Real analysis.

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1 Examples on Fourier series

Definition 1.1. *Let $f \in L^1[-\pi, \pi]$. A (trigonometric) Fourier series of f is a trigonometric series*

$$a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

with Fourier coefficients

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx; \tag{2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \;\; n \in \mathbb{N}, \tag{3}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \;\; n \in \mathbb{N}. \tag{4}$$

Notation:

$$f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx).$$

Partial sums of Fourier series of function f are denoted by $S_n(f, x)$.

1. If $f : [-\pi, \pi] \to \mathbb{R}$ is even then $b_n = 0$ and

$$a_0 = \frac{1}{\pi} \int_0^{\pi} f dx; \quad a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx.$$

2. If $f : [-\pi, \pi] \to \mathbb{R}$ is odd then $a_n = 0$ and

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx.$$

Fourier coefficients of arbitrary function $f \in L^1[-\pi, \pi]$ tend to 0, that is

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} b_n = 0.$$

Definition 1.2 (Complex form of Fourier series)**.** *Let $f : [-\pi, \pi] \to \mathbb{C}$, $f \in L^1[-\pi, \pi]$. We can transform a Fourier series of function f to the following form*

$$\sum_{n=-\infty}^{+\infty} c_n e^{inx} \tag{5}$$

with coefficients

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n \in \mathbb{Z}. \tag{6}$$

Then

$$f \sim \sum_{n=-\infty}^{+\infty} c_n e^{inx}.$$

If $f : [-\pi, \pi] \to \mathbb{R}$. Then

$$c_0 = a_0, \quad c_n = \frac{a_n - ib_n}{2}, \quad c_{-n} = \frac{a_n + ib_n}{2}.$$

In particular, $c_n = \bar{c}_{-n}$.

Definition 1.3 (Fourier series on arbitrary segment)**.** *Real form of Fourier series of a function $f : [a, a+2l] \to \mathbb{R}$.*

$$f \sim a_0 + \sum_{n=1}^{+\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),$$

where

$$a_0 = \frac{1}{2l} \int_a^{a+2l} f(x)dx, \quad a_n = \frac{1}{l} \int_a^{a+2l} f(x) \cos \frac{n\pi x}{l} dx,$$

$$b_n = \frac{1}{l} \int_a^{a+2l} f(x) \sin \frac{n\pi x}{l} dx.$$

Complex form of Fourier series

$$f \sim \sum_{n=-\infty}^{+\infty} c_n e^{int/l},$$

where

$$c_n = \frac{1}{2l} \int_a^{a+2l} f(x) e^{-inx/l} dx, \quad n \in \mathbb{Z}.$$

Corollary 1.3.1. *A function $f \in L^1[0, l]$ can be continued to the segment $[-l, l]$ in even and odd way and be decomposed by* $\sin$ *and* $\cos$ *system*

$$f \sim a_0 + \sum_{n=1}^{+\infty} a_n \cos \frac{n\pi x}{l},$$

where

$$a_0 = \frac{1}{l} \int_0^l f(x)dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

and

$$f \sim \sum_{n=1}^{+\infty} b_n \sin \frac{n\pi x}{l},$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx.$$

1.1 Derivative and integration of Fourier series

Theorem 1.4. *Let $f \in C[-\pi, \pi]$ be differentiable with $f' \in L^1[-\pi, \pi]$, $f(\pi) = f(-\pi)$ and*

$$f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$$

Then

$$f' \sim \sum_{n=1}^{+\infty} (nb_n \cos nx - na_n \sin nx) .$$

Theorem 1.5. *Let $f \in L_1$ and*

$$f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx) .$$

Then

$$\int_0^x f(t)dt = a_0 x + \sum_{n=1}^{+\infty} \left(a_n \frac{\sin nx}{n} + b_n \frac{1 - \cos nx}{n} \right) .$$

1.2 Parseval's identity

Let $f : [-\pi, \pi] \to \mathbb{R}, f \in L^2[-\pi, \pi]$,

$$f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx).$$

Then

$$\|f\|_{L^2}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^2(x) dx = a_0^2 + \sum_{n=1}^{+\infty} \frac{a_n^2 + b_n^2}{2}.$$

