Complex analysis.

Lecturer Aleksei Savelev, spring 2024.

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5 Homotopy. Cauchy theorem.

In this part we let D be a domain in $\mathbb{R}^2$ or in $\mathbb{C}$ and for simplicity assume that all paths are defined on a segment $I = [0, 1]$ (see property **L3**).

Definition 5.1. *Two paths $\gamma_0, \gamma_1 : I \to D$ with common endpoints*

$$\gamma_0(0) = \gamma_1(0) = \alpha, \quad \gamma_0(1) = \gamma_1(1) = \alpha$$

are homotopic in domain D as paths with common (or fixed) endpoints if there exists a map $\Gamma \in C(I \times I \to D)$ such that

1. $\Gamma(0, t) = \gamma_0(t)$ *and* $\Gamma(1, t) = \gamma_1(t)$ *for every* $t \in I$;
2. $\Gamma(s, 0) = \alpha$ *and* $\Gamma(s, 1) = \alpha$ *for every* $s \in I$;

Definition 5.2. *Two closed paths $\gamma_0, \gamma_1 : I \to D$ are homotopic in domain D as closed paths if there exists a map $\Gamma \in C(I \times I \to D)$ such that*

1. $\Gamma(0, t) = \gamma_0(t)$ *and* $\Gamma(1, t) = \gamma_1(t)$ *for every* $t \in I$;
2. $\Gamma(s, 0) = \Gamma(s, 1)$ *for every* $s \in I$;

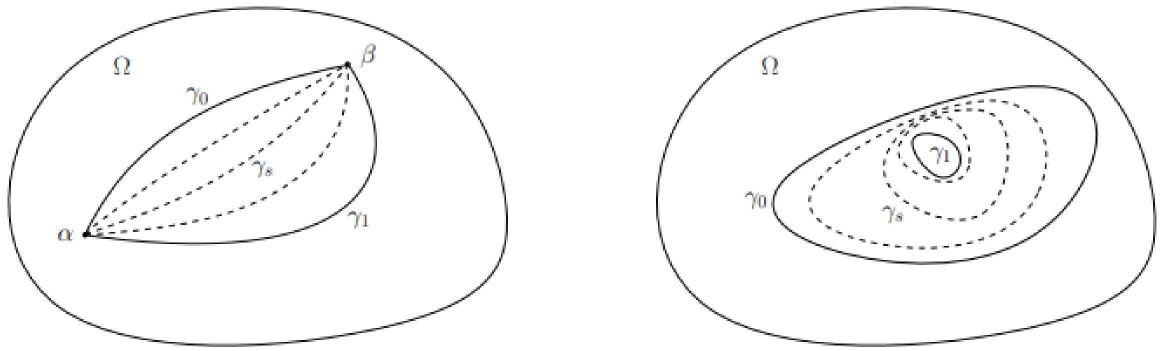

Figure 1: Left: homotopy of curves with same endpoints. Right: homotopy of closed curves.

Remark 5.3. *In both cases map Γ is called a homotopy of paths γ_0 and γ_1. An intermediate path is denoted by $\gamma_s(\cdot) = \Gamma(s, \cdot)$.*

Remark 5.4. *Homotopy is equivalence relation. Equivalent (in sense of reparametrization) paths are homotopic.*

Definition 5.5. *A path $\gamma : I \to \mathbb{C}$ is a constant path if $\gamma(t)$ is constant, $\gamma(t) = \gamma(0)$ for every $t \in I$.*

A closed path is contractible if it is homotopic to a constant path, i.e. exists a map $\Gamma \in C(I \times I \to D)$ and a point $z_0 \in D$ such that

1. $\Gamma(0, t) = \gamma_0(t)$ *and* $\Gamma(1, t) = z_0$ *for every* $t \in I$;
2. $\Gamma(s, 0) = \Gamma(s, 1)$ *for every* $s \in I$;

Definition 5.6. *A domain D is simply connected if every closed path in D is contractible.*

Definition 5.7. *A domain D is star-shaped if there exists a point $z \in D$ such that for every $w \in D$ a segment that connects z and w is* *contained in D, i.e.*

$$\exists z \in D : tw + (1-t)z \in D \text{ for every } z \in D \text{ and } t \in [0,1].$$

Example 5.1. *Every star-shaped domain (in particular, a disk) is simply connected. Every convex domain is star-shaped and, consequently, simply connected.*

Example 5.2. *Let $0 \le r < R \le \infty$, $z_0 \in \mathbb{C}$. An annulus is a set*

$$K_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R.$$

Numbers r, R are inner and outer radii and z_0 is a center of the annulus $K_{r,R}(z_0)$.

Annulus is not simply connected.

Lemma 5.8. *In a simply connected domain any two paths with common endpoints are homotopic.*

Remark. Let D be a bounded domain. TFAE

1. D is simply connected;
2. ∂D is connected;
3. D^c is connected;

Theorem 5.9 (Cauchy's theorem on homotopy.)**.** *Let f be holomorphic in domain D and γ_0, γ_1 be two paths homotopic in D. Then*

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz.$$

Proof. Let

$$\gamma_s(t) = \Gamma(s,t) : I \to D$$

is a homotopy of paths γ_0 and γ_1. Let

$$J(s) := \int_{\gamma_s} f dz \quad \text{for } s \in I.$$

To prove that $J(1) = J(0)$ is enough to show that $J(s)$ is locally constant on I, that is every point $s_0 \in I$ has a neighborhood $v = v(s_0) \subset I$ such that $J(s) = J(s_0)$ for every $s \in v$.

Let $\Phi : I \to \mathbb{C}$ be an arbitrary antiderivative of function f along path γ_{s_0}. Consider partition of a segment I by the points

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

into segments $I_j = [t_{j-1}, t_j]$ such that there exist

1. disks $U_j \subset D$ such that $\gamma_{s_0}(I_j) \subset U_j$;
2. antiderivatives $F_j \in \mathcal{O}(U_j)$ of functions f in U_j such that

$$\Phi = F_j \circ \gamma_{s_0} \quad \text{on } I_j \quad \text{for every } j = 1, \ldots, n.$$

In particular, the second condition implies that $F_j \equiv F_{j-1}$ on $U_j \cap U_{j-1}$. Moreover, the uniform continuity of $\Gamma(s,t)$ on $I \times I$ implies that there exists a neighborhood $v \subset I$ of s_0 such that $\gamma(v \times I_j) \subset U_j$ for every j.

Consider a family of functions $\Phi_s : I \to \mathbb{C}$ of a variable t letting

$$\Phi_s := F_j \circ \gamma_s \quad \text{on } I_j \quad \text{for } j = 1, \ldots, n.$$

Then for every $s \in v$ function Φ_s is continuous on I and coincides with $F(\gamma_s(t))$ in some neighborhood $t_0 \in I$ for some antiderivative F

of function f in the neighborhood of $\gamma(t_0)$ (recall that $F_j \equiv F_{j-1}$ on $U_j \cap U_{j-1}$!. Thus Φ_s is an antiderivative of f along γ_s.

By the Newton-Leibniz formula (or by the definition of $\int_{\gamma_s} f dz$ for continuous paths γ_s) we see that

$$J(s) := \int_{\gamma_s} f dz = \Phi_s(1) - \Phi_s(0).$$

We will prove that this function doesn't depend on $s \in v$ which will finalize the proof of the Theorem.

Consider cases of closed paths and paths with common endpoints independently.

1. Assume that γ_0 and γ_1 are homotopic as paths with common endpoints (s.t. $\gamma_s(0) = a$ and $\gamma_s(1) = b$ for every $s \in I$). Then values

$$\Phi_s(0) = F_1(\gamma_s(0)) = F_1(a) \quad \text{and} \quad \Phi_s(1) = F_n(\gamma_s(1)) = F_n(b)$$

do not depend on $s \in v$. Consequently, their difference $J(s)$ also doesn't depend on $s \in v$.

2. Assume that γ_0 and γ_1 are homotopic as closed paths (s.t. $\gamma_s(0) = \gamma_s(1)$ for every $s \in I$), then functions (that do not depend on s) F_1 and F_n as two antiderivatives of f in the neighborhood $U_1 \cap U_n$ of a point $z_s := \gamma_s(0) = \gamma_s(1)$ differ by a constant (that doesn't depend on s)

$$F_n(z) - F_1(z) = C \quad \text{for every} \quad z \in U_1 \cap U_n.$$

Hence,

$$J(s) = F_n(\gamma_s(1)) - F_1(\gamma_s(0)) = F_n(z_s) - F_1(z_s) = C$$

doesn't depend on $s \in v$. $\square$

Corollary 5.9.1 (Cauchy-Goursat's theorem for a contractible path)**.** *Let f be holomorphic in D and $\gamma : I \to D$ be contractible. Then*

$$\int_\gamma f dz = 0.$$

In particular, in the simply connected domain D the integral of function $f \in H(D)$ along every closed path $\gamma : I \to D$ is equal to zero.

The proof follows from the theorem on homotopy since the integral over the constant path is always zero.

Corollary 5.9.2. *Let $D \subset \mathbb{C}$ be simply connected.Then every function f holomorphic in D has antiderivative.*

Proof. Let $a \in D$. for every $z \in D$ consider a piecewise smooth path $\gamma : I \to D$ that connects a with z and let

$$F(z) := \int_\gamma f(\zeta) d\zeta.$$

The value $F(z)$ doesn't depend on γ. Indeed, if γ_1, γ_2 are two such paths then the integral of f along the closed path $\gamma_1 \cup \gamma_2^{-1}$ is equal to zero by the previous corollary

$$\int_{\gamma_1} f(\zeta) d\zeta - \int_{\gamma_2} f(\zeta) d\zeta = 0$$

In particular, if $z_0 \in D$ and U is a disk centered at z_0 contained in D then for $z \in U$ a function $F(z)$ can be written in the following form

$$F(z) = \int_{\gamma_0} f(\zeta) d\zeta + \int_{z_0}^{z} f(\zeta) d\zeta = F(z_0) + \int_{z_0}^{z} f(\zeta) d\zeta,$$

where the integral $\int_{z_0}^{z} f(\zeta)d\zeta$ is taken over the segment that connects z_0 and z and γ_0 is any path that connects a and z_0. Hence F is differentiable in U and

$$F'(z) = f(z) \quad \text{for every} \quad z \in U.$$

Since z_0 is arbitrary this implies that F is the antiderivative of function f in domain D. $\square$

5.1 Cauchy's theorem for multiple connected domains.

Recall that the bounded domain $D \subset \mathbb{C}$ is a domain with a simple boundary if its boundary is a union of a finite number of nonintersecting piecewise smooth simple closed curves $\gamma_0, \gamma_1, \ldots, \gamma_n$, where γ_0 denotes the outer boundary of domain D, and $\gamma_1, \ldots, \gamma_n$ are inner components of ∂D (see. Figure 15). For function f defined on ∂D the integral along the boundary is defined as follows

$$\int_{\partial D} f dz = \int_{\gamma_0} f dz + \sum_{j=1}^{n} \int_{\gamma_j} f dz$$

Theorem 5.10 (Cauchy-Goursat's theorem for multiple connected domain)**.** *Suppose $D \subset \mathbb{C}$ is a bounded domain with simple boundary, f is a holomorphic function in some domain $G \supset \overline{D}$. Then*

$$\int_{\partial D} f dz = 0.$$

Proof. Consider in domain D a finite number of "slits" $\lambda_1, \ldots, \lambda_n$ such that λ_k connects a point on curve γ_{k-1} with a point on γ_k and denote by

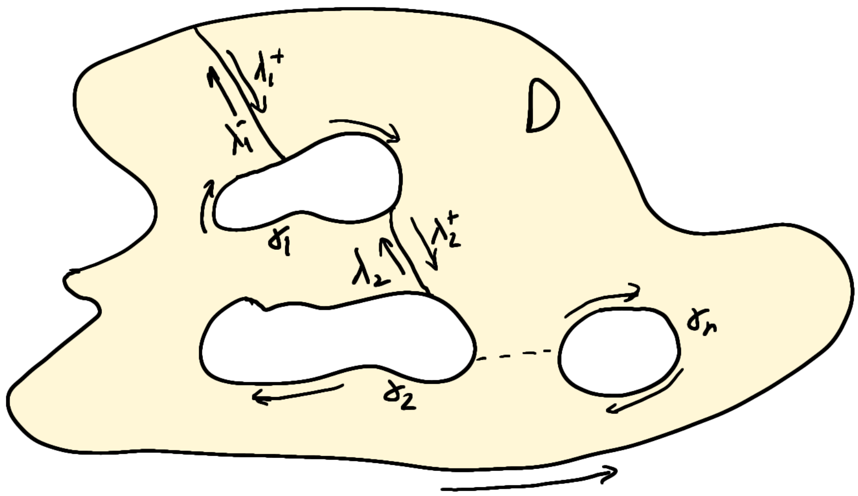

Figure 2: Multiple connected domain.

λ_k^+ the path oriented from γ_{k-1} to γ_k and by λ_k^- the opposite path. We can choose slits such that the closed path Γ composed of arcs of boundary ∂D and paths $\lambda_k^{\pm}$ is contractible in G.

Then, by Cauchy-Goursat's theorem we see that

$$0 = \int_\Gamma f dz = \int_{\partial D} f dz + \sum_{j=1}^{n} \int_{\lambda_j^+} f dz + \sum_{j=1}^{n} \int_{\lambda_j^-} f dz == \int_{\partial D} f dz.$$

Since

$$\int_{\lambda_j^+} f dz = -\int_{\lambda_j^-} f dz.$$

□

5.2 Cauchy's Integral Formula

Theorem 5.11 (Cauchy's Integral Formula). *Let $D \subset \mathbb{C}$ be an open set and let $f \in H(D)$. Let G be a domain with oriented boundary γ*

such that $\overline{G} \subset D$*. Then for any* $z \in G$ *we have*

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi.$$

Integral in the right hand side of this formula is called Cauchy integral.

Proof. Let $z \in D$ and concider a circle

$$U_r := \{\xi \in \mathbb{C} : |\xi - z| < r\}.$$

Then $\overline{U}_r \subset D$ if $r > 0$ is small enough.

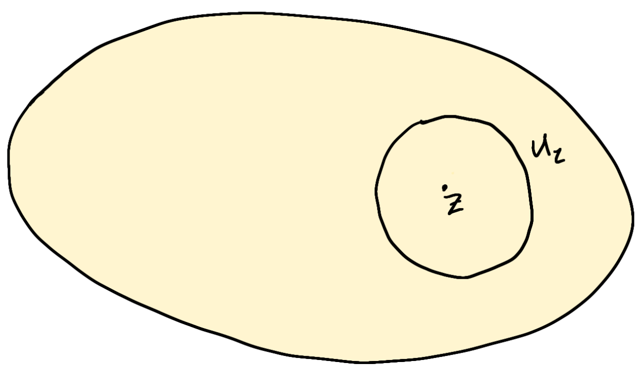

Figure 3: Domain D with a disk U_r of z.

Applying Cauchy theorem to the domain $D_r := D \backslash \overline{U}_r$ and the function

$$g(\zeta) = \frac{f(\zeta)}{\zeta - z}$$

that is holomorphic in this domain

$$\int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial U_r} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1}$$

In particular, the integral in the right-hand side doesn't depend on r. We will show that it is equal to $2\pi i f(z)$. Indeed,

$$2\pi i f(z) - \int_{\partial U_r} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial U_r} \frac{f(z) - f(\zeta)}{\zeta - z} d\zeta, \tag{2}$$

where we applied the identity

$$\int_{\partial U_r} \frac{d\zeta}{\zeta - z} = 2\pi i.$$

We will prove that the right hand side of (2) is equal to 0. Indeed,

$$\begin{aligned} \left| \int_{\partial U_r} \frac{f(z) - f(\zeta)}{\zeta - z} d\zeta \right| &\leqslant \max_{\zeta \in \partial U_r} \frac{|f(\zeta) - f(z)|}{r} \cdot 2\pi r \\ &\leqslant 2\pi \max_{\zeta \in U_r} |f(\zeta) - f(z)| \to 0, \ r \to 0, \end{aligned}$$

since f is uniformly continuous.

Since the left-hand side of (2) doesn't depend on r it is equal to 0. Consequently, the left-hand side of (1) is equal to $2\pi i f(z)$. □

Remark 1. If in terms of the theorem $z \notin \bar{G}$ then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = 0.$$

Remark 2. The Cauchy integral formula expresses the function holomorphic in the closure of domain by its values on the boundary. In particular, the holomorphic function is completely determined by its values on the boundary.

Theorem 5.12 (Mean value theorem for holomorphic function)**.** *Let $f \in H(D)$. The value of a function f that is D at any point $a \in D$ is equal to the mean value of this function over any circle*

$$U_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$$

centered at a and compactly supported in D

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta$$

Proof. By the integral formula for the disk $U_r(a) \subset D$ we see that

$$f(a) = \frac{1}{2\pi i} \int_{\partial U_r(a)} \frac{f(\zeta) d\zeta}{\zeta - a}.$$

Applying the parametrization

$$\zeta = a + re^{i\theta}, \quad d\zeta = ire^{i\theta} d\theta,$$

we prove the assertion of the theorem. □

6 Taylor series

6.1 Reminder

Recall some definitions and properties from the theory on numerical and functional series that can be applied to the complex case.

Definition 6.1. *1. The series $\sum_{n=1}^{\infty} a_n$ with complex terms converges to a sum $s \in \mathbb{C}$ if a sequence of partial sums $s_n = \sum_{j=1}^{n} a_j$ conveges* *to s*

$$\lim_{n\to\infty}\left|s-\sum_{j=1}^{n}a_j\right|=0$$

2. *The series* $\sum\limits_{n=1}^{\infty} f_n(z)$ *of complex-valued functions* $f_n : K \to \mathbb{C}$ *defined on a set* $K \subset \overline{\mathbb{C}}$ *converges to a function* $f : K \to \mathbb{C}$ *uniformly on* K *if the sequence of partial sums converges uniformly on*

$$\lim_{n\to\infty}||f-\sum_{j=1}^{n}f_j||_K=0, \quad ||\varphi||_K:=\sup_{z\in K}|\varphi(z)|$$

As in the real case we can prove the following properties of uniformly convergent series

1. Integration of the uniformly convergent series. Let $\gamma : I \to \mathbb{C}$ be a piecewise smooth path, $f_n : \gamma(I) \to \mathbb{C}$ be continuous and the series $\sum\limits_{n=1}^{\infty} f_n(z)$ converges uniformly on $\gamma(I)$. Hence its sum $f(z)$ is also continuous on $\gamma(I)$ and

$$\int_\gamma f dz=\lim_{n\to\infty}\int_\gamma f_n dz$$

2. Weierstrass criterion for uniform convergence Consider a series $\sum\limits_{n=1}^{\infty} f_n(z)$ of complex-valued functions $f_n : K \to \mathbb{C}$ which are defined on a set $K \subset \overline{\mathbb{C}}$. Assume that

$$\|f_n\|_K \leqslant c_n$$

such that a series $\sum\limits_{n=1}^{\infty} c_n$ is convergent. Then $\sum\limits_{n=1}^{\infty} f_n(z)$ is uniformly convergent on K.

6.2 Decomposition of a holomorphic function into a Taylor series.

Theorem 6.2. *Assume that a function f is holomorphic in domain $D \subset \mathbb{C}$ and $U_R(a) = \{z \in \mathbb{C} : |z - a| \leq R\}$ is a subset of D. Let*

$$c_n := \frac{1}{2\pi i} \int_{|\zeta - a| = r} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}}, \quad n = 0, 1, 2, \ldots, \quad 0 < r < R.$$

Then numbers c_n do not depend on r and are call Taylor coefficients of function f at point a. The power series

$$\sum_{n=0}^{\infty} c_n (z - a)^n$$

is called a Taylor series of function f centered at a. It converges for every $z \in U_R(a)$ and its sum is equal to $f(z)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \quad |z - a| < R. \tag{3}$$

Proof. Independence of c_n of the choice of r follows from the Cauchy theorem on homotopy since every two circles

$$\{|\zeta - a| = r_1\} \quad \text{and} \quad \{|\zeta - a| = r_2\} \quad \text{with } 0 < r_1 < r_2 < R$$

are homotopic in D as closed path.

