# Equations of Straight Line on Plane

Assume that some plane $\alpha$ in the space $\mathbb{E}$ is chosen and fixed.

On this plane we choose arbitrary coordinate system: origin $O$, and a pair of unit vectors.

We derived and start discussing several forms of equation of the line.

Vectorial parametric equation of this line is

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a}.$$

Here $\boldsymbol{r}_0$ is radius vector of arbitrary initial point $A \in a$, and $\boldsymbol{a} \parallel a$ is non-zero direction vector. $t$ is parameter.

Vectorial normal equation of this line is

$$\boldsymbol{r} \cdot \boldsymbol{n} = D.$$

Here $\boldsymbol{n} \perp a$ is non-zero normal vector, and $D = \boldsymbol{r}_0 \cdot \boldsymbol{n}$, and $\boldsymbol{r}_0$ is radius vector of arbitrary initial point $A \in a$.

Coordinate parametric equations of the line are

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t. \end{cases}$$

Here $a_x$ and $a_y$ are coordinates of direction vector and may not be zeros in the same time.

Canonical equation of the line is:

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y}$$

It also has form for line passing through pint $B(x_1, y_1)$:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0},$$

and form for two intercept points:

$$\frac{x}{a} + \frac{y}{b} = 1$$

Here $(a, 0)$, and $(o, b)$ are intercept points.

In Cartesian coordinate system we also derived slope-based forms of these equations. Slope-point form:

$$(y - y_0) = m(x - x_0),$$

and slope-intercept form:

$$y = mx + b.$$

Finally we expressed general equation of the straight line:

$$AX + By + C = 0,$$

and its normal form yielding nice technique to calculate distances between lines:

$$x\cos\varphi + y\sin\varphi - p = 0$$

During today class we first continue discussing problems involving these forms of straight line equation.

# 1 Problems corner (continued)

**Problem 7**
Find the equations of the lines parallel to the line $12x - 5y - 15 = 0$ and at a perpendicular distance from it numerically equal to 4.

**Solution**
First, we bring our equation to normal form:

$$\pm\sqrt{A^2 + B^2} = \pm\sqrt{144 + 25} = \pm\sqrt{169} = \pm 13$$

This yields expression for the distance from our line to point $P(x', y')$ in 4 units:

$$\frac{12x' - 5y' - 15}{13} = \pm 4$$

We fixed $\pm$ before 13 with described above rule, but added it before 4 to express lines laying by both sides of line in the question.
Simplifying and dropping primes, the required equations are

$$12x - 5y - 67 = 0$$

and

$$12x - 5y + 37 = 0.$$

While we recap deriving of the general line equation we remember that only free term is determined by initial point. Thus, parallel transposition keeps terms by $x$ and $y$ as is.

**Problem 8**
Given the triangle $A(-2, 1)$, $B(5, 4)$, $C(2, -3)$, determine the length of the altitude through $A$, and the area of the triangle.

**Solution**

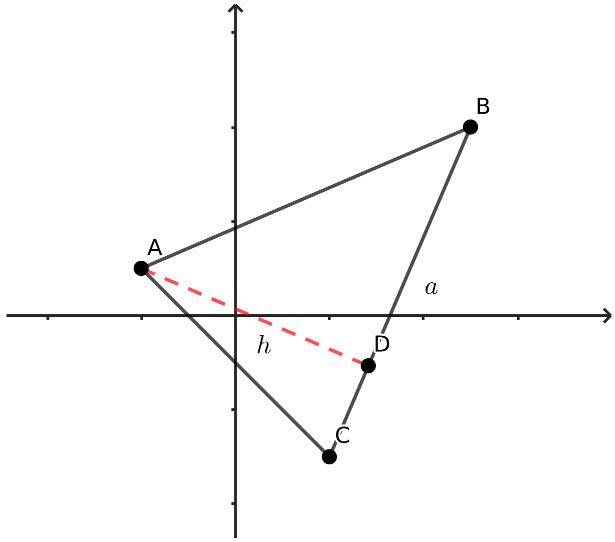


Figure 1: Altitude of triangle

We call *altitude of a triangle* is a line segment through a vertex and perpendicular to a line containing the base (the side opposite the vertex).