Assume that $f : [-\pi, \pi] \to \mathbb{C}$, $f \in L^2[-\pi, \pi]$,

$$f \sim \sum_{n=-\infty}^{+\infty} c_n e^{int}.$$

Then

$$\|f\|_{L^2}^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2(x) dx = \sum_{n=-\infty}^{+\infty} |c_n|^2.$$

1.3 Localization principle.

Theorem 1.6. *Let $f, g \in L_1$, $x \in \mathbb{R}$, $\delta \in (0, \pi)$. Assume that functions f and g coincide on the interval $(x - \delta, x + \delta)$. Then Fourier series of functions f and g behave at point x in the same way. That is*

$$S_n(f, x) - S_n(g, x) \to 0, \ n \to \infty.$$

Theorem 1.7 (Dini test for convergence of Fourier series.)**.** *Let $f \in L_1$, $x \in \mathbb{R}$, $S \in \mathbb{C}$ be such*

$$\int_0^{\pi} \frac{|f(x+t) - 2S + f(x-t)|}{t} dt < \infty. \tag{7}$$

Then Fourier series of function f converges at point $x \in \mathbb{R}$ to the sum S, that is

$$S_n(f, x) \to S, \ n \to \infty.$$

Corollary 1.7.1. *Let $f \in L_1, \ x \in \mathbb{R}$. Assume that there exist limits*

$$f(x\pm) = \lim_{t \to x\pm} f(t); \quad \alpha_{\pm} = \lim_{t \to x\pm} \frac{f(x+t) - f(x\pm)}{t}.$$

Then Fourier series of function f converges at point $x \in \mathbb{R}$ to the sum $S = \frac{f(x+)+f(x-)}{2}$. In particular if function f is continuous and has one sided derivatives ($\alpha_{\pm}$) at x then the Fourier series converges to $f(x)$.

1.4 Types of problems.

- Find Fourier series of a function (by direct calculations, by application of theorems on derivative and integration of Fourier series, by decomposition of a function into a power series on a circle).
- Applications of Fourier series to calculation of numerical series (by evaluating a power series at a point, by Parseval's identity).

Example 1. Find Fourier series for a function $f(x) = \frac{\pi - x}{2}$ on the interval $(0, 2\pi)$.

Solution.

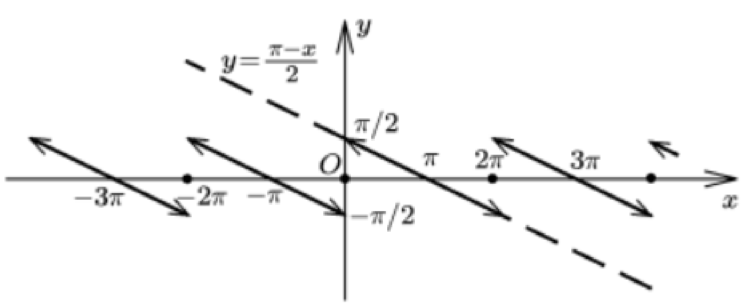
$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx = 0$$
$$a_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\cos nx dx = \frac{\pi - x}{2\pi}\frac{\sin nx}{n}\Big|_0^{2\pi} + \frac{1}{4\pi}\int_0^{2\pi} \sin nx dx = 0$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\sin nx dx = \frac{\pi - x}{2\pi}\frac{-\cos nx}{n}\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \cos nx dx = \frac{1}{n}$$
$$\frac{\pi - x}{2} \sim \sum_{n=1}^{+\infty} \frac{\sin nx}{n}$$

Applying Dini's condition we see that

$$\frac{\pi - x}{2} \sim \sum_{n=1}^{+\infty} \frac{\sin nx}{n}, \quad 0 < x < 2\pi,$$

and for $x = \frac{\pi}{2}$ we see that

$$\frac{\pi}{4} = f\left(\frac{\pi}{2}\right) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{2n-1}.$$



Example 2. Find Fourier series for a function $f(x) = e^{ax}$ in $(-h, h)$.

