Points and Coordinates on Plane and in Space

1 Base problems of analytic geometry

While building the axiomatic system we defined primitive notion of geometric figure as "*locus for collection of points*" and postulated that "*for any explicit (and verifiable) law acting on a collection of points exists a figure which contains all points satisfying underlined law and only these points*".

Locus here (Latin word for "place", "location") is a set of all points, whose location in space satisfies or is determined by one or more specified conditions.

Two base problems analytic geometry most often deals with are:

1. Given an equation, to find the corresponding locus.

2. Given a locus defined by some geometrical condition, to find the corresponding equation.

Before we start composing and solving these equations, we need to organize our knowledge about coordinates of points and their relations on plane and in space as we heave done before for the straight line.

We will operate most often with *Cartesian coordinates*, but sometimes also with skew-angular bases. This cases will be underlined.

We usually express position of the point on plane with coordinates (pair of numbers on plane or triplet in space) of corresponding radius vector.

Key advantage of the rectangular Cartesian coordinate system is correspondence between vectorial-natured operations and well known from high-school geometry relations in right triangle.

Let us organize that elementary actions with coordinates before we start with equations.

2 Points on plane

We use two real numbers (coordinates) to describe position of the point on plane. We will most often express first coordinate in the basis as x and sometimes call it **abscissa**. Second coordinate will be denoted as y, and sometimes called **ordinate**.

In rectangular Cartesian coordinate system the point (P) shapes rectangle with sides length equal with coordinates (fig. 1 a). In skew-angular coordinates this pictures transforms into parallelogram (fig. 1 b). For this case we will assume that basis vectors are unit ones and smaller angle from first vector to second is $\angle xOy = \varphi$.

It may sometimes be convenient to choose the unit of measure for the abscissa of a point different from the unit of measure for the ordinate. Thus, if the same unit, say one cm, were taken for abscissa and ordinate, the point (3, 48) might fall beyond the limits of the paper. To avoid this we may lay off the ordinate on a scale of one mm. When different units are used, the unit used on each axis should always be indicated in the drawing. When nothing is said to the contrary, the units for abscissas and ordinates are always understood to be the same.

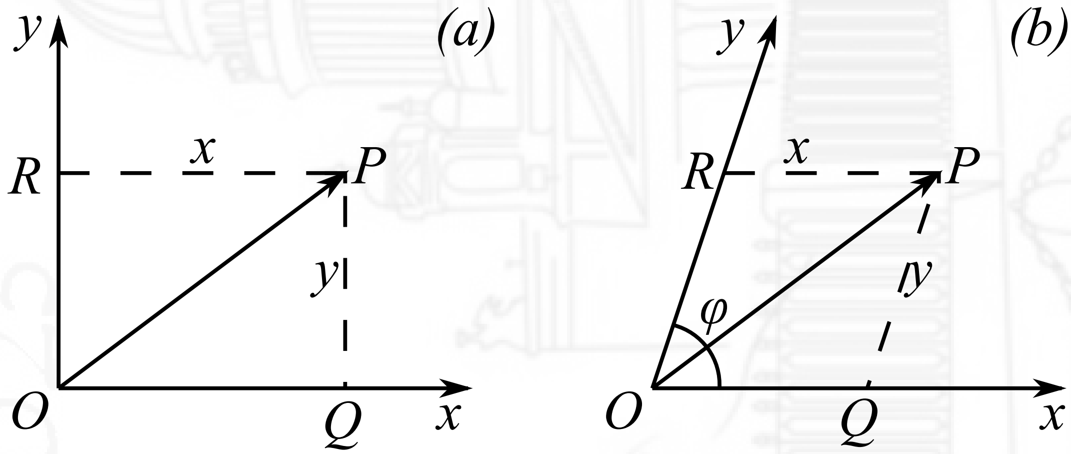


Figure 1: Point in rectangular (a) and skew-angular coordinates (b)

3 Distance of a Point from the Origin

This distance is length of corresponding radius vector.

$$r = OP = |\overrightarrow{OP}| = \sqrt{\overrightarrow{OP} \cdot \overrightarrow{OP}}$$

For rectangular coordinates its calculation is equal with next procedure

For the distance $r = OP$ of the point P from the origin O we calculate length of vector have from the right-angled triangle OQP:

$$r = \sqrt{x^2 + y^2}$$

If the basis is skew-angular we have, from the triangle OQP, in which the angle at Q is equal to $\pi - \varphi$ by the cosine law of trigonometry

In both cases two sides of triangle are parallel with one of coordinate axis.

$$r = \sqrt{x^2 + y^2 + 2xy\cos\varphi}$$

4 Distance between Two Points

Distance $d = P_1P_2$ between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ corresponds with the length of vector $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$ or $\overrightarrow{P_2P_1} = -\overrightarrow{P_2P_1}$

Expression in rectangular coordinates means length of hypotenuse in right-angled triangle $\triangle P_1P_2Q$, one of legs P_1Q or P_2Q is parallel with Ox and second is parallel with Oy

