Differential Geometry. Home Assignments

Assignment 6

Problem 1

Find the geodesic curves on the surface of the right circular cone defined by the equation $x^2 + y^2 = z^2$ in three-dimensional Euclidean space.

Problem 2

Consider a surface with the linear element given by

$$ds^2 = v(du^2 + dv^2),$$

where (u, v) are Cartesian coordinates on the parametric plane. Show that the geodesic curves on this surface correspond to parabolas in the u-v plane.

Problem 3

Find the geodesic curvature of the curve $u = \sinh v$, where $0 \leq v \leq v_0$, on the helicoid given by the parametric representation:

$$\boldsymbol{r}(u, v) = (u \cos v, u \sin v, v).$$

HW6. Week 16th.

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Problem 1

Find the geodesic curves on the surface of the right circular cone defined by the equation $x^2 + y^2 = z^2$ in three-dimensional Euclidean space.

Sol: apply the parametrization $\begin{cases} x = \frac{a\cos\theta}{\sqrt{2}} \\ y = \frac{a\sin\theta}{\sqrt{2}} \end{cases}$. $\vec{r}(a,\theta) = \left(\frac{a\cos\theta}{\sqrt{2}}, \frac{a\sin\theta}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$. $a>0$.

$\vec{r}_a = \left(\frac{1}{\sqrt{2}}\cos\theta, \frac{1}{\sqrt{2}}\sin\theta, \frac{1}{\sqrt{2}}\right)$ $E = 1$. $F = 0$ $G = \frac{a^2}{2}$

$\vec{r}_\theta = \left(-\frac{a}{\sqrt{2}}\sin\theta, \frac{a}{\sqrt{2}}\cos\theta, 0\right)$.

Christoffel symbol:

$\vec{r}_{aa} = (0,0,0)$ $\Gamma^1_{11} = 0$ $\Gamma^2_{11} = 0$ $A = -\frac{a}{2}(\theta')^2$

$\vec{r}_{a\theta} = \left(-\frac{1}{\sqrt{2}}\sin\theta, \frac{1}{\sqrt{2}}\cos\theta, 0\right)$ $\Gamma^1_{12} = 0$ $\Gamma^2_{12} = \frac{1}{a}$ $B = \frac{2}{a}(\theta' a')$

$\vec{r}_{\theta\theta} = \left(-\frac{a}{\sqrt{2}}\cos\theta, -\frac{a}{\sqrt{2}}\sin\theta, 0\right)$ $\Gamma^1_{22} = -\frac{a}{2}$ $\Gamma^2_{22} = 0$

the differential equation of geodesic curve is $\begin{cases} a'' + A = 0 \\ \theta'' + B = 0 \end{cases}$

that is. $\begin{cases} a'' - \frac{a}{2}(\theta')^2 = 0 & (1) \\ \theta'' + \frac{2}{a}(\theta' a') = 0. & (2) \end{cases}$

(2) $\Rightarrow$ $(\ln\theta')' + 2(\ln a)' = 0$

$\Rightarrow$ $\ln\theta' + 2\ln a = c$. $\Rightarrow$ $\theta' = \frac{c}{a^2}$

then (1) $\Rightarrow$ $a'' - \frac{a}{2}\cdot\frac{c^2}{a^4}$ $\Rightarrow$ $a'' - \frac{k^2}{2a^3} = 0$

(2) $\Rightarrow$ $2a'a'' - a'\cdot\frac{k^2}{a^3} = 0 \Rightarrow (a')^2 + \frac{k^2}{2a^2} = C_1$.

thus. $\frac{da}{d\theta} = \frac{\sqrt{C_1 - \frac{k^2}{2a^2}}}{\frac{c}{a^2}}$ $\Rightarrow$ $a^2\cos^2\left(\frac{\theta - C_2}{\sqrt{2}}\right) = \frac{k^2}{2C_1}$

Problem 2

Consider a surface with the linear element given by

$$ds^2 = v(du^2 + dv^2),$$

where (u, v) are Cartesian coordinates on the parametric plane. Show that the geodesic curves on this surface correspond to parabolas in the u-v plane.