Solution.

$$a_0 = \frac{1}{2h}\int_{-h}^{h} f(x)dx = \frac{e^{ah} - e^{-ah}}{2ah} = \frac{\sinh ah}{ah};$$

$$a_n = \frac{1}{h}\int_{-h}^{h} e^{ax}\cos\frac{n\pi x}{h}dx = e^{ax}\frac{\sin\frac{n\pi x}{h}}{\frac{n\pi}{h}}\Big|_{-h}^{h} - \frac{a}{n\pi}\int_{-h}^{h} e^{ax}\sin\frac{n\pi x}{h}dx =$$

$$+\frac{a}{n\pi}e^{ax}\frac{\cos\frac{n\pi x}{h}}{\frac{n\pi}{h}}\Big|_{-h}^{h} - \frac{a^2}{n\pi}\int_{-\pi}^{\pi} e^{ax}\frac{\cos\frac{n\pi x}{h}}{\frac{n\pi}{h}} = \frac{(-1)^n 2ah}{n^2\pi^2}\sinh ah - \frac{a^2h^2}{n^2\pi^2}a_n;$$

$$a_n = \frac{(-1)^n 2ah}{n^2\pi^2 + a^2h^2}\sinh ah; \quad b_n = \frac{(-1)^{n+}2\pi n}{n^2\pi^2 + a^2h^2}\sinh ah$$

$$f \sim 2\sinh(ah)\left[\frac{1}{2ah} + \sum_{n=1}^{+\infty}(-1)^n\frac{ah\cos\frac{n\pi x}{h} - \pi n\sin\frac{n\pi x}{h}}{(ah)^2 + (\pi n)^2}\right].$$

Example 3. Find Fourier series for a function f on $(-\pi, \pi)$, where

$$f(x) = \begin{cases} 1, |x| < \alpha, \\ 0, \alpha < |x| < \pi \end{cases}, \alpha > 0.$$

Solution. Function f is even and, since its Fourier series is cos-series.

$$a_0 = \frac{\alpha}{\pi}, \quad a_n = \frac{1}{\pi}\int_{-\alpha}^{\alpha}\cos nx dx = \frac{2\sin(n\alpha)}{n\pi}, \quad b_n = 0.$$

Then

$$f \sim \frac{\alpha}{\pi} + 2\sum_{n=1}^{+\infty}\frac{\sin n\alpha}{n\pi}\cos nx.$$

Then by Parseval's identity

$$\frac{\alpha}{\pi} = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} 1dx = \frac{\alpha^2}{\pi^2} + 2\sum_{n=1}^{+\infty}\frac{\sin^2 n\alpha}{\pi^2 n^2},$$

$$\frac{\alpha(\pi-\alpha)}{2}=\sum_{n=1}^{+\infty}\frac{\sin^2 n\alpha}{n^2}.$$

Example 4. Find Fourier series for a function $f(x)=x^2$

1. in $(-\pi,\pi)$ by cos system;
2. in $(0,\pi)$ by sin system;
3. in $(0,2\pi)$;

Solution. **1.**

$$\pi a_n=\int_{-\pi}^{\pi}x^2\cos nx dx=x^2\frac{\sin nx}{n}\Big|_{-\pi}^{\pi}+\frac{2}{n}\int_{-\pi}^{\pi}x\sin nx dx=$$
$$\frac{2}{n^2}x\cos nx\Big|_{-\pi}^{\pi}+\frac{2}{n^2}\int_{-\pi}^{\pi}\cos nx dx=\frac{4\pi(-1)^n}{n^2},\quad n\neq 0;$$
$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx=\frac{\pi^2}{3}.$$

Hence,

$$x^2=\frac{\pi^2}{3}+4\sum_{n=1}^{+\infty}\frac{(-1)^n\cos nx}{n^2},\quad x\in(-\pi,\pi).$$

Applying this identity at $x=0$ we see that

$$\frac{\pi^2}{12}=\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2}$$

and Parseval's identity

$$\frac{\pi^4}{5}=\frac{1}{2\pi}\int_{-\pi}^{\pi}x^4dx=\frac{\pi^4}{9}+8\sum_{n=1}\frac{1}{n^4}.$$

