Equations of mathematical physics

Content

Introduction

The course includes the main sections:

— Fundamentals of operational calculus;

— Classification of partial differential equations;

— Hyperbolic equations;

— Parabolic equations;

— Elliptical equations.

1. FUNDAMENTALS OF OPERATIONAL CALCULUS

1.1. The concepts of the original and the Laplace image. Properties of the Laplace transform

Definition 1. *An original function* is any complex-valued function $f(t)$ of a valid argument t that satisfies the conditions:

1) $f(t)$ is Riemann integrable on any finite interval of the t axis (locally integrable);

2) $f(t)=0$ for all $t<0$;

3) $M>0$ and $\alpha>0$ are constants for which

$$\left|f(t)\right| \le Me^{\alpha t}\,. \tag{1.1}$$

The lower edge α_0 of all numbers α for which the inequality (1.1) is valid is called the *growth index* of the function $f(t)$.

The first condition in definition 1 is sometimes formulated as follows: on any finite interval of the t axis, the function $f(t)$ is continuous, except, perhaps, a finite number of discontinuity points of the first kind.

The simplest original function is the Heaviside function:

$$\theta(t)=\begin{cases}0, & t<0,\\ 1, & t\ge 1.\end{cases}$$

Obviously, for any function $\varphi(t)$ it is true:

$$\varphi(t)\theta(t)=\begin{cases}0, & t<0,\\ \varphi(t), & t\ge 1.\end{cases}$$

If, for $t \geq 0$, the function $\varphi(t)$ satisfies conditions 1 and 3 of definition 1, then the function $\varphi(t)\theta(t)$ is the original. In the future, to shorten the record, we will, as a rule, write $\varphi(t)$ instead of $\varphi(t)\theta(t)$, assuming that the functions we are considering are continued by zero for negative values of the argument t.

Definition 2. *The image of the function* $f(t)$ *according to Laplace* is called the function $F(p)$ of the complex variable $p = s + i\sigma$, defined by the equality

$$F(p) = \int_0^{+\infty} f(t) e^{-pt} dt \,. \qquad (1.2)$$

Theorem 1 (on the analyticity of the image). For any original $f(t)$, its image $F(p)$ is defined and is an analytical function of the variable p in the half—plane $\operatorname{Re} p > \alpha_0$, where α_0 is the growth index of the function $f(t)$, while the equality is valid:

$$\lim_{\operatorname{Re} p \to +\infty} |F(p)| = 0 \,.$$

Theorem 2 (uniqueness). The Laplace image $F(p)$ is unique in the sense that two functions $f_1(t)$ and $f_2(t)$ having the same images coincide at all points of continuity at $t > 0$.

There are several ways to record the correspondence between the original and the image:

$$f(t) \leftrightarrow F(p)\,,\ f(t) \doteq F(p)\,,\ L\{f(t)\} = F(p)\,.$$

Example 1.

Using the definition, find the image of the function $f(t) = sin3t$.

Solution:

For the function $f(t) = sin3t$, we have $\alpha_0 = 0$. Therefore, the image $F(p)$ will be defined and analytically in the half-plane $Re\,p > 0$. Let us apply formula (1.2) to a given function, using the rule of integration in parts and the restriction on the set of values of the variable p, which ensures the convergence of the integral, when performing transformations:

$$F(p) = \int_0^{+\infty} e^{-pt}\sin 3t dt = -\frac{1}{p}e^{-pt}\sin 3t\Big|_0^{+\infty} + \frac{3}{p}\int_0^{+\infty} e^{-pt}\cos 3t dt =$$

$$= \frac{3}{p}\left(-\frac{1}{p}e^{-pt}\cos 3t\Big|_0^{+\infty} - \frac{3}{p}\int_0^{+\infty} e^{-pt}\sin 3t dt\right) = \frac{3}{p^2} - \frac{9}{p^2}F(p)\,.$$

Got equality

$$F(p) = \frac{3}{p^2} - \frac{9}{p^2}F(p)\,.$$

From here we find

$$F(p) = \frac{3}{p^2+9}\,.$$

Thus, the following correspondence is valid:

$$\sin 3t \leftrightarrow \frac{3}{p^2+9}, \ \mathrm{Re}\, p > 0.$$

Properties of the Laplace transform

1. **Linearity.** If $f(t) \leftrightarrow F(p)$, $g(t) \leftrightarrow G(p)$, then for any complex λ and μ it is performed

$$\lambda f(t) + \mu g(t) \leftrightarrow \lambda F(p) + \mu G(p), \ \mathrm{Re}\, p > \max(\alpha_0, \beta_0),$$

here and further, α_0, β_0 are the growth indicators of the function $f(t), g(t)$, respectively.

2. **Similarity.** If $f(t) \leftrightarrow F(p)$, then for $\forall \alpha > 0$ it is true

$$f(at) \leftrightarrow \frac{1}{a} F\left(\frac{p}{a}\right), \ \mathrm{Re}\, p > a\alpha_0.$$

3. **Differentiation of the original.** If $f(t), f'(t), \ldots, f^{(n)}(t)$ are originals and $f(t) \leftrightarrow F(p)$ for $\mathrm{Re}\, p > \alpha_0$, then

$$f^{(n)}(t) \leftrightarrow p^n F(p) - p^{n-1} f(+0) - p^{n-2} f'(+0) - \ldots - p f^{(n-2)}(+0) - f^{(n-1)}(+0),$$

where

$$f^{(k)}(+0) = \lim_{t \to +0} f^{(k)}(t), \ k = 0, 1, \ldots, n-1.$$

4. **Image differentiation.** If $f(t) \leftrightarrow F(p)$, then

$$F^{(n)}(p) \leftrightarrow (-t)^n f(t), \ \mathrm{Re}\, p > \alpha_0.$$

5. **Integration of the original.** If $f(t) \leftrightarrow F(p)$, then

$$\int_0^t f(\tau)d\tau \leftrightarrow \frac{F(p)}{p}, \ \operatorname{Re} p > \alpha_0 .$$

6. **Image integration.** If $f(t) \leftrightarrow F(p)$ and $\frac{f(t)}{t}$ are the original, then

$$\int_p^\infty F(\xi)d\xi \leftrightarrow \frac{f(t)}{t}, \ \operatorname{Re} p > \alpha_0 .$$

7. **The delay property.** If $f(t) \leftrightarrow F(p)$ and $f(t)=0$ for $t<\tau$, where $\tau>0$, then

$$f(t-\tau) \leftrightarrow e^{-\tau p}F(p), \ \operatorname{Re} p > \alpha_0 .$$

<u>Remark.</u> The following formulation of the delay property is possible: if $f(t) \leftrightarrow F(p)$, then for any $\tau>0$ there is

$$f(t-\tau)\theta(t-\tau) \leftrightarrow e^{-\tau p}F(p), \ \operatorname{Re} p > \alpha_0 .$$

8. **The displacement property.** If $f(t) \leftrightarrow F(p)$, then for any complex λ

$$e^{\lambda t}f(t) \leftrightarrow F(p-\lambda), \ \operatorname{Re} p > \alpha_0 + \operatorname{Re}\lambda .$$

9. **The image of the convolution.** *The convolution of functions* f *and* g is a function that is denoted by $f \cdot g$ and is defined by equality

$$(f \cdot g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau .$$

The convolution of functions has the property of symmetry, that is,

$$(f \cdot g)(t) = (g \cdot f)(t).$$

If $f(t) \leftrightarrow F(p)$ and $g(t) \leftrightarrow G(p)$, then

$$(f \cdot g)(t) \leftrightarrow F(p)G(p), \ \operatorname{Re} p > \max(\alpha_0, \beta_0).$$

Here is a table of originals and images of some elementary functions:

Original $f(t)$	Image $F(p)$	Original $f(t)$	Image $F(p)$
1	$\frac{1}{p}$	$sh\, at$	$\frac{a}{p^2 - a^2}$
e^{-at}	$\frac{1}{p+a}$	$ch\, at$	$\frac{p}{p^2 - a^2}$
t	$\frac{1}{p^2}$	$e^{-at} \cos \omega t$	$\frac{p+a}{(p+a)^2 + \omega^2}$
$\sin at$	$\frac{a}{p^2 + a^2}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(p+a)^2 + \omega^2}$
$\cos at$	$\frac{p}{p^2 + a^2}$	$e^{at}\, sh\, \omega t$	$\frac{\omega}{(p-a)^2 - \omega^2}$
$t^n, n \in Z$	$\frac{n!}{p^{n+1}}$	$e^{at}\, ch\, \omega t$	$\frac{p-a}{(p-a)^2 - \omega^2}$
$t^n e^{at}$	$\frac{n!}{(p-a)^{n+1}}$	$t\, sh\, \omega t$	$\frac{2\omega p}{(p^2 - \omega^2)^2}$
$t \sin \omega t$	$\frac{2p\omega}{(p^2 + \omega^2)^2}$	$t\, ch\, \omega t$	$\frac{p^2 + \omega^2}{(p^2 - \omega^2)^2}$
$t \cos \omega t$	$\frac{p^2 - \omega^2}{(p^2 + \omega^2)^2}$	$e^{at}\, t \sin \omega t$	$\frac{2\omega(p-a)}{((p-a)^2 + \omega^2)^2}$
$e^{at}\, t \cos \omega t$	$\frac{(p-a)^2 - \omega^2}{((p-a)^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	$\frac{1}{(p^2 + \omega^2)^2}$
$\frac{1}{2\omega^3}(\omega t\, ch\, \omega t - sh\, \omega t)$	$\frac{1}{(p^2 - \omega^2)^2}$	$\sin(\omega t \pm \varphi)$	$\frac{\omega \cos \varphi \pm p \sin \varphi}{p^2 + \omega^2}$
$\cos(\omega t \pm \varphi)$	$\frac{p \cos \varphi \mp \omega \sin \varphi}{p^2 + \omega^2}$		

Example 2.

Using the properties of the Laplace transform and the table of the main originals and images, find images of the following functions:

1) $f(t)=e^{-4t}\sin 3t\cos 2t$;
2) $f(t)=e^{(t-2)}\sin(t-2)$;
3) $f(t)=t^2e^{3t}$;
4) $f(t)=\dfrac{\sin^2 t}{t}$.

<u>Solution:</u>

1) Let's transform the expression for the function $f(t)$ as follows:

$$f(t)=e^{-4t}\sin 3t\cos 2t=\frac{1}{2}e^{-4t}\left(\sin 5t+\sin t\right)=\frac{1}{2}e^{-4t}\sin 5t+\frac{1}{2}e^{-4t}\sin t\,.$$

Since $\sin t\leftrightarrow\dfrac{1}{p^2+1}$ and $\sin 5t\leftrightarrow\dfrac{5}{p^2+25}$, then, using the properties of linearity and displacement, for the image of the function $f(t)$ we will have

$$F(p)=\frac{1}{2}\left(\frac{5}{(p+4)^2+25}+\frac{1}{(p+4)^2+1}\right).$$

2) Since $\sin t\leftrightarrow\dfrac{1}{p^2+1}$, $e^t\sin t\leftrightarrow\dfrac{1}{(p-1)^2+1}$, then, using the delay property, we will have

$$f(t)=e^{(t-2)}\sin(t-2)\leftrightarrow F(p)=\frac{e^{-2p}}{(p-1)^2+1}\,.$$

3) Since $t^2\leftrightarrow\dfrac{2}{p^3}$, then by the displacement property we have

$$f(t) = t^2 e^{3t} \leftrightarrow F(p) = \frac{2}{(p-3)^3}.$$

For comparison, we present a method for constructing an image of function $f(t) = t^2 e^{3t}$ using the image differentiation property:

$$e^{3t} \leftrightarrow \frac{1}{p-3}; \; te^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{p-3}\right) = \frac{1}{(p-3)^2};$$

$$t^2 e^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{(p-3)^2}\right) = \frac{2}{(p-3)^3}.$$

We got the same result.

4. <u>HOMEWORK (using the image integration property)</u>

For a function defined as follows:

$$f(t) = \begin{cases} 0, & t < t_1, \\ f_1(t), & t_1 \le t < t_2, \\ f_2(t), & t_2 \le t < t_3, \\ \ldots & \\ f_{n-1}(t), & t_{n-1} \le t < t_n, \\ f_n(t), & t \ge t_n, \end{cases}$$

using the Heaviside function, you can write an analytical form that is convenient to use when constructing the corresponding image.

It is easy to verify that for a function $g_k(t)$ equal to

$$g_k(t) = \begin{cases} 0, & t < t_k, \\ f_k(t), & t_k \le t < t_{k+1}, \\ 0, & t \ge t_{k+1}, \end{cases}$$

the following representation is valid using the Heaviside function:

$$g_k(t) = f_k(t)\theta(t - t_k) - f_k(t)\theta(t - t_{k+1}) . \qquad (1.3)$$

And for the function

$$g_n(t) = \begin{cases} 0, & t < t_n, \\ f_n(t), & t \geq t_n, \end{cases}$$

there is

$$g_n(t) = f_n(t)\theta(t - t_n) . \qquad (1.4)$$

Assuming that k varies from 1 to $n-1$, the function $f(t)$ can be considered as the sum of the functions $g_k(t)$ and $g_n(t)$:

$$f(t) = \sum_{k=1}^{n-1} g_k(t) + g_n(t) .$$

And then, using expressions (1.3) and (1.4), we get

$$f(t) = f_1(t)\theta(t - t_1) - \sum_{k=2}^{n} \left(f_k(t) - f_{k-1}(t) \right) \theta(t - t_k) . \qquad (1.5)$$

Example 3.

Build an image for the function $f(t)$:

$$f(t) = \begin{cases} 0, & t < a, \\ \varphi(t), & a \leq t < b, \\ 0, & t \geq b. \end{cases}$$

<u>Solution:</u>

Let's write an expression for the function $f(t)$ using the Heaviside function:

$$f(t) = \varphi(t)\theta(t-a) - \varphi(t)\theta(t-b).$$

Since

$$\varphi(t) = \varphi(t-a+a) \text{ и } \varphi(t) = \varphi(t-b+b),$$

then, having found the images for the functions $\varphi(t+a)$ and $\varphi(t+b)$,

$$\varphi(t+a) \leftrightarrow \Phi_1(p), \; \varphi(t+b) \leftrightarrow \Phi_2(p),$$

we construct an image for the function $f(t)$, taking into account the lag property

$$f(t) \leftrightarrow F(p) = \Phi_1(p)e^{-ap} - \Phi_2(p)e^{-bp}.$$

Example 4.

Find the image $F(p)$ of the function $f(t)$:

$$f(t) = \begin{cases} 0, & t \in (-\infty, 0), \\ 1, & t \in (0, a), \\ \dfrac{2a-t}{a}, & t \in (a, 3a), \\ \dfrac{t-4a}{a}, & t \in [3a, \infty). \end{cases}$$

<u>Solution:</u>

Let's find an image of the function $f(t)$, having previously written an expression for it using the Heaviside function $\theta(t)$. To do this, use the formula (1.5). Since for a given function

$t_1 = 0$, $t_2 = a$, $t_3 = 3a$ and

$$f_1(t)=1\,,\ f_2(t)=\frac{2a-t}{a}\,,\ f_3(t)=\frac{t-4a}{a}\,,$$

then we will have

$$f(t)=\theta(t)-\left(\frac{2a-t}{a}-1\right)\theta(t-a)+\left(\frac{t-4a}{a}-\frac{2a-t}{a}\right)\theta(t-3a)=$$

$$=\theta(t)-\frac{t-a}{a}\theta(t-a)+\frac{2(t-3a)}{a}\theta(t-3a)\,.$$

Applying the properties of linearity and delay to the constructed expression, we find the desired image $F(p)$:

$$F(p)=\frac{1}{p}-\frac{1}{ap^2}e^{-ap}+\frac{2}{ap^2}e^{-3ap}\,.$$

CONTROL TASKS

Example 1.

1) Check if the following functions are originals and find their growth index:

a) $f(t)=2e^{3t}\sin at,\ a\in R$

b) $f(t)=e^{3+it^2}$

c) $f(t)=\frac{1}{t}$

Solution:

a) The function $f(t)=2e^{3t}\sin at, a\in R$ is continuous at any finite interval $[0,B], B>0$. Therefore, it is integrable on $[0,B]$. Since

$$\left|f(t)\right|=2e^{3t}\left|\sin at\right|\le 2e^{3t},$$

$M=2, \alpha_0=3$, then $f(t)$ is a function of bounded growth with a growth index of $\alpha_0=3$. Therefore, the $f(t)$ -function is the original.

b) The function $f(t)=e^{3+it^2}$ is continuous and, therefore, integrable on any finite interval $[0,B]$.

$$\left|e^{3+it^2}\right|=e^3\left|\cos t^2+i\sin t^2\right|=e^3\le e^3e^{0\cdot t},$$

where $M=e^3, \alpha_0=0$.

The function $f(t)$ is the original with the growth index $\alpha_0=0$.

c) Function $f(t)=\frac{1}{t}$ is not the original. Integral $\int_0^B \frac{1}{t}\mathrm{d}t=\ln t\Big|_0^B=+\infty$ diverges. Function $f(t)$ is not integrable. The first condition for defining the original function has been violated.

Example 2.

2) The function $F(p)$ is given. Can it be an image of some original in some area? If so, specify this area.

a) $F(p)=1$,

b) $F(p)=\sin p$,

c) $F(p)=\dfrac{p}{p^2-2p+5}$

<u>Solution:</u>

a) Since

$$\lim_{\mathrm{Re}(p)\to+\infty} F(p)=1,$$

the necessary sign of the existence of an image is not fulfilled for $F(p)$ (Theorem 1). Function $F(p)$ is not an image.

b) Since

$$\lim_{\mathrm{Re}(p)\to+\infty} \sin p$$

does not exist, the necessary sign of the existence of the image is not fulfilled for $F(p)$. Function $F(p)$ is not an image.

c) The necessary indication of the existence of the image

$$\lim_{\mathrm{Re}(p)\to+\infty} \frac{p}{p^2-2p+5}=0$$

has been fulfilled. Function $F(p)$ is analytical in the entire domain except for the zeros of the denominator.

Solving the equation

$$p^2-2p+5=0,$$

we get the simple poles

$$p_{1,2}=1\pm 2i$$

of the function $F(p)$.

Therefore, $F(p)$ will be an image in the region $Re(p)>1$.

In order to verify the correctness of calculations, limiting ratios are used in operational calculus.

Theorem 3 (*on limiting ratios*).

If $f(t), f'(t)$ are originals and $f(t)\leftrightarrow F(p)$, then

$$\lim_{\mathrm{Re}\,p\to+\infty} pF(p)=\lim_{t\to+0} f(t)=f(0), \tag{2}$$

if there is a finite limit of $\lim\limits_{t\to+\infty} f(t)$, then

$$\lim_{p\to 0} pF(p)=\lim_{t\to+\infty} f(t). \tag{3}$$

Example 3.

3) Using the definition, find images of the following functions

a) $f(t)=\theta(t)$

b) $f(t)=e^{4t}$

c) $f(t)=sin t$

<u>Solution:</u>

a) Function $f(t)=\theta(t)$ is the original with a growth index of $\alpha_0=0$.

$$F(p)=\int_0^{+\infty} 1\cdot e^{-pt}dt=\lim_{B\to+\infty}\int_0^B 1\cdot e^{-pt}dt=\lim_{B\to+\infty}\left(-\frac{1}{p}\cdot e^{-pt}\Big|_0^B\right)=\frac{1}{p}.$$

b) The function $f(t)=e^{4t}$ is the original with the growth index $\alpha_0=4$

$$F(p)=\int_0^{+\infty} e^{4t}\cdot e^{-pt}dt=\lim_{B\to+\infty}\int_0^B e^{-(p-4)t}dt=-\lim_{B\to+\infty}\left(\frac{1}{p-4}e^{-(p-4)t}\Big|_0^B\right)=$$

$$=\lim_{B\to+\infty}\left(\frac{1}{p-4}-\frac{e^{-(p-4)B}}{p-4}\right)=\frac{1}{p-4}.$$

Let's check the calculations using the limit ratios (2) and (3). In this case, $\lim_{t\to+\infty} e^{4t}=+\infty$ (the final limit $f(t)$ does not exist), and condition (2) is fulfilled by

$$\lim_{\operatorname{Re} p\to+\infty} pF(p)=\lim_{\operatorname{Re} p\to+\infty}\frac{p}{p-4}=1=\lim_{t\to+0} e^{4t}=f(0).$$

c) The function $f(t)=\sin t$ is the original with the growth index $\alpha_0=0$

$$F(p)=\int_0^{+\infty}\sin t\, e^{-pt}\,dt=\begin{bmatrix} u=e^{-pt}, & dv=\sin t\,dt, \\ du=-pe^{-pt}dt, & v=-\cos t \end{bmatrix}=$$

$$=-e^{-pt}\cos t\Big|_0^{+\infty}-p\int_0^{+\infty}\cos t\, e^{-pt}\,dt=\begin{bmatrix} u=e^{-pt}, & dv=\cos t\,dt, \\ du=-pe^{-pt}dt, & v=\sin t \end{bmatrix}=$$

$$=1-p\left(pe^{-pt}\sin t\Big|_0^{+\infty}-p\int_0^{+\infty}\sin t\, e^{-pt}\,dt\right)=1-p^2\int_0^{+\infty}\sin t\, e^{-pt}\,dt.$$

From here

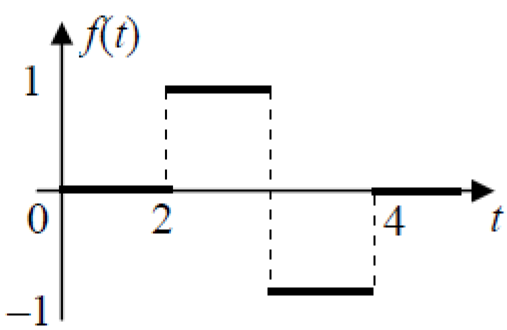
$$\int_0^{+\infty}\sin t\, e^{-pt}\,dt=1-p^2\int_0^{+\infty}\sin t\, e^{-pt}\,dt.$$

From the obtained equality, we express the desired integral:

$$F(p)=\int_0^{+\infty}\sin t\, e^{-pt}\,dt=\frac{1}{p^2+1}.$$

Example 4.

4) Find the image of the function given as follows:



<u>Solution:</u>

The function can be written in analytical form

$$f(t)=\begin{cases} 0, & t\le 2, \quad t>4, \\ 1, & 2<t\le 3, \\ -1, & 3<t\le 4. \end{cases}$$

Using the Laplace transform formula:

$$F(p)=\int_0^{+\infty} e^{-pt} f(t)\mathrm{d}t=\int_2^3 e^{-pt}\mathrm{d}t-\int_3^4 e^{-pt}\mathrm{d}t=$$

$$=\frac{1}{p}\left(-e^{-3p}+e^{-2p}+e^{-4p}-e^{-3p}\right)=$$

$$=\frac{e^{-2p}}{p}\left(1-2e^{-p}+e^{-2p}\right)=\frac{\left(e^{-p}\left(1-e^{-p}\right)\right)^2}{p}=\frac{\left(e^{-p}-e^{-2p}\right)^2}{p}.$$

Example 5. (*Using tables and properties of the Laplace transform*)

5) Using the properties of *linearity* and *similarity*, find images of the following functions:

a) $f(t)=\cos t$

b) $f(t)=2-5\cos 2t$

<u>Solution:</u>

a) According to Euler's theorem, $\cos t=\dfrac{e^{it}+e^{-it}}{2}$. Since according to the image table $e^{it}\leftrightarrow\dfrac{1}{p-i}$, $e^{-it}\leftrightarrow\dfrac{1}{p+i}$, then according to the linearity property

$$f(t)=\cos t\leftrightarrow \frac{1}{2}\left(\frac{1}{p-i}+\frac{1}{p+i}\right)=\frac{p}{p^2+1}=F(p).$$

b) By the property of *linearity* and *similarity*

$$f(t)=2-5\cos 2t \leftrightarrow \frac{2}{p}-5\frac{p}{p^2+4}=F(p).$$

Example 6. (*The displacement property*)

6) Find images of the following functions:

a) $f(t)=e^{-3t}\mathrm{ch}2t$,

b) $f(t)=e^{2t}\cos nt$

Solution:

a) According to the image table, we have $ch2t \leftrightarrow \frac{p}{p^2-4}$. The presence of a multiplier e^{-3t} implies the use of the displacement theorem (*displacement property*). Therefore:

$$e^{-3t}ch2t \leftrightarrow \frac{p+3}{(p+3)^2-4}=F(p)$$

b) Since $\cos nt \leftrightarrow \frac{p}{p^2+n^2}$, then

$$f(t)=e^{2t}\cos nt \leftrightarrow \frac{p-2}{(p-2)^2-n^2}=F(p)$$

Example 7. (*Image differentiation.)*

7) Find images of the following functions

a) $f(t)=te^{at}$,

b) $f(t)=te^{t}\cos t$

c) $f(t)=t^{2}\sin t$

<u>Solution:</u>

a) The presence of a multiplier t indicates the need to apply the image differentiation theorem:

Since $e^{at}\leftrightarrow\dfrac{1}{p-a}$, then

$$f(t)=te^{at}\leftrightarrow(-1)^{1}\left(\frac{1}{p-a}\right)'=\frac{1}{(p-a)^{2}}=F(p).$$

b) To find the image, we apply the theorems of image differentiation and displacement

$$\cos t\leftrightarrow\frac{p}{p^{2}+1},$$

$$t\cos t\leftrightarrow-\left(\frac{p}{p^{2}+1}\right)'=-\frac{p^{2}+1-2p^{2}}{(p^{2}+1)^{2}}=\frac{p^{2}-1}{(p^{2}+1)^{2}}.$$

$$f(t)=te^{t}\cos t\leftrightarrow\frac{(p-1)^{2}-1}{((p-1)^{2}+1)^{2}}=\frac{p^{2}-2p}{(p^{2}-2p+2)^{2}}=F(p)$$

c) The presence of a multiplier t^{2} indicates the need to apply the image differentiation theorem

$$\sin t \leftrightarrow \frac{1}{p^2+1},$$

$$t^2 \sin t \leftrightarrow (-1)^2 \left(\frac{1}{p^2+1} \right)''$$

$$\left(\frac{1}{p^2+1} \right)' = \frac{-2p}{(p^2+1)^2},$$

$$\left(\frac{1}{p^2+1} \right)'' = \left(\frac{-2p}{(p^2+1)^2} \right)' = \frac{6p^2-2}{(p^2+1)^3}.$$

$$f(t) = t^2 \sin t \leftrightarrow \frac{6p^2-2}{(p^2+1)^3} = F(p).$$

Example 8. (*Image integration.)*

8) Find images of the following functions:

a) $f(t) = \dfrac{e^t-1}{t}$,

b) $f(t) = \dfrac{1-\cos t}{t}$

Solution:

a) The function $f(t)$ is continuous for all $t>0$ and is bounded in the vicinity of zero (according to L'hopital's rule $\lim\limits_{t\to+0} \dfrac{e^t-1}{t} = 1$).

Since

$$e^t - 1 \leftrightarrow \frac{1}{p-1} - \frac{1}{p},$$

then by the image integration theorem we obtain

$$f(t)=\frac{e^t-1}{t}\leftrightarrow\int_p^{\infty}\left(\frac{1}{z-1}-\frac{1}{z}\right)dz=\left(ln|z-1|-ln|z|\right)\Big|_p^{\infty}=$$

$$=ln\left|\frac{z-1}{z}\right|_p^{\infty}=ln\frac{p}{p-1}=F(p).$$

b) Since $\lim\limits_{t\to+0}\frac{1-\cos t}{t}=\lim\limits_{t\to+0}\frac{2\sin^2\frac{t}{2}}{t}=\lim\limits_{t\to+0}\sin\frac{t}{2}=0$, then $f(t)$ is continuous and bounded at $t>0$. Let's apply the image integration theorem. Since $1-\cos t\leftrightarrow\frac{1}{p}-\frac{p}{p^2+1}$, then

$$f(t)=\frac{1-\cos t}{t}\leftrightarrow\int_p^{\infty}\left(\frac{1}{z}-\frac{z}{z^2+1}\right)dz=\left(ln\,z-\frac{1}{2}ln\left(z^2+1\right)\right)\Big|_p^{\infty}=$$

$$=ln\frac{z}{\sqrt{z^2+1}}\Big|_p^{\infty}=ln\frac{\sqrt{p^2+1}}{p}=F(p).$$

Example 9. *(Differentiation of the original.)*

9) Find images of the following functions:

a) $f(t)=\sin^2 t$,

b) $f(t)=te^t$.

<u>Solution:</u>

a) Let $f(t)\leftrightarrow F(p)$. Since $f(0)=0$, then $f'(t)\leftrightarrow pF(p)-f(0)=pF(p)$. Calculate the derivative of the function $f(t)$ and find the image for $f'(t)$

$$f'(t)=(sin^2 t)'=2\,sin\,t\,cos\,t=sin\,2t\leftrightarrow\frac{2}{p^2+4}$$

Thus, according to the original differentiation theorem, to

determine the image $F(p)$ we have the equation $pF(p)=\frac{2}{p^2+4}$,

solving which we get $F(p)=\frac{2}{p(p^2+4)}$.

b) Let $f(t)\leftrightarrow F(p)$. Since $f(0)=0$, then

$f'(t)\leftrightarrow pF(p)-f(0)=pF(p)$.

Let's find the image for the derivative:

$$f'(t)=\left(te^t\right)'=e^t+te^t\leftrightarrow=\frac{1}{p-1}+F(p)$$

Thus, to determine $F(p)$, we have the equation

$$\frac{1}{p-1}+F(p)=pF(p)$$

Therefore

$$F(p)=\frac{1}{(p-1)^2}$$

Example 10. *(Integrating the original).*

10) Find images of the following functions:

a) $f(t)=\int_0^t \sin\tau d\tau$,

b) $f(t)=\int_0^t \tau^2 e^{-\tau} d\tau$

<u>Solution:</u>

a) Since $\sin t \leftrightarrow \frac{1}{p^2+1}$, then by the original integration theorem

$$\int_0^t \sin\tau\, d\tau \leftrightarrow \frac{1}{p}\cdot\frac{1}{p^2+1}=\frac{1}{p(p^2+1)}$$

b) By the delay theorem $t^2 e^{-t} \leftrightarrow \frac{2!}{(p+1)^3}$. Then we get

$$\int_0^t \tau^2 e^{-\tau} d\tau \leftrightarrow \frac{1}{p}\cdot\frac{2}{(p+1)^3}=\frac{2}{p(p+1)^3}$$

Theorem *(on convolution, Borel's theorem).*

The convolution of originals

$$(f_1\cdot f_2)(t)=\int_0^t f_1(\tau)f_2(t-\tau)d\tau=\int_0^t f_1(t-\tau)f_2(\tau)d\tau$$

corresponds to the product of images

$$(f_1 \cdot f_2)(t) \leftrightarrow F_1(p)F_2(p).$$

Notation for examples (The convolution of functions):

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau = \int_0^t f_1(t-\tau) f_2(\tau)\,\mathrm{d}\tau$$

Example 11.

11) Find the convolution and the image of the convolution (by the properties of the Laplace transform and by the convolution theorem).

a) $t * e^t$

b) $\sin t * t$.

<u>Solution:</u>

a) Let's find the convolution using the formula

$$t * e^t = \int_0^t (t-\tau)e^\tau\,\mathrm{d}\tau = t\left(e^t - 1\right) - \int_0^t \tau e^\tau\,\mathrm{d}\tau = \begin{vmatrix} u = \tau & \mathrm{d}v = e^\tau \mathrm{d}\tau \\ \mathrm{d}u = \mathrm{d}\tau & v = e^\tau \end{vmatrix} =$$

$$= t\left(e^t - 1\right) - \tau e^\tau\Big|_0^t + \int_0^t e^\tau\,\mathrm{d}\tau = t\left(e^t - 1\right) - \left(\tau \cdot e^\tau - e^\tau\right)\Big|_0^t =$$

$$= te^t - t - te^t + e^t - 1 = e^t - t - 1.$$

Let's find the convolution image using the properties of linearity, displacement

$$t*e^t = e^t - t - 1 \leftrightarrow \frac{1}{p-1} - \frac{1}{p^2} - \frac{1}{p} = \frac{1}{p^2(p-1)}$$

Let's find the convolution image according to Borel's theorem

$$t \leftrightarrow \frac{1}{p^2},\quad e^t \leftrightarrow \frac{1}{p-1},\quad t*e^t \leftrightarrow \frac{1}{p^2(p-1)}$$

b) <u>HOMEWORK №1.</u>

Example 12.

12) Find images of the following functions:

a) $f(t) = \int_0^t \cos(t-\tau)e^{2\tau}d\tau$,

b) $f(t) = \int_0^t e^{2(\tau-t)}\tau^2 d\tau$

<u>Solution:</u>

a) The function $f(t)$ is a convolution of $f(t) = f_1(t) * f_2(t)$, where $f_1(t) = \cos t,\ f_2(t) = e^{2t}$.

Since $\cos t \leftrightarrow \frac{p}{p^2+1},\ e^{2t} \leftrightarrow \frac{1}{p-2}$, then

$$\int_0^t \cos(t-\tau)e^{2\tau}d\tau = \cos t * e^{2t} \leftrightarrow \frac{p}{p^2+1}\cdot\frac{1}{p-2} = \frac{p}{(p^2+1)(p-2)}.$$

b) HOMEWORK №2.

Delay theorem *(delay property, Remark).*

If $f(t)\theta(t) \leftrightarrow F(p)$ and $\tau > 0$, then

$$f(t-\tau)\theta(t-\tau) \leftrightarrow e^{-p\tau} F(p)$$

Example 13.

13) Find images of the following functions:

a) $f(t) = e^{t-3}\,\theta(t-3)$

b) $f(t) = (t-1)^2\,\theta(t-1)$

Solution:

a) For the function $e^t\,\theta(t) \leftrightarrow \dfrac{1}{p-1}$. By the delay theorem

$$e^{t-3}\,\theta(t-3) \leftrightarrow \frac{e^{-3p}}{p-1}.$$

It should be noted that $e^{t-3}\,\theta(t) = e^{-3}\,e^t\,\theta(t) \leftrightarrow \dfrac{e^{-3}}{p-1}$

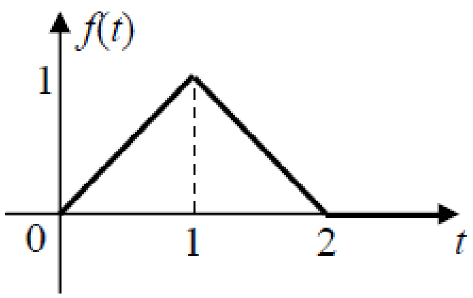
b) For the function $t^2\,\theta(t) \leftrightarrow \dfrac{2}{p^3}$. By the delay theorem

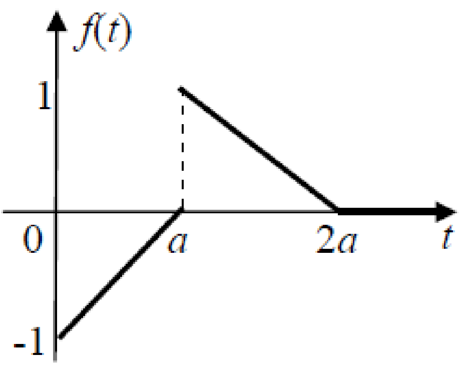
$$(t-1)^2\,\theta(t-1)\leftrightarrow\frac{2e^{-p}}{p^3}.$$

It should be noted that $(t-1)^2\,\theta(t)=(t^2-2t+1)\theta(t)\leftrightarrow\frac{2}{p^3}-\frac{2}{p^2}+\frac{1}{p}.$

Example 14.

14) Find images of the following functions defined graphically:

a) 

b) 

<u>Solution:</u>

a) The function can be written in analytical form

$$f(t)=\begin{cases}0, & t\le 0,\quad t\ge 2,\\ t, & 0<t\le 1,\\ 2-t, & 1<t\le 2.\end{cases}$$

Since

$$f_1(t)=\begin{cases} t, & t\in[0,1] \\ 0, & t\notin[0,1] \end{cases}$$

$$f_1(t)=t\theta(t)-t\theta(t-1),$$

$$f_2(t)=\begin{cases} 2-t, & t\in[1,2] \\ 0, & t\notin[1,2] \end{cases}$$

$$f_2(t)=(2-t)\theta(t-1)-(2-t)\theta(t-2),$$

the composite function $f(t)=f_1(t)+f_2(t)$ is represented by one analytical expression in the form:

$$f(t)=t\theta(t)-t\theta(t-1)+(2-t)\theta(t-1)-(2-t)\theta(t-2)=$$

$$=t\theta(t)-2(t-1)\theta(t-1)+(t-2)\theta(t-2).$$

Applying the delay theorem, we find the image of the function

$$f(t)\leftrightarrow\frac{1}{p^2}-\frac{2e^{-p}}{p^2}+\frac{e^{-2p}}{p^2}=\frac{1}{p^2}(1-e^{-p})^2.$$

b) <u>HOMEWORK №3.</u>

The property of partial degeneracy of the original (the advance theorem).

If $f(t) \leftrightarrow F(p)$ and $\tau > 0$, then

$$f(t+\tau) \leftrightarrow e^{p\tau}\left(F(p) - \int_0^{\tau} f(t) e^{-pt}\, dt \right).$$

Here $f(t+\tau) = f(t+\tau)\theta(t)$.

Figure 1 shows graphs of the original functions $f(t)$, $f(t-\tau)\theta(t-\tau)$, $f(t+\tau)$, where $\tau > 0$. To calculate the images of functions $f(t-\tau)\theta(t-\tau)$, $f(t+\tau)$ from the known image $F(p) \leftrightarrow f(t)$, the delay and advance theorems are used, respectively.

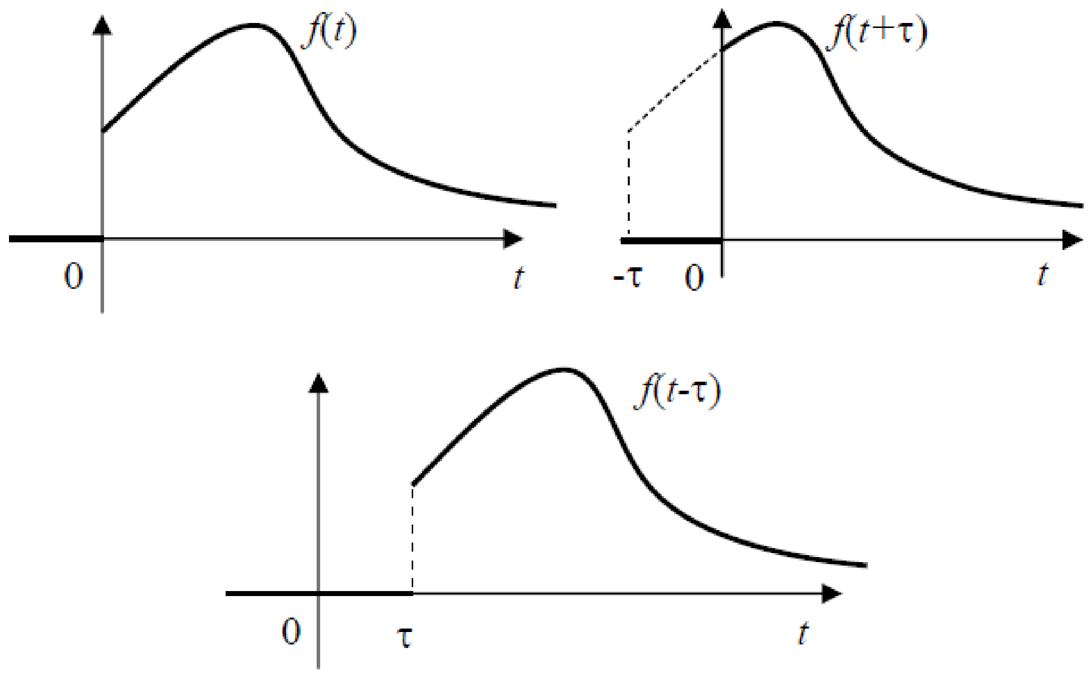


Fig. 1. Graphs of the original functions

Example 15.

