Aug. 31st.

6) $\dot{x}-tx^2=2tx$

$\Leftrightarrow \frac{dx}{dt}=t(x^2+2x)$

$\Leftrightarrow \frac{dx}{x(x+2)}=t\cdot dt$

$\Leftrightarrow \int(\frac{1}{x}-\frac{1}{x+2})dx=\int t\cdot dt$

$\Leftrightarrow \frac{1}{2}\ln|\frac{x}{x+2}|=\frac{1}{2}t^2$

$\Leftrightarrow t^2+C=\ln|\frac{x}{x+2}|$

7) $e^{-x}(1+\dot{x})=1.$

$\Leftrightarrow 1+\frac{dx}{dt}=e^x$

$\Leftrightarrow \frac{dx}{e^x-1}=dt$

$\Leftrightarrow \frac{de^x}{e^x(e^x-1)}=dt$

$\Leftrightarrow \ln|\frac{e^x-1}{e^x}|=t+C$

8) $\dot{x}=10^{t+x}$

$\Leftrightarrow \frac{dx}{dt}=10^{t+x}$

$\Leftrightarrow 10^{-x}\cdot dx=10^t\cdot dt$

$-\frac{10^{-x}}{\ln 10}=\frac{10^t}{\ln 10}+\tilde{C}$.

$\Leftrightarrow -10^{-x}=10^t+C$

9) $x\cdot\dot{x}+t=1$

$x\cdot\frac{dx}{dt}=t-1$

$\Leftrightarrow x\cdot dx=(t-1)dt$

$\Leftrightarrow \frac{1}{2}x^2=\frac{1}{2}t^2-t+\hat{C}$

$\Leftrightarrow x^2=t^2-2t+C$

10) $\dot{x}=\cos(x-t)$

Let $y=x-t$. $y'=\dot{x}-1$

$\Leftrightarrow y'+1=\cos y$

$\frac{dy}{dt}+1=\cos y$

$\frac{dy}{\cos y-1}=dt$

$-\csc^2\frac{y}{2}\,d\frac{y}{2}=dt$

$\Leftrightarrow \cot y=t+c \Leftrightarrow \cot(x-t)=t+c$

11) $\dot{x}-x=2t-3.$

$\frac{dx}{dt}-x=2t-3.$

$\Leftrightarrow dx=(2t-3+x)dt.$

$\Leftrightarrow x=t^2-3t+xt+C$

Sept 1.

(1) $(x+2y)dx-xdy=0.$

let $y=tx$.

$(x+2tx)dx-x(tdx+xdt)=0$

$\Leftrightarrow (x+tx)dx-x^2dt=0$

$\Leftrightarrow x(t+1)dx=x^2dt \quad \Rightarrow x\equiv 0$

$\Rightarrow \frac{dx}{x}=\frac{dt}{t+1}$

$\ln|x|=\ln|t+1|+\tilde{c}$

$\Leftrightarrow y=cx^2-x$ or $x\equiv 0$

(2) $(x-y)dx+(x+y)dy=0$

$y=tx$

$\Leftrightarrow (x-tx)dx+(x+tx)(tdx+xdt)=0$

$\Leftrightarrow x(t^2+1)dx+x^2(t+1)dt=0 \Rightarrow x\equiv 0$

$-\frac{dx}{x}=\frac{t+1}{t^2+1}dt$

$-\ln|x|=\frac{1}{2}\ln(t^2+1)+\arctan t+c$

$\Leftrightarrow -\ln|x|=\frac{1}{2}\ln|\frac{y^2}{x^2}+1|+\arctan\frac{y}{x}+C$ or $x\equiv 0$

(3) $(y^2-2xy)dx+x^2dy=0$

$(y^2-2xy+x^2)dx+x^2dy-x^2dx=0$

$(x-y)^2dx=x^2d(x-y)$

Let $x-y=t.$

$t^2dx=x^2dt$

$\Leftrightarrow \frac{dx}{x^2}=\frac{dt}{t^2}$

$\Leftrightarrow \frac{1}{x}=\frac{1}{x-y}+c$

(4) $2x^3\dot{y}=y(2x^2-y^2)$

let $y=tx$

$2x^3(tdx+xdt)=(2x^2-t^2x^2)\cdot txdx$

$t^3x^3dx+2x^4dt=0$

$\Leftrightarrow x\equiv 0$ or $-\frac{dx}{x}=\frac{2dt}{t^3}$

$\Leftrightarrow x\equiv 0$ or $\ln|x|=\frac{x^2}{y^2}+C$

(5) $(x^2+y^2)\dot{y}=2xy$

Let $y=tx$

$(x^2+t^2x^2)(tdx+xdt)=2tx^2dx$

$x^3dt+t^3xdx+x^3t\,dt=t^2x^2dx$

$x^3(t+1)dt=tx$

Sept 7th.

125. $2y' + x = 4\sqrt{y}$.

126. $y' = y^2 - \frac{2}{x^2}$.

127. $2xy' + y = y^2\sqrt{x - x^2y^2}$.

$t = \frac{z}{x} = \frac{1}{xy}$

P125. Let $y = z^{2m}$

$\Leftrightarrow \quad 4m z^{2m-1} \cdot z' + x = 4 \cdot z^m$

$2m-1 = 1 = m \Rightarrow m = 1.$

$4z\,dz + x\,dx = 4z\,dx.$

$\Leftrightarrow z = tx.$

$4 \cdot tx(t\,dx + x\,dt) + x\,dx = 4tx\,dx.$

$4x^2t\,dt = x(4t - 4t^2 - 1)\,dx. \ (x \equiv 0)$

$4xt\,dt = -(2t-1)^2\,dx$

$-\frac{4t\,dt}{(2t-1)^2} = \frac{dx}{x}$

$\Leftrightarrow \quad -\ln|2t-1| - \frac{1}{(2t-1)} = \ln|x| + C$

$\Leftrightarrow -\ln\left|2\frac{\sqrt{y}}{x} - 1\right| - \frac{1}{2\frac{\sqrt{y}}{x} - 1} = \ln|x| + C$

or $x \equiv 0$

P126. Let $y = z^m$.

$\Leftrightarrow \quad m \cdot z^{m-1} \cdot z' = z^{2m} - 2 \cdot x^{-2}$

$m-1 = 2m = -2 \Rightarrow m = -1.$

$-z^{-2} \cdot z' = z^{-2} - 2 \cdot x^{-2}$

$\Leftrightarrow$ let $z = tx$.

$\frac{(t\,dx + x\,dt)}{t^2x^2} + \frac{dx}{t^2x^2} = \frac{2\,dx}{x^2}$

$(t^{-1}+2)(t^{-1}-1)x^{-2}dx + t^{-2}x^{-1}dt = 0$

$x \neq 0 \Rightarrow x^{-1}\,dx = \frac{dt}{(2t+1)(t-1)}$

$\Leftrightarrow \quad \ln|x| = \frac{1}{3}\ln\left|\frac{t-1}{2t+1}\right| + C$

$\Leftrightarrow \quad \ln|x| = \frac{1}{3}\ln\left|\frac{xy-1}{xy+2}\right| + C$

P127. $y = z^m$

$2m x \cdot z^{m-1} \cdot z' + z^m = z^{2m}\sqrt{x - x^2 \cdot y^{2m}}$

$\Leftrightarrow m = -\frac{1}{2}$

$-x \cdot z^{-\frac{3}{2}} \cdot z' + z^{-\frac{1}{2}} = \frac{\sqrt{x - x^2 z^{-1}}}{z}$

$\Leftrightarrow \quad -x \cdot z^{-\frac{1}{2}} z' + z^{\frac{1}{2}} = x^{\frac{1}{2}}\sqrt{1 - x \cdot z^{-1}}$

Let $z = tx$.

$-\frac{1}{\sqrt{t}} \cdot \sqrt{x}\left(\frac{x\,dt + t\,dx}{dx}\right) + \sqrt{tx} = \sqrt{x}\sqrt{1 - t^{-1}}$

$x \equiv 0$ or $-\frac{1}{\sqrt{t}}(x\,dt + t\,dx) + \sqrt{t}\,dx = \sqrt{1 - t^{-1}}\,dx$

$\Leftrightarrow \quad -\frac{x}{\sqrt{t}}\,dt = \sqrt{1 - t^{-1}}\,dx$

$-\frac{1}{\sqrt{t-1}}\,dt = \frac{dx}{x}$

$\Leftrightarrow -2\sqrt{t-1} = \ln|x| + \hat{C}$

$\Leftrightarrow \quad \ln|x| + 2\sqrt{\frac{1}{xy^2} - 1} + C = 0$

or $x \equiv 0$

Sept. 8th.

186. $2xy\,\mathrm{d}x + (x^2 - y^2)\,\mathrm{d}y = 0.$

Check: $\dfrac{\partial(2xy)}{\partial y} = 2x, \dfrac{\partial(x^2-y^2)}{\partial x} = 2x.$

is in total differentials.

$u'_x = 2xy \Rightarrow u = x^2y + C(y)$

$u'_y = x^2 - y^2 \Rightarrow u = x^2y - \frac{1}{3}y^3 + C(x).$

$\Rightarrow u = x^2y - \frac{1}{3}y^3 + C$

Let $y = tx$.

$u = tx^3 - \frac{1}{3}t^3x^3 + C = \text{const.}$

$\frac{1}{3}t^3 = t \qquad t = \pm\sqrt{3}.$

$y = \pm\sqrt{3}x$

187. $(2 - 9xy^2)x\,\mathrm{d}x + (4y^2 - 6x^3)y\,\mathrm{d}y = 0.$

Check: $\dfrac{\partial(2-9xy^2)x\,dx}{\partial y} = 18x^2y$

$\dfrac{\partial(4y^2-6x^3)y}{\partial x} = 18x^2y$

$u'_x = (2-9xy^2)x \qquad u_x = x^2 - 3x^3y^2 + C(y)$

$u'_y = (4y^2-6x^3)y \quad u_y = y^4 - 3y^2x^3 + C(x)$

$u = x^2 + y^4 - 3x^3y^2 + C$

188. $\mathrm{e}^{-y}\,\mathrm{d}x - (2y + x\mathrm{e}^{-y})\,\mathrm{d}y = 0.$

Check: $\dfrac{\partial e^{-y}}{\partial y} = -e^{-y} \qquad \dfrac{\partial(-2y - x\cdot e^{-y})}{\partial x} = -e^{-y}$

$u'_x = e^{-y} \qquad u_x = xe^{-y} + C(y)$

$u'_y = -(2y + xe^{-y}) \qquad u_y = -y^2 + xe^{-y} + C(x).$

$u = -y^2 + x\cdot e^{-y} + C$

189. $\frac{y}{x}\,\mathrm{d}x + (y^3 + \ln x)\,\mathrm{d}y = 0.$ $x > 0.$

Check. $\dfrac{\partial(\frac{y}{x})}{\partial y} = \frac{1}{x} \qquad \dfrac{\partial(y^3+\ln x)}{\partial x} = \frac{1}{x}.$

$u'_x = \frac{y}{x} \qquad u_x = y\ln|x| + C(y)$

$u'_y = y^3 + \ln x \qquad u_y = \frac{1}{4}y^4 + y\ln x + C(y)$

$u = y\ln x + \frac{1}{4}y^4 + C \quad (x>0)$

190. $\dfrac{3x^2 + y^2}{y^2}\,\mathrm{d}x - \dfrac{2x^3 + 5y}{y^3}\,\mathrm{d}y = 0.$ $y \neq 0.$

Check: $\dfrac{\partial(\frac{3x^2+y^2}{y^2})}{\partial y} = -\frac{6x^2}{y^3} \qquad \dfrac{-\partial(\frac{2x^3+5y}{y^3})}{\partial x} = -\frac{6x^2}{y^3}$

$u_x = \dfrac{x^3 + y^2x}{y^2} + C(y) \qquad u_y = \dfrac{x^3}{y^2} + \dfrac{5}{y} + C(x)$

$u = \dfrac{x^3}{y^2} + x + \dfrac{5}{y} + C$

Sept. 12th

195. $(x^2+y^2+x)\,dx + y\,dy = 0.$

$(x^2+y^2)\,dx + \frac{1}{2}d(x^2+y^2) = 0$

Let $x^2+y^2 = z$ $(z \geq 0)$

$z\,dx = \frac{1}{2}dz.$

$x = \frac{1}{2}\ln|z| + \tilde{C}$

$\Leftrightarrow \quad x - \ln\sqrt{x^2+y^2} + C = 0$

196. $(x^2+y^2+y)\,dx - x\,dy = 0.$

$(\frac{x^2}{y^2}+1)\,dx + d(\frac{x}{y}) = 0$

Let $\frac{x}{y} = z$

$(z^2+1)\,dx + dz = 0$

$\Leftrightarrow \quad dx = -\frac{dz}{z^2+1}$

$x + \arctan\frac{x}{y} + c = 0$

197. $y\,dy = (x\,dy + y\,dx)\sqrt{1+y^2}.$

$d(\sqrt{1+y^2}) = d(xy)$

$xy - \sqrt{1+y^2} + C = 0.$

198. $xy^2(xy'+y) = 1.$

$xy^2(x\,dy + y\,dx) = dx.$

$x^2y^2\,d(xy) = x\,dx$

$x \neq 0, \quad \frac{1}{3}d(x^3y^3) = \frac{1}{2}dx^2$

$\frac{1}{3}x^3y^3 - \frac{1}{2}x^2 + C = 0.$

199. $y^2\,dx - (xy+x^3)\,dy = 0.$

$\frac{y\,dx - x\,dy}{y^2} = \frac{x^3}{y^3}dy.$

Let $\frac{x}{y} = z.$

$\Rightarrow \quad dz = z\,dy.$

$\ln|z| = y + c.$

$\Rightarrow \quad y + \ln|\frac{y}{x}| + C = 0$

200. $\left(y - \frac{1}{x}\right)dx + \frac{dy}{y} = 0.$

$xy^2dx + x\,dy - y\,dx = 0.$

$\Leftrightarrow \quad x\,dx = d(\frac{y}{x}).$

$\Leftrightarrow \quad \frac{1}{2}x^2 - \frac{y}{x} + C = 0.$

202. $y^2\,dx + (xy + \operatorname{tg} xy)\,dy = 0.$

$y\,dx + x\,dy = -\frac{\tan xy\,dy}{y}.$

$d(xy) = -\frac{\tan xy}{y}dy$

$z = xy$

$dz = -\frac{\tan z}{y}dy$

$\cot z\,dz = -\frac{dy}{y}$

$\ln|\sin z| + \ln|y| + C = 0$

$\Rightarrow \quad \ln|\sin xy| + \ln|y| + C = 0.$

203. $y(x+y)\,dx + (xy+1)\,dy = 0.$

$(x+y)\,dx + x\,dy + \frac{dy}{y} = 0$

$d(\frac{x^2}{2}) + d(xy) + d(\ln|y|) = 0$

$\Rightarrow \quad \frac{x^2}{2} + xy + \ln|y| + C = 0$

Sept. 19th.

221. а) $y' = x - y^2, \quad y(0) = 0.$

б) $y' = y^2 + 3x^2 - 1, \quad y(1) = 1.$

в) $y' = y + e^{y-1}, \quad y(0) = 1.$

г) $y' = 1 + x\sin y, \quad y(\pi) = 2\pi.$

a) $\varphi_0 \equiv 0.$

$\varphi_1 = 0 + \int_0^x (\tau - \varphi_0^2(\tau))\,d\tau = \frac{x^2}{2}$

$\varphi_2 = 0 + \int_0^x (\tau - \varphi_1^2(\tau))\,d\tau = \frac{x^2}{2} - \frac{x^5}{20}$

$\varphi_3 = 0 + \int_0^x (\tau - \varphi_2^2(\tau))\,d\tau = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400}$

b) $\varphi_0 \equiv 1.$

$\varphi_1 = 1 + \int_1^x (\varphi_0^2(\tau) + 3\tau^2 - 1)\,d\tau = x^3$

$\varphi_2 = 1 + \int_1^x (\varphi_1^2(\tau) + 3\tau^2 - 1)\,d\tau = \frac{x^7}{7} + x^3 - x + \frac{6}{7}$

c) $\varphi_0 \equiv 1$

$\varphi_1 = 1 + \int_0^x (\varphi_0(\tau) + e^{\varphi_0(\tau)-1})\,d\tau = 2x+1$

$\varphi_2 = 1 + \int_0^x (\varphi_1(\tau) + e^{\varphi_0(\tau)-1})\,d\tau = x^2 + x + \frac{e^{2x}}{2} + \frac{1}{2}$

d) $\varphi_0 \equiv 2\pi$

$\varphi_1 = 2\pi + \int_\pi^x (1 + \tau\sin\varphi_0(\tau))\,d\tau = x + \pi$

$\varphi_2 = 2\pi + \int_\pi^x (1 + \tau\sin\varphi_1(\tau))\,d\tau = 2\pi + (\tau + \tau\cos\tau - \sin\tau)\Big|_\pi^x = x + x\cos x - \sin x + 2\pi.$

Sept. 21st.