Consequently,

$$\sum_{n=1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2. To decompose by sin system we calculate

$$\frac{\pi}{2} b_n = \int_0^\pi x^2 \sin nx dx = -x^2 \frac{\cos nx}{n}\Big|_0^\pi + \frac{2}{n}\int_0^\pi x \cos nx dx =$$

$$(-1)^{n+1}\frac{\pi^2}{n} + \frac{2}{n^2} x \sin nx \Big|_0^\pi - \frac{2}{n^2}\int_0^\pi \sin nx dx = \begin{cases} \frac{(-1)^{n+1}\pi^2}{n} & n = 2k; \\ \frac{(-1)^{n-1}\pi^2}{n} - \frac{4}{(2k+1)^3} & n = 2k+1; \end{cases}$$

3. To decompose by trigonometric system in $(0, 2\pi)$ we calculate

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{4\pi^2}{3};$$

$$\pi a_n = \int_0^{2\pi} x^2 \cos nx dx = x^2 \frac{\sin nx}{n}\Big|_0^{2\pi} - \frac{2}{n}\int_0^{2\pi} x \sin nx dx = \frac{4\pi}{n^2};$$

$$\pi b_n = \int_0^{2\pi} x^2 \sin nx dx = -\, x^2 \frac{\cos nx}{n}\Big|_0^{2\pi} + \frac{2}{n}\int_0^{2\pi} x \cos nx dx =$$

$$\frac{4\pi^2}{n} + \frac{2}{n^2} x \sin nx \Big|_0^{2\pi} - \frac{2}{n^2}\int_0^{2\pi} \sin nx dx = -\frac{4\pi^2}{n};$$

$$x^2 \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{+\infty} \frac{\cos nx}{n^2} + 4\pi \sum_{n=1}^{+\infty} (-1)^{n+1} \frac{\sin nx}{n}.$$

Example 5. Find Fourier series for a function f on $(-\pi, \pi)$, where

$$f(x) = \frac{a \sin x}{1 - 2a \cos x + a^2}, |a| < 1.$$

Solution. We use Euler's formula to transform this function to function of complex variable $z = e^{ix}$

$$\cos x = \frac{z + 1/z}{2} = \frac{z^2+1}{2z}, \quad \sin x = \frac{z - 1/z}{2i} = \frac{z^2-1}{2zi}.$$

Then

$$f(x) = \frac{a(z^2-1)}{2i(1-az)(z-a)} = \frac{1}{2i}\left(\frac{1}{1-az} - \frac{1}{1-a/z}\right).$$

Since $|az| = \left|ae^{ix}\right| = |a| < 1$, $|a/z| = |a| < 1$, then

$$f(x) = \frac{1}{2i}\left(\sum_{n=0}^{+\infty} a^n z^n - \sum_{n=0}^{+\infty} \frac{a^n}{z^n}\right) = \sum_{n=0}^{+\infty} a^n \frac{e^{inx} - e^{-inx}}{2i} = \sum_{n=0}^{+\infty} a^n \sin nx.$$

Example 6. Find Fourier series for a function f on $(-\pi, \pi)$, where

$$f(x) = \ln\left|\sin\frac{x}{2}\right|.$$

Solution. Let $z = e^{ix}$, $-\pi < x < \pi$. Then

$$\ln\left|\sin\frac{x}{2}\right| = \ln\left|\frac{1-e^{-ix}}{2ie^{-ix/2}}\right| = -\ln 2 + \ln\left|1 - \frac{1}{z}\right|;$$

$$\ln(1 - 1/z) = -\sum_{n=1}^{+\infty} \frac{z^{-n}}{n} = -\sum_{n=1}^{+\infty} \frac{\cos nx}{n} + i\sum_{n=1}^{+\infty} \frac{\sin nx}{n}.$$

$$\ln\left|\sin\frac{x}{2}\right| = \operatorname{Re}\ln\left(1 - \frac{1}{z}\right) = -\ln 2 - \sum_{n=1}^{+\infty} \frac{\cos nx}{n} =$$

$$-\ln 2 - \sum_{n=1}^{+\infty} \frac{\cos nx}{n}, \quad x \neq 2\pi k, \ k \in \mathbb{Z}.$$

