Chapter 12. Introduction to ordinary differential equations.

Initial-value problem (Cauchy's problem) for first-order equation:

Find a solution $y(x)$ of the equation

$$y' = f(x,y) \qquad (1)$$

which satisfies the initial condition

$$y(a) = y_0\ , \qquad (2)$$

where $f(x,y)$ is a given function in the domain $a \le x \le b$, $-\infty < y < \infty$.

Theorem 1.

If functions $f(x,y)$ and $\partial f(x,y)/\partial y$ are continuous and bounded in the domain $a \le x \le b$, $-\infty < y < \infty$, then there exists a unique solution $y(x)$ of problem (1),(2) **(see proof in a course of Diff. Equations)**

Euler's algorithm of calculations:

Let us divide the segment $a \le x \le b$ into n subsegments of length h each by the points

$$x_0=a, \quad x_1, \quad x_2, \ldots, \quad x_n=b \qquad\qquad h=(b-a)/n$$

As shown in Chapter 12,

$$y'(x_i) = [y(x_{i+1}) - y(x_i)]/h + O(h)$$

$$y(x_{i+1}) - y(x_i) = h\,y'(x_i) - h\cdot O(h)$$

Now we replace $y'(x_i)$ **by the right-hand side of equation (1):**

$$y(x_{i+1}) - y(x_i) = h\, f(x_i, y(x_i)) - \underline{h \cdot O(h)}$$

If we omit the error $h \cdot O(h)$, **then obtain formula for calculation of the approximate solution:**

$$y_{i+1} = y_i + h\, f(x_i, y_i) \qquad (3)$$

$$i = 0, 1, 2, \ldots, n-1$$

Theorem 2.

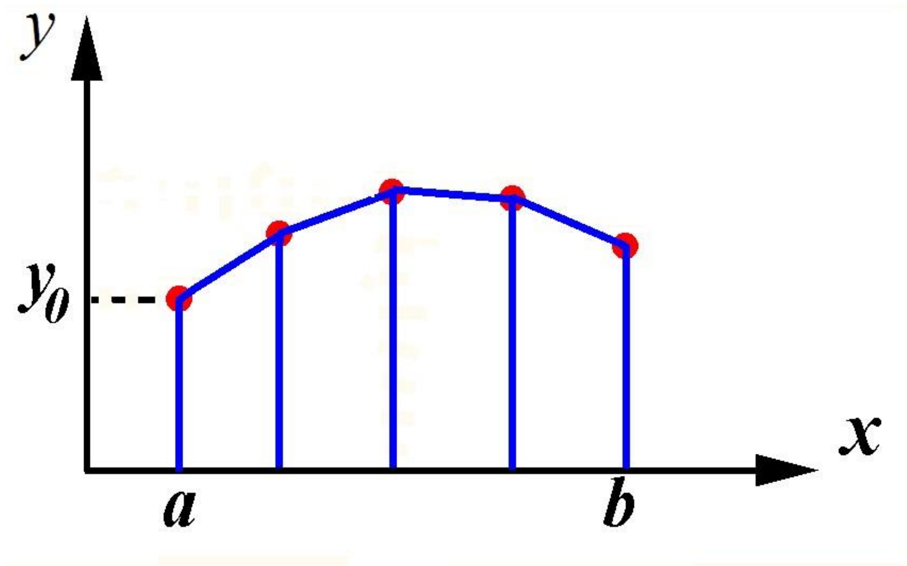
Values y_i of the approximate solution of problem (1), (2) at points x_i can be calculated using (3). The error of the solution is estimated by:

approximate solution

$$\max_{i=0,1,\ldots,n} |\, y(x_i) - y_i \,| = O(h)$$

exact solution guaranteed by Theorem 1

A broken line illustrates the approximate solution (keep in mind the linear interpolation)



Example.

$$y' = y \cdot \sin(3x), \qquad y(0)=1, \qquad 0 \le x \le 5$$

Scilab

```
 clear
n=50   // number of subsegments
a=0
b=5
h= (b-a)/n
for i=1: n+1
x(i)= a+ h*(i-1)
y(i)=1
end
for i=1:n
    f=y(i)*sin(3*x(i))
    y(i+1)= y(i)+h*f
end
plot(x,y,'ob')
```

