<u>Chapter 10.</u> Minimization of functionals

A variable I is called a <u>functional</u> , which depends on a function $y(x)$ given on a segment $a \le x \le b$, if each function $y(x)$ determines a value of I.

Examples:

$$I\,[\,y(x)] = \int_a^b y(x)dx$$

$$I\,[\,y(x)] = \int_a^b [\,x^2 + y^3(x)\,]\,dx$$

$$I\,[\,y(x)] = \int_a^b [\,x^2 + y^3(x) - y'(x)\,]\,dx$$

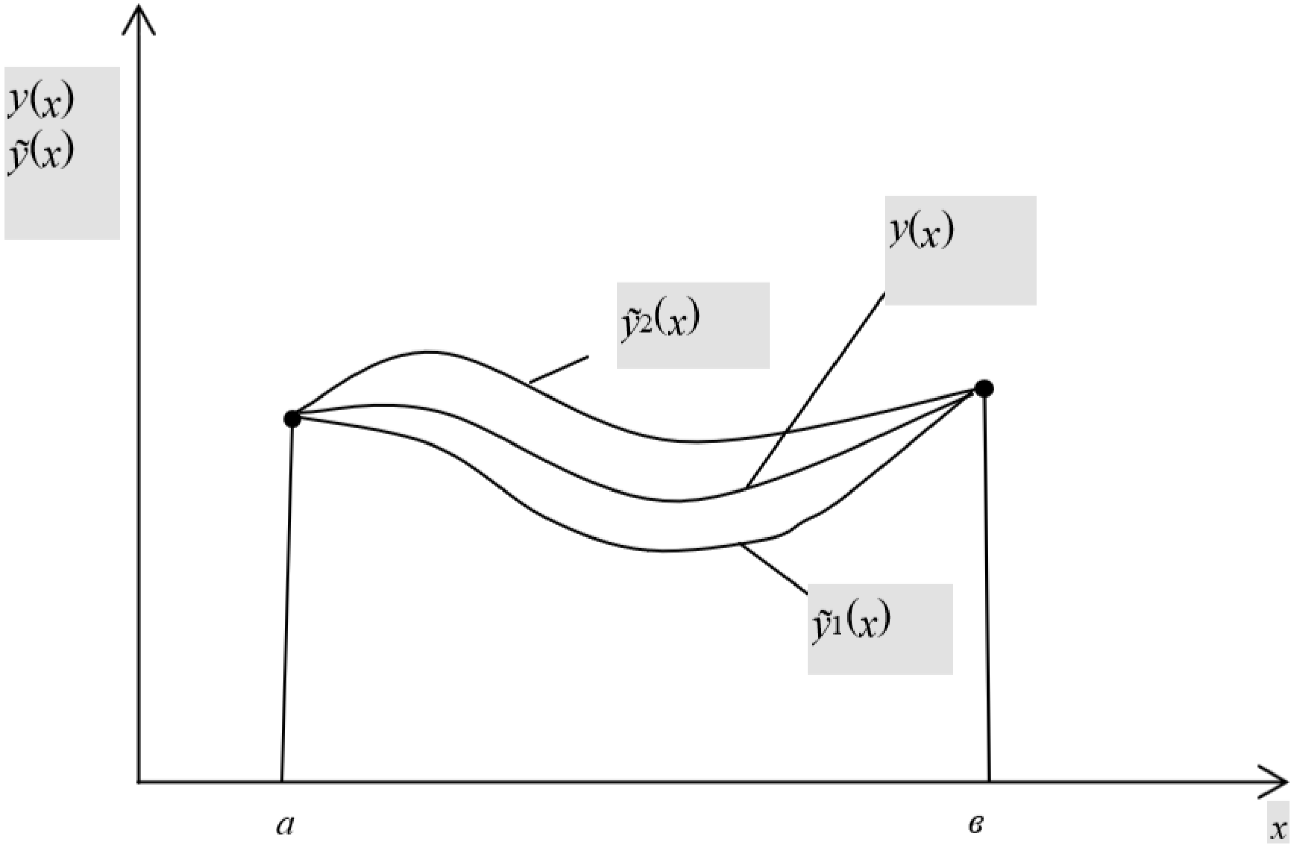
The problem of minimization for a functional

