The property of partial degeneracy of the original (the advance theorem).

If $f(t) \leftrightarrow F(p)$ and $\tau > 0$, then

$$f(t+\tau) \leftrightarrow e^{p\tau}\left(F(p) - \int_0^{\tau} f(t)e^{-pt}\,dt\right).$$

Here $f(t+\tau) = f(t+\tau)\theta(t)$.

Figure 1 shows graphs of the original functions $f(t)$, $f(t-\tau)\theta(t-\tau)$, $f(t+\tau)$, where $\tau > 0$. To calculate the images of functions $f(t-\tau)\theta(t-\tau)$, $f(t+\tau)$ from the known image $F(p) \leftrightarrow f(t)$, the delay and advance theorems are used, respectively.

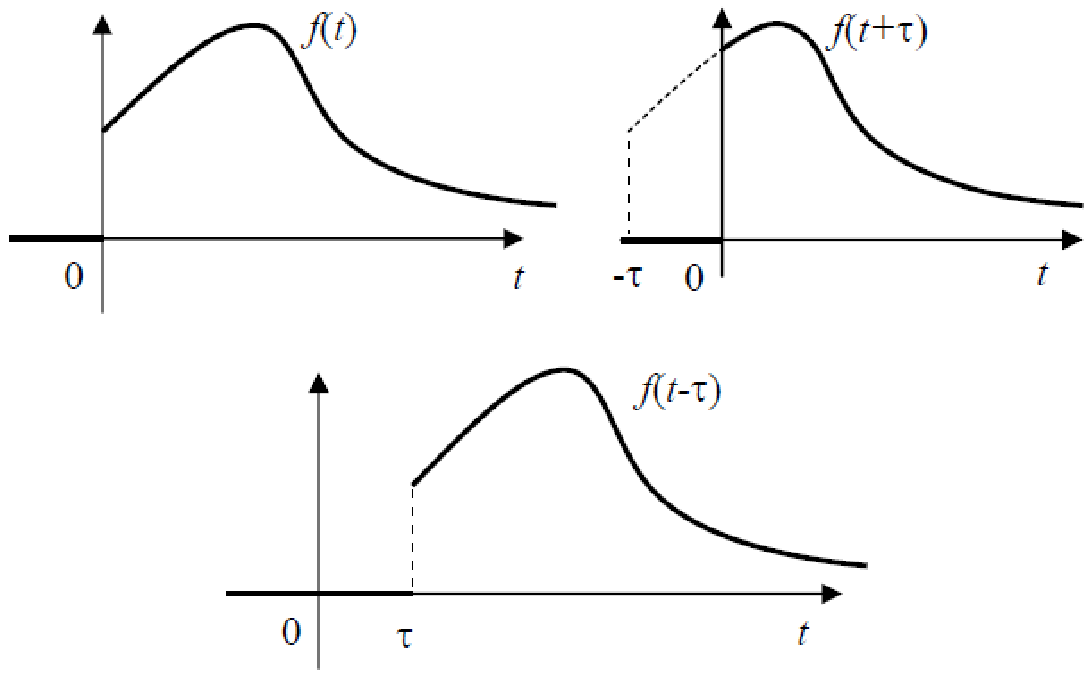


Fig. 1. Graphs of the original functions

Example 15.

15) Find images of the following functions:

a) $f(t)=\sin(t+\tau), \quad \tau>0$,

b) $f(t)=\cos(t+\tau), \quad \tau>0$.

