Chapter 3

1.1. The Taylor Formula for Polynomials

Theorem 1: For any x_0 a polynomial $P(x)$ of degree n can be represented as

$$P(x) = P(x_0) + \frac{P'(x_0)}{1!}(x - x_0) + ... + \frac{P^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= \sum_{k=0}^{n} \frac{P^{(k)}(x_0)}{k!}(x - x_0)^k, \tag{1}$$

where $P'(x_0)$, $P''(x_0)$, … are the derivatives of $P(x)$ at the point x_0.

Note: Formula (1) is called the **Taylor Formula for polynomials**.

Proof: Any polynomial of degree n can be written as follows:

$$P(x) = \sum_{k=0}^{n} a_k (x - x_0)^k . \tag{2}$$

Therefore, we have to prove that $a_k = \dfrac{P^{(k)}(x_0)}{k!}$ for $0 \le k \le n$.

First, the equality $a_0 = P(x_0)$ follows from (2) when $x = x_0$.

Then let us find the *k*th derivative of the polynomial $P(x)$ at the point $x = x_0$.

One can easily see that sum (2) contains just one term, whose *k*th derivative at the point $x = x_0$ is not equal to zero: $\left(a_k (x - x_0)^k\right)^{(k)} = a_k\, k!$

The *k*th derivative of other terms of this sum either equals zero for any x or contains the factor $(x - x_0)$, which vanishes as ever $x = x_0$.

Thus, $P^{(k)}(x_0) = a_k\, k!$ and hence, the theorem.

Example: Represent the polynomial $P(x)$ in powers of x, if

$$P(x) = 1 + 8(x - 2) + 6(x - 2)^2 + (x - 2)^3.$$

Solution: The Taylor Formula with $x_0 = 0$ gives the answer in the general form:

$$P(x) = P(0) + P'(0)\,x + \frac{P''(0)}{2}x^2 + \frac{P'''(0)}{6}x^3.$$

It remains to find $P(0)$ and $P^{(k)}(0)$:

- $P(x) = 1 + 8(x - 2) + 6(x - 2)^2 + (x - 2)^3 \quad \Rightarrow \quad P(0) = 1.$

- $P'(x) = 8 + 12(x-2) + 3(x-2)^2 \quad \Rightarrow \quad P'(0) = -4$.
- $P''(x) = 12 + 6(x-2) \quad \Rightarrow \quad P''(0) = 0$.
- $P'''(x) = 6 \quad \Rightarrow \quad P'''(0) = 6$.

Thus, $\qquad P(x) = 1 - 4x + x^3$.

1.2. The Taylor Formula with the Remainder

Theorem 2: Let a function $f(x)$ be n times differentiable at a point x_0. Then $f(x)$ can be represented by the Taylor Formula

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x), \tag{3}$$

where $f^{(0)}(x_0) = f(x_0)$ by definition, and $R_n(x)$ is a function such that

$$R_n(x_0) = R_n'(x_0) = R_n''(x_0) = \ldots = R_n^{(n)}(x_0) = 0. \tag{4}$$

Note that $R_n(x)$ is called the **remainder**.

Proof: The remainder $R_n(x)$ is the difference between $f(x)$ and the polynomial

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

By the argument used in the proof of Theorem 1, we obtain

$$P_n(x_0) = f(x_0) \quad \text{and} \quad P_n^{(k)}(x_0) = \frac{f^{(k)}(x_0)}{k!}k! = f^{(k)}(x_0).$$

Therefore,

$$R_n^{(k)}(x_0) = f^{(k)}(x_0) - P_n^{(k)}(x_0) = f^{(k)}(x_0) - f^{(k)}(x_0) = 0$$

for $0 \le k \le n$, which implies formulas (4).

In a special case when $x_0 = 0$, the Taylor formula is named the Maclaurin formula:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + R_n(x) \tag{5}$$

Now let us come back to formula (3) and rewrite it in terms of differentials. We need to recall the relevant definitions.

The difference $x - x_0 = \Delta x$ can be considered as an increment of the argument; then $f(x) - f(x_0) = \Delta f(x)$ is the corresponding increment of the function.

By definition $dx = \Delta x$, that is, the differential of the argument equals the increment. The *k*th differential of the argument is defined as $dx^k \equiv (dx)^k$.

The differential of $f(x)$ at the point $x = x_0$ is $df(x_0) = f'(x_0)dx$, and the kth differential of $f(x)$ at this point is $d^k f(x_0) = f^{(k)}(x_0)dx^k$.

The Taylor Formula has the simplest form in terms of differentials:

$$\Delta f(x) = df(x_0) + \frac{d^2 f(x_0)}{2!} + \frac{d^3 f(x_0)}{3!} + \ldots + \frac{d^n f(x_0)}{n!} + R_n(x). \qquad (6)$$

The Taylor Formula has diverse applications. Most often it is used tor approximation of transcendental functions by polynomials. In this case the polynomial $P_n(x)$ is an approximation to $f(x)$, whereas $R_n(x)$ is an error of the approximation. Such conclusion has the following background.

If $f(x)$ is a continuous function on some interval, then so is the remainder $R_n(x)$. In view of the fact that $R_n(x_0) = R_n'(x_0) = 0$, the remainder is small enough in some vicinity of the point x_0. Moreover, the remainder $R_n(x)$ is an infinitesimal whose order of smallness is greater than n as $x \to x_0$, that is,

$$\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = 0.$$

This statement can be easily proved by applying the L'Hopital rule n times and taking into account equalities (4):

$$\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = \lim_{x \to x_0} \frac{R_n'(x)}{n(x - x_0)^{n-1}} = \ldots = \lim_{x \to x_0} \frac{R_n^{(n)}(x)}{n!} = 0.$$

Whenever we deal with approximations, we need to control the errors.

One of the ways is based on the **Lagrange form of the remainder**:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \qquad (7)$$

where c is some point between x and x_0.

If $|x - x_0| < 1$ and $f^{(n+1)}(x) \le M$, then

$$\left|R_n(x)\right| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}\right| \le \left|M\,\frac{(x-x_0)^{n+1}}{(n+1)!}\right| \;\to\; 0$$

very quickly as $n \to \infty$.

Therefore, the more n, the better approximation to $f(x)$ by the polynomial $P_n(x)$.

The Taylor Formula with the Lagrange form of the remainder can be written as follows:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \qquad (8)$$

In a special case when $n = 0$, this formula implies the Mean Value Theorem over the interval $[x, x_0]$:

$$f(x) = f(x_0) + f'(c)(x - x_0).$$

1.3. Applications of the Taylor Formula

All the formulas below follow from the Maclaurin Formula. All we need to find the expansion for a specific function $f(x)$ is the general form of the nth derivative of $f(x)$. The remainders in all the cases are written in the Lagrange form.

1) Let $f(x) = e^x$. Then

$$f^{(n)}(x) = e^x \quad \text{and} \quad f^{(n)}(0) = 1 \quad \text{for } n \ge 0.$$

Therefore,

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \text{Error.}} \qquad (9)$$

$$\text{Error} = R_n(x) = \frac{e^c}{(n+1)!}\,x^{n+1},$$

where c is a point between zero and x.

