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Example 2

Find a solution to the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\ 0<x<l,\ t>0,$$

satisfying the initial conditions

$$u(x,0)=\varphi(x)=\begin{cases} x, & 0<x\le l/2, \\ l-x, & l/2\le x<l \end{cases}$$

and the boundary conditions

$$u(0,t)=u(l,t)=0.$$

<u>Solution:</u>

Coefficients a_k :

$$a_k=\frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi=\frac{2}{l}\int_0^{l/2}\xi\sin\frac{k\pi}{l}\xi d\xi+\frac{2}{l}\int_{l/2}^{l}(l-\xi)\sin\frac{k\pi}{l}\xi d\xi.$$

We integrate by parts, assuming

$$u=\xi,$$

$$dv=\sin\frac{k\pi}{l}\xi d\xi,$$

$$du=d\xi,$$

$$v = -\frac{l}{k\pi}\cos\frac{k\pi}{l}\xi d\xi\,;$$

we get

$$a_k = \frac{2}{l}\left(-\frac{l\xi}{k\pi}\cos\frac{k\pi}{l}\xi + \frac{l^2}{k^2\pi^2}\sin\frac{k\pi}{l}\xi\right)\Bigg|_0^{l/2} +$$

$$+\frac{2}{l}\left(-\frac{l^2}{k\pi}\cos\frac{k\pi}{l}\xi + \frac{l\xi}{k\pi}\cos\frac{k\pi}{l}\xi - \frac{l^2}{k^2\pi^2}\sin\frac{k\pi}{l}\xi\right)\Bigg|_{l/2}^{l} = \frac{4l}{k^2\pi^2}\sin\frac{k\pi}{2}\,.$$

Therefore, the desired solution according to the formula (4.5) has the form

$$u(x,t) = \frac{4l}{\pi^2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}e^{-\frac{(2n+1)^2\pi^2}{l^2}t}\sin\frac{(2n+1)\pi}{l}x\,.$$

Example 3

Find the temperature distribution in a rod of length l with a thermally insulated side surface if the temperature of its ends is kept equal to zero, and the initial temperature is set by the function $\varphi(x)$. Consider the case when

$\varphi(x) = Ax(l-x)/l^2$

<u>Solution:</u>

$$\begin{cases} u_t = a^2 u_{xx} \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \dfrac{Ax(l-x)}{l^2} \end{cases}$$

We will look for a solution in the form of:

$$u(x,t) = X(x)T(t)$$

$$X(x)T'(t) = a^2 X''(x)T(t)$$

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Consider the equation:

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$X''(x) + \lambda X(x) = 0$$

$$X(0) = 0$$

$$X(l) = 0$$

This is the task of Sturm-Liouville theory.

$$X_n(x) = A_n \sin\left(\frac{\pi n}{l} x\right)$$

$$\lambda = \left(\frac{\pi n}{l}\right)^2$$

Consider the equation:

$$\frac{T'(t)}{a^2 T(t)} = -\lambda$$

$$T'(t) + a^2 \lambda T(t) = 0$$

$$T'(t) = -\left(\frac{\pi n a}{l}\right)^2 T(t)$$

$$\frac{dT}{dt} = -\left(\frac{\pi n a}{l}\right)^2 T$$

$$\frac{dT}{T} = -\left(\frac{\pi n a}{l}\right)^2 dt$$

Integrating the left and right parts:

$$ln|T| = -\left(\frac{\pi n a}{l}\right)^2 t + C$$

$$T(t) = B_n e^{-\left(\frac{\pi n a}{l}\right)^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-\left(\frac{\pi n a}{l}\right)^2 t} \sin\left(\frac{\pi n}{l} x\right) \qquad \textbf{(*)}$$

Initial conditions:

$$u\left(x,0\right)=\sum_{n=1}^{\infty}\alpha_n \sin\left(\frac{\pi n}{l}x\right)=\frac{A}{l^2}x\left(l-x\right)$$

The initial condition is decomposed into a Fourier series in terms of sines:

Multiply by a sine $\sin\left(\frac{\pi m}{l}x\right)$, integrate from zero to l:

$$\sum_{n=1}^{\infty}\alpha_n\int_0^l \sin\left(\frac{\pi n}{l}x\right)\sin\left(\frac{\pi m}{l}x\right)dx=\frac{A}{l^2}\int_0^l x\left(l-x\right)\sin\left(\frac{\pi m}{l}x\right)dx$$

