Complex Integration.

§1. Path.

- domain / region : nonempty, open, connected

  Convex set.

  star-shaped set.

- curve. a graph of continuous function.

  parametric form: $\gamma(t) = (x(t), y(t))$. $t \in [a,b]$. $a \to b$

  closed - $\gamma(a) = \gamma(b)$

  reverse curve : $\gamma^{-1}(t) = \gamma(b+a-t)$. $t \in [a,b]$

  orientation : anti-clockwise + clockwise -

- path (contour): piecewise smooth, cont. diff, mapping $[a,b] \to \mathbb{C}$.

  simple - no intersection closed - $\gamma(a) = \gamma(b)$

△ a curve but not a path.

$f(t) = \begin{cases} t^2 \sin\frac{1}{t} & t \neq 0. \\ 0 & t = 0. \end{cases}$ curve $\gamma(t) = t + if(t)$. $t \in [-\pi, \pi]$. f' not cont. on $[-\pi, 0]$. $[0, \pi]$

Path / Contour Integral

path $\gamma : [a,b] \to \mathbb{C}$. f is cont. on the graph of γ

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

equivalent path

independent for parametrization. dependent for orientation

bounds: γ - path. $\ell(\gamma)$. - length. f is cont. on γ. s.t. $|f(z)| \leq M$ for all z on γ.

then $\left| \int_\gamma f(z)\,dz \right| \leq M\,\ell(\gamma)$

§. Antiderivative.

thm: region $\Omega \in \mathbb{C}$. $f \in C(\Omega)$ ∴, TFAE

(1) $\exists F \in H(\Omega)$. s.t. $f(z) = F'(z)$ for all $z \in \Omega$.

(2) $\forall z_1, z_2 \in \Omega$ $\forall \gamma$. that connects z_1 to z_2. $\int_\gamma f(z)\,dz$ independent of the path.

(3). $\oint_\gamma f\,dz \equiv 0$. on Ω.

Moreover. if (1) holds. $\forall \gamma$ in Ω join z_1 to z_2. $\int_\gamma f(z)\,dz = F(z_2) - F(z_1)$

"global antiderivative along path" 对解析函数而言一定存在且唯一

"global antiderivative" 仅在圆盘上存在，可能差一个常数.

§. Cauchy Integral thm.

. (Cauchy - Goursat). $f \in H(U)$. region $U \subset \mathbb{C}$. $\forall \Delta \subset U$. $\int_{\partial\Delta} f(z)\,dz = 0$.

(Jordan - Arc thm.) U is a region that contains a simple non-closed path γ. Then $U \setminus \gamma$ is also a region. (open and connected)

1). (for simple path). $f \in H(U)$. region $U \subset \mathbb{C}$. Contains a simple closed path and its interior. and $f' \in C(U)$.

then $\int_\gamma f(z)\,dz = 0$.

2). (for star-shaped region). $f \in H(V)$. V is open star-shaped. contains a simple closed path. γ.
(一定包括内部)

then $\int_\gamma f(z)\,dz = 0$

3) (for simply connected region). $f \in H(\Omega)$. Ω is a simply connected region. contains a closed path. γ.
(不simple就从交点分两条)

then $\int_\gamma f(z)\,dz = 0$

△ informal restatement: if f is analytic on a simple closed path and it's interior then the integral of f around this path is 0.

4) (for multiply connected regions). $\Omega = \Omega_0 \setminus (\bar{\Omega}_1 \cup \cdots \cup \bar{\Omega}_n)$ $n \in \mathbb{N}$. $\Omega_1, \ldots, \Omega_n$ be pairwise disjoint simply connected

Ω_0 is simply connected and contains the union $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$.

C_j be simple closed paths with positive orientation whose interior regions contain $\bar{\Omega}_j$.