15) Find images of the following functions:

a) $f(t)=\sin(t+\tau), \quad \tau>0,$

b) $f(t)=\cos(t+\tau), \quad \tau>0.$

Solution:

a) For the function $sin\, t \leftrightarrow \dfrac{1}{p^2+1}$. By the advance theorem

$$sin(t+\tau) \leftrightarrow e^{p\tau}\left(\frac{1}{p^2+1}-\int_0^{\tau}\sin t e^{-pt}\,dt\right).$$

Since

$$\int_0^{\tau}\sin t e^{-pt}\,dt=\left[\begin{array}{c}\text{integration}\\ \text{by parts,}\\ \text{twice}\end{array}\right]=\frac{-p\sin t-\cos t}{p^2+1}e^{-pt}\Big|_0^{\tau}=$$

$$=\frac{-p\sin\tau-\cos\tau}{p^2+1}e^{-p\tau}-\frac{1}{p^2+1},$$

then according to the advance theorem

$$\sin(t+\tau) \mapsto \frac{p\sin\tau+\cos\tau}{p^2+1}.$$

b) For the function $cos\, t \leftrightarrow \dfrac{p}{p^2+1}$. According to the advance theorem

$$cos(t+\tau) \leftrightarrow e^{p\tau}\left(\frac{p}{p^2+1}-\int_0^{\tau}\cos t e^{-pt}\,dt\right).$$

Since

$$\int_0^{\tau} \cos t e^{-pt} \mathrm{d}t = \left[\begin{array}{c} \text{integration} \\ \text{by parts,} \\ \text{twice} \end{array} \right] = \frac{-p\cos t + \sin t}{p^2+1} e^{-pt} \Bigg|_0^{\tau} =$$

$$= \frac{-p\cos\tau + \sin\tau}{p^2+1} e^{-p\tau} + \frac{p}{p^2+1},$$

then we will get

$$sin(t+\tau) \leftrightarrow \frac{p\cos\tau - \sin\tau}{p^2+1}.$$

The image of the periodic function.

Let the original function $f(t)$ have a period T.

Then if $f_0(t) \leftrightarrow F_0(p)$, where

$$f_0(t) = \begin{cases} f(t) & \text{when } 0 < t < T, \\ 0 & \text{when } t < 0 \text{ and } t > T, \end{cases}$$

then

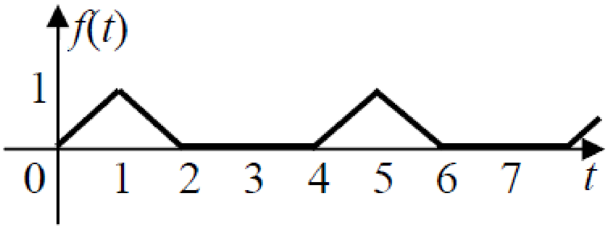
$$f(t) \leftrightarrow \frac{F_0(p)}{1-e^{-pT}}.$$

Example 16.

Find images of the following periodic functions:

a) $f(t) = |\cos t|,$

b) $f(t)=|\sin t|$.



c)

Solution:

a) The function $f(t)=|\cos t|$ is periodic with a period $T=\pi$. Consider the function

$$f_0(t)=\begin{cases}|\cos t|, & 0\le t\le \pi,\\ 0, & t<0,\ t>\pi.\end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t)\leftrightarrow \int_0^{\pi}|\cos t|e^{-pt}\mathrm{d}t=\int_0^{\pi/2}\cos t e^{-pt}\mathrm{d}t-\int_{\pi/2}^{\pi}\cos t e^{-pt}\mathrm{d}t=$$

$$=\left[\int \cos t e^{-pt}\mathrm{d}t=\frac{e^{-pt}}{p^2+1}(-p\cos t+\sin t)\right]=$$

$$=\frac{1}{p^2+1}\left(2e^{-\frac{\pi}{2}p}+p\left(1-e^{-\pi p}\right)\right).$$

According to the formula for the image of the periodic function

$$|\cos t|\leftrightarrow \frac{2e^{-\frac{\pi}{2}p}+p\left(1-e^{-\pi p}\right)}{\left(p^2+1\right)\left(1-e^{-\pi p}\right)}.$$

b) The function $f(t)=|sin t|$ is periodic with a period $T=\pi$.

Consider the function

$$f_0(t)=\begin{cases}|\sin t|, & 0\le t\le \pi,\\ 0, & t<0,\ t>\pi.\end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t)\leftrightarrow \int_0^{\pi}|\sin t|e^{-pt}\mathrm{d}t=\int_0^{\pi}\sin t e^{-pt}\mathrm{d}t=$$

$$=\left[\int \sin t e^{-pt}\mathrm{d}t=-\frac{e^{-pt}}{p^2+1}(p\sin t+\cos t)\right]=\frac{1}{p^2+1}\left(1+e^{-\pi p}\right)$$

According to the formula for the image of the periodic function

$$|sin t|\leftrightarrow \frac{1+e^{-\pi p}}{\left(p^2+1\right)\left(1-e^{-\pi p}\right)}.$$

c) The function is periodic with period $T=4$. Consider the function

$$f_0(t)=\begin{cases}t, & 0\le t\le 1,\\ 2-t, & 1<t\le 2,\\ 0, & 2<t\le 4.\end{cases}$$

Let's find an image for $f_0(t)$.

$$f_0(t)\leftrightarrow \int_0^{1}te^{-pt}\mathrm{d}t+\int_1^{2}(2-t)e^{-pt}\mathrm{d}t=$$

$$= -e^{-pt}\left(\frac{t}{p}+\frac{1}{p^2}\right)\Bigg|_0^1 + e^{-pt}\left(\frac{t-2}{p}+\frac{1}{p^2}\right)\Bigg|_1^2 =$$

$$= -e^{-p}\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{1}{p^2}+\frac{e^{-2p}}{p^2}+e^{-p}\left(\frac{1}{p}-\frac{1}{p^2}\right)=\frac{\left(1-e^{-p}\right)^2}{p^2}.$$

According to the formula for the image of the periodic function

$$f(t) \leftrightarrow \frac{\left(1-e^{-p}\right)^2}{p^2\left(1-e^{-4p}\right)}$$

The theorem of differentiation by parameter.

If, for any fixed x, the function $f(x,t)$ is the original, and $F(p,x)$ is its image, and if in the integral $F(p,x)=\int\limits_0^{+\infty} f(x,t)e^{-pt}dt$ differentiation by parameter x under the sign of the integral is possible, then $\frac{\partial f(x,t)}{\partial x} \leftrightarrow \frac{\partial F(p,x)}{\partial x}$.

This property is used in solving partial differential equations.

1.2. RESTORING THE ORIGINAL IMAGE

1.2.1 The elementary method

In many cases, a given image can be converted to a form where the original is easily restored directly using the properties of the Laplace transform and the table of originals and images.

In this case, the method of decomposing a rational fraction into the sum of the simplest ones is widely used to transform the image.

Let $F(p)$ be a rational function, to find the original, we represent the function $F(p)$ as the sum of the simplest fractions of the form

$$\frac{A}{p-a},\ \frac{Ap+B}{(p-a)^2+b^2},\ \frac{A}{(p-a)^k},\ \frac{Ap+B}{\left((p-a)^2+b^2\right)^k},\ k=2,3,\ldots$$

(A,B,a,b are some constants), for each of which we can construct the corresponding original.

Indeed, using the displacement property and the table of originals and images, we find

$$\frac{A}{p-a} \leftrightarrow Ae^{at},\ \frac{A}{(p-a)^k} \leftrightarrow \frac{A}{(k-1)!}t^{k-1}e^{at},\ k=2,3,\ldots;$$

$$\frac{Ap+B}{(p-a)^2+b^2} = \frac{A(p-a)+B+Aa}{(p-a)^2+b^2} \leftrightarrow Ae^{at}\cos bt + \frac{B+Aa}{b}e^{at}\sin bt\,.$$

Let $S_k(p) \leftrightarrow s_k(t)$. Let's build the original for the image

$$S_k(p) = \frac{Ap+B}{\left((p-a)^2+b^2\right)^k},\ k=2,3,\ldots$$

Consider the expression for image $S_2(p)$. Since

$$S_2(p)=\frac{Ap+B}{(p-a)^2+b^2}\frac{1}{(p-a)^2+b^2},$$

then, applying the convolution image property, we construct the corresponding original:

$$s_2(t)=\frac{1}{b}\int_0^t s_1(t-\tau)e^{a\tau}\sin b\tau d\tau .$$

Here the original $s_1(t)$ is defined by the expression

$$s_1(t)=Ae^{at}\cos bt+\frac{B+Aa}{b}e^{at}\sin bt .$$

Further, since

$$S_3(p)=S_2(p)\frac{1}{(p-a)^2+b^2},$$

then

$$s_3(t)=\frac{1}{b}\int_0^t s_2(t-\tau)e^{a\tau}\sin b\tau d\tau .$$

Similar reasoning leads to the following relation:

$$s_k(t)=\frac{1}{b}\int_0^t s_{k-1}(t-\tau)e^{a\tau}\sin b\tau d\tau ,\ k\geq 2 .$$

Example 1.

Find the original corresponding to the image

$$F(p)=\frac{1}{p^3-p}.$$

Solution:

Decomposing a given image into the sum of the simplest fractions

$$\frac{1}{p^3-p}=\frac{1}{p(p-1)(p+1)}=-\frac{1}{p}+\frac{1}{2(p-1)}+\frac{1}{2(p+1)},$$

we'll find the original

$$f(t)=-1+\frac{1}{2}e^t+\frac{1}{2}e^{-t}=-1+\operatorname{ch}t.$$

Example 2.

Find the original corresponding to the image

$$F(p)=\frac{1}{(p^2+4)^2}.$$

Solution:

Applying the convolution image property, we will have

$$F(p)=\frac{1}{(p^2+4)^2}=\frac{1}{p^2+4}\frac{1}{p^2+4}\leftrightarrow\frac{1}{4}\int_0^t\sin 2(t-\tau)\sin 2\tau d\tau.$$

Having calculated the integral, we get the desired expression for the original

$$f(t)=\frac{1}{16}\sin 2t-\frac{1}{8}t\cos 2t.$$

Example 3.

Find the original corresponding to the image

$$F(p)=\frac{e^{-\frac{p}{2}}}{p(p+1)(p^2+4)}.$$

Solution:

Let's imagine the fraction included in the expression as the simplest fractions:

$$\frac{1}{p(p+1)(p^2+4)}=\frac{A}{p}+\frac{B}{p+1}+\frac{Cp+D}{p^2+4}.$$

Applying the method of undefined coefficients to the decomposition,

we obtain

$$A=\frac{1}{4},\ B=D=-\frac{1}{5},\ C=-\frac{1}{20}.$$

The image has the form

$$F(p)=\frac{1}{4}\frac{e^{-\frac{p}{2}}}{p}-\frac{1}{5}\frac{e^{-\frac{p}{2}}}{p+1}-\frac{1}{20}\frac{pe^{-\frac{p}{2}}}{p^2+4}-\frac{1}{5}\frac{e^{-\frac{p}{2}}}{p^2+4}. \qquad (*)$$

Using the ratios

$$\frac{1}{p}\leftrightarrow\theta(t),\ \frac{1}{p+1}\leftrightarrow e^{-t}\theta(t),\ \frac{p}{p^2+4}\leftrightarrow\cos 2t\theta(t),\ \frac{1}{p^2+4}\leftrightarrow\frac{1}{2}\sin 2t\theta(t)$$

and given the delay property, we get the desired original for the image (*)

$$f(t)=\left(\frac{1}{4}-\frac{1}{5}e^{-\left(t-\frac{1}{2}\right)}-\frac{1}{20}\cos(2t-1)-\frac{1}{10}\sin(2t-1)\right)\theta\left(t-\frac{1}{2}\right).$$

Example 4.

Find the original corresponding to the image

$$F(p)=\frac{e^{-\frac{p}{3}}}{p(p^2+1)}.$$

<u>Solution:</u>

Applying the convolution property and the correspondence table of originals and images, we get

$$\frac{1}{p(p^2+1)}\leftrightarrow\int_0^t \sin\tau d\tau=-\cos\tau\Big|_0^t=(1-\cos t)\theta(t).$$

When constructing the original for a given image, we apply the delay property and get

$$f(t)=\left(1-\cos\left(t-\frac{1}{3}\right)\right)\theta\left(t-\frac{1}{3}\right).$$

1.2.2 The conversion formula. Decomposition theorems

Theorem 1 (Laplace transform conversion formula, Riemann-Mellin formula).

Let $f(t)$ be the original and $F(p)$ be its image. If the function $f(t)$ is continuous at point t and has finite one-sided derivatives at this point, then

$$f(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{pt}F(p)dp. \qquad (1.6)$$

The improper integral (1.6) is taken along any straight line $\operatorname{Re} p = b > \alpha_0$, where α_0 is the growth index of the function $f(t)$ and is understood in the sense of the main value, that is

$$f(t) = \int_{b-i\infty}^{b+i\infty} e^{pt} F(p) dp = \lim_{R\to+\infty} \int_{b-iR}^{b+iR} e^{pt} F(p) dp .$$

The Riemann-Mellin formula (1.6) is the inverse of the formula $F(p) = \int_0^{+\infty} f(t) e^{-pt}\, dt$ and is called *the inverse Laplace transform*.

The direct application of the conversion formula to restore the original $f(t)$ from the $F(p)$ image is difficult. Decomposition theorems are usually used to find the original.

Theorem 2 (the first decomposition theorem).

Let the function $F(p)$ be regular at point $p = \infty, F(\infty) = 0$ and its Laurent series in the vicinity of point $p = \infty$ has the form

$$F(p) = \sum_{k=0}^{\infty} \frac{c_k}{p^{k+1}} = \frac{c_0}{p} + \frac{c_1}{p^2} + \frac{c_2}{p^3} + \ldots,$$

then the function

$$f(t) = \sum_{k=0}^{\infty} c_k \cdot \frac{t^k}{k!} = c_0 + c_1 t + c_2 \cdot \frac{t^2}{2!} + \ldots, \quad t \geq 0$$

is the original with the image $F(p)$.

Definition. A function $F(p)$ is called *meromorphic* in the complex plane if it is regular in any bounded region of the complex plane, with the possible exception of a finite number of singular points of the pole type.

Theorem 3 (the second decomposition theorem).

Let the meromorphic function $F(p)$ be regular in the half-plane $\operatorname{Re} p = \alpha$ and satisfy the conditions:

1) there is a system of circles

$$C_n : |p| = R_n, \; R_1 < R_2 < \ldots < R_n \to \infty \; (n \to \infty)$$

such that $\max\limits_{p \in C_n} |F(p)| \to 0 \; (n \to \infty)$;

2) for $\forall a > \alpha$, the integral $\int\limits_{-\infty}^{\infty} |F(a + i\sigma)| d\sigma$ converges.

Then $F(p)$ is an image, the original for which is the function

$$f(t) = \sum_{p_k} \underset{p=p_k}{res} \left[F(p) e^{pt} \right],$$

where the sum is taken over all poles p_k of the function $F(p)$.

<u>Consequence.</u> If $F(p) = \dfrac{A_n(p)}{B_m(p)}$, where $A_n(p), B_m(p)$ are polynomials of degree n and m, respectively, having no common zeros, and if $n < m$, then

$$f(t)=\sum_{k=1}^{l}\frac{1}{(m_k-1)!}\frac{d^{m_k-1}}{dp^{m_k-1}}\left\{F(p)e^{pt}(p-p_k)^{m_k}\right\}\Big|_{p=p_k}, \qquad (1.7)$$

where $p_1,\ldots,p_l$ are different zeros of the polynomial $B_m(p)$, and m_k is the multiplicity of zero $p_k:\ \sum_{k=1}^{l}m_k=m$.

In particular, if all poles of the function $F(p)$ are simple, then formula (1.7) takes the form:

$$f(t)=\sum_{k=1}^{m}\frac{A_n(p_k)}{B'_m(p_k)}e^{p_k t}\ . \qquad (1.8)$$

Example 5.

Find the original corresponding to the image

$$F(p)=\frac{p^2+2}{p^3-p^2-6p}\ .$$

<u>Solution:</u>

Since $p^3-p^2-6p=p(p-3)(p+2)$, the function $F(p)$ has three simple poles: $p_1=0$, $p_2=3$, $p_3=-2$. Let's construct the corresponding original using the formula (1.8):

$$f(t)=\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=0}+\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=3}+\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\Big|_{p=-2}=-\frac{1}{3}+\frac{11}{5}e^{3t}+\frac{3}{5}e^{-2t}.$$

1.2. RESTORING THE ORIGINAL IMAGE

To find the original function for a given image, it requires knowledge of the tables of correspondence between the originals and images, the application of the properties of the Laplace transform, the decomposition of the image into the simplest fractions, the use of decomposition theorems.

Using the properties of the Laplace transform

First of all, it is necessary to bring the function to a simpler, "tabular" form.

If the denominator of a fraction contains a square trinomial, then the full square is allocated in it.

It is convenient to use the original integration theorem to find the original fraction $\frac{F(p)}{p^n}$ if the original $f(t)$ of the $F(p)$ image is known.

The presence of the $e^{-p\tau}, \tau > 0$ multiplier in image $F(p)$ indicates the need to apply the delay theorem.

If the image is represented as $F(p) = F_1(p)F_2(p)$ or $F(p) = pF_1(p)F_2(p)$ and the originals $f_1(t) \leftrightarrow F_1(p)$, $f_2(t) \leftrightarrow F_2(p)$ are known, then *Borel's theorem* and *Duhamel's integral* are used to find the original $f(t) \leftrightarrow F(p)$, respectively.

Duhamel's integral

If

$$f_1 * f_2 = \int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau \leftrightarrow F_1(p)F_2(p)$$

then

$$f_1(t)\, f_2(0) + \int_0^t f_1(\tau)\, f_2'(t-\tau)\, d\tau \leftrightarrow pF_1(p)F_2(p).$$

Comment

1) Due to the symmetry of the convolution $f_1 * f_2 = f_2 * f_1$

$$f_1(t)\, f_2(0) + \int_0^t f_1(\tau)\, f_2'(t-\tau)\, d\tau = f_1(t)\, f_2(0) + \int_0^t f_1(t-\tau)\, f_2'(\tau)\, d\tau$$

2) Obviously

$$f_1(0)\, f_2(t) + \int_0^t f_1'(\tau) f_2(t-\tau)\, d\tau = f_1(0)\, f_2(t) + \int_0^t f_1'(t-\tau) f_2(\tau)\, d\tau$$

Example 6.

Find the original corresponding to the image (using the Duhamel integral):

a) $F(p) = \dfrac{2p^2}{\left(p^2+1\right)^2}$,

b) $F(p) = \dfrac{p^3}{\left(p^2+1\right)\left(p^2+4\right)}$,

c) $F(p)=\dfrac{2p}{(p-1)\left(p^2-2p-3\right)}$.

<u>Solution:</u>

a) Let's write an image in the form

$$\frac{2p^2}{\left(p^2+1\right)^2}=2p\cdot\frac{1}{p^2+1}\cdot\frac{p}{p^2+1}.$$

Since

$$\frac{1}{p^2+1}\leftrightarrow \sin t=f_1(t),\ \frac{p}{p^2+1}\leftrightarrow \cos t=f_2(t),$$

$$f_1(0)=\sin 0=0,\ f_1'(t)=\cos t,$$

then based on the Duhamel formula we have

$$2p\cdot\frac{1}{p^2+1}\cdot\frac{p}{p^2+1}\leftrightarrow 0+2\int_0^t\cos\tau\cos(t-\tau)\,d\tau=$$

$$=2\int_0^t\frac{\cos t+\cos(t-2\tau)}{2}d\tau=\left(\tau\cos t-\frac{1}{2}\sin(t-2\tau)\right)\Bigg|_0^t=t\cos t+\sin t.$$

b) Let's write an image in the form

$$\frac{p^3}{\left(p^2+1\right)\left(p^2+4\right)}=p\cdot\frac{p}{p^2+1}\cdot\frac{p}{p^2+4}.$$

$$\frac{p}{p^2+1}\leftrightarrow \cos t=f_1(t),\ \frac{p}{p^2+4}\leftrightarrow \cos 2t=f_2(t),$$

$$f_1(0)=\cos 0=1,\quad f_1'(t)=-\sin t.$$

Then we get

$$f(t)=f_1(0)f_2(t)+\int_0^t f_1'(\tau)f_2(t-\tau)\,d\tau=$$

$$=\cos 0\cdot\cos 2t+\int_0^t \cos'\tau\cdot\cos 2(t-\tau)\,d\tau=$$

$$=\cos 2t-\int_0^t \sin\tau\cos 2(t-\tau)\,d\tau=$$

$$=\cos 2t-\frac{1}{2}\int_0^t\left(\sin(3\tau-2t)+\sin(2t-\tau)\right)d\tau=$$

$$=\cos 2t+\frac{1}{6}\cos(3\tau-2t)\Big|_0^t-\frac{1}{2}\cos(2t-\tau)\Big|_0^t=\frac{4}{3}\cos 2t-\frac{1}{3}\cos t.$$

c) <u>HOMEWORK №1</u>

Example 7.

Using the properties of the Laplace transform to find the original corresponding to the image:

a) $F(p)=\dfrac{p}{p^2-2p+26}$,

b) $F(p)=\dfrac{1}{p\left(p^2+4\right)}$,

c) $F(p)=\dfrac{e^{-p}}{p+1}$,

d) $F(p)=\dfrac{p}{(p^2+4)^2}$

<u>Solution:</u>

a) Let's transform the image by highlighting the full square in the denominator. To find the original, we will use the displacement theorem, the linearity property and the image table.

$$\frac{p}{p^2-2p+26}=\frac{(p-1)+1}{(p-1)^2+25}=\frac{p-1}{(p-1)^2+25}+\frac{1}{(p-1)^2+25}\leftrightarrow$$

$$\leftrightarrow e^t\left(\cos 5t+\frac{1}{5}\sin 5t\right)$$

b) From the table of images, we have $\frac{2}{p^2+4}\leftrightarrow \sin 2t$.

Using the linearity and integration properties of the original, we find

$$F(p)=\frac{1}{p\left(p^2+4\right)}=\frac{1}{2}\cdot\frac{1}{p}\cdot\frac{2}{p^2+4}\leftrightarrow$$

$$\frac{1}{2}\int_0^t \sin 2\tau d\tau=-\frac{1}{4}\cos 2\tau\Big|_0^t=\frac{1}{4}(1-\cos 2t).$$

You can also find the original by representing the original function as the sum of the simplest fractions,

$$F(p)=\frac{1}{4}\left(\frac{1}{p}-\frac{p}{p^2+4}\right)\leftrightarrow$$

$$\leftrightarrow\frac{1}{4}(1-\cos 2t)$$

c) From the table of images we have $\frac{1}{p+1}\leftrightarrow e^{-t}$. The presence of a multiplier e^{-p} indicates the need to apply the delay theorem. Therefore

$$\frac{e^{-p}}{p+1}\leftrightarrow e^{-(t-1)}\theta(t-1)$$

d) Let's write an image in the form

$$F(p)=\frac{p}{(p^2+4)^2}=\frac{1}{p^2+4}\cdot\frac{p}{p^2+4},$$

$$\frac{1}{p^2+4}\leftrightarrow \sin 2t,\ \frac{p}{p^2+4}\leftrightarrow \cos 2t\,.$$

Let's apply the image multiplication theorem (Borel's theorem)

$$F(p)\leftrightarrow \sin 2t*\cos 2t=\int_0^t \sin 2\tau\cos 2(t-\tau)d\tau=$$

$$=\frac{1}{2}\int_0^t(\sin(4\tau-2t)+\sin 2t)d\tau=\frac{1}{2}\left(-\frac{\cos(4\tau-2t)}{4}+\tau\cdot\sin 2t\right)\Bigg|_0^t=$$

$$=\frac{1}{2}\left(-\frac{\cos 2t}{4}+t\sin 2t+\frac{\cos(-2t)}{4}\right)=\frac{t}{2}\sin 2t.$$

1.2.1 The elementary method

Example 8.

Find the original corresponding to the image:

a) $$F(p)=\frac{-5}{p(p-1)\left(p^2+4p+5\right)},$$

b) $$F(p)=\frac{1}{p(p-1)\left(p^2+4\right)}.$$

<u>Solution:</u>

a) Let's imagine $F(p)$ as the sum of elementary fractions:

$$F(p)=\frac{-5}{p(p-1)\left(p^2+4p+5\right)}=\frac{A}{p}+\frac{B}{p-1}+\frac{Cp+D}{p^2+4p+5}.$$

To find A,B,C,D we have the equation

$$A(p-1)\left(p^2+4p+5\right)+Bp\left(p^2+4p+5\right)+(Cp+D)p(p-1)=-5\,.$$

Substituting different values of p, we obtain a system for determining the coefficients

$$p=0:\quad -5A=-5,\qquad p=1:\quad 10B=-5,$$
$$p=-1:\quad -4A-2B-2(-C+D)=-5,$$
$$p=-2:\quad -3A-2B+6(-2C+D)=-5.$$

We find the coefficients:

$$A=1,\quad B=-\frac{1}{2},\quad C=-\frac{1}{2},\quad D=-\frac{3}{2}.$$

$$F(p)=\frac{1}{p}-\frac{1}{2}\cdot\frac{1}{p-1}-\frac{1}{2}\cdot\frac{p+3}{p^2+4p+5}=$$

$$=\frac{1}{p}-\frac{1}{2}\frac{1}{p-1}-\frac{1}{2}\left(\frac{p+2}{(p+2)^2+1}+\frac{1}{(p+2)^2+1}\right)\leftrightarrow$$

$$1-\frac{1}{2}e^{t}-\frac{1}{2}e^{-2t}(\cos t+\sin t).$$

b) HOMEWORK №2

1.2.2 The conversion formula. Decomposition theorems

Theorem 1 (Riemann-Mellin).

Let the function $f(t)$ be the original with the growth index α_0, and $F(p)$ be its image. Then at any point t of the continuity of the original $f(t)$, the Riemann-Mellin formula is valid

$$f(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}F(p)e^{pt}dp \qquad (1.6)$$

where integration is performed along any straight line $\operatorname{Re} p = b$, $b > \alpha_0$, and the integral is understood in the sense of the principal value.

Equality takes place at every point at which $f(t)$ is continuous. At the point t_0, which is the point of discontinuity of the 1st kind of the function $f(t)$, the right side of the Riemann-Mellin formula is equal to

$$\frac{1}{2}\left(f(t_0-0)+f(t_0+0)\right).$$

The Riemann-Mellin formula (1.6) is the inverse of the formula $F(p)=\int_0^{+\infty} f(t)e^{-pt}\,dt$ and is called *the inverse Laplace transform*.

The direct application of the conversion formula to restore the original $f(t)$ from the $F(p)$ image is difficult. Decomposition theorems are usually used to find the original.

Theorem 2 (the first decomposition theorem).

If the function $F(p)$ in the vicinity of point $p=\infty$ can be represented as a Laurent series (point $p=\infty$ is the zero of the function $F(p)$ and $F(p)$is analytic in the vicinity of this point)

$$F(p)=\sum_{k=0}^{\infty}\frac{c_k}{p^{k+1}}=\frac{c_0}{p}+\frac{c_1}{p^2}+\frac{c_2}{p^3}+\ldots,$$

then the function

$$f(t)=\sum_{k=0}^{\infty} c_k \cdot \frac{t^k}{k!}=c_0+c_1t+c_2\cdot\frac{t^2}{2!}+\dots,\quad t\ge 0$$

is the original with the image $F(p)$.

Example 9.

Find the original corresponding to the image using the first decomposition theorem:

a) $F(p)=\dfrac{p}{p^2+1}$,

b) $F(p)=\dfrac{1}{p\left(p^4+1\right)}$,

c) $F(p)=\dfrac{1}{p}e^{\frac{1}{p^2}}$.

<u>Solution:</u>

a) Decompose the function $F(p)$ into a Laurent series

$$\frac{p}{p^2+1}=\frac{1}{p\left(1+\frac{1}{p^2}\right)}=\frac{1}{p}\left(1-\frac{1}{p^2}+\frac{1}{p^4}-\dots\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{2n+1}},\quad |p|>1.$$

Since $\dfrac{1}{p^{2n+1}}\leftrightarrow\dfrac{t^{2n}}{(2n)!}$, then according to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{2n+1}}\leftrightarrow\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}=\cos t=f(t).$$

b) Decompose the function $F(p)$ into a Laurent series

$$\frac{1}{p\left(p^4+1\right)}=\frac{1}{p^5\left(1+\frac{1}{p^4}\right)}=\frac{1}{p^5}\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n}}=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n+5}}, \quad |p|>1 .$$

Since $\frac{1}{p^{4n+5}} \leftrightarrow \frac{t^{4(n+1)}}{(4(n+1))!}$, then according to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{(-1)^n}{p^{4n+5}} \leftrightarrow \sum_{n=0}^{\infty}\frac{(-1)^n t^{4(n+1)}}{(4(n+1))!}=f(t)$$

c) Using the power series expansion of the function

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

we obtain

$$\frac{1}{p}e^{\frac{1}{p^2}}=\frac{1}{p}\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n}}=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}} .$$

According to the first decomposition theorem

$$F(p)=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}} \leftrightarrow \sum_{n=0}^{\infty}\frac{t^{2n}}{n!(2n)!}=f(t) .$$

$$F(p)=\sum_{n=0}^{\infty}\frac{1}{n!\,p^{2n+1}} \leftrightarrow \sum_{n=0}^{\infty}\frac{t^{2n}}{n!(2n)!}=f(t)$$

Theorem 3 (the second decomposition theorem).

Let the function $F(p)$ of the complex variable p be analytic in the entire plane, with the exception of a finite number of isolated singularity points $p_1, p_2, \ldots, p_n$, located in the half-plane $\operatorname{Re} p < \alpha_0$.

If $\lim\limits_{p\to\infty} F(p) = 0$, and $F(p)$ is absolutely integrable along any vertical line $\operatorname{Re} p = b,\ b > \alpha_0$, then $F(p)$ is an image, and the original $f(t)$ corresponding to the image $F(p)$ is determined by the formula

$$F(p) \leftrightarrow \sum_{k=1}^{n} \operatorname*{Res}_{p=p_k}\left[F(p)e^{pt}\right] = f(t)$$

If p_k is a pole of order m_k, then

$$\operatorname*{Res}_{p=p_k}\left[F(p)e^{pt}\right] = \lim_{p\to p_k}\left\{\frac{d^{m_k-1}}{dp^{m_k-1}}\left((p-p_k)^{m_k} F(p)e^{pt}\right)\right\} =$$
$$= \sum_{j=0}^{m_k-1} \frac{t^{m_k-1-j}}{j!(m_k-1-j)!} \lim_{p\to p_k}\left\{\frac{d^j}{dp^j}\left((p-p_k)^{m_k} F(p)\right)\right\}.$$

If $F(p) = \dfrac{P(p)}{Q(p)}$ is a rational regular irreducible fraction, p_k are poles of the order m_k, $(k = 1,2,\ldots,n)$ of the function $F(p)$, then the original $f(t)$ corresponding to the image $F(p)$ is determined by the formula

$$F(p) \leftrightarrow$$

$$\sum_{k=1}^{n} \frac{1}{(m_k-1)!} \lim_{p\to p_k}\left\{\frac{d^{m_k-1}}{dp^{m_k-1}}\left((p-p_k)^{m_k} F(p)e^{pt}\right)\right\} = f(t). \tag{1.7}$$

In particular, if $p_1, p_2, \ldots, p_n$ are the simple poles of $F(p)$, then the function

$$f(t)=\sum_{k=1}^{n}\frac{P(p_k)}{Q'(p_k)}e^{p_k t} \qquad (1.8)$$

Example 10.

Find the original corresponding to the image

$$F(p)=\frac{p^2+2}{p^3-p^2-6p}.$$

<u>Solution:</u>

Since $p^3-p^2-6p=p(p-3)(p+2)$, the function $F(p)$ has three simple poles: $p_1=0$, $p_2=3$, $p_3=-2$. Let's construct the corresponding original using the formula (1.8):

$$f(t)=\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=0}+\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=3}+\left.\frac{(p^2+2)e^{pt}}{3p^2-2p-6}\right|_{p=-2}=-\frac{1}{3}+\frac{11}{5}e^{3t}+\frac{3}{5}e^{-2t}.$$

Example 11.

Using the second decomposition theorem, find the original corresponding to the image

a) $F(p)=\dfrac{p^2+p-1}{(p-2)(p-5)(p+4)}$

b) $F(p)=\dfrac{p-1}{(p+1)\left(p^2+4\right)}$

<u>Solution:</u>

a) Function $F(p)=\dfrac{p^2+p-1}{(p-2)(p-5)(p+4)}$ has simple poles (zeros of the denominator) $p_1=2, p_2=5, p_3=-4$. Let's denote

$$P(p)=p^2+p-1, \quad Q(p)=p^3-3p^2-18p+40,$$

$$Q'(p)=3p^2-6p-18.$$

Then for $p_1=2$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_1=2}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_1=2}=-\frac{5}{18},$$

for $p_2=5$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_2=5}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_2=5}=\frac{29}{27},$$

for $p_3=-4$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_3=-4}=\left.\frac{p^2+p-1}{3p^2-6p-18}\right|_{p_3=-4}=\frac{11}{54}.$$

Therefore, according to the formula (1.8)

$$F(p)=\frac{p^2+p-1}{(p-2)(p-5)(p+4)}\leftrightarrow$$

$$-\frac{5}{18}e^{2t}+\frac{29}{27}e^{5t}+\frac{11}{54}e^{-4t}=\frac{1}{54}\left(11e^{-4t}+58e^{5t}-15e^{2t}\right)=f(t).$$

b) <u>HOMEWORK №3</u>

1.3. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF DIFFERENTIAL EQUATIONS AND SYSTEMS

The method of solving various classes of equations and other problems using the Laplace transform is called the *operational method*.

1.3.1.Differential equations and systems with constant coefficients

Consider an n -th order linear differential equation with constant coefficients:

$$L(x) \equiv x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_{n-1} x'(t) + a_n x(t) = f(t). \qquad (1.10)$$

Let's set the Cauchy problem: to find a solution to equation (1.10) satisfying the conditions:

$$x(0) = x_0, \ x'(0) = x_1, .., \ x^{(n-1)}(0) = x_{n-1}, \qquad (1.11)$$

where x_i are the specified constants, $i = 0,1,...,n-1$.

Assuming that the function $f(t)$ is the original, we will look for the solution $x(t)$ of the problem (1.10)–(1.11) on the set of originals.

Let $X(p) \leftrightarrow x(t), F(p) \leftrightarrow f(t)$. According to the rule of differentiation of the original and the property of linearity, passing to images in equation (1.10), due to the conditions (1.11), we obtain an equation for an unknown image $X(p)$, which we will call the *operator equation*

$$A(p)X(p) - B(p) = F(p),$$

where

$$A(p) = p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n ,$$

$$B(p) = x_0 (p^{n-1} + a_1 p^{n-2} + \ldots + a_{n-1}) +$$
$$+ x_1 (p^{n-2} + a_1 p^{n-3} + \ldots + a_{n-2}) + \ldots + x_{n-2} (p + a_1) + x_{n-1} .$$

Hence

$$X(p) = \frac{B(p) + F(p)}{A(p)} .$$

To find the required solution $x(t)$ of the problem (1.10)– (1.11), it is necessary to restore the original $x(t)$ from its image $X(p)$.

Similarly, the operational method is applied to solving systems of differential equations with constant coefficients.

Example 1

Solve the Cauchy problem:

a) $x' - x = 1,$

$x(0) = -1,$

b) $x'' + x = 2\cos t,$

$x(0) = 0, \quad x'(0) = -1,$

c) $x'' + 2x = t + \dfrac{t^3}{3},$

$x(0) = x'(0) = 0$

d) $x'' - 3x' + 2x = 2e^{3t},$

$x(0)=1,\ x'(0)=3.$

e) $x''+x'+6x=3(\cos 3t-\sin 3t),$

$x(0)=0, x'(0)=3.$

<u>Solution:</u>

a) Let $x(t)\leftrightarrow X(p)$.

Then, according to the original differentiation theorem, we get

$x'(t)\leftrightarrow$ $pX(p)-x(0)=pX(p)+1$.

Let's apply the Laplace transform to both parts of the equation. Let's write out the resulting operator equation

$$pX(p)+1-X(p)=\frac{1}{p}.$$

We get

$$X(p)=-\frac{1}{p}.$$

Thus

$$x(t)=-1.$$

b) Let's move on from the originals to the images

$x(t)\leftrightarrow X(p),$

$x'(t)\leftrightarrow$ $pX(p)-x(0)=pX(p),$

$x''(t) \leftrightarrow \quad p^2X(p) - px(0) - x'(0) = p^2X(p) + 1,$

$\cos t \leftrightarrow \quad \dfrac{p}{p^2+1}.$

Let's write down the equation for the images

$$p^2X(p) + 1 + X(p) = \frac{2p}{p^2+1}.$$

Let's solve the equation for images

$$X(p) = \frac{2p}{\left(p^2+1\right)^2} - \frac{1}{p^2+1}.$$

According to the image differentiation theorem, we will find the original of the first term

$$\frac{2p}{\left(p^2+1\right)^2} = -\left(\frac{1}{p^2+1}\right)' \leftrightarrow t\sin t.$$

Therefore, the solution has the form

$$x(t) = t\sin t - \sin t = (t-1)\sin t.$$

c) Let $x(t) \leftrightarrow X(p)$.

Let's move on to the images in the equation

$$p^2X(p) - px(0) - x'(0) + 2X(p) = \frac{1}{p^2} + \frac{1}{3}\frac{3!}{p^4}.$$

Since $x(0) = x'(0) = 0$, then

$$\left(p^2+2\right)X=\frac{p^2+2}{p^4}$$

$$X(p)=\frac{1}{p^4}.$$

Having found the original for this image, we get a solution to the Cauchy problem

$$X(p)=\frac{1}{p^4}=\frac{3!}{3!p^4}\leftrightarrow\frac{t^3}{3!}=\frac{t^3}{6}=x(t).$$

d) Let's move on from the originals to the images

$$x(t)\leftrightarrow X(p)$$

$$x'(t)\leftrightarrow pX(p)-1,$$

$$x''(t)\leftrightarrow p^2X(p)-p-3,$$

$$e^{3t}\leftrightarrow\frac{1}{p-3}.$$

Let's write down the equation for the images

$$p^2X(p)-p-3-3pX(p)+3+2X(p)=\frac{2}{p-3}.$$

Let's solve the equation for images

$$\left(p^2-3p+2\right)X(p)=\frac{2}{p-3}+p,$$

$$X(p)=\frac{p^2-3p+2}{(p-3)\left(p^2-3p+2\right)}=\frac{1}{p-3}.$$

Let's find the original for the function $X(p)$

$$X(p)=\frac{1}{p-3}\leftrightarrow e^{3t}=x(t)$$

e) <u>HOMEWORK</u>

Example 2

Find a solution of the differential equation

$$x'(t)+x(t)=e^{-t},$$

satisfying the condition $x(0)=1$ (Cauchy problem).

<u>Solution:</u>

Let $x(t)\leftrightarrow X(p)$. Since

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-1,$$

$$e^{-t}\leftrightarrow\frac{1}{p+1},$$

applying the Laplace transform to a given equation using the linearity property, we obtain an algebraic equation for $X(p)$:

$$pX(p) - 1 + X(p) = \frac{1}{p+1}.$$

From where we find for $X(p)$:

$$X(p) = \frac{1}{(p+1)^2} + \frac{1}{p+1}.$$

Since

$$\frac{1}{p+1} \leftrightarrow e^{-t}, \ \frac{1}{(p+1)^2} \leftrightarrow te^{-t},$$

we have

$$X(p) \leftrightarrow x(t) = te^{-t} + e^{-t}.$$

<u>Check.</u>

We show that the found function is indeed a solution to the Cauchy problem. We substitute the expression for the function $x(t)$ and its derivative

$$x'(t) = -te^{-t} + e^{-t} - e^{-t} = -te^{-t}$$

into the given equation

$$-te^{-t} + te^{-t} + e^{-t} = e^{-t}.$$

After addition of similar terms in the left part of the equation, we get the correct identity: $e^{-t} \equiv e^{-t}$.

