Continuous and Discrete Dynamical Systems

Dynamical systems (DS) describe how things change and interact over time. There are two types of dynamical systems: discrete and continuous.

Continuous DS

Let's consider a system of ordinary differential equations, having the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t) \tag{1}$$

It will define a *continuous dynamical system*. The solution of the system describes the continuous evolution in the phase space starting from the initial conditions. We will consider function $\mathbf{f}$ which is not depend on t,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) \tag{1-a}$$

Such equations are called *autonomous*.

On notation

We will use a regular font for real numbers and real-valued functions (t, x, f) and bold font for vectors and *vector-valued functions* $(\mathbf{x}, \mathbf{f})$. We will denote the components of a vector-valued function by a subscript and write them as columns.

Here, $\mathbf{x}$ is a vector-valued function $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^n$, i.e.

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} \frac{dx_1}{dt}(t) \\ \frac{dx_2}{dt}(t) \\ \vdots \\ \frac{dx_n}{dt}(t) \end{pmatrix}$$

where x_i are *coordinate functions*, $x_i\colon \mathbb{R} \to \mathbb{R}$.

Next, $\mathbf{f}$ is a vector-valued multivariable function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$, i.e.

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} f_1(\mathbf{u}) \\ f_2(\mathbf{u}) \\ \vdots \\ f_n(\mathbf{u}) \end{pmatrix}$$

where f_i are *coordinate functions*, $x_i\colon \mathbb{R}^n \to \mathbb{R}$. So, equation 1 is equal to

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \mathbf{f} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} f_1(x_1(t), x_2(t), \dots, x_n(t)) \\ f_2(x_1(t), x_2(t), \dots, x_n(t)) \\ \vdots \\ f_n(x_1(t), x_2(t), \dots, x_n(t)) \end{pmatrix}$$

In other words, $\dot{x}_i(t)$ is some combination of $x_1(t), x_2(t), \dots, x_n(t)$.

A space $\mathbb{R}^n = (x_1, x_2 \dots, x_n)$ is called *phase space*,

A space $\mathbb{R}^{n+1} = (t, x_1, x_2 \dots, x_n)$ is called *extended phase space*,

A function $\boldsymbol{\varphi}(t)\colon \mathbb{R} \to \mathbb{R}^n$ such that $\dot{\boldsymbol{\varphi}}(t) = \mathbf{f}(\boldsymbol{\varphi}(t))$ is a *general solution* of 1.

A function $\boldsymbol{\varphi}(t)$ such that $\boldsymbol{\varphi}(t^0) = \mathbf{x}^{\mathbf{0}}$ for some $t^0 \in \mathbb{R}$ and $\mathbf{x}^{\mathbf{0}} \in \mathbb{R}^n$ is a *solution* with initial conditions:

$$\begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{f}(\mathbf{x}(t)) \\ \mathbf{x}(t^0) & = \mathbf{x}^{\mathbf{0}} \end{cases} \tag{2}$$

The image of the solution of 2 (i.e. the curve in phase space) is called *the phase curve*. The graph of the solution of 2 (i.e. the curve in extended phase space) is called *the integral curve*. Every system of differential equations 1 corresponds to the *direction field* $\mathbf{f}(\mathbf{x}, t)$ on extended phase space.

In 1-dim case,

$$\frac{dx}{dt} = f(x) \to \frac{dt}{dx} = \frac{1}{f(x)}, \text{ for } f(x) \neq 0$$

Integrating, we get

$$t = \int \frac{1}{f(x)} \tag{Int}$$

or with initial conditions

$$t - t_0 = \int_{x_0}^{x} \frac{1}{f(\xi)}, \text{ for } f(x) \neq 0$$

- A point $\mathbf{p}$ is a *point of equlibrium* if there exists a solution $\boldsymbol{\varphi}$ such that $\boldsymbol{\varphi}(t) \equiv \mathbf{p}$ (so, this is a solution have to satisfy initial conditions $\boldsymbol{\varphi}(t^0) = \mathbf{p}$).
- A point $\mathbf{p}$ is a *point of stable equlibrium* if there exists a neighborhood $V(\mathbf{p}) \subset \mathbb{R}^n$ such that if $\boldsymbol{\varphi}(t) \in V(\mathbf{p})$ for some t, then $\lim_{t\to\infty} = \mathbf{p}$.
- A point $\mathbf{p}$ is a *point of unstable equlibrium* if there exists a neighborhood $V(\mathbf{p}) \subset \mathbb{R}^n$ such that if $\boldsymbol{\varphi}(t) \in V(\mathbf{p})$ for some t, then $\lim_{t\to-\infty} = \mathbf{p}$.

