

## 23 秋中俄班数学分析(3)期末试题解答

### Problem 1

(5 points)

Find partial derivatives and  $\frac{\partial^2 z}{\partial x \partial y}$  for the implicit function  $z$  defined by the equation

$$xyz = x + y + z.$$

Solution:  $yz + xy \frac{\partial z}{\partial x} = 1 + \frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial x} = \frac{1 - yz}{xy - 1}.$

$$z + y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} + xy \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y},$$

$$\frac{\partial^2 z}{\partial x \partial y} = - \frac{z + y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x}}{xy - 1}.$$

$$xz + xy \frac{\partial z}{\partial y} = 1 + \frac{\partial z}{\partial y}, \quad \frac{\partial z}{\partial y} = \frac{1 - xz}{xy - 1}.$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x \partial y} &= - \frac{z + y \frac{1 - xz}{xy - 1} + x \frac{1 - yz}{xy - 1}}{xy - 1} = \frac{-xyz + z - y + xyz - x + xyz}{(xy - 1)^2} \\ &= \frac{xyz - x - y + z}{(xy - 1)^2} \end{aligned}$$

### Problem 2

(5 points)

Check the differentiability of the function  $f(x, y) = \sqrt[3]{xy}$  at the point  $(0, 0)$ .

Solution A:  $f'_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0.$

Similarly,  $f'_y(0, 0) = 0.$

$$\lim_{\rho \rightarrow 0^+} \frac{f(\Delta x, \Delta y) - f(0, 0) - f'_x(0, 0)\Delta x - f'_y(0, 0)\Delta y}{\rho}$$

$$= \lim_{\rho \rightarrow 0^+} \frac{\sqrt[3]{\Delta x \Delta y}}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \stackrel{\Delta y = \Delta x}{=} \lim_{\Delta x \rightarrow 0} \frac{(\Delta x)^{\frac{2}{3}}}{\sqrt{2}|\Delta x|} = \frac{1}{\sqrt{2}} \lim_{\Delta x \rightarrow 0} \frac{1}{|\Delta x|^{\frac{1}{3}}} = +\infty$$

Hence,  $f(x, y)$  is not differentiable at point  $(0, 0)$ .

Solution B:

$$\lim_{\rho \rightarrow 0^+} \frac{f\left(0 + \rho \frac{\sqrt{2}}{2}, 0 + \rho \frac{\sqrt{2}}{2}\right) - f(0, 0)}{\rho} = \lim_{\rho \rightarrow 0^+} \frac{\sqrt[3]{\frac{1}{2}\rho^2}}{\rho} = \sqrt[3]{\frac{1}{2}} \lim_{\rho \rightarrow 0^+} \frac{1}{\rho^{\frac{1}{3}}} = +\infty$$

, which means that along the direction  $\left\{\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right\}$ , function  $f(x, y)$

doesn't have directional derivative at point  $(0, 0)$ . Hence,  $f(x, y)$  is not differentiable at point  $(0, 0)$ .

Problem 3

(5 points)

Find extremal points of the function

$$f(x, y, z) = x^2 + y^2 + z^2 + 12xy + 2z.$$

Solution:  $df = 2x dx + 2y dy + 2z dz + 12(y dx + x dy) + 2 dz$ ,

$$d^2 f = 2 dx^2 + 2 dy^2 + 2 dz^2 + 12(dx dy + dy dx) = 2 dx^2 + 2 dy^2 + 2 dz^2 + 24 dx dy$$

Set  $\mathbf{A} = \begin{bmatrix} 2 & 12 & 0 \\ 12 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ , then

$$|\mathbf{A} - \lambda \mathbf{E}| = \begin{vmatrix} 2 - \lambda & 12 & 0 \\ 12 & 2 - \lambda & 0 \\ 0 & 0 & 2 - \lambda \end{vmatrix} = [(\lambda - 2)^2 - 12^2] (2 - \lambda) \\ = (\lambda - 14) (\lambda + 10) (2 - \lambda) = 0$$

$\lambda_1 = 14$ ,  $\lambda_2 = -10$ ,  $\lambda_3 = 2$ . Hence  $\mathbf{A}$  is neither positive definite nor negative definite, i.e.,  $f(x, y, z)$  has no extremal point.

