Home Work 22

1. **Green's formula.**
2. **The Gauss-Ostrogradsky theorem.**
3. **The Stokes theorem.**

TASKS

Using Green's formula, calculate the curvilinear integral over a closed curve Γ, traversed so that its interior remains on the left $(1-8)$.

1. $\int_\Gamma (xy+x+y)dx+(xy+x-y)dy$, if: Γ - circle $x^2+y^2=ax$.
2. $\int_\Gamma (2xy-y)dx+x^2dy, \Gamma$ - ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.
3. $\int_\Gamma \frac{xdy+ydx}{x^2+y^2}, \Gamma$ - circle $(x-1)^2+(y-1)^2=1$.
4. $\int_\Gamma (x+y)^2dx-\left(x^2+y^2\right)dy, \Gamma-$ the boundary of a triangle with vertices $(1;1),(3;2),(2;5)$.
5. $\int_\Gamma e^x[(1-\cos y)dx+(\sin y-y)dy], \Gamma-$ the border of the area $0<x<\pi$, $0<y<\sin x$.
6. $\int_\Gamma e^{y^2-x^2}(\cos 2xydx+\sin 2xydy, \quad \Gamma$ - circle $x^2+y^2=R^2$.
7. $\int_\Gamma \left(e^x\sin y-y\right)dx+\left(e^x\cos y-1\right)dy, \Gamma$ - the border of the area $x^2+y^2<ax$, $y>0$.
8. $\int_\Gamma \frac{dx-dy}{x+y}, \Gamma$ - the border of a square with vertices $(1;0),(0;1),(-1;0),(0;-1)$.

After making sure that the integral expression is a complete differential, calculate the curved integral along the curve Γ with the beginning at point A and the end at point $B(9-13)$.

9. $\int_\Gamma xdx+ydy, A(-1;0), B(-3;4)$.
10. $\int_\Gamma 2xydx+x^2dy, A(0;0), B(-2;-1)$.
11. $\int_\Gamma \left(x^2+2xy-y^2\right)dx+\left(x^2-2xy-y^2\right)dy, \quad A(3;0), \quad B(0;-3)$.
12. $\int_\Gamma \left(3x^2-2xy+y^2\right)dx+\left(2xy-x^2-3y^2\right)dy,\ A(-1;2), B(1;-2)$.
13. $\int_\Gamma \frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}, A\in S_1, B\in S_2$, where S_1 - sphere $x^2+y^2+z^2=R_1^2$, S_2- sphere $x^2+y^2+z^2=R_2^2\,(R_1>0, R_2>0)$.

Using the Gauss-Ostrogradsky theorem, calculate the integrals (14-16).

14. $\iint_S (1+2x)dydz + (2x+3y)dzdx + (3y+4z)dxdy$, where S the outer side of the pyramid surface $x/a + y/b + z/c \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$

15. $\iint_S zdxdy + (5x+y)dydz$, where S :

    (a) the inner side of the ellipsoid $x^2/4 + y^2/9 + z^2 = 1$;

    (b) the outer side of the area boundary $1 < x^2 + y^2 + z^2 < 4$.

16. $\iint_S x^2dydz + y^2dzdx + z^2dxdy$, where S : the inner side of the parallelepiped surface $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$

Using the Stokes formula, calculate the integrals (17-18).

17. $\int_L y^2dx + z^2dy + x^2dz, L-$ the boundary of a triangle with vertices at points $(a;0;0), (0;a;0), (0;0;a)$, oriented positively with respect to the vector $(0;1;0)$.

18. $\int_L ydx - zdy + xdz, L$ - curve $x^2 + y^2 + 2z^2 = 2a^2, y - x = 0$, oriented positively with respect to the vector $(1;0;0)$.