$$I\,[\,y(x)] = \int_a^b F(x, y(x), y'(x))\,dx$$

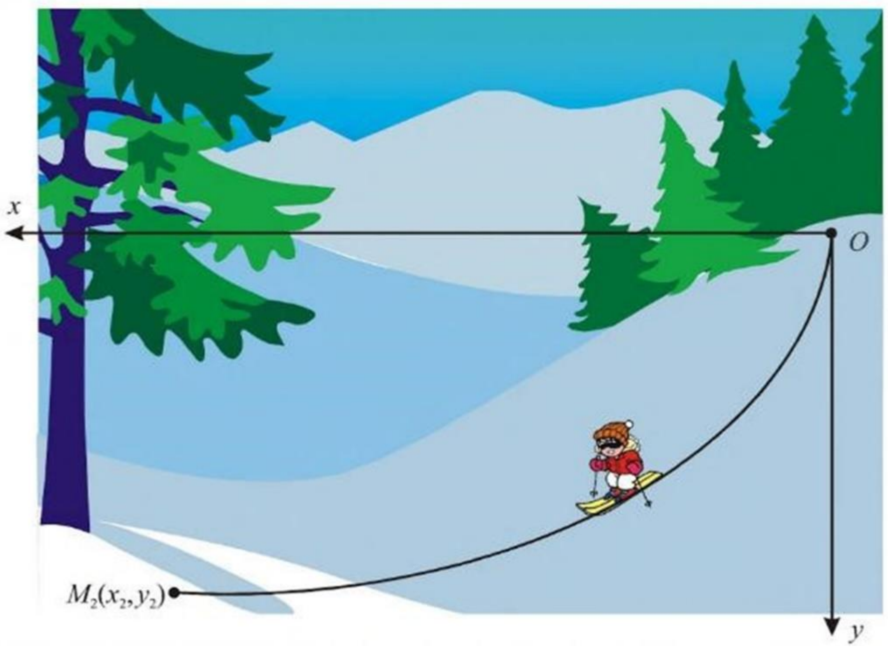
often involves conditions for $y(x)$ at endpoints of the segment:

