1. $\iint_S \frac{x^2 sinxy}{y} dxdy$, where $S = \{(x,y) \in R^2 : ay \leqslant x^2 \leqslant by,\ px \leqslant y^2 \leqslant qx,\ x > 0,\ y > 0\}$

Solution: Let $\begin{cases} u = \dfrac{x^2}{y} \\ v = \dfrac{y^2}{x} \end{cases}$, then

$$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{2x}{y} & -\dfrac{x^2}{y^2} \\ -\dfrac{y^2}{x^2} & \dfrac{2y}{x} \end{vmatrix} = 3, \quad \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{3}.$$

$S = \{(u,v) | a \leqslant u \leqslant b,\ p \leqslant v \leqslant q\}.$

$$\iint_S \frac{x^2 \sin(xy)}{y} \mathrm{d}x\,\mathrm{d}y$$
$$= \iint_S u \sin(uv) \cdot \frac{1}{3} \mathrm{d}u\,\mathrm{d}v = \frac{1}{3}\int_a^b \mathrm{d}u \int_p^q u \sin(uv)\,\mathrm{d}v$$
$$= \frac{1}{3}\int_a^b \left[-\cos(uv)\Big|_p^q\right] \mathrm{d}u = \frac{1}{3}\int_a^b [\cos(pu) - \cos(qu)]\,\mathrm{d}u$$
$$= \frac{1}{3}\left[\frac{\sin(pb) - \sin(pa)}{p} - \frac{\sin(qb) - \sin(qa)}{q}\right]$$

2. $\iiint_V xdxdydz$, where $V = \{(x,y,z) \in R^3 : o \leqslant y \leqslant h,\ x + z \leqslant a,\ 0 \leqslant x,\ 0 \leqslant z\}$

Solution:

$$\iiint_V x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_0^a x\,\mathrm{d}x \int_0^{a-x} \mathrm{d}z \int_0^h \mathrm{d}y = h\int_0^a x(a-x)\,\mathrm{d}x = h\left[\left(\frac{a}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^a\right] = \frac{1}{6}a^3 h$$

3. Calculate the Curve Intergal of a first kind $\int_L (x^3+y^3)dl$, where $L=\{(x,y):(x^2+y^2)^2=2a^2xy,\ 0\leqslant x,\ 0\leqslant y\}$

Solution: $L=\left\{(\theta,r)\,\middle|\,0\leqslant\theta\leqslant\frac{\pi}{2},r=a\sqrt{\sin 2\theta}\right\}$, $r(\theta)=a\sqrt{\sin 2\theta}$,

$$\int_L (x^3+y^3)\mathrm{d}l$$

$$=\int_0^{\frac{\pi}{2}}[r^3(\theta)\cos^3\theta+r^3(\theta)\sin^3\theta]\sqrt{r^2(\theta)+r'^2(\theta)}\,\mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}a^3\sin^{\frac{3}{2}}2\theta(\cos^3\theta+\sin^3\theta)\sqrt{a^2\sin 2\theta+a^2\frac{\cos^2 2\theta}{\sin 2\theta}}\,\mathrm{d}\theta$$

$$=a^4\int_0^{\frac{\pi}{2}}\sin 2\theta(\cos^3\theta+\sin^3\theta)\mathrm{d}\theta$$

$$=2a^4\left(\int_0^{\frac{\pi}{2}}\sin\theta\cos^4\theta\,\mathrm{d}\theta+\int_0^{\frac{\pi}{2}}\sin^4\theta\cos\theta\,\mathrm{d}\theta\right)$$

$$=2a^4\left(\frac{1}{5}+\frac{1}{5}\right)=\frac{4}{5}a^4$$

4. Calculate the Curve Intergal of a second kind $\int_L(-x^2y dx+xy^2dy)$, where $L=\{(x,y):x^2+y^2=r^2\}$ with positive direction