Example (real-valued function)

Let's consider the next ODE with initial conditions:

$$\begin{cases} \dot{x} = x \\ x(0) = 1 \text{ [i.e. } t^0 = 0,\ x^0 = 1] \end{cases}$$

Phase space: $\mathbb{R}$
Extended phase space: $\mathbb{R} \times \mathbb{R}$
General solution: $\varphi(t) = Ce^t$
Integral curve corresponds to initial conditions: $\varphi(t) = e^t$

$$\frac{dx}{dt} = x \to \frac{dt}{dx} = \frac{1}{x},\ x \neq 0$$

By Int,

$$t = \int_1^x \frac{1}{\xi},\ x \neq 0$$

$$t = \ln|x| \to x = e^t,\ x > 0$$

Example (vector-valued function)

Let's consider the next ODE

$$\begin{cases} \dot{x} &= x - xy \\ \dot{y} &= -y + xy \end{cases}$$

with different initial conditions, $(x(0), y(0))$ equals to

$$(1, 1) \tag{A}$$

$$(1 \pm \varepsilon, 1) \tag{B}$$

$$(10, 1) \tag{C}$$

Here,

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x - xy \\ -y + xy \end{pmatrix}$$

Phase space: $\mathbb{R}^2$
Phase curves: $x - \ln x + y - \ln y = C$

Remark (on uniqness of solution)

Let's consider the next ODE with initial conditions:

$$\begin{cases} \dot{x} = \sqrt{x} \\ x(0) = 0 \end{cases}$$

It has many solutions:

$$x(t) = \begin{cases} 0 & t \in [0, c] \\ \frac{1}{4}(t - c)^2 & t > c \end{cases}$$

Discrete Dynamical System

Let's consider a real-valued function f of one real variable $f\colon \mathbb{R} \to \mathbb{R}$. (Also, we will consider $f\colon X \to X$, where the set X can be $[0, 1], \mathbb{R}^2, \mathbb{R}^n$ or even an abstract space.) As well, $\mathbb{R}$ is called *a phase space*. Starting from some point $x_0 \in X$ let's define the iteration procedure:

$$x_1 = f(x_0)$$

$$x_2 = f(x_1)$$

$$\dots$$

$$x_{n+1} = f(x_n) = f(\underbrace{f(\dots f}_{n \text{ times}}(x_0)))$$

This iteration procedure defines a *discrete dynamical system*. *The forward orbit* or *the forward trajectory* of a point x is the following set:

$$O^+(x) = \{x, f(x), f^2(x), \dots\}$$

If f is a homeomorphism, the following sets are called *the backward orbit* and *the full orbit* of a point x, respectively:

$$O^-(x) = \{x, f^{-1}(x), f^{-2}(x), \ldots\}$$

$$O(x) = O^-(x) \bigcup O^+(x)$$

Here $f^n(x) = f(f(\ldots f(x)))$

The orbit describes the discrete evolution in the phase space starting from the initial position.

Periodicity and stability

Let's recall a few more basic definitions from the discrete dynamics. Stability.

- A point x is a *fixed point* for f if $f(x) = x$.
- A point x is a *periodic point of period* n if $f^n(x) = x$.
- A point x is *eventually periodic of period* n if x is not periodic but there exists $m > 0$ such that $f^{n+i}(x) = f^i(x)$ for all $i \geqslant m$. That is, $f^i(x)$ is periodic for $i \geqslant m$.
- Let p be a periodic point of period n. The point p is *hyperbolic* if $|(f^n)'(p)| \neq 1$. The point p is called an *attracting (repelling)* point if $|(f^n)'(p)| < 1$ ($|(f^n)'(p)| > 1$).
- Let p be a periodic point of period n. A point x is *forward asymptotic* to p if $\lim_{i\to\infty} f^{in}(x) = p$.
- The set of points forward asymptotic to p is called *the stable set of* p and is denoted by $W^s(p)$.
- Let p be a periodic point of period n and f be an invertible function. A point x is *backward asymptotic* to p if $\lim_{i\to-\infty} f^{in}(x) = p$.
- The set of points backwards asymptotic to p is called *the unstable set of* p and is denoted by $W^u(p)$.

Classical result: Sarkovskii's theorem

Theorem 1 (Period three theorem)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that f has a periodic point of period three. Then f has periodic points of all other periods.*

Sarkovskii's Theorem describes the question of existence of periods for continuous maps of $\mathbb{R}$. More precisely, it gives a description of complete accounting of which periods imply which other periods. Consider the following ordering of the natural numbers:

$$
\begin{aligned}
&3 \lhd 5 \lhd 7 \lhd \dots \\
&2 \cdot 3 \lhd 2 \cdot 5 \lhd 2 \cdot 7 \lhd \dots \\
&2^2 \cdot 3 \lhd 2^2 \cdot 5 \lhd 2^2 \cdot 7 \lhd \dots \\
&\dots \\
&2^n \cdot 3 \lhd 2^n \cdot 5 \lhd 2^n \cdot 7 \lhd \dots \\
&\dots \\
&\dots \lhd 2^n \dots \lhd 2^3 \lhd 2^2 \lhd 2 \lhd 1
\end{aligned}
$$

