Tropical Optimization Problems

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Tropical Mathematics: Introduction

- Tropical (idempotent) mathematics deals with the theory and application of semirings (semifields) with idempotent operations
- An operation is idempotent, if applied to operands of the same value, it returns this value as output (example: $\max(x, x) = x$)
- Methods of tropical mathematics find applications in many areas to provide new solutions to various old and novel problems in
  - *project scheduling, location analysis, decision making,*
  - *discrete event systems, neural networks, cryptographic protocols, pattern recognition and other fields*

- Tropical mathematics has its origins in 1960s in the works of R. A. Cuninghame-Green, B. Giffler, A. J. Hoffman, S. N. N. Pandit, N. N. Vorobyev, I. V. Romanovsky
- First researches concentrated on the ability to rename such operation as $\max$ into a generalized idempotent addition $\oplus$
- These formal tricks allowed one to replace the polish postfix notation by the standard infix notation: $\max(x,y) = x \oplus y$
- Moreover, after translation into the tropical language, many problems that are not linear in the ordinary sense became linear
- This offers a potential for the use of the concept of linearity and related results to study nonlinear problems

- If the operation is idempotent, it is not invertible, and hence a subtraction as the inversion of $\oplus$ is undefined in tropical algebra
- Because of lack of subtraction, most of the techniques available in linear mathematics cannot be translated into the tropical language
- This leads to the need to develop new approaches to the solution of tropical analogues of many traditional problems
- At the same time, tropical solutions normally appear to be less complicated than that in the conventional mathematics
- Application of methods of tropical mathematics can offer complete analytical solutions to a range of classical and new problems

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- References at http://www.math.spbu.ru/user/krivulin/
- Papers in open access archive arXiv at http://arxiv.org/a/krivulin_n_1

Examples of Applications

- Temporal project scheduling in project management
- Minimax location on the plane and in multidimensional space
- Rating alternatives from pairwise comparisons in decision making

Project Scheduling: Constraints and Objectives

- Project scheduling is aimed at the development of optimal schedules of activities in a project, subject to various constraints
- The scheduling objectives are usually set in terms of time-oriented criteria to optimize, such as makespan, lateness and tardiness
- In real-world problems other objectives can be added, taking into account the project cost, profit, resource allocation or consumption
- Scheduling constraints may include temporal constraints in the form of time bounds for and relationships between activities
- The constraints may be formulated as material and manpower resource requirements, budget limitations and others restrictions

Temporal Project Scheduling Problems

- Project scheduling problems with constraints of different types may be rather complicated and even known to be NP-hard to solve
- Solution approaches involve methods of mixed integer linear programming, combinatorial and discrete optimization
- The temporal scheduling problems with only time-oriented objectives and constraints, can be formulated as linear programs
- These problems are solved using algorithms of linear programming which offer quite efficient numerical techniques
- Linear programming typically provides efficient numerical solutions, but does not allow to derive all solutions analytically
- In the framework of tropical algebra, many temporal project scheduling problems can be analytically solved in explicit form

Start-Finish Relations

- Consider a project that involves n activities (tasks, operations, jobs) performed in parallel, subject to a set of temporal constraints
- The start-finish relations specify the minimum allowed time lag between the start of one activity and finish of another
- Each activity finishes when all constraints for its finish are fulfilled
- For each activity $i = 1, \ldots, n$, the following notation is used:

x_i, *the unknown start time*;
y_i, *the unknown finish time*;
a_{ij}, *the given minimum possible time lag between the start of activity $j = 1, \ldots, n$ and finish of i ($a_{ij} = -\infty$ if unspecified)*

- The start-finish constraints take the form of the following inequalities (where at least one inequality holds as an equality):

$$y_i \geq x_j + a_{ij}, \qquad i = 1, \ldots, n$$

Scalar Representation of Model

- Combining all start-finish relations for activity i yields the equation

$$y_i = \max(x_1 + a_{i1}, \ldots, x_n + a_{in}), \qquad i = 1, \ldots, n$$

- After replacing the operations $\max$ by $\oplus$ and $+$ by $\otimes$, we obtain

$$y_i = a_{i1} \otimes x_1 \oplus \cdots \oplus a_{in} \otimes x_n, \qquad i = 1, \ldots, n$$

- The multiplication sign $\otimes$, as usual, can be eliminated to write

$$y_i = a_{i1} x_1 \oplus \cdots \oplus a_{in} x_n, \qquad i = 1, \ldots, n$$

- The last equation is very similar to the ordinary linear expression

$$y_i = a_{i1} x_1 + \cdots + a_{in} x_n, \qquad i = 1, \ldots, n$$

Vector Representation of Model

- We introduce the matrix and vectors

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

- The model is represented in the form of the vector equation

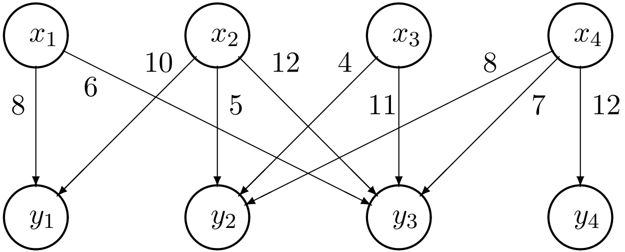
$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

- The vector equation corresponds the system of scalar equations

$$\begin{aligned} y_1 &= a_{11} \otimes x_1 \oplus \cdots \oplus a_{1n} \otimes x_n, \\ &\vdots \\ y_n &= a_{n1} \otimes x_1 \oplus \cdots \oplus a_{nn} \otimes x_n \end{aligned}$$

Graph and Matrix of Project

- Example of the graph of a project



- The corresponding matrix of the project ($\mathbb{0} = -\infty$):

$$
\boldsymbol{A} = \begin{pmatrix} 8 & 10 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 5 & 4 & 8 \\ 6 & 12 & 11 & 7 \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & 12 \end{pmatrix}
$$

Due Dates

- Suppose that due dates are given for activity in the project, which specify the time by which the activities should be finished
- For each activity $i = 1, \ldots, n$, the following notation is used:

  p_i, *the given due date*

- Let us introduce the vector notation:

$$\boldsymbol{p} = \begin{pmatrix} p_1 & \ldots & p_n \end{pmatrix}^T$$

Scheduling Problem

- Consider the problem to find the start time x_i of each activity i, for which the completion time y_i coincides with the due dates p_i
- The solution of the problem corresponds to solving the following vector equation (in terms of algebra with $\oplus = \max$ and $\otimes = +$)

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p} \qquad \text{(one-sided equation)}$$

Start-Start Relations

- Consider a project with start-start relations that specify the minimum allowed time lag between the start time of two activities
- For each activity $i = 1, \ldots, n$, we use the following notation:

x_i, *the unknown start time*;
b_{ij}, *the given minimum possible time lag between the start of activity $j = 1, \ldots, n$ and start of i ($b_{ij} = -\infty$ if unspecified)*

- The start-start relations are written as the inequalities

$$x_i \geq x_j + b_{ij}, \qquad i = 1, \ldots, n$$

- All relations for activity i are combined into one inequality

$$x_i \geq \max(b_{i1} + x_1, \ldots, b_{in} + x_n) \quad \text{(in ordinary notation)},$$
$$x_i \geq b_{i1} x_1 \oplus \cdots \oplus b_{in} x_n \quad \text{(after replacing operations)}$$

Vector Representation

- In matrix-vector notation, we have

$$\boldsymbol{B} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Scheduling Problem

- The problem of finding the start time x_i for each i to satisfy the start-start relations, corresponds to solving the inequality

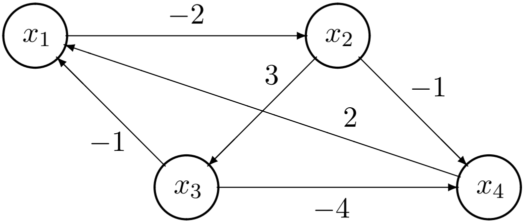
$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}$$

- If each activity starts immediately as soon as all its start-start relations are satisfied, the problem reduces to the equation

$$\boldsymbol{B}\boldsymbol{x} = \boldsymbol{x} \qquad \text{(homogeneous two-sided equation)}$$

Graph and Matrix of Project

- Example of the graph of a project



- The corresponding matrix of the project ($\mathbb{0} = -\infty$):

$$
\boldsymbol{B} = \begin{pmatrix} 0 & -2 & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & 0 & 3 & -1 \\ -1 & \mathbb{0} & 0 & -4 \\ 2 & \mathbb{0} & \mathbb{0} & 0 \end{pmatrix}
$$

Release Dates

- Suppose that release dates are given for activities in the project, which specify the earliest allowed start time for each activity
- For each activity $i = 1, \ldots, n$, we additionally define

  g_i, *the given release date*

- The release date constraints take the form of inequalities

$$x_i \geq g_i, \qquad i = 1, \ldots, n$$

- The start-start relations and release dates yield the inequalities

$$x_i \geq \max(b_{i1} + x_1, \ldots, b_{in} + x_n, g_i) \quad \text{(in ordinary notation)},$$

$$x_i \geq b_{i1} x_1 \oplus \cdots \oplus b_{in} x_n \oplus g_i \quad \text{(after replacing operations)}$$

Vector Representation

- We introduce the vector notation

$$\boldsymbol{g} = \begin{pmatrix} g_1 & \dots & g_n \end{pmatrix}^T$$

Scheduling Problem

- Consider the problem to find the start time x_i of each activity i to satisfy both the start-start relations and release dates constraints
- The solution of the problem corresponds to solving the inequality

$$\boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} \leq \boldsymbol{x}$$

- If each activity starts immediately as soon as all its start-start relations are satisfied, the problem reduces to the equation

$$\boldsymbol{B}\boldsymbol{x} \oplus \boldsymbol{g} = \boldsymbol{x} \qquad \text{(nonhomogenious two-sided equation)}$$

Scheduling with Mixed Constraints

- Consider a project with a matrix $\boldsymbol{A}$ of start-finish relations and a vector $\boldsymbol{p}$ of due dates, which result in the constraint in the form

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{p}$$

- Further assume that start-start constraints with a matrix $\boldsymbol{B}$ are also imposed, which yield the inequality constraint

$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}$$

Scheduling Problem

- As scheduling problem of interest, one can consider the derivation of the vector $\boldsymbol{x}$ of start time, which satisfies the system

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} &= \boldsymbol{p}, \\ \boldsymbol{B}\boldsymbol{x} &\leq \boldsymbol{x} \end{aligned}$$

Optimality Criteria

Project Makespan

- Consider a project with constraints given by start-finish relations
- Suppose we need to minimize the project makespan (the overall duration of the project) as the optimality criterion for scheduling
- For each activity $i = 1, \ldots, n$, we use the following notation:

  x_i, *the unknown start time*;

  y_i, *the unknown finish time*;

  a_{ij}, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and finish of* i *(*$a_{ij} = -\infty$ *if unspecified)*

- Furthermore, we introduce the matrix and vectors

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{y} = (y_i)$$

- We use the obvious identity $\min(u,v) = -\max(-u,-v)$ to represent the overall duration of the project as the difference

$$\max_{1\leq i\leq n} y_i - \min_{1\leq i\leq n} x_i = \max_{1\leq i\leq n} y_i + \max_{1\leq i\leq n} (-x_i)$$

- Consider the column vector $\boldsymbol{x}$ and define its conjugate row vector

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \boldsymbol{x}^{-} = \begin{pmatrix} -x_1 & \dots & -x_n \end{pmatrix}$$

- We also define the vector of arithmetic zeros and its conjugate as

$$\mathbf{1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \mathbf{1}^{-} = \mathbf{1}^{T} = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix}$$

- Consider the objective function representing the project makespan

$$\max_{1\le i\le n} y_i + \max_{1\le i\le n}(-x_i)$$

- After replacing the operations $\max$ by $\oplus$ and $+$ by $\otimes$, we obtain

$$(y_1 \oplus \cdots \oplus y_n)((-x_1) \oplus \cdots \oplus (-x_n))$$

- In vector notation, taking into account that $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we have

$$\mathbf{1}^T \boldsymbol{y}\boldsymbol{x}^- \mathbf{1} = \mathbf{1}^T \boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^- \mathbf{1}$$

Scheduling Problem

- The problem is to derive a vector $\boldsymbol{x}$ of start time, which attains

$$\min_{\boldsymbol{x}} \quad \mathbf{1}^T \boldsymbol{A}\boldsymbol{x}\boldsymbol{x}^- \mathbf{1}$$

Maximum Deviation From Due Dates

- Consider a project with start-finish constraints and due dates
- Let us define the maximum deviation from due dates as the optimality criterion for scheduling, which has to be minimized
- For each activity $i = 1, \ldots, n$, we use the following notation:

x_i, *the unknown start time*;
y_i, *the unknown finish time*;
a_{ij}, *the given minimum possible time lag between the start of activity $j = 1, \ldots, n$ and finish of i ($a_{ij} = -\infty$ if unspecified)*;
p_i, *the given due date*

- We introduce the matrix and vectors

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{y} = (y_i), \qquad \boldsymbol{p} = (p_i)$$

- We use the identity $|u| = \max(-u, u)$ to represent the maximum deviation of the elements of $\boldsymbol{y}$ from the elements of $\boldsymbol{p}$ as follows:

$$\begin{aligned}\max_{1\leq i\leq n} |y_i - p_i| &= \max_{1\leq i\leq n} \max(y_i - p_i, p_i - y_i) \\ &= \max\left(\max_{1\leq i\leq n}(y_i + (-p_i)), \max_{1\leq i\leq n}(p_i + (-y_i))\right)\end{aligned}$$

- Consider the vector of finish time and vector of due dates

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \qquad \boldsymbol{p} = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

- For these two vectors, define their conjugate row vectors

$$\boldsymbol{y}^{-} = \begin{pmatrix} -y_1 & \ldots & -y_n \end{pmatrix}, \qquad \boldsymbol{p}^{-} = \begin{pmatrix} -p_1 & \ldots & -p_n \end{pmatrix}$$

- Consider the expression of the maximum deviation

$$\max(\max_{1\leq i\leq n}(y_i+(-p_i)), \max_{1\leq i\leq n}(p_i+(-y_i)))$$

- After replacing the operations $\max$ by $\oplus$ and $+$ by $\otimes$, we obtain

$$(y_1\otimes(-p_1)\oplus\cdots\oplus y_n\otimes(-p_n))\oplus(p_1\otimes(-y_1)\oplus\cdots\oplus p_n\otimes(-y_n))$$

- In vector notation, with the substitution $\boldsymbol{y}=\boldsymbol{A}\boldsymbol{x}$, we obtain

$$\boldsymbol{p}^{-}\boldsymbol{y}\oplus\boldsymbol{y}^{-}\boldsymbol{p}=\boldsymbol{p}^{-}\boldsymbol{A}\boldsymbol{x}\oplus(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$$

Scheduling Problem

- The scheduling problem is to find a vector $\boldsymbol{x}$ that provides

$$\min_{\boldsymbol{x}}\quad \boldsymbol{p}^{-}\boldsymbol{A}\boldsymbol{x}\oplus(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p}$$

Maximum Flowtime

- Consider a project with start-finish and start-start constraints
- We define the maximum flowtime (maximum total time, cycle time) of activities as the optimality criterion, which has to be minimized
- For each activity $i = 1, \ldots, n$, we use the following notation:

  x_i, *the unknown start time*;
  y_i, *the unknown finish time*;
  a_{ij}, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and finish of* i;
  b_{ij}, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and start of* i

- The flowtime of activity i is given by the difference

$$y_i - x_i, \qquad i = 1, \ldots, n$$

- We introduce the matrices and vectors

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{B} = (b_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{y} = (y_i)$$

- Let us consider the maximum flowtime over all activities

$$\max(y_1 - x_1, \ldots, y_n - x_n) \qquad \text{(in ordinary notation)}$$
$$y_1 \otimes (-x_1) \oplus \cdots \oplus y_n \otimes (-x_n) \qquad \text{(after replacing operations)}$$

- In vector notation, with the substitution $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we obtain

$$\boldsymbol{x}^{-}\boldsymbol{y} = \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

Scheduling Problem

- The scheduling problem is to find a vector $\boldsymbol{x}$ that attains

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \end{aligned}$$

Maximum Deviation of Finish Time

- Consider a project with start-finish and start-start constraints
- Suppose that the optimal schedule has to minimize the maximum deviation of finish time
- For each activity $i = 1, \ldots, n$, we use the following notation:

  x_i, *the unknown start time*;
  y_i, *the unknown finish time*;
  a_{ij}, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and finish of* i;
  b_{ij}, *the given minimum possible time lag between the start of activity* $j = 1, \ldots, n$ *and start of* i

- The maximum deviation of finish time over all activities is given by

$$\max_{1 \leq i \leq n} y_i - \min_{1 \leq i \leq n} y_i = \max_{1 \leq i \leq n} y_i + \max_{1 \leq i \leq n} (-y_i)$$

- We represent the maximum deviation of finish time as follows:

$$\max_{1\le i\le n} y_i + \max_{1\le i\le n}(-y_i) \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{i=1}^{n} y_i \otimes \bigoplus_{j=1}^{n}(-y_j) \qquad \text{(after replacing operations)}$$

- In vector notation, with the substitution $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, we have

$$\mathbf{1}^T\boldsymbol{y}\boldsymbol{y}^{-}\mathbf{1} = \mathbf{1}^T\boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\mathbf{1}, \qquad \mathbf{1} = (0,\dots,0)^T$$

Scheduling Problem

- The problem is to find a vector $\boldsymbol{x}$ that provides the minimum

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \mathbf{1}^T\boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\mathbf{1}, \\ \text{s. t.} \quad & \boldsymbol{B}\boldsymbol{x} \le \boldsymbol{x} \end{aligned}$$

Tropical Algebra: Max-Algebra

- Max-algebra is the set of nonnegative reals $\mathbb{R}_{+} = \{x \in \mathbb{R} | x \geq 0\}$ with binary operations of addition $\oplus$ and multiplication $\otimes$
- Addition is defined as taking maximum

$$x \oplus y = \max\{x, y\} \qquad \forall x, y \in \mathbb{R}_{+}$$

- Addition possesses the idempotency property

$$x \oplus x = \max\{x, x\} = x \qquad \forall x \in \mathbb{R}_{+}$$

- Multiplication is defined as usual: $x \otimes y = x \times y$
- The neutral elements with respect to addition $\mathbb{0}$ and multiplication $\mathbb{1}$ coincide with the arithmetic zero 0 and one 1
- The multiplicative inverse and power have the usual meaning
- The additive inverse does not exists, and subtraction is undefined

Examples

- The operations $\oplus$ and $\otimes$ are defined on nonnegative reals $\mathbb{R}_+$
- Addition:

$$2 \oplus 0 = 2 \qquad (\max(2,0) = 2)$$
$$1 \oplus 3 = 3 \qquad (\max(1,3) = 3)$$

- Multiplication:

$$1 \otimes 0 = 0 \qquad (1 \times 0 = 0)$$
$$2 \otimes (1/3) = 2/3 \qquad (2 \times (1/3) = 2/3)$$

- Exponentiation:

$$2^2 = 4 \qquad (2^2 = 4)$$
$$8^{1/3} = 2 \qquad (8^{1/3} = 2)$$

- Inversion:

$$1^{-1} = 1 \qquad (1^{-1} = 1)$$
$$2^{-1} = 1/2 \qquad (2^{-1} = 1/2)$$

Max-Plus Algebra

- Max-plus algebra is the extended set of reals $\mathbb{R} \cup \{-\infty\}$ with binary operations of addition $\oplus$ and multiplication $\otimes$
- Addition is idempotent and defined as

$$x \oplus y = \max\{x, y\} \qquad \forall x, y \in \mathbb{R} \cup \{-\infty\}$$

- Multiplication is invertible and defined as arithmetic addition

$$x \otimes y = x + y \qquad \forall x, y \in \mathbb{R} \cup \{-\infty\}$$

- The neutral elements are given by

$$\mathbb{0} = -\infty, \qquad \mathbb{1} = 0$$

- For each $x \in \mathbb{R}$ its inverse x^{-1} coincides with the opposite number $-x$ in the standard arithmetic
- The power x^y corresponds to the arithmetic product $x \times y$

Examples

- The operations $\oplus$ and $\otimes$ are defined on $\mathbb{R} \cup \{-\infty\}$
- Addition:

$$2 \oplus 0 = 2 \qquad (\max(2,0) = 2)$$

$$1 \oplus (-3) = 1 \qquad (\max(1,-3) = 1)$$

- Multiplication:

$$1 \otimes 0 = 1 \qquad (1 + 0 = 1)$$

$$2 \otimes (-3) = -1 \qquad (2 + (-3) = -1)$$

- Exponentiation:

$$2^2 = 4 \qquad (2 \times 2 = 4)$$

$$(-2)^{1/3} = -2/3 \qquad ((-2) \times (1/3) = -2/3)$$

- Inversion:

$$1^{-1} = -1 \qquad (1 \times (-1) = -1)$$

$$(-2)^{-1} = 2 \qquad ((-2) \times (-1) = 2)$$

Idempotent Semifield

- Idempotent semifield is the algebraic system $\langle \mathbb{X}, \mathbb{0}, \mathbb{1}, \oplus, \otimes \rangle$
- The carrier set $\mathbb{X}$ includes the zero $\mathbb{0}$ and one $\mathbb{1}$, $\mathbb{0} \neq \mathbb{1}$
- The set $\mathbb{X}$ is closed under addition $\oplus$ and multiplication $\otimes$
- Both operations $\oplus$ and $\otimes$ are associative and commutative
- Multiplication $\otimes$ distributes over addition $\oplus$
- Addition is idempotent: $x \oplus x = x$ for all $x \in \mathbb{X}$
- Multiplication is invertible: for each $x \neq \mathbb{0}$ there exists inverse x^{-1}
- Idempotent addition induces a partial order on $\mathbb{X}$ by to the rule

$$x \leq y \qquad \text{if and only if} \qquad x \oplus y = y$$

Idempotent Semifield (cont.)

