Combinatorics

Lecture 5

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Estimates for n!

Estimates of the function n are important in many branches of mathematics. Let's prove some of them.

Proposition 1.

If $n \geq 6$
$$\left(\frac{n}{3}\right)^n \leq n! \leq \left(\frac{n}{2}\right)^n$$

Proof. We prove the theorem by induction on n. For $n = 6$, the validity of the inequalities is verified directly. Let us prove the upper bound. Let's assume that $n! \leq \left(\frac{n}{2}\right)^n$, then

$$\left(\frac{n+1}{2}\right)^{n+1} = \left(\frac{n+1}{2}\right)^n \frac{n+1}{2} = \left(\frac{n}{2}\right)^n \left(1 + \frac{1}{n}\right)^n \frac{n+1}{2} \geq$$

$$\geq n!\left(1 + 1 + C_n^2 \frac{1}{n^2} + \ldots + C_n^n \frac{1}{n^n}\right) \frac{n+1}{2} > 2n! \frac{n+1}{2} = (n+1)!$$

Now we prove the lower bound in a similar way. Let's assume that $\left(\frac{n}{3}\right)^n \le n!$. Then, taking into account that $n! \ge 2^{n-1}$ for $n \ge 2$ (trivial exercise!), we have

$$\left(\frac{n+1}{3}\right)^{n+1} = \left(\frac{n+1}{3}\right)^n \frac{n+1}{3} = \left(\frac{n}{3}\right)^n \left(1 + \frac{1}{n}\right)^n \frac{n+1}{3} \le$$

$$\le n!\left(1 + 1 + C_n^2 \frac{1}{n^2} + \ldots + C_n^n \frac{1}{n^n}\right) \frac{n+1}{3} <$$

$$< n!\left(1 + 1 + \frac{n^2}{2!}\frac{1}{n^2} + \ldots + \frac{n^n}{n!}\frac{1}{n^n}\right) \frac{n+1}{3} <$$

$$< n!\left(1 + 1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}\right) \frac{n+1}{3} < (n+1)! \blacksquare$$

Proposition 2.

If $n \geq 1$
$$e\left(\frac{n}{e}\right)^n \leq n! \leq ne\left(\frac{n}{e}\right)^n$$

Proof. For $n = 1, 2$ the validity of the inequalities is checked substituting these values. Further, it is easy to see that, for $k \geq 2$, the following inequalities hold for $ln(k)$:

$$\int_{k-1}^{k} ln(x)dx < ln(k) < \int_{k}^{k+1} ln(x)dx$$

Hence
$$\int_{1}^{n} ln(x)dx < ln(n!) < \int_{2}^{n} ln(x)dx$$

Transforming the right inequality under the condition $n \geq 3$, we get that $ln(n!) < \int_{2}^{n+1} ln(x)dx = (xln(x) - x)\Big|_{2}^{n+1} =$

$$= (n+1)ln(n+1) - (n+1) - 2ln(2) + 2 = (n+1)ln(\tfrac{n+1}{e}) - 2ln(2) + 2 =$$

$$= (n+1)ln(\frac{n}{e}) + (n+1)ln(\frac{1}{n}+1) - 2ln(2) + 2 < (n+1)ln(\frac{n}{e}) + 2$$

Hence,

$$n! < ne\left(\frac{n}{e}\right)^n$$

Similarly, we transform the left inequality

$$ln(n!) > \int_1^n ln(x)dx = (xln(x) - x)\Big|_1^n = nln(n) - n + 1 = nln(\frac{n}{e}) + 1$$

Thus,

$$n! > e\left(\frac{n}{e}\right)^n \blacksquare$$

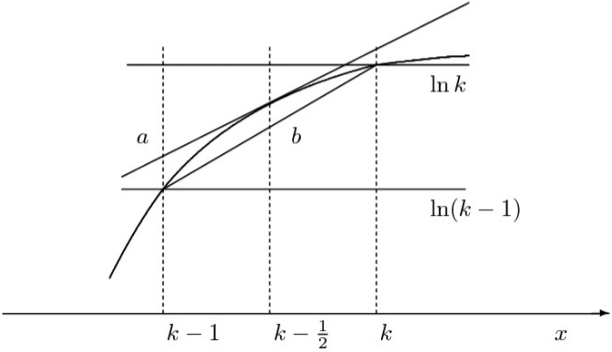
Let us establish slightly more precise inequalities for logarithms.

Lemma 3.

