Equations of Line in Space.

Assume that some straight line a in the space $\mathbb{E}$ is chosen and fixed. In order to study various equations determining this line we choose some coordinate system O, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in the space.

Vectorial parametric equation of this line is:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a} \cdot t \tag{1}$$

Coordinate parametric equations of this line are:

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \\ z = z_0 + a_z t \end{cases} \tag{2}$$

General-case vectorial equation of line has form

$$(\boldsymbol{r} - \boldsymbol{r}_0) \times \boldsymbol{a} = \boldsymbol{0}. \tag{3}$$

or

$$\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{b}, \tag{4}$$

Exclusion t from parametric equations yields canonical equation of line

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z}, \tag{5}$$

and its form for line passing through two points

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0} \tag{6}$$

Final studied form of the equation of line is expression of line with two crossing planes

$$\begin{cases} A_1 x + B_1 y + C_1 z + D_1 = 0 \\ A_2 x + B_2 y + C_2 z + D_2 = 0, \end{cases} \tag{7}$$

including form in projections

$$\begin{cases} x = Mz + b_x \\ y = Nz + b_y \end{cases} \tag{8}$$

1 Distance from the line to a point in space

Suppose that line a in space expressed with any form of its equation, hence direction vector of this line $\boldsymbol{a} \mapsto (a_x, a_y, a_z)$ and initial point $A(x_0, y_0, z_0)$ may be restored.

Suppose $M(x_1, y_1, z_1)$ is a point not laying on the line.

Definition. Distance from the line to the point is a length of segment perpendicular to the line and connecting point and line.

Let us calculate this length.

There are different approaches to this problem.

1.1 Approach one. Utilizing dot product

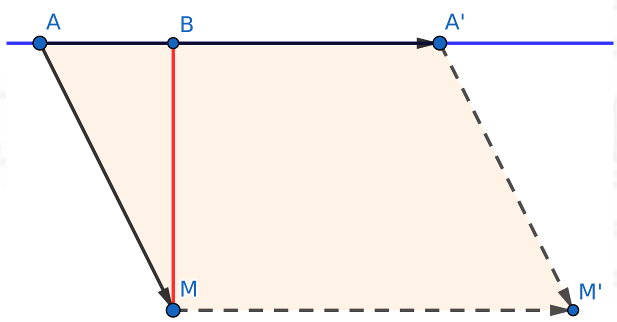

Figure 1: Distance from the line to a point in space

Pair of non-collinear vectors $\overrightarrow{AA'} = \boldsymbol{a}$ and $\overrightarrow{AM}$ established from the point A on the line obviously govern a pair of crossing lines, hence shape the plane. Perpendicular segment $MB \perp a$, $B \in a$ also lies on this plane (its endpoints both lay on plane) and is altitude of the parallelogram.

There are at least two approaches to find area of parallelogram $MAA'M'$.

By the one hand, area of parallelogram is modulus of the cross product of vectors governing its sides

$$S = |\overrightarrow{AM} \times \overrightarrow{AA'}| = |\overrightarrow{AM} \times \boldsymbol{a}|.$$

By the other hand, area of parallelogram is product between length of its side and altitude perpendicular to that side:

$$S = MB \cdot AA' = d|\boldsymbol{a}|,$$

d is distance from the line to pint in question.

Therefore,

$$|\overrightarrow{AM} \times \boldsymbol{a}| = d|\boldsymbol{a}|,$$

$$d = \frac{|\overrightarrow{AM} \times \boldsymbol{a}|}{|\boldsymbol{a}|}.$$

Problem 1

Find distance from line

$$\frac{x-3}{2} = \frac{y-1}{1} = \frac{z+1}{2}$$

to point $M(0,2,3)$.

Basis is right orthonormal.

Solution

Direction vector of the line $\boldsymbol{a}$ has coordinates $(2,1,2)$. $|\boldsymbol{a}| = \sqrt{4+1+4} = 3$

Initial point of the line is $(3,1,-1)$. Coordinates of $\overrightarrow{AM}$ are $(-3,1,4)$.