MA. HW 22

1. $\int_\Gamma (xy+x+y)dx + (xy+x-y)dy$, if: Γ - circle $x^2+y^2=ax$.

$$= \iint_J \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint (y+1)-(x+1)\,dx\,dy = \iint y-x\,dx\,dy$$

$$\underline{\begin{cases} x = \frac{a}{2} + r\cos\theta \\ y = r\sin\theta \end{cases}} \quad \iint \left[r(\sin\theta-\cos\theta) - \frac{a}{2}\right] r\,dr\,d\theta = \int_0^{2\pi} (\sin\theta - \cos\theta)\,d\theta \int_0^{\frac{a}{2}} r^2\,dr - \int_0^{2\pi} d\theta \int_0^{\frac{a}{2}} \frac{a}{2} r\,dr$$

$$= -\frac{a^3\pi}{8}$$

✓

2. $\int_\Gamma (2xy-y)dx + x^2dy, \Gamma$ - ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$= \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_J (2x - 2x + 1)\,dx\,dy = \iint_S dx\,dy = ab\pi.$$

✓

~~3.~~ $\int_\Gamma \frac{xdy+ydx}{x^2+y^2}, \Gamma$ - circle $(x-1)^2+(y-1)^2=1$.

$(x,y) \neq (0,0)$. $\frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ exist.

$$\int_J \frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \int_J \frac{y}{2(x+y)-1}\,dx + \frac{x}{2(x+y)-1}\,dy$$

$$\begin{cases} x = 1 + r\cos\theta \\ y = 1 + r\sin\theta. \end{cases}$$

$$= \iint_J \frac{2(x+y)-1-2x-[2(x+y)-1-2y]}{(2(x+y)-1)^2}\,dx\,dy$$

$$= \iint_J \; 2(y-x)$$

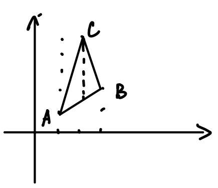
轮换对称性 $\iint_J \frac{y^2-x^2}{(x^2+y^2)^2} = \iint_J \frac{x^2-y^2}{(x^2+y^2)^2}$

① + ② = 0

① = ② = 0.

$$= \iint_J \left[\frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} - \frac{(y^2+x^2)-2y^2}{(x^2+y^2)^2}\right] dx\,dy = 2\iint_J \frac{y^2-x^2}{(x^2+y^2)^2}\,dx\,dy$$

4. $\int_\Gamma (x+y)^2 dx - (x^2+y^2)\,dy, \Gamma-$ the boundary of a triangle with vertices $(1;1), (3;2), (2;5)$.



AC: $y = 4x-3$

AB: $y = \frac{1}{2}x + \frac{1}{2}$.

BC: $y = -3x+11$

$$\int_J = \iint_J \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \iint_J [-2x - 2(x+y)]\,dx\,dy = -2\iint_J (2x+y)\,dx\,dy$$

$$= -2\left[\int_1^2 dx \int_{\frac{1}{2}(x+1)}^{4x-3} (2x+y)dy + \int_2^3 dx \int_{\frac{1}{2}(x+1)}^{-3x+11} (2x+y)\,dy\right]$$

$$= -\int_1^2 \left[4x(4x-3) + (4x-3)^2 - 2x(x+1) - \frac{1}{4}(x+1)^2\right] dx - \int_2^3 \left[4x(-3x+11) + (3x-11)^2 - 2x(x+1) - \frac{1}{4}(x+1)^2\right] dx$$

$$= \int_1^3 2x(x+1) + \frac{1}{4}(x+1)^2\,dx - \int_1^2 (32x^2 - 36x + 9)\,dx - \int_2^3 -3x^2 - 22x + 121\,dx$$

$$= 30 - \left(\frac{32}{3}x^3 - 18x^2 + 9x \Big|_1^2\right) - \left(-x^3 - 11x^2 + 121x \Big|_2^3\right) = 30 - 47 - \frac{89}{3} = -\frac{140}{3}$$

✓

5. $\int_\Gamma e^x[(1-\cos y)dx + (\sin y - y)dy], \Gamma-$ the border of the area $0 < x < \pi$, $0 < y < \sin x$.

$$\int_\Gamma = \iint_\Gamma \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \iint \left[e^x(\sin y - y) - e^x \sin y \right] dx\,dy = -\int_0^\pi e^x\,dx \int_0^{\sin x} y\,dy$$

$$= -\frac{1}{2}\int_0^\pi e^x \cdot \sin^2 x\,dx = \frac{1}{4}\int_0^\pi e^x(\cos 2x - 1)\,dx = \frac{1}{4}\int_0^\pi e^x \cos 2x - \frac{1}{4}\int_0^\pi e^x\,dx$$

$$= \frac{1}{4}\,\frac{e^x \cos 2x + 2e^x \sin 2x}{5}\Big|_0^\pi - \frac{1}{4}(e^\pi - 1)$$