This is called *the Sarkovskii ordering of the natural numbers.*

Theorem 2 (Sarkovskii's theorem)**.** *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose f has a periodic point of prime period k. If $k \lhd l$ in the above ordering, then f also has a periodic point of period l.*

Correspondence Between Continuous and Discrete DS

Again, let's consider the next ODE with initial conditions:

$$
\begin{cases}
\dot{x} = x \\
x(0) = 1
\end{cases}
$$

Let's define *discrete derivative (or forward difference)*:

$$\Delta x = x(n+1) - x(n)$$

Now, transform differential equation above to the *difference equation*:

$$
\begin{cases}
x(n+1) - x(n) = x(n) \\
x(0) = 1
\end{cases}
$$

So, $x(n+1) = 2x(n)$, or $x_{n+1} = 2x_n$. It means, that corresponding discrete DS defined by $f(x) = 2x$ with $x_0 = 1$.

Logistic function and logistic mapping

> A *logistic function* or *logistic curve* is a common S-shaped curve (*sigmoid curve*) with the equation
> $$s(u) = \frac{1}{1+e^{-u}}$$
> A *logistic map* is a quadratic function
> $$l(u) = u(1-u)$$
> The main question:
>
> **What do they have in common?**

Sigmoid curve

Let's solve the equation
$$\dot{x} = x(1-x)$$
By Int, we get:
$$t - t_0 = \int_{x_0}^{x} \frac{1}{\xi(1-\xi)} = \ln\left(\frac{x}{1-x}\right) - \underbrace{\ln\left(\frac{x_0}{1-x_0}\right)}_{\tilde{C}},\ x \in (0,1)$$

Or just simply
$$t = \int \frac{1}{x(1-x)} = \ln\left(\frac{x}{1-x}\right) + C,\ x \in (0,1)$$
So,
$$x = \frac{e^{t-C}}{e^{t-C}+1},\ x \in (0,1)$$
For initial conditions $x(0) = 1/2$ we get
$$x = \frac{e^{t}}{e^{t}+1},\ x \in (0,1)$$

Remainer: logistic family

Logistic family is a discrete DS
$$l(x) = \mu x(1-x)$$
considering for different μ.

Stability and structural stability

Stability

Let's recall the definitions of stability.

Continuous case:

- A point $\mathbf{p}$ is a *point of equlibrium* if there exists a solution $\boldsymbol{\varphi}$ such that $\boldsymbol{\varphi}(t) \equiv \mathbf{p}$ (so, this is a solution have to satisfy initial conditions $\boldsymbol{\varphi}(t^0) = \mathbf{p}$).
- A point $\mathbf{p}$ is a *point of stable equlibrium* if there exists a neighborhood $V(\mathbf{p}) \subset \mathbb{R}^n$ such that if $\boldsymbol{\varphi}(t) \in V(\mathbf{p})$ for some t, then $\lim_{t\to\infty} = \mathbf{p}$.
- A point $\mathbf{p}$ is a *point of (totally) unstable equlibrium* if there exists a neighborhood $V(\mathbf{p}) \subset \mathbb{R}^n$ such that if $\boldsymbol{\varphi}(t) \in V(\mathbf{p})$ for some t, then $\lim_{t\to-\infty} = \mathbf{p}$.
- A point $\mathbf{p}$ is a *point of unstable equlibrium* if for every neighborhood $V(\mathbf{p}) \subset \mathbb{R}^n$ there exists $\boldsymbol{\varphi}(t) \in V(\mathbf{p})$ for some t, such that $\lim_{t\to-\infty} = \mathbf{p}$.

Discrete case:

- Let p be a periodic point of period n. The point p is *hyperbolic* if $|(f^n)'(p)| \neq 1$. The point p is called an *attracting (repelling)* point if $|(f^n)'(p)| < 1$ ($|(f^n)'(p)| > 1$).
- Let p be a periodic point of period n. A point x is *forward asymptotic* to p if $\lim_{i\to\infty} f^{in}(x) = p$.
- The set of points forward asymptotic to p is called *the stable set of* p and is denoted by $W^s(p)$.
- Let p be a periodic point of period n and f be an invertible function. A point x is *backward asymptotic* to p if $\lim_{i\to-\infty} f^{in}(x) = p$.
- The set of points backwards asymptotic to p is called *the unstable set of* p and is denoted by $W^u(p)$.

Chaos

- Function $f\colon J \to J$ is said to be *topologically transitive* if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \varnothing$.
- Function $f\colon J \to J$ *has sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geqslant 0$ such that $|f^n(x) - f^n(y)| > \delta$.
- Let V be a set. Function $f\colon \to V$ is said to be *chaotic* on V if
  1. f has sensitive dependence on initial conditions.
  2. f is topologically transitive.
  3. periodic points of f are dense in V.