If $x < 0$, then $e^c < 1$ and

$$|\,\text{Error}\,| < \frac{1}{(n+1)!}\,|\,x\,|^{n+1}.$$

2) If $f(x) = \sin x$, then

$$f^{(k)}(x) = \sin(x + \frac{k\pi}{2}) \quad \text{and} \quad f^{(k)}(0) = \sin\frac{k\pi}{2}.$$

If $k = 2n-1$, then

$$f^{(2n-1)}(0) = (-1)^{n-1},$$

while if $k = 2n$, then

$$f^{(2n)}(0) = 0.$$

Therefore,

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \text{Error.}} \tag{10}$$

$$|\,\text{Error}\,| < \frac{|x|^{2n}}{(2n)!} \quad \text{for} \quad x < 0,$$

$$|\,\text{Error}\,| < \frac{|x|^{2n+1}}{(2n+1)!} \quad \text{for} \quad x > 0.$$

3) If $f(x) = \cos x$, then

$$f^{(k)}(x) = \cos(x + \frac{k\pi}{2}) \quad \text{and} \quad f^{(k)}(0) = \cos\frac{k\pi}{2},$$

that is, $\quad f^{(2n+1)}(0) = 0 \quad$ and $\quad f^{(2n)}(0) = (-1)^n$.

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n\frac{x^{2n}}{(2n)!} + \text{Error.}} \tag{11}$$

$$|\,\text{Error}\,| < \frac{|x|^{2n+2}}{(2n+2)!} \quad \text{for any } x.$$

4) Let $f(x) = \arctan x$. Then

$$\boxed{\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \text{Error.}} \tag{12}$$

If $\quad 0 < x < 1,\quad$ then

$$|\,\text{Error}\,| < \frac{1}{2n+1}|x|^{2n+1}.$$

5) Let $f(x) = \ln(1+x)$, where $x > -1$. Then $f(0) = 1$,

$$f^{(n)}(x) = \frac{(-1)^{n-1}}{(1+x)^n}(n-1)!, \text{ and } f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

$$\boxed{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \text{Error.}} \qquad (13)$$

$$\text{Error} = R_n(x) = (-1)^n \frac{x^{n+1}}{n+1}(1+c)^{-n-1},$$

where c is a point between zero and x.

$$\text{If} \quad 0 < x \le 1, \quad \text{then} \quad |\,\text{Error}\,| < \frac{1}{n+1}x^{n+1}.$$

6) Let $f(x) = (1+x)^m$, where m is any rational number. Then

$$f^{(n)}(x) = m(m-1)\ldots(m-n+1)(1+x)^{m-n},$$

$$f^{(n)}(0) = m(m-1)\ldots(m-n+1).$$

Therefore,

$$\boxed{\begin{aligned}(1+x)^m &= 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^n \\ &\quad + \text{Error}\end{aligned}} \qquad (14)$$

$$\text{If} \quad m = n, \quad \text{then} \quad \text{Error} = 0.$$

Example 1: Calculate approximately $\sqrt[3]{e}$.

Solution: Formula(9) for $n = 1, 2, 3$ and $x = 1/3$ yields successively:

1) $e^x \approx 1 + x \quad \Rightarrow \quad \sqrt[3]{e} = e^{1/3} \approx 1 + \frac{1}{3} = \frac{4}{3} \approx 1.333;$

2) $e^x \approx 1 + x + \frac{x^2}{2} \quad \Rightarrow \quad \sqrt[3]{e} \approx 1 + \frac{1}{3} + \frac{1}{2!}\frac{1}{3^2} = \frac{25}{18} \approx 1.389;$

3) $e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \quad \Rightarrow$

$$\sqrt[3]{e} \approx 1 + \frac{1}{3} + \frac{1}{2!}\frac{1}{3^2} + \frac{1}{3!}\frac{1}{3^3} = \frac{113}{81} \approx 1.395.$$

Compare the answers with the exact result $\sqrt[3]{e} = 1.3956\ldots$

Example 2: Calculate approximately $\sin 18°$.

Solution: First, it is necessary to convert degrees to radians: $18° = \pi/10$.

Then formula (10) for $n = 1, 2$ and $x = \pi/10$ yields successively:

1) $\sin x \approx x \quad \Rightarrow \quad \sin\frac{\pi}{10} \approx \frac{\pi}{10} \approx 0.314$;

2) $\sin x = x - \frac{x^3}{3!} \quad \Rightarrow \quad \sin\frac{\pi}{10} \approx \frac{\pi}{10} - \frac{1}{6}\left(\frac{\pi}{10}\right)^3 \approx 0.3089$.

The exact result is $0.3090...$

Example 3: Calculate approximately $\sqrt[3]{10}$.

Solution: It is necessary to transform the problem for applying formula (14):

$$\sqrt[3]{10} = \sqrt[3]{8+2} = \sqrt[3]{8(1+1/4)} = 2\sqrt[3]{1+1/4}.$$

Now formula (14) for $n = 2$ and $x = 1/4$ yields

$$\sqrt[3]{1+1/4} = (1+1/4)^{1/3} \approx 1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{2}\frac{1}{3}(\frac{1}{3}-1)\frac{1}{4^2} = \frac{155}{144}.$$

Therefore, $\sqrt[3]{10} = 2\sqrt[3]{1+1/4} \approx 155/72 \approx 2.1528$.

The exact result is $2.1544...$

Example 4: Suppose we need to calculate $\sqrt{e}$, using an approximating polynomial.

In order to estimate an error bound, we can use the Lagrange form of the remainder. Since e^x is the increasing function and $0 < c < 0.5$, so $e^c < 2$.

Therefore,

$$R_n(\frac{1}{2}) = \frac{e^c}{(n+1)!}\cdot\frac{1}{2^{n+1}} < \frac{1}{(n+1)!\,2^n},$$

which yields

$$R_1(0.5) < 0.25, \quad R_2(0.5) < 1/24 \approx 0.04, \quad R_3(0.5) < 1/192 \approx 0.005,$$

etc.

Thus, the approximating polynomial of the third degree yields a value of $\sqrt{e}$ with an error bound of at most 0.005.

Summary: The table below contains a list of approximating formulas for some functions in a vicinity of the point $x = 0$. The formulas are illustrated by drawings.

Functions	First Approximation	Close Approximation
e^x	$1 + x$	$1 + x + \frac{x^2}{2}$
$\sin x$	x	$x - \frac{x^3}{6}$
$\cos x$	$1 - \frac{x^2}{2}$	$1 - \frac{x^2}{2} + \frac{x^4}{24}$
$\tan x$	x	$x + \frac{x^3}{3}$
$\ln(1 + x)$	x	$x - \frac{x^2}{2}$
$\arctan x$	x	$x - \frac{x^3}{3}$
$\frac{1}{1 + x}$	$1 - x$	$1 - x + x^2$
$\sqrt{1 + x}$	$1 + \frac{x}{2}$	$1 + \frac{x}{2} - \frac{x^2}{8}$