Improved Euler's algorithm

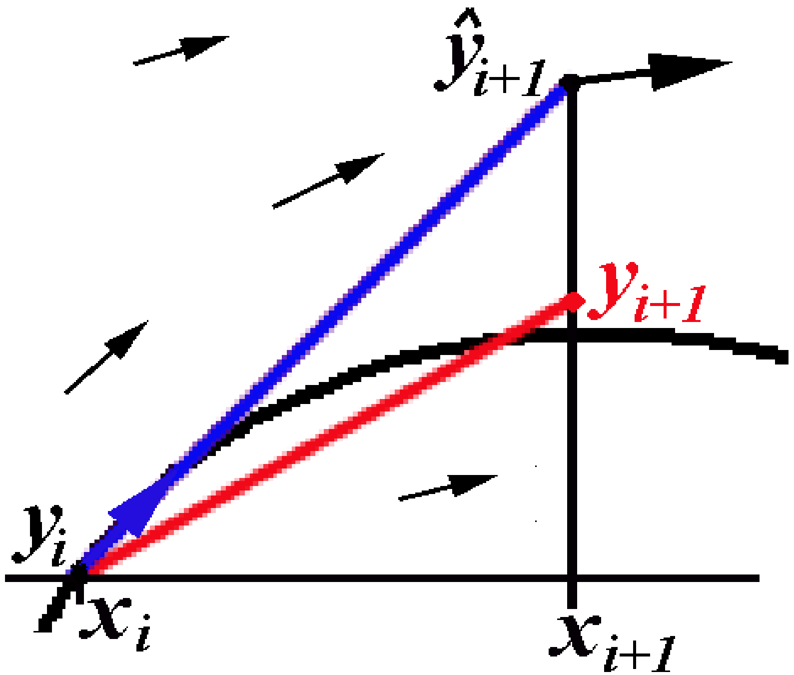
(error at 1 step is of order $O(h^3)$)

$$y' = f(x,y)$$

First, calculate $\hat{y}_{i+1} = y_i + h\, f(x_i, y_i)$ following (3)

Then make a more accurate step:

$$\boxed{y_{i+1} = y_i + h\,[\,f(x_i, y_i) + f(x_{i+1}, \hat{y}_{i+1})\,]/2} \qquad (4)$$



exact solution

Derivation of (4) is based on the expression of second-order accuracy for y'_i at inner nodes, see Chapter 12:

$$[\quad y'_i = (y_{i+1} - y_{i-1}) / (2h) + O(h^2) \quad]$$

In this case: $y'_{i+1/2} = (y_{i+1} - y_i) / (2h/2) + O(h^2/4)$

$$(y_{i+1} - y_i) / (2h/2) + O(h^2/4) = f(x_{i+1/2}, y_{i+1/2})$$

$$(y_{i+1} - y_i) / h + O(h^2) = [f(x_i, y_i) + f(x_{i+1}, y_{i+1})]/2 + \\ + O(h^2)$$