#### Problem 4

(5 points)

Calculate the integral

$$\iint_G \frac{dx dy dz}{(x + y)(x + y + z)},$$

where

$$G = \{(x, y, z) \in \mathbb{R}^3 : 1 < x < 2, 1 < x + y < 3, 1 < x + y + z < 4\}.$$

Solution: Set  $\begin{cases} u = x \\ v = x + y \\ w = x + y + z \end{cases}$ , then  $\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1$ ,

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 1, \quad G = \{(u, v, w) | 1 \leq u \leq 2, 1 \leq v \leq 3, 1 \leq w \leq 4\}.$$

$$\iiint_G \frac{dx dy dz}{(x + y)(x + y + z)} = \iiint_G \frac{du dv dw}{vw} = \int_1^2 du \int_1^3 \frac{dv}{v} \int_1^4 \frac{dw}{w} \\ = (2 - 1) (\ln 3 - \ln 1) (\ln 4 - \ln 1) = \ln 3 \cdot \ln 4$$

Problem 5

(5 points)

Find the conditional extremum

$$f(x, y) = x + 2y, \quad x^2 + y^2 = 4.$$

Solution: Set  $F(x, y) = x + 2y + \lambda(x^2 + y^2 - 4)$ , from the system of

$$\text{equations } \begin{cases} F'_x = 1 + 2\lambda x = 0 \\ F'_y = 2 + 2\lambda y = 0 \\ x^2 + y^2 - 4 = 0 \end{cases}, \text{ we obtain solutions } \begin{cases} x = \frac{2}{\sqrt{5}} \\ y = \frac{4}{\sqrt{5}} \\ \lambda = -\frac{\sqrt{5}}{4} \end{cases},$$

$$\begin{cases} x = -\frac{2}{\sqrt{5}} \\ y = -\frac{4}{\sqrt{5}} \\ \lambda = \frac{\sqrt{5}}{4} \end{cases}.$$

$$dF = dx + 2dy + \lambda(2x dx + 2y dy), \quad d^2 f = d^2 F = 2\lambda(dx^2 + dy^2),$$

$$2x dx + 2y dy = 0.$$

$$\text{At point } \left( \frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right),$$

$$d^2 f \Big|_{\left( \frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right)} = 2 \left( -\frac{\sqrt{5}}{4} \right) \left( dx^2 + \frac{1}{4} dx^2 \right) = -\frac{5\sqrt{5}}{8} dx^2 < 0, \text{ hence}$$

$$\left( \frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}} \right) \text{ is the conditional maximum.}$$

$$\text{At point } \left( -\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}} \right),$$

$$d^2 f \Big|_{\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)} = 2 \cdot \frac{\sqrt{5}}{4} \left(dx^2 + \frac{1}{4} dx^2\right) = \frac{5\sqrt{5}}{8} dx^2 > 0, \text{ hence}$$

$$\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right) \text{ is the conditional minimum.}$$

### Problem 6

(5 points)

Calculate the Curve Integral of a first kind  $\int_L (x^2 + y^2 + z^2) dl$ , where  $L = \{(x, y, z) : x = \cos t, y = \sin t, z = t, 0 \leq t \leq 2\pi\}$

Solution:

$$\begin{aligned} & \int_L (x^2 + y^2 + z^2) ds \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2) \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} dt \\ &= \int_0^{2\pi} (1 + t^2) \sqrt{2} dt = \sqrt{2} \left( t + \frac{1}{3} t^3 \right) \Big|_0^{2\pi} = \sqrt{2} \left( 2\pi + \frac{8}{3} \pi^3 \right) \end{aligned}$$

### Problem 7

(5 points)

Calculate the Curve Integral of a second kind  $\int_L (-2xy^2 dx + y^3 dy)$ , where  $L$  - part of the curve  $x^3 + 2x^2 + y^2 = 3$  from point  $A(-1, \sqrt{2})$  to point  $B(1, 0)$

Solution:

$$\begin{aligned} & \int_L (-2xy^2) dx + y^3 dy \\ &= \int_L (-2x) (3 - x^3 - 2x^2) dx + y^3 dy = \int_L (2x^4 + 4x^3 - 6x) dx + \int_L y^3 dy \\ &= \int_{-1}^1 (2x^4 + 4x^3 - 6x) dx + \int_{\sqrt{2}}^0 y^3 dy = 2 \cdot 2 \cdot \frac{1}{5} + \left( \frac{1}{4} y^4 \right) \Big|_{\sqrt{2}}^0 = \frac{4}{5} - 1 = -\frac{1}{5} \end{aligned}$$