P222.

а) $y' = 2x + z, \quad z' = y; \quad y(1) = 1, \quad z(1) = 0.$

б) $\frac{dx}{dt} = y, \quad \frac{dy}{dt} = x^2; \quad x(0) = 1, \quad y(0) = 2.$

в) $y'' + y'^2 - 2y = 0; \quad y(0) = 1, \quad y'(0) = 0.$

г) $\frac{d^2x}{dt^2} = 3tx; \quad x(1) = 2, \quad \left.\frac{dx}{dt}\right|_{t=1} = -1.$

a) $\begin{cases} y_0 \equiv 1 \\ z_0 \equiv 1 \end{cases}$ $\begin{cases} y_1 = y_0 + \int_1^x f(x, y_0, z_0)\,dt = x^2 \\ z_1 = z_0 + \int_1^x f(x, y_0, z_0)\,dt = x-1 \end{cases}$ y' $\int y''$

$\begin{cases} y_2 = y_0 + \int_1^x f(x, y_1, z_1)\,dt = \frac{3}{2}x^2 - x + \frac{1}{2} \\ z_2 = z_0 + \int_1^x f(x, y_1, z_1)\,dt = \frac{1}{3}(x^3 - 1) \end{cases}$ y''

b) $\begin{cases} x_0 \equiv 1 \\ y_0 \equiv 2 \end{cases}$ $\begin{cases} x_1 = x_0 + \int_0^x f(t, x_0, y_0)\,dt = 2t+1 \\ y_1 = y_0 + \int_0^x f(t, x_0, y_0)\,dt = t+2 \end{cases}$

$\begin{cases} x_2 = x_0 + \int_0^x f(t, x_1, y_1)\,dt = 1 + \int_0^x (t+2)\,dt = \frac{t^2}{2} + 2t + 1 \\ y_2 = y_0 + \int_0^x f(t, x_1, y_1)\,dt = 2 + \int_0^x (2t+1)^2\,dt = \frac{4}{3}t^3 + 2t^2 + t + 2 \end{cases}$

c) Let $y' = z$.

$z' = 2y - z^2$.

$\begin{cases} y_0 \equiv 1 \\ z_0 \equiv 0 \end{cases}$ $\begin{cases} y_1 = y_0 + \int_0^x \cdot 0\,dt = 1 \\ z_1 = z_0 + \int_0^x (2 - 0^2)\,dt = 2x \end{cases}$

$\begin{cases} y_2 = y_0 + \int_0^x 2t\,dt = x^2 \\ z_2 = z_0 + \int_0^x (2 - 4t^2)\,dt = 2x - \frac{4}{3}x^3 \end{cases}$

d) Let $\frac{dx}{dt} = y$. $\Rightarrow \begin{cases} x' = y \\ y' = 3tx \end{cases}$

$\begin{cases} x_0 \equiv 2 \\ y_0 \equiv -1 \end{cases}$ $\begin{cases} x_1 = x_0 + \int_1^t -1\,d\tau = 2 - t + 1 = 3 - t \\ y_1 = y_0 + \int_1^t 6\tau\,d\tau = 3t^2 - 3 - 1 = 3t^2 - 4. \end{cases}$

$\begin{cases} x_2 = x_0 + \int_1^t (3\tau^2 - 4)\,d\tau = t^3 - 4t + 3 + 2 = t^3 - 4t + 5 \\ y_2 = y_0 + \int_1^t (3\tau(3 - \tau))\,d\tau = \frac{9}{2}t^2 - t^3 - \frac{3}{2} - 1 = -t^3 + \frac{9}{2}t^2 - \frac{5}{2}. \end{cases}$

Sept. 22nd

P223.

а) $y' = x + y^3, \quad y(0) = 0.$

б) $y' = 2y^2 - x, \quad y(1) = 1.$

в) $\frac{dx}{dt} = t + e^x, \quad x(1) = 0.$

г) $\frac{dx}{dt} = y^2, \ \frac{dy}{dt} = x^2, \quad x(0) = 1, \ y(0) = 2.$

a). $(x_0, y_0) = (0,0)$. $G = \{(x,y): |x - x_0| \le 1, |y - y_0| \le 1\}$.

$|x + y^3| \le 2$. $h = \min\{1, \frac{1}{2}\} = \frac{1}{2}$.

$[-\frac{1}{2}, \frac{1}{2}]$

b) $(x_0, y_0) = (1,1)$. $G = \{(x,y): |x - x_0| \le 1, |y - y_0| \le 1\}$.

$|2y^2 - x| \le 8$ $h = \min\{1, \frac{1}{8}\} = \frac{1}{8}$

$[\frac{7}{8}, \frac{9}{8}]$

c) $(t_0, x_0) = (1, 0)$. $G = \{(t,x): |t - t_0| \le 1, |x - x_0| \le 1\}$.

$|t + e^x| \le e + 2$. $h = \min\{1, \frac{1}{e+2}\} = \frac{1}{e+2}$

$[\frac{e+1}{e+2}, \frac{e+3}{e+2}]$

d) $(t_0, \vec{x_0}) = (0, \binom{1}{2})$. $G = \{(t, \vec{x}): |t - t_0| \le 1, \|\vec{x} - \vec{x_0}\| \le 1\}$

$\|(y^2, x^2)^T\| \le \sqrt{13}$. $h = \min\{1, \frac{1}{\sqrt{13}}\} = \frac{1}{\sqrt{13}}$

$[1 - \frac{1}{\sqrt{13}}, 1 + \frac{1}{\sqrt{13}}]$

137. $(2x+1)y' = 4x + 2y.$

$(2x+1)y' - 2y = 4x.$

1) $(2x+1)y' - 2y = 0.$

$$\frac{dy}{2y} = \frac{dx}{2x+1}.$$

$\Rightarrow \ln|2y| = \ln|2x+1| + C_1$

$\Rightarrow y = C(x)\cdot(2x+1).$

2) $(2x+1)[C'(x)\cdot(2x+1) + 2C(x)] = 2C(x)(2x+1) + 4x.$

$\Rightarrow C'(x)(2x+1)^2 = 4x$

$\Rightarrow C'(x) = \frac{4x}{(2x+1)^2}$

$\Rightarrow C(x) = \ln|2x+1| + \frac{1}{2x+1} + \tilde{C}$

$y = (2x+1)(\ln|2x+1| + \frac{1}{2x+1} + \tilde{C})$

138. $y' + y \operatorname{tg} x = \sec x.$

1) $y' + y \operatorname{tg} x = 0.$

$\frac{dy}{y} = -\operatorname{tg} x \, dx \Rightarrow \ln|y| = \ln|\cos x| + \tilde{C}$

$\Rightarrow y = C(x)\cos x$

2) $C(x)(-\sin x) + C'(x)\cos x + C(x)\sin x = \sec x$

$C'(x)\cos x = \frac{1}{\cos x}$

$C(x) = \tan x + C$

$y = \sin x + C\cdot\cos x.$

139. $(xy + e^x)\,dx - x\,dy = 0.$

1). $xy - xy' + e^x = 0$

Let $xy' - xy = 0$

$x\,dy = xy\,dx.$

$x \neq 0$. or $\ln|y| = x + C_1$

$y = C(x)\cdot e^x$

2) $x\cdot[C'(x)\cdot e^x + C(x)\cdot e^x] - x\cdot C(x)\cdot e^x = e^x$

$e^x \neq 0$. $x\,dC(x) = dx.$

$C(x) = \ln|x| + C$

$y = (\ln|x| + C)\cdot e^x$

140. $x^2y' + xy + 1 = 0.$

1) $x^2y' + xy = -1.$

Let $x^2y' + xy = 0$

$\Rightarrow x \neq 0.$ $xy' + y = 0$

$\Rightarrow -\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln|y| = -\ln|x| + \tilde{C}$

$y = \frac{C(x)}{x}$ $y' = \frac{C'(x)\cdot x - C(x)}{x^2}$

2) $C'(x)x - C(x) + C(x) = -1.$

$dC(x) = -\frac{dx}{x}.$ $\Rightarrow C(x) = -\ln|x| + C$

$y = \frac{-\ln|x| + C}{x}$

Sept. 26th.

151. $y' + 2y = y^2 e^x$.

1) $y \equiv 0$

2). $\frac{dy}{dx} \cdot \frac{1}{y^2} + \frac{2}{y} = e^x$

Let. $z = \frac{1}{y}$. $dy = -\frac{1}{z^2} dz$.

$-\frac{dz}{dx} + 2z = e^x$.

Let $-\frac{dz}{dx} + 2z = 0$

$\Rightarrow \quad dx = \frac{dz}{2z}$

$\Rightarrow \quad \frac{1}{2} \ln|z| = x + C_1(x)$.

$z = C(x) \cdot e^{2x}$

$-C'(x) \cdot e^{2x} - 2e^{2x} C(x) + 2C(x) \cdot e^{2x} = e^x$

$\Rightarrow \quad C'(x) \cdot e^x + 1 = 0$.

$\Rightarrow \quad \frac{dC(x)}{dx} \cdot e^x + 1 = 0$.

$\Rightarrow \quad dC(x) = -\frac{dx}{e^x}$

$C(x) = \frac{1}{e^x} + C$

$\frac{1}{y} = e^x + C \cdot e^{2x}$ or $y \equiv 0$.

153. $y' = y^4 \cos x + y \operatorname{tg} x$.

$\Rightarrow \frac{y'}{y^4} - \frac{\operatorname{tg} x}{y^3} = \cos x$.

$z = y^{-3} \qquad y' = -\frac{1}{3} \cdot z^{-\frac{4}{3}} \cdot z'$

$-\frac{1}{3} \cdot z^{-\frac{4}{3}} \cdot z' \cdot (z^{\frac{4}{3}}) - \operatorname{tg} x \cdot z = \cos x$.

$z' + 3\operatorname{tg} x \, z = -3\cos x$.

Let $z' + 3\operatorname{tg} x \, z = 0$.

$\frac{dz}{z} = -3\operatorname{tg} x \, dx. \Rightarrow \ln|z| = 3\ln|\cos x| + \tilde{C}$

$z = C(x) \cdot \cos^3 x$

$C'(x) \cdot \cos^3 x + 3\cos x = 0$. $\cos x \neq 0$

$C'(x) = -3\sec^2 x \qquad C(x) = -3\operatorname{tg} x + C$

$\frac{1}{y^3} = (-3\operatorname{tg} x + C) \cdot \cos^3 x$. or $y \equiv 0$.

152. $(x+1)(y' + y^2) = -y$.

$(x+1) y' + y = -(x+1) y^2$

1) $y \equiv 0$

2). Let $z = y^{-1} \qquad y' = -\frac{z'}{z^2}$

$-(x+1) z' + z = -(x+1)$

$\Rightarrow (x+1) z' - z = (x+1)$

Let $(x+1) z' - z = 0$.

$\frac{dz}{z} = \frac{dx}{x+1} \Rightarrow \ln|z| = \ln|x+1| + \tilde{C}(x)$

$z = C(x)(x+1)$

$[C'(x)(x+1) + C(x)](x+1) - C(x)(x+1) = (x+1)$.

$x \neq -1$. $C'(x)(x+1) = 1 \Rightarrow dC(x) = \frac{dx}{x+1}$

$C(x) = \ln|x+1| + C$

$\frac{1}{y} = (x+1)(\ln|x+1| + C)$ or $y \equiv 0$

156. $xy' - 2x^2\sqrt{y} = 4y$.

$\Rightarrow y' \cdot \frac{1}{\sqrt{y}} - 4\sqrt{y} = 2x^2$

Let $z = \sqrt{y}$, $y' = 2z \cdot z'$

$\Rightarrow 2z' - 4z = 0. \Rightarrow \frac{dz}{2z} = dx$

$\Rightarrow \ln|z| = 2x + \tilde{C} \Rightarrow z = e^{2x} \cdot C(x)$.

$2 \cdot C'(x) \cdot e^{2x} = 2x^2 \Rightarrow dC(x) = \frac{x^2 \cdot dx}{e^{2x}}$

$\int x^2 \cdot e^{-2x} dx = x^2 \cdot (-\frac{1}{2}) \cdot e^{-2x} + \int x \cdot e^{-2x} dx$

$= -\frac{x^2}{2} \cdot e^{-2x} + x \cdot (-\frac{1}{2}) \cdot e^{-2x} + \int e^{-2x} dx$

$= -(\frac{x^2}{2} + \frac{x}{2} + \frac{1}{2}) \cdot e^{-2x} + C$

$\sqrt{y} = -\frac{1}{2}(x^2 + x + 1) + C \cdot e^{2x}$

157. $xy' + 2y + x^5y^3e^x = 0.$

$$\frac{y'}{y^3} + \frac{2}{x}\cdot\frac{1}{y^2} = -x^4\cdot e^x$$

$$z = \frac{1}{y^2}, \quad y = \frac{1}{\sqrt{z}}, \quad y' = -\frac{1}{2}\cdot z^{-\frac{3}{2}}\cdot z'$$

$$-\frac{1}{2}z' + \frac{2}{x}\cdot z = -x^4\cdot e^x$$

$$\Rightarrow \quad z' - \frac{4}{x}\cdot z = x^4\cdot e^x$$

$$\Rightarrow \quad \frac{dz}{z} = 4\frac{dx}{x}. \Rightarrow z = x^4\cdot c(x)$$

$$c'(x)\cdot x^4 = x^4\cdot e^x. \quad x \neq 0.$$

$$c(x) = e^x + c$$

$$\frac{1}{y^2} = x^4(e^x + c) \quad \text{or} \quad y \equiv 0$$

159. $y'x^3\sin y = xy' - 2y.$

$$y' = \frac{2y}{x - x^3\sin y}$$

$$x' = \frac{x - x^3\sin y}{2y}$$

$$x' - \frac{x}{2y} = -\frac{x^3\sin y}{2y}$$

$$\text{Let } z = \frac{1}{x^2} \quad x' = -\frac{1}{2}\cdot z^{-\frac{3}{2}}\cdot z'$$

$$z' + \frac{1}{y}z = \frac{\sin y}{y}.$$

$$\text{Let } z' + \frac{1}{y}z = 0. \quad \frac{dz}{dy} = -\frac{z}{y}.$$

$$z = \frac{c(y)}{y}. \quad \frac{c'(y)}{y} = \frac{\sin y}{y}$$

$$c(y) = -\cos(y) + c$$

$$\Rightarrow \quad \frac{1}{x^2} = \frac{-\cos(y) + c}{y}.$$

Sept. 28th

145. $(x + y^2)\,dy = y\,dx.$

$$\frac{dy}{dx} = \frac{y}{x + y^2}$$

$$x' - \frac{x}{y} = y$$

$$\text{Let } x' - \frac{x}{y} = 0. \quad \frac{dx}{dy} = \frac{x}{y}$$

$$x = c(y)\cdot y$$

$$c'(y)\cdot y + c(y) - c(y) = y.$$

$$c'(y) = 1 \Rightarrow c(y) = y + c$$

$$x = y^2 + cy.$$

146. $(2e^y - x)y' = 1.$

$$x' + x = 2e^y$$

$$\Rightarrow \frac{dx}{dy} + x = 0 \Rightarrow \frac{dx}{x} = -dy.$$

$$\Rightarrow \ln|x| = -y + \tilde{c} \Rightarrow x = c(y)\cdot e^{-y}$$

$$c'(y)\cdot e^{-y} = 2e^y$$

$$\Rightarrow c'(y) = 2\cdot e^{2y} \Rightarrow c(y) = e^{2y} + c$$

$$x = e^y + c\cdot e^{-y}$$

147. $(\sin^2 y + x\,\mathrm{ctg}\,y)y' = 1.$

$$x' - \mathrm{ctg}\,y\,x = \sin^2 y$$

$$\text{Let } \frac{dx}{dy} = \mathrm{ctg}\,y\,x \Rightarrow \frac{dx}{x} = \mathrm{ctg}\,y\,dy$$

$$\ln|x| = \ln|\sin y| + \tilde{c} \quad x = c(y)\cdot\sin y.$$

$$c'(y)\sin y = \sin^2 y$$

$$c(y) = -\cos y + c$$

$$x = \sin y(-\cos y + c)$$

160. $(2x^2 y \ln y - x)y' = y.$

$x' = 2x^2 \ln y - \frac{x}{y}.$

$x' + \frac{x}{y} = 2x^2 \ln y$

Let $z = \frac{1}{x}$ $\quad x' = -\frac{1}{z^2} z'$

$z' - \frac{1}{y} \cdot z = -2 \ln y.$

Let $z' - \frac{1}{y} \cdot z = 0.$

$\Rightarrow \frac{dz}{dy} = -\frac{z}{y}. \qquad z = \frac{c(y)}{y}.$

$\frac{c'(y)}{y.} = -2 \ln y.$

$c(y) = -y^2 \cdot \ln y + \frac{y^2}{2} + c$

$\frac{1}{x} = \frac{-y^2 \ln y + \frac{y^2}{2} + c}{y}$

148. $(2x + y)\,\mathrm{d}y = y\,\mathrm{d}x + 4 \ln y\,\mathrm{d}y.$

$2x + y = y \cdot x' + 4 \ln y$

$x' - \frac{2}{y} x = 1 - \frac{4 \ln y}{y}$

let $x' - \frac{2}{y} x = 0$

$\frac{dx}{dy} = \frac{2x}{y} \Rightarrow x = c(y) \cdot y^2$

$c'(y) \cdot y^2 = 1 - 4 \ln y \cdot y^{-1}$

$c'(y) = y^{-3} - 4 \ln y \cdot y^{-2}$

$c(y) = -\frac{1}{2} \cdot y^{-2} + 4 \frac{1 + \ln y}{y} + c$

$x = -\frac{1}{2} + 4y(1 + \ln y) + c \cdot y^2$

149. $y' = \frac{y}{3x - y^2}.$

$x' - \frac{3}{y} x = -y$

$\Rightarrow \frac{dx}{dy} = \frac{3x}{y} \Rightarrow x = y^3 \cdot c(y)$

$c'(y) \cdot y^3 = -y$

$\Rightarrow c'(y) = -\frac{1}{y^2} \Rightarrow c(y) = \frac{1}{y} + c$

$x = y^2 + c \cdot y^3$

Oct. 10th.