Solution: (Method I) $D=\{(x,y)|x^2+y^2\leqslant r^2\}$, by Green's formula,

$$\int_L -x^2y\,\mathrm{d}x+xy^2\mathrm{d}y=$$

$$=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y=\iint_D(y^2+x^2)\mathrm{d}x\,\mathrm{d}y$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^r\rho^3\mathrm{d}\rho=2\pi\cdot\frac{1}{4}r^4=\frac{\pi}{2}r^4$$

(Method II) $\vec{\boldsymbol{n}}=\{x,y\}$, $\vec{\boldsymbol{t}}=\{-y,x\}$, $\vec{\boldsymbol{t}}^{\,0}=\left\{-\frac{y}{r},\frac{x}{r}\right\}$, i.e., $-\frac{y}{r}=\cos\alpha$,

$$\frac{x}{r} = \sin\alpha,$$

$$\int_L - x^2 y\,\mathrm{d}x + xy^2\,\mathrm{d}y$$

$$= \int_L [- x^2 y\cos\alpha + xy^2\sin\alpha]\,\mathrm{d}s = \frac{2}{r}\int_L x^2y^2\,\mathrm{d}s$$

$$= \frac{2}{r}\int_0^{2\pi} r^4\cos^2\theta\sin^2\theta\cdot r\,\mathrm{d}\theta = 2r^4\int_0^{2\pi}\cos^2\theta\sin^2\theta\,\mathrm{d}\theta$$

$$= \frac{r^4}{2}\int_0^{2\pi}\sin^2 2\theta\,\mathrm{d}\theta = \frac{r^4}{4}\int_0^{2\pi}(1-\cos 4\theta)\,\mathrm{d}\theta = \frac{r^4}{4}(2\pi - 0) = \frac{\pi}{2}r^4$$

5. Calculate the Surface Intergal of a first kind $\iint_S (x^2+y^2)ds$, where S - the border of body $V = \{(x,y,z) : \sqrt{x^2+y^2} \leqslant z \leqslant 1\}$

Solution: $D = \{(x,y) | x^2+y^2 \leqslant 1\}$,

$S_1 = \left\{(x,y,z) | z = \sqrt{x^2+y^2}, (x,y) \in D\right\}$,

$S_2 = \{(x,y,z) | z = 1, (x,y) \in D\}$.

$$\iint_S (x^2+y^2)\,\mathrm{d}S = \iint_{S_1}(x^2+y^2)\,\mathrm{d}S + \iint_{S_2}(x^2+y^2)\,\mathrm{d}S.$$

On S_1, $z = \sqrt{x^2+y^2}$, $\dfrac{\partial z}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$, $\dfrac{\partial z}{\partial y} = \dfrac{y}{\sqrt{x^2+y^2}}$,

$$\mathrm{d}S = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\,\mathrm{d}x\,\mathrm{d}y,$$

$$\iint_{S_1}(x^2+y^2)\,\mathrm{d}S$$

$$= \iint_D (x^2+y^2)\sqrt{2}\,\mathrm{d}x\,\mathrm{d}y = \sqrt{2}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\,\mathrm{d}r = \sqrt{2}\cdot 2\pi\cdot\frac{1}{4} = \frac{\pi}{\sqrt{2}}$$

On S_2, $z = 1$, $\dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial y} = 0$,

$$\mathrm{d}S=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y=\mathrm{d}x\,\mathrm{d}y\,,$$

$$\iint_{S_2}(x^2+y^2)\mathrm{d}S$$

$$=\iint_D(x^2+y^2)\mathrm{d}x\,\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^1 r^3\mathrm{d}r=2\pi\cdot\frac{1}{4}=\frac{\pi}{2}$$

Hence, $\iint_S(x^2+y^2)\mathrm{d}S=\frac{\pi}{\sqrt{2}}+\frac{\pi}{2}$.