If we calculated length of altitude $h$ and its base $a$, area of triangle is $\frac{1}{2}ha$

Thus, perpendicular distance from the vertex to opposite side of triangle is exactly length of its altitude.

First, we write equation of $BC$:

$$\frac{y+3}{x-2} = \frac{4+3}{5-2}$$
$$7x - 3y - 23 = 0$$

Normalization multiply: $\sqrt{A^2 + B^2} = \sqrt{49+9} = \sqrt{58}$ Distance from $BC$ to $A$:

$$d = \frac{7(-2) + 3(-1) - 23}{\sqrt{58}} = \frac{-40}{\sqrt{58}}$$

Sign of this distance plays no role for us as we search some ”effective” parameter.

Length of $BC$:

$$BC = \sqrt{(5-2)^2 + (4+3)^2} = \sqrt{58}$$

Area of triangle:

$$S = \frac{1}{2}\sqrt{58}\frac{40}{\sqrt{58}} = 20$$

**Problem 9**

Determine the equations of the bisectors of the angles between the lines

$$(L_1)\, 3x - 4y + 8 = 0 \text{ and}$$
$$(L_1)\, 5x + 12y - 15 = 0.$$

## Solution

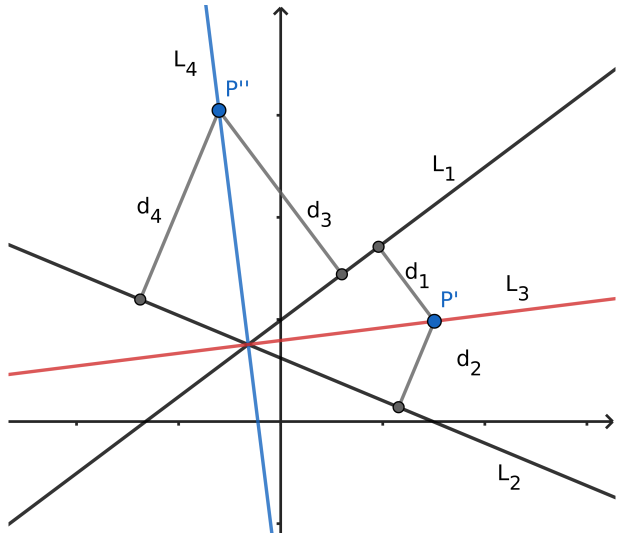


Figure 2: Bisectors of crossing lines

Normalization ratios for lines in question are

$$k_1 = \pm\sqrt{9+16} = \pm\sqrt{25} = \pm 5$$
$$k_2 = \pm\sqrt{25+144} = \pm\sqrt{169} = \pm 13$$

Standard values for these ratios are $-5$ and 13 respectively (sign$(-C)$).

Let $P'(x', y')$ be any point on bisector $L_3$. For this bisector we suppose that distances from lines to $P$ are positive. Expressions for distances are:

$$d_1 = \frac{3x' - 4y' + 8}{-5}, \text{ and}$$
$$d_2 = \frac{5x' + 12y' - 15}{13}.$$

Condition that point lies on the bisector means equality of corresponding transverse segments.

$P'$ and the origin are on the same side of $L_1$ but on opposite sides of $L_2$, Hence $d_1$ is negative and $d_2$ is positive, and

$$d_1 = d_2.$$

Then the locus of $P'$ is

$$\frac{3x' - 4y' + 8}{-5} = -\frac{5x' + 12y' - 15}{13}$$

Simplifying and dropping primes,

$$39x - 52y + 104 = 25x + 60y - 75,$$

the equation of $L_3$ is

$$14x - 112y + 179 = 0.$$

Similarly, let $P$"$(x$", $y$"$)$ be any point on bisector $L_4$ Since $P$" and the origin are on opposite sides of $L_1$ and $L_2$, $d_3$ and $d_4$ are both positive, and $d_3 = d_4$.

Then the locus of $P$" is

$$\frac{3x'' - 4y'' + 8}{-5} = \frac{5x'' + 12y'' - 15}{13}$$

Simplifying and dropping primes,

$$39x - 52y + 104 = -25x - 60y + 75,$$

the equation of $L_4$ is

$$64x + 8y + 29 = 0.$$

Slope of $L_3$ is $m_3 = -\dfrac{14}{-112} = \dfrac{1}{8}$. Slope of $L_4$ is $m_4 = -\dfrac{64}{8} = -8$.