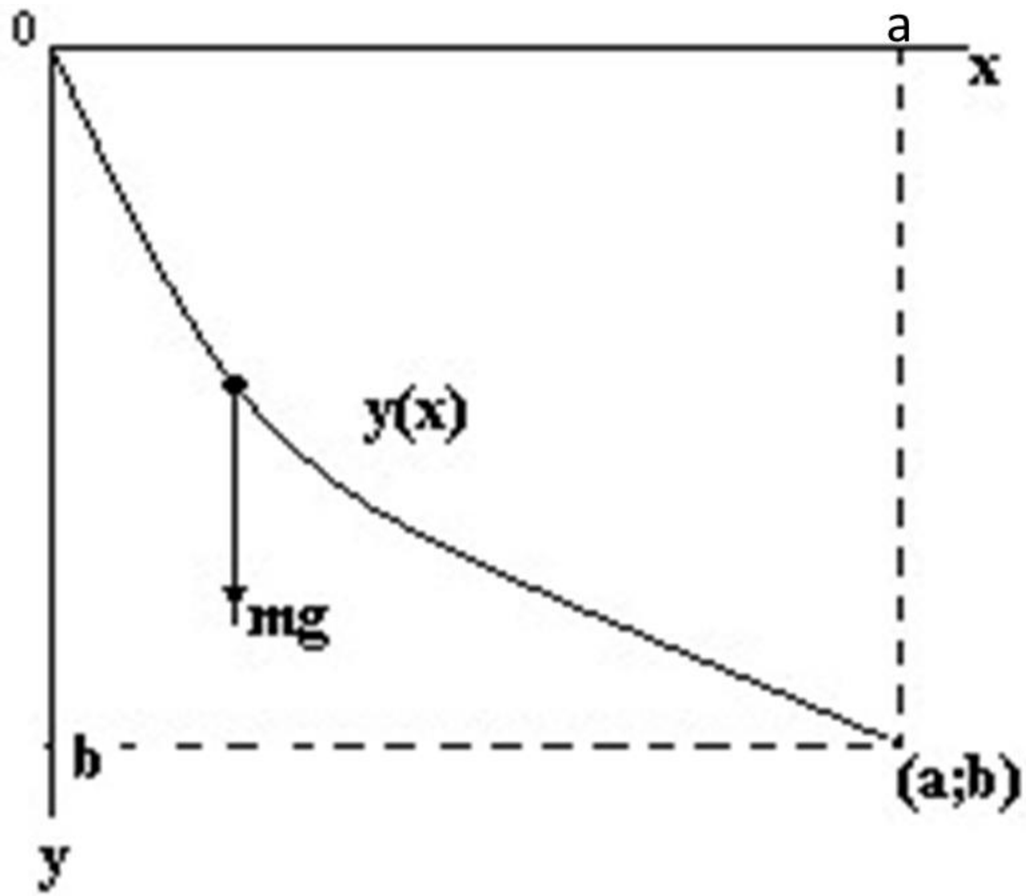
$y(a)=y_1, \quad y(b)=y_2$

(fixed endpoints).



<u>Example 1.</u> **Find the shape of a curve that begins at point $x=0$, $y=0$ and ends at $x=a$, $y=b$ using the condition of minimum time a ball rolls down along the curve under the influence of gravity (in the absence of friction).**





$$mV^2/2=mgy \quad \Rightarrow \quad V^2/2=gy$$

$$ds = V\,dt \qquad V=\sqrt{2gy} \qquad y(0)=0 \qquad y(a)=b$$

$$dt = \frac{ds}{V} = \frac{\sqrt{1+y'^2}\,dx}{\sqrt{2gy}}$$

$$t = \int_0^a \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

Johann Bernoulli, 1667 - 1748

The problem formulated in Example 1 was set by Bernoully in 1696

Example 2. Let us solve numerically a simpler problem

$$I = \int_0^1 [\, 12xy + (y')^2]dx$$

$y(0)=0 \qquad y(1)=1$

For an approximate solution, we will test 3 different polynomials.

1) **If we choose $y=x$, then easy calculations show $I_1 = 4+1=5$**

2) **Let us consider $y=ax^2+bx$ and select a, b so as to minimize the integral I:**