167. $x^2y' + xy + x^2y^2 = 4.$

find particular solution $y = \frac{2}{x}$.

$y = z + \frac{2}{x}$.

$\Rightarrow z' + \frac{5}{x}z = -z^2$

$\Rightarrow \frac{z'}{z^2} + \frac{5}{x}\cdot\frac{1}{z} = -1.$

$\frac{1}{z} = t$, $dz = -\frac{dt}{t^2}$

$\frac{dt}{dx} - \frac{5t}{x} = 1.$

Consider $\frac{dt}{t} = \frac{5dx}{x} \Rightarrow t = c(x)\cdot x^5$

$c'(x)\cdot x^5 = 1$

$c(x) = -\frac{x^{-4}}{4} + c \Rightarrow t = -\frac{x}{4} + \tilde{c}\cdot x^5$,

$z = \frac{4}{cx^5 - x}$ $\quad y = \frac{2}{x} + \frac{4}{cx^5 - x}$

168. $3y' + y^2 + \frac{2}{x^2} = 0.$

Let $y = \frac{a}{x}$.

$-\frac{3a}{x^2} + \frac{a^2}{x^2} + \frac{2}{x^2} = 0.$ $\quad a = 1$ or 2.

Let $y = z + \frac{1}{x}$.

$3z' - \frac{3}{x^2} + z^2 + 2\frac{z}{x} + \frac{1}{x^2} + \frac{2}{x^2} = 0.$

$\Rightarrow 3z' + 2\frac{1}{x}z = -z^2$

$\Rightarrow \frac{3z'}{z^2} + 2\frac{1}{x}\cdot\frac{1}{z} = -1.$ Let $\frac{1}{z} = t$.

$\frac{3dt}{dx} - \frac{2t}{x} = 1.$

Consider $\frac{3dt}{dx} = \frac{2t}{x}$.

$t = c(x)\cdot x^{\frac{2}{3}}$

$c'(x)\cdot x^{\frac{2}{3}} = 1.$ $\quad c(x) = \frac{2}{5}\cdot x^{\frac{5}{2}} + c$

$\frac{x}{xy-1} = \left(\frac{2}{5}\cdot x^{\frac{5}{2}} + c\right)\cdot x^{\frac{2}{3}}$

169. $xy' - (2x+1)y + y^2 = -x^2.$

find particular solution $y = x$

Let $y = z + x$.

$x(z'+1) - (2x+1)(z+x) + (z+x)^2 = -x^2$

$\Rightarrow xz' - (2x+1)z + z^2 + 2zx = 0$

$\Rightarrow z' - \frac{1}{x}z = -\frac{z^2}{x}$

$\Rightarrow \frac{z'}{z^2} - \frac{1}{x}\cdot\frac{1}{z} = -\frac{1}{x}.$ Let $\frac{1}{z} = t$.

$\Rightarrow \frac{dt}{dx} = \frac{1-t}{x}$

$\Rightarrow \frac{dt}{1-t} = \frac{dx}{x}$

$\Rightarrow \tilde{c} - \ln|1-t| = \ln|x|$

$\Rightarrow cx = \frac{1}{1-t} \Rightarrow z = \frac{cx}{cx-1}$

$y = x + \frac{cx}{cx-1}$

170. $y' - 2xy + y^2 = 5 - x^2.$

Let $y = ax + b$.

$a - 2x(ax+b) + (ax+b)^2 = 5 - x^2$

$\begin{cases} a + b^2 = 5 \\ -2b + 2ab = 0 \\ -2a + a^2 = -1 \end{cases} \Rightarrow \begin{cases} a = 1 \\ b = \pm 2. \end{cases}$

Let $y = x + 2 + z$

$1 + z' - 2x(x+2+z) + (x+z+2)^2 = 5 - x^2$

$z' - 4x - 2zx + z^2 + 2xz + 4z + 4x = 0$

$z' + 4z + z^2 = 0$

$-\frac{dz}{z(z+4)} = dx$

$x = \frac{1}{4}\ln\left|\frac{z+4}{z}\right| + c$

$= \frac{1}{4}\ln\left|1 + \frac{4}{y-x-2}\right| + c$

171. $y' + 2ye^x - y^2 = e^{2x} + e^x.$

A particular solution: $y = e^x$

$$y = e^x + z$$

$$e^x + z' + 2e^{2x} + 2z \cdot e^x - e^{2x} - 2 \cdot e^x z - z^2 = e^{2x} + e^x$$

$$\frac{dz}{dx} = z^2$$

$$x = -\frac{1}{z} + c \Rightarrow z = \frac{1}{c-x}$$

$$y = e^x + \frac{1}{c-x}$$

Question list.

1. Eq. with separable variable.

2. Homogenuous equation

Affine change $f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right).$

Power change

1. $y' - xy^2 = 2xy \qquad y(0)=1$

$\frac{dy}{dx} = xy(y+2)$ (check $y\equiv 0$ or $y \equiv 2$ impossible)

$\frac{dy}{y(y+2)} = x\,dx.$

$\Rightarrow c + \ln\left|\frac{y}{y+2}\right| = x^2$

$y(0)=1 \Rightarrow c = \ln 3.$

$x^2 = \ln\left|\frac{3y}{(y+2)}\right|$

2. $xy' = \sqrt{x^2-y^2} + y$

Let $y = tx.$

$x(t\,dx + x\,dt) = [x\cdot\sqrt{1-t^2} + tx]\,dx$

$x\equiv 0$ means $y\equiv 0.$

$x \neq 0 \Rightarrow \quad x\,dt = \sqrt{1-t^2}\,dx.$

$1-t^2=0. \quad t=\pm 1 \quad y = \pm x.$

$t^2-1\neq 0 \Rightarrow \quad \frac{dt}{\sqrt{1-t^2}} = \frac{dx}{x}.$

$\arcsin\frac{y}{x} = \ln|x| + C. \quad y = \pm x$

3. find $\begin{cases} y+2=0 \\ x+y-1=0 \end{cases} \Rightarrow \begin{cases} y=-2. \\ x=3. \end{cases}$

$\begin{cases} u = y+2. \\ v = x-3. \end{cases}$

$\frac{du}{dv} = 2\left(\frac{u}{u+v}\right)^2$

Let $u = tv.$

$\frac{t\,dv + v\,dt}{dv} = 2\left(\frac{t}{1+t}\right)^2$

$t^3dv + 2t^2dv + t\,dv + t^2v\,dt + 2vt\,dt + v\,dt = 2t^2\,dv$

$t(t^2+1)\,dv + v(t^2+2t+1)\,dt = 0$

$\frac{dv}{v} = -\frac{(t+1)^2}{t(t^2+1)}\,dt$

$\ln|v| = -\ln|t| - 2\arctan t. + \tilde{C}$

$\ln|u| = -2\arctan\left(\frac{u}{v}\right) + \tilde{C}$

$y = e^{-2\arctan\left(\frac{y+2}{x-3}\right)} + C$

$$2xy' + y = y^2\sqrt{x - x^2y^2}.$$

$$y = z^m. \qquad y = \frac{1}{\sqrt{z}}$$

$$m\,2x \cdot z^{m-1} \cdot z' + z^m = z^{2m}\sqrt{x - x^2 \cdot z^{2m}}$$

$$\begin{cases} 2+2m = 1. \\ \frac{1}{2} + 2m = m = m. \end{cases} \Rightarrow m = -\frac{1}{2}.$$

$$-x \cdot z^{-\frac{3}{2}} \cdot z' + z^{-\frac{1}{2}} = z^{-1}\sqrt{x - x^2 z^{-1}}$$

$$-x \cdot z' + z = \sqrt{xz - x^2}$$

Let $z = tx$.

$$-x(t\,dx + x\,dt) + tx \cdot dx = x\sqrt{t-1}\,dx$$

$$-x^2 dt = x\sqrt{t-1}\,dx$$

$$x \neq 0. \qquad -\frac{dt}{\sqrt{t-1}} = \frac{dx}{x}$$

$$\Rightarrow -2\sqrt{t-1} + \tilde{C} = \ln|x|$$

$$x = c \cdot e^{-2\sqrt{t-1}}$$

$$\Rightarrow x = c \cdot e^{-2\sqrt{\frac{1-xy^2}{xy^2}}}$$

Oct. 19th.

533. $y''-2y'-3y=e^{4x}$.

$P(\lambda)=\lambda^2-2\lambda-3=0 \quad \lambda=3 \text{ or } -1$

$\psi(t)=ce^{4t}$

$16ce^{4t}-8ce^{4t}-3ce^{4t}=e^{4t}$

$\Rightarrow \quad c=\frac{1}{5}$

$\varphi(t)=c_1e^{3t}+c_2e^{-t}+\frac{1}{5}e^{4t}$

$x''-x=2e^t$

535. $y''-y=2e^x$

$P(\lambda)=\lambda^2-1=0 \quad \Rightarrow \lambda=\pm1.$

$\psi(t)=cte^t \quad \psi'(t)=c(t+1)e^t \quad \psi''(t)=c(t+2)e^t$

$\Rightarrow \quad c(t+2)e^t-cte^t=2e^t$

$c=1$

particular solution $\psi(t)=te^t$

consider $x''-x=0. \quad \varphi_1=e^t \quad \varphi_2=e^{-t}$

general solution $\varphi(t)=c_1e^t+c_2e^{-t}+te^t$

534. $y''+y=4xe^x$.

$P(\lambda)=\lambda^2+1=0 \quad \lambda=\pm i$

$\psi(x)=a_1xe^x+a_2e^x$

$(a_1+2)xe^x+a_2e^x+a_1xe^x+a_2e^x=4xe^x$

$\Rightarrow \begin{cases} a_1=1 \\ a_2=0 \end{cases} \quad \psi(x)=x\cdot e^x$

$\varphi(x)=c_1\cos x+c_2\sin x+x\cdot e^x$

536. $y''+y'-2y=3xe^x$.

$P(\lambda)=\lambda^2+\lambda-2 \quad \Rightarrow \lambda=-2 \text{ or } 1.$

$P(\lambda_0)=0. \quad d=1$

$\psi(x)=x(a_1xe^x+a_2e^x)$

$a_1(x^2+4x+2)e^x+a_2(x+2)e^x+a_1(x^2+2x)e^x+a_2(x+1)e^x-2a_1x^2e^x-2a_2xe^x$
$=3xe^x$

$\begin{cases} 2a_1+2a_2+a_2=0 \\ 4a_1+a_2+2a_1+a_2-2a_2=3. \end{cases} \Rightarrow \begin{cases} a_1=\frac{1}{2} \\ a_2=-\frac{1}{3} \end{cases}$

$\varphi(x)=c_1e^{-2x}+c_2e^x+\frac{x^2}{2}e^x-\frac{x}{3}e^x$

537. $y''-3y'+2y=\sin x$.

$P(\lambda)=y^2-3y+2=0 \Rightarrow 1 \text{ or } 2. \quad \lambda_0=i.$

Let. $\psi(x)=(a_1\cos x+a_2\sin x)$

$\begin{cases} -a_1+3a_2+2a_1=0 \\ -a_2-3a_1+2a_2=1 \end{cases} \Rightarrow \begin{cases} a_1=-\frac{3}{10} \\ a_2=\frac{1}{10} \end{cases}$

$\varphi(x)=c_1e^x+c_2e^{2x}+(-\frac{3}{10}\cos x+\frac{1}{10}\sin x).$

538. $y''+y=4\sin x$.

$P(\lambda)=\lambda^2+1 \Rightarrow \lambda=\pm i. \quad \lambda_0=i$

$\psi(x)=x(a\cos x+b\sin x)$

$\psi'(x)=a\cos x+b\sin x+x(b\cos x-a\sin x)$

$\psi''(x)=-a\sin x+b\cos x+b\cos x-a\sin x$
$-x(a\cos x+b\sin x)$

$\begin{cases} -a-a=4 \\ 2b=0 \end{cases} \Rightarrow \begin{cases} a=-2 \\ b=0. \end{cases}$

$\varphi(x)=c_1\cos x+c_2\sin x-2x\cos x$

539. $y''-5y'+4y=4x^2e^{2x}$.

$P(\lambda)=\lambda^2-5\lambda+4 \quad \lambda=1 \text{ or } 4.$

$\psi(x)=(a_2x^2+a_1x+a_0)\cdot e^{2x}$

$\psi'=(2a_2x+a_1)e^{2x}+2(a_2x^2+a_1x+a_0)\cdot e^{2x}$
$=[2a_2x^2+(2a_2+2a_1)x+a_1+2a_0]\cdot e^{2x}$

$\psi''=[4a_2x^2+(8a_2+4a_1)x+4a_1+4a_0+2a_2]\cdot e^{2x}$

$\begin{cases} a_2=1 \\ a_1=-2 \\ a_0=1.5 \end{cases} \Rightarrow \quad \varphi(x)=c_1e^{4x}+c_2e^x+(x^2-2x+1.5)\cdot e^{2x}$

540. $y''-3y'+2y=x\cos x.$

$P(\lambda)=\lambda^2-3\lambda+2=0 \quad \lambda=1 \text{ or } 2. \quad \lambda_0=i$

$\psi(x)= (a_1x+b_1)\cos x+(a_2x+b_2)\sin x.$

$\psi'(x)= a_1\cos x - a_1x\sin x - b_1\sin x + a_2\sin x + a_2x\cos x + b_2\cos x$

$= (a_1+b_2)\cos x + (a_2-b_1)\sin x + a_2x\cos x - a_1x\sin x.$

$\psi''(x) = -(a_1+b_2)\sin x + (a_2-b_1)\cos x + a_2\cos x - a_2x\sin x - a_1\sin x - a_1x\cos x$

$$\begin{cases} -a_1-3a_2+2a_1=1 \\ -a_2+3a_1+2a_2=0 \\ -(a_1+b_2)-a_1-3(a_2-b_1)+2b_2=0 \\ (a_2-b_1)+a_2-3(a_1+b_2)+2b_1=0 \end{cases} \Rightarrow \begin{cases} a_1=\frac{1}{10} \\ a_2=-\frac{3}{10} \\ b_1=-\frac{3}{25} \\ b_2=-\frac{17}{50} \end{cases}$$

$\varphi(x)=C_1e^x+C_2e^{2x}+(\frac{1}{10}x-\frac{3}{25})\cos x+(-\frac{3}{10}x-\frac{17}{50})\sin x.$

541. $y''+3y'-4y=e^{-4x}+xe^{-x}.$

① $y''+3y'-4y=e^{-4x}.$

$P_1(\lambda)=\lambda^2+3\lambda-4\lambda \qquad \lambda=-4 \text{ or } 1.$

$\psi_1(x)=cx\cdot e^{-4x}$

$\psi_1'(x)=c\cdot e^{-4x}+cx\cdot(-4)\cdot e^{-4x}=c(-4x+1)\cdot e^{-4x}$

$\psi_1''(x)=-4c\cdot e^{-4x}+-4c(-4x+1)\cdot e^{-4x}$

$=-4c(-4x+1)\cdot e^{-4x}$

$$\begin{cases} 16c-12c-4c=0 \\ -4c+3c=1 \end{cases} \Rightarrow c=-1$$

② $y''+3y'-4y=x\cdot e^{-x}. \quad \lambda_0=-1$

$\psi_2(x)=(a_1x+a_2)\cdot e^{-x}$

$\psi_2'(x)= a_1\cdot e^{-x}-(a_1x+a_2)\cdot e^{-x}=(-a_1x+a_1-a_2)\cdot e^{-x}$

$\psi_2''(x)= -a_1\cdot e^{-x}+(a_1x+a_2-a_1)\cdot e^{-x}=(a_1x+a_2-2a_1)\cdot e^{-x}$

$$\begin{cases} a_1-a_1+a_1=1 \\ a_2+a_1-a_2+a_2-2a_1=0 \end{cases} \Rightarrow \begin{cases} a_1=1 \\ a_2=1 \end{cases}$$

$\varphi(t)=c_1e^{-4t}+c_2e^{t}+-x\cdot e^{-4x}+(x+1)\cdot e^{-x}$

542. $y''+2y'-3y=x^2e^x$.