Thus, the constructed function is a solution to the equation.

Let's check if it satisfies the initial condition $x(0) = 1$:

$$x(0) = 0 \cdot e^{-0} + e^{-0} = 1.$$

Therefore, the found function is a solution to the Cauchy problem.

Example 3

Find a solution of the differential equation

$$x''(t) + 3x'(t) = e^t ,$$

satisfying the condition $x(0) = 0,\ x'(0) = -1$.

<u>Solution:</u>

We apply the Laplace transform to the equation. Using the property of linearity and considering that

$$x(t) \leftrightarrow X(p) ,$$

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - 0 = pX(p) ,$$

$$x''(t) \leftrightarrow p^2 X(p) - px(0) - x'(0) = p^2 X(p) - p \cdot 0 - (-1) = p^2 X(p) + 1 ,$$

$$e^t \leftrightarrow \frac{1}{p-1} ,$$

we obtain an algebraic equation for $X(p)$:

$$p^2 X(p) + 1 + 3pX(p) = \frac{1}{p-1} , \Leftrightarrow (p^2 + 3p)X(p) = \frac{1}{p-1} - 1 .$$

We will find a fundamental solution:

$$H(p) = \frac{1}{(p^2 + 3p)} = \frac{1}{3}\left(\frac{1}{p} - \frac{1}{p+3}\right) \leftrightarrow h(t) = \frac{1}{3}\left(1 - e^{-3t}\right).$$

Then, since

$$X(p) = \left(\frac{1}{p-1} - 1\right) H(p) = \frac{1}{p-1} H(p) - H(p) ,$$

using the convolution image property, we will write the solution of the given equation in the form

$$x(t)=\frac{1}{3}\int_0^t e^{t-\tau}(1-e^{-3\tau})d\tau-\frac{1}{3}(1-e^{-3t}).$$

Having calculated the integrals and addition of similar terms, we get the final answer:

$$x(t)=-\frac{2}{3}+\frac{1}{4}e^{t}+\frac{5}{12}e^{-3t}.$$

<u>Check.</u>

We have

$$x(t)=-\frac{2}{3}+\frac{1}{4}e^{t}+\frac{5}{12}e^{-3t},\ x'(t)=\frac{1}{4}e^{t}-\frac{5}{4}e^{-3t},\ x''(t)=\frac{1}{4}e^{t}+\frac{15}{4}e^{-3t}.$$

We substitute everything into a given equation

$$\frac{1}{4}e^{t}+\frac{15}{4}e^{-3t}+3\left(\frac{1}{4}e^{t}+\frac{5}{4}e^{-3t}\right)\equiv e^{t}.$$

As a result, we get the identity $e^{t}\equiv e^{t}$. Therefore, the found function is a solution to the equation. Let's check the fulfillment of the initial conditions:

$$x(0)=-\frac{2}{3}+\frac{1}{4}e^{0}+\frac{5}{12}e^{-0}=0;\ x'(0)=\frac{1}{4}e^{0}-\frac{5}{4}e^{-0}=-1.$$

Therefore, the found function is a solution of the Cauchy problem.

Example 4

Find a solution of the differential equation

$$x'''(t) + 2x''(t) + 5x'(t) = 0 \,,$$

satisfying the conditions: $x(0) = -1, \; x'(0) = 2, \; x''(0) = 0$.

<u>Solution:</u>

Let $x(t) \leftrightarrow X(p)$.

Since, take into account the given conditions, we have

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - (-1) = pX(p) + 1 \,,$$

$$x''(t) \leftrightarrow p^2X(p) - px(0) - x'(0) = p^2X(p) - p(-1) - 2 = p^2X(p) + p - 2 \,,$$

$$x'''(t) \leftrightarrow p^3X(p) - p^2x(0) - px'(0) - x''(0) =$$

$$= p^3X(p) - p^2(-1) - p2 - 0 = p^3X(p) + p^2 - 2p \,,$$

then, after applying the Laplace transform for a given equation, we obtain the following operator equation:

$$p^3X(p) + p^2 - 2p + 2p^2X(p) + 2p - 4 + 5pX(p) + 5 = 0 \,,$$

or after the transformations:

$$X(p)(p^3 + 2p^2 + 5p) = -p^2 - 1 \,.$$

Solving this equation for $X(p)$, we obtain

$$X(p) = \frac{-p^2 - 1}{p(p^2 + 2p + 5)} \,.$$

The resulting expression is decomposed into the simplest fractions:

$$\frac{-p^2 - 1}{p(p^2 + 2p + 5)} = \frac{A}{p} + \frac{Bp + C}{p^2 + 2p + 5} \,.$$

Using the method of undefined coefficients, we find A, B, C.

To do this, we bring the fractions to a common denominator and equate the coefficients with the same degrees of p:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{Ap^2+2Ap+5A+Bp^2+Cp}{p(p^2+2p+5)}.$$

We obtain a system of algebraic equations for A,B,C:

$$A+B=-1,\ 2A+C=0,\ 5A=-1,$$

the solution of which will be: $A=-\frac{1}{5}, B=-\frac{4}{5}, C=\frac{2}{5}$.

Then

$$X(p)=-\frac{1}{5p}+\frac{1}{5}\frac{-4p+2}{p^2+2p+5}.$$

To find the original of the second fraction, select the full square in its denominator: $p^2+2p+5=(p+1)^2+4$, then select the summand $p+1$ in the numerator:

$$-4p+2=-4(p+1)+6,$$

and decompose the fraction into the sum of two fractions:

$$\frac{1}{5}\frac{-4p+2}{p^2+2p+5}=-\frac{4}{5}\frac{p+1}{(p+1)^2+4}+\frac{3}{5}\frac{2}{(p+1)^2+4}.$$

Next, using the displacement property and the table of correspondence between images and originals, we obtain a solution to the original equation:

$$x(t)=-\frac{1}{5}-\frac{4}{5}e^{-t}\cos 2t+\frac{3}{5}e^{-t}\sin 2t.$$

Example 5

Find a solution to the system:

$$\begin{cases} \dfrac{dx_1}{dt} = x_1 + 2x_2 + \sin t, \\ \dfrac{dx_2}{dt} = -x_1 + x_2 + 1, \end{cases}$$

satisfying the initial conditions $x_1(0) = 1,\ x_2(0) = 0$.

Solution:

Let's construct a solution using the Laplace transform, first reducing the system to one equivalent second-order equation.

Let's express the unknown function $x_2(t)$ from the first equation of the system

$$x_2 = \frac{1}{2}\left(\frac{dx_1}{dt} - x_1 - \sin t\right),$$

$$\frac{dx_2}{dt} = \frac{1}{2}\left(\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt} - \cos t\right)$$

and substitute it into the second equation

$$\frac{1}{2}\left(\frac{d^2x_1}{dt^2} - \frac{dx_1}{dt} - \cos t\right) = -x_1 + \frac{1}{2}\left(\frac{dx_1}{dt} - x_1 - \sin t\right) + 1 .$$

Let's transform the resulting equation by entering the notation $f(t)$ for the right side:

$$\frac{d^2x_1}{dt^2} - 2\frac{dx_1}{dt} + 3x_1 = \cos t - \sin t \equiv f(t) . \qquad (*)$$

Let's find the initial conditions

$$x_1(t)\big|_{t=0}=1\,;\;\; x_1'(t)\big|_{t=0}=(x_1+2x_2+\sin t)\big|_{t=0}=1\,. \qquad (**)$$

Let's apply the Laplace transform to the equation (*) with initial conditions (**).

Let $X_1(p)\leftrightarrow x_1(t)$, $F(p)\leftrightarrow f(t)$, then

$$p^2X_1(p)-p-1-2pX_1(p)+2+3X_1(p)=F(p)\,,$$

$$X_1(p)(p^2-2p+3)=F(p)+p-1\,,$$

$$X_1(p)=\frac{F(p)}{p^2-2p+3}+\frac{p-1}{p^2-2p+3}\,.$$

We will find a fundamental solution:

$$h(t)\leftrightarrow H(p)=\frac{1}{p^2-2p+3}=\frac{1}{(p-1)^2+2}\leftrightarrow\frac{1}{\sqrt{2}}e^t\sin\sqrt{2}t\,.$$

Let's find the original $x_1(t)$, given that $h'(t)\leftrightarrow pH(p)-h(0)=pH(p)$,

$$x_1(t)=\int_0^t h(t-\tau)f(\tau)d\tau+h'(t)-h(t)\,.$$

The expression for the function $x_2(t)$ can be constructed using the second equation of a given system, substituting the found expression for the function $x_1(t)$ into it:

$$x_2=\frac{1}{2}\left(\frac{dx_1}{dt}-x_1-\sin t\right).$$

As a result

$$x_1(t)=\frac{2}{3}-\frac{1}{2}t+\frac{1}{3}e^t\cos\sqrt{2}t+\frac{7\sqrt{12}}{12}e^t\sin\sqrt{2}t\,,$$

$$x_2(t)=-\frac{1}{3}-\frac{1}{4}\cos t-\frac{1}{4}\sin t+\frac{7}{12}e^t\cos\sqrt{2}t-\frac{\sqrt{2}}{6}e^t\sin\sqrt{2}t\,.$$

Example 4

Find a solution of the differential equation

$$x'''(t)+2x''(t)+5x'(t)=0\,,$$

satisfying the conditions: $x(0)=-1,\ x'(0)=2,\ x''(0)=0$.

Solution:

Let $x(t)\leftrightarrow X(p)$.

Since, take into account the given conditions, we have

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-(-1)=pX(p)+1\,,$$

$$x''(t)\leftrightarrow p^2X(p)-px(0)-x'(0)=p^2X(p)-p(-1)-2=p^2X(p)+p-2\,,$$

$$x'''(t)\leftrightarrow p^3X(p)-p^2x(0)-px'(0)-x''(0)=$$

$$=p^3X(p)-p^2(-1)-p2-0=p^3X(p)+p^2-2p\,,$$

then, after applying the Laplace transform for a given equation, we obtain the following operator equation:

$$p^3X(p)+p^2-2p+2p^2X(p)+2p-4+5pX(p)+5=0\,,$$

or after the transformations:

$$X(p)(p^3+2p^2+5p)=-p^2-1\,.$$

Solving this equation for $X(p)$, we obtain

$$X(p)=\frac{-p^2-1}{p(p^2+2p+5)}\,.$$

The resulting expression is decomposed into the simplest fractions:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{A}{p}+\frac{Bp+C}{p^2+2p+5}\,.$$

Using the method of undefined coefficients, we find A, B, C.

To do this, we bring the fractions to a common denominator and equate the coefficients with the same degrees of p:

$$\frac{-p^2-1}{p(p^2+2p+5)} = \frac{Ap^2+2Ap+5A+Bp^2+Cp}{p(p^2+2p+5)}.$$

We obtain a system of algebraic equations for A,B,C:

$$A+B=-1,\ 2A+C=0,\ 5A=-1,$$

the solution of which will be: $A=-\frac{1}{5}, B=-\frac{4}{5}, C=\frac{2}{5}$.

Then

$$X(p) = -\frac{1}{5p} + \frac{1}{5}\frac{-4p+2}{p^2+2p+5}.$$

To find the original of the second fraction, select the full square in its denominator: $p^2+2p+5=(p+1)^2+4$, then select the summand $p+1$ in the numerator:

$$-4p+2=-4(p+1)+6,$$

and decompose the fraction into the sum of two fractions:

$$\frac{1}{5}\frac{-4p+2}{p^2+2p+5} = -\frac{4}{5}\frac{p+1}{(p+1)^2+4} + \frac{3}{5}\frac{2}{(p+1)^2+4}.$$

Next, using the displacement property and the table of correspondence between images and originals, we obtain a solution to the original equation:

$$x(t) = -\frac{1}{5} - \frac{4}{5}e^{-t}\cos 2t + \frac{3}{5}e^{-t}\sin 2t.$$

Example 5

Solve the Cauchy problem:

$$x''' - x'' - 6x' = 0,$$

$$x(0) = 15,\ x'(0) = 2,\ x''(0) = 56.$$

Solution:

Let's move on from the originals to the images:

$x(t) \leftrightarrow X(p)$,

$x'(t) \leftrightarrow pX(p) - 15$,

$x'''(t) \leftrightarrow p^3 X(p) - 15p^2 - 2p - 56$.

Let's solve the equation for images

$$(p^3 - p^2 - 6p)X(p) = 15p^2 - 13p - 36,$$

$$X(p) = \frac{15p^2 - 13p - 36}{p(p-3)(p+2)}.$$

The function $X(p)$ is a proper rational irreducible fraction for which the points $p_1 = 0,\ p_2 = 3,\ p_3 = -2$ are simple poles. Since

$$P(p) = 15p^2 - 13p - 36,$$

$$Q(p) = p^3 - p^2 - 6p,$$

$$Q'(p) = 3p^2 - 2p - 6,$$

that's for

$p_1 = 0$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_1=0} = \frac{-36}{-6} = 6\,,$$

for

$p_2 = 3$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_2=3} = \frac{60}{15} = 4\,,$$

For

$p_3 = -2$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_3=-2} = \frac{50}{10} = 5\,,$$

and by the second decomposition theorem we get

$$x(t) = 6 + 5e^{-2t} + 4e^{3t}\,.$$

<u>Remark</u>

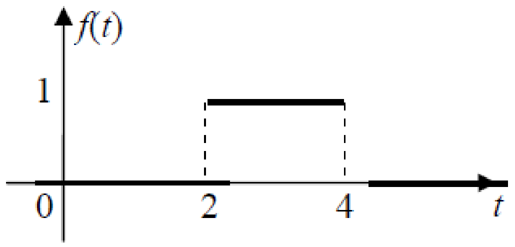
In many practical problems, the right side of the differential equation is given graphically. In this case, the solution algorithm does not change, and the delay theorem and methods from properties are used to find the image of the original given by the graph.

Example 6

Solve the Cauchy problem for a differential equation with the right-hand side given graphically:

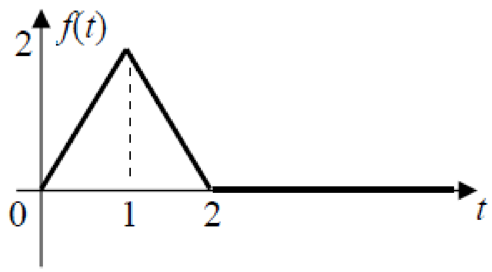
a)

$$x''+x=f(t),$$
$$x(0)=x'(0)=0,$$



b)

$$x''+4x=f(t),$$
$$x(0)=x'(0)=0,$$



<u>Solution:</u>

a) Let's move on from the originals to the images:

$$x(t)\leftrightarrow X(p)$$

$$x'(t)\leftrightarrow pX(p)$$

$$x''(t)\leftrightarrow p^2X(p)$$

$$f(t)=\theta(t-2)-\theta(t-4)\leftrightarrow\frac{1}{p}\left(e^{-2p}-e^{-4p}\right)$$

Let's solve the equation for images

$$\left(p^2+1\right)X(p)=\frac{1}{p}\left(e^{-2p}-e^{-4p}\right),$$

$$X(p)=\frac{1}{p(p^2+4)}\left(e^{-2p}-e^{-4p}\right).$$

Since

$$\frac{1}{p(p^2+1)}=\frac{1}{p}-\frac{p}{p^2+1}\leftrightarrow 1-cos\,t\,,$$

then

$$x(t)=(1-cos(t-2))\theta(t-2)-(1-cos(t-4))\theta(t-4).$$

The solution of the Cauchy problem can be presented in an analytical form:

$$x(t)=\begin{cases} 0, & t<2, \\ 1-\cos(t-2), & 2\le t<4, \\ \cos(t-4)-\cos(t-2), & t\ge 4. \end{cases}$$

b) Let's move on from the originals to the images:

$$x(t)\leftrightarrow X(p)$$

$$x'(t)\leftrightarrow pX(p)$$

$$x''(t)\leftrightarrow p^2X(p)$$

$$f(t)=2t\theta(t)-2t\theta(t-1)+(4-2t)\theta(t-1)-(4-2t)\theta(t-2)=$$

$$=2t\theta(t)-4(t-1)\theta(t-1)+2(t-2)\theta(t-2)\leftrightarrow$$

$$\leftrightarrow\frac{2}{p^2}-\frac{4}{p^2}e^{-p}+\frac{2}{p^2}e^{-2p}=\frac{2}{p^2}\left(1-2e^{-p}+e^{-2p}\right)$$

Let's solve the equation for images

$$\left(p^2+4\right)X(p)=\frac{2}{p^2}\left(1-2e^{-p}+e^{-2p}\right),$$

$$X(p)=\frac{2}{p^2\left(p^2+4\right)}\left(1-2e^{-p}+e^{-2p}\right).$$

Since

$$\frac{2}{p^2\left(p^2+4\right)}=\frac{1}{2}\left(\frac{1}{p^2}-\frac{1}{p^2+4}\right)\leftrightarrow\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right).$$

Then

$$x(t)=\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)\theta(t)-\left(t-1-\frac{1}{2}\sin 2(t-1)\right)\theta(t-1)+$$

$$+\frac{1}{2}\left(t-2-\frac{1}{2}\sin 2(t-2)\right)\theta(t-2).$$

<u>Remark</u>

If $t=t_0\neq 0$ is taken for the initial time in the Cauchy problem, then a new variable $\tau=t-t_0$ is introduced. Then $\tau=0$ for $t=t_0$.

Example 7

Solve the Cauchy problem

a)

$$x''+x'=t,$$
$$x(1)=1,\quad x'(1)=0,$$

b)

$$x'' + x = -2\sin t,$$
$$x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1.$$

Solution:

a) Let $t = \tau + 1,\ x(t) = x(\tau + 1) = z(\tau)$.

Then the equation and the initial conditions will take the form

$$z'' + z = \tau + 1, \quad z(0) = 1, \quad z'(0) = 0.$$

Let's move on from the originals to the images

$$z(\tau) \leftrightarrow Z(p)$$

$$z'(\tau) \leftrightarrow pZ(p) - 1$$

$$z''(\tau) \leftrightarrow p^2 Z(p) - p$$

$$\tau + 1 \leftrightarrow \frac{1}{p^2} + \frac{1}{p}.$$

Let's write down the equation for the images

$$p^2 Z(p) - p + pZ(p) - 1 = \frac{1}{p^2} + \frac{1}{p}.$$

Solving the operator equation and moving on to the originals, we get

$$Z(p) = \frac{1}{p^3} + \frac{1}{p} \leftrightarrow 1 + \frac{\tau^2}{2} = z(\tau).$$

Returning to the original variable t, we obtain a solution to the Cauchy problem

$$x(t) = 1 + \frac{(t-1)^2}{2}.$$

b) Let

$$t=\tau+\frac{\pi}{2}, \quad x(t)=x\left(\tau+\frac{\pi}{2}\right)=z(\tau).$$

Then the equation and the initial conditions will take the form

$$z''+z=-2\sin\left(\tau+\frac{\pi}{2}\right),$$

$$z(0)=0, \quad z'(0)=1.$$

Let's move on from the originals to the images

$$z(\tau)\leftrightarrow Z(p)$$

$$z'(\tau)\leftrightarrow pZ(p)$$

$$z''(\tau)\leftrightarrow p^2Z(p)-1$$

$$-2\sin\left(\tau+\frac{\pi}{2}\right)=-2\cos\tau\leftrightarrow\frac{-2p}{p^2+1}.$$

Let's write down the equation for the images

$$p^2Z(p)-1+Z(p)=\frac{-2p}{p^2+1}.$$

Let's solve the equation for images

$$Z(p)=\frac{1}{p^2+1}-\frac{2p}{\left(p^2+1\right)^2}.$$

Turning to the originals, we get

$$\frac{1}{p^2+1}\leftrightarrow\sin\tau$$

$$\frac{p}{\left(p^2+1\right)^2} \leftrightarrow \sin\tau * \cos\tau = \frac{1}{2}\tau\sin\tau$$

$$z(\tau) = (1-\tau)\sin\tau \, .$$

Returning to the original variable t, we obtain a solution to the original Cauchy problem

$$x(t) = \left(1 - t + \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right) = \left(t - 1 - \frac{\pi}{2}\right)\cos t \, .$$

Example 8

Solve systems of differential equations with given initial conditions:

a)

$$\begin{cases} x' + y = 2e^t, \\ y' + x = 2e^t, \end{cases}$$

$$x(0) = y(0) = 1,$$

b)

$$\begin{cases} x'' + x' + y'' - y = e^t, \\ x' + 2x - y' + y = e^{-t}, \end{cases}$$

$$x(0) = y(0) = y'(0) = 0, \; x'(0) = 1.$$

<u>Solutions:</u>

a) Let

$$x(t) \leftrightarrow X(p)$$

$$y(t) \leftrightarrow Y(p)$$

Considering that

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - 1$$

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p) - 1$$

$$e^t \leftrightarrow \frac{1}{p-1}$$

we obtain an operator system of linear equations

$$\begin{cases} pX(p) - 1 + Y(p) = \dfrac{2}{p-1}, \\ pY(p) - 1 + X(p) = \dfrac{2}{p-1}. \end{cases} \Rightarrow \begin{cases} pX(p) + Y(p) = \dfrac{p+1}{p-1}, \\ pY(p) + X(p) = \dfrac{p+1}{p-1}. \end{cases}$$

Solving the system, we get $X(p) = Y(p) = \dfrac{1}{p-1}$.

Using the image table, we will find $x(t) = y(t) = e^t$.

b) We have

$$x(t) \leftrightarrow X(p)$$

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p)$$

$$x''(t) \leftrightarrow p^2 X(p) - x'(0) = pX(p) - 1$$

$$y(t) \leftrightarrow Y(p)$$

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p)$$

$$y''(t) \leftrightarrow p^2 Y(p) - y'(0) = p^2 Y(p)$$

$$e^{t} \leftrightarrow \frac{1}{p-1}$$

$$e^{-t} \leftrightarrow \frac{1}{p+1}$$

Let's write down a system of operator equations

$$\begin{cases} p^2X-1+pX+p^2Y-Y=\dfrac{1}{p-1}, \\ pX+2X-pY+Y=\dfrac{1}{p+1}. \end{cases} \Rightarrow \begin{cases} \left(p^2+p\right)X+\left(p^2-1\right)Y=\dfrac{p}{p-1}, \\ (p+2)X+(1-p)Y=\dfrac{1}{p+1}. \end{cases}$$

Let's solve a system of linear equations with respect to X and Y using Kramer's formulas:

$$\Delta=\begin{vmatrix} p^2+p & p^2-1 \\ p+2 & 1-p \end{vmatrix}=p(p+1)(1-p)-(p+2)\left(p^2-1\right)=$$

$$=2(1+p)^2(1-p),$$

$$\Delta_x=\begin{vmatrix} \dfrac{p}{p-1} & p^2-1 \\ \dfrac{1}{p+1} & 1-p \end{vmatrix}=1-2p, \quad \Delta_y=\begin{vmatrix} p^2+p & \dfrac{p}{p-1} \\ p+2 & \dfrac{1}{p+1} \end{vmatrix}=\frac{3p}{1-p}.$$

Thus

$$X(p)=\frac{\Delta_x}{\Delta}=\frac{1-2p}{2(p+1)^2(1-p)}=\frac{1}{8}\frac{1}{p-1}+\frac{3}{4}\frac{1}{(p+1)^2}-\frac{1}{8}\frac{1}{p+1},$$

$$Y(p)=\frac{\Delta_y}{\Delta}=\frac{3p}{2(p+1)^2(1-p)^2}=\frac{3p}{2\left(p^2-1\right)^2}.$$

Let's move on to the originals. Since

$e^{-t} \leftrightarrow \dfrac{1}{p+1}$ and $\operatorname{sh} t \leftrightarrow \dfrac{1}{p^2-1}$

then, according to the image differentiation theorem, we find

$$\left(\frac{1}{p+1}\right)' = -\frac{1}{(p+1)^2} \leftrightarrow -te^{-t}$$

$$\left(\frac{1}{p^2-1}\right)' = -\frac{2p}{(p^2-1)^2} \leftrightarrow -t\,\mathrm{sh}\,t$$

Therefore, the solution of the system will be

$$x(t) = \frac{1}{4}\mathrm{sh}\,t + \frac{3}{4}te^{-t}, \quad y(t) = \frac{3}{4}t\,\mathrm{sh}\,t .$$

Example 9 (HOMEWORK)

Solve system of differential equations with given initial conditions:

$$\begin{cases} x'(t) = -x(t) + y(t) + e^t, \\ y'(t) = x(t) - y(t) + e^t \end{cases}$$

$$x(0) = y(0) = 1 .$$

* This homework and homework from 09.09.24 should be done by September 15, 2024.

1.3.2. DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

Consider an equation of the form

$$a_0(t)x^{(n)}(t)+a_1(t)x^{(n-1)}(t)+\ldots+a_n(t)x(t)=f(t)\,, \qquad (1.12)$$

where $a_i(t)$, $i=0,1,...,n$ - polynomials of degree m_i, function $f(t)$ is the original.

Let's denote $m=\max\{m_0,m_1,...,m_n\}$.

We will assume that the Cauchy problem for equation (1.12) with the conditions

$$x(0)=x_0\,,\ x'(0)=x_1\,,\ \ldots,\ x^{(n-1)}(0)=x_{n-1}$$

has a solution on the set of originals.

Let $x(t)\leftrightarrow X(p)$.

According to the image differentiation rule, we have

$$t^k x^{(s)}(t)\leftrightarrow(-1)^k\frac{d^k}{dp^k}\left(L\left\{x^{(s)}(t)\right\}\right)=(-1)^k\frac{d^k}{dp^k}\left(p^sX(p)-p^{s-1}x_0-\ldots-x_{s-1}\right).$$

Thus, applying the Laplace transform to both parts of equation (1.12), equation (1.12) is transformed into an m-th order differential equation with respect to the image $X(p)$. After that, the task of integrating equation (1.12) is simplified.

Example 1

Find a solution to the equation

$$ty''(t)-(1+t)y'(t)+y(t)=0$$

<u>Solution:</u>

$y(t) \leftrightarrow Y(p)$

$y'(t) \leftrightarrow pY(p)-y(0)$

Using the image differentiation property ($t\,f(t) \leftrightarrow -F'(p)$), we have

$$t\,y'(t) \leftrightarrow -\left(pY(p)-y(0)\right)' = -\left(Y(p)+pY'(p)\right) = -p\frac{dY(p)}{dp}-Y(p),$$

(because $y(0)=\text{const}$).

$y''(t) \leftrightarrow p^2Y(p)-py(0)-y'(0)$

$$t\,y''(t) \leftrightarrow -\left(p^2Y(p)-py(0)-y'(0)\right)' = -\left(2pY(p)+p^2Y'(p)-y(0)\right)$$

We substitute the images for these terms into the equation and multiply each term by (-1) to get rid of the numerous minus signs.

We have

$$t\,y''(t)-y'(t)-t\,y'(t)+y(t)=0$$

we substitute and get

$$2pY(p)+p^2Y'(p)-y(0)-\left(-pY(p)+y(0)\right)-\left(Y(p)+pY'(p)\right)-Y(p)=0$$

$$2pY(p)+p^2Y'(p)-y(0)+pY(p)-y(0)-Y(p)-pY'(p)-Y(p)=0$$

$$\left(p^2-p\right)Y'(p)+(3p-2)Y(p)=2y(0)=2C_0$$

let's denote an unknown quantity by $C_0=\text{const}$.

In the image space, we did not get an algebraic equation, as before, we get a first-order differential equation (linear inhomogeneous). First, we solve the corresponding homogeneous equation, when in the right part, instead of the term $2C_0$, there is zero.

$$\left(p^2-p\right)Y'(p)+(3p-2)Y(p)=0$$

The resulting equation is an equation with separable variables.

Since $Y'(p)=\dfrac{dY}{dp}$ then

$$\left(p^2-p\right)\frac{dY}{dp}=-(3p-2)Y(p)$$

$$\frac{dY}{Y}=-\frac{(3p-2)}{\left(p^2-p\right)}dp$$

We can integrate and get:

$$\ln Y(p)=-\int\frac{(3p-2)}{p^2-p}dp=-\int\frac{(2p-1)}{p^2-p}dp-\int\frac{(p-1)}{p^2-p}dp=$$

$$==-\int\frac{d\left(p^2-p\right)}{\left(p^2-p\right)}-\int\frac{dp}{p}=-\ln\left(p^2-p\right)-\ln p=-\ln\left(p^3-p^2\right)+\ln C$$

We have received

$$Y_0(p) = \frac{C}{p^3 - p^2}$$

We have set the index to zero, since this is not a solution to our equation.

We use the method of variation of parameters (variation of constants), the Lagrange method to

$$\left(p^2 - p\right)Y'(p) + (3p - 2)Y(p) = 2y(0) = 2C_0$$

The solution of an inhomogeneous equation is sought in the same form as the solution of a homogeneous one, only instead of an arbitrary constant C, a new unknown function $C(p)$ is put.

$$Y_0(p) = \frac{C}{p^3 - p^2} \to Y(p) = \frac{C(p)}{p^3 - p^2}$$

This change of variables is substituted into the inhomogeneous equation. Only first you need to calculate $Y'(p)$:

$$Y'(p) = \frac{C'(p)}{p^3 - p^2} - \frac{C(p)\left(3p^2 - 2p\right)}{\left(p^3 - p^2\right)^2}$$

$$\left(p^2 - p\right)\left[\frac{C'(p)}{p^3 - p^2} - \frac{C(p)\left(3p^2 - 2p\right)}{\left(p^3 - p^2\right)^2}\right] + (3p - 2)\left[C(p)\frac{1}{p^3 - p^2}\right] = 2C_0$$

$$\frac{C'(p)}{p} - \frac{C(p)\left(3p^2 - 2p\right)}{p\left(p^3 - p^2\right)} + \frac{(3p - 2)C(p)}{p^3 - p^2} = 2C_0$$

$$C'(p) = 2C_0 p$$

$$C(p) = C_0 p^2 + C_1$$

Substituting the found function in $Y(p) = \frac{C(p)}{p^3 - p^2}$ and get

$$Y(p) = \frac{C_0 p^2 + C_1}{p^3 - p^2} = \frac{C_0}{p-1} + \frac{C_1}{(p-1)p^2}$$

From the image of the solution, you need to calculate the inverse Laplace transform (find the original)

$$\frac{1}{p-1} \leftrightarrow e^t$$

For the second term, let's use the property ($\int_0^t f(u)\,du \leftrightarrow \frac{F(p)}{p}$):

$$\frac{1}{p(p-1)} \leftrightarrow \int_0^t e^u du = e^t - 1$$

$$\frac{1}{p^2(p-1)} \leftrightarrow \int_0^t (e^u - 1) du = e^t - 1 - t$$

Our solution is

$$y(t) = C_0 e^t + C_1 (e^t - 1 - t)$$

Let's denote

$$C_0 + C_1 = \tilde{C}_0$$

$$-C_1 = \tilde{C}_1$$

$$y(t) = e^t\left(C_0 + C_1\right) + C_1\left(-1 - t\right)$$

$$y(t) = \tilde{C}_0\, e^t + \tilde{C}_1\left(1 + t\right)$$

Example 2

Find a solution to the equation

$$tx''(t) - (1+t)x'(t) + 2(1-t)x(t) = 0 \; .$$

<u>Solution:</u>

Let $x(t) \leftrightarrow X(p)$.

Then, using the property of differentiating the original and differentiating the image, we write:

$$x'(t) \leftrightarrow pX(p) - x(0) \; ,$$

$$x''(t) \leftrightarrow p^2X(p) - px(0) - x'(0) \, ,$$

$$tx(t) \leftrightarrow -\frac{dX(p)}{dp} \; ,$$

$$tx'(t) \leftrightarrow -\frac{d}{dp}\{pX(p) - x(0)\} = -p\frac{dX}{dp} - X(p) \; ,$$

$$tx''(t) \leftrightarrow -\frac{d}{dp}\{p^2X(p) - px(0) - x'(0)\} = -p^2\frac{dX}{dp} - 2pX(p) + x(0) \, .$$

Applying the Laplace transform to a given equation, we obtain the following operator equation:

$$-p^2\frac{dX}{dp}-2pX(p)+x(0)-$$

$$-pX(p)+x(0)+p\frac{dX(p)}{dp}+X(p)+2X(p)+2\frac{dX(p)}{dp}=0\ ,$$

which can be easily reduced to the form

$$(p^2-p-2)\frac{dX}{dp}+3(p-1)X(p)=2x(0)\,.$$

Having solved the obtained ordinary differential equation, for example, by the method of variation of parameters, we construct its general solution:

$$X(p)=\frac{x(0)}{p-2}+\frac{c}{(p-2)(p+1)^2}\,.$$

Here c is an arbitrary constant. Further, since

$$\frac{1}{p-2}\leftrightarrow e^{2t}\ ,$$

$$\frac{1}{(p+1)^2}\leftrightarrow te^{-t}\ ,$$

$$\frac{1}{(p-2)(p+1)^2}\leftrightarrow\int_0^t \tau e^{-\tau}e^{2(t-\tau)}d\tau=\frac{1}{9}\left(e^{2t}-(3t+1)e^{-t}\right),$$

then the general solution of the given equation will have the form

$$x(t)=x(0)e^{2t}+c\left(\frac{1}{9}e^{2t}-\frac{1}{9}(3t+1)e^{-t}\right)=\left(x(0)+c\right)e^{2t}-c\left(3t+1\right)e^{-t}\,.$$

1.4. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF DIFFERENTIAL EQUATIONS WITH A DELAYED ARGUMENT

Consider a linear differential equation with a delayed argument with constant coefficients:

$$x^{(n)}(t)=\sum_{k=0}^{n-1}a_k x^{(k)}(t-\tau_k)+f(t)\,,\ 0<t<+\infty\,, \qquad (1.13)$$

where $a_k=\text{const}$, $\tau_k=\text{const}\ge 0$.

Let's assume that

$$x(t)=x'(t)=\ldots=x^{(n-1)}(t)\equiv 0\,,$$

$$\text{for } \forall t<0\,.$$

Let it be required to find a solution to equation (1.13) satisfying the initial conditions:

$$x(0)=x'(0)=\ldots=x^{(n-1)}(0)=0\,. \qquad (1.14)$$

Applying the Laplace transform to both parts of equation (1.13) and taking into account the delay property of the original, we obtain the operator equation for the image $X(p)\leftrightarrow x(t)$:

$$p^n X(p)=\sum_{k=0}^{n-1}a_k p^k X(p)e^{-\tau_k p}+F(p)\,, \qquad (1.15)$$

where $F(p)\leftrightarrow f(t)$.

From (1.15) for $X(p)$ we will have

$$X(p)=\frac{F(p)}{p^n-\sum_{k=0}^{n-1}a_k p^k e^{-\tau_k p}}. \qquad (1.16)$$

The original for the image (1.16) defines the solution of equation (1.13) satisfying the conditions (1.14).

Let's formulate a problem for an equation with a delayed argument describing a *process with an aftereffect*. It is required to find a continuously differentiable solution $x(t)$ for $t>t_0$ of the equation

$$x'(t)=f\left(t,x(t),x(t-\tau)\right),\ \tau=\text{const}>0, \qquad (1.17)$$

if it is known that

$$x(t)=\varphi(t),\ t\in[t_0-\tau,t_0]. \qquad (1.18)$$

The initial function $\varphi(t)$ is a given continuously differentiable function.

The segment $[t_0-\tau,t_0]$ on which the function $\varphi(t)$ is defined is called the *initial set.*

If equation (1.17) is linear, then its solution satisfying condition (1.18) can be found using the Laplace transform. Let $t_0=0$, then when constructing the corresponding operator equation, it should be taken into account that for the image of the function $x(t-\tau)$ we have

$$x(t-\tau)\leftrightarrow\int_0^\infty e^{-pt}x(t-\tau)dt=\int_{-\tau}^\infty e^{-p(\eta+\tau)}x(\eta)d\eta=$$

$$=\int_{-\tau}^0 e^{-p(\eta+\tau)}x(\eta)d\eta+\int_0^\infty e^{-p(\eta+\tau)}x(\eta)d\eta=e^{-p\tau}\int_{-\tau}^0 e^{-p\eta}\varphi(\eta)d\eta+e^{-p\tau}X(p).$$

When restoring originals from known images, you can use the following decomposition:

$$\frac{1}{1-\dfrac{\gamma e^{-np}}{(p+a)^m}}=1+\frac{\gamma e^{-np}}{(p+a)^m}+\left(\frac{\gamma e^{-np}}{(p+a)^m}\right)^2+\ldots=\sum_{k=0}^{\infty}\left(\frac{\gamma e^{-np}}{(p+a)^m}\right)^k, \quad (1.19)$$

which is true for any $n, m \in N$, on condition $\operatorname{Re} p > 0$.

Example 1

Find a solution of the equation

$$x'(t)=x(t-1)+1, \; x(0)=0.$$

<u>Solution:</u>

Assuming that $x(t)\equiv 0$ for $t\in[-1,0]$, applying the Laplace transform to a given equation, we obtain the following operator equation:

$$pX(p)=X(p)e^{-p}+\frac{1}{p}.$$

Where from

$$X(p)=\frac{1}{p}\frac{1}{pe^{-p}}=\frac{1}{p^2}\frac{1}{1-\dfrac{e^{-p}}{p}}.$$

Further, applying the formula (1.19), we have

$$X(p)=\frac{1}{p^2}\sum_{k=0}^{\infty}\left(\frac{e^{-p}}{p}\right)^k=\sum_{k=0}^{\infty}\frac{e^{-pk}}{p^{k+2}}.$$

Given the delay property, we construct an expression for the corresponding original $x(t)$ in the form

$$x(t)=\sum_{k=0}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\theta(t-k).$$

Example 2

Find a solution of the equation

$$x'(t)=x(t-1),$$

if $x(t)\equiv 2$ for $\forall t\in[-1,0]$.

Solution:

Let $x(t)\leftrightarrow X(p)$.

It follows from the condition that $x(0)=2$, so we have

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-2.$$

We apply the Laplace transform to both parts of the given equation. For the right side of the equation, we have

$$x(t-1)\leftrightarrow\int_{0}^{\infty}e^{-pt}x(t-1)dt=\int_{-1}^{\infty}e^{-p(z+1)}x(z)dz=$$

$$=\int_{-1}^{0}e^{-p(z+1)}x(z)dz+\int_{0}^{\infty}e^{-p(z+1)}x(z)dz=$$

$$=2\int_{-1}^{0}e^{-p(z+1)}dz+e^{-p}X(p)=\frac{2}{p}\left(1-e^{-p}\right)+e^{-p}X(p).$$

Therefore, the corresponding operator equation has the form

$$pX(p)-2=\frac{2}{p}\left(1-e^{-p}\right)+e^{-p}X(p).$$

From here we get

$$X(p)=2\frac{p+1-e^{-p}}{p^2-pe^{-p}}=\frac{2}{p}+\frac{2}{p(p-e^{-p})}.$$

Using the result of the previous example, we will construct the original in the form

$$x(t)=2\left(\theta(t)+\sum_{k=0}^{\infty}\frac{(t-k)^{k+1}}{(k+1)!}\theta(t-k)\right).$$

Example 3

Find a solution of the equation

$$x'(t)+2x(t)-x(t-1)=f(t),$$

if $x(0)=0$ and $x(t)\equiv 0$ for $\forall t<0$.

Solution:

Let $x(t)\leftrightarrow X(p)$, $f(t)\leftrightarrow F(p)$.