$$y_{i+1} - y_i = h\,[f(x_i, y_i) + f(x_{i+1}, y_{i+1})]/2 + O(h^3)$$

$$y_{i+1} - y_i = h[f(x_i, y_i) + f(x_{i+1}, \hat{y}_{i+1})]/2 + O(h^3)$$

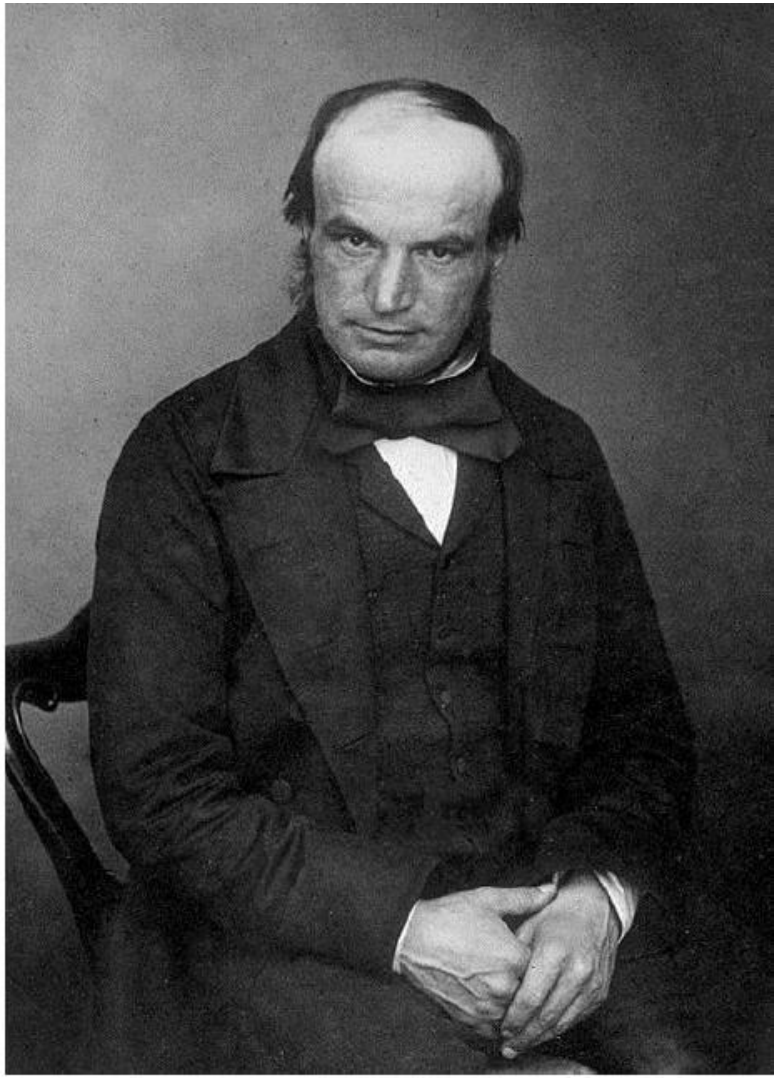
(details will be explained in a lecture in November)

```
n=50                          // improved Euler
a=0
b=5
h= (b-a)/n
for i=1: n+1
x(i)= a+ h*(i-1)
end
y(1)= 1
for i=1:n
f1=y(i)*sin(3*x(i))
ysimple= y(i)+h*f1
f2 = ysimple*sin(3*x(i+1))
y(i+1)=y(i)+h*0.5*(f1+f2)
end
plot(x,y,'r')
```

Scilab: y =ode(y0, x0, x, right2)

```
function  myfunction=f(x,y)
   myfunction = y*sin(3*x)
endfunction
y0 = 1;
x0 = 0;
x = 0:0.1: 5;
y = ode(y0, x0, x, f);
plot(x,y,'k')
```

by default it uses a predictor-corrector Adams method



John Adams (photo ≈ 1870)

y =ode("rk", y0, x0, x, right2)

Runge-Kutta method

Carle Runge

Martin Wilhelm Kutta

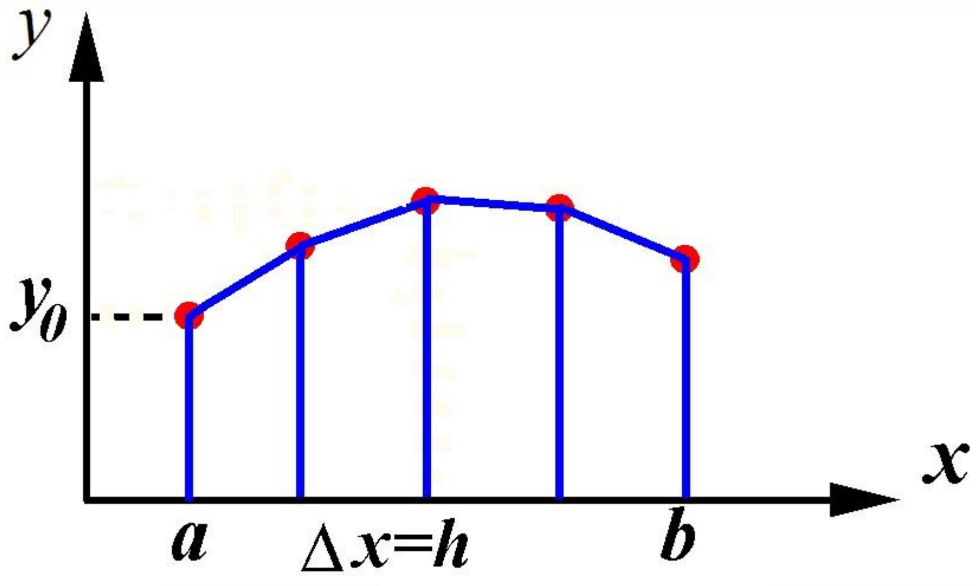
Initial-value problem for a system of **two** first-order equations

