Naive set theory

Relation (R or $\sim$) between A and B $(A\times B)$ / on A $(A\times A)$

Definition 1. A binary relation between two sets A and B is a subset R of $A \times B$ - i.e., is a set of ordered pairs $(x, y) \in A \times B$.

If $A = B$, so that the relation R is a subset of $A \times A$, we say that R is a relation on A.

If R is a relation between A and B (i.e., if $R \subseteq A \times B$), we often write xRy or $x \sim y$ instead of $(x, y) \in R$.

Equivalent relation

Definition 2. Let R be a binary relation on a set A :

- R is reflexive if $a \sim a$ for all $a \in A$.
- R is symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in A$.
- R is transitive if $a \sim b$ and $b \sim c$ imply $a \sim c$ for all $a, b, c \in A$

anti- if $(a\sim b)\wedge(b\sim a) \Rightarrow a=b$

a- if $a\sim b \Rightarrow$ not $b\sim a$.

We say that R is an **equivalence relation** if it satisfies all of the three properties defined above.

relation 是在集合中找有某种同性质的元素.(形成子集)

Equivalent class (对应 equivalence relation)

Definition 3. If A is a set, $a \in A$, and R is an equivalence relation on A, then the equivalence class of a, written $[a]$ (a subset of A), is the set of all $x \in A$ such that $x \sim a$. → $x \in [a]$

If C is an equivalence class for R and $x \in C$, then one frequently says that x is a representative for the equivalence class C.

Remark. If A is a set and R is an equivalence relation on A, then the collection of all $R-$ equivalence classes is a set.

Theorem 1.

Let A be a set, suppose that $x, y \in A$, and let R be an equivalence relation on A. Then either the equivalence classes $[x]$ and $[y]$ are disjoint (不相交的) or they are equal.

(Pf: Assume joint. $\exists z \in [x]\cap[y]$, $z\sim x$, $z\sim y \Rightarrow x\sim y$)

Corollary 2.

The equivalence classes of an equivalence relation on A form a family of pairwise disjoint subsets whose union is all of A. (所有元素都可以被用来找 equivalent class)

Quotient

Definition 4.

Let $\sim$ be an equivalence relation on a set A. The quotient set, denoted $A/\sim$, is the set of all $\sim$ −equivalence classes - i.e., $A/\sim = \{[x] \mid x \in A\}$. The map $\pi : A \to A/\sim$ given by $\pi(x) = [x]$ is called quotient projection.

class 作为 element

surjective

Proposition 3. (pairwise disjoint subset → equivalent relation)

Let A be a set, and let $\mathcal{C} = (C_i)_{i \in I}$ be a family of subsets of A such that
(i) the subsets in $\mathcal{C}$ are pairwise disjoint
(ii) the union of the subsets in $\mathcal{C}$ is equal to A
Then there is an equivalence relation $\sim$ on A whose equivalence classes are the sets in the family $\mathcal{C}$. In other words, $A/\sim = \mathcal{C}$.

(Pf: construct relation. shows it's equivalent)

Proposition 4.

Let A be a set equipped with a equivalence relation $\sim$, B be a second set, and $f : A \to B$. If f is constant along equivalence classes of $\sim$, there is a unique function $\tilde{f} : A/\sim \to B$ such that $\tilde{f} \circ \pi = f$, where π is the quotient projection. In particular, we have the equality $Im(f) = Im(\tilde{f})$ between images.

$\tilde{f}([x]) = f(x)$

$\Uparrow$ π is surjective. ($Im(\pi) = A$).

Corollary 5.

Let A and B be sets and $f : A \to B$ be a function. If $\sim$ is defined via f (i.e., $x \sim y$ iff $f(x) = f(y)$), then there is a unique injective function $f : A/\sim \to B$ such that $\tilde{f} \circ \pi = f$, where is the quotient projection. In particular, we have the equality $Im(f) = Im(\tilde{f})$.

Pf: $\tilde{f}[x] = \tilde{f}[y] \Rightarrow f(x) = f(y) \Rightarrow x \sim y \Rightarrow [x] = [y]$

Definition 5. (偏序)
(i) A relation $\preceq$ on a set X is called a partial order, if it is reflexive, antisymmetric 反对称 (i.e., if $a \preceq b$ and $b \preceq a$, then $a = b$), and transitive.
(ii) A relation $\prec$ on a set X is called a strict partial order, if it is irreflexive (NOT $a \prec a$), asymmetric 不对称 (i.e., if $a \prec b$ then not $b \prec a$), and transitive.
A set X together with a partial ordering $\preceq$ is called a partially ordered set, or poset, and is denoted by $(X, \preceq)$.

Definition 6.

偏序集中可以建立关系的元素

- The elements a and b of a poset $(X, \preceq)$ are comparable if either $a \preceq b$ or $b \preceq a$ holds. When a and b are elements of X so that neither $a \preceq$ nor $b \preceq a$ holds, then a and b are called incomparable.
- If any two elements of X are comparable, then X is called a linearly ordered set (the term chain are also used). (线性有序集 / 全序集)

Definition 7. Given two partially ordered sets $(X_1, \preceq_1)$ and $(X_2, \preceq_2)$, the lexicographic ordering on $X_1 \times X_2$ is defined by specifying when (a_1, a_2) is less than (b_1, b_2), written, $(a_1, a_2) \prec (b_1, b_2)$, which holds either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ and $a_2 \prec_1 b_2$ holds.
This definition can be easily extended to a lexicographic ordering on strings.

