3 The fundamental theorem of algebra

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3.1 Cubic equations and Cardano's method

Consider a cubic equation

$$ax^3 + bx^2 + cx + d = 0, \quad a \neq 0.$$

First, divide it by a:

$$x^3 + \tilde{b}x^2 + \tilde{c}x + \tilde{d} = 0.$$

Then change the variable $x = y - \frac{\tilde{b}}{3}$ in order to get zero coefficient at y^2:

$$(y - \tfrac{\tilde{b}}{3})^3 + \tilde{b}(y - \tfrac{\tilde{b}}{3})^2 + \tilde{c}(y - \tfrac{\tilde{b}}{3}) + \tilde{d} = y^3 - \cancel{\tilde{b}y^2} + \tfrac{\tilde{b}^2}{3}y - \tfrac{\tilde{b}^3}{27} + \cancel{\tilde{b}y^2} - \tfrac{2\tilde{b}^2}{3}y + \tfrac{\tilde{b}^3}{9} + \tilde{c}y - \tfrac{\tilde{c}\tilde{b}}{3} + \tilde{d} = 0$$

which gives

$$y^3 + py + q = 0.$$

This is called a "depressed" equation. Set $y = u + v$ and get

$$u^3 + v^3 + 3uv(u + v) + p(u + v) + q = 0.$$

This can be solved if we set

$$\begin{cases} u^3 + v^3 = -q \\ 3uv = -p \end{cases}.$$

Cubing the second equation and substituting it into the first one yields

$$u^3 - \frac{p^3}{27u^3} = -q.$$

This is a quadric equation in $t = u^3$:

$$t^2 + qt - \frac{p^3}{27} = 0.$$

One of its solution is

$$t = -\frac{q}{2} \pm \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}}$$

which gives

$$u = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Similarly one can find

$$v = \sqrt[3]{-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

The condition $3uv = -p$ implies that if

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

then

$$v=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

whence

$$y=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}.$$

There are three cube roots of every number, thus the above formula produces 9 values. We have to match the cube roots so that their product equals $-p/3$. Then only 3 values remain. Finally, we put $x=y-\frac{\tilde{b}}{3}$.

Remark. If another root of the quadratic equation in t is chosen, the values of u_1,u_2,u_3 will be different. So will the values of v_1,v_2,v_3

Remark. The determinant of the depressed cubic $f(y)=y^3+py+q$ is $D(f)=-4p^3-27q^2$. We can see it in the above formula under the square root.

The solutions of the general cubic equation $(j=1,2,3)$ are

$$x_j=-\frac{b}{3a}+\frac{\varepsilon^j\sqrt[3]{\sqrt{\left(-27a^2d+9abc-2b^3\right)^2+4\left(3ac-b^2\right)^3}-27a^2d+9abc-2b^3}}{3\sqrt[3]{2}a}$$
$$-\frac{\bar{\varepsilon}^j\sqrt[3]{2}\left(3ac-b^2\right)}{3a\sqrt[3]{\sqrt{\left(-27a^2d+9abc-2b^3\right)^2+4\left(3ac-b^2\right)^3}-27a^2d+9abc-2b^3}},$$

where $\varepsilon=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ is a cube root of unity.

Example. To solve

$$z^3+6z^2+9z+3=0,$$

change the variable $w=z+2$ which gives

$$(w-2)^3+6(w-2)^2+9(w-2)+3=w^3-3w+1=0$$

If we find u,v with

$$uv=1,\quad u^3+v^3=-1.$$

then $w=u+v$. The above equations imply

$$\left(t-u^3\right)\left(t-v^3\right)=t^2+t+1,$$

and u^3,v^3 are the roots of the equation

$$t^2+t+1=0.$$

Solving it we obtain $t=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$. Fix one of them, $t=-\frac{1}{2}+i\frac{\sqrt{3}}{2}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$. This gives

$$u_1=\cos\tfrac{2\pi}{9}+i\sin\tfrac{2\pi}{9},\quad u_2=\cos\tfrac{8\pi}{9}+i\sin\tfrac{8\pi}{9},\quad u_3=\cos\tfrac{-4\pi}{9}+i\sin\tfrac{-4\pi}{9}$$

and $v_j = 1/u_j, j = 1, 2, 3$, whence

$$v_1 = \cos\tfrac{2\pi}{9} - i\sin\tfrac{2\pi}{9}, \quad v_2 = \cos\tfrac{8\pi}{9} - i\sin\tfrac{8\pi}{9}, \quad v_3 = \cos\tfrac{-4\pi}{9} - i\sin\tfrac{-4\pi}{9}.$$

Therefore

$$w_1 = u_1 + v_1 = 2\cos\tfrac{2\pi}{9}, \quad w_2 = u_2 + v_2 = 2\cos\tfrac{8\pi}{9}, \quad w_3 = u_3 + v_3 = 2\cos\tfrac{4\pi}{9}.$$