C_j be pairwise disjoint. C be simple closed paths with positive orientation

in Ω. C's interior region contains all C_j.

$$f \in H(\Omega), \quad \int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz.$$

Important results.

$$\int_C \frac{1}{z-z_0}\,dz = \begin{cases} 0 & z_0 \text{ lies in exterior of } C \\ 2\pi i & z_0 \text{ lies in interior of } C. \end{cases}$$

准备：圆周

$$\int_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & (n=1), \\ 0 & (n \neq 1, 且为整数), \end{cases}$$

这里 C 表示以 a 为圆心，ρ 为半径的圆周.（**注意**：积分值与 a，ρ 均无关，a 可为（

复连通区域的柯西定理.

定理 3.10 设 D 是由复周线

$$C = C_0 + C_1^- + C_2^- + \cdots + C_n^-$$

所围成的有界 $n+1$ 连通区域，函数 $f(z)$ 在 D 内解析，在 $\overline{D} = D + C$ 上连续，则

$$\int_C f(z)\,dz = 0,$$

或写成

$$\int_{C_0} f(z)\,dz + \int_{C_1^-} f(z)\,dz + \cdots + \int_{C_n^-} f(z)\,dz = 0, \tag{3.13}$$

或写成

$$\int_{C_0} f(z)\,dz = \int_{C_1} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz. \tag{3.14}$$

（沿外边界积分等于沿内边界积分之和.）

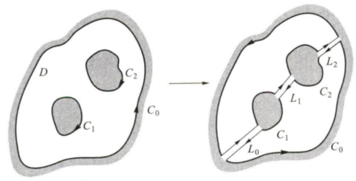


图 3.10

推广：圆周 → 周线.

周线：逐段光滑的简单闭曲线

例 3.11 设 a 为周线 C 内部一点，则

$$\int_C \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i & (n=1), \\ 0 & (n \neq 1, 且为整数). \end{cases}$$

证 以 a 为圆心画圆周 C'，使 C' 全含于 C 的内部，则由(3.14)，

$$\int_C \frac{dz}{(z-a)^n} = \int_{C'} \frac{dz}{(z-a)^n},$$

推广：复连通区域 重要积分

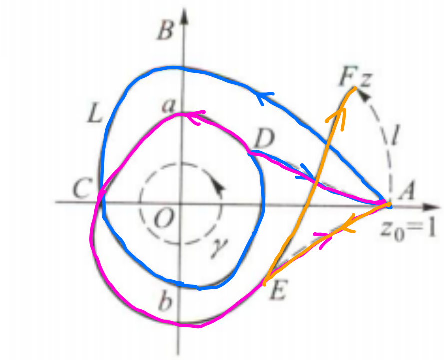
例 3.13（多连通区域内的不定积分或变上限积分） 试证

$$\int_1^z \frac{d\zeta}{\zeta} = \mathrm{Ln}\, z \quad (z \in G: z \neq 0, \infty),$$

其中积分路径是不过原点，且连接点 $z_0 = 1$ 和点 z 的任意逐段光滑曲线.

$$\int_L \frac{d\zeta}{\zeta} = \ln z + 2n\pi i.$$

（n是正向过负半轴次数？）



§. Zero and Singularity.

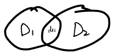
def. zero of order : point z_0 is zero of order $m \in \mathbb{Z}_+$ of f.

$$f(z) = (z - z_0)^m g(z_0)$$

解析延拓. (解析原理: 延拓函数时不破坏函数的解析性).

定义. $f\in H(D)$. $D\subset G$. $F(z)\in H(G)$. $\forall z\in D$. $f(z)=F(z)$. $F(z)$是$f(z)$在G内解析延拓.

解析函数元素 (f, D).

直接解析延拓 (DAC). "互为" 

解析延拓链 $(f_1, D_1)\ (f_2, D_2)\ (f_3, D_3)\cdots(f_n, D_n)$ 相邻: 互为直接解析延拓. 不相邻: 互为间接解析延拓.

定理 2 (唯一性)①. 如果两个函数 $f_1, f_2 \in \mathcal{O}(D)$ 在集合 E 中相同, 而 E 至少有一个属于 D 的极限点 a, 则在整个 D 上 $f_1 = f_2$.