$\lambda=-3$ or 1.

$\psi(x) = x(ax^2+bx+c)\cdot e^x$

$\psi'(x) = (3ax^2+2bx+c+ax^3+bx^2+cx)\cdot e^x$

$= [ax^3+(3a+b)\cdot x^2+(2b+c)x+c]\cdot e^x$

$\psi''(x) = \{[3ax^2+(6a+2b)x+2b+c]+[ax^3+(3a+b)x^2+(2b+c)x+c]\}\cdot e^x$

$= (ax^3+(6a+b)x^2+(6a+4b+c)x+2(b+c))\cdot e^x$

$$\begin{cases} a+2a-3a = 0 \\ b+6a+2(b+3a)-3b = 1 \\ 6a+4b+c+2(2b+c)-3c=0 \\ 2(b+c)+2c-3c=0 \end{cases} \Rightarrow \begin{cases} a=\frac{1}{12} \\ b=-\frac{1}{16} \\ c=\frac{1}{8} \end{cases}$$

$\psi(x) = c_1e^{-3x}+c_2e^x+\left(\frac{x^3}{12}-\frac{x^2}{16}+\frac{x}{8}\right)\cdot e^x$

Oct. 20th.

575. $y''-2y'+y=\frac{e^x}{x}$.

$P(\lambda) = \lambda^2-2\lambda+1=0 \quad \lambda=1.$

$\varphi_1=e^x \quad \varphi_2=x\cdot e^x$

$\psi = c_1(x)\cdot e^x + c_2(x)x\cdot e^x$

$$\begin{cases} \dot{c}_1e^x+\dot{c}_2x\cdot e^x=0 \\ \dot{c}_1e^x+\dot{c}_1xe^x+c_2e^x=\frac{e^x}{x} \end{cases} \Rightarrow \begin{cases} c_2=\ln|x| \\ c_1=-x \end{cases}$$

576. $y''+3y'+2y=\frac{1}{e^x+1}$.

$P(\lambda) = \lambda^2+3\lambda+2=0. \quad \lambda=-1$ or -2.

$\psi = c_1(x)\cdot e^{-x}+c_2(x)\cdot e^{-2x}$

$$\begin{cases} \dot{c}_1(x)\cdot e^{-x}+\dot{c}_2(x)\cdot e^{-2x}=0 \\ -\dot{c}_1(x)\cdot e^{x}-2\dot{c}_2(x)\cdot e^{-2x}=\frac{1}{e^x+1} \end{cases}$$

$-\dot{c}_2(x)\cdot e^{-2x}=\frac{1}{e^x+1}$

$\dot{c}_2(x) = -\frac{e^{2x}}{e^x+1} = -\frac{e^{2x}+2e^x+1}{e^x+1}+\frac{2(e^x+1)}{e^x+1}-\frac{1}{e^x+1}$

$= -e^x+1-\frac{1}{e^x+1}$

$c_2(x) = -e^x+x-\ln\left|\frac{e^x}{e^x+1}\right|+c_2$

$\dot{c}_1(x)\cdot e^{-x} = \frac{1}{e^x+1}$

$\dot{c}_1(x) = 1-\frac{1}{e^x+1}$

$c_1(x) = x+\ln|1+e^{-x}|+c_1.$

$\psi(x) = e^{-x}(x+\ln|1+e^{-x}|+c_1)$

$+e^{-2x}\left(-e^{x}+x-\ln\left|\frac{e^x}{e^x+1}\right|+c_2\right).$

579 $\cancel{578}$

577. $y''+y=\dfrac{1}{\sin x}$.

$\lambda = \pm i$

$\psi(x) = C_1(x)\cos x + C_2 \sin x$

$$\begin{cases} \dot{C_1}\cos x + \dot{C_2}\sin x = 0 \\ -\dot{C_1}\sin x + \dot{C_2}\cos x = \dfrac{1}{\sin x} \end{cases} \Rightarrow \begin{cases} C_1(x) = -x + C_1 \\ C_2(x) = \ln|\sin x| + C_2 \end{cases}$$

$\psi(x) = (-x+C_1)\cos x + (\ln|\sin x| + C_2)\cdot \sin x$

579. $y''+2y'+y=3e^{-x}\sqrt{x+1}$.

$\lambda = -1$.

$\psi(x) = C_1(x)\cdot e^{-x} + C_2(x)\cdot x\cdot e^{-x}$

$$\begin{cases} \dot{C_1}\cdot e^{-x} + \dot{C_2}\, x\cdot e^{-x} = 0 \\ -\dot{C_1}\cdot e^{-x} + \dot{C_2}\,(1-x)\cdot e^{-x} = 3e^{-x}\sqrt{x+1} \end{cases} \Rightarrow \begin{cases} C_1(x) = -2x(x+1)^{\frac{3}{2}} - \frac{4}{5}(x+1)^{\frac{5}{2}} + C_1 \\ C_2(x) = 2(x+1)^{\frac{3}{2}} + C_2 \end{cases}$$

$\Rightarrow \quad \dot{C_1} = -x\,\dot{C_2}$

578. $y''+4y=2\operatorname{tg} x$.

$\lambda = \pm 2i$

$\psi(x) = C_1(x)\cos 2x + C_2(x)\sin 2x$

$$\begin{cases} \dot{C_1}\cos 2x + \dot{C_2}\sin 2x = 0 \\ -2\dot{C_1}\sin 2x + 2\dot{C_2}\cos 2x = 2\operatorname{tg} x \end{cases} \Rightarrow \begin{cases} \dot{C_1} = -\dfrac{\sin x}{\cos x}\cdot \sin 2x. \\ \dot{C_2} = \dfrac{\sin x}{\cos x}\cdot \cos 2x \end{cases}$$

Oct. 24th.

589. $x^2y''-4xy'+6y=0.$

Let $x=e^t$. $\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}=e^t\cdot\frac{dy}{dx}$ $\quad \frac{dy}{dx}=e^{-t}\cdot\frac{dy}{dt}$

$$\frac{d^2y}{dx^2}=\frac{d\left(\frac{dy}{dx}\right)}{dx}=\frac{d\left(e^{-t}\cdot\frac{dy}{dt}\right)}{dt}\cdot\frac{dt}{dx}=\frac{-e^{-t}dt\cdot\frac{dy}{dt}+e^{-t}\cdot\frac{d^2y}{dt}}{\cdot dt}\cdot e^{-t}$$

$$=-e^{-2t}\frac{dy}{dt}+e^{-2t}\frac{d^2y}{dt^2}$$

$$(e^{-2t}\cdot y''_{tt}-e^{-2t}y'_t)e^{2t}-4e^t(e^{-t}y'_t)+6y=0.$$

$$y''_{tt}-5y'_t+6=0$$

$$\Rightarrow \varphi(t)=c_1e^{2t}+c_2e^{3t} \qquad \varphi(x)=c_1x^2+c_2x^3.$$

590. $x^2y''-xy'-3y=0.$

$x=e^t.$

$$(e^{-2t}y''_{tt}-e^{-2t}y'_t)\cdot e^{2t}-e^t(e^{-t}y'_t)-3y=0.$$

$$y''_{tt}-2y'_t-3=0$$

$$\Rightarrow \quad \varphi(x)=c_1x^3+c_2x^{-1}$$

591. $x^3y'''+xy'-y=0.$

$$\frac{d^3y}{dx^3}=\frac{d\left(\frac{d^2y}{dx}\right)}{dt}\cdot\frac{dt}{dx}=\frac{d\left(e^{-2t}\cdot\frac{d^2y}{dt^2}-e^{-2t}\cdot\frac{dy}{dt}\right)}{dt}\cdot e^{-t}$$

$$=\left[-2\cdot e^{-2t}\cdot\frac{d^2y}{dt^2}+2e^{-2t}\frac{dy}{dt}+e^{-2t}\cdot\frac{d^3y}{dt^3}-e^{-2t}\frac{d^2y}{dt^2}\right)\cdot e^{-t}$$

$$=e^{-3t}\frac{d^3y}{dt^3}-3\cdot e^{-3t}\cdot\frac{d^2y}{dt^2}+2\cdot e^{-3t}\frac{dy}{dt}$$

$$y'''-3y''+3y'-y=0. \qquad \Rightarrow \lambda=1. \text{ 3-multiple.}$$

$$\varphi(t)=c_1t^2e^t+c_2te^t+c_3e^t$$

$$\varphi(x)=c_1x\ln^2|x|+c_2x\ln|x|+c_3\cdot x.$$

592. $x^2y''' = 2y'$.

1) $x \equiv 0$.

2). $x \neq 0$. $\quad x^3 y''' - 2x y' = 0$

$$y''' - 3y'' + 2y' - 2y' = 0.$$

$$\begin{cases} \lambda = 0 \Rightarrow 2 \text{ multiple.} \\ \lambda = 3 \end{cases}$$

$$\varphi(t) = C_1 e^{3t} + C_2 t + C_3$$

$$\varphi(x) = C_1 x^3 + C_2 \ln|x| + C_3.$$

593. $x^2y'' - xy' + y = 8x^3$.

Consider $x^2y'' - xy' + y = 0 \quad x = e^t$

$\Rightarrow y''_{tt} - y'_t - y'_t + y = 0$

$\lambda = 1$. 2-multiple. $\quad \varphi_1(t) = e^t \quad \varphi_2(t) = t \cdot e^t$

$$\begin{cases} \dot{c}_1(t) \cdot e^t + \dot{c}_2(t) \cdot t \cdot e^t = 0 \\ \dot{c}_1(t) \cdot e^t + \dot{c}_2(t) \cdot (1+t) e^t = 8e^{3t} \end{cases} \Rightarrow \begin{cases} \dot{c}_2(t) = 8e^{2t}. \\ \dot{c}_1(t) = -8t \cdot e^{2t} \end{cases}$$

$$c_1(t) = -2(2t-1) \cdot e^{2t} + C_1$$

$$c_2(t) = 4 \cdot e^{2t} + C_2.$$

$$y = \left[-4x^2 \ln|x| + 2x^2 + C_1\right] \cdot x + \left[4x^2 + C_2\right] \cdot x \ln|x|$$

$$= 2x^3 + C_1 x + C_2 x \ln|x|$$

Oct. 26th.

681. $(2x+1)y''+4xy'-4y=0.$

A particular solution $y=x$.

$$\Rightarrow y''+\frac{4x}{2x+1}y'-\frac{4}{2x+1}y=0.$$

By Liouville's formula.

$$W(x)=c\cdot e^{-\int\frac{4x}{2x+1}dx}=c\cdot(2x+1)\cdot e^{-2x}$$

$$\begin{vmatrix}\varphi_1 & \varphi_2\\ \varphi_1' & \varphi_2'\end{vmatrix}=c\cdot(2x+1)\cdot e^{-2x}$$

$$\Rightarrow x\cdot\varphi_2'-\varphi_2=c\cdot(2x+1)\cdot e^{-2x}.$$

$$\Rightarrow \varphi_2'-\frac{1}{x}\varphi_2=c\cdot\frac{(2x+1)}{x}\cdot e^{-2x}$$

$$\Rightarrow \varphi_2'-\frac{\varphi_2}{x}=0.\quad \varphi_2'=c(x)\cdot x.$$

$$\Rightarrow c'(x)\cdot x=c\cdot\frac{(2x+1)}{x}\cdot e^{-2x}$$

$$c'(x)=c\cdot\frac{(2x+1)\cdot e^{-2x}}{x^2}$$

$$c(x)=-\frac{c}{e^{2x}\cdot x}+C_2$$

$$\Rightarrow y=C_1\cdot e^{-2x}+C_2x.$$

683. $xy''-(2x+1)y'+(x+1)y=0.$

particular solution $y=e^x$.

$$\Rightarrow y''-\frac{2x+1}{x}y'+\frac{x+1}{x}y=0$$

$$W(x)=c\cdot e^{\int\frac{2x+1}{x}dx}=cx\cdot e^{2x}$$

$$\begin{vmatrix}e^x & \varphi_2\\ e^x & \varphi_2'\end{vmatrix}=cx\cdot e^{2x}$$

$$\varphi_2'-\varphi_2=cx\cdot e^x$$

$$\Rightarrow \varphi_2=c(x)\cdot e^x$$

$$c'(x)e^x=cx\cdot e^x$$

$$\tilde{c}_1=\frac{cx^2}{2}+C_0$$

$$y=\frac{C_2x^2}{2}\cdot e^x+C_1e^x$$

684. $xy''+2y'-xy=0;\ y_1=\dfrac{e^x}{x}.$

$$y''+\frac{2}{x}y'-y=0.$$

$$\Rightarrow W(x)=c\cdot e^{-\int\frac{2}{x}dx}=\frac{c}{x^2}$$

$$\begin{vmatrix}\frac{e^x}{x} & \varphi_2\\ \frac{xe^x-e^x}{x^2} & \varphi_2'\end{vmatrix}=\frac{c}{x^2}$$

$$\frac{e^x}{x}\left(\varphi_2'-\varphi_2\cdot\frac{x-1}{x}\right)=\frac{c}{x^2}$$

$$\Rightarrow \varphi_2'-\varphi_2\frac{x-1}{x}=\frac{c}{xe^x}$$

Consider $\dfrac{d\varphi_2}{dx}=\dfrac{\varphi_2(x-1)}{x}$

$$\varphi_2=\frac{e^x}{x}\cdot c'(x)$$

$$c'(x)\cdot\frac{e^x}{x}=\frac{c}{xe^x}$$

$$\Rightarrow c'(x)=c\cdot e^{-2x}$$

$$c(x)=-\frac{\tilde{c}}{2}\cdot e^{-2x}+\tilde{C}_1$$

$$\Rightarrow y=C_1\cdot\frac{1}{xe^x}+C_2\cdot\frac{e^x}{x}.$$

704. $(x^2-1)y''+4xy'+2y=6x;\ y_1=x,\ y_2=\dfrac{x^2+x+1}{x+1}.$

$$y''+\frac{4x}{x^2-1}y'+\frac{2}{x^2-1}y=\frac{6x}{x^2-1}$$

$$\varphi_1=y_2-y_1=\frac{x^2+x+1-x^2-x}{x+1}=\frac{1}{x+1}.$$

$$W(x)=Ce^{-\int\frac{4x}{x^2-1}dx}=Ce^{-2\ln|x^2-1|}=\frac{C}{(x^2-1)^2}$$

$$W(x)=\begin{vmatrix}\frac{1}{x+1} & \varphi_2\\ -\frac{1}{(x+1)^2} & \varphi_2'\end{vmatrix}=\frac{C}{(x^2-1)^2}$$

$$\frac{\varphi_2'}{x+1}+\frac{\varphi_2}{(x+1)^2}=\frac{C}{(x^2-1)^2}\Rightarrow\varphi_2'+\frac{\varphi_2}{(x+1)}=\frac{C}{(x-1)^2(x+1)}$$

Consider $\dfrac{d\varphi_2}{dx}=-\dfrac{\varphi_2}{x+1}\Rightarrow\dfrac{d\varphi_2}{\varphi_2}=-\dfrac{dx}{x+1}\Rightarrow\varphi_2=\dfrac{C(x)}{x+1}$

$$\frac{C'(x)}{x+1}=\frac{C}{(x-1)^2(x+1)}\Rightarrow C(x)=-\frac{C}{x-1}+\tilde{C_2}$$

$$\varphi_2=\frac{\tilde{C}}{x^2-1}+\frac{\tilde{C_2}}{x+1}\Rightarrow y=C_1\frac{1}{x^2-1}+C_2\cdot\frac{1}{x+1}+x.$$

718. 证明，当 $q(x)<0$ 时方程 $y''+p(x)y'+q(x)y=0$ 的解不可能有正的极大值.

