

## 4 Roots of unity and their applications

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### 4.1 Roots of unity

**Theorem 4.1.** *Let  $w = r(\cos \varphi + i \sin \varphi) \in \mathbb{C}^*$ ,  $n \in \mathbb{N}$ . There are exactly  $n$  different complex numbers  $z$  such that  $z^n = w$ :*

$$z_k = \sqrt[n]{r} \left( \cos\left(\frac{\varphi+2\pi k}{n}\right) + i \sin\left(\frac{\varphi+2\pi k}{n}\right) \right), \quad k = 0, 1, \dots, n-1.$$

*Proof.* If  $z = |z| \cdot (\cos \psi + i \sin \psi)$  then  $z^n = |z|^n \cdot (\cos(n\psi) + i \sin(n\psi)) = r(\cos \varphi + i \sin \varphi)$ . It is equivalent to the fact that  $|z| = \sqrt[n]{r}$  and  $\psi = (\varphi + 2\pi k)/n$ .

It remains to count the number of different  $z_k$ . Notice that  $z_k = z_l$  if and only if the angles  $(\varphi + 2\pi k)/n$  and  $(\varphi + 2\pi l)/n$  are equal. This is equivalent to  $(\varphi + 2\pi k)/n = (\varphi + 2\pi l)/n + 2\pi t$  for  $t \in \mathbb{Z}$ , or  $k - l = tn$ . Clearly, if  $0 \leq k, l \leq n-1$ , this equality does not hold and if  $l$  does not lie in this interval, there exists  $0 \leq k \leq n-1$ , for which this equality is satisfied. It implies that there are exactly  $n$  different  $z_k$ , namely  $z_0, \dots, z_{n-1}$ .  $\square$

**Corrolary 4.2.** *If  $z_0, \dots, z_{n-1}$  are the  $n$ th roots of  $w \in \mathbb{C}$  then  $\prod_{k=0}^{n-1} (x - z_k) = x^n - w$ .*

**Definition.** Let  $n \in \mathbb{N}$ . The complex number  $z \in \mathbb{C}$  is called an  **$n$ th root of unity** if  $z^n = 1$ .

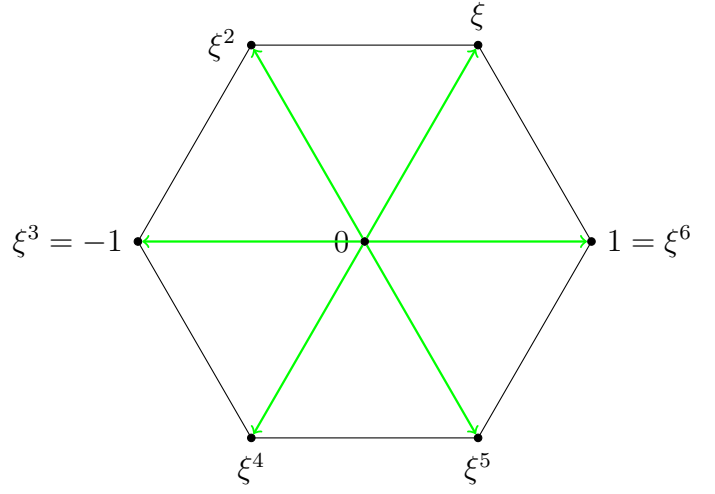
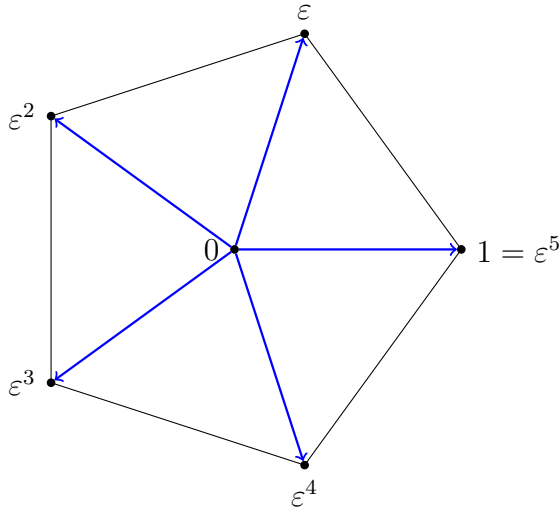
**Proposition 4.3.** *For every  $n \in \mathbb{N}$  there are exactly  $n$   $n$ th roots of unity; namely  $\varepsilon_0, \dots, \varepsilon_{n-1}$ , where*