Hence, we checked that bisectors for these lines are perpendicular.

**Problem 10**

Find the point of intersection of the bisectors of the interior angles of the triangle whose sides are:

$$\begin{aligned}(L_1)\, 7x - y + 11 = 0,\\ (L_2)\, x + y - 15 = 0,\\ (L_3)\, 7x + 17y + 65 = 0.\end{aligned}$$

**Solution**

Distances from sides of the triangle to arbitrary point are:

$$\begin{aligned}d_1(x, y) &= \frac{7x - y + 11}{-\sqrt{50}},\\ d_2(x, y) &= \frac{x + y - 15 = 0}{\sqrt{2}},\\ d_3(x, y) &= \frac{7x + 17y + 65}{-\sqrt{338}}.\end{aligned}$$

Sign depends on disposition of the point and origin.

The point of intersection $(h, k)$ is the center of the circle inscribed in the triangle. Thus, distances to $(h, k)$ are numerically equal.

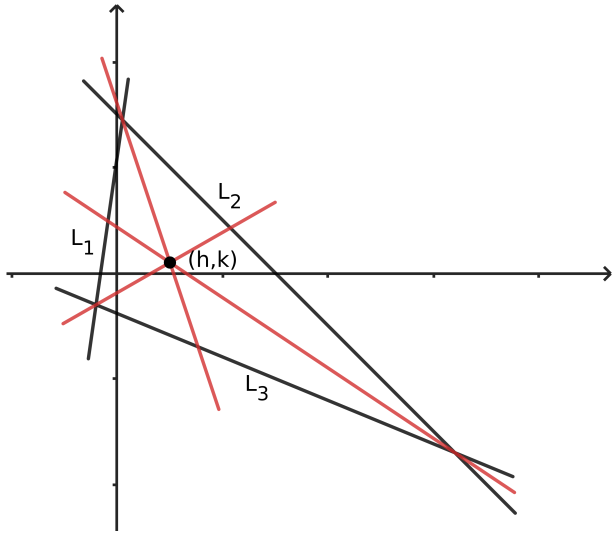


Figure 3: Cross point for bisectors in triangle

The distances are all negative, since the point and the origin are on the same side of each line. Then, $d_1(h,k) = d_2(h,k) = d_3(h,k)$. This yields system of two equations:

$$\begin{cases} \dfrac{7h - k + 11}{-5\sqrt{2}} = \dfrac{h + k - 15 = 0}{\sqrt{2}} \\ \dfrac{7h - k + 11}{-5\sqrt{2}} = \dfrac{7x + 17y + 65}{-13\sqrt{2}} \end{cases}$$

Simplifying,

$$\begin{cases} 3h + k = 16 \\ 4h - 7k = 13 \end{cases}$$

$$\begin{cases} h = 5 \\ k = 1 \end{cases}$$

## 2 System of the lines

While solving the problems on lines we often begin with formulation the condition which is satisfied for variety of lines.

**Definition.** We call **system of the lines** any set of lines satisfying arbitrary condition.

Examples of the systems of lines:

- Lines whose slope is $-4$.

  Let $k$ be $y$-intercept of the system of lines whose slope is $-4$. From $y = mx + b$, the required equation is $y = -4x + k$, or $4x + y - k = 0$.

- Lines passing through (4,1).

  Let $k$ be slope of the system of lines passing through $(4, 1)$. Substituting it in $y - y_1 = m(x - x_1)$, the required equation is $y - 1 = k(x - 4)$, or $kx - y + 1 - 4k = 0$.

- Lines whose y-intercept is 7.

  Let $k$ be slope of the system of lines whose $y$-intercept is 7. From $y = mx + b$, the required equation is $y = kx + 7$, or $kx - y + 7 = 0$.

- Lines whose x-intercept is 5.

  Let $k$ be slope of the system of lines whose $x$-intercept is 5. From $y - y_1 = m(x - x_1)$, the required equation is $y = k(x - 5)$, or $kx - y - 5k = 0$.

- Lines the sum of whose intercepts is 8.