$$I = \int_0^1 [\, 12\,(ax^3+bx^2\,) + (2ax+b\,)^2\,]\,dx =$$

$$= 3a + 4b + \int (4a^2x^2+4abx+b^2\,)\,dx$$

$$= 3a + 4b + 4a^2/3+2ab+b^2$$

where $a+b=1$, $b=1-a$. $\qquad = 3a + 4(1-a) + 4a^2/3+2a(1-a)+(1-a)^2$

$$= 3a + 4(1-a) + 4a^2/3 + 2a(1-a) + (1-a)^2$$

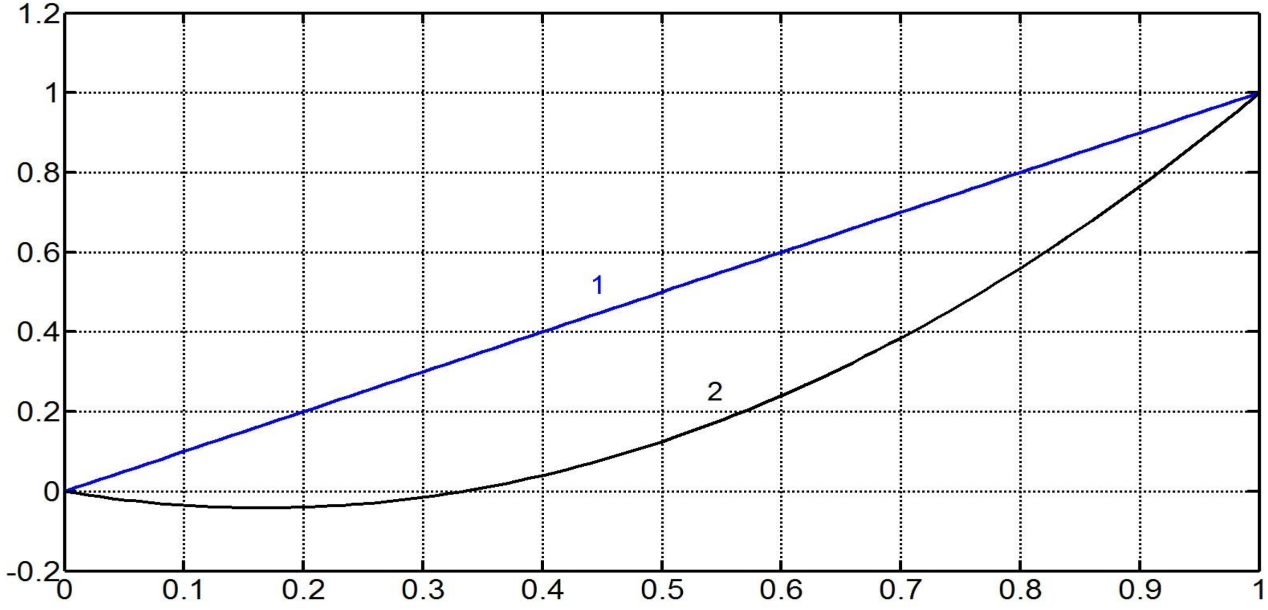
The equation $dI/da=0$:

$$3 - 4 + 8a/3 + 2 - 4a + 2(1-a)(-1) = 0, \quad -1 + 8a/3 - 2a = 0$$

and we find $a=1.5$, $b=-0.5$.

Therefore, a minimum is realized at the function $y=1.5x^2-0.5x$,

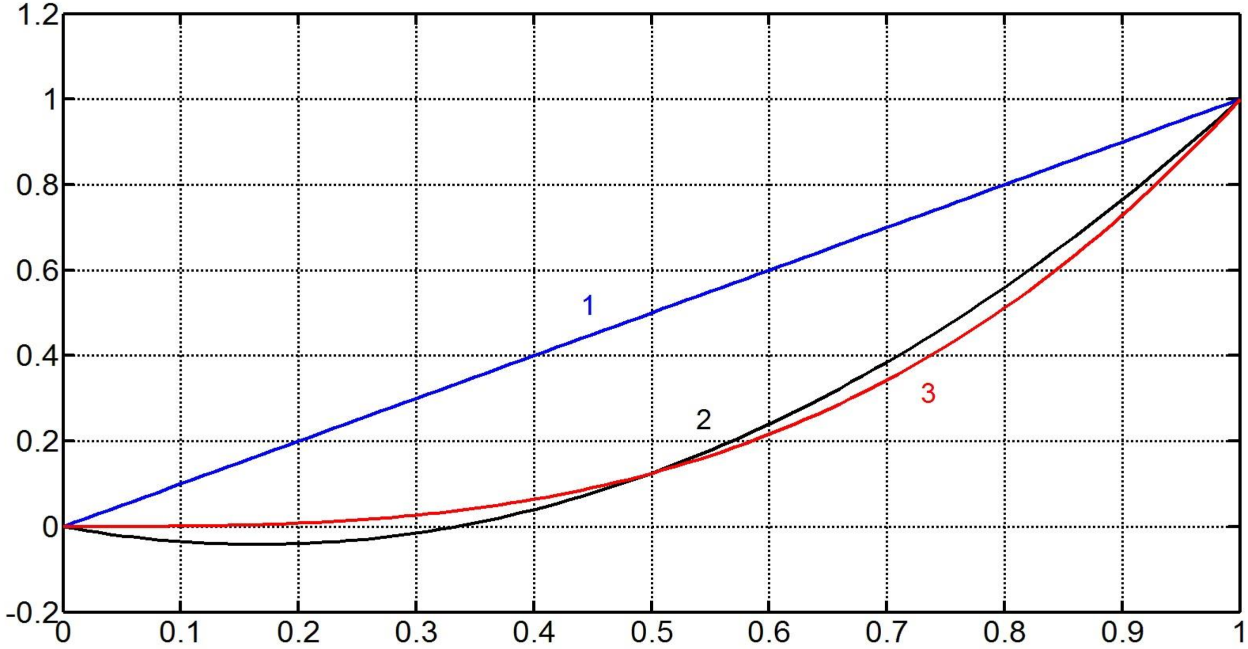
and this minimum is $I_2=3a+4b+4a^2/3+2ab+b^2 = 4.25$