<u>Solution:</u>

a) For the function $sin\,t \leftrightarrow \dfrac{1}{p^2+1}$. By the advance theorem

$$sin(t+\tau) \leftrightarrow e^{p\tau}\left(\frac{1}{p^2+1}-\int_0^\tau \sin t e^{-pt}\,dt\right).$$

Since

$$\int_0^\tau \sin t e^{-pt}\,dt = \left[\begin{array}{c}\text{integration}\\ \text{by parts,}\\ \text{twice}\end{array}\right] = \left.\frac{-p\sin t-\cos t}{p^2+1}e^{-pt}\right|_0^\tau =$$

$$= \frac{-p\sin\tau-\cos\tau}{p^2+1}e^{-p\tau} - \frac{1}{p^2+1},$$

then according to the advance theorem

$$\sin(t+\tau) \mapsto \frac{p\sin\tau+\cos\tau}{p^2+1}.$$

b) For the function $cos\,t \leftrightarrow \dfrac{p}{p^2+1}$. According to the advance theorem

$$cos(t+\tau) \leftrightarrow e^{p\tau}\left(\frac{p}{p^2+1}-\int_0^\tau \cos t e^{-pt}\,dt\right).$$

Since

$$\int_0^{\tau}\cos t e^{-pt}\,dt = \left[\begin{array}{c}\text{integration}\\ \text{by parts,}\\ \text{twice}\end{array}\right] = \frac{-p\cos t+\sin t}{p^2+1}e^{-pt}\Bigg|_0^{\tau} =$$

$$= \frac{-p\cos\tau+\sin\tau}{p^2+1}e^{-p\tau}+\frac{p}{p^2+1},$$

then we will get

$$sin(t+\tau)\leftrightarrow \frac{p\cos\tau-\sin\tau}{p^2+1}.$$

The image of the periodic function.

Let the original function $f(t)$ have a period T.

Then if $f_0(t)\leftrightarrow F_0(p)$, where

$$f_0(t)=\begin{cases} f(t) & \text{when } 0<t<T, \\ 0 & \text{when } t<0 \text{ and } t>T, \end{cases}$$

then

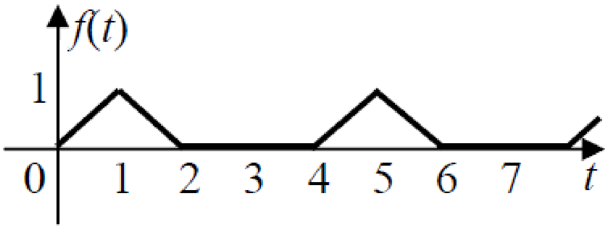
$$f(t)\leftrightarrow\frac{F_0(p)}{1-e^{-pT}}.$$

Example 16.

Find images of the following periodic functions:

a) $f(t)=|\cos t|$,

b) $f(t) = |\sin t|$.



c)

Solution:

a) The function $f(t) = |\cos t|$ is periodic with a period $T = \pi$. Consider the function

$$f_0(t) = \begin{cases} |\cos t|, & 0 \le t \le \pi, \\ 0, & t < 0,\ t > \pi. \end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t) \leftrightarrow \int_0^{\pi} |\cos t| e^{-pt} dt = \int_0^{\pi/2} \cos t e^{-pt} dt - \int_{\pi/2}^{\pi} \cos t e^{-pt} dt =$$

$$= \left[\int \cos t e^{-pt} dt = \frac{e^{-pt}}{p^2+1}(-p\cos t + \sin t) \right] =$$

$$= \frac{1}{p^2+1}\left(2e^{-\frac{\pi}{2}p} + p\left(1 - e^{-\pi p}\right) \right).$$

According to the formula for the image of the periodic function

$$|\cos t| \leftrightarrow \frac{2e^{-\frac{\pi}{2}p} + p\left(1 - e^{-\pi p}\right)}{\left(p^2+1\right)\left(1 - e^{-\pi p}\right)}.$$

b) The function $f(t)=|\sin t|$ is periodic with a period $T=\pi$.