Example 7. Find Fourier series of a function

$$f(x) = \ln\left|\tan\frac{x}{2}\right|$$

on $(-\pi, \pi)$.
Solution. First

$$\tan\frac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{1}{i}\frac{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}}{e^{i\frac{x}{2}} + e^{-i\frac{x}{2}}} = \frac{1}{i}\frac{1 - e^{-ix}}{1 + e^{-ix}}$$

Let $z = e^{-ix}$, $-\pi < x < \pi$. Then

$$f(x) = \ln\left|\frac{1-z}{1+z}\right| = \mathrm{Re}(\ln(1-z) - \ln(1+z)),\ |z| = 1,\ z \neq \pm 1.$$

Consequently,

$$\ln(1-z) - \ln(1+z) = -\sum_{n=1}^{\infty}\frac{z^n}{n} - \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}z^n = -\sum_{n=1}^{\infty}\frac{1+(-1)^{n-1}}{n}z^n =$$
$$-\sum_{n=1}^{\infty}\frac{2}{2n+1}z^{2n+1} = -\sum_{n=1}^{\infty}\frac{2}{2n+1}e^{-(2n+1)ix};$$

and

$$f(x) = -\sum_{n=1}^{\infty}\frac{\cos(2n+1)x}{2n+1}, \quad x \in (-\pi, \pi).$$

Example 8. Find Fourier series of functions

$$f(x) = e^{\cos x}\cos\sin x, \quad f(x) = e^{\cos x}\sin\sin x, x \in [-\pi, \pi].$$

Solution. Notice that

$$f(x) + ig(x) = e^{cosx}(\cos\sin x + i\sin\sin x) = e^{\cos x}e^{i\sin x} = e^{e^{ix}} = \\ e^z = \sum_{n=0} \frac{z^n}{n!} = \sum_{n=0} \frac{e^{inx}}{n!} = \sum_{n=0} \frac{\cos nx + i\sin nx}{n!},$$

where $z = e^{ix}$. Hence,

$$f(x) = \sum_{n=0} \frac{\cos nx}{n!}, \quad f(x) = \sum_{n=0} \frac{\sin nx}{n!}, \quad x \in [-\pi, \pi].$$

Example 9 Calculate a sum of a series

$$u(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n}.$$

Solution. This series converges for $x \neq 2\pi k$, $k \in \mathbb{Z}$. Consider also a series

$$v(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

that is convergent on the whole real line. Consider

$$f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z) = \ln\frac{1}{1-z}, \quad |z| \leqslant 1, \quad z \neq 1.$$

Consequently,

$$u(x) + iv(x) = f(e^{ix}) = \ln\frac{1}{1-e^{ix}}, \quad 0 < x < 2\pi.$$

Transforming the expression under the logarithm:

$$\frac{1}{1-e^{ix}} = \frac{1}{(1-\cos x) - i\sin x} =$$

$$= \frac{1}{2\sin^2(x/2) - 2i\sin(x/2)\cos(x/2)} = \frac{1}{2\sin(x/2)} \frac{1}{\sin(x/2) - i\cos(x/2)} =$$

$$= \frac{1}{2\sin(x/2)} \left(\sin\frac{x}{2} + i\cos\frac{x}{2}\right) =$$

$$= \frac{1}{2\sin(x/2)} \left[\cos\left(\frac{\pi}{2} - \frac{x}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{x}{2}\right)\right].$$

Hence, absolute value of $1/\left(1-e^{ix}\right)$ is equal to $1/2\sin(x/2)$ and argument to $(\pi - x)/2$. Hence

$$u(x) + iv(x) = \ln\frac{1}{1-e^{ix}} = \ln\frac{1}{2\sin(x/2)} + i\left(\frac{\pi}{2} - \frac{x}{2}\right) =$$

$$= \ln\left(2\sin\frac{x}{2}\right) + \frac{i(\pi - x)}{2}$$

Consequently,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln\left(2\sin\frac{x}{2}\right), \quad 0 < x < 2\pi,$$

and

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2}, \quad 0 < x < 2\pi.$$

2 Examples on Fourier transform

2.1 Theoretical background

Definition 2.1. *LEt $f \in L^1(\mathbb{R})$. The function $\hat{f}$ defined by*

$$\hat{f}(y) = \int_{-\infty}^{+\infty} f(t)e^{-2\pi iyt}dt \tag{8}$$

is called Fourier transform of function f and is also denoted by $\mathcal{F}[f]$.