6. Calculate the Surface Intergal of a second kind $\iint_S(y^2+z^2)dxdy$, where S - upper part of $z=\sqrt{a^2-x^2}$ $(0\leqslant y\leqslant b)$

Solution: (Method I) $D_{xy}=\{(x,y)|-a\leqslant x\leqslant a,0\leqslant y\leqslant b\}$,

$$\iint_S(y^2+z^2)\mathrm{d}x\,\mathrm{d}y$$

$$=\iint_{D_{xy}}(y^2+a^2-x^2)\mathrm{d}x\,\mathrm{d}y$$

$$=2\int_0^a\mathrm{d}x\int_0^b y^2\mathrm{d}y+a^2\cdot 2a\cdot b-2\int_0^a x^2\mathrm{d}x\int_0^b\mathrm{d}y$$

$$=2\cdot a\cdot\frac{1}{3}b^3+2a^3b-2\cdot\frac{1}{3}a^3\cdot b=\frac{2}{3}ab^3+\frac{4}{3}a^3b$$

(Method II) $S_1=\left\{(x,y,z)|y=0,0\leqslant z\leqslant\sqrt{a^2-x^2}\right\}$, leftward;

$S_2=\left\{(x,y,z)|y=b,0\leqslant z\leqslant\sqrt{a^2-x^2}\right\}$, rightward;

$S_3=\{(x,y,z)|z=0,-a\leqslant x\leqslant a,0\leqslant y\leqslant b\}$, downward.

$$\iint_S(y^2+z^2)\mathrm{d}x\,\mathrm{d}y$$

$$=\oiint_{S+S_1+S_2+S_3}(y^2+z^2)\mathrm{d}x\,\mathrm{d}y-\sum_{i=1}^{3}\iint_{S_i}(y^2+z^2)\mathrm{d}x\,\mathrm{d}y$$

Set $V=\left\{(x,y,z)|0\leqslant z\leqslant\sqrt{a^2-x^2},0\leqslant y\leqslant b\right\}$, by Gauss' formula,

$$\oiint_{S+S_1+S_2+S_3}(y^2+z^2)\mathrm{d}x\,\mathrm{d}y$$

$$=\iiint_V 2z\,\mathrm{d}V=2\int_0^b \mathrm{d}y\iint_{\sigma_y} z\,\mathrm{d}x\,\mathrm{d}z$$

$$=2b\cdot 2\int_0^{\frac{\pi}{2}}\sin\theta\,\mathrm{d}\theta\int_0^a r^2\mathrm{d}r=4b\cdot 1\cdot\frac{a^3}{3}=\frac{4}{3}a^3b$$

$$\iint_{S_1}(y^2+z^2)\mathrm{d}x\,\mathrm{d}y=\iint_{S_2}(y^2+z^2)\mathrm{d}x\,\mathrm{d}y=0,$$

$$\iint_{S_3}(y^2+z^2)\mathrm{d}x\,\mathrm{d}y$$

$$=-\iint_{D_{xy}}(y^2+0^2)\mathrm{d}x\,\mathrm{d}y=-2\int_0^a \mathrm{d}x\int_0^b y^2\mathrm{d}y$$

$$=(-2)\cdot a\cdot\frac{1}{3}b^3=\left(-\frac{2}{3}\right)ab^3$$

Thus, $\iint_S(y^2+z^2)\mathrm{d}x\,\mathrm{d}y=\frac{4}{3}a^3b+\frac{2}{3}ab^3.$

7. Using Green's formula, calculate the curvilinear integral over a closed curve $\int_L xdy+ydx$, L - part of the curve $y=\begin{cases}x^2 sin\frac{1}{x}+\frac{4}{\pi^2}, x\neq 0\\ \frac{4}{\pi^2}, x=0\end{cases}$ from point $A(0;\frac{4}{\pi^2})$ to point $B(\frac{2}{\pi};\frac{8}{\pi^2})$

