Complex Analysis 2024. Homework 4.

1. Find all values of $\operatorname{Ln} z$ at points

$$z_1 = 1 + i; \quad z_2 = -3; \quad z_2 = -1 + i\sqrt{3}.$$

Solution. Since $1 + i = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2}e^{i\frac{\pi}{4}}$ then

$$\operatorname{Ln}(1+i) = \frac{1}{2}\ln 2 + \frac{\pi}{4}i + 2\pi k i, \quad k \in \mathbb{Z};$$

Since $-3 = 3e^{i\pi}$ then

$$\operatorname{Ln}(-3) = \ln 3 + \pi(2k+1)i, \quad k \in \mathbb{Z};$$

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Since $z_2 = -1 + i\sqrt{3} = 2\left(-\frac{1}{2} + \frac{sqrt3}{2}i\right) = 2e^{i\frac{5\pi}{6}}$ then

$$\operatorname{Ln}(-1 + i\sqrt{3}) = \ln 2 + \frac{5\pi}{6}i + 2\pi k i, \quad k \in \mathbb{Z}.$$

2. Calculate

$$(1+i)^i; \quad 3^{2i/\pi}; \quad (ei)^{sqrt2}.$$

Solution.

$$(1+i)^i = e^{i\operatorname{Ln}(1+i)} = e^{i(\frac{1}{2}\ln 2 + \frac{\pi}{4}i + 2\pi k i)} = e^{-\frac{\pi}{4} - 2\pi k}\left(\cos\frac{\ln 2}{2} + i\sin\frac{\ln 2}{2}\right).$$

$$3^{2i/\pi} = e^{\frac{2\operatorname{Ln}(3)}{\pi}i} = e^{\frac{2\ln 3 + 2\pi k i}{\pi}i} = e^{-2k}\left(\cos\frac{\ln 9}{\pi} + i\sin\frac{\ln 9}{\pi}\right).$$

3. Prove that the principal value of logarithm

$$\ln z = \ln|z| + i\arg z, \quad \arg z \in (-\pi, \pi),$$

is conformal in a slit domain $\mathbb{C}\backslash(-\infty, 0]$.

Solution. We already know from the previous homeworks that

$$\ln(z)' = \frac{1}{z} \neq 0, \ z \neq 0.$$

Hence, $\ln z$ is conformal in every disk that doesn't contain 0 and, hence, in domain $\mathbb{C}\backslash(-\infty, 0]$.

4. Prove that Jacobian of a conformal map $f : D \to G$ is equal to

$$J_f = |f'(z)|^2 .$$

Solution. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Then

$$J_f = \left|\det\begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix}\right| = \left|u'_x v'_y - v'_x u'_y\right| = (u'_x)^2 + (v'_x)^2 = \left|u'_x + iv'_x\right|^2 = |f'_x|^2 = |f'_z|^2$$

since f satisfies Cauchy-Riemann equations

$$u'_x = v'_y, \quad u'_y = -v'_x;$$

$$f'_z = \frac{1}{2}(f'_x - if'_y) = \frac{1}{2}((u'_x + v'_y) + i(v'_x - u'_y)) = u'_x + iv'_x.$$

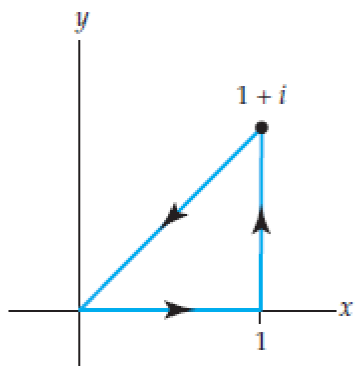
5. Evaluate the integral $\int_\gamma |z|^2\, dz$, where γ is $x = t^2$, $y = 1/t$, $1 \le t \le 2$.