$$\varepsilon_k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$$

*and  $\varepsilon_k = \varepsilon_1^k$ .*

*Proof.* The formula for  $\varepsilon_k$  immediately follows from Theorem 4.1 given that  $|1| = 1$  and  $\arg(1) = 0$ .  $\square$

*Remark* (Geometric interpretation of roots of unity). From the formula proved above, we see that the moduli of all  $n$ roots of unity are 1, and the arguments are  $0, 2\pi/n, 4\pi/n, \dots, 2(n-1)\pi/n$ , that is, they form an arithmetic progression with a common difference of  $2\pi/n$ . Thus, the points  $\varepsilon_k$  lie on the unit circle with center at 0, and the angle between any two adjacent points is  $2\pi/n$ . Then the points  $\varepsilon_0, \dots, \varepsilon_{n-1}$  are the vertices of a regular  $n$ -gon with center at 0. Moreover, since  $\varepsilon_0 = 1$ , the number 1 is one of its vertices. The picture below shows the 5th and the 6th roots of unity.



**Proposition 4.4.** Let  $z_0$  be a (partial) solution to the equation  $z^n = w$ . Then the general solution of this equation is the set  $\{z_0\varepsilon_0, z_0\varepsilon_1, \dots, z_0\varepsilon_{n-1}\}$  where  $\{\varepsilon_0 = 1, \varepsilon_1, \dots, \varepsilon_{n-1}\}$  are the  $n$ th root of unity.

*Proof.* Let  $z^n = w$ , then  $(z/z_0)^n = z^n/z_0^n = w/w = 1$ . In other words,  $z/z_0$  is an  $n$ th root of unity. Therefore,  $z/z_0 = \varepsilon_k$  for some  $k$ , and  $z = z_0\varepsilon_k$ .

If  $z = z_0\varepsilon_k$  for some  $k$ , then  $z^n = z_0^n(\varepsilon_k)^n = w \cdot 1 = w$ .  $\square$

## 4.2 Applications to trigonometric identities

**Problem 4.5.** Find  $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$

*Solution.* Let  $\varepsilon = \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}$  with  $\varepsilon^n = -1$ . Then  $\varepsilon^{-1} = \cos \frac{\pi}{n} - i \sin \frac{\pi}{n}$  and  $\sin \frac{k\pi}{n} = \frac{1}{2i}(\varepsilon^k - \varepsilon^{-k})$ . Now

$$\begin{aligned} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} &= \frac{1}{(2i)^{n-1}} \prod_{k=1}^{n-1} (\varepsilon^k - \varepsilon^{-k}) = \frac{1}{(2i)^{n-1}} \varepsilon^{-1-2-\dots-(n-1)} \prod_{k=1}^{n-1} (\varepsilon^{2k} - 1) \\ &= \frac{1}{(2i)^{n-1}} \varepsilon^{-\frac{n(n-1)}{2}} \prod_{k=1}^{n-1} (\varepsilon^{2k} - 1) = \frac{1}{(-2)^{n-1}} \prod_{k=1}^{n-1} (\xi^k - 1) = \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} (1 - \xi^k), \end{aligned}$$

where  $\xi = \varepsilon^2 = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$

Now note that  $1, \xi, \dots, \xi^{n-1}$  are all the  $n$ th roots of unity, thus  $(x-1)(x^{n-1} + \dots + x + 1) = x^n - 1 = \prod_{k=0}^{n-1} (x - \xi^k)$ . Cancelling out  $(x-1)$  we have  $\prod_{k=1}^{n-1} (x - \xi^k) = x^{n-1} + \dots + x + 1$ . Finally, substitute  $x = 1$  and get  $\prod_{k=1}^{n-1} (1 - \xi^k) = n$ . Therefore  $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$ .  $\square$