Structural Stability (informal definition)

D.S. is *structurally stable*, if it is stable or persistence under small changes or perturbations. In other words, a map f is structurally stable if every "nearby" map has essentially the same dynamics. If, no matter how we perturb f or change f slightly, we get an equivalent dynamical system, then the dynamical structure of f is stable.

The notion of structural stability is extremely important in applications. Suppose our dynamical system, mathematical model, comes from a real world physical system. Our model can be only an approximation to reality. If the dynamical system in question is not structurally stable, then the small errors and approximations made in the model have a chance of dramatically changing the structure of the real solution to the system. That is, our "solution" could be radically wrong or unstable. If, on the other hand, the dynamical system in question is structurally stable, then the small errors introduced by approximations and experimental errors may not matter at all: the solution to the model system may be equivalent to the actual solution.

Logistic family (continuous case)

Let's take a logistic function, but now let's add a parameter μ.

1. First try

$$s_\mu(x) = \mu x(1 - x)$$

2. Second try - interesting!

$$s_\mu(x) = x(1-x) - \mu$$

Now we get the next cases:

- $\mu < 1/4$
- $\mu = 1/4$
- $\mu > 1/4$

3. Third try - fix the second try!

$$s_\mu(x) = x(1-x) - px,\ p \in (0,1)$$

Logistic family (discrete case)

Now let's consider discrete case of logistic family,

1. First try

$$s_\mu(x) = x(1-x) - \mu$$

2. Second try - interesting!

$$l_\mu(x) = \mu x(1-x)$$

Here we get the next cases:

- $0 < \mu < 1$
- $1 < \mu < 3$
- $\mu = 2$
- $3 < \mu < 4$
- $\mu = 4$
- $\mu > 4$

Structural Stability

Let $f\colon A \to A$ and $g\colon B \to B$ be two maps. Then f and g are said to be *topologically conjugate* if there exists a homeomorphism $h\colon A \to B$ such that, $h \circ f = g \circ h$. The homeomorphism h is called a *topological conjugacy*.

Topological conjugacy is an *equivalence relation*

Let $f\colon J \to J$. Then f is said to be C^r*-structurally stable on* J, if there exists $\varepsilon > 0$ such that whenever $d_r(f, g) < \varepsilon$ for $g\colon J \to J$, it follows that f is topologically conjugate to g.

Bifurcation

We will consider a one-parameter family of functions f_λ.
Examples:

- λe^x;
- $\lambda \sin(x)$;
- $\mu x(1-x)$.

Theorem 3. *Let f_λ be a one-parameter family of functions and suppose that*

1. $f_{\lambda_0}(x_0) = x_0$;
2. $f'_{\lambda_0}(x_0) \neq 1$.

Then there are intervals I about x_0 and N about λ_0 and a smooth function $p\colon N \to I$ such that $p(\lambda_0) = x_0$ and $f_\lambda(p(\lambda)) = p(\lambda)$. Moreover, f_λ has no other fixed points in I.

Proof. Consider the function defined by $G(x, \lambda) = f\lambda(x) - x$. By hypothesis, $G(x_0, \lambda_0) = 0$ and

$$\frac{\partial G}{\partial x}(x_0, \lambda_0) = f_{\lambda_0}(x_0) - 1 \neq 0$$

By the Implicit Function Theorem, there are intervals I about x_0 and N about λ_0, and a smooth function $p\colon N \to I$ such that $p(\lambda_0) = x_0$ and $G(p(\lambda), \lambda) = 0$ for all $\lambda \in N$. Moreover, $G(x, \lambda) \neq 0$ unless $x = p(\lambda)$. This concludes the proof. □

Theorem 4 (The saddle-node bifurcation)**.** *Suppose that*

1. $f_{\lambda_0}(0) = 0$
2. $f'_{\lambda_0}(0) = 1$
3. $f''_{\lambda_0}(0) \neq 0$
4. $\left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}(0) \neq 0$.

Then there exists an interval I about 0 and a smooth function $p : I \to \mathbb{R}$ satisfying $p(0) = \lambda_0$ and such that

$$f_{p(x)}(x) = x.$$

Moreover, $p'(0) = 0$ and $p''(0) \neq 0$.

Proof. Define $G(x, \lambda) = f_\lambda(x) - x$. Note that $G(x, \lambda) = 0$ implies that f_λ has a fixed point at x. We will apply the Implicit Function Theorem to G.