  Let $k$ be $x$-intercept of system of lines. Then $8 - k$ is $y$-intercept of the system. From equation

  $$\frac{x}{a} + \frac{y}{b} = 1,$$

  equation for this system of lines is

  $$\frac{x}{k} + \frac{y}{8 - k} = 1,$$

  or

  $$(8 - k)x + ky - 8k + k^2 = 0.$$

- Lines whose y-intercept is twice the x-intercept.

  From equation

  $$\frac{x}{a} + \frac{y}{b} = 1,$$

  equation for this system of lines is

  $$\frac{x}{k/2} + \frac{y}{k} = 1,$$

  or

  $$2x + y - k = 0.$$

- Lines having one intercept numerically twice the other intercept.

  Slope of the line is negative relation of $y$-intercept and $x$-intercept.

  When the $x$-intercept is numerically $(\pm)$ twice the $y$-intercept,slope of the line is $\pm\frac{1}{2}$.

When the $y$-intercept is numerically ($\pm$) twice the $x$-intercept,slope of the line is $\pm 2$.

Let $k$ be $y$-intercept.

System of line is expressed with two families of equations:

$$y = \pm\frac{1}{2}x + k,$$

and

$$y = \pm 2x + k.$$

**Definition.** We call the **bundle of lines** in the plane the system of lines that pass through a fixed point, or even the system of lines parallel to a given line.

A bundle of lines is said to be **proper** if each of its lines passes through the same point, called the **center** or **support** of the bundle.

This point is identified by the intersection of any two lines in the bundle.

A proper bundle of lines is described by an equation similar to that of a single straight line, but in which the constants depend on a parameter $k$; each value of $k$ corresponds to a line in the bundle.

All straight lines in a bundle, except the vertical straight line of equation $x = x_0$, can be parameterized by making the slope $m$ and the intercept $b$ depend on the parameter $k$:

$$y = m(k)x + b(k)$$

If the center of the bundle has coordinates $(x_0, y_0)$ so $b(k) = y_0 - m(k)x_0$, and the equation can also be written as

$$y - y_0 = m(k)(x - x_0)$$

In most basic case it may be said that $m(k) = k$.

A bundle of lines is said to be **improper** if its straight lines are all parallel to each other.

As in the case of a proper bundle, all the lines of an improper bundle can be parameterized by observing that now the slope of the lines is constant ($m = \text{const}$). The bundle of lines can be parameterized as

$$y = mx + b(k),$$

or

$$a(x - x_0) + b(y - y_0) = k$$

**Problem 1**

Derive the equation of the line perpendicular to $4x + y - 1 = 0$ and passing through the point of intersection of lines $2x - 5y + 3 = 0$ and $x - 3y - 7 = 0$.

**Solution**

By the one hand, slope of $4x + y - 1 = 0$ is $m = -\dfrac{A}{B} = -4$. Then, required line is part of improper bundle with slope $\dfrac{1}{4}$.

By the other hand, required line passes through the point of intersection of $2x - 5y + 3 = 0$ and $x - 3y - 7 = 0$, which is

$$\begin{cases} 2x - 5y + 3 = 0 \\ x - 3y - 7 = 0 \end{cases}$$

$$\begin{cases} y = -17 \\ x = -44 \end{cases}$$

Then, required line is in proper bundle with center $(-17, -44)$.

Equation for this bundle is:

$$y + 17 = k(x + 44).$$

For the slope $\dfrac{1}{4}$, the exact equation is

$$y + 17 = \frac{1}{4}(x + 44),$$

$$y = \frac{1}{4}x - 6.$$

# 3 Intersection of three lines

Suppose lines governed with equations $A_1x + B_1y + C_1 = 0$, $A_2x + B_2y + C_2 = 0$ and $A_3x + B_3y + C_3 = 0$ are passing through the same point $P$. This condition means that system of equations

$$\begin{cases} A_1x + B_1y + C_1 = 0 \\ A_2x + B_2y + C_2 = 0 \\ A_3x + B_3y + C_3 = 0 \end{cases} \tag{1}$$

has a solution $(x, y)$.

This condition may be checked manually by looking for intersection, say, first and second line and checking if this point lies on the third line.

But mo rational approach is to investigate the system (??).

