Equations of Mathematical Physics Homework 7

Version 1

Find canonical form of equation $u_{xx} - yu_{yy} = 0$
Solution:
Firstly,compute $\Delta = b^2 - ac = 0 - 1 \cdot (-y) = y$, then the characteristic equation is $(\mathrm{d}y)^2 - y(\mathrm{d}x)^2 = 0$.
1) $y > 0$ **hyperbolic case**

$$\frac{\mathrm{d}y}{\pm\sqrt{y}} = \mathrm{d}x \Longrightarrow \begin{cases} \xi(x,y) = x + 2\sqrt{y} \\ \eta(x,y) = x - 2\sqrt{y} \end{cases}$$

compute the partial derivatives,we get:

$$\begin{cases} u_x = U_\xi + U_\eta \\ u_y = (\frac{1}{\sqrt{y}}) \cdot U_\xi + (-\frac{1}{\sqrt{y}}) \cdot U_\eta \\ u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta} \\ u_{yy} = \frac{1}{y} \cdot U_{\xi\xi} + (-\frac{2}{y}) \cdot U_{\xi\eta} + \frac{1}{y} \cdot U_{\eta\eta} + (-\frac{1}{2} \cdot y^{-\frac{3}{2}}) \cdot U_\eta + (\frac{1}{2} \cdot y^{-\frac{3}{2}}) \cdot U_\xi \end{cases}$$

then we put the new variables in the original equation,that is:

$$U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta} - y(\frac{1}{y} \cdot U_{\xi\xi} + (-\frac{2}{y}) \cdot U_{\xi\eta} + \frac{1}{y} \cdot U_{\eta\eta} + (-\frac{1}{2} \cdot y^{-\frac{3}{2}}) \cdot U_\eta + (\frac{1}{2} \cdot y^{-\frac{3}{2}}) \cdot U_\xi) = 0$$

we simplify this equation and replace y by ξ and η , by the replacement, we have $y^{-\frac{1}{2}} = \frac{4}{\xi-\eta}$

$$\Longrightarrow 4U_{\xi\eta} = (-\frac{1}{2} \cdot y^{-\frac{1}{2}}) \cdot U_\eta + (\frac{1}{2} \cdot y^{-\frac{1}{2}}) \cdot U_\xi \quad \Longrightarrow U_{\xi\eta} = -\frac{1}{2}\frac{U_\xi - U_\eta}{\xi - \eta}$$

2) $y < 0$ **elliptical case**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\sqrt{y} = \pm\sqrt{-y}\,i \Longrightarrow x \pm 2 \pm \sqrt{-y}\,i = c \Longrightarrow \begin{cases} \alpha(x,y) = x \\ \beta(x,y) = 2\sqrt{-y} \end{cases}$$

compute the partial derivatives,we get:

$$\begin{cases} \alpha_x = 1 \quad \alpha_y = 0 \\ \beta_x = 0 \quad \beta_y = -\frac{1}{\sqrt{-y}} \end{cases} \qquad \begin{cases} u_x = U_\alpha \\ u_y = -\frac{1}{\sqrt{-y}} \cdot U_\beta \end{cases}$$

$$\begin{cases} u_{xx} = \alpha_x U_{\alpha\alpha} + \beta_x U_{\alpha\beta} = U_{\alpha\alpha} \\ u_{yy} = \beta_y(\alpha_y U_{\alpha\beta} + \beta_y U_{\beta\beta}) + U_\beta \cdot \beta_{yy} = \frac{1}{-y} U_{\beta\beta} - \frac{1}{2}(-y)^{-\frac{3}{2}} U_\beta \end{cases}$$

then we put the new variables in the original equation,that is:

$$U_{\alpha\alpha} - y(\frac{1}{-y} U_{\beta\beta} - \frac{1}{2}(-y)^{-\frac{3}{2}} U_\beta) = 0$$

we simplify this equation and replace y by α and β , by the replacement, we have $\frac{1}{\beta} = \frac{1}{2}(-y)^{-\frac{1}{2}}$

$$\Longrightarrow U_{\alpha\alpha} + U_{\beta\beta} = -\frac{1}{2}(-y)^{-\frac{3}{2}} U_\beta \cdot y \Longrightarrow U_{\alpha\alpha} + U_{\beta\beta} = \frac{1}{2}(-y)^{-\frac{1}{2}} U_\beta \Longrightarrow U_{\alpha\alpha} + U_{\beta\beta} = \frac{1}{\beta} U_\beta$$

3) $y = 0$ **parabolic case**
when $y = 0$ the original equation can be reduced into: $u_{xx} = 0$.
That is already satisfied the form of canonical form of parabolic type of equation.

In general,we get the canonical form of the equation:

$$\begin{cases} U_{\xi\eta} = -\frac{1}{2}\frac{U_\xi - U_\eta}{\xi - \eta} & \text{when } y > 0 \text{ , hyperbolic type} \\ U_{\alpha\alpha} + U_{\beta\beta} = \frac{1}{\beta} U_\beta & \text{when } y < 0 \text{ , elliptical type} \\ U_{\xi\xi} = 0 & \text{when } y = 0 \text{ , parabolic type} \end{cases}$$