Since under the given conditions $x(t-1)\leftrightarrow e^{-p}X(p)$, the operator equation corresponding to the given one has the form

$$pX(p)+2X(p)-e^{-p}X(p)=F(p).$$

The solution of this equation is written as a product:

$$X(p)=\frac{1}{p+2-e^{-p}}F(p).$$

Let's build the original for the function

$$Y(p)=\frac{1}{p+2-e^{-p}},$$

by performing the following transformations:

$$Y(p)=\frac{1}{p+2-e^{-p}}=\frac{1}{p+2}\left(\frac{1}{1-\dfrac{e^{-p}}{p+2}}\right)=\sum_{k=0}^{\infty}\frac{e^{kp}}{(p+2)^{k+1}}.$$

The last equality is written, taking into account the formula (1.19). Turning to the originals for the summands of the sum of the series, using $t^k \leftrightarrow \dfrac{k!}{p^{k+1}}$, $t<0$ and the delay property, we find

$$Y(p)\leftrightarrow y(t)=\sum_{k=0}^{\infty}\frac{(t-k)^k}{k!}e^{-2(t-k)}\theta(t-k).$$

The solution to this problem will be the function $x(t)$, which is a convolution of the functions $f(t)$ and $y(t)$:

$$x(t)=\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^t f(t-\tau)(\tau-k)^k e^{-2(\tau-k)}\theta(\tau-k)d\tau.$$

1.5. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF INTEGRAL EQUATIONS AND SYSTEMS

1.5.1.The Volterra equation of the second kind

Consider the Volterra linear integral equation of the second kind with a kernel $K(t)$ of the form

$$y(t) = f(t) + \int_0^t K(t-\tau) y(\tau) d\tau , \qquad (1.20)$$

where $K(t), f(t)$ - given functions, $y(t)$ - the desired function.

Let $y(t) \leftrightarrow Y(p)$, $f(t) \leftrightarrow F(p)$, $K(t) \leftrightarrow K^*(p)$. Passing to the images in equation (1.20) and using the convolution image property, we obtain the corresponding operator equation

$$Y(p) = F(p) + K^*(p) Y(p) .$$

From here

$$Y(p) = \frac{F(p)}{1 - K^*(p)} .$$

The original for image $Y(p)$ has the desired solution to equation (1.20).

1.5.2. The Volterra equation of the first kind

Consider a linear integral Volterra equation of the first kind with a kernel $K(t)$ of the form

$$\int_0^t K(t-\tau) y(\tau) d\tau = f(t) , \qquad (1.21)$$

where $K(t), f(t)$ - given functions, $y(t)$ - the desired function.

Let $y(t) \leftrightarrow Y(p)$, $f(t) \leftrightarrow F(p)$, $K(t) \leftrightarrow K^*(p)$. Then, applying the Laplace transform to equation (1.21), we obtain the operator equation.

$$K^*(p)Y(p) = F(p) \Rightarrow Y(p) = \frac{F(p)}{K^*(p)}.$$

The original for $Y(p)$ gives the desired solution to equation (1.21).

1.5.3.Systems of Volterra integral equations

Consider a system of Volterra integral equations of the form

$$y_i(t) = f_i(t) + \sum_{k=1}^{s} \int_0^t K_{ik}(t-\tau) y_k(\tau) d\tau, \; i = 1,2,...s, \qquad (1.22)$$

where $K_{ik}(t)$, $f_i(t)$ - given functions, $i,k = 1,2,...,s$.

Let

$$F_i(p) \leftrightarrow f_i(t), \; K_{ik}^*(p) \leftrightarrow K_{ik}(t), \; Y_i(p) \leftrightarrow y_i(t).$$

Applying the Laplace transform to both parts of the equations (1.22), we obtain a system of operator equations

$$Y_i(p) = F_i(p) + \sum_{k=1}^{s} K_{ik}^*(p) Y_k(p), \; i = 1,2,...s, \qquad (1.23)$$

linear with respect to the images $Y_i(p)$. Solving the system (1.23), we find $Y_i(p)$, the originals for which will be the solution of the original system of integral equations (1.22).

What will be useful to us:

$$\int_0^\infty y(t)e^{-pt}\,dt = Y(p)$$ - Laplace transform

$$\int_0^\infty y'(t)e^{-pt}\,dt = pY(p) - y(0)$$ - Laplace transform of the derivative

$$\int_0^\infty y^{(n)}(t)e^{-pt}\,dt = p\int_0^\infty y^{(n-1)}(t)e^{-pt}dt - y^{(n-1)}(0)$$ - Laplace transform from high order derivatives

$$\int_0^\infty \left(\int_0^t y(\tau)\,d\tau\right)e^{-pt}\,dt = \frac{Y(p)}{p}$$ - Laplace transform from the integral

$$\int_0^\infty \left(\int_0^x g(x-t)\,y(t)\,dt\right)e^{-px}\,dx = G(p)Y(p)$$ - Laplace transform from a convolution type integral

Let we have $K(x-t)$ - the kernel of the integral operator (difference kernel).

The Volterra integral equation of the second kind looks like this:

$$y(x) = f(x) + \int_0^x K(x-t)\,f(t)\,dt$$

Let's move everything to the left side:

$$y(x) - f(x) - \int_0^x K(x-t)\,f(t)\,dt = 0$$

$$\int_0^\infty \left[y(x) - f(x) - \int_0^x K(x-t)\, f(t)\, dt \right] e^{-px} dx = 0$$

We have obtained three Laplace transformations:

$$\int_0^\infty y(x) e^{-px} dx = \int_0^\infty f(x) e^{-px} dx + \int_0^\infty \left(\int_0^x K(x-t)\, y(t)\, dt \right) e^{-px} dx$$

The image of the first integral $Y(p)$, the second is $F(p)$, and the third is a convolution type integral.

The integral equation turns into an algebraic equation for images:

$$Y(p) = F(p) + K^*(p) Y(p)$$

$$Y(p) = \frac{F(p)}{1 - K^*(p)} \ ,$$

where $F(p), K^*(p) -$ we know.

Example 1

Solve the integral equation

$$y(x) = \sin x + \int_0^x (x-t)y(t)dt\ .$$

<u>Solution:</u>

Let $y(x) \leftrightarrow Y(p)$.

Since the integral included in the given equation is a convolution of two functions t and $y(t)$, then its image will be the product of images of these functions, that is $\frac{1}{p^2}Y(p)$. Applying the Laplace transform to the equation, we obtain the following operator equation:

$$Y(p) = \frac{1}{p^2+1} + \frac{1}{p^2}Y(p)\ .$$

His solution has the form

$$Y(p) = \frac{p^2}{(p^2-1)(p^2+1)} = \frac{p}{(p^2-1)}\frac{p}{(p^2+1)}\ .$$

Since

$$\frac{p}{\left(p^2-1\right)} \leftrightarrow \operatorname{ch} x\,,\ \frac{p}{\left(p^2+1\right)} \leftrightarrow \cos x\,,$$

then the original corresponding to the image $Y(p)$ is a convolution of two functions — $\operatorname{ch} x$ and $\cos x$:

$$y(x) = \int_0^x \operatorname{ch}(x-t)\cos t dt$$

Having calculated the integral, we get the desired solution:

$$y(x) = \frac{1}{2}\sin x + \frac{1}{4}e^{x} - \frac{1}{4}e^{-x}\ .$$

Example 2

Solve the integral equation

$$y(x) = \cos x + \int_0^x (x-t)\,y(t)\,dt\ ;$$

Solution:

$$\cos x \leftrightarrow \frac{p}{p^2+1}$$

$$Y(p) = \frac{p}{p^2+1} + \frac{1}{p^2}Y(p)$$

$$Y(p) = \frac{p^3}{(p^2-1)(p^2+1)} = p\cdot\frac{p}{p^2-1}\cdot\frac{p}{p^2+1}$$

$$\frac{p}{p^2-1} \leftrightarrow ch(t) = f_1(t)$$

$$\frac{p}{p^2+1} \leftrightarrow \cos(t) = f_2(t)$$

We found the original corresponding to the image using the Duhamel integral.

Duhamel integral:

$$f(t) = f_1(0)\,f_2(t) + \int_0^t f_1'(\tau)\,f_2(t-\tau)\,d\tau$$

We have

$$y(t) = 1 \cdot cos(t) + \int_0^t sh(\tau) cos(t-\tau) d\tau$$

Using twice integration by parts in the integral, we have

$$\int sh(\tau) cos(\tau - t) d\tau = \frac{sh(\tau) sin(\tau - t) + ch(\tau) cos(\tau - t)}{2} + C =$$

$$= \frac{e^{-\tau}\left[\left(e^{2\tau} - 1\right) sin(\tau - t) + \left(e^{2\tau} + 1\right) cos(\tau - t)\right]}{2} + C$$

And

$$\int_0^t sh(\tau) cos(\tau - t) d\tau = \frac{e^{-t}\left(e^{2t} + 1\right)}{4} - \frac{cos(t)}{2}$$

$$y(t) = 1 \cdot cos(t) + \frac{e^{-t}\left(e^{2t} + 1\right)}{4} - \frac{cos(t)}{2} = \frac{e^{-t}\left(e^{2t} + 1\right)}{4} + \frac{cos(t)}{2}$$

Our solution is

$$y(x) = \frac{e^{-x}\left(e^{2x} + 1\right)}{4} + \frac{cos(x)}{2}$$

Example 3

Solve a system of integral equations:

$$\begin{cases} y(x) = e^x + \int\limits_0^x y(t)dt - \int\limits_0^x e^{(x-t)} z(t)dt, \\ z(x) = -x - \int\limits_0^x (x-t) y(t)dt - \int\limits_0^x z(t)dt. \end{cases}$$

<u>Solution:</u>

Let $y(x) \leftrightarrow Y(p)$, $z(x) \leftrightarrow Z(p)$.

We apply the Laplace transform to each equation of the system. Using the properties on the integration of the original and on convolution to construct images of the original equations, we obtain

$$\begin{cases} Y(p) = \frac{1}{p-1} + \frac{Y(p)}{p} - \frac{Z(p)}{p-1}, \\ Z(p) = -\frac{1}{p^2} - \frac{Y(p)}{p^2} - \frac{Z(p)}{p}. \end{cases}$$

Solving a system of algebraic equations, we find the images

$$Y(p) = \frac{1}{p-2},$$

$$Z(p) = -\frac{1}{p(p-2)} = \frac{1}{2}\left(\frac{1}{p} - \frac{1}{p-2}\right),$$

which correspond to the originals:

$$y(x) = e^{2x},$$

$$z(x) = \frac{1}{2} - \frac{1}{2}e^{2x}.$$

Example 4

Solve the integral-differential equation

$$y''(x)+2y'(x)-2\int_0^x \sin(x-t)y'(t)dt=\cos x\ ,\ \ y(0)=y'(0)=0\ .$$

Solution:

Let $y(x)\leftrightarrow Y(p)$.

We apply the Laplace transform to a given equation:

$$p^2Y(p)+2pY(p)-2\frac{1}{p^2+1}pY(p)=\frac{p}{p^2+1}\ .$$

Solving the equation with respect to $Y(p)$, we obtain

$$Y(p)=\frac{1}{p(p+1)^2}=\frac{1}{p}-\frac{1}{(p+1)^2}-\frac{1}{p+1}\leftrightarrow y(x)=1-e^{-x}x-e^{-x}\ .$$

In that way

$$y(x)=1-e^{-x}x-e^{-x}\ .$$

Example 5

Solve the integral-differential equation

$$y''(x)-2y'(x)+y(x)+2\int_0^x \cos(x-t)y''(t)dt+$$

$$+2\int_0^x \sin(x-t)y'(t)dt=\cos x\ ,\ \ y(0)=y'(0)=0\ .$$

Solution:

Let $y(x) \leftrightarrow Y(p)$.

We apply the Laplace transform to a given equation:

$$p^2Y(p) - 2pY(p) + Y(p) + 2\frac{p}{p^2+1}p^2Y(p) + \frac{2}{p^2+1}pY(p) = \frac{p}{p^2+1}.$$

Solving the equation with respect to $Y(p)$, we obtain

$$Y(p) = \frac{p}{p^2+1}\frac{1}{p^2+1} \leftrightarrow y(x) = \int_0^x \cos(x-t)\sin t\,dt = \frac{x}{2}\sin x.$$

When switching to the original, the convolution image property was used.

1.5. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF INTEGRAL EQUATIONS AND SYSTEMS

Example 1

Solve the integral equation

$$\int_0^t e^{2(t-\tau)} y(\tau)\,d\tau = t^2 e^t$$

Solution:

In this equation $K(t-\tau) = e^{2(t-\tau)}$, therefore $K(t) = e^{2t}$.

Let's find an image of this function $\frac{1}{p-2} \leftrightarrow e^{2t}$.

Let's find the image of the right side of the equation, that is, the function $t^2 e^t$:

$$\frac{2}{(p-1)^3} \leftrightarrow t^2 e^t$$

Let's write down the equation

$$Y(p)\frac{1}{p-2} = \frac{2}{(p-1)^3}$$

From here

$$Y(p) = \frac{2p-4}{(p-1)^3}$$

Using the method of undetermined coefficients, we will find the decomposition of a fraction into the simplest fractions:

$$Y(p) = \frac{2p-4}{(p-1)^3} = \frac{A}{(p-1)^3} + \frac{B}{(p-1)^2} + \frac{C}{p-1}$$

Let's bring the right part to the common denominator and equate the numerators of the resulting and the original fraction:

$$A+B(p-1)+C(p-1)^2 \equiv 2p-4$$

Let's find the coefficients A, B, C.

$$p=1 \qquad A=-2$$

$$p=2 \qquad A+B+C=0$$

$$p=0 \qquad A-B+C=-4$$

$$2B=4 \quad \Rightarrow \quad B=2;$$

$$2A+2C=-4 \quad \Rightarrow \quad A+C=-2 \quad \Rightarrow \quad C=0$$

In that way

$$Y(p)=\frac{2p-4}{(p-1)^3}=-\frac{2}{(p-1)^3}+2\frac{1}{(p-1)^2}$$

We will find the original corresponding to the image:

$$y(t)=-t^2e^t+2te^t=te^t(2-t)$$

So, the solution of this integral equation is the function

$$y(t)=te^t(2-t)$$

Example 2

Solve the integral equation

$$y(t)=1+t+\int_0^t cos(t-\tau)\,y(\tau)\,d\tau$$

Solution:

In this case $\qquad f(t)=1+t \quad \Rightarrow \quad F(p)=\frac{1}{p}+\frac{1}{p^2}$;

$$K(t)=cos\,t \quad \Rightarrow \quad K^*(p)=\frac{p}{p^2+1}$$

The integral $\int_0^t \cos(t-\tau)\,y(\tau)\,d\tau$ is a convolution of the function $\cos t$ and $y(t)$.

Image of the equation:

$$Y(p)=\frac{1}{p}+\frac{1}{p^2}+Y(p)\frac{p}{p^2+1}$$

We'll find $Y(p)$.

$$Y(p)\left(1-\frac{p}{p^2+1}\right)=\frac{1}{p}+\frac{1}{p^2} \quad \Rightarrow$$

$$Y(p)\frac{p^2-p+1}{p^2+1}=\frac{p+1}{p^2} \quad \Rightarrow$$

$$Y(p)=\frac{(p^2+1)(p+1)}{p^2(p^2-p+1)}$$

Let's imagine the image of the solution as the sum of the simplest fractions:

$$Y(p)=\frac{(p^2+1)(p+1)}{p^2(p^2-p+1)}=\frac{A}{p^2}+\frac{B}{p}+\frac{Cp+D}{p^2-p+1}$$

Let's find the decomposition coefficients.

$$A(p^2-p+1)+Bp(p^2-p+1)+Cp^3+Dp^2 \equiv p^3+p^2+p+1$$

We equate the coefficients with the same degrees p in the right and left parts of the identity:

p^3	$B+C=1$
p^2	$A-B+D=1$
p	$-A+B=1$
p^0	$A=1$

We will get $A=1,\ B=2,\ C=-1,\ D=2.$

Therefore, the decomposition has the form:

$$Y(p)=\frac{(p^2+1)(p+1)}{p^2(p^2-p+1)}=\frac{1}{p^2}+\frac{2}{p}+\frac{2-p}{p^2-p+1}\ .$$

Let's convert all fractions into table fractions:

$$Y(p)=\frac{1}{p^2}+\frac{2}{p}+\frac{2-p}{p^2-p+1}=\frac{1}{p^2}+2\frac{1}{p}+\frac{\frac{3}{2}+\frac{1}{2}-p}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$=\frac{1}{p^2}+2\frac{1}{p}+\sqrt{3}\frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}$$

We will find the corresponding original:

$$y(t)=t+2+e^{\frac{t}{2}}\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}t-\cos\frac{\sqrt{3}}{2}t\right)$$

This function is the solution of a given integral equation.

Example 3

Solve the integral equation

$$y(t)=cos\,t-\int_0^t e^{t-\tau}y(\tau)\,d\tau$$

<u>Solution:</u>

Here $f(t)=\cos t;\quad K(t)=e^t$.

Since $\int_0^t e^{t-\tau}y(\tau)\,d\tau$ is a convolution of functions e^t and $y(t)$, that is,

$$e^t * y(t)=\int_0^t e^{t-\tau}y(\tau)d\tau \leftrightarrow K^*(p)Y(p)\text{, where}$$

$$K(t)\leftrightarrow K^*(p)=\frac{1}{p-1},\ \ f(t)\leftrightarrow F(p)=\frac{p}{p^2+1};$$

$$y(t)\leftrightarrow Y(p).$$

The image of the integral equation takes the form

$$Y(p)=\frac{p}{p^2+1}-\frac{1}{p-1}Y(p)$$

$$Y(p)+\frac{1}{p-1}Y(p)=\frac{p}{p^2+1} \quad \Rightarrow$$

$$Y(p)\frac{p}{p-1}=\frac{p}{p^2+1} \qquad \Rightarrow$$

$$Y(p)=\frac{p-1}{p^2+1}$$

Based on the image of the solution, we will find its original:

$$Y(p)=\frac{p-1}{p^2+1}=\frac{p}{p^2+1}-\frac{1}{p^2+1} \quad \leftrightarrow \quad \cos t-\sin t=y(t)$$

So, the solution of the integral equation is the function:

$$y(t)=\cos t-\sin t$$

Example 4 (HOMEWORK 5, the deadline is September 24th)

№1	№2
$y(x)=x+\int_0^x \sin(x-t)y(t)dt$;	$y''(x)-2y'(x)+y(x)+2\int_0^x \cos(x-t)y''(t)dt+$ $+2\int_0^x \sin(x-t)y'(t)dt=\sin x,\ y(0)=y'(0)=0$;

2. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

2.1. PARTIAL DIFFERENTIAL EQUATIONS

Denote by D the region of the n -dimensional space R^n of points $x=(x_1,x_2,...,x_n)$, $x_1,x_2,...,x_n$, $n \ge 2$ — Cartesian coordinates of point x.

An equation of the form

$$F\left(x,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},...,\frac{\partial^k u}{\partial x_1^{i_1}...\partial x_n^{i_n}},\frac{\partial u}{\partial x_n^m}\right)=0,\ x\in D, \tag{2.1}$$

$$\sum_{j=1}^{n} i_j = k,\ k=0,1,...,m,\ m\ge 1$$

is called a *partial differential equation of the order* m with respect to an unknown function $u=u(x)$, where $F=F\left(x,u,\frac{\partial u}{\partial x_1},...\right)-$ is a given real function of points $x \in D$, an unknown function u and its partial derivatives. The left side of equality (2.1) is called a *partial differential operator of order* m.

The real function $u=u(x_1,x_2,...,x_n)$, defined in the domain D of the assignment of equation (2.1), continuous together with its partial derivatives included in this equation and converting it into an identity, is called the *classical (regular) solution* of equation (2.1).

Equation (2.1) is called linear if F depends linearly on all variables of the form

$$\frac{\partial^k u}{\partial x_1^{i_1}...\partial x_n^{i_n}},\ 0\le k\le m.$$

The linear equation can be written as

$$\sum_{k=0}^{m}\sum_{i_1,\dots,i_n} a_{i_1\dots i_n}(x)\frac{\partial^k u}{\partial x_1^{i_1}\dots\partial x_n^{i_n}} = f(x)\,,\ \sum_{j=1}^{n} i_j = k\ ,\ x\in D$$

or in the form of

$$Lu = f(x)\,,\ x\in D\,,$$

where $L-$ linear differential operator of order m:

$$L\equiv\sum_{k=0}^{m}\sum_{i_1,\dots,i_n} a_{i_1\dots i_n}(x)\frac{\partial^k}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}\,,\ \sum_{j=1}^{n} i_j = k\ .$$

A linear equation is called homogeneous if $f(x)\equiv 0$, inhomogeneous if $f(x)\neq 0$.

Equation (2.1) of order m is called quasilinear if F linearly depends only on partial derivatives of order m:

$$\frac{\partial^m u}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}\,,\ \sum_{j=1}^{n} i_j = m\ .$$

Control tasks

Example 1

Find out if the following equalities are partial differential equations:

1. $\cos\left(u_x+u_y\right)-\cos u_x\cos u_y+\sin u_x\sin u_y=0$;

   Solution:

1. By converting the cosine of the sum into the product of cosines and sines, we obtain the identity

$$\cos\left(u_x+u_y\right)=\cos u_x\cdot\cos u_y-\sin u_x\cdot\sin u_y$$

$$\cos u_x \cdot \cos u_y - \sin u_x \cdot \sin u_y - \cos u_x \cdot \cos u_y + \sin u_x \cdot \sin u_y = 0$$

$$0 = 0$$

This is not a differential equation.

2. $u_{xx}^2 + u_{yy}^2 - \left(u_{xx} - u_{yy}\right)^2 = 0$;

<u>Solution:</u>

We open the brackets, give similar terms, and get

$$u_{xx}^2 + u_{yy}^2 - \left(u_{xx}^2 - 2u_{xx}u_{yy} + u_{yy}^2\right) = 0$$

$$u_{xx}^2 + u_{yy}^2 - u_{xx}^2 + 2u_{xx}u_{yy} - u_{yy}^2 = 0$$

$$2u_{xx}u_{yy} = 0$$

Equation (2) is a differential equation.

3. $\sin^2\left(u_{xx} + u_{xy}\right) + \cos^2\left(u_{xx} + u_{xy}\right) - u = 1$;

<u>Solution:</u>

Using the basic trigonometric identity, we obtain

$$\sin^2\left(u_{xx} + u_{xy}\right) + \cos^2\left(u_{xx} + u_{xy}\right) = 1$$

$$1 - u = 1$$

$$-u = 0$$

Equation (3) is not a differential equation.

Example 2

Determine the order of the equations:

1. $\ln\left|u_{xx}u_{yy}\right| - \ln\left|u_{xx}\right| - \ln\left|u_{yy}\right| + u_x + u_y = 0$

Solution:

1. Converting the sum of the logarithms into the logarithm of the product and giving similar terms, we get

$$\ln\left|u_{xx}\right| + \ln\left|u_{yy}\right| = \ln\left|u_{xx}\,u_{yy}\right|$$

$$\ln\left|u_{xx}u_{yy}\right| - \ln\left|u_{xx}u_{yy}\right| + u_x + u_y = 0$$

$$u_x + u_y = 0$$

The order of the differential equation (1) is the first.

2. $u_x u_{xy}^2 + \left(u_{xx}^2 - 2u_{xy}^2 + u_y\right)^2 - 2xy = 0$

Solution:

We open the brackets, give similar terms, and get

$$\left(u_{xx}^2 - 2u_{xy}^2 + u_y\right)^2 = u_{xx}^4 - 4u_{xx}^2 u_{xy}^2 + 4u_{xy}^4 + 2u_{xx}^2 u_y - 4u_{xy}^2 u_y + u_y^2$$

$$u_x u_{xy}^2 + u_{xx}^4 - 4u_{xx}^2 u_{xy}^2 + 4u_{xy}^4 + 2u_{xx}^2 u_y - 4u_{xy}^2 u_y + u_y^2 - 2xy = 0$$

The order of the differential equation (2) is second.

Example 3

Find out which of the following equations are linear and which are nonlinear (quasilinear):

1. $2\sin(x+y)u_{xx} - x\cos y u_{xy} + xyu_x - 3u + 1 = 0$;

<u>Solution:</u>

In this equation, the coefficients before the second and first derivatives are functions of x and y, so the equation is linear. The function $f(x,y)=1$, that is, the equation is inhomogeneous.

Equation (1) is linear and inhomogeneous.

2. $x^2yu_{xxy} + 2e^x y^2 u_{xy} - (x^2y^2+1)u_{xx} - 2u = 0$;

<u>Solution:</u>

In this equation, the coefficients before the second and third derivatives are functions of x and y, so the equation is linear, of the third order.

The function $f(x,y)=0$, that is, the equation is homogeneous.

Equation (2) is linear and homogeneous.

3. $3u_{xy} - 6u_{xx} + 7u_y - u_x + 8x = 0$;

<u>Solution:</u>

In this equation, the coefficients before the second and first derivatives are constant values, so the equation is linear with constant coefficients. The function $f(x,y)=8x$, that is, the equation is inhomogeneous.

Equation (3) is linear with constant coefficients and inhomogeneous.

4. $u_x u_{xy}^2 + 2xuu_{yy} - 3xyu_y - u = 0$;

<u>Solution:</u>

The first term can be represented as follows: $u_x u_{xy}^2 = u_x u_{xy} u_{xy}$. It was found that the coefficient before the highest second derivative also depends on the second derivative u_{xy}, that is, the equation is nonlinear.

Equation (4) is nonlinear.

2.2. EXAMPLES OF THE SIMPLEST PARTIAL DIFFERENTIAL EQUATIONS

Let's look at some examples of partial differential equations.

Example 1

Find the function $u = u(x,y)$ satisfying the differential equation:

$$\frac{\partial u}{\partial x} = 1$$

Solution:

Integrating, we get

$$u = x + \varphi(y),$$

where $\varphi(y)$ - an arbitrary function. This is the general solution of this differential equation.

Example 2

Solve the equation

$$\frac{\partial^2 u}{\partial y^2} = 6y,$$

where $u = u(x,y)$.

Solution:

Integrating twice by y, we get

$$\frac{\partial u}{\partial y} = 3y^2 + \varphi(x),$$

$$u = y^3 + y\varphi(x) + \psi(x),$$

where $\varphi(x)$ and $\psi(x)$ an arbitrary functions.

Example 3

Solve the equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0.$$

Solution:

Integrating the equation with respect to x, we have

$$\frac{\partial u}{\partial y} = f(y).$$

Integrating obtained result by y, we find

$$u = \varphi(x) + \psi(y),$$

where $\psi(y) = \int f(y)dy$, $\varphi(x)$ and $\psi(y)$ - arbitrary functions.

Example 4

Solve the equation

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 2x \frac{\partial u}{\partial y} = 0, \ x \neq 0.$$

Solution:

This equation can be reduced to the form

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial y}\right) = 0 .$$

Integrating the equation with respect to the variable x, we obtain

$$x^2 \frac{\partial u}{\partial y} = f(y) ,$$

where $f(y)$ - arbitrary function.

Integrating the result obtained with respect to the variable y, we find

$$u = \varphi(x) + \psi(y) ,$$

where $\psi(y) = \frac{1}{x^2}\int f(y)dy$, $\varphi(x)$ and $\psi(y)$ - are arbitrary functions.

2.3. FIRST-ORDER DIFFERENTIAL EQUATIONS, LINEAR WITH RESPECT TO PARTIAL DERIVATIVES

Partial differential equation of the first order

Some problems of classical mechanics, continuum mechanics, acoustics, optics, hydrodynamics, and radiation transfer are reduced to partial differential equations of the first order. Analytical methods developed in the theory of ordinary differential equations are applicable to the solution of some of them.

1. Basic concepts. Classification of equations

A partial differential equation of the first order is an equation of the form [1]

$$F\left(x_1,\dots,x_n,u,\frac{\partial u}{\partial x_1},\dots,\frac{\partial u}{\partial x_n}\right)=0, \qquad (1)$$

where $x_1,\dots,x_n$ − are independent variables, $u=u(x_1,\dots,x_n)$ − is an unknown function, $F(x_1,x_2,\dots,x_n,u,p_1,p_2,\dots,p_n)$ − is a given continuously differentiable function [2] in some region $G\subset\mathbb{R}^{2n+1}$, and at each point in the region G

$$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial p_i}\right)^2\neq 0.$$

Equation (1) can be abbreviated as [3]

$$F(x,u,\operatorname{grad} u)=0, \qquad (1')$$

where $x=(x_1,...,x_n)$ and $\operatorname{grad} u=\left(\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_n}\right)$.

---

[1] Next, another entry of the first-order partial derivative will be used too. For example, for the function $u=u(x,y,z)$:

$$u'_x=\frac{\partial u}{ux},\ u'_y=\frac{\partial u}{\partial y},\ u'_z=\frac{\partial u}{\partial z}.$$

[2] Here, for the function F, the arguments p_i, $i=1,...,n,$ denote partial derivatives u'_{x_i}.

[3] Often, to write the gradient of the function $u(x)$, the operator ∇ («nabla») is used, that is, instead of $\operatorname{grad} u$, they write ∇u. Then equation $(1')$ can be written in the form $F(x,u,\nabla u)=0$.

Definition 1. The function $u=\varphi(x_1,...,x_n)$ given in the domain $D\subset\mathbb{R}^n$ is called the **solution of equation** (1) if:

1) $\varphi(x_1,...,x_n)-$ is a continuously differentiable function in D,

2) for all points $x=(x_1,...,x_n)\in D$ point $\left(x,\varphi,\varphi'_{x_1},...,\varphi'_{x_n}\right)\in G$,

3) $F\left(x_1,...,x_n,\varphi,\varphi'_{x_1},...,\varphi'_{x_n}\right)\equiv 0$ for any $(x_1,...,x_n)\in D$.

The solution of equation (1) in the $(n+1)$ - dimensional space of variables $x_1,...,x_n,u$ defines some smooth surface of dimension n, which is called the **integral surface of equation** (1).

Depending on how the unknown function u and its partial derivatives enter equation (1), **linear** and **nonlinear** equations are distinguished.

A linear partial differential equation of the first order is an equation of the form

$$A_1(x)\frac{\partial u}{\partial x_1} + \ldots + A_n(x)\frac{\partial u}{\partial x_n} = B(x)u + f(x), \qquad (2)$$

where $A_i(x)$ $(i=1,\ldots,n)$, $B(x)$ and $f(x)$ - the given functions of point $x=(x_1,\ldots,x_n)\in D$, moreover, $\sum_{i=1}^{n} A_i^2(x)\neq 0$ for any $x\in D$.

The functions A_i and B are called the coefficients of the equation. The linearity of the equation is determined by the fact that the unknown function $u(x)$ and all its partial derivatives enter the equation linearly.

If $f(x)\equiv 0$, then equation (2) is called **homogeneous**, otherwise- **inhomogeneous**.

If equation (1) cannot be written as (2), then it is called **nonlinear**. If in it the function F is linear with respect to all derivatives of the unknown function $\frac{\partial u}{\partial x_i}$, then equation (1) is called **quasi-linear**. The quasi-linear equation can be written as follows

$$A_1(x,u)\frac{\partial u}{\partial x_1} + \ldots + A_n(x,u)\frac{\partial u}{\partial x_n} = B(x,u). \qquad (3)$$

2. Homogeneous linear equation

Let the point $x=(x_1,\ldots,x_n)$ belong to the domain $D\subset\mathbb{R}^n$, $n\geq 2$. In the domain D, consider a homogeneous linear partial differential equation of the first order of the form

$$A_1(x_1,\dots,x_n)\frac{\partial u}{\partial x_1}+\dots+A_n(x_1,\dots,x_n)\frac{\partial u}{\partial x_n}=0. \qquad (4)$$

Let the coefficients $A_i(x)$ $(i=1,...,n)-$ be continuously differentiable functions in D, for which

$$\sum_{i=1}^{n} A_i^2(x) \neq 0, \quad \forall\, x \in D.$$

Equation (4) can be given the following geometric interpretation. If we consider the coefficients $A_i(x)$ to be components of the vector $\mathbf{A}(x)$ in n-dimensional space, then equation (4) means that the derivative of the function $u(x)$ is equal to zero along the direction of vector $\mathbf{A}$.

Obviously, equation (4) has a solution of the form $u \equiv C$, where C is a constant. But equation (4) also has infinitely many solutions other than the constant.

For example, the solution to the equation $\frac{\partial u}{\partial x_1}=0$ is any continuous function Φ that does not depend on x_1, that is, $u(x_1,...,x_n)=\Phi(x_2,...,x_n)$ (this solution is obtained by integrating the equation with respect to the variable x_1). In general, the search for solutions to equation (4) is reduced to constructing solutions to a system of ordinary differential equations.

Let us compare equation (4) with a system of ordinary differential equations called **equations of characteristics:**

$$\frac{dx_1}{A_1(x_1,\ldots,x_n)} = \frac{dx_2}{A_2(x_1,\ldots,x_n)} = \ldots = \frac{dx_n}{A_n(x_1,\ldots,x_n)}. \qquad (5)$$

This system is called a system of differential equations in a *symmetric form* corresponding to a homogeneous linear partial differential equation (4) (or a **characteristic system**).

In the case of two independent variables, it consists of a single equation. Under the assumptions made regarding the coefficients $A_1(x_1,\ldots,x_n),\ldots,A_n(x_1,\ldots,x_n)$, system (5) has exactly $n-1$ independent first integrals.

Definition 2. The first integral of the system (5) is called the function $\psi(x_1,\ldots,x_n)$, which differs from the constant, which is identically equal to some constant at all points $(x_1,\ldots,x_n)$ of the integral curve of the system (5).

Often, the first integral is not called the function ψ, but the ratio $\psi = C$, where C is a constant.

The integral curves of the system of equations (5) are called the **characteristics of the partial differential equation** (4).

THEOREM 1. Every solution $\varphi(x_1,...,x_n)$ of equation (4) is the first integral of system (5), and, conversely, every first integral $\psi(x_1,...,x_n)$ of system (5) is the solution of equation (4).

For example, the equation

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} - z\frac{\partial u}{\partial z} = 0 \qquad (*)$$

corresponds to the system of differential equations

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dz}{-z},$$

which has the following integrals

$$\psi_1 = xz, \qquad \psi_2 = x\sqrt{y}.$$

Then the functions $u_1 = xz$ and $u_2 = x\sqrt{y}$ are solutions to this equation. You can verify by substituting the expressions for u_1 and u_2 into the equation (*).

Let's assume that $A_n(x) \neq 0$ in D and $n-1$ independent first integrals of the system (5) are found:

$$\psi_1(x_1,\ldots,x_n),\ldots,\psi_{n-1}(x_1,\ldots,x_n). \qquad (6)$$

The condition for the independence of integrals (6) is the difference from the zero of the Jacobian [4]

$$J = \frac{D(\psi_1, \ldots, \psi_{n-1})}{D(x_1, \ldots, x_{n-1})} \neq 0, \quad \forall x \in D. \qquad (7)$$

---

[4] The Jacobian of functions $\psi_i(x_1,\ldots,x_m,x_{m+1},\ldots,x_n)$, $i=1,2,\ldots,m,$ is a determinant of order m, the i -th row of which contains partial derivatives of the first order of the function ψ_i with respect to variables $x_1,\ldots,x_m$. It is briefly indicated by the symbol $\frac{D(\psi_1,\ldots,\psi_m)}{D(x_1,\ldots,x_m)}$.

Let's introduce new independent variables

$$\begin{aligned} &\xi_1 = \psi_1(x_1, \ldots, x_n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &\xi_{n-1} = \psi_{n-1}(x_1, \ldots, x_n), \\ &\xi_n = \psi_n(x_1, \ldots, x_n), \end{aligned} \qquad (8)$$

where the function $\psi_n(x_1,\ldots,x_n)$ can be any continuously differentiable function in D, but in which the transformation (8) is non-degenerate, and a new notation with $\upsilon = \upsilon(\xi_1,\ldots,\xi_n)$ is used for the dependent variable, and $u(x) = \upsilon(\psi_1(x),\ldots,\psi_n(x))$.

$$v = v(\xi_1, \ldots, \xi_n),$$

$$u(x) = v(\psi_1(x), \ldots, \psi_n(x)). \qquad (9)$$

We show that when replacing (8)-(9), equation (4) is reduced to the simplest form when it is easy to construct its solution.

Indeed, we will express the derivatives included in equation (4) in terms of new variables using the rule of differentiation of a complex function:

$$\frac{\partial u}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial v}{\partial \xi_i}\frac{\partial \xi_i}{\partial x_k} = \sum_{i=1}^{n} \frac{\partial v}{\partial \xi_i}\frac{\partial \psi_i}{\partial x_k}, \quad k = 1, \ldots, n.$$

Substituting these expressions into equation (4) and grouping the terms, we obtain the equation

$$\sum_{i=1}^{n-1} \left(A_1(x)\frac{\partial \psi_i}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_i}{\partial x_n} \right) \frac{\partial v}{\partial \xi_i} +$$
$$+ \left(A_1(x)\frac{\partial \psi_n}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_n}{\partial x_n} \right) \frac{\partial v}{\partial \xi_n} = 0$$

Since the functions ψ_i for $i = 1,...,n-1$ are the first integrals of the system (5), then, according to Theorem 1, they are solutions of equation (4). Therefore, the last equation takes the form

$$\left(A_1(x)\frac{\partial \psi_n}{\partial x_1} + \ \ldots \ + A_n(x)\frac{\partial \psi_n}{\partial x_n} \right) \frac{\partial v}{\partial \xi_n} = 0.$$

And since the transformation (8) is non-degenerate, the function ψ_n cannot be a solution to equation (4), and therefore we will have

$$\frac{\partial v}{\partial \xi_n} = 0. \tag{10}$$

Thus, using the non-degenerate transformation (8), equation (4) is reduced to the form (10), which is called canonical.

Integrating equation (10) by ξ_n, we obtain its solution

$$v(\xi_1, \ldots, \xi_n) = \Phi(\xi_1, \ldots, \xi_{n-1}),$$

where Φ is an arbitrary function that does not depend on ξ_n and has continuous derivatives with respect to the variables $\xi_1,\ldots,\xi_{n-1}$. Returning to the old variables, we obtain the solution of equation (4).

THEOREM 2. Any solution $u=\varphi(x_1,\ldots,x_n)$ of equation (4) is represented as

$$u = \Phi(\psi_1(x_1,\ldots,x_n),\ldots,\psi_{n-1}(x_1,\ldots,x_n)), \tag{11}$$

where $\Phi(\psi_1,\ldots,\psi_{n-1})$ is some differentiable function of its arguments $\psi_1,\ldots,\psi_{n-1}$, and $\psi_i(x_1,\ldots,x_n)$ $(i=1,\ldots,n-1)$ are the first integrals of the system (5) satisfying the independence condition (7).

Formula (11) represents **the general solution** of equation (4).

Thus, the problem of constructing a general solution to equation (4) is equivalent to the problem of finding n− 1 independent first integrals of the corresponding system of ordinary differential equations (5).

Example 1

Find the general solution $u=u(x,y)$ of the equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0.$$

<u>Solution:</u>

Let's write down the equation of characteristics

$$\frac{dx}{y} = \frac{dy}{-x}.$$

This equation has a solution

$$x^2 + y^2 = C.$$

Therefore, the first integral is the function:

$$\psi = x^2 + y^2.$$

Then the general solution has the form

$$u = \Phi(x^2 + y^2),$$

and it represents a family of surfaces of rotation with the axis of rotation Ou. In particular, for $\Phi(\psi) = \psi$ we obtain a paraboloid of rotation:

$$u = x^2 + y^2,$$

when $\Phi(\psi) = \sqrt{\psi}$, we get a cone

$$u = \sqrt{x^2 + y^2}.$$

It is often possible to construct the first integrals of the characteristic system (5) for the case $n > 2$ by finding integrable combinations. *An integrable combination* is called a differential equation, which is a consequence of the system of equations (5) and is integrated in quadratures. The first integral of the system (5) is obtained from each integrable combination.