### Problem 8

Calculate the Surface Integral of a first kind  $\iint_S (x^2 + y^2 + z) dS$ , where

$S$  - upper hemisphere  $x^2 + y^2 + z^2 = a^2$ ,  $0 \leq z$

Solution: (Solution A)  $S: z = \sqrt{a^2 - x^2 - y^2}$ ,

$$(x, y) \in D_{xy} = \{(x, y) | x^2 + y^2 \leq a^2\}. \quad \frac{\partial z}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}},$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}},$$

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy$$

$$= \sqrt{1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}} dx dy = \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy$$

$$\iint_S (x^2 + y^2 + z) dS$$

$$= \iint_S (x^2 + y^2) dS + \iint_S z dS$$

$$= \iint_{D_{xy}} (x^2 + y^2) \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy + \iint_{D_{xy}} \sqrt{a^2 - x^2 - y^2} \frac{a}{\sqrt{a^2 - x^2 - y^2}} dx dy$$

$$= a \iint_{D_{xy}} \frac{r^2}{\sqrt{a^2 - r^2}} r dr d\theta + a \iint_{D_{xy}} dx dy = a \int_0^{2\pi} d\theta \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} dr + a \cdot \pi a^2$$

$$= 2\pi a \int_0^a \frac{r^3}{\sqrt{a^2 - r^2}} dr + \pi a^3 = 2\pi a \int_0^{\frac{\pi}{2}} \frac{a^3 \sin^3 \theta}{a \cos \theta} a \cos \theta d\theta + \pi a^3$$

$$= 2\pi a^4 \int_0^{\frac{\pi}{2}} \sin^3 \theta d\theta + \pi a^3 = 2\pi a^4 \cdot \frac{2}{3} + \pi a^3 = \frac{4\pi}{3} a^4 + \pi a^3$$

(Solution B)  $S: \rho = a$ ,  $0 \leq \varphi \leq \frac{\pi}{2}$ ,  $0 \leq \theta \leq 2\pi$ .

$$dS = a^2 \sin \varphi d\varphi d\theta.$$

$$\begin{aligned}
& \iint_S (x^2 + y^2 + z) dS \\
&= \iint_S (x^2 + y^2) dS + \iint_S z dS \\
&= \iint_S a^2 \sin^2 \varphi \cdot a^2 \sin \varphi d\varphi d\theta + \iint_S a \cos \varphi \cdot a^2 \sin \varphi d\varphi d\theta \\
&= a^4 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin^3 \varphi d\varphi + a^3 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi \cos \varphi d\varphi \\
&= a^4 \cdot 2\pi \cdot \frac{2}{3} + a^3 \cdot 2\pi \int_0^{\frac{\pi}{2}} \sin \varphi d(\sin \varphi) \\
&= \frac{4\pi}{3} a^4 + a^3 \cdot 2\pi \cdot \left( \frac{1}{2} \sin^2 \varphi \Big|_0^{\frac{\pi}{2}} \right) = \frac{4\pi}{3} a^4 + \pi a^3
\end{aligned}$$

(Solution C)  $\vec{n}^0 = \left\{ \frac{x}{a}, \frac{y}{a}, \frac{z}{a} \right\},$

$$\begin{aligned}
& \iint_S (x^2 + y^2 + z) dS \\
&= a \iint_S \left( x \cdot \frac{x}{a} + y \cdot \frac{y}{a} + \frac{z}{a} \right) dS = a \iint_S x dy dz + y dz dx + dx dy \\
&= a \iint_{D_{xy}} \left[ x \left( -\frac{\partial z}{\partial x} \right) + y \left( -\frac{\partial z}{\partial y} \right) + 1 \right] dx dy \\
&= a \left[ \iint_{D_{xy}} \left( x \cdot \frac{x}{\sqrt{a^2 - x^2 - y^2}} + y \cdot \frac{y}{\sqrt{a^2 - x^2 - y^2}} \right) dx dy + \iint_{D_{xy}} dx dy \right] \\
&= a \iint_{D_{xy}} \frac{x^2 + y^2}{\sqrt{a^2 - x^2 - y^2}} dx dy + a \cdot \pi a^2 = \frac{4\pi}{3} a^4 + \pi a^3
\end{aligned}$$