Pf: the maximal extremum point satisfies $y'(x_0)=0.\quad y''(x_0)\leq 0$

$y''(x_0)+q(x_0)y=0$

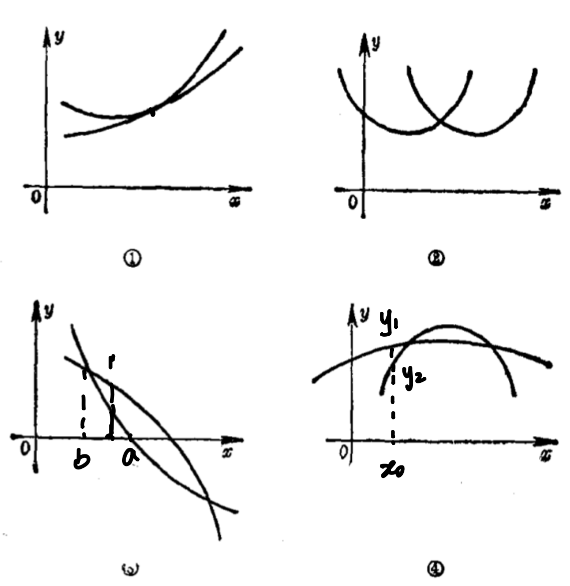
i) $y''(x_0)=0.\quad q(x_0)\neq 0\Rightarrow y(x_0)=0.$

ii) $y''(x_0)<0.\quad q(x_0)y(x_0)>0.\quad q(x_0)<0$

$\Rightarrow y(x_0)<0$

$\Rightarrow y(x_0)\leq 0.$

720. 方程 $y''+q(x)y=0$（函数 $q(x)$ 连续）的两个解的图形能够处于图 3①, 图 3②, 图 3③, 图 3④ 的位置吗？



① Set the Cauchy Problem on the intersect point.
the uniqueness doesn't satisfy.

② can be

③ for fixed x. we have fixed. $q(x)$.
$y''=-\dfrac{q(x)}{y(x)}.\quad x\in(b,a).$

we know y'' need to be same sign. in different solution.
Can't. be. a convex downward and another convex upward.

④ Can't be. $y''(x)=-\dfrac{q(x)}{y(x)}\quad(y\neq 0)$
at fixed x_0. $y_1>y_2$. $\quad y_1''>y_2''$
but y_2 is "more convex".

Oct. 31st.

421. $x^2y''=y'^2$.

Let $y'=z$

$\Rightarrow x^2\cdot\frac{dz}{dx}=z^2$

$\Rightarrow \frac{dz}{z^2}=\frac{dx}{x^2}$

$\Rightarrow -\frac{1}{z}=-\frac{1}{x}+\hat{c}$

$\Rightarrow y'=\frac{x}{1+c_1x}$

1) $c_1=0$. $y=\frac{x^2}{2}+C$

2) $c_1\neq0$. $y=\frac{x}{c_1}-\frac{1}{c_1^2}\ln|c_1x+1|+c_2$

422. $2xy'y''=y'^2-1$.

Let $y'=z$.

$2xz\cdot z'=z^2-1$.

$\Rightarrow \frac{2z\,dz}{z^2-1}=\frac{dx}{x}$

$\Rightarrow z^2-1=c_1x$.

$\Rightarrow y'=\sqrt{1+c_1x}$.

1) $c_1\neq0$ $y=\frac{2}{3c_1}(1+c_1x)^{\frac{3}{2}}+c_2$

2) $c_1=0$ $y=\pm x+C$

424. $y'^2+2yy''=0$.

Let. $y'=P(y)$. $y''=P'(y)\cdot y'=P'(y)P(y)$.

$P^2(y)+2y\cdot\frac{dP(y)}{dy}P(y)=0$.

1). $P(y)=0$. $\Rightarrow y=C$ ①

2) $P(y)\neq0$

$\Rightarrow P(y)+2\frac{dP(y)}{dy}y=0\Rightarrow P(y)=-\frac{dP(y)}{dy}y$

$\Rightarrow \frac{dP(y)}{P(y)}=-\frac{dy}{y}\Rightarrow P(y)=\frac{c_1}{y}$.

1) $c_1=0$. $\Rightarrow P(y)=0\Rightarrow y=C$

2). $c_1\neq0$. $\frac{dy}{dx}=\frac{c_1}{y}$. $\Rightarrow y=e^{c_1x}+c_2$. ②

423. $y^3y''=1$.

Let $y'=P(y)$.

$y^3\cdot\frac{dP(y)}{dy}\cdot P(y)=1$.

$\frac{1}{2}P^2(y)+\tilde{c}_1=-\frac{1}{2}y^{-2}$

$\Rightarrow P^2(y)+y^{-2}-c_1=0$. , $c_1=0$. impossible.

$\Rightarrow \frac{dy}{dx}=\sqrt{c_1-y^{-2}}$

$x=\int\frac{dy}{\sqrt{c_1-y^{-2}}}=\int\frac{y\,dy}{\sqrt{c_1y^2-1}}$

$=\frac{1}{2c_1}\int\frac{d(c_1y^2)}{\sqrt{c_1y^2-1}}=\frac{\sqrt{c_1y^2-1}}{c_1}+c_2$

$(c_1\neq0)$

425. $y''=2yy'$.

Let. $\frac{dy}{dx}=P(y)$

$\frac{dP(y)}{dy}P(y)=2y\,P(y)$

1) $P(y)=0$ $\Rightarrow y=C$ ①

2) $P(y)\neq0$. $dP(y)=2y\,dy$.

$\Rightarrow P(y)=y^2+\tilde{c}_1$

$\Rightarrow \frac{dy}{dx}=y^2+\tilde{c}_1\Rightarrow\frac{dy}{y^2+\tilde{c}_1}=dx$

1) $\tilde{c}_1=0$ $\Rightarrow x=-y^{-1}+\tilde{C}$ ②

2) $\tilde{c}_1\neq0$

i) $\tilde{c}_1>0$

$x=c_1\cdot\arctan(\frac{y}{c_1})+c_2$. ③

ii) $\tilde{c}_1<0$.

$x=\frac{1}{2c_1}\ln\left|\frac{y-c_1}{y+c_1}\right|+c_1$

Nov. 2nd.

463. $xyy'' - xy'^2 = yy'$.

Let $y' = yz$.

$y'' = y'z + yz' = yz^2 + yz'$

$xy(yz^2 + yz') - x(yz)^2 = y^2 z$.

$y \equiv 0$. $\Rightarrow xz' = z$

$\Rightarrow \frac{dz}{z} = \frac{dx}{x} \Rightarrow z = cx$.

$\frac{y'}{y} = cx. \Rightarrow y = \hat{c}\, e^{\tilde{c}x^2}$

464. $yy'' = y'^2 + 15y^2\sqrt{x}$.

$y' = yz$.

$y(yz^2 + yz') = (yz)^2 + 15y^2\sqrt{x}$.

① $y \equiv 0$ ② $z' = 15\sqrt{x}$.

$z = 10 \cdot x^{\frac{3}{2}} + c$.

$\frac{dy}{dx} = (10x^{\frac{3}{2}} + c)y$

$\Rightarrow \frac{dy}{y} = (10x^{\frac{3}{2}} + c)\,dx$

$\Rightarrow \ln|y| = 4x^{\frac{5}{2}} + cx. + \hat{c}$

or $y \equiv 0$

465. $(x^2+1)(y'^2 - yy'') = xyy'$.

$y' = yz$.

$(x^2+1)((yz)^2 - y(yz^2 + yz')) = xy(yz)$

$-(x^2+1)\, y^2 z' = xy^2 z$.

$y \equiv 0$. , $-(x^2+1)\frac{dz}{dx} = x \cdot z$.

$\Rightarrow \frac{x\,dx}{x^2+1} = -\frac{dz}{z}$.

$\Rightarrow -\ln|z| = \frac{1}{2}\ln|x^2+1|$

$z = \frac{c}{\sqrt{x^2+1}}$

$\Rightarrow \frac{dy}{dx} = \frac{cy}{\sqrt{x^2+1}} \Rightarrow \frac{dy}{y} = \frac{c\,dx}{\sqrt{x^2+1}}$

$\ln|y| = c\ln|x + \sqrt{x^2+1}| + \hat{c}$

$y = \tilde{c}\,(x + \sqrt{x^2+1})^c$

466. $xyy'' + xy'^2 = 2yy'$.

$y' = yz$.

$xy(yz^2 + yz') + x(yz)^2 = 2y(yz)$

$\Rightarrow 2xy^2z^2 + xy^2z' = 2y^2 z$

$y \equiv 0$. $xz' + 2xz^2 - 2z. = 0$.

$z' - 2\frac{1}{x}z = -2z^2$

Let $t = \frac{1}{z}$.

$\Rightarrow t' + 2\frac{1}{x} \cdot t = 2$

$\Rightarrow \frac{dt}{dx} = -\frac{2t}{x} \Rightarrow \frac{dt}{t} = -\frac{2dx}{x}$.

$t = \frac{c(x)}{x^2}$, $\frac{c'(x)}{x^2} = 2$. $c(x) = \frac{2}{3}x^3 + c$.

$\Rightarrow \frac{1}{z} = \frac{2x}{3} + \frac{c}{x^2} \Rightarrow \frac{y\,dx}{dy} = \frac{\frac{2}{3}x^3 + c}{x^2}$.

$\frac{dy}{y} = \frac{x^2 dx}{\frac{2}{3}x^3 + c} = \frac{\frac{1}{2}d(\frac{2}{3}x^3 + c)}{\frac{2}{3}x^3 + c}$

$\ln|y| = \frac{1}{2}\ln|\frac{2}{3}x^3 + c|$

$\Rightarrow y = \hat{c}\,(\frac{2}{3}x^3 + c)^{\frac{1}{2}}$ or $y \equiv 0$

Nov. 3rd.

457. $yy''=y'(y'+1)$.

① $y\cong 0$ $\quad \dfrac{y''}{y'+1}=\dfrac{y'}{y}$

$\Rightarrow [\ln(y'+1)]'=[\ln y]'$

$\ln(y'+1)=\ln y+\tilde{c}$

$y'=cy-1 \Rightarrow \dfrac{dy}{dx}=cy-1$ if $c=0$ $y=-x+C$

consider $\dfrac{dy}{dx}=cy$. $y=c(x)\cdot e^{cx}$

$\Rightarrow c'(x)=-e^{-cx}$ $\quad c(x)=\frac{1}{c}e^{-cx}+\hat{c}$

$y=\frac{1}{c}+\hat{c}e^{cx}$ or $y\equiv 0$. or $y=-x+C$

456. $y'y'''=2y''^2$.

$\dfrac{y^{(3)}}{y^{(2)}}=2\dfrac{y^{(2)}}{y^{(1)}}$.

$\Rightarrow y^{(2)}=2\tilde{c}_1 y^{(1)}$

$\Rightarrow \dfrac{dy^{(1)}}{dx}=2\tilde{c}_1 y^{(1)}$ or $y^{(2)}=0$

$\Rightarrow y^{(1)}=C_2 e^{C_1x}$

$\Rightarrow y=\dfrac{C_2}{C_1}e^{C_1x}+C_3$

or $y=C_1x+C_2$

458. $5y'''^2-3y''y^{(4)}=0$.

$5\dfrac{y'''}{y''}=3\dfrac{y^{(4)}}{y'''}$ $\quad$ ① $y''=0$

$\frac{5}{3}(\ln y^{(2)})'=(\ln y^{(3)})'$

$\frac{5}{3}\ln y^{(2)}=\ln y^{(3)}+\tilde{c}$

$\Rightarrow [y^{(2)}]^{\frac{5}{3}}=\hat{C}_1 y^{(3)}$

$\hat{C}_1\neq 0$ $\quad [y^{(2)}]^{\frac{5}{3}}=\dfrac{\hat{C}_1 d(y^{(2)})}{dx}$

$dx=\dfrac{\hat{C}_1 d(y^{(2)})}{[y^{(2)}]^{\frac{5}{3}}}$

$x=\bar{C}_1\cdot y^{(2)\,-\frac{2}{3}}+\tilde{C}_4$

$y^{(2)}=C_1(x+C_4)^{-\frac{3}{2}}$

$y^{(1)}=\tilde{C}_1(x+C_4)^{-\frac{1}{2}}+C_2$

$y=C_1(x+C_4)^{\frac{1}{2}}+C_2x+C_3$ or $y=C_1x^2+C_2x+C_3$.

455. $yy'''+3y'y''=0$.

$\Rightarrow \dfrac{y'''}{y''}=-\dfrac{3y'}{y}$. $\quad y\cong 0$

$\Rightarrow y^{(2)}=C_1\cdot\dfrac{1}{y^3}$; $C_1=0$, $y=C_1x+C_2$

Let $y'=P(y)$. $\quad y''=P'(y)\cdot y'=P'(y)\cdot P(y)$.

$\dfrac{dP(y)}{dy}\cdot P(y)=C_1\cdot\dfrac{1}{y^3}$

$P(y)\,dP(y)=\tilde{C}_1\dfrac{dy}{y^3}$

$P^2(y)=C_1y^{-2}+C_2$

$P(y)=\pm\sqrt{C_1y^{-2}+C_2}$

$\dfrac{y\,dy}{\pm\sqrt{C_1+C_2y^2}}=dx$

$x=\pm\dfrac{1}{C_2}\sqrt{C_1+C_2y^2}+C_3$

or $y=C_1x+C_2$

Nov. 7th.

786. $\begin{cases} \dot{x} = 2x + y, \\ \dot{y} = 3x + 4y. \end{cases}$

$\begin{vmatrix} 2-\lambda & 1 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5. \quad \lambda = 1 \text{ or } 5.$

find eigenvector: $\lambda_1 = 1 \quad v_1 = \binom{1}{-1} \Rightarrow \begin{pmatrix} c_1 e^t \\ -c_1 e^t \end{pmatrix}$

$\lambda_2 = 3 \quad V_2 = \binom{1}{3} \Rightarrow \begin{pmatrix} c_2 e^{5t} \\ 3c_2 e^{5t} \end{pmatrix}$

787. $\begin{cases} \dot{x} = x - y, \\ \dot{y} = y - 4x. \end{cases}$

$\begin{vmatrix} 1-\lambda & -1 \\ -4 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 4 = (\lambda-3)(\lambda+1)$

$\lambda_1 = 3 \quad \lambda_2 = -1.$

$\lambda_1 = 3. \quad \begin{cases} -2x_1 - x_2 = 0 \\ -4x_2 - 2x_2 = 0. \end{cases} \Rightarrow v_1 = \binom{1}{-2}$

$\lambda_2 = -1 \quad \begin{cases} 2x_1 - x_2 = 0 \\ -4x_2 + 2x_2 = 0 \end{cases} \Rightarrow V_2 = \binom{1}{2}$

$\begin{cases} x = c_1 e^{3t} + c_2 \cdot e^{-t} \\ y = -2c_1 e^{3t} + 2c_2 \cdot e^{-t}. \end{cases}$

788. $\begin{cases} \dot{x} + x - 8y = 0, \\ \dot{y} - x - y = 0. \end{cases} \Rightarrow \begin{cases} \dot{x} = -x + 8y \\ \dot{y} = x + y. \end{cases}$

$\begin{vmatrix} -1-\lambda & 8 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 1 - 8 \quad \lambda_1 = 3 \quad \lambda_2 = -3.$

$\begin{cases} -4x_1 + 8x_2 = 0 \\ x_1 - 2x_2 = 0 \end{cases} \Rightarrow V_1 = \binom{2}{1}$

$\begin{cases} 2x_1 + 8x_2 = 0 \\ x_1 + 4x_2 = 0 \end{cases} \Rightarrow V_2 = \binom{2}{-1}$

$\begin{cases} x = 2c_1 e^{3t} + 2c_2 e^{-3t} \\ y = c_1 e^{3t} - c_2 e^{-3t}. \end{cases}$

796. $\begin{cases} \dot{x} = x + z - y, \\ \dot{y} = x + y - z, \\ \dot{z} = 2x - y \end{cases}$ $\quad \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix}$

$(\lambda_1 = 1, \ \lambda_2 = 2, \ \lambda_3 = -1)$.