字典序，先比前一个，前一个相等比后一个

$\boxed{\text{Definition 8.}}$

Let A be a poset.

- We say that an element $a \in A$ is a least element of A if $a \preceq b$ for all $b \in A$.
- We say that a is a minimal element of A if $b \preceq a$ implies $b = a$.
- We say that $a \in A$ is a greatest element if $b \preceq a$ for all $b \in A$.
- We say that $a \in A$ is a maximal element if $a \preceq b$ implies $b = a$.

$$\text{Least} \quad \underset{\not\longleftarrow}{\longrightarrow} \quad \text{Minimal}$$

(不一定能建立 relation)

Lemma 6.

Let A be a finite partially ordered set. If A is nonempty, then A has at least one minimal element.

Pf. Choose $a_0 \in A$ $\to$ minimal

$\to$ not minimal. $\exists a_1$. $a_0 \succeq a_1$ (finite procedure

Remark. If A is a linearly ordered set, then every minimal element of A is a least element of A. Using Lemma 6, we deduce that if A is finite and nonempty, then it contains a least element. The same argument shows that A has a greatest element.

线性有序集中 least / minimal 一致.

Proposition 7. (有限线性有序集与有限自然数集的双射)

Let A be a finite linearly ordered set. Then there is a unique order-preserving bijection $\epsilon : \{1, 2, \dots, n\} \to A$, for some $n \in \mathbb{N}$.

Pf: induction on n. (A must have greatest elements)

Proposition 8.

可建立互逆映射

Let A be a partially ordered set. Then A is isomorphic (as a partially ordered set) to a subset of $\mathcal{P}(X)$, for some set X.

Construct. $\phi : A \to P(A)$ 1) ϕ is injective. 2) For $a, b \in A$. we have $a \leq b \Leftrightarrow \phi(a) \sqsubseteq \phi(b)$

$\phi(a) = A_{\leq a}$ (构成了一个"≤"关系的 equivalent class，必满射)

$\boxed{\text{Definition 9.}}$ Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be posets. We say that a map $\phi : A \to B$ is order-preserving, or monotone, if $a \preceq_A a'$ implies $\phi(a) \preceq_B \phi(a')$. (映射的保序)

Proposition 9.

Let A be a finite poset. Then there exists an order-preserving bijection $\phi : A \to B$, where B is a linearly ordered set. (infinite A also have this property).

Pf: induction on the number of elements of A.

extract. max. a. construct $\epsilon : A \setminus \{a\} \to \{1, 2, \dots, n-1\}$ new set also poset.

公理互证

所有非空子集到原集合的映射

Axiom of Choice A choice function on a set X is a function $f : \mathcal{P}(X) \setminus \{\varnothing\} \to X$ such that $f(S) \in S$ for every non-empty $S \subset X$. The **Axiom of Choice** asserts that on every set there is a choice function.

(non-empty)

Informally, the axiom of choice says that it is possible to choose an element from every set. $\to f(S)$

Zorn's lemma asserts that if P is a non-empty poset in which each chain 可互相比较的元素形成的子集 has an upper bound, then P has a maximal element.

We say that an element u is an upper bound for a linearly ordered C if $x \preceq u$ for each $x \in C$.

(注：upper bound 可以不属于 chain，但要属于给定的全集)

Well-ordering principle A linearly ordered P is called **well-ordered** if every non-empty subset $S \subset P$ has a minimum. The well-ordering principle asserts that every set can be well-ordered by a suitable relation.

Introduction of mathematics logic

$\underbrace{\text{elements}}_{\text{alphabet}} \to$ words $\to$ texts

Definition 1. Let $\mathcal{A}$ be any abstract set. We call $\mathcal{A}$ an alphabet.
Finite sequences (可重复排列) of elements of $\mathcal{A}$ are called words in $\mathcal{A}$. can be empty (ϵ)
Finite sequences of words are called texts.
The length of the word ω, denoted $lh(\omega)$, is the length of ω as a sequence of symbols.

Definition 2. If α and β are words in the alphabet $\mathcal{A}$, then the word $\alpha\beta$ (the result of adding the word β to the end of the word α) is called the concatenation of the words α and β.

Remark. If α is a word and $n \in \mathbb{N}$, then α^n stands for the word $\underbrace{\alpha\alpha\ldots\alpha}_{n}$.

The set of all words in the alphabet $\mathcal{A}$ is denoted by $\mathcal{A}^*$.
Here is a simple lemma on the cardinality of $\mathcal{A}^*$.

Proposition 1.
If the alphabet $\mathcal{A}$ is finite or countable, then the set $\mathcal{A}^*$ is countable.

First-ordered language

元(常元/变元) $\Longrightarrow$ 词
项 $\Longrightarrow$ 公式

Definition 3. The vocabulary or alphabet of a first-order language contains the following symbols (and only the following symbols):

逻辑符
1. $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ (联结词)
2. $\forall, \exists$ (量词 quantifier)
3. ("variables") $v_0, v_1, \ldots$
4. $\equiv$

非逻辑符
5. for every $n \geq 1$ a (possibly empty) set of $n-$ary predicates P_i^n $(i \in \mathbb{N})$ (n元谓词集.)
6. for every $n \geq 1$ a (possibly empty) set of $n-$ary (n元 (arity)) function signs f_i^n $(i \in \mathbb{N})$ (函数符. $f_1^n, \ldots f_n^n$ 对应一个函数)
7. a (possibly empty) set of constants c_i (常元符) $(i \in \mathbb{N})$ and parentheses as auxiliary symbols.