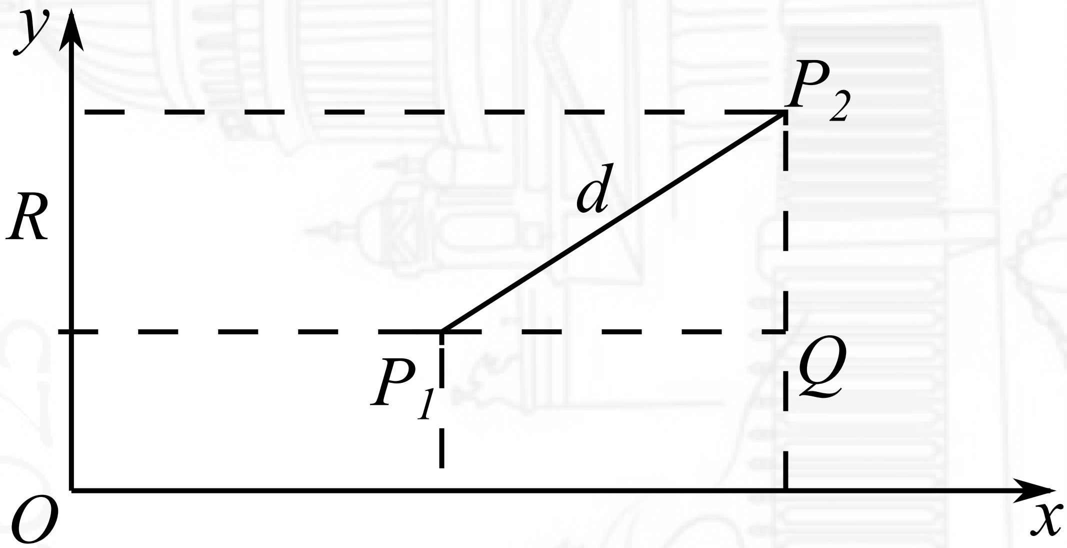


Figure 2: Distance between two points

$$P_1Q = x_2 - x_1, QP_2 = y_2 - y_1$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

The distance between any two points is equal to the square root of the sum of the squares of the differences between their corresponding coordinates

For skew-angular basis we have formulas

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\varphi} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + 2\Delta x \Delta y \cos\varphi}$$

5 Ratio of Division

If two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are given by their coordinates, the coordinates x, y of any point P on the line P_1P_2 can be found if the division ratio $\frac{P_1P}{P_1P_2} = k$ is known in which the point P divides the segment P_1P_2.

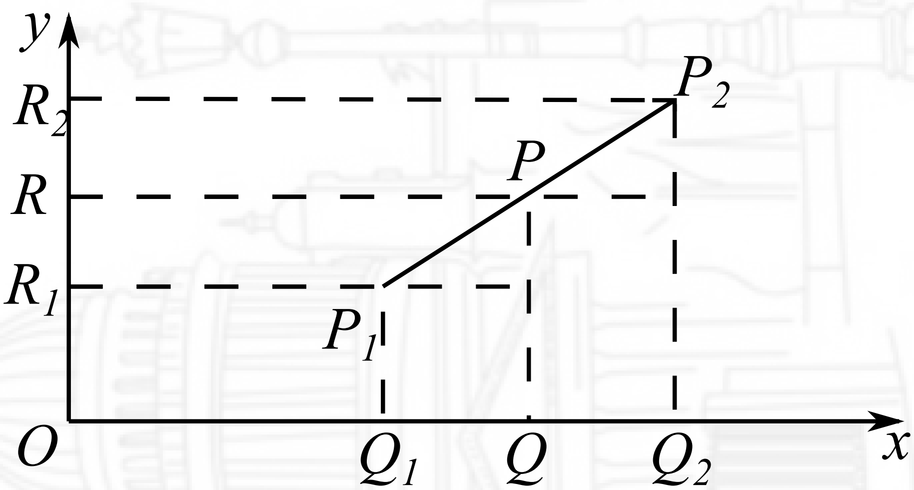


Figure 3: Ratio of division

Observe abscissa. Let $OQ_1 = x_1$, $OQ_2 = x_2$, $OQ = x$. Then $x = x_1 + k(x_2 - x_1)$
The same for ordinate $y = y_1 + k(y_2 - y_1)$
Or, with differences Δx and Δy

$$x = x_1 + k\Delta x \quad y = y_1 + k\Delta y$$

If the division ratio k is negative, the two segments P_1P_2 and P_1P must have opposite sense, so that the points P and P_2 must lie on opposite sides of the point P_1

Ratio of division may also be expressed in form $\frac{P_1P}{PP_2} = r$

$$\frac{Q_1Q}{QQ_2} = \frac{x - x_1}{x - x_2} = r \quad x = \frac{x_1 + rx_2}{1 + r}$$
$$\frac{R_1R}{RR_2} = \frac{y - y_1}{y - y_2} = r \quad y = \frac{y_1 + ry_2}{1 + r}$$

Since P_1P and PP_2 are read in the same direction on the line, the ratio will be positive. If the point of division $P(x, y)$ were on the segment extended in either direction, then the ratio would be negative because P_1P and PP_2 would then have opposite directions.

6 Midpoint of a Segment

The midpoint P of a segment P_1P_2 has for its coordinates the arithmetic means of the corresponding coordinates of P_1 and P_2

Coefficient $k = \frac{1}{2}$

$$x = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2) \quad y = y_1 + \frac{1}{2}(y_2 - y_1) = \frac{1}{2}(y_1 + y_2)$$

Coefficient $r = 1$ in this case

7 Transition of Coordinates

The change from one set of axes to a new set is called a **transformation of coordinates**.

Instead of the origin and the axes Ox, Oy, let us select a new origin $0'$ and new axes $0'x'$ $O'y'$, parallel to the old axes.

Then any point P whose coordinates with reference to the old axes are $OQ = x$, $QP = y$ will have with reference to the new axes the coordinates $O'Q' = x'$, $Q'P' = y'$, and if h, k are the coordinates of the new origin O', then

$$x = x' + h$$
$$y = y' + k$$

In the present case, where the new axes are parallel to the old, this transformation can be said to consist in a translation of the axes.

8 Example. Area of Triangle

For example, area of triangle is $\frac{1}{2}$ of the length of cross product of any two free vectors coincide two its sides:

This area may also be explained as area of rectangle minus areas of tree non-shaded triangles (see fig 4). Let one vertex of a triangle be the origin O, and let the other vertices be $P_1(x_1, y_1)$ and $P_2(x_1, y_2)$.