Example 2

Find the general solution $u = u(x,y,z)$ of the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

<u>Solution:</u>

Let's write down the equations of characteristics in a symmetric form

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Solving: $\frac{dy}{y} = \frac{dx}{x}$ and $\frac{dz}{z} = \frac{dx}{x}$.

we find the first two integrals:

$$\frac{y}{x} = C_1 \text{ and } \frac{z}{x} = C_2$$

Then the general solution of the given equation has the form

$$u(x,y,z) = \Phi\left(\frac{y}{x}, \frac{z}{x}\right)$$

To make integrable combinations of system (5), you can use the following rule of **equal fractions.**

If there are equal fractions

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots = \frac{a_n}{b_n},$$

and arbitrary numbers $\lambda_1,...,\lambda_n$ such that $\lambda_1 b_1+...+\lambda_n b_n \neq 0$, then

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\ldots=\frac{a_n}{b_n}=\frac{\lambda_1 a_1+\ldots+\lambda_n a_n}{\lambda_1 b_1+\ldots+\lambda_n b_n}.$$

Example 3

Find a solution of the equation

$$(x+z)u'_x+(y+z)u'_y+(x+y)u'_z=0.$$

<u>Solution:</u>

To find the independent first integrals, we make up the equations of characteristics in a symmetric form

$$\frac{dx}{x+z}=\frac{dy}{y+z}=\frac{dz}{x+y}.$$

By the property of equal fractions, we have

$$\frac{dx-dz}{z-y}=\frac{dy-dz}{z-x} \Rightarrow (x-z)d(x-z)=(y-z)d(y-z).$$

Integrating the last equality, we get the first integral

$$\psi_1(x,y,z)=(x-z)^2-(y-z)^2=(x-y)(x+y-2z).$$

According to the property of equal fractions, we will make another equality

$$\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{x-y} \Leftrightarrow \frac{d(x+y+z)}{x+y+z}=\frac{2d(x-y)}{x-y},$$

the integration of which gives another first integral

$$\psi_2(x,y,z) = \frac{x+y+z}{(x-y)^2}$$

Then the general solution of the given equation has the form:

$$u(x,y,z) = \Phi\left((x-y)(x+y-2z), \frac{x+y+z}{(x-y)^2}\right).$$

where $\Phi(a,b)-$ is an arbitrary continuously differentiable function.

Example 4

Find a solution of the equation

$$xu'_x + yu'_y + xyu'_z = 0, \quad x \neq 0, \ y \neq 0.$$

<u>Solution:</u>

Let's make up a characteristic system

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy}. \tag{$*$}$$

We will find the first integral by solving the equation

$$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \frac{x}{y} = C_1.$$

So, $\psi_1(x,y,z) = \dfrac{x}{y}$.

We will find another first integral by considering the second equation of the characteristic system (*)

$$\frac{dy}{y} = \frac{dz}{xy},$$

excluding x from it using the already found first integral ψ_1. Since $x = C_1 y$, we will have

$$\frac{dy}{y} = \frac{dz}{C_1 y^2} \quad \Rightarrow \quad C_1 y dy = dz \quad \Rightarrow \quad C_1 y^2 - 2z = C_2 \quad \Rightarrow \quad xy - 2z = C_2.$$

So, $\psi_2(x,y,z) = xy - 2z$ and the general solution of the given equation will be written as

$$u(x,y,z) = \Phi\left(\frac{x}{y}, xy - 2z\right).$$

2.3. FIRST-ORDER DIFFERENTIAL EQUATIONS, LINEAR WITH RESPECT TO PARTIAL DERIVATIVES

The Cauchy problem for a homogeneous linear equation

To isolate a single particular solution from the general solution, additional conditions must be set. Such conditions, for example, include initial conditions. Initial conditions are often set by fixing one of the independent variables.

We will consider the initial problem, or Cauchy problem, for equation (4) in the following formulation. Among all the solutions of equation (4), find such a solution

$$u = F(x_1, \dots, x_n), \tag{13}$$

which satisfies the initial conditions:

$$u = \varphi(x_1, \dots, x_{n-1}) \quad \text{при} \quad x_n = x_n^{(0)}, \tag{14}$$

where φ – is a given continuously differentiable function of variables $x_1,\dots,x_{n-1}$.

In the case where the desired function depends on two independent variables, Cauchy problem is to find a solution

$$u = F(x, y),$$

which satisfies the initial conditions:

$$u = \varphi(y)$$

at $x = x_0,$

where $\varphi(y)-$ the given function from y.

Geometrically, this means that among all integral surfaces, we are looking for an integral surface $u=F(x,y)$ that passes through a given curve $u=\varphi(y)$ lying in the plane $x=x_0$ parallel to the plane yOu.

Taking into account the general solution of the equation $u=\Phi(\psi_1,...,\psi_{n-1})$, the solution of the Cauchy problem is reduced to determining the type of function Φ such that

$$\Phi(\psi_1,\ldots,\psi_{n-1})\big|_{x_n=x_n^{(0)}} = \varphi(x_1,\ldots,x_{n-1}). \tag{15}$$

Let's denote

$$\begin{cases} \psi_1(x_1,\ldots,x_{n-1},x_n^{(0)}) = \overline{\psi}_1, \\ \psi_2(x_1,\ldots,x_{n-1},x_n^{(0)}) = \overline{\psi}_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \psi_{n-1}(x_1,\ldots,x_{n-1},x_n^{(0)}) = \overline{\psi}_{n-1}, \end{cases} \tag{16}$$

then equality (15) can be rewritten as

$$\Phi(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}) = \varphi(x_1,\ldots,x_{n-1}). \tag{17}$$

The system (16) is solvable with respect to $x_1,...,x_{n-1}$ at least in some neighborhood of the point $\left(x_1^{(0)},...,x_n^{(0)}\right)$ if $A_n\left(x_1^{(0)},...,x_n^{(0)}\right)\neq 0$, which we assume. Resolving the system (16) with respect to $x_1,...,x_{n-1}$, we obtain:

$$\begin{cases} x_1 = \omega_1(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \\ x_2 = \omega_2(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ x_{n-1} = \omega_{n-1}(\overline{\psi}_1,\ldots,\overline{\psi}_{n-1}), \end{cases} \tag{18}$$

If we now take the function

$$\Phi(\psi_1,\dots,\psi_{n-1}) = \varphi(\omega_1(\psi_1,\dots,\psi_{n-1}),\dots,\omega_{n-1}(\psi_1,\dots,\psi_{n-1})),$$

as Φ, then condition (17) will be fulfilled.

Therefore, the function gives the desired solution to the Cauchy problem. Here, the function φ is the function that participates in the initial conditions.

Thus, we come to the following algorithm for solving the Cauchy problem:

1) create the corresponding system of ordinary differential equations and find $n-1$ independent integrals:

$$\begin{cases} \psi_1(x_1,\dots,x_n), \\ \psi_2(x_1,\dots,x_n), \\ \dots\dots\dots\dots\dots\dots, \\ \psi_{n-1}(x_1,\dots,x_n). \end{cases} \tag{19}$$

2) replace the independent variable in integrals (19) with its specified value $x_n^{(0)}$:

$$\begin{cases} \psi_1(x_1,\dots,x_{n-1},x_n^{(0)}) = \overline{\psi}_1, \\ \psi_2(x_1,\dots,x_{n-1},x_n^{(0)}) = \overline{\psi}_2, \\ \dots\dots\dots\dots\dots\dots, \\ \psi_{n-1}(x_1,\dots,x_{n-1},x_n^{(0)}) = \overline{\psi}_{n-1}. \end{cases} \tag{20}$$

3) solve the system of equations (20) with respect to $x_1,\dots,x_{n-1}$:

$$\begin{cases} x_1 = \omega_1(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}), \\ x_2 = \omega_2(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}), \\ \dots\dots\dots\dots\dots\dots, \\ x_{n-1} = \omega_{n-1}(\overline{\psi}_1,\dots,\overline{\psi}_{n-1}). \end{cases} \tag{21}$$

4) construct a function

$$u = \varphi(\omega_1(\psi_1, \ldots, \psi_{n-1}), \ldots, \omega_{n-1}(\psi_1, \ldots, \psi_{n-1})), \qquad (22)$$

that gives a solution to the Cauchy problem.

Example 1

Find a solution to the Cauchy problem

$$xu'_x + yu'_y + xyu'_z = 0, \quad u(x, y, 0) = x^2 + y^2.$$

Solution:

Let's make a characteristic system

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{xy}. \qquad (*)$$

We will find the first integral by solving the equation

$$\frac{dx}{x} = \frac{dy}{y} \quad \Rightarrow \quad \frac{x}{y} = C_1.$$

So, $\psi_1(x,y,z) = \frac{x}{y}$.

We will find another first integral by considering the second equation of the characteristic system (*)

$$\frac{dy}{y} = \frac{dz}{xy},$$

excluding x from it using the already found first integral ψ_1. Since $x = C_1 y$, we will have

$$\frac{dy}{y} = \frac{dz}{C_1 y^2} \quad \Rightarrow \quad C_1 y dy = dz \quad \Rightarrow \quad C_1 y^2 - 2z = C_2 \quad \Rightarrow \quad xy - 2z = C_2.$$

So, $\psi_2(x,y,z) = xy - 2z$.

The characteristic system corresponding to the equation has the following two first integrals

$$\psi_1(x,y,z) = \frac{x}{y}, \quad \psi_2(x,y,z) = xy - 2z.$$

Considering them at $z = 0$, we make up a system of equations

$$\frac{x}{y} = \overline{\psi}_1, \quad xy = \overline{\psi}_2,$$

from which we find

$$y^2 = \frac{\overline{\psi}_2}{\overline{\psi}_1}, \quad x^2 = \overline{\psi}_1 \overline{\psi}_2.$$

Then for a given function $\varphi(x,y) = x^2 + y^2$ we will have

$$\varphi(\overline{\psi}_1, \overline{\psi}_2) = \left(\overline{\psi}_1 + \frac{1}{\overline{\psi}_1}\right) \overline{\psi}_2.$$

Therefore, the solution to the Cauchy problem has the form

$$u(x,y,z) = \left(\frac{x}{y} + \frac{y}{x}\right)(xy - 2z).$$

Quasi-linear equations

Let the point $x=(x_1,\dots,x_n)$ belong to the domain $D\subset\mathbb{R}^n$. Consider a quasi-linear equation

$$A_1(x,u)\frac{\partial u}{\partial x_1}+\ldots+A_n(x,u)\frac{\partial u}{\partial x_n}=B(x,u), \qquad (23)$$

assuming that $A_i(x,u)$ $(i=1,\dots,n)$ and $B(x,u)$ are differentiable functions of the arguments x,u in some domain $G\subset\mathbb{R}^{n+1}$.

Equation (23) corresponds to the following linear equation

$$A_1(x,u)\frac{\partial v}{\partial x_1}+\ldots+A_n(x,u)\frac{\partial v}{\partial x_n}+B(x,u)\frac{\partial v}{\partial u}=0, \qquad (24)$$

with an unknown function $\upsilon=\upsilon(x,u)$.

The method of solving a quasi-linear equation is based on the following theorem:

THEOREM 3.

Let $\upsilon=V(x,u)$ be the solution of equation (24). Let equation $V(x,u)=0$ define in the domain of D variables $x=(x_1,\dots,x_n)$ some differentiable function $u=\varphi(x)$, and let $\frac{\partial V}{\partial u}\neq 0$ in D for $u=\varphi$. Then $u=\varphi(x)$ is the solution of equation (23).

We describe *an algorithm for constructing a solution to a quasi-linear equation.*

1) write out the characteristic system for the linear equation (24):

$$\frac{dx_1}{A_1(x,u)} = \ldots = \frac{dx_n}{A_n(x,u)} = \frac{du}{B(x,u)} \qquad (25)$$

The characteristics of the linear equation (24) are called **the characteristics of the quasi-linear equation** (23).

2) find the n independent first integrals of the system (25):

$$\psi_1(x,u), \ldots, \psi_n(x,u). \qquad (26)$$

(or we can write $\psi_1(x,u) = c_1, \ldots, \psi_n(x,u) = c_n$).

3) using formula (11), construct a general solution to equation (24):

$$v(x,u) = \Phi(\psi_1(x,u), \ldots, \psi_n(x,u)).$$

(or we can write $\upsilon = V(\psi_1(x,u), \ldots, \psi_n(x,u))$).

4) assuming $\upsilon = 0$, write down the equation to determine the set of solutions to equation (23):

$$\Phi(\psi_1(x,u), \ldots, \psi_n(x,u)) = 0. \qquad (27)$$

(or we can write $V(\psi_1(x,u), \ldots, \psi_n(x,u)) = 0$).

The expression (27) is called the **general integral**, or the **general solution**, of equation (23). If u is included only in one of the first integrals (26), for example, in the last one, then the general solution can be written as follows:

$$\psi_n(x,u) = F(\psi_1,\ldots,\psi_{n-1}), \tag{28}$$

where $F-$ an arbitrary differentiable function. If it is possible to resolve equality (28) with respect to u, then we obtain a general solution of equation (23) in explicit form.

Comment 1

It is possible that there may be solutions to equation (23) for which equation (24) is not satisfied identically in (x,u), but only when $u=\varphi(x)$ is identical in x. Such solutions are not contained in formula (27) and are called *special*. A special solution is an exceptional case, and therefore we will not consider them further.

Comment 2

The solution of linear equation (2) can also be constructed in the described way.

Comment 3

When constructing the first integrals of the system (25), in some cases it may turn out that the variable u will enter only one of them:

$$\psi_1(x) = c_1,\ \ldots,\ \psi_{n-1}(x) = c_{n-1},\ \psi_n(x,\ u) = c_n.$$

Then the general solution will be from the ratio

$$V\big(\psi_1(x),\ \ldots,\ \psi_{n-1}(x),\ \psi_n(x,\ u)\big) = 0,$$

which can be rewritten by the implicit function theorem in the form

$$\psi_n(x,u) = F\big(\psi_1(x,u),\ldots,\psi_{n-1}(x)\big) \tag{29}$$

By resolving equality (29) with respect to u, we obtain a general solution of equation (23) in explicit form.

The characteristic system (25) will be written in this case as

$$\frac{dx_1}{A_1(x)} = \ldots = \frac{dx_n}{A_n(x)} = \frac{du}{0}.$$

A system of n independent first integrals can be chosen as follows

$$\psi_1(x) = c_1, \ \ldots, \ \psi_{n-1}(x) = c_{n-1}, \ u = c_n.$$

We see that the variable u will enter only the last first integral. The solution of the equation can be written using the formula (29).

Example 2

Solve the equation

$$x_2 \frac{\partial u}{\partial x_1} + x_1 \frac{\partial u}{\partial x_2} = x_1 - x_2.$$

Solution:

The characteristic system has the form

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1} = \frac{du}{x_1 - x_2}.$$

The first equality

$$\frac{dx_1}{x_2} = \frac{dx_2}{x_1}$$

it will be written in the form of

$$x_1 dx_1 = x_2 dx_2$$

and leads to the first integral

$$x_1^2 - x_2^2 = c_1.$$

To obtain another first integral, we use the property of adding proportions:

$$\frac{d(x_2 - x_1)}{x_1 - x_2} = \frac{du}{x_1 - x_2},$$

where do we get the first integral:

$$u + x_1 - x_2 = c_2.$$

Since u is included in only one of the first integrals obtained, we obtain an explicit solution to the equation

$$u = x_2 - x_1 + \mathrm{F}\left(x_1^2 - x_2^2\right),$$

The function $F(y)-$ is an arbitrary function of class C^1.

Example 3

Solve the equation

$$\left(x_2 + 2u^2\right)\frac{\partial u}{\partial x_1} - 2x_1^2 u \frac{\partial u}{\partial x_2} = x_1^2.$$

<u>Solution:</u>

The characteristic system has the form

$$\frac{dx_1}{x_2 + 2u^2} = \frac{dx_2}{-2x_1^2 u} = \frac{du}{x_1^2}.$$

From the second ratio we have

$$dx_2 = -2udu,$$

where do we get the first integral:

$$x_2 + u^2 = c_1. \qquad (*)$$

Consider the ratio

$$\frac{dx_1}{x_2 + 2u^2} = \frac{du}{x_1^2}$$

Substituting $x_2 = c_1 - u^2$ into the ratio, we get

$$\frac{dx_1}{c_1 + u^2} = \frac{du}{x_1^2}$$

from which

$$x_1^2\, dx_1 = \left(c_1 + u^2\right) du,$$

and therefore

$$\frac{x_1^3}{3} = c_1 u + \frac{u^3}{3} + c_2.$$

Then the general solution of the equation will be written in an implicit form

$$V\left(x_2 + u^2,\ x_1^3 - 3\left(x_2 + u^2\right)u - u^3\right) = 0,$$

where the function V is an arbitrary function of class C^1 and such that

$$\frac{\partial V\left(x_2 + u^2,\ x_1^3 - 3\left(x_2 + u^2\right)u - u^3\right)}{\partial u} \neq 0.$$

Example 4

Find a solution to the equation

$$\sin y \cdot u_x + e^x \cdot u_y = 2x \sin y \cdot u^2.$$

<u>Solution:</u>

Obviously, the given equation is quasi-linear. To construct a general solution, we find two independent first integrals of the system of equations:

$$\frac{dx}{\sin y} = \frac{dy}{e^x} = \frac{du}{2x \sin y \cdot u^2}.$$

Considering two equations

$$\frac{dx}{\sin y} = \frac{dy}{e^x} \quad \text{and} \quad \frac{dx}{\sin y} = \frac{du}{2x \sin y \cdot u^2},$$

we obtain

$$\psi_1(x,y,u) = e^x + \cos y, \qquad \psi_2(x,y,u) = x^2 + \frac{1}{u}.$$

Then the general integral of the given equation has the form:

$$\Phi\left(e^x + \cos y, x^2 + \frac{1}{u}\right) = 0.$$

Solving this equation with respect to the second argument, we obtain

$$x^2 + \frac{1}{u} = F(e^x + \cos y) \quad \Rightarrow \quad u = \left(F(e^x + \cos y) - x^2\right)^{-1}.$$

where F is an arbitrary continuously differentiable function.

Example 5

Find a solution to the equation

$$(2y - u)\,u'_x + y\,u'_y = u.$$

Solution:

Let's make up a characteristic system

$$\frac{dx}{2y-u} = \frac{dy}{y} = \frac{du}{u}.$$

Solving the equation

$$\frac{dy}{y} = \frac{du}{u} \quad \Rightarrow \quad \frac{u}{y} = C_1,$$

we find the first integral

$$\psi_1(x,y,u) = \frac{u}{y}.$$

Using the rule of equal fractions, we will make an integrable combination

$$\frac{dx}{2y-u} = \frac{2dy - du}{2y-u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

Where do we get another first integral

$$\psi_2(x, y, u) = x - 2y + u.$$

Therefore, the general integral of the given equation has the form

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0,$$

where Φ is an arbitrary continuously differentiable function.

Example 1.

To find a general solution of a quasi-linear inhomogeneous partial differential equation of the first order

$$x^2 u \frac{\partial u}{\partial x} + y^2 u \frac{\partial u}{\partial y} = x + y$$

<u>Solution:</u>

The characteristic system, which corresponds to this quasi-linear equation, in a symmetric form, has the form:

$$\frac{dx}{x^2 u} = \frac{dy}{y^2 u} = \frac{dz}{x+y}$$

The first integrated combination

$$\frac{dx}{x^2 u} = \frac{dy}{y^2 u}$$

after the reduction by u:

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

and integration, it gives

$$\frac{1}{x} - \frac{1}{y} = C_1.$$

To obtain another first integral, we make an integrable combination

$$\frac{dx - dy}{x^2 u - y^2 u} = \frac{du}{x+y}$$

Where from

$$\frac{d(x-y)}{u(x^2-y^2)}=\frac{du}{x+y}$$

or

$$\frac{d(x-y)}{x-y}=udu$$

After integration, we get

$$\ln|x-y|-\frac{u^2}{2}=C_2\,.$$

Therefore, the general solution of this quasi-linear differential equation will be:

$$\Phi\left(\frac{1}{x}-\frac{1}{y},\ln|x-y|-\frac{u^2}{2}\right)=0\,,$$

where Φ is an arbitrary function. Since u is included in only one of the first integrals, the general solution can be written as

$$\ln|x-y|-\frac{u^2}{2}=F\left(\frac{1}{x}-\frac{1}{y}\right),$$

where F is an arbitrary function.

Example 2.

Find a general solution to a quasi-linear partial differential equation of the first order

$$xy\frac{\partial u}{\partial x}+(x-2u)\frac{\partial u}{\partial y}=yu\,.$$

Solution:

The characteristic system corresponding to this quasi-linear equation takes the form in a symmetric form:

$$\frac{dx}{xy}=\frac{dy}{x-2u}=\frac{du}{yu}$$

From the first integrable combination

$$\frac{dx}{xy}=\frac{du}{yu}$$

From the first integrable combination we obtain

$$\frac{u}{x}=C_1\,.$$

To find the second independent integral of the characteristic system, we rewrite it as:

$$\begin{cases} x'=xy, \\ y'=x-2u, \\ u'=yu. \end{cases}$$

Let's differentiate the second equation of the system

$$y''=x'-2u',$$

substitute instead of x' and y' their expressions from the first and third equations of the system:

$$y''=x'-2u'=xy-2yu=y\left(x-2u\right)=yy',$$

as a result, we obtain a second-order equation

$$y'' = yy'.$$

By replacing $y' = p$, where $p = p(y)$, $y'' = pp'$, we lower the order of the equation:

$$pp' = yp.$$

From where we have two equations: $p = 0$ and $p' = y$. The first one gives a trivial solution

$$y' = 0, \ \ y = const,$$

which we are not interested in. Solving the second equation, we get

$$p = \frac{y^2}{2} + \tilde{C}_2$$

or

$$y' = \frac{y^2}{2} + \tilde{C}_2$$

$$x - 2u = \frac{y^2}{2} + \tilde{C}_2$$

$$2x - 4u = y^2 + 2\tilde{C}_2$$

$$2x - 4u - y^2 = 2\tilde{C}_2 = C_2.$$

Thus, the general solution of the original quasi-linear differential equation is

$$\Phi\left(\frac{u}{x}, 2x - 4u - y^2\right) = 0,$$

where Φ is an arbitrary function.

Example 3.

Find a general solution to the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2.$$

Solution:

Let's make up the equations of characteristics:

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-x^2 - y^2}.$$

The first equation of this system can be solved separately from the second, since it does not contain u (the variable u, which stands in the left and right sides of this equation, is reduced). From equality

$$\frac{dx}{x} = \frac{dy}{y}$$

we get by integrating

$$\ln|x| = \ln|y| + \ln C,$$

where do we find the first integral of the system:

$$\frac{y}{x} = C_1.$$

The second equation of this system contains all three variables

To exclude the variable y, let's use the first integral found. Since the desired integral curve lies on one of the surfaces defined by the first integral found, at each point of this curve $y = C_1 x$ (the value of the constant C_1 is the same at all points of the desired integral curve, but may differ if you switch to another integral curve). Replace y with $C_1 x$ in the second equation, after the transformations we get

$$-(1 + C_1^2)x\,dx = u\,du.$$

Integrating, we find the dependence

$$(1 + C_1^2)x^2 + u^2 = C_2.$$

This ratio containing C_2 is not the first integral, since it also contains an arbitrary constant C_1. Given that for the found curve $C_1 = \frac{y}{x}$, we rewrite the ratio as

$$x^2 + y^2 + u^2 = C_2.$$

In this form of notation, the relation is performed for any of the integral curves, that is, it is the first integral.

The general solution of the first-order equation has the form (in an implicit form)

$$\Phi\left(\frac{y}{x}, x^2 + y^2 + u^2\right) = 0,$$

where Φ is an arbitrary differentiable function. It is possible to get an explicit solution from the last expression:

$$u = \pm\sqrt{f\left(\frac{y}{x}\right) - x^2 - y^2},$$

where f is an arbitrary differentiable function.

Comment

When finding the first integrals of a system written in symmetric form, derivative proportions are often used to obtain integrable combinations, for example

$$\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}.$$

In the above example, we will rewrite the system in the form

$$\frac{x\,dx}{x^2} = \frac{y\,dy}{y^2} = \frac{u\,du}{-x^2-y^2},$$

then we will use the derived proportion by adding the numerators

and denominators of the first and second ratios:

$$\frac{x\,dx + y\,dy}{x^2+y^2} = \frac{x\,dx}{x^2} = \frac{y\,dy}{y^2} = \frac{u\,du}{-x^2-y^2}.$$

Comparing the first relation with the last one, we get

$$x\,dx + y\,dy = -u\,du,$$

hence the first integral

$$x^2 + y^2 + u^2 = C_2.$$

FINDING A PARTICULAR SOLUTION (CAUCHY PROBLEM)

In order to single out one definite solution from the infinite set of solutions given by formula

$$\Phi\left(\psi_1(x,y,u),\psi_2(x,y,u)\right)=0,$$

it is necessary to find the function Φ included in the solution.

This can be done under additional conditions. We formulate the problem of finding a partial solution to equation

$$P(x,y,u)\frac{\partial u}{\partial x}+Q(x,y,u)\frac{\partial u}{\partial y}=R(x,y,u) \qquad (1)$$

— the Cauchy problem.

(*A general view of a first-order quasilinear equation with two independent variables, where $u=u(x,y)$ is the desired function; $P(x,y,u)$, $Q(x,y,u)$, $R(x,y,u)$ are continuous changes in the variables of the function in the considered area that do not vanish at the same time.*) We just wrote it down in this form.

Along some curve L_1 of the plane (x,y), the values of the desired function are set

$$\begin{cases} y=f(x), \\ u=g(x), \end{cases}$$

where $f(x)$, $g(x)$ are differentiable functions. Find such a solution $u=u(x,y)$ of equation (1) in the vicinity of the line L_1 so that $u=u(x,f(x))=g(x)$.

Geometric illustration of the Cauchy problem: through a space curve L with a continuously varying tangent (smooth curve), draw the integral surface of equation (1) (Fig. 1).

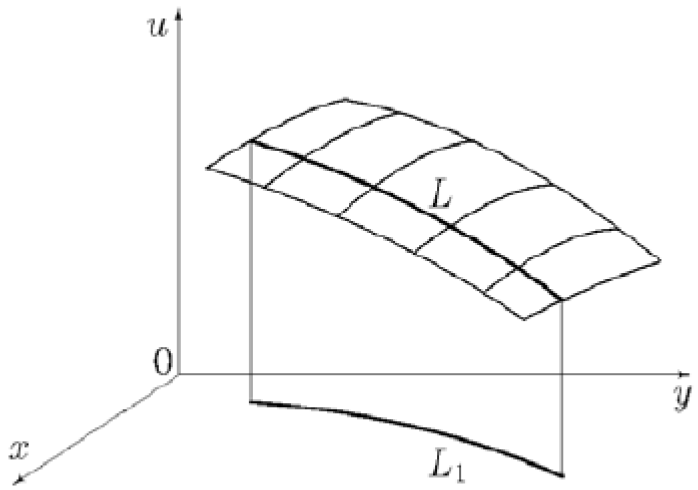


Fig.1.

The L line can be defined in a more general way:

$$\begin{cases} x = \varphi(\sigma), \\ y = \varkappa(\sigma), \\ u = \chi(\sigma). \end{cases} \qquad (2)$$

The first two of the equations (2) define the curve L_1 in parametric form, all three equations define the curve L in space (x,y,u), for which L_1 is a projection onto the plane (x,y). In this case, the condition

$$u(\varphi(\sigma), \varkappa(\sigma)) = \chi(\sigma)$$

must be fulfilled.

The geometric solution to the Cauchy problem is obvious:

a characteristic must be drawn through each point of a given line L.

The set of characteristics passing through all points of the line L form the desired integral surface (see Fig. 1).

A system of differential equations

$$\frac{dx}{P(x,y,u)} = \frac{dy}{Q(x,y,u)} = \frac{du}{R(x,y,u)}$$

can be integrated without knowledge of the integral surface. The general integral of the system

$$\begin{cases} \psi_1(x,y,u) = C_1, \\ \psi_2(x,y,u) = C_2 \end{cases}$$

defines a family of characteristics (depending on two parameters C_1 and C_2), which has the following property: one characteristic passes through each point of the domain where the conditions for the existence and uniqueness of the solution are met. Constants C_1 and C_2 are independent, each of them can be assigned any values; a two-parameter family of characteristics is obtained as a result of the intersection of each surface of one family with each surface of the other (Fig.2).

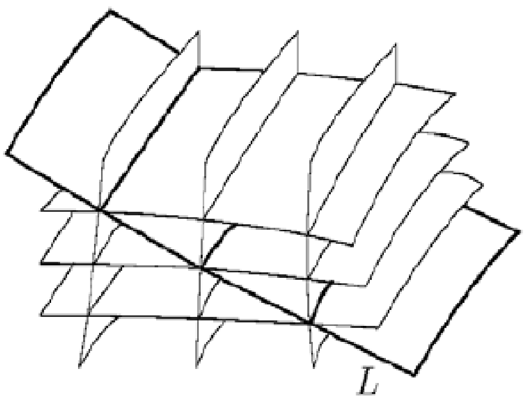


Fig.2.

Of all the surfaces defined by the first integrals, we need to leave only such pairs of surfaces whose intersection lines pass through the points of the line L; for this, we must learn how to select pairs C_1 and C_2 accordingly. In other words, between arbitrary constants C_1 and C_2 in the general integral of the system, it is necessary to establish some dependence $\Phi(C_1, C_2) = 0$.

A surface from the family $\psi_1(x,y,u) = C_1$ can be drawn through each point of the line L. Substituting the coordinates of this point, given as functions of the parameter σ, into the equation of the surface, we establish a relationship between the value of the constant C_1 defining the surface and the value of the parameter σ corresponding to the point of intersection of the line L with this surface:

$$C_1 = \psi_1(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_1(\sigma). \qquad (3)$$

A similar ratio

$$C_2 = \psi_2(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_2(\sigma) \qquad (4)$$

gives the dependence between the parameter σ of the point of the curve L and the constant C_2 of the surface of another family intersecting the line L at this point. Any pair of values C_1 and C_2 calculated by formulas (3) and (4) for one value σ will determine a pair of surfaces whose intersection line (characteristic) passes through the point of the line L corresponding to this value σ. Therefore, formulas (3) and (4) taken together define in parametric form the desired dependence between C_1 and C_2:

$$\begin{cases} C_1 = C_1(\sigma), \\ C_2 = C_2(\sigma). \end{cases} \qquad (5)$$

Thus, the system of equations

$$\begin{cases} \psi_1(x, y, u) = C_1(\sigma), \\ \psi_2(x, y, u) = C_2(\sigma) \end{cases} \qquad (6)$$

defines a family of characteristics passing through the line L. This family of characteristics, depending on one parameter σ, forms the desired integral surface (solution of the Cauchy problem).

Excluding the parameter σ from equations (5), can obtain a dependence of the form:

$$\Phi(C_1, C_2) = 0. \qquad (7)$$

Accordingly, the solution of the Cauchy problem will be presented as

$$\Phi(\psi_1(x, y, u), \psi_2(x, y, u)) = 0. \qquad (8)$$

To exclude σ, for example, equation (3) with respect to σ should be resolved and the expression $\sigma(C_1)$ should be substituted into the left part of the relation (4). This is possible when σ enters the left parts of equations (3) and (4).

If the entire curve L lies, for example, on the surface $\psi_1(x,y,u) = C_1^0$, then the ratio

$$\psi_1(\varphi(\sigma), \varkappa(\sigma), \chi(\sigma)) = C_1^0$$

cannot be resolved relative to the σ parameter. But then this surface itself $\psi_1(x,y,u)=C_1^0$ is the integral surface in the Cauchy problem.

Finally, if a given curve L lies simultaneously on two surfaces of different families, then it itself is a characteristic, and the parameter σ is not included in any of the relations (3), (4). The Cauchy problem becomes indefinite, since each characteristic belongs to an infinite set of integral surfaces. Indeed, if only the constants C_1^0 and C_2^0 satisfy equation

$$\Phi(C_1, C_2) = 0,$$

where Φ is an arbitrary differentiable function, then equation

$$\Phi(\psi_1(x,y,u), \psi_2(x,y,u)) = 0$$

defines an integral surface passing through the line L. Thus, countless integral surfaces can be drawn through a given line L.

Example 4.

Find the integral surface of the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2,$$

passing through the curve

$$\begin{cases} x = a, \\ u = \sqrt{y^2 + a^2}, \end{cases}$$

where a is a constant.

<u>Solution:</u>

In Example 3, the first integrals of the equations of characteristics are found,

$$\frac{y}{x} = C_1, \qquad x^2 + y^2 + u^2 = C_2.$$

The first equation defines a set of planes passing through the Oz -axis, the second one defines spheres of different radii centered at the origin. The characteristics of the original partial differential equation can be represented as meridians on spheres.

In the equations of a given curve, we take y as an independent variable.

Substituting the coordinates of points into the first integrals of the system, we obtain

$$\begin{cases} C_1 = \dfrac{y}{a}, \\ C_2 = a^2 + y^2 + (y^2 + a^2). \end{cases}$$

From here

$$C_2 = 2a^2(1 + C_1^2).$$

Replacing C_1 and C_2 in this dependence with the functions on the left sides of the first integrals, we find the desired solution

$$x^2 + y^2 + u^2 = 2a^2 \frac{x^2 + y^2}{x^2}.$$

Example 5.

Find the integral surface of the equation

$$(x^2 - y^2 - u^2)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 2xu,$$

passing through the curve L: $x = 0, y = 2a\cos t, u = 2a\sin t$.

<u>Solution:</u>

Integrating a system of equations

$$\frac{dx}{x^2 - y^2 - u^2} = \frac{dy}{2xy} = \frac{du}{2xu}.$$

From the second equation

$$\frac{dy}{y} = \frac{du}{u}$$

we get the first integral

$$\frac{u}{y} = C_1.$$

To obtain another first integral, we present the system as

$$\frac{x\,dx}{x^2 - y^2 - u^2} = \frac{y\,dy}{2y^2} = \frac{u\,du}{2u^2},$$

and then we apply the derivative proportion, adding up all the numerators and all the denominators of the relations:

$$\frac{x\,dx + y\,dy + u\,du}{x^2 + y^2 + u^2} = \frac{dy}{2y}.$$

Thus, an integrable combination is obtained

$$\frac{d(x^2 + y^2 + u^2)}{x^2 + y^2 + u^2} = \frac{dy}{y},$$

where from

$$\frac{x^2+y^2+u^2}{y}=C_2.$$

The first found integrals define planes passing through the Ox axis and spheres centered on the Oy axis passing through the origin. The characteristics will be circles passing through the origin, for which the Ox axis is tangent.

The desired surface will be obtained by rotating a circle of radius a around the Ox axis, touching the axis at the origin. Let's find the equation of this surface. Substituting the coordinates of the points of a given line, expressed in terms of t, into the first integrals, we obtain

$$C_1=\operatorname{tg}t, \qquad C_2=\frac{2a}{\cos t}.$$

Excluding t, we find the relationship between C_1 and C_2:

$$4a^2(C_1^2+1)=C_2^2,$$

and then the solution of the Cauchy problem

$$4a^2(u^2+y^2)=(x^2+y^2+u^2)^2.$$

In spherical coordinates $(x=r\sin\theta,\ y=r\cos\theta\cos\varphi,\ u=r\cos\theta\sin\varphi)$ the equation of the surface takes the form $r=2a\cos\theta$.

September 30th, 2024

Example 1

Find the general integral of the equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u\ .$$

<u>Solution:</u>

Consider a system of equations

$$\frac{dx}{x}=\frac{dy}{y}=\frac{du}{u}\ .$$

Solving the equation

$$\frac{dx}{x}=\frac{dy}{y},$$

we get

$$\frac{y}{x}=C_1;$$

the solution of the equation

$$\frac{dx}{x}=\frac{du}{u}$$

is

$$\frac{u}{x}=C_2\ .$$

Now we can find the general integral of the given equation:

$$\Phi\left(\frac{y}{x},\frac{u}{x}\right)=0$$

or

$$\frac{u}{x}=\psi\left(\frac{y}{x}\right),$$

so

$$u = x\psi\left(\frac{y}{x}\right),$$

where ψ is an arbitrary function.

Example 2

Find the general integral of the equation

$$\left(x^2 + y^2\right)\frac{\partial u}{\partial x} + 2xy\frac{\partial u}{\partial y} = 0 .$$

<u>Solution:</u>

Let's write down a system of equations

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy} = \frac{du}{0} .$$

Using the property of proportion, we present the equation

$$\frac{dx}{x^2 + y^2} = \frac{dy}{2xy}$$

as

$$\frac{dx + dy}{x^2 + y^2 + 2xy} = \frac{dx - dy}{x^2 + y^2 - 2xy} ,$$

$$\frac{d(x+y)}{(x+y)^2} = \frac{d(x-y)}{(x-y)^2} .$$

Integrating, we get

$$-\frac{1}{x+y} = -\frac{1}{x-y} + C_1 ,$$

$$\frac{1}{x-y} - \frac{1}{x+y} = C_1 ,$$

$$\frac{2y}{x^2 - y^2} = C_1 .$$

The last equality can be rewritten as

$$\frac{y}{x^2 - y^2} = C_1 .$$

The second equation of the system:

$$du = 0 .$$

$$u = C_2 .$$

The general integral of a given equation has the form

$$\Phi\left(\frac{y}{x^2 - y^2}, u\right) = 0 ,$$

or

$$u = \psi\left(\frac{y}{x^2 - y^2}\right),$$

where ψ is an arbitrary function.

Example 3

Find solutions to the equation

$$(2y - u)\, u'_x + y\, u'_y = u.$$

Solution:

Let's make up a characteristic system

$$\frac{dx}{2y - u} = \frac{dy}{y} = \frac{du}{u}.$$

Solving the equation

$$\frac{dy}{y} = \frac{du}{u} \quad \Rightarrow \quad \frac{u}{y} = C_1,$$

we find the first integral

$$\psi_1(x, y, u) = \frac{u}{y}.$$

Using the rule of equal fractions, we will make an integrable combination

$$\frac{dx}{2y-u} = \frac{2dy-du}{2y-u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

From where we get another first integral

$$\psi_2(x, y, u) = x - 2y + u.$$

Therefore, the general integral of the given equation has the form

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0,$$

where Φ is an arbitrary continuously differentiable function.

Example 4

Find a general solution to the equation

$$xu\frac{\partial u}{\partial x} + yu\frac{\partial u}{\partial y} = -x^2 - y^2.$$

<u>Solution:</u>

Let's make up the equations of characteristics:

$$\frac{dx}{xu} = \frac{dy}{yu} = \frac{du}{-x^2-y^2}.$$

The first equation of this system can be solved separately from the second, since it does not contain u (the variable u, which stands in the left and right sides of this equation, is reduced).

From equality,

$$\frac{dx}{x} = \frac{dy}{y}$$

we obtain by integrating

$$\ln|x| = \ln|y| + \ln C,$$

from where we find the first integral of the system

$$\frac{y}{x} = C_1.$$

The second equation of this system

$$\frac{dx}{xu} = \frac{du}{-x^2 - y^2}$$

contains all three variables. To exclude the variable y, let's use the first integral found.

Since the desired integral curve lies on one of the surfaces defined by the first integral found, at each point of this curve $y = C_1 x$ (the value of the constant C_1 is the same at all points of the desired integral curve, but may differ if you switch to another integral curve).

Replace y with $C_1 x$ in the second equation, and after the transformations we get

$$-(1 + C_1^2)x\,dx = u\,du.$$

Integrating, we find the dependence

$$(1 + C_1^2)x^2 + u^2 = C_2.$$

This ratio containing C_2 is not the first integral, since it also contains an arbitrary constant C_1. Given that for the found curve

$$C_1 = \frac{y}{x},$$

we rewrite the ratio as

$$x^2 + y^2 + u^2 = C_2.$$

In this form of notation, the relation is performed for any of the integral curves, that is, it is the first integral.

The general solution of the first-order equation has the form (in an implicit form)

$$\Phi\left(\frac{y}{x}, x^2 + y^2 + u^2\right) = 0,$$

where Φ is an arbitrary differentiable function. It is possible to get an explicit solution from the last expression:

$$u = \pm\sqrt{f\left(\frac{y}{x}\right) - x^2 - y^2},$$

where f is an arbitrary differentiable function.