Note that $G(0, \lambda_0) = 0$ and that

$$\frac{\partial G}{\partial \lambda}(0, \lambda_0) = \left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}(0) \neq 0.$$

Hence there exists a smooth function $p(x)$ satisfying $G(x, p(x)) = 0$. From the chain rule, we have

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial \lambda} p'(x) = 0$$

Therefore

$$p'(x) = \frac{-\frac{\partial G}{\partial x}(x, p(x))}{\frac{\partial G}{\partial \lambda}(x, p(x))}$$

Differentiating this expression and using the above gives

$$\begin{aligned} p''(0) &= \frac{-\frac{\partial^2 G}{\partial x^2}(0) \left.\frac{\partial G}{\partial \lambda}\right|_{\lambda=\lambda_0}(0)}{\left(\frac{\partial G}{\partial \lambda}\right)^2} \\ &= -\frac{f''_{\lambda_0}(0)}{\left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=\lambda_0}(0)} \end{aligned}$$

This completes the proof. □

Theorem 5 (Period-doubling bifurcation)**.** *Suppose*

1. *$f_\lambda(0) = 0$ for all λ in an interval about λ_0.*
2. *$f'_{\lambda_0}(0) = -1$.*
3. *$\left.\frac{\partial \left(f_\lambda^2\right)'}{\partial \lambda}\right|_{\lambda=\lambda_0}(0) \neq 0$.*

Then there is an interval I about 0 and a function $p : I \to \mathbb{R}$ such that

$$f_{p(x)}(x) \neq x$$

but

$$f^2_{p(x)}(x) = x$$

Moreover, $p'(0) = 0$.

Proof. For this proof we define $G(x,\lambda) = f^2_\lambda(x) - x$. We cannot apply the Implicit Function Theorem directly because

$$\frac{\partial G}{\partial \lambda}(0,\lambda_0) = 0.$$

Thus we set

$$H(x,\lambda) = \begin{cases} \frac{G(x,\lambda)}{x} & x \neq 0 \\ \frac{\partial G}{\partial x}(0,\lambda) & x = 0. \end{cases}$$

One checks easily that H is smooth and satisfies

$$\begin{aligned} \frac{\partial H}{\partial x}(0,\lambda_0) &= \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(0,\lambda_0) \\ \frac{\partial^2 H}{\partial x^2}(0,\lambda_0) &= \frac{1}{3}\frac{\partial^3 G}{\partial x^3}(0,\lambda_0). \end{aligned}$$

We now apply the Implicit Function Theorem to H. Note that

$$\begin{aligned} H(0,\lambda_0) &= \frac{\partial G}{\partial x}(0,\lambda_0) \\ &= \left(f^2_{\lambda_0}\right)'(0) - 1 \\ &= f'_{\lambda_0}(0) \cdot f'_\lambda(0) - 1 \\ &= 0 \end{aligned}$$

We have by assumption that

$$\begin{aligned} \frac{\partial H}{\partial \lambda}(0,\lambda_0) &= \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=\lambda_0} \left(\left(f^2_\lambda\right)'(0) - 1\right) \\ &= \frac{\partial \left(f^2_\lambda\right)'}{\partial \lambda}(0) \\ &\neq 0 \end{aligned}$$

Hence there is a smooth function $p(x)$ defined on a neighborhood of 0 and satisfying $p(0) = \lambda_0$ and $H(x,p(x)) = 0$. In particular,

$$\frac{1}{x}G(x,p(x)) = 0$$

for $x \neq 0$ and it follows that x is a period two point for $f_{p(x)}$. Note that x is not fixed by $f_{p(x)}$ because of the previous theorem.

As above, we compute

$$p'(0) = \frac{-\frac{\partial H}{\partial x}(0, \lambda_0)}{\frac{\partial H}{\partial \lambda}(0, \lambda_0)} = 0$$

since

$$\left(f_{\lambda_0}^2\right)''(0) = f_{\lambda_0}''(0) \cdot \left(f_{\lambda_0}'(0)\right)^2 + f_{\lambda_0}''(0) \cdot f_{\lambda_0}'(0) = 0$$

where we have used $f_{\lambda_0}'(0) = -1$. This completes the proof. $\square$

We may get more information about the configuration of the curve of periodic points as follows. Using the notation of the previous proof, one computes

$$\begin{aligned} p''(0) &= \frac{-\frac{\partial^2 H}{\partial x^2}(0, \lambda_0) \cdot \frac{\partial H}{\partial \lambda}(0, \lambda_0)}{\left(\frac{\partial H}{\partial \lambda}(0, \lambda_0)\right)^2} \\ &= \frac{\frac{2}{3} f_{\lambda_0}'''(0) + \left(f_{\lambda_0}''(0)\right)^2}{\frac{\partial}{\partial \lambda}\big|_{\lambda = \lambda_0} (f_\lambda^2)'(0)} \end{aligned}$$

Note that the numerator of this expression is precisely $(-2/3) S f_{\lambda_0}(0)$ since $f_{\lambda_0}'(0) = -1$, where $Sf(x)$ is a *Schwarzian derivative* ...

Maps of the circle

An *orientation-preserving diffeomorphisms* of a circle is a diffeomorphisms $f\colon S^1 \to S^1$ which preserve the order of points on the circle.

The most important invariant associated to a circle map is its rotation number. This number, between 0 and 1, essentially measures the average amount points are rotated by an iteration of the map.