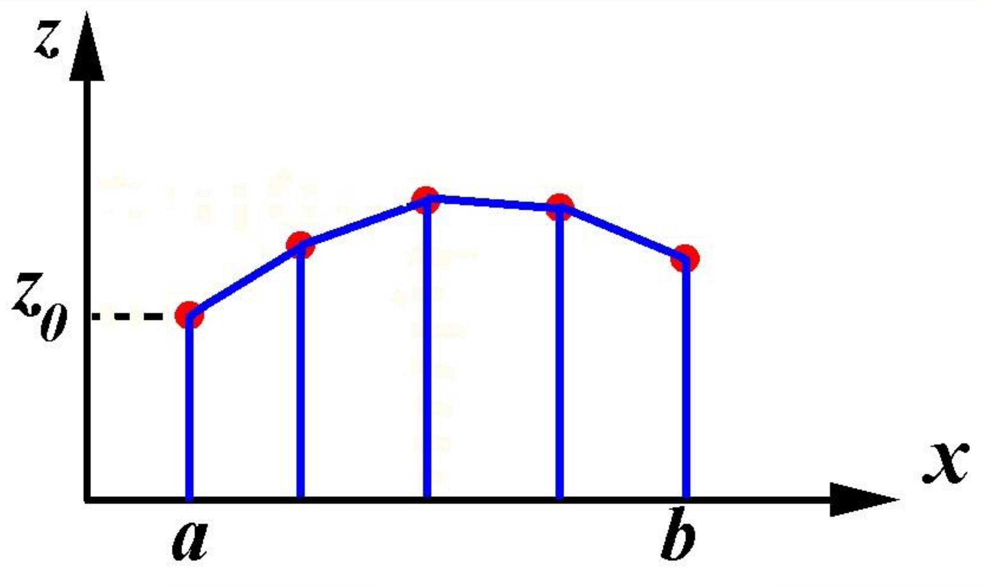
Problem: find a solution $y(x)$, $z(x)$ of the system

$$\begin{cases} y' = f(x,y,z) \\ z' = g(x,y,z) \end{cases}$$

which satisfies the initial condition $y(a)=y_0$, $z(a)=z_0$,

where functions f **and** g **are given in the domain** $a \le x \le b$, $-\infty < y, z < \infty$.





$$y' = f(x, y, z) \qquad z' = g(x, y, z)$$

<u>Euler's method:</u>

$$\Delta y/\Delta x = f(x_i, y_i, z_i) \qquad \Delta z/\Delta x = g(x_i, y_i, z_i)$$

$$(y_{i+1} - y_i)/h = f(x_i, y_i, z_i) \qquad (z_{i+1} - z_i)/h = g(x_i, y_i, z_i)$$

$$i = 0, 1, 2, \ldots, n-1$$

$$y_{i+1} = y_i + h\, f(x_i, y_i, z_i) \qquad z_{i+1} = z_i + h\, g(x_i, y_i, z_i)$$

<u>Example</u>.

$$y' = z(x+y) \qquad y(0)=0$$

$$z' = zx+3 \qquad z(0)=0$$

$$x_1 = h$$
$$y_1 = 0 + h\cdot 0 = 0$$
$$z_1 = 0 + h\cdot 3 = 3h$$

$$x_2 = 2h$$
$$y_2 = y_1 + h\, f(x_1, y_1, z_1) = 0 + h3h(h+0) = 3h^3$$
$$z_2 = z_1 + h\, g(x_1, y_1, z_1) = 3h + h(3hh+3) = 6h + 3h^3$$