- $\mathcal{A} = \{(1); (2); (3); (4)\}$ is fixed: these symbols are contained in every first-order language.
- $\mathcal{S} = \{(5); (6); (7)\}$ is optional: the choice of $\mathcal{S}$ determines the specific character of a first-order language.
- $\mathcal{A}_\mathcal{S} = \mathcal{A} \bigcup \mathcal{S}$ is the actual alphabet of the first-order language that is determined by $\mathcal{S}$.

此处1-7对应左侧，即每个一阶语言的逻辑符集相同，非逻辑符可选不同
谓词集. subset. e.g. $x<y$ defines a predicates set. element.(x.y) $\{(x,y) \mid x<y\}$
一个谓词符号表示一种性质.(各个领域)

Examples. 1.) $\mathcal{S}_{Equ} = \{\underbrace{\sim}_{P_0^2}\}$ determines the first-order language of equivalence structures.

函数 $D^n \to D$
也可示作一个子集.(图像的集合. element $(x, f(x))$)

2.) $\mathcal{S}_{Gr} = \{\underbrace{\circ}_{f_0^2}, \underbrace{e}_{c_0}\}$ determines the first-order language of group theory.

函数符集 常元符集.

e.g. 初等算术语言 $\mathcal{A}$. 常元符 $\{0\}$. 函数符集 $\{S, +, \cdot\}$. 谓词符号集为 $\{<\}$

Term(项)

Definition 4. Let $\mathcal{S}$ be the specific symbol set of a first-order language. $\mathcal{S}-$terms are $\boxed{\text{precisely}}$ those words over $\mathcal{A}_\mathcal{S}$ ($\in \mathcal{A}_\mathcal{S}^*$) that can be generated according to the following rules. (term 仅限于以下三种情况; denote by T_S)

(T1) Every variable is an $\mathcal{S}-$term.

(T2) Every constant in $\mathcal{S}$ is an $\mathcal{S}-$term.

(T3) If $t_1, \ldots, t_n$ are $\mathcal{S}-$terms and f is an $n-$ary function sign in $\mathcal{S}$, then $f(t_1, \ldots, t_n)$ is an $\mathcal{S}-$term.

This means:
$t \in \mathcal{A}_\mathcal{S}^*$ is an $\mathcal{S}-$term if there is a sequence $u_1, \ldots, u_k$of elements of $\mathcal{A}_\mathcal{S}^*$, (词集.) such that $u_k = t$ and for all u_i with $1 \leq i \leq k$ it is the case that:
- u_i is a variable or
- u_i is a constant in $\mathcal{S}$ or
- $u_i = f(t_1, \ldots, t_n)$ and $t_1, \ldots, t_n \in \{u_1, \ldots, u_{i-1}\}$.

Formula (公式)

Definition 5. Let $\mathcal{S}$ be the specific symbol set of a first-order language. $\mathcal{S}-$formulas are precisely those strings over $\mathcal{A}_\mathcal{S}$ that can be generated according to the following rules: (denote by F_S)

公式的实质是一个有穷字符串. 是一个符号集

(F1) $\equiv (t_1, t_2)$ (for $\mathcal{S}-$terms t_1, t_2)
(F2) $P(t_1, \ldots, t_n)$ (for $\mathcal{S}-$terms $t_1, t_2, \ldots, t_n$, for n-ary $P \in \mathcal{S}$)
(Formulas which can be generated solely on basis of (F1) and (F2) are called atomic.) 原子公式
(F3) $\neg\phi$ (ϕ is formula)
(F4) $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$, $\phi \leftrightarrow \psi$ (ϕ, ψ are formulas)
(F5) $\forall x \phi$, $\exists x \phi$ (for arbitrary variables x) x is bounded variable

Lemma 2.

For all symbol sets $\mathcal{S}$, $\mathcal{T}_\mathcal{S}$ and $\mathcal{F}_\mathcal{S}$ are countable.

自由变元

$\boxed{\textit{Definition 6.}}$ Let $\phi \in \mathcal{F}_\mathcal{S}$ (for arbitrary symbol set $\mathcal{S}$): $free(\phi)$, i.e., the set of variables which occur freely in ϕ, can be defined recursively as follows:

不被量词所限制

$var(x) = \{x\}$.
$var(c_i) = \phi$
$var(f(t_1, \ldots t_n)) = \bigcup_{i=1}^{n} var(t_i)$
若 $var(t) = \phi$. t为闭项
$x \in var(t)$, x为t的自由变元

- $free(t_1 \equiv t_2) := var(t_1) \cup var(t_2)$ (let $var(t)$ be the set of variables in t)
- $free(P(t_1, \ldots, t_n) = var(t_1) \cup \ldots \cup var(t_n)$
- $free(\neg\phi) := free(\phi)$
- $free(\phi \wedge \psi) := free(\phi) \cup free(\psi)$ (analogously for $\vee, \rightarrow, \leftrightarrow$)
- $free(\forall \phi x) := free(\phi) \setminus \{x\}$ (analogously for $\exists$)

e.g. $free(\exists x P(x)) = free(P(x)) \setminus \{x\} = \{x\} \setminus \{x\} = 0$

A formula ϕ without free variables, i.e., for which $free(\phi) = \varnothing$, is called a sentence.