Applying derived formula

$$\overrightarrow{AM} \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ -3 & 1 & 4 \\ 2 & 1 & 2 \end{vmatrix} = \boldsymbol{e}_1((-1)\cdot 2 - (-4)\cdot 1) - \boldsymbol{e}_2(3\cdot 2 - (-4)\cdot 2) + \boldsymbol{e}_3(3\cdot 1 - (-1)\cdot 2) = -$$

Hence, $|\overrightarrow{AM} \times \boldsymbol{a}| = \sqrt{(-2)^2 + 14^2 + (-5)^2} = \sqrt{4+196+25} = \sqrt{225} = 15.$

$$d = \frac{|\overrightarrow{AM} \times \boldsymbol{a}|}{|\boldsymbol{a}|} = \frac{15}{3} = 5.$$

1.2 Approach two. Investigation of projection

Let us take a look on the right-angled triangle $\triangle ABM$ (see sketch for previous approach).

In this triangle leg $AB = AM \cos\angle A$. Hence, AB is *projection* of the leg AM on the line in question.

Cosines of the angle $\angle A$ we may obtain with calculation of the dot product:

$$\overrightarrow{AA'} \cdot \overrightarrow{AM} = \boldsymbol{a} \cdot \overrightarrow{AM} = |\boldsymbol{a}||\overrightarrow{AM}| \cos\angle A$$

$$\cos\angle A = \frac{\boldsymbol{a} \cdot \overrightarrow{AM}}{|\boldsymbol{a}||\overrightarrow{AM}|} = \frac{\boldsymbol{a} \cdot \overrightarrow{AM}}{|\boldsymbol{a}|AM}$$

If we fortunately find $\cos\angle A = 0$, point A itself is the base of perpendicular segment!

$$AB = AM\frac{\boldsymbol{a} \cdot \overrightarrow{AM}}{|\boldsymbol{a}|AM} = \frac{\boldsymbol{a} \cdot \overrightarrow{AM}}{|\boldsymbol{a}|}$$

Vectors $\overrightarrow{AB}$ and $\overrightarrow{AA'} = \boldsymbol{a}$ are collinear, hence we can express one through other:

$$\overrightarrow{AB} = AB\frac{\boldsymbol{a}}{|\boldsymbol{a}|} = \boldsymbol{a}\frac{\boldsymbol{a} \cdot \overrightarrow{AM}}{\boldsymbol{a}^2}$$

Therefore, knowing coordinates of A, we restore position of B and calculate distance BM.

Let us try this approach for problem 1.

Recap: direction vector of the line $\boldsymbol{a}$ has coordinates $(2,1,2)$, $|\boldsymbol{a}| = \sqrt{4+1+4} = 3$ initial point of the line is $(3,1,-1)$, coordinates of $\overrightarrow{AM}$ are $(-3,1,4)$. Point in question $M(0,2,3)$.

$$\boldsymbol{a}\cdot\overrightarrow{AM} = 2\cdot(-3)+1\cdot 1+2\cdot 4 = 3,$$

$$\overrightarrow{AB} = \boldsymbol{a}\frac{\boldsymbol{a}\cdot\overrightarrow{AM}}{\boldsymbol{a}^2} = \boldsymbol{a}\frac{3}{9} = \frac{1}{3}\boldsymbol{a}.$$

Hence, coordinates of $\overrightarrow{AB}$ are $\left(\frac{2}{3},\frac{1}{3},\frac{2}{3}\right) = (x_B - x_A, y_B - y_A, z_B - z_A)$.

Coordinate of B are $\left(\frac{11}{3},\frac{4}{3},-\frac{1}{3}\right)$

$$BM = \sqrt{\left(0-\frac{11}{3}\right)^2+\left(2-\frac{4}{3}\right)^2+\left(3+\frac{11}{3}\right)^2} = \sqrt{25} = 5.$$

1.3 Approach three. Normal plane

While segment connecting line and point, and having length equal with distance between them is perpendicular with line, it lies on a plane perpendicular with line.

We write equation of this line in form

$$A(x-x_1)+B(y-y_1)+C(z-z_1) = 0,$$

x_0

To find covariant coordinates of the plane normal A, B, and C, we notice that normal of this plane corresponds with known direction vector of the line.