To prove the convergence of a Taylor series and of the equality (3) we fix a point $z \in U_R(a)$ and a number $0 < r < R$ such that $|z-a| < r < R$. Then by the Cauchy integral formula we see that

$$f(z) = \frac{1}{2\pi i} \int_{\partial U_{r(a)}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Since $|z-a| < r = |\zeta - a|$ for every $\zeta \in \partial U_r(a)$ we can decompose the integrand into a geometric series

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - a) - (z - a)} = \frac{f(\zeta)}{\zeta - a} \cdot \frac{1}{1 - \frac{z-a}{\zeta - a}} = \sum_{n=0}^{\infty} \frac{(z-a)^n f(\zeta)}{(\zeta - a)^{n+1}}. \quad (4)$$

An absolute value of the $n-$th term of this series can be estimated as following

$$\left| \frac{(z-a)^n f(\zeta)}{(\zeta - a)^{n+1}} \right| \leqslant \frac{M(r)}{r} \left(\frac{|z-a|}{r} \right)^n, \text{ where } M(r) := \max_{|\zeta - a| = r} |f(\zeta)|$$

Consequently, by the Weierstrass criterion this series converges uniformly with respect to $\zeta \in \partial U_r(a)$. Consequently, the equation (4) may be integrated along $\partial U_r(a)$. Dividing both parts by $2\pi i$ we obtain equation (3). □

6.3 Cauchy inequalities

Theorem 6.3. *In terms of the previous theorem for $0 < r < R$ and $n = 0, 1, 2, \ldots$ the following estimates are satisfied*

$$|c_n| \leqslant \frac{M(r)}{r^n}, \qquad \textit{where } M(r) := \max_{|\zeta - a| = r} |f(\zeta)|.$$

Proof.

$$|c_n| = \left|\frac{1}{2\pi i}\int_{\partial U_r(a)} \frac{f(\zeta)}{(\zeta - a)^{n+1}} d\zeta\right| \leqslant \frac{1}{2\pi} \cdot \frac{M(r)}{r^{n+1}} \cdot 2\pi r = \frac{M(r)}{r^n}.$$

□

6.4 Liuville theorem

Theorem 6.4. *Assume that f is holomorphic and bounded in $\mathbb{C}$, that is there exists $M > 0$ such that*

$$|f(z)| \leq M, \quad z \in \mathbb{C}.$$

Then f is constant.

Proof. Let c_n be Taylor's coefficients of function f at $a = 0$. Then

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{C}.$$

Then by Cauchy inequalities

$$|c_n| \leqslant \frac{M}{r^n}$$

for every $r > 0$ and $n = 0, 1, 2, \ldots$. Leting $r \to \infty$ we see that $c_n = 0$ for $n = 1, 2, \ldots$. Consequently, $f(z) = c_0$. □

6.5 The fundamental theorem of algebra.

Theorem 6.5. *Every polynomial of degree $n \geq 1$ has n roots.*

Proof. Assume that $P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ does not have a root. Then $g(z) := 1/P(z)$ is an entire function. Furthermore, g is bounded since

$$\lim_{|z| \to \infty} \frac{|P(z)|}{|z|^n} = |a_n| \Rightarrow \lim_{|z| \to \infty} \frac{1}{P(z)} = 0$$

By Liouville's theorem, $1/P(z)$ must be a constant equal to zero, which is not possible. Hence, P has at least one root α, and we can write

$$P(z) = (z - \alpha) Q(z).$$

Repeating the steps for Q, we find that P must eventually have n roots. $\square$

6.6 A set of convergence of a power series

Definition 6.6. *Let $\{b_n\}$ be a complex sequence, $a \in \mathbb{C}$. Consider a power series*

$$\sum_{n=0}^{\infty} b_n (z - a)^n.$$

A value $R = [0, +\infty]$

$$R = \left(\overline{\lim} \sqrt[n]{|b_n|} \right)^{-1} \tag{5}$$

is called a radius of convergence and the disk

$$U_R(a)\{z \in \mathbb{C} : |z - a| < R\}$$

is called a disk of convergence of this power series. Formula (5) is called Cauchy-Hadamart's formula.

Theorem 6.7. *A power series*

$$\sum_{n=0}^{\infty} b_n(z-a)^n$$

converges for every $z \in U_R(\ a)$ *moreover this convergence is uniform on every compact subset* $K \subset U_R(a)$. *This series diverges for every* $z \in \mathbb{C}\backslash\overline{U_R(a)}$.

Proof. Assume that $0 < R < \infty$ (consider cases $R = 0, \infty$ by yourself). Then the definition R is equivalent to

$$\overline{\lim_{n\to\infty}} \left(|b_n|\, R^n\right)^{1/n} = 1. \tag{6}$$

Convergence. Let $z \in U_R(a)$ that is $|z-a| < R$. Then by (6) for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that

$$|b_n|\, R^n < (1+\varepsilon)^n \quad \text{for } n \geqslant n_0.$$

Let ε be such that

$$\frac{|z-a|}{R}(1+\varepsilon) =: q < 1.$$

Then

$$|b_n(z-a)^n| < \frac{(1+\varepsilon)^n}{R^n}|z-a|^n = q^n \quad \text{for } n \geqslant n_0.$$

Consequently,

$$\sum_{n=0}^{\infty} b_n(z-a)^n$$

converges by Weierstrass criterion.

Let $K \subset U_R(a)$ be compact. Then

$$\max_{z \in K} |z-a| =: r < R.$$

Let $\varepsilon > 0$ be such that

$$\frac{r}{R}(1+\varepsilon) =: q < 1.$$

Consequently,

$$|b_n(z-a)^n| < q^n \quad \text{for every} \quad z \in K \quad \text{and } n \geqslant n_0.$$

Consequently, the series converges uniformly on K by the Weierstrass criterion for uniform convergence.

Divergence. Let $z \in \mathbb{C} \setminus \overline{U_R(a)}$, that is $|z-a| > R$. Then by (6) and definition of upper limit for every $\varepsilon > 0$ there exists a sequence $n_k \to \infty$ such that

$$|b_n|\, R^n > (1-\varepsilon)^n \quad \text{for every } n = n_k.$$

Let $\varepsilon > 0$ be such that

$$\frac{|z-a|}{R}(1-\varepsilon) =: q > 1.$$

Then for every $n = n_k$

$$|b_n(z-a)^n| > \frac{(1-\varepsilon)^n}{R^n}|z-a|^n = q^n \to \infty,$$

and the series

$$\sum_{n=0}^{\infty} b_n(z-a)^n$$

diverges since its terms do not tend to 0. $\square$

Corollary 6.7.1 (Uniqueness of Taylor's series)**.** *Assume that f is holomorphic in $U = \{z : |z-a| < r\}$ and is defined by a power series*

$$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n. \tag{7}$$

Then it coincides with Taylor series of function f.

Proof. By assumption of the theorem the disk $U := \{|z-a| < r\}$ is contained in a disk of convergence of the series (7). Consequently it converges on the compacts of U. Hence, for every fixed $k = 0, 1, \dots$ and $\rho \in (0, r)$ the series

$$\frac{1}{(z-a)^{k+1}} \sum_{n=0}^{\infty} b_n (z-a)^n = \frac{f(z)}{(z-a)^{k+1}}$$

converges uniformly on the circle $\{|z-a| = \rho\}$. Then the integration of the series along $\{|z-a| = \rho\}$ we obtain

$$2\pi b_k = 2\pi c_k,$$

where c_k is the $k-$th Taylor coefficient of function f. □

6.7 Holomorphity of a sum of power series

Theorem 6.8. *The sum of a power series*

$$f(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$$

is holomorphic in its disk of convergence. Moreover

$$f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} b_n z^{n-m}.$$

Proof. Let R be radius of convergence, $r < R$ and $z, w \in B(0, r)$. Then

$$\frac{f(w) - f(z)}{w - z} = \sum_{k=1}^{\infty} c_k \frac{w^k - z^k}{w - z} = \sum_{k=1}^{\infty} c_k (z^{k-1} + z^{k-2} w + \ldots + z w^{k-2} + w^{k-1}). \tag{8}$$

Here we see that

$$\left| c_k (z^{k-1} + z^{k-2} w + \ldots + z w^{k-2} + w^{k-1}) \right| \leq k \, |c_k| \, r^{k-1}$$

and a series $\sum_{k=1}^{\infty} k \, |c_k| \, r^{k-1}$ converges. Consequently, the series in the right hand side of (8) converges uniformly for $w \in B(0, r) \setminus \{z\}$ and by the theorem on the limit of uniformly convergent series

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \sum_{k=1}^{\infty} c_k \lim_{w \to z} (z^{k-1} + z^{k-2} w + \ldots + z w^{k-2} + w^{k-1}) = \sum_{k=1}^{\infty} k c_k z^{k-1}.$$

$\square$

6.8 Infinite differentiability of holomorphic functions.

Theorem 6.9. *The function f, holomorphic in an arbitrary domain $D \subset \mathbb{C}$, has derivatives of all orders in D, which are also holomorphic in D. In this case, the Taylor series of the $n - th$ derivative $f^{(n)}(z)$ centered at an arbitrary point $a \in D$ is obtained by n-fold differentiation of the Taylor series for $f(z)$ centered at a.*

Proof. Consider a disk of convergence $U_R(a)$ of a Taylor series of function $f(z)$ centered at a. By the previous theorem $f^{(m)}(z)$ can be expressed in

$U_R(a)$ by a power series that is obtained by term-by-term differentiation of Taylor series for $f(z)$. Consequently, function $f^{(m)}$ is holomorphic in $U_R(a)$ and its series is the Taylor series centered at a (by theorem on uniqueness of Taylor series). $\square$

6.9 Coefficients of Taylor series

Theorem 6.10. *Let*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

be holomorphic in $U_R(a) = \{|z-a| < R\}$*. Then its Taylor coefficients can be calculated as*

$$c_n = \frac{f^{(n)}(a)}{n!} \quad n = 0, 1, 2, \ldots .$$

Proof. By theorem 6.9

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} c_k (z-a)^{k-n}.$$

Applying this for $z = a$ we see that $f^{(n)}(a) = n! c_n$. $\square$

6.10 Cauchy integral formula for derivatives.

Theorem. Let $D \subset \mathbb{C}$ be a domain with a simple boundary and the function f is holomorphic in the neighborhood of the closure $\bar{D}$ of the domain D. Then for all $n = 0, 1, 2, \ldots$ and all $z \in D$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}$$

Proof. Consider an arbitrary point $z \in D$ and choose $r > 0$ so that the circle $U_r = U_r(z) \Subset D$ with the center in z compactly belongs to D. For the coefficients of the Taylor series of the function f in the circle U_r, we have obtained two different formulas (see paragraphs 6.2, 6.8):

$$c_n = \frac{1}{2\pi i}\int_{\partial U_r} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}} \quad \text{and} \quad c_n = \frac{f^{(n)}(z)}{n!}.$$

Equating their right-hand sides, we get that

$$\frac{f^{(n)}(z)}{n!} = \frac{1}{2\pi i}\int_{\partial U_r} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+1}}$$

According to Cauchy's theorem for a multiply connected domain (see section 5.2), the integral of ∂U_r in this formula can be replaced by an integral of ∂D, which gives the required statement.

6.11 Morer's theorem.

Morer's theorem. If the function f is continuous in the domain D and the integral of f along the boundary of any triangle $\Delta \Subset D$ is zero, then f is holomorphic in D.

Proof. It is enough to prove the holomorphism of f in an arbitrary circle $U \Subset D$. But according to the theorem on the existence of a primitive (proposition 4.2 of clause 4.4), f has a primitive F in U. Since the derivative of the holomorphic function F is itself holomorphic (see clause 6.7), we obtain from here the holomorphism of f in the circle U.

6.12 Three equivalent definitions of a holomorphic function.

Theorem. Each of the following conditions is equivalent to the holomorphism of the function f at the point $a \in \mathbb{C}$:

(1) the function f is $\mathbb{C}$-differentiable in some neighborhood U of the point a;

(2) the function f is analytic at point a, i.e. decomposes into a power series centered at point a, converging in some neighborhood U of point a;

(3) The function f is continuous in some neighborhood U of point a and the integral of f along the boundary of any triangle $\Delta \Subset U$ is zero.

Proof. $(1) \Rightarrow (2)$: Taylor series expansion theorem (clause 6.2).

$(2) \Rightarrow (1)$: the holomorphy theorem of the sum of a power series (clause 6.6).

($(1) \Rightarrow (3)$: Cauchy's integral theorem in any of the forms of paragraphs 5.1, 5.2; even Gurse's lemma (paragraph 4.3) is sufficient.

$(3) \Rightarrow (1)$: Morer's theorem (clause 6.10).

6.13 Decomposition of a holomorphic function in neighbourhood of its zero.

Theorem 6.11. *Let f be holomorphic at $a \in \mathbb{C}$, $f(a) = 0$ but f is not identically zero in any neighborhood of a. Then in some neighborhood U of a point a function f can be expressed in the following form*

$$f(z) = (z - a)^n g(z),$$

where n is some natural number, and function g is holomorphic and doesn't vanish in U.

Proof. The function f can be expressed by it's Taylor's series

$$f(z) = \sum_{k=1}^{\infty} c_k (z-a)^k$$

in the disk of convergence $U_R(a)$ (notice that $c_0 = f(a) = 0$). Let

$$n := \min \{m \geqslant 1 : c_m \neq 0\}.$$

This definition is correct since if $c_m = 0$ for every $m \geqslant 1$ then $f(z) \equiv 0$ in $U_R(a)$ which contradicts the assumption of the theorem. Consider a series

$$g(z) := c_n + c_{n+1}(z-a) + \cdots .$$

That has the same radius of convergence as the Taylor series of function f. Hence, its sum $g(z)$ is holomorphic in $U_R(a)$. Moreover,

$$f(z) = (z-a)^n g(z), \quad \in U_R(a).$$

Since g is continuous in $U_R(a)$ and $g(a) = c_n \neq 0$ then there exists a neighborhood $U \subset U_R(a)$ of a such that $g(z) \neq 0$ for $z \in U$. □

Definition 6.12. *The number*

$$n = \min \{m \geqslant 1 : c_m \neq 0\} = \min \left\{ m \geqslant 1 : f^{(m)}(a) \neq 0 \right\}$$

from the theorem is called multiplicity of zero of holomorphic function f at point a. In other words, multiplicit of zero of function f at point a is the minimal number n such that $f^{(n)}(a) \neq 0$, or, equivalently, it is unique number n such that in some neighborhood U of point a

$$f(z) = (z-a)^n g(z)$$

for some $g \in \mathcal{O}(U)$ such that $g(a) \neq 0$.

Corollary 6.12.1. *If function f is holomorphic in domain D and has zero at $a \in D$ then either $f \equiv 0$ in some neighborhood of a, or there exists a neighborhood $U \subset D$ of a point a such that $f(z) \neq 0$ for every $z \in U \setminus \{a\}$.*

6.14 Uniqueness theorem.

Theorem 6.13. *Let $f, g \in H(D)$. Assume that a set*

$$E = \{z \in D : f(z) = g(z)\}$$

has a limit point in D. Then $f \equiv g$ in D.

Proof. Let

$$A = \{z \in D : f = g \text{ in some neighborhood of } g\}.$$

Then A is

- open (from the definition);
- closed. Since if $z_0 \in D$ is a limit point of A then z_0 is a nonisolated zero of function $f - g$ and $f - g$ is identicaly zero in some neighborhood of z_0.
- not empty since the limit point of E is not isolated zero of $f - g$ and thus belongs to A.

Consequently, since D is connected, A coincides with D, which implies the assertion of theorem. $\square$

6.15 Properties of elementary functions of complex variable

Assume that

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \quad g(z) = \sum_{k=0}^{\infty} d_k z^k, \quad |z| < R.$$

then for $\lambda \in \mathbb{R}$

$$(f+g)(z) = \sum_{k=0}^{\infty} (c_k + d_k) z^k; \quad \lambda f(z) = \sum_{k=0}^{\infty} \lambda c_k z^k.$$

And since both series are convergent absolutely for $|z| < R$ the Cauchy's rule for the product of two series implies that

$$(fg)(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} c_j d_{k-j} \right) z^k.$$

Definition 6.14.

0. Let $D = \mathbb{C}$. A function defined by a series $\sum_{k=1}^{n} \frac{z^k}{k!}$ is a unique entire function that coincides with real exponent on $\mathbb{R}$. Analogously functions defined by corresponding series define unique entire functions that continue sin and cos from $(0, \pi)$ to $\mathbb{C}$. That is, for every $z \in \mathbb{C}$

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!};$$
$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1};$$
$$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k};$$

1. e^z, $\cos z$, $\sin z$ are infinitely differentiable on $\mathbb{C}$ and

$$(e^z)' = e^z, \quad (\sin z)' = \cos z, \quad (\cos z)' = -\sin z.$$

2. Main property of the exponent

$$e^{z_1+z_2} = e^{z_1} e^{z_2}.$$

Proof.

$$e^{z_1} e^{z_2} = \left(\sum_{k=0}^{\infty} \frac{z_1^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{z_2^k}{k!}\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{z_1^j}{j!} \frac{z_2^{k-j}}{(k-j)!}\right) =$$
$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z_1^j z_2^{k-j} = \sum_{k=0}^{\infty} \frac{(z_1+z_2)^k}{k!} = e^{z_1+z_2}.$$

□

3. Trigonometric identities Since $\mathbb{R}$ has limit points in $\mathbb{C}$ then, by uniqueness theorem the trigonometric identities

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2;$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2;$$
$$\sin^2(z) + \cos^2(z) = 1$$

are true for every $z, z_1, z_2 \in \mathbb{C}$.

Corollary 6.14.1. *Let $A \in \mathbb{C}$. If $f(p) = A$ then p is called A-point of function f. Notice that if f is not constant then for every $A \in \mathbb{C}$ A-points of function f are isolated.*

6.16 Maximum modulus principle

Theorem 6.15. *Let $f \in H(D)$. If there exists a point $p \in D$ such that $|f(z)| \leq |f(p)|$ for every $z \in D$ then f is constant.*

Proof. Denote $|f(p)| = M$. Let $\rho > 0$ be such that $B(z_0, \rho) \subset D$. Then by the mean value theorem for every $r \in [0, \rho)$

$$M = |f(z_0)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(z_0 + re^{it}\right) dt \right| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f\left(z_0 + re^{it}\right) \right| dt \leqslant M.$$

Hence
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left(M - \left| f\left(z_0 + re^{it}\right) \right| \right) dt = 0.$$

The integrand is nonnegative and continuous, consequently it is identicaly zero, that is $|f(z)| = M$ for every $z : |z - z_0| = r$. Since $r \in [0, \rho)$ is arbitrary then $|f(z)| = M$ in $B(z_0, \rho)$. Consequently, f is constant in this disk an, consequently, in D. $\square$

Corollary 6.15.1. *An absolute value of a nonconstant holomorphic function does not have a maximum (even a local maximum) in the domain.*

Proof. Indeed, if p is a local maximum of $|f|$ then f is constant in some neighborhood of p, consequently, f is constant in D. $\square$

Corollary 6.15.2. *Let $D \subset \mathbb{C}$ be a bounded domain, $f \in H(D) \cap C(\bar{D})$. Then $|f|$ obtains its maximum on the boundary of D.*

Corollary 6.15.3. *Let $f \in H(D)$, f is not constant, f does not vanish in D. Then $|f|$ does not have a minimum (even a local minimum) in the domain.*

7 Laurent series and singular points

The Taylor series discussed in the previous chapter make it easy to deduce the basic properties of functions holomorphic in a circle, but are poorly adapted for studying the features of such functions. A convenient tool for investigating isolated singularities of holomorphic functions is the Laurent series studied in this lecture.

7.1 Decomposition of a holomorphic function into a Laurent series.

Theorem. Let the function $f(z)$ be holomorphic in a ring

$$V = \{z \in \mathbb{C} : r < |z - a| < R\}$$

centered at a point $a \in \mathbb{C}$ $c_0 \leqslant r < R \leqslant +\infty$. Let's say

$$c_n := \frac{1}{2\pi i} \int_{|\zeta - a| = \rho} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}} \quad \text{for all } n \in \mathbb{Z} \quad \text{and} \quad r < \rho < R$$

The numbers c_n are independent of ρ and are called the Laurent coefficients of the function f in the ring V. The series

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

called the Laurent series of the function f in the ring V, converges for all $z \in D$ and its sum is $f(z)$:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n \quad \text{for} \quad r < |z-a| < R \tag{7.1}$$

The part of the Laurent series set next to

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

it is called the regular part of the Laurent series, and the remaining part

$$\sum_{n=-\infty}^{-1} c_n(z-a)^n$$

its main part. We emphasize that here and further the convergence of the Laurent series

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n$$

by definition, it means that the series converge separately

$$\sum_{n=0}^{\infty} c_n(z-a)^n \quad \text{and} \quad \sum_{n=-\infty}^{-1} c_n(z-a)^n$$

representing, respectively, the regular and main parts of the Laurent series.

Proof. The independence of the coefficients c_n from ρ follows from Cauchy's theorem, since any two circles

$$\{|\zeta - a| = \rho_1\} \quad \text{and} \quad \{|\zeta - a| = \rho_2\} \quad \text{with} \quad r < \rho_1 < \rho_2 < R$$

homotopy in V as closed paths.

