

Equations of mathematical physics

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Introduction

The course includes the main sections:

- Fundamentals of operational calculus;
- Classification of partial differential equations;
- Hyperbolic equations;
- Parabolic equations;
- Elliptical equations.

1. FUNDAMENTALS OF OPERATIONAL CALCULUS

1.1. The concepts of the original and the Laplace image. Properties of the Laplace transform

Definition 1. An *original function* is any complex-valued function $f(t)$ of a valid argument t that satisfies the conditions:

- 1) $f(t)$ is Riemann integrable on any finite interval of the t axis (locally integrable);
- 2) $f(t)=0$ for all $t < 0$;
- 3) $M > 0$ and $\alpha > 0$ are constants for which

$$|f(t)| \leq Me^{\alpha t}. \quad (1.1)$$

The lower edge α_0 of all numbers α for which the inequality (1.1) is valid is called the *growth index* of the function $f(t)$.

The first condition in definition 1 is sometimes formulated as follows: on any finite interval of the t axis, the function $f(t)$ is continuous, except, perhaps, a finite number of discontinuity points of the first kind.

The simplest original function is the Heaviside function:

$$\theta(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

Obviously, for any function $\varphi(t)$ it is true:

$$\varphi(t)\theta(t) = \begin{cases} 0, & t < 0, \\ \varphi(t), & t \geq 0. \end{cases}$$

If, for $t \geq 0$, the function $\varphi(t)$ satisfies conditions 1 and 3 of definition 1, then the function $\varphi(t)\theta(t)$ is the original. In the future, to shorten the record, we will, as a rule, write $\varphi(t)$ instead of $\varphi(t)\theta(t)$, assuming that the functions we are considering are continued by zero for negative values of the argument t .

Definition 2. *The image of the function $f(t)$ according to Laplace is called the function $F(p)$ of the complex variable $p = s + i\sigma$, defined by the equality*

$$F(p) = \int_0^{+\infty} f(t)e^{-pt} dt. \quad (1.2)$$

Theorem 1 (on the analyticity of the image). For any original $f(t)$, its image $F(p)$ is defined and is an analytical function of the variable p in the half—plane $\operatorname{Re} p > \alpha_0$, where α_0 is the growth index of the function $f(t)$, while the equality is valid:

$$\lim_{\operatorname{Re} p \rightarrow +\infty} |F(p)| = 0.$$

Theorem 2 (uniqueness). The Laplace image $F(p)$ is unique in the sense that two functions $f_1(t)$ and $f_2(t)$ having the same images coincide at all points of continuity at $t > 0$.

There are several ways to record the correspondence between the original and the image:

$$f(t) \leftrightarrow F(p), \quad f(t) \doteq F(p), \quad L\{f(t)\} = F(p).$$

Example 1.

Using the definition, find the image of the function $f(t) = \sin 3t$.

Solution:

For the function $f(t) = \sin 3t$, we have $\alpha_0 = 0$. Therefore, the image $F(p)$ will be defined and analytically in the half-plane $\operatorname{Re} p > 0$. Let us apply formula (1.2) to a given function, using the rule of integration in parts and the restriction on the set of values of the variable p , which ensures the convergence of the integral, when performing transformations:

$$\begin{aligned} F(p) &= \int_0^{+\infty} e^{-pt} \sin 3t dt = -\frac{1}{p} e^{-pt} \sin 3t \Big|_0^{+\infty} + \frac{3}{p} \int_0^{+\infty} e^{-pt} \cos 3t dt = \\ &= \frac{3}{p} \left(-\frac{1}{p} e^{-pt} \cos 3t \Big|_0^{+\infty} - \frac{3}{p} \int_0^{+\infty} e^{-pt} \sin 3t dt \right) = \frac{3}{p^2} - \frac{9}{p^2} F(p). \end{aligned}$$

Got equality

$$F(p) = \frac{3}{p^2} - \frac{9}{p^2} F(p).$$

From here we find

$$F(p) = \frac{3}{p^2 + 9}.$$

Thus, the following correspondence is valid:

$$\sin 3t \leftrightarrow \frac{3}{p^2 + 9}, \operatorname{Re} p > 0.$$

Properties of the Laplace transform

- 1. Linearity.** If $f(t) \leftrightarrow F(p)$, $g(t) \leftrightarrow G(p)$, then for any complex λ and μ it is performed

$$\lambda f(t) + \mu g(t) \leftrightarrow \lambda F(p) + \mu G(p), \operatorname{Re} p > \max(\alpha_0, \beta_0),$$

here and further, α_0, β_0 are the growth indicators of the function $f(t), g(t)$, respectively.

