The forth homework.

1. (2 points). Let the distribution of (X, Y) is given by the following table:

X/Y	Y=0	Y=1	Y=2
X=-1	0.1	p	0.15
X=2	0.2	0.3	0.05

Find p, $corr(X, Y), E(X|Y), E(Y^2|X)$.

2. (4 points).

Let X,Y are i.i.d.r.v's with the following density function

$$f(x, \sigma) = \sigma^2 x \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0, \sigma > 0.$$

Find the distribution of the following random variables and check their independence:
$U = \frac{X}{2Y}, \quad V = X^2 + Y^2$;

2. (6 points).

a)(2 points) Find the distribution of $\sum_{i=1}^{k} X_i$, if X_i are i.i.d.r.v's $\sim Exp(a)$;

b)(2 points) Find the distribution of $\frac{Y}{X}$, if X,Y are independent and $X \sim N(0, 4), Y \sim N(0, 5)$;

c)(2 points) Find the distribution of $\frac{2X}{3Y}$, if X,Y are independent and $X \sim N(0, \sigma^2), Y \sim Exp(a)$;

d)* (1 point) Find the distribution of $\sqrt{X^2 + Y^2}$, if X,Y are independent and
$X \sim Norm(0, \sigma^2), \quad Y \sim Norm(0, 10)$.

3. (8 points) Let (X, Y) be given by the following joint density function:

$$f(x, y) = \begin{cases} c(x + y), & x \in [0, 1], \quad y \in [0, 2]. \\ 0, & \text{else.} \end{cases}$$

a)(4 points). Find c, $F_{X,Y}(x, y)$, $corr(X, Y)$, $E(X|Y)$, $E(X^2|Y), E(Y|X)$ and $E(Y^2|X)$.

b) (4 points). Find the joint distribution $U = \frac{X}{Y}, \quad V = X - Y$ and check their independence.

4. (8 points)

Let (X, Y) be given by the following joint density function:

$$f(x, y) = \begin{cases} cxy, & x, y \in [0, 1], \ x + y > 1 \\ 0, & \text{else.} \end{cases}$$

a) (2 points) Find the joint distribution of $U = X + Y, V = \frac{Y}{X}$ and check their independence.
b) (4 points). Find the joint distribution of $U = 2X + Y, V = \frac{X^2}{2Y}$ and check their independence.
c) (2 points). Find $E(X|Y)$ and $E(Y|X)$.

5. (2 points) Let random vector (X,Y,Z) is given by some joint density $f_{x,y,z}(\cdot)$.
a) Find the expression $E(g(X)|\frac{Y}{Z} = t)$.
b)* Prove

$$\frac{D(E(Y|X)}{D(Y)} = corr^2(E(Y|X), Y).$$

1. (2 points). Let the distribution of (X, Y) is given by the following table:

X/Y	Y=0	Y=1	Y=2	
X=-1	0.1	p	0.15	0.45.
X=2	0.2	0.3	0.05	0.55
	0.3	0.5	0.2	

Find p, $corr(X, Y), E(X|Y), E(Y^2|X)$.

$p = 1 - 0.1 - 0.2 - 0.3 - 0.15 - 0.05 = 0.2$

$E(X) = -1 \times 0.45 + 2 \times 0.55 = 0.65$

$E(X^2) = 1 \times 0.45 + 4 \times 0.55 = 2.65$

$D(X) = E(X^2) - (E(X))^2 = 2.2275$

$E(Y) = 0 \times 0.3 + 1 \times 0.5 + 2 \times 0.2 = 0.9$

$E(Y^2) = 0 \times 0.3 + 1 \times 0.5 + 4 \times 0.2 = 1.3$

$D(Y) = E(Y^2) - (E(Y))^2 = 0.49.$

$E(XY) = 0 \cdot 0.3 + (-1) \cdot 0.2 + 2 \cdot 0.3 + (-2) \cdot 0.15 + 4 \cdot 0.05$
$= 0.3.$

$$corr(X,Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{D(X) \cdot D(Y)}} = \frac{0.3 - 0.65 \cdot 0.9}{\sqrt{2.2275 \times 0.49}} \approx -0.27$$