Consider the function

$$f_0(t)=\begin{cases}|\sin t|, & 0\le t\le \pi,\\ 0, & t<0,\ t>\pi.\end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t)\leftrightarrow \int_0^{\pi}|\sin t|e^{-pt}\,dt=\int_0^{\pi}\sin t e^{-pt}\,dt=$$

$$=\left[\int \sin t e^{-pt}\,dt=-\frac{e^{-pt}}{p^2+1}(p\sin t+\cos t)\right]=\frac{1}{p^2+1}\left(1+e^{-\pi p}\right)$$

According to the formula for the image of the periodic function

$$|\sin t|\leftrightarrow \frac{1+e^{-\pi p}}{\left(p^2+1\right)\left(1-e^{-\pi p}\right)}.$$

c) The function is periodic with period $T=4$. Consider the function

$$f_0(t)=\begin{cases}t, & 0\le t\le 1,\\ 2-t, & 1<t\le 2,\\ 0, & 2<t\le 4.\end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t)\leftrightarrow \int_0^{1}te^{-pt}\,dt+\int_1^{2}(2-t)e^{-pt}\,dt=$$

$$= -e^{-pt}\left(\frac{t}{p}+\frac{1}{p^2}\right)\Bigg|_0^1 + e^{-pt}\left(\frac{t-2}{p}+\frac{1}{p^2}\right)\Bigg|_1^2 =$$

$$= -e^{-p}\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{1}{p^2}+\frac{e^{-2p}}{p^2}+e^{-p}\left(\frac{1}{p}-\frac{1}{p^2}\right)=\frac{\left(1-e^{-p}\right)^2}{p^2}.$$

According to the formula for the image of the periodic function

$$f(t) \leftrightarrow \frac{\left(1-e^{-p}\right)^2}{p^2\left(1-e^{-4p}\right)}$$

The theorem of differentiation by parameter.

If, for any fixed x, the function $f(x,t)$ is the original, and $F(p,x)$ is its image, and if in the integral $F(p,x)=\int\limits_0^{+\infty} f(x,t)e^{-pt}dt$ differentiation by parameter x under the sign of the integral is possible, then $\frac{\partial f(x,t)}{\partial x} \leftrightarrow \frac{\partial F(p,x)}{\partial x}$.

This property is used in solving partial differential equations.

1.2. RESTORING THE ORIGINAL IMAGE

1.2.1 The elementary method

In many cases, a given image can be converted to a form where the original is easily restored directly using the properties of the Laplace transform and the table of originals and images.

In this case, the method of decomposing a rational fraction into the sum of the simplest ones is widely used to transform the image.

Let $F(p)$ be a rational function, to find the original, we represent the function $F(p)$ as the sum of the simplest fractions of the form

$$\frac{A}{p-a},\ \frac{Ap+B}{(p-a)^2+b^2},\ \frac{A}{(p-a)^k},\ \frac{Ap+B}{\left((p-a)^2+b^2\right)^k},\ k=2,3,\ldots$$

(A,B,a,b are some constants), for each of which we can construct the corresponding original.

Indeed, using the displacement property and the table of originals and images, we find

$$\frac{A}{p-a} \leftrightarrow Ae^{at},\ \frac{A}{(p-a)^k} \leftrightarrow \frac{A}{(k-1)!}t^{k-1}e^{at},\ k=2,3,\ldots;$$

$$\frac{Ap+B}{(p-a)^2+b^2} = \frac{A(p-a)+B+Aa}{(p-a)^2+b^2} \leftrightarrow Ae^{at}\cos bt + \frac{B+Aa}{b}e^{at}\sin bt\ .$$

Let $S_k(p) \leftrightarrow s_k(t)$. Let's build the original for the image

$$S_k(p) = \frac{Ap+B}{\left((p-a)^2+b^2\right)^k},\ k=2,3,\ldots$$

Consider the expression for image $S_2(p)$. Since

$$S_2(p) = \frac{Ap+B}{(p-a)^2+b^2}\frac{1}{(p-a)^2+b^2},$$

then, applying the convolution image property, we construct the corresponding original:

$$s_2(t) = \frac{1}{b}\int_0^t s_1(t-\tau)e^{a\tau}\sin b\tau d\tau .$$

Here the original $s_1(t)$ is defined by the expression

$$s_1(t) = Ae^{at}\cos bt + \frac{B+Aa}{b}e^{at}\sin bt .$$

Further, since

$$S_3(p) = S_2(p)\frac{1}{(p-a)^2+b^2},$$

then

$$s_3(t) = \frac{1}{b}\int_0^t s_2(t-\tau)e^{a\tau}\sin b\tau d\tau .$$

Similar reasoning leads to the following relation:

$$s_k(t) = \frac{1}{b}\int_0^t s_{k-1}(t-\tau)e^{a\tau}\sin b\tau d\tau,\ k \geq 2 .$$

Example 1.

