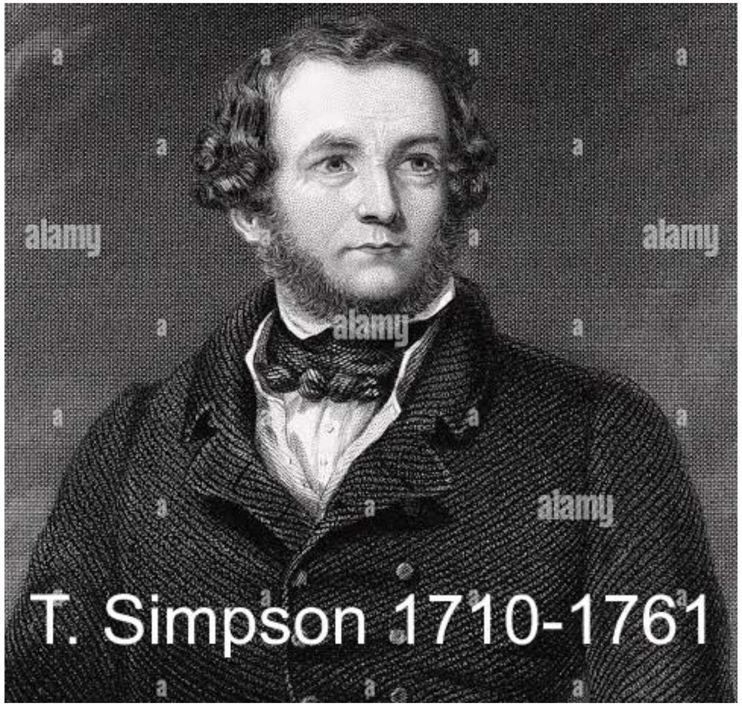
Chapter 4. Simpson's method for calculation of definite integrals



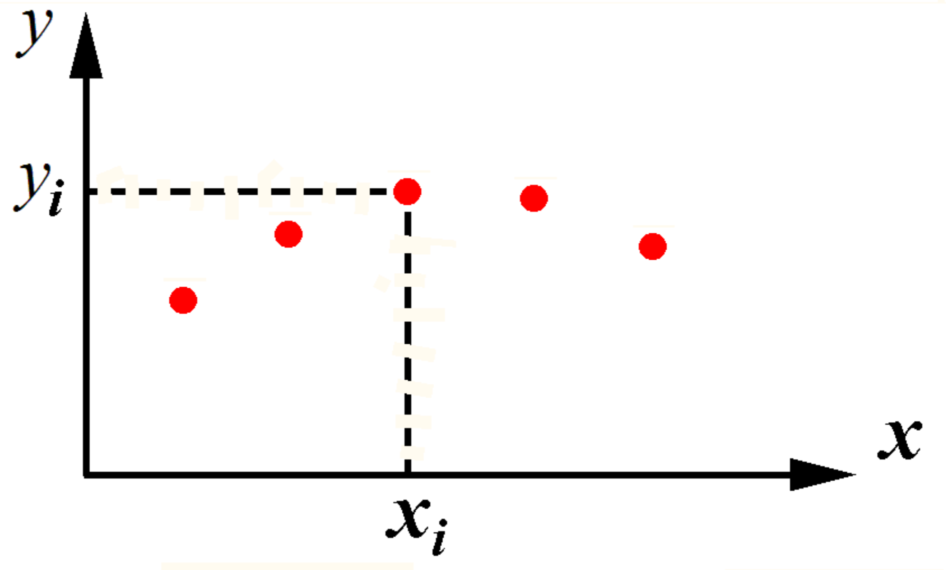
$$\int_a^b f(x)dx = ?$$

Again, we assume that values of $f(x)$ are given at a finite number of points of the segment [a,b]:

$$x_0, x_1, x_2, \ldots, x_n \qquad y_0, y_1, y_2, \ldots, y_n$$

In addition, we assume for simplicity that

$$x_{i+1} - x_i = \text{const} = h$$



$$\int_{x_0}^{x_n} f(x)\,dx = \ ?$$

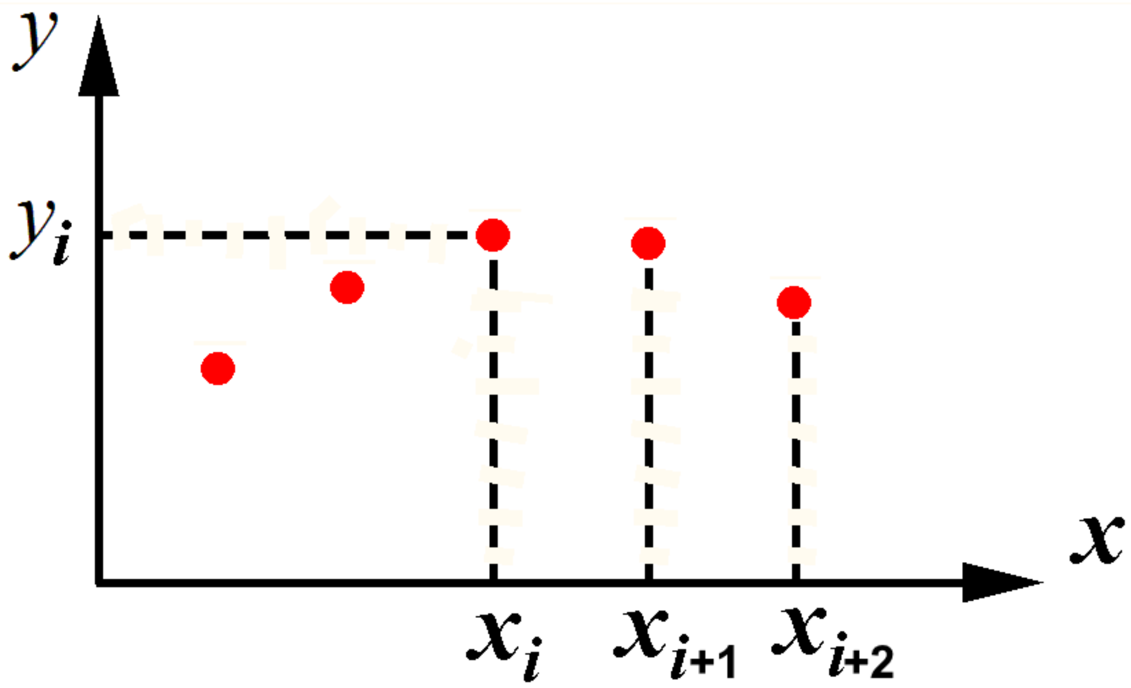
Thomas Simpson's formula:

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx \approx h[\, y_0 + y_n + 2(y_2 + y_4 + \ldots + y_{n-2}) + 4(y_1 + y_3 + \ldots + y_{n-1})\,] / 3}$$

It provides a higher accuracy (smaller error) of integral calculation than trapezoid formula.

The way of obtaining Simpson's formula:

we approximate *f(x)* by pieces of <u>parabolas</u> which pass through three neighboring points (x_i, y_i) (x_{i+1}, y_{i+1}) (x_{i+2}, y_{i+2}) (not by straight segments as in trapezoids method).



$$f(x) \approx y_i (x - x_{i+1})(x - x_{i+2}) / (2h^2) -$$
$$- y_{i+1} (x - x_i)(x - x_{i+2}) / h^2 +$$
$$+ y_{i+2} (x - x_i)(x - x_{i+1}) / (2h^2)$$

$$f(x) \approx y_i\,(x-x_{i+1})\,(x-x_{i+2})/\,(2h^2) -$$
$$-\,y_{i+1}\,(x-x_i)\,(x-x_{i+2})/\,h^2 +$$
$$+\,y_{i+2}\,(x-x_i)\,(x-x_{i+1})/\,(2h^2)$$

Suppose x_0 =0, then on segment [0, 2h]:

$$f(x) \approx y_0\,(x-h)\,(x-2h)/\,(2h^2) -$$
$$-\,y_1\,(x-0)\,(x-2h)/\,h^2 +$$
$$+\,y_2\,(x-0)\,(x-h)/\,(2h^2)$$

Integral of parabola can be expressed in analytical form

$$\int_{x_0}^{x_2} f(x)\,dx \approx h(y_0 + 4y_1 + y_2)/3$$

$$\int_{x_0}^{x_2} f(x)\,dx \approx h(y_0 + 4y_1 + y_2)/3$$

$$\int_{x_0}^{x_4} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx \approx$$

$$\approx h(y_0 + 4y_1 + y_2)/3 + h(y_2 + 4y_3 + y_4)/3 =$$

$$= h(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)/3$$

$$\int_{x_0}^{x_6} f(x)\,dx \approx h(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)/3$$

and so on. Eventually:

$$\int_{x_o}^{x_n} f(x)\,dx \approx h[\, y_0 + y_n + 2(y_2 + y_4 + \ldots + y_{n-2}) + 4(y_1 + y_3 + \ldots + y_{n-1}) \,] / 3$$