Def.

在该的运算规则下封闭

We call $X \subseteq \mathcal{A}^*_\mathcal{S}$ closed under the rules of the term calculus (for a given symbol set $\mathcal{S}$) if and only if:

1.) all variables are contained in X
2.) all constants in $\mathcal{S}$ are contained in X
3.) if $t_1, \ldots, t_n$ are in X, then $f(t_1, \ldots, t_n)$ is in X (where f is an arbitrary function sign in $\mathcal{S}$).

Lemma 3.

$$\mathcal{T}_\mathcal{S} = \bigcap_{X \text{ closed}} X$$

Thus, $\mathcal{T}_\mathcal{S}$ is the least subset of $\mathcal{A}^*_\mathcal{S}$ that is closed under the rules of the term calculus.

Consider the most important first-order languages: the Zermelo–Fraenkel language of set theory L_1Set, and the Peano language of arithmetic L_1Ar.

Translation from L_1Set to "ordinary language".

$\forall x(\neg(x \in \varnothing))$:

"for all (sets) x it is false that x is an element of (the set) $\varnothing$"

$\forall x(y \in z)$:

任意x → 影响 y, → 满足 z.

The literal translation "for all x it is true that y is an element of z" sounds a little strange. Later we shall see that from the point of view of "truth" or "deducibility," such a formula is equivalent to the formula $y \in z$.

一阶局限性

Syntactically, this is reflected in the prohibition against forming expressions such as $\forall p(p(x))$, where p is a relation of degree 1; relations must stand for fixed rather than variable properties. 量词只对个体使用，不对集合或谓词使用

$\forall x(p(x))$ ✓

中文书:
-阶语言 $\mathcal{L}$ M 论域 I 解释
结构 $M=(M,I)$
$I(P)$ 是 M 上一个 n 元关系. $I(c). I(f). I(P) \to C_M . f_M . P_M$

模型 - 结构与赋值的偶对

Definition 1. Let $\mathcal{S}$ be an arbitrary symbol set:
An $\mathcal{S}$−model is an ordered pair $\mathfrak{M} = (\boxed{D}, \mathfrak{I})$, such that: "论域"
↳ "解释" 定义域为 D 的映射.

1.) D is a set; $D \neq \varnothing$ (domain)

2.) $\mathfrak{I}$ is defined on $\mathcal{S}$ as follows (interpretation of $\mathcal{S}$): **Semantic**

- for n−ary predicates P in $\mathcal{S}$: $\mathfrak{I}(P) \subseteq D^n$ → $(t_1, \dots t_n) \in \mathfrak{I}(P)$, $(t_1, \dots t_n)$ satisfied P.
- for n−ary function signs f in $\mathcal{S}$: $\mathfrak{I}(f) : D^n \to D$ * 对 0 元谓词 P. $\mathfrak{I}(P) \in \{T, F\}$
- for every constant c in $\mathcal{S}$: $\mathfrak{I}(c) \in D$

(定义域为变元集合, 值域为 D) 二元组

Definition 2. A variable assignment over a model $\mathfrak{M} = (D, \mathfrak{I})$ is a function $s : \{v_0, v_1, \dots\} \to D$.

"再赋值" 记作赋值 $s[x:=d]$. 将 d 赋予变元 x. 其余同赋值 s

Analogously for $\exists x\phi$ and the existence of an element $d \in D$. It is also useful to have a formal way of changing variable assignments: Let s be a variable assignment over $\mathfrak{M} = (D, \mathfrak{I})$, let $d \in D$:
We define

$$s\frac{d}{x} : \{v_0, v_1, \dots\} \to D$$

$$s\frac{d}{x}(y) = \begin{cases} d, & \text{if } y = x; \\ s(y), & \text{if } y \neq x. \end{cases}$$

(where x is some variable in $\{v_0, v_1, \dots\}$).

Given an $\mathcal{S}$−model together with a variable assignment over this model, we can define the semantic value of a term/formula:

Definition 3. Let $\mathfrak{M} = (D, \mathfrak{I})$ be an $\mathcal{S}$−model. Let s be a variable assignment over $\mathfrak{M}$: → actually. model : $(\mathfrak{M}, s)$
项的语义函数 (V1-3) 公式语义函数 (输出 T/F) (V4-V12)
$Val_{\mathfrak{M},s}$ ("semantic value function") is defined on $\mathcal{T}_{\mathcal{S}} \cup \mathcal{F}_{\mathcal{S}}$, such that: domain

(V1) $Val_{\mathfrak{M},s}(x) = s(x)$ 变元被赋值

(V2) $Val_{\mathfrak{M},s}(c) = \mathfrak{I}(c)$ 常元被解释.