From the general course of algebra we recap that system of this type have a solution only and only if

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0$$

**Problem**

Make sure that lines

$$\begin{aligned} 3x - y - 1 &= 0 \\ 2x - y + 3 &= 0 \\ x - y + 7 &= 0 \end{aligned}$$

Are crossing in a single point

**Solution**

$$\begin{vmatrix} 3 & -1 & -1 \\ 2 & -1 & 3 \\ 1 & -1 & 7 \end{vmatrix} = 0$$

# 4 Example of complex problem requiring several approaches

**Problem**

Lines $3x + 4y - 30 = 0$ and $3x - 4y + 12 = 0$ are tangent for the circle of radius $R = 5$.

Find the coordinates of the center of circle, and area of the figure shaped with the segments of tangent lines form tangent points to the lines intersection and corresponding radiuses of the circle.

**Solution**

Tangent line is perpendicular with radius established to the tangent point. Thus, distance from the tangent line to the center of the circle is exactly the radius.

Standard normalizing ratio for both lines is $+5$, for thirst line, and $-5$ for second line.

We have four possible positions for the circle and combinations for position of the center with respect to the lines and origin:

$$\begin{cases} \dfrac{3x+4y-30}{5} = 5 \\ \dfrac{3x-4y+12}{-5} = 5, \end{cases}$$

or

$$\begin{cases} \dfrac{3x+4y-30}{5} = -5 \\ \dfrac{3x-4y+12}{-5} = -5, \end{cases}$$

or

$$\begin{cases} \dfrac{3x+4y-30}{5} = -5 \\ \dfrac{3x-4y+12}{-5} = 5, \end{cases}$$

or

$$\begin{cases} \dfrac{3x+4y-30}{5} = 5 \\ \dfrac{3x-4y+12}{-5} = -5, \end{cases}$$

Neutralizing denominator, we yield:

$$\begin{cases} 3x+4y-30 = 25 \\ 3x-4y+12 = -25, \end{cases}$$

or

$$\begin{cases} 3x+4y-30 = -25 \\ 3x-4y+12 = 25, \end{cases}$$

or

$$\begin{cases} 3x+4y-30 = 25 \\ 3x-4y+12 = 25, \end{cases}$$

or

$$\begin{cases} 3x+4y-30 = -25 \\ 3x-4y+12 = -25, \end{cases}$$

This yields four systems:

$$\begin{cases} 3x+4y-31 = 0 \\ 3x-4y+13 = 0, \end{cases}$$

or

$$\begin{cases} 3x + 4y - 29 = 0 \\ 3x - 4y + 11 = 0, \end{cases}$$

or

$$\begin{cases} 3x + 4y - 31 = 0 \\ 3x - 4y + 11 = 0, \end{cases}$$

or

$$\begin{cases} 3x + 4y - 29 = 0 \\ 3x - 4y + 13 = 0, \end{cases}$$

Taking sum of each pair of equations we yield

$$\begin{cases} 6x = 18,\ x = 3, \\ y = \dfrac{23}{2} \end{cases}$$

or

$$\begin{cases} 6x = 18,\ x = 3, \\ y = -1 \end{cases}$$

or

$$\begin{cases} 6x = 20,\ x = \dfrac{34}{3}, \\ y = \dfrac{21}{4} \end{cases}$$

or

$$\begin{cases} 6x = 16,\ x = -\dfrac{16}{3}, \\ y = \dfrac{21}{4} \end{cases}$$

Lines in questions intersect in point $O$

$$\begin{cases} 3x + 4y - 30 = 0 \\ 3x - 4y + 12 = 0, \end{cases}$$

$$\begin{cases} 6x = 18,\ x = 3 \\ y = \dfrac{21}{4} \end{cases}$$

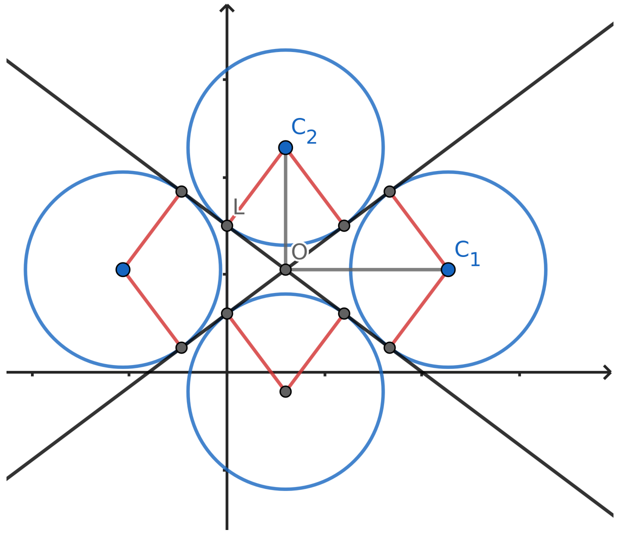


Figure 4: Sketch of intersecting lines and tangent circles

All our bundle is symmetric with respect to this point and also with lines $x = 3$, and $y = \dfrac{21}{4}$.