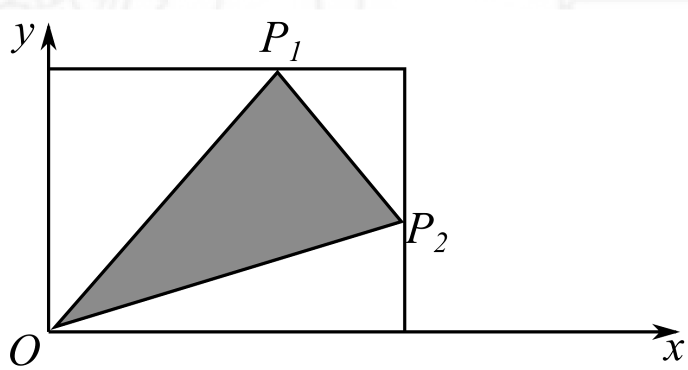


Figure 4: Area of triangle

$$S = x_1y_2 - \frac{1}{2}(x_1y_1 + x_2y_2 + (x_2 - x_1)(y_2 - y_1)) = \frac{1}{2}(x_1y_2 - x_2y_1) = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

Suppose third vertex of triangle is point $O'(x_3, y_3)$. Transition of the origin into this point yields.

$$x_1' = x_1 - x_3 \quad y_1' = y_1 - y_3$$
$$x_1' = x_1 - x_3 \quad y_1' = y_1 - y_3$$

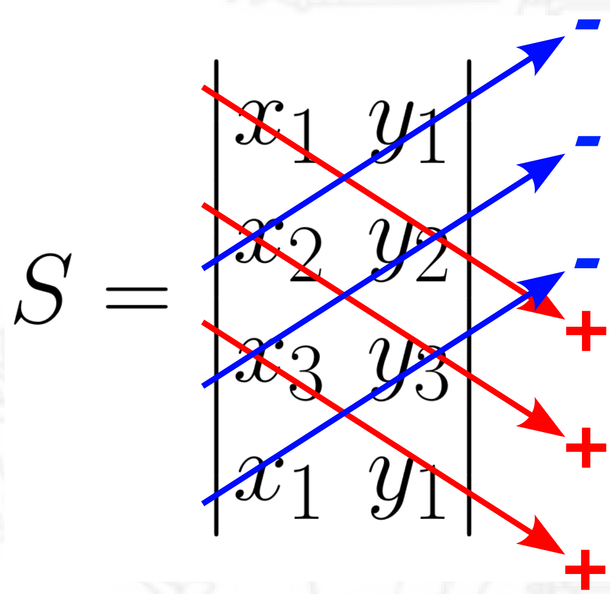
and

$$S = \begin{vmatrix} x_1 - x_3 & x_1 - x_3 \\ x_1 - x_3 & y_1 - y_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Simplification of the expression yields

$$S = \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_3y_2 - x_2y_1)$$

This expression has nice interpretation: Area in this formula is signed and corresponds with



direction of cross product: "to us", or "from us".

Area of any polygon may be expressed as sum of areas of triangles.

In using analytic geometry to prove general geometric propositions, it is generally convenient to select as origin a prominent point in the geometric figure, and as axes of coordinates prominent lines of the figure. But sometimes greater symmetry and elegance is gained by taking the coordinate system in a general position.

9 Inclination and Slope of Line

The **inclination** of a line L(not parallel to the x-axis) is defined as the smallest positive angle measured from the positive direction of the x-axis in a counterclockwise direction to L.

Unless otherwise stated the positive direction of L will be considered upward.

If L is parallel to the x-axis its inclination is defined as zero.

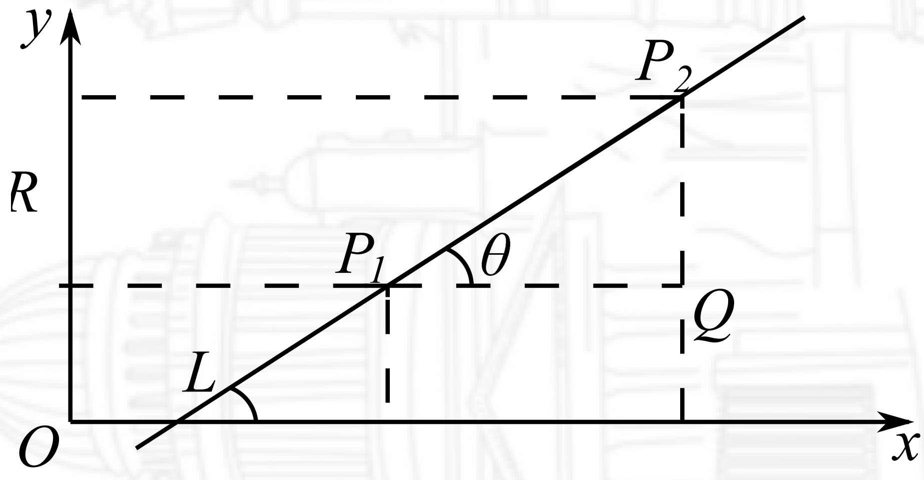


Figure 5: Inclination and slope of Line

The **slope** of a line is defined as the tangent of the angle of inclination.

Thus, $m = \tan\theta$ where θ is the inclination, and m is the slope.

The slope of a line passing through two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

If two lines are parallel, their slopes are equal.

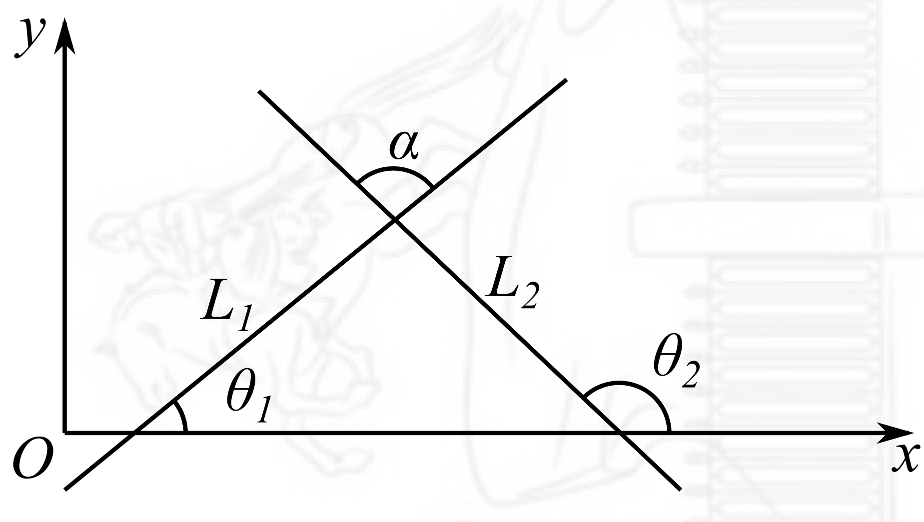


Figure 6: Crossing lines

For a pair of crossing lines, the tangent of angle α measured in a positive direction (counterclockwise), from the line L_1 whose slope is m_1, to the line L_2 whose slope is m_2 is

