Real Analysis 2024. Homework 3.

1. Consider a function $f : [0,1] \to [0,1]$ defined as following:

$$f(x) = \begin{cases} x + 1/2, & 0 \le x < 1/2; \\ x - 1/2, & 1/2 \le x \le 1. \end{cases}$$

Find the image of Lebesgue measure with respect to this transform.

Solution. Let $E \subset [0,1]$ be measurable. Then

$$f^{-1}(E) = A_1 \cup A_2,$$

where

$$A_1 = \{x + 1/2 : x \in E \cap [0, 1/2]\}$$

and

$$A_2 = \{x - 1/2 : x \in E \cap [1/2, 1)\}.$$

Since $\mu(A_1) = \mu(E \cap [0, 1/2])$ and $\mu(A_2) = \mu(E \cap [1/2, 1))$. Consequently,

$$f_*(\mu)(E) = \mu(f^{-1}(E)) = \mu(A_1) + \mu(A_2) = \mu(E \cap [0, 1/2]) + \mu(E \cap [1/2, 1)) = \mu(E)$$

and Lebesgue measure is preserved under map f.

2. Consider a measure defined as linear combination of Dirac measures

$$\mu = \delta_0 + \delta_1 + 2\delta_2$$

on $\mathcal{A}_{max}$ on $\mathbb{R}$. Consider a function $f(x) = x^2$. Prove that it is measurable map with respect to $\sigma-$algebra $\mathcal{A}_{max}$. Find $f_*(\mu)$.

Solution.

$$f_*(\mu)(E) = \delta_0(f^{-1}(E)) + \delta_1(f^{-1}(E)) + 2\delta_2(f^{-1}(E)) =$$

$$\begin{cases} 0, & 0,1,2 \notin f^{-1}(E); \\ 1, & 0 \in f^{-1}(E),\ 1,2 \notin f^{-1}(E); \\ 1, & 1 \in f^{-1}(E),\ 0,2 \notin f^{-1}(E); \\ 2, & 0,1 \in f^{-1}(E),\ 2 \notin f^{-1}(E); \\ 2, & 2 \in f^{-1}(E),\ 0,1 \notin f^{-1}(E); \\ 4, & 0,1,2 \in f^{-1}(E) \end{cases} =$$

$$\begin{cases} 0, & 0,1,4 \notin E; \\ 1, & 0 \in E,\ 1,4 \notin E; \\ 1, & 1 \in E,\ 0,4 \notin E; \\ 2, & 0,1 \in E,\ 4 \notin E; \\ 2, & 4 \in E,\ 0,1 \notin E; \\ 4, & 0,1,4 \in E \end{cases}$$

Consequently,

$$f_*(\mu) = \delta_0 + \delta_1 + 2\delta_4.$$

3. Calculate the integral of function $f(x) = x^3$ on $[0, +\infty)$ with respect measure

$$\mu = \frac{\delta_1 + 2\delta_2 + 3\delta_3}{6}.$$

Solution.

$$\int_{[0,+\infty)} f(x) d\mu = \frac{1}{6}(f(1) + 2f(2) + 3f(3)) = \frac{1}{6}(1 + 16 + 81) = \frac{49}{3}.$$

4. Explain why the counting measure on $\mathbb{R}$ is not $\sigma-$finite.

Solution. The counting measure ν of a set $A \subset \mathbb{R}$ is finite if and only if A is finite. Consequently, if

$$E = \bigcup_{k=1}^{\infty} A_k, \quad \nu(A_k) < \infty,$$

then E is at most countable, while $\mathbb{R}$ is uncountable.

5. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let E_j, $1 \leq j \leq k$, be measurable sets. Prove that

$$\sum_{i=1}^{k} \mu E_i \leqslant \mu \left(\bigcup_{i=1}^{k} E_i \right) + \sum_{i<j} \mu \left(E_i \cap E_j \right).$$

Proof. First, by additivity of a measure

$$\mu E_1 + \mu E_2 = \mu E_1 + \mu \left(E_2 \backslash E_1 \right) + \mu \left(E_1 \cap E_2 \right) = \mu \left(E_1 \cup E_2 \right) + \mu \left(E_1 \cap E_2 \right).$$

We will prove that the identity by induction over k. Let

$$\sum_{i=1}^{k} \mu E_i \leqslant \mu \left(\bigcup_{i=1}^{k} E_i \right) + \sum_{1 \leq i < j \leq k} m \left(E_i \cap E_j \right).$$

Then

$$\begin{aligned}
\sum_{i=1}^{k+1} \mu E_i = \sum_{i=1}^{k} \mu E_i + \mu E_{k+1} \leqslant \mu \left(\bigcup_{i=1}^{k} E_i \right) + \sum_{1<i<j<k} \mu \left(E_i \cap E_j \right) + \mu E_{k+1} = \\
= \mu \left(\bigcup_{i=1}^{k+1} E_i \right) + \mu \left(\left(\bigcup_{i=1}^{k} E_i \right) \cap E_{k+1} \right) + \sum_{1<i<j \leqslant k} \mu \left(E_i \cap E_j \right) \leqslant \\
\leqslant \mu \left(\bigcup_{i=1}^{k+1} E_i \right) + \sum_{1 \leqslant i < k} \mu \left(E_i \cap E_{k+1} \right) + \sum_{1<i<j<k} \mu \left(E_i \cap E_j \right) = \\
= \mu \left(\bigcup_{i=1}^{k+1} E_i \right) + \sum_{1<i<j \leqslant k+1} \mu \left(E_i \cap E_j \right).
\end{aligned}$$

$\square$