- Integer powers are defined for each $x \neq \mathbb{0}$ and natural n by

$$x^0 = \mathbb{1}, \quad x^n = x^{n-1} \otimes x, \quad x^{-n} = (x^{-1})^n, \quad \mathbb{0}^n = \mathbb{0}$$

- Algebraic completeness: the equation $x^n = a$ is solvable for each $a \in \mathbb{X}$ and natural n (existence of rational exponents)
- Linear order: the partial order induced by idempotent addition by the rule $x \leq y \Longleftrightarrow x \oplus y = y$ is extendable to a total order
- Absorption rule: $x \otimes \mathbb{0} = \mathbb{0}$ for all $x \in \mathbb{X}$
- In what follows, the multiplication sign $\otimes$, as usual, is omitted

Examples of Idempotent Semifields

- Max-algebra:
$$\mathbb{R}_{\max} = \langle \mathbb{R}_+ \cup \{0\}, 0, 1, \max, \times \rangle$$
- Max-plus algebra:
$$\mathbb{R}_{\max,+} = \langle \mathbb{R} \cup \{-\infty\}, -\infty, 0, \max, + \rangle$$
- Min-algebra:
$$\mathbb{R}_{\min} = \langle \mathbb{R}_+ \cup \{+\infty\}, +\infty, 1, \min, \times \rangle$$
- Min-plus algebra:
$$\mathbb{R}_{\min,+} = \langle \mathbb{R} \cup \{+\infty\}, +\infty, 0, \min, + \rangle$$
- The semifields $\mathbb{R}_{\max,\times}$, $\mathbb{R}_{\max,+}$, $\mathbb{R}_{\min,\times}$, $\mathbb{R}_{\min,+}$ are isomorphic

Isomorphism of Idempotent Semifields

$$\begin{tikzcd}
\mathbb{R}_{\max,+} \arrow[r, shift left, "y=-x"] \arrow[d, shift left, "y=e^x"] & \mathbb{R}_{\min,+} \arrow[l, shift left, "y=-x"] \arrow[d, shift left, "y=e^x"] \\
\mathbb{R}_{\max} \arrow[u, shift left, "y=\ln x"] \arrow[r, shift left, "y=1/x"] & \mathbb{R}_{\min} \arrow[u, shift left, "y=\ln x"] \arrow[l, shift left, "y=1/x"]
\end{tikzcd}$$

- Isomorphism of the semifields $\mathbb{R}_{\max,+}$, $\mathbb{R}_{\min,+}$, $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$

Examples of Idempotent Semirings

- Max-min algebra:

$$\mathbb{R}_{\max,\min} = \langle \mathbb{R} \cup \{-\infty, +\infty\}, -\infty, +\infty, \max, \min \rangle$$

- Algebra defined on the set $\mathbb{X}$ of all subsets of a compact set S :

$$\mathbb{X}_{\cup,\cap} = \langle \mathbb{X}, S, \emptyset, \cup, \cap \rangle$$

Properties of Operations

- The extremal property of addition (majority law):

$$x \leq x \oplus y, \quad y \leq x \oplus y, \qquad \forall x, y \in \mathbb{X}$$

- The monotonicity of addition and multiplication:

$$x \leq y \quad \Longrightarrow \quad x \oplus z \leq y \oplus z, \quad xz \leq yz, \qquad \forall x, y, z \in \mathbb{X}$$

- The equivalence of inequalities:

$$x \oplus y \leq z \quad \Longleftrightarrow \quad x \leq z, \quad y \leq z, \qquad \forall x, y, z \in \mathbb{X}$$

- The monotonicity of powers:

$$x \leq y \quad \Longrightarrow \quad \begin{cases} x^q \geq y^q, & \text{if } q < 0; \\ x^q \leq y^q, & \text{if } q \geq 0; \end{cases} \qquad \forall x, y \in \mathbb{X} \setminus \{\mathbb{0}\}$$

Binomial Identity

- A tropical analogue of binomial identity:

$$(x \oplus y)^\alpha = x^\alpha \oplus y^\alpha \qquad \forall x, y \in \mathbb{X}, \quad \alpha > 0$$

- Extension of the identity to n terms:

$$(x_1 \oplus \cdots \oplus x_n)^\alpha = x_1^\alpha \oplus \cdots \oplus x_n^\alpha \qquad \forall x_1, \ldots, x_n \in \mathbb{X}, \quad \alpha \geq 0$$

- A tropical analogue of the inequality between arithmetic and geometric means:

$$x \oplus y \geq (xy)^{1/2}, \qquad \forall x, y \in \mathbb{X}$$

- Extension of the inequality to n terms:

$$x_1 \oplus \cdots \oplus x_n \geq (x_1 \cdots x_n)^{1/n}, \qquad \forall x_1, \ldots, x_n \in \mathbb{X}$$

Linear Function: Definition and Properties

- A tropical analogue of linear function $f : \mathbb{X} \to \mathbb{X}$ is given by

$$f(x) = ax \oplus b, \qquad a, b \in \mathbb{X}$$

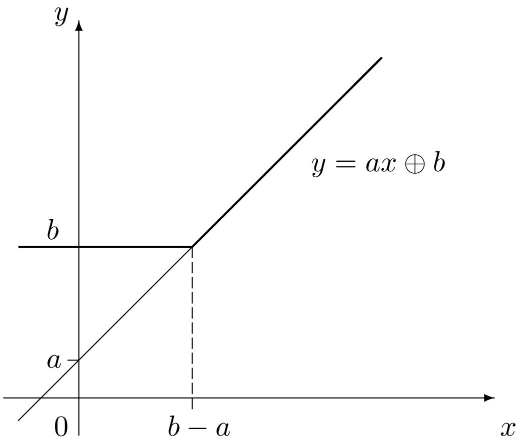
- If $b = \mathbb{0}$ the function is called homogeneous
- The additive property of the function:

$$f(x_1 \oplus x_2) = a(x_1 \oplus x_2) \oplus b = (ax_1 \oplus b) \oplus (ax_2 \oplus b) = f(x_1) \oplus f(x_2)$$

- As in the conventional algebra, we have

$$b = f(\mathbb{0}), \qquad a = \lim_{x \to \infty} x^{-1} f(x)$$

Graph of Linear Function in $\mathbb{R}_{\max,+}$



- Graph of Linear Function in the framework of $\mathbb{R}_{\max,+}$

Linear Equation in One Variable

- The general linear equation in one variable takes the form

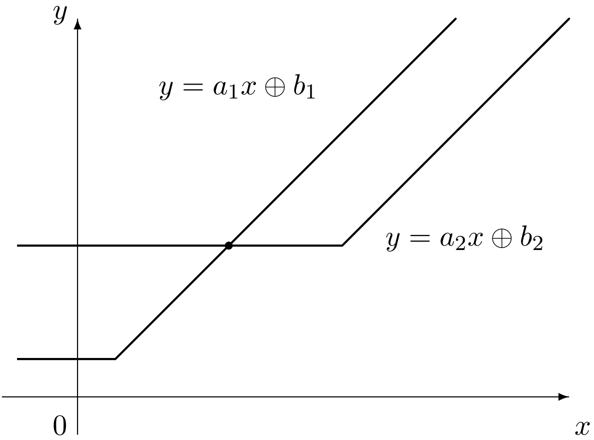
$$a_1x \oplus b_1 = a_2x \oplus b_2$$

- This equation cannot be reduced as follows:

$$ax = b$$

- In the framework of $\mathbb{R}_{\max,+}$, it can be solved graphically

Graphical Solution of Linear Equation



- An example of the solution of linear equation

Proposition

The following statements hold:

1. *If* $a_1 < a_2$ *and* $b_2 < b_1$ *, or* $a_2 < a_1$ *and* $b_1 < b_2$ *, then there is a unique solution*
$$x = (a_1 \oplus a_2)^{-1}(b_1 \oplus b_2);$$
2. *If* $a_1 \neq a_2$ *and* $b_1 \neq b_2$ *and both conditions of the previous case do not hold, then the equation has no solution;*
3. *If* $a_1 = a_2$ *and* $b_1 \neq b_2$ *, then the solution is given by the inequality*
$$x \geq a_1^{-1}(b_1 \oplus b_2);$$
4. *If* $a_1 \neq a_2$ *and* $b_1 = b_2$ *, then the solution is given by the inequality*
$$x \leq (a_1 \oplus a_2)^{-1} b_1;$$
5. *If* $a_1 = a_2$ *and* $b_1 = b_2$ *, then any* $x \in \mathbb{X}$ *is a solution*

Vector Algebra

- The matrix and vector operations follow the standard rules, where the operations $+$ and $\times$ are replaced by $\oplus$ and $\otimes$
- Addition of vectors $\boldsymbol{a} = (a_j)$ and $\boldsymbol{b} = (b_j)$, and multiplication by scalar x are given by the entrywise formulas

$$\{\boldsymbol{a} \oplus \boldsymbol{b}\}_j = a_j \oplus b_j, \qquad \{x\boldsymbol{a}\}_j = x a_j$$

- Zero vector has all components equal to $\mathbb{0}$ and it is denoted $\mathbf{0}$
- A vector without zero components is called regular
- For any nonzero column vector $\boldsymbol{a} = (a_j)$, its multiplicative conjugate transpose is the row vector $\boldsymbol{a}^{-} = (a_j^{-})$, where

$$a_j^{-} = \begin{cases} a_j^{-1}, & \text{if } a_j \neq \mathbb{0}; \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

Examples

- Vector operations in the framework of $\mathbb{R}_{\max,+}$:
- Vector addition

$$\begin{pmatrix} -1 \\ 0 \\ \mathbb{0} \end{pmatrix} \oplus \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

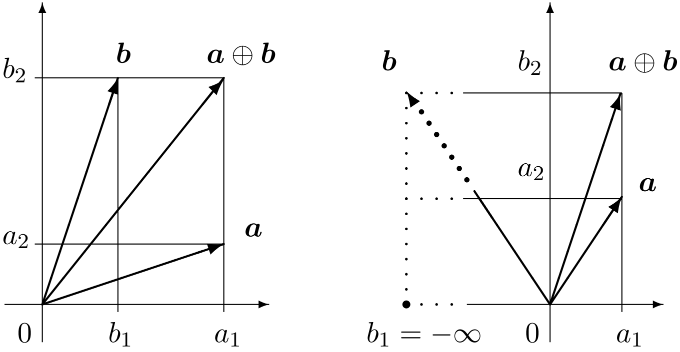
- Scalar multiplication

$$(-1) \begin{pmatrix} \mathbb{0} \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} \mathbb{0} \\ 1 \\ -2 \end{pmatrix}$$

- Multiplicative conjugation

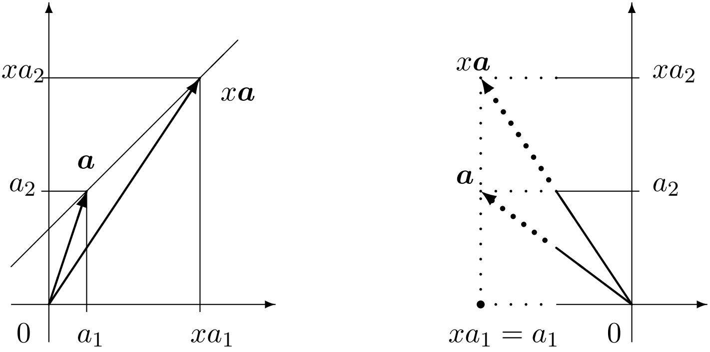
$$\begin{pmatrix} \mathbb{0} \\ -1 \\ 3 \end{pmatrix}^{-} = \begin{pmatrix} \mathbb{0} & 1 & -3 \end{pmatrix}$$

Graphical Illustration of Vector Addition in $\mathbb{R}^2_{\max,+}$



- Addition of regular vectors (left) and with an irregular vector (right)
- Addition follows a Rectangle Rule instead of Parallelogram Rule

Graphical Illustration of Scalar Multiplication in $\mathbb{R}^2_{\max,+}$



- Scalar multiplication of a regular vector (left) and of an irregular vector (right)

Linear Dependence of Vectors

- Linear combination of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ with coefficients $x_1, \ldots, x_n \in \mathbb{X}$ is defined as the sum $x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n$
- A vector $\boldsymbol{b}$ is linearly dependent on vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, if there are scalars $x_1, \ldots, x_n \in \mathbb{X}$ such that

$$\boldsymbol{b} = x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n$$

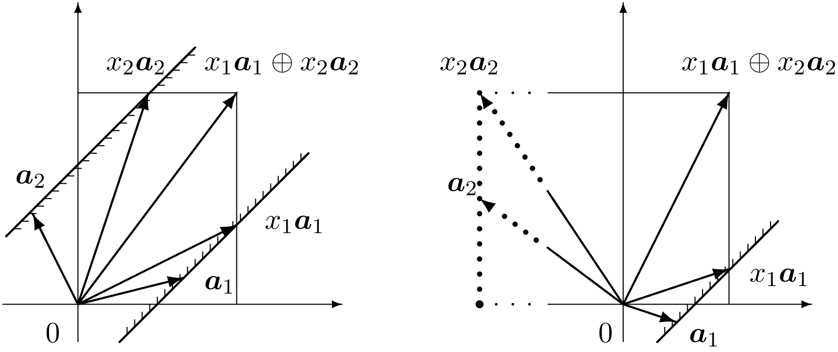
- Vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear if $\boldsymbol{b} = x\boldsymbol{a}$ for some $x \in \mathbb{X}$
- The linear span of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is given by

$$\operatorname{span}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \{x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n | \ x_1, \ldots, x_n \in \mathbb{X}\}$$

  and forms a tropical linear space generated by the vectors
- Any vector $\boldsymbol{y}$ from this space is represented by the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ and vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ in the form

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

Graphical Illustration of Linear Span of Vectors in $\mathbb{R}^2_{\max,+}$



- The linear span of two regular vectors is a band (left) and of regular and irregular vectors is a half-plane (right)

Minimal Generating System

- If $\boldsymbol{b}$ is dependent on a system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, but independent of any its subsystem, the system is a minimal generating system for $\boldsymbol{b}$
- Let us verify that the representation of a regular vector as a linear combination of vectors of its minimal generating system is unique
- Suppose there are two different representations of the vector $\boldsymbol{b}$:

$$\boldsymbol{b} = x_1 \boldsymbol{a}_1 \oplus \cdots \oplus x_n \boldsymbol{a}_n = x_1' \boldsymbol{a}_1 \oplus \cdots \oplus x_n' \boldsymbol{a}_n,$$

- Assume for definiteness that $x_i' < x_i$ for some $i = 1, \ldots, n$
- Then, $\boldsymbol{b} \geq x_i \boldsymbol{a}_i > x_i' \boldsymbol{a}_i$, which means that $x_i' \boldsymbol{a}_i$ does not affect $\boldsymbol{b}$
- Therefore, the vector $\boldsymbol{b}$ does not depend on the vector $\boldsymbol{a}_i$, which contradicts with the minimality of the system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$

Matrix Algebra

- For conforming matrices $\boldsymbol{A} = (a_{ij})$, $\boldsymbol{B} = (b_{ij})$ and $\boldsymbol{C} = (c_{ij})$, and a scalar x, the matrix operations are given by

$$\{\boldsymbol{A} \oplus \boldsymbol{B}\}_{ij} = a_{ij} \oplus b_{ij}, \quad \{\boldsymbol{A}\boldsymbol{C}\}_{ij} = \bigoplus_{k} a_{ik} c_{kj}, \quad \{x\boldsymbol{A}\}_{ij} = x a_{ij}$$

- The zero matrix has all components equal to $\mathbb{0}$ and is denoted $\mathbf{0}$
- A matrix without zero columns (rows) is column (row) regular
- For any nonzero matrix $\boldsymbol{A} = (a_{ij})$, its multiplicative conjugate transpose is the matrix $\boldsymbol{A}^{-} = (a_{ij}^{-})$, where

$$a_{ij}^{-} = \begin{cases} a_{ji}^{-1}, & \text{if } a_{ji} \neq \mathbb{0}; \\ \mathbb{0}, & \text{otherwise} \end{cases}$$

Examples

- Matrix operations in the framework of $\mathbb{R}_{\max,+}$:
- Matrix addition

$$\begin{pmatrix} -1 & 1 \\ 0 & -2 \\ 2 & \mathbb{0} \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 3 & \mathbb{0} \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \\ 2 & 0 \end{pmatrix}$$

- Matrix multiplication

$$\begin{pmatrix} 1 & -1 & \mathbb{0} \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} 1 & \mathbb{0} \\ 0 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 2 & 0 \end{pmatrix}$$

- Scalar multiplication

$$2\begin{pmatrix} -1 & 1 & \mathbb{0} \\ 0 & 2 & -2 \\ 1 & \mathbb{0} & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & \mathbb{0} \\ 2 & 4 & 0 \\ 3 & \mathbb{0} & -1 \end{pmatrix}$$

- Multiplicative conjugation

$$\begin{pmatrix} -1 & 1 & \mathbb{0} \\ 0 & 2 & -2 \\ 1 & \mathbb{0} & -3 \end{pmatrix}^{-} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & -2 & \mathbb{0} \\ \mathbb{0} & 2 & 3 \end{pmatrix}$$

Square Matrices

- The identity matrix has the usual diagonal form

$$I = \begin{pmatrix} \mathbb{1} & & \mathbb{0} \\ & \ddots & \\ \mathbb{0} & & \mathbb{1} \end{pmatrix}$$

- The identity matrix in max-plus algebra $\mathbb{R}_{\max,+}$

$$I = \begin{pmatrix} 0 & & -\infty \\ & \ddots & \\ -\infty & & 0 \end{pmatrix}$$

- The identity matrix in max-algebra $\mathbb{R}_{\max}$ has the conventional form with the arithmetic 1's on the diagonal and 0's elsewhere

Square Matrices (cont.)