If $k \geq 2$,

$$\int_{k-1}^{k} ln(x)dx + ln(\frac{2k}{2k-1}) \leq ln(k) \leq \int_{k-1}^{k} ln(x)dx + \frac{1}{2}ln(\frac{k}{k-1})$$

Proof. The figure shows a part of the graph of the function $ln(x)$ located between the points $x = k - 1$ and $x = k$. In this figure, tangent line a is drawn to the curve $ln(x)$ at point $x = k - \frac{1}{2}$, the segment b connects the points $(k-1, ln(k-1))$ and $(k, ln(k))$. Its easy to see that $\int_{k-1}^{k} ln(x)dx$ exceeds the difference between $ln(k)$ and the area of the triangle bounded by by segment b and lines $x = k - 1$ and $y = ln(k)$.



Acrobat.jpg

Рис.: Graph

Hence

$$\int_{k-1}^{k} ln(x)dx \geq ln(k) - \frac{1}{2}(ln(k) - ln(k-1))$$

Clearly, $\int_{k-1}^{k} ln(x)dx$ does not exceed the area of the trapezoid, bounded by the line a and the lines $x = k-1$, $x = k$ and $y = 0$. Since the area area of the trapezoid is equal to the product of the length of the centre line, which in this case is $ln(k - \frac{1}{2})$ and the height equal to one, then

$$\int_{k-1}^{k} ln(x)dx \leq ln(k-\frac{1}{2}) = ln(k)+ln(1-\frac{1}{2k}) = ln(k)-ln(2k)+ln(2k-1)$$

Comparing the obtained inequalities for $ln(k)$, we obtain the required estimates. ■

Proposition 4.

$$\frac{4}{5}e\sqrt{n}\left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$$

Proof. To prove the proposition, it suffices to sum the inequalities of Lemma 3 over all k from 2 to n. Summing up the right inequalities, we see that

$$\sum_{k=2}^{n} ln(k) \leq \int_1^n ln(x)dx + \frac{1}{2}(ln(n) - ln(1)) = nln(n) - n + 1 + \frac{1}{2}ln(n)$$

Hence,

$$n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$$

To estimate the sum of left inequalities, we put

$$a_1 = \sum_{k=2}^{n}(ln(2k) - ln(2k-1)), \quad a_2 = \sum_{k=2}^{n}(ln(2k+1) - ln(2k))$$

Then $a_1 > a_2$, and $a_1 + a_2 = ln(2n+1) - ln(3)$, so

$$a_1 > \frac{1}{2}ln(2n+1) - \frac{1}{2}ln(3) > \frac{1}{2}ln(n) - \frac{1}{2}ln(\frac{3}{2})$$

Thus,

$$\sum_{k=2}^{n} ln(k) > nln(n) - n + 1 + \frac{1}{2}ln(n) - \frac{1}{2}ln(\frac{3}{2})$$

Now note that $\sqrt{2/3} > \frac{4}{5}$ and consequently

$$n! > \frac{4}{5}e\sqrt{n}\left(\frac{n}{e}\right)^n \blacksquare$$

The obtained inequalities are still quite weak - we will try to improve them. We start with the following lemma

Lemma 5.

$$\int_0^{\frac{\pi}{2}} sin^{2n}(x)dx = \frac{(2n-1)(2n-3)\dots 3\cdot 1}{2n(2n-2)\dots 4\cdot 2}\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} sin^{2n+1}(x)dx = \frac{2n(2n-2)\dots 4\cdot 2}{(2n+1)(2n-1)\dots 3\cdot 1}$$

Proof. Denote the integral $\int\limits_0^{\frac{\pi}{2}} sin^{2n}(x)dx$ by I_n. Then

$$I_n = -\int_0^{\frac{\pi}{2}} sin^{n-1}(x)d(cos(x)) =$$

$$= -sin^{n-1}(x)cos(x)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} cos(x)d(sin^{n-1}(x)) =$$

$$= (n-1)\int_0^{\frac{\pi}{2}} cos^2(x)sin^{n-2}(x)dx = (n-1)\int_0^{\frac{\pi}{2}}(1-sin^2(x))sin^{n-2}(x)dx =$$

$$= (n-1)(I_{n-2} - I_n)$$

Therefore, the folowing formula holds

$$I_n = \frac{n-1}{n} I_{n-2},$$

iterated application of which to the integrals I_{2n} and I_{2n+1} gives the following equalities:

$$I_{2n} = \frac{(2n-1)(2n-3)\dots 3\cdot 1}{2n(2n-2)\dots 4\cdot 2} I_0$$

and

$$I_{2n+1} = \frac{2n(2n-2)\dots 4\cdot 2}{(2n+1)(2n-1)\dots 3\cdot 1} I_1$$

Since $I_0 = \frac{\pi}{2}$ and $I_1 = 1$, substituting these values into the formulas above, we obtain the required equalities. ■

Notation (Double factorial). $n!!$ is the product of all the positive integers up to n that have the same parity (odd or even) as n.