Sol: we have $E = G = v$. and $1 = v\cdot\left[\left(\frac{du}{ds}\right)^2 + \left(\frac{dv}{ds}\right)^2\right]$

Christoffel symbol: $\Gamma^2_{11} = -\frac{1}{2v}$ $\Gamma^1_{12} = \frac{1}{2v}$ $\Gamma^2_{22} = \frac{1}{2v}$ others $= 0$

$A = \frac{1}{v}u'v'$ $B = -\frac{1}{2v}u'^2 + \frac{1}{2v}v'^2$ $\begin{cases} A + u'' = 0 \\ B + v'' = 0 \end{cases}$ $\Rightarrow$ $\begin{cases} u''v + u'v' = 0 & (1) \\ 2vv'' = u'^2 - v'^2 & (2) \end{cases}$

(1) $\Rightarrow$ $(u'v)' = 0$ $\Rightarrow$ $u'v = \text{const}$. let $u' = \frac{k}{v}$. (2) $\Rightarrow$ $v'' - \frac{v'^2}{2v} + \frac{k^2}{2v^3} = 0$

let. $w = v'$. $\Rightarrow 2v\, w\cdot\frac{dw}{dv} = \frac{k^2}{v^2} - w^2$

then $r = w^2 \Rightarrow$ $v\cdot\frac{dr}{dv} + r = \frac{k^2}{v^2}$ $\Rightarrow$ $\frac{d(rv)}{dv} = \frac{k^2}{v^2}$

$\Rightarrow vr + \frac{k^2}{v} = C_1$. $\Rightarrow$ $v' = \pm\frac{\sqrt{C_1 v - k^2}}{v}$

$\Rightarrow$ $t + C_2 = \pm\frac{2}{C_1}\left[\frac{1}{3}(C_1 v - k^2)^{3/2} + k^2(C_1 v - k^2)^{1/2}\right]$

then apply to u. $\frac{du}{dv} = \frac{u'}{v'} = \pm\frac{k}{\sqrt{C_1 v - k^2}}$

$\Rightarrow u + C_3 = \pm\frac{2k}{C_1}\sqrt{C_1 v - k^2}$ $\Rightarrow$ $(u - C_3)^2 = \frac{4k^2}{C_1^2}v - \frac{4k^4}{C_1^2}$

after simplify those const. we have $(u - u_0)^2 = \tilde{k}(v - v_0)$.

which are parabolas

Problem 3

Find the geodesic curvature of the curve $u = \sinh v$, where $0 \le v \le v_0$, on the helicoid given by the parametric representation:

$$\boldsymbol{r}(u, v) = (u \cos v, u \sin v, v).$$

Sol: $\vec{r}_u = (\cos v, \sin v, 0)$ $\qquad E = 1 \quad F = 0 \quad G = u^2+1$

$\vec{r}_v = (-u\sin v, u\cos v, 1)$

for the curve $u = \sinh v$. $\quad u' = \cosh v. \quad u'' = \sinh v.$

$v' = 1 \quad v'' = 0.$

$$\kappa = \frac{\sqrt{G}}{(u'^2 + G v'^2)^{3/2}} \left(u''v' - v''u' - \frac{1}{2} G_u v'^3 - \frac{1}{2} G_v G^{-1} \cdot u'v'^2 - \frac{G_u}{G} u'^2 v' \right). = \frac{\sqrt{\sinh^2 v + 1}}{(\cosh^2 v + \sinh^2 v + 1)^{3/2}} \left(u'' - u - \frac{2u}{u^2+1} \cdot u'^2 \right).$$

$$= \frac{\cosh v}{|2\sqrt{2}\cosh^3 v|} \left(\sinh v - \sinh v - \frac{2\sinh v}{\cosh^2 v} \cosh^2 v \right) = \frac{\sinh v}{\sqrt{2}\cosh^2 v}. \qquad v \in [0, v_0].$$