Hence, equation of the plane is

$$a^x(x-x_1)+a^y(y-y_1)+a^z(z-z_1) = 0,$$

where a^x, a^y and a^z are covariant coordinates of direction vector $\boldsymbol{a}$

Now we express piercing point of line and plane while solving the system:

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \\ z = z_0 + a_z t \\ a^x(x-x_1)+a^y(y-y_1)+a^z(z-z_1) = 0. \end{cases}$$

x, y and z are explicitly expressed with t, hence

$$a^x(x_0 + a_x t - x_1) + a^y(y_0 + a_y t - y_1) + C(z_0 + a_z t - z_1) = 0$$
$$a^x(x_0 - x_1) + a^y(y_0 - y_1) + a^z(z_0 - z_1) + t(a^x a_x + a^y a_y + a^z a_z) = 0$$
$$t_0 = -\frac{a^x(x_0 - x_1) + a^y(y_0 - y_1) + a^z(z_0 - z_1)}{a^x a_x + a^y a_y + a^z a_z} = -\frac{\overrightarrow{AM} \cdot \boldsymbol{a}}{\boldsymbol{a}^2}$$

$$x = x_0 - a_x \frac{\overrightarrow{AM} \cdot \boldsymbol{a}}{\boldsymbol{a}^2}$$
$$y = y_0 - a_y \frac{\overrightarrow{AM} \cdot \boldsymbol{a}}{\boldsymbol{a}^2}$$
$$z = z_0 - a_z \frac{\overrightarrow{AM} \cdot \boldsymbol{a}}{\boldsymbol{a}^2}$$

Distance in question now we calculate with length of vector connecting piercing point and point in question.

$$d = \sqrt{(x_1 - x_0 - a_x t_0)^2 + (y_1 - y_0 - a_y t_0)^2 + (z_1 - z_0 - a_z t_0)^2}$$

Problem 2

Find distance form the line

$$\boldsymbol{r} = 2\boldsymbol{e}_1 + \boldsymbol{e}_2 + (\boldsymbol{e}_2 + 2\boldsymbol{e}_3)t$$

to the origin.

Basis is right orthonormal.

Solution

Vectorial parametric equation of the line written here via notation for basis vectors.
Hence, initial point is $A(2, 1, 0)$, coordinates of direction vector are $(0, 1, 2)$.
Desired plane passes through origin and has direction vector of line as normal vector.
While basis is right orthonormal, this equation is

$$y + 2z = 0$$

Substitute $y = 2 + t$, $z = 2t$ from parametric equation o line into it:

$$1 + t + 4t = 0$$
$$t = -\frac{1}{5},$$

$$\begin{cases} x = 2 \\ y = 1 - \dfrac{1}{5} = \dfrac{4}{5} \\ z = 2 \cdot \left(-\dfrac{1}{5}\right) = -\dfrac{2}{5} \end{cases}$$

$$d = \sqrt{2^2 + \left(\frac{4}{5}\right)^2 + \left(-\frac{2}{5}\right)^2} = 2\sqrt{\frac{6}{5}}$$

2 Approach four. Minimization of segment

Let us try to solve problem 2 using alternative approach

Any point of the line satisfies equations:

$$\begin{cases} x = 2 \\ y = 1 + t \\ z = 2t \end{cases}$$

Distance of this point to the origin is

$$d(t) = \sqrt{2^2 + (1+t)^2 + (2t)^2} = \sqrt{4 + (1 + 2t + t^2) + 4t^2} = \sqrt{5t^2 + 2t + 5} = \sqrt{f(x)}$$

Perpendicular segment is the shortest one connecting a point and line, hence we investigate minimum of this polynomial function.

$$f(t) = 5t^2 + 2t + 5$$
$$\frac{\mathrm{d}f}{\mathrm{d}t} = 10t + 2$$

Letting it zero yields $t = -\dfrac{1}{5}$, which corresponds with t obtained with previous method

3 Distance between two lines

There are three possible relative dispositions of a pair of non-coinciding lines in space:

1. Lines have common point, hence lay in a single plane and intersect. Conditions

$$\begin{vmatrix} x_0' - x_0 & y_0' - y_0 & z_0' - z_0 \\ a_x & a_y & a_z \\ a_x' & a_y' & a_z' \end{vmatrix} = 0,$$

   where x_0, y_0, z_0 and x_0', y_0', z_0' are coordinates of initial points, a_x, a_y, a_z and a_x', a_y', a_z' are coordinates of direction vectors.