Next expression shows an error of Simpson's formula

$$\int_{x_o}^{x_n} f(x)\,dx = h[\, y_0 + y_n + 2(y_2 + y_4 + \ldots + y_{n-2}) + 4(y_1 + y_3 + \ldots + y_{n-1}) \,] / 3 - f^{(4)}(c)\, h^4 (x_n - x_0) / 180$$

where $x_0 < c < x_n$

Practical Runge's rule for estimation of the error:

Suppose that an integral

$$\int_a^b f(x)dx$$

is calculated using the splitting of [a, b] into *n* subsegments, and denote the result by I_n .
Then calculate the same integral using the splitting into *2n* subsegments, denote the result by I_{2n}.

Runge's rule: the estimate of the error is

$$|I^* - I_{2n}| \le q\,|I_{2n} - I_n|$$

where I^* is the exact value of integral,

q= 1/3 in case of trapezoids method,

q= 1/15 in case of Simpson's method.

Example of calculation using Simpson's formula.

$$\int_0^1 e^{x \sin(\cos(\sin x))}\, dx$$

$$\int_{x_o}^{x_n} f(x)\, dx \approx h[\, y_0 + y_n + 2(y_2 + y_4 + \ldots + y_{n-2}) + $$
$$+ 4(y_1 + y_3 + \ldots + y_{n-1})\,] / 3$$

Choose n=100, *h*=0.01:

$$\int_{x_o}^{x_{100}} f(x)\, dx \approx h[\, y_0 + y_{100} + 2(y_2 + y_4 + \ldots + y_{98}) + $$
$$+ 4(y_1 + y_3 + \ldots + y_{99})] / 3$$

For convenience of writing the code, we renumerate the points:

$$\int_{x_1}^{x_{101}} f(x)\, dx \approx h[\, y_1 + y_{101} + 2(y_3 + y_5 + \ldots + y_{99}) + 4(y_2 + y_4 + \ldots + y_{100})\,] / 3$$

Scilab:

```
clear
x= 0:0.01:1
y= exp(x.*sin(cos(sin(x))))
h=0.01
s= y(1)+y(101)
for i=1:49
s=s+2*y(2*i+1)
s=s+4*y(2*i)
end
s=s+4*y(100)
Int=h*s/3
disp(Int)
```

Answers: 1.4569240 Simpson, h=0.01
1.4569217 – trapezoid method, h=0.01;
1.4569240 – trapezoid methods, h=0.001

Scilab built-in subroutines:

integrate and intg

```
Int=integrate('exp(x*sin(cos(sin(x))))','x',0,1)        // error< 1e-13
printf("%1.12f",Int)

Int2=integrate('exp(x*sin(cos(sin(x))))','x',0,1,1e-15)        // error< 1e-15
printf("%1.12f",Int2)

clear
function [f]=myfun(x)
f=exp(x*sin(cos(sin(x))))
endfunction
[ Int3, error] = intg(0,1, myfun )     // intg !
```

Matlab:

Int=SIMPS(x,y)

Int=quad(function, a, b);
default tol 10e-6

Int2=quad(function, a, b, tol);

Calculation of double integrals

Formulae for the evaluation of a double integral can be obtained by repeatedly applying the trapezoidal and Simpson's rules

$$I = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x, y)\, dx\, dy,$$

where

$$x_{i+1} = x_i + h \quad \text{and} \quad y_{j+1} = y_j + k.$$

$$I = \int_{y_j}^{y_{j+1}} \left(\int_{x_i}^{x_{i+1}} f(x, y)\, dx \right) dy,$$

Let us apply the trapezoid formula with respect to x and then with respect to y :

$$I = \frac{h}{2} \int_{y_j}^{y_{j+1}} [\, f(x_i, y) + f(x_{i+1}, y)] \, dy$$

$$= \frac{hk}{4} [\, f(x_i, y_j) + f(x_{i+1}, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_{j+1})]$$

final trapezoid formula

Similarly, applying Simpson's formula to integral

$$I = \int_{y_{j-1}}^{y_{j+1}} \left(\int_{x_{i-1}}^{x_{i+1}} f(x, y)\, dx \right) dy,$$