- Positive integer powers of a square matrix $\boldsymbol{A}$ indicates repeated (tropical) multiplication of the matrix by itself

$$\mathbf{0}^p = \mathbf{0}, \quad \boldsymbol{A}^0 = \boldsymbol{I}, \quad \boldsymbol{A}^p = \boldsymbol{A}^{p-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{p-1}, \qquad \forall p \geq 1$$

- The entry $a_{ij}^{(k)}$ of the matrix $\boldsymbol{A}^k$ takes the form

$$a_{ij}^{(k)} = \bigoplus_{1 \leq i_1, \dots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$$

Inverse Matrix

- A matrix A^{-1} is the inverse matrix for A, if $A^{-1}A = AA^{-1} = I$
- A matrix is invertible if and only if it has only one nonzero entry in each row and column (proof by contradiction)
- The inverse matrix exists only for
  - *the strictly diagonal matrices (without zero diagonal entries),*
  - *the matrices obtained from the strictly diagonal by permutation of rows and/or columns*
- If a matrix A has an inverse, then $A^{-} = A^{-1}$
- Since the class of invertible matrices is very poor, conjugate transposition plays more important role than matrix inversion

Linear Operators: Linear Equations

Linear Equations

- Any $(m \times n)$-matrix $\boldsymbol{A}$ defines an operator from $\mathbb{X}^n$ to $\mathbb{X}^m$
- For any two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{X}^n$ and scalar $\alpha \in \mathbb{X}$, we have
  1. $\boldsymbol{A}(\boldsymbol{x} \oplus \boldsymbol{y}) = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{A}\boldsymbol{y}$ *(additivity);*
  2. $\boldsymbol{A}(\alpha\boldsymbol{x}) = \alpha\boldsymbol{A}\boldsymbol{x}$ *(multiplicativity)*
- With these properties, the operator $\boldsymbol{A}$ is a linear operator
- The general linear equation in an unknown vector $\boldsymbol{x}$ is given by

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{C}\boldsymbol{x} \oplus \boldsymbol{d}$$

Special Cases of General Equation

- One-sided equations:

$$Ax = d, \qquad Ax \oplus b = d$$

- One-sided inequalities:

$$Ax \leq d, \qquad Ax \oplus b \leq d$$

- Two-sided equations:

$$Ax = x, \qquad Ax \oplus b = x$$

- Two-sided inequalities:

$$Ax \leq x, \qquad Ax \oplus b \leq x$$

One-Sided Inequality: Definitions and Preliminaries

- Given an $(m \times n)$-matrix $\boldsymbol{A}$ and m-vector $\boldsymbol{b}$, the following inequality in an unknown n-vector $\boldsymbol{x}$ is called one-sided:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

- This inequality has the unknown vector $\boldsymbol{x}$ only on one side
- This one-sided inequality always has solutions; specifically, the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ obviously satisfies the inequality
- We obtain a solution of the inequality by applying properties of conjugate transposition and simple algebraic manipulations

Proposition (Properties of Conjugate Transposition)

The following statements hold:

1. *For any regular n-vector, the following inequality is valid:*

$$\boldsymbol{x}\boldsymbol{x}^{-} = \begin{pmatrix} x_1 x_1^{-1} & \dots & x_1 x_n^{-1} \\ \vdots & \ddots & \vdots \\ x_n x_1^{-1} & \dots & x_n x_n^{-1} \end{pmatrix} \geq \boldsymbol{I}$$

2. *For any nonzero vector $\boldsymbol{x}$, the following equality holds:*

$$\boldsymbol{x}^{-}\boldsymbol{x} = \bigoplus_{i:\, x_i \neq \mathbb{0}} x_i^{-1} x_i = \mathbb{1}$$

- Since all diagonal entries of the matrix $\boldsymbol{x}\boldsymbol{x}^{-}$ are equal to $\mathbb{1}$, and off-diagonal entries are greater than $\mathbb{0}$, we see that $\boldsymbol{x}\boldsymbol{x}^{-} \geq \boldsymbol{I}$
- The inequality $\boldsymbol{x}^{-}\boldsymbol{x} = \mathbb{1}$ is trivially holds for any nonzero $\boldsymbol{x}$

Solution of One-Sided Inequality

- Given an $(m \times n)$-matrix $\boldsymbol{A}$ and m-vector $\boldsymbol{b}$, we start with the problem to find n-vectors $\boldsymbol{x}$ that satisfy the one-sided inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

Lemma (Solution of One-Sided Inequality)

For any column-regular (without zero columns) matrix $\boldsymbol{A}$ and regular (w/o zero entries) vector $\boldsymbol{b}$, all solutions of the inequality are given by

$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

Proof

- Let us verify that the following inequalities are equivalent:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \qquad \boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

- Left multiplication of the first inequality by the matrix $(\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}$ and monotonicity of multiplication yield the result

$$(\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{b}$$

- It follows from the properties of conjugate transposition that

$$(\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{A} \geq \boldsymbol{I}, \qquad \boldsymbol{b}^{-}\boldsymbol{b} = \mathbb{1}$$

- After substitution, we obtain the second inequality as follows:

$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}\boldsymbol{b}^{-}\boldsymbol{b} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

- Left multiplication of the second inequality by $\boldsymbol{A}$ leads to the first:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} \leq \boldsymbol{b}\boldsymbol{b}^{-}\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \boldsymbol{b} \quad \blacksquare$$

Example in Two Dimensions

- Consider the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ with the matrix and vectors

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

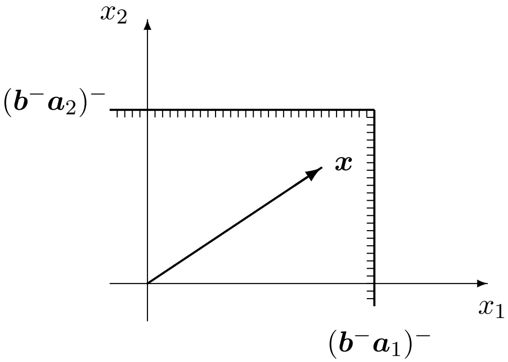
- We denote the columns of the matrix by small bold letters:

$$\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2), \qquad \boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}, \qquad \boldsymbol{a}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}$$

- We assume that $a_{11}, a_{12}, a_{21}, a_{22} > \mathbb{0}$ and $d_1, d_2 > \mathbb{0}$
- All solutions of the inequality are given by

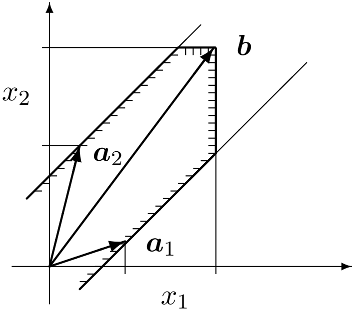
$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} (\boldsymbol{b}^{-}\boldsymbol{a}_1)^{-1} \\ (\boldsymbol{b}^{-}\boldsymbol{a}_2)^{-1} \end{pmatrix} = \begin{pmatrix} (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1} \\ (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1} \end{pmatrix}$$

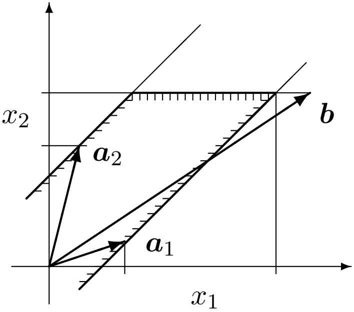
Graphical Illustration of Solution to $Ax \leq b$ in $\mathbb{R}^2_{\max,+}$



- Solution of the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ with $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$, represented in the space of solution vectors $\boldsymbol{x}$ in Cartesian coordinates

Graphical Illustration of Solution to $Ax \leq b$ in $\mathbb{R}^2_{\max,+}$





- Illustration of solutions in the space of columns in $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$
- Solutions are shown for the cases when $\boldsymbol{b}$ is inside (left) and outside (right) the linear span of the columns of $\boldsymbol{A}$

Numerical Example

- Consider an inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ defined in $\mathbb{R}^3_{\max,+}$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

- To solve the inequality, we first calculate the product

$$\boldsymbol{b}^{-}\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix}$$

- After conjugation of the obtained result, we arrive at the solution

$$\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix}$$

One-Sided Equation: Definitions and Preliminaries

- Given an $(m \times n)$-matrix $\boldsymbol{A}$ and m-vector $\boldsymbol{b}$, the one-sided equation in an unknown n-vector $\boldsymbol{x}$ is defined as follows:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

- This equation has the unknown on one side and can be referred to as an equation of the first kind (by analogy with integral equations)
- Since the equation may have no (exact) solution, we concentrate on finding a best approximate solution in the sense of some metric
- We examine the distance between a vector and a tropical vector subspace, and then apply the result to solve the equation

Generalized Metric

- We define the distance between regular vectors $\boldsymbol{x} = (x_i)$ and $\boldsymbol{y} = (y_i)$ by the following distance function:

$$d(\boldsymbol{x}, \boldsymbol{y}) = \bigoplus_i \left(x_i y_i^{-1} \oplus x_i^{-1} y_i \right) = \boldsymbol{y}^- \boldsymbol{x} \oplus \boldsymbol{x}^- \boldsymbol{y}$$

- If one of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is regular and the other is not, we put $d(\boldsymbol{x}, \boldsymbol{y}) = \infty$, where ∞ denotes an undefined value
- We observe that this function has its minimum value equal to $\mathbb{1}$
- In the context of $\mathbb{R}_{\max,+}$ (max-plus algebra), where $\mathbb{1} = 0$, the distance function d coincides with the Chebyshev metric

$$d_\infty(\boldsymbol{x}, \boldsymbol{y}) = \max_i |x_i - y_i| = \max_i \max(x_i - y_i, y_i - x_i)$$

- In $\mathbb{R}_{\max}$ (max-algebra), the function d can be considered as a generalized metric that takes values in the interval $[1, \infty)$

Distance Between Linear Span and Vector

- Consider the linear span of vectors $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n$, which is given by

$$\mathcal{A} = \operatorname{span}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n) = \{x_1\boldsymbol{a}_1 \oplus \cdots \oplus x_n\boldsymbol{a}_n |\, x_1,\ldots,x_n \in \mathbb{X}\}$$

- With the matrix $\boldsymbol{A} = (\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n)$ and vector $\boldsymbol{x} = (x_1,\ldots,x_n)^T$, any vector $\boldsymbol{y} \in \mathcal{A}$ is represented as

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$$

- Define the distance between the linear span and a vector $\boldsymbol{b}$ as

$$d(\mathcal{A},\boldsymbol{b}) = \min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x},\boldsymbol{b})$$

Proposition (Distance from Linear Span to Regular Vector)

If the vector $\boldsymbol{b}$ is regular, then

$$d(\mathcal{A},\boldsymbol{b}) = \min_{\text{regular } \boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x},\boldsymbol{b})$$

Proof

- Take a vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ such that $d(\boldsymbol{A}\boldsymbol{x},\boldsymbol{b})$ achieves its minimum
- If $\boldsymbol{y}$ is not regular, then the statement is true since $d(\boldsymbol{y},\boldsymbol{b}) = \infty$
- Suppose $\boldsymbol{y} = (y_i)$ is regular, and assume the corresponding vector $\boldsymbol{x} = (x_j)$ to have a zero component, say $x_k = \mathbb{0}$
- We define the following index set and threshold value:

$$I = \{i | a_{ij} > \mathbb{0}\} \neq \emptyset, \qquad \varepsilon = \min\{a_{ij}^{-1} y_i | i \in I\} > \mathbb{0}$$

- We replace $x_k = \mathbb{0}$ by $x_k = \varepsilon$, and note that all components of $\boldsymbol{y}$ along with the minimum value of $d(\boldsymbol{A}\boldsymbol{x},\boldsymbol{b})$ remain unchanged ■

- Given a matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, we find the distance between the linear span of the columns in $\boldsymbol{A}$ and $\boldsymbol{b}$ by solving the problem

$$\min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$$

Lemma (Evaluation of Distance)

Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ regular vector. Define the scalar

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}.$$

Then, the distance between the linear span and vector $\boldsymbol{b}$ is given by

$$d(\mathcal{A}, \boldsymbol{b}) = \min_{\boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \Delta^{1/2},$$

where the minimum is attained at

$$\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

Proof

- Assume both $\boldsymbol{A}$ and $\boldsymbol{b}$ to be regular, and consider the problem

$$\min_{\text{regular } \boldsymbol{x}} d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b})$$

- Substitution of the expression for the distance function yields

$$d(\boldsymbol{A}\boldsymbol{x}, \boldsymbol{b}) = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b}$$

- We take any regular vector $\boldsymbol{x}$, and denote the value of distance by

$$r = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b} > \mathbb{0}$$

- It follows from the extremal property of tropical addition that

$$r \geq \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x}, \qquad r \geq (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b}$$

Proof (cont.)

- Let us solve with respect to r the obtained system of inequalities

$$r \geq \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x}, \qquad r \geq (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b}$$

- The solution of the first inequality as a one-sided inequality yields

$$\boldsymbol{x} \leq r(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

- After left multiplication by $\boldsymbol{A}$ and conjugate transposition, we have

$$(\boldsymbol{A}\boldsymbol{x})^{-} \geq r^{-1}(\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}$$

- Substitution into the second inequality leads to the inequality

$$r \geq r^{-1}(\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = r^{-1}\Delta$$

- As a result, we obtain the lower bound for the objective function

$$r = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b} \geq \Delta^{1/2}$$

Proof (cont.)

- Let us verify that the following lower bound is attainable:

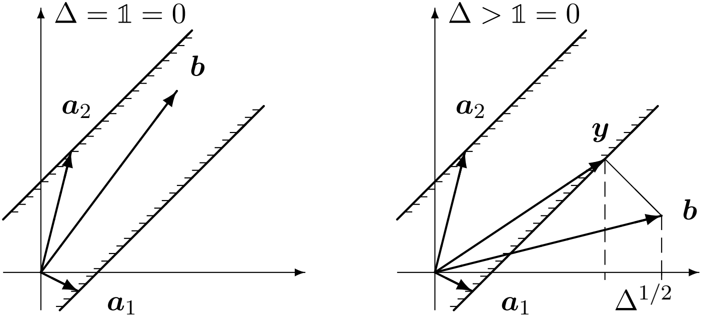
$$r = \boldsymbol{b}^{-}\boldsymbol{A}\boldsymbol{x} \oplus (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b} \geq \Delta^{1/2}$$

- Indeed, substitution of the vector $\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ gives

$$r = \Delta^{1/2}\boldsymbol{b}^{-}\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} \oplus \Delta^{-1/2}(\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \Delta^{1/2}$$

- Therefore, $\Delta^{1/2}$ is a strict (attainable) lower bound, and hence the minimum of the objective function which is the distance in question
- The vector $\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is a solution of the minimization problem that gives a closest vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ in the linear span ■

Graphical Illustration of Evaluation of Distance in $\mathbb{R}^2_{\max,+}$



- Evaluation of the distance from a vector to a linear span
- Illustration is given for the cases when $\boldsymbol{b}$ is inside (left) and outside (right) the linear span of the columns of $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$

Lemma (Linear Dependence)

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ *be vectors such that the matrix* $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ *is regular, and* $\boldsymbol{b}$ *be regular vector. Define the scalar*

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}.$$

The vector $\boldsymbol{b}$ *is linearly dependent on vectors* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ *if and only if*

$$\Delta = \mathbb{1}$$

Proof

- From geometric viewpoint, a vector $\boldsymbol{b}$ is linearly dependent on $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ if $\boldsymbol{b}$ belongs to the linear span $\mathcal{A} = \operatorname{span}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$
- By the lemma on evaluation of distance, the equality $\Delta = \mathbb{1}$ means that $\boldsymbol{b} \in \mathcal{A}$, whereas the inequality $\Delta > \mathbb{1}$ that $\boldsymbol{b} \notin \mathcal{A}$ ∎

Linearly Independent System of Vectors

- A set of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ is a linearly dependent system if at least one vector is linearly dependent on others
- Otherwise, this set forms a linearly independent system
- Two systems of vectors are equivalent systems if each vector of one system is linearly dependent on vectors of the other system
- Consider a system $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ that may have dependent vectors
- To construct an equivalent independent system, we successively reduce the system until it becomes linearly independent
- We use a procedure that applies the criterion provided by the lemma on linear dependence to examine the vectors one by one
- The procedure removes a vector if it is linearly dependent on others, or leaves the vector in the system otherwise

Solution of One-Sided Equation

- Given a matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$, we consider the equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

Theorem (Solution of One-Sided Equation)

Let $\boldsymbol{A}$ be a regular matrix and $\boldsymbol{b}$ a regular vector. Define the scalar

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b}.$$

Then, the following statements hold:

1. *If $\Delta = \mathbb{1}$, then the equation has regular solutions including*

$$\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-};$$

2. *The above solution is the maximal solution, and it is unique if the columns in $\boldsymbol{A}$ form a minimal generating set for the vector $\boldsymbol{b}$*
3. *If $\Delta \neq \mathbb{1}$, then there are no regular solutions*

Proof

- The fact that equality $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ holds for some $\boldsymbol{x}$ means that the vector $\boldsymbol{b}$ belongs to the linear span $\mathcal{A}$ of columns in the matrix $\boldsymbol{A}$
- It follows from the lemma on evaluation of distance that $\boldsymbol{b} \in \mathcal{A}$ if and only if the following condition holds:

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \mathbb{1}$$

- As another consequence of the lemma, one can see that the regular solutions of the equation (if any exists) include the vector

$$\boldsymbol{x} = \Delta^{1/2}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$$

- By the lemma on one-sided inequality, the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ is equivalent to $\boldsymbol{x} \leq (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$, and thus this solution is maximal
- The uniqueness condition follows from unique representation of a vector as a linear combination of its minimal set of generators ■