Definition 2.2. *The function $\tilde{f}$ defined by*

$$\tilde{f}(y) = \int_{-\infty}^{+\infty} f(t)e^{2\pi iyt}dt \tag{9}$$

is called inverse Fourier transform of function f and is also denoted by $\mathcal{F}^{-1}[f]$.

Properties of Fourier transform.

1. $\widehat{f} \in C(\mathbb{R})$ and $\left|\widehat{f}(y)\right| \leqslant \|f\|_1$ for every $y \in \mathbb{R}$.

2. $\widehat{f}(y) \to 0$ as $y \to \infty$.

3. If $f \in L(\mathbb{R})$ and for some $r \in \mathbb{N}$ a function $t \mapsto t^r f(t)$ is integrable on $\mathbb{R}$ then $\widehat{f} \in C^{(r)}(\mathbb{R})$ and for every $k \in [1:r]$

$$\widehat{f}^{(k)}(y) = (-2\pi i)^k \int_{\mathbb{R}} t^k f(t)e^{-2\pi iyt}dt,$$

moreover, $\widehat{f}^{(k)}(y) \to 0$ as $y \to \infty$.

4. Suppose $r \in \mathbb{N}$, $f \in C^{(r)}(\mathbb{R})$ and $f^{(k)} \in L^1(\mathbb{R})$ for every $k \in [0 : r]$. Then for every $k \in [1 : r]$
$$\widehat{f^{(k)}}(y) = (2\pi i y)^k \widehat{f}(y).$$

5. Fourier transform of a shift and scaling. Let $f_h(x) = f(x+h)$. Then
$$\widehat{f_h}(y) = e^{2\pi i h y}\widehat{f}(y), \quad \widehat{(f(a\cdot))}(y) = \frac{1}{|a|}\widehat{f}\left(\frac{y}{a}\right), \ a \neq 0. \qquad (10)$$

   Definition 2.3. *Let $f, g \in L(\mathbb{R})$. The function*
$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt$$
   is called a convolution of functions f and g.

6. Let $f, g \in L^1(\mathbb{R})$. Then
$$\widehat{f * g}(y) = \widehat{f}(y)\widehat{g}(y), \quad y \in \mathbb{R}.$$

7. If $f \in L(\mathbb{R}) \cap C(\mathbb{R})$ and $\hat{f} \in L(\mathbb{R})$ then $f = \widetilde{(\widehat{f})}$ on $\mathbb{R}$.

Definition 2.4. *The integral*
$$J(f)(x) = v.p.\int_{-\infty}^{+\infty} \widehat{f}(y)e^{2\pi i x y}dy = \lim_{A\to+\infty}\int_{-A}^{A} \widehat{f}(y)e^{2\pi i x y}dy$$
is called Fourier integral.

Theorem 2.5 (Dini's test for convergence of Fourier integral)**.** *Let $f \in L(\mathbb{R})$, $x \in \mathbb{R}$, $S \in \mathbb{R}$ and*
$$\int_0^{\delta} \frac{|f(x+t) - 2S + f(x-t)|}{t}dt < +\infty.$$
for some $\delta > 0$. Then $S = J(f)(x)$.

Corollary 2.5.1. *Suppose $f \in L(\mathbb{R})$, $x \in \mathbb{R}$ and there exist four finite limits*

$$f(x\pm) = \lim_{t \to x\pm} f(t), \quad \alpha_\pm = \lim_{t \to 0\pm} \frac{f(x+t) - f(x\pm)}{t}.$$

Then $J(f) = \frac{f(x+)+f(x-)}{2}$. In particular, if f is continuous at x and has finite right and left side derivatives then $J(f)(x) = f(x)$.

