Graph.

kinds. (V, E). V - vertices. E - relation. A - adjacency matrix.

1). directed graph $E \mapsto V \times V$ injective. $A_{ij} \in \{0,1\}$.

2). ~ without loop. $E \mapsto V \times V / \Delta_V$. $\Delta_V = \{(v,v) \mid v \in V\}$. injective.

$A_{ij} \in \{0,1\}$. diagonal - 0.

3) directed multigraph. $E \mapsto V \times V$. $A_{ij} \in \mathbb{Z}_{\geq 0}$. - number of edge with same source (i). target (j)

simple graph: indirected. without loop. without multiple edges. A - symmetric. 0 on diagonal

$\left.\begin{array}{l}\text{indirected.}\\ \text{without loop}\end{array}\right\}$ symmetry. and $\forall v \in V$. v is unrelated to itself.

(If we need reflexive. use " virtual edge " not loops.

Incidence matrix: For directed graph $G = (V, E)$ incidence matrix $I(G)$ is a $V \times E$ matrix with component equal to $0, 1$ and -1. $I(G)_{v,e} = -1$ iff v is the source for the edge e, i.e. $e = (v, u)$ for some $u \in V$. $I(G)_{ve} = 1$ iff v is the target for the edge e, that means $e = (u, v)$ for some $u \in V$. And $I(G)_{v,e} = 0$ in all the other cases.

- 对 simple graph 有类似定义. 此时.
$I(G)_{v,e} \in \{0,1\}$.
$I(G)_{v,e} = 1$. v is incident to e.

path of length: **Definition.** For two vertices $u, v \in V(G)$ in a simple (may be directed) graph G the path of length k is a sequence of vertices $x_0 = u, x_1, x_2, \ldots, x_k = v$ such that x_{i-1} is adjacent to x_i for all $i = 1 \ldots k$ (in directed case (x_{i-1}, x_i) belongs to $E(G)$). For the case of multigraph one should specify the actual choice of edges connecting x_{i-1} and x_i.

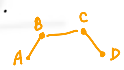
simple path: all the vertices in $\{x_0, x_1 \ldots x_k\}$. are distinct.

close path: $x_0 = x_k$.

cycle: the only repetition vertices is x_0 (x_k).

Proposition 2.1. *Let $A(G)$ be an adjacency matrix of a simple graph G. Then the components of its k-th power $A(G)^k_{u,v}$ are equal to number of paths of length k.*

不是 simple path.



从 $B \to C$ 长度为3的 path 有3条

$B \to C \to D \to C$ $B \to C \to B \to C$.
$B \to A \to B \to C$

△ 元素 $A(G)^k_{u,v}$ 的值 = $u \to v$ 路径长为 k 的路径数

△ 对有向图也成立.

degree: **Definition.** Let G be a simple graph. Then

$$\deg(v) = |\{u \in V(G) \mid \{u, v\} \in E(G) \wedge u \neq v\}| \qquad (2)$$

is the number of other vertices which are adjacent to v. It is the same as the number of edges incident to a given vertex.

其余相邻点的数目.

可分 $\deg^+(v)$. outgoing edge $\mapsto$
$\deg^-(v)$ incoming edge $\leftarrow$

Proposition 2.2. *Let $A(G)$ be the adjacency matrix of a simple graph G with vertices $v_1, v_2, \ldots, v_n$. Then $A(G) \cdot (1, 1, \ldots, 1)^T = (d_1, d_2, \ldots, d_n)^T$ where $d_i = \deg(v)$.*

Proposition 2.3. *Let $I(G)$ be an incidence matrix of directed graph G and $\overline{G}$ is the corresponding simple graph obtaining by forgetting directions of arcs. Then $I(G)I(G)^T = D(\overline{G}) - A(\overline{G})$ where and $D(G)$ is a diagonal matrix with $\deg_{\overline{G}}(v_1), \ldots, \deg_{\overline{G}}(v_n)$ on its diagonal.*

Graph mapping.

$G = (V, E)$. $G' = (V', E')$.

△此处的 adjacency 关系是有向的.

Morphism: 1) for simple graphs. $f: V \to V'$ s.t. any $u, v \in V$. adjecent. in G. $f(u)$. $f(v)$ adjecent in G'

表示为 $f: G \to G'$.

