September 30th, 2024

Example 1

Find the general integral of the equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u\,.$$

Solution:

Consider a system of equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{du}{u}\,.$$

Solving the equation

$$\frac{dx}{x}=\frac{dy}{y}\,,$$

we get

$$\frac{y}{x}=C_1\,;$$

the solution of the equation

$$\frac{dx}{x}=\frac{du}{u}$$

is

$$\frac{u}{x}=C_2\,.$$

Now we can find the general integral of the given equation:

$$\Phi\left(\frac{y}{x},\frac{u}{x}\right)=0$$

or

$$\frac{u}{x}=\psi\left(\frac{y}{x}\right),$$

so

$$u = x\psi\left(\frac{y}{x}\right),$$

where ψ is an arbitrary function.

Example 2

Find the general integral of the equation

$$\left(x^2 + y^2\right)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 0 .$$

Solution:

Let's write down a system of equations

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{du}{0} .$$

Using the property of proportion, we present the equation

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy}$$

as

$$\frac{dx + dy}{x^2 + y^2 + 2xy} = \frac{dx - dy}{x^2 + y^2 - 2xy},$$

$$\frac{d(x+y)}{(x+y)^2} = \frac{d(x-y)}{(x-y)^2}.$$

Integrating, we get

$$-\frac{1}{x+y} = -\frac{1}{x-y} + C_1,$$

$$\frac{1}{x-y} - \frac{1}{x+y} = C_1,$$

$$\frac{2y}{x^2 - y^2} = C_1 .$$

The last equality can be rewritten as

$$\frac{y}{x^2-y^2}=C_1.$$

The second equation of the system:

$$du=0.$$

$$u=C_2.$$

The general integral of a given equation has the form

$$\Phi\left(\frac{y}{x^2-y^2},u\right)=0,$$

or

$$u=\psi\left(\frac{y}{x^2-y^2}\right),$$

where ψ is an arbitrary function.

Example 3

Find solutions to the equation

$$(2y-u)\,u'_x+y\,u'_y=u.$$

Solution:

Let's make up a characteristic system

$$\frac{dx}{2y-u}=\frac{dy}{y}=\frac{du}{u}.$$

Solving the equation

$$\frac{dy}{y}=\frac{du}{u}\qquad\Rightarrow\qquad\frac{u}{y}=C_1,$$

we find the first integral

$$\psi_1(x,y,u)=\frac{u}{y}.$$

Using the rule of equal fractions, we will make an integrable combination

$$\frac{dx}{2y-u} = \frac{2dy-du}{2y-u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

From where we get another first integral

$$\psi_2(x, y, u) = x - 2y + u.$$

Therefore, the general integral of the given equation has the form

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0,$$

where Φ is an arbitrary continuously differentiable function.

Example 4

Find a general solution to the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2.$$

<u>Solution:</u>

Let's make up the equations of characteristics:

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-x^2 - y^2}.$$

The first equation of this system can be solved separately from the second, since it does not contain u (the variable u, which stands in the left and right sides of this equation, is reduced).

From equality,

$$\frac{dx}{x} = \frac{dy}{y}$$

we obtain by integrating

$$\ln|x| = \ln|y| + \ln C,$$

from where we find the first integral of the system

$$\frac{y}{x} = C_1.$$

The second equation of this system

$$\frac{dx}{xu} = \frac{du}{-x^2 - y^2}$$

contains all three variables. To exclude the variable y, let's use the first integral found.

Since the desired integral curve lies on one of the surfaces defined by the first integral found, at each point of this curve $y = C_1 x$ (the value of the constant C_1 is the same at all points of the desired integral curve, but may differ if you switch to another integral curve).