The integral on the left side we know what it is equal to. It is equal to zero when m is not equal to n, and is equal to $\frac{l}{2}$ when $m=n$.

$$\alpha_m\frac{l}{2}=\frac{A}{l^2}\int_0^l x\left(l-x\right)\sin\left(\frac{\pi m}{l}x\right)dx$$

In the right part, we put the sine under the differential, and open the brackets.

$$\alpha_m\frac{l}{2}=-\frac{A}{l^2}\frac{l}{\pi m}\int_0^l\left(lx-x^2\right)d\left(\cos\frac{\pi m}{l}x\right)=$$

integrate by parts

$$=-\frac{A}{l\pi m}\left(\left(lx-x^2\right)\cos\left(\frac{\pi m}{l}x\right)\Big|_0^l-\int_0^l\cos\left(\frac{\pi m}{l}x\right)\left(l-2x\right)dx\right)=$$

when substituting, the first term is zero

$$=\frac{A}{l\pi m}\cdot\frac{l}{\pi m}\int_0^l (l-2x)\,d\,sin\left(\frac{\pi m}{l}x\right)=$$

integration by parts again

$$=\frac{A}{(\pi m)^2}\left(\left.(l-2x)sin\left(\frac{\pi m}{l}x\right)\right|_0^l+2\int_0^l sin\left(\frac{\pi m}{l}x\right)dx\right)=$$

when substituting, the first term is zero

$$=-\frac{2A}{(\pi m)^2}\frac{l}{\pi m}\left.cos\left(\frac{\pi m}{l}x\right)\right|_0^l=-\frac{2Al}{(\pi m)^3}\left((-1)^m-1\right)$$

We get:

$$\alpha_m\frac{l}{2}=-\frac{2Al}{(\pi m)^3}\left((-1)^m-1\right)$$

Let's express α_m, then

$$\alpha_m=\frac{4A}{(\pi m)^3}\left(1-(-1)^m\right)$$

The general view of the solution is (*)

$$u(x,t)=\sum_{n=1}^{\infty}\alpha_n e^{-\left(\frac{\pi na}{l}\right)^2 t}sin\left(\frac{\pi n}{l}x\right)$$

Answer: (We change m to n, and substitute α_m in (*)) and we have:

$$u(x,t)=\sum_{n=1}^{\infty}\frac{4A}{(\pi n)^3}\left(1-(-1)^n\right)e^{-\left(\frac{\pi na}{l}\right)^2 t}\sin\left(\frac{\pi n}{l}x\right)$$

We can simplify the solution:

At $n=2k+1$:

$$u(x,t)=\sum_{k=0}^{\infty}\frac{8A}{\left(\pi(2k+1)\right)^3}e^{-\left(\frac{\pi a(2k+1)}{l}\right)^2 t}\sin\left(\frac{\pi(2k+1)}{l}x\right)$$

(When t tending to infinity, the temperature of the rod tends to zero.)

4.2.2. THE INHOMOGENEOUS EQUATION OF THERMAL CONDUCTIVITY

Consider the inhomogeneous equation of thermal conductivity (heat equation):

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+f(x,t)\,,\ 0<x<l\,,\ t>0 \qquad (4.6)$$

with an initial condition

$$u(x,0)=0\,, \qquad (4.7)$$

and boundary conditions

$$u(0,t)=0\,,\ u(l,t)=0\,,\ t\geq 0\,.$$

We will look for a solution to this problem in the form of a Fourier series of functions $\left\{\sin\frac{k\pi}{l}x\right\}$:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)\sin\frac{k\pi}{l}x\,, \tag{4.8}$$

while considering t as a parameter.