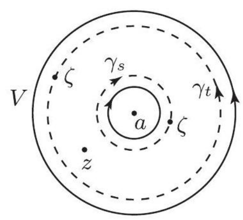

Fig. 29

To prove equality (7.1), fix $z \in V$ and choose $s, t \in (r, R)$ so that $s < |z - a| < t$ (see Fig. 29). According to the Cauchy formula for a ring $\{\zeta \in \mathbb{C} : s < |\zeta - a| < t\}$ bounded by circles

$$\gamma_s := \{|\zeta - a| = s\} \quad \text{and} \quad \gamma_t := \{|\zeta - a| = t\}$$

we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_t} \frac{f(\zeta) d\zeta}{\zeta - a} - \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta) d\zeta}{\zeta - a} =: I_1 - I_2 \tag{7.2}$$

Using the fact that $|z - a| < |\zeta - a|$ for all $\zeta \in \gamma_t$, we decompose the integrand in I_1 in a geometric progression:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - a) - (z - a)} = \frac{f(\zeta)}{\zeta - a} \cdot \frac{1}{1 - \frac{z-a}{\zeta - a}} = \sum_{n=0}^{\infty} \frac{(z-a)^n f(\zeta)}{(\zeta - a)^{n+1}}$$

The modulus of the nth term of this series does not exceed

$$\frac{M(t)}{t}\left(\frac{|z-a|}{t}\right)^n, \quad \text{where } M(t) := \max_{|\zeta-a|=t} |f(\zeta)|.$$

Therefore, by the property 2° from clause 6.1, this series converges uniformly over $\zeta \in \gamma_t$. Integrating it directly by γ_t, we get

$$I_1 = \sum_{n=0}^{\infty} c_n(z-a)^n, \quad \text{where} \quad c_n := \frac{1}{2\pi i}\int_{\gamma_t} \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} \tag{7.3}$$

that is, the integral I_1 coincides with the regular part of the Laurent series. Note that the first part of the proof is carried out in the same way as the proof of Theorem 6.2 on the decomposition of a holomorphic function into a power series.

Let us now turn to the integral I_2. For $\zeta \in \gamma_s$ we have $|\zeta-a| <|z-a|$, so

$$\frac{1}{\zeta-z} = \frac{1}{(\zeta-a)-(z-a)} = \frac{1}{z-a}\cdot\frac{1}{\frac{\zeta-a}{z-a}-1} = -\sum_{m=0}^{\infty}\frac{(\zeta-a)^m}{(z-a)^{m+1}}$$

Multiplying this series by $f(\zeta)$ and integrating it by γ_s, we get that

$$I_2 = \sum_{m=0}^{\infty} b_m(z-a)^{-(m+1)}$$

where is the coefficient

$$b_m = -\frac{1}{2\pi i}\int_{\gamma_s}(\zeta-a)^m f(\zeta)d\zeta$$

matches $-c_{-(m+1)}$. Thus, the integral I_2 is equal to the minus sign of the main part of the Laurent series. Substituting this expression for I_2 and the expression for I_1 given by formula (7.3) into formula (7.2), we obtain the required formula (7.1).

7.2 Convergence of series by integer degrees $z-a$.

Theorem. For an arbitrary set of $\{c_n : n \in \mathbb{Z}\}$ complex numbers, we put

$$R := \left\{ \overline{\lim_{n\to\infty}} |c_n|^{1/n} \right\}^{-1}, \quad r := \overline{\lim_{n\to\infty}} |c_{-n}|^{1/n}.$$

Then Laurent's row

$$f(z) := \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

converges absolutely and uniformly on compacts in the ring $\{r <|z-a| < R\}$, and its sum $f(z)$ is holomorphic in this ring and satisfies the relation

$$\frac{1}{2\pi i} \int_{|\zeta-a|=\rho} \frac{f(\zeta) d\zeta}{(\zeta-a)^{n+1}} = c_n \quad \text{for all } n \in \mathbb{Z} \quad \text{and} \quad r < \rho < R. \tag{7.4}$$

If $|z-a| < r$, then the main part of the Laurent series is

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n$$

It's diverging. If $|z-a| > R$, then its regular part diverges

$$\sum_{n=0}^{\infty} c_n(z-a)^n$$

Note that equality (7.4) implies the uniqueness property of the Laurent coefficients.

Proof. Statements about convergence and divergence follow from the Cauchy-Hadamard formula applied to power series

$$f_1(z) := \sum_{n=0}^{\infty} c_n(z-a)^n$$

and

$$f_2(z) := \sum_{n=-\infty}^{-1} c_n(z-a)^n = \sum_{m=1}^{\infty} c_{-m} Z^m, \quad \text{where} \quad Z := \frac{1}{z-a}.$$

The holomorphism of $f(z)$ in the ring $r < |z-a| < R$ follows from the fact that the function $f_1(z)$ is holomorphic at $|z-a| < R$ by virtue of clause 6.6 , and the function $f_2(z)$ is holomorphic at $|Z| < r^{-1}$, i.e. at $|z-a| > r$.

To prove the formula (7.4), it is necessary to multiply the equality

$$f(\zeta) := \sum_{n=-\infty}^{\infty} c_n(\zeta-a)^n$$

by $(\zeta-a)^{-(m+1)}$ and integrate the soil along the circle $|\zeta-a| = \rho$ (this is legal due to the uniform convergence of the series). According to example 4.1 of clause 4.1, the integrals from all terms of the series with

$n \neq m$ are zero, and the integral from the term with $n = m$ is equal to $2\pi i$, which gives (7.4).

7.3 Cauchy inequalities for Laurent coefficients.

Proposal. Let the function

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

be holomorphic in the ring $\{r < |z-a| < R\}$. Then for all $n \in \mathbb{Z}$ and all $\rho \in (r, R)$ the inequalities are valid

$$|c_n| \leqslant \frac{M(\rho)}{\rho^n}, \quad \text{where} \quad M(\rho) := \max_{|\zeta - a| = \rho} |f(\zeta)|.$$

The proof repeats the proof of Cauchy inequalities.

7.4 A note on the Laurent and Fourier series.

Each convergent Laurent series can be considered as a Fourier series. If, for example, the function f is holomorphic in the ring $\{1 - \varepsilon < |z| < 1 + \varepsilon\}$ for some $\varepsilon > 0$, then the n-th Laurent coefficient c_n of the function f can be written in kind of

$$c_n = \frac{1}{2\pi i} \int_{|\zeta|=1} f(\zeta)\zeta^{-(n+1)} d\zeta = \frac{1}{2\pi} \int_0^{2\pi} f\left(e^{it}\right) e^{-int} dt.$$

In other words, it coincides with the nth Fourier coefficient of the function $\varphi(t) = f\left(e^{it}\right)$ on the segment $0 \leqslant t \leqslant 2\pi$. In particular, by virtue of the theorem in clause 7.2, it follows that the Fourier series of the function $f\left(e^{it}\right)$ converges to it uniformly on the segment $0 \leqslant t \leqslant 2\pi$.

Note, however, that the reverse transition from the Fourier series to the Laurent series is not always possible, more precisely, not every Fourier series is a Laurent series of some function. In more detail, for each function $\varphi \in L^1(0, 2\pi)$ it is possible to determine its Fourier coefficients using the formula

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t) e^{-int} dt$$

If the function φ is smooth enough (for example, of the class $C^2(0, 2\pi)\big)$, and both φ itself and its derivatives φ' and φ'' take the same values at points 0 and 2π, then the Fourier series of this function

$$\sum_{n=-\infty}^{\infty} c_n e^{int}$$

converges to φ absolutely and uniformly by $[0, 2\pi]$. However, for the convergence of the Laurent series

$$\sum_{n=-\infty}^{\infty} c_n z^n$$

in any ring $\{1 - \varepsilon < |z| < 1 + \varepsilon\}$, $\varepsilon > 0$, it is necessary that the function φ be real analytic (this follows from Theorem 7.2 and the results of Chapter 6).

7.5 Isolated singular points. Definition.

Definition. A point $a \in \mathbb{C}$ is called an isolated singular point (of unambiguous character) for the function $f(z)$ if f is holomorphic in some punctured neighborhood $V = \{0 <|z-a| < \varepsilon\}, \varepsilon > 0$, points a. The isolated singular point a of the function f is called:

(1) fixable if there is a (finite) limit

$$\lim_{z\to a} f(z) \in \mathbb{C};$$

(2) by a pole, if there is

$$\lim_{z\to a} f(z) = \infty;$$

(3) essentially a singular point in all other situations, i.e. when there is no (finite or infinite) limit of $f(z)$ for $z \to a$.

Examples.

The point $a = 0$ is:

(1) the fixable singular point for the function

$$f(z) = \frac{\sin z}{z}$$

(2) A pole for the function

$$f(z) = \frac{1}{z}$$

(3) essentially a singular point for the function

$$f(z) = e^{1/z}$$

(indeed, for $z = x \to 0$, the limit on the right is $+\infty$, the limit on the left is 0, and for $z = iy \to 0$, the function $e^{-i/y}$ $=\cos(1/y) - i\sin(1/y)$ has no limit at all);

(4) an unisolated singular point for the function

$$f(z) = \operatorname{ctg} \frac{1}{z}$$

having poles at points $z_n = (\pi n)^{-1}$.

7.6 Description of fixable singular points.

Theorem. For a function f holomorphic in a punctured neighborhood $V = \{0 < |z-a| < \varepsilon\}$ of a point a, the following statements are equivalent:

(1) $z = a$ is a fixable singular point;

(2) $f(z)$ is bounded in some punctured neighborhood $V' = \{0 < |z-a| < \varepsilon'\}, \varepsilon' > 0$, of the point a;

(3) the Laurent coefficients c_n of the function f in the punctured neighborhood $V = \{0 < |z-a| < \varepsilon\}$ satisfy the condition

$$c_n = 0 \quad \text{at} \quad n < 0$$

(4) it is possible to define the function $f(z)$ for $z =$a in such a way that the resulting function f becomes holomorphic in the complete neighborhood of $U = \{|z-a| < \varepsilon\}$ of the point a.

Proof. (1) $\Rightarrow$ (2). Obviously.

(2) $\Rightarrow$ (3). If

$$|f(z)| \leqslant M \quad \text{for} \quad 0 < |z-a| < \varepsilon'$$

then according to the Cauchy inequalities (clause 7.3) we have

$$|c_{-k}| \leqslant M\rho^k \quad \text{for all } k = 1, 2, \ldots \text{and all} \rho \in (0, \varepsilon').$$

Aiming $\rho \to 0$, we get that $c_{-k} = 0$ for every $k = 1, 2, \ldots$. (3) $\Rightarrow$ (4). By condition we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \quad for \quad 0 < |z-a| < \varepsilon$$

If we put $f(a) = c_0$, then this equality will be true for $|z-a| < \varepsilon$. According to the theorem from clause 6.6, the function f is holomorphic in the neighborhood of $\{|z-a| < \varepsilon\}$.

(4) $\Rightarrow$ (1). Obviously.

7.7 Description of the poles.

Theorem.

Point a is the pole of the f function holomorphic in the punctured neighborhood $V = \{0 < |z-a| < \varepsilon\}$ of this point if and only if the main part of the Loran expansion of f in the neighborhood of V contains only a finite (but nonzero) number non-zero members. In other words, the Loran decomposition of f in the neighborhood of V has the form

$$f(z) = \sum_{n=-N}^{\infty} c_n (z-a)^n$$

for some $N \in \mathbb{N}$, with $c_{-N} \neq 0$.

Proof. $\Longrightarrow$. By definition of the pole

$$\lim_{z \to a} f(z) = \infty$$

so $f(z) \neq 0$ at $0 < |z-a| < \varepsilon'$. Therefore, the function

$$g(z) := \frac{1}{f(z)}$$

holomorphic in a punctured neighborhood $V' = \{0 < |z-a| < \varepsilon'\}$. At the same time, according to the condition

$$\lim_{z \to a} g(z) = 0.$$

According to the theorem from clause 7.6, the function g will be holomorphic in the complete neighborhood of $U' = \{|z-a| < \varepsilon'\}$ of the point a if we define it at this point, assuming $g(a)$ =0. Denote by N the order of zero $g(z)$ for $z = a$. Then for $0 < |z-a| < \varepsilon'$ we will have

$$g(z) = (z-a)^N h(z)$$

where the function h is holomorphic in the neighborhood of $U' = \{|z-a| < \varepsilon'\}$ and $h(z) \neq 0$ at $0 < |z-a| < \varepsilon''$. The function $1/h$ is holomorphic in the circle $U'' = \{|z-a| < \varepsilon''\}$ and, therefore, by the theorem from clause 6.2 decomposes there into a Taylor series:

$$\frac{1}{h(z)} = b_0 + b_1(z-a) + \cdots$$

moreover, $b_0 = 1/h(a) \neq 0$. Multiplying this decomposition by $(z-a)^{-N}$, we see that the function

$$f(z) = (z-a)^{-N} \cdot \frac{1}{h(z)}$$

has the required Laurent series in the punctured neighborhood $\{0 <|z-a| < \varepsilon\}$.

$\Longleftarrow$. By condition

$$f(z) = (z-a)^{-N} g(z),$$

where the function g is holomorphic in the neighborhood of $U = \{|z-a| < \varepsilon\}$ and $g(a) \neq 0$ in U. Therefore,

$$\lim_{z \to a} f(z) = \infty$$

Remark 7.1. The number N from this theorem, determined by the properties:

$$c_{-n} = 0 \quad \text{for} \quad n > N, \quad \text{but} \quad c_{-N} \neq 0,$$

It is called the pole order of the function $f(z)$ at the point $z = A$. It is clear from the proof that the function $f(z)$ has a pole at point A if and only if the function $1/f(z)$ is holomorphic and equal to zero at this point; in this case, the order of the pole $f(z)$ at point A coincides with the order of zero $1/f(z)$ at point A.

Remark 7.2. The description of fixable singularities and poles implies the following characterization of essentially singular points in terms of the Laurent series: the function

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

holomorphic in a punctured neighborhood $V = \{0 < |z-a| < \varepsilon\}$ points $z = a$, has an essential feature at this point if and only if there are infinitely many numbers $n \geqslant 1$ such that

$$c_{-n} \neq 0.$$

7.8 Sokhotsky's theorem.

Theorem. If $a \in \mathbb{C}$ is an essentially special point of the function f, then for any $A \in \overline{\mathbb{C}}$ one can find a sequence of points $z_n \to$a such that

$$\lim_{n \to \infty} f(z_n) = A.$$

Proof. 1°. Let $A = \infty$. According to the theorem from clause 7.6, the function f cannot be bounded in any punctured neighborhood of the point a (otherwise a would be an eliminated singular point for f). Therefore, there is a sequence $z_n \to a$ such that $f(z_n) \to \infty$ for $n \to \infty$.

2°. Let's say $A \in \mathbb{C}$. If in any punctured neighborhood of point a there is a point z with $f(z) = A$, then the statement of the theorem is obvious (you can even find the sequence $z_n \to a$ such that $f(z_n) = A$ for all n). If this is not the case, then the function

$$g(z) := \frac{1}{f(z) - A}$$

has an isolated singular point at $z = a$. The point a cannot be a pole or a removable feature for the function $g(z)$, since in both cases the function

$$f(z) = A + \frac{1}{g(z)}$$

would have a limit (possibly equal to ∞) at $z \to a$, which contradicts the definition of an essentially singular point (see clause 7.5). Therefore, a is an essentially special point for $g(z)$. But then, according to the first part of the proof, there will be a sequence $z_n \to a$ such that

$$g(z_n) \to \infty \quad \text{for} \quad n \to \infty$$

It follows that

$$f(z_n) = A + \frac{1}{g(z_n)} \to A \quad \text{for} \quad n \to \infty$$

7.9 $a = \infty$ as an isolated singular point.

In the definition and classification of isolated singular points $a \in \mathbb{C}$ from clause 7.5, the case $a = \infty$ can also be included (with minor modifications). For example, the definition of an isolated singular point ∞ looks like this.

Definition. The point $a = \infty$ is called an isolated singular point (of an unambiguous character) for the function f if $f \in \mathcal{O}(\{|z| > R\})$ for some $R > 0$.

The type of isolated singular point ∞ (pole, fixable or essentially singular point) is defined in the same way as in clause 7.5. In accordance with clause 2.6, the point $z = \infty$ is removable (pole, essentially singular) for the function $f(z)$ if and only if when the point $\zeta = 0$ is eliminated (a pole, essentially special) for the function $g(\zeta) := f(1/\zeta)$.

The results of paragraphs 7.6, 7.7 characterizing the type of singular point $a \in \mathbb{C}$ in terms of the Laurent series are also immediately transferred to the case of $a = \infty$. Namely, let $a = \infty$ be an isolated singular point of the function f, holomorphic in some ring $\{R < |z| < \infty\}$, which decomposes in this ring into a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

Then $a = \infty$ is:

(1) the fixable singular point of the function $f \Longleftrightarrow c_n = 0$ for all $n \geqslant 1$

(2) the pole of the function $f \Longleftrightarrow$ exists $N \geqslant 1$ such that

$$c_N \neq 0, \quad \textit{but} \quad c_n = 0 \quad \text{when} \quad n \geqslant N+1$$

(the number N is called the pole order in ∞);

(3) an essentially singular point of the function $f \Longleftrightarrow c_n \neq 0$ for an infinite set of natural $n \geqslant 1$.

Due to these results, the main part (i.e., the part defining the type of singular point) of the Laurent series of the function f in the punctured neighborhood of ∞ is the series

$$\sum_{n=1}^{\infty} c_n z^n$$

and its regular part is set next to

$$\sum_{n=-\infty}^{0} c_n z^n$$

7.10 Integer functions with a pole at infinity.

Definition. A function that is holomorphic in the entire complex plane $\mathbb{C}$ is called an integer.

Proposal. If the integer function f has a fixable singular point or pole for $z = \infty$, than f is a polynomial.

Proof. Denote by

$$P(z) = \sum_{n=1}^{N} c_n z^n$$

the main part of the Laurent series of the function f in the punctured neighborhood of the point ∞, which is a polynomial according to clause 7.9. Then the function $g(z) := f(z) - P(z)$ is integer and has a removable singularity at the point ∞. Therefore, $g(z) \equiv$const by Liouville's theorem, which implies that f is a polynomial.

7.11 Meromorphic functions with a pole at infinity.

Definition. A function f is called meromorphic in the domain $D \subset \overline{\mathbb{C}}$ if it has no other features in D other than poles. In other words, there is a subset of $M \subset D$ such that $f \in \mathcal{O}(D \backslash M)$ and f has a pole at each point of $a \in M$.

Since, by definition of the pole, the set M consists of isolated points, it is no more than countable (since $M \cap K$ is finite for every compact $K \subset D$).

An example of a function that is meromorphic in $\mathbb{C}$ is the function $\mathrm{ctg}(z)$. At the same time, the function $\mathrm{ctg}(1/z)$ is not meromorphic in $\mathbb{C}$, since the point $z = 0$ is the limit for the poles of $\mathrm{ctg}(1/z)$.

Proposal. If a function f is meromorphic in $\mathbb{C}$ and has a fixable singular point or pole at $z = \infty$ (thus, f is meromorphic in $\overline{\mathbb{C}}$), then it is rational.

Proof. Since the poles of f in $\overline{\mathbb{C}}$ are isolated, their set consists of a finite number of points $a_1, \ldots, a_n$. Denote by

$$R_j(z) = \sum_{k=1}^{n_j} c_{jk} \left(z - a_j\right)^{-k}, \quad j = 1, \ldots, n$$

the main parts of the Laurent series of the function f in the punctured neighborhood of these poles, and through

$$P(z) = \sum_{k=1}^{n} c_k z^k$$

the main part of the Laurent series is f in the punctured neighborhood of ∞. Then the function

$$g := f - (P + R_1 + \cdots + R_N)$$

It is holomorphic in the entire complex plane $\mathbb{C}$ and has a removable singularity at $z = \infty$, from which it follows by Liouville's theorem that $g(z) \equiv$const.

Remark. Equality

$$f(z) = C + P(z) + R_1(z) + \cdots + R_N(z),$$

The proof of this proposition is nothing more than the decomposition of the rational function $f(z)$ into the simplest fractions.

7.12 Examples

EXAMPLE 1 Series of the Form Given in (1)

The function $f(z) = \frac{\sin z}{z^4}$ is not analytic at the isolated singularity $z = 0$ and hence cannot be expanded in a Maclaurin series. However, $\sin z$ is an entire function, and we know that its Maclaurin series,

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \cdots$$

converges for $|z| < \infty$. By dividing this power series by z^4 we obtain a series for f with negative and positive integer powers of z :

$$f(z) = \frac{\sin z}{z^4} = \overbrace{\frac{1}{z^3} - \frac{1}{3!z}}^{\substack{\text{principal}\\ \text{part}}} \overbrace{+\frac{z}{5!} - \frac{z^3}{7!} + \frac{z^5}{9!} - \cdots}^{\substack{\text{analytic}\\ \text{part}}} \tag{6}$$

The analytic part of the series in (6) converges for $|z| < \infty$. (Verify.) The principal part is valid for $|z| > 0$. Thus (6) converges for all z except at $z = 0$; that is, the series representation is valid for $0 < |z| < \infty$.