Example 5

Find a general solution to the partial differential equation

$$e^x \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = ye^x$$

<u>Solution:</u>

Creating a system

$$\frac{dx}{e^x} = \frac{dy}{y^2} = \frac{du}{ye^x}$$

From the first equation, we find one first integral

$$\frac{1}{y} - e^{-x} = C_1$$

and from the second, taking into account the equality

$$e^x = \frac{y}{1 - yC_1}$$

another first integral

$$u - \frac{\ln|y| - x}{e^{-x} - y^{-1}} = C_2$$

follows.

Thus, the general integral of this equation will be

$$\Phi\left(\frac{1}{y} - e^{-x}, \frac{\ln|y| - x}{e^{-x} - y^{-1}} - u\right) = 0$$

The general solution has the form

$$u=\frac{\ln|y|-x}{e^{-x}-y^{-1}}+\varphi\left(\frac{1}{y}-e^{-x}\right).$$

Example 6

Find a solution to the equation

$$u\frac{\partial u}{\partial x}+(u^2-x^2)\frac{\partial u}{\partial y}+x=0$$

under additional conditions:

a) $y=2x^2,\ u=x;$

b) $y=1+x^2,\quad u=x.$

Solution:

Let's write down the equations of characteristics

$$\frac{dx}{u}=\frac{dy}{u^2-x^2}=\frac{du}{-x}.$$

Let's find the first integral of the system:

$$\frac{dx}{u}=\frac{du}{-x};$$

$$x\,dx+u\,du=0;$$

$$x^2+u^2=C_1.$$

To determine the next integral, we take

$$\frac{x\,dx+u\,du}{u^2-x^2}=\frac{dy}{u^2-x^2}.$$

Comparing the first and second relations, we get

$$x\,dx+u\,du=dy,$$

or

$$d(xu)=dy.$$

Another first integral has been found

$$xu - y = C_2$$

The general solution

$$\Phi(x^2 + u^2, xu - y) = 0.$$

a) Solving the Cauchy problem, it is convenient to take x as the parameter σ on this curve. Substituting into the first integrals x,

$$y = 2x^2,\ u = x,$$

we get

$$C_1 = 2x^2,\ C_2 = -1.$$

Therefore,

$$C_1 = -2C_2.$$

Then the particular solution has the form

$$x^2 + u^2 = -2xu + 2y,$$

or

$$(x + u)^2 = 2y.$$

b) Let's solve the problem under another initial conditions.

Substituting $x,\ y = 1 + x^2,\ u = x$ into the first integrals, we get $C_1 = 2x^2$, $C_2 = -1$. The independence of C_2 from x means that the curve given by the initial conditions lies on the surface $xu - y = -1$.

At the same time, C_1 depends on x, which means that different surfaces of the first family $x^2 + u^2 = C_1$ correspond to different points of a given line. Therefore, the line specified by the initial conditions is not a characteristic, and $xu - y = -1$ is the only integral surface satisfying the initial conditions.

Homework №6. (The deadline is the 4th of October).

Solve the Cauchy problem for the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2xy,$$

satisfying the conditions $y = x,\, u = x^2.$

October 9, 2024

2.3.1. APPLICATION OF THE LAPLACE TRANSFORM TO SOLVING FIRST-ORDER LINEAR EQUATIONS

An unknown function satisfying a first-order linear partial differential equation and given conditions can be found using a one-time or two-time Laplace transform, depending on the type of conditions.

In the first case, the transformation is applied to a partial differential equation for one of the independent variables, assuming that the other remains unchanged.

The result is an operator equation with respect to the image, which is an ordinary differential equation with a parameter.

After integrating the operator equation from the image found from it, the original is found as a solution to the original equation.

In the second case, the Laplace transform is applied sequentially, resulting in an equation from which a two-fold image of the desired function is found.

Using inverse transformations, the original function is restored.

The solution of the partial differential equation found using the two-fold Laplace transform does not depend on the sequence in which the forward and reverse transformations were applied.

Example 1

In the region $x>0$, $y>0$, using the Laplace transform to find a solution to the equation

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=x+y\,,$$

satisfying the conditions: $u(0,y)=u(x,0)=1$.

<u>Solution:</u>

We apply the Laplace transform with respect to the variable x to the given equation , assuming $u(x,y)\leftrightarrow U(p,y)$. Since

$$\frac{\partial u}{\partial x}\leftrightarrow pU(p,y)-u(0,y)=pU(p,y)-1\,,$$

$$\frac{\partial u}{\partial y}\leftrightarrow\frac{\partial U(p,y)}{\partial y}\,,$$

$$u(x,0)=1\leftrightarrow U(p,0)=\frac{1}{p}\,,$$

the specified transformation gives the operator equation:

$$pU(p,y)-1+\frac{\partial U(p,y)}{\partial y}=\frac{1}{p^2}+\frac{y}{p}\,,$$

to which the condition

$$U(p,0)=\frac{1}{p}$$

should be added.

Thus, a single Laplace transform with respect to the variable x gives the problem

$$\begin{cases} pU(p,y)+\dfrac{\partial U(p,y)}{\partial y}=\dfrac{1}{p^2}+\dfrac{y}{p}+1, \quad y>0, \\ U(p,0)=\dfrac{1}{p}. \end{cases} \qquad (*)$$

The resulting equation can be considered as an ordinary first - order differential equation with constant coefficients for the function U, with an independent variable y and a parameter p. Let's solve the Cauchy problem (*) in two ways.

First, by solving a differential equation, it is possible to construct its general solution:

$$U(p,y)=Ce^{-py}+\frac{y}{p^2}+\frac{1}{p},$$

and select a solution that satisfies the given initial condition:

$$U(p,y)=\frac{y}{p^2}+\frac{1}{p}.$$

It is easy to build a corresponding original for the found image:

$$u(x,y)=yx+1.$$

The second method involves solving the problem (*) using the Laplace transform with respect to the variable y.

Assuming $U(p,y)\leftrightarrow V(p,q)$, we construct the operator equation

$$qV(p,q)-\frac{1}{p}+pV=\frac{1}{p^2q}+\frac{1}{pq^2}+\frac{1}{q},$$

from where we find

$$V(p,q) = \frac{1}{p^2q^2} + \frac{1}{pq}.$$

By performing inverse transformations, we find a solution to the problem formulated in the example condition.

Both the first method, the one—time Laplace transform, and the second, the two-time Laplace transform, give the same result.

2.4. CLASSIFICATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

A linear partial differential equation of the second order is called the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f(x)\ ,\ x \in D\ , \qquad (2.3)$$

where the coefficients are real functions of the point x in the region D:

$$a_{ij} = a_{ij}(x)\ ,\ b_i = b_i(x)\ ,\ c = c(x)\ .$$

Equation (2.3) corresponds to the characteristic form:

$$Q(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j\ ,$$

which is quadratic.

At each fixed point $x \in D$, using a non-special affine transformation of variables:

$$\lambda_i = \lambda_i(\mu_1, \mu_2, \ldots, \mu_n)\ ,\ i = 1, 2, \ldots, n$$

the quadratic form Q can be reduced to the canonical form

$$\tilde{Q}(\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i=1}^{n} \alpha_i \mu_i^2\ , \qquad (2.4)$$

where $\alpha_i \in \{-1, 0, 1\}$.

The canonical form of the quadratic form determines the type of equation (2.3).

The linear equation (2.3) will be called *elliptical* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, all $\alpha_i \neq 0$ and one sign.

Equation (2.3) will be called *hyperbolic* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, all $\alpha_i \neq 0$, but not all of the same sign.

Equation (2.3) will be called *parabolic* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, at least one of the coefficients $\alpha_k = 0$.

Example 1

Determine the type of equation for $u = u(x,y)$:

$$u_{xx} - 4u_{xy} + 8u_{yy} + u_x - 6u + y = 0 .$$

<u>Solution:</u>

The given equation corresponds to the quadratic form

$$Q(\lambda_1,\lambda_2) = \lambda_1^2 - 4\lambda_1\lambda_2 + 8\lambda_2^2,$$

which we bring to the canonical form by sequentially highlighting the complete squares:

$$Q(\lambda_1,\lambda_2) = \lambda_1^2 - 4\lambda_1\lambda_2 + 8\lambda_2^2 = \lambda_1^2 - 4\lambda_1\lambda_2 + 4\lambda_2^2 + 4\lambda_2^2 =$$

$$= (\lambda_1 - 2\lambda_2)^2 + (2\lambda_2)^2 = \mu_1^2 + \mu_2^2 = \tilde{Q}(\mu_1,\mu_2) .$$

Since both coefficients in the canonical form of a quadratic form have the same sign, the given equation has an elliptical type in the entire domain of setting the variables x, y.

Example 2

Determine the type of equation for $u = u(x, y, z)$:

$$u_{xx} - 4u_{yy} + 2u_{xz} + 4u_{yz} + 2u_x - u_y = xyz^2.$$

<u>Solution:</u>

The given equation corresponds to the quadratic form

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 - 4\lambda_2^2 + 2\lambda_1\lambda_3 + 4\lambda_2\lambda_3.$$

Let's bring it to a canonical form, sequentially highlighting the full squares:

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + 2\lambda_1\lambda_3 + \lambda_3^2 - \lambda_3^2 - 4\lambda_2^2 + 4\lambda_2\lambda_3 =$$
$$= (\lambda_1 + \lambda_3)^2 - (2\lambda_2 - \lambda_3)^2 = \mu_1^2 - \mu_2^2 = \tilde{Q}(\mu_1, \mu_2, \mu_3).$$

Since one of the coefficients in the canonical form of the quadratic form is 0 (for μ_3^2), the given equation has a parabolic type in the entire domain of setting the variables x, y, z.

2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

Consider a second-order equation, linear with respect to the higher derivatives, for an unknown function $u(x,y)$ of two independent variables x and y:

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + F(x,y,u,u_x,u_y) = 0, \qquad (2.5)$$

where the real functions $a(x,y), b(x,y), c(x,y)$ are defined in the domain D.

2.5.1. REPLACING INDEPENDENT VARIABLES

Let's introduce independent variables

$$\xi = \xi(x,y), \ \eta = \eta(x,y), \qquad (2.6)$$

where ξ, η are twice continuously differentiable functions in the domain D.

We require that the Jacobian of the transformation be nonzero:

$$\frac{D(\xi,\eta)}{D(x,y)} \neq 0 .$$

Let's try to choose the transformation (2.6) in such a way that equation (2.5) has the simplest form in the new variables. We transform equation (2.5) to new variables, assuming

$$U(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta)).$$

Then we get

$$u_x = U_\xi \xi_x + U_\eta \eta_x ,$$

$$u_y = U_\xi \xi_y + U_\eta \eta_y ,$$

$$u_{xx} = U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx} ,$$

$$u_{yy} = U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy} ,$$

$$u_{xy} = U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + U_{\eta\eta}\eta_x\eta_y + U_\xi \xi_{xy} + U_\eta \eta_{xy} .$$

In the new variables, equation (2.5) will take the form

$$\bar{a}U_{\xi\xi} + 2\bar{b}U_{\xi\eta} + \bar{c}U_{\eta\eta} + \bar{F} = 0 , \qquad (2.7)$$

where

$$\bar{a} = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 , \qquad (2.8)$$

$$\bar{c} = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 ,$$

$$\bar{b} = a\xi_x\eta_x + 2b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y ,$$

$\bar{F} = \bar{F}\left(\xi, \eta, U, U_\xi, U_\eta\right) -$ a function that does not depend on the higher derivatives.

Definition 1. Equation (2.5) has at the point (x,y):

– *hyperbolic* type if $b^2 - ac > 0$ at point (x,y),

– *elliptical* type if $b^2 - ac < 0$ at point (x,y),

– *parabolic* type if $b^2 - ac = 0$ at point (x,y).

If the type of equation is preserved at all points of the domain D, then the equation is called an equation of this type in the entire domain D.

If an equation belongs to different types at different points in the domain, then it is called a *mixed type* equation in the domain D.

2.5.2.THE EQUATION OF CHARACTERISTICS

Now let's figure out how to introduce new variables x and h so that equation (2.5) takes the simplest form.

Assumption. Equation (2.5) belongs to a certain type in the entire domain D and $a(x,y)$ and $c(x,y)$ not equal to zero at the same time.

We assume that $a(x,y) \neq 0$.

It can be seen from the relation (2.8) that in order for $\bar{a} = 0$, it is necessary as a function of $\xi(x,y)$ to take the solution of the equation:

$$az_x^2 + 2bz_x z_y + cz_y^2 = 0. \quad (2.9)$$

Definition 2. Equation (2.9) is called the characteristic equation for equation (2.5).

Lemma. Let the function $z(x,y)$ be continuously differentiable in the domain D and such that $z_y \neq 0$. In order for the family of curves $z(x,y) = C$ to represent the characteristics of equation (2.5), it is necessary and sufficient that the expression $z(x,y)=C$ be the general integral of the ordinary differential equation

$$a(x,y)(dy)^2 - 2b(x,y)dxdy + c(x,y)(dx)^2 = 0\,. \qquad (2.10)$$

Definition 3.

Equation (2.10) is called the *equation of characteristics* for equation (2.5).

Assuming $\xi=\varphi(x,y)$, where $\varphi(x,y)=C$ is the integral of equation (2.10), we nullify the coefficient at $U_{\xi\xi}$ in equation (2.7).

If $\psi(x,y)=C$ is another integral of equation (2.10), independent of $\varphi(x,y)$, then assuming $\eta=\psi(x,y)$, we also zero the coefficient at $U_{\eta\eta}$.

Equation (2.10) splits into two equations:

$$\frac{dy}{dx}=\frac{b+\sqrt{b^2-ac}}{a}\,, \qquad (2.11)$$

$$\frac{dy}{dx}=\frac{b-\sqrt{b^2-ac}}{a}\,. \qquad (2.12)$$

Definition 4. Solutions of equations (2.11), (2.12) are called *characteristics* for equation (2.5).

2.5.3.CANONICAL FORMS OF EQUATIONS

Consider the region D, at all points of which equation (2.5) has the same type.

1. For an equation of hyperbolic type $b^2 - ac > 0$, the right-hand sides of equations (2.11) and (2.12) are valid and different.

Their general integrals, $\varphi(x,y) = C_1$ and $\psi(x,y) = C_2$, define families of characteristics that do not touch each other.

Choosing $\xi = \varphi(x,y), \eta = \psi(x, y)$, we get $\bar{a} = 0$, $\bar{c} = 0$.

Therefore, equation (2.7), after division by $\bar{b} \neq 0$, takes the form

$$U_{\xi\eta} = \bar{F}(\xi, \eta, U, U_\xi, U_\eta). \tag{2.13}$$

Definition 5. The form of equation (2.13) is called *the first canonical form* of the hyperbolic type equation.

Another canonical form is often used, which can be obtained by replacing:

$$\alpha = \frac{1}{2}(\xi - \eta),\ \beta = \frac{1}{2}(\xi + \eta).$$

In this case, the equation takes the form

$$U_{\alpha\alpha} - U_{\beta\beta} = \bar{F}_1(\xi, \eta, U, U_\xi, U_\eta).$$

2. Let the equation (2.5) be of elliptical type in the domain D, that is, $b^2 - ac < 0$.

Then the equations of characteristics (2.11) and (2.12) with real coefficients a,b,c have complex conjugate right-hand sides. All the characteristics will be complex.

Assuming that the coefficients a,b,c are defined in the complex domain, and making a formal substitution:

$$\xi = \xi(x,y)\,,\ \eta = \xi^{*}(x,y)\,,$$

where $\xi(x,y) = C_1$ and $\xi^{*}(x,y) = C_2$ - complex conjugate integrals (2.11) and (2.12), we obtain the equation

$$U_{\xi\eta} = \bar{F}_2(\xi,\eta,U,U_{\xi},U_{\eta}) \qquad (2.14)$$

in the complex domain.

If we make another replacement:

$$\alpha = \frac{1}{2}(\xi+\eta) = \operatorname{Re}\xi\,,\ \beta = -\frac{i}{2}(\xi-\eta) = \operatorname{Im}\xi\,,$$

then equation (2.14) will take the form

$$U_{\alpha\alpha} + U_{\beta\beta} = \bar{F}_3(\xi,\eta,U,U_{\xi},U_{\eta}) \qquad (2.15)$$

already in the real domain.

Definition 6. The form (2.15) of the transformed equation (2.5) is the canonical form of an elliptic equation.

3. Finally, let us consider a parabolic equation in the region D:

$$b^2 - ac = 0 .$$

In this case, there is only one equation of characteristics

$$\frac{dy}{dx} = \frac{b}{a} .$$

Let $\xi(x,y) = C$ be its integral. Let's take an arbitrary twice differentiable function $\eta(x,y)$ such that the condition

$$\frac{D(\xi,\eta)}{D(x,y)} \neq 0$$

is satisfied.

Then, when replacing $\xi = \xi(x,y), \eta = \eta(x,y)$, equation (2.7) takes the form

$$U_{\eta\eta} = \overline{F}_4(\xi, \eta, U, U_\xi, U_\eta) . \qquad (2.16)$$

Definition 7.

The form (2.16) of the transformed equation (2.5) is the *canonical form* of a parabolic equation.

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2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

We are considering $u(x, y)$ - unknown function of two independent variables.

$$a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}+F\left(x,y,u,u_x,u_y\right)=0 \quad \textbf{(*)}$$

where $a(x,y),\ b(x,y),\ c(x,y)$ – are defined in the domain D.

$b^2-ac>0$ - *hyperbolic type*

$b^2-ac=0$ - *parabolic type*

$b^2-ac<0$ - *elliptical type*

Depending on what type of equation, you can find such a coordinate transformation:

It is possible to move from coordinates (x,y) to coordinates (ξ,η):

$$(x,y)\to(\xi,\eta).$$

You can find such a transformation and make it.

$$\xi=\varphi(x,y)$$

$$\eta=\psi(x,y)$$

Then the equation (*) can be written much more simply (depending on the type of equation).

And depending on the equation, it will be clear how to solve it.

The coefficients $a(x,y),\ b(x,y),\ c(x,y)$ – may depend on $x,\ y$, so if they are constant, then the equation will always have the same type for all $x,\ y$.

If the coefficients are variable, if they depend on $x,\ y$, then it may happen that the equation in one part of the plane has a hyperbolic type, and in another part of the plane has an elliptical type.

And then in each separate part it is necessary to solve it separately.

It will be necessary to find its canonical form separately for each part.

To bring the equation to a canonical form, it is <u>necessary to determine its type.</u>

Usually, the problem asks you to determine the type of equation.

And then we write <u>the characteristic equation:</u>

$$a(x,y)dy^2 - 2b(x,y)dx\,dy + c(x,y)dx^2 = 0$$

This is an ordinary second-order differential equation.

It can be solved as a quadratic equation with respect to dy.

If we solve such a quadratic equation:

$$dy = \frac{b \pm \sqrt{b^2 - ac}}{a}dx$$

Convert, multiply by a :

$$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$$

And then we see that the solution will depend on what type of equation it is.

If the **equation is hyperbolic**, then there is a positive number under the square root, everything is fine, we have two solutions.

There will be two equations:

$$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0 \quad \Rightarrow \begin{cases} \varphi(x,y) = C_1 \\ \psi(x,y) = C_2 \end{cases}$$

The fact is that if we now use these two independent integrals (these are the functions φ and ψ that we found) in order to replace the variable:

$$\xi = \varphi(x,y)$$

$$\eta = \psi(x,y)$$

Then, when substituting these variables into the equation (*), the equation becomes much simpler.

In the case of a **parabolic type** equation, under the square root of 0, therefore, two equations will not work. Let's look at a specific example.

In the case of an **elliptic type,** there will be a negative number under the square root, respectively, there will be complex numbers. Let's show you a specific example.

You can always find the functions φ, ψ, and make a replacement that will bring the original equation to a much simpler form.

And after that, you can write down the *canonical form* of this equation.

To create a new function from ξ and η, we should theoretically replace it ($u(x,y)$) with a some function $U(\xi,\eta)$:

$$u(x,y) \to U(\xi,\eta) = U(\varphi(x,y),\psi(x,y)) = u(x,y)$$

In theory, we should replace one function $u(x,y)$ with some other function $U(\xi,\eta)$.

Example 1

To bring the following differential equation to a canonical form:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 0$$

Solution:

The type of equation:

$$a = 1, \ \ b = 1, \ c = -3$$

$$b^2 - ac = 1 + 3 = 4 \ \ > 0$$

This equation has a hyperbolic type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$dy^2 - 2dydx - 3dx^2 = 0$$

We solve this equation as a quadratic one.

$$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$$

$$dy - \left(1 \pm \sqrt{4}\right)dx = 0$$

$$\begin{cases} dy - 3dx = 0 \\ dy + dx = 0 \end{cases}$$

It is good that we have an equation with constant coefficients: a, b, c do not depend on x, y.

$$\begin{cases} y-3x=C_1 \\ y+x=C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi=y-3x \\ \eta=y+x \end{cases}$$

When moving to these variables, the equation (*) is greatly simplified.

$$u(x,y)\to U(\xi,\eta)$$

The derivative of the whole function:

$$u_x=U_\xi\xi_x+U_\eta\eta_x$$
$$u_y=U_\xi\xi_y+U_\eta\eta_y$$

Take the second derivatives of $u_{x,}\ u_y$.

$$u_x=U_\xi\cdot(\text{-}3)+U_\eta\cdot 1$$

$$u_y=U_\xi\cdot 1+U_\eta\cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx}=-3\left(U_{\xi\xi}\cdot\xi_x+U_{\xi\eta}\cdot\eta_x\right)+1\left(U_{\eta\xi}\cdot\xi_x+U_{\eta\eta}\cdot\eta_x\right)=$$

$$=-3\left(U_{\xi\xi}\cdot(-3)+U_{\xi\eta}\cdot 1\right)+1\left(U_{\eta\xi}\cdot(-3)+U_{\eta\eta}\cdot 1\right)=$$

$$=9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}$$

$$u_{xy}=-3\left(U_{\xi\xi}\cdot\xi_y+U_{\xi\eta}\cdot\eta_y\right)+1\left(U_{\eta\xi}\cdot\xi_y+U_{\eta\eta}\cdot\eta_y\right)=$$

$$= -3\left(U_{\xi\xi}\cdot 1 + U_{\xi\eta}\cdot 1\right) + 1\left(U_{\eta\xi}\cdot 1 + U_{\eta\eta}\cdot 1\right) =$$

$$= -3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = 1\left(U_{\xi\xi}\cdot\xi_y + U_{\xi\eta}\cdot\eta_y\right) + 1\left(U_{\eta\xi}\cdot\xi_y + U_{\eta\eta}\cdot\eta_y\right) =$$

$$= 1\cdot U_{\xi\xi} + U_{\xi\eta}\cdot 1 + U_{\eta\xi}\cdot 1 + U_{\eta\eta}\cdot 1 =$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Substituting everything into our equation:

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} + 2\left(-3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - 3\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) +$$

$$+2\left(U_{\xi}\cdot(-3) + U_{\eta}\right) + 6\left(U_{\xi} + U_{\eta}\right) = 0$$

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} - 6U_{\xi\xi} - 4U_{\xi\eta} + 2U_{\eta\eta} - 3U_{\xi\xi} - 6U_{\xi\eta} - 3U_{\eta\eta} -$$

$$-6U_{\xi} + 2U_{\eta} + 6U_{\xi} + 6U_{\eta} = 0$$

$$-16U_{\xi\eta} + 8U_{\eta} = 0$$

$$-U_{\xi\eta} = \frac{1}{2}U_{\eta}$$

This expression is already the canonical form of a hyperbolic equation. This is the answer to this task.

In principle, in general, the **canonical form of a hyperbolic equation** will be as follows:

$$U_{\xi\eta} = \Phi\left(\xi, \eta, U, U_{\xi}, U_{\eta}\right).$$

When we moved on to the new variables, there should be a single derivative of this order (mixed) in the left part, with a coefficient equal to one; in the right part - some kind of function Φ, which contains independent variables $\xi, \eta, U, U_{\xi}, U_{\eta}$.

Example 2

$$\frac{\partial^2 u}{\partial x^2} - 6\frac{\partial^2 u}{\partial x \partial y} + 13\frac{\partial^2 u}{\partial y^2} = 0$$

Solution:

The type of equation:

$$a = 1,\ \ b = \text{-}3,\ \ c = 13$$

$$b^2 - ac = 9 - 13 = -4 \ \ < 0$$

This equation has an elliptical type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$1 \cdot dy^2 + 6dxdy + 13dx^2 = 0$$

$$dy - \left(-3 \pm \sqrt{-4}\right)dx = 0$$

$$dy - (-3 \pm 2i)\,dx = 0$$

Here is a complex number.

There are two equations here, but they merge into each other during the operation of complex conjugation.

Recall that such a complex conjugation is:

A complex number $z = a + ib$

A complex conjugate number $\overline{z} = a - ib$

Here, one equation can be obtained from another by complex conjugation. Therefore, it does not make much sense to consider the second equation. We have a real and imaginary part of this equation.

Let's make one equation:

$$dy - (-3 + 2i)\,dx = 0$$

We will integrate:

$$y - (-3 + 2i)\,x = A = C_1 + iC_2$$

the real part of the equation $y + 3x = C_1$

the imaginary part of the equation $-2x = C_2$

$$\begin{cases} y+3x=C_1 \\ -2x=C_2 \end{cases}$$

These are the functions that we can use in this case, as a substitute for variables.

$$\begin{cases} \xi = y+3x \\ \eta = -2x \end{cases}$$

Replace $u(x,y) \to U(\xi,\eta)$.

$$u_x = U_\xi \cdot \xi_x + U_\eta \cdot \eta_x = U_\xi \cdot 3 + U_\eta \cdot (-2)$$

$$u_y = U_\xi \cdot \xi_y + U_\eta \cdot \eta_y = U_\xi \cdot 1 + U_\eta \cdot 0$$

$$u_x = 3U_\xi - 2U_\eta$$

$$u_y = U_\xi$$

$$u_{xx} = 3\left(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x\right) - 2\left(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x\right) =$$

$$= 3\left(U_{\xi\xi} \cdot 3 + U_{\xi\eta} \cdot (-2)\right) - 2\left(U_{\eta\xi} \cdot 3 + U_{\eta\eta} \cdot (-2)\right) =$$

$$= 9U_{\xi\xi} - 12U_{\xi\eta} + 4U_{\eta\eta}$$

$$u_{xy} = 3\left(U_{\xi\xi}\xi_y + U_{\xi\eta}\eta_y\right) - 2\left(U_{\eta\xi}\xi_y + U_{\eta\eta}\eta_y\right) =$$

$$= 3\left(U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0\right) - 2\left(U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 0\right) =$$

$$= 3U_{\xi\xi} - 2U_{\eta\xi}$$

$$u_{yy} = U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0 = U_{\xi\xi}$$

We substitute all derivatives:

$$9U_{\xi\xi} - 12U_{\xi\eta} + 4U_{\eta\eta} - 18U_{\xi\xi} + 12U_{\xi\eta} + 13U_{\xi\xi} = 0$$

$$4U_{\xi\xi} + 4U_{\eta\eta} = 0$$

$$U_{\xi\xi} + U_{\eta\eta} = 0$$

This is the canonical form of the elliptical type.

General view (canonical view) elliptic equations:

$$U_{\xi\xi} + U_{\eta\eta} = \Phi\left(\xi, \eta, U, U_{\xi}, U_{\eta}\right)$$

with a coefficient equal to one (on the left side). If you get something else, you should look for an error.

Example 3

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + 1\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial u}{\partial x} + \beta\frac{\partial u}{\partial y} + cu = 0$$

<u>Solution:</u>

The type of equation:

$$a = 1,\ \ b = \text{-}1,\ \ c = 1$$

$$b^2 - ac = (-1)^2 - 1 = 0$$

<u>This equation has a parabolic type.</u>

Next, we need to bring it to a canonical form.

We write the <u>characteristic equation:</u>

$$dy^2 + 2dxdy + dx^2 = 0$$

$$(dy + dx)^2 = 0$$

$$dy + dx = 0$$

$$y + x = C_1$$

That is, $\xi = y + x$.

Where do we get η from?

In the case of parabolic type equations, the function η can generally be taken any (linearly independent, which is written for ξ):

$$\eta = x.$$

Usually take x, the constant will not be linearly independent.

That is,

$$\begin{cases} \xi = y + x \\ \eta = x \end{cases}$$

If you doubt that these functions are linearly independent (or if ξ is a composite function), then we remember about replacing variables, and turn to topics about integrals if we do some kind of transformation (replacing variables from $(x,y) \to (\xi,\eta)$).

This transformation will be non-degenerate, so the functions will be linearly independent.

The Jacobian of the transition (determinant) will not be equal to 0.

We can make such a determinant, calculate, the transformation is non-degenerate, therefore we can choose the function as $\eta = x$.

$$(x,y) \to (\xi,\eta)$$

The transformation is non-degenerate and the Jacobian of the transition is non-zero.

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

The transformation is non-degenerate.

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi \cdot 1 + U_\eta \cdot 1;$$
$$u_y = U_\xi;$$
$$u_{xx} = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 1 + U_{\eta\xi} \cdot 1 + U_{\eta\eta} =$$
$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$
$$u_{xy} = U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 0 + U_{\eta\xi} \cdot 1 =$$
$$= U_{\xi\xi} + U_{\eta\xi}$$
$$u_{yy} = U_{\xi\xi}$$

All derivatives have been found.

Let's substitute it into the original equation:

$$U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta} - 2U_{\xi\xi} - 2U_{\eta\xi} + U_{\xi\xi} + \alpha U_\xi + \alpha U_\eta + \beta U_\xi + cU = 0$$

$$U_{\eta\eta} = -(\alpha + \beta)U_\xi - \alpha U_\eta - cU$$

A parabolic type equation has been reduced to a canonical form.

In general, the canonical form of a parabolic equation is:

$$U_{\eta\eta} = \Phi\left(\xi, \eta, U, U_\xi, U_\eta\right).$$

Example 4 (EQUATION WITH VARIABLE COEFFICIENTS)

$$y^2\frac{\partial^2 u}{\partial x^2}-x^2\frac{\partial^2 u}{\partial y^2}-2x\frac{\partial u}{\partial x}=0$$

Solution:

The type of equation:

$$a=y^2,\;\; b=0,\; c=-x^2$$

$$b^2-ac=x^2y^2 \;\; >0, \qquad x\neq 0,\, y\neq 0$$

The hyperbolic type is everywhere on the entire plane (except the axes).

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$y^2dy^2-x^2dx^2=0$$

$$(ydy)^2=(xdx)^2$$

$$ydy=\pm xdx$$

$$2ydy=\pm 2xdx$$

We will integrate:

$$\begin{cases} y^2=x^2+C_1 \\ y^2=-x^2+C_2 \end{cases}$$

$$\begin{cases} y^2-x^2=C_1 \\ y^2+x^2=C_2 \end{cases}$$

Substitution:

$$\begin{cases} \xi = y^2 - x^2 \\ \eta = y^2 + x^2 \end{cases}$$

We substitute all this, and we get the canonical form:

$$\xi_x = -2x$$
$$\xi_y = 2y$$
$$\eta_x = 2x$$
$$\eta_y = 2y$$

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi(-2x) + U_\eta(2x) = 2x(-U_\xi + U_\eta)$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y = U_\xi(2y) + U_\eta(2y) = 2y(U_\xi + U_\eta)$$
$$u_{xx} = \left[2x \cdot (-U_\xi + U_\eta)\right]'_x = 2(-U_\xi + U_\eta) + 2x\left[-U_\xi + U_\eta\right]'_x =$$
$$= 2(-U_\xi + U_\eta) + 2x(-U_{\xi\xi}\xi_x - U_{\xi\eta}\eta_x + U_{\eta\xi}\xi_x + U_{\eta\eta}\eta_x) =$$
$$= 2(-U_\xi + U_\eta) + 2x(-U_{\xi\xi}(-2x) - U_{\xi\eta}(2x) + U_{\eta\xi}(-2x) + U_{\eta\eta}(2x)) =$$
$$= 2(-U_\xi + U_\eta) + 4x^2(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta})$$

We used the formula: $(u\upsilon)' = u'\upsilon + \upsilon' u$

$\left[-U_\xi + U_\eta\right]'_x$ - and here we take it as a derivative of a complicated function

u_{xy} we don't need it, it's not in the equation.

$$u_{yy}=2\left(U_\xi+U_\eta\right)+2y\left(U_{\xi\xi}\xi_y+U_{\xi\eta}\eta_y+U_{\eta\xi}\xi_y+U_{\eta\eta}\eta_y\right)=$$
$$=2\left(U_\xi+U_\eta\right)+2y\left(U_{\xi\xi}(2y)+U_{\xi\eta}(2y)+U_{\eta\xi}(2y)+U_{\eta\eta}(2y)\right)=$$
$$=2\left(U_\xi+U_\eta\right)+4y^2\left(U_{\xi\xi}+2U_{\xi\eta}+U_{\eta\eta}\right)$$

We substitute it into the equation:

$$2y^2\left(-U_\xi+U_\eta\right)+4x^2y^2\left(U_{\xi\xi}-2U_{\xi\eta}+U_{\eta\eta}\right)-$$
$$-2x^2\left(U_\xi+U_\eta\right)-4x^2y^2\left(U_{\xi\xi}+2U_{\xi\eta}+U_{\eta\eta}\right)-$$
$$-4x^2\left(-U_\xi+U_\eta\right)=0$$

$$-16x^2y^2U_{\xi\eta}+\left(-2y^2+2x^2\right)U_\xi+\left(2y^2-6x^2\right)U_\eta=0$$

Let's express x^2 and y^2:

$$\begin{cases}\xi=y^2-x^2 & (1)\\ \eta=y^2+x^2 & (2)\end{cases}$$

let's add two equations:

$$y^2=\frac{\xi+\eta}{2}$$

Subtract from (2) equation (1) equation:

$$x^2=\frac{\eta-\xi}{2}$$

$$-16\frac{\eta^2-\xi^2}{4}U_{\xi\eta}+\left(-\xi-\eta+\eta-\xi\right)U_{\xi}+\left(\xi+\eta-3\eta+3\xi\right)U_{\eta}=0$$

$$-4\left(\eta^2-\xi^2\right)U_{\xi\eta}-2\xi U_{\xi}+\left(4\xi-2\eta\right)U_{\eta}=0$$

$$U_{\xi\eta}=\frac{1}{4\left(\eta^2-\xi^2\right)}\left(-2\xi U_{\xi}+\left(4\xi-2\eta\right)U_{\eta}\right)$$

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2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

Algorithm

1) Find $b^2 - ac$, determine the type of equation.
2) We find the first integrals of the characteristic equations:

in the case when $a \neq 0$: $$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a},$$

in the case when $c \neq 0$: $$\frac{dx}{dy} = \frac{b \pm \sqrt{b^2 - ac}}{c}$$

3) The first integrals have the form:

in the case of hyperbolic type: $\varphi(x,y) = C, \ \psi(x,y) = C;$

in the case of elliptic type: $\alpha(x,y) \pm i\beta(x,y) = C$,

in the case of parabolic type: $\delta(x,y) = C$.

4) We replace variables:

in the case of hyperbolic type: $$\begin{cases} \xi = \varphi(x,y); \\ \eta = \psi(x,y). \end{cases}$$

in the case of elliptical type: $$\begin{cases} \xi = \alpha(x,y); \\ \eta = \beta(x,y). \end{cases}$$

in the case of parabolic type: $$\begin{cases} \xi = \delta(x,y); \\ \eta = \varepsilon(x,y). \end{cases}$$

where $\varepsilon(x,y)$ is any function of C^1 such that $\begin{vmatrix} \delta_x & \delta_y \\ \varepsilon_x & \varepsilon_y \end{vmatrix} \neq 0$.

The result of the replacement will be the canonical form of the equation.

Example 1

Bring it to a canonical form

$$u_{xx} + 2u_{xy} - 3u_{yy} + u_x + u_y = 0$$

Solution:

The type of equation:

$a = 1,\ \ b = 1,\ \ c = \text{-}3$

$b^2 - ac = 1 + 3 = 4 \ \ > 0$

This equation has a hyperbolic type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$dy^2 - 2dydx - 3dx^2 = 0$

We solve this equation as a quadratic one.

$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$

$dy - \left(1 \pm \sqrt{4}\right)dx = 0$

$$\begin{cases} dy-3dx=0 \\ dy+dx=0 \end{cases}$$

It is good that we have an equation with constant coefficients: a, b, c do not depend on x, y.

$$\begin{cases} y-3x=C_1 \\ y+x=C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi = y-3x \\ \eta = y+x \end{cases}$$

When moving to these variables, the equation (*) is greatly simplified.

$$u(x,y) \to U(\xi,\eta)$$

The derivative of the whole function:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

Take the second derivatives of $u_{x,}$ u_y.

$$u_x = U_\xi \cdot (\text{-}3) + U_\eta \cdot 1$$

$$u_y = U_\xi \cdot 1 + U_\eta \cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx} = -3\left(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x\right) + 1\left(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x\right) =$$

$$= -3\left(U_{\xi\xi}\cdot(-3)+U_{\xi\eta}\cdot 1\right)+1\left(U_{\eta\xi}\cdot(-3)+U_{\eta\eta}\cdot 1\right)=$$

$$= 9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}$$

$$u_{xy} = -3\left(U_{\xi\xi}\cdot\xi_y+U_{\xi\eta}\cdot\eta_y\right)+1\left(U_{\eta\xi}\cdot\xi_y+U_{\eta\eta}\cdot\eta_y\right)=$$

$$= -3\left(U_{\xi\xi}\cdot 1+U_{\xi\eta}\cdot 1\right)+1\left(U_{\eta\xi}\cdot 1+U_{\eta\eta}\cdot 1\right)=$$

$$= -3U_{\xi\xi}-2U_{\xi\eta}+U_{\eta\eta}$$

$$u_{yy} = 1\left(U_{\xi\xi}\cdot\xi_y+U_{\xi\eta}\cdot\eta_y\right)+1\left(U_{\eta\xi}\cdot\xi_y+U_{\eta\eta}\cdot\eta_y\right)=$$

$$= 1\cdot U_{\xi\xi}+U_{\xi\eta}\cdot 1+U_{\eta\xi}\cdot 1+U_{\eta\eta}\cdot 1=$$

$$= U_{\xi\xi}+2U_{\xi\eta}+U_{\eta\eta}$$

Substituting everything into our equation:

$$u_{xx}+2u_{xy}-3u_{yy}+u_x+u_y=0$$

$$\begin{gathered}9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}+2\left(-3U_{\xi\xi}-2U_{\xi\eta}+U_{\eta\eta}\right)-\\-3\left(U_{\xi\xi}+2U_{\xi\eta}+U_{\eta\eta}\right)+U_{\xi}\cdot(-3)+U_{\eta}\cdot 1+U_{\xi}\cdot 1+U_{\eta}\cdot 1=0\end{gathered}$$

$$\begin{gathered}9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}-6U_{\xi\xi}-4U_{\xi\eta}+2U_{\eta\eta}-\\-3U_{\xi\xi}-6U_{\xi\eta}-3U_{\eta\eta}-3U_{\xi}+U_{\eta}+U_{\xi}+U_{\eta}=0\end{gathered}$$

$$-16U_{\xi\eta}+2U_{\eta}-2U_{\xi}=0$$

$$U_{\xi\eta} = \frac{2U_\xi - 2U_\eta}{-16}$$

Example 2 EQUATION WITH VARIABLE COEFFICIENTS

To bring to a canonical form in each area where the type is preserved, the equation

$$yu_{xx} + u_{yy} = 0$$

Solution:

Step 1. The type of equation:

$$a = y,\ \ b = 0,\ c = 1$$

$$\Delta = b^2 - ac = 0 - y \cdot 1 = -y$$

Therefore,

1) in the half-plane $y < 0$, $\Delta > 0$ $\Rightarrow$ means hyperbolic type,

2) in the half-plane $y > 0$, $\Delta < 0$ $\Rightarrow$ means elliptical type,

3) on the straight $y = 0$ discriminant $\Delta = 0$ $\Rightarrow$ means parabolic type.

