frac{3}{2}\ln|\frac{y^2-x^2+2}{x^2+y^2}| + C = 0$$

$$(x^2-1)^2 (y^2-x^2+2)^3 = C (x^2+y^2).$$

$$3. \frac{x}{y} \frac{dy}{dx} = f(xy)$$

$$\Rightarrow dy = d(\frac{u}{x}) = \frac{xdu - udx}{x^2}$$

$$\frac{x^2}{u} \cdot \frac{xdu - udx}{x^2 \cdot dx} = f(u)$$

$$x du = u(f(u) + 1) dx$$

$$\Rightarrow \frac{du}{u(f(u)+1)} = \frac{dx}{x}$$

$$(1) f(xy) = (1+x^2y^2),$$

$$\frac{du}{u(u^2+2)} = \frac{dx}{x}.$$

$$\frac{1}{4} \ln \frac{u^2}{u^2+2} = \ln|x| + \tilde{C}$$

$$\frac{u^2}{(u^2+2)} = |x|^4 \cdot C$$

$$y^2 = Cx^2 (x^2 y^2 + 2)$$

$$(2) f(u) = \frac{2+u^2}{2-u^2} = \frac{-(2-u^2)+4}{2-u^2} = -1 + \frac{4}{2-u^2}$$

$$\frac{2-u^2}{4u} du = \frac{dx}{x}$$

$$\frac{1}{2} \ln|xy| - \frac{(xy)^2}{8} = \ln|x| + C$$

$$(3) \frac{dy}{dx} = \frac{x-y+5}{x-y-2}$$

$$\text{Let } x-y-2=t, \quad y = x-t-2.$$

$$\frac{dx - dt}{dx} = \frac{t+7}{t}$$

$$tdx - tdt = (t+7)dx$$

$$-tdt = 7dx$$

$$-\frac{t^2}{2} = 7x + \tilde{C}$$

$$(x-y-2)^2 + 14x + C = 0.$$

$$(b) \frac{dy}{dx} = \frac{2x^3 + 3xy^2 + x}{3x^2 y + 2y^3 - y}$$

$$\Rightarrow \frac{dy^2}{dx^2} = \frac{2x^2 + 3y^2 + 1}{3x^2 + 2y^2 - 1}$$

$$\begin{cases} 2x^2 + 3y^2 + 1 = 0 \\ 3x^2 + 2y^2 - 1 = 0 \end{cases} \Rightarrow \begin{cases} t = x^2 - 1 \\ s = y^2 + 1 \end{cases}$$

$$\frac{ds}{dt} = \frac{2t+3s}{3t+2s}$$

$$s = kt.$$

$$t(3+2k) + (tdk + kdt) = t(2+3k) dt \quad t>0 \Rightarrow \text{no soln.}$$

$$(3+2k)t + dk + (3+2k) \cdot kdt = (2+3k)dt$$

$$(3+2k)t dk + 2(k^2 - 1) dt = 0$$

$$-\frac{(3+2k)dk}{2(k^2-1)} = \frac{dt}{t}$$

$$-\frac{1}{2} \cdot \frac{d(k^2-1)+3}{k^2-1} dk = \frac{dt}{t}$$

$$2\ln|t| + \ln|k^2-1| + \frac{3}{2}\ln|\frac{k-1}{k+1}| + \tilde{C} = 0$$

$$2\ln|x^2-1| + \ln|\frac{y^2-1}{x^2-1}-1| + \frac{3}{2}\ln|\frac{y^2-x^2+2}{x^2+y^2}| + C = 0$$

$$(x^2-1)^2 (y^2-x^2+2)^3 = C (x^2+y^2).$$

$$4. \text{ if } f(x). \quad F'(x) = f(x).$$

$$F'(x) (F(x) - F(0)) = 1.$$

$$\frac{dF(x)}{dx} (F(x) - F(0)) = 1.$$

$$(F(x) - F(0)) dF(x) = dx$$

$$\frac{F^2(x)}{2} - F(0) F(x) = x + \tilde{C}$$

$$F^2(x) - 2F(0) F(x) + F^2(0) = 2x + C$$

$$F(x) = F(0) \pm \sqrt{2x + C}.$$

$$f(x) = \pm \frac{1}{\sqrt{2x + C}}$$

Mid

1. §. Peano thm.
§. Uniqueness
§. Equation in symmetric form
§. Integral of the equ.
2. §. Lip. condition
§. Picard approximation
§
3. § linear. diff. equation
§. L.i. function
§. linear (non) homo
§. fundamental system of solution.
* §. L.s. with constant coeff.

$\psi_1, \psi_2 \in C^1(a, b)$. i.e. $W(t) \neq 0$.

Prove that. 1) $\exists t_1, t_2$. $\psi_1(t_1) = 0$, $\psi_2(t_2) = 0$

2) $\exists (\alpha, \beta)$. s.t. ψ_1, ψ_2 are l.d. on (α, β)

$$(1) W = \begin{vmatrix} \psi_1(t) & \psi_2(t) \\ \psi'_1(t) & \psi'_2(t) \end{vmatrix}$$

$$\psi_1(t) \psi'_2(t) - \psi_2(t) \psi'_1(t) = \psi'_1(t) \psi_2(t).$$

$$c_1 \psi_1(t) + c_2 \psi_2(t) = 0$$

$$c_1 \psi'_1(t) + c_2 \psi'_2(t) = 0.$$

Assume $\psi_2 \neq 0$.

denote $\psi = \frac{\psi_1}{\psi_2}$. $\psi' = 0$. $\psi = c \neq 0$.

$$\psi_1 = c \psi_2. \Rightarrow c \cdot \psi_1 - c \psi_2 = 0 \Rightarrow \psi_1, \psi_2 \text{ l.d.}$$

(2) By. we need to show $\exists (\alpha, \beta) \subset [a, b]$. s.t. $\psi_2 \neq 0$. or $\psi_1 \neq 0$.

since ψ_1, ψ_2 is cont. diff.

not. $\psi_1, \psi_2 \equiv 0$ (otherwise. $c_1 \psi_1(t) + c_2 \psi_2(t) = 0$. $c_2 \in \mathbb{R}$).

(3). def of stability \Rightarrow def of instability (with ε and δ).

$\exists \varepsilon > 0$. $\forall \delta > 0$. we can find a solution. $\psi(t)$

s.t. $|\psi(t_0) - \psi(t_0)| < \delta$. we have $|\psi(t) - \psi(t_0)| < \varepsilon$
for any $t \in [t_0, +\infty)$

Final - exam test

1. Linear differential equations/systems
2. stability kind of equilibrium

Topics We Have

1. first order equation
2. equation solved respect to
3. linear differential equations
fundamental systems of solution
wronskian
4. linear systems
Wronskian
Fundamental system of solution
Liouville formulas
5. Theorem about integral Continuity
6. Stability
7. Autonomous systems (types of $x=0$ trajectory)