(V3) $Val_{\mathfrak{M},s}(f(t_1, \dots, t_n)) = \mathfrak{I}(f)(Val_{\mathfrak{M},s}(t_1), \dots, Val_{\mathfrak{M},s}(t_n))$ → 先把 f 解释为 D 中 n 元函数 $\mathfrak{I}(f)$. 分别求 $Val_{\mathfrak{M},s}(t_i)$ (同理变元赋值. 常元解释) 再求 $\mathfrak{I}(f)$ 在 ($Val_{\mathfrak{M},s}(t_i)$) 的值

(V4) $Val_{\mathfrak{M},s}(t_1 \equiv t_2) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(t_1) = Val_{\mathfrak{M},s}(t_2)$

(V5) $Val_{\mathfrak{M},s}(P(t_1, \dots, t_n)) := 1 \Leftrightarrow (Val_{\mathfrak{M},s}(t_1), \dots, Val_{\mathfrak{M},s}(t_n)) \in \mathfrak{I}(P)$

(V6) $Val_{\mathfrak{M},s}(\neg\phi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = 0$

(V7) $Val_{\mathfrak{M},s}(\phi \wedge \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = Val_{\mathfrak{M},s}(\psi) = 1$

(V8) $Val_{\mathfrak{M},s}(\phi \vee \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = 1$ or $Val_{\mathfrak{M},s}(\psi) = 1$

(V9) $Val_{\mathfrak{M},s}(\phi \to \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = 0$ or $Val_{\mathfrak{M},s}(\psi) = 1$

(V10) $Val_{\mathfrak{M},s}(\phi \leftrightarrow \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = Val_{\mathfrak{M},s}(\psi)$

(V11) $Val_{\mathfrak{M},s}(\forall x\phi) := 1 \Leftrightarrow$ for all $d \in D$: $Val_{\mathfrak{M},s\frac{d}{x}}(\phi) = 1$

(V12) $Val_{\mathfrak{M},s}(\exists x\phi) := 1 \Leftrightarrow$ there is a $d \in D$, such that $Val_{\mathfrak{M},s\frac{d}{x}}(\phi) = 1$

For (V4) - (V12): in case the "iff" condition is not satisfied, the corresponding semantic value is defined to be 0.

项t 语义 决定 因素:
- $\mathfrak{I}(c_i)$ — **Interpretation on constant**
- $f(t_1, \dots t_n)$ — **Function signs occur in t**
- $s(t)$ — **Assignment function assigns to the variables that occur in t**

公式 语义 决定 因素 结果只有 $\{0,1\}$:
- $\mathfrak{I}(c_i)$ — **Interpretation on constant**
- $f(t_i)$ — **Function signs occur in t** (在公式中 仅未考虑)
- $P(t_i)$ — **Interpretation on predicates**
- ϕ — **Assignment function assigns to the variables that occur in ϕ freely** (非 free 的都被 $s\frac{d}{x}$ 重新赋值了)

术语
Terminology:

$Val_{\mathfrak{M},s}(t)$ and $Val_{\mathfrak{M},s}(\phi)$ are the semantic values of t and ϕ respectively (relative to $\mathfrak{M}$), where

- $Val_{\mathfrak{M},s}(t) \in D$
- $Val_{\mathfrak{M},s}(\phi) \in \{1, 0\}$

Instead of writing that $Val_{\mathfrak{M},s}(\phi) = 1$, we may also say:
- ϕ is true at $\mathfrak{M}, s$
- $\mathfrak{M}, s$ make ϕ true
- $\mathfrak{M}, s$ satisfy ϕ
- briefly $\mathfrak{M}, s \models \phi$ (逻辑推理关系)

We will also write for sets Φ of formulas:
$\mathfrak{M}, s \models \Phi \iff$ for all $\phi \in \Phi$ $\mathfrak{M}, s \models \phi$

所有满足 $Val_{\mathfrak{M},s}(\varphi) = 1$ 的公式 φ 组成集合 Φ

Lemma 1 (Coincidence Lemma).

Let $\mathcal{S}_1, \mathcal{S}_2$ be two symbol sets. Let $\mathfrak{M}_1 = (D, \mathfrak{I}_1)$ be an $\mathcal{S}_1-$model , $\mathfrak{M}_2 = (D, \mathfrak{I}_2)$ be an $\mathcal{S}_2-$model. Let s_1 be a variable assignment over $\mathfrak{M}_1$, s_2 a variable assignment over $\mathfrak{M}_2$. Finally, let $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$.

1.) For all terms $t \in \mathcal{T}_{\mathcal{S}}$: 符号集交集中的所有项/公式

If $\mathfrak{I}_1(c) = \mathfrak{I}_2(c)$ for all c in t 在语义函数下等价的判断条件

$\mathfrak{I}_1(f) = \mathfrak{I}_2(f)$ for all f in t

$s_1(x) = s_2(x)$ for all x in t

then $Val_{\mathfrak{M}_1, s_1}(t) = Val_{\mathfrak{M}_2, s_2}(t)$

2.) For all formulas $\phi \in \mathcal{F}_{\mathcal{S}}$:

If $\mathfrak{I}_1(c) = \mathfrak{I}_2(c)$ for all c in ϕ

$\mathfrak{I}_1(f) = \mathfrak{I}_2(f)$ for all f in ϕ

$\mathfrak{I}_1(P) = \mathfrak{I}_2(P)$ for all P in ϕ

$s_1(x) = s_2(x)$ for all x in $free(\phi)$

then $Val_{\mathfrak{M}_1, s_1}(\phi) = Val_{\mathfrak{M}_2, s_2}(\phi)$

Corollary 2. $free(\varphi) = \emptyset$

Let ϕ be an $\mathcal{S}-$sentence, let s_1, s_2 be variable assignments over an $\mathcal{S}-$model $\mathfrak{M}$:

It follows that $Val_{\mathfrak{M}, s_1}(\phi) = Val_{\mathfrak{M}, s_2}(\phi)$. (在考虑 sentence 的语义时，赋值函数s的选取与结果无关)
即结果对某一赋值函数成立时，对任意赋值函数均成立

Remark. We see that as far as sentences are concerned, it is irrelevant which variable assignment we choose in order to evaluate them: a sentence ϕ is true in a model $\mathfrak{M}$ relative to some variable assignment over $\mathfrak{M}$ iff ϕ is true in $\mathfrak{M}$ relative to all variable assignments over $\mathfrak{M}$.
Therefore we are entitled to write for sentences ϕ and sets Φ of sentences:

$$\mathfrak{M} \models \phi \text{ and } \mathfrak{M} \models \Phi$$

without mentioning a variable assignment s at all.

在所有 I, M 下均成立的公式.

Definition 4. For all $\phi \in \mathcal{F}_{\mathcal{S}}$, $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$: ψ是Φ的逻辑结论 Φ ⊨ ψ 有效

ϕ follows logically from Φ, briefly: $\Phi \models \phi$ iff for all $\mathcal{S}-$models $\mathfrak{M}$, for all variable assignments s over $\mathfrak{M}$:

$$\text{if } \mathfrak{M}, s \models \Phi, \text{ then } \mathfrak{M}, s \models \phi$$

We also say equivalently:
Φ logically implies ϕ ; or ϕ is a logical consequence of Φ.

Definition 5. For all $\phi \in \mathcal{F}_{\mathcal{S}}$: ϕ is logically true iff for all $\mathcal{S}-$models $\mathfrak{D}$, for all variable assignments s over $\mathfrak{D}$, 永真性 可计为 ⊨ ψ

$$\mathfrak{M}, s \models \phi$$

永真公式，与模型无关

在部分解释/赋值下成立的公式.

对"⊨". 若 s, 𝔐 不出现, 即此式对任意 s, 𝔐 均成立

$\boxed{\textit{Definition 6.}}$ For all $\phi \in \mathcal{F_S}$, $\Phi \subseteq \mathcal{F_S}$:
ϕ is satisfiable iff there is an $\mathcal{S}-$model $\mathfrak{M}$ and a variable assignment s over $\mathfrak{M}$, such that: $\mathfrak{M}, s \models \phi$.
$\Phi \subseteq \mathcal{F_S}$ is (simultaneously) satisfiable iff there are $\mathfrak{M}, s$ such that $\mathfrak{M}, s \models \phi$.

Lemma 3.

For all $\phi \in \mathcal{F_S}$, $\Phi \subseteq \mathcal{F_S}$:
1. ϕ is logically true iff $\varnothing \models \phi$.
2. $\Phi \models \phi$ iff $\Phi \cup \{\neg\phi\}$ is not satisfiable.
3. ϕ is logically true iff $\neg\phi$ is not satisfiable.

Proof. 1.) $\varnothing \models \phi \Leftrightarrow$ for all $\mathfrak{M}, s$: if $\mathfrak{M}, s \models \varnothing$, then $\mathfrak{M}, s \models \phi \Leftrightarrow$ for all $\mathfrak{M}, s$: $\mathfrak{M}, s \models \phi \Leftrightarrow \phi$ is logically true.
→ 该模型对于 ϕ 内所有公式有 Val = 1. 但 ϕ 内没有公式. 所以任意模型都成立.

$\boxed{\textit{Definition 7.}}$ For all $\phi, \psi \in \mathcal{F_S}$: ϕ is logically equivalent to ψ iff $\phi \models \psi$ and $\psi \models \phi$.

For all $\mathfrak{M}, s$. $\quad \mathfrak{M}, s \models \varphi \Leftrightarrow \mathfrak{M}, s \models \psi$

Proposition 4.

Let $\mathcal{S}, \mathcal{S}'$ be symbol sets, such that $\mathcal{S} \subseteq \mathcal{S}'$. Let ϕ be an $\mathcal{S}-$formula ($\Rightarrow \phi$ is also an $\mathcal{S}'$-formula). Then:
ϕ is $\mathcal{S}-$satisfiable iff ϕ is $\mathcal{S}'$-satisfiable.

"⇒" 只需要善定义 $S' \setminus S$ 部分的模型. $D' = D$. $\mathfrak{I}'|_S = \mathfrak{I}$. $\mathfrak{I}$ on $S' \setminus S$ is arbitrary. $s' = s$
"⇐" simply "forget" about interpretation of symbol in $S' \setminus S$ (the reduct of $\mathfrak{M}'$)

Substitution

$\boxed{\textit{Definition 1.}}$ Let $\mathcal{S}$ be an arbitrary symbol set. Let $t_0, \ldots, t_n \in \mathcal{T_S}$, let $x_0, \ldots, x_n$ be pairwise distinct variables. We define the substitution function $\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n}$ on $\mathcal{T_S} \cup \mathcal{F_S}$ as follows:

$$[x]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} = \begin{cases} t_i, & \text{for } x = x_i; \\ x, & \text{else.} \end{cases} \qquad [c]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} = c$$

$$[f(t_1',\ldots,t_m')]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} := f([t_1']\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n},\ldots,[t_m']\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n})$$

$$[t_1' \equiv t_2']\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} := [t_1']\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} \equiv [t_2']\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n}$$