$$\tan\alpha = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Proof of this statement is simple expression for tangent of sum:

$$\theta_2 = \alpha + \theta_1 \quad \alpha = \theta_2 - \theta_1$$

$$\tan\alpha = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1 \tan\theta_2} = \frac{m_2 - m_1}{1 - m_1 m_2}$$

In expression $\tan\varphi = \dfrac{\sin\varphi}{\cos\varphi}$ right angle yields zero in denominator.

If two lines L_1 and L_2 are perpendicular, the slope of one of the lines is the negative reciprocal of the slope of the other line. Thus, if m_1 is the slope of l_1 and m_2 is the slope of L_2, then $m_1 = -\dfrac{1}{m_2}$, or $m_1 m_2 = -1$.

Let points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ lay on a line L, and points $P_1'(x_1', y_1')$ and $P_2'(x_2', y_2')$ lay on a line L. The fact that lines L and L' are perpendicular has expression:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{x_1' - x_2'}{y_2' - y_1'}$$

10 Points and coordinates on plane: problems corner

Problem 1

Determine the distance between (a) (-2,3) and (5,1), (b) (6,-1) and (-4,-3)

Solution

(a) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(5+2)^2 + (1-3)^2} = \sqrt{49+4} = \sqrt{53}$
(b) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-4-6)^2 + (-3+1)^2} = \sqrt{104} = 2\sqrt{26}$

Problem 2

(a) Show that the points $A(7,5)$, $B(2,3)$, $C(6,-7)$ are the vertices of a right triangle (b) Find the area of the right triangle

Solution

$AB = \sqrt{(7-2)^2 + (5-3)^2} = \sqrt{29}$ $\quad BC = \sqrt{(2-6)^2 + (3+7)^2} = \sqrt{116}$
$AC = \sqrt{(7-6)^2 + (5+7)^2} = \sqrt{145}$
$AB^2 + BC^2 = 29 + 116 = 146 = AC^2$. Triangle is right. Alternative evidence:
$\overrightarrow{BA} \cdot \overrightarrow{BC} = (7-2)(6-2) + (5-3)(-7-3) = 5 \cdot 4 - 2 \cdot 10 = 0$

Problem 3

Show that the following points lie in a straight line: $A(-3,-2)$, $B(5,2)$, $C(9,4)$.

Solution

Laying on a single line for a triplet of points means that triangle inequality for them becomes a valid equality. To check it we calculate segments lengths
$AB = \sqrt{(5+3)^2 + (2+2)^2} = 4\sqrt{5}$ $\quad BC = \sqrt{(9-5)^2 + (4-2)^2} = 2\sqrt{5}$
$AC = \sqrt{(9+3)^2 + (4+2)^2} = 6\sqrt{5}$
$AB + BC = 4\sqrt{5} + 2\sqrt{5} = 6\sqrt{5} = AC$

Problem 4

Determine the point which is equidistant from points $A(1,7)$, $B(8,6)$, $C(7,-1)$.

Solution

Let $P(x,y)$ be the required point. Then $PA = PB = PC$, or $PA = PB$ and $PA = PC$.
We will write squared equations to make them more clear.
$PA^2 = PB^2$ means $(x-1)^2 + (y-7)^2 = (x-8)^2 + (y-6)^2$
$PA^2 = PC^2$ means $(x-1)^2 + (y-7)^2 = (x-7)^2 + (y+1)^2$
This yields system of equations:

$$\begin{cases} x^2 - 2x + 1 + y^2 - 14y + 49 - x^2 + 16x - 64 - y^2 + 12y - 36 = 0 \\ x^2 - 2x + 1 + y^2 - 14y + 49 - x^2 + 14x - 49 - y^2 - 2y - 1 = 0 \end{cases}$$

$$\begin{cases} 14x - 2y - 50 = 0 \\ 12x - 16y = 0 \end{cases}$$

$$\begin{cases} 7x - y - 25 = 0 \\ 3x - 4y = 0 \end{cases}$$

To solve it we multiply first equation by 4 and replace $4y$ with $3x$:

$$28x - 3x - 100 = 0$$
$$x = 4$$
$$y = 3$$

Hence, the required point is $(4, 3)$.

Problem 5

Determine the coordinates of the point $P(x, y)$ which divides the line segment from $P_1(-2, 1)$ to $P_2(3, -4)$ externally in the ratio $r = -8/3$.

Solution

Since the ratio is negative, P_1P and PP_2 have opposite directions outside segment P_1P_2 (external division)

$$x = \frac{x_1 + rx_2}{1 + r} = \frac{-2 + 3\left(-\frac{8}{3}\right)}{1 - \frac{8}{3}} = 6$$
$$y = \frac{y_1 + ry_2}{1 + r} = \frac{1 - 4\left(-\frac{8}{3}\right)}{1 - \frac{8}{3}} = -7$$

P has coordinates $(6, -7)$

Problem 6

A circle with center at $P_1(-4, 1)$ has one end of a diameter at $P_2(2, 6)$. Determine the coordinates $P(x, y)$ of the other end of diameter.