$\lambda_1 = 1$

$\begin{cases} -x_2 + x_3 = 0 \\ x_1 - x_3 = 0 \\ 2x_1 - x_2 - x_3 = 0 \end{cases} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2.$

$\begin{cases} -x_1 - x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \\ 2x_1 - x_2 - 2x_3 = 0 \end{cases} \Rightarrow V_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_3 = -1$

$\begin{cases} 2x_1 - x_2 + x_3 = 0 \\ x_1 + 2x_2 - x_3 = 0 \\ 2x_1 - x_2 + x_3 = 0 \end{cases} \Rightarrow V_3 = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$

$\begin{cases} x = c_1 e^t + c_2 e^{2t} + c_3 e^{-t} \\ y = c_1 e^t + c_3 e^{-3t} \\ z = c_1 e^t + c_2 e^{2t} + c_3 e^{-3t}. \end{cases}$

797. $\begin{cases} \dot{x}=x-2y-z, \\ \dot{y}=y-x+z, \\ \dot{z}=x-z \end{cases}$ $\begin{vmatrix} 1-\lambda & -2 & -1 \\ -1 & 1-\lambda & 1 \\ 1 & 0 & -1-\lambda \end{vmatrix}$

$(\lambda_1=0,\ \lambda_2=2,\ \lambda_3=-1)$.

$\lambda_1=0$ $\begin{cases} x_1-2x_2-x_3=0 \\ -x_1+x_2+x_3=0 \\ x_1 \quad\quad -x_3=0 \end{cases} \Rightarrow v_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$

$\lambda_3=-1$ $\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0 \Rightarrow V_3=\begin{pmatrix}0\\1\\-2\end{pmatrix}$

$\lambda_2=2$ $\begin{cases} -x_1-2x_2-x_3=0 \\ -x_1-x_2+x_3=0 \\ x_1 - 3x_3=0 \end{cases} \Rightarrow V_2=\begin{pmatrix}-3\\2\\-1\end{pmatrix}$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=C_1\begin{pmatrix}1\\0\\1\end{pmatrix}+C_2\begin{pmatrix}-3\\2\\-1\end{pmatrix}e^{2t}+C_3\begin{pmatrix}0\\1\\-2\end{pmatrix}e^{-t}$$

793. $\begin{cases} \dot{x}=3x-y, \\ \dot{y}=4x-y. \end{cases} \Rightarrow \begin{vmatrix} 3-\lambda & -1 \\ 4 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-3)+4$

$= \lambda^2-2\lambda+1$

$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\alpha_1+\beta_1 t\\ \alpha_2+\beta_2 t\end{pmatrix}e^{t} \Rightarrow \begin{cases} \alpha_1+\beta_1=3\alpha_1-\alpha_2 \\ \beta_1=3\beta_1-\beta_2 \\ \alpha_2+\beta_2=4\alpha_1-\alpha_2 \\ \beta_2=4\beta_1-\beta_2 \end{cases} \Rightarrow \begin{cases} \alpha_1=C_1 \\ \alpha_2=C_2 \\ \beta_1=2C_1-C_2. \\ \beta_2=4C_1-2C_2 \end{cases}$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}C_1+(2C_1-C_2)t\\ C_2+(4C_1-2C_2)t\end{pmatrix}e^{t}$$

792. $\begin{cases} \dot{x}=2x+y, \\ \dot{y}=4y-x. \end{cases} \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} = (\lambda-2)(\lambda-4)+1=(\lambda-3)^2$

$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}\alpha_1+\beta_1 t\\ \alpha_2+\beta_2 t\end{pmatrix}e^{3t} \Rightarrow \beta_1+3(\alpha_1+\beta_1 t)=2(\alpha_1+\beta_1 t)+\alpha_2+\beta_2 t$

$\beta_2+3(\alpha_2+\beta_2 t)=4(\alpha_2+\beta_2 t)-(\alpha_1+\beta_1 t)$

$\begin{cases} \beta_1+3\alpha_1=2\alpha_1+\alpha_2 \\ 3\beta_1=2\beta_1+\beta_2 \\ \beta_2+3\alpha_2=4\alpha_2-\alpha_1 \\ 3\beta_2=4\beta_2-\beta_1 \end{cases} \Rightarrow \begin{cases} \beta_1=\beta_2 \\ \alpha_2-\alpha_1=\beta_1 \end{cases}$

Let $\alpha_1=C_1$ $\alpha_2=C_2$. $\beta_1=\beta_2=C_2-C_1$

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}C_1+(C_2-C_1)t\\ C_2+(C_2-C_1)t\end{pmatrix}e^{3t}$$

804. $\begin{cases} \dot{x}=4x-y-z, \\ \dot{y}=x+2y-z, \\ \dot{z}=x-y+2z \end{cases}$ $\begin{vmatrix} 4-\lambda & -1 & -1 \\ 1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}.$

$(\lambda_1=2,\ \lambda_2=\lambda_3=3).$

$\lambda_1=2.$ $\begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0 \Rightarrow v_1=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$ $\lambda_2=\lambda_3=3.$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t}+\begin{pmatrix} \alpha_1+\beta_1 t \\ \alpha_2+\beta_2 t \\ \alpha_3+\beta_3 t \end{pmatrix}e^{3t}$

$$\begin{cases} \beta_1+3(\alpha_1+\beta_1 t)=4(\alpha_1+\beta_1 t)-(\alpha_2+\beta_2 t)-(\alpha_3+\beta_3 t) \\ \beta_2+3(\alpha_2+\beta_2 t)=(\alpha_1+\beta_1 t)+2(\alpha_2+\beta_2 t)-(\alpha_3+\beta_3 t) \\ \beta_3+3(\alpha_3+\beta_3 t)=(\alpha_1+\beta_1 t)-(\alpha_2+\beta_2 t)+2(\alpha_3+\beta_3 t). \end{cases}$$

$$\Rightarrow \begin{cases} \beta_1+3\alpha_1=4\alpha_1-\alpha_2-\alpha_3 \\ 3\beta_1=4\beta_1-\beta_2-\beta_3 \\ \beta_2+3\alpha_2=\alpha_1+2\alpha_2-\alpha_3 \\ 3\beta_2=\beta_1+2\beta_2-\beta_3 \\ \beta_3+3\alpha_3=\alpha_1-\alpha_2+2\alpha_3. \\ 3\beta_3=\beta_1-\beta_2+2\beta_3 \end{cases} \Rightarrow \begin{cases} \beta_1=\beta_2=\beta_3=0 \\ \alpha_1=\alpha_2+\alpha_3. \end{cases}$$

Let $C_2=\alpha_2.$ $C_3=\alpha_3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}e^{2t}+\begin{pmatrix} C_2 \\ C_3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}e^{3t}$$

805. $\begin{cases} \dot{x}=2x-y-z, \\ \dot{y}=3x-2y-3z, \\ \dot{z}=2z-x+y \end{cases}$ $\begin{vmatrix} 2-\lambda & -1 & -1 \\ 3 & -2-\lambda & -3 \\ -1 & 1 & 2-\lambda \end{vmatrix}.$

$(\lambda_1=0,\ \lambda_2=\lambda_3=1).$

$\lambda_1=0$ $\begin{pmatrix} 2 & -1 & -1 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=0 \Rightarrow v_1=\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

$\lambda_1=\lambda_2=1.$ $$\begin{cases} \beta_1+\alpha_1+\beta_1 t=2(\alpha_1+\beta_1 t)-(\alpha_2+\beta_2 t)-(\alpha_3+\beta_3 t) \\ \beta_2+\alpha_2+\beta_2 t=3(\alpha_1+\beta_1 t)-2(\alpha_2+\beta_2 t)-3(\alpha_3+\beta_3 t) \\ \beta_3+\alpha_3+\beta_3 t=-(\alpha_1+\beta_1 t)+(\alpha_2+\beta_2 t)+2(\alpha_3+\beta_3 t) \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1+\beta_1=2\alpha_1-\alpha_2-\alpha_3. \\ \alpha_2+\beta_2=3\alpha_1-2\alpha_2-3\alpha_3 \\ \alpha_3+\beta_3=-\alpha_1+\alpha_2+2\alpha_3 \\ \beta_1=2\beta_1-\beta_2-\beta_3 \\ \beta_2=3\beta_1-2\beta_2-3\beta_3 \\ \beta_3=-\beta_1+\beta_2+2\beta_3 \end{cases} \Rightarrow \begin{cases} \beta_1=\beta_2=\beta_3=0. \\ \alpha_1=\alpha_2+\alpha_3. \end{cases}$$

Let $\alpha_2=C_2.$ $\alpha_3=C_3.$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=C_1\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}+\begin{pmatrix} C_2+C_3 \\ C_2 \\ C_3 \end{pmatrix}e^t$$

Nov. 9th.

789. $\begin{cases} \dot{x}=x+y, \\ \dot{y}=3y-2x. \end{cases}$ $\begin{vmatrix} 1-\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = \lambda^2-4\lambda+5$

$\Rightarrow \lambda = 2\pm i.$

$\lambda=2+i.$ $\begin{pmatrix} -1-i & 1 \\ -2 & 1-i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$

$$\begin{pmatrix} e^{(2+i)t} \\ (1+i)e^{(2+i)t} \end{pmatrix} = \begin{pmatrix} e^{2t}(\cos t + i\sin t) \\ e^{2t}(1+i)(\cos t + i \sin t) \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + i e^{2t} \begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{2t}\begin{pmatrix} \cos t \\ \cos t - \sin t \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} \sin t \\ \cos t + \sin t \end{pmatrix}$$

790. $\begin{cases} \dot{x}=x-3y, \\ \dot{y}=3x+y. \end{cases}$ $\begin{vmatrix} 1-\lambda & -3 \\ 3 & 1-\lambda \end{vmatrix} = \lambda^2-2\lambda+10.$

$\lambda = 1\pm 3i.$

$\lambda=1+3i.$ $\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\begin{pmatrix} e^{(1+3i)t} \\ -i e^{(1+3i)t} \end{pmatrix} = e^t\begin{pmatrix} \cos 3t + i \sin 3t \\ i(\cos 3t + i\sin 3t) \end{pmatrix}.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^t \begin{pmatrix} \cos 3t \\ -\sin 3t \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$$

801. $\begin{cases} \dot{x}=x-y-z, \\ \dot{y}=x+y, \\ \dot{z}=3x+z \end{cases}$ $\begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix}$

$(\lambda_1=1,\ \lambda_{2,3}=1\pm 2i).$

1) $\lambda_1=1.$ $\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = 0$ $\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$

2) $\lambda_2 = 1+2i$ $\begin{pmatrix} -2i & -1 & -1 \\ 1 & -2i & 0 \\ 3 & 0 & -2i \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0$ $\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2}i \\ -\frac{3}{2}i \end{pmatrix}$

$$e^{(1+2i)t}\begin{pmatrix}1\\-\frac{i}{2}\\-\frac{3i}{2}\end{pmatrix}=e^t\begin{pmatrix}\cos 2t+i\sin 2t\\ \frac{1}{2}\sin 2t+i(-\frac{1}{2}\cos 2t)\\ \frac{3}{2}\sin 2t+i(-\frac{3}{2}\cos 2t)\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=C_1e^t\begin{pmatrix}0\\1\\-1\end{pmatrix}+C_2e^t\begin{pmatrix}\cos 2t\\ \frac{1}{2}\sin 2t\\ \frac{3}{2}\sin 2t\end{pmatrix}+C_3e^t\begin{pmatrix}\sin 2t\\ -\frac{1}{2}\cos 2t\\ -\frac{3}{2}\cos 2t\end{pmatrix}$$

802. $\begin{cases}\dot{x}=2x+y,\\ \dot{y}=x+3y-z,\\ \dot{z}=2y+3z-x\end{cases}$ $\begin{vmatrix}2-\lambda & 1 & 0\\ 1 & 3-\lambda & -1\\ -1 & 2 & 3-\lambda\end{vmatrix}$

$(\lambda_1=2,\ \lambda_{2,3}=3\pm i).$

1) $\lambda_1=2$ $\begin{pmatrix}0&1&0\\1&1&-1\\-1&2&1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}1\\0\\1\end{pmatrix}$

2) $\lambda_2=3+i$ $\begin{pmatrix}-i&1&0\\1&-i&-1\\-1&2&-i\end{pmatrix}\begin{pmatrix}\alpha_1\\ \beta_1\\ \gamma_1\end{pmatrix}=\begin{pmatrix}1\\1+i\\2-i\end{pmatrix}$

$$e^{(3+i)t}\begin{pmatrix}1\\1+i\\2-i\end{pmatrix}=e^{3t}\begin{pmatrix}\cos t+i\sin t\\(1+i)(\cos t+i\sin t)\\(2-i)(\cos t+i\sin t)\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=C_1e^{2t}\begin{pmatrix}1\\0\\1\end{pmatrix}+C_2e^{3t}\begin{pmatrix}\cos t\\ \cos t-\sin t\\ 2\cos t+\sin t\end{pmatrix}+C_3e^{3t}\begin{pmatrix}\sin t\\ \cos t+\sin t\\ 2\sin t-\cos t\end{pmatrix}$$

827. $\begin{cases}\dot{x}=y-\underline{5\cos t},\\ \dot{y}=2x+y.\end{cases}$ $\rightarrow$ $P(t)\,e^{\alpha t}$

$P(t)\,e^{\alpha t}\cos\beta t$

$P(t)\,e^{\alpha t}\sin\beta t.$

(1) solve the homo- systems

$\lambda_1=2$ $V_1=\begin{pmatrix}1\\2\end{pmatrix}$ $V_2=\begin{pmatrix}1\\-1\end{pmatrix}$

$\lambda_2=-1$

$$\begin{pmatrix}x\\y\end{pmatrix}=c_1e^{2t}\begin{pmatrix}1\\2\end{pmatrix}+c_2e^{t}\begin{pmatrix}1\\-1\end{pmatrix}$$

(2) find particular solution of n.h.s.

since $\alpha+\beta i\neq\lambda_1,\lambda_2.$

$\begin{cases}x=A\cos t+B\sin t\\ y=C\cos t+D\sin t\end{cases}\Rightarrow\begin{cases}A=-1\\B=-2\\C=3\\D=1\end{cases}$

$$\begin{pmatrix}x\\y\end{pmatrix}=c_1e^{2t}\begin{pmatrix}1\\2\end{pmatrix}+c_2e^{t}\begin{pmatrix}1\\-1\end{pmatrix}+\begin{pmatrix}-\cos t-2\sin t\\3\cos t+\sin t\end{pmatrix}$$

828. $\begin{cases} \dot{x} = 3x + 2y + 4e^{5t}, \\ \dot{y} = x + 2y. \end{cases}$

$$\begin{vmatrix} 3-\lambda & 2 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 \qquad \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 4. \end{cases}$$

1) $\lambda_1 = 1$ $\begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$ $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2) $\lambda_2 = 4$ $\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

since $5 \neq \lambda_1, \lambda_2$.

$$\begin{cases} x = A e^{5t} \\ y = B e^{5t} \end{cases} \Rightarrow \begin{cases} 5A = 3A + 2B + 4 \\ 5B = A + 2B \end{cases} \Rightarrow \begin{cases} A = 3 \\ B = 1 \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3e^{5t} \\ e^{5t} \end{pmatrix}$$

833. $\begin{cases} \dot{x} = 2x + y + e^{t}, \\ \dot{y} = -2x + 2t. \end{cases}$

$$\begin{vmatrix} 2-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2$$

$$\lambda = 1 \pm i.$$

$\lambda = 1 + i$

$$\begin{pmatrix} 1-i & 1 \\ -2 & -1-i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \quad v_1 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

$$\begin{pmatrix} e^{(1+i)t} \\ (-1+i)\, e^{(1+i)t} \end{pmatrix} = \begin{pmatrix} e^{t}(\cos t + i \sin t) \\ (-1+i)\, e^{t}(\cos t + i\sin t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos t \\ -e^{t}\cos t - e^{t}\sin t \end{pmatrix} + i \begin{pmatrix} e^{t}\sin t \\ e^{t}\cos t - e^{t}\sin t \end{pmatrix}$$

since $\alpha + \beta i \neq \lambda_1, \lambda_2$.