Finally,

$$z_1 = w_1 - 2 = 2\cos\tfrac{2\pi}{9} - 2, \quad z_2 = w_2 - 2 = 2\cos\tfrac{8\pi}{9} - 2 = -2\cos\frac{\pi}{9} - 2, \quad z_3 = w_3 - 2 = 2\cos\tfrac{4\pi}{9} - 2.$$

3.2 Quartic equations and Ferrari's method

To solve a quartic equation

$$ay^4 + by^3 + cy^2 + dy + e = 0$$

one proceeds as follows. First, we divide both sides by a

$$y^4 + \tilde{b}y^3 + \tilde{c}y^2 + \tilde{d}y + \tilde{e} = 0$$

and make the substitution $y = z - \tilde{b}/4$ to eliminate $\tilde{b}y^2$:

$$(z - \tfrac{\tilde{b}}{4})^4 + \tilde{b}(z - \tfrac{\tilde{b}}{4})^3 + \tilde{c}(z - \tfrac{\tilde{b}}{4})^2 + \tilde{d}(z - \tfrac{\tilde{b}}{4}) + \tilde{e} = 0.$$

Thus we reduce to an equation of the form

$$z^4 + pz^2 + qz + r = 0.$$

Now isolate the term z^4 and put the other terms on the right side, then add to both sides $t^2z^2 + t^4/4$, to get

$$z^4 + t^2z^2 + \tfrac{t^4}{4} = (t^2 - p)z^2 - qz + (\tfrac{t^4}{4} - r).$$

The left side is a perfect square, namely $(z^2 + \frac{t^2}{2})^2$, and we can solve the equation easily if we can choose t so that the right side is also a perfect square. It is equivalent to the condition

$$4(t^2 - p)(\tfrac{t^4}{4} - r) - (-q)^2 = 0$$

or

$$t^6 - pt^4 - 4rt^2 + (4pr - q^2) = 0.$$

Setting $t^2 = x$ yields

$$x^3 - px^2 - 4rx + (4pr - q^2) = 0,$$

a cubic that we already know how to solve!

Example. To solve

$$48z^4 - 72z^2 + 16\sqrt{6}z - 1 = 0,$$

note that the coefficient of z^3 already is zero, so a shift of the variable is not needed. Then we introduce a variable t:

$$3(4z^2 + t)^2 = 48z^4 + 24z^2t + 3t^2 = (24t + 72)z^2 - 16\sqrt{6}z + (3t^2 + 1),$$

which gives the condition

$$(-16\sqrt{6})^2 - 4(24t + 72)(3t^2 + 1) = 0$$

or

$$(t + 3)(3t^2 + 1) - 16 = 3t^3 + 9t^2 + t - 13 = 0.$$

Clearly, $t = 1$ is a root of this equation, therefore

$$3(4z^2 + 1)^2 = 96z^2 - 16\sqrt{6}z + 4 = (4\sqrt{6}z + 2)^2.$$

Then

$$\left(\sqrt{3}(4z^2+1)-(4\sqrt{6}z+2)\right)\left(\sqrt{3}(4z^2+1)+(4\sqrt{6}z+2)\right) = 48\left(z^2-\sqrt{2}z+\tfrac{3+2\sqrt{3}}{12}\right)\left(z^2+\sqrt{2}z+\tfrac{3-2\sqrt{3}}{12}\right) = 0$$

Its solutions are

$$z = -\tfrac{\sqrt{2}}{2} \pm \tfrac{1}{2}\sqrt{\tfrac{2}{\sqrt{3}} + 1}; \quad \tfrac{\sqrt{2}}{2} \pm \tfrac{i}{2}\sqrt{\tfrac{2}{\sqrt{3}} - 1}$$

Problem 3.1. *Solve* $x^4 - 6x^3 + 6x^2 + 27x - 56 = 0$

Theorem 3.2 (Abel's impossibility theorem)**.** *There is no solution in radicals (an expression involving only the coefficients of the equation, and the operations of addition, subtraction, multiplication, division, and nth root extraction) to general polynomial equations of degree five or higher with arbitrary coefficients.*

3.3 The fundamental theorem

Theorem 3.3 (Fundamental theorem of algebra)**.** *Every polynomial* $p \in \mathbb{C}[x]$ *of degree* ≥ 1 *has a root in* $\mathbb{C}$*.*

A polynomial $p \in \mathbb{C}[z]$ may be written as $p(z) = p_1(x, y) + ip_2(x, y)$, where $p_1, p_2 \in \mathbb{R}[x, y]$. Then $|p(z)| = \sqrt{p_1(x, y)^2 + p_2(x, y)^2}$. Since p_1 and p_2 are real polynomials in two variables, $|p(z)|$ is a continuous function of x and y as a composition of continuous functions.

A basic fact from calculus is that a function continuous on a closed disk $D = \{(x, y) \mid x^2+y^2 \leq R\}$ has a minimum value in D.