2) for multigraphs: $\begin{cases} f_0: V \to V' \\ f_1: E \to E' \end{cases}$ $\forall v \in V$, $e \in E$. v incident to e; $f_0(v)$ is incident to $f_1(e)$

Fact: composition of graph morphism is also graph morphism.

Isomorphism: Morphism $f: G \to G'$. if exists. morphism $g: G \to G'$ s.t. $f_0: V \to V'$, $g_0: V' \to V$ are inverse mapping (for simple graph). and $f_1: E \to E'$. $g_1: E' \to E$ inverse mutually (for multigraph).

Graph G and G' is isomorphic if $\exists f: G \to G'$ is a. Isomorphism

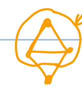
方法. 判断图之间是否同构: 找"invariant". 可以是图上定义的任意函数. 若一个"不变量"对2图不同. 此2图不同构.

常见"invariant": 顶点数. 边数. 给定度数的顶点数. 最小 cycle 长. 最大的 simple path 长.

induced subgraph:

Definition. When H is a subgraph of G and for any two vertices $u, v \in V(H)$ they are adjacent in H if and only if the are adjacent in G then H is said to be **induced subgraph** of G.

从图形上. 子图是原图被一个凸集切割而成. 并不要求保持连通性(可以连顶点一起切掉).



子图 H 可以通过 G. 递归删去边. 和 孤立点、

non-degenerated edge (in G).

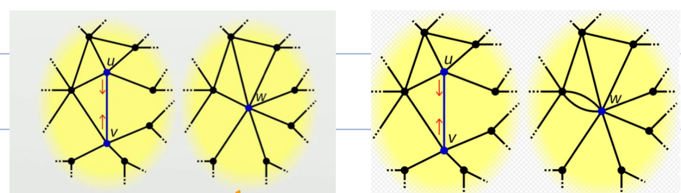
Definition. Let G be a simple graph and $\{u, v\}$ is a non-degenerate edge in G. Consider a graph G' such that $V(G') = (V(G) \setminus \{u, v\}) \cup \{w\}$ where w is a new vertex with adjacency relation defined by:

- For $x, y \neq u, v, w$ vertices x and y adjacent to each other in G' if and only they are adjacent in G;
- For $x \neq w$ vertices x and w are adjacent in G' if and only if either x and u are adjacent or x and v are adjacent in G.

Then the graph homomorphism $G \to G'$ which maps vertices u, v into w and all other vertices of the graph G into the same vertices of the graph G' is called **edge contraction**.

△退化边的判别, 是通过边的压缩实现的.

Then the graph homomorphism $G \to G'$ which maps vertices u, v into w and all other vertices of the graph G into the same vertices of the graph G' is called **edge contraction**.



simple

multigraph.

Handshaking Lemma.

directed graph. $$\sum_{v \in V(G)} \deg^+(v) = \sum_{v \in V(G)} \deg^-(v) = |E(G)|$$

simple graph. $$\sum_{v \in V(G)} \deg(v) = 2|E(G)|.$$

degree sequence. **Definition.** For a simple graph G with vertex set $V = \{v_1, \ldots, v_n\}$ its **degree sequence** or degree vector is the vector $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ where the vertices are renumbered in such a way that the sequence is decreasing $\deg(v_1) \geqslant \deg(v_2) \geqslant \ldots \geqslant \deg(v_n)$.

判别:

△ non-isomorphic graph 可有相同 degree sequence.

complete graph. It is easy to see that there is a unique up to isomorphism graph with n vertices whose degree vector is $(n-1, n-1, \dots, n-1)$. In this graph every vertex is adjacent to all others. This graph is called a **complete graph** on n vertices and denoted by K_n.

△即所有点都相邻.

Corollary 2.4. *In a simple graph G there is an even number of odd degree vertices.*

regular graph. **Definition.** A graph G is called **regular** if all the vertices have the same degree. When k is the common degree of all its vertices then G is called k-regular.