$$E(X|Y=0) = (-1) \cdot \frac{0.1}{0.3} + 2 \cdot \frac{0.2}{0.3} = 1$$

$$E(X|Y=1) = (-1) \cdot \frac{0.2}{0.5} + 2 \cdot \frac{0.3}{0.5} = \frac{4}{5}.$$

$$E(X|Y=2) = (-1) \cdot \frac{0.15}{0.2} + 2 \cdot \frac{0.05}{0.2} = -\frac{1}{4}.$$

$$E(Y^2|X=-1) = 1^2 \cdot \frac{0.2}{0.45} + 2^2 \cdot \frac{0.15}{0.45} = \frac{16}{9}$$

$$E(Y^2|X=2) = 1^2 \cdot \frac{0.3}{0.55} + 2^2 \cdot \frac{0.05}{0.55} = \frac{10}{11}$$

2. (4 points).
Let X,Y are i.i.d.r.v's with the following density function

$$f(x, \sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad x > 0, \sigma > 0.$$

Find the distribution of the following random variables and check their independence:
$U = \frac{X}{2Y}, \quad V = X^2 + Y^2;$

Sol: since X, Y independent $f_{X,Y}(x,y,\sigma) = \sigma^{-4} xy\, e^{-\frac{x^2+y^2}{2\sigma^2}}$

$$d\left(-\frac{x^2}{2\sigma^2}\right). \qquad \frac{4U^2}{4U^2+1}$$

$$\begin{cases} U = \frac{X}{2Y} \\ V = X^2 + Y^2 \end{cases} \Rightarrow \begin{cases} X = 2U \cdot \sqrt{\frac{V}{4U^2+1}} \\ Y = \sqrt{\frac{V}{4U^2+1}} \end{cases}$$

$$|J|^{-1} = \begin{vmatrix} \frac{1}{2Y} & -\frac{X}{2Y^2} \\ 2X & 2Y \end{vmatrix} = 1 + \frac{X^2}{Y^2} = \frac{X^2+Y^2}{Y^2} = \frac{V}{\frac{V}{4U^2+1}} = 4U^2+1.$$

$$f_U(u) = \int_{-\infty}^{\infty} f\left(2u \cdot \sqrt{\frac{v}{4u^2+1}}\right) f\left(\sqrt{\frac{v}{4u^2+1}}\right) \cdot \frac{1}{4u^2+1}\, dv = \int_0^{\infty} \sigma^{-4} \cdot \frac{2uv}{(4u^2+1)^2} \cdot e^{-\frac{v}{2\sigma^2}}\, dv.$$

$$= \frac{2u\sigma^{-4}}{(4u^2+1)^2} \left[\int_0^{\infty} v e^{-\frac{v}{2\sigma^2}}\, dv\right] = \frac{2u\sigma^{-4}}{(4u^2+1)^2} \cdot (2\sigma^2)^2 = \frac{8u}{(4u^2+1)^2}.$$

$$f_V(v) = \int_0^{\infty} \sigma^{-4} \cdot \frac{2uv}{(4u^2+1)^2} e^{-\frac{v}{2\sigma^2}}\, du. = \sigma^{-4} v \cdot e^{-\frac{v}{2\sigma^2}} \int_0^{\infty} \frac{d(4u^2+1)}{4(4u^2+1)^2} = \frac{\sigma^{-4}}{4} v \cdot e^{-\frac{v}{2\sigma^2}}$$

$$f_{U,V}(u,v) = f\left(2u \cdot \sqrt{\frac{v}{4u^2+1}}\right) f\left(\sqrt{\frac{v}{4u^2+1}}\right) \cdot \frac{1}{4u^2+1} = \sigma^{-4} \cdot \frac{2uv}{(4u^2+1)^2} e^{-\frac{v}{2\sigma^2}}$$

thus we have $f_{U,V}(u,v) = f_U(u) f_V(v)$. U . V are independent.