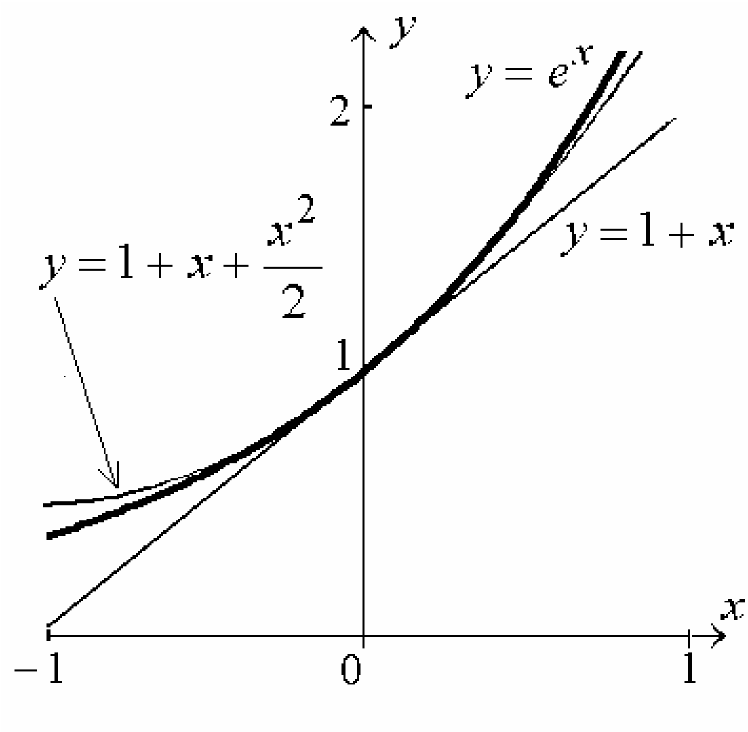

y
$y = e^x$
2
$y = 1 + x + \frac{x^2}{2}$
$y = 1 + x$
1
x
-1
0
1

Fig. 1

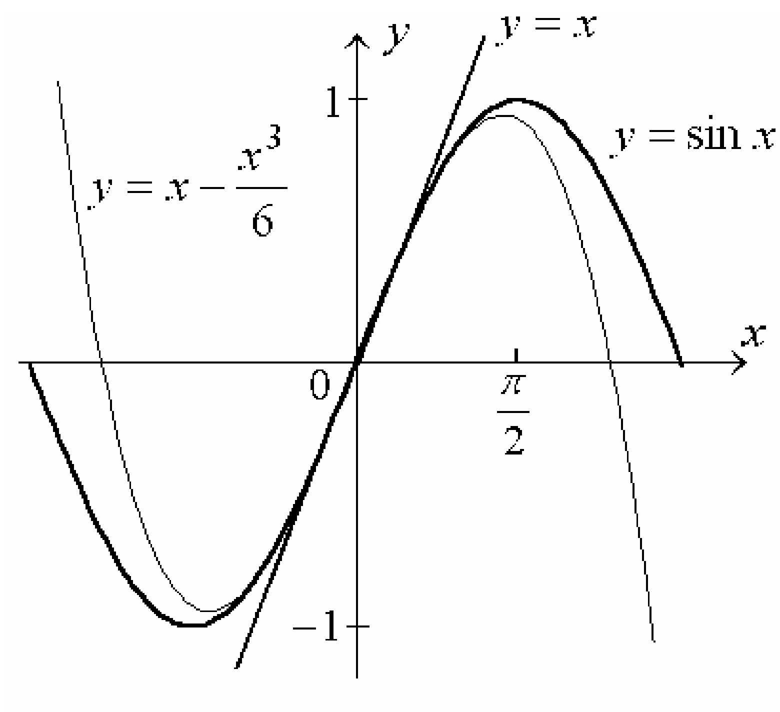

y
$y = x$
1
$y = \sin x$
$y = x - \frac{x^3}{6}$
x
0
$\frac{\pi}{2}$
-1

Fig. 2

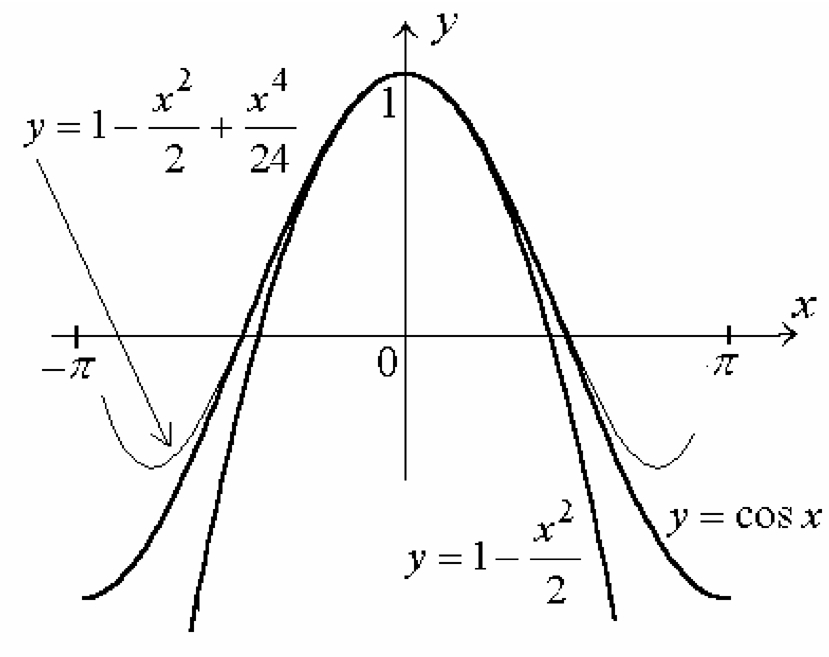


Fig. 3

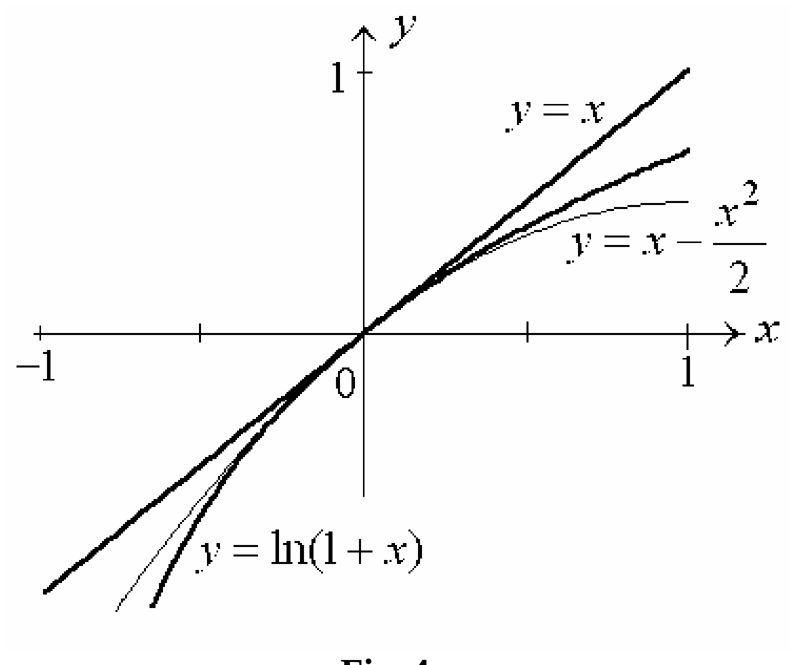


Fig. 4

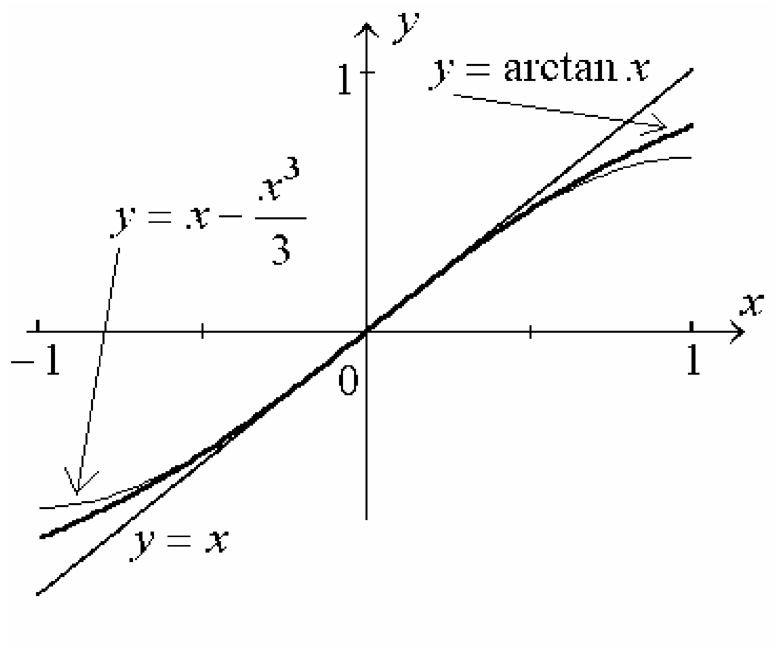


Fig. 5

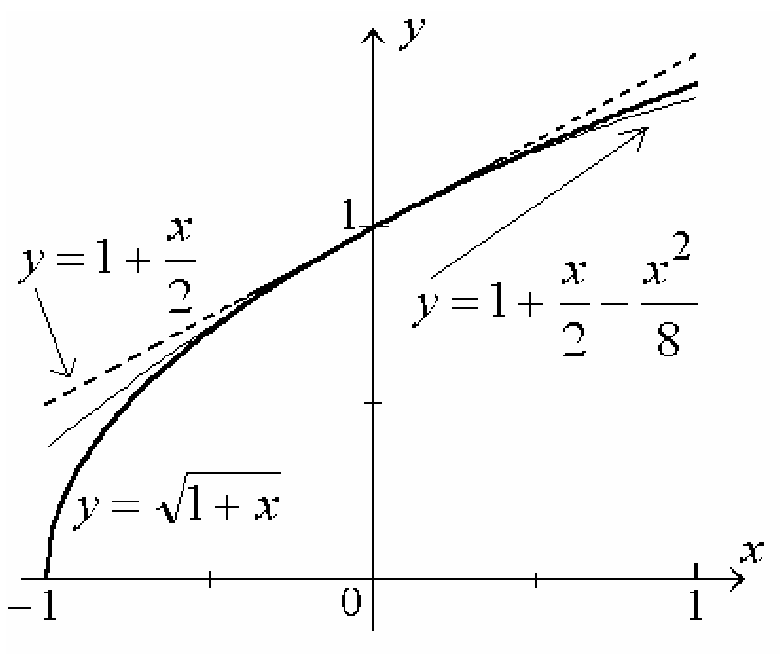


Fig. 6

Chapter 3

TAYLOR AND MACLAURIN SERIES

Let the function $f(x)$ at point x_0 have derivatives of any order. Power series

$$f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\ldots=\sum_{j=0}^{\infty}\frac{f^{(j)}(x_0)}{j!}(x-x_0)^j$$

(1)

the power series is called the **Taylor series** of the function $f(x)$ at point x_0.

The Taylor series at $x_0=0$, that is, the series

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{j=0}^{\infty}\frac{f^{(j)}(0)}{j!}x^j \qquad \textbf{(2)}$$

is called the **Maclaurin series.**

Since the Maclaurin series is a special case of the Taylor series, information for the Taylor series will be provided below. This information, of course, will apply to the Maclaurin series.

If for any x from some set $X\{x\}$, the series (1) converges to $f(x)$, then it is said that on the set X, the function $f(x)$ *decomposes into a Taylor series in degrees* $x-x_0$.

Let the function $f(x)$ in some neighborhood of point x_0 have derivatives of any order, or, as they say, *infinitely differentiable* in this neighborhood. We write down the Taylor formula with a residual term in the Lagrange form for any fixed x from the neighborhood under consideration:

$$f(x)=f(x_0)+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+r_n(x),$$

$$r_n(x)=\frac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1},$$

where θ satisfies the inequality $0<\theta<1$.

Theorem 1.

In order for the function $f(x)$ to allow expansion into the Taylor series (1) in the specified neighborhood, it is necessary and sufficient that for any x from this neighborhood the condition $\lim\limits_{n\to\infty} r_n(x)=0$ is fulfilled.

In particular, this will be done if the derivatives of any order of the function f are *uniformly bounded* in the neighborhood under consideration, that is, if there exists such a number M, $M > 0$, that for any x from this neighborhood and for any natural number n , the inequality $\left| f^{(n)}(x) \right| < M$. takes place.