Step 2.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$ydy^2 + dx^2 = 0$$

We solve this equation as a quadratic one.

Since $c = 1 \neq 0$, the characteristic equations have the form:

$$\frac{dx}{dy} = \frac{b \pm \sqrt{\Delta}}{c}, \quad \text{that is,} \quad \frac{dx}{dy} = \pm\sqrt{-y}$$

This is an equation with separable variables. We solve them:

1. in the half-plane $y < 0$

$$dx = \pm\sqrt{-y}dy \qquad \Rightarrow \qquad x + c = \mp\frac{2}{3}(-y)^{\frac{3}{2}}.$$

Therefore, the first integrals have the form:

$$\boxed{\varphi(x, y) = x + \tfrac{2}{3}(-y)^{\frac{3}{2}}} = c, \qquad \boxed{\psi(x, y) = x - \tfrac{2}{3}(-y)^{\frac{3}{2}}} = c$$

2. in the half-plane $y > 0$

$$dx = \pm i\sqrt{y}dy \qquad \Rightarrow \qquad x + c = \pm i\frac{2}{3}y^{\frac{3}{2}}.$$

Therefore, the first integrals have the form:

$$\alpha(x, y) \pm i\beta(x, y) = c,$$

where

$$\boxed{\alpha(x,\, y) = x}, \qquad \boxed{\beta(x,\, y) = \tfrac{2}{3}\, y^{\frac{3}{2}}}$$

3. on the straight $y = 0$

$$dx = 0 \cdot dy \qquad \Rightarrow \qquad x = c$$

Therefore, the first integral (the only linearly independent one) has the form:

$$\boxed{\delta(x,\, y) = x.}$$

Step 3.

Replacing variables.

According to the algorithm, it is necessary to carry out a replacement.

1. in the half-plane $y < 0$

$$\begin{cases} \xi = x + \frac{2}{3}(-y)^{\frac{3}{2}}; \\ \eta = x - \frac{2}{3}(-y)^{\frac{3}{2}}. \end{cases}$$

$$\xi_x = 1,$$
$$\xi_y = -\sqrt{-y},$$
$$\eta_x = 1,$$
$$\eta_y = \sqrt{-y}$$

Then by entering the function $U(\xi, \eta)$, we get:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

Take the second derivatives of $u_{x,}\ u_y$.

$$u_x = U_\xi \cdot 1 + U_\eta \cdot 1 = U_\xi + U_\eta$$

$$u_y = U_\xi \cdot \left(-\sqrt{-y}\right) + U_\eta \cdot \left(\sqrt{-y}\right) = \sqrt{-y}\left(-U_\xi + U_\eta\right)$$

From these derivatives we take the second derivatives:

$$u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = -y\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - \frac{1}{2\sqrt{-y}}\left(-U_\xi + U_\eta\right)$$

Substituting the found derivatives into the original equation, we get:

$$yu_{xx} + u_{yy} = 0$$

$$y\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) - y\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - \frac{1}{2\sqrt{-y}}\left(-U_\xi + U_\eta\right) =$$

$$= y\left[4U_{\xi\eta} - \frac{1}{2(-y)^{\frac{3}{2}}}\left(-U_\xi + U_\eta\right)\right] = 0$$

Dividing by $4y$ and expressing $2(-y)^{\frac{3}{2}} = \frac{3}{2}(\xi - \eta)$, we get the canonical form:

$$U_{\xi\eta} - \frac{1}{6(\xi - \eta)}\left(-U_\xi + U_\eta\right) = 0$$

2. in the half-plane $y > 0$

$$\begin{cases} \xi = x; \\ \eta = \frac{2}{3} y^{\frac{3}{2}}. \end{cases}$$

$$\xi_x = 1,$$
$$\xi_y = 0,$$
$$\eta_x = 0,$$
$$\eta_y = \sqrt{y}$$

Then by entering the function $U(\xi, \eta)$, we get:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

$$u_x = U_\xi + U_\eta \cdot 0 = U_\xi$$
$$u_y = U_\xi \cdot 0 + U_\eta \sqrt{y} = U_\eta \sqrt{y}$$

$$u_{xx} = U_{\xi\xi}$$
$$u_{yy} = U_{\eta\eta} y + \frac{1}{2\sqrt{y}} U_\eta$$

Substituting the found derivatives into the original equation, we get:

$$y u_{xx} + u_{yy} = 0$$

$$yU_{\xi\xi}+U_{\eta\eta}y+\frac{1}{2\sqrt{y}}U_\eta=y\left(U_{\xi\xi}+U_{\eta\eta}\right)+\frac{1}{2\sqrt{y}}U_\eta=$$

$$=y\left(U_{\xi\xi}+U_{\eta\eta}+\frac{1}{2y^{\frac{3}{2}}}U_\eta\right)=\left[2y^{\frac{3}{2}}=3\eta\right]=$$

$$=y\left(U_{\xi\xi}+U_{\eta\eta}+\frac{1}{3\eta}U_\eta\right)=0$$

Dividing by y, we get the canonical form:

$$U_{\xi\xi}+U_{\eta\eta}+\frac{1}{3\eta}U_\eta=0$$

3. on the straight $y=0$

$$\begin{cases}\xi=x;\\ \eta=y.\end{cases}$$

We need to arbitrarily choose $\eta(x,y)$ so that the functions ξ,η form a linearly independent pair.

$\xi_x=1,$
$\xi_y=0,$
$\eta_x=0,$
$\eta_y=1$

Then by entering the function $U(\xi,\eta)$, we get:

$$u_x=U_\xi\xi_x+U_\eta\eta_x$$
$$u_y=U_\xi\xi_y+U_\eta\eta_y$$

$$u_x = U_\xi$$
$$u_y = U_\eta$$

$$u_{xx} = U_{\xi\xi}$$
$$u_{yy} = U_{\eta\eta}$$

Substituting the found derivatives into the original equation for $y = 0$, we get:

$$u_{yy} = U_{\eta\eta} = 0$$

So, the canonical form of the original equation on the line $y = 0$:

$U_{\eta\eta} = 0$ or, what is the same, $u_{yy} = 0$.

<u>Answer:</u>

$$\begin{cases} U_{\xi\eta} - \dfrac{1}{6(\xi - \eta)}\left(-U_\xi + U_\eta\right) = 0 \quad \textit{in the area} \quad y < 0, \textit{ hyperbolic type} \\ U_{\xi\xi} + U_{\eta\eta} + \dfrac{1}{3\eta} U_\eta = 0 \quad \textit{in the area } y > 0, \textit{ elliptical type} \\ U_{\eta\eta} = 0 \quad \textit{in the area } y = 0, \textit{ parabolic type.} \end{cases}$$

REDUCTION TO THE CANONICAL FORM OF PARTIAL DIFFERENTIAL EQUATIONS OF THE 2ND ORDER WITH CONSTANT COEFFICIENTS

In this section, we will consider second-order partial differential equations with constant coefficients and n independent variables:

$$\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + f(x_1,\ \ldots,\ x_n;\ u,\ u_{x_1},\ \ldots,\ u_{x_n}) = 0, \tag{1}$$

$a_{ij} = const \in \mathbb{R},\ i, j = \overline{1,n}$.

Definition 1

The characteristic quadratic form of equation (1) is the expression:

$$Q(\lambda_1,\ \ldots,\ \lambda_n) = \sum_{i,j=1}^{n} a_{ij}\, \lambda_i \lambda_j. \tag{2}$$

The normal form of the quadratic form (2) is its form:

$$\tilde{Q}(\mu_1,\ \ldots,\ \mu_n) = \sum_{k=1}^{n} \beta_k\, \mu_k^2, \qquad \beta_k \in \{-1,\ 0,\ 1\}. \tag{3}$$

The canonical form of equation (1) is the form in which its characteristic quadratic form takes the normal (or canonical) form:

$$\sum_{k=1}^{n} \beta_k \, u_{x_k x_k} + g(x_1, \ldots, x_n; \; u, u_{x_1}, \ldots, u_{x_n}) = 0. \qquad \textbf{(4)}$$

Definition 2

Equation (1) refers to

1) **hyperbolic type**, if the coefficients β_k are different from zero and not all of the same sign;

2) **elliptical type**, if all coefficients β_k are nonzero and all of the same sign;

3) **parabolic type,** if at least one of the coefficients β_k is zero.

Algorithm

1) We reduce the characteristic quadratic form to the canonical (normal) form (3) (by the method of selecting complete squares). We write out the transformation matrix that performs this process:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \mu_n \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \vdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \vdots & \alpha_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \vdots & \alpha_{nn} \end{pmatrix}}_{A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \lambda_n \end{pmatrix}, \qquad \det A \neq 0.$$

2) We find the matrix Γ of the substitution of variables according to the law:

$$\Gamma = \left(A^T\right)^{-1}.$$

3) We replace variables:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma_{11} & \gamma_{12} & \vdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \vdots & \gamma_{2n} \\ \dots & \dots & \ddots & \dots \\ \gamma_{n1} & \gamma_{n2} & \vdots & \gamma_{nn} \end{pmatrix}}_{\Gamma} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

The result of the replacement will be the canonical form (4) of equation (1).

Example 3

To bring the equation to a canonical form:

$$u_{xx} + 2u_{xy} + 5u_{yy} - 32u = 0.$$

<u>Solution:</u>

Step 1

The characteristic quadratic form of this equation has the form

$$Q(\lambda_1,\ \lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + 5\lambda_2^2.$$

Let's bring it to the canonical form:

$$Q(\lambda_1,\ \lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + 5\lambda_2^2 = (\lambda_1 + \lambda_2)^2 + (2\lambda_2)^2 = \mu_1^2 + \mu_2^2,$$

where

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_2; \\ \mu_2 = 2\lambda_2 \end{cases}$$

that is

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}}_{A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Step 2

Let's find the matrix of substitution of variables Γ:

$$\Gamma = \left(A^T\right)^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

Step 3

We replace the variables:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

that is,

$$\begin{cases} \xi = x; \\ \eta = \frac{1}{2}(-x + y). \end{cases}$$

To put new variables in the original equation, let's put

$$U(\xi,\eta) = u(x,y)$$

and find $u_x, u_y, u_{xx}, u_{xy}, u_{yy}$ as derivatives of a complicated function $U(\xi(x,y),\eta(x,y))$:

$$u_x = U_\xi - \frac{1}{2}U_\eta$$

$$u_y = \frac{1}{2}U_\eta$$

$$u_{xx} = U_{\xi\xi} - U_{\xi\eta} + \frac{1}{4}U_{\eta\eta}$$

$$u_{xy} = \frac{1}{2}U_{\xi\eta} - \frac{1}{4}U_{\eta\eta}$$

$$u_{yy} = \frac{1}{4}U_{\eta\eta}$$

Substituting the found derivatives into the left side of the original equation and giving similar ones, we get:

$$u_{xx} + 2u_{xy} + 5u_{yy} - 32u = \left(U_{\xi\xi} - U_{\xi\eta} + \frac{1}{4}U_{\eta\eta}\right) + 2\left(\frac{1}{2}U_{\xi\eta} - \frac{1}{4}U_{\eta\eta}\right) +$$
$$+5\left(\frac{1}{4}U_{\eta\eta}\right) - 32U = U_{\xi\xi} + U_{\eta\eta} - 32U$$

<u>Answer:</u>

the equation has an elliptical type,

$$U_{\xi\xi} + U_{\eta\eta} - 32U = 0, \text{ where } \xi = x, \ \eta = \frac{1}{2}(-x + y).$$

HOMEWORK 7 (The deadline is October 14, 2024)

Version 1	**Version 2**
$u_{xx} - yu_{yy} = 0\,.$	$xu_{xx} - 2\sqrt{xy}u_{xy} + yu_{yy} + \frac{1}{2}u_y = 0\,.$
Wang Jiahe	Yan Shukun
Guan Haochen	Liu Yudong
Liu Tianxing	Wang Youshen
Ma Yueyang	Lu Qibo
Wang Changzhi	Chen Langbo
Yang Zihao	An Junhao
Zhong Yuhao	Yu Hang
Wu Haonan	Ni Zhongshuo
Yan Sensen	Li Kangjian
Wang Yudong	Lin Enbei
Li Sicheng	Xia Xinglin
Li Kaiyan	Huang Yifan
Yang Guowei	Shen Xingye
Kong Xiangning	Wang Haojun
Liu Jiashan	Li Jiashen
Zhao Yixiao	Chen Shiwen
Li Xinyi	Wang Leihan
Zhao Xiaohui	Yang Yuhao
Qu Linfeng	Liu Xingyu
Zhou Zixin	Qian Keqing
Yu Rongyi	Wu Jiaxin
Mei Mingzhe	Lu Mingyu
Zhang Hongbo	
Yan Shukun	

14.10.2024

2.6. BASIC EQUATIONS OF MATHEMATICAL PHYSICS

14.10.2024

2.6. BASIC EQUATIONS OF MATHEMATICAL PHYSICS

The subject of the theory of equations of mathematical physics is the study of differential, integral and functional equations describing natural phenomena. The construction of a mathematical model of the process begins with the establishment of values that are decisive for the process under study. Further, using physical laws (principles) expressing the relationship between these quantities, an equation (system of equations) in partial derivatives is constructed and additional conditions (initial and boundary) to the equation (system) are drawn up.

We will mainly study second-order partial differential equations with one unknown function, in particular the wave equation, the heat equation and the Laplace equation, commonly called the classical equations of mathematical physics.

2.6.1. THE OSCILLATION EQUATION

Many problems of mechanics (vibrations of strings, rods, membranes and three-dimensional volumes) and physics (electromagnetic oscillation) lead to an oscillation equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \operatorname{div}(k\operatorname{grad}u) - qu + F(x,t),$$

where the unknown function $u = u(x,t)$ depends on n $(n = 1,2,3)$ spatial variables $x = (x_1, x_2, \ldots, x_n)$ and time t, coefficients ρ, k, q are determined by the properties of the medium, $F(x,t)$ is the density of the external disturbance:

$$\operatorname{div}(k\operatorname{grad}u)=\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(k\frac{\partial u}{\partial x_i}\right).$$

Consider a stretched string fixed at the ends. By string we mean a thin thread that does not exert any resistance to changing its shape, unrelated to changing its length. The tension force T_0 acting on the string is assumed to be significant, so the effect of gravity can be ignored.

Let the string be directed along the x axis in the equilibrium position.

We will consider only the *transverse vibrations* of the string, assuming that the movement occurs in the same plane and that all points of the string move perpendicular to the x axis.

Let's denote by $u(x,t)$ the displacement of the string points at time t from the equilibrium position.

Considering further only *small vibrations* of the string, we will assume that the displacement $u(x,t)$, as well as the derivative $\frac{\partial u}{\partial x}$, are so small that their squares and products can be neglected compared to the quantities themselves.

For each fixed value of t, the graph of the function $u(x,t)$ obviously gives the shape of the string at this point in time (Fig. 1).

Denote by $F(x,t)$ the density of external forces acting on the string at point x at time t and directed perpendicular to the x axis.

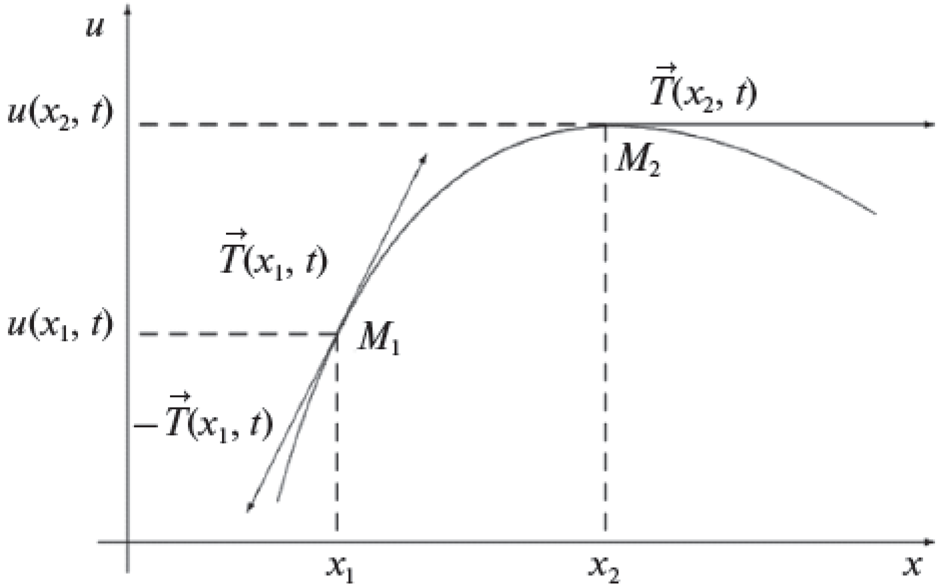

Fig. 1. Instantaneous profile of the string section (x_1, x_2) at time t

Let $\rho(x)$ be the linear density of the string, then the function $u(x,t)$ satisfies the differential *equation of string vibrations*:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + F(x,t).$$

If $\rho(x) = \rho = \mathrm{const}$, that is, in the case of a homogeneous string, the equation is usually written as

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),\ a = \sqrt{\frac{T_0}{\rho}},\ f(x,t) = \frac{F(x,t)}{\rho}.$$

This equation will be called the *one-dimensional wave equation.*

If there is no external force, then we have: $F(x,t) = 0$ and get the *equation of free vibrations of the string*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

The equation of small transverse vibrations of the membrane A =0 is similar:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + F(x,t).$$

If the density ρ is constant, then the membrane oscillation equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + f(x,t),\ a = \sqrt{\frac{T_0}{\rho}},\ f(x,t) = \frac{F(x,t)}{\rho}.$$

The last equation will be called the *two-dimensional wave equation*.

Three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) + f(x,t)$$

describes the processes of sound propagation in a homogeneous medium and electromagnetic waves in a homogeneous nonconducting medium. This equation is satisfied by the density of the gas, its pressure and velocity potential, as well as the components of the electric and magnetic field strengths and the corresponding potentials.

We will write the wave equations using the single formula

$$\frac{\partial^2 u}{\partial t^2} = a^2\Delta u + f,$$

where Δ is the Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

2.6.2. THE EQUATION OF THERMAL CONDUCTIVITY (HEAT EQUATION)

The processes of heat propagation or particle diffusion in the medium are described by the heat equation

$$\rho\frac{\partial u}{\partial t} = \mathrm{div}(k\,\mathrm{grad}u) - qu + F(x,t).$$

Let's derive the equation of heat propagation.

Denote by $u(x,t)$ the temperature of the medium at point $x=(x_1,x_2,x_3)$ at time t, and by $\rho(x)$, $c(x)$ and $k(x)$, respectively, its density, specific density and thermal conductivity coefficient at point x.

Let $F(x,t)$ be the intensity of heat sources at point x at time t.

Let's calculate the heat balance in an arbitrary volume V over a period of time $(t,t+\Delta t)$. Denote by S the boundary of V and let $\vec{n}-$ be the external normal to it.

According to Fourier's law, through the surface S, an amount of heat

$$Q_1 = \iint_S k(x)\frac{\partial u}{\partial n}dS\Delta t = \Delta t\iint_S (k(x)\mathrm{grad}u,\vec{n})dS,$$

enters the volume V, equal, by virtue of the Gauss's-Ostrogradsky's formula (theorem):

$$Q_1 = \iiint_V \mathrm{div}(k(x)\mathrm{grad}u)dx\Delta t.$$

Due to the thermal sources in volume V the amount of heat

$$Q_2 = \iiint_V F(x,t)dx\Delta t.$$

Since the temperature in volume V has increased by

$$u(x,t+\Delta t)-u(x,t)\approx\frac{\partial u}{\partial t}\Delta t,$$

over a period of time $(t,t+\Delta t)$, it is necessary to expend the amount of heat

$$Q_3=\iiint_V c(x)\rho(x)\frac{\partial u}{\partial t}dx\Delta t.$$

On the other hand, $Q_3=Q_1+Q_2$ and therefore

$$\iiint_V\left[\operatorname{div}(k(x)\operatorname{grad}u)+F-c(x)\rho(x)\frac{\partial u}{\partial t}\right]dx\Delta t=0,$$

from where, due to the arbitrariness of the volume V, we obtain the equation of heat propagation

$$c(x)\rho(x)\frac{\partial u}{\partial t}=\operatorname{div}(k(x)\operatorname{grad}u)+F(x,t). \tag{2.17}$$

If the medium is homogeneous, that is, $c(x)$, $\rho(x)$ and $k(x)$ are constants, then equation (2.17) takes the form

$$\frac{\partial u}{\partial t}=a^2\Delta u+f, \tag{2.18}$$

where

$$a^2=\frac{k}{c\rho},\ f=\frac{F}{c\rho}.$$

Equation (2.18) is called the heat equation or the *diffusion equation.*

2.6.3. THE STATIONARY EQUATION

For stationary processes $F(x,t) = F(x),\, u(x,t) = u(x)$, and the equations of oscillations and heat take the form

$$-\text{div}(k\,\text{grad}\,u) + qu = F(x)\,. \tag{2.19}$$

For $k = const,\, q = 0$, equation (2.19) is called the Poisson equation:

$$\Delta u = -f\ ,\ f = \frac{F}{k}\,. \tag{2.20}$$

For $f = 0$, the equation (2.20) is called the Laplace equation:

$$\Delta u = 0\,.$$

Let us consider the potential flow of fluid without sources, namely: let inside a certain volume V with a boundary S, which has a stationary flow of an incompressible fluid (density $\rho = \text{const}$), characterized by a velocity $\vec{v}(x_1, x_2, x_3)$. If the fluid flow is not vortex ($\text{rot}\,\vec{v} = 0$), then the velocity $\vec{v}$ is a potential vector, that is,

$$\vec{v} = \text{grad}\,u\,, \tag{2.21}$$

where u is a scalar function called the *velocity potential*.

If there are no sources, then

$$\text{div}\,\vec{v} = 0\,. \tag{2.22}$$

Now from formulas (2.21) and (2.22) we get

$$\text{div}\,\text{grad}\,u = 0\,,$$

or

$$\Delta u = 0\,,$$

that is, the velocity potential satisfies the Laplace equation.

2.7. FORMULATION OF BASIC BOUNDARY VALUE PROBLEMS FOR A SECOND-ORDER DIFFERENTIAL EQUATION

2.7.1.CLASSIFICATION OF BOUNDARY VALUE PROBLEMS

As shown, the linear equation of the second order

$$\rho\frac{\partial^2 u}{\partial t^2}=\mathrm{div}(k\,\mathrm{grad}u)-qu+F(x,t) \qquad (2.23)$$

describes the processes of vibrations, equation

$$\rho\frac{\partial u}{\partial t}=\mathrm{div}(k\,\mathrm{grad}u)-qu+F(x,t) \qquad (2.24)$$

describes the processes of diffusion, and equation

$$-\mathrm{div}(k\,\mathrm{grad}u)+qu=F(x) \qquad (2.25)$$

describes stationary processes.

Let $G\subset R^n$ be the area where the process takes place and S be its boundary. Thus, G is the domain of setting equation (2.25). The domain of setting equations (2.23) and (2.24) will be considered cylinder $\Omega_T=G\times(0,T)$ height T and base G. Its boundary consists of the lateral surface $S\times(0,T)$ and two bases: the lower $\bar{G}\times\{0\}$ and the upper $\bar{G}\times\{T\}$ (Fig. 2).

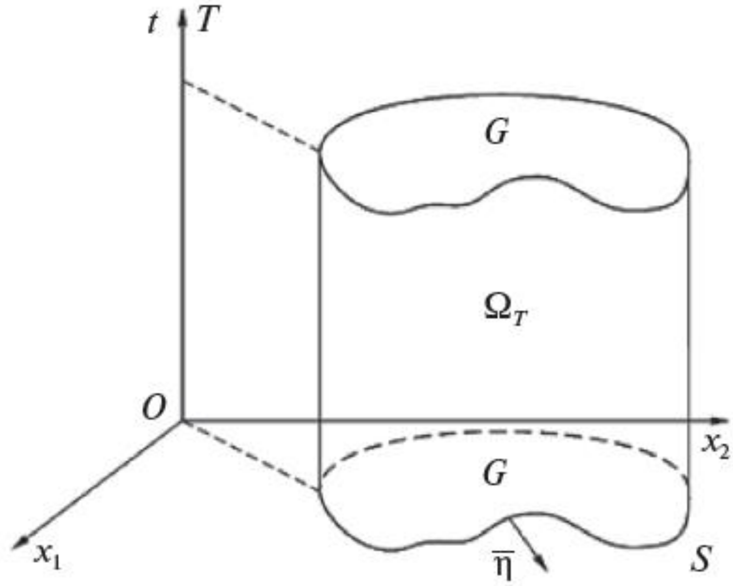

Fig. 2. The field of setting the equations of oscillations and diffusion

We will assume that the coefficients ρ, k, q of equations (2.23)– (2.25) do not depend on time t; further, in accordance with their physical meaning, we will assume that $\rho(x) > 0,\ k(x) > 0,\ q(x) \geq 0, x \in \overline{G}$.

Under these assumptions, the oscillation equation (2.23) is of the hyperbolic type, the diffusion (heat equation) equation (2.24) is of the parabolic type, and the stationary equation (2.25) is of the elliptical type.

Further, in order to fully describe the physical process, it is necessary, in addition to the equation describing this process, to specify the initial state of this process (initial conditions) and the regime at the boundary of the region in which the process occurs (boundary conditions).
There are three types of problems for differential equations.

1) The Cauchy problem for hyperbolic and parabolic equations: initial conditions are specifying, the region G coincides with the entire space R^n, there are no boundary conditions.

2) Boundary value problem for elliptic type equations: boundary conditions are specifying at the boundary S, the initial conditions, of course, are absent.

3) A mixed problem for hyperbolic and parabolic equations: both initial and boundary conditions are specifying, $G \neq R^n$.

Let us describe in more detail each of the listed boundary value problems for the equations (2.23)– (2.25) under consideration.

2.7.2.THE CAUCHY PROBLEM

For the oscillation equation (2.23), the Cauchy problem is posed as follows: find a function $u(x,t)$ of class $C^2(t>0)\cap C^1(t\geq 0)$ satisfying equation (2.23) in the half-space $t>0$ and the initial conditions at $t=0$:

$$u\big|_{t=0} = u_0(x), \ \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x). \qquad (2.26)$$

At the same time, it is necessary:

$$F \in C(t>0), \ u_0 \in C^1\left(R^n\right), \ u_1 \in C\left(R^n\right).$$

For the thermal conductivity equation (heat equation) (2.24), the Cauchy problem is posed as follows: find a function $u(x,t)$ of class $C^2(t>0)\cap C^1(t\geq 0)$ satisfying equation (2.24) in the half-space $t>0$ and the initial conditions at $t=0$:

$$u\big|_{t=0} = u_0(x). \qquad (2.27)$$

At the same time, it is necessary that

$$F \in C(t>0),\ u_0 \in C\left(R^n\right).$$

The above statement of the Cauchy problem admits the following generalization. Let the differential equations of the 2nd order be given:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_{i0}\frac{\partial^2 u}{\partial x_i \partial t} + \Phi\left(x, t, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t}\right), \quad (2.28)$$

piecewise smooth surface Σ: $t = \sigma(x)$ and functions u_0 and u_1 on Σ. The Cauchy problem for equation (2.28) consists in finding, in some part of the domain $t > \sigma(x)$ adjacent to the surface Σ, a solution $u(x,t)$ satisfying the boundary conditions on Σ

$$u\big|_{\Sigma} = u_0,\ \left.\frac{\partial u}{\partial n}\right|_{\Sigma} = u_1,$$

where $\vec{n}$ is the normal to Σ directed towards increasing t.

2.7.3. BOUNDARY VALUE PROBLEM FOR ELLIPTIC TYPE EQUATIONS. A MIXED TASK

The boundary value problem for equation (2.25) consists in finding a function $u(x)$ of class $C^2(G)\cap C^1(\bar{G})$ satisfying in the domain G equation (2.25) and a boundary condition on S of the form

$$\alpha u + \beta \frac{\partial u}{\partial n}\bigg|_S = \nu \text{ , } (2.29)$$

where $\alpha, \beta, \nu -$ are given continuous functions on S, and

$$\alpha \geq 0 \text{ , } \beta \geq 0 \text{ , } \alpha + \beta > 0 \text{ .}$$

The following types of boundary conditions are distinguished (2.29).

Boundary condition of the first kind $(\alpha = 1, \beta = 0)$:

$$u\big|_S = u_0 \text{ .}$$

The boundary condition of the second kind $(\alpha = 0, \beta = 1)$:

$$\frac{\partial u}{\partial n}\bigg|_S = u_1 \text{ .}$$

Boundary condition of the third kind $(\alpha \geq 0, \beta = 1)$:

$$\alpha u + \frac{\partial u}{\partial n}\bigg|_S = u_2 \text{ .}$$

The corresponding boundary value problems are called problems of *the I, II and III kind.* For the Laplace and Poisson equations, the boundary value problem of the first kind:

$$\Delta u = -f \ , \ u\big|_S = u_0$$

is called the *Dirichlet problem*; the boundary value problem of the second kind:

$$\Delta u = -f \ , \ \frac{\partial u}{\partial n}\bigg|_S = u_1$$

is called the *Neumann problem.*

For the oscillation equation (2.23), the mixed problem is posed as follows: find a function $u(x,t)$ of class $C^2(\Omega_\infty) \cap C^1(\bar{\Omega}_\infty)$ satisfying equation (2.23) in the cylinder Ω_∞, the initial conditions (2.26) at $t=0$ and the boundary condition (2.29) at $x \in S, \ t \geq 0$.

Similarly, for the diffusion equation (2.24), the mixed problem is posed as follows: find a function $u(x,t)$ of class $C^2(\Omega_\infty) \cap C^1(\bar{\Omega}_\infty)$ satisfying equation (2.24) in the cylinder Ω_∞, the initial condition (2.27) at $t=0$ and the boundary condition (2.29) at $x \in S, \ t \geq 0$.

2.7.4. THE CORRECTNESS OF THE FORMULATION OF MATHEMATICAL PHYSICS PROBLEMS

Since the problems of mathematical physics describe real physical processes, the mathematical formulation of these problems must meet the following requirements:

1) the solution exists in some class of M1 functions;

2) the solution is the only one in a certain class of M2 functions;

3) the solution continuously depends on the data of the problem (initial and boundary data, free term, coefficients of the equation, and so on).

The continuous dependence of the solution u on the data of the problem $\tilde{u}$ means the following: let the sequence $\tilde{u}_k, k=1,2,...$, in some sense tends to $\tilde{u}$ and $\tilde{u}_k, k=1,2,...,$ u are the corresponding solutions to the problem; then $u_k \to u,\ k \to \infty$ in the sense of convergence, appropriately chosen.

The requirement of continuous dependence of the solution is due to the fact that the data of a physical problem, as a rule, are determined from experiment approximately, and therefore one must be sure that the solution of the problem will not significantly depend on measurement errors.

A problem satisfying the listed requirements 1)-3) is called *correctly posed*, and the corresponding set of functions $M_1 \cap M_2$ is a *correctness class.*

Consider the following system of differential equations with N unknown functions $u_1, u_2, ..., u_N$:

$$\frac{\partial^{k_i} u_i}{\partial t^{k_k}} = \Phi_i\left(x, t, u_1, u_2, \ldots, u_N, \ldots, \frac{\partial^{\alpha_0+\alpha_1+\ldots+\alpha_n} u_j}{\partial t^{\alpha_0} \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}, \ldots \right), \qquad (2.30)$$

where $i = 1,2,\ldots,N$.

Here, the right-hand sides of Φ_i do not contain derivatives of order higher than k_i and derivatives with respect to t of order higher than $k_i - 1$, that is

$$\alpha_0 + \alpha_1 + \ldots + \alpha_n \le k_i, \; \alpha_0 \le k_i - 1.$$

For system (2.30), we set the following Cauchy problem: to find a solution $u_1, u_2, \ldots, u_N$ of this system satisfying the initial conditions at $t = t_0$:

$$\left.\frac{\partial^k u_i}{\partial t^k}\right|_{t=t_0} = \varphi_{ik}(x), \; k = 0,1,\ldots,k_i - 1, \; i = 1,2,\ldots,N, \qquad (2.31)$$

where $\varphi_{ik}(x)$ are the given functions in some domain $G \subset R^n$.

16.10.24

3. HYPERBOLIC EQUATIONS

3.1. THE STRING OSCILLATION EQUATION AND ITS SOLUTION BY THE D'ALEMBERT METHOD

The study of methods for solving boundary value problems for hyperbolic equations begins with the Cauchy problem for the equation of free vibrations of a string:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 , \qquad (3.1)$$

$$\begin{cases} u(x,0) = \varphi(x), \\ \dfrac{\partial u(x,0)}{\partial t} = \psi(x). \end{cases} \qquad (3.2)$$

3.1.1. D'ALEMBERT FORMULA

We transform equation (3.1) to a canonical form containing a mixed derivative. The equation of characteristics

$$\left[\frac{dx}{dt}\right]^2 - a^2 = 0$$

splits into two equations:

$$\frac{dx}{dt} - a = 0 , \ \frac{dx}{dt} + a = 0 ,$$

the integrals of which are

$$x - at = C_1 , \ x + at = C_2 .$$

Now, assuming

$$\xi = x + at\ ,\ \ \eta = x - at\ ,$$

equation (3.1) is transformed to the form

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0\,. \tag{3.3}$$

The general solution of equation (3.3) is determined by the formula

$$u = f_1(\xi) + f_2(\eta)\,,$$

where $f_1(\xi)$ and $f_2(\eta)$ are arbitrary functions. Returning to the variables x, t, we get

$$u = f_1(x + at) + f_2(x - at)\,. \tag{3.4}$$

The resulting solution depends on two arbitrary functions f_1 and f_2. It is called the *D'Alembert solution*.

Next, substituting formula (3.4) into (3.2), we will have

$$f_1(x) + f_2(x) = \varphi(x)\,, \tag{3.5}$$

$$af_1'(x) - af_2'(x) = \psi(x)\,, \tag{3.6}$$

from where, integrating the second equality (3.6), we get

$$f_1(x) - f_2(x) = \frac{1}{a}\int\limits_{x_0}^{x} \psi(y)dy + C\,, \tag{3.7}$$

where x_0 and C are constants. From the formulas (3.5) and (3.7) we find

$$f_1(x)=\frac{1}{2}\left[\varphi(x)+\frac{1}{a}\int_{x_0}^{x}\psi(y)dy+C\right],$$

$$f_2(x)=\frac{1}{2}\left[\varphi(x)-\frac{1}{a}\int_{x_0}^{x}\psi(y)dy-C\right].$$

At the same time, taking into account the formula (3.4), we have

$$u(x,t)=\frac{1}{2}\left[\varphi(x+at)+\frac{1}{a}\int_{x_0}^{x+at}\psi(y)dy+C+\varphi(x-at)-\frac{1}{a}\int_{x_0}^{x-at}\psi(y)dy-C\right]$$

and finally we get the formula

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)dy\,. \qquad (3.8)$$

The formula (3.8) is called the *D'Alembert's formula.*

It is not difficult to verify that formula (3.8) satisfies equation (3.1) and initial conditions (3.2) given that $\varphi(x)\in C^2(R)$ and $\psi(x)\in C^1(R)$. Thus, the described method proves both the uniqueness and the existence of a solution to the problem.

Example 1

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2},$$

if

$$u\big|_{t=0}=x^2,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=0\,.$$

Solution:

We know that $u_{tt} - a^2 u_{xx} = 0$.

Then $a = 1$.

We know that $\begin{cases} u(x;0) = \varphi(x) \\ \dfrac{\partial u}{\partial t}(x;0) = \psi(x) \end{cases}$

Since $u(x;0) = x^2$, then $\varphi(x) = x^2$.

Since $u_t'(x;0) = 0$, then $\psi(x) = 0$.

$$f_1 = \frac{1}{2}\left[x^2 + \frac{1}{1}\int_{x_0}^{x} 0dy + C\right] = \frac{1}{2}\left[x^2 + C\right]$$
$$f_2 = \frac{1}{2}\left[x^2 - C\right]$$

We know that $u = f_1(x - at) + f_2(x + at)$ and $a = 1$:

$$u = f_1(x - t) + f_2(x + t) = \frac{1}{2}\left[(x - t)^2 + C\right] + \frac{1}{2}\left[(x + t)^2 - C\right] =$$
$$= \frac{1}{2}\left[x^2 - 2xt + t^2 + x^2 + 2xt + t^2\right] = x^2 + t^2$$

We have $u = x^2 + t^2$.

OR:

$$u(x,t) = \frac{\varphi(x + at) + \varphi(x - at)}{2},$$

where $a = 1,\ \varphi(x) = x^2$.

$$u(x,t) = \frac{(x+t)^2 + (x-t)^2}{2} = \frac{x^2 + 2xt + t^2 + x^2 - 2xt + t^2}{2} =$$

$$= \frac{2(x^2 + t^2)}{2} = x^2 + t^2$$

<u>Let's check three conditions:</u>

1) $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$

$$u_x = 2x$$
$$u_{xx} = 2$$
$$u_t = 2t$$
$$u_{tt} = 2$$

2) $u\big|_{t=0} = x^2$

$$u = x^2 + 0 = x^2$$

3) $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$

$$u_t = 2t$$

$$u_t(0) = 2 \cdot 0 = 0$$

Example 2

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

if

$$u\big|_{t=0}=0\,,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=x\,.$$

Solution:

We know that $u_{tt}-a^2u_{xx}=0$.

Then $a=2$.

We know that $\begin{cases} u(x;0)=\varphi(x) \\ \dfrac{\partial u}{\partial t}(x;0)=\psi(x) \end{cases}$

Since $u(x;0)=0$, then $\varphi(x)=0$.

Since $u_t'(x;0)=x$, then $\psi(x)=x$.

We know that

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)dy$$

and $a=2$:

$$u(x,t)=\frac{1}{4}\int_{x-2t}^{x+2t}ydy=\frac{1}{8}y^2\Big|_{x-2t}^{x+2t}=\frac{1}{8}\Big[(x+2t)^2-(x-2t)^2\Big]=$$

$$=\frac{1}{8}\Big[x^2+4xt+4t^2-x^2+4xt-4t^2\Big]=\frac{1}{8}[8xt]=xt$$

We have $u(x,t)=xt$.

Let's check three conditions:

1) $\dfrac{\partial^2 u}{\partial t^2} = 4\dfrac{\partial^2 u}{\partial x^2}$

$u_x = t$

$u_{xx} = 0$

$u_t = x$

$u_{tt} = 0$

2) $u\big|_{t=0} = 0$

$u = x \cdot 0 = 0$

3) $\dfrac{\partial u}{\partial t}\bigg|_{t=0} = x$

$u_t = x$

3.1.2. THE INHOMOGENEOUS EQUATION

Consider the Cauchy problem for an inhomogeneous oscillation equation:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t),\ x \in R,\ t > 0, \qquad (3.9)$$

$$u(x,0) = \varphi(x),\ \frac{\partial u(x,0)}{\partial t} = \psi(x),\ x \in R. \qquad (3.10)$$

It is easy to verify that the solution of the problem (3.9), (3.10) $u = u(x,t)$ is represented in the form

$$u = \upsilon + \omega, \qquad (3.11)$$

where υ is the solution of the Cauchy problem (3.1), (3.2), and ω is the solution of the following problem:

$$\begin{cases} \dfrac{\partial^2 \omega}{\partial t^2} = a^2 \dfrac{\partial^2 \omega}{\partial x^2} + f(x,t), & x \in R, \quad t > 0, \\ \omega(x,0) = 0, \quad \dfrac{\partial \omega(x,0)}{\partial t} = 0, & x \in R. \end{cases} \tag{3.12}$$

Let $W(x,t;\tau)$ be the solution of the auxiliary Cauchy problem:

$$\begin{cases} \dfrac{\partial^2 W}{\partial t^2} = a^2 \dfrac{\partial^2 W}{\partial x^2}, & x \in R, \quad t > \tau, \\ W(x,t;\tau)\big|_{t=\tau} = 0, \quad \dfrac{\partial W(x,0)}{\partial t}\bigg|_{t=\tau} = f(x,\tau). \end{cases} \tag{3.13}$$

We show that the solution of the $\omega(x,t)$ problem (3.12) is determined by the formula

$$\omega(x,t) = \int_0^t W(x,t;\tau)\,d\tau\,, \tag{3.14}$$

where $W(x,t;\tau)-$ is the solution to the problem (3.13).