Corollary 2.5.2. *Suppose $f \in L(\mathbb{R}) \cap C(\mathbb{R})$. If, moreover, $\widehat{f} \in L(\mathbb{R})$ then $f = J(f)$, or, equivalently, $f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$.*

2.2 Cosine and Sine Fourier transforms

Definition 2.6. *The Fourier cosine transform of function $f \in L^1(\mathbb{R})$ is defined by*

$$\mathcal{F}_c[f](y) = \int_{-\infty}^{+\infty} f(x)\cos(2\pi xy)dx.$$

The Fourier sine transform of function $f \in L^1(\mathbb{R})$ is defined by

$$\mathcal{F}_s[f](y) = \int_{-\infty}^{+\infty} f(x)\sin(2\pi xy)dx.$$

Notice that

$$\mathcal{F}[f](y) = \mathcal{F}_c[f](y) - i\mathcal{F}_s[f](y).$$

If function f is even then $\mathcal{F}_s[f](y) = 0$ and

$$\mathcal{F}[f](y) = \mathcal{F}_c[f](y) = 2\int_{0}^{+\infty} f(x)\cos(2\pi xy)dx.$$

If function f is odd then $\mathcal{F}_c[f](y) = 0$, $\mathcal{F}[f](y) = -i\mathcal{F}_s[f](y)$ and

$$\mathcal{F}_s[f](y) = 2\int_{0}^{+\infty} f(x)\sin(2\pi xy)dx.$$

Definition 2.7 (Sine and cosine transforms of a function $f \in L^1(0,+\infty)$).
The Fourier sine transform of function $f \in L^1(0,+\infty)$ is defined by

$$\mathcal{F}_s[f](y) = 2\int_0^{+\infty} f(x)\sin(2\pi xy)dx.$$

The Fourier cosine transform of function $f \in L^1(0,+\infty)$ is defined by

$$\mathcal{F}_c[f](y) = 2\int_0^{+\infty} f(x)\cos(2\pi xy)dx.$$

Theorem 2.8. *Suppose $f \in L^1(0,+\infty) \cap C[0,+\infty)$. If, moreover, $\widehat{f} \in L^1(0,+\infty)$ then*

$$\mathcal{F}_c[\mathcal{F}_c(f)](x) = f(x), \quad x \geq 0; \tag{11}$$
$$\mathcal{F}_s[\mathcal{F}_s(f)](x) = f(x), \quad x > 0. \tag{12}$$

2.3 Examples.

Example 1 Express function $f(x) = e^{-\alpha|x|}$, $\alpha > 0$, by the Fourier integral.
Solution. Since f is a continuous even function then we let

$$\mathcal{F}[f](y) = \mathcal{F}_c[f](y) = 2\int_0^{+\infty} e^{-\alpha t}\cos(2\pi yt)dt = \frac{2\alpha}{\alpha^2 + (2\pi y)^2},$$

$$e^{-\alpha|x|} = \mathcal{F}_c\left[\mathcal{F}_c[f]\right] = 2\alpha\int_0^{+\infty} \frac{\cos(2\pi yx)}{\alpha^2 + (2\pi y)^2}dy = \frac{2\alpha}{\pi}\int_0^{+\infty} \frac{\cos tx}{\alpha^2 + t^2}dt, \quad x \in \mathbb{R}.$$

Example 2. Express function

$$f(x) = \begin{cases} -1, & -1 < x < 0, \\ 1, & 0 < x < 1, \\ 0, & |x| > 1, \end{cases}$$

by the Fourier integral.
Solution. Function f is odd and continuous on R except points $x=-1$, $x=0$, $x=1$. Then

$$\mathcal{F}_s[f](y) = 2\int_0^1 \sin(2\pi y t)dt = \frac{1-\cos(2\pi y)}{\pi y},$$

$$f(x) = \frac{2}{\pi}\int_0^{+\infty} \frac{1-\cos(2\pi y)}{y}\sin(2\pi x y)dy = \frac{2}{\pi}\int_0^{+\infty}\frac{1-\cos z}{z}\sin(xz)dz$$

for $x \neq 0, x \neq \pm 1$. At $x=1$ we have

$$\frac{f(1-0)+f(1+0)}{2} = \frac{1}{2} = \frac{2}{\pi}\int_0^{+\infty}\frac{1-\cos z}{z}\sin z dz.$$

Letting $z=2t$ we obtain that

$$\int_0^{+\infty}\frac{\sin^3 t\cos t}{t}dt = \frac{\pi}{16}.$$

Letting $x=1/2$ we see that

$$f\left(\frac{1}{2}\right) = 1 = \frac{2}{\pi}\int_0^{\infty}\frac{1-\cos z}{z}\sin\frac{z}{2}dz$$

and performing the change $z=2t$ we obtain

$$\int_0^{+\infty}\frac{\sin^3 t}{t}dt = \frac{\pi}{4}$$

Example 3. Calculate the Fourier transforms of functions

1. $f(x) = x^2 e^{-|x|}$;
2. $g(x) = \frac{d^3}{dx^3}\left(\frac{1}{1+x^2}\right)$.