Theorem 2.

If an infinitely differentiable function $f(x)$ in some neighborhood of point x_0 is the sum of a power series $\sum_{j=0}^{\infty} a_j (x - x_0)^j$, then this series is its Taylor series (1), that is, $a_j = \frac{f^{(j)}(x_0)}{j!}$, $j = 0,1,2,\ldots$.

Maclaurin series expansions of functions e^x, $\sin x$, $\cos x$, $ln(1+x)$, $arctg\, x$, binomial expansion and decomposition of $\frac{1}{1-q}$ as the sum of an infinite geometric progression (Maclaurin series expansion by degrees of A) are presented. After each decomposition, we indicate the interval of argument change at which decomposition is possible (the resulting series converges to the decomposable function):

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, -\infty < x < +\infty, \qquad \textbf{(3)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, -\infty < x < +\infty, \qquad \textbf{(4)}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, -\infty < x < +\infty, \qquad \textbf{(5)}$$

$$ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, -1 < x \le 1, \qquad \textbf{(6)}$$

$$arctg\, x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, -1 \le x \le 1, \qquad \textbf{(7)}$$

binomial expansion:

$$(1+x)^{\mu} = 1 + \mu x + \frac{\mu(\mu-1)}{2!} x^2 + \frac{\mu(\mu-1)(\mu-2)}{3!} x^3 + \ldots =$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\mu(\mu-1)(\mu-2)\ldots(\mu-n+1)}{n!}, \mu \in \mathrm{R}, -1 < x < 1, \qquad \textbf{(8)}$$

$$\frac{1}{1-q} = 1 + q + q^2 + q^3 + q^4 + \ldots = \sum_{n=0}^{\infty} q^n, -1 < q < 1. \qquad \textbf{(9)}$$

Comment

Equalities (3)-(9) are convenient to use in the practical decomposition of functions into power series. The direct application of decompositions (1) and (2) requires that sets of argument values be found on which the resulting decompositions converge to the original decomposable functions. This, for example, can be done using Theorem 1. As a rule, such studies are quite difficult. Therefore, it is better to use ready-made "standard" decompositions (3)-(9).

Example 1.

Decompose the function $f(x)=\dfrac{2x-1}{x^2-7x+12}$ by degrees of $t=x-1$.