It must be noted that two opposite centers lie on horizontal line $y = \dfrac{21}{4}$ and two on vertical line $x = 3$.

Thus, we overlook general calculation of the lengths and yield hypotenuses $OC_1$, $OC_2$ which are $\dfrac{25}{4}$ and $\dfrac{25}{3}$ respectively

Second leg in these triangle are $\dfrac{1}{4}\sqrt{25^2 - 20^2} = \dfrac{15}{4}$ and $\dfrac{1}{3}\sqrt{25^2 - 15^2} = \dfrac{20}{4}$.

Area of quadrilateral is $\dfrac{75}{4}$, or $\dfrac{100}{3}$.

# 5 Equation of line in polar coordinates

In polar coordinates we use length of radius vector $r$ and minimal counterclockwise angle $\varphi$ radius vector shapes with reference direction (usually positive direction of first coordinates).

Transition from Cartesian to polar coordinates correspond with formulas:

$$x = r\cos\varphi \tag{2}$$
$$y = r\sin\varphi. \tag{3}$$

Suppose general equation of line is

$$Ax + By + C = 0 \tag{4}$$

Substituting (**??**)int (**??**) we yield

$$Ar\cos\varphi + Br\sin\varphi + C = 0$$

Factor out the $r$:

$$r(A\cos\varphi + B\sin\varphi) + C = 0$$

Solving for $r$:

$$r = \frac{-C}{A\cos\varphi + B\sin\varphi}$$

**Problem 1**

Write in polar coordinates equation for line $y = x - 4$ and plot the graph in polar coordinates.

**Solution**

General form for this line is:

$$x - y - 4 = 0$$

Polar equation of this line solved for $r$ is

$$r = \frac{4}{\cos\varphi + sin\varphi}$$

To sketch the graph of a polar equation we use a different strategy based on the equation given. To sketch the graph of a line in polar form, we can create a table of values just like we did with the rectangular counterpart. We would plug in for $\varphi$ and get a value for $r$.

Alternatively, we can grab points in rectangular form and convert them over to polar form.

If we think about the equation in rectangular form, notice that the $y$-intercept would occur at $B(0, -4)$ on the rectangular plane. If we think about this with polar coordinates, this would be $(4; 3\pi/2)$.

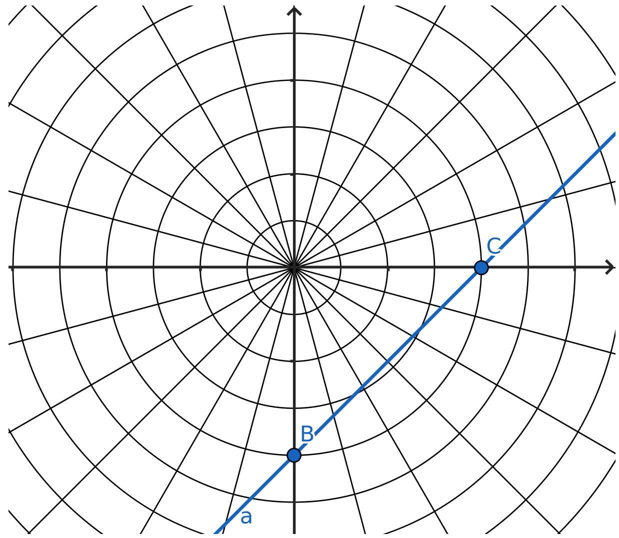


Figure 5: Line $x - y - 4 = 0$ plotted in polar coordinates

Additionally, our $x$-intercept would occur at $(4, 0)$ on the rectangular plane. If we think about this with polar coordinates, this would be $(4; 0)$.