Corollary 2.5. *Let G be a k-regular graph with n vertices. The the number of edges is equal to $\frac{kn}{2}$.*

Remark. **Proposition** $\boxed{2.2}$ says that the column $(1, 1, \dots, 1)^T$ is an eigencolumn for adjacency matrix of a k-regular graph corresponding to eigenvalue k.

Proposition 2.6. *Any 2-regular graph is a disjoint union of one or more cycles.*

cubic graph. **Definition.** Any 3-regular graph is usually called **cubic** graph.

bipartite graph. **Definition.** The graph G is called **bipartite** if $V(G)$ can be decomposed into disjoint union $V(G) = V_0(G) \sqcup V_1(G)$ such that any two vertices belonging to the same part are not adjacent to each other, In other words, any edge $e \in E(G)$ has one ends in $V_0(G)$ and other end in $V_1(G)$.

△用边分开. 不相邻的2部分(非子图. 仅分点).

Remark. Let G be a graph such that G is k-regular and G is simultaneously bipartite. Then $V_0(G)$ and $V_1(G)$ has the same size.

Since every subgraph of a bipartite graph is also bipartite then every cycle in bipartite graph should have even length since the cycle satisfy the condition of the above remark.

Cycle有偶数长(即偶数个点被分2组)

$\forall$ cycle has even length $\rightleftarrows$ bipartite graph.

Theorem 2.7. *Consider a simple graph G such that all the vertices have even degrees. Then every edge e belongs to an odd number of cycles.*

Lemma 2.9. *(Switching lemma) Let G be a graph and $u, v \in V(G)$ such that $\deg(u) > \deg(v)$ then there exists a vertex w such that w is adjacent to u and non-adjacent to v. Deleting the edge $\{uw\}$ and adding the edge $\{vw\}$ we obtain a graph G' such that $\deg_{G'}(u) = \deg_G(u) - 1$ and $\deg_{G'}(v) = \deg_G(v) + 1$.*

△判别 degree sequence 的存在性:

Problem 2.8. *Is there exists a graph with degree vector* $(3, 3, 3, 2, 2, 1)$.

Suppose that required graph does exist. Consider the vertex of degree 3. Deleting it we obtain a graph with five vertex whose degree sequence can be obtained by decreasing by 1 three members in the sequence $(3, 3, 2, 2, 1)$. The crucial point is that one can decrease four biggest member. Since if there exists a graph with non yet ordered degree sequence $3-1, 3, 2-1, 2, 1-1$ then by switching an edge once one can obtain a graph with a degree sequence $\{3-1, 3-1, 2, 2, 1-1\}$ and then switching second time we obtain degree sequence $\{3-1, 3-1, 2, 2-1, 1\} = \{3-1, 3-1, 2-1, 2, 1\}$.

△从最大的开始删.(若不是.. switch一下就等价了).

Therefore the sequence $(3, 3, 3, 2, 2, 1)$ is realizable as degree sequence of some six vertex graph if and only if the degree sequence $\{3-1, 3-1, 2-1, 2, 1\} = (2, 2, 2, 1, 1)$ is realizable as degree sequence of five vertex graph.

The second step is reducing the question to realizibility of degree sequence $\{2-1, 2-1, 1, 1\} = (1, 1, 1, 1)$ of four vertex graph. One can do another step or just realize that four-vertex graph all whose vertex have degree 1 is just a disjoint union of two edges.

3. Connectivity.

connected components: (of vertex v):

→ equivalent class.

Definition. In a simple graph two vertices $x, y \in V(G)$ are called to be connected to each other if there is path starting with x and ending with y. This is obviously an equivalence condition. The equivalent classes are called in this cases **connected components**. That means that connected component of the vertex v is an induced subgraph on the vertex set $\{x \mid v \text{ is connected to } x\}$. → 可以由这个导出. v连x. 再找x连.

Obviously, every graph is a disjoint union of its connected components. If there is only one connected components in a graph then it is called **connected**. That means that for every two vertices in this graph there exists a path joining them.

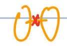
reachability.

For directed graph one can define **reachability** relation on the vertex. It is reflexive transitive relation. For vertices $x, y \in V(G)$ to be mutually reachable one from another is the equivalence relation. Equivalence classes in this case are called **strongly connected component**. ($\forall$ ordered pair. $\forall x.y \in V \times V$. $\exists$ path $x \to y$.) → 对有向图是最大强连接子图.

bridge:

Definition. A **bridge** (or isthmus) in a simple graph G is an edge such that its deleting strongly increases the number of connected components.