2. (6 points).

a)(2 points) Find the distribution of $\sum_{i=1}^{k} X_i$, if X_i are i.i.d.r.v's $\sim Exp(a)$;

b)(2 points) Find the distribution of $\frac{Y}{X}$, if X,Y are independent and $X \sim N(0,4), Y \sim N(0,5)$;

c)(2 points) Find the distribution of $\frac{2X}{3Y}$, if X,Y are independent and $X \sim N(0,\sigma^2), Y \sim Exp(a)$;

d)* (1 point) Find the distribution of $\sqrt{X^2+Y^2}$, if X,Y are independent and $X \sim Norm(0,\sigma^2), \quad Y \sim Norm(0,10)$.

a). denote $X = \sum^{n} X_i$. $\quad f_{X_i}(x_i) = a e^{-ax_i} \quad x_i \geq 0$.

if $n=2$. $f_X(x) = \int_0^x a^2 e^{-at} \cdot e^{-a(x-t)} dt = a^2 e^{-ax} \cdot \int_0^x dt = a^2 x e^{-ax}$

if $n=3$. $f_X(x) = \int_0^x t a^3 e^{-at} \cdot e^{-a(x-t)} dt = a^3 e^{-ax} \cdot \int_0^x t\, dt = a^3 \frac{x^2}{2} e^{-ax}$

repeat the process for $n=k$. $f_X(x) = a^n \cdot \frac{x^{n-1}}{(n-1)!} \cdot e^{-ax}$, $x \geq 0$.

b) let $X=u$. $Y=uv$. $\Rightarrow \begin{cases} u = X \\ v = \frac{Y}{X} \end{cases}$ $\quad |J| = \begin{vmatrix} 1 & v \\ 0 & u \end{vmatrix} = u$.

$$f_{U,V}(u,v) = f_X(u) \cdot f_Y(uv) \cdot J = \frac{1}{4\pi\sqrt{5}} e^{-(\frac{u^2}{8}+\frac{u^2v^2}{10})} |u|$$

$$f_V(v) = \int_{-\infty}^{\infty} f(u,v)\,du = \frac{1}{2\pi\sqrt{5}} \int_0^{\infty} u \cdot e^{-u^2(\frac{1}{8}+\frac{v^2}{10})} du = \frac{1}{2\pi\sqrt{5}} \cdot \frac{1}{2(\frac{1}{8}+\frac{v^2}{10})} = \frac{2\sqrt{5}}{\pi(5+4v^2)}$$

c). Let $\begin{cases} u = Y \\ v = \frac{2X}{3Y} \end{cases} \Rightarrow \begin{cases} X = \frac{3vu}{2} \\ Y = u \end{cases}$ $\quad |J| = \begin{vmatrix} \frac{3v}{2} & \frac{3u}{2} \\ 1 & 0 \end{vmatrix} = \frac{3}{2}|u|$

$$f_{U,V}(u,v) = f_X(\tfrac{3vu}{2}) \cdot f_Y(u) \cdot |J| = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{9u^2v^2}{8\sigma^2}} \cdot a \cdot e^{-au} \cdot \frac{3}{2}u.$$

$$f_V(v) = \frac{3a}{2\sqrt{2\pi}\sigma} \int_0^{+\infty} e^{-(\frac{9v^2}{8\sigma^2}\cdot u^2 + au)} \cdot u\,du.$$

$$= \frac{3a}{2\sqrt{2\pi}\sigma} \left[\int_0^{+\infty} e^{-(\frac{9v^2}{8\sigma^2}\cdot u^2+au)} \cdot (\frac{9v^2}{4\sigma^2}u + a) \cdot \frac{4\sigma^2}{9v^2} du - \frac{4\sigma^2 a}{9v^2} \int_0^{+\infty} e^{-(\frac{9v^2}{8\sigma^2}u^2+au)} du \right]$$

$$= \frac{\sqrt{2}\sigma}{3\sqrt{\pi}v^2} - \frac{\sqrt{2}a^2\sigma}{3\sqrt{\pi}v^2} \int_0^{+\infty} e^{-(\frac{9v^2}{8\sigma^2}u^2+au)} du$$ → seems can't find elementary function as it's sol.