Lemma 6.

$$\frac{2^{2n}}{\sqrt{\pi n}} e^{-1/4n} \leq C_{2n}^{n} \leq \frac{2^{2n}}{\sqrt{\pi n}}$$

Proof. To begin with, we note that
$sin^{2n+1}(x) \leq sin^{2n}(x) \leq sin^{2n-1}(x)$ for $0 \leq x \leq \frac{\pi}{2}$.

$$\int_0^{\frac{\pi}{2}} sin^{2n+1}(x)dx \leq \int_0^{\frac{\pi}{2}} sin^{2n}(x)dx \leq \int_0^{\frac{\pi}{2}} sin^{2n-1}(x)dx$$

From Lemma 3 we get

$$\frac{2n!!}{(2n+1)!!} \leq \frac{(2n-1)!!}{2n!!}\frac{\pi}{2} \leq \frac{(2n-2)!!}{(2n-1)!!}$$

which, as is easy to see, are transformed into the form

$$\frac{2n!!2n!!}{(2n+1)!!(2n-1)!!} \leq \frac{\pi}{2} \leq \frac{2n!!(2n-2)!!}{(2n-1)!!(2n-1)!!}$$

Taking the square roots of the new inequalities, we get that

$$\frac{1}{\sqrt{2n+1}}\frac{2n!!}{(2n-1)!!} \leq \sqrt{\frac{\pi}{2}} \leq \frac{1}{\sqrt{2n}}\frac{2n!!}{(2n-1)!!}$$

Next, we divide all the terms of the resulting inequalities by $2n!!$ and $\sqrt{\pi/2}$ and multiply by $(2n-1)!!$ and 2^{2n}. Then

$$\frac{1}{\sqrt{1+1/2n}}\frac{2^{2n}}{\sqrt{\pi n}} \leq 2^{2n}\frac{(2n-1)!!}{2n!!} \leq \frac{2^{2n}}{\sqrt{\pi n}}$$

Let us also note that

$$\frac{(2n-1)!!}{2n!!} = \frac{(2n-1)!!2n!!}{2n!!2n!!} = \frac{(2n)!}{2^{2n}n!n!} = C_{2n}^{n}2^{-2n}$$

Now, given that $e^{-x} < \frac{1}{(1+x)}$ for $0 < x < 1$, we substitute the last equality into the previous one. We are done ■

Theorem 7.

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{-1/4n} \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/4n}$$

Proof. $n! = \frac{2n(2n-1)\dots(n+1)}{C_{2n}^n}$. We will use the inequalities from Lemma 3

$$ln(k) \ge \int\limits_{k-1}^{k} ln(x)dx + ln(2k) - ln(2k-1)$$

$$ln(k) \le \int\limits_{k-1}^{k} ln(x)dx + \frac{1}{2}(ln(k) - ln(k-1))$$

Summing up the inequalities over all k from $n+1$ to $2n$, we see that

$$\sum_{k=n+1}^{2n} ln(k) \leq \int_n^{2n} ln(x)dx + \frac{1}{2}(ln(2n) - ln(n)) =$$

$$= \int_n^{2n} ln(x)dx + \frac{1}{2}ln(2) = nln(n) + 2nln(2) - n + \frac{1}{2}ln(2).$$

Let's introduce the following notation again

$$a_1 = \sum_{k=n+1}^{2n} (ln(2k) - ln(2k-1)), \quad a_2 = \sum_{k=n+1}^{2n} (ln(2k+1) - ln(2k))$$

Then $a_1 + a_2 = ln(4n+1) - ln(2n+1)$, and $a_1 > a_2$, so

$$a_1 > \frac{1}{2}(ln(4n+1) - ln(2n+1)) = \frac{1}{2}ln\left(2 \cdot \frac{2n+1/2}{2n+1}\right) =$$

$$= \frac{1}{2} ln(2) + \frac{1}{2} ln(1 - \frac{1}{4n+2}) \geq \frac{1}{2} ln(2) - \frac{1}{4n}.$$