△删去, connected component 数目↑↑.



set of bridge → cut set. ("strongly increases the number of connected components.", 并不要求删去每一边都这样的效果.

Proposition 3.1. *Minimal (by inclusion) cut-set (for example, any bridge) could increase the number of connected components only by* 1.

Proposition 3.2. *For any edge e in a graph G such that e is not a bridge there exists a cycle in G containing e.* (不是桥就是 cycle 的部分★).

Proposition 3.3. *Let G be a graph without cycles. Then the number of connected components in G is equal to $c(G) = \underset{\text{点数}}{v(G)} - \underset{\text{边数.}}{e(G)}$ where $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For arbitrary graph G one has inequality $c(G) \geqslant v(G) - e(G)$.*

Corollary 3.4. *The following conditions are equivalent for a graph G.*

- *Graph G is connected and has no cycles;*
- *Graph G is connected and $e(G) = v(G) - 1$;*
- *Graph G has no cycles and $e(G) = v(G) - 1$.*

tree & forest.

Definition. A graph G satisfying one of the equivalent condition of **Corollary** 3.4 is called a **tree**. Arbitrary graph without cycles is called a **forest**. 可以全离散点.

△ forest 中 ∀ connected component 必须是一个 tree.

Corollary 3.5. *In any tree (and forest) there exists a vertex of degree* 1.

△除了全离散点的情况.

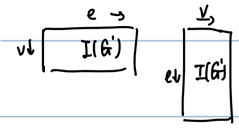
circuit rank (of G):

Definition. For any graph G the number $c(G) - v(G) + e(G)$ is called a cyclomatic number or circuit rank of G and denoted usually by $\mathfrak{z}(G)$. → $z(G) \geq 0$. $= 0 \Leftrightarrow$ G是 forest.

Theorem 3.6. *Let G be a simple graph and G' be a directed graph being obtained form G by choosing somehow an orientation on every edge. Then the nullity of the incidence matrix $I(G')$ is equal to the cyclomatic number.* 有向量. 改方向是对 I 的某列 ×-1. 不改秩.

$e - \text{null}(I) = \text{rank}(I)$ $\qquad \text{null}(I) = c - v + e \Leftrightarrow \text{rank}(I) = v - c.$

Corollary 3.7. *The number of connected components $c(G)$ is equal to $v(G) - \mathrm{rank}\,(L(G))$ where $L(G)$ is a Laplacian matrix for graph G.*



$L(G) = I(G')\, I(G')^T$ - non negative definite, real, symmetric.

Remark. Laplacian matrix $L(G)$ has zero eigenvalue. The corresponding eigenvector is $(1, 1 \ldots, 1)^T$. When G is connected the multiplicity of a zero eigenvalue is 1 and all other eigenvalues of $L(G)$ are strictly positive.

Definition. Two graphs G_1 and G_2 with the same set of vertices $V = V(G_1) = V(G_2)$ where $|V| = n$ are called **complimentary** to each other if $E(K_n) = E(G_1) \sqcup E(G_2)$.

点相同，边关于全图互补.

Proposition 3.8. *Let G be a simple graph. Then G is a tree if and only of for any two vertices $x, y \in V(G)$ there exists a unique simple path joining x and y.*

Spanning Forest.

Definition. For any graph G the maximal by inclusion of the edge-set subgraph in G with the same vertex set which contains mo cycles is called **spanning forest** of G.

遍历导出的子图，与原图点集相同.
(原图删去尽可能多的 cycle 也，不改连通性).

Proposition 3.9. *Spanning forests of G has exactly the same connected components as G and therefore has $v(G) - c(G)$ edges.*

Proposition 3.10. *For any subgraph with the same vertex set number of connected components is at least $c(G)$. Therefore by* **Proposition** $\boxed{3.3}$ *the number of edge in any subgraph of G without cycles is at most $v(G) - c(G)$. Actually for spanning forest here is an equality.*

对任意和原图点集相同的子图，G_0
$c(G_0) \geq c(G)$
对任意无 cycle 子图
$e(G_0) \leq v(G) - c(G)$.
$\Rightarrow$ spanning forest
$c(G_0) = c(G)$
$e(G_0) = v(G) - c(G)$.