A *canonical covering map* is a function $\pi\colon \mathbb{R} \to S^1$ defined as:

$$\pi(x) = \exp(2\pi i x) = \cos(2\pi x) + i\sin(2\pi x)$$

A function $F\colon \mathbb{R} \to \mathbb{R}$ is a *lift* of $f\colon S^1 \to S^1$ if

$$\begin{cases} \pi \circ F = f \circ \pi \\ F(0) \in [0, 1) \end{cases}$$

The *rotation number* of f is

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n}$$

Theorem 6 (On rotation number). *Let $f\colon S^1 \to S^1$ be orientation-preserving diffeomorphism. Then $\rho(f)$ exists and is independent of x. Moreover, $\rho(f)$ is irrational if and only if f has no periodic points.*

An orientation-preserving diffeomorphism of a circle is called *the Morse-Smale diffeomorphism* if it has rational rotation number and all of its periodic points are hyperbolic.

Theorem 7. *Let f be an orientation-preserving diffeomorphism of a circle S^1. For any $\varepsilon > 0$, there is a C^1 Morse-Smale diffeomorphism g which is $C^1 - \varepsilon$ close to f.*

We will divide the proof of this theorem into a sequence of several steps. Firstly, we may assume that f has rational rotation number $\rho(f)$. We will in fact assume that $\rho(f) = 0$, so that f has only fixed points. The proof in the more general periodic point case is analogous. Then, we will show that any diffeomorphism map may be approximated by one with isolated periodic points. Finally, we will perturb f again so that all of the isolated periodic points become hyperbolic.

Theorem 8 (The closing lemma)**.** *Suppose f is a diffeomorphism of a circle S^1 with an irrational rotation number. Then, for any $\varepsilon > 0$, there exists a diffeomorphism $g\colon S^1 \to S^1$ which is $C^r - \varepsilon$ close to f and which has rational rotation number.*

Theorem 9. *Let $f\colon S^1 \to S^1$ be an orientation-preserving diffeomorphism which has an interval of periodic points, that is $f(\theta) = \theta$ for all θ in the interval $|\theta - \theta_0| \leqslant 2\pi\delta$ for some θ_0. Then, for any $\varepsilon > 0$, there exists a diffeomorphism g, $C^r - \varepsilon$ close to f which satisfies:*

1. $g(\theta) = f(\theta)$ *if* $|\theta - \theta_0| \leqslant 2\pi\delta$;
2. $g(\theta_0) = \theta_0$;
3. $g(\theta) \neq \theta if 0 < |\theta - \theta_0| < 2\pi\delta$.

Theorem 10. *Suppose f is an orientation-preserving diffeomorphism of the circle with the rotation number equal to zero, $\rho(f) = 0$. Then there is a C^1 diffeomorphism g which is arbitrarily close to f with respect to the C^1-distance and which has only isolated fixed points.*

Theorem 11. *Let f be an orientation-preserving diffeomorphism of the circle which has isolated periodic points. There is a diffeomorphism g which is $C^r - \varepsilon$ close to f and which has only hyperbolic periodic points.*

Theorem 12. *A Morse-Smale diffeomorphism of S^1 is C^1-structurally stable.*

Fixed Point Theorem and Root-Finding Algorithms

Fixed Point Theorem

Let (X, d) be a metric space and let $f\colon X \to X$. Function f is *contractive* (with a contraction constant q), provided there exists a constant $q \in [0, 1)$ such that $d(f(x), f(y)) \leqslant qd(x, y)$ for every $x, y \in X$.

Theorem 13. *If X is a complete metric space and $f\colon X \to X$ is a contractive mapping, then f has a unique fixed point.*

Proof. Fix an $x \in X$ and notice that $\{f^n(x)\}$ is a Cauchy sequence, since the series formed by the distances $d\left(f^n(x), f^{n+1}(x)\right) \leq q^n d(x, f(x))$ is convergent as it is bounded by the geometric series $\sum_{n=0}^{\infty} d(x, f(x))q^n$, where $q \in [0, 1)$ is a contraction constant for f.

So the sequence converges to a point $\xi \in X$, which is a fixed point, since

$$\begin{aligned} d(\xi, f(\xi)) &= \lim_{n\to\infty} d\left(f^n(x), f\left(f^n(x)\right)\right) \\ &= \lim_{n\to\infty} d\left(f^n(x), f^{n+1}(x)\right) = d(\xi, \xi) = 0 \end{aligned}$$

implying that $f(\xi) = \xi$. Contraction property implies the uniqueness of ξ. □

Remind the Mean Value Theorem

Theorem 14 (Mean Value Theorem). *Let f be differentiable on $[a, b]$. Then there exists a $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The Mean Value Theorem does not, in general, tell us where the point c is located: it only guarantees the existence of at least one such c.

