Combinatorics

Lecture 3

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Inclusion–exclusion principle

Suppose that we have two sets A and B. The size of the union is certainly at most $|A| + |B|$. This way, however, we are counting twice all elements in $A \cap B$, the intersection of the two sets. To correct for this, we subtract $|A \cap B|$ to obtain the following formula:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, the principle of inclusion-exclusion says that the cardinality of the union of n sets is the sum of the cardinalities of the individual sets, minus the cardinality of their pairwise intersection, plus the cardinality of their 3−way intersections, and so on, with the plus and minus signs alternating with intersections of larger number of sets.

Theorem 1 (Inclusion–exclusion principle)

For any collection of finite sets $A_1, A_2, \ldots, A_n$, we have

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\varnothing \neq I \subset \{1,2,\ldots,n\}} (-1)^{|I|+1} \left|\bigcap_{i \in I} A_i\right|$$

Proof. Assume that $A_1, A_2, \ldots, A_n \subset X$. For each set A_i, define the characteristic function $\chi_i(x)$ where $\chi_i(x) = 1$ if $x \in A_i$ and $\chi_i(x) = 0$ if $x \notin A_i$. We consider the following formula:

$$\Xi(x) = \prod_{i=1}^{n} (1 - \chi_i(x))$$

Observe that this is the characteristic function of the complement of $\bigcup_{i=1}^{n} A_i$: it is 1 iff x is not in any of the sets A_i. Hence,

$$\sum_{x \in X} \Xi(x) = \left|X \setminus \bigcup_{i=1}^{n} A_i\right|$$

Now we write $\Xi(x)$ differently, by expanding the product into 2^n terms:

$$\Xi(x) = \sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \prod_{i \in I} \chi_i(x)$$

Observe that $\prod_{i \in I} \chi_i(x)$ is the characteristic function of $\bigcap_{i \in I} A_i$.
Therefore, we get

$$\sum_{x \in X} \Xi(x) = \sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \sum_{x \in X} \prod_{i \in I} \chi_i(x) = \sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

Comparing both sums, we see that

$$\left| X \setminus \bigcup_{i=1}^{n} A_i \right| = |X| - |\bigcup_{i=1}^{n} A_i| = \sum_{I \subset \{1,2,\ldots,n\}} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|$$

Therefore, the theorem follows.■

Example I. How many integers between 1 and 100 are multiples of either 2 or 3?
Let us define a couple of sets:
$A = \{1 \leq i \leq 100 | 2 | i\}$ and $B = \{1 \leq i \leq 100 | 3 | i\}$
Notice that $|A| = 50, |B| = 33$, and $A \cap B = \{1 \leq i \leq 100 | 6 | i\}$
Thus, $|A \cap B| = 16$, and by the principle of inclusion-exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B| = 50 + 33 - 16 = 67.$$

Example II (Derangements). How many ways can n items be permuted so that none of the items are in their original position? Such permutations are called derangements.

Some preliminaries:

— There are $n!$ permutations of n elements.

— There are $(n-1)!$ permutations where element i stays in its original position (and some other elements might, but don't have to).

— There are $(n-2)!$ where any two elements stay in their original position (again with maybe some others staying).

— And so on: $(n-k)!$ ways to permute where a specific k elements stay in their original positions.

— Let's use F_I to denote the number of permutations where the elements of I are fixed (for example, $F_{\{1\}} = (n-1)!$ as we said above).

When counting, we can start by counting all of the permutations, and subtract the ones where the sets of size one are fixed, and add the ones where the sets of size two are fixed, then subtract the ones where the sets of size three are fixed, and so on. That will be every derangement counted exactly once.

So, the number of derangements is,

$$D_n = F_{\varnothing} - F_{\{1\}} - \ldots - F_{\{n\}} +$$

$$+F_{\{1,2\}} + F_{\{1,3\}} + \ldots + F_{\{n-1,n\}} - \ldots + (-1)^n F_{\{1,2,\ldots,n-1,n\}} =$$

$$= n! - C_n^1(n-1)! + C_n^2(n-2)! - C_n^3(n-3)! + \ldots + C_n^n(-1)^n =$$

$$= n!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{n!}{n!}) =$$

$$= n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$$

Example III. Suppose that we have two sets $|A| = m$ and $|B| = n$. How many surjections?