Solution:

From the last equality, we find $x=t+1$ and substitute into the expression of the function f. We will have $f=\dfrac{2t+1}{t^2-5t+6}$. Decompose the resulting correct rational fraction into the simplest fractions: $\dfrac{2t+1}{t^2-5t+6}=\dfrac{A}{t-2}+\dfrac{B}{t-3}$, from where $2t+1=A(t-3)+B(t-2)$. Assuming $t=2$ and $t=3$ in the last equality, we find $A=-5$ and $B=7$. Hence

$$f=\frac{5}{2-t}-\frac{7}{3-t}=\frac{5}{2}\cdot\frac{1}{1-\frac{t}{2}}-\frac{7}{3}\cdot\frac{1}{1-\frac{t}{3}}.$$

Now using the decomposition (9), putting for the first term in the right part of the last equality $q=\dfrac{t}{2}$, for the second- $q=\dfrac{t}{3}$. We get

$$f=\frac{5}{2}\sum_{n=0}^{\infty}\left(\frac{t}{2}\right)^n-\frac{7}{3}\sum_{n=0}^{\infty}\left(\frac{t}{3}\right)^n=\sum_{n=0}^{\infty}\left(\frac{5}{2^{n+1}}-\frac{7}{3^{n+1}}\right)t^n=\sum_{n=0}^{\infty}\left(\frac{5}{2^{n+1}}-\frac{7}{3^{n+1}}\right)(x-1)^n,$$

while from the condition $-1<q<1$ we have

$$\begin{cases}\left|\dfrac{t}{2}\right|<1,\\[2mm] \left|\dfrac{t}{3}\right|<1,\end{cases}\Leftrightarrow -2<t<2\Leftrightarrow -1<x<3.$$

Answer:

$$f(x)=\sum_{n=0}^{\infty}\left(\frac{5}{2^{n+1}}-\frac{7}{3^{n+1}}\right)(x-1)^n,\ -1<x<3.$$

Example 2.

Decompose the function $f(x)=\sin 2x$ by degrees of $t=x-\frac{\pi}{4}$.

Solution:

Substitute the argument $x=t+\frac{\pi}{4}$ into the expression of the function f, and we get $f=\cos 2t$. We use decomposition (5), replacing x in it with the expression $2t$:

$$f=\sum_{n=0}^{\infty}(-1)^n\frac{(2t)^{2n}}{(2n)!}=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}t^{2n}=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n}.$$

Considering that the decomposition (5) takes place for any real value of the argument, we come to the inequality $-\infty<x<+\infty$.

Answer:

$$f=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}}{(2n)!}\left(x-\frac{\pi}{4}\right)^{2n},\quad -\infty<x<+\infty.$$

Example 3.

Decompose the function $f(x)=\sqrt{6-x}$ by degrees of $t=x-2.$

Solution:

Substituting the argument $x=t+2$ into the expression of the function f, we get $f=\sqrt{4-t}=2\left(1-\frac{t}{4}\right)^{\frac{1}{2}}$. Using decomposition (8) for $\mu=\frac{1}{2}$ and for $x=-\frac{t}{4}$, we get:

$$f=2\left(1+\frac{1}{2}\left(-\frac{t}{4}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{t}{4}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{t}{4}\right)^3+\ldots\right.$$

$$\left.\ldots+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!}\left(-\frac{t}{4}\right)^n+\ldots\right)=$$

$$=2\left(1-\frac{1}{8}t-\frac{1}{128}t^2-\ldots-\frac{1\cdot3\cdot5\ldots(2n-3)}{2^{3n}\,n!}t^n-\ldots\right)=$$

$$=2\left(1-\frac{1}{8}(x-2)-\frac{1}{128}(x-2)^2-...-\frac{1\cdot 3\cdot 5...(2n-3)}{2^{3n}\,n!}(x-2)^n-...\right);$$

$$-1<\frac{t}{4}<1\Leftrightarrow -2<x<6.$$

Answer:

$$f(x)=2\left(1-\frac{1}{8}(x-2)-\frac{1}{128}(x-2)^2-...-\frac{1\cdot 3\cdot 5...(2n-3)}{2^{3n}\,n!}(x-2)^n-...\right),\ -2<x<6.$$

Example 4.

To calculate approximately the integral $I=\int_0^{\frac{1}{2}}\frac{\sin x}{x}dx$ with an accuracy of 10^{-4} .

Solution:

Using decomposition (4), we obtain

$$I=\int_0^{\frac{1}{2}}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-...\right)dx=\int_0^{\frac{1}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}dx.$$

Decomposition (4) takes place at $-\infty<x<+\infty$. Based on the theorem of direct integration and direct differentiation of a power series, the expression under the sign of the integral can be directly integrated over the interval $\left[0,\frac{1}{2}\right]$:

$$I=\left(x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-...\right)\Bigg|_0^{\frac{1}{2}}=\left(\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)(2n+1)!}\right)\Bigg|_0^{\frac{1}{2}}=$$

$$=\frac{1}{2}-\frac{1}{3\cdot 3!}\cdot\frac{1}{2^3}+\frac{1}{5\cdot 5!}\cdot\frac{1}{2^5}-...+(-1)^n\frac{1}{(2n+1)\cdot(2n+1)!}\cdot\frac{1}{2^{2n+1}}...=$$

$$=\frac{1}{2}-\frac{1}{144}+\frac{1}{19200}-...=\frac{71}{144}+r.$$

The resulting numerical series is an alternating series satisfying the conditions of Leibniz's theorem. As a consequence of this theorem, the modulus of the sum of the remainder of a series is less than the modulus of its first term, the sign of the remainder coincides with the sign of its first term. Therefore $0<r<\frac{1}{19200}<10^{-4}$.

Answer:

$$\int_{0}^{\frac{1}{2}} \frac{\sin x}{x}\, dx = \frac{71}{144} + r, \quad 0 < r < \frac{1}{19200}.$$

Chapter 4

FUNCTIONS OF SEVERAL VARIABLES

4.1. Introduction

The basic concepts of the theory of functions of several variables are the same or can be formulated like that of a single variable. Many definitions of a function of one variable can be easily generalized to functions of two or more than two variables.
However, some complications arise in the computation and interpretation of results.
Let us begin from the simplest concepts.

Distance Between Points

Any point P in the $xy-$plane can be described by the ordered pair (x, y) of real numbers. The distance between two points $P(x, y)$ and $P_0(x_0, y_0)$ is $\rho(P, P_0) = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.

In order to describe a point in three-dimensional space, it is necessary to operate with a triplet (x, y, z) of numbers, so that the distance between points $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$ is

$$\rho(P, P_0) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} .$$

In a similar way a point in multidimensional space can be represented by n numbers $x_1, x_2, ..., x_n$. The generalized formula for the distance between points $P(x_1, x_2, ..., x_n)$ and $P(a_1, a_2, ..., a_n)$ looks like above:

$$\rho(P, P_0) = \sqrt{(x_1 - a_1)^2 + (x_2 - a_2)^2 + ... + (x_n - a_n)^2} . \qquad (1)$$

Definition of Functions

Let $P(x_1, x_2, ..., x_n)$ be a point of some set D.
If each point of D is associated with one value of a variable u, then it is said that a function u of variables $x_1, x_2, ..., x_n$ is defined on the set D.
Recall that a function of one variable is denoted as $y = f(x)$. A function of several variables is denoted just in the same manner using the function notation by the equality

$$u = f(x_1, x_2, ..., x_n)$$

or in a short form as $u = f(P)$.

The set D is called the **domain** of definition, and the set of all values of u is called the **range** of a function.

In particular, a function of two independent variables is usually denoted as $z = f(x, y)$. The equation $z = f(x, y)$ can be interpreted graphically as a surface in three-dimensional space.

The domain of definition of a function of two variables is some set of points in $xy-$plane.

Example: The domain D of the function

$$z = \sqrt{x^2 + y^2 - 1} + \sqrt{2 - x^2 - y^2}$$

is the ring domain $D = (x| \ (x^2 + y^2 \ge 1) \cap (x^2 + y^2 \le 2))$, that means any values of x and y such that $1 \le x^2 + y^2 \le 2$.

Some examples of domains are shown in Fig. 1.