Theorem 15. *Let $f \in C^1[a, b]$. Suppose that $\bar{x} \in (a, b)$ is a fixed point of f such that $|f'(\bar{x})| < 1$. Then there exists an open interval $J \subset [a, b]$ containing $\bar{x}$ such that for any $x \in J$,*

$$\lim_{n\to\infty} f^n(x) = \bar{x}$$

In other words, $\bar{x}$ is an attractive fixed point and the interval I belongs to the basin of attraction of $\bar{x}$.

Remark. The result can be written in the following way: for any $x_0 \in I$, the iteration sequence $x_{n+1} = f(x_n), n \geq 0$, converges to the fixed point:

$$\lim_{n\to\infty} x_n = \bar{x}$$

Proof. Since $f'(x)$ is assumed to be continuous, the fact that $|f'(\bar{x})| < 1$ implies that there exists an open interval J containing $\bar{x}$ and a constant $0 \leq K < 1$ such that

$$|f'(x)| \leq K < 1 \quad \text{for all} \quad x \in J$$

Now choose any two points $x, y \in J$. We now wish to compare the distance $|f(x) - f(y)|$ to the distance $|x - y|$. From the Mean Value Theorem it follows

$$\frac{f(x) - f(y)}{x - y} = f'(c)$$

for some c between x and y. So, we obtain

$$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq K|x - y|, \quad x, y \in I$$

Therefore, mapping f is contractive on J and therefore it has a unique fixed point. □

Root-Finding Algorithms

Let's find the root of equation:

$$g(x) = 0$$

Let's transform it:

$$pg(x) = 0, \text{ for some constant } p \neq 0$$

$$x - x + pg(x) = 0$$

$$x = \underbrace{x - pg(x)}_{f(x)}$$

Now, if we have some estimation of $f'(x)$ on some interval J which contains a root of equation $g(x) = 0$ we can apply f iteratively (see remark after Theorem 15) to any point from this interval to calculate exact value of the root.

Examples

- Square root: $x^2 = 2 \;\Rightarrow\; x = 1/2(x + 2/x)$
- Golden ratio: $x^2 - x - 1 = 0 \;\Rightarrow\; x = 1 + 1/x$
- Newton's method: $x = x - \frac{g(x)}{g'(x)}, \quad p = \frac{1}{g'(x)}$

What if we extend Newton's method to the complex plane? We get complicated holomorphic dynamics.

Higher dimensional dynamics. Introduction to holomorphic dynamics

Linear dynamics

Let's our dynamics defined by linear mapping: $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$. We will consider 2-dimensional and 3-dimensional cases.

Let's recall *the standard form* for the linear map.

• **2-dimensional case.** Let our linear map represented by 2×2 matrix A. Then there exists a real matrix G such that $G^{-1}AG$ assumes one of the three forms with $\beta \neq 0$

$$1.\ \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \qquad 2.\ \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

$$3.\ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

• **3-dimensional case.** Let our linear map represented by 3×3 matrix A. Then there exists a real 3×3 matrix G such that $G^{-1}AG$ assumes one of the four forms

$$1.\ \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad 2.\ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \eta \end{pmatrix}$$

$$3.\ \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad 4.\ \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

where all entries are real and $\beta \neq 0$.

Remark. We can replace 1-s above by any $\varepsilon \neq 0$.

It all means, that the linear map topologically conjugates to the one with standard form.

Examples

- L_1 **mapping**

$$L_1(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbf{x}$$

That is, if

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

then

$$L_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2} y \end{pmatrix}$$

- L_2 **mapping**

$$L_2(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{x}.$$

- L_3 **mapping**

$$L_3(\mathbf{x}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \mathbf{x}$$

- L_4 **mapping**

$$L(\mathbf{x}) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x}.$$

This linear map has eigenvalues $\frac{3}{2} + \frac{\sqrt{5}}{2} > 1$ and $0 < \frac{3}{2} - \frac{\sqrt{5}}{2} < 1$. The eigenvectors corresponding to these eigenvalues are

$$y = \left(\frac{\sqrt{5} - 1}{2} \right) x$$
$$y = - \left(\frac{\sqrt{5} + 1}{2} \right) x$$

respectively.

- L_5 **mapping**

$$L_4(\mathbf{x}) = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \mathbf{x}$$

Here the eigenvalues are $\pm i/2$.

• L_6 **mapping**

$$L_5(\mathbf{x}) = \begin{pmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Here the eigenvalues are $\pm i/2$ and 2.

General results

Theorem 16. *Suppose $L : \mathbf{R}^3 \to \mathbf{R}^3$ has all eigenvalues less than one in absolute value. Then $L^n(\mathbf{x}) \to 0$ as $n \to \infty$ for all $\mathbf{x} \in \mathbf{R}^3$. Suppose that all eigenvalues of L have absolute value larger than one. Then $L^n(\mathbf{x}) \to 0$ as $n \to -\infty$.*

Proof. Consider the real-valued function $V(x, y, z) = x^2 + y^2 + z^2$. We claim that there exists $\nu < 1$ such that, if $\epsilon > 0$ is small enough, we have

$$V \circ L(\mathbf{x}) \leq \nu V(\mathbf{x})$$

with equality if and only if $\mathbf{x} = 0$.