Example in Two Dimension

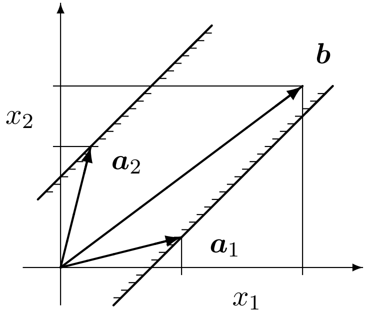
- Consider the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ with the matrix and vectors

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Assume that $a_{11}, a_{12}, a_{21}, a_{22} > \mathbb{0}$ and $b_1, b_2 > \mathbb{0}$
- Suppose the condition $\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \mathbb{1}$ holds
- The maximal solution takes the form

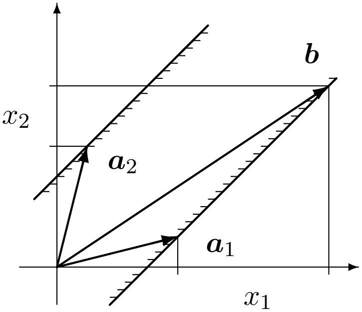
$$\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} (\boldsymbol{b}^{-}\boldsymbol{a}_1)^{-1} \\ (\boldsymbol{b}^{-}\boldsymbol{a}_2)^{-1} \end{pmatrix} = \begin{pmatrix} (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1} \\ (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1} \end{pmatrix}$$

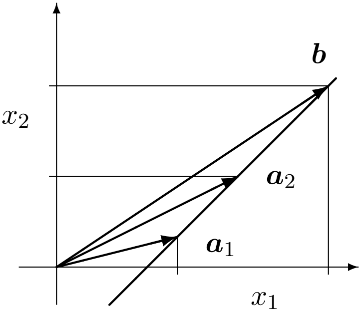
Graphical Illustration of Unique Solution in $\mathbb{R}^2_{\max,+}$



- If the vector $\boldsymbol{b}$ is not collinear to any of the vectors $\boldsymbol{a}_1$ or $\boldsymbol{a}_2$, then the solution vector $\boldsymbol{x}$ of the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is unique

Graphical Illustration of Nonunique Solutions in $\mathbb{R}^2_{\max,+}$





- If the vector $\boldsymbol{b}$ is collinear with only one vectors from $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ (left), or with both vectors (right), the solution is nonunique

Representation of Nonunique Solutions

- Suppose that the vector $\boldsymbol{b}$ is collinear to $\boldsymbol{a}_1$, but not to $\boldsymbol{a}_2$
- Then, the solution is any vector $\boldsymbol{x}$ with the components

$$x_1 = (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1},$$
$$x_2 \leq (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1}$$

- Assume both vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ to be collinear to each other
- In this case, there are two solution sets that consist of vectors $\boldsymbol{x}' = (x_1', x_2')^T$ and $\boldsymbol{x}'' = (x_1'', x_2'')^T$, where

$$x_1' = (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1}, \qquad x_1'' \leq (b_1^{-1}a_{11} \oplus b_2^{-1}a_{21})^{-1},$$
$$x_2' \leq (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1}; \qquad x_2'' = (b_1^{-1}a_{12} \oplus b_2^{-1}a_{22})^{-1}$$

All Solutions of One-Sided Equation

- Let $\boldsymbol{A} = (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ be a matrix, $\boldsymbol{b}$ a vector, and I be a subset of column indices of the matrix $\boldsymbol{A}$ such that $\boldsymbol{b} \in \operatorname{span}\{\boldsymbol{a}_i | i \in I\}$
- Then, any vector $\boldsymbol{x} = (x_i)$ with components

$$\begin{aligned} x_i &= (\boldsymbol{b}^- \boldsymbol{a}_i)^-, &\quad \text{if } i \in I; \\ x_i &\leq (\boldsymbol{b}^- \boldsymbol{a}_i)^-, &\quad \text{if } i \notin I \end{aligned}$$

  is a solution to the one-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
- To obtain all solutions to the equation, one has to find all minimal subsets of columns in $\boldsymbol{A}$ that generate the vector $\boldsymbol{b}$
- A generating subset of columns in the matrix $\boldsymbol{A}$ is minimal if it contains no proper subset that generates the vector $\boldsymbol{b}$
- To represent the solution for a minimal set given by I, we replace the equations in $\boldsymbol{x} = (\boldsymbol{b}^- \boldsymbol{A})^-$ for x_i with $i \notin I$ by inequalities

Numerical Examples

- Consider an equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ defined in $\mathbb{R}^3_{\max,+}$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

- To verify the condition $\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \mathbb{1}$, we calculate

$$\boldsymbol{b}^{-}\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix},$$

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^{-}\boldsymbol{A})^{-})^{-}\boldsymbol{b} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 1 \neq \mathbb{1} = 0$$

- Consider an equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ defined in $\mathbb{R}^3_{\max,+}$, where

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 & -1 \\ \mathbb{0} & -1 & 3 \\ 0 & \mathbb{0} & 1 \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbb{0} = -\infty$$

- We verify that the condition $\Delta = (\boldsymbol{A}(\boldsymbol{b}^- \boldsymbol{A})^-)^- \boldsymbol{b} = \mathbb{1}$ is true:

$$\boldsymbol{b}^- \boldsymbol{A} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & -1 \\ \mathbb{0} & -1 & 3 \\ 0 & \mathbb{0} & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}(\boldsymbol{b}^- \boldsymbol{A})^- = \begin{pmatrix} 2 & 0 & -1 \\ \mathbb{0} & -1 & 3 \\ 0 & \mathbb{0} & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\Delta = (\boldsymbol{A}(\boldsymbol{b}^- \boldsymbol{A})^-)^- \boldsymbol{b} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 = \mathbb{1}$$

- Since the condition $\Delta = \mathbb{1}$ holds, we conclude that the equation has solutions, including the maximal solution

$$\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

- We can describe all solutions by finding all minimal sets of columns in the matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3)$ that generate the vector $\boldsymbol{b}$
- If all columns in $\boldsymbol{A}$ form the minimal generating set (no column can be dropped), then the vector $\boldsymbol{x} = (\boldsymbol{b}^{-}\boldsymbol{A})^{-}$ is unique solution
- To see if we can drop a column, say the first column, to have $\boldsymbol{b} \in \operatorname{span}(\boldsymbol{a}_2, \boldsymbol{a}_3)$, we need to verify the condition

$$\Delta_{(1)} = (\boldsymbol{A}_{(1)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(1)})^{-})^{-}\boldsymbol{b} = \mathbb{1}, \qquad \boldsymbol{A}_{(1)} = (\boldsymbol{a}_2, \boldsymbol{a}_3)$$

- If $\Delta_{(1)} = \mathbb{1}$, we further verify that $\boldsymbol{b} \in \operatorname{span}(\boldsymbol{a}_2)$ and $\boldsymbol{b} \in \operatorname{span}(\boldsymbol{a}_3)$

- We form with the matrices

$$\boldsymbol{A}_{(1)} = \begin{pmatrix} 0 & -1 \\ -1 & 3 \\ \mathbb{0} & 1 \end{pmatrix}, \quad \boldsymbol{A}_{(2)} = \begin{pmatrix} 2 & -1 \\ \mathbb{0} & 3 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{A}_{(3)} = \begin{pmatrix} 2 & 0 \\ \mathbb{0} & -1 \\ 0 & \mathbb{0} \end{pmatrix}$$

- We check whether $\Delta_{(i)} = (\boldsymbol{A}_{(i)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(i)})^{-})^{-}\boldsymbol{b} = \mathbb{1}$ for $i = 1, 2, 3$:

$$\boldsymbol{b}^{-}\boldsymbol{A}_{(1)} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 3 \\ \mathbb{0} & 1 \end{pmatrix} = \begin{pmatrix} -1 & 2 \end{pmatrix},$$

$$\boldsymbol{A}_{(1)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(1)})^{-} = \begin{pmatrix} 0 & -1 \\ -1 & 3 \\ \mathbb{0} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix},$$

$$\Delta_{(1)} = (\boldsymbol{A}_{(1)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(1)})^{-})^{-}\boldsymbol{b} = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 = \mathbb{1}$$

- In the same way, we obtain

$$\Delta_{(2)} = (\boldsymbol{A}_{(2)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(2)})^{-})^{-}\boldsymbol{b} = 0 = \mathbb{1},$$
$$\Delta_{(3)} = (\boldsymbol{A}_{(3)}(\boldsymbol{b}^{-}\boldsymbol{A}_{(3)})^{-})^{-}\boldsymbol{b} = 1 \neq \mathbb{1}$$

- Since $\Delta_{(1)} = \Delta_{(2)} = \mathbb{1}$, the set of all columns in $\boldsymbol{A}$ is not minimal
- Taking into account that both $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are not collinear to $\boldsymbol{b}$, the set $(\boldsymbol{a}_2, \boldsymbol{a}_3)$ cannot be further reduced, and hence is minimal
- By the same argument, we conclude that $(\boldsymbol{a}_1, \boldsymbol{a}_3)$ is a minimal set
- All solutions of the equation form two subsets given by

$$\begin{aligned} x_1 &\leq -1, & \qquad x_1 &= -1, \\ x_2 &= 1, & x_2 &\leq 1, \\ x_3 &= -2, & x_3 &= -2 \end{aligned}$$

Two-Sided Inequality: Definitions and Preliminaries

- Given an $(n \times n)$-matrix $\boldsymbol{A}$, the following inequality in an unknown n-vector $\boldsymbol{x}$ is called two-sided:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

- This inequality has the unknown vector $\boldsymbol{x}$ on both sides
- This two-sided inequality always has solutions; specifically, the trivial solution $\boldsymbol{x} = \mathbf{0}$ obviously satisfies the inequality
- We obtain a solution of the inequality by applying a tropical analogue of matrix determinant and Kleene (star) matrix operator

Trace and Determinant of Matrix

- The trace of a square matrix $\boldsymbol{A} = (a_{ij})$ of order n is given by

$$\operatorname{tr} \boldsymbol{A} = a_{11} \oplus \cdots \oplus a_{nn} = \bigoplus_{i=1}^{m} a_{ii}$$

- For any matrix $\boldsymbol{A} = (a_{ij})$ of order n, a tropical analogue of the matrix determinant is a trace function of matrix powers defined as

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^{n} = \bigoplus_{k=1}^{n} \operatorname{tr} \boldsymbol{A}^{k}$$

- The determinant is the sum of cyclic products of matrix entries

$$\operatorname{Tr}(\boldsymbol{A}) = \bigoplus_{1 \leq k \leq n} \bigoplus_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$$

Kleene Star Operator

- For any square matrix $\boldsymbol{A}$, a Kleene star operator is defined which maps the matrix $\boldsymbol{A}$ into the infinite some of integer powers
$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2 \oplus \cdots = \bigoplus_{k \geq 0} \boldsymbol{A}^k$$

Lemma (Extremal Property of Kleene Star)

For any $(n \times n)$*-matrix* $\boldsymbol{A}$ *with* $\operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$*, the next statements hold:*

1. *For any integer* $k \geq 0$*, the following inequality is valid:*
$$\boldsymbol{A}^k \leq \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1};$$
2. *The Kleene star matrix reduces to the finite sum of powers*
$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} = \bigoplus_{k=0}^{n-1} \boldsymbol{A}^k$$

Proof

- We verify that if $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, then for all integers $k \geq 0$, we have
$$\boldsymbol{A}^k \leq \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}$$
- The entries of the power $\boldsymbol{A}^k$ is defined by entries in $\boldsymbol{A} = (a_{ij})$ as
$$\{\boldsymbol{A}^k\}_{ij} = \bigoplus_{1 \leq i_1,\dots,i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$$
- Consider a product under summation and denote it by
$$P = a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$$
- We rearrange multipliers to write $P = P_{\mathsf{c}} P_{\mathsf{a}}$, where P_{c} consists of cyclic subproducts of P and P_{a} does not have cyclic subproducts
- We first extract from P all cyclic subproducts of length 1 (of the form a_{kk}), then the subproducts of length 2 ($a_{kl} a_{lk}$) and so on
- We continue this until the subproducts of length n are extracted

Proof (cont.)

- Since any cyclic product of length from 1 to n is not greater than $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, we see that the inequality $P_{\mathsf{c}} \leq \mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ is valid
- After extracting all cyclic products in P, we denote the remaining subproduct by P_{a} to represent the original product as $P = P_{\mathsf{c}} P_{\mathsf{a}}$
- We note that P_{a} is acyclic with a length not exceeding $n-1$
- Since each product of length $l \leq n-1$, starting from index i and ending with j is bounded from above by $\{\boldsymbol{A}^{l}\}_{ij}$, we have

$$P_{\mathsf{a}} \leq \{\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}\}_{ij}$$

- As a results, we arrive at the upper bound for P:

$$a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = P = P_{\mathsf{c}} P_{\mathsf{a}} \leq \{\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}\}_{ij}$$

- This bound holds for all products under summation, and therefore

$$\{\boldsymbol{A}^{k}\}_{ij} \leq \{\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}\}_{ij} \qquad \blacksquare$$

Solution of Two-Sided Inequality

- Given a $(n \times n)$-matrix $\boldsymbol{A}$, we solve the problem to find n-vectors $\boldsymbol{x}$ that satisfy the two-sided inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

Theorem

The following statements hold:

1. *If* $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, *then all solutions of the inequality are given by*

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}, \qquad \boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1},$$

where $\boldsymbol{u}$ *is a vector of parameters;*

2. *If* $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, *the inequality has only trivial solution* $\boldsymbol{x} = \boldsymbol{0}$

Proof of Statement 1

- Let us show that under the condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the vector $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ satisfies the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with any vector $\boldsymbol{u}$
- Indeed, since $\boldsymbol{A}\boldsymbol{A}^{*} = \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n} \leq \boldsymbol{A}^{*}$, we have

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}(\boldsymbol{A}^{*}\boldsymbol{u}) = (\boldsymbol{A}\boldsymbol{A}^{*})\boldsymbol{u} \leq \boldsymbol{A}^{*}\boldsymbol{u} = \boldsymbol{x}$$

- Suppose now that $\boldsymbol{x}$ is a solution of the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$, and verify that the equality $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ holds for some vector $\boldsymbol{u}$
- Left multiplication of two-sided inequality by $\boldsymbol{A}$ yields the inequality $\boldsymbol{A}^{k}\boldsymbol{x} \leq \boldsymbol{x}$ for all integers $k \geq 1$, and therefore,

$$\boldsymbol{A}^{*}\boldsymbol{x} = (\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1})\boldsymbol{x} \leq \boldsymbol{x}$$

- Because $\boldsymbol{A}^{*} \geq \boldsymbol{I}$, the inequality $\boldsymbol{A}^{*}\boldsymbol{x} \geq \boldsymbol{x}$ is valid as well
- Both inequalities result in the equality $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ with $\boldsymbol{u} = \boldsymbol{x}$ ■

Example in Two Dimensions

- Consider the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with the matrix and vector

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Suppose that the condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ holds
- We calculate the matrix

$$\boldsymbol{A}^2 = \begin{pmatrix} a_{11}^2 \oplus a_{12}a_{21} & a_{11}a_{12} \oplus a_{12}a_{22} \\ a_{21}a_{11} \oplus a_{22}a_{21} & a_{12}a_{21} \oplus a_{22}^2 \end{pmatrix}$$

- Consider the condition

$$\mathrm{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{A}^2 = a_{11} \oplus a_{22} \oplus a_{12}a_{21} \leq \mathbb{1}$$

- It follows from this condition, that the next inequalities are valid:

$$a_{11} \leq \mathbb{1}, \qquad a_{22} \leq \mathbb{1}, \qquad a_{12}a_{21} \leq \mathbb{1}$$

- Since $a_{11}, a_{22} \leq \mathbb{1}$, the Kleene star matrix takes the form

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix} \oplus \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}$$

- All solutions of the two-sided inequality are given by

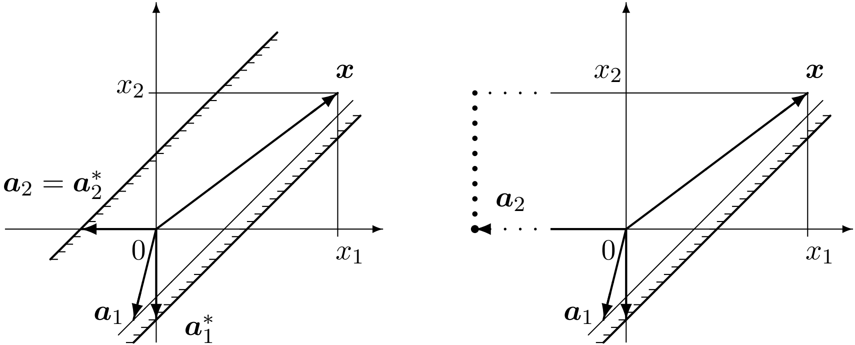
$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

  where $\boldsymbol{u}$ is a vector of parameters

- In scalar form, the solution is written as

$$x_1 = u_1 \oplus a_{12} u_2,$$
$$x_2 = a_{21} u_1 \oplus u_2$$

Graphical Illustration of Solution to $Ax \leq x$ in $\mathbb{R}^2_{\max,+}$



- Solutions of a two-sided inequality for a matrix $\boldsymbol{A} = (\boldsymbol{a}_1, \boldsymbol{a}_2)$ without (left) and with (right) zero entries

Numerical Example

- Consider an inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ defined in $\mathbb{R}^3_{\max,\times}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix}$$

- To verify the existence condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, we calculate

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix},$$

$$\boldsymbol{A}^3 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}$$

- Since $\mathrm{Tr}(\boldsymbol{A}) = \mathrm{tr}\,\boldsymbol{A} \oplus \mathrm{tr}\,\boldsymbol{A}^2 \oplus \mathrm{tr}\,\boldsymbol{A}^3 = 1 = \mathbb{1}$, the condition holds

- Calculation of the Kleene star matrix $\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2$ yields

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix}$$

- All solutions of the two-sided inequality are given by

$$\boldsymbol{x} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad u_1, u_2, u_3 \geq 0$$

- In terms of conventional algebra, the solution is written as

$$x_1 = \max\left(u_1, \frac{1}{2}u_2, 2u_3\right), \qquad x_2 = \max(2u_1, u_2, 4u_3),$$
$$x_3 = \max\left(\frac{1}{3}u_1, \frac{1}{6}u_2, u_3\right)$$

Representation of Generating Matrix

- Consider an inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with a matrix $\boldsymbol{A}$ of order n
- Suppose that $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ and examine the solution defined by the Kleene matrix $\boldsymbol{A}^* = (\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_n^*)$ and vector $\boldsymbol{u} = (u_1, \ldots, u_n)^T$ as

$$\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{u} = u_1\boldsymbol{a}_1^* \oplus \cdots \oplus u_n\boldsymbol{a}_n^*$$