Let's imagine the function $f(x,t)$ as a Fourier series:

$$f(x,t)=\sum_{k=1}^{\infty}f_k(t)\sin\frac{k\pi}{l}x\,,\ f_k(t)=\frac{2}{l}\int_0^l f(\xi,t)\sin\frac{k\pi}{l}\xi d\xi\,. \tag{4.9}$$

Substituting the series (4.8) and (4.9) into the original equation (4.6), we will have

$$\sum_{k=1}^{\infty}\left[\left(\frac{ak\pi}{l}\right)^2u_k(t)+\frac{du_k(t)}{dt}-f_k(t)\right]\sin\frac{k\pi}{l}x=0\,.$$

This equation will be satisfied if all the expansion coefficients are zero, that is

$$\frac{du_k(t)}{dt}=-\left(\frac{ak\pi}{l}\right)^2u_k(t)+f_k(t)\,. \tag{4.10}$$

To determine $u_k(t)$, we obtained an ordinary differential equation with constant coefficients. Further, the initial conditions (4.7) give

$$u(x,0)=\sum_{k=1}^{\infty}u_k(0)\sin\frac{k\pi}{l}x=0\,,$$

therefore,

$$u_k(0) = 0. \tag{4.11}$$

The condition (4.11) completely determines the solution (4.10), namely

$$u_k(t) = \int_0^t e^{-\left(\frac{ak\pi}{l}\right)^2 (t-\tau)} f_k(\tau)\, d\tau. \tag{4.12}$$

Thus, the solution of the initial problem according to the formulas (4.8) and (4.12) will be written as

$$u(x,t) = \sum_{k=1}^{\infty} \int_0^t e^{-\left(\frac{ak\pi}{l}\right)^2 (t-\tau)} f_k(\tau)\, d\tau \sin\frac{k\pi}{l} x. \tag{4.13}$$

Further, using the expression (4.9) for $f_k(t)$, the found solution (4.13) can be represented using the instantaneous point source function $G(x,\xi,t)$ as follows:

$$u(x,t) = \int_0^t \int_0^l G(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau.$$

4.3. PROBLEMS ON AN INFINITE LINE FOR THE EQUATION OF THERMAL CONDUCTIVITY

4.3.1. CAUCHY PROBLEM

Let's consider a problem with initial data on an infinite line (Cauchy problem): find the function $u(x,t)$ $(t>0, -\infty<x<\infty)$ satisfying the equation of thermal conductivity (heat equation):

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (4.14)$$

and the initial condition

$$u(x,0) = \varphi(x), -\infty < x < \infty \qquad (4.15)$$

where $\varphi(x)-$ is a continuous and bounded function.

Example 1

$$\begin{cases} u_t = a^2 u_{xx} \\ u(x,0) = \varphi(x) \end{cases} \qquad \begin{pmatrix} -\infty < x < \infty \\ 0 < t < +\infty \end{pmatrix}$$

Solution:

$$u_t = a^2 u_{xx}$$

$$u(x,t) = X(x)T(t)$$

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

There is an infinite line here. There are no boundary conditions. This is the main difference. We write λ^2 for convenience.

$$X'' + \lambda^2 X = 0$$

We know the general solution.

$$X(x) = A\sin(\lambda x) + B\cos(\lambda x) = C_\lambda e^{i\lambda x} + D_\lambda e^{-i\lambda x}$$

$$T' + a^2\lambda^2 T = 0$$

$$T(t) = e^{-a^2\lambda^2 t}$$

$$u_\lambda(x,t) = e^{-a^2\lambda^2 t}\left(C_\lambda e^{i\lambda x} + D_\lambda e^{-i\lambda x}\right)$$

If we knew which kind of λ, then this would be the solution.

λ is not fixed in any way. It can be anything at all.

It is necessary to make such a sum in which all possible values of λ will be.

This is generally an integral. Each λ has its own constant.

$$u(x,t)=\int_0^\infty e^{-a^2\lambda^2 t}\left(C(\lambda)e^{i\lambda x}+D(\lambda)e^{-i\lambda x}\right)d\lambda=$$

$$=\int_0^\infty e^{-a^2\lambda^2 t}C(\lambda)e^{i\lambda x}d\lambda+\int_0^\infty e^{-a^2\lambda^2 t}D(\lambda)e^{-i\lambda x}d\lambda=$$

In the second integral, we will replace the variable. ($-\lambda=\nu$)

$$=\int_0^\infty e^{-a^2\lambda^2 t}C(\lambda)e^{i\lambda x}d\lambda+(-1)\int_0^\infty e^{-a^2\lambda^2 t}D(\lambda)e^{-i\lambda x}d(-\lambda)=$$

$$=\int_0^\infty e^{-a^2\lambda^2 t}C(\lambda)e^{i\lambda x}d\lambda+\int_{-\infty}^0 e^{-a^2\nu^2 t}D(-\nu)e^{i\nu x}d\nu=$$

Let $D(-\nu)=C(\nu)$.