Solution

Points P and P_2 lay on opposite sides of the point P_1, thus $k < 0$. Center of the circle is midpoint for its diameter, thus $k = \dfrac{P_1P}{PP2} = -\dfrac{1}{2}$

$$x = \frac{x_1 + rx_2}{1+r} = \frac{-4 + 2\left(-\frac{1}{2}\right)}{1 - \frac{1}{2}} = -10$$

$$y = \frac{y_1 + ry_2}{1+r} = \frac{1 + 6\left(-\frac{1}{2}\right)}{1 - \frac{1}{2}} = -4$$

Problem 7

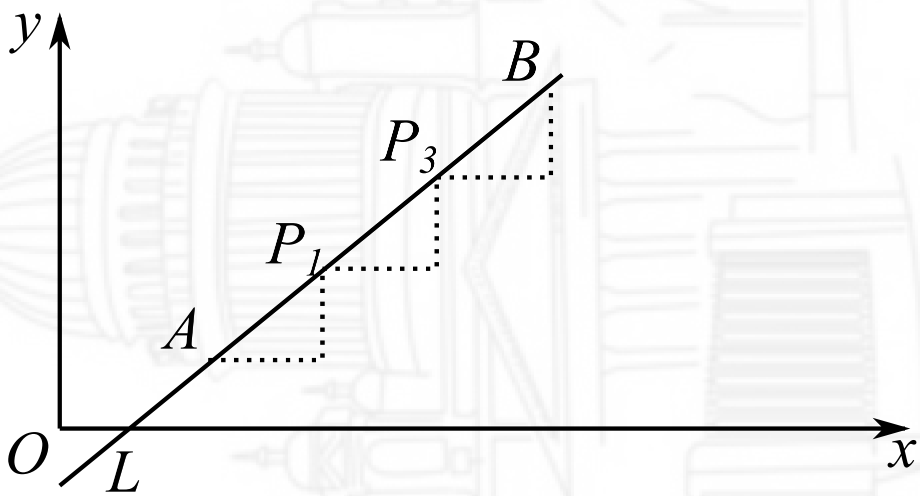


Figure 7: Points of trisection: $AP_1 = P_1P_2 = P_2B$

Determine the two points of trisection, $P_1(x_1, y_2)$ and $P_2(x_2, y_2)$, of the line segment joining $A(3, -1)$ and $B(9, 7)$.

Solution
Looking for P_1:

$$r_1 = \frac{AP_1}{P_1B} = \frac{1}{2}$$

$$x_1 = \frac{3 + 9\frac{1}{2}}{1 + \frac{1}{2}} = 5 \quad y_1 = \frac{-1 + 7\frac{1}{2}}{1 + \frac{1}{2}} = \frac{5}{3}$$

Looking for P_2:

$$r_2 = \frac{AP_1}{P_1B} = \frac{2}{1}$$

$$x_2 = \frac{3 + 9 \cdot 2}{1+2} = 7 \quad y_2 = \frac{-1 + 7 \cdot 2}{1+2} = \frac{13}{3}$$

Problem 8

The medians of a triangle intersect in a point $P(x, y)$, called the *centroid of the triangle*, which is $2/3$ of the distance from any vertex to the midpoint of the opposite side. Find the coordinates of $P(x, y)$ if the vertices of the triangle are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$

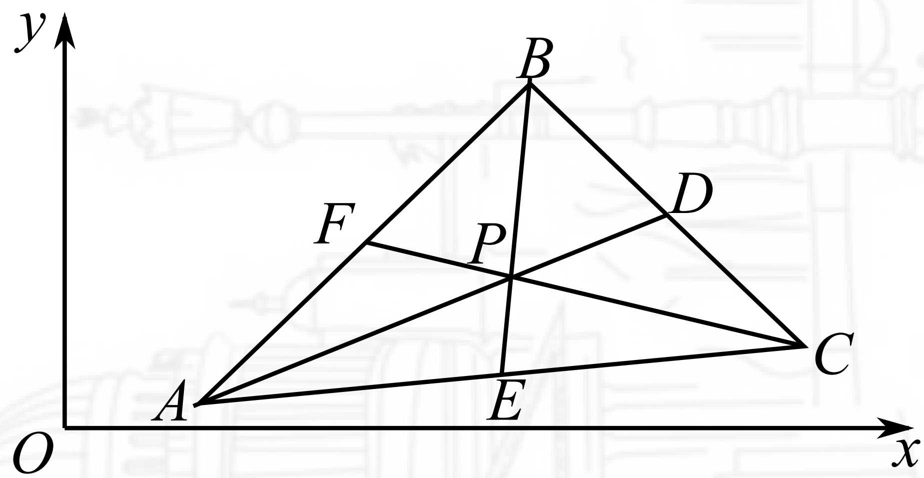


Figure 8: Centroid of triangle

Solution

Let us observe median AD, $P \in AD$, $\dfrac{AP}{AD} = \dfrac{2}{3} = k$.

The coordinates of D are $\left(\dfrac{x_2 + x_3}{2}, \dfrac{y_2 + y_3}{2}\right)$, as it is a midpoint of BC

$$x = x_1 + k\Delta x = x_1 + \frac{2}{3}\left(\frac{x_2 + x_3}{2} - x_1\right) = \frac{1}{6}(6x_1 + 2x_2 + 2x_3 - 4x_1) = \frac{x_1 + x_2 + x_3}{3}$$
$$y = \frac{y_1 + y_2 + y_3}{3}$$

Problem 9

Find the slope m and the angle of inclination θ of the lines through each of the following pairs of points $A(-11, 4)$ and $B(-11, 10)$, and $C(8, 6)$ and $D(14, 6)$

Solution

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For first pair we have 0 in denominator, thus angle is $\theta = \frac{\pi}{2}$
For second pair we have 0 in numerator, thus angle is $\theta = 0$

Problem 10

Find internal angles for a triangle $\triangle ABC$ with vertices $A(-3, -2)$, $B(2, 5)$ and $C(4, 2)$