- 2. Similarity.** If $f(t) \leftrightarrow F(p)$, then for $\forall \alpha > 0$ it is true

$$f(at) \leftrightarrow \frac{1}{a} F\left(\frac{p}{a}\right), \operatorname{Re} p > a\alpha_0.$$

- 3. Differentiation of the original.** If $f(t), f'(t), \dots, f^{(n)}(t)$ are originals and $f(t) \leftrightarrow F(p)$ for $\operatorname{Re} p > \alpha_0$, then

$$f^{(n)}(t) \leftrightarrow p^n F(p) - p^{n-1} f(+0) - p^{n-2} f'(+0) - \dots - p f^{(n-2)}(+0) - f^{(n-1)}(+0),$$

where

$$f^{(k)}(+0) = \lim_{t \rightarrow +0} f^{(k)}(t), \quad k = 0, 1, \dots, n-1.$$

- 4. Image differentiation.** If $f(t) \leftrightarrow F(p)$, then

$$F^{(n)}(p) \leftrightarrow (-t)^n f(t), \operatorname{Re} p > \alpha_0.$$

- 5. Integration of the original.** If $f(t) \leftrightarrow F(p)$, then

$$\int_0^t f(\tau) d\tau \leftrightarrow \frac{F(p)}{p}, \operatorname{Re} p > \alpha_0.$$

6. **Image integration.** If $f(t) \leftrightarrow F(p)$ and $\frac{f(t)}{t}$ are the original, then

$$\int_p^\infty F(\xi) d\xi \leftrightarrow \frac{f(t)}{t}, \operatorname{Re} p > \alpha_0.$$

7. **The delay property.** If $f(t) \leftrightarrow F(p)$ and $f(t)=0$ for $t < \tau$, where $\tau > 0$, then

$$f(t-\tau) \leftrightarrow e^{-\tau p} F(p), \operatorname{Re} p > \alpha_0.$$

Remark. The following formulation of the delay property is possible: if $f(t) \leftrightarrow F(p)$, then for any $\tau > 0$ there is

$$f(t-\tau)\theta(t-\tau) \leftrightarrow e^{-\tau p} F(p), \operatorname{Re} p > \alpha_0.$$

8. **The displacement property.** If $f(t) \leftrightarrow F(p)$, then for any complex λ

$$e^{\lambda t} f(t) \leftrightarrow F(p-\lambda), \operatorname{Re} p > \alpha_0 + \operatorname{Re} \lambda.$$

9. **The image of the convolution.** The convolution of functions f and g is a function that is denoted by $f \cdot g$ and is defined by equality

$$(f \cdot g)(t) = \int_0^t f(\tau) g(t-\tau) d\tau.$$

The convolution of functions has the property of symmetry, that is,

$$(f \cdot g)(t) = (g \cdot f)(t).$$

If $f(t) \leftrightarrow F(p)$ and $g(t) \leftrightarrow G(p)$, then

$$(f \cdot g)(t) \leftrightarrow F(p)G(p), \operatorname{Re} p > \max(\alpha_0, \beta_0).$$

Here is a table of originals and images of some elementary functions:

Original $f(t)$	Image $F(p)$	Original $f(t)$	Image $F(p)$
1	$\frac{1}{p}$	$sh at$	$\frac{a}{p^2 - a^2}$
e^{-at}	$\frac{1}{p + a}$	$ch at$	$\frac{p}{p^2 - a^2}$
t	$\frac{1}{p^2}$	$e^{-at} \cos \omega t$	$\frac{p + a}{(p + a)^2 + \omega^2}$
$\sin at$	$\frac{a}{p^2 + a^2}$	$e^{-at} \sin \omega t$	$\frac{\omega}{(p + a)^2 + \omega^2}$
$\cos at$	$\frac{p}{p^2 + a^2}$	$e^{at} sh \omega t$	$\frac{\omega}{(p - a)^2 - \omega^2}$
$t^n, n \in \mathbb{Z}$	$\frac{n!}{p^{n+1}}$	$e^{at} ch \omega t$	$\frac{p - a}{(p - a)^2 - \omega^2}$
$t^n e^{at}$	$\frac{n!}{(p - a)^{n+1}}$	$t sh \omega t$	$\frac{2\omega p}{(p^2 - \omega^2)^2}$
$t \sin \omega t$	$\frac{2p\omega}{(p^2 + \omega^2)^2}$	$t ch \omega t$	$\frac{p^2 + \omega^2}{(p^2 - \omega^2)^2}$
$t \cos \omega t$	$\frac{p^2 - \omega^2}{(p^2 + \omega^2)^2}$	$e^{at} t \sin \omega t$	$\frac{2\omega(p - a)}{((p - a)^2 + \omega^2)^2}$
$e^{at} t \cos \omega t$	$\frac{(p - a)^2 - \omega^2}{((p - a)^2 + \omega^2)^2}$	$\frac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$	$\frac{1}{(p^2 + \omega^2)^2}$
$\frac{1}{2\omega^3}(\omega t ch \omega t - sh \omega t)$	$\frac{1}{(p^2 - \omega^2)^2}$	$\sin(\omega t \pm \varphi)$	$\frac{\omega \cos \varphi \pm p \sin \varphi}{p^2 + \omega^2}$
$\cos(\omega t \pm \varphi)$	$\frac{p \cos \varphi \mp \omega \sin \varphi}{p^2 + \omega^2}$		

Example 2.

Using the properties of the Laplace transform and the table of the main originals and images, find images of the following functions:

1) $f(t) = e^{-4t} \sin 3t \cos 2t$;

2) $f(t) = e^{(t-2)} \sin(t-2)$;

3) $f(t) = t^2 e^{3t}$;

4) $f(t) = \frac{\sin^2 t}{t}$.

Solution:

1) Let's transform the expression for the function $f(t)$ as follows:

$$f(t) = e^{-4t} \sin 3t \cos 2t = \frac{1}{2} e^{-4t} (\sin 5t + \sin t) = \frac{1}{2} e^{-4t} \sin 5t + \frac{1}{2} e^{-4t} \sin t.$$

Since $\sin t \leftrightarrow \frac{1}{p^2 + 1}$ and $\sin 5t \leftrightarrow \frac{5}{p^2 + 25}$, then, using the properties of linearity and displacement, for the image of the function $f(t)$ we will have

$$F(p) = \frac{1}{2} \left(\frac{5}{(p+4)^2 + 25} + \frac{1}{(p+4)^2 + 1} \right).$$

2) Since $\sin t \leftrightarrow \frac{1}{p^2 + 1}$, $e^t \sin t \leftrightarrow \frac{1}{(p-1)^2 + 1}$, then, using the delay property, we will have

$$f(t) = e^{(t-2)} \sin(t-2) \leftrightarrow F(p) = \frac{e^{-2p}}{(p-1)^2 + 1}.$$

3) Since $t^2 \leftrightarrow \frac{2}{p^3}$, then by the displacement property we have

$$f(t) = t^2 e^{3t} \leftrightarrow F(p) = \frac{2}{(p-3)^3}.$$

For comparison, we present a method for constructing an image of function

$f(t) = t^2 e^{3t}$ using the image differentiation property:

$$e^{3t} \leftrightarrow \frac{1}{p-3}; \quad te^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{p-3}\right) = \frac{1}{(p-3)^2};$$

$$t^2 e^{3t} \leftrightarrow -\frac{d}{dp}\left(\frac{1}{(p-3)^2}\right) = \frac{2}{(p-3)^3}.$$

We got the same result.