3) Now we consider $y = ax^3 + bx^2 + cx$ and select a, b, c so as to minimize the integral I under condition $a + b + c = 1$

$$I = \int_0^1 [\, 12\,(ax^4 + bx^3 + cx^2) + (3ax^2 + 2b\,x + c)^2 \,]\, dx = \ldots\ldots\ldots\ldots\ldots$$

Using Excel or condition $\partial I/\partial a = \partial I/\partial b = 0$ for finding a minimum, we get $a=1$, $b=c=0$, $y = x^3$, $I_3 = 0.25$



Trigonometric functions can also be used in searches of minimizing function.

Example 3 Find a minimum of functional

$$I = \int_0^1 [\, x^2 + y^2 + (y')^2 \,]\, dx \qquad y(0)=1 \qquad y(1)=2$$

1) $y = x+1$

$$I_1 = \int_0^1 [\, x^2 + y^2 + (y')^2 \,]\, dx = \int [\, x^2 + (x+1)^2 + 1 \,]\, dx = \int (2x^2 + 2x + 2 \,]\, dx =$$

$$= 2/3 + 1 + 2 = 3.66666666$$

2) Now choose $y = x + 1 + a \sin(\pi x)$ and select a to minimize I:

$$I_2 = \int_0^1 [\, x^2 + (x + 1 + a \sin(\pi x))^2 + (1 + a\, \pi \cos(\pi x))^2 \,]\, dx =$$

$$= \int [\, x^2 + (x+1)^2 + 2(x+1)\, a \sin(\pi x) + a^2 \sin^2(\pi x) + 1 + \cancel{2a\, \pi \cos(\pi x)} + a^2 \pi^2 \cos^2(\pi x)]\, dx$$

$$dI/da = \int [\, 2(x+1) \sin(\pi x) + 2a \sin^2(\pi x) + 2a\, \pi^2 \cos^2(\pi x)]\, dx = 0$$

$$\int_0^1 \left[\, 2(x+1)\sin(\pi x) + a\,(1-\cos(2\pi x)) + a\,\pi^2(1+\cos(2\pi x)) \,\right] dx = 0$$

$$\int_0^1 \left[\, 2(x+1)\sin(\pi x) + a + a\,\pi^2 \,\right] dx = 0$$

$$\int_0^1 \left[\, 2(x+1)\sin(\pi x) \,\right] dx + a\,(1+\pi^2) = 0$$

Scilab:

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integrate('2*(x+1)*sin(%pi*x)','x',0,1)
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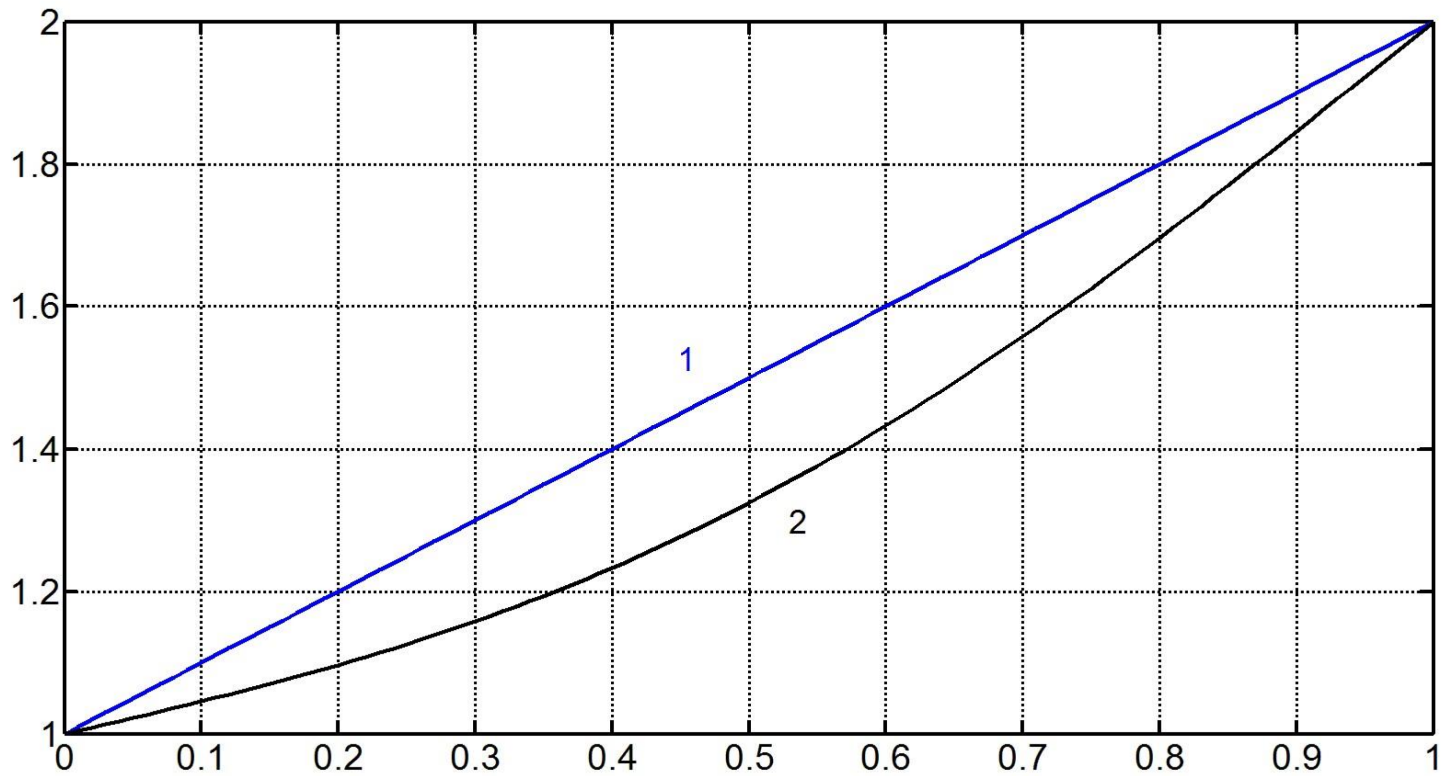
ans = 1.9098593

$$1.9098593 + a(1+\pi^2) = 0$$

$$a = -\,1.9098593/\,(1+\pi^2) = -0.17570642$$

Again Scilab: calculate the minimum

$$I_2 = \int_0^1 [\, x^2 + (x+1+a\sin(\pi x))^2 + (1+a\,\pi\cos(\pi x))^2 \,]\, dx = 3.4988794$$