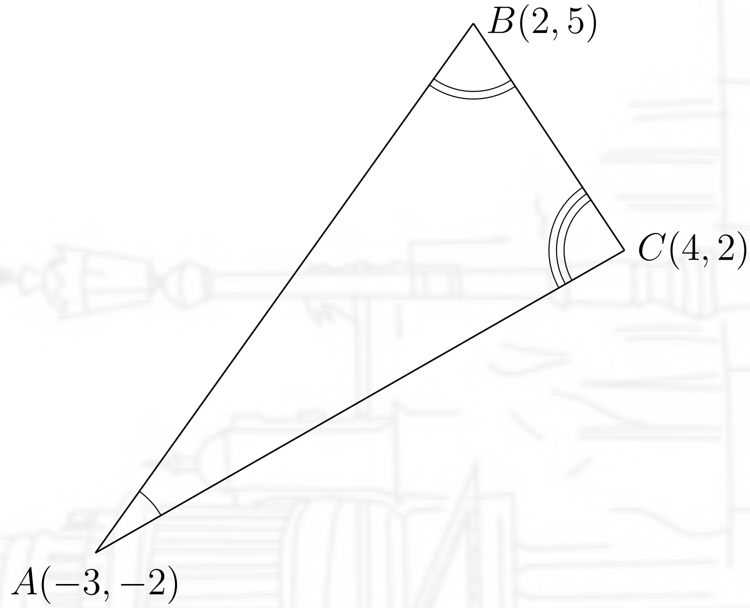


Figure 9: Angles of triangle

Solution
We will stop with tangents.
Slopes of the sides are

$$m_{AB}=\frac{5+2}{2+3}=\frac{7}{5}\quad m_{BC}=\frac{2-5}{4-2}=-\frac{3}{2}\quad m_{CA}=\frac{2+2}{4+3}=\frac{4}{7}$$

$$\tan A=\frac{m_{AB}-m_{CA}}{1+m_{AB}m_{CA}}=\frac{\frac{7}{5}-\frac{4}{7}}{1+\frac{7}{5}\cdot\frac{4}{7}}=\frac{29}{63}$$
$$\tan B=\frac{m_{BC}-m_{AB}}{1+m_{BC}m_{AB}}=\frac{29}{11}$$
$$\tan C=\frac{m_{CA}-m_{BC}}{1+m_{CA}m_{BC}}=\frac{29}{2}$$

After obtaining these tangents we are free to estimate angles and check if their sum is π.

Problem 11
Find the area S of the pentagon whose vertices are $(-5,-2)$, $(-2,5)$, $(2,7)$, $(5,1)$, $(2,-4)$

Solution
We split the polygon into triangles: first one with vertices $(x_1,y_1)=(-5,-2)$, $(x_2,y_2)=(-2,5)$, $(x_3,y_3)=(2,7)$, second one with vertices $(x_3,y_3)=(2,7)$, $(x_4,y_4)=(5,1)$, $(x_5,y_5)=(2,-4)$, and third one with vertices $(x_1,y_1)=(-5,-2)$ $(x_3,y_3)=(2,7)$, $(x_5,y_5)=(2,-4)$ While we calculate each area, we obtain

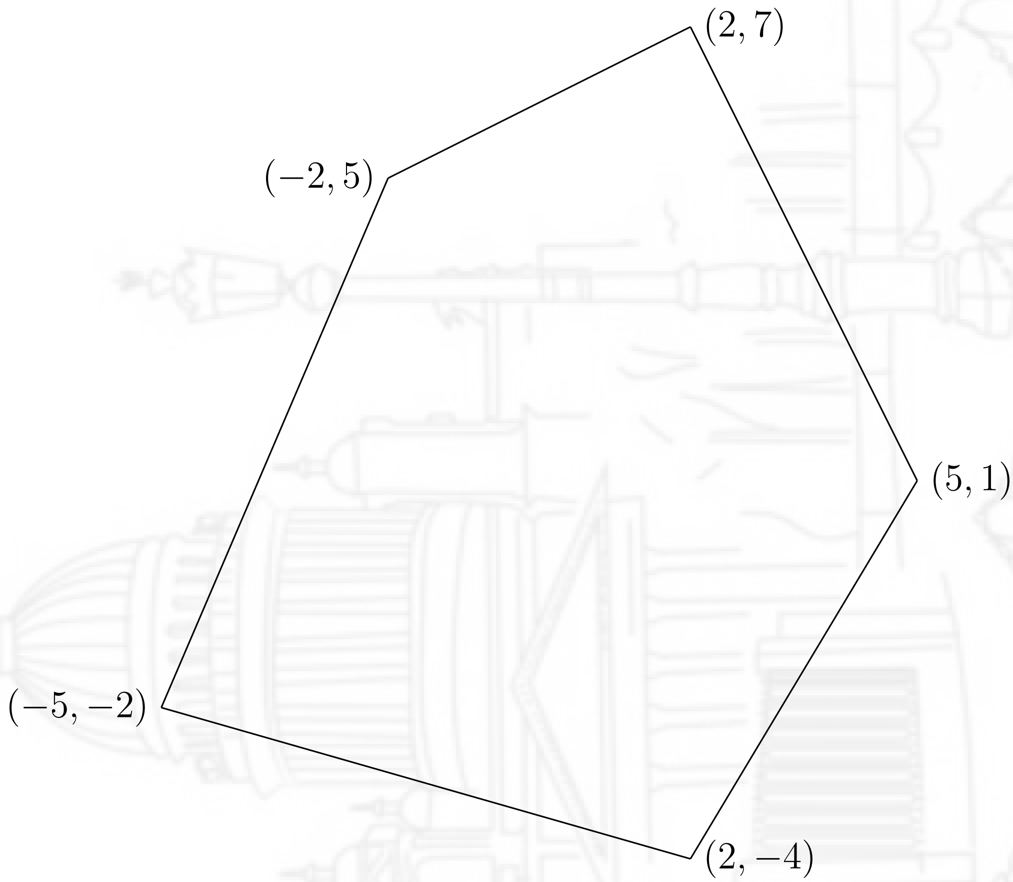


Figure 10: Pentagon

$$2S = 2S_1 + 2S_2 + 2S_3 = (x_1y_2 + x_2y_3 + x3_y1 - x_1y_3 - x_3y_2 - x_2y_1) + (x_4y_4 + x_4y_5 + x5_y3 - x_3y_5$$