A series representation of a function f that has the form given in (6) is called a Laurent series or a Laurent expansion of f about z_0 on the annulus $r < |z - z_0| < R$.

EXAMPLE 2 Four Laurent Expansions

Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains,

(a) $0 < |z| < 1$

(b) $1 < |z|$

(c) $0 < |z - 1| < 1$

(d) $1 < |z - 1|$

In parts (a) and (b) we want to represent f in a series involving only negative and nonnegative integer powers of z, whereas in parts (c) and

(d) we want to represent f in a series involving negative and nonnegative integer powers of $z-1$.

(a) By writing

$$f(z) = -\frac{1}{z}\frac{1}{1-z}$$

we can use (6) of Section 6.1 to write $1/(1-z)$ as a series:

$$f(z) = -\frac{1}{z}\left[1 + z + z^2 + z^3 + \cdots\right]$$

The infinite series in the brackets converges for $|z| < 1$, but after we multiply this expression by $1/z$, the resulting series

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$

converges for $0 < |z| < 1$.

(b) To obtain a series that converges for $1 < |z|$, we start by constructing a series that converges for $|1/z| < 1$. To this end we write the given function f as

$$f(z) = \frac{1}{z^2}\frac{1}{1-\frac{1}{z}}$$

and again use (6) of Section 6.1 with z replaced by $1/z$:

$$f(z) = \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right]$$

The series in the brackets converges for $|1/z| < 1$ or equivalently for $1 < |z|$. Thus the required Laurent series is

$$f(z)=\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\dots$$

(c) This is basically the same problem as in part (a), except that we want all powers of $z-1$. To that end, we add and subtract 1 in the denominator and use (7) of Section 6.1 with z replaced by $z-1$:

$$\begin{aligned} f(z) &= \frac{1}{(1-1+z)(z-1)} \\ &= \frac{1}{z-1}\frac{1}{1+(z-1)} \\ &= \frac{1}{z-1}\left[1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right] \\ &= \frac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots \end{aligned}$$

The requirement that $z\neq 1$ is equivalent to $0<|z-1|$, and the geometric series in brackets converges for $|z-1|<1$. Thus the last series converges for z satisfying $0<|z-1|$ and $|z-1|<1$, that is, for $0<|z-1|<1$.

(d) Proceeding as in part (b), we write

$$\begin{aligned} f(z) &= \frac{1}{z-1}\frac{1}{1+(z-1)}=\frac{1}{(z-1)^2}\frac{1}{1+\frac{1}{z-1}} \\ &= \frac{1}{(z-1)^2}\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots\right] \\ &= \frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\frac{1}{(z-1)^5}+\cdots \end{aligned}$$

Because the series within the brackets converges for $|1/(z-1)|<1$, the final series converges for $1<|z-1|$.

EXAMPLE 3 Laurent Expansions

Expand $f(z) = \frac{1}{(z-1)^2(z-3)}$ in a Laurent series valid for (a) $0 < |z-1| < 2$ and (b) $0 < |z-3| < 2$.

Solution (a) As in parts (c) and (d) of Example 2, we want only powers of $z-1$ and so we need to express $z-3$ in terms of $z-1$. This can be done by writing

$$f(z) = \frac{1}{(z-1)^2(z-3)} = \frac{1}{(z-1)^2}\frac{1}{-2+(z-1)} = \frac{-1}{2(z-1)^2}\frac{1}{1-\frac{z-1}{2}}$$

and then using (6) of Section 6.1 with the symbol z replaced by $(z-1)/2$,

$$\begin{aligned} f(z) &= \frac{-1}{2(z-1)^2}\left[1+\frac{z-1}{2}+\frac{(z-1)^2}{2^2}+\frac{(z-1)^3}{2^3}+\cdots\right] \\ &= -\frac{1}{2(z-1)^2}-\frac{1}{4(z-1)}-\frac{1}{8}-\frac{1}{16}(z-1)-\cdots \end{aligned} \tag{16}$$

(b) To obtain powers of $z-3$, we write $z-1 = 2+(z-3)$ and We now factor 2 from this expression

$$\begin{aligned} f(z) &= \frac{1}{(z-1)^2(z-3)} = \frac{1}{z-3}\overbrace{[2+(z-3)]^{-2}} \\ &= \frac{1}{4(z-3)}\left[1+\frac{z-3}{2}\right]^{-2} \end{aligned}$$

At this point we can obtain a power series for $\left[1+\frac{z-3}{2}\right]^{-2}$ by using the binomial expansion, ′

$$f(z) = \frac{1}{4(z-3)}\left[1 + \frac{(-2)}{1!}\left(\frac{z-3}{2}\right) + \frac{(-2)(-3)}{2!}\left(\frac{z-3}{2}\right)^2 + \frac{(-2)(-3)(-4)}{3!}\left(\frac{z-3}{2}\right)^3 + \cdots\right].$$

The binomial series in the brackets is valid for $|(z-3)/2| < 1$ or $|z-3| < 2$. Multiplying this series by $\frac{1}{4(z-3)}$ gives a Laurent series that is valid for $0 < |z-3| < 2$:

$$f(z) = \frac{1}{4(z-3)} - \frac{1}{4} + \frac{3}{16}(z-3) - \frac{1}{8}(z-3)^2 + \cdots$$

EXAMPLE 4 A Laurent Expansion

Expand $f(z) = \frac{8z+1}{z(1-z)}$ in a Laurent series valid for $0 < |z| < 1$.

Solution By partial fractions we can rewrite f as

$$f(z) = \frac{8z+1}{z(1-z)} = \frac{1}{z} + \frac{9}{1-z}$$

Then by (6) of Section 6.1 ,

$$\frac{9}{1-z} = 9 + 9z + 9z^2 + \cdots$$

The foregoing geometric series converges for $|z| < 1$, but after we add the term $1/z$ to it, the resulting Laurent series

$$f(z) = \frac{1}{z} + 9 + 9z + 9z^2 + \cdots$$

is valid for $0 < |z| < 1$.

[0†] For α real, the binomial series $(1+z)^\alpha = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}z^3 + \cdots$ is valid for $|z| < 1$.

In the preceding examples the point at the center of the annular domain of validity for each Laurent series was an isolated singularity of the function f.

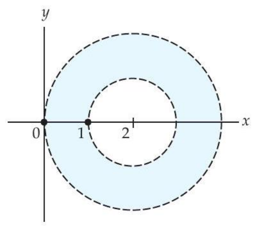

Figure 6.9 Annular domain for Example 5

EXAMPLE 5 A Laurent Expansion

Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for $1 < |z-2| < 2$.

Solution The specified annular domain is shown in Figure 6.9. The center of this domain, $z = 2$, is the point of analyticity of the function f. Our goal now is to find two series involving integer powers of $z-2$, one converging for $1 < |z-2|$ and the other converging for $|z-2| < 2$. To accomplish this, we proceed as in the last example by decomposing f into partial fractions:

$$f(z) = -\frac{1}{z} + \frac{1}{z-1} = f_1(z) + f_2(z) \tag{17}$$

Now,

$$
\begin{aligned}
f_1(z) &= -\frac{1}{z} = -\frac{1}{2+z-2} \\
&= -\frac{1}{2}\frac{1}{1+\frac{z-2}{2}} \\
&= -\frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \cdots\right] \\
&= -\frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdots
\end{aligned}
$$

This series converges for $|(z-2)/2| < 1$ or $|z-2| < 2$. Furthermore,

$$
\begin{aligned}
f_2(z) = \frac{1}{z-1} = \frac{1}{1+z-2} &= \frac{1}{z-2}\frac{1}{1+\frac{1}{z-2}} \\
&= \frac{1}{z-2}\left[1 - \frac{1}{z-2} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} + \cdots\right] \\
&= \frac{1}{z-2} - \frac{1}{(z-2)^2} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^4} + \cdots
\end{aligned}
$$

converges for $|1/(z-2)| < 1$ or $1 < |z-2|$. Substituting these two results in (17) then gives

$$
f(z) = \cdots - \frac{1}{(z-2)^4} + \frac{1}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{1}{z-2} - \frac{1}{2} + \frac{z-2}{2^2} - \frac{(z-2)^2}{2^3} + \frac{(z-2)^3}{2^4} - \cdot
$$

This representation is valid for z satisfying $|z-2| < 2$ and $1 < |z-2|$; in other words, for $1 < |z-2| < 2$.

EXAMPLE 6 A Laurent Expansion

Expand $f(z) = e^{3/z}$ in a Laurent series valid for $0 < |z| < \infty$.

Solution From (12) of Section 6.2 we know that for all finite z, that is, $|z| < \infty$,

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{18}$$

We obtain the Laurent series for f by simply replacing z in (18) by $3/z$, $z \neq 0$,

$$e^{3/z} = 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots \tag{19}$$

This series (19) is valid for $z \neq 0$, that is, for $0 < |z| < \infty$.

Remarks (i) In conclusion, we point out a result that will be of special interest to us in Sections 6.5 and 6.6. Replacing the complex variable s with the usual symbol z, we see that when $k - 1$, formula (8) for the Laurent series coefficients yields $a_{-1} = \frac{1}{2\pi i} \oint_C f(z)dz$, or more important,

$$\oint_C f(z)dz = 2\pi i a_{-1} \tag{20}$$

(ii) Regardless how a Laurent expansion of a function f is obtained in a specified annular domain it is the Laurent series; that is, the series we obtain is unique.

8 Deductions (Residues)

8.1 Cauchy's theorem on deductions.

Let the function f be holomorphic in the punctured neighborhood $V = \{0 < |z - a| < \varepsilon\}$ of the point $a \in \mathbb{C}$, so that a is its isolated feature.

Definition. The deduction of the function f at an isolated singular point $a \in \mathbb{C}$ is a number

$$\operatorname{res}_a f = \frac{1}{2\pi i} \int_{|\zeta - a| = r} f(\zeta) d\zeta, \qquad \text{where } 0 < r < \varepsilon$$

(according to Cauchy's theorem, this integral does not depend on the choice of r).

Cauchy's theorem on deductions. Let $D \subset \mathbb{C}$ be a region with a simple boundary and G be some region in $\mathbb{C}$ containing the closure $\bar{D}$ of the domain D. Suppose that the function f is holomorphic everywhere in the domain G, except for a finite number of singular points $a_1, \ldots, a_n \in D$. Then

$$\int_{\partial D} f(\zeta) d\zeta = 2\pi i \sum_{j=1}^{n} \operatorname{res}_{a_j} f$$

Proof. Let's choose $\varepsilon > 0$ so that the circles

$$B_j := \{z \in \mathbb{C} : |z - a_j| < \varepsilon\}, \quad j = 1, \ldots, n.$$

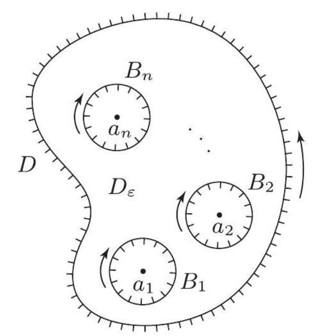

Fig. 30

They did not intersect in pairs, and their closures were contained in D (see Fig. 30). Then by Cauchy's theorem for a multiply connected domain (clause 5.2)

$$D_\varepsilon := D \backslash \bigcup_{j=1}^{n} \bar{B}_j$$

we will have

$$\begin{aligned} 0 = \int_{\partial D_\varepsilon} f(\zeta) d\zeta &= \int_{\partial D} f(\zeta) d\zeta - \sum_{j=1}^{n} \int_{\partial B_j} f(\zeta) d\zeta \\ &= \int_{\partial D} f(\zeta) d\zeta - \sum_{j=1}^{n} 2\pi i \operatorname{res}_{a_j} f \end{aligned}$$

which was exactly what needed to be proved.

8.2 Deduction in terms of the Laurent series.

Proposal. If the function

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

is holomorphic in the punctured neighborhood of $V = \{0 < |z - a| < \varepsilon\}$ of the point a $\in \mathbb{C}$, then

$$\operatorname{res}_a f = c_{-1}.$$

Proof. The relation being proved is a special case of equality (7.4) from clause 7.2, corresponding to $n = -1$. However, the direct proof is so simple that it is worth repeating it here:

$$\begin{aligned}\operatorname{res}_a f &= \frac{1}{2\pi i}\int_{|z-a|=r} f(z)dz \\ &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} c_n \int_{|z-a|=r}(z-a)^n dz = \frac{1}{2\pi i}\cdot 2\pi i c_{-1} = c_{-1},\end{aligned}$$

where we used the definition of deduction, the uniform convergence of the Laurent series for f on the circle $|z-a| = r$, $0 < r < \varepsilon$, and example 4.1 from clause 4.1; according to this example integral $\int_{|z-a|=r}(z-a)^n dz$ is zero for $n \neq -1$ and $2\pi i$ for $n = -1$.

Consequence. The deduction at the eliminated singular point $a \in \mathbb{C}$ is zero.

Note, however, that from the equality of the zero deduction of f at some special point, it does not follow at all that this point is an eliminated feature for f. Indeed, the vanishing of the Loran coefficient $c_{-1} = 0$ does not mean that the coefficients $c_{-2}, c_{-3}, \ldots$ vanish. For example, the subtraction at zero of the function z^{-2} is zero, but the function itself has a 2nd-order pole at this point.

8.3 Formulas for calculating deductions.

Case 1: deduction in a simple pole. Let a be the simple pole (i.e., the pole of the 1st order) of the function f. The Loran decomposition of f at the point a has the form

$$f(z) = \frac{c_{-1}}{z-a} + \sum_{n=0}^{\infty} c_n (z-a)^n$$

That means

$$c_{-1} = \operatorname{res}_a f = \lim_{z \to a} (z-a) f(z)$$

Consider a typical example of a function having a simple pole. Suppose that the function f is represented in the punctured neighborhood of the point a in the form

$$f(z) = \frac{\varphi(z)}{\psi(z)}$$

where the functions $\varphi(z), \psi(z)$ are holomorphic in the neighborhood of a, and

$$\varphi(a) \neq 0, \quad \psi(a) = 0, \quad \text{BUT} \quad \psi'(a) \neq 0.$$

In this situation

$$\operatorname{res}_a f = \lim_{z \to a} (z-a) \frac{\varphi(z)}{\psi(z)} = \lim_{z \to a} \varphi(z) \cdot \frac{z-a}{\psi(z) - \psi(a)} = \frac{\varphi(a)}{\psi'(a)}.$$

The case of the pole of the nth order. Let a be the pole of the nth order of the function f. Then its Loran decomposition at the point a has the form

$$f(z) = \frac{c_{-n}}{(z-a)^n} + \cdots + \frac{c_{-1}}{z-a} + \sum_{m=0}^{\infty} c_n (z-a)^m$$

To "extract" c_{-1} from here, you need to multiply $f(z)$ by $(z-a)^n$ and take the derivative of the order $n-1$ from the resulting function at $z = a:$

$$\operatorname{res}_a f = c_{-1} = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \{(z-a)^n f(z)\}$$

8.4 Deduction at the point $a = \infty$.

Let the function f be holomorphic in the exterior of some circle $\{|z| \leqslant R_0\}$ and have ∞ as its isolated singular point.

Definition. The Deduction of f at infinity is a number

$$\operatorname{res}_\infty f = \frac{1}{2\pi i} \int_{\gamma_R^{-1}} f dz$$

where the integral is taken along a circle $\gamma_R = \{|z| = R\}$ of a sufficiently large radius $R > R_0$, traversed clockwise.

It is easy to see that the deduction at infinity of the function f given in the domain $\{|z| > R_0\}$ by the Loran decomposition

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$

equal

$$\operatorname{res}_\infty f = -c_{-1}$$

To prove it, it is enough to integrate the Loran decomposition of f in γ_R^{-1} in detail.

Remark. The above formula shows, in particular, that the deduction at infinity differs from the deduction at the common point $a \in \mathbb{C}$. For example, the previously proven statement: the deduction at a singular point is zero if it is fixable - is not true for the deduction at infinity.

This apparent difference disappears in the context of the general theory of Riemann surfaces. At this stage, we only note that the real motivation for the above definition of deduction at infinity is the theorem on the total amount of deductions, proved in the next paragraph.

8.5 The theorem on the total amount of deductions.

The theorem on the total amount of deductions. Let the function f be holomorphic in the entire plane $\mathbb{C}$, except for a finite number of points $\{a_\nu\}$. Then the sum of the deductions at the points $\{a_\nu\}$ and at infinity is zero:

$$\operatorname{res}_\infty f + \sum_\nu \operatorname{res}_{a_\nu} f = 0$$

Proof. Let $U_R = \{|z| < R\}$ be a circle of sufficiently large radius containing all singular points $\{a_\nu\}$. Applying Cauchy's deduction theorem to this circle (clause 8.1), we obtain that

$$\frac{1}{2\pi i} \int_{\partial U_R} f dz = \sum_\nu \operatorname{res}_{a_\nu} f$$

It remains to be noted that the left side of this equality coincides with $-\operatorname{res}_\infty f$.

8.6 Jordan's lemma.

In the practical calculation of integrals in the complex domain, the following is often useful

Jordan's Lemma. Let the function f be defined and continuous on the set

$$\{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0, |z| \geqslant R_0\}.$$

Let it be $R \geqslant R_0$

$$M(R) := \max_{z \in \gamma_R} |f(z)|$$

where γ_R is a semicircle of the form (see Fig. 31)

$$\gamma_R = \left\{z = Re^{i\theta} : 0 \leqslant \theta \leqslant \pi\right\}.$$

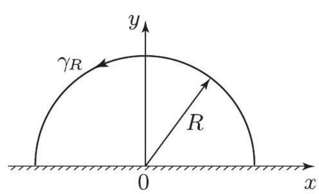

Fig. 31

Suppose that f tends to zero at infinity so that

$$\lim_{R\to\infty} M(R) = 0$$

Then for every $t > 0$ the ratio is valid

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)e^{itz}dz = 0$$

Proof. We have

$$
\begin{aligned}
\left|\int_{\gamma_R} f(z)e^{itz}dz\right| &= \left|\int_0^\pi f\left(Re^{i\theta}\right)e^{-tR\sin\theta+itR\cos\theta} iRe^{i\theta}d\theta\right| \\
&\leqslant \int_0^\pi M(R)Re^{-tR\sin\theta}d\theta
\end{aligned}
$$

To evaluate the last integral, let's use the inequality

$$
\sin\theta \geqslant \frac{2}{\pi}\theta \quad \text{for} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}
$$

(the graph of the sine over the segment $[0, \pi/2]$ lies above the chord, see Fig. 32).

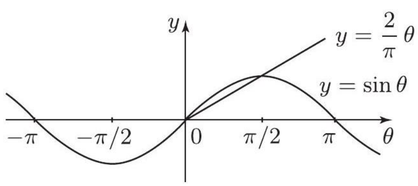

Fig. 32

By replacing $\tau := 2R\theta/\pi$, we get that

$$
\begin{aligned}
\int_0^\pi Re^{-tR\sin\theta}d\theta = 2\int_0^{\frac{\pi}{2}} Re^{-tR\sin\theta}d\theta &\leqslant 2\int_0^{\pi/2} Re^{-2tR\theta/\pi}d\theta \\
&= \pi\int_0^R e^{-t\tau}d\tau = \frac{\pi}{t}\left(1 - e^{-tR}\right),
\end{aligned}
$$

hence the desired result.

8.7 An example for calculating the Fourier transform of rational functions.

Let $a > 0, t > 0$. Integral

$$I(t) := \int_{-\infty}^{\infty} \frac{x \sin tx}{x^2 + a^2} dx$$

it fits if you understand it as

$$\lim_{R \to +\infty} \int_{-R}^{R} \frac{x \sin tx}{x^2 + a^2} dx$$

(This can be shown using the Abel-Dirichlet convergence feature, but it will also be independently proved in the course of our calculation of $I(t)$.) Consider the following limit

$$J(t) := \lim_{R \to +\infty} \int_{-R}^{R} \frac{x e^{itx}}{x^2 + a^2} dx$$

Let's prove that it exists and find its meaning. Denote by

$$f(z) := \frac{z e^{itz}}{z^2 + a^2}$$

continuation of the integrand into the complex plane.

The border of the region

$$D_R := \{z \in \mathbb{C} : \operatorname{Im} z > 0, |z| < R\}$$

it consists of a segment $[-R, R]$ and a semicircle $\gamma_R := \{z \in \mathbb{C} : \operatorname{Im} z \geqslant 0, |z| = R\}$ (see Fig. 33).