2. Lines are parallel.

   Lemma 3.1. *Parallel lines in plane lay in a common plane.*

   Proof.
   - Let $\boldsymbol{a}$ and $\boldsymbol{a}'$ be direction vectors of these lines. For parallel lines these vectors are collinear, hence $\boldsymbol{a}' = \alpha\boldsymbol{a}$, $\alpha \neq 0$.
   - Axiomatics of parallel lines grants, in particular case, that each transverse segment between two lines perpendicular to any of lines is also perpendicular with second line, hence *lines in question have common normal vector* which is parallel with their common perpendicular $\boldsymbol{n}^*$.
   - $\boldsymbol{a}$ and $\boldsymbol{n}^*$ govern a plane, and $\boldsymbol{a}'$ lays in it, as in has valid representation:

   $$\boldsymbol{a}' = \alpha\boldsymbol{a} + 0 \cdot \boldsymbol{n}^*$$

   - $\boldsymbol{a}$ also has representation:

   $$\boldsymbol{a} = \frac{1}{\alpha}\boldsymbol{a}' + 0 \cdot (-\frac{1}{\alpha})\boldsymbol{n}^*$$

   - And they both lay in the same plane

   □

   To measure distance between parallel lines we take a point on one if lines and measure distance from this point to a second line. It is most natural to take initial point as such reference.

   Problem 1

   Find distance between lines

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{y-1}{1}$$

and

$$\frac{x-2}{2}=\frac{y+1}{2}=\frac{y-3}{2}$$

Basis is right orthonormal.

Solution

Initial point of first line is $A(1,1,1)$. Direction vectors have coordinates $(1,1,1)$, $(2,2,2)$ and are proportional.

Parametric equations of second line are:

$$\begin{cases} x=2+2t \\ y=-1+2t \\ z=3+2t \end{cases}$$

Function of the distance to the point $(1,1,1)$ from the line is

$$f(t)=(2+2t-1)^2+(-1+2t-1)^2+(3+2t-1)^2=(1+2t)^2-2(1-t)^2+2(1+t)^2=$$

$$\frac{\mathrm{d}f}{\mathrm{d}dt}=24t+4.$$

Letting derivative be zero yields

$$\begin{aligned} 24t+4&=0 \\ t&=-\frac{1}{6} \end{aligned}$$

Therefore, corresponding point on second line is

$$\begin{aligned} x&=2+2\cdot(-\frac{1}{6})=\frac{5}{3} \\ y&=-1+2\cdot(-\frac{1}{6})=-\frac{4}{3} \\ z&=3+2\cdot(-\frac{1}{6})=\frac{8}{3} \end{aligned}$$

$$d=\sqrt{\left(1-\frac{5}{3}\right)^2+\left(1+\frac{4}{3}\right)^2+\left(1-\frac{5}{3}\right)^2}$$

3. Lines do not intersect and are non-parallel

Definition. Two non-parallel lines in space are called **skew lines** if they do not have common points

Lemma 3.2. *Skew lines lay on parallel planes*

Proof. Let L_1 and L_2 be these skew lines.

Suppose line L_3 intersects L_1 and $L_3 \parallel L_2$. As direction vectors of L_3 and L_2 are equal, L_1 and L_3 govern plane Π_1 parallel with L_2.

Line L_4 intersecting L_2 and parallel with L_1 shapes opposite plane Π_1, to which L_1 is parallel.

Therefore, direction vectors of lines correspond:

$$\begin{aligned} \boldsymbol{a}_1 &= \boldsymbol{a}_4 \\ \boldsymbol{a}_2 &= \boldsymbol{a}_3 \end{aligned}$$

Hence, normal vectors of the planes are:

$$\begin{aligned} \boldsymbol{n}_1 &= \boldsymbol{a}_1 \times \boldsymbol{a}_3 \\ \boldsymbol{n}_2 &= \boldsymbol{a}_4 \times \boldsymbol{a}_2 = \boldsymbol{n}_1, \end{aligned}$$

and planes are parallel. □ □

Definition. There are two skew lines L_1 and L_2. Two points $c_1 \in L_1$ and $c_2 \in L2$ forming shortest segment are called the **nearest points of skew lines**.

Definition. Distance between skew lines is a distance between the nearest points.

Lemma 3.3. *Nearest points of skew lines are endpoints of common perpendicular segment containing parallel planes governed by skew lines.*

Proof. Let us revisit picture of previous lemma.

Suppose $L_2' \parallel L_2$ is *perpendicular projection* of L_2 on Π_1. To do so we select two distant points on L_2 and establish lines from them perpendicular with Π_1. Line connecting piercing pints is L_2'. Let P be intersection point of L_1 and L_2'.