The last step. In the second integral, replace the letter ν with the letter λ. Anyway these are integration variables. What difference does it make which letter it is marked with. And let's assemble this integral into one.

$$=\int_{-\infty}^{+\infty} C(\lambda)e^{-a^2\lambda^2 t}e^{i\lambda x}d\lambda$$

We have

$$u(x,t)==\int_{-\infty}^{+\infty} C(\lambda)e^{-a^2\lambda^2 t}e^{i\lambda x}d\lambda \qquad (**)$$

<u>Let's check this solution:</u>

Let's prove that (**) is the solution of this equation $u_t=a^2u_{xx}$.

Let's take the derivative of t:

$$u_t = -a^2 \int_{\infty}^{\infty} \lambda^2 C(\lambda) e^{-a^2\lambda^2 t} e^{i\lambda x} dx$$

$$u_{xx} = (i)^2 \int_{-\infty}^{\infty} \lambda^2 C(\lambda) e^{-a^2\lambda^2 t}\, e^{i\lambda x}\, d\lambda = -\int_{-\infty}^{\infty} \lambda^2 C(\lambda) e^{-a^2\lambda^2 t}\, e^{i\lambda x}\, d\lambda$$

It is easy to see that they are equal.

This function $C(\lambda)$ is integrable and defined on the entire infinite line.

In order to find $C(\lambda)$, you need to substitute the initial conditions:

$$u(x,0) = \int_{-\infty}^{\infty} C(\lambda) e^{i\lambda x} d\lambda = \varphi(x)$$

Let's now use the formula for the inverse transformation of the Fourier integral (additional information):

$$C(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi) e^{-i\lambda\xi} d\xi$$

$$u(x,t) = \int_{-\infty}^{\infty} C(\lambda) e^{-a^2\lambda^2 t} e^{i\lambda x} d\lambda = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi) e^{-i\lambda\xi} d\xi\, e^{-a^2\lambda^2 t} e^{i\lambda x} d\lambda =$$

let's change the limits of integration

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi) \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} \cdot e^{i\lambda(x-\xi)} d\lambda d\xi =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi) \left(\int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} \cdot e^{i\lambda(x-\xi)} d\lambda \right) d\xi =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi) F(t,x,\xi) d\xi$$

$$F(t,x,\xi) = \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} \cdot e^{i\lambda(x-\xi)} d\lambda$$

Let's calculate this integral:

$$F(t,x,\xi) = \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} \cdot e^{i\lambda(x-\xi)} d\lambda$$

$$F_x(t,x,\xi) = i \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} \lambda e^{i\lambda(x-\xi)} d\lambda = \frac{i}{2} \frac{1}{a^2 t} \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} e^{i\lambda(x-\xi)} d\left(\lambda^2 \cdot a^2 t\right) =$$

we will put the exponent under the differential

$$= -\frac{i}{2a^2 t} \int_{-\infty}^{\infty} e^{i\lambda(x-\xi)} d e^{-a^2\lambda^2 t} =$$

integration by parts

$$= -\frac{i}{2a^2 t} \left(e^{-a^2\lambda^2 t} e^{i\lambda(x-\xi)} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} d e^{i\lambda(x-\xi)} \right) =$$

the first term is zero

$$= \frac{i \cdot i\left(x-\xi\right)}{2a^2 t}\left(\int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} e^{i\lambda\left(x-\xi\right)} d\lambda\right)$$

That is, we got

$$F_x\left(t,x,\xi\right) = -\frac{\left(x-\xi\right)}{2a^2 t} F\left(t,x,\xi\right)$$