4. HOMEWORK (using the image integration property)

For a function defined as follows:

$$f(t) = \begin{cases} 0, & t < t_1, \\ f_1(t), & t_1 \leq t < t_2, \\ f_2(t), & t_2 \leq t < t_3, \\ \dots \\ f_{n-1}(t), & t_{n-1} \leq t < t_n, \\ f_n(t), & t \geq t_n, \end{cases}$$

using the Heaviside function, you can write an analytical form that is convenient to use when constructing the corresponding image.

It is easy to verify that for a function $g_k(t)$ equal to

$$g_k(t) = \begin{cases} 0, & t < t_k, \\ f_k(t), & t_k \leq t < t_{k+1}, \\ 0, & t \geq t_{k+1}, \end{cases}$$

the following representation is valid using the Heaviside function:

$$g_k(t) = f_k(t)\theta(t-t_k) - f_k(t)\theta(t-t_{k+1}). \quad (1.3)$$

And for the function

$$g_n(t) = \begin{cases} 0, & t < t_n, \\ f_n(t), & t \geq t_n, \end{cases}$$

there is

$$g_n(t) = f_n(t)\theta(t-t_n). \quad (1.4)$$

Assuming that k varies from 1 to $n-1$, the function $f(t)$ can be considered as the sum of the functions $g_k(t)$ and $g_n(t)$:

$$f(t) = \sum_{k=1}^{n-1} g_k(t) + g_n(t).$$

And then, using expressions (1.3) and (1.4), we get

$$f(t) = f_1(t)\theta(t-t_1) - \sum_{k=2}^n (f_k(t) - f_{k-1}(t))\theta(t-t_k). \quad (1.5)$$

Example 3.

Build an image for the function $f(t)$:

$$f(t) = \begin{cases} 0, & t < a, \\ \varphi(t), & a \leq t < b, \\ 0, & t \geq b. \end{cases}$$

Solution:

Let's write an expression for the function $f(t)$ using the Heaviside function:

$$f(t) = \varphi(t)\theta(t-a) - \varphi(t)\theta(t-b).$$

Since

$$\varphi(t) = \varphi(t-a+a) \text{ и } \varphi(t) = \varphi(t-b+b),$$

then, having found the images for the functions $\varphi(t+a)$ and $\varphi(t+b)$,

$$\varphi(t+a) \leftrightarrow \Phi_1(p), \quad \varphi(t+b) \leftrightarrow \Phi_2(p),$$

we construct an image for the function $f(t)$, taking into account the lag property

$$f(t) \leftrightarrow F(p) = \Phi_1(p)e^{-ap} - \Phi_2(p)e^{-bp}.$$

Example 4.

Find the image $F(p)$ of the function $f(t)$:

$$f(t) = \begin{cases} 0, & t \in (-\infty, 0), \\ 1, & t \in (0, a), \\ \frac{2a-t}{a}, & t \in (a, 3a), \\ \frac{t-4a}{a}, & t \in [3a, \infty). \end{cases}$$

Solution:

Let's find an image of the function $f(t)$, having previously written an expression for it using the Heaviside function $\theta(t)$. To do this, use the formula (1.5). Since for a given function

$$t_1 = 0, \quad t_2 = a, \quad t_3 = 3a \quad \text{and}$$

$$f_1(t)=1, \quad f_2(t)=\frac{2a-t}{a}, \quad f_3(t)=\frac{t-4a}{a},$$

then we will have

$$\begin{aligned} f(t) &= \theta(t) - \left(\frac{2a-t}{a} - 1 \right) \theta(t-a) + \left(\frac{t-4a}{a} - \frac{2a-t}{a} \right) \theta(t-3a) = \\ &= \theta(t) - \frac{t-a}{a} \theta(t-a) + \frac{2(t-3a)}{a} \theta(t-3a). \end{aligned}$$

Applying the properties of linearity and delay to the constructed expression, we find the desired image $F(p)$:

$$F(p) = \frac{1}{p} - \frac{1}{ap^2} e^{-ap} + \frac{2}{ap^2} e^{-3ap}.$$