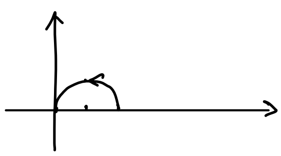
$$= \frac{1}{4}\cdot\frac{4}{5}(e^\pi - 1) = \frac{1}{5}(e^\pi - 1)$$ ✓

6. $\int_\Gamma e^{y^2-x^2}(\cos 2xy dx + \sin 2xy dy)$ Γ - circle $x^2+y^2=R^2$.

$$\int_\Gamma = \iint_\Gamma e^{y^2-x^2}\left(\cos 2xy \cdot 2y + \sin 2xy \cdot (-2x)\right) - e^{y^2-x^2}\left(-\sin 2xy \cdot 2x + 2y \cdot \cos 2xy\right) \cdot dx\,dy$$

$$= 0$$ ✓

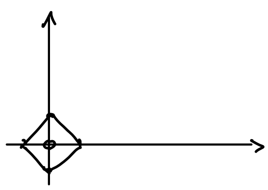
7. $\int_\Gamma (e^x \sin y - y)\,dx + (e^x \cos y - 1)\,dy, \Gamma$ - the border of the area $x^2+y^2 < ax$, $y > 0$.



$$\int_\Gamma = \iint_\Gamma \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint_\Gamma \left(e^x \cos y - (e^x \cos y - 1) \right) dx\,dy$$

$$= \iint_\Gamma dx\,dy = \frac{a^2\pi}{4} \times \frac{1}{2} = \frac{a^2\pi}{8}$$ ✓

* 8. $\int_\Gamma \frac{dx-dy}{x+y}, \Gamma$ - the border of a square with vertices $(1;0), (0;1), (-1;0), (0;-1)$.



$$\int_\Gamma = \iint_\Gamma \frac{1}{(x+y)^2} + \frac{1}{(x+y)^2}\,dx\,dy = 2\iint_\Gamma \frac{1}{(x+y)^2}\,dx\,dy.$$

$$= 4\iint_{\Gamma_1} \frac{1}{(x+y)^2}\,dx\,dy$$

$$= 4\int_{-1}^{1} dx \int_{\varepsilon_1}^{1-|x|} \frac{1}{(x+y)^2}$$

$$\int_\Gamma$$

$$\int_{-1}^{\varepsilon_-} dx \int_0^{x+1} \frac{1}{(x+y)^2} dy = -\int_{-1}^{\varepsilon} dx \frac{1}{(x+y)} \Big|_0^{x+1} = \int_{-1}^{\varepsilon_-} \left(\frac{1}{x} - \frac{1}{2x+1}\right) dx =$$

$$= \ln|x| - \frac{1}{2}\ln|2x+1| \Big|_{-1}^{\varepsilon_-} = \ln\left|\frac{x}{\sqrt{2x+1}}\right| \Big|_{-1}^{\varepsilon_-} = \ln\left|\frac{\varepsilon}{\sqrt{2\varepsilon+1}}\right| = \ln|\varepsilon|$$

$$\int_{\varepsilon_+}^1 dx \cdot -\frac{1}{x+y}\Big|_0^{1-x} = \int_{\varepsilon_+}^1 \left(\frac{1}{x} - 1\right) dx = \ln|x| - x \Big|_{\varepsilon_+}^1 = -1 - \ln|\varepsilon_+| + \varepsilon$$

$$= 4\left[\int_{-1}^0 dx \int_0^{x+1} \frac{1}{(x+y)^2} dy + \int_0^1 dx \int_0^{1-x} \frac{1}{(x+y)^2} dy\right] = -4 \quad ✓ \ ?$$

After making sure that the integral expression is a complete differential, calculate the curved integral along the curve Γ with the beginning at point A and the end at point $B(9-13)$.

$y = -2x$

9. $\int_\Gamma x dx + y dy, A(-1;0), B(-3;4)$.
10. $\int_\Gamma 2xy dx + x^2 dy, A(0;0), B(-2;-1)$.
11. $\int_\Gamma \left(x^2+2xy-y^2\right) dx + \left(x^2-2xy-y^2\right) dy, \quad A(3;0), \quad B(0;-3)$.
12. $\int_\Gamma \left(3x^2-2xy+y^2\right) dx + \left(2xy-x^2-3y^2\right) dy, A(-1;2), B(1;-2)$.
13. $\int_\Gamma \frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}, A \in S_1, B \in S_2$, where S_1 - sphere $x^2+y^2+z^2=R_1^2$, S_2- sphere $x^2+y^2+z^2=R_2^2\,(R_1>0, R_2>0)$.