Or, denote  $S' : z = 0, (x, y) \in D_{xy}$ , downward.

$$\begin{aligned}
& \iint_S (x^2 + y^2 + z) dS \\
&= a \iint_S \left( x \cdot \frac{x}{a} + y \cdot \frac{y}{a} + \frac{z}{a} \right) dS = a \iint_S x dy dz + y dz dx + dx dy \\
&= a \left( \oiint_{S+S'} x dy dz + y dz dx + dx dy - \iint_{S'} x dy dz + y dz dx + dx dy \right) \\
&= a \iiint_{\Omega} (1 + 1 + 0) d\Omega - a(-1) \iint_{D_{xy}} dx dy \\
&= a \cdot 2 \cdot \frac{1}{2} \cdot \frac{4\pi}{3} a^3 + a \cdot \pi a^2 = \frac{4\pi}{3} a^4 + \pi a^3
\end{aligned}$$

### Problem 9

(5 points)

Calculate the Integral  $\int_L (x^2 + y) dx + (y^2 + z) dy + (z^2 + x) dz$ , where L - ellipse  $x^2 + y^2 = 4$ ,  $x + z = 2$  positive oriented on the upper side of the plane

Solution: Set  $\Sigma: z = 2 - x$ ,  $(x, y) \in D_{xy} = \{(x, y) | x^2 + y^2 \leq 4\}$ ,

upward.

$$\begin{aligned}
& \oint_L (x^2 + y) dx + (y^2 + z) dy + (z^2 + x) dz \\
&= \iint_{\Sigma} \begin{vmatrix} dy dz & dz dx & dx dy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y & y^2 + z & z^2 + x \end{vmatrix} = \iint_{\Sigma} (-1) dy dz + (-1) dz dx + (-1) dx dy \\
&= \iint_{D_{xy}} \left[ (-1) \left( -\frac{\partial z}{\partial x} \right) + (-1) \left( -\frac{\partial z}{\partial y} \right) + (-1) \right] dx dy \\
&= \iint_{D_{xy}} [(-1) \cdot 1 + (-1) \cdot 0 + (-1)] dx dy = \iint_{D_{xy}} (-2) dx dy = (-2) \cdot 4\pi = -8\pi
\end{aligned}$$



## Problem 10

(5 points)

Formulate the Next Theorems and Definitions:

(a) Gamma Function Definition:

We recall that the gamma function is defined for  $x > 0$  by the formula

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (1)$$

(b) Weierstrass approximation Theorem:

Let  $f \in C(\mathbb{R}^m)$ . Then for any  $R > 0$  and  $\varepsilon > 0$  there exists a polynomial  $P$  in  $m$  variables such that

$$|f(x) - P(x)| < \varepsilon \quad \text{for all } x \text{ in } \bar{B}(R).$$

(c) The first definition of a smooth manifold

A set  $M, M \subset \mathbb{R}^m$ , is called a *k-dimensional manifold of class  $C^r$*  if every point  $p$  from  $M$  has a neighborhood  $U$  such that the intersection  $U \cap M$  is a simple  $k$ -dimensional manifold of class  $C^r$ . Its parametrization is called a local parametrization of the manifold  $M$  in the vicinity of the point  $p$ .

(d) Definition of potential vector field

A vector field  $V = (V_1, \dots, V_m)$  defined in a domain  $\mathcal{O}$  is called potential if there exists a smooth function  $F$  on  $\mathcal{O}$  (a potential of  $V$ ) such that  $V(x) = \text{grad } F(x)$  for all points  $x \in \mathcal{O}$ .

(e) The Gauss-Ostrogradski formula

Let  $f$  be a function smooth on a standard compact set  $K \subset \mathbb{R}^m$ .

Then for every unit vector  $e \in \mathbb{R}^m$ , one has

$$\int_K \frac{\partial f}{\partial e}(x) dx = \int_{\partial K} f(x) \langle v(x), e \rangle d\sigma(x).$$