we obtain

$$I = \frac{h}{3} \int_{y_{j-1}}^{y_{j+1}} [\, f(x_{i-1}, y) + 4\, f(x_i, y) + f(x_{i+1}, y)]\, dy$$

$$= \frac{hk}{9} [\, f(x_{i-1}, y_{j-1}) + 4\, f(x_{i-1}, y_j) + f(x_{i-1}, y_{j+1})$$

$$+ 4\{ f(x_i, y_{j-1}) + 4\, f(x_i, y_j) + f(x_i, y_{j+1})\}$$

$$+ f(x_{i+1}, y_{j-1}) + 4\, f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})]$$

A numerical example is given here.

Example 6.20 Evaluate

$$I = \int_0^1 \int_0^1 e^{x+y}\, dx\, dy,$$

using the trapezoidal and Simpson's rules. With $h = k = 0.5$, we have the following table of values of e^{x+y}.

	x		
y	0	0.5	1.0
0	1	1.6487	2.7183
0.5	1.6487	2.7183	4.4817
1.0	2.7183	4.4817	7.3891

Using the 'trapezoidal rule' from Eq. (6.94) repeatedly, we obtain

$$I = \frac{0.25}{4}[1.0 + 4(1.6487) + 6(2.7183) + 4(4.4817) + 7.3891]$$

$$= \frac{12.3050}{4}$$

$$= 3.0762.$$

Using 'Simpson's rule' given in Eq. (6.96) repeatedly, we obtain

$$I = \frac{0.25}{9}[1.0 + 2.7183 + 7.3891 + 2.7183 + 4(1.6487 + 4.4817 + 4.4817 + 1.6487) + 16(2.7183)]$$

$$= \frac{26.59042}{9}$$

$$= 2.9545.$$

The exact value I^* of the integral is $(e-1)^2$ = 2.9524924 . . .
It is seen that the error in result given by Simpson's rule is 0.002, the error in result given by trapezoid rule is 0.123, i.e., about 60 times larger.

Scilab subroutine: int2d( )

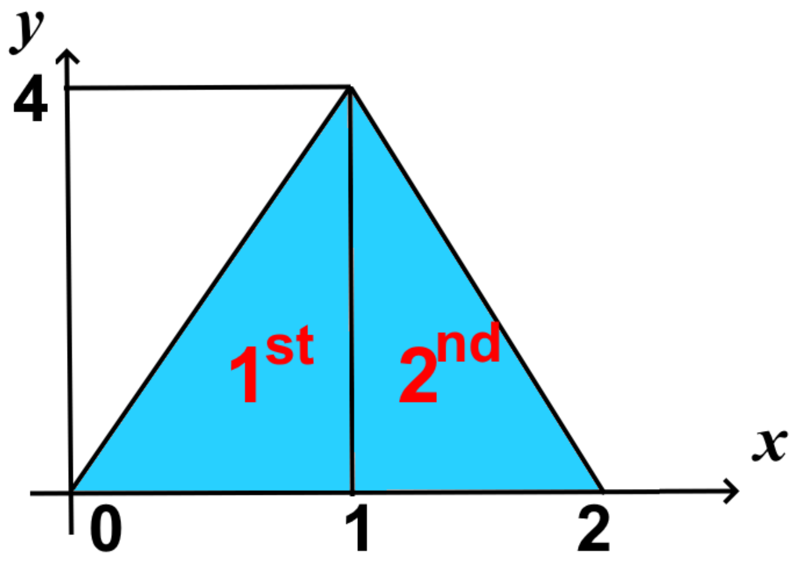
Example:

[Int, err] = int2d(X,Y,’exp(x+y)’)

X,Y – coordinates of the vertices of triangles, which constitute the domain

Coordinates:

X		Y	
0	1	0	0
1	2	0	0
1	1	4	4
1^{st} triangle	2^{nd} triangle	1^{st} triangle	2nd triangle



Command window:

```
X = [0  1; 1  2 ;  1  1]
Y = [0  0; 0  0 ;  4  4]
deff('z=f(x,y)', 'z=cos(x+y)')
[I,e] = int2d(X, Y, f)
```

Matlab subroutine: dblquad ( )

dblquad(fun,xmin,xmax,ymin,ymax,tol,method)

Calculates the integral over the rectangle

$$x_{min} \leq x \leq x_{max}$$

$$y_{min} \leq y \leq y_{max}$$