Corollary 3.11. *The cyclomatic number $z(G)$ is equal to a minimal number of edges we must delete form G in order to obtain* 无环的 *acyclic graph.*

denote $t(G)$ – number of spanning tree in a connected graph G.

(对一个图，看成环的也有多少种删法不改连通性，且消环).

Proposition 3.12. *(Cut and join equation for $t(G)$). For any connected graph or multigraph G and an edge $e \in E(G)$ the number of spanning trees satisfies equation:*

$$t(G) = t(G \setminus e) + t(G * e) \tag{5}$$

where $G * e$ is a multigraph graph obtained from G by contracting the edge e and for disconnected graph $t(G)$ is supposed to be zero.

$t(G) = 2 \times 2 = 4$.

$t(G|e) = 2$

$t(G*e) = t(\ ) = 2$.

Kirchoff matrix tree theorem.

Theorem 3.13. *The number of spanning trees in every multigraph G is equal to every cofactor of its Laplacian matrix.*

Laplacian matrix. $\det(L) = 0$.

$L \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 0$ (每一行/列元素之和为0).

$L = \begin{pmatrix} d_1 & & \cdots \\ & d_2 & \\ \cdots & & \ddots & d_n \end{pmatrix}$ ← -1 或 0. 可通过 d 推，点标号和可直接通过图列出.
(相邻 2 点就对应 -1)

$\det(L_{1,1}) = t(G)$.

对角是度数.

Definition. Matrix $D(G) - A(G)$ for an oriented graph or even for a multigraph G without loops is called **Laplacian** matrix of G an will be denoted by $L(G)$.

△可证 L 的 n-1 阶代数余子式总是相等

$L \cdot \mathrm{adj}(L) = \det(L) \cdot I = 0$

$\mathrm{rank}(L) = n-1$ $\quad \mathrm{adj}(L) = 1.$ 且 $L \cdot \begin{pmatrix} \vdots \end{pmatrix} = 0.$

$\mathrm{adj}\, L = \begin{pmatrix} k_1 & \cdots & k_n \\ k_1 & \cdots & k_n \\ \vdots \\ k_1 & \cdots & k_n \end{pmatrix}$ $\quad \mathrm{adj}\, L = (\mathrm{adj}\, L)^T.$ $\quad k_i = k_j.$

Remark. Another purely algebraic formula for counting spanning trees

$$t(G) = \frac{1}{n}\lambda_2\lambda_3\ldots\lambda_n$$

where $0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_n$ denote the eigenvalues of Laplacian matrix $L(G)$. It is easy to prove computing coefficient before t in the characteristic polynomial $\det(A - tE) = \prod_{i=1}^{n}(\lambda_i - t)$ in two ways.

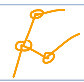
Corollary 3.14. *(Caley formula) The number of labelled trees with n vertices is equal to n^{n-2}.*

→ spanning tree of $t(K_n)$.

证 $L(K_n)_{1,1} \sim nE_{n-1} - I_{n-1}$.

Block and Articulation Point

Definition. An **articulation point** (also called a cut vertex or separating vertex) is a vertex in a graph whose removal would increase the number of connected components in the graph.

e.g. $\forall$ tree each vertex $\deg \geqslant 2$. is articulation. (无环、不是端点的点 ).

Theorem 3.15. *Let G be a connected graph. Then for any $v \in V(G)$ the following conditions are equivalent :*

1. *v is an articulation point in G;*
2. *There exist $u, w \in V(G)$ such that $u, w \neq v$ and any path joining u and w contains v.* ("没别的路了，必须往这里过").
3. *There exist two disjoint subsets $U, W \subset V(G)$ such that $U \sqcup W = V(G) \setminus \{v\}$ and any path joining a vertex in U to a vertex in W contains v.*

Graph is called **biconnected** or 2-connected if it is connected and does not have any articulation vertices.