By the same way we build line $L_1' \parallel L_2$ on Π_2 and set Q as intersection point of L_1' and L_2.

Hence, P and Q are projections of each other, lay on common perpendicular of planes and on the lines themselves.

While perpendicular is the shortest segment between planes, PQ is the distance between lines, P and Q are the nearest points □

Let skew lines be governed with vectorial parametric equations:

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a}$$
$$\boldsymbol{r} = \boldsymbol{r}_0' + \tau\boldsymbol{a}'$$

Normal unit vector of containing them planes may be defined with equation is

$$\boldsymbol{n} = \frac{\boldsymbol{a} \times \boldsymbol{a}'}{|\boldsymbol{a} \times \boldsymbol{a}'|}.$$

Hence, distance between lines is

$$d = |\boldsymbol{n} \cdot (\boldsymbol{r}_0 - \boldsymbol{r}_0')|$$

Problem 2

Find distance between lines:

$$\frac{x-9}{4} = \frac{y+2}{-3} = \frac{z}{1}$$

and

$$\frac{x}{-2} = \frac{y+7}{9} = \frac{z-2}{2}.$$

Basis is right orthonormal.

Solution

Normal vector of the parallel planes is

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 4 & -3 & 1 \\ -2 & 9 & 2 \end{vmatrix} = -15\boldsymbol{e}_1 - 10\boldsymbol{e}_2 + 30\boldsymbol{e}_3$$

length of this vector is 35, hence unit normal vector has coordinates $\left(\frac{3}{7}, \frac{2}{7}, -\frac{6}{7}\right)$

$$d = \left| \begin{pmatrix} \frac{3}{7} \\ \frac{2}{7} \\ -\frac{6}{7} \end{pmatrix} \cdot \begin{pmatrix} 0-9 \\ -7-(-2) \\ 2-0 \end{pmatrix} \right| =$$
$$= \left| \begin{pmatrix} \frac{3}{7} \\ \frac{2}{7} \\ -\frac{6}{7} \end{pmatrix} \cdot \begin{pmatrix} -9 \\ -5 \\ 2 \end{pmatrix} \right| = 7$$

Problem 3

Derive equation of the plane equidistant from a pair of skew lines

$$\frac{x - x_1}{a_x} = \frac{y - y_1}{a_y} = \frac{z - z_1}{a_z}$$

and

$$\frac{x - x_2}{b_x} = \frac{y - y_2}{b_y} = \frac{z - z_2}{b_z}$$

Basis is right orthonormal

Solution

Plane in question must be parallel with both lines, hence it is parallel with planes governed by these lines.

First, we derive normal vector of plane:

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y\, b_z - a_z\, b_y)\ \boldsymbol{e}_1 + (a_x\, b_z - a_z\, b_x)\ \boldsymbol{e}_2 + (a_x\, b_y - a_y\, b_x)\ \boldsymbol{e}_3 = A\boldsymbol{e}_1 + B\boldsymbol{e}_2 + C\boldsymbol{e}_3$$

Normalization ratio is $\sqrt{A^2 + B^2 + C^2}$. Let a, b, c be coefficients in normal equation of the plane. Unit normal vector $\boldsymbol{n}$ has coordinates (a, b, c).

Radius vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of initial points are (x_1, y_1, z_1) and (x_2, y_2, z_2).

Equations of the planes governed by skew lines:

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

and

$$a(x - x_2) + b(y - y_2) + c(z - z_2) = 0$$

Equation of the plane in question:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Distance between lines

$$d = |\boldsymbol{n} \cdot (\boldsymbol{r}_1 - \boldsymbol{r}_2)|$$

To find initial point of this plane, we build perpendicular between planes and find its mid-point.