$[P(t_1',\ldots,t_m')]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n}$ is defined analogously to the case of ϕ

$$[\neg\phi]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} := \neg[\phi]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n}$$

$$[\phi \vee \psi]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} := ([\phi]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} \vee [\psi]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n})$$

Let $x_{i_1}, \ldots, x_{i_k}$ be those variables x_i among $x_0, \ldots, x_n$ for which it holds that: (i) $x_i \in free(\exists x \phi)$ (ii) $x_i \neq t_i$

Call these variables the relevant variables of the substitution.

$$[\exists x \phi] \frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} := \exists u [\phi] \frac{t_{i_1}, \ldots, t_{i_k}, u}{x_{i_1}, \ldots, x_{i_k}, x}$$

where $u := x$, if x does not occur in $t_{i_1}, \ldots, t_{i_k}$

else: let u be the first variable in $v_0, v_1, v_2, \ldots$ that does not occur in $\phi, t_{i_1}, \ldots, t_{i_k}$

△ Nothing can be substituted for bound variables.

For variable x

- $[\exists y\ y < x]\frac{y}{x} = \exists u [y < x]\frac{y, u}{x, y} =$ since y occurs within t_{i_1}, i.e., within y, it follows from our definition that u must be distinct from $x, y =$)
$= \exists u [y]\frac{y, u}{x, y} < [x]\frac{y, u}{x, y} = \exists u\ u < y$

For variable y

- $[\exists y\ y < x]\frac{x}{y}$ (since y is not free in $\exists y\ y < x$, only the substitution for u is going to remain)
$= \exists u\ [y < x]\frac{u}{y} =$ (since the number k of relevant variables is in this case 0, there are no $t_{i_1}, \ldots, t_{i_k}$ in which y could occur, thus it follows that $u = y$)
$= \exists y\ [y < x]\frac{y}{y} = \exists y\ y < x$. We see that nothing can be substituted for bound variables.

Substitution Lemma.

Let $\mathfrak{M}$ be an $\mathcal{S}$-model:

1.) For all terms $t \in \mathcal{T}_\mathcal{S}$:

For all variable assignments s over $\mathfrak{M}$, for all terms $t_0, \ldots, t_n \in \mathcal{T}_\mathcal{S}$, for all pairwise distinct variables $x_0, \ldots, x_n$:

$$Val_{\mathfrak{M},s}\left(t \frac{t_0, \ldots, t_n}{x_0, \ldots, x_n}\right) = Val_{\mathfrak{M},s\frac{Val_{\mathfrak{M},s}(t_0), \ldots, Val_{\mathfrak{M},s}(t_n)}{x_0, \ldots, x_n}}(t)$$

2.) For all formulas $\phi \in \mathcal{F}_\mathcal{S}$:

For all variable assignments s over $\mathfrak{M}$, for all terms $t_0, \ldots, t_n \in \mathcal{T}_\mathcal{S}$, for all pairwise distinct variables $x_0, \ldots, x_n$:

$$\mathfrak{M}, s \models \phi \frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} \text{ iff } \mathfrak{M}, s \frac{Val_{\mathfrak{M},s}(t_0), \ldots, Val_{\mathfrak{M},s}(t_n)}{x_0, \ldots, x_n} \models \phi$$

Calculus

Let $\mathcal{S}$ be an arbitrary symbol set. An $\mathcal{S}$-sequent is a finite sequence $\phi_1 \phi_2 \ldots \phi_n$ of $\mathcal{S}$-formulas.

- $\phi_1 \phi_2 \ldots \phi_{n-1}$ the antecedent (先例) of the sequent ("assumptions")
- ϕ_n the consequent of the sequent (i.e., the formula for which we want to claim that it follows from the assumptions)

Correctness preserving rules

- Show that both $\phi_1 \phi_2 \ldots \phi_k \neg\psi\rho$ and $\phi_1 \phi_2 \ldots \phi_k \neg\psi\neg\rho$ are correct
- Conclude that $\phi_1 \phi_2 \ldots \phi_k \psi$ is correct

$\boxed{\textit{Definition 2.}}$ For all $\mathcal{S}$-sequents $\Gamma\ \phi$:

$\Gamma\ \phi$ is correct iff $\{\psi | \psi$ is sequence member of $\Gamma\} \models \phi$

$\boxed{\textit{Definition 3.}}$ Let Φ be a set of $\mathcal{S}$-formulas, let ϕ be an $\mathcal{S}$-formula:

ϕ is derivable from Φ (briefly $\Phi \vdash \phi$) if and only if there are $\phi_1, \ldots, \phi_n \in \Phi$, such that $\phi_1, \ldots, \phi_n \phi$ is derivable in the sequent calculus.

Lemma 1.

$\Phi \vdash \phi \iff$ there is a finite set $\Phi' \subseteq \Phi$ such that $\Phi' \vdash \phi$.

Antecedent rule :

$$\frac{\Gamma\phi}{\Gamma'\phi} \quad \text{for} \quad \Gamma \subseteq \Gamma'$$

(Γφ is correct then Γ'φ is correct)

Assumption rule:

$$\frac{}{\Gamma\phi}$$ for ϕ being a sequence member of Γ, then $\Gamma\varphi$ is correct.