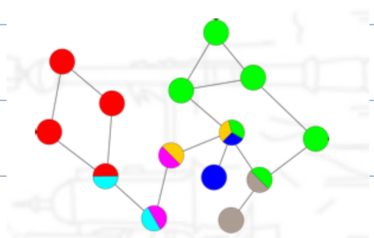
(对应图/子图来说)

最大连通分支.

Definition. A **block** in a graph G is a maximal (by inclusion) biconnected subgraph in G. Sometimes blocks are called biconnected components.

or cycle / cycles.

Proposition 3.16. *Any two blocks in the graph can have at most one vertex in common. If the vertex is a common point of two blocks then it is an articulation point.*



Corollary 3.17. *Let G_0 be a biconnected subgraph of a given graph G and $v(G_0) \geqslant 2$. Then there exists a unique block of G containing G_0.*

Theorem 3.18. *If a biconnected graph G contains at least 3 vertices then for any two vertices $u, w \in V(G)$ there exist at least two simple paths joining u and w which do not intersect each other at inner points.*

3个点的 biconnected 必有 cycle.

Proposition 3.19. *In every biconnected graph G for any edge e and two distinct vertices u and w which are not incident to e there exist a simple path joining u and w and using the edge e.*

Corollary 3.20. *In a biconnected graph every two edges lie on the same cycle.*

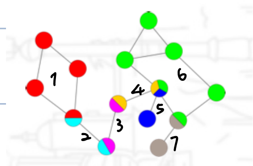
Corollary 3.21. *For any three distinct vertices u, v, w in a biconnected graph there exists a simple path joining u and w and containing v.*

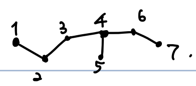
Block graph.

Definition. If G is any simple graph, the block graph of G, denoted $B(G)$, is the intersection graph of the blocks of G: vertices in $B(G)$ are blocks in the given graph G, and two vertices of $B(G)$ are adjacent if the corresponding two blocks meet at an articulation vertex.

定义一种新图.
顶点 - 原图的块.
边 - 原图有一个共同节点
反映哪些块相交.

e.g.





Block-cut tree.

顶点有两类. 块/割点
是二分图. 在连通G中必为树.
边: 割点属于某块 (即割点之间. 块之间 都不可能连).

Definition. For a connected graph its **block-cut tree** is a bipartite graph that contains a node for each block of a given graph and for each articulation point. An edge connects node representing an articulation point to the node representing block if and only if articulation point belongs to this block.

Proposition 3.22. *The block-cut tree for every connected graph is indeed a tree.*

Definition. A **leaf** block in a graph is a block having exactly one articulation point.

Lemma 3.23. *Let G be non-complete biconnected graph such that each vertex has degree at least 3. Then there exist a length two path abc such that a is not adjacent to c and $G \setminus \{a, b\}$ is connected.*

Proposition 3.24. *Let G be a connected graph with blocks $B_1, B_2, \ldots, B_k$. Then $t(G) = t(B_1) \cdot t(B_2) \cdot \cdots \cdot t(B_k)$.*

4. Hamiltonian and Eulerian Graphs.

Definition. A path in a non-necessary simple connected graph is called an **Euler** path if it uses every edge of the given graph exactly once. A **Hamiltonian** path is that visits every vertex exactly once.

欧拉 - 用每条边一次
汉密顿 - 用每个点一次

Obviously, every Hamilton path is a simple path. If first and last vertex in a Hamilton path are adjacent to each other then adding the edge joining them one obtains a **Hamilton** cycle.

Usually, Euler path is not simple. Closed Euler path is called **Euler circuit**.

A Graph is said to be **Eulerian** if it admits an Euler circuit. A graph is called Hamiltonian if there is a Hamiltonian cycle in this graph.

Hamilton cycle is a subgraph of a given graph. Euler circuit defines a homomorphism of a cycle graph C_n to a given graph G that is bijective on the edge sets.