Find the original corresponding to the image

$$F(p) = \frac{1}{p^3-p}.$$

Solution:

Decomposing a given image into the sum of the simplest fractions

$$\frac{1}{p^3-p}=\frac{1}{p(p-1)(p+1)}=-\frac{1}{p}+\frac{1}{2(p-1)}+\frac{1}{2(p+1)},$$

we'll find the original

$$f(t)=-1+\frac{1}{2}e^{t}+\frac{1}{2}e^{-t}=-1+\mathrm{ch}\,t\,.$$

Example 2.

Find the original corresponding to the image

$$F(p)=\frac{1}{(p^2+4)^2}\,.$$

Solution:

Applying the convolution image property, we will have

$$F(p)=\frac{1}{(p^2+4)^2}=\frac{1}{p^2+4}\frac{1}{p^2+4}\leftrightarrow\frac{1}{4}\int_0^t \sin 2(t-\tau)\sin 2\tau d\tau\,.$$

Having calculated the integral, we get the desired expression for the original

$$f(t)=\frac{1}{16}\sin 2t-\frac{1}{8}t\cos 2t\,.$$

Example 3.

Find the original corresponding to the image

$$F(p) = \frac{e^{-\frac{p}{2}}}{p(p+1)(p^2+4)}.$$

Solution:

Let's imagine the fraction included in the expression as the simplest fractions:

$$\frac{1}{p(p+1)(p^2+4)} = \frac{A}{p} + \frac{B}{p+1} + \frac{Cp+D}{p^2+4}.$$

Applying the method of undefined coefficients to the decomposition,

we obtain

$$A = \frac{1}{4},\ B = D = -\frac{1}{5},\ C = -\frac{1}{20}.$$

The image has the form

$$F(p) = \frac{1}{4}\frac{e^{-\frac{p}{2}}}{p} - \frac{1}{5}\frac{e^{-\frac{p}{2}}}{p+1} - \frac{1}{20}\frac{pe^{-\frac{p}{2}}}{p^2+4} - \frac{1}{5}\frac{e^{-\frac{p}{2}}}{p^2+4}. \qquad (*)$$

Using the ratios

$$\frac{1}{p} \leftrightarrow \theta(t),\ \frac{1}{p+1} \leftrightarrow e^{-t}\theta(t),\ \frac{p}{p^2+4} \leftrightarrow \cos 2t\theta(t),\ \frac{1}{p^2+4} \leftrightarrow \frac{1}{2}\sin 2t\theta(t)$$

and given the delay property, we get the desired original for the image (*)

$$f(t) = \left(\frac{1}{4} - \frac{1}{5}e^{-\left(t-\frac{1}{2}\right)} - \frac{1}{20}\cos(2t-1) - \frac{1}{10}\sin(2t-1)\right)\theta\left(t - \frac{1}{2}\right).$$

Example 4.

Find the original corresponding to the image

$$F(p)=\frac{e^{-\frac{p}{3}}}{p(p^2+1)}.$$

Solution:

Applying the convolution property and the correspondence table of originals and images, we get

$$\frac{1}{p(p^2+1)}\leftrightarrow\int_0^t \sin\tau d\tau=-\cos\tau\Big|_0^t=(1-\cos t)\theta(t).$$

When constructing the original for a given image, we apply the delay property and get

$$f(t)=\left(1-\cos\left(t-\frac{1}{3}\right)\right)\theta\left(t-\frac{1}{3}\right).$$

1.2.2 The conversion formula. Decomposition theorems

Theorem 1 (Laplace transform conversion formula, Riemann-Mellin formula).