- This representation means that each solution is a linear combination of columns $\boldsymbol{a}_1^*, \ldots, \boldsymbol{a}_n^*$, which generate all solutions
- If a column in $\boldsymbol{A}^*$ is linearly dependent on others, it can be removed from the set of generators without losing solutions
- To eliminate dependent columns, we apply the procedure of constructing an equivalent linear independent system of vectors

Numerical Example

- Consider the solution in the last example, generated by the matrix

$$\boldsymbol{A}^{*} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix}$$

- Since the first and second columns are collinear, one of them, say the second, can be removed to represent the solution as

$$\boldsymbol{x} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 1/3 & 1 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1, u_2 \geq 0$$

- In terms of standard algebra, the solution is written as

$$x_1 = \max(u_1, 2u_2), \quad x_2 = \max(2u_1, 4u_2), \quad x_3 = \max\left(\frac{1}{3}u_1, u_2\right)$$

Two-Sided Equation: Definitions and Preliminaries

- Given an $(n \times n)$-matrix $\boldsymbol{A}$, the following equation in an unknown n-vectors $\boldsymbol{x}$ is called a two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$$

- The equation has the unknown on both sides and can be called an equation of the second kind (by analogy with integral equations)
- This equation is also referred to as the Bellman equation
- The two-sided equation always has the trivial solution $\boldsymbol{x} = \boldsymbol{0}$
- Existence conditions for nontrivial solutions can be represented in terms of the function (determinant) $\mathrm{Tr}(\boldsymbol{A}) = \mathrm{tr}\,\boldsymbol{A} \oplus \cdots \oplus \mathrm{tr}\,\boldsymbol{A}^n$
- We describe all solutions in a parametric form that is based on calculation of the Kleene star matrix $\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}$

Proposition (Solution Set of Two-Sided Equation)

The set of solutions of the two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ is closed under vector addition and scalar multiplication

Proof

- If $\boldsymbol{x}$ and $\boldsymbol{y}$ are vectors such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ and $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{y}$, and α and β are scalars, then for the vector $\boldsymbol{z} = \alpha\boldsymbol{x} \oplus \beta\boldsymbol{y}$, we have

$$\boldsymbol{A}\boldsymbol{z} = \boldsymbol{A}(\alpha\boldsymbol{x} \oplus \beta\boldsymbol{y}) = \alpha\boldsymbol{A}\boldsymbol{x} \oplus \beta\boldsymbol{A}\boldsymbol{y} = \alpha\boldsymbol{x} \oplus \beta\boldsymbol{y} = \boldsymbol{z} \quad \blacksquare$$

- We now can conclude that the set of solutions is a tropical vector space, which can be described by its generating matrix
- Below, we show how this generating matrix can be constructed

Kleene Star and Kleene Plus Matrices

- For any square matrix $\boldsymbol{A}$, the Kleene Star and Kleene Plus matrices are defined as infinite sums given by

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^{2} \oplus \cdots, \qquad \boldsymbol{A}^{+} = \boldsymbol{A}\boldsymbol{A}^{*} = \boldsymbol{A} \oplus \boldsymbol{A}^{2} \oplus \cdots$$

- It follows from the extremal property of the Kleene star that if $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, then for any $k \geq 0$, the following inequality holds:

$$\boldsymbol{A}^{k} \leq \boldsymbol{A}^{*}$$

- As a result, when $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the infinite sums become finite to define the Kleene star and Kleene plus matrices in the form

$$\boldsymbol{A}^{*} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1}, \qquad \boldsymbol{A}^{+} = \boldsymbol{A}\boldsymbol{A}^{*} = \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n}$$

Proposition

If the condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$ *holds, then the following equality is valid:*

$$\boldsymbol{I} \oplus \boldsymbol{A}^{+} = \boldsymbol{A}^{*}$$

Proof

- Since $\boldsymbol{A}^{k} \leq \boldsymbol{A}^{*}$ for all integers $k > 0$, we immediately obtain
$$\boldsymbol{I} \oplus \boldsymbol{A}^{+} = \boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^{n-1} \oplus \boldsymbol{A}^{n} = \boldsymbol{A}^{*} \oplus \boldsymbol{A}^{n} = \boldsymbol{A}^{*} \qquad \blacksquare$$

Remarks

- If the equality $\boldsymbol{I} \oplus \boldsymbol{A}^{+} = \boldsymbol{A}^{*}$ holds, then $\boldsymbol{A}^{+} \leq \boldsymbol{A}^{*}$
- In the matrices $\boldsymbol{A}^{*} = (a_{ij}^{*})$ and $\boldsymbol{A}^{+} = (a_{ij}^{+})$, the corresponding entries a_{ij}^{*} and a_{ij}^{+} coincide except for diagonal entries
- The diagonal entries satisfy the conditions $a_{ii}^{*} = \mathbb{1}$ and $a_{ii}^{+} \leq \mathbb{1}$

Proposition

If the condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ *is valid, then the following statements hold:*

1. *The Kleene matrices* $\boldsymbol{A}^{*} = (\boldsymbol{a}_i^{*})$ *and* $\boldsymbol{A}^{+} = (\boldsymbol{a}_i^{+})$ *have common columns that coincide;*
2. *The equality* $\boldsymbol{a}_i^{*} = \boldsymbol{a}_i^{+}$ *holds if and only if* $a_{ii}^{(m)} = \mathbb{1}$*, where* $a_{ii}^{(m)}$ *is a diagonal entry in the matrix* $\boldsymbol{A}^{m} = (a_{ij}^{(m)})$ *for some* $m = 1, \ldots, n$

Proof

- If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, the off-diagonal entries in $\boldsymbol{A}^{*}$ and $\boldsymbol{A}^{+}$ coincide
- The condition $\mathrm{Tr}(\boldsymbol{A}) = \mathrm{tr}\,\boldsymbol{A} \oplus \cdots \oplus \mathrm{tr}\,\boldsymbol{A}^{n} = \mathbb{1}$ means that the equality $\mathrm{tr}\,\boldsymbol{A}^{m} = \mathbb{1}$ is valid for at least one $m = 1, \ldots, n$
- The last equality holds if and only if $a_{ii}^{(m)} = \mathbb{1}$ for some index i
- In this case, we have $a_{ii}^{*} = a_{ii}^{+} = \mathbb{1}$, and thus $\boldsymbol{a}_i^{*} = \boldsymbol{a}_i^{+}$ ∎

Matrix $A^{\times}$

- In order to describe solutions of the two-sided equation in a compact vector form, we introduce a matrix $\boldsymbol{A}^{\times}$ as follows
- Let $\boldsymbol{A}$ be a square $(n \times n)$-matrix such that $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$
- Let $\boldsymbol{A}^{*}$ and $\boldsymbol{A}^{+}$ be the Kleene star and Kleene plus matrices for $\boldsymbol{A}$ with columns $\boldsymbol{a}_1^{*}, \ldots, \boldsymbol{a}_n^{*}$ and $\boldsymbol{a}_1^{+}, \ldots, \boldsymbol{a}_n^{+}$ respectively
- We define a matrix $\boldsymbol{A}^{\times}$ of the same size as $\boldsymbol{A}$ with the columns

$$\boldsymbol{a}_i^{\times} = \begin{cases} \boldsymbol{a}_i^{*}, & \text{if } \boldsymbol{a}_i^{*} = \boldsymbol{a}_i^{+}; \\ \boldsymbol{0}, & \text{if } \boldsymbol{a}_i^{*} \neq \boldsymbol{a}_i^{+}; \end{cases} \qquad i = 1, \ldots, n$$

- If $\mathrm{Tr}(\boldsymbol{A}) \neq \mathbb{1}$, we put $\boldsymbol{A}^{\times} = \boldsymbol{0}$

Solution of Two-Sided Equation

- Given a $(n \times n)$-matrix $\boldsymbol{A}$, we solve the problem of finding n-vectors $\boldsymbol{x}$ that satisfy the two-sided equation

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$$

Lemma (Solution of Two-Sided Equation)

If the condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ *holds, then any vector given by*

$$\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{v}, \qquad \boldsymbol{v} > \mathbf{0},$$

satisfies the two-sided equation

Proof

- If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then the matrices $\boldsymbol{A}^*$ and $\boldsymbol{A}^+$ have common columns that are the same, say columns $\boldsymbol{a}_i^* = \boldsymbol{a}_i^+$
- Since the equality $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^+$ always holds, we can write

  $$\boldsymbol{A}\boldsymbol{a}_i^* = \boldsymbol{a}_i^+ = \boldsymbol{a}_i^*,$$

  which means that the column $\boldsymbol{a}_i^*$ satisfies the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$
- We observe that all common columns of the matrices $\boldsymbol{A}^*$ and $\boldsymbol{A}^+$ form nonzero columns in the matrix $\boldsymbol{A}^\times$
- The vector $\boldsymbol{x} = \boldsymbol{A}^\times \boldsymbol{v}$ for any vector $\boldsymbol{v} > \boldsymbol{0}$ is a linear combination of columns in $\boldsymbol{A}^\times$, and thus satisfies the two-sided equation ■

Irreducible Matrices

- A matrix $\boldsymbol{A}$ is reducible if simultaneous row-column permutations can put it into a block-triangular form, and irreducible otherwise
- The lower triangular normal form of a matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{0} & \dots & \boldsymbol{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \\ \boldsymbol{A}_{s1} & \boldsymbol{A}_{s2} & \dots & \boldsymbol{A}_{ss} \end{pmatrix},$$

where $\boldsymbol{A}_{ii}$ is either an irreducible or zero matrix for all $i = 1, \dots, s$

Proposition

If a matrix $\boldsymbol{A}$ is irreducible, then any nontrivial solution $\boldsymbol{x} \neq \boldsymbol{0}$ of the two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ has no zero entries

Proof

- Consider the equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ with an irreducible matrix $\boldsymbol{A}$
- Suppose that a nontrivial vector $\boldsymbol{x} = (x_i)$ is a solution of the equation, and verify that $\boldsymbol{x}$ does not have zero entries
- Let $\boldsymbol{x}$ have one zero entry, $x_k = \mathbb{0}$, whereas $x_j > \mathbb{0}$ for all $j \neq k$
- The scalar equation corresponding to row k in $\boldsymbol{A}$ takes the form

$$a_{k1}x_1 \oplus \cdots \oplus a_{kn}x_n = \mathbb{0}$$

- Since $x_j > \mathbb{0}$ for $j \neq k$, the equation holds only if $a_{kj} = \mathbb{0}$, $j \neq k$
- By swapping rows 1 and k, and columns 1 and k, we obtain a matrix with a zero block in the first row, which is a contradiction
- The assumption that the solution vector $\boldsymbol{x}$ has more than one (but not all) zero entries is examined in an analogues way ■

Proposition (Existence of Nontrivial Solutions)

The two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ with irreducible matrix $\boldsymbol{A}$ has nontrivial solutions if and only if the condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ holds

Proof

- The sufficiency of the condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ follows from the lemma on the solution of two-sided equation
- To verify the necessity of the condition, assume that $\boldsymbol{x}$ is a nontrivial solution, and show that then $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$
- Let us take an arbitrary cyclic sequence of indices $i_0, \ldots, i_m$, where $i_m = i_0$ and $1 \leq m \leq n$
- It follows from the equations $a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n = x_i$ for all i that

$$a_{i_0 i_1} x_{i_1} \leq x_{i_0}, \qquad a_{i_1 i_2} x_{i_2} \leq x_{i_1}, \quad \ldots, \quad a_{i_{m-1} i_m} x_{i_m} \leq x_{i_{m-1}}$$

Proof (cont.)

- Consider the inequalities

$$a_{i_0 i_1} x_{i_1} \leq x_{i_0}, \qquad a_{i_1 i_2} x_{i_2} \leq x_{i_1}, \quad \ldots, \quad a_{i_{m-1} i_m} x_{i_m} \leq x_{i_{m-1}}$$

- Side-by-side multiplication of inequalities yields

$$a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{m-1} i_m} x_{i_1} x_{i_2} \cdots x_{i_m} \leq x_{i_0} x_{i_1} \cdots x_{i_{m-1}} = x_{i_1} x_{i_2} \cdots x_{i_m}$$

- By reducing by the common factor $x_{i_1} \cdots x_{i_m} \neq \mathbb{0}$, we obtain

$$a_{i_0 i_1} \cdots a_{i_{m-1} i_m} \leq \mathbb{1}$$

- Considering an arbitrary choice of $i_0, \ldots, i_{m-1}$, we have

$$\operatorname{tr} \boldsymbol{A}^m \leq \mathbb{1}, \qquad m = 1, \ldots, n$$

- As a result, the following inequality holds:

$$\operatorname{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr} \boldsymbol{A}^n \leq \mathbb{1}$$

Proof (cont.)

- It remains to verify that the inequality $\mathrm{Tr}(\boldsymbol{A}) \geq \mathbb{1}$ holds as well
- It follows from the scalar equations $a_{i1}x_1 \oplus \cdots \oplus a_{in}x_n = x_i$ that for any index i, there is an index j such that $a_{ij}x_j = x_i$
- Let us take an arbitrary index i_0 and construct a sequence $i_0, i_1, i_2, \ldots$ by choosing indices that satisfy the equalities

$$a_{i_0 i_1} x_{i_1} = x_{i_0}, \qquad a_{i_1 i_2} x_{i_2} = x_{i_1}, \qquad \ldots$$

- We select a cyclic subsequence $i_l, \ldots, i_{l+m}$ with $i_l = i_{l+m}$, $m \leq n$
- After side-by-side multiplication of equalities that correspond to the subsequence, and reduction by $x_{i_l} \cdots x_{i_{l+m}} \neq \mathbb{0}$, we obtain

$$a_{i_l i_{l+1}} \cdots a_{i_{l+m-1} i_{l+m}} = \mathbb{1}$$

- As a consequence of the last equality, we have

$$\mathrm{Tr}(\boldsymbol{A}) \geq \operatorname{tr} \boldsymbol{A}^m \geq a_{i_l i_{l+1}} \cdots a_{i_{l+m-1} i_{l+m}} = \mathbb{1} \qquad \blacksquare$$

Complete Solution of Equation with Irreducible Matrix

- A complete solution of the two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ with an irreducible matrix is provided by the next statement

Theorem (Complete Solution)

Let $\boldsymbol{A}$ is an irreducible matrix. Then, the following statements hold:

1. *If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then all regular solutions are given by*

$$\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{u},$$

   where $\boldsymbol{u} > \boldsymbol{0}$ is a vector of parameters;

2. *If $\mathrm{Tr}(\boldsymbol{A}) \neq \mathbb{1}$, then the equation has only trivial solution $\boldsymbol{x} = \boldsymbol{0}$*

Example in Two Dimensions

- Consider the equation $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ with the matrix and vector

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Suppose that the existence condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$ holds
- We calculate the matrix

$$\boldsymbol{A}^2 = \begin{pmatrix} a_{11}^2 \oplus a_{12}a_{21} & a_{11}a_{12} \oplus a_{12}a_{22} \\ a_{21}a_{11} \oplus a_{22}a_{21} & a_{12}a_{21} \oplus a_{22}^2 \end{pmatrix}$$

- Consider the existence condition and represent it as follows:

$$\mathrm{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{A}^2 = a_{11} \oplus a_{22} \oplus a_{12}a_{21} = \mathbb{1}$$

- As a consequence of this condition, we have the inequalities

$$a_{11} \leq \mathbb{1}, \qquad a_{22} \leq \mathbb{1}, \qquad a_{12}a_{21} \leq \mathbb{1}$$

- Since $a_{11}, a_{22} \leq \mathbb{1}$, the Kleene matrices take the form

$$\boldsymbol{A}^* = \boldsymbol{I} \oplus \boldsymbol{A} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} \end{pmatrix} \oplus \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix},$$

$$\boldsymbol{A}^+ = \boldsymbol{A} \oplus \boldsymbol{A}^2 = \boldsymbol{A}\boldsymbol{A}^* = \begin{pmatrix} a_{11} \oplus a_{12}a_{21} & a_{12} \end{pmatrix}$$

- To obtain the solution $\boldsymbol{x} = \boldsymbol{A}^\times \boldsymbol{v}$, we need to derive the matrix $\boldsymbol{A}^\times$
- If $a_{11} = \mathbb{1}$, $a_{22} < \mathbb{1}$ and $a_{12}a_{21} < \mathbb{1}$, then we have

$$\boldsymbol{A}^+ = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & a_{12}a_{21} \oplus a_{22} \end{pmatrix}, \qquad \boldsymbol{A}^\times = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ a_{21} & \mathbb{0} \end{pmatrix}$$

- By removing the second column of $\boldsymbol{A}^\times$, we write the solution as

$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} \\ a_{21} \end{pmatrix} v, \quad v \in \mathbb{X}$$

- If $a_{11} < \mathbb{1}$, $a_{22} = \mathbb{1}$ and $a_{12}a_{21} < \mathbb{1}$, then we have

$$\boldsymbol{A}^{*} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}, \qquad \boldsymbol{A}^{+} = \begin{pmatrix} a_{11} \oplus a_{12}a_{21} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}$$

- From the matrices $\boldsymbol{A}^{*}$ and $\boldsymbol{A}^{+}$, we obtain the generating matrix

$$\boldsymbol{A}^{\times} = \begin{pmatrix} \mathbb{0} & a_{12} \\ \mathbb{0} & \mathbb{1} \end{pmatrix}$$

- The corresponding solution can be written as

$$\boldsymbol{x} = \begin{pmatrix} a_{12} \\ \mathbb{1} \end{pmatrix} v, \quad v \in \mathbb{X}$$

- Provided that at least one of the conditions $a_{11} = a_{22} = \mathbb{1}$ and $a_{12}a_{21} = \mathbb{1}$ is satisfied, then we obtain

$$\boldsymbol{A}^{\times} = \boldsymbol{A}^{+} = \boldsymbol{A}^{\times} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}$$

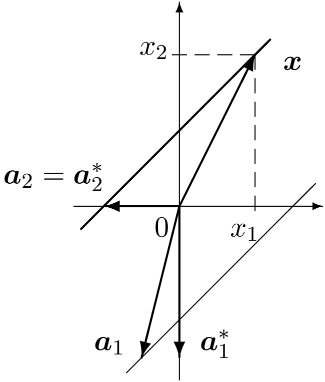
- In the case when $a_{12}, a_{21} \neq \mathbb{1}$, the solution is given by

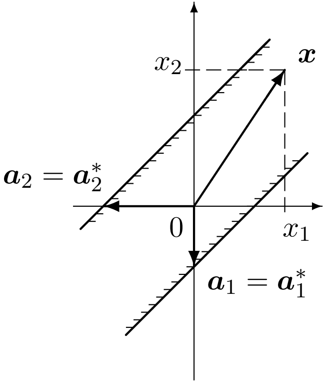
$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix} \boldsymbol{v}, \quad \boldsymbol{v} \in \mathbb{X}^{2}$$

- Under the condition $a_{12} = a_{21} = \mathbb{1}$, we have the solution

$$\boldsymbol{x} = \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} v, \quad v \in \mathbb{X}$$

Graphical Illustration of Solution to $Ax = x$ in $\mathbb{R}^2_{\max,+}$





- Examples of the solution generated by one column (left) and solution given by the linear span of both columns of $\boldsymbol{A}$ (right)

Numerical Example

- Consider an equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ defined in $\mathbb{R}^3_{\max,\times}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 3 \\ 1/3 & 0 & 1/2 \end{pmatrix}$$

- To verify the existence condition $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, we calculate

$$\boldsymbol{A}^2 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}, \qquad \boldsymbol{A}^3 = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}$$

- Since $\mathrm{Tr}(\boldsymbol{A}) = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr} \boldsymbol{A}^2 \oplus \operatorname{tr} \boldsymbol{A}^3 = 1 = \mathbb{1}$, the condition holds

- Calculation of the Kleene star and Kleene plus matrices yields

$$\boldsymbol{A}^{*} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 1 \end{pmatrix}, \qquad \boldsymbol{A}^{+} = \begin{pmatrix} 1 & 1/2 & 2 \\ 2 & 1 & 4 \\ 1/3 & 1/6 & 2/3 \end{pmatrix}$$

- Since the first two columns in the matrices coincide, we obtain

$$\boldsymbol{A}^{+} = \begin{pmatrix} 1 & 1/2 & \mathbb{0} \\ 2 & 1 & \mathbb{0} \\ 1/3 & 1/6 & \mathbb{0} \end{pmatrix}$$

- All solutions of the two-sided equation are given by

$$\boldsymbol{x} = \begin{pmatrix} 1 & 1/2 \\ 2 & 1 \\ 1/3 & 1/6 \end{pmatrix} \boldsymbol{u}, \qquad \boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \qquad u_1, u_2 \geq 0$$

- Consider the generating matrix of the solution

$$\begin{pmatrix} 1 & 1/2 \\ 2 & 1 \\ 1/3 & 1/6 \end{pmatrix}$$

- Since both columns in the matrix are collinear, we can drop one of them, say the second, to write the solution as

$$\boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 1/3 \end{pmatrix} u, \qquad u > \mathbb{0}$$

- In terms of conventional algebra, the solution is written as

$$x_1 = u, \qquad x_2 = 2u, \qquad x_3 = \frac{1}{3}u$$

Nonhomogeneous Two-Sided Equation

- Given an $(n \times n)$-matrix $\boldsymbol{A}$ and n-vector $\boldsymbol{b}$, the following equation is called nonhomogeneous two-sided equation:

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$$

- The equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ is a homogeneous two-sided equation

Lemma

The nonhomogeneous equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ with irreducible matrix $\boldsymbol{A}$ has solutions if and only if at least one of the following conditions hold:

1. $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$;
2. $\boldsymbol{b} = \boldsymbol{0}$.