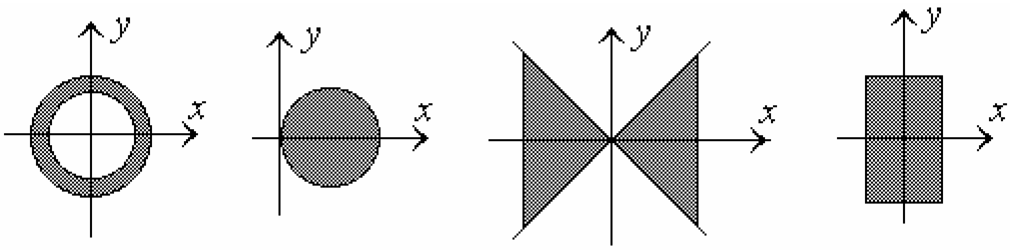


Fig. 1

4.2. Limits of Functions of Several Variables

The mathematical statement

$$\lim_{x \to a} f(x) = A$$

for a function of the single variable means that the difference between $f(x)$ and A vanishes as the distance between points x and a on the number line is getting smaller and smaller.

The definition as well as the properties of limits of a function of one variable can be easily generalized to functions of more than one variable.

Moreover, the limit of a function of several independent variables can be defined just in the same way as in case of a function of one variable.

Let $f(P)$ be a function of several variables, which is defined in some vicinity of a point P_0.

The limit of $f(P)$ as P tends to P_0 is equal to A if and only if for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that $f(P)$ obeys the inequality

$$|f(P) - A| < \varepsilon,$$

whenever the distance $\rho(P, P_0)$ between points P and P_0 obeys the inequality

$$|\rho(P, P_0)| < \delta .$$

This statement is denoted as

$$\lim_{P \to P_0} f(P) = A . \tag{2}$$

In a particular case of a function of two variables one uses the natural notation

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = A \tag{3}$$

If limit (3) exists, then

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = \lim_{x \to a} \lim_{y \to b} f(x, y) = \lim_{y \to b} \lim_{x \to a} f(x, y) . \tag{4}$$

All properties of limits hold for functions of several variables:

--- If there exists $\lim_{P \to P_0} f(P)$ and c is any number, then

$$\lim_{P \to P_0} cf(P) = c \lim_{P \to P_0} f(P) . \tag{5}$$

--- If there exist both limits, $\lim_{P \to P_0} f(P)$ and $\lim_{P \to P_0} g(P)$, then there exist the limits of the sum, product and quotient of functions such that

- The limit of the sum of functions is the sum of the limits of the functions:

$$\lim_{P \to P_0} (f(P) \pm g(P)) = \lim_{P \to P_0} f(P) \pm \lim_{P \to P_0} g(P) . \tag{6}$$

- The limit of the product of functions is the product of the limits of the functions:

$$\lim_{P \to P_0} f(P)g(P) = \lim_{P \to P_0} f(P) \lim_{P \to P_0} g(P) . \tag{7}$$

- The limit of the quotient of functions is the quotient of the limits of the functions, provided $\lim_{P \to P_0} g(P) \neq 0$:

$$\lim_{P \to P_0} \frac{f(P)}{g(P)} = \frac{\lim_{P \to P_0} f(P)}{\lim_{P \to P_0} g(P)} . \tag{8}$$

Example: Find the limit of the function $f(x,y)=\dfrac{\sin xy}{x(1+y)}$ as $(x,y)\to(0,3)$.

Solution:

1) In view of (4) we have to hold fixed one of the variables in order to take the limit with respect to the second variable.

Let us hold fixed y as x approaches zero:

$$\lim_{x\to 0}\frac{\sin xy}{x(1+y)}=\frac{1}{(1+y)}\lim_{x\to 0}\frac{\sin xy}{x}=\frac{y}{(1+y)}.$$

2) Now let $y\to 3$:

$$\lim_{y\to 3}\frac{y}{(1+y)}=\frac{3}{4}.$$

It does not matter whether we hold fixed x or y. By interchanging of the order of a passage to the limit we obtain the same result as above:

$$\lim_{y\to 3}\frac{\sin xy}{x(1+y)}=\frac{\sin 3x}{4x} \qquad \Rightarrow$$

$$\lim_{x\to 0}\lim_{y\to 3}\frac{\sin xy}{x(1+y)}=\lim_{x\to 0}\frac{\sin 3x}{4x}=\frac{3}{4}.$$

Thus, the given function tends to $3/4$ as (x,y) approaches $(0,3)$.

Naturally, there are such functions, which have no limits at some points. For instance, consider the limit of the function $f(x)=\dfrac{x^2-y^2}{x^2+y^2}$ as $(x,y)\to(0,0)$.

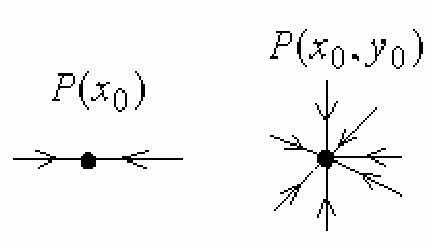
Note that

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}=\lim_{x\to 0}1=1,$$

while

$$\lim_{y\to 0}\lim_{x\to 0}\frac{x^2-y^2}{x^2+y^2}=\lim_{y\to 0}(-1)=-1.$$

The results differ from each other. Hence, the given function has no a limit at the point $(0,0)$.



It is appropriate to mention here that the limit of a function of one variable exists if and only if the left-hand and right-hand limits equal with each other.
So, as for the example above, there is nothing new; the only difference is that there is an infinite number of directions of passages to the limit point but not just two.

4.3. Continuity of Functions of Several Variables

The concept of continuity of functions of one variable does not require any modification with reference to functions of several variables.
A function $f(P)$ is called continuous at a point P_0 if there exists the finite limit of $f(P)$ that equals the value of the function at the point P_0: $\lim\limits_{P \to P_0} f(P) = f(P_0)$.
A function $f(P)$ is said to be continuous on some set D, if it is continuous at each point of D. Otherwise, if $f(P)$ is not continuous, e.g., at a point P_1, it is said that the function $f(P)$ is discontinuous at the point P_1 or that $f(P)$ has a discontinuity at the point P_1.
The points of discontinuity can form lines or surfaces.

Examples:

- The function $z = \tan xy$ is not defined on the lines $xy = (2k+1)\pi/2$, where k is any integer. The lines of discontinuity are the set of hyperbolas.
- The function $u = \dfrac{x - z^2}{2x + y - 3x}$ is not defined in the plane $2x + y - 3z = 0$, which is the plane of discontinuity.

Continuous functions have the same properties, no matter how many of variables.

--- The sum of a finite number of continuous functions is a continuous function as well as the product of a finite number of continuous functions is a continuous function.

--- The quotient of two continuous functions is a continuous function wherever the denominator is non-zero.

--- All elementary functions are continuous in their domains.

4.4. Partial Derivatives

The derivative of a function of one variable is defined as the limit of the quotient of the increment $\Delta f = f(x+\Delta x) - f(x)$ of the function to an increment Δx of the argument as $\Delta x \to 0$:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

The partial derivatives of a function of several variables are defined in a similar way.

For convenience sake consider a function of two independent variables.

The partial derivative of $u = f(x, y)$ with respect to x is defined as

$$\frac{\partial f(x, y)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}. \qquad (9)$$

The definition of the partial derivative of $z = f(x, y)$ with respect to y looks like above:

$$\frac{\partial f(x, y)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta\, y}. \qquad (10)$$

In a short form partial derivatives are denoted by the symbols f'_x, f'_y or u'_x, u'_y.