Let $f(t)$ be the original and $F(p)$ be its image. If the function $f(t)$ is continuous at point t and has finite one-sided derivatives at this point, then

$$f(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{pt}F(p)dp. \qquad (1.6)$$

The improper integral (1.6) is taken along any straight line $\operatorname{Re} p = b > \alpha_0$, where α_0 is the growth index of the function $f(t)$ and is understood in the sense of the main value, that is

$$f(t) = \int_{b-i\infty}^{b+i\infty} e^{pt} F(p) dp = \lim_{R\to+\infty} \int_{b-iR}^{b+iR} e^{pt} F(p) dp \, .$$

The Riemann-Mellin formula (1.6) is the inverse of the formula $F(p) = \int_0^{+\infty} f(t) e^{-pt}\, dt$ and is called *the inverse Laplace transform*.

The direct application of the conversion formula to restore the original $f(t)$ from the $F(p)$ image is difficult. Decomposition theorems are usually used to find the original.

Theorem 2 (the first decomposition theorem).

Let the function $F(p)$ be regular at point $p = \infty, F(\infty) = 0$ and its Laurent series in the vicinity of point $p = \infty$ has the form

$$F(p) = \sum_{k=0}^{\infty} \frac{c_k}{p^{k+1}} = \frac{c_0}{p} + \frac{c_1}{p^2} + \frac{c_2}{p^3} + \dots,$$

then the function

$$f(t) = \sum_{k=0}^{\infty} c_k \cdot \frac{t^k}{k!} = c_0 + c_1 t + c_2 \cdot \frac{t^2}{2!} + \dots, \quad t \geq 0$$

is the original with the image $F(p)$.

Definition. A function $F(p)$ is called *meromorphic* in the complex plane if it is regular in any bounded region of the complex plane, with the possible exception of a finite number of singular points of the pole type.

Theorem 3 (the second decomposition theorem).

Let the meromorphic function $F(p)$ be regular in the half-plane $\operatorname{Re} p = \alpha$ and satisfy the conditions:

1) there is a system of circles

$$C_n : |p| = R_n ,\ R_1 < R_2 < \ldots < R_n \to \infty \ (n \to \infty)$$

such that $\max\limits_{p \in C_n} |F(p)| \to 0 \ (n \to \infty)$;

2) for $\forall a > \alpha$, the integral $\int\limits_{-\infty}^{\infty} |F(a + i\sigma)| d\sigma$ converges.

Then $F(p)$ is an image, the original for which is the function

$$f(t) = \sum_{p_k} \underset{p=p_k}{res} \left[F(p) e^{pt} \right],$$

where the sum is taken over all poles p_k of the function $F(p)$.

<u>Consequence.</u> If $F(p) = \dfrac{A_n(p)}{B_m(p)}$, where $A_n(p), B_m(p)$ are polynomials of degree n and m, respectively, having no common zeros, and if $n < m$, then

$$f(t)=\sum_{k=1}^{l}\frac{1}{(m_k-1)!}\frac{d^{m_k-1}}{dp^{m_k-1}}\left\{F(p)e^{pt}(p-p_k)^{m_k}\right\}\Big|_{p=p_k}, \qquad (1.7)$$

where $p_1,...,p_l$ are different zeros of the polynomial $B_m(p)$, and m_k is the multiplicity of zero $p_k:\ \sum_{k=1}^{l}m_k=m$.

In particular, if all poles of the function $F(p)$ are simple, then formula (1.7) takes the form:

$$f(t)=\sum_{k=1}^{m}\frac{A_n(p_k)}{B'_m(p_k)}e^{p_k t}. \qquad (1.8)$$

Example 5.

Find the original corresponding to the image

$$F(p)=\frac{p^2+2}{p^3-p^2-6p}.$$

<u>Solution:</u>

Since $p^3-p^2-6p=p(p-3)(p+2)$, the function $F(p)$ has three simple poles: $p_1=0$, $p_2=3$, $p_3=-2$. Let's construct the corresponding original using the formula (1.8):

$$f(t)=\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=0}+\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=3}+\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=-2}=-\frac{1}{3}+\frac{11}{5}e^{3t}+\frac{3}{5}e^{-2t}.$$