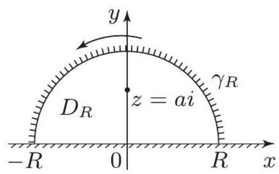

Fig. 33

We claim that the integral

$$\int_{-R}^{R} f(x)dx = \int_{\partial D_R} f(z)dz - \int_{\gamma_R} f(z)dz$$

It is equal to $\pi i e^{-at} + o(1)$ for $R \to \infty$.

Indeed, if $R > a$, then the integral of ∂D_R is equal to

$$2\pi i \operatorname{res}_{z=ai} f(z) = \pi i e^{-at}$$

by the deduction theorem, and the integral by γ_R is $o(1)$ for $R \to \infty$ by Jordan's lemma (it is important here that $t > 0$!). Aiming at $R \to \infty$, we get that

$$J(t) = \pi i e^{-at} \quad \text{for } t > 0.$$

Therefore,

$$I(t) = \operatorname{Im} J(t) = \pi e^{-at} \quad \text{for} \quad t > 0.$$

Due to the odd $\sin tx$ by t, the final answer has the form:

$$I(t) = \begin{cases} \pi e^{-at} & \text{for } t > 0 \\ 0 & \text{for} \quad t = 0 \\ -\pi e^{at} & \text{for } t < 0 \end{cases}$$

Remark 8.1. Since the function

$$\operatorname{Re} \frac{xe^{itx}}{x^2+a^2} = \frac{x \cos tx}{x^2+a^2}$$

odd on the segment $[-R, R]$, then

$$J(t) = iI(t)$$

Thus, our reasoning provides an independent proof of the convergence of the integral $I(t)$ at $t > 0$, independent of the Abel-Dirichlet sign. Moreover, returning at $t < 0$ to the formula

$$\int_{-R}^{R} f(x)dx = \int_{\partial D_R} f(z)dz - \int_{\gamma_R} f(z)dz$$

we see from it now that

$$\lim_{R \to \infty} \int_{\gamma_R} \frac{ze^{itz}}{z^2+a^2} dz = \pi i e^{at} + \pi i e^{-at} \neq 0$$

In other words, Jordan's lemma at $t < 0$ is, generally speaking, incorrect.

Remark 8.2. The described method also allows us to calculate the Fourier transform of any rational function $\frac{P(x)}{Q(x)}$ with $\deg P \leqslant \deg Q - 1$.

8.8 Practical examples

EXAMPLE 1 Residues

a) We know that $z = 1$ is a pole of order two of the function $f(z) = \frac{1}{(z-1)^2(z-3)}$. From the Laurent series obtained in that example valid for the deleted neighborhood of $z = 1$ defined by $0 < |z-1| < 2$

$$f(z) = \frac{-1/2}{(z-1)^2} + \overbrace{\frac{-1/4}{z-1}}^{a_{-1}} - \frac{1}{8} - \frac{z-1}{16} - \cdots$$

we see that the coefficient of $1/(z-1)$ is $a_{-1} = \operatorname{Res}(f(z), 1) = -\frac{1}{4}$.

(b) In Example we saw that $z = 0$ is an essential singularity of $f(z) = e^{3/z}$. Inspection of the Laurent series obtained in that example,

$$e^{3/z} = 1 + \overbrace{\frac{3}{z}}^{a_{-1}} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots$$

$0 < |z| < \infty$, shows that the coefficient of $1/z$ is $a_{-1} = \operatorname{Res}(f(z), 0) = 3$.

EXAMPLE 2 Residue at a Pole

The function $f(z) = \frac{1}{(z-1)^2(z-3)}$ has a simple pole at $z = 3$ and a pole of order 2 at $z = 1$.

Solution Since $z = 3$ is a simple pole, we use (1):

$$\operatorname{Res}(f(z), 3) = \lim_{z\to 3}(z-3)f(z) = \lim_{z\to 3}\frac{1}{(z-1)^2} = \frac{1}{4}$$

Now at the pole of order 2, the result in (2) gives

$$\begin{aligned}\operatorname{Res}(f(z), 1) &= \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}(z-1)^2 f(z) \\ &= \lim_{z\to 1}\frac{d}{dz}\frac{1}{z-3} \\ &= \lim_{z\to 1}\frac{-1}{(z-3)^2} = -\frac{1}{4}\end{aligned}$$

When f is not a rational function, calculating residues by means of (1) or (2) can sometimes be tedious. It is possible to devise alternative residue formulas. In particular, suppose a function f can be written as a quotient $f(z)=g(z)/h(z)$, where g and h are analytic at $z=z_0$. If $g\left(z_0\right) \neq 0$ and if the function h has a zero of order 1 at z_0, then f has a simple pole at $z=z_0$ and

$$\operatorname{Res}\left(f(z), z_0\right)=\frac{g\left(z_0\right)}{h^{\prime}\left(z_0\right)} \tag{4}$$

To derive this result we shall use the definition of a zero of order 1 , the definition of a derivative, and then (1). First, since the function h has a zero of order 1 at z_0, we must have $h\left(z_0\right)=0$ and $h^{\prime}\left(z_0\right) \neq 0$. Second, by definition of the derivative

$$h^{\prime}\left(z_0\right)=\lim _{z \rightarrow z_0} \frac{h(z)-\overbrace{h\left(z_0\right)}^{0}}{z-z_0}=\lim _{z \rightarrow z_0} \frac{h(z)}{z-z_0}$$

We then combine the preceding two facts in the following manner in (1):

$$\operatorname{Res}\left(f(z), z_0\right)=\lim _{z \rightarrow z_0}\left(z-z_0\right) \frac{g(z)}{h(z)}=\lim _{z \rightarrow z_0} \frac{g(z)}{\frac{h(z)}{z-z_0}}=\frac{g\left(z_0\right)}{h^{\prime}\left(z_0\right)}$$

EXAMPLE 3 Using (4) to Compute Residues

The polynomial z^4+1 can be factored as $\left(z-z_1\right)\left(z-z_2\right)\left(z-z_3\right)\left(z-z_4\right)$, where z_1, z_2, z_3, and z_4 are the four distinct roots of the equation $z^4+1=0$ (or equivalently, the four fourth roots of -1). The function

$$f(z)=\frac{1}{z^4+1}$$

has four simple poles. Now we have $z_1=e^{\pi i/4}$, $z_2=e^{3\pi i/4}$, $z_3=e^{5\pi i/4}$, and $z_4=e^{7\pi i/4}$. To compute the residues, we use (4) of this section along with Euler's formula:

$$\begin{aligned}
\operatorname{Res}(f(z),z_1)&=\frac{1}{4z_1^3}=\frac{1}{4}e^{-3\pi i/4}=-\frac{1}{4\sqrt{2}}-\frac{1}{4\sqrt{2}}i\\
\operatorname{Res}(f(z),z_2)&=\frac{1}{4z_2^3}=\frac{1}{4}e^{-9\pi i/4}=\frac{1}{4\sqrt{2}}-\frac{1}{4\sqrt{2}}i\\
\operatorname{Res}(f(z),z_3)&=\frac{1}{4z_3^3}=\frac{1}{4}e^{-15\pi i/4}=\frac{1}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}i\\
\operatorname{Res}(f(z),z_4)&=\frac{1}{4z_4^3}=\frac{1}{4}e^{-21\pi i/4}=-\frac{1}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}i
\end{aligned}$$

Of course, we could have calculated each of the residues in Example 3 using formula (1). But the procedure in this case would have entailed substantially more algebra. For example, we first use the factorization of z^4+1 to write f as:

$$f(z)=\frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

By (1) the residue at, say, the pole z_1 is

$$\begin{aligned}\operatorname{Res}(f(z), z_1) &= \lim_{z\to z_1}(z-z_1)\frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\\ &= \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}\\ &= \frac{1}{\left(e^{\pi i/4}-e^{3\pi i/4}\right)\left(e^{\pi i/4}-e^{5\pi i/4}\right)\left(e^{\pi i/4}-e^{7\pi i/4}\right)}\end{aligned}$$

Then we face the daunting task of simplifying the denominator of the last expression. Finally, we must do this process three more times.

EXAMPLE 4 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{1}{(z-1)^2(z-3)}dz$, where

(a) the contour C is the rectangle defined by $x=0, x=4, y=-1, y=1$,

(b) and the contour C is the circle $|z|=2$.

Solution (a) Since both $z=1$ and $z=3$ are poles within the rectangle we have:

$$\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i[\operatorname{Res}(f(z),1)+\operatorname{Res}(f(z),3)]$$

We found these residues in Example 2. Therefore,

$$\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i\left[\left(-\frac{1}{4}\right)+\frac{1}{4}\right]=0$$

(b) Since only the pole $z=1$ lies within the circle $|z|=2$, we have:

$$\oint_C \frac{1}{(z-1)^2(z-3)}dz = 2\pi i\operatorname{Res}(f(z),1) = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2}i$$

EXAMPLE 5 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{2z+6}{z^2+4}dz$, where the contour C is the circle $|z-i|=2$.

Solution By factoring the denominator as $z^2+4=(z-2i)(z+2i)$ we see that the integrand has simple poles at $-2i$ and $2i$. Because only $2i$ lies within the contour C, it follows that

$$\oint_C \frac{2z+6}{z^2+4}dz = 2\pi i \operatorname{Res}(f(z), 2i)$$

But

$$\begin{aligned}\operatorname{Res}(f(z), 2i) &= \lim_{z\to 2i}(z-2i)\frac{2z+6}{(z-2i)(z+2i)} \\ &= \frac{6+4i}{4i} = \frac{3+2i}{2i}\end{aligned}$$

Hence,

$$\oint_C \frac{2z+6}{z^2+4}dz = 2\pi i\left(\frac{3+2i}{2i}\right) = \pi(3+2i)$$

EXAMPLE 6 Evaluation by the Residue Theorem

Evaluate $\oint_C \frac{e^z}{z^4+5z^3}dz$, where the contour C is the circle $|z|=2$.

Solution Writing the denominator as $z^4+5z^3=z^3(z+5)$ reveals that the integrand $f(z)$ has a pole of order 3 at $z=0$ and a simple pole at $z=-5$. But only the pole $z=0$ lies within the given contour and so we have,

$$\oint_C \frac{e^z}{z^4+5z^3}dz = 2\pi i\,\mathrm{Res}(f(z),0) = 2\pi i\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}z^3\cdot\frac{e^z}{z^3(z+5)}$$
$$= \pi i\lim_{z\to 0}\frac{\left(z^2+8z+17\right)e^z}{(z+5)^3} = \frac{17\pi}{125}i$$

EXAMPLE 7 Evaluation by the Residue Theorem

Evaluate $\oint_C \tan z dz$, where the contour C is the circle $|z| = 2$.

Solution The integrand $f(z) = \tan z = \sin z/\cos z$ has simple poles at the points where $\cos z = 0$. We knew that the only zeros $\cos z$ are the real numbers $z = (2n+1)\pi/2, n = 0, \pm1, \pm2, \ldots$. Since only $-\pi/2$ and $\pi/2$ are within the circle $|z| = 2$, we have

$$\oint_C \tan z dz = 2\pi i\left[\mathrm{Res}\left(f(z), -\frac{\pi}{2}\right) + \mathrm{Res}\left(f(z), \frac{\pi}{2}\right)\right]$$

With the identifications $g(z) = \sin z, h(z) = \cos z$, and $h'(z) = -\sin z$, we see from (4) that

$$\mathrm{Res}\left(f(z), -\frac{\pi}{2}\right) = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$$
$$\mathrm{Res}\left(f(z), \frac{\pi}{2}\right) = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$$

and

Therefore,

$$\oint_C \tan z dz = 2\pi i[-1-1] = -4\pi i.$$

EXAMPLE 8 Evaluation by the Residue Theorem

Evaluate $\oint_C e^{3/z} dz$, where the contour C is the circle $|z| = 1$.

Solution As we have seen, $z = 0$ is an essential singularity of the integrand $f(z) = e^{3/z}$ and so neither formulas (1) and (2) are applicable to find the residue of f at that point. Nevertheless, we saw in Example 1 that the Laurent series of f at $z = 0$ gives $\operatorname{Res}(f(z), 0) = 3$. Hence we have

$$\oint_C e^{3/z} dz = 2\pi i \operatorname{Res}(f(z), 0) = 2\pi i(3) = 6\pi i$$

9 Some Consequences of the Residue Theorem

In this section we shall see how residue theory can be used to evaluate real integrals of the forms

$$\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta \tag{1}$$

$$\int_{-\infty}^{\infty} f(x) dx \tag{2}$$

$$\int_{-\infty}^{\infty} f(x) \cos\alpha x dx \text{ and } \int_{-\infty}^{\infty} f(x) \sin\alpha x dx \tag{3}$$

where F in (1) and f in (2) and (3) are rational functions. For the rational function $f(x) = p(x)/q(x)$ in (2) and (3), we will assume that the polynomials p and q have no common factors.

In addition to evaluating the three integrals just given, we shall demonstrate how to use residues to evaluate real improper integrals that require integration along a branch cut.

The discussion ends with the relationship between the residue theory and the zeros of an analytic function and a consideration of how residues can, in certain cases, be used to find the sum of an infinite series.

9.1 Evaluation of Real Trigonometric Integrals

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta$

The basic idea here is to convert a real trigonometric integral of form (1) into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin.

To do this we begin with (10) of Section 2.2 to parametrize this contour by $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$. We can then write

$$dz = ie^{i\theta} d\theta, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

The last two expressions follow from (2) and (3) of Section 4.3. Since $dz = ie^{i\theta} d\theta = iz d\theta$ and $z^{-1} = 1/z = e^{-i\theta}$, these three quantities are equivalent to

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}\left(z + z^{-1}\right), \quad \sin\theta = \frac{1}{2i}\left(z - z^{-1}\right) \tag{4}$$

The conversion of the integral in (1) into a contour integral is accomplished by replacing, in turn, $d\theta, \cos\theta$, and $\sin\theta$ by the expressions in (4):

$$\oint_C F\left(\frac{1}{2}\left(z+z^{-1}\right), \frac{1}{2i}\left(z-z^{-1}\right)\right)\frac{dz}{iz}$$

where C is the unit circle $|z|=1$.

EXAMPLE 1 A Real Trigonometric Integral

Evaluate $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta$

Solution When we use the substitutions given in (4), the given trigonometric integral becomes the contour integral

$$\oint_C \frac{1}{\left(2+\frac{1}{2}\left(z+z^{-1}\right)\right)^2}\frac{dz}{iz} = \oint_C \frac{1}{\left(2+\frac{z^2+1}{2z}\right)^2}\frac{dz}{iz}$$

Carrying out the algebraic simplification of the integrand then yields

$$\frac{4}{i}\oint_C \frac{z}{\left(z^2+4z+1\right)^2}dz$$

From the quadratic formula we can factor the polynomial z^2+4z+1 as $z^2+4z+1=(z-z_1)(z-z_2)$, where $z_1=-2-\sqrt{3}$ and $z_2=-2+\sqrt{3}$. Thus, the integrand can be written

$$\frac{z}{\left(z^2+4z+1\right)^2} = \frac{z}{(z-z_1)^2(z-z_2)^2}$$

Because only z_2 is inside the unit circle C, we have

$$\oint_C \frac{z}{\left(z^2+4z+1\right)^2}dz = 2\pi i \operatorname{Res}(f(z), z_2)$$

To calculate the residue, we first note that z_2 is a pole of order 2 and so we use (2) of Section 6.5 :

$$\begin{aligned} \operatorname{Res}\left(f(z), z_2\right) &= \lim_{z \to z_2} \frac{d}{dz}\left(z - z_2\right)^2 f(z) = \lim_{z \to z_2} \frac{d}{dz} \frac{z}{\left(z - z_1\right)^2} \\ &= \lim_{z \to z_2} \frac{-z - z_1}{\left(z - z_1\right)^3} = \frac{1}{6\sqrt{3}} \end{aligned}$$

Hence, $\frac{4}{i} \oint_C \frac{z}{\left(z^2+4z+1\right)} dz = \frac{4}{i} \cdot 2\pi i \operatorname{Res}\left(f(z), z_1\right) = \frac{4}{i} \cdot 2\pi i \cdot \frac{1}{6\sqrt{3}}$
and, finally,

$$\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta = \frac{4\pi}{3\sqrt{3}}$$

9.2 Evaluation of Real Improper Integrals

Integrals of the Form $\int_{-\infty}^{\infty} f(x)dx$

Suppose $y = f(x)$ is a real function that is defined and continuous on the interval $[0, \infty)$. In elementary calculus the improper integral $I_1 = \int_0^{\infty} f(x)dx$ is defined as the limit

$$I_1 = \int_0^{\infty} f(x)dx = \lim_{R \to \infty} \int_0^R f(x)dx \tag{5}$$

If the limit exists, the integral I_1 is said to be convergent; otherwise, it is divergent. The improper integral $I_2 = \int_{-\infty}^{0} f(x)dx$ is defined similarly:

$$I_2 = \int_{-\infty}^{0} f(x)dx = \lim_{R \to \infty} \int_{-R}^{0} f(x)dx \tag{6}$$

Finally, if f is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)dx$ is defined to be

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = I_1 + I_2 \tag{7}$$

provided both integrals I_1 and I_2 are convergent. If either one, I_1 or I_2, is divergent, then $\int_{-\infty}^{\infty} f(x)dx$ is divergent. It is important to remember that the right-hand side of (7) is not the same as

$$\lim_{R\to\infty} \left[\int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx \right] = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{8}$$

For the integral $\int_{-\infty}^{\infty} f(x)dx$ to be convergent, the limits (5) and (6) must exist independently of one another. But, in the event that we know (a priori) that an improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, we can then evaluate it by means of the single limiting process given in (8):

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{9}$$

On the other hand, the symmetric limit in (9) may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is divergent. For example, the integral $\int_{-\infty}^{\infty} xdx$ is divergent since $\lim_{R\to\infty} \int_{0}^{R} xdx = \lim_{R\to\infty} \frac{1}{2}R^2 = \infty$. However, (9) gives

$$\lim_{R\to\infty} \int_{-R}^{R} xdx = \lim_{R\to\infty} \frac{1}{2} \left[R^2 - (-R)^2 \right] = 0 \tag{10}$$

The limit in (9), if it exists, is called the Cauchy principal value (P.V.) of the integral and is written

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \tag{11}$$

In (10) we have shown that P.V. $\int_{-\infty}^{\infty} x dx = 0$. To summarize:

Cauchy Principal Value When an integral of form (2) converges, its Cauchy principal value is the same as the value of the integral. If the integral diverges, it may still possess a Cauchy principal value (11).

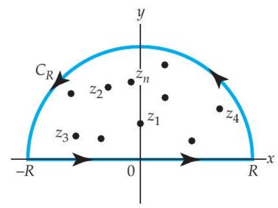

Figure 6.11 Semicircular contour

One final point about the Cauchy principal value: Suppose $f(x)$ is continuous on $(-\infty, \infty)$ and is an even function, that is, $f(-x) = f(x)$. Then its graph is symmetric with respect to the y-axis and as a consequence

$$\int_{-R}^{0} f(x)dx = \int_{0}^{R} f(x)dx \tag{12}$$

and $\int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx.$

From (12) and (13) we conclude that if the Cauchy principal value (11) exists, then both $\int_{0}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ converge. The values of the integrals are

$\int_{0}^{\infty} f(x)dx = \frac{1}{2}$ P.V. $\int_{-\infty}^{\infty} f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx =$ P.V. $\int_{-\infty}^{\infty} f(x)dx$.

To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where the rational function $f(x) = p(x)/q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = p(z)/q(z)$ in the upper half-plane $\text{Im}(z) > 0$. See Figure 6.11. By Theorem 6.16 of Section 6.5 we have

$$\oint_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_k\right),$$

where $z_k, k = 1, 2, \ldots, n$ denotes poles in the upper half-plane. If we can show that the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\left(f(z), z_k\right) \quad (14)$$

EXAMPLE 2 Cauchy P.V. of an Improper Integral

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)} dx$.