Radius vector of the point on second plane corresponding with point (x_1, y_1, z_1) is

$$\boldsymbol{r}_3 = \boldsymbol{r}_1 \pm d\boldsymbol{n}$$

To select proper point (x_3, y_3, z_3)among two purposed candidates, condition of perpendicularity with $\boldsymbol{n}$ must be checked:

$$\begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \\ z_2 - z_3 \end{pmatrix} \cdot \boldsymbol{n} = 0$$

Finally, plane in question is

$$a\left(x - \frac{x_1 + x_3}{2}\right) + b\left(y - \frac{y_1 + y_3}{2}\right) + c\left(z - \frac{z_1 + z_3}{2}\right) = 0$$

Problem 4
There is triangle $\triangle ABC$ with vertices $A(4, 1, -2)$, $B(2, 0, 0)$ and $C(-2, 3, -5)$.
Measure its altitude established from B

Solution
Length of altitude is equal with distance from the point B to line AC.
Direction vector of line AB $\boldsymbol{a}$ has coordinates $(-6, 2, -4)$, initial point $A(4, 1, -2)$.

$$d = \frac{|\overrightarrow{AB \times \boldsymbol{a}}|}{|\boldsymbol{a}|}$$

Coordinates of AB are $(-2, -1, 2)$
Coordinates of cross product are $(-1 - 18 - 10)$

$$d = \frac{\sqrt{1^2 + 18^2 + 10^2}}{\sqrt{6^2 + 2^2 + 4^2}} = \sqrt{\frac{425}{56}}$$

Problem 5
Given equations of two faces of a parallelepiped: $2x + y3z + 5 = 0$ and $4x + 2y - 6x + 9 = 0$. Find the height of the parallelepiped.

Solution
Since coordinates of normal vectors are proportional, the case in hand is about parallel faces. Then, the height of the parallelepiped is the distance from any point of the first plane to the second plane. Let us take a point $A(0, -5, 0)$ in the first plane and find its distance from the second plane by the formula for the distance from a point to a plane:

$$d = \frac{\sqrt{|4 \cdot 0 + 2 \cdot (-5) - 6 \cdot 0 + 9|}}{\sqrt{16 + 4 + 36}} = \frac{19}{2\sqrt{14}}$$

Problem 6
Given two lines:

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$$

and

$$\frac{x+1}{3} = \frac{y-2}{2} = \frac{z-1}{0}$$

1. Proof that lines are skew ones
2. Write equations of the parallel planes governed by lines
3. Find distance between them

Basis is right orthonormal.

Solution
Coordinates of direction vectors are $(3, -2, 2)$ and $(3, 2, 0)$. Hence, lines are not parallel. Known points on the lines are $(2, -1, 0)$ on first line, and $(-1, 2, 1)$ on second line. Connecting them vector has coordinates $(-3, 3, 1)$. Checking that vectors are non-coplanar:

$$\begin{vmatrix} -3 & 3 & 1 \\ 3 & -2 & 2 \\ 3 & 2 & 0 \end{vmatrix} = 42 \neq 0$$

Direction vectors of this skew lines govern parallel planes containing them.
Canonical equations of planes:

$$\begin{vmatrix} x-2 & y+1 & z \\ 3 & -2 & 2 \\ 3 & 2 & 0 \end{vmatrix}$$

or

$$2x - 3y - 6z - 7 = 0,$$

and

$$\begin{vmatrix} x+1 & y-2 & z-1 \\ 3 & -2 & 2 \\ 3 & 2 & 0 \end{vmatrix}$$

or

$$2x - 3y - 6z + 14 = 0.$$

Normalization ratio is $\sqrt{4+9+26} = \sqrt{49} = 7$.

To calculate distance between lines we take arbitrary point on one plane and calculate distance to opposite plane.

We select point $\left(\frac{7}{2}, 0, 0\right)$ on a first plane.

$$d = \frac{7+14}{7} = 3.$$

Problem 7

Find the projection of the straight line

$$\begin{cases} 2x - y + z - 3 = 0 \\ x + y + 2z + 1 = 0 \end{cases}$$

onto plane

$$3x - y + z - 4 = 0$$

Basis is right orthonormal.

Solution

The projection lies on the intersection of two planes: the plane $3x - y + z - 4 = 0$ and the plane passing through the given straight line perpendicularly to the given plane. To find the equation of it, we represent it as a plane in a proper beam defined by the given straight line:

$$p(2x - y + z - 3) + q(x + y + 2z + 1) = 0,$$

or

$$x(2p+q) + y(-p+q) + z(p+2q) - 3p + q = 0.$$

Hence, normal vector of this plane has coordinates $(2p+q, -p+q, p+2q)$.