Proposition 4.1. *Suppose that graph G is Eulerian. Then it is connected and every vertex has even degree.*

Corollary 4.2. *If an Euler path exists in a graph or multigraph then the number of vertices of odd degree is 0 or 2.*

Theorem 4.3. *The necessary and sufficient condition for connected graph G to be Eulerian is that all vertex degree are even.*

Corollary 4.4. *There exists an open Euler path in a connected Graph if and only if there are exactly two vertices of odd degree. The start and the end of any Euler path are necessary these two vertices of odd degree.* 恰有2个奇度顶点为起止点

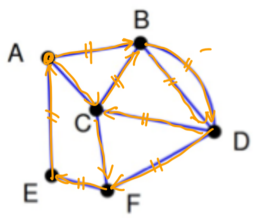
Fleury's Algorithm for finding the Eulerian circuit

Start at any vertex u and traverse the edges step by step, subject only to the following rules:

- erase the edges as they are traversed, and if any isolated vertices result, erase them too; 删边/孤立点.
- at each step, use a bridge only if there is no alternative. 尽量走非桥边.

边耗尽得 Euler 回路(删边的顺序).

Example. Find an Euler path using Fleury algorithm in a graph below:



Another standard application of Euler circuits is to so called *De Bruijn* sequence.

De Brujin sequence.

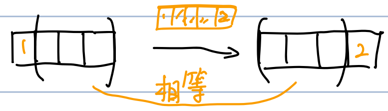
Definition. A binary De Bruijn sequence of order n is a cyclic $0-1$ sequence in which every possible length-n string A occurs exactly once as a substring.

每个长度为 n 的串恰好出现一次

例: 00010111 3-order

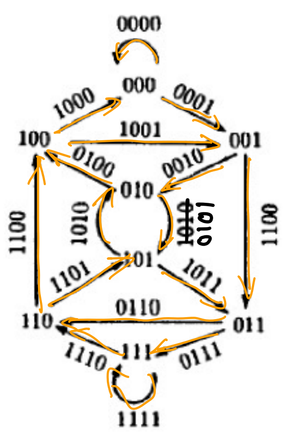
000 001 010

Definition. A De Bruijn graph is a directed graph $DB_n = (\{0,1\}^{n-1}, \{0,1\}^n)$, that is, $V(DB_n) = \{0,1\}^{n-1}$ and $E(DB_n) = \{0,1\}^n$ where there is an edge from $v = (a_1, ..., a_{n-1})$ to $v' = (a'_1, \ldots, a'_{n-1})$ if and only if $a_2 = a'_1, a_3 = a'_2, \ldots, a_{n-1} = a'_{n-2}$.



De Bruijn graph 任意顶点有 indegree, outdegree 2. ⇒ 是有向欧拉图.

Euler circuits in De Bruijn graph ⇔ De Bruijn sequence. ⇒ De Bruijn 总是存在.



De Bruijn sequence 0000100110101111

(首尾相接后 所有n阶串出现一次.

0000100110101111

(走一遍就是 Euler circuit.)

Chinese postman problem.

Chinese postman problem

There is a classical problem in the branch of Discrete optimization where one can easy apply Euler theory.

Given some net of streets how the postman could walk all the streets such that the total distance walked by the postman was as short as possible?

More formally speaking, given a weighted graph with some weight function $w : E(G) \to \mathbb{R}_{\geqslant 0}$ we need to find a closed path $f : C \to G$ that visit every edge at least once and minimize $\sum_{e \in E(C)} w(f_1(e))$. 不仅一笔画, 要回端点

If given graph is Eulerian then all Euler circuits has the same total length that is the sum of weight of all edges in G. But if G is not Eulerian we has to walk through some edges at least twice. 无奇度点

① 求顶点度数

② 对奇数点配对. 取最小的路径 "double".

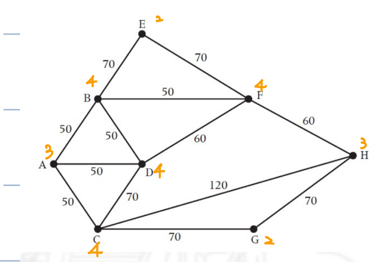
(如2个奇度点 AB.

总长 = $\sum$ total weight + min (AB).

4个奇度点 A.B.C.D.

比较 min (AB+CD, AC+BD, AD+BC). ...)