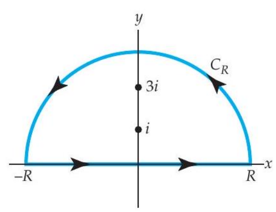

Figure 6.12 Contour for Example 2

Solution Let $f(z) = 1/\left(z^2+1\right)\left(z^2+9\right)$. Since

$$\left(z^2+1\right)\left(z^2+9\right) = (z-i)(z+i)(z-3i)(z+3i)$$

we take C be the closed contour consisting of the interval $[-R, R]$ on the x-axis and the semicircle C_R of radius $R > 3$. As seen from Figure 6.12,

$$
\begin{aligned}
&\oint_C \frac{1}{\left(z^2+1\right)\left(z^2+9\right)} dz = \\
&= \int_{-R}^{R} \frac{1}{\left(x^2+1\right)\left(x^2+9\right)} dx + \int_{C_R} \frac{1}{\left(z^2+1\right)\left(z^2+9\right)} dz = \\
&= I_1 + I_2
\end{aligned}
$$

$$I_1 + I_2 = 2\pi i[\operatorname{Res}(f(z), i) + \operatorname{Res}(f(z), 3i)].$$

At the simple poles $z = i$ and $z = 3i$ we find, respectively, so that

$$\operatorname{Res}(f(z), i) = \frac{1}{16i} \quad \text{and} \quad \operatorname{Res}(f(z), 3i) = -\frac{1}{48i}$$

$$I_1 + I_2 = 2\pi i \left[\frac{1}{16i} + \left(-\frac{1}{48i}\right)\right] = \frac{\pi}{12} \tag{15}$$

We now want to let $R \to \infty$ in (15). Before doing this, we use the inequality (10) of Section 1.2 to note that on the contour C_R,

$$
\begin{aligned}
\left|\left(z^2+1\right)\left(z^2+9\right)\right| = \left|z^2+1\right| \cdot \left|z^2+9\right| \geq \left||z^2| - 1\right| \cdot \left||z^2| - 9\right| = \\
= \left(R^2-1\right)\left(R^2-1\right)\left(R^2-9\right).
\end{aligned}
$$

Since the length L of the semicircle is πR, it follows from the ML-inequality, Theorem 5.3 of Section 5.2, that

$$|I_2| = \left|\int_{C_R} \frac{1}{\left(z^2+1\right)\left(z^2+9\right)} dz\right| \leq \frac{\pi R}{\left(R^2-1\right)\left(R^2-9\right)}$$

This last result shows that $|I_2| \to 0$ as $R \to \infty$, and so we conclude that $\lim_{R\to\infty} I_2 = 0$. It follows from (15) that $\lim_{R\to\infty} I_1 = \pi/12$; in other words,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{1}{(x^2+1)(x^2+9)}dx=\frac{\pi}{12} \text{ or P.V. } \int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+9)}dx=\frac{\pi}{12}$$

Because the integrand in Example 2 is an even function, the existence of the Cauchy principal value implies that the original integral converges to $\pi/12$.

It is often tedious to have to show that the contour integral along C_R approaches zero as $R\to\infty$. Sufficient conditions under which this behavior is always true are summarized in the next theorem.

Theorem 6.17 Behavior of Integral as $R\to\infty$
Suppose $f(z)=\frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m\geq n+2$. If C_R is a semicircular contour $z=Re^{i\theta}, 0\leq\theta\leq\pi$, then $\int_{C_R} f(z)dz\to 0$ as $R\to\infty$.

In other words, the integral along C_R approaches zero as $R\to\infty$ when the denominator of f is of a power at least 2 more than its numerator. The proof of this fact follows in the same manner as in Example 2. Notice in that example, the conditions stipulated in Theorem 6.17 are satisfied, since degree of $p(z)=1$ is 0 and the degree of $q(z)=\left(z^2+1\right)\left(z^2+9\right)$ is 4 .

EXAMPLE 3 Cauchy P.V. of an Improper Integral
Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$

Solution By inspection of the integrand we see that the conditions given in Theorem 6.17 are satisfied. Moreover, we know from Example 3 of Section 6.5 that $f(z)=1/\left(z^4+1\right)$ has simple poles in the upper half-plane at $z_1=e^{\pi i/4}$ and $z_2=e^{3\pi i/4}$. We also saw in that example that the residues at these poles are

$$\operatorname{Res}(f(z), z_1) = -\frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i \quad \text{and} \quad \operatorname{Res}(f(z), z_2) = \frac{1}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}i$$

Thus, by (14),

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^4+1} dx = 2\pi i \left[\operatorname{Res}(f(z), z_1) + \operatorname{Res}(f(z), z_2)\right] = \frac{\pi}{\sqrt{2}}$$

Since the integrand is an even function, the original integral converges to $\pi/\sqrt{2}$

Integrals of the Form $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$ Because improper integrals of the form $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$ are encountered in applications of Fourier analysis, they often are referred to as Fourier integrals. Fourier integrals appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$ In view

of Euler's formula $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, where α is a positive real number, we can write

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x dx \qquad (16)$$

whenever both integrals on the right-hand side converge. Suppose $f(x) = p(x)/q(x)$ is a rational function that is continuous on $(-\infty, \infty)$. Then both Fourier integrals in (10) can be evaluated at the same time by considering the complex integral $\int_C f(z)e^{i\alpha z}dz$, where $\alpha > 0$, and the contour C again consists of the interval $[-R, R]$ on the real axis and a semicircular contour C_R with radius large enough to enclose the poles of $f(z)$ in the upper-half plane.

Before proceeding, we give, without proof, sufficient conditions under which the contour integral along C_R approaches zero as $R \to \infty$.

Theorem 6.18 Behavior of Integral as $R \to \infty$
Suppose $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where the degree of $p(z)$ is n and the degree of $q(z)$ is $m \geq n+2$. If C_R is a semicircular contour $z = Re^{i\theta}, 0 \leq \theta \leq \pi$, and $\alpha > 0$, then $\int_{C_R} f(z)e^{i\alpha z}dz \to 0$ as $R \to \infty$.

EXAMPLE 4 Using Symmetry
Evaluate the Cauchy principal value of $\int_0^\infty \frac{x \sin x}{x^2+9} dx$.

Solution First note that the limits of integration in the given integral are not from $-\infty$ to ∞ as required by the method just described. This can be remedied by observing that since the integrand is an even function of x (verify), we can write

$$\int_0^\infty \frac{x \sin x}{x^2+9} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin x}{x^2+9} dx \tag{17}$$

With $\alpha = 1$ we now form the contour integral

$$\oint_C \frac{z}{z^2+9} e^{iz} dz$$

where C is the same contour shown in Figure 6.12. By Theorem 6.16,

$$\int_{C_R} \frac{z}{z^2+9} e^{iz} dz + \int_{-R}^{R} \frac{x}{x^2+9} e^{ix} dx = 2\pi i \operatorname{Res}\left(f(z)e^{iz}, 3i\right)$$

where $f(z) = z/\left(z^2+9\right)$, and

$$\operatorname{Res}\left(f(z)e^{iz}, 3i\right) = \left.\frac{ze^{iz}}{2z}\right|_{z=3i} = \frac{e^{-3}}{2}$$

Then, from Theorem 6.18 we conclude $\int_{C_R} f(z)e^{iz}dz \to 0$ as $R \to \infty$, and so

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x}{x^2+9} e^{ix} dx = 2\pi i \left(\frac{e^{-3}}{2} \right) = \frac{\pi}{e^3} i$$

But by (16),

$$\int_{-\infty}^{\infty} \frac{x}{x^2+9} e^{ix} dx = \int_{-\infty}^{\infty} \frac{x \cos x}{x^2+9} dx + i \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{e^3} i$$

Equating real and imaginary parts in the last line gives the bonus result

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \cos x}{x^2+9} dx = 0 \quad \text{along with} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{e^3} \tag{18}$$

Finally, in view of the fact that the integrand is an even function, we obtain the value of the prescribed integral:

$$\int_{0}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{\pi}{2e^3}$$

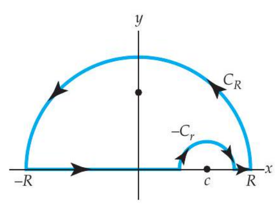

Figure 6.13 Indented contour

Indented Contours The improper integrals of forms (2) and (3) that we have considered up to this point were continuous on the interval $(-\infty, \infty)$.

In other words, the complex function $f(z) = p(z)/q(z)$ did not have poles on the real axis. In the situation where f has poles on the real axis, we must modify the procedure illustrated in Examples 2-4. For example, to evaluate $\int_{-\infty}^{\infty} f(x)dx$ by residues when $f(z)$ has a pole at $z = c$, where c is a real number, we use an indented contour as illustrated in Figure 6.13. The symbol C_r denotes a semicircular contour centered at $z = c$ and oriented in the positive direction. The next theorem is important to this discussion.

Theorem 6.19 Behavior of Integral as $r \to 0$

Suppose f has a simple pole $z = c$ on the real axis. If C_r is the contour defined by $z = c + re^{i\theta}, 0 \le \theta \le \pi$, then

$$\lim_{r\to 0} \int_{C_r} f(z)dz = \pi i \operatorname{Res}(f(z), c)$$

Proof Since f has a simple pole at $z = c$, its Laurent series is

$$f(z) = \frac{a_{-1}}{z-c} + g(z)$$

where $a_{-1} = \operatorname{Res}(f(z), c)$ and g is analytic at the point c. Using the Laurent series and the parametrization of C_r we have

$$\int_{C_r} f(z)dz = a_{-1}\int_0^{\pi} \frac{ire^{i\theta}}{re^{i\theta}}d\theta + ir\int_0^{\pi} g\left(c + re^{i\theta}\right)e^{i\theta}d\theta = I_1 + I_2 \quad (19)$$

First, we see that

$$I_1 = a_{-1}\int_0^{\pi} \frac{ire^{i\theta}}{re^{i\theta}}d\theta = a_{-1}\int_0^{\pi} id\theta = \pi i a_{-1} = \pi i \operatorname{Res}(f(z), c)$$

Next, g is analytic at c, and so it is continuous at this point and bounded in a neighborhood of the point; that is, there exists an $M > 0$ for which $\left|g\left(c + re^{i\theta}\right)\right| \leq M$. Hence,

$$|I_2| = \left|ir \int_0^{\pi} g\left(c + re^{i\theta}\right) d\theta\right| \leq r \int_0^{\pi} M d\theta = \pi r M$$

It follows from this last inequality that $\lim_{r\to 0} |I_2| = 0$ and consequently $\lim_{r\to 0} I_2 = 0$. By taking the limit of (19) as $r \to 0$, the theorem is proved.

EXAMPLE 5 Using an Indented Contour

Evaluate the Cauchy principal value of $\int_{-\infty}^{\infty} \frac{\sin x}{x\left(x^2-2x+2\right)} dx$.

Solution Since the integral is of the type given in (3), we consider the contour integral

$$\oint_C \frac{e^{iz}}{z\left(z^2 - 2z + 2\right)} dz$$

The function $f(z) = 1/z\left(z^2 - 2z + 2\right)$ has a pole at $z = 0$ and at $z = 1 + i$ in the upper half-plane. The contour C, shown in Figure 6.14, is indented at the origin. Adopting an obvious condensed notation, we have

$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i \operatorname{Res}\left(f(z)e^{iz}, 1+i\right) \qquad (20)$$

where $\int_{-C_r} = -\int_{C_r}$. If we take the limits of (20) as $R \to \infty$ and as $r \to 0$, it follows from Theorems 6.18 and 6.19 that

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x\left(x^2 - 2x + 2\right)} dx - \pi i \operatorname{Res}\left(f(z)e^{iz}, 0\right) = 2\pi i \operatorname{Res}\left(f(z)e^{iz}, 1+i\right)$$

Now,

$$\operatorname{Res}\left(f(z)e^{iz},0\right)=\frac{1}{2} \quad \text{and} \quad \operatorname{Res}\left(f(z)e^{iz},1+i\right)=-\frac{e^{-1+i}}{4}(1+i)$$

Therefore,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x\left(x^2-2x+2\right)}dx=\pi i\left(\frac{1}{2}\right)+2\pi i\left(-\frac{e^{-1+i}}{4}(1+i)\right)$$

Using $e^{-1+i}=e^{-1}(\cos 1+i\sin 1)$, simplifying, and then equating real and imaginary parts, we get from the last equality

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{x\left(x^2-2x+2\right)}dx=\frac{\pi}{2}e^{-1}(\sin 1+\cos 1)$$
$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{x\left(x^2-2x+2\right)}dx=\frac{\pi}{2}\left[1+e^{-1}(\sin 1-\cos 1)\right]$$

9.3 Integration along a Branch Cut

Branch Point at $z=0$ In the next discussion we examine integrals of the form $\int_0^\infty f(x)dx$, where the integrand $f(x)$ is algebraic. But similar to Example 5, these integrals require a special type of contour because when $f(x)$ is converted to a complex function, the resulting integrand $f(z)$ has, in addition to poles, a nonisolated singularity at $z=0$. Before proceeding, the reader is encouraged to review the discussion on branch cuts in Sections 2.6 and 4.1.

In the example that follows we consider a special case of the real integral

$$\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx \tag{21}$$

where α is a real constant restricted to the interval $0 < \alpha < 1$. Observe that when $\alpha = \frac{1}{2}$ and x is replaced by z, the integrand of (12) becomes the multiple-valued function

$$\frac{1}{z^{1/2}(z+1)} \tag{22}$$

The origin is a branch point of (22) since $z^{1/2}$ has two values for any $z \neq 0$. If you envision traveling in a complete circle around the origin $z = 0$, starting from a point $z = re^{i\theta}, r > 0$, you return to the same starting point z, but θ has increased by 2π. Correspondingly, the value of $z^{1/2}$ changes from $z^{1/2} = \sqrt{r}e^{i\theta/2}$ to a different value or different branch:

$$z^{1/2} = \sqrt{r}e^{i(\theta+2\pi)/2} = \sqrt{r}e^{i\theta/2}e^{i\pi} = -\sqrt{r}e^{i\theta/2}$$

Recall, we can force $z^{1/2}$ to be single valued by restricting θ to some interval of length 2π. For (22), if we choose the positive x-axis as a branch cut, in other words by restricting θ to $0 < \theta < 2\pi$, we then guarantee that $z^{1/2} = \sqrt{r}e^{i\theta/2}$ is single valued. See page 126 .

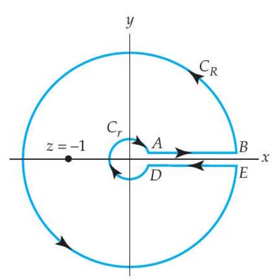

Figure 6.15 Contour for Example 6

EXAMPLE 6 Integration along a Branch Cut

Evaluate $\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx$

Solution First observe that the real integral is improper for two reasons. Notice an infinite discontinuity at $x = 0$ and the infinite limit of integration. Moreover, it can be argued from the facts that the integrand behaves like $x^{-1/2}$ near the origin and like $x^{-3/2}$ as $x \to \infty$, that the integral converges.

We form the integral $\oint_C \frac{1}{z^{1/2}(z+1)}dz$, where C is the closed contour shown in Figure 6.15 consisting of four components: C_r and C_R are portions of circles, and AB and ED are parallel horizontal line segments running along opposite sides of the branch cut. The integrand $f(z)$ of the contour integral is single valued and analytic on and within C, except for the simple pole at $z = -1 = e^{\pi i}$. Hence we can write

or

$$\oint_C \frac{1}{z^{1/2}(z+1)}dz = 2\pi i \operatorname{Res}(f(z), -1)$$

$$\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \operatorname{Res}(f(z), -1). \tag{23}$$

Despite what is shown in Figure 6.15, it is permissible to think that the line segments AB and ED actually rest on the positive real axis, more precisely, AB coincides with the upper side of the positive real axis for which $\theta = 0$ and ED coincides with the lower side of the positive real axis for which $\theta = 2\pi$. On $AB, z = xe^{0i}$, and on $ED, z = xe^{(0+2\pi)i} = xe^{2\pi i}$, so that

$$\int_{ED} = \int_R^r \frac{\left(xe^{2\pi i}\right)^{-1/2}}{xe^{2\pi i}+1}\left(e^{2\pi i}dx\right) = -\int_R^r \frac{x^{-1/2}}{x+1}dx = \int_r^R \frac{x^{-1/2}}{x+1}dx \quad (24)$$

and

$$\int_{AB} = \int_r^R \frac{\left(xe^{0i}\right)^{-1/2}}{xe^{0i}+1}\left(e^{0i}dx\right) = \int_r^R \frac{x^{-1/2}}{x+1}dx \qquad (25)$$

Now with $z = re^{i\theta}$ and $z = Re^{i\theta}$ on C_r and C_R, respectively, it can be shown, by analysis similar to that given in Example 2 and in the proof of Theorem 6.17, that $\int_{C_r} \to 0$ as $r \to 0$ and $\int_{C_R} \to 0$ as $R \to \infty$. Thus from (23), (24), and (25) we see that

$$\lim_{\substack{r\to 0\\ R\to\infty}} \left[\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \operatorname{Res}(f(z), -1)\right]$$

is the same as

$$2\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = 2\pi i \operatorname{Res}(f(z), -1) \qquad (26)$$

Finally, from (4) of Section 6.5,

$$\operatorname{Res}(f(z), -1) = z^{-1/2}\Big|_{z=e^{\pi i}} = e^{-\pi i/2} = -i$$

and so (26) yields the result

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)}dx = \pi$$

9.4 The Argument Principle and Rouché's Theorem

Argument Principle Unlike the foregoing discussion in which the focus was on the evaluation of real integrals, we next apply residue theory to the location of zeros of an analytic function. To get to that topic, we must first consider two theorems that are important in their own right.

In the first theorem we need to count the number of zeros and poles of a function f that are located within a simple closed contour C; in this counting we include the order or multiplicity of each zero and pole. For example, if

$$f(z)=\frac{(z-1)(z-9)^4(z+i)^2}{\left(z^2-2z+2\right)^2(z-i)^6(z+6i)^7} \tag{27}$$

and C is taken to be the circle $|z|=2$, then inspection of the numerator of f reveals that the zeros inside C are $z=1$ (a simple zero) and $z=-i$ (a zero of order or multiplicity 2). Therefore, the number N_0 of zeros inside C is taken to be $N_0=1+2=3$. Similarly, inspection of the denominator of f shows, after factoring z^2-2z+2, that the poles inside C are $z=1-i$ (pole of order 2), $z=1+i$ (pole of order 2), and $z=i$ (pole of order 6). The number N_p of poles inside C is taken to be $N_p=2+2+6=10$.

Theorem 6.20 Argument Principle

Let C be a simple closed contour lying entirely within a domain D. Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z)\neq 0$ on C. Then

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz=N_0-N_p \tag{28}$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C. In determining N_0 and N_p, zeros and poles are counted according to their order or multiplicities.

Proof We start with a reminder that when we use the symbol $\oint_C$ for a contour, this signifies that we are integrating in the positive direction around the closed curve C.

The integrand $f'(z)/f(z)$ in (28) is analytic in and on the contour C except at the points in the interior of C where f has a zero or a pole. If z_0 is a zero of order n of f inside C, then by (5) of Section 6.4 we can write $f(z) = (z - z_0)^n \phi(z)$, where ϕ is analytic at z_0 and $\phi(z_0) \neq 0$. We differentiate f by the product rule,

$$f'(z) = (z - z_0)^n \phi'(z) + n(z - z_0)^{n-1} \phi(z)$$

and divide this expression by f. In some punctured disk centered at z_0, we have

$$\frac{f'(z)}{f(z)} = \frac{(z - z_0)^n \phi'(z) + n(z - z_0)^{n-1} \phi(z)}{(z - z_0)^n \phi(z)} = \frac{\phi'(z)}{\phi(z)} + \frac{n}{z - z_0} \qquad (29)$$

The result in (29) shows that the integrand $f'(z)/f(z)$ has a simple pole at z_0 and the residue at that point is

$$\begin{aligned} \operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_0\right) &= \lim_{z \to z_0} (z - z_0) \left[\frac{\phi'(z)}{\phi(z)} + \frac{n}{z - z_0}\right] \\ &= \lim_{z \to z_0} \left[(z - z_0)\frac{\phi'(z)}{\phi(z)} + n\right] = 0 + n = n \end{aligned} \qquad (30)$$

which is the order of the zero z_0.

Now if z_p is a pole of order m of f within C we can write $f(z) = g(z)/\left(z - z_p\right)^m$, where g is analytic at z_p and $g\left(z_p\right) \neq 0$. By differentiating, in this case $f(z) = \left(z - z_p\right)^{-m} g(z)$, we have

$$f'(z) = \left(z - z_p\right)^{-m} g'(z) - m\left(z - z_p\right)^{-m-1} g(z)$$

Therefore, in some punctured disk centered at z_p,

$$\frac{f'(z)}{f(z)} = \frac{\left(z - z_p\right)^{-m} g'(z) - m\left(z - z_p\right)^{-m-1} g(z)}{\left(z - z_p\right)^{-m} g(z)} = \frac{g'(z)}{g(z)} + \frac{-m}{z - z_p} \quad (31)$$

We see from (31) that the integrand $f'(z)/f(z)$ has a simple pole at z_p. Proceeding as in (30), we also see that the residue at z_p is equal to $-m$, which is the negative of the order of the pole of f.