d). let $\begin{cases} X = r\cos\varphi \\ Y = r\sin\varphi \end{cases}$ $\quad J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = |r|$

$$f_{R,\Phi}(r,\varphi) = f_X(r\cos\varphi) \cdot f_Y(r\sin\varphi) \cdot |r| = \frac{1}{2\pi\sigma \cdot \sqrt{10}} \cdot e^{-\frac{r^2\cos^2\varphi}{2\sigma^2}} \cdot e^{-\frac{r^2\sin^2\varphi}{20}} |r|.$$

$$f_R(|r|) = \frac{|r|}{2\pi\sigma\sqrt{10}} \int_0^{2\pi} e^{-r^2(\frac{\cos^2\varphi}{2\sigma^2}+\frac{\sin^2\varphi}{20})} d\varphi$$

i) $\sigma^2 = 10$ $\quad f_R(r) = \frac{r}{10} \cdot e^{-\frac{r^2}{20}}$. $\quad r > 0$.

ii) $\sigma^2 \neq 10$. $\quad f_R(r) = \frac{r}{2\pi\sqrt{10}\sigma} \int_0^{2\pi} e^{-\frac{r^2}{2}(\frac{\cos^2\varphi}{\sigma^2}+\frac{\sin^2\varphi}{10})} d\varphi$. $\quad r>0$. the integral no elementary function expression.

3. (8 points) Let (X, Y) be given by the following joint density function:

$$f(x,y) = \begin{cases} c(x+y), & x \in [0,1], \quad y \in [0,2]. \\ 0, & \text{else.} \end{cases}$$

a)(4 points). Find c, $F_{X,Y}(x,y)$, $corr(X,Y)$, $E(X|Y)$, $E(X^2|Y), E(Y|X)$ and $E(Y^2|X)$.

b) (4 points). Find the joint distribution $U = \frac{X}{Y}$, $\quad V = X - Y$ and check their independence.

a) $\int_0^1 \int_0^2 c(x+y)\, dy\, dx = \int_0^1 (2cx+2c)dx = cx^2 + 2cx \Big|_0^1 = 3c \qquad 3c = 1 \quad \Rightarrow c = \frac{1}{3}.$

$F_{X,Y}(x,y) = \frac{1}{3}\int_0^x \int_0^y (u+v)\, du\, dv. = \frac{1}{3}\int_0^x (\frac{y^2}{2} + yv)\, dv = \frac{xy}{6}(x+y). \qquad x \in [0,1] \quad y \in [0,2]$

$f_X(x) = \int_0^2 \frac{1}{3}(x+y)\, dy = \frac{1}{3}\cdot(xy + \frac{y^2}{2})\Big|_0^2 = \frac{2(x+1)}{3} \qquad x \in [0,1]$

$f_Y(y) = \int_0^1 \frac{1}{3}(x+y)\, dx = \frac{1}{3}\cdot(xy + \frac{x^2}{2})\Big|_0^1 = \frac{2y+1}{6} \qquad y \in [0,2]$

$E(XY) = \int_0^2 \int_0^1 \frac{x+y}{3}\cdot xy\, dx\, dy = \frac{1}{3}\int_0^2 y\left(\int_0^1 (x^2+xy)\, dx\right) dy = \frac{1}{3}\int_0^2 y\left(\frac{1}{3} + \frac{y}{2}\right) dy = \frac{1}{3}\cdot\left(\frac{1}{6}y^2 + \frac{1}{6}y^3\Big|_0^2\right) = \frac{2}{3}.$

$E(X) = \int_0^1 \frac{2x^2+2x}{3}\, dx = \frac{1}{3}\cdot(\frac{2}{3}x^3 + x^2)\Big|_0^1 = \frac{5}{9} \qquad E(X^2) = \int_0^1 \frac{2x^3+2x^2}{3}\, dx = \frac{1}{3}\cdot\left(\frac{2}{4}x^4 + \frac{2}{3}x^3\right)\Big|_0^1 = \frac{7}{18}$