The solution to Chinese postman problem is Eulerian circuit in some multigraph $\tilde{G}$ that is obtained from G by doubling several edges. In the given graph there are only 2 odd degree vertices A and H. Hence the complimentary graph G' with the edge set $E(\tilde{G}) \setminus E(G)$ aslo would have these two odd degree vertices. Therefore, G' consist of a simple path from A to H and possibly a few cycles. Since our aim to minimize sum of weights then we can assume that G' is just a simple path. The simple path of minimal total length here is $ABFH$. So the minimal length of the Chinese postman route is 1000.



Hamiltonian graph.

Proposition 4.6. *For any orientation of the complete gtaph K_n there exist a directed hamiltonian path.*

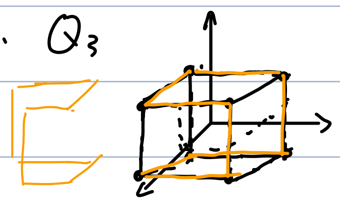
Theorem 4.7. *Let G be an orientation of complete graph K_n such that G is strongly connected then G admits a directed Hamiltonian cycle.*

$\forall x, y$, $\exists$ path $x \to y$.

Remark. There is a name for a vertex of indegree zero — **source** and fro a vertex of outdegree zero — **target**

Definition. A graph Q_n is defined as follows. Its vertices are 2^n points in $\mathbb{R}^n$ with coordinates $(\pm 1, \pm 1, \ldots, \pm 1)$ and two vertices are adjacent if the corresponding points differs only at one position.

e.g. Q_3

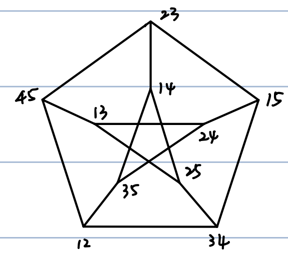


对 Q_n. 每个顶点. deg = n.

Problem 4.8. *Prove that Petersen graph is not Hamiltonian.*

Proof. Petersen graph is a complement to the intersection graph of $\binom{5}{2}$ 2-element subsets in 5-element set. That means that its vertices are just ten unordered pairs $\{i, j\} \subset \{a, b, c, d, e\}$ and two pairs are adjacent to each other if they have an empty intersection. Therefore it is easy to check that for every two non-adjacent vertices in Petersen graph there is a unique path of length 2 joining them. Indeed let $\{ab\}$ and $\{ac\}$ be two non-adjacent point then the unique point adjacent to both is a 2-element subset having empty intersection with $\{a, b, c\}$. Therefore there are no 4-cycles in Petersen graph. For the same reason there is no 3-cycles.

Let us assume that Petersen graph is Hamiltonian. Presenting it as a cycle $v_0, v_1, \ldots, v_9$ with chords we conclude that there five extra chords and any vertex is joint to only one vertex by a chord. Since there are no cycles of length 3 and 4 the vertex v_0 ca not be joined with v_2, v_3, v_8, v_7. If all chords would be the diameters of a cycle then the graph obviously has 4-cycle. Hence one can assume that v_0 is joint with v_4. If v_5 is adjacent to v_1 then there is obvious 4-cycle. If v_5 is adjacent to v_9 then there is also obvious 4-cycle. We obtain a contradiction which proves that Petersen graph is Non-Hamiltonian. □



先把  10 边形 (decagon) 画出来

依据 no 3, 4 cycle 证明无法中间连.

deg = 3 for each vertex.

no 4 cycle

no 3 cycle

{de} — {a,b} — {c d}

| |

{ce} — {?} ? a,b.

Theorem 4.9. *Let G be a graph where all the vertices have odd degree. Choose an edge e. Then the number of Hamiltonian cycles containing e is even.*

Property of Hamiltonian graph.

- Hamiltonian graph does not have any articulation points.
- Removing k distinct points in a Hamiltonian graph one obtains a graph with at most k connected components.

Corollary 4.10. *If a 3-regular graph is Hamiltonian then it has at least 3 different Hamiltonian cycles.*

Corollary 4.11. *In an arbitrary simple graph G and chosen vertex u there is even number of Hamilton paths starting with u and ending in a (non-fixed) vertex of even degree.*

Theorem 4.12. *Let G be a graph with n vertices. Assume that $\deg(v) \geqslant \dfrac{n}{2}$ for every $v \in V(G)$. Then G is Hamiltonian.*