Proposition 3.4. *Let* $f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \in \mathbb{C}[x]$*. For every* $M \geq 0$*, if* $|z| \geq M + 1 + |a_{n-1}| + \ldots + |a_1| + |a_0|$*, then* $|f(z)| > M$*.*

Proof. First we prove that if $|z| \geq 1$, then

$$|f(z)| \geq |z| - (|a_{n-1}| + \cdots + |a_1| + |a_0|)$$

by induction on $n = \deg f$.

If $\deg f = 1$, then $f(z) = z + a_0$, so $|f(z)| \geq |z| - |a_0|$ by the triangle inequality. If $\deg f = n > 1$, let $f(z) = zf_1(z) + a_0$ with $f_1(z) = z^{n-1} + a_{n-1}z^{n-2} + \cdots + a_2z + a_1$.

By induction assumption if $|z| \geq 1$, then $|f_1(z)| \geq |z| - (|a_{n-1}| + \cdots + |a_1|)$. Then

$$\begin{aligned}|f(z)| &= |zf_1(z) + a_0| \geq |zf_1(z)| - |a_0| = |z||f_1(z)| - |a_0| \\ &\geq |f_1(z)| - |a_0| \geq |z| - (|a_{n-1}| + \cdots + |a_1| + |a_0|)\end{aligned}$$

Now if $|z| \geq M+(1+|a_{n-1}|+\cdots+|a_1|+|a_0|)$, then $|f(z)| \geq |z|-(|a_{n-1}|+\cdots+|a_1|+|a_0|) \geq M+1$. $\square$

Corrolary 3.5. *If r is a root of f, then*

$$|r| < 1 + |a_{n-1}| + \cdots + |a_1| + |a_0|.$$

Our proof that $p(z)$ has a root in $\mathbb{C}$ has two parts:

I. There is $z_0 \in \mathbb{C}$ such that $|p(z_0)| < |p(z)|$ for all $z \in \mathbb{C}$.

   Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Using Proposition 3.4, choose $M = 1 + |a_0|$ and put $R = M + 1 + |a_{n-1}| + \ldots + |a_1| + |a_0|$. Then $|p(z)| \geq M$ if $|z| > R$. Let $D = \{z \in \mathbb{C} : |z| \leq R\}$. There is $z_0 \in D$ such that $|p(z_0)| \leq |p(z)|$ for all $z \in D$.

   Now $|p(z_0)| \leq |p(z)|$ for all z. Indeed if $z \notin D$ then $|z| > R$, so $|p(z)| \geq M = 1 + |a_0| > |a_0| = |p(0)| \geq |p(z_0)|$ since $0 \in D$.

II. If $|p(z_0)| < |p(z)|$ for all $z \in \mathbb{C}$, then $p(z_0) = 0$.

   First make a change of variables $w = z - z_0$. Then $p(z) = p(w + z_0) = q_1(w) \in \mathbb{C}[w]$ and $|q_1(0)| = |p(z_0)| \leq |p(z)| = |q_1(w)|$ for all $w \in \mathbb{C}$. We want to show that $q_1(0) = p(z_0) = 0$. If that is the case, we are done. Assume that $q_1(0) = a_0 \neq 0$.

   Let $q_2(w) = \frac{1}{a_0} q_1(w)$. Then $|q_2(w)|$ has a minimum at $w = 0$ and $q_2(0) = 1$. Now q_2 has the form

$$q_2(w) = 1 + bw^m + b_1 w^{m+1} + \cdots + b_k w^{m+k}$$

   for some $m \geq 1$, where $b \in \mathbb{C}, b \neq 0$ and $m + k = n = \deg q_2 = \deg p$.

   Finally, let $r \in \mathbb{C}, r^m = -\frac{1}{b}$ and $q(u) = q_2(ru)$. Then $|q(u)|$ has a minimum at $u = 0$, $q(0) = q_2(0) = 1$ and

$$q(u) = q_2(ru) = 1 + b(ru)^m + b_1(ru)^{m+1} + \cdots + b_k(ru)^{m+k} = 1 - u^m + u^m Q(u)$$

   where $Q(u) = c_1 u + \cdots + c_k u^k \in \mathbb{C}[u]$, with $c_j = b_j r^{m+j}$ for $1 \leq j \leq k$.

   Let $t \in \mathbb{R}, t > 0$. Then

$$|Q(t)| = |c_1 t + \cdots + c_k t^k| \leq |c_1| t + \cdots + |c_k| t^k \to 0 \quad \text{when} \quad t \to 0$$

   Therefore there is $0 < t_0 < 1$ so that $|Q(t_0)| < 1$. Then

$$|q(t_0)| = |1 - t_0^m + t_0^m Q(t_0)| \leq |1 - t_0^m| + t_0^m |Q(t_0)| < 1 - t_0^m + t_0^m = 1$$

   which is a contradiction.