Initial-value problem (Cauchy's problem) for a <u>second-order</u> differential equation

Formulation of the problem: find a solution $y(x)$ of equation

$$y''=f(x, y, y')$$

$$d^2y/dx^2=f(x, y, dy/dx)$$

which satisfies the initial condition $y(a)=y_0$, $y'(a)=y'_0$

where function f **is given at** $a \le x \le b$, $-\infty < y, y' < \infty$

<u>Solution:</u>

Let us denote $y'(x)=z(x)$.

Then the problem transforms to the above considered problem for the system of 2 equations:

$$\begin{cases} z' = f(x, y, z) \\ y' = z \end{cases}$$

with initial data:

$z(a)= y'_0$, $y(a)= y_0$

Initial-value problem (Cauchy's problem) for a system of 2 second-order differential equations

$$\begin{cases} y'' = f(x, y, y', z, z') \\ z'' = g(x, y, y', z, z') \end{cases}$$

We introduce 2 new functions:

$$u(x) = y'(x), \quad \mathrm{v}(x) = z'(x)$$

and obtain the system of 4 first-order equations

$$\begin{cases} u' = f(x, y, u, z, \mathrm{v}) \\ \mathrm{v}' = g(x, y, u, z, \mathrm{v}) \\ y' = u \\ z' = \mathrm{v} \end{cases}$$

Example: rotation of a small mass in a gravity field

We denote by t (instead of x) the independent variable, which is time, in the next example.

Equations of mass rotation in the plane (*y*,*z*):

$$\begin{cases} d^2y/dt^2 = -ky/(y^2+z^2)^{3/2} \\ d^2z/dt^2 = -kz/(y^2+z^2)^{3/2} \end{cases} \quad \text{where} \quad k=5+0.1\cdot t$$

The system transforms to

$$\begin{cases} dy/dt = u \\ dz/dt = \mathrm{v} \\ du/dt = -ky/(y^2+z^2)^{3/2} \\ d\mathrm{v}/dt = -kz/(y^2+z^2)^{3/2} \end{cases}$$

Initial data: **y=0, z=1, u=1, v=1 at t=0**

```
clear
function f=right3(t, z)
// z(1)=y    z(2)=z   z(3)=u   z(4)=v
    k=5+0.1*t
    a=z(1)*z(1)+z(2)*z(2) ;
    b=a*sqrt(a)  ;
f(1) =z(3)       ;     // right hand side of eq. 1
f(2) =z(4)       ;     // right hand side of eq. 2
f(3) = -k*z(1)/b  ;   // right hand side of eq. 3
f(4) = -k*z(2)/b  ;   // right hand side of eq. 4
endfunction
t=0 : 0.002 : 15
z=ode([0;1; 1;1], 0, t, right3) ;
comet( z(1,:), z(2,:))
```