Finally, suppose that $z_{0_1}, z_{0_2}, \ldots, z_{0_r}$ and $z_{p_1}, z_{p_2}, \ldots, z_{p_s}$ are the zeros and poles of f within C and suppose further that the order of the zeros are $n_1, n_2, \ldots, n_r$ and that order of the poles are $m_1, m_2, \ldots, m_s$. Then each of these points is a simple pole of the integrand $f'(z)/f(z)$ with corresponding residues $n_1, n_2, \ldots, n_r$ and $-m_1, -m_2, \ldots, -m_s$. It follows from the residue theorem (Theorem 6.16) that $\oint_C f'(z)dz/f(z)$ is equal to $2\pi i$ times the sum of the residues at the poles:

$$\begin{aligned}
\oint_C \frac{f'(z)}{f(z)}dz &= 2\pi i\left[\sum_{k=1}^{r} \operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_{0_k}\right) + \sum_{k=1}^{s} \operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_{p_k}\right)\right] \\
&= 2\pi i\left[\sum_{k=1}^{r} n_k + \sum_{k=1}^{s}\left(-m_k\right)\right] = 2\pi i\left[N_0 - N_p\right]
\end{aligned}$$

Dividing by $2\pi i$ establishes (28).

To illustrate Theorem 6.20, suppose the simple closed contour is $|z| = 2$ and the function f is the one given in (27). The result in (28) indicates that in the evaluation of $\oint_C f'(z)dz/f(z)$, each zero of f within C contributes $2\pi i$ times the order of multiplicity of the zero and each pole contributes $2\pi i$ times the negative of the order of the pole:

$$\oint_C \frac{f'(z)}{f(z)}dz = \overbrace{[2\pi i(1) + 2\pi i(2)]}^{\substack{\text{contribution}\\ \text{of zeros of } f}} + \overbrace{[2\pi i(-2) + 2\pi i(-2) + 2\pi i(-6)]}^{\substack{\text{contribution}\\ \text{of poles of } f}} = -14\pi i$$

Why the Name? Why is Theorem 6.20 called the argument principle? This question may have occurred to you since no reference is made in the proof of the theorem to any arguments of complex quantities. But in point of fact there is a relation between the number $N_0 - N_p$ in Theorem 6.20 and $\arg(f(z))$. More precisely,

$N_0 - N_p = \frac{1}{2\pi}$ [change in $\arg(f(z))$ as z traverses C once in the positive direction].

This principle can be easily verified using the simple function $f(z) = z^2$ and the unit circle $|z| = 1$ as the simple closed contour C in the z-plane. Because the function f has a zero of multiplicity 2 within C and no poles, we have $N_0 - N_p = 2$. Now, if C is parametrized by $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$, then its image C' in the w-plane under the mapping $w = z^2$ is $w = e^{i2\theta}, 0 \leq \theta \leq 2\pi$, which is the unit circle $|w| = 1$. As z traverses C once starting at $z = 1(\theta = 0)$ and finishing at $z = 1(\theta = 2\pi)$, we see $\arg(f(z)) = \arg(w) = 2\theta$ increases from 0 to 4π. Put another way, w traverses or winds around the circle $|w| = 1$ twice. Thus, $\frac{1}{2\pi}$[change in $\arg(f(z))$ as z traverses C once in the positive direction] $= \frac{1}{2\pi}[4\pi - 0] = 2$.

Rouché's Theorem The next result follows as a consequence of the argument principle. The theorem is helpful in determining the number of zeros of an analytic function.

Theorem 6.21 Rouché's Theorem
Let C be a simple closed contour lying entirely within a domain D. Suppose f and g are analytic in D. If the strict inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C, then f and g have the same number of zeros (counted according to their order or multiplicities) inside C.

Proof We start with the observation that the hypothesis "the inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C" indicates that both f and g have no zeros on the contour C.

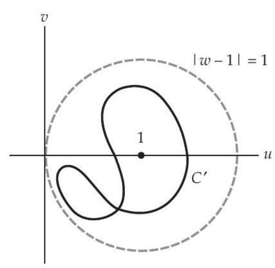

Figure 6.16 Image of C lies within the disk $|w - 1| < 1$.

From $|f(z) - g(z)| = |g(z) - f(z)|$, we see that by dividing the inequality by $|f(z)|$ we have, for all z on C,

$$|F(z) - 1| < 1 \tag{32}$$

where $F(z) = g(z)/f(z)$. The inequality in (32) shows that the image C' in the w-plane of the curve C under the mapping $w = F(z)$ is a closed path and must lie within the unit open disk $|w - 1| < 1$ centered at $w = 1$. See Figure 6.16. As a consequence, the curve C' does not enclose $w = 0$, and therefore $1/w$ is analytic in and on C'. By the Cauchy-Goursat theorem,

$$\int_{C'} \frac{1}{w} dw = 0 \quad \text{or} \quad \oint_C \frac{F'(z)}{F(z)} dz = 0 \tag{33}$$

since $w = F(z)$ and $dw = F'(z)dz$. From the quotient rule,

$$F'(z) = \frac{f(z)g'(z) - g(z)f'(z)}{[f(z)]^2}$$

we get

$$\frac{F'(z)}{F(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$$

Using the last expression in the second integral in (33) then gives

$$\oint_C \left[\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right] dz = 0 \quad \text{or} \quad \oint_C \frac{g'(z)}{g(z)} dz = \oint_C \frac{f'(z)}{f(z)} dz$$

It follows from (28) of Theorem 6.20 , with $N_p = 0$, that the number of zeros of g inside C is the same as the number of zeros of f inside C.

EXAMPLE 7 Location of Zeros

Locate the zeros of the polynomial function $g(z) = z^9 - 8z^2 + 5$.

Solution We begin by choosing $f(z) = z^9$ because it has the same number of zeros as g. Since f has a zero of order 9 at the origin $z = 0$, we begin our search for the zeros of g by examining circles centered at $z = 0$. In other words, if we can establish that $|f(z) - g(z)| < |f(z)|$ for all z on some circle $|z| = R$, then Theorem 6.21 states that f and g have the same number of zeros within the disk $|z| < R$. Now by the triangle inequality (6) of Section 1.2

$$|f(z) - g(z)| = \left|z^9 - \left(z^9 - 8z^2 + 5\right)\right| = \left|8z^2 - 5\right| \leq 8|z|^2 + 5$$

Also, $|f(z)| = |z|^9$. Observe that $|f(z) - g(z)| < |f(z)|$ or $8|z|^2 + 5 < |z|^9$ is not true for all points on the circle $|z| = 1$, so we can draw no conclusion. However, by expanding the search to the larger circle $|z| = \frac{3}{2}$ we see

$$|f(z) - g(z)| \leq 8|z|^2 + 5 = 8\left(\frac{3}{2}\right)^2 + 5 = 23 < \left(\frac{3}{2}\right)^9 = |f(z)| \quad (34)$$

since $\left(\frac{3}{2}\right)^9 \approx 38.44$. We conclude from (34) that because f has a zero of order 9 within the disk $|z| < \frac{3}{2}$, all nine zeros of g lie within the same disk.

By slightly subtler reasoning, we can demonstrate that the function g in Example 7 has some zeros inside the unit disk $|z| < 1$. To see this suppose we choose $f(z) = -8z^2 + 5$. Then for all z on $|z| = 1$,

$$|f(z) - g(z)| = \left|\left(-8z^2 + 5\right) - \left(z^9 - 8z^2 + 5\right)\right| = \left|-z^9\right| = |z|^9 = (1)^9 = 1 \quad (35)$$

But from (10) of Section 1.2 we have, for all z on $|z| = 1$,

$$|f(z)| = |-f(z)| = \left|8z^2 - 5\right| \geq |8|z|^2 - |-5|| = |8 - 5| = 3 \quad (36)$$

The values in (35) and (36) show, for all z on $|z| = 1$, that $|f(z) - g(z)| < |f(z)|$. Because f has two zeros within $|z| < 1$ (namely, $\pm\sqrt{\frac{5}{8}} \approx \pm 0.79\Big)$, we can conclude from Theorem 6.21 that two zeros of g also lie within this disk.

We can continue the reasoning of the previous paragraph. Suppose now we choose $f(z) = 5$ and $|z| = \frac{1}{2}$. Then for all z on $|z| = \frac{1}{2}$,

$$|f(z) - g(z)| = \left|5 - \left(z^9 - 8z^2 + 5\right)\right| = \left|-z^9 + 8z^2\right| \leq$$

$$\leq |z|^9 + 8|z|^2 = \left(\frac{1}{2}\right)^9 + 2 \approx 2.002$$

We now have $|f(z) - g(z)| < |f(z)| = 5$ for all z on $|z| = \frac{1}{2}$. Since f has no zeros within the disk $|z| < \frac{1}{2}$, neither does g. At this point we are able to conclude that all nine zeros of $g(z) = z^9 - 8z^2 + 5$ lie within the annular region $\frac{1}{2} < |z| < \frac{3}{2}$; two of these zeros lie within $\frac{1}{2} < |z| < 1$.

9.5 Summing Infinite Series

Using cot πz In some specialized circumstances, the residues at the simple poles of the trigonometric function cot πz enable us to find the sum of an infinite series.

In Section 4.3 we saw that the zeros of $\sin z$ were the real numbers $z = k\pi$,

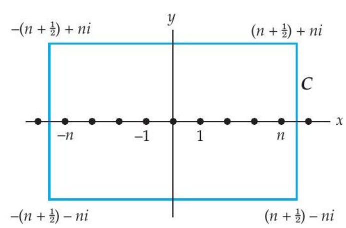

Figure 6.17 Rectangular contour C enclosing poles of (37)

$k = 0, \pm 1, \pm 2, \ldots$

Thus the function cot πz has simple poles at the zeros of $\sin \pi z$, which are $\pi z = k\pi$ or $z = k, k = 0, \pm 1, \pm 2, \ldots$. If a polynomial function $p(z)$ has (i) real coefficients, (ii) degree $n \geq 2$, and (iii) no integer zeros, then the function

$$f(z) = \frac{\pi \cot \pi z}{p(z)} \tag{37}$$

has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \ldots$ from cot πz and a finite number of poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$ from the zeros of $p(z)$. The closed rectangular contour C shown in Figure 6.17 has vertices $\left(n + \frac{1}{2}\right) + ni, -\left(n + \frac{1}{2}\right) + ni, -\left(n + \frac{1}{2}\right) - ni$, and $\left(n + \frac{1}{2}\right) - ni$, where n is taken large
enough so that C encloses the simple poles $z = 0, \pm 1, \pm 2, \ldots, \pm n$ and all of the poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$. By the residue theorem,

$$\oint_C \frac{\pi \cot \pi z}{p(z)} dz = 2\pi i \left[\sum_{k=-n}^{n} \text{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k \right) + \sum_{j=1}^{r} \text{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j} \right) \right] \tag{38}$$

In a manner similar to that used several times in the discussion in Subsection 6.6.2, it can be shown that $\oint_C \pi \cot \pi z dz/p(z) \to 0$ as $n \to \infty$ and so (38) becomes $0 = \sum_k$ residues $+ \sum_j$ residues. That is,

$$\sum_{k=-\infty}^{\infty} \text{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k \right) = -\sum_{j=1}^{r} \text{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j} \right) \tag{39}$$

Now from (4) of Section 6.5 (with the identifications $g(z) = \pi \cos \pi z / p(z)$, $h(z) = \sin \pi z, h'(z) = \pi \cos \pi z$), it is a straightforward task to compute the residues at the simple poles $0, \pm 1, \pm 2, \ldots$:

$$\text{Res}\left(\frac{\pi \cot \pi z}{p(z)}, k \right) = \frac{\pi \cos k\pi / p(k)}{\pi \cos k\pi} = \frac{1}{p(k)} \tag{40}$$

By combining (40) and (39) we arrive at our desired result

$$\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}, z_{p_j}\right) \tag{41}$$

Using $\csc \pi z$ There exist several more summation formulas similar to (41). If $p(z)$ is a polynomial function satisfying the same assumptions $(i)-(iii)$ given above, then the function

$$f(z) = \frac{\pi \csc \pi z}{p(z)} \tag{42}$$

has an infinite number of simple poles $z = 0, \pm 1, \pm 2, \ldots$ from $\csc \pi z$ and a finite number of poles $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$ from the zeros of $p(z)$. In this case it can be shown that

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{p(k)} = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \csc \pi z}{p(z)}, z_{p_j}\right) \tag{43}$$

In our last example we show how to use the result in (41) to find the sum of an infinite series.

EXAMPLE 8 Summing an Infinite Series

Find the sum of the series $\sum_{k=0}^{\infty} \frac{1}{k^2+4}$.

Solution Observe that if we identify $p(z) = z^2 + 4$, then the three assumptions (i)-(iii) preceding (37) hold true. The zeros of $p(z)$ are $\pm 2i$ and correspond to simple poles of $f(z) = \pi \cot \pi z / \left(z^2 + 4\right)$. According to the formula in (41),

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+4} = -\left[\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+4}, -2i\right) + \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2+4}, 2i\right)\right] \tag{44}$$

Now again by (4) of Section 6.5 we have

$$\operatorname{Res}\left(\frac{\pi\cot\pi z}{z^2+4},-2i\right)=\frac{\pi\cot 2\pi i}{4i}\quad\text{and}\quad\operatorname{Res}\left(\frac{\pi\cot\pi z}{z^2+4},2i\right)=\frac{\pi\cot 2\pi i}{4i}$$

The sum of the residues is $(\pi/2i)\cot 2\pi i$. This sum is a real quantity because from (27) of Section 4.3 :

$$\frac{\pi}{2i}\cot 2\pi i=\frac{\pi}{2i}\frac{\cosh(-2\pi)}{(-i\sinh(-2\pi))}=-\frac{\pi}{2}\coth 2\pi$$

Hence (44) becomes

$$\sum_{k=-\infty}^{\infty}\frac{1}{k^2+4}=\frac{\pi}{2}\coth 2\pi \tag{45}$$

This is not quite the desired result. To that end we must manipulate the summation $\sum_{k=-\infty}^{\infty}$ in order to put it in the form $\sum_{k=0}^{\infty}$. Observe

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{1}{k^2+4}&=\sum_{k=-\infty}^{-1}\frac{1}{k^2+4}+\overbrace{\frac{1}{4}}^{\substack{k=0\\ \text{term}}}+\sum_{k=1}^{\infty}\frac{1}{k^2+4}=\\&=\sum_{k=1}^{\infty}\frac{1}{(-k)^2+4}+\frac{1}{4}+\sum_{k=1}^{\infty}\frac{1}{k^2+4}=\\&=2\sum_{k=1}^{\infty}\frac{1}{k^2+4}+\frac{1}{4}=2\sum_{k=0}^{\infty}\frac{1}{k^2+4}-\frac{1}{4}\end{aligned}$$

Finally, we obtain the sum of the original series by combining (45) with (46),

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+4} = 2\sum_{k=0}^{\infty} \frac{1}{k^2+4} - \frac{1}{4} = \frac{\pi}{2}\coth 2\pi$$

and solving for $\sum_{k=0}^{\infty}$:

$$\sum_{k=0}^{\infty} \frac{1}{k^2+4} = \frac{1}{8} + \frac{\pi}{4}\coth 2\pi \tag{47}$$

With the help of calculator, we find that the right side of (47) is approximately 0.9104

10 Applications

In other courses in mathematics or engineering you may have used the Laplace transform of a real function f defined for $t \geq 0$,

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \tag{1}$$

In the application of (1) we face two problems:

(i) The direct problem: Given a function $f(t)$ satisfying certain conditions, find its Laplace transform.

When the integral in (1) converges, the result is a function of s. It is common practice to emphasize the relationship between a function and its transform by using a lowercase letter to denote the function and the corresponding uppercase letter to denote its Laplace transform, for example $\mathscr{L}\{f(t)\} = F(s), \mathscr{L}\{y(t)\} = Y(s)$, and so on.

(ii) The inverse problem: Find the function $f(t)$ that has a given transform $F(s)$.

The function $F(s)$ is called the inverse Laplace transform and is denoted by $\mathscr{L}^{-1}\{F(s)\}$

The Laplace transform is an invaluable aid in solving solve certain kinds of applied problems involving differential equations. In these problems we deal with the transform $Y(s)$ of an unknown function $y(t)$. The determination of $y(t)$ requires the computation of $\mathscr{L}^{-1}\{Y(s)\}$. In the case when $Y(s)$ is a rational function of s, you may recall employing partial fractions, operational properties, or tables to compute this inverse.

We will see in this section that the inverse Laplace transform is not merely a symbol but actually another integral transform. The reason why you did not use this inverse integral transform in previous courses is that it is a special type of complex contour integral.

We begin with a review of the notion of integral transform pairs. The section concludes with a brief introduction to the Fourier transform.

Integral Transforms Suppose $f(x, y)$ is a real-valued function of two real variables. Then a definite integral of f with respect to one of the variables leads to a function of the other variable. For example, if we hold y constant, integration with respect to the real variable x gives $\int_1^2 4xy^2 dx = 6y^2$. Thus a definite integral such as $F(\alpha) = \int_a^b f(x)K(\alpha, x)dx$ transforms a function f of the variable x into a function F of the variable α. We say that

$$F(\alpha) = \int_a^b f(x)K(\alpha, x)dx \tag{2}$$

is an integral transform of the function f. Integral transforms appear in transform pairs. This means that the original function f can be recovered by another integral transform

$$f(x) = \int_c^d F(\alpha)H(\alpha, x)d\alpha \tag{3}$$

called the inverse transform. The function $K(\alpha, x)$ in (2) and the function $H(\alpha, x)$ in (3) are called the kernels of their respective transforms. We note that if α represents a complex variable, then the definite integral (3) is replaced by a contour integral.

The Laplace Transform Suppose now in (2) that the symbol α is replaced by the symbol s, and that f represents a represents a real function that is defined on the unbounded interval $[0, \infty)$. Then (2) is an improper integral and is defined as the limit

$$\int_0^\infty K(s,t)f(t)dt = \lim_{b\to\infty} \int_0^b K(s,t)f(t)dt \tag{4}$$

If the limit in (4) exists, we say that the integral exists or is convergent; if the limit does not exist the integral does not exist and is said to be divergent. The choice $K(s,t) = e^{-st}$, where s is a complex variable, for the kernel in (4) gives the Laplace transform $\mathscr{L}\{f(t)\}$ defined previously in (1). The integral that defines the Laplace transform may not converge for certain kinds of functions f. For example, neither $\mathscr{L}\left\{e^{t^2}\right\}$ nor $\mathscr{L}\{1/t\}$ exist. Also, the limit in (4) will exist for only certain values of the variable s.

EXAMPLE 1 Existence of a Laplace Transform

The Laplace transform of $f(t) = 1, t \geq 0$ is

$$
\begin{aligned}
\mathscr{L}\{1\} &= \int_0^\infty e^{-st}(1)dt = \lim_{b\to\infty}\int_0^b e^{-st}dt \\
&= \lim_{b\to\infty} \left.\frac{-e^{-st}}{s}\right|_0^b = \lim_{b\to\infty}\frac{1-e^{-sb}}{s} \qquad (5)
\end{aligned}
$$

If s is a complex variable, $s = x + iy$, then recall

$$e^{-sb} = e^{-bx}(\cos by + i\sin by) \qquad (6)$$

From (6) we see in (5) that $e^{-sb} \to 0$ as $b \to \infty$ if $x > 0$. In other words, (5) gives $\mathscr{L}\{1\} = \frac{1}{s}$, provided $\mathrm{Re}(s) > 0$.

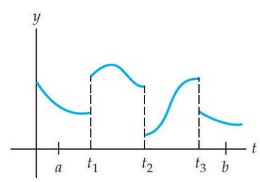

Figure 6.20 Piecewise continuity $[0, \infty)$

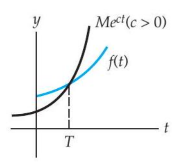

Figure 6.21 Exponential order

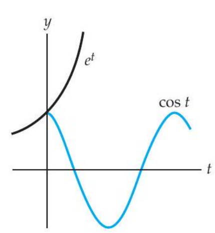

Figure 6.22$f(t) = \cos t$ is of exponential order $c = 0$.

Existence of $\mathscr{L}\{f(t)\}$ Conditions that are sufficient to guarantee the existence of $\mathscr{L}\{f(t)\}$ are that f be piecewise continuous on $[0, \infty)$ and that f be of exponential order. Recall from elementary calculus, piecewise continuity on $[0, \infty)$ means that on any interval there are at most a finite number of points $t_k, k = 1, 2, \dots, n, t_{k-1} < t_k$, at which f has finite discontinuities and is continuous on each open interval $t_{k-1} < t < t_k$. See Figure 6.20. A function f is said to be exponential order c if there exist constants $c, M > 0$, and $T > 0$ so that $|f(t)| \leq Me^{ct}$, for $t > T$. The condition $|f(t)| \leq Me^{ct}$ for $t > T$ states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}. See Figure 6.21. Alternatively, $e^{-ct}|f(t)|$ is bounded; that is, $e^{-ct}|f(t)| \leq M$ for $t > T$. As can be seen in Figure 6.22, the function $f(t) = \cos t, t \geq 0$ is of exponential order $c = 0$ for $t > 0$. Indeed, it follows that all bounded functions are necessarily of exponential order $c = 0$.