9. $\frac{\partial Q}{\partial x} = 0 = \frac{\partial P}{\partial y}$

$L_1: y = -2x-2$. $\quad \int_\Gamma x dx + y dy = \int_\Gamma x dx + (-2x-2)\cdot(-2) dx = \int_{-1}^{-3} (5x+4) dx.$

$$= \frac{5}{2}x^2 + 4x \Big|_{-1}^{-3} = 12 \ ✓$$

10. $\frac{\partial Q}{\partial x} = 2x = \frac{\partial P}{\partial y}$

$L_1: y = \frac{1}{2}x \qquad \int_\Gamma 2xy dx + x^2 dy = \int_0^{-2} x^2 dx + \frac{x^2}{2} dx = \frac{x^3}{2}\Big|_0^{-2} = -4.$ ✓

11. $\frac{\partial Q}{\partial x} = 2x-2y = \frac{\partial P}{\partial y}$

$-b+b \quad +b-b \qquad 9$

$$\int_\Gamma (x^2+2xy-y^2) dx + (x^2-2xy-y^2) dy = \int_3^0 x^2+2x(x-3)-(x-3)^2+x^2-2x(x-3)-(x-3)^2 dx$$

$y = x-3$

$$= \int_3^0 12x - 18 \, dx = 6x^2 - 18x \Big|_3^0 = 0. ✓$$

12. $\dfrac{\partial P}{\partial y} = -2x+2y = \dfrac{\partial Q}{\partial x}$

$L_1: y = -2x$ $\qquad 11x^2 + 34x^2$

$45x^2 \qquad 15x^3\Big|_{-1}^{1} = 30$

$$\int_J = \int_{-1}^{1}\left(3x^2+4x^2+4x^2 \overset{-2}{\left[-6x^2-x^2-12x^2\right]}\right)dx = -\int_{-1}^{1} 6x^2\,dx = -2x^3\Big|_{-1}^{1} = -4$$

×

$\exists\, u(x,y,z) = \sqrt{x^2+y^2+z^2} + C$

s.t. $du = \dfrac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$

$(x,y,z) \neq (0,0,0)$

13. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$.

consider $C(x_1, y_1, z_2)$. A, C are on the plane $P_1: x = x_1$.

$$\int_{J_{P_1}} \frac{y\,dy + z\,dz}{\sqrt{x_1^2+y^2+z^2}} \quad . \quad \frac{\partial P}{\partial z} = \frac{0 - y\cdot\frac{2z}{2\sqrt{x_1^2+y^2+z^2}}}{(x_1^2+y^2+z^2)} = -\frac{yz}{(x_1^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial Q}{\partial y}.$$

L_1: line AC. $$\int_{z_1}^{z_2} \frac{z\,dz}{\sqrt{x_1^2+y_1^2+z^2}} = \frac{1}{2}\int_{z_1}^{z_2}\frac{d(x_1^2+y_1^2+z^2)}{\sqrt{x_1^2+y_1^2+z^2}} = \sqrt{x_1^2+y_1^2+z^2}\,\Big|_{z_1}^{z_2}$$

B, C are on the plane $P_2: z = z_2$.

$$\int_{J_{P_2}} \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2+z_2^2}} \qquad \frac{\partial P}{\partial y} = \frac{-xy}{(x^2+y^2+z_2^2)^{\frac{3}{2}}} = \frac{\partial Q}{\partial x}.$$

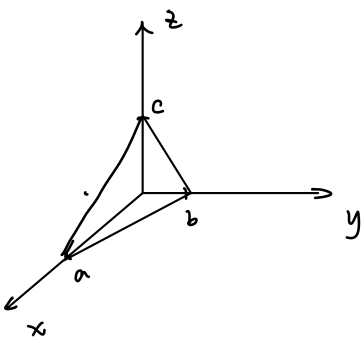
Let $D(x_1, y_2, z_2)$.