**Problem 2**

Plot in polar coordinates' locus of equation

$$\varphi = \frac{2\pi}{3},$$

and find the corresponding equation in Cartesian coordinates.

**Solution**

Written equation says us that for any length of radius vector it shapes constant angle $2pi/3$ with $Ox$.

Thus, this is a line passing through the origin and inclination of the line is $2pi/3$.

In polar coordinates it line constructed with ray $Oa$ shaping angle $2\pi/3$ with $Ox$, and ray $Oa'$ shaping angle $5\pi/3$ with $Ox$.

Knowing inclination of the line we calculate its slope:

$$m = \tan\frac{2\pi}{3} = -\sqrt{3},$$

and write equation of the line in Cartesian coordinates:

$$y = -x\sqrt{3}$$

**Definition.** Line passing through the origin has in polar coordinates equation in form

$$\varphi = \varphi_0,$$

here $\varphi_0$ is inclination of this line.

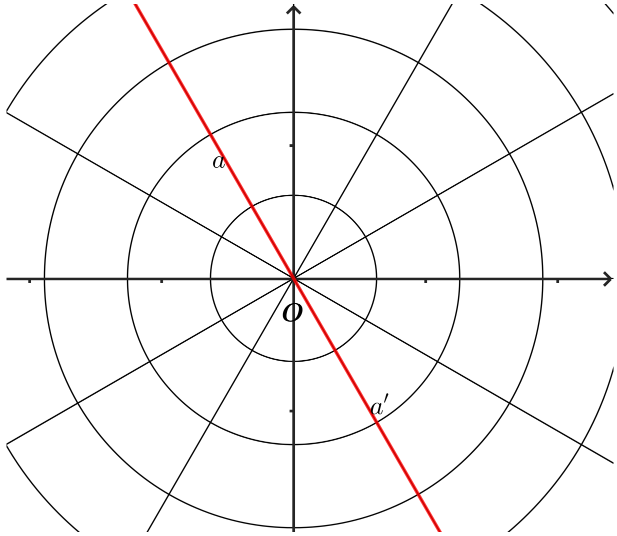


Figure 6: Line passing through the origin in polar coordinates

# 6 Horizontal and vertical lines in polar form

We will also come across horizontal and vertical lines in polar form. Let's begin with a horizontal line.

$$y = b$$

Substitute $y = r\sin\varphi$:

$$r\sin\varphi = b$$

$$r = \frac{b}{sin\varphi}$$

$$r = b\csc\varphi$$

Let's now think about a vertical line.

$$y = a$$

Substitute $x = r\cos\varphi$

$$r\cos\varphi = a$$

$$r = \frac{a}{\cos\varphi}$$

$$r = a\sec\varphi$$

# 7 Normal equation of the line in polar coordinates

Normal coordinate equation of line

$$x\cos\omega + y\sin\omega - p = 0,$$

expresses line with length of the perpendicular established to the line from origin $p$,and angle this perpendicular shapes with positive direction of axis $Ox$

Substituting expressions of polar coordinates with Cartesian

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

yields us with this equation

$$r\cos\varphi\cos\omega + r\sin\varphi\sin\omega - p = 0,$$

With recap formula for cosines of difference of two angles we yield **normal equation of the linein polar coordinates**:

$$r\cos(\varphi - \omega) = p. \tag{5}$$

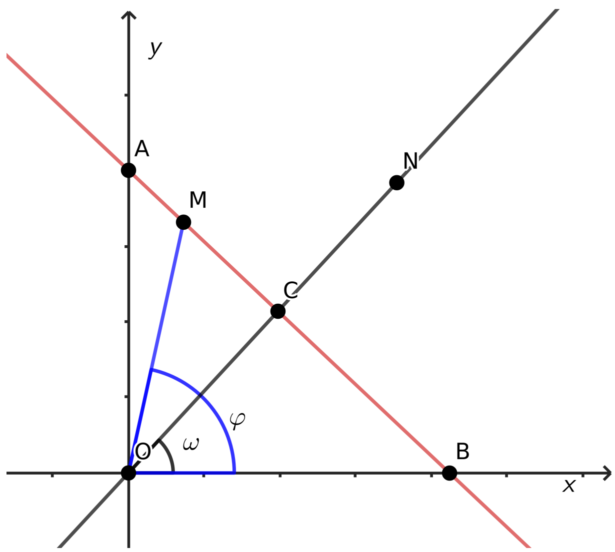


Figure 7: Normal equation of the line in polar coordinates

Any point $M$ of the line in question distant form the base of origin-line perpendicular $C$ is a hypotenuse in right-angled triangle $OCM$, and $\angle MOC = \varphi - \omega$. Thus, we're *projecting any point of the line onto point* $C$.