$E(Y) = \int_0^2 \frac{2y^2+y}{6}\, dy = \frac{1}{6}\cdot(\frac{2}{3}y^3 + \frac{y^2}{2})\Big|_0^2 = \frac{11}{9} \qquad E(Y^2) = \int_0^2 \frac{2y^3+y^2}{6}\, dy = \frac{1}{6}\cdot\left(\frac{1}{2}y^4 + \frac{1}{3}y^3\right)\Big|_0^2 = \frac{16}{9}$

$D(X) = E(X^2) - (E(X))^2 = \frac{7}{18} - \frac{25}{81} = \frac{63-50}{162} = \frac{13}{162}$

$D(Y) = E(Y^2) - (E(Y))^2 = \frac{16}{9} - \frac{121}{81} = \frac{144-121}{81} = \frac{23}{81}$

$corr(X,Y) = \frac{cov(X,Y)}{\sqrt{D(X)\cdot D(Y)}} = \frac{\frac{2}{3} - \frac{5}{9}\cdot\frac{11}{9}}{\sqrt{\frac{13}{162}\cdot\frac{23}{81}}} = \frac{-\frac{1}{81}}{\frac{1}{81}\cdot\sqrt{13\cdot 23/2}} = -\frac{1}{\sqrt{13\cdot 23/2}} = -0.0082$

$E(X|Y) = g(y) = \int_{-\infty}^{+\infty} x\frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx = \int_0^1 x\frac{\frac{1}{3}(x+y)}{\frac{2y+1}{6}}\, dx = \frac{2}{2y+1}\int_0^1 x^2 + xy\, dx = \frac{2}{2y+1}\left(\frac{1}{3} + \frac{y}{2}\right) = \frac{2+3y}{3(2y+1)}$

$E(X^2|Y) = \int_{-\infty}^{+\infty} x^2\cdot\frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx = \frac{2}{2y+1}\cdot\int_0^1 (x^3 + x^2y)dx = \frac{2}{2y+1}\left(\frac{1}{4} + \frac{y}{3}\right) = \frac{3+4y}{6(2y+1)}$

$E(Y|X) = \int_0^2 y\cdot\frac{f_{X,Y}(x,y)}{f_X(x)}\, dy = \frac{1}{2(X+1)}\int_0^2 y(x+y).\, dy = \frac{1}{2(X+1)}\left[2x + \frac{8}{3}\right] = \frac{3x+4}{3(x+1)}$

$E(Y^2|X) = \int_0^2 y^2\cdot\frac{f_{X,Y}(x,y)}{f_X(x)}\, dy = \frac{1}{2(X+1)}\int_0^2 y^2(x+y)\, dy = \frac{1}{2(X+1)}\left[\frac{8}{3}x + 4\right] = \frac{4x+6}{3(x+1)}$

b). $\begin{cases} U = \frac{X}{Y} \\ V = X - Y. \end{cases} \Rightarrow \begin{cases} X = \frac{UV}{U-1} \\ Y = \frac{V}{U-1} \end{cases} \qquad |J| = \begin{vmatrix} -\frac{V}{(U-1)^2} & \frac{U}{U-1} \\ -\frac{V}{(U-1)^2} & \frac{1}{U-1} \end{vmatrix} = \left|\frac{-V+VU}{(U-1)^3}\right| = \frac{|V|}{(U-1)^2} \qquad U \in \mathbb{R}_+. \quad V \in [-1,1].$

$f_{U,V}(u,v) = f_{X,Y}\left(\frac{UV}{U-1}, \frac{V}{U-1}\right)\cdot|J| = \frac{V}{3}|V|\cdot\frac{U+1}{(U-1)^3}$

$\begin{cases} 0 \le \frac{UV}{U-1} \le 1 \\ 0 \le \frac{V}{U-1} \le 2. \end{cases}$ i) $U \ge 1$ $\begin{cases} 0 \le UV \le U-1 \\ 0 \le V \le 2(U-1). \end{cases} \Rightarrow 0 \le V \le \frac{U-1}{U}$ ii) $U \le 1.$ $\begin{cases} 0 \ge UV \ge U-1 \\ 0 \ge V \ge 2(U-1) \end{cases} \Rightarrow 0 \ge V \ge \max\left\{\frac{U-1}{U}, 2(U-1)\right\}.$

Thus we have. the domain of V is dependent with U.