This is a simple computation which we will make in case that matrix representation A of L has the next form

$$A = \begin{pmatrix} \lambda & \epsilon & 0 \\ 0 & \lambda & \epsilon \\ 0 & 0 & \lambda \end{pmatrix}$$

where all entries are real and $\epsilon \neq 0$. (The other three cases can be considered similarly.)

$$\begin{aligned} V \circ L(\mathbf{x}) &= \lambda^2 \left(x^2 + y^2 + z^2\right) + \epsilon^2 \left(y^2 + z^2\right) + 2\lambda\epsilon(xy + yz) \\ &\leq \left(\lambda^2 + \epsilon^2\right) \left(x^2 + y^2 + z^2\right) + 2|\lambda\epsilon|(|xy| + |yz| + |xz|) \\ &\leq \left(\lambda^2 + \epsilon^2 + 4|\lambda\epsilon|\right) (V(\mathbf{x})) \end{aligned}$$

since $|xy| \leq x^2 + y^2$. Consequently, we may choose ϵ small enough so that the inequality holds with $\nu = \lambda^2 + 4|\lambda\epsilon| + \epsilon^2$.

It follows that if $\mathbf{x} \neq \mathbf{0}$, then

$$V \circ L^n(\mathbf{x}) \leq \nu^n V(\mathbf{x}).$$

Hence $V \circ L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$. But $V(\mathbf{x}) = 0$ if $\mathbf{x} = \mathbf{0}$. Thus $L^n(\mathbf{x}) \to \mathbf{0}$ as required.

In case all eigenvalues of L have absolute value larger than one the arguments above may be altered. $\square$

The function V is called *a Liapounov function.*

We now turn our attention to the case of mixed eigenvalues: some with absolute value larger than one and some smaller.

Theorem 17. *Suppose the eigenvalues of L are λ_1, λ_2, and λ_3 with*

1. $|\lambda_1|, |\lambda_2| < 1$
2. $|\lambda_3| > 1$.

Then there is a plane W^s and a line W^u on which

1. *if $\mathbf{x} \in W^s$, then $L(\mathbf{x}) \in W^s$ and $L^n(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$.*
2. *if $\mathbf{x} \in W^u$, then $L(\mathbf{x}) \in W^u$ and $L^{-n}(\mathbf{x}) \to \mathbf{0}$ as $n \to \infty$.*
3. *if $x \notin W^u \cup W^s$ then $|L^n(\mathbf{x})| \to \infty$ as $n \to \pm\infty$.*

Proof. The standard form of $L(\mathbf{x}) = A\mathbf{x}$ is either

$$A = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \lambda_1 & * & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where $*$ is either positive (only if $\lambda_1 = \lambda_2$) or zero. (In the first case, $\lambda_1 = \alpha + i\beta$.) In this form, W^s is the x, y-plane and W^u is the z-axis. Application of the previous theorem to either of these subspaces yields the result. $\square$

Special case

Let

$$L(\mathbf{x}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$$

A has eigenvalues $\pm i$ and L is a rotation through 90°. Hence all points except $\mathbf{0}$ are periodic with period 4 . In general, if A is of the form

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

with $\alpha^2+\beta^2 = 1$, then L is a rotation of the plane through angle $\arctan(\beta/\alpha)$.

If we treat our plane $\mathbb{R}^2$ as a complex plane $\mathbb{C}$, this will be equivalent to complex multiplication, i.e. if we define $z = x_1 + ix_2$ for $x = (x_1, x_2)$ then $A(x)$ will be equivalent to $f(z) = cz$, where $c = e^{\mathrm{arctg}(\beta/\alpha)}$.

Nonlinear dynamics

Examples

- The Horseshoe map
- The Hénon map
$$H_{ab}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - ax^2 + y \\ bx \end{pmatrix}$$
- Quadratic complex map: $f(z) = z^2$, $f(z) = z^2 + c$

Let's look at the third example. The mapping $f(z) = z^2$ is an expansion of the corresponding map of the circle $\theta \to \theta^2$. It has two invariant sets $\mathbb{R}$ and $J = S^1$. We can see that the unit circle J can be viewed as a separator or boundary set. It acts as the boundary between two dynamical processes:
for $|z| < 1$ points travel toward the attractive fixed point 0,
for $|z| > 1$ points travel outward, absolute value tends to infinity.

The set J is called *the Julia set* of the map $f(z) = z^2$ in honour of Gaston Julia, a French mathematician who made many important contributions to the study of complex dynamics.

Theorem 18. *Let z_0 be a point on the unit circle J, the Julia set of the map $f(z) = z^2$. Suppose that U is any neighourhood of z_0. Then for each $z \in \mathbb{C}, z \neq 0$, there is an m such that $z \in f^m(U)$.*