Solution.

$$\int_\gamma |z|^2\, dz = \int_1^2 \left(t^4 + t^{-2}\right)(2t - it^{-2})dt = \int_1^2 (2t^5 + 2t^{-1})dt - i\int_1^2 (t^2 + t^{-4})dt =$$

$$21 + \ln 4 - \frac{21}{8} i.$$

6. Evaluate the integral $\oint_\gamma \overline{z}^2 dz$ along the contour γ given in the figure

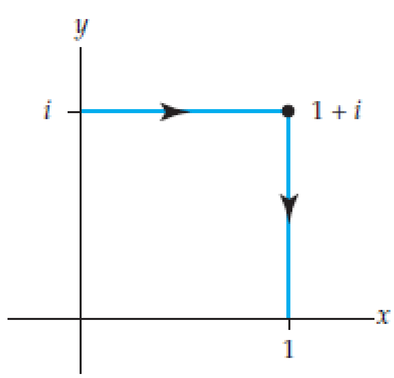


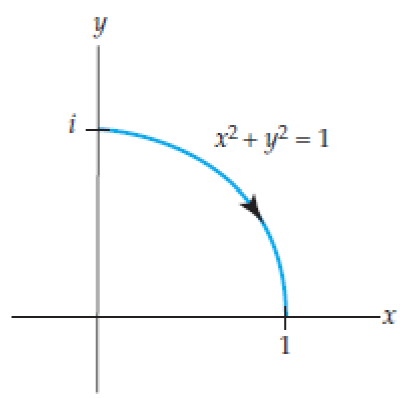
$$\oint_\gamma \overline{z}^2 dz = \underbrace{\int_0^1 (x - i0)^2 dx}_{z=x,\ dz=dx} + \underbrace{\int_0^1 \overline{(0+iy)}^2 i dy}_{z=1+iy;\ dz=idy} + \underbrace{\int_0^1 (\overline{(1+i)(1-t)})^2(-(1+i))dt}_{z=(1-t)(1+i);\ dz=-(1+i)dt} =$$

$$\int_0^1 x^2 dx - i\int_0^1 y^2 dy - (1-i)^2(1+i)\int_0^1 (1-t)^2 dt =$$

$$\frac{1}{3}(1 - i + 2(1-i)) = 1 - i.$$

7. Evaluate the integral $\int_\gamma (z^2 - z + 2)dz$ along the contours γ given in the figures (see page 2)





First contour

$$\int_\gamma (z^2 - z + 2)dz = \underbrace{\int_0^1 ((t+i)^2 - (t+i) + 2)dt}_{z=t+i,\ dz=dt} +$$

$$\underbrace{\int_0^1 ((1+(1-t)i)^2 - (1+(1-t)i) + 2)(-i)dt}_{z=1+i(1-t),\ dz=-idt} = \frac{5}{6} - \frac{5}{3}i.$$

Also we can notice that $F'(z) = z^2 - z + 2$ for $F(z) = z^3/3 - z^2/2 + 2z$ and for both contours

$$\int_\gamma (z^2 - z + 2)dz = F(1) - F(i) = 11/6 - (-i/3 + 1/2 + 2i) = 5/6 - 5i/3.$$

8. Prove Cauchy-Goursat theorem for rectangles.

   Solution. The proof is obtained word by word from the proof of Cauchy-Goursat theorem for triangles by dividing the rectangles into four equal parts.

9. Suppose that a function f is holomorphic in domain D and that G is a domain bounded by a simple closed smooth path γ such that $\overline{G} \subset D$. Assume, moreover, that f' is continuous in D. Prove, using Green's formula that
$$\int_\gamma f dz = 0.$$

   Solution. By Green's formula and Cauchy-Riemann identities we see that

$$\int_\gamma f dz = \int_\gamma (u+iv)dx + (-v+iu)dy) = \int_G ((-v+iu)'_x - (u+iv)'_y)dxdy =$$

$$\int_G ((u'_y - v'_x) + i(u'_x + v_y))dxdy = 0$$