# 8 Examples of polar equation of lines

**Problem 3**

Sketch the given equation on the polar grid and find the corresponding equation in Cartesian coordinates.

$$r = \csc(\varphi + \frac{\pi}{4})$$

**Solution**

Let's begin by writing cosecant in terms of sine ($\csc\alpha = \sin^- 1\alpha$):

$$r = \frac{1}{\sin(\varphi + \frac{\pi}{4})}$$

With multiplying both sides by $\sin(\varphi + \frac{\pi}{4})$ we yield

$$r\sin(\varphi + \frac{\pi}{4}) = 1$$

With formula $\cos(\frac{\pi}{2} + \alpha) = -\sin(\alpha)$ we rewrite:

$$r\cos(\varphi + \frac{3\pi}{4}) = -1,$$

or

$$r\cos(\varphi - \frac{\pi}{4}) = 1.$$

Hence, we derived normal equation of the line in polar coordinates. To sketch this line we need just build segment of length 1 shaping angle $\frac{\pi}{4}$ with positive $Ox$ direction, and build line perpendicular to it.

Or we easily can find intercepts $(\sqrt{2}; 0)$, and $(\sqrt{2}; \frac{\pi}{2})$.

Normal equation of this line in Cartesian coordinates is

$$x\cos\frac{\pi}{4} + y\sin\frac{\pi}{4} - 1 = 0$$

or

$$x\frac{1}{\sqrt{2}} + y\frac{1}{\sqrt{2}} - 1 = 0$$

$$x + y - \sqrt{2} = 0$$

**Problem 4**

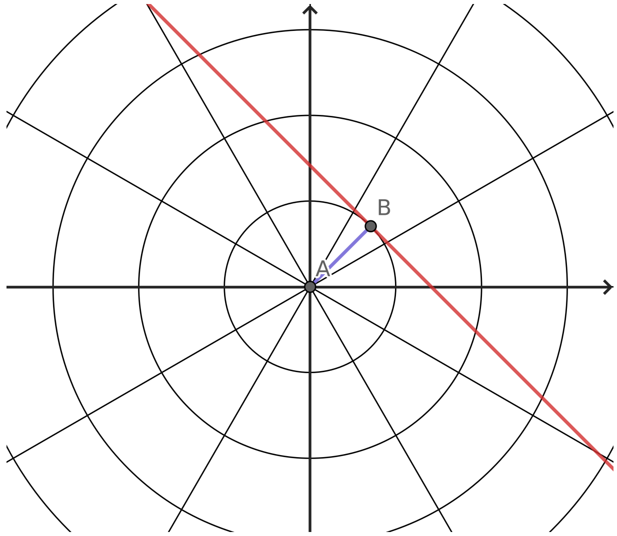


Figure 8: Line expressed with equation $r\cos(\varphi - \frac{\pi}{4}) = 1$

Express equation of the line in polar coordinate for given inclination $\beta$ and length of perpendicular to line from origin

**Solution**
There is two cases:

1. $\beta < \frac{\pi}{2}$ Thus, $\beta$ itself for positive $p$, or angle opposite with $\beta$ is the opposite angle in $\triangle OMB$ ($O$ is origin, $B$ is intercept with axis, and $M$ is endpoint of perpendicular from the origin to line). Inclination of $OM$ is $\frac{pi}{2} - \beta$

   Equation is

$$r\sin(\varphi = \beta) = p$$

2. $\beta > \frac{\pi}{2}$ Thus, $\pi - \beta$ itself for positive $p$, or angle opposite with it is the opposite angle in $\triangle OMB$ ($O$ is origin, $B$ is intercept with axis, and $M$ is endpoint of perpendicular from the origin to line). Inclination of $OM$ is $\frac{pi}{2} - \pi + \beta = \beta - \frac{pi}{2}$

$$r\sin(\beta - \varphi) = -p$$