Therefore

$$\sum_{k=n+1}^{2n} ln(k) > nln(n) + 2nln(2) - n + \frac{1}{2} ln(2) - \frac{1}{4n} \quad \text{whence}$$

$$\sqrt{2}\left(\frac{n}{e}\right)^n 2^{2n} e^{-1/4n} \leq 2n(2n-1)\ldots(n+1) \leq \sqrt{2}\left(\frac{n}{e}\right)^n 2^{2n}.$$

Lemma 6 implies

$$\frac{\sqrt{\pi n}}{2^{2n}} < \frac{1}{C_{2n}^n} < \frac{\sqrt{\pi n}}{2^{2n}} e^{1/4n}$$

multiply the last inequalities and obtain

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{-1/4n} \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/4n} \blacksquare$$

Corollary 8 (Stirling's formula).

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + o(1))$$

Remark. More precise estimates for $n!$ can be found in Graham, Knuth, Patashnik, "Concrete Mathematics - A foundation for computer science" (1994).

Proposition 9.

For all integers n and k such that $n \geq k \geq 1$, we have that

$$\left(\frac{n}{k}\right)^k \leq C_n^k$$

Proof. Fix integers n, k such that $n \geq k \geq 1$. We observe that for all $i \in \{0, \dots, k-1\}$, we have that $\frac{n-i}{k-i} \geq \frac{n}{k}$, and so

$$C_n^k = \prod_{i=0}^{k-1} \frac{n-i}{k-i} \geq \prod_{i=0}^{k-1} \frac{n}{k} = \left(\frac{n}{k}\right)^k$$

which is what we needed. ■

Lemma 10.

Fix integers n and k such that $n \geq k \geq 1$. Then for all real numbers x such that $0 < x \leq 1$, we have that

$$\sum_{i=0}^{k} C_n^i \leq \frac{(1+x)^n}{x^k}.$$

Proof. By the Binomial theorem, we have that

$$(1+x)^n = \sum_{i=0}^{n} C_n^i x^i \geq \sum_{i=0}^{k} C_n^i x^i \quad \text{since } n \geq k \text{ and } x > 0.$$

Dividing by x^k, we then obtain

$$\frac{(1+x)^n}{x^k} \geq \sum_{i=0}^{k} C_n^i \frac{1}{x^{k-i}} \geq \sum_{i=0}^{k} C_n^i \quad \text{because} \quad 0 < x \leq 1 \quad \blacksquare$$

Proposition 11.

For all integers n and k such that $n \geq k \geq 1$, we have that:

$$\sum_{i=0}^{k} C_n^i \leq \left(\frac{en}{k}\right)^k$$

Proof. Apply the Lemma 10 to $x := \frac{k}{n}$, we obtain

$$\sum_{i=0}^{k} C_n^i \leq \left(1+\frac{k}{n}\right)^n \left(\frac{n}{k}\right)^k \leq (e^{\frac{k}{n}})^n \left(\frac{n}{k}\right)^k = \left(\frac{en}{k}\right)^k$$

Here the second inequality follows from $1 + x \leq e^x$ ■

Consider the set of integers $\mathbb{Z}$. We begin our walk at 0, and at each step we move at random either one step to the left or one step to the right.

We would like to estimate the number of times that we return to the origin in such a walk. Obviously, we can only return to the origin after an even number of steps. There are 2^{2m} random walks of length $2m$, and exactly C_{2m}^{m} of those walks end at the origin (indeed, we must go left exactly m times, and right exactly m times. Out of $2m$ moves, we have C_{2m}^{m} ways of selecting the m leftward moves, and the other m moves are rightward).

So, the probability of returning to the origin after exactly $2m$ steps is

$$\frac{C_{2m}^{m}}{2^{2m}}.$$

By Lemma 6, we have that

$$\sum_{m=1}^{\infty} \frac{C_{2m}^{m}}{2^{2m}} \geq \sum_{m=1}^{\infty} \frac{1}{2\sqrt{m}} = \infty$$

Thus, we can expect that in an infinite one-dimensional random walk starting at the origin, we will return to the origin an infinite number of times.

Exercise. Let $n \to \infty$ and $k = o(n^{2/3})$, then

$$C_n^k = \frac{n^k e^{\frac{-k^2}{2n}}}{k!}(1 + o(1))$$