$$\int_{J_{P_2}} = \int_{J_{CB}} = \int_{J_{CD+DB}} \qquad \int_{J_{CD}} = \int_{y_1}^{y_2} \frac{y\,dy}{\sqrt{x_1^2+y^2+z_2^2}} = \sqrt{x_1^2+y^2+z_2^2}\,\Big|_{y_1}^{y_2}$$

$$\int_{J_{DB}} = \int_{x_1}^{x_2} \frac{x\,dx}{\sqrt{x^2+y_2^2+z_2^2}} = \sqrt{x^2+y_2^2+z_2^2}\,\Big|_{x_1}^{x_2}$$

$$\int_{J_{AB}} = R_2^2 - R_1^2$$ ✓

14. $\iint_S (1+2x)dydz + (2x+3y)dzdx + (3y+4z)dxdy$, where S the outer side of the pyramid surface $x/a + y/b + z/c \leqslant 1$, $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$



$$\iint_S = \iiint (2+3+4)\,dxdydz$$

$$= 9\cdot\frac{1}{6}abc = \frac{3}{2}abc$$ ✓

15. $\iint_S z dxdy + (5x+y)dydz$, where S :

(a) the inner side of the ellipsoid $x^2/4 + y^2/9 + z^2 = 1$;

(b) the outer side of the area boundary $1 < x^2 + y^2 + z^2 < 4$.

(a) $\iint_S = -\iiint_V (1+5)\,dxdydz = -6\cdot\frac{4}{3}\pi\cdot 2\times 3\times 1 = -48\pi.$ ✓

(b) $\iint_S = \iiint_{V_1-V_2} = 6\left(\frac{4}{3}\pi(2)^3 - \frac{4}{3}\pi(1)\right) = 56\pi$ ✓

16. $\iint_S x^2dydz + y^2dzdx + z^2dxdy$, where S : the inner side of the parallelepiped surface $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $0 \leqslant z \leqslant c$

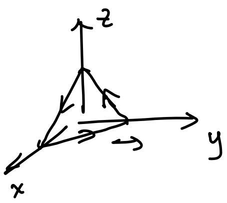
$$\iint_S = -2\iiint_V (x+y+z)\,dxdydz. = -2\int_0^a dx\int_0^b dy\int_0^c (x+y+z)\,dz$$

$$= -2\int_0^a dx\int_0^b \left[c(x+y) + \frac{1}{2}c^2\right]dy = -2\int_0^a bcx + \frac{1}{2}bc(b+c)$$

$$= -2\cdot\frac{a^2}{2}bc + \frac{1}{2}abc(b+c) = -abc(a+b+c)$$ ✓

17. $\int_L \overset{P}{y^2}dx + \overset{Q}{z^2}dy + \overset{R}{x^2}dz$, $L-$ the boundary of a triangle with vertices at points $(a;0;0), (0;a;0), (0;0;a)$, oriented positively with respect to the vector $(0;1;0)$.

$\vec{n} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



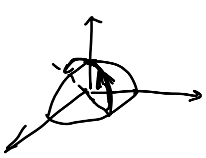
$$\int_L = \iint_\Sigma \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R. \end{vmatrix}$$

$$= \iint_\Sigma -2z\,dydz + (-2x)\,dzdx + (-2y)\,dxdy$$

$$= -\frac{2}{\sqrt{3}}\iint_\Sigma (x+y+z)\,dS$$

$$= -\frac{2a}{\sqrt{3}}\iint_\Sigma dS = -\frac{2a}{\sqrt{3}}\cdot(\sqrt{2}a)^2\cdot\frac{\sqrt{3}}{4} = -a^3$$ ✓

18. $\int_L \overset{P}{y}dx - \overset{Q}{z}dy + \overset{R}{x}dz$, L - curve $x^2 + y^2 + 2z^2 = 2a^2, y - x = 0$, oriented positively with respect to the vector $(1;0;0)$.



$$\int_L ydx - zdy + xdz = \iint_\Sigma \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R. \end{vmatrix} = \iint \left[1\cdot\frac{1}{\sqrt{2}} + (-1)\cdot\left(-\frac{1}{\sqrt{2}}\right) + (-1)\cdot 0\right]dS$$

$\vec{n_0} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$$= \sqrt{2}\iint_\Sigma dS$$

$$= \sqrt{2}\cdot\pi\cdot a\cdot\sqrt{2}a = 2a^2\pi$$ ✓