If the equation has solutions, then $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b}$ is its minimal solution

Proof (Sufficiency)

- Under the condition $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, the iterations of the equation yield

$$\begin{aligned}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{A}(\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b}) \oplus \boldsymbol{b} = \boldsymbol{A}^2\boldsymbol{x} \oplus (\boldsymbol{I} \oplus \boldsymbol{A})\boldsymbol{b} \\ = \boldsymbol{A}^3\boldsymbol{x} \oplus (\boldsymbol{I} \oplus \boldsymbol{A} \oplus \boldsymbol{A}^2)\boldsymbol{b} = \cdots = \boldsymbol{A}^n\boldsymbol{x} \oplus \boldsymbol{A}^*\boldsymbol{b}\end{aligned}$$

- As a result, the equation reduces to that in the equivalent form

$$\boldsymbol{A}^n\boldsymbol{x} \oplus \boldsymbol{A}^*\boldsymbol{b} = \boldsymbol{x}$$

- As a consequence of the last equation, we have the inequality

$$\boldsymbol{x} \geq \boldsymbol{A}^*\boldsymbol{b}$$

- Let us verify that the vector $\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b}$ is a solution of the equation

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{A}(\boldsymbol{A}^*\boldsymbol{b}) \oplus \boldsymbol{b} = (\boldsymbol{I} \oplus \boldsymbol{A} \oplus \cdots \oplus \boldsymbol{A}^n)\boldsymbol{b} = \boldsymbol{A}^*\boldsymbol{b} = \boldsymbol{x}$$

- Taking into account the above inequality, this solution is minimal
- Note that if $\boldsymbol{b} = \boldsymbol{0}$, the equation always has a solution $\boldsymbol{x} = \boldsymbol{0}$ ∎

General Solution of Equation

- The set of all possible solutions of an equation (inequality) is called the general solution of the equation (inequality)
- The general solution of the two-sided inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$ is given in parametric form by

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{X}^{n}$$

- The general solution of the homogeneous two-sided equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$ with irreducible matrix is given in parametric form by

$$\boldsymbol{x} = \boldsymbol{A}^{\times}\boldsymbol{v}, \qquad \boldsymbol{v} \in \mathbb{X}^{n}$$

- Every single solution of an equation (inequality) is referred to as a particular solution of the equation (inequality)

- The next statement establishes a connection between the solutions of the nonhomogeneous and homogeneous equations

Lemma

Let $\boldsymbol{u}$ be the minimal (particular) solution of a nonhomogeneous equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ with irreducible matrix $\boldsymbol{A}$ and $\boldsymbol{v}$ be the general solution of the homogeneous equation $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}$.

Then, the general solution of the nonhomogeneous equation is given by

$$\boldsymbol{x} = \boldsymbol{u} \oplus \boldsymbol{v}$$

- Given a $(n \times n)$-matrix $\boldsymbol{A}$ and n-vector $\boldsymbol{b}$, we find n-vectors $\boldsymbol{x}$ that satisfy the nonhomogeneous two-sided equation

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$$

- Combining the lemmas on the existence of solutions and general solution of nonhomogeneous equation yields the next statements

Theorem

Suppose that the nonhomogeneous equation with irreducible matrix has solutions, and let $\boldsymbol{x}$ be the general solution of the equation.

Then, the following statements hold:

1. *If $\mathrm{Tr}(\boldsymbol{A}) < \mathbb{1}$, then there is a single solution $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b}$;*
2. *If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b} \oplus \boldsymbol{A}^{\times}\boldsymbol{v}$ for any vector $\boldsymbol{v} \in \mathbb{X}^{n}$;*
3. *If $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, then the equation has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ (when $\boldsymbol{b} = \boldsymbol{0}$)*

Example in Two Dimensions

- Consider the equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ with

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

- Suppose that $a_{11}, a_{12}, a_{21}, a_{22} > \mathbb{0}$
- Let us calculate the Klene matrix $\boldsymbol{A}^*$ and then the vector $\boldsymbol{A}^*\boldsymbol{b}$:

$$\boldsymbol{A}^* = \begin{pmatrix} \mathbb{1} & a_{12} \\ a_{21} & \mathbb{1} \end{pmatrix}, \qquad \boldsymbol{A}^*\boldsymbol{b} = \begin{pmatrix} b_1 \oplus a_{12}b_2 \\ a_{21}b_1 \oplus b_2 \end{pmatrix}$$

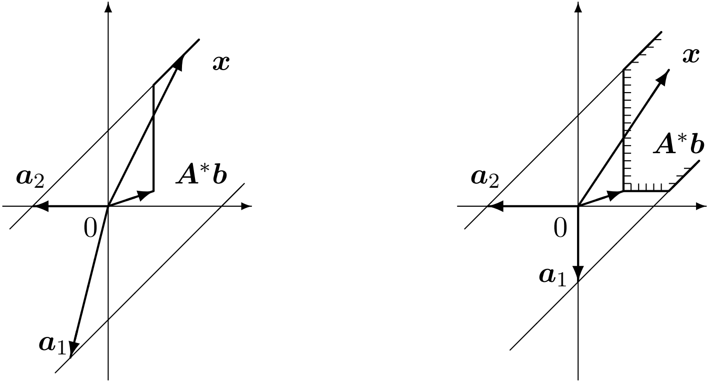
- If $\mathrm{Tr}(\boldsymbol{A}) < \mathbb{1}$, then

$$\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b}$$

- If $\mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}$, then

$$\boldsymbol{x} = \boldsymbol{A}^*\boldsymbol{b} \oplus \boldsymbol{A}^\times \boldsymbol{v}, \qquad \boldsymbol{v} \in \mathbb{X}^2$$

Illustration of Solution to $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{x}$ in $\mathbb{R}^2_{\max,+}$ if $\mathrm{Tr}(\boldsymbol{A}) = 0$



- Examples of solutions when the vector $\boldsymbol{b}$ is outside the solution set of homogeneous equation (left), and inside the set (right)

Solution of Two-Sided Inequalities

- Suppose that a n-matrix $\boldsymbol{A}$ and n-vector $\boldsymbol{b}$ are given
- By analogy with two-sided equations, the following inequality can be referred to as a homogeneous two-sided inequality:

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

- In the same way, the following inequality can be referred to as a nonhomogeneous two sided inequality

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$$

- We now show that these inequalities can be solved by converting into corresponding two-sided equations using auxiliary variables

- Given a $(n \times n)$-matrix $\boldsymbol{A}$, the problem is to find n-vectors $\boldsymbol{x}$ that satisfy the homogeneous two-sided inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

Lemma

The general solution of the homogeneous two-sided inequality with irreducible matrix is given by the following statements.

1. *If* $\operatorname{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, *then* $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ *for any vector* $\boldsymbol{u}$ *;*
2. *If* $\operatorname{Tr}(\boldsymbol{A}) > \mathbb{1}$, *then there is only trivial solution* $\boldsymbol{x} = \boldsymbol{0}$

Proof

- If a vector $\boldsymbol{x}$ solves the inequality $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$, it is also a solution of the following equation in two unknown vectors $\boldsymbol{x}$ and $\boldsymbol{u}$:

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{u} = \boldsymbol{x}$$

- For each fixed $\boldsymbol{u}$, this is a nonhomogeneous equation in $\boldsymbol{x}$

Proof (cont.)

- Consider the obtained nonhomogeneous equation (where $\boldsymbol{b} = \boldsymbol{u}$)

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{u} = \boldsymbol{x}$$

- After solving this equation with respect to $\boldsymbol{x}$, we have the solution

$$\boldsymbol{x} = \begin{cases} \boldsymbol{A}^{*}\boldsymbol{u}, & \text{if } \mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}; \\ \boldsymbol{A}^{*}\boldsymbol{u} \oplus \boldsymbol{A}^{\times}\boldsymbol{v} & \text{if } \mathrm{Tr}(\boldsymbol{A}) = \mathbb{1}; \end{cases}$$

  where $\boldsymbol{u}$ and $\boldsymbol{v}$ are any vectors of parameters
- Since nonzero columns in the matrix $\boldsymbol{A}^{\times}$ coincide with the same columns in $\boldsymbol{A}^{*}$, we can combine both solutions as

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$$

- If $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, the equation $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{u} = \boldsymbol{x}$ can have only trivial solution $\boldsymbol{x} = \boldsymbol{0}$ which requires that $\boldsymbol{u} = \boldsymbol{0}$ ■

Lemma

The nonhomogeneous inequality $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$ with irreducible matrix has solutions if and only if at least one of the following conditions hold:

1. $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$;
2. $\boldsymbol{b} = \boldsymbol{0}$.

If the equation has solutions, then $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{b}$ is its minimal solution

Theorem

Suppose that the nonhomogeneous equation with irreducible matrix has solutions, and let $\boldsymbol{x}$ be the general solution of the equation.

Then, the following statements hold:

1. *If $\mathrm{Tr}(\boldsymbol{A}) \leq \mathbb{1}$, then $\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}$ for any vector $\boldsymbol{u} \geq \boldsymbol{b}$;*
2. *If $\mathrm{Tr}(\boldsymbol{A}) > \mathbb{1}$, then the inequality has only the trivial solution $\boldsymbol{x} = \boldsymbol{0}$ (when $\boldsymbol{b} = \boldsymbol{0}$)*

Proof of Theorem

- We use an auxiliary variable $\boldsymbol{u}$ to transform the inequality $\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{x}$ into the equation in both $\boldsymbol{x}$ and $\boldsymbol{u}$ in the form

$$\boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \oplus \boldsymbol{u} = \boldsymbol{x}$$

- We solve the equation with respect to $\boldsymbol{x}$ as a nonhomogeneous equation where $\boldsymbol{b}$ is replaced by $\boldsymbol{b} \oplus \boldsymbol{u}$, which yields the result

$$\boldsymbol{x} = \boldsymbol{A}^{*}(\boldsymbol{b} \oplus \boldsymbol{u})$$

- Since the vector $\boldsymbol{u}$ contributes to the solution only when it has entries greater than in $\boldsymbol{b}$, we can represent the solution as

$$\boldsymbol{x} = \boldsymbol{A}^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \geq \boldsymbol{b} \qquad \blacksquare$$

Eigenvalues and Eigenvectors: Introduction

- A scalar λ is an eigenvalue of an $(n \times n)$-matrix $\boldsymbol{A}$, if there exists a nonzero n-vector $\boldsymbol{x}$ such that the following equality holds:

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$

- Any nonzero vector $\boldsymbol{x}$ that satisfies this equality, is an eigenvector of the matrix $\boldsymbol{A}$, which corresponds to the eigenvalue λ
- The set of all eigenvectors of a matrix $\boldsymbol{A}$ together with the zero vector form a tropical eigenspace of $\boldsymbol{A}$
- The eigenvalue problem can be examined in the context of tropical analogues of the characteristic polynomial and equation of matrix
- The eigenvector problem can be reduced to a two-sided equation

Irreducible and Reducible Matrices

- A matrix $\boldsymbol{A}$ is called reducible if simultaneous permutations of its rows and columns transform it into a block-triangular normal form
- Otherwise, the matrix $\boldsymbol{A}$ is referred to as irreducible
- The lower block-triangular normal form of a matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & & \mathbf{0} \\ \vdots & \vdots & \ddots & \\ \boldsymbol{A}_{s1} & \boldsymbol{A}_{s2} & \dots & \boldsymbol{A}_{ss} \end{pmatrix},$$

  where $\boldsymbol{A}_{ii}$ is an irreducible square matrix for all $i = 1, \dots, s$
- We denote by λ_i an eigenvector of the diagonal block $\boldsymbol{A}_{ii}$

Further Results on Eigenvalues (without proof)

- Any square matrix $\boldsymbol{A}$ of order n has at least one eigenvalue, which is called the spectral radius of $\boldsymbol{A}$ and given by

$$\lambda = \bigoplus_{m=1}^{n} \mathrm{tr}^{1/m}(\boldsymbol{A}^m)$$

- If the matrix $\boldsymbol{A}$ is irreducible, it has no other eigenvalues
- A reducible matrix may have more than one eigenvalues
- Each eigenvalue of a reducible matrix given in the block-triangular form is one of the eigenvalues $\lambda_1, \dots, \lambda_s$ of the diagonal blocks
- However, some of the eigenvalues of diagonal blocks of an reducible matrix may not be eigenvalues of the matrix
- The spectral radius of the matrix is the maximum eigenvalue:

$$\lambda = \lambda_1 \oplus \cdots \oplus \lambda_s$$

Tropical Optimization Problems: Overview

- Tropical optimization problems are formulated and solved within the framework of tropical mathematics
- Many problems have objective functions defined on vectors over idempotent semifields (semirings with multiplicative inverses)
- Both unconstrained problems and problems with constraints in the form of linear inequalities and equalities are considered
- There are many problems that can be solved analytically; for other problems, only algorithmic solutions are known in the literature
- The problems find applications in many areas including
  - *project scheduling,*
  - *location analysis,*
  - *transportation networks,*
  - *decision making and other fields*

Location Analysis: Minimax Location Problem

- The problem is to locate a new point in a feasible area to minimize the maximum Chebyshev distance (with addends) to given points
- The Chebyshev distance (maximum or l_∞ -metric) between two vectors $\boldsymbol{r} = (r_1, \ldots, r_n)^T$ and $\boldsymbol{s} = (s_1, \ldots, s_n)^T$ in $\mathbb{R}^n$ is given by

$$d(\boldsymbol{r}, \boldsymbol{s}) = \max_{1 \leq i \leq n} |r_i - s_i|$$

- Suppose there is a set of vectors $\boldsymbol{r}_k = (r_{1k}, \ldots, r_{nk})^T \in \mathbb{R}^n$ for all $k = 1, \ldots, m$ and a vector of addends $\boldsymbol{w} = (w_1, \ldots, w_m)^T \in \mathbb{R}^m$
- The location problem is to minimize the maximum distance (with addends) from a new vector $\boldsymbol{x} = (x_1, \ldots, x_n)^T$ to the vectors $\boldsymbol{r}_k$:

$$\min_{\boldsymbol{x}} \quad \max_{1 \leq k \leq m} (d(\boldsymbol{r}_k, \boldsymbol{x}) + w_k)$$

Tropical Representation

- Scalar representation of the Chebyshev metric in terms of $\mathbb{R}_{\max,+}$

$$
\begin{aligned}
d(\boldsymbol{r},\boldsymbol{s}) &= \max_{1\leq i\leq n} |r_i - s_i| \\
&= \max_{1\leq i\leq n} \max(r_i - s_i, s_i - r_i) && \text{(in ordinary notation)}
\end{aligned}
$$

$$
\begin{aligned}
d(\boldsymbol{r},\boldsymbol{s}) &= \bigoplus_{1\leq i\leq n} (s_i^{-1} r_i \oplus r_i^{-1} s_i) \\
&= \bigoplus_{1\leq i\leq n} s_i^{-1} r_i \oplus \bigoplus_{1\leq i\leq n} r_i^{-1} s_i && \text{(after replacing operations)}
\end{aligned}
$$

- Vector representation of the Chebyshev metric

$$
d(\boldsymbol{r},\boldsymbol{s}) = \boldsymbol{s}^{-}\boldsymbol{r} \oplus \boldsymbol{r}^{-}\boldsymbol{s}
$$

Representation of Objective Function

- The objective function of the problem is written as

$$\max_{1\leq k\leq m}(d(\boldsymbol{r}_k,\boldsymbol{x})+w_k) \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{1\leq k\leq m} w_k(\boldsymbol{x}^-\boldsymbol{r}_k\oplus\boldsymbol{r}_k^-\boldsymbol{x})$$

$$=\bigoplus_{1\leq k\leq m} w_k\boldsymbol{x}^-\boldsymbol{r}_k\oplus\bigoplus_{1\leq k\leq m} w_k\boldsymbol{r}_k^-\boldsymbol{x} \qquad \text{(after replacing operations)}$$

- Consider a matrix that consists of the vectors $\boldsymbol{r}_k$ as columns

$$\boldsymbol{R}=(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_m)$$

- With this matrix, we can write

$$\bigoplus_{1\leq k\leq m} w_k\boldsymbol{r}_k=\boldsymbol{R}\boldsymbol{w}, \qquad \bigoplus_{1\leq k\leq m} w_k\boldsymbol{r}_k^-=\boldsymbol{w}^T\boldsymbol{R}^-$$

- Vector representation of the objective function

$$\bigoplus_{1\leq k\leq m} w_k(\boldsymbol{x}^{-}\boldsymbol{r}_k \oplus \boldsymbol{r}_k^{-}\boldsymbol{x}) = \boldsymbol{x}^{-}\boldsymbol{R}\boldsymbol{w} \oplus \boldsymbol{w}^T\boldsymbol{R}^{-}\boldsymbol{x}$$

