Real Analysis 2024. Homework 1.

1. Prove that any open subset of a real line can be expressed as at most countable union of disjoint open intervals.

   Proof. Any open set in $\mathbb{R}$ can be considered as the union of connected components, which are open intervals. These components do not intersect are disjoint and each contains a rational number, consequently, any open set in $\mathbb{R}$ has no more than countable number of components. $\square$

2. Prove that Borel σ-algebra in $\mathbb{R}$ can be generated by the family of open rays $\{(-\infty, a) : a \in \mathbb{R}\}$.

   Proof. It is enough to notice that for $-\infty < a < b$ the open interval can be obtained from set of open rays by countable number of set-operations

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b\right) = \bigcup_{n=1}^{\infty} \left((-\infty, b) \setminus \left(-\infty, a + \frac{1}{n}\right]\right).$$

   And by the previous consideration any open set is at most countable union of disjoint intervals. Hence, $\sigma-$algebra generated by the family open rays contains open sets, and consequently Borel $\sigma-$algebra. $\square$

3. Let $(X, \mathcal{A})$ be measurable space, X' be a set, $\mathcal{A}'_{\text{min}} = \{\varnothing, X'\}$ be a minimal σ-algebra on X'. Prove that every map $f : X \to X'$ is measurable.

   Proof. Let $E \in \mathcal{A}'_{\text{min}}$. Then we have only two cases

$$f^{-1}(E) = \begin{cases} \varnothing, & E = \varnothing; \\ X, & E = X'. \end{cases}$$

   Since $\varnothing, X$ belong to any $\sigma-$algebra on X this means measurability of map f. $\square$