Combining of terms yields approach similar with area of triangle:

$$S = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_1 & y_1 \end{vmatrix}$$

11 Points in space

We use three real numbers (coordinates) to describe position of the point space. We will most often express first coordinate in the basis as x and sometimes call it **abscissa**. Second coordinate will be denoted as y, and sometimes called **ordinate**. Third coordinate will be denoted as z and sometimes called **applicate**.

Point shapes in coordinate system parallepiped with legs parallel to corresponding coordinate axes.

Direction of the radius vector of point is defined with directing cosines which are cosines of angles between the vector and positive direction of each axis.

Directing cosines satisfy Pythagorean relation:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Coordinates of arbitrary point may be expressed as

$$x = \rho\cos\alpha \quad y = \rho\cos\beta \quad z = \rho\cos\gamma$$

Here ρ is length of radius vector.

Sometimes coordinates of any radius vector codirected with investigated are called **direction numbers**. Directions cosines in this terms are direction numbers of unit radius vector.

12 Distance of a point from the origin

Distance of a point from the origin is exactly length of corresponding radius vector. In Cartesian coordinates this distance for point $P(x, y, z)$ is expressed as

$$OP = |\overrightarrow{OP}^2| = \sqrt{x^2 + y^2 + z^2}$$

Thus, expression for directing cosines is

$$\cos\alpha = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \quad \cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \quad \cos\gamma = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

13 Distance between two points.

Distance $d = P_1P_2$ between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ corresponds with the length of vector $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$ or $\overrightarrow{P_2P_1} = -\overrightarrow{P_2P_1}$

Expression in rectangular coordinates means length of diagonal in right-angled parallepiped with legs parallel with coordinate axes.

$$P_1Q = x_2 - x_1, QP_2 = y_2 - y_1$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

The distance between any two points is equal to the square root of the sum of the squares of the differences between their corresponding coordinates

14 Direction of arbitrary line

If points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ direction cosines of this line defined like

$$\cos\alpha = \frac{x_2 - x_1}{d} \quad \cos\beta = \frac{y_2 - y_1}{d} \quad \cos\gamma = \frac{z_2 - z_1}{d}$$

They are direction cosines of the equal directed segment established from the origin of coordinate system.

15 Ratio of Division

If two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are given by their coordinates, the coordinates x, y, z of any point P on the line P_1P_2 can be found if the division ratio $\frac{P_1P}{P_1P_2} = k$ is known in which the point P divides the segment P_1P_2.

Observe abscissa. Let $OQ_1 = x_1$, $OQ_2 = x_2$, $OQ = x$. Then $x = x_1 + k(x_2 - x_1)$

The same for ordinate $y = y_1 + k(y_2 - y_1)$ and for applicate $y = z_1 + k(z_2 - z_1)$

Or, with differences Δx, Δy and Δz

$$x = x_1 + k\Delta x \quad y = y_1 + k\Delta y \quad z = z_1 + k\Delta z$$

If the division ratio k is negative, the two segments P_1P_2 and P_1P must have opposite sense, so that the points P and P_2 must lie on opposite sides of the point P_1

Ratio of division may also be expressed in form $\frac{P_1P}{PP_2} = r$

$$x = \frac{x_1 + rx_2}{1 + r}$$
$$y = \frac{y_1 + ry_2}{1 + r}$$
$$z = \frac{z_1 + rz_2}{1 + r}$$

Since P_1P and PP_2 are read in the same direction on the line, the ratio will be positive. If the point of division $P(x, y)$ were on the segment extended in either direction, then the ratio would be negative because P_1P and PP_2 would then have opposite directions.

16 Midpoint of a Segment

The midpoint P of a segment P_1P_2 has for its coordinates the arithmetic means of the corresponding coordinates of P_1 and P_2

Coefficient $k = \dfrac{1}{2}$

$$x = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$$
$$y = y_1 + \frac{1}{2}(y_2 - y_1) = \frac{1}{2}(y_1 + y_2)$$
$$z = z_1 + \frac{1}{2}(z_2 - z_1) = \frac{1}{2}(z_1 + z_2)$$

Coefficient $r = 1$ in this case

17 Angle between lines

Note. The angle between two lines that do not meet is defined as the angle between two intersecting lines, each of which is parallel to one of the given lines.

Now we expect that lines are really crossing.

Let OP_1 and OP_2 be two lines through the origin parallel to the two given lines, and let θ be the angle between the lines. From the vectorial point of view it means that we're investigating directed segments colinear with that lines.

Angle between that vectors may be expressed with dot product:

$$\cos\theta = \frac{\overrightarrow{OP_1} \cdot \overrightarrow{OP_2}}{|OP_1||OP_1|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\rho_1\rho_2}$$

Comparison with expressions for directing cosines yields:

$$\cos\theta = \cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2$$

Evidence for parallel lines: $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$. Thus, we formed Pythagorean relation:

$$\cos\theta = \cos\alpha_1\cos\alpha_1 + \cos\beta_1\cos\beta_1 + \cos\gamma_1\cos\gamma_1 = 1$$

Evidence for perpendicular lines: $\cos\theta = 0$

$$\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2 = 0$$

18 Points and coordinates in space: problems corner

Problem 1

Find distance from the origin and perpendicular distances from the axes for the point $A(6, 2, 3)$

Solution

"Perpendicular distance" is a length of segment connecting the point and it's projection on one of coordinate axes.