<u>Boundary-value problems</u> for a second-order differential equation

Problem: find a solution $y(x)$ of equation

$$y''=f(x, y, y')$$

at $a \le x \le b$ that satisfies the boundary conditions

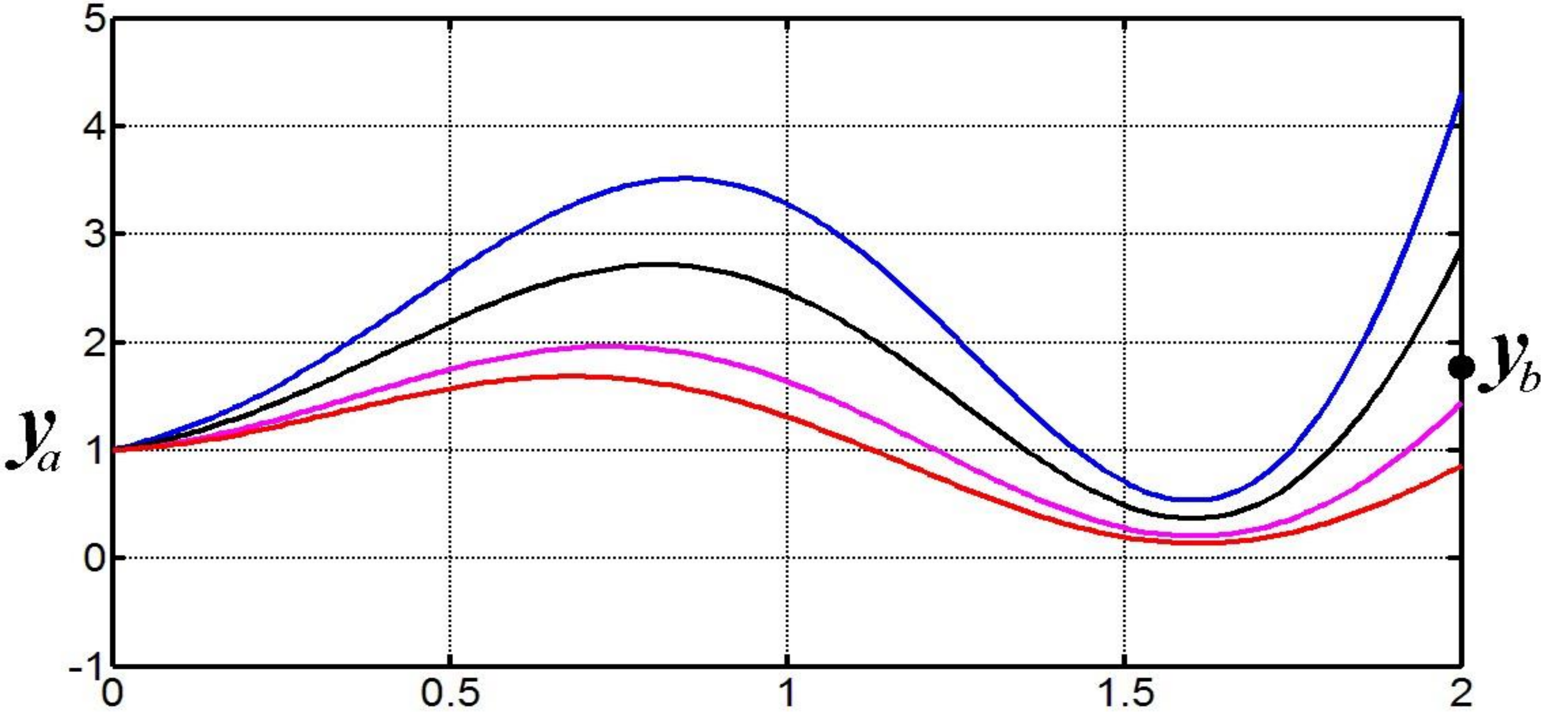
$y(a)=y_a$, $y(b)= y_b$ at opposite ends of $[a,b]$

Shooting method

Let us assign some $y'(a)$ ***and solve the problem with initial data***

$$y(a) = y_a\,, \quad y'(a)$$

As a result, we obtain some y_b **at** $x = b$.



Compare obtained y_n with given y_b, and calculate $|y_n - y_b|$.

We correct the value of $y'(a)$, again solve the initial value problem and in this way usually choose $y'(a)$ so as to provide $|y_n - y_b| < \varepsilon$.

Example 1. $y'' = -y - 1, \qquad y(0) = y(1) = 0$

In some cases, however, the solution of boundary-value problem does not exist.

Example 2. $y'' = (y')^2 + 1 \qquad 0 \le x \le 4 \quad y(0) = 0 \quad y(4) = 1$

(why does not Theorem 1 work in this case?)