Really,

$$\omega(x,0) = 0\,,\ \frac{\partial \omega(x,t)}{\partial t} = W(x,t;t) + \int_0^t \frac{\partial W(x,t;\tau)}{\partial t}\,d\tau\,,$$

therefore, $\dfrac{\partial \omega(x,0)}{\partial t} = 0$ by virtue of the initial condition (3.13).

And finally:

$$\frac{\partial^2 \omega}{\partial t^2} - a^2 \frac{\partial^2 \omega}{\partial x^2} = \frac{\partial W(x,t;\tau)}{\partial t}\bigg|_{t=\tau} + \int_0^t \frac{\partial^2 W(x,t;\tau)}{\partial t^2} - a^2 \frac{\partial^2 W(x,t;\tau)}{\partial x^2}\,d\tau = f(x,t)\,.$$

The solution of the problem (3.13) is determined by the D'Alembert's formula:

$$W(x,t;\tau)=\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi\,. \qquad (3.15)$$

Now, using the formulas (3.8), (3.11), (3.14) and (3.15), we find that the solution of the initial problem (3.9), (3.10) is given by the formula

$$u(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)dy+\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)d\xi d\tau\,.$$

Example 3

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+x\sin t\,,$$

if

$$u\big|_{t=0}=\sin x\,,\ \left.\frac{\partial u}{\partial t}\right|_{t=0}=\cos x\,,\ x\in R\,.$$

<u>Solution:</u>

$$u_{tt}-a\,u_{xx}=f\left(x;t\right)$$

$$a=1$$

$$f\left(x;t\right)=x\cdot\sin t$$

$$u\big|_{t=0}=\sin x : u\left(x;0\right)=\sin x, \quad \text{so} \quad \varphi\left(x\right)=\sin x$$
$$u'_t\big|_{t=0}=\cos x : u'_t\left(x;0\right)=\cos x, \quad \text{so} \quad \psi\left(x\right)=\cos x$$

Since $u(x;t)=\upsilon(x;t)+\omega(x;t)$, we will find $\upsilon(x;t)$ and $\omega(x;t)$.

Let $\upsilon(x;t)=\upsilon_1+\upsilon_2$.

$$\upsilon_1(x,t)=\frac{\varphi(x+at)+\varphi(x-at)}{2}$$

$$\upsilon_1=\frac{1}{2}\big(\varphi(x+t)+\varphi(x-t)\big)=\frac{1}{2}\big(\sin(x+t)+\sin(x-t)\big)$$

$$\upsilon_2=\frac{1}{2a}\int_{x-at}^{x+at}\psi(y)\,dy=\frac{1}{2}\int_{x-t}^{x+t}\cos y\,dy=\frac{1}{2}\sin y\Big|_{x-t}^{x+t}=$$

$$=\frac{1}{2}\big(sin(x+t)-sin(x-t)\big)$$

$$\upsilon(x;t)=\frac{1}{2}\big(\sin(x+t)+\sin(x-t)\big)+\frac{1}{2}\big(sin(x+t)-sin(x-t)\big)=$$

$$=\sin(x+t)$$

We have $\upsilon(x;t)=\sin(x+t)$.

Now let's find $\omega(x;t)$.

Since $f(x;t)=x\cdot\sin t$, then

$$\omega=\frac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi;\tau)\,d\xi\,d\tau=\frac{1}{2}\int_0^t\int_{x-t+\tau}^{x+t-\tau}\xi\cdot\sin\tau\,d\xi\,d\tau=\frac{1}{2}\int_0^t\sin\tau\,d\tau\,\frac{\xi^2}{2}\Big|_{x-t+\tau}^{x+t-\tau}=$$

$$=\frac{1}{4}\int_0^t\sin\tau\Big[(x+t-\tau)^2-(x-t+\tau)^2\Big]d\tau=$$

$$=\frac{1}{4}\int_0^t\sin\tau\big[(x+t-\tau+x-t+\tau)(x+t-\tau-x+t-\tau)\big]d\tau=$$

$$=x\int_0^t\sin\tau\cdot(t-\tau)\,d\tau=x(t-\sin t)$$

We get

$$u = \sin(x+t) + xt - x\sin t$$

3.1.3. THE CONTINUATION METHOD

The first boundary value problem

The first boundary value problem for the oscillation equation on a half-line with a homogeneous boundary condition is set as follows: find a solution to the oscillation equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \; x > 0, \; t > 0, \qquad (3.16)$$

satisfying the boundary condition

$$u(0,t) = 0, \; t > 0 \qquad (3.17)$$

and the initial conditions

$$u(x,0) = \varphi(x), \; \frac{\partial u(x,0)}{\partial t} = \psi(x), \; x \geq 0. \qquad (3.18)$$

Let's add the conjugation conditions

$$\varphi(0) = 0, \; \psi(0) = 0$$

to ensure the continuity of the functions $u(x,t)$ and $\frac{\partial u(x,t)}{\partial t}$ at zero.

Let's define the functions $\varphi(x)$ and $\psi(x)$ in an odd way on the entire line by specifying new functions Φ and Ψ:

$$\Phi(x)=\begin{cases}\varphi(x), & x\ge 0,\\ -\varphi(-x), & x<0,\end{cases}$$

$$\Psi(x)=\begin{cases}\psi(x), & x\ge 0,\\ -\psi(-x), & x<0.\end{cases}$$

Consider a modified Cauchy problem:

$$\begin{cases}\dfrac{\partial^2 U}{\partial t^2}=a^2\dfrac{\partial^2 U}{\partial x^2}, & -\infty<x<\infty,\quad t>0,\\ U(x,0)=\Phi(x), \quad \dfrac{\partial U(x,0)}{\partial t}=\Psi(x).\end{cases}$$

In this case, to find $U(x,t)$, we can apply the D'Alembert formula:

$$U(x,t)=\frac{\Phi(x+at)+\Phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(y)dy\,.$$

Let's take the function $U(x,t)$ as the function we need $u(x,t)$ for $x\ge 0, t\ge 0$. Obviously, conditions (3.16) and (3.18) for $x\ge 0, t\ge 0$ are fulfilled immediately — this follows from the definition of the functions $\Phi(x)$ and $\Psi(x)$. The fulfillment of condition (3.17) follows from the following transformations:

$$u(0,t)=U(0,t)=\frac{\Phi(at)+\Phi(-at)}{2}+\frac{1}{2a}\int_{-at}^{at}\Psi(y)dy\,.$$

Due to the odd number of functions $\Phi(x)$ and $\Psi(x)$, the first and second terms vanish, which gives the fulfillment of condition (3.17).

We express $\Phi(x)$ and $\Psi(x)$ through the original functions $\varphi(x)$ and $\psi(x)$, respectively:

$$x \ge at\ ,\ \begin{cases} \Phi(x+at) = \varphi(x+at), \\ \Phi(x-at) = \varphi(x-at), \\ \Psi(y) = \psi(y), \quad y \in [x-at, x+at]; \end{cases}$$

$$x < at\ ,\ \begin{cases} \Phi(x+at) = \varphi(x+at), \\ \Phi(x-at) = -\varphi(at-x). \end{cases}$$

Now let's write down an auxiliary formula for solving the first boundary value problem:

at $x < at$,

$$\int_{x-at}^{x+at} \Psi(y)dy = \int_{x-at}^{0} \Psi(y)dy + \int_{0}^{x+at} \Psi(y)dy =$$

$$= \int_{x-at}^{0} -\psi(-y)dy + \int_{0}^{x+at} \psi(y)dy =$$

{let's put $-y = y$ }

$$= \int_{at-x}^{0} \psi(y)dy + \int_{0}^{x+at} \psi(y)dy = \int_{at-x}^{x+at} \psi(y)dy\,.$$

Then the general formula will be as follows:

$$u(x,t) = \begin{cases} \dfrac{\varphi(x+at)+\varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(y)dy, \quad x \ge at, \\ \dfrac{\varphi(x+at)-\varphi(at-x)}{2} + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} \psi(y)dy, \quad x < at. \end{cases}$$

The second boundary value problem

The second boundary value problem for the equation of oscillations on a half-line with a homogeneous boundary condition is set as follows: find a solution to the equation of oscillations

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \ x > 0, \ t > 0, \qquad (3.19)$$

satisfying the boundary condition

$$\frac{\partial u(0,t)}{\partial x} = 0, \ t \geq 0, \qquad (3.20)$$

and the initial conditions:

$$u(x,0) = \varphi(x), \ \frac{\partial u(x,0)}{\partial t} = \psi(x), \ x \geq 0. \qquad (3.21)$$

We will act in the same way as in the previous case, however, only an even continuation will suit us here:

$$\Phi(x) = \begin{cases} \varphi(x), & x \geq 0, \\ \varphi(-x), & x < 0, \end{cases}$$

$$\Psi(x) = \begin{cases} \psi(x), & x \geq 0, \\ \psi(-x), & x < 0. \end{cases}$$

The new Cauchy problem and the solution for it according to the D'Alembert formula will look the same as in the previous case:

$$U(x,t) = \frac{\Phi(x+at) + \Phi(x-at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \Psi(y) dy.$$

Similarly, let the function $u(x,t)=U(x,t)$ for $x>0,\ t>0$. Then the fulfillment of conditions (3.19) and (3.21) is again obvious. Let's check the condition (3.20). Differentiating the D'Alembert formula and using the fact that the derivative of an even function is odd, we get

$$\frac{\partial u(0,t)}{\partial x}=\frac{\partial U(0,t)}{\partial x}=\frac{\Phi'(at)+\Phi'(-at)}{2}+\frac{1}{2a}\left[\Psi(at)-\Psi(-at)\right].$$

From the odd $\Phi'(t)$ and the parity $\Psi(t)$, it can be seen that both terms are equal to zero.

The general formula is obtained similarly:

$$u(x,t)=\begin{cases}\dfrac{\varphi(x+at)+\varphi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(y)dy, & x\geq at,\\ \dfrac{\varphi(x+at)+\varphi(at-x)}{2}+\dfrac{1}{2a}\left[\displaystyle\int_{0}^{at-x}\psi(y)dy+\int_{0}^{x+at}\psi(y)dy\right], & x<at.\end{cases}$$

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3.2.THE EQUATION OF STRING VIBRATION AND ITS SOLUTION BY THE METHOD OF SEPARATION OF VARIABLES (FOURIER METHOD)

The method of separation of variables, or the Fourier method, is one of the most common methods for solving partial differential equations. We will present this method for the problem of vibrations of a string fixed at the ends.

3.2.1. THE EQUATION OF FREE VIBRATIONS OF THE STRING

The following mixed problem is considered.

Task 1.

Let it be required to find a solution:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \ 0 < x < l, \ t > 0,$$

satisfying the initial and boundary conditions:

$$u(x,0) = \varphi(x),$$

$$\frac{\partial u(x,0)}{\partial t} = \psi(x),$$

$$u(0,t) = 0,$$

$$u(l,t) = 0, \ t \ge 0.$$

We are looking for a solution to this problem in the form of a product:

$$u(x,t) = X(x)T(t),$$

substituting which into this equation, we have

$$X(x)T''(t) = a^2 X''(x)T(t).$$

Dividing both parts of this equation by $a^2 X(x)T(t)$, we obtain

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}. \tag{3.22}$$

The right side of equality (3.22) is a function of only variable x, and the left side is only t, so the right and left sides of equality (3.22), when changing their arguments. retain a constant value. It is convenient to denote this value by $-\lambda$, that is

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

$$X''(x) + \lambda X(x) = 0,$$

$$T''(t) + \lambda a^2 T(t) = 0.$$

The general solutions of these equations have the form

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

$$T(t) = C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t,$$

where $A,B,C,D-$ are arbitrary constants, and the function $u(x,t)$ is

$$u(x,t) = \left(A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\right)\left(C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t\right).$$

Constants A and B can be found using the boundary conditions of Task 1. Since

$$T(t) \not\equiv 0,$$

then

$$X(0)=0,\ X(l)=0.$$

$$X(0)=A=0,$$

$$X(l)=A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l=0,$$

that is,

$$A=0,$$

$$B\sin\sqrt{\lambda}l=0.$$

From where

$$\sqrt{\lambda}=\frac{k\pi}{l},\ k=1,2,\dots.$$

So,

$$X(x)=B\sin\frac{k\pi}{l}x.$$

The values $\lambda=\frac{k^2\pi^2}{l^2}$ found are called *eigenvalues* for a given boundary value Task 1, and the functions $X(x)=B\sin\frac{k\pi}{l}x$ are called *eigenfunctions.*

With the values of λ found, we get

$$T(t)=C\cos\frac{ak\pi}{l}t+D\sin\frac{ak\pi}{l}t,$$

$$u_k(x,t) = \sin\frac{k\pi}{l}x\left(a_k\cos\frac{ak\pi}{l}t + b_k\sin\frac{ak\pi}{l}t\right),\ k = 1,2,\dots .$$

Since the equation is linear and homogeneous, the sum of the solutions is also a solution that can be represented as a series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{ak\pi}{l}t + b_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x .$$

In this case, the solution must satisfy the initial condition:

$$u(x,0) = \sum_{k=1}^{\infty} a_k\sin\frac{k\pi}{l}x = \varphi(x) .$$

If the function $\varphi(x)$ decomposes into a Fourier series in the interval $(0,l)$ in terms of sines, then

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,dx . \qquad (3.23)$$

From the initial condition

$$\frac{\partial u(x,0)}{\partial t} = \psi(x)$$

we have

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = \sum_{k=1}^{\infty}\frac{ak\pi}{l}b_k\sin\frac{k\pi}{l}x = \psi(x) .$$

We determine the Fourier coefficients of this series:

$$\frac{ak\pi}{l}b_k = \frac{2}{l}\int_0^l \psi(x)\sin\frac{k\pi}{l}x\,dx ,$$

from where

$$b_k = \frac{2}{ak\pi}\int_0^l \psi(x)\sin\frac{k\pi}{l}xdx\,. \qquad (3.24)$$

Thus, the solution of the string oscillation equation can be represented as the sum of an infinite series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty}\left(a_k \cos\frac{ak\pi}{l}t + b_k \sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x\,, \qquad (3.25)$$

where a_k, b_k are determined by formulas (3.23) and (3.24).

Theorem. Let $\varphi(x) \in C^2([0,l])$, $\psi(x) \in C^1([0,l])$, in addition, $\varphi(x)$ has a third, and $\psi(x)$ has a second piecewise continuous derivative and the relations are fulfilled: $\varphi(0)=\varphi(l)=0,\ \varphi''(0)=\varphi''(l)=0,\ \psi(0)=\psi(l)=0$. Then the sum of the series (3.25) with coefficients defined by formulas (3.23), (3.24) is the solution to Task 1.

Example 1

Find the deviation $u(x;t)$ from the equilibrium position of a homogeneous horizontal string fixed at the ends $x=0$ and $x=l$, if at the initial moment the string had the shape $\frac{1}{8}\sin\frac{3\pi x}{l}$, and the initial velocities were absent.

Solution:

$$\begin{cases} u_{tt} = a^2 u_{xx} & (*) \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \dfrac{1}{8}\sin\dfrac{3\pi x}{l} \\ u_t(x,0) = 0 \end{cases}$$

The method of separation of variables, the Fourier method:

we will look for a solution in the form

$$u(x,t) = T(t)\cdot X(x)$$

$$u_{tt}(x,t) = T''(t)\cdot X(x)$$

$$u_{xx}(x,t) = T(t)\cdot X''(x)$$

We substitute it into equation (*):

$$T''(t)X(x) = a^2 T(t)X''(x)$$

Divide by $a^2T(t)X(x)$, we get

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda\ .$$

Consider

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0 \\ X(l)=0 \end{cases}$$

We obtain an ordinary differential equation of the second order. This is the task of Sturm-Liouville theory.

We are solving this problem:

1) $\lambda=0 \quad \Rightarrow \quad X''(x)=0$

$$X(x)=C_1x+C_2$$

Substituting into the boundary conditions, we get:

$$C_1=0$$
$$C_2=0$$

That is, for $\lambda=0$, the only solution is: $X(x)\equiv 0$. This solution does not suit us.

2) $\lambda<0 \quad \Rightarrow \quad X(x)\equiv 0$

3) $\lambda>0 \quad \Rightarrow \quad X(x)=A\sin\left(\sqrt{\lambda}\,x\right)+B\cos\left(\sqrt{\lambda}\,x\right)$

Substitute the boundary conditions, we get:

$$X(0)=B=0$$
$$X(l)=A\sin\left(\sqrt{\lambda}\,l\right)=0$$

There are two options:

If $A=0$, then $X(x)\equiv 0$. It doesn't suit us.

If $A \neq 0$, then

$$sin\left(\sqrt{\lambda}\, l\right) = 0$$

$$\sqrt{\lambda}\, l = \pi n, \quad n \in \mathbb{Z}$$

$$\sqrt{\lambda} = \frac{\pi n}{l},$$

$$\lambda = \left(\frac{\pi n}{l}\right)^2$$

$$X(x) = A \sin\left(\frac{\pi n}{l} x\right)$$

The solution is definitely ambiguous. $A-$ can be any, $n-$ can be different numbers.

So

$$X_n(x) = A_n \sin\left(\frac{\pi n}{l} x\right)$$

We have found Sturm-Liouville's *eigenfunctions.*

$$T''(t) + a^2 \lambda T(t) = 0$$

$$T''(t) + \left(\frac{a\pi n}{l}\right)^2 T(t) = 0$$

$$T(t) = C\, sin\left(\frac{a\pi n}{l} t\right) + D\, cos\left(\frac{a\pi n}{l} t\right)$$

$$T_n(t)=C_n\sin\left(\frac{a\pi n}{l}t\right)+D_n\cos\left(\frac{a\pi n}{l}t\right)$$

$$u_n(x,t)=\left(C_n\sin\left(\frac{a\pi n}{l}t\right)+D_n\cos\left(\frac{a\pi n}{l}t\right)\right)\cdot A_n\sin\left(\frac{\pi n}{l}x\right)=$$

$$=\left(a_n\sin\left(\frac{a\pi n}{l}t\right)+b_n\cos\left(\frac{a\pi n}{l}t\right)\right)\cdot\sin\left(\frac{\pi n}{l}x\right)$$

$$u_n(x,t)=\left(a_n\sin\left(\frac{a\pi n}{l}t\right)+b_n\cos\left(\frac{a\pi n}{l}t\right)\right)\cdot\sin\left(\frac{\pi n}{l}x\right)$$

We don't know these constants a_n and b_n, there are an infinite number of them.

Consider the initial conditions (initial amplitude and initial velocities).

$$u(x;0)=\sum_{n=1}^{\infty}b_n\,sin\left(\frac{\pi n}{l}x\right)=\frac{1}{8}sin\left(\frac{3\pi}{l}x\right)$$

$$u_t(x;0)=\sum_{n=1}^{\infty}a_n\cdot\frac{a\pi n}{l}\cdot 1\cdot sin\left(\frac{\pi n}{l}x\right)=0$$

We get that $a_n=0\quad\forall n$.

$$b_1\,sin\left(\frac{\pi}{l}x\right)+b_2\,sin\left(\frac{2\pi}{l}x\right)+b_3\,sin\left(\frac{3\pi}{l}x\right)+b_4\,sin\left(\frac{4\pi}{l}x\right)+...=\frac{1}{8}sin\left(\frac{3\pi}{l}x\right).$$

We get that $b_3=\frac{1}{8},\quad b_n=0\quad n\neq 3$.

<u>Answer:</u> $u(x,t)=\frac{1}{8}cos\left(\frac{3a\pi}{l}t\right)sin\left(\frac{3\pi}{l}x\right)$.

Example 2

Let the initial rejection of the string fixed at points $x=0$ and $x=l$ be zero, and the initial velocity

$$\frac{\partial u}{\partial t}=\begin{cases} v_0, & \left|x-\frac{l}{2}\right|<\frac{h}{2}, \\ 0, & \left|x-\frac{l}{2}\right|>\frac{h}{2}. \end{cases}$$

Determine the shape of the string for any time t.

<u>Solution:</u>

Here $\varphi(x)=0$, and $\psi(x)=v_0$ in the interval $\left(\frac{l-h}{2},\frac{l+h}{2}\right)$, and $\psi(x)=0$ outside this interval.

Therefore,

$$a_k=0,$$

$$b_k=\frac{2}{ak\pi}\int_{(l-h)/2}^{(l+h)/2} v_0\sin\frac{k\pi}{l}x\,dx=-\frac{2v_0}{ak\pi}\frac{l}{k\pi}\cos\frac{k\pi}{l}x\Bigg|_{(l-h)/2}^{(l+h)/2}=$$

$$=\frac{2v_0 l}{ak^2\pi^2}\left[\cos\frac{k\pi(l-h)}{2l}-\cos\frac{k\pi(l+h)}{2l}\right]=\frac{4v_0 l}{ak^2\pi^2}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2l}.$$

Hence

$$u(x,t)=\frac{4v_0 l}{a\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2l}\sin\frac{ak\pi t}{l}\sin\frac{k\pi x}{l}.$$

Example 3

A string is given, fixed at the ends $x=0$ and $x=l$. Let's assume that at the initial moment the shape of the string has the form of a polyline OAB, shown in Fig. Find the shape of the string for any time t if there are no initial velocities.

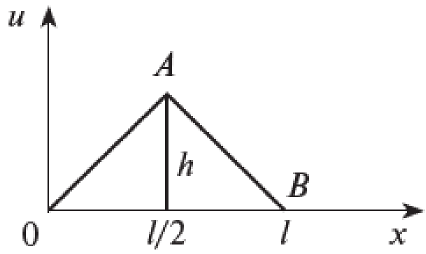

Fig. The shape of the string at the initial moment of time

Solution:

The angular coefficient of the straight line OA is equal to $\frac{h}{l/2}$, the equation of this straight line is $u=\frac{2h}{l}x$. The straight line AB cuts off the segments: l and $2h$ on the coordinate axes, which means that the equation of the straight line AB: $u=\frac{2h}{l}(l-x)$. So,

$$\varphi(x)=\begin{cases}\frac{2h}{l}x, & 0\le x\le\frac{l}{2}\\ \frac{2h}{l}(l-x), & \frac{l}{2}\le x\le l,\end{cases}$$

$$\psi(x)=0.$$

We find

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}dx =$$

$$= \frac{4h}{l^2}\int_0^{l/2} x\sin\frac{k\pi x}{l}dx + \frac{4h}{l^2}\int_{l/2}^{l}(l-x)\sin\frac{k\pi x}{l}dx\ ,$$

$$b_k = 0\,.$$

Integrating by parts, we get

$$a_k = -\frac{4h}{k\pi l}x\cos\frac{k\pi x}{l}\Bigg|_0^{l/2} + \frac{4h}{k\pi l}\int_0^{l/2}\cos\frac{k\pi x}{l}dx -$$

$$-\frac{4h}{k\pi l}(l-x)\cos\frac{k\pi x}{l}\Bigg|_{l/2}^{l} - \frac{4h}{k\pi l}\int_{l/2}^{l}\cos\frac{k\pi x}{l}dx =$$

$$= -\frac{2h}{k\pi}\cos\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\sin\frac{k\pi x}{l}\Bigg|_0^{l/2} + \frac{2h}{k\pi}\cos\frac{k\pi}{2} - \frac{4h}{k^2\pi^2}\sin\frac{k\pi x}{l}\Bigg|_{l/2}^{l} =$$

$$= \frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2} = \frac{8h}{k^2\pi^2}\sin\frac{k\pi}{2}\,.$$

Therefore,

$$u(x,t) = \frac{8h}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{2}\sin\frac{k\pi x}{l}\cos\frac{ak\pi t}{l}\,.$$

Example 4

The string fixed at the ends $x = 0$ and $x = l$ has the shape of a parabola

$$u(x,0) = \frac{4}{l^2}\,x(l-x)\,.$$

at the initial moment of time. Find the shape of the string at any given time if there are no initial velocities.

Solution:

Here

$$\varphi(x) = \frac{4}{l^2} x\left(l - x\right),$$

$$\psi(x) = 0\,.$$

Therefore, we have

$$a_k = \frac{2}{l} \int_0^l \frac{4}{l^2} x(l - x) \sin\frac{k\pi x}{l} dx,$$

Applying the integration by parts method twice, we get

$$a_k = \frac{8}{l^3} \int_0^l (lx - x^2) \sin\frac{k\pi x}{l} dx =$$

$$= \frac{8}{l^3}\left(-\frac{l\left(lx - x^2\right)}{k\pi} \cos\frac{k\pi x}{l}\bigg|_0^l + \frac{l}{k\pi} \int_0^l \left(l - 2x\right) \cos\frac{k\pi x}{l} dx \right) =$$

$$= \frac{8}{l^2 k\pi} \int_0^l \left(l - 2x\right) \cos\frac{k\pi x}{l} dx =$$

$$= \frac{8}{l^2 k\pi}\left(\frac{l\left(l - 2x\right)}{k\pi} \sin\frac{k\pi x}{l}\bigg|_0^l + \frac{2l}{k\pi} \int_0^l \sin\frac{k\pi x}{l} dx \right) =$$

$$= \frac{16}{lk^2\pi^2} \int_0^l \sin\frac{k\pi x}{l} dx = \frac{16}{lk^2\pi^2}\left(-\frac{l}{k\pi} \cos\frac{k\pi x}{l} \right)\bigg|_0^l =$$

$$= -\frac{16}{k^3\pi^3}\left(\cos k\pi - 1\right) = \frac{16}{k^3\pi^3}\left(1 - (-1)^k\right) = \left[\begin{array}{ll} \dfrac{32}{k^3\pi^3}, & k = 2n + 1, \\ 0, & k = 2n. \end{array}\right.$$

Then the solution of the problem will take the following form:

$$u(x,t) = \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{\left(2n+1\right)^3} \sin\frac{\left(2n+1\right)\pi x}{l} \cos\frac{a\left(2n+1\right)\pi t}{l}.$$

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3. HYPERBOLIC EQUATIONS

3.2.2. THE INHOMOGENEOUS EQUATION

The following mixed problem is considered.

Task 2.

Let it be necessary to find a solution to the inhomogeneous equation of string vibrations:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \; 0 < x < l, \; t > 0, \tag{3.26}$$

satisfying the initial and boundary conditions:

$$u(x,0) = \varphi(x), \; \frac{\partial u(x,0)}{\partial t} = \psi(x), \tag{3.27}$$

$$u(0,t) = 0, \; u(l,t) = 0, \; t \geq 0.$$

We will look for a solution to Task 2 in the form of a Fourier series expansion in x:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin\frac{k\pi}{l} x, \tag{3.28}$$

considering t as a parameter.

Let's imagine the function $f(x,t)$ as a Fourier series:

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) \sin\frac{k\pi}{l} x,$$

$$f_k(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}x\,dx. \qquad (3.29)$$

Substituting series (3.28) and (3.29) into the original equation (3.26):

$$\sum_{k=1}^{\infty}\left[u_k''(t) + a^2\left(\frac{k\pi}{l}\right)^2 u_k(t) - f_k(t)\right]\sin\frac{k\pi}{l}x = 0,$$

we see that it will be satisfied if all the expansion coefficients are equal:

$$u_k''(t) + a^2\left(\frac{k\pi}{l}\right)^2 u_k(t) = f_k(t). \qquad (3.30)$$

To determine $u_k(t)$, we obtained an ordinary differential equation with constant coefficients.

Further, the initial conditions (3.27) give:

$$\varphi(x) = \sum_{k=1}^{\infty} u_k(0)\sin\frac{k\pi}{l}x,$$

$$\psi(x) = \sum_{k=1}^{\infty} u_k'(0)\sin\frac{k\pi}{l}x,$$

therefore,

$$u_k(0) = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi}{l}x\,dx,$$

$$u_k'(0) = \frac{2}{l}\int_0^l \psi(x)\sin\frac{k\pi}{l}x\,dx, \qquad (3.31)$$

$$\varphi_k = u_k(0), \quad \psi_k = u_k'(0).$$

The conditions (3.31) completely determine the solution (3.30):

$$u_k(t) = \varphi_k \cos\frac{ak\pi}{l}t + \frac{l}{ak\pi}\psi_k \sin\frac{ak\pi}{l}t + \frac{l}{ak\pi}\int_0^t f_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)d\tau . \quad (3.32)$$

Thus, the desired solution to Task 2, according to formulas (3.28) and (3.32), will be written in the form

$$u(x,t) = \sum_{k=1}^{\infty}\left\{\varphi_k \cos\frac{ak\pi}{l}t + \frac{l}{ak\pi}\psi_k \sin\frac{ak\pi}{l}t + \frac{l}{ak\pi}\int_0^t f_k(\tau)\sin\frac{ak\pi}{l}(t-\tau)d\tau\right\}\times$$
$$\times \sin\frac{k\pi}{l}x,$$

where the values $\varphi_k, \psi_k, f_k(\tau)$ are calculated by (3.31) and (3.29), respectively.

Example 5.

Find a solution to the boundary value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 2b,\ 0 < x < l,\ t > 0,$$

$$u(x,0) = 0,\ \frac{\partial u(x,0)}{\partial t} = 0,$$

$$u(0,t) = 0,\ u(l,t) = 0,\ t \ge 0.$$

<u>Solution:</u>

Here

$$\varphi(x) = 0,\ t,\ \psi(x) = 0,\ f(x,t) = 2b,\ a = 1.$$

Therefore,

$\varphi_k = 0$, $\psi_k = 0$.

$$f_k(t) = \frac{2}{l}\int_0^l f(x,t)\sin\frac{k\pi}{l}x dx.$$

$$f_k(t) = \frac{2}{l}\int_0^l 2b\sin\frac{k\pi}{l}x dx = -\frac{4b}{l}\frac{l}{k\pi}\cos\frac{k\pi}{l}x\bigg|_0^l =$$

$$= -\frac{4b}{k\pi}\left[(-1)^k - 1\right] = \begin{cases} 0, & k = 2n, \\ \dfrac{8b}{k\pi}, & k = 2n+1. \end{cases}$$

Further

$$\int_0^t f_{2n+1}(\tau)\sin\frac{(2n+1)\pi}{l}(t-\tau)d\tau = \frac{8b}{(2n+1)\pi}\int_0^t \sin\frac{(2n+1)\pi}{l}(t-\tau)d\tau =$$

$$= \frac{8b}{(2n+1)\pi}\frac{l}{(2n+1)\pi}\cos\frac{(2n+1)\pi}{l}(t-\tau)\bigg|_0^t = \frac{8bl}{(2n+1)^2\pi^2}\left[1-\cos\frac{(2n+1)\pi}{l}t\right].$$

Hence

$$u(x,t) = \sum_{n=0}^{\infty}\frac{8bl^2}{(2n+1)^3\pi^3}\left[1-\cos\frac{(2n+1)\pi}{l}t\right]\sin\frac{(2n+1)\pi}{l}x.$$

4. PARABOLIC EQUATIONS

4.1. ONE-DIMENSIONAL EQUATION OF THERMAL CONDUCTIVITY. SETTING BOUNDARY VALUE PROBLEMS

The process of temperature distribution in a rod, thermally insulated from the sides and thin enough that at any given time the temperature at all points of the cross section can be considered a single one, can be described by the function $u(x,t)$, representing the temperature in the cross section x at time t. This function $u(x,t)$ is the solution of the equation:

$$c\rho\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+F(x,t),$$

called *the heat equation.*

Here $\rho(x),c(x),k(x)-$ are, respectively, the density, specific heat capacity and thermal conductivity coefficient of the rod at point x, and $F(x,t)-$ is the intensity of heat sources at point x at time t.

To identify the only one solution to the heat equation, it is necessary to attach the initial and boundary conditions to the equation.

The initial condition, unlike a hyperbolic equation, consists only in setting the values of the function u at the initial moment of time t_0:

$$u(x,t_0)=\varphi(x).$$

The main types of boundary conditions are boundary value problems of the first, second and third types.

The first boundary value problem is set if the temperature at the end of the rod $x = 0$ is maintained according to a certain law, for example:

$$u(0,t) = \mu(t) ,$$

where $\mu(t)$ is a given function of time.

The second boundary value problem is posed if the heat flow q is set at the end of the rod $x = l$, for example:

$$q(l,t) = -k\frac{\partial u(l,t)}{\partial x} ,$$

therefore, the boundary condition has the form

$$\frac{\partial u(l,t)}{\partial x} = \upsilon(t) = -\frac{1}{k}q(l,t) .$$

In particular, in the case of a thermally insulated end, there is no heat flow through it, that is, $\upsilon(t) = 0$.

The third boundary value problem is formulated when heat exchange with the ambient occurs at the end of the rod $x = l$ according to *Newton's law:*

$$q(l,t) = H\left(u(l,t) - \theta(t)\right),$$

where $\theta(t)$ is the ambient temperature, H is the heat exchange coefficient, that is, the amount of heat that has passed through a single section of the rod per unit time with a temperature change of one degree.

The boundary condition has the form

$$\frac{\partial u(l,t)}{\partial x} = -\lambda\left(u(l,t) - \theta(t)\right),$$

where $\lambda = \frac{H}{k}$.

Some limiting cases are also considered. For example, if the process of thermal conductivity is studied in a very long rod. For a short period of time, the influence of the temperature regime set at the boundary in the central part of the rod has a very weak effect, and the temperature in this area is mainly determined only by the initial temperature distribution. In problems of this type, it is usually assumed that the rod has an infinite length. Thus, a problem with initial conditions (the Cauchy problem) is posed on the temperature distribution on an infinite line: to find a solution to the thermal conductivity equation in the region $-\infty < x < \infty$ and $t \geq t_0$ satisfying the condition

$$u(x,t_0) = \varphi(x), \ -\infty < x < \infty,$$

where $\varphi(x)$ is a given function.

Similarly, if the section of the rod whose temperature we are interested in is located near one end and far from the other, then in this case the temperature is practically determined by the temperature regime of the near end and the initial conditions. In problems of this type, it is usually assumed that the rod is semi-infinite, and the coordinate measured from the end varies within $0 \leq x < \infty$. Let us give as an example the formulation of the first boundary value problem for a semi—infinite rod: to find a solution to the thermal conductivity equation in the region $-0 < x < \infty$ and $t \geq t_0$ satisfying the conditions

$$u(x,t_0) = \varphi(x),\ 0 < x < \infty,$$

$$u(0,t) = \mu(t),\ t \geq t_0,$$

where $\varphi(x)$ and $\mu(t)$ are given functions.

4.2. A METHOD FOR SEPARATING VARIABLES FOR THE EQUATION OF THERMAL CONDUCTIVITY. INSTANT POINT SOURCE FUNCTION

4.2.1. HOMOGENEOUS BOUNDARY VALUE PROBLEM

The following first boundary value problem is considered.

Find a solution to a homogeneous equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < l,\ 0 < t \leq T, \tag{4.1}$$

satisfying the initial condition

$$u(x,0) = \varphi(x),\ 0 \leq x \leq l \tag{4.2}$$

and homogeneous boundary conditions

$$u(0,t) = 0,\ u(l,t) = 0,\ 0 \leq t \leq T. \tag{4.3}$$

We are looking for a solution to this problem in the form of a product

$$u(x,t) = X(x)T(t),$$

substituting it into equation (4.1), we have

$$X(x)T'(t) = a^2 X''(x)T(t).$$

Dividing both parts of this equation by $a^2 X(x)T(t)$, we get

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}. \qquad (4.4)$$

The right side of equality (4.4) is a function of only the variable x, and the left side is only t, so the right and left sides of equality (4.4) retain a constant value when changing their arguments. It is convenient to denote this value by $-\lambda$, that is,

$$\frac{T'(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

$$X''(x) + \lambda X(x) = 0,$$

$$T'(t) + \lambda a^2 T(t) = 0.$$

The general solutions of these equations have the form

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

$$T(t) = Ce^{-a^2\lambda t},$$

where $A, B, C -$ are arbitrary constants, and the function $u(x,t)$ is

$$u(x,t) = \left(A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\right)Ce^{-a^2\lambda t}.$$

Constants A and B can be found using the boundary conditions (4.3) of the problem. Since

$$T(t) \not\equiv 0,$$

then

$$X(0)=0,\ X(l)=0.$$

$$X(0)=A=0,$$

$$X(l)=A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l=0,$$

that is,

$$A=0 \text{ и } B\sin\sqrt{\lambda}l=0.$$

From where

$$\sqrt{\lambda}=\frac{k\pi}{l},\ k=1,2,\ldots.$$

So,

$$X(x)=B\sin\frac{k\pi}{l}x.$$

The values $\lambda=\frac{k^2\pi^2}{l^2}$ found are called *eigenvalues* for a given boundary value problem, and the functions $X(x)=B\sin\frac{k\pi}{l}x$ are called *eigenfunctions.*

When the values of λ are found, we get

$$T(t) = Ce^{-\frac{a^2k^2\pi^2}{l^2}t},$$

$$u_k(x,t) = a_k e^{-\frac{a^2k^2\pi^2}{l^2}t} \sin\frac{k\pi}{l}x,\ k = 1,2,\dots .$$

Since equation (4.1) is linear and homogeneous, the sum of the solutions is also a solution that can be represented as a series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} a_k e^{-\frac{a^2k^2\pi^2}{l^2}t} \sin\frac{k\pi}{l}x .$$

In this case, the solution must satisfy the initial condition (4.2):

$$u(x,0) = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi}{l}x = \varphi(x) .$$

If the function $\varphi(x)$ decomposes into a Fourier series in the interval $(0,l)$ according to the sine, then

$$a_k = \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi .$$

Thus, the solution of the heat equation can be represented as the sum of an infinite series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi e^{-\frac{a^2k^2\pi^2}{l^2}t} \sin\frac{k\pi}{l}x . \qquad (4.5)$$

Theorem. Let $\varphi(x)\in C^1([0,l]), \varphi(0)=\varphi(l)=0$. Then there is a unique solution to the problem (4.1)–(4.3), which is represented as an absolutely and uniformly converging series (4.5).

The solution (4.5) can be represented as

$$u(x,t)=\int_0^l G(x,\xi,t)\varphi(\xi)d\xi \ ,$$

where the function

$$G(x,\xi,t)=\frac{2}{l}\sum_{k=1}^{\infty} e^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi x}{l}\sin\frac{k\pi\xi}{l} \ ,$$

is introduced, called the *instantaneous point source function.*

The physical meaning of the function $G(x,\xi,t)$ is that, as a function of the argument x, it represents the temperature distribution in the rod $0\le x\le l$ at time t, if at $t=0$ the temperature was zero, and at this moment at point $x=\xi$ a certain amount of heat Q was instantly released, and at the ends of the rod is constantly maintained the temperature is zero.

Example 1

A thin homogeneous rod $0\le x\le l$ is given , the side surface of which is thermally insulated. Find the temperature distribution $u(x,t)$ in the rod if the ends of the rod are maintained at zero temperature, and the initial temperature $u(x,0)=u_0=const$.

Solution:

The problem is reduced to solving the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\ 0<x<l,\ t>0$$

under conditions

$u(x,0)=u_0=\text{const}$,

$u(0,t)=u(l,t)=0$.

Let's calculate:

$$\frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{k\pi}{l}\xi d\xi=\frac{2}{l}\int_0^l u_0\sin\frac{k\pi}{l}\xi d\xi=-\frac{2u_0}{k\pi}\cos\frac{k\pi}{l}\xi\Big|_0^l=$$

$$=-\frac{2u_0}{k\pi}\left((-1)^k-1\right)=\begin{cases}\dfrac{4u_0}{k\pi}, & k=2n+1,\\ 0, & k=2n.\end{cases}$$

Then the solution will take the form (according to the formula (4.5)):

$$u(x,t)=\frac{4u_0}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}e^{-\frac{a^2(2n+1)^2\pi^2}{l^2}t}\sin\frac{(2n+1)\pi}{l}x.$$