Proof by cases:

$$\frac{\Gamma\psi\phi \qquad \Gamma\neg\psi\phi}{\Gamma\phi}$$

Thus, if we can show ϕ both under the assumption ψ and under the assumption $\neg\psi$ (and since one of these two assumptions must actually be the case), we are allowed to conclude ϕ without assuming anything about ψ or $\neg\psi$.

Contradiction:

$$\frac{\Gamma\neg\psi\rho \qquad \Gamma\neg\psi\neg\rho}{\Gamma\psi}$$

So, if assuming $\neg\psi$ leads to a contradiction, then we are allowed to infer ψ

$\vee$- Introduction in the antecedent:

$$\frac{\Gamma\phi\rho \qquad \Gamma\psi\rho}{\Gamma(\phi \vee \psi)\rho}$$

Disjunctions $\phi \vee \psi$ in the antecedent allow for being treated in terms of two cases - case ϕ on the one hand and case ψ on the other.

$\vee$- Introduction in the consequent:

1.) $\frac{\Gamma\phi}{\Gamma(\phi\vee\psi)}$

2.) $\frac{\Gamma\psi}{\Gamma(\phi\vee\psi)}$

So, we are always allowed to weaken consequents by introducing disjunctions.

Excluded middle

$$\frac{}{\phi \vee \neg\phi}$$

Triviality

$$\frac{\Gamma\phi \qquad \Gamma\neg\phi}{\Gamma\psi}$$

Chain syllogism

$$\frac{\begin{array}{c}\Gamma\phi\psi\\ \Gamma\phi\end{array}}{\Gamma\psi}$$

Disjunctive syllogism

$$\frac{\begin{array}{c}\Gamma(\phi\vee\psi)\\ \Gamma\neg\phi\end{array}}{\Gamma\psi}$$

Contraposition

1.) $\dfrac{\Gamma\phi\psi}{\Gamma\neg\psi\neg\phi}$ 3.) $\dfrac{\Gamma\neg\phi\psi}{\Gamma\neg\psi\phi}$

2.) $\dfrac{\Gamma\phi\neg\psi}{\Gamma\psi\neg\phi}$ 4.) $\dfrac{\Gamma\neg\phi\neg\psi}{\Gamma\psi\phi}$

$\exists$ - Introduction in the consequent

$$\frac{\Gamma\phi\frac{t}{x}}{\Gamma\ \exists x\phi}$$

Here x and y are arbitrary variables.

$\exists$ - Introduction in the antecedent:

$$\frac{\Gamma\phi\frac{y}{x}\psi}{\Gamma\ \exists x\phi\psi}$$

Reflexivity

$$\frac{}{t\equiv t}$$

Substitution rule:

$$\frac{\Gamma\phi\frac{t}{x}}{\Gamma t\equiv t'\phi\frac{t'}{x}}$$

Symmetry

$$\frac{\Gamma\ t_1\equiv t_2}{\Gamma\ t_2\equiv t_1}$$

Contradiction:

$$\frac{\begin{array}{c}\Gamma\ t_1\equiv t_2\\ \Gamma\ t_2\equiv t_3\end{array}}{\Gamma\ t_1\equiv t_3}$$

Main Theorems

Soundness Theorem

For all $\Phi\subseteq\mathcal{F}_S$, for all $\phi\in\mathcal{F}_S$, it holds:
If $\Phi\vdash\phi$, then $\Phi\models\phi$.

Remark. The sequent calculus does not only contain correct rules for $\neg,\vee,\exists,\equiv$, but also for $\wedge,\rightarrow,\leftrightarrow,\forall$ by means of the metalinguistic abbreviations that we considered earlier (E.g. $\phi\rightarrow\psi:=\neg\phi\vee\psi$)

根据下表，找一个两侧等价(同时T/F)关系即可

X	Y	$\mathbf{B}_\vee(X,Y)$	$\mathbf{B}_\wedge(X,Y)$	$\mathbf{B}_\rightarrow(X,Y)$	$\mathbf{B}_\leftrightarrow(X,Y)$
T	T	T	T	T	T
T	F	T	F	F	F
F	T	T	F	T	F
F	F	F	F	T	T

Modus Ponens

(分离规则)

$$\frac{\begin{array}{c}\Gamma\ \phi\rightarrow\psi\\ \Gamma\ \phi\end{array}}{\Gamma\ \psi}$$

$\boxed{\textit{Definition 1.}}$ For all $\phi \in \mathcal{F}_S$:
ϕ is provable iff the (one-element) sequent ϕ is derivable in the sequent calculus (briefly: $\vdash \phi$).

$\boxed{\textit{Definition 2.}}$ For all $\phi \in \mathcal{F}_S$, $\Phi \subseteq \mathcal{F}_S$:

ϕ is consistent iff there is no $\psi \in \mathcal{F}_S$ with: $\{\phi\} \vdash \psi$, $\{\phi\} \vdash \neg\psi$.

Φ is consistent iff there is no $\psi \in \mathcal{F}_S$ with: $\Phi \vdash \psi$, $\Phi \vdash \neg\psi$.

Lemma 1.

Φ is consistent iff there is a $\psi \in \mathcal{F}_S$, such that $\Phi \nvdash \psi$.

Lemma 2.

Φ is consistent iff every finite subset $\Phi' \subseteq \Phi$ is consistent.

Soundness Theorem: Second Version

For all $\Phi \subseteq \mathcal{F}_S$, if Φ is satisfiable, then Φ is consistent.