In a particular case of the linear equation

$$y''+p(x)y'+q(x)\,y = f(x) \qquad\qquad (*)$$

$$y(a)= y_a\,, \qquad y(b)= y_b$$

(where $p,\ q,\ f$ – given functions) **the solution certainly exist and it is unique if $q\le 0$ (see a course of ode).**

Moreover, this solution can be calculated using only two shooting. Indeed, it can be represented in the form

$$y(x) = \hat{y}(x) + C\cdot\tilde{y}(x)$$

where $\hat{y}$ – solution of equation (*) with initial condition

$$\hat{y}(a)=y_a\,, \qquad \hat{y}'(a)=0 \qquad \textbf{(first shooting);}$$

$\tilde{y}$ - solution of equation $(*)$ with $f=0$ and initial condition $\tilde{y}(a)=0, \quad \tilde{y}'(a)=1$ (second shooting),

$$C = [y_b - \hat{y}(b)] / \tilde{y}(b)$$

Finite-Difference Method of solving boundary-value problem for the linear second-order differential equation

(by transition to algebraic equations)

$$y'' + p(x)y' + q(x)\,y = f(x) \qquad (1)$$

$$y(a) = y_a\,, \qquad y(b) = y_b \qquad (2)$$

By dividing the segment $a \le x \le b$ into n subsegments, we obtain as usual the nodal points

$$x_0 = a,\quad x_1,\quad x_2, \ldots,\quad x_n = b\,, \qquad h = (b-a)/n$$

We can write down equation (*) at inner points, and replace y'', y' by $y(x_{i-1}),\ y(x_i),\ y(x_{i+1})$ according to formulae discussed in Chapter 11:

$$[\, y(x_{i+1}) - 2y(x_i) + y(x_{i-1}) \,] / h^2 + O(h^2) +$$
$$+ p(x_i)[y(x_{i+1}) - y(x_{i-1})] / (2h) + O(h^2) +$$
$$+ q(x_i)\, y(x_i) = f(x_i) \qquad (3)$$

at $i = 1, 2, \ldots, n-1.$

If $y(x)$ is the solution of problem (1),(2), then it must satisfy expression (3).

Now, omitting errors $O(h^2)$, we get the system of algebraic equations for the approximate solution $y_1, \ y_2, \ldots, \ y_{n-1}$ **:**

$$(y_{i+1} - 2y_i + y_{i-1})/h^2 + \\ + p(x_i)(y_{i+1} - y_{i-1})/(2h) + \\ + q(x_i)\, y_i = f(x_i) \qquad (4)$$

at $i = 1, 2, \ldots, n-1.$

We notice that $y_0 = y_a$ **and** $y_n = y_b$ **are known due to boundary conditions (2).**

Multiplying **(4)** by h^2 , we obtain:

$$y_{i+1} - 2y_i + y_{i-1} + \\ + p(x_i)(y_{i+1} - y_{i-1})h/2 + \\ + h^2 q(x_i)\, y_i = h^2 f(x_i)$$

This system can be transformed to standard form:

$$a_{11} y_1 + a_{12} y_2 = b_1,$$

$$a_{21} y_1 + a_{22} y_2 + a_{23} y_3 = b_2,$$

$$a_{32} y_2 + a_{33} y_3 + a_{34} y_4 = b_3,$$

- -

$$a_{n-2,n-3} y_{n-3} + a_{n-2,\, n-2} y_{n-2} + a_{n-2,\, n-1} y_{n-1} = b_{n-2}$$

$$a_{n-1,\, n-2} y_{n-2} + a_{n-1,\, n-1} y_{n-1} = b_{n-1}$$

where a_{ij} **are known coefficients,**

y_i **are unknowns**

Usually this system can be solved using Gaussian's elimination, see Chapter 3:

$$a_{11}\, y_1 + a_{12}\, y_2 \qquad = b_1\,,$$

$$\tilde{a}_{22}\, y_2 + a_{23}\, y_3 \qquad = \beta_2\,,$$

$$\tilde{a}_{33}\, y_3 + a_{34}\, y_4 \qquad = \beta_3\,,$$

- -

$$\tilde{a}_{n-2,\, n-2}\, y_{n-2} + a_{n-2,\, n-1}\, y_{n-1} = \beta_{n-2}$$

$$\tilde{a}_{n-1,\, n-1}\, y_{n-1} = \beta_{n-1}$$

Other boundary conditions which can occur in practice:

(+) $y(a) = c_1 , \quad y'(b) = c_2$

(++) $y'(a) = c_3 , \quad y(b) = c_4$

(+++) $y(a) = c_5 , \qquad y(b) + c_6\, y'(b) = c_7$

where values of $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are given.