Replace y with $C_1 x$ in the second equation, and after the transformations we get

$$-(1 + C_1^2)x\,dx = u\,du.$$

Integrating, we find the dependence

$$(1 + C_1^2)x^2 + u^2 = C_2.$$

This ratio containing C_2 is not the first integral, since it also contains an arbitrary constant C_1. Given that for the found curve

$$C_1 = \frac{y}{x},$$

we rewrite the ratio as

$$x^2 + y^2 + u^2 = C_2.$$

In this form of notation, the relation is performed for any of the integral curves, that is, it is the first integral.

The general solution of the first-order equation has the form (in an implicit form)

$$\Phi\left(\frac{y}{x}, x^2 + y^2 + u^2\right) = 0,$$

where Φ is an arbitrary differentiable function. It is possible to get an explicit solution from the last expression:

$$u = \pm\sqrt{f\left(\frac{y}{x}\right) - x^2 - y^2},$$

where f is an arbitrary differentiable function.

Example 5

Find a general solution to the partial differential equation

$$e^x \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = ye^x$$

Solution:

Creating a system

$$\frac{dx}{e^x} = \frac{dy}{y^2} = \frac{du}{ye^x}$$

From the first equation, we find one first integral

$$\frac{1}{y} - e^{-x} = C_1$$

and from the second, taking into account the equality

$$e^x = \frac{y}{1 - yC_1}$$

another first integral

$$u - \frac{\ln|y| - x}{e^{-x} - y^{-1}} = C_2$$

follows.

Thus, the general integral of this equation will be

$$\Phi\left(\frac{1}{y} - e^{-x}, \frac{\ln|y| - x}{e^{-x} - y^{-1}} - u\right) = 0$$

The general solution has the form

$$u = \frac{\ln|y| - x}{e^{-x} - y^{-1}} + \varphi\left(\frac{1}{y} - e^{-x}\right).$$

Example 6

Find a solution to the equation

$$u\frac{\partial u}{\partial x} + (u^2 - x^2)\frac{\partial u}{\partial y} + x = 0$$

under additional conditions:

a) $y = 2x^2,\ u = x;$

b) $y = 1 + x^2, \quad u = x.$

<u>Solution:</u>

Let's write down the equations of characteristics

$$\frac{dx}{u} = \frac{dy}{u^2 - x^2} = \frac{du}{-x}.$$

Let's find the first integral of the system:

$$\frac{dx}{u} = \frac{du}{-x};$$

$$x\,dx + u\,du = 0;$$

$$x^2 + u^2 = C_1.$$

To determine the next integral, we take

$$\frac{x\,dx + u\,du}{u^2 - x^2} = \frac{dy}{u^2 - x^2}.$$

Comparing the first and second relations, we get

$$x\,dx + u\,du = dy,$$

or

$$d(xu) = dy.$$

Another first integral has been found

$$xu - y = C_2$$

The general solution

$$\Phi(x^2 + u^2, xu - y) = 0.$$

a) Solving the Cauchy problem, it is convenient to take x as the parameter σ on this curve. Substituting into the first integrals x,

$$y = 2x^2,\ u = x,$$

we get

$$C_1 = 2x^2,\ C_2 = -1.$$

Therefore,

$$C_1 = -2C_2.$$

Then the particular solution has the form

$$x^2 + u^2 = -2xu + 2y,$$

or

$$(x + u)^2 = 2y.$$

b) Let's solve the problem under another initial conditions.

Substituting $x,\ y = 1 + x^2,\ u = x$ into the first integrals, we get $C_1 = 2x^2$, $C_2 = -1$. The independence of C_2 from x means that the curve given by the initial conditions lies on the surface $xu - y = -1$.

At the same time, C_1 depends on x, which means that different surfaces of the first family $x^2 + u^2 = C_1$ correspond to different points of a given line. Therefore, the line specified by the initial conditions is not a characteristic, and $xu - y = -1$ is the only integral surface satisfying the initial conditions.

Homework №6. (The deadline is the 4th of October).

Solve the Cauchy problem for the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xy,$$

satisfying the conditions $y = x, u = x^2$.