1) consider

$$\begin{cases} \dot{x} = 2x + y + e^{t} \\ \dot{y} = -2x \end{cases} \Rightarrow \begin{cases} x = A e^{t} \\ y = B e^{t} \end{cases} \Rightarrow \begin{cases} A = 2A + B + 1 \\ B = -2A. \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -2 \end{cases}$$

2) consider

$$\begin{cases} \dot{x} = 2x+y \\ \dot{y} = -2x+2t \end{cases} \Rightarrow \begin{cases} x = C+Dt \\ y = E+Ft \end{cases} \Rightarrow \begin{cases} D = 2C+2Dt+E+Ft \\ F = -2C-2Dt+2t \end{cases}$$

$$\Rightarrow \begin{cases} F+2D=0 \\ D=2C+E \\ F=-2C. \\ -2D+2=0 \end{cases} \Rightarrow \begin{cases} C=1 \\ D=1 \\ E=-1 \\ F=-2 \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 e^t \begin{pmatrix} \cos t \\ -\cos t-\sin t \end{pmatrix} + C_2 e^t \begin{pmatrix} \sin t \\ \cos t-\sin t \end{pmatrix} + \begin{pmatrix} e^t+t+1 \\ -2e^t-t-2 \end{pmatrix}$$

829. $\begin{cases} \dot{x} = 2x-4y+4e^{-2t}, \\ \dot{y} = 2x-2y. \end{cases}$ $\begin{vmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix} = \lambda^2-4+8 = \lambda^2+4$

$\lambda = \pm 2i.$

$\lambda = 2i$ $\begin{pmatrix} 2-2i & -4 \\ 2 & -2-2i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow V_1 = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$

$$e^{2it}\begin{pmatrix} 2 \\ 1-i \end{pmatrix} = \begin{bmatrix} 2(\cos 2t + i\sin 2t) \\ (1-i)(\cos 2t + i\sin 2t) \end{bmatrix} = \begin{pmatrix} 2\cos 2t \\ \cos 2t+\sin 2t \end{pmatrix} + i\begin{pmatrix} 2\sin 2t \\ \sin 2t-\cos 2t \end{pmatrix}$$

Since $2i \neq -2$.

$$\begin{cases} x = Ae^{-2t} \\ y = Be^{-2t} \end{cases} \Rightarrow \begin{cases} -2A = 2A-4B+4 \\ -2B = 2A-2B \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 2\cos 2t \\ \cos 2t+\sin 2t \end{pmatrix} + C_2 \begin{pmatrix} 2\sin 2t \\ \sin 2t-\cos 2t \end{pmatrix} + \begin{pmatrix} 0 \\ e^{-2t} \end{pmatrix}$$

831. $\begin{cases} \dot{x}=2y-x+1, \\ \dot{y}=3y-2x. \end{cases}$ $\begin{vmatrix} -1-\lambda & 2 \\ -2 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda+1)+4 = \lambda^2-2\lambda+1 = (\lambda-1)^2$

$\lambda_{1,2}=1.$ $\begin{cases} x=(A+Bt)e^t \\ y=(C+Dt)e^t \end{cases} \Rightarrow \begin{cases} A+B+Bt = -A-Bt+2C+2Dt \\ C+D+Dt = -2A-2Bt+3C+3Dt \end{cases}$

$\Rightarrow \begin{cases} A+B=-A+2C \\ B=-B+2D \\ C+D=-2A+3C \\ D=-2B+3D \end{cases} \Rightarrow \begin{cases} B=D. \\ B=2C-2A \end{cases}$

Let $A=C_1$, $C=C_2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^t \begin{pmatrix} C_1+2(C_2-C_1)t \\ C_2+2(C_2-C_1)t \end{pmatrix}$$

Let $\begin{cases} x=\alpha \\ y=\beta \end{cases} \Rightarrow \begin{cases} 2\beta-\alpha+1=0 \\ 3\beta-2\alpha=0. \end{cases} \Rightarrow \begin{cases} \alpha=-3 \\ \beta=-2 \end{cases}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^t \begin{pmatrix} C_1+2(C_2-C_1)t \\ C_2+2(C_2-C_1)t \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \end{pmatrix}$$

830. $\begin{cases} \dot{x}=4x+y-e^{2t}, \\ \dot{y}=y-2x. \end{cases}$

$\begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2-5\lambda+6.$

$\lambda_1=2$ $v_1=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2=3$ $v_2=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

since $\lambda_0=\lambda_1$.

$\begin{cases} x=(A+Bt)e^{2t} \\ y=(C+Bt)e^{2t} \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \\ C=0 \\ B=-2 \end{cases}$

835. $\begin{cases} \dot{x}=2x-4y, \\ \dot{y}=x-3y+3e^t. \end{cases}$ $\begin{vmatrix} 2-\lambda & -4 \\ 1 & -3-\lambda \end{vmatrix} = \lambda^2+\lambda-2$

$\lambda_1 = -2 \quad \lambda_2 = 1.$

1) $\begin{pmatrix} 4 & -4 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2) $\begin{pmatrix} 1 & -4 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$\lambda_0 = \lambda_1$

$\begin{cases} x = (A+Bt)e^t \\ y = (C+Dt)e^t \end{cases} \Rightarrow \begin{cases} A+B+Bt = 2(A+Bt)-4(C+Dt) \\ C+D+Dt = (A+Bt)-3(C+Dt)+3 \end{cases}$

$\Rightarrow \begin{cases} B = A-4C \\ 0 = B-4D \\ D = A-4C+3 \\ D = B-3D \end{cases} \Rightarrow \begin{cases} A-4C=-4 \\ B=-4 \\ D=-1 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=-4 \\ C=1 \\ D=-1 \end{cases}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 1 \\ 4 \end{pmatrix}e^t + \begin{pmatrix} -4t \\ 1-t \end{pmatrix}e^t$$

838. $\begin{cases} \dot{x}=2x+4y-8, \\ \dot{y}=3x+6y. \end{cases}$ $\begin{vmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{vmatrix} = \lambda^2-8\lambda$

$\lambda_1 = 0 \quad \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda_2 = 8 \quad \begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda_1 = \lambda_0.$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A+Bt \\ C+Dt \end{pmatrix}. \quad \begin{cases} B = 2(A+Bt)+4(C+Dt)-8 \\ D = 3(A+Bt)+6(C+Dt) \end{cases} \Rightarrow \begin{cases} B = 2A+4C-8 \\ 0 = 2B+4D \\ D = 3A+6C \\ 0 = 3B+6D. \end{cases}$

$\Rightarrow \begin{cases} A+2C=1 \\ B=-6 \\ D=3. \end{cases} \Rightarrow \begin{cases} A=1 \\ B=-6 \\ C=0 \\ D=3 \end{cases}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + C_2 e^{8t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -6t+1 \\ 3t \end{pmatrix}.$$

840. $\begin{cases} \dot{x} = x - y + 2\sin t, \\ \dot{y} = 2x - y. \end{cases}$ $\begin{vmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2+1.$

$\lambda = i.$ $\begin{pmatrix} 1-i & -1 \\ 2 & -1-i \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$

$e^{it}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = \begin{pmatrix} (i\sin t + \cos t) \\ (1-i)(i\sin t + \cos t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix}.$

we have i here

$\lambda_1 = \lambda_0.$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A_1 t + B_1)\cos t + (C_1 t + D_1)\sin t \\ (A_2 t + B_2)\cos t + (C_2 t + D_2)\sin t \end{pmatrix}$

$$\begin{cases} A_1\cos t - (A_1 t + B_1)\sin t + C_1 \sin t + (C_1 t + D_1)\cos t \\ = [(A_1 t + B_1)\cos t + (C_1 t + D_1)\sin t] - [(A_2 t + B_2)\cos t + (C_2 t + D_2)\sin t] + 2\sin t. \\ A_2\cos t - (A_2 t + B_2)\sin t + C_2\sin t + (C_2 t + D_2)\cos t \\ = 2[(A_1 t + B_1)\cos t + (C_1 t + D_1)\sin t] - [(A_2 t + B_2)\cos t + (C_2 t + D_2)\sin t] \end{cases}$$

$$\begin{cases} C_1 = A_1 - A_2 \\ -A_1 = C_1 - C_2 \\ A_1 + D_1 = B_1 - B_2 \\ -B_1 + C_1 = D_1 - D_2 + 2. \\ C_2 = 2A_1 - A_2 \\ -A_2 = 2C_1 - C_2 \\ A_2 + D_2 = 2B_1 - B_2 \\ -B_2 + C_2 = 2D_1 - D_2. \end{cases} \Rightarrow \begin{cases} A_1 = -1. \\ A_2 = -2 \\ C_1 = 1 \\ C_2 = 0 \\ D_1 = B_1 - B_2 + 1 \\ D_2 = 2B_1 - B_2 + 2. \end{cases} \xRightarrow{\text{Let } B_1 = B_2 = 0} \begin{cases} D_1 = 1 \\ D_2 = 2 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = C_1\begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \sin t - \cos t \end{pmatrix} + \begin{pmatrix} -t\cos t + t\sin t + \sin t \\ -2t\cos t + 2\sin t \end{pmatrix}$$

why here has no i.

843. $\begin{cases} \dot{x} = 2x + y + 2e^t, \\ \dot{y} = x + 2y - 3e^{4t}. \end{cases}$ $\qquad \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3.$

$\lambda_1 = 1. \quad \lambda_2 = 3. \qquad$ 1) $\lambda_1 = 1 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_0 = \lambda_1. \qquad$ 2) $\lambda_2 = 3 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{cases} \dot{x} = 2x + y + 2e^t \\ \dot{y} = x + 2y \end{cases} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} At + B \\ Ct + D \end{pmatrix} e^t$

$$\Rightarrow \begin{cases} A + At + B = 2(At + B) + (Ct + D) + 2 \\ C + Ct + D = (At + B) + 2(Ct + D) \end{cases} \Rightarrow \begin{cases} A + B = 2B + D + 2 \\ A = 2A + C \\ C = A + 2C \\ C + D = B + 2D. \end{cases}$$

$$\Rightarrow \begin{cases} A = 1 \\ C = -1 \\ B + D = -1 \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t-1 \\ -t \end{pmatrix} e^t$$

$\begin{cases} \dot{x} = 2x + y \\ \dot{y} = x + 2y - 3e^{4t} \end{cases} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} e^{4t} \quad \begin{cases} 4A = 2A + B \\ 4B = A + 2B - 3. \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = -2 \end{cases}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t} + \begin{pmatrix} (t-1)e^t - e^{4t} \\ -te^t - 2e^{4t} \end{pmatrix}$$

826. $\begin{cases} \dot{x} = y + 2e^t, \\ \dot{y} = x + t^2. \end{cases}$ $\qquad \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1$

$\lambda_1 = 1. \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -1 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \Rightarrow V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_0 = \lambda_1 \quad \begin{cases} \dot{x} = y + 2e^t \\ \dot{y} = x \end{cases} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A + Bt \\ C + Dt \end{pmatrix} e^t \Rightarrow \begin{cases} A + B + Bt = C + Dt + 2 \\ C + D + Dt = A + Bt. \end{cases}$

$$\Rightarrow \begin{cases} A + B = C + 2 \\ B = D \\ C + D = A \\ B = D \end{cases} \Rightarrow \begin{cases} A = C + 1 \\ B = 1 \\ D = 1 \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + t \\ t \end{pmatrix} e^t$$

$\lambda_0' \neq \lambda_2. \quad \begin{cases} \dot{x} = y \\ \dot{y} = x + t^2 \end{cases} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 t^2 + b_1 t + c_1 \\ a_2 t^2 + b_2 t + c_2 \end{pmatrix} \Rightarrow \begin{cases} 2a_1 t + b_1 = a_2 t^2 + b_2 t + c_2 \\ 2a_2 t + b_2 = a_1 t^2 + b_1 t + c_1 + t^2 \end{cases}$

$$\Rightarrow \begin{cases} a_2 = 0 \\ 2a_1 = b_2 \\ b_1 = c_2 \\ a_1+1 = 0 \\ 2a_2 = b_1 \\ b_2 = c_1 \end{cases} \Rightarrow \begin{cases} a_1 = -1 \\ a_2 = 0 \\ b_1 = 0 \\ b_2 = -2 \\ c_1 = -2 \\ c_2 = 0 \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -t^2-2 \\ -2t \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \begin{pmatrix} (1+t)e^{t} - (t^2+2) \\ te^{t} - 2t \end{pmatrix}$$

881. 用利亚普诺夫稳定性定义说明下列各方程在给定初始条件下的解是不是稳定的.

① $3(t-1)\dot{x}=x,\ x(2)=0$. ② $\dot{x}=4x-t^2x,\ x(0)=0$.

③ $\dot{x}=t-x,\ x(0)=1$. ④ $2t\dot{x}=x-x^3,\ x(1)=0$.

① $3(t-1)\dfrac{dx}{dt} = x \quad x\equiv 0.$ For any $\delta<0$.

$3\dfrac{dx}{x} = \dfrac{dt}{t-1}$ $\quad \|\psi(2)-x(2)\|<\delta \Rightarrow 0<|C|<\delta.$

$\tilde{C} + \ln|x|^3 = \ln|t-1|$ $\quad \|\psi(t)-x(t)\| = |c(t-1)^{\frac{1}{3}}| \xrightarrow[t\to\infty]{} \infty$

$x = C(t-1)^{\frac{1}{3}}$ $\quad$ unstable

$x(2)=0.\quad C_0=0.$

② $\dfrac{dx}{dt} = (4-t^2)x. \Rightarrow \dfrac{dx}{x} = (4-t^2)\,dt. \Rightarrow x = ce^{4t-\frac{1}{3}t^3}$

$x(0)=(0) \Rightarrow C_0=0 \Rightarrow x\equiv 0.$

$\forall \varepsilon>0.\ \exists\delta=\varepsilon \quad \|\psi(0)-\varphi(0)\| = |c|$

$\|\psi(t)-\varphi(t)\| = |ce^{4t-\frac{1}{3}t^3}| \le |c\cdot e^{t(4-\frac{t^2}{3})}| \to \infty$ $\quad$ it's stable.

③ $\dot{x}+x=0. \Rightarrow \dfrac{dx}{dt}=-x. \Rightarrow x=c(t)\cdot e^{-t} \Rightarrow c'(t)e^{-t}=t$

$c(t) = \int te^{t} = (t+1)e^{t}+C \quad x=(t+1)+c\cdot e^{-t}$

$x(0)=1.\quad C_0=0.$

$\|\psi(0)-\varphi(0)\| = |C+1-1| = |C|.$

$|\psi(t)-\varphi(t)| = |c\cdot e^{-t}| \xrightarrow[t\to\infty]{} 0$ $\quad$ stable.

④ $2t\dot{x}=x-x^3,\ x(1)=0.$ $x\equiv 0$ or $x\equiv 1.$

$$\frac{2dx}{x(1-x^2)} = \frac{dt}{t}$$

$$\ln\frac{x^2}{x^2-1} = \ln t + \tilde{c}$$

$$\frac{x^2}{x^2-1} = ct. \quad x^2=\frac{ct}{ct-1} = 1+\frac{1}{ct-1}. \Rightarrow \varphi(t)\equiv 0$$

$$\|\psi(1)-\varphi(1)\| = \left|\sqrt{1+\frac{1}{c-1}}\right|$$

$$\|\psi(t)-\varphi(t)\| = \left|\sqrt{1+\frac{1}{ct-1}}\right| \geqslant 1 \quad \text{for any } t>\frac{1}{c}.$$

unstable.

899. $\begin{cases}\dot{x}=2xy-x+y,\\ \dot{y}=5x^4+y^3+2x-3y.\end{cases}$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 2 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}2xy\\ 5x^4+y^3\end{pmatrix}$$

denote $g(x,y) = \begin{pmatrix}2xy\\ 5x^4+y^3\end{pmatrix}.$ $\lim\limits_{(x,y)\to(0,0)}\frac{\|g(x,y)\|}{\|(x,y)^T\|} = 0.$

$$\begin{vmatrix}-1-\lambda & 1\\ 2 & -3-\lambda\end{vmatrix} = (\lambda+1)(\lambda+3)-2 = \lambda^2+4\lambda+1. \quad \lambda=-2\pm\sqrt{3}$$

$\mathrm{Re}\,\lambda<0.$ Thus $\begin{pmatrix}x\\ y\end{pmatrix}\equiv 0$ is asymptotical stable.

901. $\begin{cases}\dot{x}=e^{x+2y}-\cos 3x,\\ \dot{y}=\sqrt{4+8x}-2e^y.\end{cases}$

$$e^{x+2y}-\cos 3x = 1+(x+2y)+o(\rho)-\left(1-\frac{(3x)^2}{2}+o(\rho^2)\right) = x+2y+o(\rho)$$

$$\sqrt{4+8x}-2e^y = 2\left[1+\frac{1}{2}(2x)+o(\rho)\right]-2(1+y+o(\rho)) = 2x-2y+o(\rho)$$

thus $\frac{\|g(t,x)\|}{\|x\|} \to 0.$ $\begin{vmatrix}1-\lambda & 2\\ 2 & -2-\lambda\end{vmatrix} = (\lambda+2)(\lambda-1)-4 = \lambda^2+\lambda-6$

$\lambda=-3$ or $2.$ $\exists\lambda_2=2>0.$ unstable.