**Problem 4.6.** Find  $\sum_{k=1}^n \tan^2 \frac{\pi k}{2n+1}$  and  $\prod_{k=1}^n \tan \frac{\pi k}{2n+1}$

*Solution.* Denote  $\eta_k = \tan \frac{\pi k}{2n+1}$  for  $-n \leq k \leq n$ . Define  $\varepsilon = \cos \frac{\pi}{2n+1} + i \sin \frac{\pi}{2n+1}$ ,  $\xi = \varepsilon^2 = \cos \frac{2\pi}{2n+1} + i \sin \frac{2\pi}{2n+1}$ . Then  $\varepsilon^k = \cos \frac{\pi k}{2n+1} + i \sin \frac{\pi k}{2n+1}$ ,  $\varepsilon^{-k} = \cos \frac{\pi k}{2n+1} - i \sin \frac{\pi k}{2n+1}$  and

$$\eta_k = \tan \frac{\pi k}{2n+1} = \frac{\varepsilon^k - \varepsilon^{-k}}{i(\varepsilon^k + \varepsilon^{-k})} = \frac{\xi^k - 1}{i(\xi^k + 1)}$$

whence

$$\xi^k = \frac{1 + i\eta_k}{1 - i\eta_k} = \frac{i - \eta_k}{i + \eta_k}$$

and

$$(\eta_k - i)^{2n+1} + (\eta_k + i)^{2n+1} = 0.$$

Then  $\{\eta_k \mid -n \leq k \leq n\}$  are zeros of the polynomial  $P(x) = (x + i)^{2n+1} + (x - i)^{2n+1}$  which implies that  $P$  divides  $\prod_{k=-n}^n (x - \eta_k)$ .

Since  $\deg P = 2n + 1$  and the leading coefficient of  $P$  is 2, one has

$$P(x) = 2 \prod_{k=-n}^n (x - \eta_k) = 2x \prod_{k=1}^n (x^2 - \eta_k^2)$$

because  $\eta_{-k} = -\eta_k$ . On the other hand

$$\begin{aligned} P(x) &= \sum_{\ell=0}^{2n+1} \binom{2n+1}{\ell} i^{2n+1-\ell} x^\ell + \sum_{\ell=0}^{2n+1} \binom{2n+1}{\ell} (-i)^{2n+1-\ell} x^\ell \\ &= \sum_{\ell=0}^{2n+1} \binom{2n+1}{\ell} x^\ell i^{2n+1-\ell} (1 - (-1)^\ell) = 2x \sum_{k=0}^n \binom{2n+1}{2k+1} x^{2k} (-1)^{n-k}. \end{aligned}$$

It follows that

$$\prod_{k=1}^n (x^2 - \eta_k^2) = \sum_{k=0}^n \binom{2n+1}{2k+1} x^{2k} (-1)^{n-k}$$

or

$$\prod_{k=1}^n (t - \eta_k^2) = \sum_{k=0}^n \binom{2n+1}{2k+1} t^k (-1)^{n-k}.$$

Now comparing the coefficients at  $t^{n-1}$  on both sides we get

$$\sum_{k=1}^n \eta_k^2 = \binom{2n+1}{2n-1} = \binom{2n+1}{2} = n(2n+1)$$

or

$$\sum_{k=1}^n \tan^2 \frac{\pi k}{2n+1} = n(2n+1).$$

Comparing the constant terms, we get

$$\prod_{k=1}^n \eta_k^2 = \binom{2n+1}{1} = 2n+1$$

and since  $\prod_{k=1}^n \eta_k > 0$

$$\prod_{k=1}^n \tan \frac{\pi k}{2n+1} = \sqrt{2n+1}.$$

□

**Problem 4.7.** Find

$$\tan \theta + \tan\left(\theta + \frac{\pi}{n}\right) + \tan\left(\theta + \frac{2\pi}{n}\right) + \cdots + \tan\left(\theta + \frac{(n-1)\pi}{n}\right)$$