证明. 令函数 $f = f_1 - f_2 \in \mathcal{O}(D)$; 可以证明在 D 中 $f \equiv 0$, 即集合 $F = \{z \in D : f(z) = 0\} \supset E$ 与 D 重合. 极限点 a 是 f 的零点 (由于 f 的连续性). 根据定理 1, 在 a 的一个邻域内函数 $f \equiv 0$, 不然的话, 这个点便不可能是 f 的零点的极限点.

因此, 集合 F 的核 $\mathring{F}$ (即它的内点集) 非空: 它包含了点 a. 按构造, $\mathring{F}$ 为开, 但它同时也闭 (相对于区域 D 的拓扑). 事实上, 如果 $b \in D$ 是 $\mathring{F}$ 的极限点, 则由同一个定理 1 知, 在点 b 的某个邻域内 $f \equiv 0$, 即 $b \in \mathring{F}$. 由于按区域的定义 D 连通, 故根据第 4 小节的定理 2, 我们有 $\mathring{F} = D$. □

这个定理也指出了函数的全纯概念与在实分析意义下的可微性概念实质上的不同. 事实上, 两个无限可微的实变函数可以在部分定义域上重合而不恒同. 然而所证明的定理表明, 两个全纯函数只要在任意一个在它们全纯的区域中还有一个极限点的集合上重合 (譬如属于区域的一个小圆, 一段弧) 时, 则在整个区域恒同.

求解析延拓.

幂级数方法. 给定 (f, D). $z_1\in D$. $f(z)$可在$U(z_1)$内展开为 $f_1(z)=\sum_{n=1}^{\infty} C_n^{(1)}(z-z_1)^n$. $C_n^{(1)}=\frac{1}{n!}f^{(n)}(z_1)$.

若$R=+\infty$. $f_1(z)=\sum_{n=1}^{\infty} C_n^{(1)}(z-z_1)$ 是 f 在 $\mathbb{C}$ 上的解析延拓.

解析函数唯一性

f_1, f_2在D内解析. D内有一收敛于$a\in D$的点列$\{z_n\}$ $(z_n\neq a)$. $f_1(z_n)=f_2(z_n)$. 则 $f_1(z)=f_2(z)$ on D.

$f_1(z)$. $f_2(z)$在D内解析, 且在某一子区域内相等, 则必在区域D内相等.

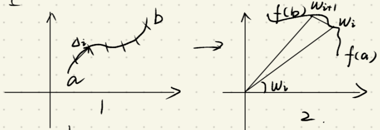
全纯与解析 (定义上有区别, 但等价).

全纯: 在a点的某一邻域内可以展开成关于$z-a$的幂级数.

复分析可视化方法.

$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\frac{dz}{dt}\cdot dt$

$= \lim \sum f\,\Delta$

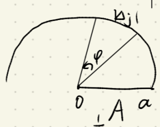


$\tilde{\Delta}_i = w_i\Delta_i$

$= |w_i|\Delta_i\, \arg w_i$

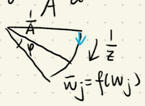
$z \to \frac{1}{z}$

$r \to \frac{1}{r}$



$f = \frac{1}{z}$

$|\Delta_j| = A\varphi.$



$\tilde{\Delta}_j = w_j\cdot\Delta_j = \frac{1}{A}(e^{-i(\varphi)})A\varphi e^{i\varphi+\frac{\pi}{2}}$

$\tilde{\Delta}_1 = w_1\cdot\Delta_1 = \frac{1}{A}iA\varphi$

$= i\varphi$

$w_1 \to 0$ 没有角度 $\frac{1}{A}$

$\Delta_i \to 0$ 没有实部

例: $z \to \frac{1}{z}$ 复反演(函数)

$\tilde{\Delta}_i := w_i\Delta_i.$ $w_i = f(z_i)$ $|\Delta_j| = A\phi$

函数值. 分割

$z = A\cdot e^{i\theta} \xrightarrow{\frac{1}{z}} \frac{1}{A}\cdot e^{-i\theta}$

留数定理 (留数, 环绕数).

洛朗级数