$\Rightarrow U.V$ is not independent.

$$f_{U,V}(u,v) = \begin{cases} \frac{V^2}{3}\frac{U+1}{(U-1)^3} & U \ge 1,\ V \in [0, \frac{U-1}{U}]. \\ -\frac{V^2}{3}\cdot\frac{U+1}{(U-1)^3} & U \in (0, \frac{1}{2}],\ V \in [2(U-1), 0) \text{ or } U \in (\frac{1}{2}, 1).\ V \in [\frac{U-1}{U}, 0) \end{cases}$$

4. (8 points)
Let (X, Y) be given by the following joint density function:

$$f(x,y) = \begin{cases} cxy, & x, y \in [0,1],\ x+y>1 \\ 0, & \text{else.} \end{cases}$$

a) (2 points) Find the joint distribution of $U = X+Y, V = \frac{Y}{X}$ and check their independence.
b) (4 points). Find the joint distribution of $U = 2X+Y, V = \frac{X^2}{2Y}$ and check their independence.
c) (2 points). Find $E(X|Y)$ and $E(Y|X)$.

find c: $\int_0^1 \int_{1-x}^1 c(xy).\ dy\ dx = c\int_0^1 \left[x\frac{y^2}{2}\Big|_{1-x}^{1}\right] dx = c\int_0^1 x^2 - \frac{x^3}{2}\ dx = c\cdot\left[\frac{1}{3}-\frac{1}{8}\right] = 1 \Rightarrow c = \frac{24}{5}$

a). $\begin{cases} U = X+Y \\ V = \frac{Y}{X} \end{cases} \Rightarrow \begin{cases} X = \frac{U}{V+1} \\ Y = \frac{UV}{V+1} \end{cases}$ $\quad |J| = \begin{vmatrix} \frac{1}{V+1} & -\frac{U}{(V+1)^2} \\ \frac{V}{V+1} & \frac{U}{(V+1)^2} \end{vmatrix} = \frac{U}{(V+1)^2}$ $\quad V>0$, $U>1$.

$$f_{U,V}(u,v) = f_{X,Y}\left(\frac{u}{v+1}, \frac{uv}{v+1}\right)|J| = \frac{24u^3v}{5(v+1)^4}$$

$\begin{cases} 0 \le \frac{U}{V+1} \le 1 \\ 0 \le \frac{UV}{V+1} \le 1 \\ U>1 \end{cases} \Rightarrow \begin{cases} 2 \ge U > 1 \\ \frac{1}{U-1} \ge V \ge U-1 \end{cases}$

v has domain dependent with u. thus. u, v. not independent

$f_{U,V}(u,v) = \frac{24u^3v}{5(v+1)^4}$. $\quad \begin{cases} u \in (1,2] \\ v \in [U-1, \frac{1}{U-1}]. \end{cases}$

b). $\begin{cases} U = 2X+Y \\ V = \frac{X^2}{2Y} \end{cases} \Rightarrow \begin{cases} X = \sqrt{4V^2+2VU} - 2V. \\ Y = U+4V-2\sqrt{4V^2+2VU}. \end{cases}$ $\quad |J|^{-1} = \begin{vmatrix} 2 & 1 \\ \frac{X}{Y} & -\frac{X^2}{2Y^2} \end{vmatrix} = \frac{X^2+XY.}{Y^2}$

$|J| = \frac{Y^2}{X^2+XY} = \frac{Y^2}{2VY+XY} = \frac{Y}{2V+X} = \frac{U+4V}{\sqrt{4V^2+2VU}} - 2.$

$$f_{U,V}(u,v) = f_{X,Y}(x,y)|J| = \frac{24}{5}\cdot xy\cdot\frac{y^2}{x(x+y)} = \frac{24}{5}\cdot\frac{y^3}{x+y} = \frac{24}{5}\cdot\frac{(u+4v-2\sqrt{2v(2v+u)})^3}{(2v+u)-\sqrt{2v(2v+u)}}$$

$X+Y>1 \Rightarrow u+2v-\sqrt{2v(2v+u)} > 1 \Rightarrow (u-1)^2 > (2u-4)v$.

if $u>2$. $0<v<\frac{(u-1)^2}{2u-4}$ $\quad$ v has domain depend on u.