So

$$y' = -\frac{\left(x-a\right)}{b} y$$

$$\frac{dy}{y} = -\frac{\left(x-a\right)}{b} dx$$

$$y = Ce^{-\frac{\left(x-a\right)^2}{2b}}$$

$$F(t,x,\xi) = G(t,\xi)e^{-\frac{(x-\xi)^2}{4a^2t}}$$

if $x = \xi$:

$$F(t,\xi,\xi) = G(t,\xi)$$

$$F(t,\xi,\xi) = \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} d\lambda = \frac{1}{a\sqrt{t}} \int_{-\infty}^{\infty} e^{-a^2\lambda^2 t} d\left(a\lambda\sqrt{t}\right) =$$

let $\left| a\lambda\sqrt{t} = z \right|$

$$= \frac{1}{a\sqrt{t}} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{\sqrt{\pi}}{a\sqrt{t}}$$

$$F(t,x,\xi) = \frac{\sqrt{\pi}}{a\sqrt{t}} e^{-\frac{(x-\xi)^2}{4a^2t}}$$

The final answer:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\xi) \frac{\sqrt{\pi}}{a\sqrt{t}} e^{-\frac{(x-\xi)^2}{4a^2t}} d\xi =$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\frac{(x-\xi)^2}{4a^2t}} d\xi$$

This is called the Poisson formula.

Useful additional information (Topic Fourier integral from mathematical analysis):

The classical Fourier series

$f(x)$ $[-l;l]$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{\pi n}{l}x\right)+b_n\sin\left(\frac{\pi n}{l}x\right)\right)$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(\xi)d\xi$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(\xi)\cos\left(\frac{\pi n}{l}\xi\right)d\xi$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(\xi)\sin\left(\frac{\pi n}{l}\xi\right)d\xi$$

$$\cos\alpha=\frac{e^{i\alpha}+e^{-i\alpha}}{2}$$

$$\sin\alpha=\frac{e^{i\alpha}-e^{-i\alpha}}{2i}=\frac{i}{i}\cdot\frac{e^{i\alpha}-e^{-i\alpha}}{2i}=i\,\frac{e^{-i\alpha}-e^{i\alpha}}{2}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\frac{a_n-ib_n}{2}e^{i\frac{\pi n}{l}x}+\frac{a_n+ib_n}{2}e^{-i\frac{\pi n}{l}x}\right)=$$

$$=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(\alpha_n e^{i\frac{\pi n}{l}x}+\beta_n e^{i\frac{\pi(-n)}{l}x}\right)=$$

Let

$$\frac{a_0}{2} = \alpha_0$$

$$\beta_n = \alpha_{-n}$$

$$= \sum_{n=-\infty}^{\infty} \alpha_n e^{i\frac{\pi n}{l}x}$$

We know that

$$a_n = \frac{1}{l}\int_{-l}^{l} f(\xi) cos\left(\frac{\pi n}{l}\xi\right) d\xi$$

and

$$\alpha_n = \frac{a_n - ib_n}{2} = \frac{1}{2l}\int_{-l}^{l} f(\xi)\left(cos\left(\frac{\pi n}{l}\xi\right) - i\,sin\left(\frac{\pi n}{l}\xi\right)\right) d\xi =$$

Euler's formula:

$$e^{i\varphi} = cos\,\varphi + i\,sin\,\varphi$$

$$e^{-i\varphi} = cos\,\varphi - i\,sin\,\varphi$$

$$= \frac{1}{2l}\int_{-l}^{l} f(\xi) e^{-i\frac{\pi n}{l}\xi} d\xi$$

For negative n, this also works.

The decomposition of the function was obtained:

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i\frac{\pi n}{l}x}$$

$$\alpha_n = \frac{1}{2l}\int_{-l}^{l} f(\xi) e^{-i\frac{\pi n}{l}\xi} d\xi$$

Let's move on to the Fourier integral:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2l} \int_{-l}^{l} f(\xi) e^{-i\frac{\pi n}{l}\xi}\, d\xi\, e^{i\frac{\pi n}{l}x} =$$

Let

$$\lambda_n = \frac{\pi n}{l}$$

$$\triangle \lambda_n = \lambda_{n+1} - \lambda_n = \frac{\pi}{l}$$

$$\frac{1}{l} = \frac{\triangle \lambda_n}{\pi}$$

Let's write everything down in these terms:

$$= \sum_{n=-\infty}^{\infty} \frac{\triangle \lambda_n}{2\pi} \int_{-l}^{l} f(\xi) e^{-i\lambda_n \xi} d\xi\, e^{i\lambda_n x}$$

$$[-l; l] \to (-\infty; \infty)$$

$$f(x) = \lim_{\substack{l \to \infty \\ (\triangle \lambda_n \to 0)}} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-l}^{l} f(\xi) e^{-i\lambda_n \xi} d\xi\, e^{i\lambda_n x}\, \triangle \lambda_n =$$

we recall the mathematical analysis of the second year (recall the definition of definite integral)