- We introduce the vectors

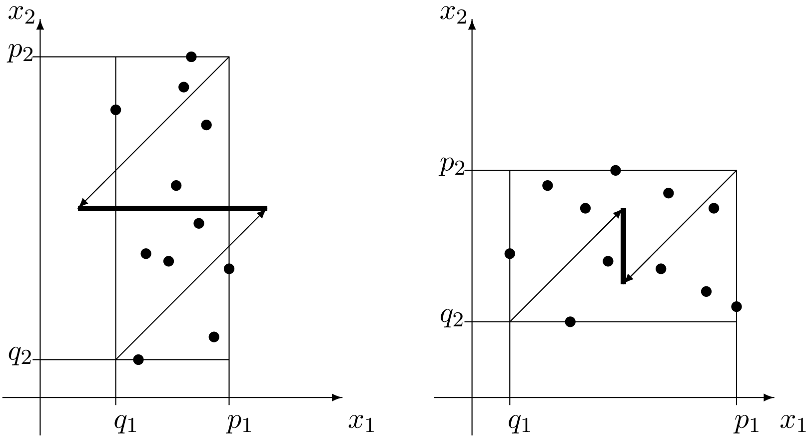
$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{w}, \qquad \boldsymbol{q}^{-} = \boldsymbol{w}^T\boldsymbol{R}^{-}$$

Location Problem

- The problem is to find a vector $\boldsymbol{x}$ that attains the minimum

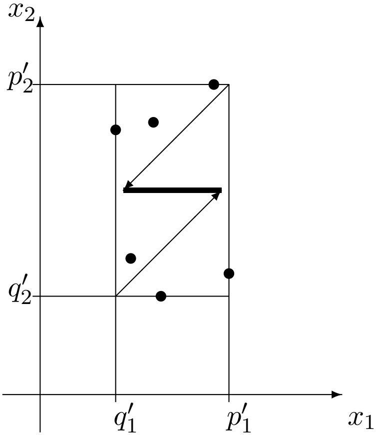
$$\min_{\boldsymbol{x}} \quad \boldsymbol{x}^{-}\boldsymbol{p} \oplus \boldsymbol{q}^{-}\boldsymbol{x}$$

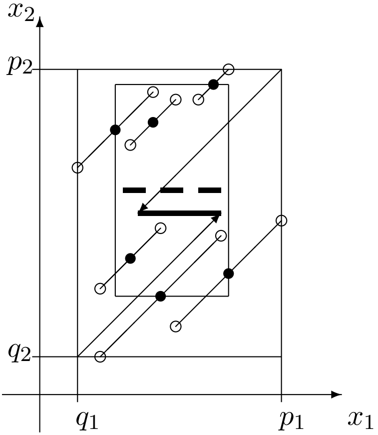
Solution of a problem with $w_k = 0$ in $\mathbb{R}^2$



- The solution is a segment on the line drawn across the minimal enclosing rectangle through the center points of its long sides

Solution of a problem with $w_k > 0$ in $\mathbb{R}^2$





- Each given point $\boldsymbol{r}_k$ (left) is replaced with two points $w_k \boldsymbol{r}_k$ and $w_k^{-1} \boldsymbol{r}_k$ to produce a new minimum enclosing rectangle (right)

Constrained Location Problem

- Suppose the following matrix and vectors are given:

$$\boldsymbol{B} = (b_{ij}) \in \mathbb{R}^{n\times n}, \qquad \boldsymbol{g} = (g_i) \in \mathbb{R}^n, \quad \boldsymbol{h} = (h_i) \in \mathbb{R}^n$$

- The feasible location area is defined by the inequalities

$$\begin{aligned} b_{ij} + x_j &\leq x_i, \\ g_i \leq x_i &\leq h_i, \qquad i, j = 1, \ldots, n \end{aligned}$$

- The feasible area is an intersection of the half-spaces given by $b_{ij} + x_j \leq x_i$, and the hyper-rectangle given by $g_i \leq x_i \leq h_i$

- The inequalities $b_{ij} + x_j \leq x_i$ for all $j = 1, \ldots, n$ combine into

$$\max_{1 \leq j \leq n} (b_{ij} + x_j) \leq x_i \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{1 \leq j \leq n} b_{ij} x_j \leq x_i \qquad \text{(after replacing operations)}$$

- Vector representation of constraints is of the form

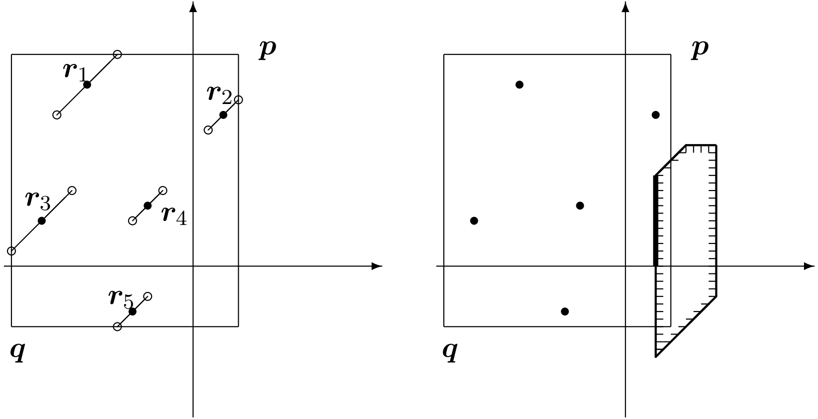
$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \qquad \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}$$

Constrained Location Problem

- The problem is to find a vector $\boldsymbol{x}$ that attains the minimum

$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-}\boldsymbol{p} \oplus \boldsymbol{q}^{-}\boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}, \quad \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h} \end{aligned}$$

Solution to Constrained Problem in $\mathbb{R}^2$



- The minimal enclosing rectangle of a problem (left) and the solution to the problem under constraints (right)

Decision Making: Ranking by Pairwise Comparisons

Ranking by Pairwise Comparisons

- Consider a problem to evaluate ratings (scores, priorities, weights) of alternatives from the results of their pairwise comparisons
- Outcome of comparisons is given by a matrix $\boldsymbol{A} = (a_{ij})$, where a_{ij} shows by how much times alternative i is preferable than j
- A pairwise comparison matrix $\boldsymbol{A}$ is consistent if its entries are transitive to satisfy the condition $a_{ij} = a_{ik}a_{kj}$ for all i, j, k
- Each consistent matrix $\boldsymbol{A}$ has the entries $a_{ij} = x_i/x_j$ given by a positive vector $\boldsymbol{x} = (x_j)$ that entirely specifies the matrix $\boldsymbol{A}$
- If a comparison matrix $\boldsymbol{A}$ is consistent, its vector $\boldsymbol{x}$ (up to a positive factor) defines the individual ratings of alternatives

Approximation Problem

- The pairwise comparison matrices which are encountered in real-world decision-making problems are usually inconsistent
- If a matrix $\boldsymbol{A}$ is inconsistent, approximation problem arises to find approximating consistent matrices $\boldsymbol{X} = (x_{ij})$ with $x_{ij} = x_i/x_j$
- The approximation with approximation error measured in linear scale involves optimization problems that are difficult to solve
- Evaluating the approximation error on a logarithmic scale may simplify the analysis and even provide a direct analytical solution
- As an example, one can consider log-Chebyshev approximation which uses the Chebyshev metric in logarithmic scale

Log-Chebyshev Approximation

- For matrices $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{X} = (x_{ij})$, the log-Chebyshev distance with a logarithm to a base greater than 1 is given by
$$d(\boldsymbol{A}, \boldsymbol{X}) = \max_{1 \leq i,j \leq n} |\log a_{ij} - \log x_{ij}|$$
- It follows from the monotonicity of logarithm that
$$d(\boldsymbol{A}, \boldsymbol{X}) = \log \max_{1 \leq i,j \leq n} \max \left\{ \frac{a_{ij}}{x_{ij}}, \frac{x_{ij}}{a_{ij}} \right\}$$
- Taking into account that $a_{ij} = 1/a_{ji}$ and $x_{ij} = x_i/x_j$, we have
$$d(\boldsymbol{A}, \boldsymbol{X}) = \log \max_{1 \leq i,j \leq n} \max \left\{ \frac{a_{ij} x_j}{x_i}, \frac{a_{ji} x_i}{x_j} \right\} = \log \max_{1 \leq i,j \leq n} \frac{a_{ij} x_j}{x_i}$$
- Since logarithm is monotone, the minimization of the logarithm is equivalent to minimizing its argument, which leads to the problem
$$\min_{\boldsymbol{x}} \quad \max_{1 \leq i,j \leq n} \frac{a_{ij} x_j}{x_i}$$

Tropical Representation

- Representation of the objective function in terms of algebra with addition $\oplus$ defined as $\max$ and multiplication $\otimes$ as usual

$$\max_{1 \leq i,j \leq n} \frac{a_{ij} x_j}{x_i} \qquad \text{(in ordinary notation)}$$

$$\bigoplus_{1 \leq i,j \leq n} x_i^{-1} a_{ij} x_j \qquad \text{(after replacing operations)}$$

- Vector representation of the objective function

$$\bigoplus_{1 \leq i,j \leq n} x_i^{-1} a_{ij} x_j = \boldsymbol{x}^{-} \boldsymbol{A} \boldsymbol{x}$$

Pairwise Comparison Problem

- The problem is to find a vector $\boldsymbol{x}$ that provides the minimum

$$\min_{\boldsymbol{x}} \quad \boldsymbol{x}^{-} \boldsymbol{A} \boldsymbol{x}$$

Constrained Rating

- Given a matrix $\boldsymbol{B} = (b_{ij})$ with nonnegative entries, suppose that the final ratings must satisfy the inequalities
$$b_{ij} x_j \leq x_i, \qquad i, j = 1, \ldots, n$$
- These constraints may require, for instance, that the rating of alternative j must be at least in two times higher than that of i
- Combining the inequalities $b_{ij} x_j \leq x_i$ for $j = 1, \ldots, n$ gives
$$\max(b_{i1} x_1, \ldots, b_{in} x_n) \leq x_i \qquad \text{(in ordinary notation)}$$
$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \qquad \text{(after replacing operations)}$$

Constrained Pairwise Comparison Problem

- The problem is to find a vector $\boldsymbol{x}$ that provides the minimum
$$\begin{aligned} \min_{\boldsymbol{x}} \quad & \boldsymbol{x}^{-} \boldsymbol{A} \boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \end{aligned}$$

Various Examples: Linear Objective Functions

Linear Objective Functions

- Hoffman (1963), Superville (1978), U. Zimmermann (1981)

$$\begin{aligned} \min\quad & \boldsymbol{p}^T\boldsymbol{x}, \\ \mathrm{s.\,t.}\quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

(direct solution)

- K. Zimmermann (1984, 1992, 2003, 2006)

$$\begin{aligned} \min\quad & \boldsymbol{p}^T\boldsymbol{x}, \\ \mathrm{s.\,t.}\quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{d}, \quad \boldsymbol{C}\boldsymbol{x} \geq \boldsymbol{b}, \\ & \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h} \end{aligned}$$

(algorithmic solution)

- Butkovič (1984, 2010), Butkovič and Aminu (2009)

$$\begin{aligned} \min\quad & \boldsymbol{p}^T\boldsymbol{x}, \\ \mathrm{s.\,t.}\quad & \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} = \boldsymbol{C}\boldsymbol{x} \oplus \boldsymbol{d} \end{aligned}$$

(algorithmic solution)

Nonlinear Objective Functions

- Cuninghame-Green (1962, 1979), Engel and Schneider (1975), Elsner and van den Driessche (2004, 2010), K. (2014, 2015)

$$\min \quad \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

(direct solution)

- Cuninghame-Green (1976), U. Zimmermann (1981)

$$\begin{aligned} \min \quad & (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{b}, \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \end{aligned}$$

(direct solution)

- K. Zimmermann (1984)

$$\begin{aligned} \min \quad & (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \oplus \boldsymbol{p}^{-}\boldsymbol{A}\boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h} \end{aligned}$$

(algorithmic solution)

Nonlinear Objective Functions

- K. (2004, 2009, 2012)

$$\min \quad (\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p} \oplus \boldsymbol{q}^{-}\boldsymbol{A}\boldsymbol{x}$$ (direct solution)

- Butkovič and Tam (2009), K. (2013, 2015)

$$\min \quad \mathbf{1}^{T}\boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\mathbf{1};$$ (direct solution)

$$\max \quad \mathbf{1}^{T}\boldsymbol{A}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\mathbf{1}$$ (direct solution)

- Gaubert, Katz and Sergeev (2012)

$$\begin{aligned} \min \quad & (\boldsymbol{p}^{T}\boldsymbol{x} \oplus r)(\boldsymbol{q}^{T}\boldsymbol{x} \oplus s)^{-1}, \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \oplus \boldsymbol{b} \leq \boldsymbol{C}\boldsymbol{x} \oplus \boldsymbol{d} \end{aligned}$$ (algorithmic solution)

Examples: Minimization of Conjugate Quadratic Form

- Given an $(n \times n)$-matrix $\boldsymbol{A}$, the problem is to find regular n-vectors $\boldsymbol{x}$ that attain the minimum

$$\min_{\boldsymbol{x} > \mathbf{0}} \quad \boldsymbol{x}^{-} \boldsymbol{A} \boldsymbol{x}$$

- The following result offers a complete solution of the problem

Theorem (K. 2013, 2015)

Let $\boldsymbol{A}$ be a matrix with spectral radius $\lambda > \mathbb{0}$. Then, the minimum is equal to λ, and all solutions are given in parametric form by

$$\boldsymbol{x} = (\lambda^{-1} \boldsymbol{A})^{*} \boldsymbol{u},$$

where $\boldsymbol{u} \neq \mathbf{0}$ is a vector of parameters

Proof

- Denote the minimum of $\boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$ by θ and consider the equation

$$\boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x} = \theta > \mathbb{0}$$

- Since θ is assumed to be the minimum, the set of solutions does not change if we replace the equation by the inequality

$$\boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x} \leq \theta$$

- We solve the inequality with respect to $\boldsymbol{A}\boldsymbol{x}$ by the lemma on one-sided inequality to obtain the equivalent inequality

$$\boldsymbol{A}\boldsymbol{x} \leq \theta\boldsymbol{x}$$

- After multiplying both sides by θ^{-1}, we have two-sided inequality

$$\theta^{-1}\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

Proof (cont.)

- Let us consider the two-sided inequality obtained

$$\theta^{-1}\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{x}$$

- By the theorem on the two-sided inequality, the above inequality has regular solutions if the following existence condition holds:

$$\mathrm{Tr}(\theta^{-1}\boldsymbol{A}) \leq \mathbb{1}$$

- Under this condition, all regular solutions are given in the form

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A})^{*}\boldsymbol{u}, \qquad \boldsymbol{u} \neq \boldsymbol{0}$$

- We use the definition of the tropical determinant Tr and apply properties of trace to write the existence condition as

$$\mathrm{Tr}(\theta^{-1}\boldsymbol{A}) = \theta^{-1}\,\mathrm{tr}\,\boldsymbol{A} \oplus \cdots \oplus \theta^{-n}\,\mathrm{tr}\,\boldsymbol{A}^{n} \leq \mathbb{1}$$

Proof (cont.)

- Let us examine the obtained inequality

$$\theta^{-1} \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \theta^{-n} \operatorname{tr} \boldsymbol{A}^n \leq \mathbb{1}$$

- This inequality is equivalent to the system of inequalities,

$$\theta^{-k} \operatorname{tr} \boldsymbol{A}^k \leq 1, \qquad k = 1, \ldots, n$$

- Solving each inequality with respect to θ yields

$$\theta \geq \operatorname{tr}^{1/k}(\boldsymbol{A}^k), \qquad k = 1, \ldots, n$$

Proof (cont.)

- Consider the system of inequalities,

$$\theta \geq \operatorname{tr}^{1/k}(\boldsymbol{A}^k), \qquad k = 1, \dots, n$$

- This system is equivalent to one inequality with the right-hand side that takes the form of the spectral radius of the matrix $\boldsymbol{A}$,

$$\theta \geq \operatorname{tr} \boldsymbol{A} \oplus \cdots \oplus \operatorname{tr}^{1/n}(\boldsymbol{A}^n) = \lambda$$

- Since θ denotes the minimum of the objective function, the last inequality can be replaced by the equality

$$\theta = \lambda$$

- Substitution of λ for θ in $\boldsymbol{x} = (\theta^{-1}\boldsymbol{A})^* \boldsymbol{u}$ completes the proof ∎

- Given $(n \times n)$-matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we need to find regular n-vectors $\boldsymbol{x}$ that solve the problem

$$\begin{aligned} \min_{\boldsymbol{x} > \boldsymbol{0}} \quad & \boldsymbol{x}^{-} \boldsymbol{A} \boldsymbol{x}, \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \end{aligned}$$

Theorem (K. 2015, 2017)

Let $\boldsymbol{A}$ be a matrix with spectral radius $\lambda > \mathbb{0}$ and $\boldsymbol{B}$ a matrix such that $\operatorname{Tr}(\boldsymbol{B}) \leq \mathbb{1}$. Then, the minimum in the problem is equal to

$$\theta = \bigoplus_{1 \leq k \leq n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \operatorname{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k}),$$

and all solutions are given in parametric form by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*}\boldsymbol{u},$$

where $\boldsymbol{u} \neq \boldsymbol{0}$ is a vector of parameters

Lemma (Binomial Identity for Trace)

For any matrices $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *, and* $m \geq 1$ *, the following identity holds:*

$$\operatorname{tr}(\boldsymbol{A} \oplus \boldsymbol{B})^m = \operatorname{tr} \boldsymbol{B}^m \oplus \bigoplus_{k=1}^{m} \bigoplus_{i_1+\cdots+i_k=m-k} \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k})$$

Proof

- The power $(\boldsymbol{A} \oplus \boldsymbol{B})^m$ is the sum of 2^m products of k matrices $\boldsymbol{A}$ and $m-k$ matrices $\boldsymbol{B}$ coming in any order, $k = 0, 1, \ldots m$
- We calculate the trace of the matrix $(\boldsymbol{A} \oplus \boldsymbol{B})^m$ and consider those summands which have at least one matrix $\boldsymbol{A}$ inside
- Since $\operatorname{tr}(\boldsymbol{B}\boldsymbol{A}) = \operatorname{tr}(\boldsymbol{A}\boldsymbol{B})$, these summands can be written as

  $$\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k},$$

  where $i_1, \ldots, i_k$ are integers such that $i_1 + \cdots + i_k = m - k$ ∎

Proof of Theorem

- We introduce an auxiliary variable and reduce the optimization problem to the solution of a parametrized inequality
- Let θ denote the minimum value of the objective function and note that this minimum satisfies the condition $\theta \geq \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x} \geq \lambda > \mathbb{0}$
- The set of solutions of the problem is defined by the system

$$\begin{aligned} \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x} &= \theta, \\ \boldsymbol{B}\boldsymbol{x} &\leq \boldsymbol{x} \end{aligned}$$

- Since θ is assumed to be the minimum, the first equality can be replaced by an inequality to rewrite the system as

$$\begin{aligned} \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x} &\leq \theta, \\ \boldsymbol{B}\boldsymbol{x} &\leq \boldsymbol{x} \end{aligned}$$

Proof (cont.)