$u<2$ $\quad 0>v>\frac{(u-1)^2}{2u-4}$ $\quad$ thus U. V not independent.

c). $f_X(x) = \frac{24}{5}\int_{1-x}^1 xy\ dy = \frac{24}{5}\cdot x\cdot\frac{y^2}{2}\Big|_{1-x}^{1} = \frac{24}{5}\cdot\left[\frac{x}{2}-\frac{x}{2}(1-x)^2\right] = \frac{24}{5}(x^2-\frac{1}{2}x^3) = \frac{24}{5}x^2 - \frac{12}{5}x^3.$

$f_Y(y) = f_X(y) = \frac{24}{5}x^2 - \frac{12}{5}x^3$ by symmetry of x. y.

$$E(X|Y) = \int_{1-y}^1 x\cdot\frac{f_{X,Y}(x,y)}{f_Y(y)}dx = 2\int_{1-y}^1 x\cdot\frac{xy}{2y^2-y^3.} = \frac{2}{y(2-y)}\int_{1-y}^1 x^2\,dx = \frac{2}{y(2-y)}\cdot\frac{y(3-3y+y^2)}{3} = \frac{2(y^2-3y+3)}{3(2-y)}$$

$$E(Y|X) = \int_{1-x}^1 y\,\frac{f_{x,y}(x,y)}{f_X(x,y)}\,dy = \frac{2(x^2-3x+3)}{3(2-x).}$$

5. (2 points) Let random vector (X,Y,Z) is given by some joint density $f_{x,y,z}(\cdot)$.
a) Find the expression $E(g(X)|\frac{Y}{Z}=t)$.
b)* Prove

$$\frac{D(E(Y|X)}{D(Y)} = corr^2(E(Y|X), Y).$$

a). Let $\begin{cases} W = \frac{Y}{Z.} \\ Z = Z. \end{cases} \Rightarrow \begin{cases} Y = WZ \\ Z = Z. \end{cases} \Rightarrow |J| = \begin{vmatrix} z & w \\ 0 & 1 \end{vmatrix} = |z|.$

$f_{X,W,Z} = f_{X,Y,Z}(x, wz, z)\cdot|z|.$

$f_{X|W=t}(x) = \frac{f_{X,W}}{f_W} = \frac{\int f_{X,Y,Z}(X, WZ, Z)\cdot|Z|\,dz.}{\iint f_{X,Y,Z}(X,WZ,Z)\cdot|Z|\,dz\,dx.}$ $\qquad E(g(X)|\frac{Y}{Z}=t) = \int g(x)\cdot f_{X|W=t}(x)\,dx.$

b) $\frac{D(E(Y|X))}{D(Y)} = corr^2(E(Y|X), Y)$.

$$RHS = \frac{Cov^2(E(Y|X), Y)}{D(E(Y|X)) \cdot D(Y)}$$

it suffices to check $Cov^2(E(Y|X), Y) = D^2(E(Y|X))$. since variance > 0. eliminate the square.

$$Cov(E(Y|X), Y) = E(E(Y|X) \cdot Y) - E(E(Y|X)) \cdot E(Y).$$

$$D(E(Y|X)) = E(E(Y|X)^2) - [E(E(Y|X))]^2$$

by the property of conditional expectation. $E(E(Y|X)) = E(Y)$. thus. $E(E(Y|X)) \cdot E(Y) = [E(E(Y|X))]^2$

$$E(E(Y|X) \cdot Y) = E[E(E(Y|X) \cdot Y | X)] \xlongequal[E(Y|X) = const.]{\text{when } X \text{ is fixed}} E[E(Y|X) \cdot E(Y|X)] = E(E^2(Y|X)).$$

thus we have $Cov(E(Y|X), Y) = D(E(Y|X)) \Rightarrow LHS = RHS$.