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) e^{-i\lambda \xi} d\xi e^{i\lambda x} d\lambda$$

On an infinite line, the function can be decomposed:

$$f(x) == \int_{-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) e^{-i\lambda\xi} d\xi\, e^{i\lambda x} d\lambda =$$

$$= \int_{-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\xi) e^{-i\lambda\xi} d\xi \right) e^{i\lambda x} d\lambda =$$

$$= \int_{-\infty}^{\infty} C(\lambda) e^{i\lambda x} d\lambda$$

$$C(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) e^{-i\lambda\xi} d\xi$$

Additional information shows how this formula was obtained.

Example 1

$$4u_t = u_{xx}$$

$$u\big|_{t=0} = e^{2x-x^2}$$

Solution:

$$\begin{cases} u_t = \dfrac{1}{4} u_{xx} \\ u(x,0) = e^{2x-x^2} \end{cases}$$

Let's use the Poisson formula and write down the answer:

$$u\left(x,t\right)=\frac{1}{2\sqrt{\pi\frac{1}{4}t}}\int_{-\infty}^{\infty}e^{2\xi-\xi^2}e^{-\frac{\left(x-\xi\right)^2}{4\frac{1}{4}t}}d\xi=$$

$$=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{2\xi-\xi^2-\frac{x^2}{t}+\frac{2\xi x}{t}-\frac{\xi^2}{t}}d\xi$$

$$2\xi-\xi^2-\frac{x^2}{t}+\frac{2\xi x}{t}-\frac{\xi^2}{t}=-\left(\xi^2\left(1+\frac{1}{t}\right)-2\xi\left(1+\frac{x}{t}\right)+\frac{x^2}{t}\right)=$$

$$=-\frac{t+1}{t}\left(\xi^2-2\left(\frac{t+x}{t+1}\right)\xi+\frac{x^2}{t+1}\right)=$$

$$=-\frac{t+1}{t}\left[\xi^2-2\left(\frac{t+x}{t+1}\right)\xi+\left(\frac{t+x}{t+1}\right)^2-\left(\frac{t+x}{t+1}\right)^2+\frac{x^2}{t+1}\right]=$$

$$=-\frac{t+1}{t}\left[\left(\xi-\frac{t+x}{t+1}\right)^2--\left(\frac{t+x}{t+1}\right)^2+\frac{x^2}{t+1}\right]$$

$$u(x,t)=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{2\xi-\xi^2-\frac{x^2}{t}+\frac{2\xi x}{t}-\frac{\xi^2}{t}}d\xi=$$

$$=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{t+1}{t}\left[\left(\xi-\frac{t+x}{t+1}\right)^2-\left(\frac{t+x}{t+1}\right)^2+\frac{x^2}{t+1}\right]}d\xi=$$

$$=\frac{1}{\sqrt{\pi t}}e^{-\frac{t+1}{t}\left[\frac{x^2}{t+1}-\left(\frac{t+x}{t+1}\right)^2\right]}\int_{-\infty}^{\infty} e^{-\frac{t+1}{t}\left(\xi-\frac{t+x}{t+1}\right)^2}d\left(\xi-\frac{x+t}{t+1}\right)=$$

$$=\frac{1}{\sqrt{\pi t}}e^{-\frac{t+1}{t}\left[\frac{x^2}{t+1}-\left(\frac{t+x}{t+1}\right)^2\right]}\cdot\sqrt{\frac{t}{t+1}}\int_{-\infty}^{\infty} e^{-\left[\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)\right]^2}d\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)$$

$$\int_{-\infty}^{\infty} e^{-\left[\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)\right]^2}d\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)=\sqrt{\pi}$$

$$u(x,t)=\frac{1}{\sqrt{t+1}}e^{-\frac{t+1}{t}\left(\frac{x^2}{t+1}-\left(\frac{x+t}{t+1}\right)^2\right)}$$

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4.3.2. BOUNDARY VALUE PROBLEM FOR A SEMI-BOUNDED LINE

4.3.3. APPLICATION OF THE LAPLACE. TRANSFORM TO SOLVING BOUNDARY VALUE PROBLEMS

5. ELLIPTIC EQUATIONS

5.1. THE LAPLACE EQUATION. SETTING BOUNDARY VALUE PROBLEMS