- Consider the system of inequalities obtained

$$\begin{aligned} \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x} &\leq \theta, \\ \boldsymbol{B}\boldsymbol{x} &\leq \boldsymbol{x} \end{aligned}$$

- After solving the first inequality with respect to $\boldsymbol{A}\boldsymbol{x}$ as one-sided inequality, and multiplying both sides by θ^{-1}, the system becomes

$$\begin{aligned} \theta^{-1}\boldsymbol{A}\boldsymbol{x} &\leq \boldsymbol{x}, \\ \boldsymbol{B}\boldsymbol{x} &\leq \boldsymbol{x} \end{aligned}$$

- It follows from properties of idempotent addition that this system is equivalent to one two-sided inequality, which is given by

$$(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})\boldsymbol{x} \leq \boldsymbol{x}$$

Proof (cont.)

- Consider the two-sided inequality obtained

$$(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})\boldsymbol{x} \leq \boldsymbol{x}$$

- By the theorem on two-sided inequality, the above inequality has regular solutions if and only if the following condition holds:

$$\mathrm{Tr}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B}) \leq \mathbb{1}$$

- Let us examine the left-hand side and represent it as

$$\mathrm{Tr}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B}) = \bigoplus_{m=1}^{n} \mathrm{tr}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{m}$$

Proof (cont.)

- We observe that from the binomial identity it follows that

$$\operatorname{tr}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^m = \operatorname{tr} \boldsymbol{B}^m \oplus \bigoplus_{k=1}^{m} \bigoplus_{i_1+\cdots+i_k=m-k} \theta^{-k} \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k})$$

- After substitution of this identity, we obtain

$$\begin{aligned}\operatorname{Tr}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B}) &= \bigoplus_{m=1}^{n} \operatorname{tr}(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^m \\ &= \operatorname{Tr}(\boldsymbol{B}) \oplus \bigoplus_{m=1}^{n} \bigoplus_{k=1}^{m} \bigoplus_{i_1+\cdots+i_k=m-k} \theta^{-k} \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k})\end{aligned}$$

Proof (cont.)

- Consider the representation obtained

$$\mathrm{Tr}(\theta^{-1}\boldsymbol{A}\oplus\boldsymbol{B}) = \mathrm{Tr}(\boldsymbol{B})\oplus\bigoplus_{m=1}^{n}\bigoplus_{k=1}^{m}\bigoplus_{i_1+\cdots+i_k=m-k}\theta^{-k}\,\mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1}\cdots\boldsymbol{A}\boldsymbol{B}^{i_k})$$

- We rearrange the summands in the second sum on the right to put them in the order of exponents of θ^{-k} with $k=1,\ldots,n$ to write

$$\mathrm{Tr}(\theta^{-1}\boldsymbol{A}\oplus\boldsymbol{B}) = \mathrm{Tr}(\boldsymbol{B})\oplus\bigoplus_{k=1}^{n}\bigoplus_{0\leq i_1+\cdots+i_k\leq n-k}\theta^{-k}\,\mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1}\cdots\boldsymbol{A}\boldsymbol{B}^{i_k})$$

- We now solve with respect to θ the inequality

$$\mathrm{Tr}(\theta^{-1}\boldsymbol{A}\oplus\boldsymbol{B}) \leq \mathbb{1}$$

Proof (cont.)

- Consider the exsitence condition, which is written in the form

$$\mathrm{Tr}(\boldsymbol{B}) \oplus \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \theta^{-k} \, \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k}) \leq \mathbb{1}$$

- Since $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$ by assumption, the above inequality reduces to

$$\bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \theta^{-k} \, \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k}) \leq \mathbb{1}$$

- The last inequality is equivalent to the system of inequalities

$$\bigoplus_{0 \leq i_1 + \cdots + i_k \leq n-k} \theta^{-k} \, \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k}) \leq \mathbb{1}, \qquad k = 1, \ldots, n$$

Proof (cont.)

- We solve with respect to θ the inequalities

$$\bigoplus_{0\leq i_1+\cdots+i_k\leq n-k} \theta^{-k}\,\mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{i_1}\cdots\boldsymbol{A}\boldsymbol{B}^{i_k}) \leq \mathbb{1}, \qquad k=1,\dots,n$$

- As a result, we obtain the system of inequalities

$$\theta \geq \bigoplus_{0\leq i_1+\cdots+i_k\leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1}\cdots\boldsymbol{A}\boldsymbol{B}^{i_k}), \qquad k=1,\dots,n$$

- The inequalities obtained are equivalent to one inequality

$$\theta \geq \bigoplus_{k=1}^{n}\bigoplus_{0\leq i_1+\cdots+i_k\leq n-k} \mathrm{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1}\cdots\boldsymbol{A}\boldsymbol{B}^{i_k})$$

Proof (cont.)

- Consider the inequality obtained

$$\theta \geq \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1+\cdots+i_k \leq n-k} \operatorname{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k})$$

- Since θ is assumed to be a minimum value of the objective function, the last inequality must hold as the equality

$$\theta = \bigoplus_{k=1}^{n} \bigoplus_{0 \leq i_1+\cdots+i_k \leq n-k} \operatorname{tr}^{1/k}(\boldsymbol{A}\boldsymbol{B}^{i_1} \cdots \boldsymbol{A}\boldsymbol{B}^{i_k})$$

- By the theorem on the two-sided inequality, all solutions of the inequality $(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})\boldsymbol{x} \leq \boldsymbol{x}$ and thus of the problem are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*}\boldsymbol{u}, \qquad \boldsymbol{u} > \boldsymbol{0} \qquad \blacksquare$$

Application to Project Scheduling

- Consider a project that involves n activities performed in parallel under “start-finish” and “start-start” constraints
- Let x_i denote the start time, and y_i the finish time of activity i
- The “start-finish” constraints take the form of equalities

$$\max(x_1 + a_{1i}, \ldots, x_n + a_{ni}) = y_i, \qquad i = 1, \ldots, n$$

- The “start-start” constraints take the form of inequalities

$$\max(x_1 + b_{1i}, \ldots, x_n + b_{ni}) \leq x_i, \qquad i = 1, \ldots, n$$

- For each activity, the flowtime (cycle time) is defined as

$$y_i - x_i$$

- The problem is to find the start time x_i for each activity i to minimize the maximum flowtime over all activities

- We introduce the following matrices and vectors

$$\boldsymbol{A} = (a_{ij}), \qquad \boldsymbol{B} = (b_{ij}), \qquad \boldsymbol{x} = (x_i), \qquad \boldsymbol{y} = (y_i)$$

- The "start-finish" constraints are written in terms of $\mathbb{R}_{\max,+}$ as

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{y}$$

- The "start-start" constraints are represented as

$$\boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x}$$

- The maximum flowtime over all activities is given by

$$\boldsymbol{x}^{-}\boldsymbol{y} = \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}$$

- The problem of project scheduling is formulated in the form

$$\begin{aligned} \text{min} \quad & \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\ \mathrm{s.\,t.} \quad & \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{x} \end{aligned}$$

Example

- Consider a problem that is formulated in terms of the semifield $\mathbb{R}_{\max,+}$ (max-plus algebra) with $n = 2$ and the matrices

$$\boldsymbol{A} = \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix}$$

- To verify the condition $\mathrm{Tr}(\boldsymbol{B}) \leq \mathbb{1}$, we calculate the matrix

$$\boldsymbol{B}^2 = \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -10 \\ 4 & -6 \end{pmatrix}$$

- Next, we evaluate the traces and obtain

$$\mathrm{Tr}(\boldsymbol{B}) = \mathrm{tr}\,\boldsymbol{B} \oplus \mathrm{tr}\,\boldsymbol{B}^2 = 0 = \mathbb{1}$$

  which means that the above condition is satisfied

Example (cont.)

- Furthermore, we find the matrix

$$\boldsymbol{A}^2 = \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix} \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -7 & -6 \end{pmatrix}$$

- The spectral radius of the matrix $\boldsymbol{A}$ is equal to

$$\lambda = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr}^{1/2}(\boldsymbol{A}^2) = 0 = \mathbb{1}$$

- The minimum value of the objective function takes the form

$$\theta = \operatorname{tr} \boldsymbol{A} \oplus \operatorname{tr}^{1/2}(\boldsymbol{A}^2) \oplus \operatorname{tr}(\boldsymbol{AB}) = \lambda \oplus \operatorname{tr}(\boldsymbol{AB})$$

Example (cont.)

- To evaluate the minimum θ, we calculate the matrix

$$\boldsymbol{A}\boldsymbol{B} = \begin{pmatrix} 0 & -2 \\ -7 & -3 \end{pmatrix} \begin{pmatrix} 0 & -10 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -5 \\ 1 & -6 \end{pmatrix}$$

- The evaluation of the minimum leads to

$$\theta = \lambda \oplus \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}) = 2$$

- Let us construct the matrices

$$\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B} = \begin{pmatrix} 0 & -4 \\ 4 & -3 \end{pmatrix},$$

$$(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*} = \boldsymbol{I} \oplus \theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}$$

Example (cont.)

- All solutions of the problem are given by

$$\boldsymbol{x} = (\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*}\boldsymbol{u} = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}\boldsymbol{u}, \qquad \boldsymbol{u} \in \mathbb{R}^{2}$$

- We note that the columns of the matrix $(\theta^{-1}\boldsymbol{A} \oplus \boldsymbol{B})^{*}$ are collinear:

$$\begin{pmatrix} 0 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} -4 \\ 0 \end{pmatrix}$$

- After deleting the second column, the solution becomes simpler

$$\boldsymbol{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} u, \qquad u \in \mathbb{R}$$

- In terms of the ordinary operations, the solution is represented as

$$x_{1} = u, \quad x_{2} = u + 4, \qquad u \in \mathbb{R}$$

- Given a $(n \times n)$-matrix $\boldsymbol{A}$ and n-vectors $\boldsymbol{g} \leq \boldsymbol{h}$, we need to find regular n-vectors $\boldsymbol{x}$ that solve the problem

$$
\begin{aligned}
\min_{\boldsymbol{x}>\boldsymbol{0}} \quad & \boldsymbol{x}^{-}\boldsymbol{A}\boldsymbol{x}, \\
\text{s.t.} \quad & \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}
\end{aligned}
$$

Theorem (K. 2015, 2017)

Let $\boldsymbol{A}$ be a matrix with spectral radius $\lambda > \mathbb{0}$, $\boldsymbol{g}$ be a vector and $\boldsymbol{h}$ a regular vector. Then, the minimum of the objective function is equal to

$$
\theta = \bigoplus_{k=1}^{n} \operatorname{tr}^{1/k}(\boldsymbol{A}^{k}) \oplus \bigoplus_{k=1}^{n} (\boldsymbol{h}^{-}\boldsymbol{A}^{k}\boldsymbol{g})^{1/k} = \lambda \oplus \bigoplus_{k=1}^{n} (\boldsymbol{h}^{-}\boldsymbol{A}^{k}\boldsymbol{g})^{1/k},
$$

and all solutions are given in parametric form by

$$
\boldsymbol{x} = (\theta^{-1}\boldsymbol{A})^{*}\boldsymbol{u}, \qquad \boldsymbol{g} \leq \boldsymbol{u} \leq (\boldsymbol{h}^{-}(\theta^{-1}\boldsymbol{A})^{*})^{-}
$$

Chebyshev Approximation

Given n-vectors $\boldsymbol{p}$, $\boldsymbol{q}$, $\boldsymbol{g}$ and $\boldsymbol{h}$, the problem is to find regular n-vectors $\boldsymbol{x}$ that attain the minimum

$$\begin{aligned} \text{min} \quad & \boldsymbol{q}^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{p}, \\ \text{s.t.} \quad & \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h} \end{aligned}$$

Theorem (K. and Zimmerman, 2013)

Let $\boldsymbol{p}, \boldsymbol{g}$ be nonzero vectors and $\boldsymbol{q}, \boldsymbol{h}$ be regular vectors, and $\boldsymbol{g} \leq \boldsymbol{h}$. Then the minimum of the objective function is equal to

$$\mu = \sqrt{\boldsymbol{q}^{-}\boldsymbol{p}} \oplus \boldsymbol{q}^{-}\boldsymbol{g} \oplus \boldsymbol{h}^{-}\boldsymbol{p},$$

and all solutions are given by

$$\mu^{-1}\boldsymbol{p} \oplus \boldsymbol{g} \leq \boldsymbol{x} \leq (\mu^{-1}\boldsymbol{q}^{-} \oplus \boldsymbol{h}^{-})^{-}$$

Proof

- Let us show that μ is a lower bound for the objective function
- For an arbitrary regular vector $\boldsymbol{x}$, consider the value

$$r = \boldsymbol{q}^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{p}$$

- It follows from this equality that the following inequalities hold:

$$r \geq \boldsymbol{q}^{-}\boldsymbol{x}, \qquad r \geq \boldsymbol{x}^{-}\boldsymbol{p}$$

- The first inequality and the lower bound $\boldsymbol{x} \geq \boldsymbol{g}$ yield

$$r \geq \boldsymbol{q}^{-}\boldsymbol{x} \geq \boldsymbol{q}^{-}\boldsymbol{g}$$

- The solution of the first inequality results in the inequality

$$\boldsymbol{x} \leq r\boldsymbol{q}$$

- After conjugate transposition, the above inequality becomes

$$\boldsymbol{x}^{-} \leq r^{-1}\boldsymbol{q}^{-}$$

Proof (cont.)

- Substitution of $\boldsymbol{x}^{-} \leq r^{-1}\boldsymbol{q}^{-}$ into the second inequality results in

$$r \geq \boldsymbol{x}^{-}\boldsymbol{p} \geq r^{-1}\boldsymbol{q}^{-}\boldsymbol{p}$$

- It follows from the inequality obtained that

$$r \geq \sqrt{\boldsymbol{q}^{-}\boldsymbol{p}}$$

- The second inequality and the upper bound (which is represented in the form $\boldsymbol{x}^{-} \geq \boldsymbol{h}^{-}$) yield

$$r \geq \boldsymbol{x}^{-}\boldsymbol{p} \geq \boldsymbol{h}^{-}\boldsymbol{p}$$

- By combining all bounds on r, we have

$$r \geq \sqrt{\boldsymbol{q}^{-}\boldsymbol{p}} \oplus \boldsymbol{q}^{-}\boldsymbol{g} \oplus \boldsymbol{h}^{-}\boldsymbol{p} = \mu$$

- To find all solution of the problem, we need to solve the equation

$$\boldsymbol{q}^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{p} = \mu$$

Proof (cont.)

- Let us find regular solutions of the equation

$$\boldsymbol{q}^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{p} = \mu$$

- Since μ is a lower bound, we replace the equation by inequality

$$\boldsymbol{q}^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{p} \leq \mu$$

- This inequality is equivalent to the following inequalities:

$$\boldsymbol{q}^{-}\boldsymbol{x} \leq \mu, \qquad \boldsymbol{x}^{-}\boldsymbol{p} \leq \mu$$

- The solutions of the inequalities are represented as

$$\boldsymbol{x} \leq \mu\boldsymbol{q}, \qquad \boldsymbol{x} \geq \mu^{-1}\boldsymbol{p}$$

Proof (cont.)

- Consider the solution written in the form of the double inequality

$$\mu^{-1}\boldsymbol{p} \leq \boldsymbol{x} \leq \mu\boldsymbol{q}$$

- It follows from the inequality $\mu^2 \geq \boldsymbol{q}^-\boldsymbol{p}$ that $\mu\boldsymbol{q} \geq \mu^{-1}\boldsymbol{p}$, and thus the above double inequality has regular solutions
- Since the lower bound μ is attainable, this bound is the minimum of the objective function in the problem
- By combining the obtained double inequality with the given bounds $\boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}$, we arrive at the desired result:

$$\mu^{-1}\boldsymbol{p} \oplus \boldsymbol{g} \leq \boldsymbol{x} \leq (\mu^{-1}\boldsymbol{q}^- \oplus \boldsymbol{h}^-)^- \quad \blacksquare$$

Application to Location Problems

- Suppose that $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ are given points in $\mathbb{R}^n$
- We find a point $\boldsymbol{x}$ that minimizes the maximum Chebyshev distance from $\boldsymbol{x}$ to $\boldsymbol{r}$ and $\boldsymbol{s}$ under the constraints $\boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h}$
- We first represent the maximum distance in terms of $\mathbb{R}_{\max,+}$ as

$$\max\{\rho(\boldsymbol{x},\boldsymbol{r}),\rho(\boldsymbol{x},\boldsymbol{s})\} = \rho(\boldsymbol{x},\boldsymbol{r}) \oplus \rho(\boldsymbol{x},\boldsymbol{s}) =$$
$$= (\boldsymbol{r}^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{r}) \oplus (\boldsymbol{s}^{-}\boldsymbol{x} \oplus \boldsymbol{x}^{-}\boldsymbol{s}) = (\boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-})\boldsymbol{x} \oplus \boldsymbol{x}^{-}(\boldsymbol{r} \oplus \boldsymbol{s})$$

- The problem can now be formulated as follows:

$$\begin{aligned} \text{min} \quad & (\boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-})\boldsymbol{x} \oplus \boldsymbol{x}^{-}(\boldsymbol{r} \oplus \boldsymbol{s}), \\ \text{s.t.} \quad & \boldsymbol{g} \leq \boldsymbol{x} \leq \boldsymbol{h} \end{aligned}$$

- This problem takes a form of the previous problem with

$$\boldsymbol{p} = \boldsymbol{r} \oplus \boldsymbol{s}, \qquad \boldsymbol{q}^{-} = \boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-}$$

Example

- Consider a location problem in $\mathbb{R}^3$ with vectors given by

$$\boldsymbol{r} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{s} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \quad \boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{h} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- First, we calculate the vectors

$$\boldsymbol{p} = \boldsymbol{r} \oplus \boldsymbol{s} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \qquad \boldsymbol{q}^{-} = \boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-} = \begin{pmatrix} 3 & -1 & 2 \end{pmatrix}$$

- Furthermore, we evaluate

$$\sqrt{(\boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-})(\boldsymbol{r} \oplus \boldsymbol{s})} = 2, \quad (\boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-})\boldsymbol{g} = 3, \quad \boldsymbol{h}^{-}(\boldsymbol{r} \oplus \boldsymbol{s}) = 2$$

Example (cont.)

- The minimum of the objective function is equal to

$$\mu = \sqrt{(\boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-})(\boldsymbol{r} \oplus \boldsymbol{s})} \oplus (\boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-})\boldsymbol{g} \oplus \boldsymbol{h}^{-}(\boldsymbol{r} \oplus \boldsymbol{s}) = 3$$

- We evaluate the vectors

$$\mu^{-1}(\boldsymbol{r} \oplus \boldsymbol{s}) \oplus \boldsymbol{g} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (\mu^{-1}(\boldsymbol{r}^{-} \oplus \boldsymbol{s}^{-}) \oplus \boldsymbol{h}^{-})^{-} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

- The minimum is attained at any vector $\boldsymbol{x}$ such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leq \boldsymbol{x} \leq \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$