*Solution.* First, derive a formula which expresses  $\tan nx$  via  $\tan x$ . By de Moivre's formula,  $\cos nx + i \sin nx = (\cos x + i \sin x)^n = \sum_{k=0}^n i^{n-k} \cos^k x \sin^{n-k} x$ . Then

$$\begin{aligned} \sin nx &= \binom{n}{1} \cos^{n-1} x \sin x - \binom{n}{3} \cos^{n-3} x \sin^3 x + \binom{n}{5} \cos^{n-5} x \sin^5 x - \cdots \\ &= \cos^n x \left( \binom{n}{1} \tan x - \binom{n}{3} \tan^3 x + \binom{n}{5} \tan^5 x - \cdots \right) \end{aligned}$$

and

$$\begin{aligned} \cos nx &= \cos^n x - \binom{n}{2} \cos^{n-2} x \sin^2 x + \binom{n}{4} \cos^{n-4} x \sin^4 x - \cdots \\ &= \cos^n x \left( 1 - \binom{n}{2} \tan^2 x + \binom{n}{4} \tan^4 x - \cdots \right) \end{aligned}$$

Finally

$$\tan nx = \frac{\binom{n}{1} \tan x - \binom{n}{3} \tan^3 x + \binom{n}{5} \tan^5 x - \cdots}{1 - \binom{n}{2} \tan^2 x + \binom{n}{4} \tan^4 x - \cdots}$$

Let  $\theta_k = \theta + \frac{k\pi}{n}$ ,  $0 \leq k \leq n-1$ . Then  $\tan n\theta_k = \tan n\theta$  and  $\tan \theta_0, \dots, \tan \theta_{n-1}$  are roots of the polynomial

$$f(t) = \left( \binom{n}{1} t - \binom{n}{3} t^3 + \binom{n}{5} t^5 - \cdots \right) - \left( 1 - \binom{n}{2} t^2 + \binom{n}{4} t^4 - \cdots \right) \tan n\theta.$$

If  $n = 2m$  then

$$\begin{aligned} f(t) &= \left( \binom{2m}{1} t - \binom{2m}{3} t^3 + \cdots + (-1)^{m-1} \binom{2m}{2m-1} t^{2m-1} \right) \\ &\quad - \left( 1 - \binom{2m}{2} t^2 + \cdots + (-1)^m t^{2m} \right) \tan 2m\theta \end{aligned}$$

and by Vieta's formulas,

$$\tan \theta_0 + \cdots + \tan \theta_{n-1} = \frac{(-1)^{m-1} \binom{2m}{2m-1}}{(-1)^m \tan 2m\theta} = -\frac{2m}{\tan 2m\theta}.$$

If  $n = 2m+1$  then

$$\begin{aligned} f(t) &= \left( \binom{2m+1}{1} t - \binom{2m+1}{3} t^3 + \cdots + (-1)^m t^{2m+1} \right) \\ &\quad - \left( 1 - \binom{2m+1}{2} t^2 + \cdots + (-1)^m \binom{2m+1}{2m} t^{2m} \right) \tan(2m+1)\theta \end{aligned}$$

and

$$\tan \theta_0 + \cdots + \tan \theta_{n-1} = \frac{(-1)^m \binom{2m+1}{2m} \tan(2m+1)\theta}{(-1)^m} = (2m+1) \tan(2m+1)\theta.$$

Thus

$$\tan \theta + \tan\left(\theta + \frac{\pi}{n}\right) + \tan\left(\theta + \frac{2\pi}{n}\right) + \cdots + \tan\left(\theta + \frac{(n-1)\pi}{n}\right) = \begin{cases} -\frac{n}{\tan n\theta}, & \text{if } n \text{ is even} \\ n \tan n\theta, & \text{if } n \text{ is odd} \end{cases}$$