This plane is desired to be perpendicular with a plane with normal $(3, -1, t)$, therefore, p and q have relation:

$$3(2p+q) - (-p+q) + (p+2q) = 0$$

$$8p + 4q = 0,$$
$$q = -2p$$

While we are interested only in proposition, we take $p = 1$, $q = -2$.
Desired equation now is

$$3y + 3z + 5 = 0$$

Projection now has form

$$\begin{cases} 3y + 3z + 5 = 0 \\ 3x - y + z - 4 = 0 \end{cases}$$

Problem 8
Given a cube with side 1.
Find the shortest distance between a cube diagonal and a face diagonal not intersecting it.

Solution
In this problem we are not given wit any coordinate system, so we shape it by our own will.
Coordinate system and cube are provided on sketch

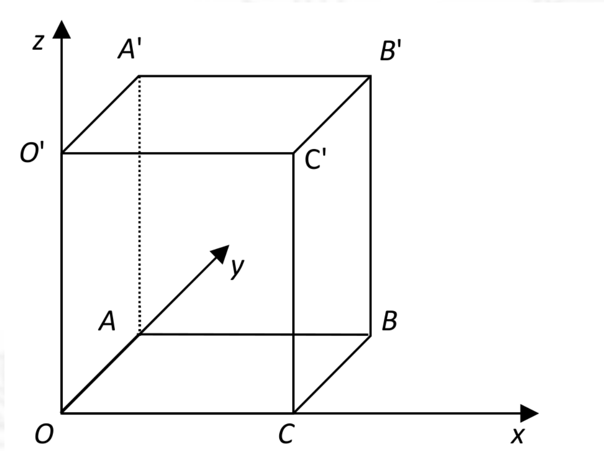

Figure 2: Cube with introduced coordinate system.

Selected system has origin in a cube vertex, right orthonormal basis with basis vectors coincide with cube edges.

The shortest distance is the distance between skew lines. Let the straight line OB' for which we have two points $O(0,0,0)$, $B'(1,1,1)$ and the directional vector $\overrightarrow{OB'}$ with coordinates $(1,1,1)$ be taken as the cube diagonal.

The straight line AC for which we have two points $A(0,1,0)$, $C(1,0,0)$, and the directional vector $\overrightarrow{AC}$ with coordinates $(1,-1,0)$ is taken as the face diagonal.

Common normal vector of lines:

$$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \boldsymbol{e}_1 + \boldsymbol{e}_2 - 2\boldsymbol{e}_3$$

Normalization ratio is $\sqrt{1+1+4} = \sqrt{6}$

We select point O on first line and point C on second line $\overrightarrow{OC}$ has coordinates $(1, 0, 0)$. Hence,

$$d = \frac{|\boldsymbol{n} \cdot \overrightarrow{OC}|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

Problem 9

A triangle is formed by intersection lines of the plane

$$9x + 12y + 20z - 60 = 0$$

and coordinate planes. Find its altitude drawn from the vertex laying on the Oz axis and write equations of the straight line containing this altitude.

Solution

If the triangle vertex A lies on the Oz axis and in the given plane, its coordinates are $(0, 0, 3)$.

One should find the distance from this point to the opposite triangle side, i.e., to the line

$$\left\{ 9x + 12y + 20z - 60 = 0z = 0 \right.$$

Normal vectors of these planes have coordinates $(9, 12, 20)$ and $(0, 0, 1)$. Hence, direction vector which is their dot product is

$$\boldsymbol{a} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 9 & 12 & 20 \\ 0 & 0 & 1 \end{vmatrix} = 12\boldsymbol{e}_1 - 9\boldsymbol{e}_2$$

and is codirected with vector having coordinates $(4, -3, 0)$.

$|\boldsymbol{a}| = \sqrt{16+9} = 5$.

As a point on a line we take intercept $B(0, 5, 0)$. $\overrightarrow{AB}$ has coordinates $(0, 5 - 3)$.

$$d = \frac{\overrightarrow{AB} \times \boldsymbol{a}}{|\boldsymbol{a}|} = \frac{1}{5} \left\| \begin{matrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 0 & 5 & 3 \\ 4 & -3 & 0 \end{matrix} \right\| = \frac{1}{5} | -9\boldsymbol{e}_1 - 12\boldsymbol{e}_2 - 20\boldsymbol{e}_3| =$$
$$= \frac{\sqrt{81 + 144 + 400}}{5} = \frac{\sqrt{625}}{5} = \frac{25}{5} = 5$$