Partial derivatives have the same properties as ordinary derivative as well as all rules of differentiation hold.

Note that when one takes the partial derivative, e.g., with respect to x, it is necessary to hold the other variables as constants.

Example: Find the partial derivatives of $f(x, y)$ with respect to x and y, if $f(x, y) = x^2 y^3 + 5\sin x - e^{\sqrt{y}}$.

Solution:

$$f'_x = 2xy^3 + 5\cos x,$$

$$f'_x = 3x^2 y^2 - e^{\sqrt{y}} \frac{1}{2\sqrt{y}}.$$

Partial derivatives of higher orders are defined in a similar way as ordinary higher derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

They are also denoted as f''_{x^2}, f''_{y^2}, f''_{xy}, f''_{yx} correspondingly.

Partial derivatives like f''_{xy}, f''_{yx} are called mixed partial derivatives.

There exists the theorem according to that mixed partial derivatives do not depend on the order of differentiation provided that the partial derivatives are continuous functions. We are going to consider only functions, which obey such conditions.

Therefore,

$$u''_{xy} = u''_{yx}\ , \qquad u'''_{x^2 y} = u'''_{xyx} = u'''_{yx^2}\ , \qquad etc.$$

4.5. Total Differentials

Let $u = f(x, y)$ be a function of two independent variables.

Increments of the argument are called the differentials of the independent variables:

$$dx = \Delta x \qquad \text{and} \qquad dy = \Delta y\,.$$

The total differential of a function $u = f(x, y)$ is defined as

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\,. \tag{11}$$

This definition of the differential can be easily generalized for a function u of n independent variables:

$$du = \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}dx_k = \frac{\partial u}{\partial x_1}dx_1 + \frac{\partial u}{\partial x_2}dx_2 + \ldots + \frac{\partial u}{\partial x_n}dx_n\,. \tag{12}$$

The properties of differentials of functions of several variables differ nothing from that of one variable:

$$d(u \pm v) = du \pm du\,,$$

$$d(u \cdot v) = udv + vdu\,,$$

$$d(\frac{u}{v}) = \frac{udv - vdu}{v^2}\,.$$

Theorem: Let the functions $A(x, y)$ and $B(x, y)$ have continuous partial derivatives to the second order inclusive. Then the expression of the form

$$A(x, y)dx + B(x, y)dy$$

is the total differential of some function $u = f(x, y)$ if and only if

$$\frac{\partial A}{\partial y} = \frac{\partial B}{\partial x}\,. \tag{13}$$

Proof: Let us prove the necessity of condition (13).

Assume that

$$du = A(x, y)dx + B(x, y)dy\,.$$

Then from definition (11) it follows that

$$A(x, y) = \frac{\partial u}{\partial x} \quad \text{and} \quad B(x, y) = \frac{\partial u}{\partial y}\,.$$

Therefore,

$$\frac{\partial A(x, y)}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} \quad \text{and} \quad \frac{\partial B(x, y)}{\partial x} = \frac{\partial^2 u}{\partial y \partial x}\,.$$

However, the mixed partial derivatives u''_{xy} and u''_{yx} equal each other because they are continuous functions.

Hence, $A'_y = B'_x$.

4.6. Differentials of Higher Orders

The nth differentials of arguments are the nth power of the first differentials:

$$dx^2 = (dx)^2, \qquad dy^2 = (dy)^2, \ldots \quad, \quad dx^n = (dx)^n, \quad dy^n = (dy)^n\,.$$

The second differential of a function is the differential of the first differential; the third differential is the differential of the second differential, and so on:

$$d^2 u = d(du), \qquad d^3 u = d(d^2 u), \ldots, \; d^n u = d(d^{n-1} u)\,.$$

If $u = f(x, y)$, then

$$\begin{aligned} d^2 u &= d\left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy\right) = d\left(\frac{\partial u}{\partial x} dx\right) + d\left(\frac{\partial u}{\partial y} dy\right) \\ &= \left(\frac{\partial^2 u}{\partial x^2} dx^2 + \frac{\partial^2 u}{\partial y \partial x} dydx\right) + \left(\frac{\partial^2 u}{\partial x \partial y} dxdy + \frac{\partial^2 u}{\partial y^2} dy^2\right) \qquad (14) \\ &= \frac{\partial^2 u}{\partial x^2} dx^2 + 2\frac{\partial^2 u}{\partial x \partial y} dxdy + \frac{\partial^2 u}{\partial y^2} dy^2 \end{aligned}$$

due to equality of the mixed partial derivatives.

The nth differential of a function can be simply obtained by the following formal rule:

$$d^n u = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial x}\right)^n u\,. \qquad (15)$$

In such a way the sum has to be raised to nth power. Then the parentheses have to be removed, putting the symbol u from the right of each of the symbols like $\frac{\partial}{\partial x}$. Finally, we have to interpret the exponents as the orders of derivatives and differentials.

Example: Find the third differential of a function of two variables.

Solution: The first step:

$$d^3u = (dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y})^3 u\,.$$

The second step:

$$d^3u = (dx^3\frac{\partial^3}{\partial x^3} + 3dx^2\,dy\frac{\partial^3}{\partial x^2\partial y} + 3dx\,dy^2\frac{\partial^3}{\partial x\partial y^2} + dy^3\frac{\partial^3}{\partial y^3})u\,.$$

The final step:

$$d^3u = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2\partial y}dx^2\,dy + 3\frac{\partial^3 u}{\partial x\partial y^2}dx\,dy^2 + \frac{\partial^3 u}{\partial y^3}dy^3.$$

4.7. Derivatives of Composite Functions

Let $u = f(x_1, x_2, ..., x_n)$ be a composite function of the variables $x_1, x_2, ..., x_n$, where $x_1 = x_1(t)$, $x_2 = x_2(t)$, …, $x_n = x_n(t)$ all are functions of the variable t. Then the complete derivative is

$$\frac{du}{dt} = \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{dx_k}{dt}\,. \tag{16}$$

If the function u is also an explicit function of t, that is, $u = f(x_1, x_2, ..., x_n, t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \sum_{k=1}^{n}\frac{\partial u}{\partial x_k}\frac{dx_k}{dt}\,. \tag{17}$$

In particular, let $u = f(x, y, t)$ with $x = x(t)$ and $y = y(t)$. Then

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}\,. \tag{18}$$

Example: Find $\frac{du}{dt}$, if $u = e^{5x}y^3$ with $x = \sin t$ and $y = t^2$.

Solution:

$$\frac{du}{dt} = 5e^{5x}\cos t\cdot y^3 + e^{5x}3y^2\,2t = 5e^{5\sin t}\cos t\cdot t^6 + 6e^{5\sin t}t^5.$$

4.8. Derivatives of Implicit Functions

- Let a function $y = y(x)$ be defined by an implicit function:

$$F(x, y) = 0. \tag{19}$$

Then the total differential of F is $dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0$.