Theorem 6.22 Sufficient Conditions for Existence
Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t > T$. Then $\mathscr{L}\{f(t)\}$ exists for $\mathrm{Re}(s) > c$.

Proof By the additive interval property of definite integrals,

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)dt + \int_T^\infty e^{-st} f(t)dt = I_1 + I_2$$

The integral I_1 exists since it can be written as a sum of integrals over intervals on which $e^{-st}f(t)$ is continuous. To prove the existence of I_2, we let s be a
complex variable $s = x+iy$. Then using $|e^{-st}| = |e^{-xt}(\cos yt - i \sin yt)| = e^{-xt}$ and the definition of exponential order that $|f(t)| \leq Me^{ct}, t > T$, we get

$$|I_2| \leq \int_T^\infty \left|e^{st}f(t)\right| dt \leq M \int_T^\infty e^{-xt}e^{ct}dt =$$

$$= M \int_T^\infty e^{-(x-c)t}dt = -\, M\frac{e^{-(x-c)t}}{x-c}\bigg|_T^\infty = M\frac{e^{-(x-c)T}}{x-c}$$

for $x = \mathrm{Re}(s) > c$. Since $\int_T^\infty Me^{-(x-c)t}dt$ converges, the integral $\int_T^\infty |e^{-st}f(t)|\, dt$ converges by the comparison test for improper integrals. This, in turn, implies that I_2 exists for $\mathrm{Re}(s) > c$. The existence of I_1 and I_2 implies that $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$ exists for $\mathrm{Re}(s) > c$.

With the foregoing concepts in mind we state the next theorem without proof.

Theorem 6.23 Analyticity of the Laplace Transform
Suppose f is piecewise continuous on $[0, \infty)$ and of exponential order c for $t \geq 0$. Then the Laplace transform of f,

$$F(s) = \int_0^\infty e^{-st}f(t)dt$$

is an analytic function in the right half-plane defined by $\mathrm{Re}(s) > c$.

The Inverse Laplace Transform Although Theorem 6.23 indicates that the complex function $F(s)$ is analytic to the right of the line $x = c$ in the complex plane, $F(s)$ will, in general, have singularities to the left of that line. We are now in a position to give the integral form of the inverse Laplace transform.

Theorem 6.24 Inverse Laplace Transform
If f and f' are piecewise continuous on $[0, \infty)$ and f is of exponential order c for $t \geq 0$, and $F(s)$ is a Laplace transform, then the inverse Laplace transform $\mathscr{L}^{-1}\{F(s)\}$ is

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{\gamma-iR}^{\gamma+iR} e^{st}F(s)ds \tag{7}$$

where $\gamma > c$.

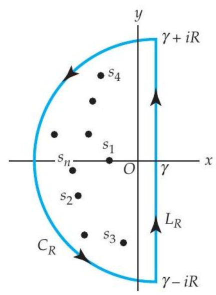

Figure 6.23 Possible contour that could be used to evaluate (7)

The limit in (7), which defines a principal value of the integral, is usually written as

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds \tag{8}$$

where the limits of integration indicate that the integration is along the infinitely long vertical-line contour $\text{Re}(s) = x = \gamma$. Here γ is a positive real constant greater than c and greater than all the real parts of the singularities in the left half-plane. The integral in (8) is called a Bromwich contour integral. Relating (8) back to (3), we see that the kernel of the inverse transform is $H(s,t) = e^{st}/2\pi i$.

The fact that $F(s)$ has singularities $s_1, s_2, \ldots, s_n$ to the left of the line $x = \gamma$ makes it possible for us to evaluate (7) by using an appropriate closed contour encircling the singularities. A closed contour C that is commonly used consists of a semicircle C_R of radius R centered at $(\gamma, 0)$ and a vertical line segment L_R parallel to the y-axis passing through the point $(\gamma, 0)$ and extending from $y = \gamma - iR$ to $y = \gamma + iR$. See Figure 6.23 . We take the radius R of the semicircle to be larger than the largest number in set of moduli of the singularities $\{|s_1|, |s_2|, \ldots, |s_n|\}$, that is, large enough so that all the singularities lie within the semicircular region. With the contour C chosen in this manner, (7) can often be evaluated using Cauchy's residue theorem. If we allow the radius R of the semicircle to approach ∞, the vertical part of the contour approaches the infinite vertical line that is the contour in (8).

We use the contour just described in the proof of the following theorem.

Theorem 6.25 Inverse Laplace Transform

Suppose $F(s)$ is a Laplace transform that has a finite number of poles $s_1, s_2, \ldots, s_n$ to the left of the vertical line $\text{Re}(s) = \gamma$ and that C is the contour illustrated in Figure 6.23. If $sF(s)$ is bounded as $R \to \infty$, then

$$\mathscr{L}^{-1}\{F(s)\} = \sum_{k=1}^{n} \text{Res}\left(e^{st}F(s), s_k\right) \tag{9}$$

Proof From Figure 6.23 and Cauchy's residue theorem, we have

$$\int_{C_R} e^{st}F(s)ds + \int_{L_R} e^{st}F(s)ds = 2\pi i \sum_{k=1}^{n} \text{Res}\left(e^{st}F(s), s_k\right)$$

$$\text{or } \frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} e^{st}F(s)ds = \sum_{k=1}^{n} \text{Res}\left(e^{st}F(s), s_k\right) - \frac{1}{2\pi i}\int_{C_R} e^{st}F(s)ds \tag{10}$$

The theorem is justified by letting $R \to \infty$ in (10) and showing that $\lim_{R\to\infty} \int_{C_R} e^{st}F(s)ds = 0$. Now if the semicircle C_R is parametrized by

$s=\gamma+Re^{i\theta}, \pi/2 \leq \theta \leq 3\pi/2,$ then $ds = \mathrm{Ri}^{i\theta}\, d\theta = (s-\gamma)id\theta,$ and so,

$$\frac{1}{2\pi i}\int_{C_R} e^{st}F(s)ds = \frac{1}{2\pi i}\int_{\pi/2}^{3\pi/2} e^{\gamma t+Rte^{i\theta}}F\left(\gamma+Re^{i\theta}\right)Rie^{i\theta}d\theta$$

$$\frac{1}{2\pi}\left|\int_{C_R} e^{st}F(s)ds\right| \leq \frac{1}{2\pi}\int_{\pi/2}^{3\pi/2} \left|e^{\gamma t+Rte^{i\theta}}\right|\left|F\left(\gamma+Re^{i\theta}\right)\right|\left|Rie^{i\theta}\right|d\theta \tag{11}$$

To find an upper bound for the expression in (11) we examine the three moduli of the integrand of the right-hand side. First,

$$\left|e^{\gamma t+Rte^{i\theta}}\right| = \left|e^{\gamma t}e^{Rt(\cos\theta+i\sin\theta)}\right| \overset{\downarrow \text{ since } \left|e^{iRt\sin\theta}\right|=1}{=} e^{\gamma t}e^{Rt\cos\theta}$$

Next, for $|s|$ sufficiently large, we can write

$$\left|\mathrm{Ri}^{i\theta}\right| = |s-\gamma||i| \leq |s|+|\gamma| < |s|+|s| = 2|s| \text{ and } |sF(s)| < M$$

The first of these two inequalities follows from the triangle inequality, and the second from the hypothesis that $sF(s)$ is bounded as $R\to\infty$. Thus the inequality in (11) continues as

$$\frac{1}{2\pi}\left|\int_{C_R} e^{st}F(s)ds\right| \leq \frac{M}{\pi}e^{\gamma t}\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}d\theta \tag{12}$$

If we let $\theta=\phi+\pi/2$, then the integral on the right-hand side of (12) becomes $\int_0^\pi e^{-Rt\sin\phi}d\phi$. Because the integrand is symmetric about the line $\theta=\pi/2$, we have

$$\int_0^{\pi} e^{-Rt\sin\phi} d\phi = 2\int_0^{\pi/2} e^{-Rt\sin\phi} d\phi \tag{13}$$

Now since $\sin\phi \geq 2\phi/\pi$, * it follows that

$$2\int_0^{\pi/2} e^{-Rt\sin\phi} d\phi \leq 2\int_0^{\pi/2} e^{-2Rt\phi/\pi} d\phi = -\frac{\pi}{Rt} e^{-2Rt\phi/\pi}\Big|_0^{\pi/2} = \frac{\pi}{Rt}\left[1-e^{-Rt}\right] \tag{14}$$

Thus (11), (12), (13), and (14) together give

$$\frac{1}{2\pi}\left|\int_{C_R} e^{st}F(s)ds\right| \leq \frac{Me^{\gamma t}}{Rt}\left[1-e^{-Rt}\right] \tag{15}$$

Since the right-hand side of (15) approaches zero as $R \to \infty$ for $t > 0$ we conclude that $\lim_{R\to\infty}\int_{C_R} e^{st}F(s)ds = 0$. Finally, as $R \to \infty$ we see from (10) that

$$\mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds = \sum_{k=1}^{n} \operatorname{Res}\left(e^{st}F(s), s_k\right)$$

EXAMPLE 2 Inverse Laplace Transform

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}, \operatorname{Re}(s) > 0$.

Solution Considered as a function of a complex variable s, the function $F(s) = 1/s^3$ has a pole of order 3 at $s = 0$. Thus by (9) and (2) of Section 6.5 :

$$\begin{aligned} f(t) = \mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} &= \operatorname{Res}\left(e^{st}\frac{1}{s^3}, 0\right) = \frac{1}{2}\lim_{s\to 0}\frac{d^2}{ds^2}(s-0)^3\frac{e^{st}}{s^3} \\ &= \frac{1}{2}\lim_{s\to 0}\frac{d^2}{ds^2}e^{st} = \frac{1}{2}\lim_{s\to 0} t^2 e^{st} = \frac{1}{2}t^2. \end{aligned}$$

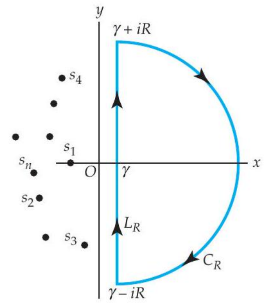

Figure 6.24 Contour for inversion integral (7) for $t < 0$

Those readers familiar with the Laplace transform recognize that the answer in Example 1 is consistent (for $n = 2$) with the result $\mathscr{L}\left\{t^{n}\right\} = n!/s^{n+1}$ found in all tables of Laplace transforms.

The Laplace transform (1) utilizes only the values of a function $f(t)$ for $t > 0$, and so f is often taken to be 0 for $t < 0$. This is no major handicap because the functions we deal with in applications are for the most part defined only for $t > 0$. Although we shall not delve into details, the inversion integral (7) can be derived from a result known as the Fourier integral formula. In that analysis it is shown that

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds = \begin{cases} f(t), & t > 0 \\ 0, & t < 0 \end{cases} \tag{16}$$

This result is hinted at in the proof of Theorem 6.25. Notice from (15) that the conclusion $\lim_{R\to\infty}\int_{C_R} e^{st}F(s)ds = 0$ is not valid for $t < 0$. However, if we close the contour to the right for $t < 0$, as shown in Figure 6.24, then $\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)ds = 0$, which is consistent with (16). We use these results in the next example.

EXAMPLE 3 Inverse Laplace Transform

Evaluate $\mathscr{L}^{-1}\left\{\frac{e^{-2s}}{(s-1)(s-3)}\right\}$, $\operatorname{Re}(s) > 3$.

Solution Before we calculate the residues at the simple poles at $s = 1$

and $s = 3$, we note, after combining the two exponential functions and replacing
the symbol t by $t - 2$, that (16) gives

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{s(t-2)}}{(s-1)(s-3)}ds = \begin{cases} f(t), & t-2>0 \\ 0, & t-2<0 \end{cases} \tag{17}$$

Thus from (17), (9), and (1) of Section 6.5

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left\{\frac{e^{-2s}}{(s-1)(s-3)}\right\} = \operatorname{Res}\left(e^{st}\frac{e^{-2s}}{(s-1)(s-3)}, 1\right) + \\ &+ \operatorname{Res}\left(e^{st}\frac{e^{-2s}}{(s-1)(s-3)}, 3\right) = \\ &= \lim_{s\to 1}(s-1)\frac{e^{s(t-2)}}{(s-1)(s-3)} + \lim_{s\to 3}(s-3)\frac{e^{s(t-2)}}{(s-1)(s-3)} \\ &= -\frac{1}{2}e^{t-2} + \frac{1}{2}e^{3(t-2)} \end{aligned}$$

In other words,

$$f(t) = \begin{cases} -\frac{1}{2}e^{t-2} + \frac{1}{2}e^{3(t-2)}, & t>2 \\ 0, & t<2 \end{cases} \tag{18}$$

In the study of the Laplace transform the unit step function,

$$\mathcal{U}(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t<a \end{cases}$$

proves to be extremely useful when working with piecewise continuous functions. The discontinuous function in (18) can be written as

$$f(t)=-\frac{1}{2}e^{t-2}\mathfrak{u}(t-2)+\frac{1}{2}e^{3(t-2)}\mathfrak{u}(t-2)$$

Fourier Transform Suppose now that $f(x)$ is a real function defined on the interval $(-\infty,\infty)$. Another important transform pair is the Fourier transform

$$\mathfrak{F}\{f(x)\}=\int_{-\infty}^{\infty}f(x)e^{i\alpha x}dx=F(\alpha) \tag{19}$$

and the inverse Fourier transform

$$\mathfrak{F}^{-1}\{F(\alpha)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\alpha)e^{-i\alpha x}d\alpha=f(x) \tag{20}$$

Matching (19) and (20) with (2) and (3), we see that the kernel of the Fourier transform is $K(\alpha,x)=e^{i\alpha x}$, whereas the kernel of the inverse transform is $H(\alpha,x)=e^{-i\alpha x}/2\pi$. In (19) and (20) we assume that α is a real variable. Also, observe that in contrast to (7), the inverse transform (20) is not a contour integral.

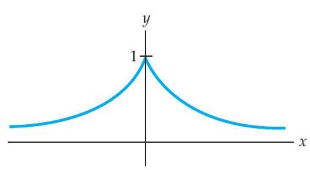

Figure 6.25 Graph of f in Example 4

EXAMPLE 4 Fourier Transform

Find the Fourier transform of $f(x)=e^{-|x|}$.

Solution The graph of f,

$$f(x)=\begin{cases} e^{x}, & x<0 \\ e^{-x}, & x\geq 0 \end{cases} \tag{21}$$

is given in Figure 6.25. From the expanded definition of f in (21), it follows from (19) that the Fourier transform of f is

$$\mathfrak{F}\{f(x)\} = \int_{-\infty}^{0} e^{x} e^{i\alpha x} dx + \int_{0}^{\infty} e^{-x} e^{i\alpha x} dx = I_1 + I_2 \tag{22}$$

We shall begin by evaluating the improper integral I_2. One of several ways of proceeding is to write:

$$I_2 = \lim_{b\to\infty} \int_0^b e^{-x(1-\alpha i)} dx = \lim_{b\to\infty} \left. \frac{e^{-x(1-\alpha i)}}{\alpha i - 1} \right|_0^b = \lim_{b\to\infty} \frac{e^{-b(1-\alpha i)} - 1}{\alpha i - 1} =$$
$$= \frac{1}{\alpha i - 1} \lim_{b\to\infty} \left[e^{-b} \cos b\alpha + i e^{-b} \sin b\alpha - 1 \right] = \frac{1}{1 - \alpha i}$$

Here we have used $\lim_{b\to\infty} e^{-b} \cos b\alpha = 0$ and $\lim_{b\to\infty} e^{-b} \sin b\alpha = 0$ for $b > 0$.

The integral I_1 can be evaluate in the same manner to obtain

$$I_1 = \frac{1}{1 + \alpha i}$$

Adding I_1 and I_2 gives the value of the Fourier transform (22):

$$\mathfrak{F}\{f(x)\} = \frac{1}{1 - \alpha i} + \frac{1}{1 + \alpha i} \quad \text{or} \quad F(\alpha) = \frac{2}{1 + \alpha^2}$$

EXAMPLE 5 Inverse Fourier Transform

Find the inverse Fourier transform of $F(\alpha) = \frac{2}{1+\alpha^2}$.

Solution The idea here is to recover the function f in Example 4 from the inverse transform (20),

$$\mathfrak{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1 + \alpha^2} e^{-i\alpha x} d\alpha = f(x) \tag{23}$$

To evaluate (23), we let z be a complex variable and introduce the contour integral $\oint_C \frac{1}{\pi\left(1+z^2\right)} e^{-izx} dz$. Note that the integrand has simple poles at

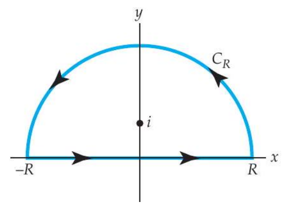

Figure 6.26 First contour used to evaluate (23)

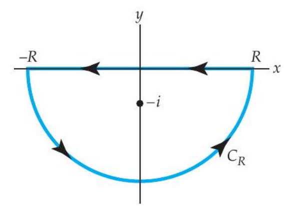

Figure 6.27 Second contour used to evaluate (23) $z = \pm i$.

From here on the procedure used is basically the same as that used to evaluate trigonometric integrals in the preceding section by the theory of residues. The contour C shown in Figure 6.26 encloses the simple pole $z = i$ in the upper plane and consists of the interval $[-R, R]$ on the real axis and a semicircular contour C_R, where $R > 1$. Formally, we have

$$\oint_C \frac{1}{\pi\left(1+z^2\right)} e^{-izx} dz = 2\pi i \operatorname{Res}\left(\frac{1}{\pi\left(1+z^2\right)} e^{-izx}, i\right) = e^x \qquad (24)$$

Obviously the result in (24) is not the function f that we started with in Example 4. A more detailed analysis in this case would reveal that the contour integral along C_R approaches zero as $R \to \infty$ only if we assume that $x < 0$. In other words, the answer in (24) is actually $e^x, x < 0$.

If we consider $\oint_C \frac{1}{\pi\left(1+z^2\right)} e^{-izx} dz$, where C is the contour in Figure 6.27, it can be shown that the integral along C_R now approaches zero as $R \rightarrow \infty$ when x is assumed to be positive. Hence,

$\oint_C \frac{1}{\pi\left(1+z^2\right)} e^{-izx} dz = -2\pi i \operatorname{Res}\left(\frac{1}{\pi\left(1+z^2\right)} e^{-izx}, -i\right) = e^{-x}, x > 0.$

Note the extra minus sign appearing in front of the factor $2\pi i$ on the right side of (25). This sign comes from the fact that on C in Figure 6.27, $\oint_C = \int_{C_R} + \int_R^{-R} = \int_{C_R} - \int_{-R}^{R} = 2\pi i \operatorname{Res}(z=-i)$. As $R \rightarrow \infty, \int_{C_R} \rightarrow 0$ for $x > 0$, we then have $-\lim_{R\rightarrow\infty} \int_{-R}^{R} = 2\pi i \operatorname{Res}(z=-i)$ or $\lim_{R\rightarrow\infty} \int_{-R}^{R} = -2\pi i \operatorname{Res}(z=-i)$. By combining (23), (24), and (25), we arrive at

$$\mathfrak{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+\alpha^2} e^{-i\alpha x} d\alpha = \begin{cases} e^{x}, x < 0 \\ e^{-x}, x > 0 \end{cases}$$

which agrees with (21). Note that when $x = 0$ in (23) conventional integration gives the value 1 , which is $f(0)$ in (21).

Remarks

(i) The two conditions of piecewise continuity and exponential order are sufficient but not necessary for the existence of $F(s) = \mathscr{L}\{f(t)\}$. For example, the function $f(t) = t^{-1/2}$ is not piecewise continuous on $[0, \infty)$ (Why not?); nevertheless $\mathscr{L}\left\{t^{-1/2}\right\}$ exists.

(ii) We have assumed that $F(s)$ has a finite number of poles in the complex plane. This is usually the case when $F(s)$ arises from the solution of an ordinary differential equations. In the solution of applied problems involving a partial differential equation it is not uncommon to obtain a function $F(s)$ with an infinite number of poles. Although the proof of Theorem 6.19 is not valid when $F(s)$ has an infinite number of poles in the left half-plane $\text{Re}(s) < c$, the result stated in the theorem is valid. In this case the value of the integral is an infinite series obtained from the infinite sum of the residues.

(iii) Although we have illustrated the use of (1) when the singularities of $F(s)$ are poles, its principal use is to compute inverse transforms of more complicated functions such as $F(s) = \left(s^2 + a^2\right)^{-1/2}$.

(iv) We did not mention conditions under which the Fourier transform (19) of a function $f(x)$ exists. These conditions are considerably more demanding than those stated for the existence of the Laplace transform. For example, $\mathscr{L}\{1\} = 1/s$ but $\mathfrak{F}\{1\}$ does not exist. For more information on the theory and applications of the Fourier integral you are urged to consult texts on Fourier analysis or advanced engineering mathematics. $\prime$