902. $\begin{cases} \dot{x} = \ln(4y + e^{-3x}), \\ \dot{y} = 2y - 1 + \sqrt[3]{1-6x}. \end{cases}$

$$\ln(4y + e^{-3x}) = \ln(4y + e^{-3x}) + 3x - 3x = \ln(4y + e^{-3x}) + \ln e^{3x} - 3x$$
$$= \ln(4y \cdot e^{3x} + 1) - 3x = 4y \cdot e^{3x} + o(\rho) - 3x$$
$$= 4y \cdot (1 + 3x + o(\rho)) - 3x = -3x + 4y + o(\rho)$$

$$2y - 1 + \sqrt[3]{1-6x} = 2y - 1 + 1 + \frac{1}{3}(-6x) + o(\rho) = -2x + 2y + o(\rho)$$

$$\begin{vmatrix} -3-\lambda & 4 \\ -2 & 2-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2) + 8 = \lambda^2 + \lambda + 2 \qquad \lambda = -\frac{1}{2} \pm \frac{\sqrt{7}}{2} i.$$

all $\mathrm{Re}\,\lambda < 0$. asymptotically stable.

903. $\begin{cases} \dot{x} = \ln(3e^{y} - 2\cos x) \\ \dot{y} = 2e^{x} - \sqrt[3]{8+12y}. \end{cases}$

$$\ln(3e^y - 2\cos x) = \ln\left(3 + 3y + o(\rho) - 2\left(1 - \frac{1}{2}x^2 + o(\rho^2)\right)\right)$$
$$= \ln(1 + 3y + o(\rho))$$
$$= 3y + o(\rho)$$

$$2e^x - \sqrt[3]{8+12y} = 2(1 + x + o(\rho)) - 2\left(1 + \frac{1}{3}\left(\frac{3}{2}y\right) + o(\rho)\right)$$
$$= 2x - y + o(\rho)$$

$$\begin{vmatrix} -\lambda & 3 \\ 2 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda - 6 = 0 \qquad \lambda = -3 \text{ or } 2. \quad \exists\, \mathrm{Re}\,\lambda > 0. \text{ unstable}$$

904. $\begin{cases} \dot{x} = \mathrm{tg}(y - x), \\ \dot{y} = 2^{y} - 2\cos\left(\frac{\pi}{3} - x\right). \end{cases}$

$$\mathrm{tg}(y-x) = (y-x) + o(\rho)$$

$$2^y - 2\left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right) = 2^y - \cos x - \sqrt{3}\sin x$$
$$= 1 + \ln 2\, y + o(\rho) - (1 + o(\rho)) - \sqrt{3}(x + o(\rho))$$
$$= -\sqrt{3}x + \ln 2\, y + o(\rho)$$

$$\begin{vmatrix} -1-\lambda & 1 \\ -\sqrt{3} & \ln 2 - \lambda \end{vmatrix} = (\lambda+1)(\lambda - \ln 2) + \sqrt{3} = \lambda^2 + (1 - \ln 2)\lambda + \sqrt{3} - \ln 2.$$

$\lambda = \frac{\ln 2 - 1}{2} \pm bi.$ $\quad \frac{\ln 2 - 1}{2} < 0.$ asymptotically stable.

907. $\begin{cases} \dot{x} = ax - 2y + x^2, \\ \dot{y} = x + y + xy. \end{cases}$ $\quad x^2, xy = o(\rho)$.

$$\begin{vmatrix} a-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-a)+2 = \lambda^2-(1+a)\lambda+(a+2)$$

$\lambda_1\lambda_2 = a+2 \qquad \lambda_1+\lambda_2 = a+1.$

$\Delta = (1+a)^2 - 4(a+2) = a^2-2a-7$

1) $a \in [1-2\sqrt{2}, 1+2\sqrt{2}]$. $\lambda_1, \lambda_2 \in \mathbb{R}$.

$\lambda_1, \lambda_2 < 0 \Rightarrow \begin{cases} \lambda_1\lambda_2 > 0 \\ \lambda_1+\lambda_2 < 0 \end{cases} \Rightarrow a \in (-2,-1) \cap [1-2\sqrt{2}, 1+2\sqrt{2}] = [1-2\sqrt{2}, -1)$

2) $a \in \mathbb{R} \setminus [1-2\sqrt{2}, 1+2\sqrt{2}]$.

denote $\lambda = b \pm ci \ (c \neq 0) \Rightarrow \begin{cases} \lambda_1\lambda_2 = b^2+c^2 > 0 \\ \lambda_1+\lambda_2 = 2b < 0 \end{cases} \Rightarrow a \in (-2,-1) \cap \mathbb{R}\setminus[1-2\sqrt{2}, 1+2\sqrt{2}] = (-2, 1-2\sqrt{2})$

In conclusion, $a \in (-2,-1)$

908. $\begin{cases} \dot{x} = ax + y + x^2, \\ \dot{y} = x + ay + y^2. \end{cases}$ $\quad x^2, y^2 = o(\rho)$

$$\begin{vmatrix} a-\lambda & 1 \\ 1 & a-\lambda \end{vmatrix} = \lambda^2 - 2a + a^2 - 1 = [\lambda-(a+1)][\lambda-(a-1)]$$

$\Delta = (-2a)^2 - 4(a^2-1) = 4 > 0$, $\lambda_1, \lambda_2 \in \mathbb{R}$.

$\lambda_1 = a+1 \quad \lambda_2 = a-1 \qquad$ we need $\lambda_1, \lambda_2 < 0$.

i.e. $a < -1$

909. $\begin{cases} \dot{x} = x + ay + y^2, \\ \dot{y} = bx - 3y - x^2. \end{cases}$

$$\begin{vmatrix} 1-\lambda & a \\ b & -3-\lambda \end{vmatrix} = (\lambda+3)(\lambda-1) - ab = \lambda^2+2\lambda-(ab+3)$$

$\Delta = 4 + 4(ab+3) = 4ab+16.$

1) $4ab+16 < 0$. $ab < -4$. $\lambda_1, \lambda_2 \in \mathbb{C}\setminus\mathbb{R}$.

$\lambda_1+\lambda_2 = -2 < 0$. $\mathrm{Re}\,\lambda_1 = \mathrm{Re}\,\lambda_2 = -1 \qquad \mathrm{Re}\,\lambda_1, \mathrm{Re}\,\lambda_2 < 0$. holds

2) $4ab+16 \geq 0$. $ab \geq -4$

$\begin{cases} \lambda_1+\lambda_2 < 0 \\ \lambda_1\lambda_2 > 0 \end{cases} \Rightarrow \begin{cases} -2 < 0 \\ -ab-3 > 0 \end{cases} \Rightarrow -4 \leq ab < -3$

In conclusion $ab < -3$.

915. $\begin{cases} \dot{x} = y - x^2 - x, \\ \dot{y} = 3x - x^2 - y. \end{cases}$

$\begin{cases} y - x^2 - x = 0 \\ 3x - x^2 - y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 1 \\ y = 2 \end{cases}$

for $(0,0)$. $\begin{vmatrix} -1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 3 = \lambda^2 + 2\lambda - 2 = -1 \pm \sqrt{3}$.

$\lambda_1 = -1 + \sqrt{3} > 0$ unstable

for $(1,2)$ $\begin{cases} x' = x - 1 \\ y' = y - 2 \end{cases} \Rightarrow \begin{cases} x' = (y'+2) - (x'+1)^2 - (x+1) \\ y' = 3(x'+1) - (x'+1)^2 - (y'+2) \end{cases}$

$\Rightarrow \begin{cases} x' = -3x' + y' - x'^2 \\ y' = x' - y' - x'^2 \end{cases} \quad \begin{vmatrix} -3-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)(\lambda+3) - 1 = \lambda^2 + 4\lambda + 2.$

$\lambda = -2 \pm \sqrt{2} < 0$. asymptotically stable.

916. $\begin{cases} \dot{x} = (x-1)(y-1), \\ \dot{y} = xy - 2. \end{cases}$

equilibrium $\begin{cases} x = 1 \\ y = 2 \end{cases} \quad \begin{cases} x = 2 \\ y = 1. \end{cases}$

1). for $(1,2)$. $\begin{cases} x_1 = x - 1 \\ y_1 = y - 2. \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_1(y_1+1) \\ \dot{y}_1 = (x_1+1)(y_1+2) - 2 \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_1 + x_1 y_1 \\ \dot{y}_1 = 2x_1 + y_1 + x_1 y_1. \end{cases}$

$\begin{vmatrix} 1-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 \quad \lambda_{1,2} = 1 > 0$. unstable.

2) for $(2,1)$ $\begin{cases} x_2 = x - 2 \\ y_2 = y - 1 \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = y_2 + x_2 y_2 \\ \dot{y}_2 = x_2 + 2y_2 + x_2 y_2 \end{cases}$

$\begin{vmatrix} -\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 \quad \lambda_{1,2} = 1 > 0$ unstable

917. $\begin{cases} \dot{x} = y, \\ \dot{y} = \sin(x+y). \end{cases}$

equilibrium $\begin{cases} x = k\pi \\ y = 0 \end{cases}$

k is even. $\begin{cases} x_1 = x - k\pi \\ y_1 = y. \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = \sin(x_1 - k\pi + y_1) \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = x_1 + y_1 + o(\rho). \end{cases} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda + 1.$

$Re\,\lambda > 0$. unstable.

k is odd $\begin{cases} x_1 = x - k\pi. \\ y_1 = y \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = y_1 \\ \dot{y}_1 = -x_1 - y_1 + o(\rho) \end{cases} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda + 2$

$Re\,\lambda < 0$ unstable.

918. $\begin{cases} \dot{x}=\ln(y^2-x), \\ \dot{y}=x-y-1. \end{cases}$

$$\begin{cases} \ln(y^2-x)=0 \\ x-y-1=0 \end{cases} \Rightarrow \begin{cases} y^2-x=1 \\ x-y-1=0 \end{cases} \Rightarrow \begin{cases} x=3 \\ y=2 \end{cases} \quad \begin{cases} y=0 \\ x=-1 \end{cases}$$

1) for (3,2). $\begin{cases} x_1=x-3 \\ y_1=y-2. \end{cases} \Rightarrow \begin{cases} \dot{x}_1=\ln((y_1+2)^2-(x_1+3)) \\ \dot{y}_1=x_1-y_1 \end{cases} \Rightarrow \begin{cases} \dot{x}_1=\ln(1+y_1^2+2y_1-x_1) \\ \dot{y}_1=x_1-y_1 \end{cases}$

$\Rightarrow \begin{cases} \dot{x}_1=-x_1+2y_1+o(\rho) \\ \dot{y}_1=x_1-y_1 \end{cases}$ $\quad \begin{vmatrix} -1-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2+2\lambda-1 = -1\pm\sqrt{2}$

$-1+\sqrt{2}>0.$ unstable

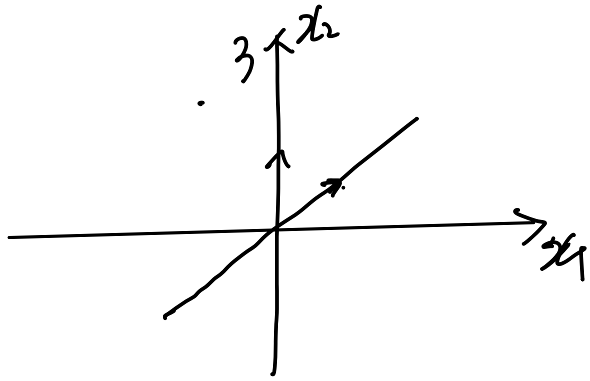
2) for $(-1,0)$ $\begin{cases} x_2=x+1 \\ y_2=y \end{cases} \Rightarrow \begin{cases} \dot{x}_2=\ln(y_2^2-(x_2-1)) \\ \dot{y}_2=x_2-y_2 \end{cases} \Rightarrow \begin{cases} \dot{x}_2=-x_1+o(\rho) \\ \dot{y}_2=x_2-y_2 \end{cases}$

$\begin{vmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)^2. \quad \lambda_{1,2}=-1<0.$ stable.

971. $\begin{cases} \dot{x}=3x, \\ \dot{y}=2x+y. \end{cases} \Rightarrow \begin{vmatrix} 3-\lambda & 0 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3).$

$\lambda_1=1 \quad \lambda_2=3. \quad \lambda_1\lambda_2>0.$ equilibrium is node.

$\lambda_1=1 \quad V_1=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\lambda_2=3$. $V_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$



972. $\begin{cases} \dot{x}=2x-y, \\ \dot{y}=x. \end{cases}$

990. $\begin{cases} \dot{x} = \ln \dfrac{y^2-y+1}{3}, \\ \dot{y} = x^2 - y^2. \end{cases}$

$\begin{cases} y^2-y+1=3. \\ x^2-y^2=0 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=2 \end{cases} \begin{cases} x=-2 \\ y=2 \end{cases} \begin{cases} x=1 \\ y=-1 \end{cases} \begin{cases} x=-1 \\ y=-1 \end{cases}$

$(2,2).$ $\begin{cases} x_1 = x-2 \\ y_1 = y-2. \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = y_1 + o(\rho) \\ \dot{y}_1 = 4x_1 - 4y_1 + o(\rho) \end{cases} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ 4 & -4-\lambda \end{vmatrix} = \lambda^2+4\lambda-4$

$\lambda_1\lambda_2 < 0$ saddle.

$(-2,2).$ $\begin{cases} x_2 = x+2 \\ y_1 = y-2 \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = y_2 + o(\rho) \\ \dot{y}_2 = -4x_2 - 4y_2 \end{cases} \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -4 & -4-\lambda \end{vmatrix} = \lambda^2+4\lambda+4.$

$\lambda = -2.$

$C = A - \lambda E = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$

$C^T z = 0 \Rightarrow \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \quad (\alpha, \beta)^T = (2,1).$

$\begin{pmatrix} 2 & 1 \\ -4 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = 0. \Rightarrow \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$ the solution of $CX=0$ and $C\hat{X}=0.$

Jordan block. $\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}.$ degenerate. node

$(1,-1)$

$\begin{cases} x_3 = x-1 \\ y_3 = y+1. \end{cases} \Rightarrow \begin{cases} \dot{x}_3 = -y_3. + o(\rho) \\ \dot{y}_3 = 2x_3 + 2y_3 + o(\rho) \end{cases} \Rightarrow \begin{vmatrix} \lambda & -1 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2.$

$\lambda = 1 \pm i.$

$1 \neq 0$ focus.

$(-1,-1)$ $\begin{cases} x_4 = x+1 \\ y_4 = x+1 \end{cases}$ $\begin{cases} \dot{x}_4 = -y_4 + o(\rho) \\ \dot{y}_3 = -2x_4 + 2x_4 + o(\rho) \end{cases} \Rightarrow \begin{vmatrix} \lambda & -1 \\ -2 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 2$

$\lambda = 1 \pm \sqrt{3}.$

$\lambda_1\lambda_2 < 0$ saddle

985. $\begin{cases} \dot{x} = x^2 - y, \\ \dot{y} = \ln(1 - x + x^2) - \ln 3. \end{cases}$

$$\begin{cases} x^2 - y = 0 \\ \frac{1-x+x^2}{3} = 1 \end{cases} \Rightarrow \begin{cases} x^2 - y = 0 \\ x^2 - x - 2 = 0 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 1 \end{cases} \quad \begin{cases} x = 2 \\ y = 4 \end{cases}$$

$(-1, 1)$. $\begin{cases} x_1 = x + 1 \\ y_1 = y - 1. \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = -2x_1 - y_1 + o(\rho) \\ \dot{y}_1 = -x_1 + o(\rho) \end{cases} \Rightarrow \begin{vmatrix} -2-\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = \lambda(\lambda+2) - 1.$

$\lambda = -1 \pm \sqrt{2}$

$\lambda_1 \lambda_2 < 0$ saddle.

$(2, 4)$ $\begin{cases} x_2 = x - 2 \\ y_2 = y - 4 \end{cases} \Rightarrow \begin{cases} \dot{x}_2 = 4x_2 - y_2 + o(\rho) \\ \dot{y}_2 = x_2 + o(\rho) \end{cases} \Rightarrow \begin{vmatrix} 4-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1$

$\lambda = 2 \pm \sqrt{3}$ $\quad \lambda_1 \lambda_2 > 0$. node.