Let F be the set of functions $A \to B$, and F_I for $I \subset B$ be the set of functions that do not map onto the elements of I. (In other words, functions $A \to B \setminus I$). Note that every function is in $F_\varnothing$.

If $|I| = k$, then there are $(n-k)^m$ functions in F_I, and $|F_B| = 0$. The number of surjections is:

$$|F_\varnothing| - C_n^1 |F_a| + C_n^2 |F_{\{a,b\}}| - \ldots + (-1)^{n-1} C_n^{n-1} |F_{B\setminus\{a\}}| =$$

$$= n^m - C_n^1 (n-1)^m + C_n^2 (n-2)^m - \ldots + (-1)^{n-1} C_n^{n-1} 1^m =$$

$$= \sum_{k=0}^{n-1} (-1)^k C_n^k (n-k)^m$$

Alternative form of inclusion-exclusion

Suppose there is a set of N elements and some "properties" $P_1, \ldots, P_n$.

Let also $N(P_i)$ denote the number of objects that satisfy the property P_i, $N(P_i P_j)$ – that simultaneously satisfy both properties P_i, P_j and so on.

Set $N(P_1' P_2' \ldots P_n')$ denote the number of elements that have none of the properties $P_1, \ldots, P_n$.

Then we will have:

$$N(P_1' P_2' \ldots P_n') = N - N(P_1) - \ldots N(P_n) +$$

$$+ N(P_1 P_2) + \ldots N(P_{n-1} P_n) - \ldots + (-1)^n N(P_1 P_2 \ldots P_n)$$

Example IV. How many ways are there to distribute six different gifts to three different children such that each child gets at least one gift?
We can apply Example III where $m = 6$ and $n = 3$:

$$3^6 - C_3^1 2^6 + C_3^2 = 729 - 192 + 3 = 540$$

Example V. $\sum_{k=0}^{n} C_n^k(-1)^k(n-k)^m = 0$ for $m < n$.

Proof. Consider $\{a_1, \ldots, a_n\}$ and choose all partial permutations of length m from n-set with repetitions (there are $\tilde{A}_n^m = n^m$ in total). The objects to which we will apply the inclusion-exclusion formula will be considered these permutations, i.e. $N = n^m$.

Let's say that a partial permutation has the property P_i if it does not contain a_i. So, $N(P_i) = (n-1)^m$, $N(P_iP_j) = (n-2)^m$, ..., $N(P_1P_2 \ldots P_n) = 0$, and $N(P_1'P_2' \ldots P_n') = 0$ (because $m < n$).

Applying the inclusion-exclusion formula, we obtain the required equality.

Example VI. How many solutions does the equation $x + y + z = 13$ have where x, y, and z are nonnegative integers less than 6?
Let P_1, P_2, P_3 be the property of the solution when
$P_1 = x \geq 6$
$P_2 = y \geq 6$
$P_13 = z \geq 6$
Then the number of total solutions $N = C_{15}^{13} = 105$
We need to find the remaining number of solutions:
$N(P_1) = N(P_2) = N(P_3) = C_9^7 = 36$
$N(P_1P_2) = N(P_2P_3) = N(P_1P_3) = 3$
$N(P_1P_2P_3) = 0$
We need to find $N(P_1'P_2'P_3')$.
$N(P_1'P_2'P_3') = N - N(P_1) - N(P_2) - N(P_3) + N(P_1P_2) + N(P_2P_3) + N(P_1P_3) - N(P_1P_2P_3) =$
$= 105 - 36 - 36 - 36 + 3 + 3 + 3 - 0 = 6$
There are 6 solutions to the problem.

Example VII. How many $m \times n$ matrices, consisting only of 0 and 1, have no row or column consisting entirely of zeros?