□

**Problem 4.8.** *Prove the identity*

$$\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \sqrt{11}.$$

*Solution.* Let  $\omega = \cos \frac{\pi}{11} + i \sin \frac{\pi}{11}$  with  $\omega^{11} = -1$ . Then

$$\sin \frac{2\pi}{11} = \frac{\omega^2 - \omega^{-2}}{2i} = \frac{\omega^4 - 1}{2i\omega^2} = \frac{\omega^4 + \omega^{11}}{2i\omega^2} = \frac{1}{2i}(\omega^2 + \omega^9)$$

and

$$\begin{aligned} \tan \frac{3\pi}{11} &= \frac{\omega^3 - \omega^{-3}}{i(\omega^3 + \omega^{-3})} = \frac{\omega^6 - 1}{i(\omega^6 + 1)} = \frac{\omega^6 - \omega^{66}}{i(\omega^6 + 1)} \\ &= \frac{1}{i}(-\omega^{60} + \omega^{54} - \omega^{48} + \omega^{42} - \omega^{36} + \omega^{30} - \omega^{24} + \omega^{18} - \omega^{12} + \omega^6) \\ &= \frac{1}{i}(\omega^5 + \omega^{10} - \omega^4 - \omega^9 + \omega^3 + \omega^8 - \omega^2 - \omega^7 + \omega + \omega^6) \end{aligned}$$

Then

$$\begin{aligned} \tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} &= \frac{1}{i}(\omega^5 + \omega^{10} - \omega^4 - \omega^9 + \omega^3 + \omega^8 - \omega^2 - \omega^7 + \omega + \omega^6) + \frac{2}{i}(\omega^2 + \omega^9) \\ &= \frac{1}{i}(\omega^{10} + \omega^9 + \omega^8 - \omega^7 + \omega^6 + \omega^5 - \omega^4 + \omega^3 + \omega^2 + \omega) \end{aligned}$$

Since  $\omega^{11} + 1 = 0$ , one has

$$\omega^{10} - \omega^9 + \omega^8 - \omega^7 + \omega^6 - \omega^5 + \omega^4 - \omega^3 + \omega^2 - \omega + 1 = 0$$

Then

$$\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \frac{1}{i}(2\omega^9 + 2\omega^5 - 2\omega^4 + 2\omega^3 + 2\omega - 1)$$

It remains to check that

$$(2\omega^9 + 2\omega^5 - 2\omega^4 + 2\omega^3 + 2\omega - 1)^2 = -11.$$

It is equivalent to the identity

$$\begin{aligned} &4\omega^{18} + 4\omega^{10} + 4\omega^8 + 4\omega^6 + 4\omega^2 + 1 \\ &+ 8\omega^{14} - 8\omega^{13} + 8\omega^{12} + 8\omega^{10} - 4\omega^9 \\ &- 8\omega^9 + 8\omega^8 + 8\omega^6 - 4\omega^5 \\ &- 8\omega^7 - 8\omega^5 + 4\omega^4 \\ &+ 8\omega^4 - 4\omega^3 \\ &- 4\omega = -11 \end{aligned}$$

or

$$12\omega^{10} - 12\omega^9 + 12\omega^8 - 12\omega^7 + 12\omega^6 - 12\omega^5 + 12\omega^4 - 12\omega^3 + 12\omega^2 - 12\omega + 12 = 0$$

which has appeared above.