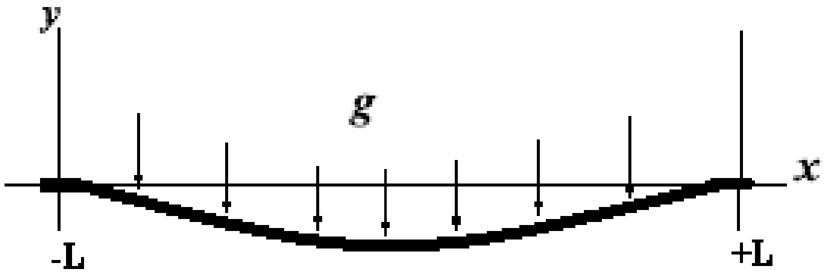
1: $y = x+1$ ➔ $I_1=3.6666666$ **2: $y = x+1+a\sin(\pi x)$ ➔ $I_2=3.4988794$**

3) Then we can try $y = x+1+a\sin(\pi x)+b\sin(2\pi x)$ and select a, b so as to minimize functional I. Probably, the obtained value I_3 will be smaller than I_2.

The more terms in the sum, the better.

A functional may depend on the second-order derivative:

Example. Equilibrium of a rigid beam



Beam is fixed at *x*=-L, *x*=L.

The potential energy depends on its shape

$$I[y(x)] = \int_{-L}^{L} (1/2\, EJy''^2 + \rho g y)\,dx, \quad y(L) = y(-L) = 0,\ y'(-L) = y'(L) = 0$$

First term is density of elastic energy of beam deformation (bending), second term is density of potential energy due to gravity.

Equilibrium corresponds to a minimum of potential energy (see a course of Physics). Using a polynomial of 4th degree for minimization of the functional, we can obtain the solution

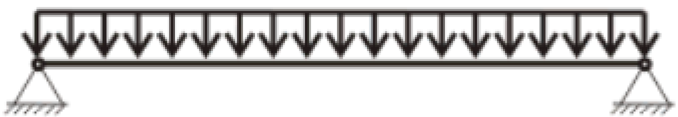
$$y(x) = -\frac{\rho g}{24EJ} x^4 + c_1 x^3 + c_2 x^2 + c_3 x + c_4$$

Constants c_1, c_2, c_3, c_4 are determined by four boundary conditions at $x = \pm L$ (details are omitted).

Finally:

$$y(x) = -\rho g/(24EJ)(x+L)^2(x-L)^2$$

Other conditions at endpoints:



The examples considered above demonstrated a Direct numerical method of functional minimization.

An idea of the direct methods is to create a sequence of approximate solutions (functions) which will converge to the exact solution.

The idea was suggested by Walter Ritz in 1908.

Walter Ritz 1878 — 1909 (Swiss mathematician and physicist)

General approach to minimization of functionals:

$$I[y(x)] = \int_a^b F(x, y(x), y'(x))dx$$

Ritz proposed to seek ***n***-the approximation to the minimizing function in the form

$$y_n(x) = \sum_{i=1}^{n} \alpha_i \varphi_i(x) \qquad (*)$$

with constant coefficients α_i and known functions $\varphi_i(x)$.

Then functional **becomes a function of *n* variables $\alpha_1, \alpha_2, \ldots, \alpha_n$**

Now in order to find a minimum, one needs either to find first-order derivatives and set them to zero, or use numerical methods.

Here the problem is unconstrained.

Functions $\varphi_i(x)$ belong to a set of functions which must be complete. This means that any continuous function $f(x)$ can be approximated by expression (*) accurately enough at sufficiently large n .

As n increases, the sequence (*) converges to a function that minimizes functional (proof is omitted).

In practice, engineers and researchers normally use polynomials

$$1,\ x,\ x^2, \ldots,\ x^n, \ldots$$

or systems of trigonometric functions.

For estimation of the error, in practice, after calculation of two approximate solutions $y_n(x)$ and $y_{n+1}(x)$, it makes sense to compare them at a few points of segment [a,b]. If the difference does not exceed the admitted tolerance, then $y_n(x)$ can be accepted as an approximate solution of the functional minimization problem.