Let F be a fixed set of k columns; we wiil count the matrices that have a 1 in every row but have no 1 in any of the columns in F. In each row there are $2^{n-k} - 1$ ways to choose a non-empty subset of the remaining $n - k$ columns in which to place ones, and there are m rows, so there are $(2^{n-k} - 1)^m$ such matrices. There are C_n^k ways to choose F, so by inclusion-exclusion formula we get:

$$\sum_{k=0}^{n} (-1)^k (2^{n-k} - 1)^m$$

Example VIII (**Euler's function.**)

Let $\phi(n)$ be the number of positive integers $x \leq n$ which are mutually prime to n i.e. have no common factors with n, other than 1 (i.e. $\phi(n) := |\{x \in \{1, 2, \dots, n\} | gcd(x, n) = 1\}|$)

Let n be any positive integer, and let $p_1, p_2, \dots, p_t$ be the prime divisors of n. Then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\dots\left(1 - \frac{1}{p_t}\right)$$

Remark. The formula above can be rewritten into the following possibly more friendly form. Let $n = p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$ be the prime factorisation of n. Then

$$\phi(n) = p_1^{a_1-1}(p_1 - 1)p_2^{a_2-1}(p_2 - 1)\dots p_t^{a_t-1}(p_t - 1)$$

Proof. Case 1: Let p be a prime number. Then $\{x \in \{1, 2, \ldots, p\} | gcd(x, p) = 1\} = \{1, 2, \ldots, p-1\}$, so $\phi(p) = p - 1$.

Case 2: Let p be a prime number, and $d \geq 1$. Then $\{x \in \{1, 2, \ldots, p^d\} | gcd(x, p^d) > 1\} = \{p, 2p, \ldots, (p^{d-1} - 1)p, p^d\}$
Hence it follows that
$\phi(p^d) = p^d - \left| \{x \in \{1, 2, \ldots, p^d\} | gcd(x, p^d) > 1\} \right| = p^{d-1}(p - 1)$

General case: We write, for $i \in \{1, \ldots, t\}$

$$A_i = \{x \in \{1, 2, \ldots, n\} | x \quad \text{is divisible by} \quad p_i\}$$

It follows that

$$A_i = \{p_i, 2p_i, \ldots, n\}$$

and it follows by direct counting that $|A_i| = \frac{n}{p_i}$

By the same principle, we have, for any distinct $i_1, \ldots, i_j \in \{1, \ldots, t\}$,

$$A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j} = \{kp_{i_1} p_{i_2} \ldots p_{i_j} | 1 \leq k \leq \frac{n}{p_{i_1} p_{i_2} \ldots p_{i_j}}\}$$

$$\left| A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j} \right| = \frac{n}{p_{i_1} p_{i_2} \ldots p_{i_j}}$$

It then follows from the inclusion-exclusion principle that

$$|A_1 \cup A_2 \cup \ldots \cup A_t| = \sum_{j=1}^{t} (-1)^{j-1} \sum_{i_1, \ldots, i_j} \left| A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_j} \right| =$$

$$= \sum_{j=1}^{t} (-1)^{j-1} \sum_{i_1, \ldots, i_j} \frac{n}{p_{i_1} p_{i_2} \ldots p_{i_j}}$$

On the other hand, note that

$$A_1 \cup A_2 \cup \ldots \cup A_t = \{x \in \{1, 2, \ldots, n\} | gcd(x, n) > 1\}$$

Hence, it follows from all the arguments outlined above that

$$\phi(n) = n - |\{x \in \{1, 2, \ldots, n\} | gcd(x, n) > 1\}| =$$

$$= n - \sum_{j=1}^{t} (-1)^{j-1} \sum_{i_1, \ldots, i_j} \frac{n}{p_{i_1} p_{i_2} \ldots p_{i_j}} =$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_t}\right)$$

Exercises

1.) How many 3−element subsets of $\{1, 2, \ldots, 100\}$ contain at least one element that is divisible by 2 and at least one element that is divisible by 5?

2.) How many permutations of the set $\{0, 1, \ldots, 9\}$ exist such that the first element is greater than 1 and the last one is less than 8.

3.) Consider the following equation: $x_1 + x_2 + \ldots x_6 = 20$, where $0 \leq x_i \leq 8$. Count the number of solutions to the equation.