Solution: Set $C\left(\frac{2}{\pi},\frac{4}{\pi^2}\right)$, since $\frac{\partial Q}{\partial x}=1=\frac{\partial P}{\partial y}$,

$$\int_L x\,\mathrm{d}y + y\,\mathrm{d}x$$

$$= \int_{\overrightarrow{AC}} \frac{4}{\pi^2}\,\mathrm{d}x + \int_{\overrightarrow{CB}} \frac{2}{\pi}\,\mathrm{d}y = \int_0^{\frac{2}{\pi}} \frac{4}{\pi^2}\,\mathrm{d}x + \int_{\frac{4}{\pi^2}}^{\frac{8}{\pi^2}} \frac{2}{\pi}\,\mathrm{d}y$$

$$= \frac{4}{\pi^2}\left(\frac{2}{\pi} - 0\right) + \frac{2}{\pi}\left(\frac{8}{\pi^2} - \frac{4}{\pi^2}\right) = \frac{16}{\pi^3}$$

8. Using Stokes formula, calculate the integral $\int_C (z - x^2 - y)dx + (x + y + z)dy + (y + 2x + z^3)dz$, where C - curve of intersection of cone $y^2 + z^2 = x^2$, $0 \leqslant x$ and hemisphere $x^2 + y^2 + z^2 = 2az$, positive oriented on outer side of right hemisphere $(0 \leqslant x)$

Solution: $\begin{cases} y^2 + z^2 = x^2 \\ x^2 + y^2 + z^2 = 2az \end{cases} \Rightarrow y^2 + z^2 = 2az - y^2 - z^2,$

$$\frac{y^2}{\frac{a^2}{4}} + \frac{\left(z - \frac{a}{2}\right)^2}{\frac{a^2}{4}} = 1, \text{ set } D_{yz} = \left\{(y,z)\,\middle|\, \frac{y^2}{\frac{a^2}{4}} + \frac{\left(z - \frac{a}{2}\right)^2}{\frac{a^2}{4}} \leqslant 1\right\},$$

$S = \left\{(x,y,z)\,|\,x = \sqrt{y^2 + z^2}, (y,z) \in D_{yz}\right\}$. By Stokes formula,

$$\int_C (z - x^2 - y)\,\mathrm{d}x + (x + y + z)\,\mathrm{d}y + (y + 2x + z^3)\,\mathrm{d}z$$

$$= \iint_S \begin{vmatrix} \mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z - x^2 - y & x + y + z & y + 2x + z^3 \end{vmatrix}$$

$$= \iint_S (-1)\,\mathrm{d}z\,\mathrm{d}x + 2\,\mathrm{d}x\,\mathrm{d}y$$

On S, $x = \sqrt{y^2 + z^2}$, $\dfrac{\partial x}{\partial y} = \dfrac{y}{\sqrt{y^2 + z^2}}$, $\dfrac{\partial x}{\partial z} = \dfrac{z}{\sqrt{y^2 + z^2}}$,

$$\iint_S (-1)\,\mathrm{d}z\,\mathrm{d}x + 2\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint_{D_{yz}} \left[(-1)\left(-\frac{\partial x}{\partial y}\right) + 2\left(-\frac{\partial x}{\partial z}\right)\right]\mathrm{d}y\,\mathrm{d}z = \iint_{D_{yz}} \left(\frac{y}{\sqrt{y^2+z^2}} - 2\frac{z}{\sqrt{y^2+z^2}} \right)\mathrm{d}y\,\mathrm{d}z$$

$$= -2\iint_{D_{yz}} \frac{z}{\sqrt{y^2+z^2}}\,\mathrm{d}y\,\mathrm{d}z = -4\int_0^{\frac{\pi}{2}} \sin\theta\,\mathrm{d}\theta \int_0^{a\sin\theta} r\,\mathrm{d}r$$

$$= -2a^2 \int_0^{\frac{\pi}{2}} \sin^3\theta\,\mathrm{d}\theta = (-2a^2)\,\frac{2}{3} = -\frac{4}{3}a^2$$

$$-2 \iint = -8 \int_0^{\frac{\pi}{2}} d\theta \int$$