□

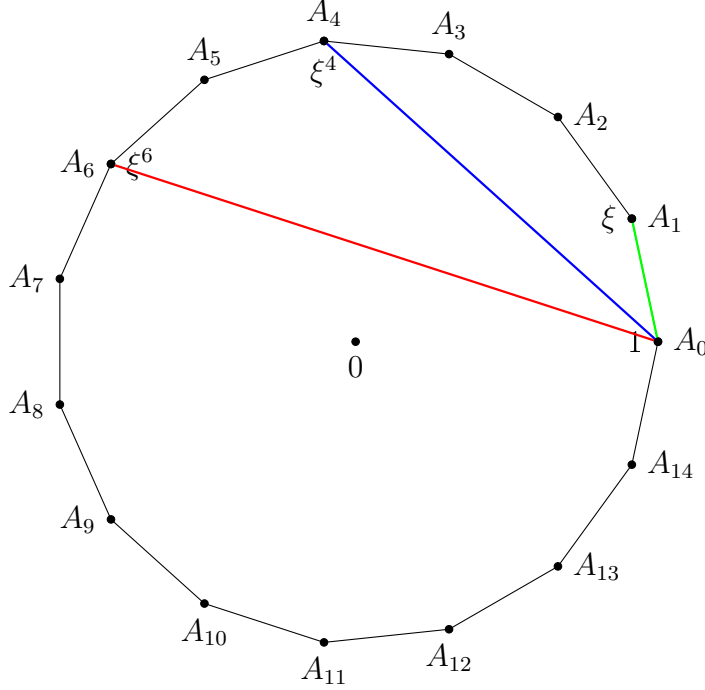
**Problem 4.9.** Prove the identity

$$\tan \frac{3\pi}{7} - 4 \sin \frac{\pi}{7} = \sqrt{7}.$$

### 4.3 Applications to geometry

**Problem 4.10.** For a regular 15-gon  $A_0A_1 \cdots A_{14}$  prove that  $|A_0A_1| = |A_0A_6| - |A_0A_4|$ .

*Solution.* One can assume that the regular 15-gon is inscribed in a circle of radius 1 so that its center and  $A_0$  are associated with the points 0 and 1 on the complex plane, respectively.



Then for  $\xi = \cos \frac{2\pi}{15} + i \sin \frac{2\pi}{15}$ , one needs to prove that

$$|\xi - 1| + |\xi^4 - 1| = |\xi^6 - 1|$$

Squaring both parts gives

$$(\xi - 1)(\bar{\xi} - 1) + (\xi^4 - 1)(\bar{\xi}^4 - 1) + 2|\xi - 1| \cdot |\xi^4 - 1| = (\xi^6 - 1)(\bar{\xi}^6 - 1)$$

since  $|z|^2 = z \cdot \bar{z}$ . This is equivalent to

$$2 - \xi - \bar{\xi} + 2 - \xi^4 - \bar{\xi}^4 + 2|\xi - 1| \cdot |\xi^4 - 1| = 2 - \xi^6 - \bar{\xi}^6$$

or

$$2|\xi^5 - \xi^4 - \xi + 1| = \xi + \xi^{14} + \xi^4 + \xi^{11} - 2 - \xi^6 - \xi^9$$

since  $\xi\bar{\xi} = |\xi|^2 = 1$ .

Further,  $0 = \xi^{15} - 1 = (\xi^5 - 1)(\xi^{10} + \xi^5 + 1)$ , whence  $\xi^{10} + \xi^5 + 1 = 0$ . It implies

$$2|\xi^5 - \xi^4 - \xi + 1| = -2\xi^6 - 2\xi^9 - 2$$

or

$$|\xi^5 - \xi^4 - \xi + 1| = -\xi^6 - \xi^9 - 1.$$

Squaring both parts gives

$$(\xi^5 - \xi^4 - \xi + 1)(\xi^{10} - \xi^{11} - \xi^{14} + 1) = \xi^{12} + \xi^3 + 1 + 2 + 2\xi^9 + 2\xi^6$$

or

$$1 - \xi - \xi^4 + \xi^5 - \xi^{14} + 1 + \cancel{\xi^8} - \xi^4 - \xi^{11} + \cancel{\xi^{12}} + 1 - \xi + \xi^{10} - \xi^{11} - \xi^{14} + 1 = \cancel{\xi^{12}} + \cancel{\xi^8} + 1 + 2 + 2\xi^9 + 2\xi^6$$

Using the identity  $\xi^{10} + \xi^5 + 1 = 0$ , one gets

$$-\xi - \xi^4 - \xi^{11} - \xi^{14} = \xi^9 + \xi^6$$

which follows from the same identity. □

**Problem 4.11.** *Prove that the length of a side of a regular 9-gon is equal to the difference of the lengths of its longest and shortest diagonals.*