Thus we are loorking for distances between A and $A(6, 0, 0)$, $A(0, 2, 0)$, $A(0, 0, 3)$

$$\begin{aligned} OA &= \sqrt{6^2 + 2^2 + 3^3} = 7 \\ AA_1 &= \sqrt{(6-6)^2 + (2-0)^2 + (3-0)^3} = \sqrt{13} \\ AA_2 &= \sqrt{(6-0)^2 + (2-2)^2 + (3-0)^3} = 3\sqrt{5} \\ AA_3 &= \sqrt{(6-0)^2 + (2-0)^2 + (3-3)^3} = 2\sqrt{10} \end{aligned}$$

Problem 2

Prove that the geometrical center or centroid or center of area, i.e., the intersection of the medians, of any triangle $A(x_1, y_2, z_3)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

The medians of the triangle ABC intersect in a point $P(x, y, z)$ such that $\dfrac{AP}{PD} = \dfrac{BP}{PE} = \dfrac{CP}{PF} = \dfrac{2}{1}$

$$x = \frac{x_1 + 2\frac{x_2 + x_3}{2}}{1+2} = \frac{x_1 + x_2 + x_3}{3}$$
$$y = \frac{y_1 + 2\frac{y_2 + y_3}{2}}{1+2} = \frac{y_1 + y_2 + y_3}{3} x = \frac{z_1 + 2\frac{z_2 + z_3}{2}}{1+2} = \frac{z_1 + z_2 + z_3}{3}$$

Problem 3

Find the interior angles of the triangle whose vertices are $A(3, -1, 4)$, $B(1, 2, -4)$, $C(-3, 2, 1)$.

Direction cosines of AB are $(\frac{-2}{\sqrt{77}}, \frac{3}{\sqrt{77}}, \frac{-6}{\sqrt{77}})$

Direction cosines of BC are $(\frac{-4}{\sqrt{41}}, 0, \frac{5}{\sqrt{41}})$

Direction cosines of AC are $(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}})$

Observing direction cosines of BA instead of AB in vectorial terms means multiply by -1, thus int means that all direction cosines change sign

Let us calculate $\cos A$:

$$\cos A = \frac{-2}{\sqrt{77}}\frac{-2}{\sqrt{6}} + \frac{3}{\sqrt{77}}\frac{1}{\sqrt{6}} + \frac{-6}{\sqrt{77}}\frac{-1}{\sqrt{6}} = \frac{15}{\sqrt{462}}$$

$\cos B$ and $\cos C$ have similar expressions

19 Equation and locus

Locus or **graph** of an equation in two variables is the curve or straight line containing all the points, and only the points, whose coordinates satisfy the equation. Before plotting the graph of an equation, it is most often very help¬ ful to determine from the form of the equation certain properties of the curve. Such properties are: *intercepts*, *symmetry*, *extent*.

Intercepts are the directed (positive or negative) distances from the origin to the points where the curve intersects the coordinate axes.

To determine intercepts we are letting firs or second coordinates to zero

Example. Observe equation $y^2 + 2x = 16$. Letting $y = 0$ yields intercept in point $(8, 0)$. Letting $x = 0$ yields intercepts in points $(0, +4)$, and $(0, -4)$. x-intercept is 8, and y-intercepts are $+4$ and -4

Two points are **symmetric with respect to a line** if that line is the perpendicular bisector of the line connecting the two points. Two points are **symmetric about a point** if that point is the midpoint of the line connecting the two given points.

- If an equation remains unchanged when x is replaced by $-x$, the graph is symmetric with respect to the y-axis. Example: $x^2 - 6y + 12 = 0$, or $x = \pm\sqrt{6y - 12}$.
- If an equation remains unchanged when y is replaced by $-y$, the graph is symmetric with respect to the x-axis. Example: $y^2 - 4x - 7 = 0$, or $y = \pm\sqrt{4x + 7}$.
- If an equation remains unchanged when x is replaced by $-x$ and y is replaced by $-y$, the graph is symmetric with respect to the origin. Example: $x^3 + x + y^3 = 0$.

If certain values of one variable cause the other variable to become *imaginary*, **such values must be excluded** (extent).

Example: $x = \pm\sqrt{6y - 12}$ for $y < 2$ value under the root is negative, therefore the curve lies to the right from line $x = 2$

20 Forms of equations

If the radius vector of a point enters an equa- tion as a whole without subdividing it into separate coordinates, such an equation is called a **vectorial equation**.

Example: $\boldsymbol{r} \times \boldsymbol{a} = \mathbf{0}$

If the radius vector of a point enters an equation through its coordinates, such an equation is called a **coordinate equation**

Example: $x^2 + y^2 + z^2 = 1$

One or two degrees of freedom can be implemented in an equation explicitly when the radius of a point is given as a function of one or two variables, which are called parameters. In this case the equation is called **parametric**.

Example: $\begin{cases} x = at; \\ y = bt; \\ z = L \end{cases}$

Non-parametric equations behave as obstructions decreasing the number of degrees of freedom from the initial three to one or two.

21 Example. Equation of circle on plane. Equation of sphere in space

Circle means figure shaped of points laying in an equal distance from the center of it.

Thus, if center of circle is point $C(x_0, y_0)$, equal distance R from eat means

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

In parametric form:

$$\begin{cases} x = x_0 + R\cos\varphi \\ y = y_0 + R\sin\varphi \end{cases}$$

Vectorial equation:

$$(\boldsymbol{r} - \boldsymbol{c}) \cdot (\boldsymbol{r} - \boldsymbol{c}) = R^2$$

Here $\boldsymbol{c}$ is radius vector for C

For the sphere as a locus of point laying in equal distance R form center $C(x_0, y_0, z_0)$ we have

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

In parametric form:

$$\begin{cases} x = x_0 + R\sin\theta\cos\varphi \\ y = y_0 + R\sin\theta\sin\varphi \\ z = z_0 + R\cos\theta \end{cases}$$

Vectorial equation matches plane case of circle

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