Real Analysis 2024. Homework 12. Solution.

1. Let $f_n \xrightarrow[n \to \infty]{} f$ in measure. Show that if $\mu(X) < +\infty$ and $g \in L^0(X)$, then $f_n g \xrightarrow[n \to \infty]{} fg$ in measure. Is this true for an infinite measure?

   Proof. Suppose that $g \in L^0(X)$. This means that f is finite a.e. Since $\mu(X) < \infty$ then

   $$\lim_{M \to +\infty} \mu \left(\bigcap_{M > 0} E(|g| > M) \right) = \mu \left(\bigcap_{M > 0} E(|g| > M) \right) = 0$$

   (this is not true when $\mu(X) = +\infty$ consider, for example, function $g_0 : [1, +\infty)$ such that $g_0(x) = n, x \in [n, n+1)$.

   Let $\varepsilon > 0$ and choose $M > 0$ such that $E(|g| > M) < \varepsilon/2$.

   Let $\delta > 0$. Since $f_n \xrightarrow[n \to \infty]{} f$ then there exists $N \in \mathbb{N}$ such that for every $n > N$ we have
   $$E\left(|f_n - f| > \delta/M\right) < \varepsilon/2.$$

   Finally

   $$E(|f_n g - fg| > \delta) \subset E(|g| > M) \bigcup (E(|g| > M) \cup E(|f_n - f| > \delta/M))$$

   and for $n > N$ we have

   $$\mu E(|f_n g - fg| > \delta) \leq \mu E(|g| > M) + \mu E(|f_n - f| > \delta/M) < \varepsilon.$$

   The statement is not true for $X = [1, +\infty)$, function g_0 mentioned above and sequence $f_n \equiv 1/n$ that converges to 0 uniformly. $\square$

2. Let $f_k^{(n)}(x) = \cos^{2k}(\pi n! x)(x \in \mathbb{R})$. Show that:

   (a) for every $x \in \mathbb{R}$, the limit $g_n(x) = \lim_{k \to \infty} f_k^{(n)}(x)$ exists;

   (b) $g_n(x) \xrightarrow[n \to \infty]{} \chi(x)$ everywhere on $\mathbb{R}$ (here $\chi = \chi_{\mathbb{Q}}$ is the Dirichlet function);

   (c) there is no sequence of continuous functions (and, in particular, no diagonal sequence $\{f_{k_n}^{(n)}\}$ that converges to the Dirichlet function pointwise on a non-degenerate interval.

Proof. (a) If $n!x \notin \mathbb{N}$ then $|\cos(\pi n!x)| < 1$ and $g_n(x) = \lim\limits_{k\to\infty} \cos^{2k}(\pi n!x) = 0$. Otherwise, $|\cos(\pi n!x)| = 1$ and $g_n(x) = \lim\limits_{k\to\infty} \cos^{2k}(\pi n!x) = 1$.

(b) If $x \notin \mathbb{Q}$ then $n!x \notin \mathbb{N}$ for every $n \in \mathbb{N}$ and $g_n(x) = 0 = \chi(x)$. If $x = p/q \in \mathbb{Q}$ then $n!x \in \mathbb{N}$ for $n > q$ and $g_n(x) = 1 = \chi(x)$, $n > q$.

$\square$

3. Assume that the measure under consideration is $\sigma-$finite and a sequence of measurable functions f_k converges to zero almost everywhere. Show that $c_k f_k \to 0$ a.e. for some numerical sequence $c_k \to +\infty$. Hint. Assuming that the sequence $\{|f_k|\}$ is decreasing, apply the diagonal sequence theorem to the functions $f_k^{(n)} = nf_k$.

Proof. **Case 1.** Suppose that the sequence $\{|f_k|\}$ is decreasing and let $f_k^{(n)} = nf_k$. Then for every $n \in \mathbb{N}$ $\lim\limits_{k\to+\infty} f_k^{(n)} = 0$ a.e.

Hence, by the diagonal sequence theorem there exists a sequence k_n such that $\lim\limits_{n\to+\infty} nf_{k_n} = 0$. a.e.

Let $c_k = n$, $k_n \leq k_{n+1} - 1$. Then $c_k |f_k| \leq nf_{k_n}$ for $k \geq k_n$. Hence $c_k f_k \to 0$ a.e.

Case 2. If $|f_n|$ is not decreasing let

$$g_n(x) = \sup_{k \geq n} |f_k(x)|.$$

Then g_n is decreasing and $g_n \to 0$ a.e. $\square$

4. Suppose that $f_n \leq g_n \leq h_n$ a.e. on E, $f_n, h_n \xrightarrow[n\to\infty]{} f$ in measure on E. Prove that $g_n \xrightarrow[n\to\infty]{} f$.

Proof. Since
$$|g_n - f| \leq \max(|h_n - f|, |f_n - f|)$$
then
$$\mu(|g_n - f| > \varepsilon) \leq \mu(|f_n - f| > \varepsilon) + \mu(|h_n - f| > \varepsilon) \to 0, \ n \to +\infty,$$
for every $\varepsilon > 0$. $\square$

5. Let $f : [0, 1]^2 \to \mathbb{R}$ is continuous by the first argument (for arbitrary fixed second). Prove that if $f(x, y) \xrightarrow[y \to 0]{} 0$ for a.e. $x \in [0, 1]$ then for every $\varepsilon > 0$ there exists $e \subset [0, 1]$ such that $\lambda(e) < \varepsilon$ and $f(x, y) \xrightarrow[y \to 0]{} 0$ uniformly on $[0, 1] \backslash e$. Hint: consider sets

$$G_n(\varepsilon) = \left\{ (x, y) : \ 0 < x < 1, \ 0 < y < \frac{1}{n}, \ |f(x, y)| > \varepsilon \right\}$$

and their projections on $0X-$axis.

Proof. Consider a sequence $g_n(x) = \sup_{y \in (0, 1/n)} f(x, y)$. Then $g_n(x) \to 0$ a.e. on $[0, 1]$ and, consequently, by Lebesgue theorem, in measure. Hence, applying Egorov theorem, there exists $e \subset [0, 1]$ such that

$$g_n \rightrightarrows 0 \text{ on } [0, 1] \backslash e,$$

which implies that $f(x, y) \underset{y \to 0}{\rightrightarrows} 0$ on $[0, 1] \backslash e$

$\square$