Solution. Notice that for $f_\alpha(x) = e^{-\alpha|x|}$, $\alpha > 0$, we have

$$\mathcal{F}[f_\alpha](y) = \frac{2\alpha}{\alpha^2 + (2\pi y)^2}.$$

Applying formulas for derivative of Fourier transform

$$\mathcal{F}[x^2 e^{-|x|}] = -\frac{1}{4\pi^2}\left(\mathcal{F}\left[e^{-|x|}\right]\right)'' = -\frac{1}{4\pi^2}\left(\frac{2}{1+(2\pi y)^2}\right)'' = 2\frac{12\pi^2 y^2 - 1}{(1+4\pi^2 y^2)^3}.$$

Applying formula for Fourier transform of derivative we see that

$$\mathcal{F}(g) = (2\pi i y)^3 F\left[\frac{1}{1+x^2}\right],$$

where

$$\mathcal{F}\left[\frac{1}{1+x^2}\right] = \mathcal{F}_c\left[\frac{1}{1+x^2}\right] = 2\int_0^{+\infty} \frac{\cos(2\pi xy)}{1+x^2}dx = \pi e^{-2\pi|y|}.$$

Hence, $\mathcal{F}[g] = -8\pi^4 i y^3 e^{-2\pi|y|}$.

Example 4. Calculate Fourier transform of a function $f(x) = \ln\left(1 + \frac{1}{x^2}\right)$.
Solution. Consider cosine transform of function $g(x) = \frac{1-e^{-\beta x}}{x}$, $\beta > 0$,

$$F(x,\beta) = \mathcal{F}_c[g](x) = 2\int_0^{+\infty} \frac{1-e^{-\beta t}}{t}\cos(2\pi x t)dt.$$

To calculate this integral we differentiate with respect to β under the integral

$$\frac{\partial}{\partial\beta}F(x,\beta) = 2\int_0^{+\infty} e^{-\beta t}\cos(2\pi x t)dt = \frac{2\beta}{\beta^2 + 4\pi^2 x^2}, \quad \beta \neq 0,$$

hence
$$F(x,\beta)=\ln(\beta^2+4\pi^2x^2)+C,\ \beta\neq 0.$$

Since $F(x,\beta)$ is continuous for $\beta>0$ and for $\beta=0$ $F(x,0)=0$ then $C=-\ln(4\pi^2x^2)$. Consequently
$$F(x,\beta)=\ln\left(1+\frac{\beta^2}{4\pi^2x^2}\right).$$

Hence, for $\beta=2\pi$ we have
$$F(x,2\pi)=\ln\left(1+\frac{1}{x^2}\right)=f(x).$$

Applying inversion formula for $y\geqslant 0$
$$g(y)=\frac{1-e^{-2\pi y}}{y}=\mathcal{F}_c[F(x,2\pi)](y)=\mathcal{F}_c[f](y)=\mathcal{F}[f](y)$$
we conclude that
$$\mathcal{F}[f](y)=\frac{1-e^{-2\pi y}}{y},\quad y\geq 0$$
and
$$\mathcal{F}[f](y)=\frac{1-e^{-2\pi|y|}}{|y|}$$
since $\mathcal{F}[f]$ is even.

To prove that we can differentiate by parameter and use inversion theorem notice that
$$g(x)=\begin{cases}\frac{1-e^{-\beta x}}{x}, & x\neq 0\\ -\beta, & x=0,\end{cases}$$
is integrable on $[0,+\infty)$, continuous and the integral $\int_0^{+\infty}e^{-\beta x}\cos(2\pi x)dx$ converges uniformly on a set $\beta\geqslant\varepsilon>0$.