Therefore, the derivative of y with respect to x can be expressed through the partial derivatives as follows:

$$\frac{dy}{dx} = -\frac{F'_x}{F'_y}. \tag{20}$$

- Let a function $z = z(x, y)$ of two variables be defined by an implicit function:

$$F(x, y, z) = 0.$$

As above, the total differential of F is equal to zero:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0. \tag{21}$$

In order to find, for instance, the partial derivative of z with respect to x, we divide both sides of this equality by dx and hold the variable y as a constant. In this case the ratio $\frac{dz}{dx}$ has to be considered as the partial derivative $\frac{\partial z}{\partial x}$, and hence

$$\frac{\partial z}{\partial x} = -\frac{F'_x}{F'_z} \tag{22a}$$

The other partial derivatives can be found in a similar way:

$$\frac{\partial z}{\partial y} = -\frac{F'_y}{F'_z}, \quad \frac{\partial y}{\partial x} = -\frac{F'_x}{F'_y}, \quad \frac{\partial y}{\partial z} = -\frac{F'_z}{F'_y}, \quad etc. \tag{22b}$$

Example: Find the partial derivatives of z with respect to x and y if

$$xy^2z^3 + \sqrt{z}\ln x - y/z = 0.$$

Solution: First, let us find the partial derivatives of the function

$$F(x, y, z) = xy^2z^3 + \sqrt{z}\ln x - y/z:$$

$$F'_x = y^2z^3 + \sqrt{z}/x, \qquad F'_y = 2xyz^3 - 1/z,$$

$$F'_z = 3xy^2z^2 + \ln x/(2\sqrt{z}) + y/z^2.$$

Then, we use formulas (22):

$$\frac{\partial z}{\partial x} = -\frac{y^2 z^3 + \sqrt{z}/x}{3xy^2 z^2 + \ln x/(2\sqrt{z}) + y/z^2},$$

$$\frac{\partial z}{\partial y} = -\frac{2xyz^3 - 1/z}{3xy^2 z^2 + \ln x/(2\sqrt{z}) + y/z^2}.$$

4.9. Geometric Interpretation of Partial Derivatives

Consider a function of two variables.
The equation of a surface in three-dimension space can be written as

$$z = f(x, y). \tag{23}$$

This equation can also be represented in the implicit form as follows:

$$F(x, y, z) = 0. \tag{24}$$

Assume that there exist the partial derivatives of z at some point $P_0(x_0, y_0, z_0)$ on the surface, that is, the surface is smooth enough in the vicinity of the point P_0.

There is an infinite number of lines that are tangents to the surface at this point. These lines form a plane called the **tangent plane** to the surface at the given point.

An equation of the tangent plane can be written in the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0. \tag{25}$$

Here A, B and C are components of a normal vector to the surface at the point P_0.

In order to determine this vector we consider anther way to get the equation of the tangent plane.

Let $P(x, y, z)$ be any point on the surface. If the point $P(x, y, z)$ approaches $P_0(x_0, y_0, z_0)$, that is,

$$\Delta x = x - x_0 \to 0, \ \Delta y = y - y_0 \to 0, \qquad \Delta z = z - z_0 \to 0,$$

then the vector $\Delta \mathbf{r} = \{\Delta x, \Delta y, \Delta z\}$ tends to the vector $d\mathbf{r} = \{dx, dy, dz\}$, which is coplanar to the tangent plane.

By equation (24), the differential of F at the point $P_0(x_0, y_0, z_0)$ is equal to zero. Hence, in view of formula (21)

$$\frac{\partial F(x_0, y_0, z_0)}{\partial x} dx + \frac{\partial F(x_0, y_0, z_0)}{\partial y} dy + \frac{\partial F(x_0, y_0, z_0)}{\partial z} dz = 0. \tag{26}$$

This equation states the orthogonality condition of the vectors $d\mathbf{r} = \{dx, dy, dz\}$ and $\mathbf{N} = \{F'_x(P_0), F'_y(P_0), F'_z(P_0)\}$, where the vector $d\mathbf{r}$ lies in the tangent plane.
Therefore, the partial derivatives of F at the point P_0 are the components of a normal vector to the tangent plane and so to the considered surface at this point:

$$A = \frac{\partial F(x_0, y_0, z_0)}{\partial x}, \qquad B = \frac{\partial F(x_0, y_0, z_0)}{\partial y}, \qquad C = \frac{\partial F(x_0, y_0, z_0)}{\partial z}.$$

Then formula (25) yields the equation of the tangent plane to surface (24) at the given point:

$$F'_x(P_0)(x - x_0) + F'_y(P_0)(y - y_0) + F'_z(P_0)(z - z_0) = 0. \qquad (27)$$

Now we can also write the equations of the straight line passing through the point P_0 and being perpendicular to the surface $F(x, y, z) = 0$:

$$\frac{x - x_0}{F'_x(x_0, y_0, z_0)} = \frac{y - y_0}{F'_y(x_0, y_0, z_0)} = \frac{z - z_0}{F'_z(x_0, y_0, z_0)}. \qquad (28)$$

If the surface is defined by equation (23) in the explicit form, then

$$F(x, y, z) = z - f(x, y)$$

and hence,

$$F'_x = -f'_x, \quad F'_y = -f'_y \quad \text{and} \quad F'_z = 1. \qquad (29)$$

Example: Find the equation of the tangent plane to the surface of the paraboloid of revolution $z = x^2 + y^2$ at the point $(-4, 3, 25)$.
Solution: Using formulas (29) we find the partial derivatives of

$$F(x, y, z) = z - x^2 - y^2$$

at the given point: $F'_x(P_0) = 8$, $F'_y(P_0) = -6$ and $F'_z(P_0) = 1$.
In view of (27) the equation of the tangent plane is

$$8(x + 4) - 6(y - 3) + (z - 25) = 0$$

or equivalently

$$8x - 6y + z + 25 = 0.$$

4.10. Maxima and Minima of Functions of Two Variables

The definitions of the maximum and minimum of a function of several variables are just the same as in case of function of one variable.
For instance, a function $f(P)$ has a **relative maximum** at a point P_0, if $f(P) \le f(P_0)$ for all points P in some vicinity of the point P_0.
An **extreme** point is the point where the function attains either maximum or minimum.
The problem of determining the maximum and minimum of some differentiable function can be solved by using of the Taylor Formula.
The main idea is quite clear: if the difference $\Delta f(P_0)$ holds its sign in some vicinity of P_0, then P_0 is an extreme point. Otherwise, the function $f(P)$ has neither maximum nor minimum at this point.
Let $z = f(x, y)$ be a given function. We begin with the first approximation: $\Delta f(P_0) \approx df(P_0) = f'_x(P_0)dx + f'_y(P_0)dy$.
Even if one of these partial derivatives is not equal zero, then the sign of $\Delta f(P_0)$ depends on the signs of the increments dx and dy.
Hence, all the partial derivatives of $f(P)$ either equal zero or do not exist at the extreme point.
To find critical points we need to solve the following equations simultaneously:

$$f'_x(x, y) = 0 \qquad \text{and} \qquad f'_y(x, y) = 0. \tag{30}$$

Note that the tangent planes at such critical points are parallel to the $xy-$plane.
Then we have to take into account the next term in the Taylor formula.
Using the form of the second differential one can prove the following rule.

Rule: Let the partial derivatives of the second order $f''_{x^2}(P_0)$, $f''_{xy}(P_0)$, $f''_{yx}(P_0)$ and $f''_{y^2}(P_0)$ be the elements of the determinant:

$$D = \begin{vmatrix} f''_{x^2}(P_0) & f''_{xy}(P_0) \\ f''_{yx}(P_0) & f''_{y^2}(P_0) \end{vmatrix} \tag{31}$$

where P_0 is a critical point.

- If $D > 0$ and $f''_{x^2}(P_0) > 0$, then P_0 is a point of a relative minimum.
- If $D > 0$ and $f''_{x^2}(P_0) < 0$, then P_0 is a point of a relative maximum.
- If $D < 0$, then function $f(x, y)$ has a saddle point at P_0.
- If $D = 0$, then the rule does not give any answer.