Home assignement 2

Problem 1. In a rectangular coordinate system, a second-order curves are given by the equations

$$\begin{aligned} &a.\quad 5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0;\\ &b.\quad 4x^2 - 4xy + y^2 - 2x - 14y + 7 = 0;\\ &c.\quad 12xy + 5y^2 - 12x - 22y - 19 = 0. \end{aligned}$$

1. Find the canonical form of equation.
2. Find the relations between the old coordinates x and y and the new x' and y' (canonical).
3. Find the coordinates of the center, vertices and foci of this curves relative to the initial coordinate system.
4. Find the equations of directrices, asymptotes and tangents at the vertices of this curve in the initial coordinate system.

Solution.

a. $5x^2 + 8xy + 5y^2 - 18x - 18y + 9 = 0$

Invariants:

$$S = 10, \qquad \delta = 9, \qquad \Delta = -81.$$

Characteristic equation:

$$\begin{vmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = 0.$$

Roots of characteristic equation:

$$\lambda_1 = 1, \qquad \lambda_2 = 9.$$

Reduced equation:

$$x'^2 + 9y'^2 - 9 = 0.$$

Canonical equation:

$$\frac{x'^2}{9} + \frac{y'^2}{1} = 1.$$

Direction of canonical axes of ellipse:

$$\lambda_1 = 1: \quad \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_1, \beta_1\} = \{1, -1\},$$

$$\lambda_2 = 9: \quad \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_2, \beta_2\} = \{1, 1\}.$$

Unit vectors of canonical coordinate system:

$$\mathbf{e}_1' = \left\{ \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\}, \qquad \mathbf{e}_2' = \left\{ \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\}.$$

Center:

$$\begin{cases} 5x + 4y - 9 = 0, \\ 4x + 5y - 9 = 0, \end{cases} \quad \Rightarrow \quad (1, 1).$$

Formula of coordinate changing:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix}.$$

Vertices and foci in canonical coordinate system:

$$A(\pm 3, 0), \quad B(0, \pm 1), \quad F(\pm 2\sqrt{2}, 0).$$

Vertices and foci relative to the initial coordinate system:

$$A\left(\pm\frac{3\sqrt{2}}{2} + 1, \mp\frac{3\sqrt{2}}{2} + 1 \right), \quad B\left(\pm\frac{\sqrt{2}}{2} + 1, \pm\frac{\sqrt{2}}{2} + 1 \right), \quad F_1(3, -1), \quad F_2(-1, 3).$$

Equations of directrices in canonical coordinate system:

$$x' = \pm\frac{a}{e} = \pm\frac{a^2}{c} = \pm\frac{9}{2\sqrt{2}},$$

in the initial coordinate system:

$$2x - 2y \pm 9 = 0.$$

Tangents at the vertices in canonical coordinate system:

$$x' = \pm 3, \qquad y' = \pm 1,$$

in the initial coordinate system:

$$x - y \pm 3\sqrt{2} = 0, \qquad x + y - 2 \pm \sqrt{2} = 0.$$

b. $4x^2 - 4xy + y^2 - 2x - 14y + 7 = 0$

Invariants:

$$S = 5, \qquad \delta = 0, \qquad \Delta = -225.$$

Characteristic equation:

$$\begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda(\lambda - 5) = 0.$$

Roots of characteristic equation:

$$\lambda_1 = 0, \qquad \lambda_2 = 5.$$

Reduced equation:

$$5y'^2 - 2\sqrt{-\frac{-225}{5}}x' = 0.$$

Canonical equation:

$$y'^2 = \frac{6}{\sqrt{5}}x'.$$

Direction of canonical axes of parabola:

$$\lambda_1 = 0: \quad \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_1, \beta_1\} = \{1, 2\},$$

$$\lambda_2 = 5: \quad \begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_2, \beta_2\} = \{2, -1\}.$$

Unit vectors of canonical coordinate system:

$$\mathbf{e}_1' = \left\{\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\}, \qquad \mathbf{e}_2' = \left\{\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\}.$$

Equation of parabola axis as diameter conjugate to second unit vector:

$$\alpha_2 F'_x + \beta_2 F'_y = 0 \quad \Rightarrow$$

$$\frac{2}{\sqrt{5}}(4x - 2y - 1) - \frac{1}{\sqrt{5}}(-2x + y - 7) = 0 \quad \Rightarrow$$

$$2x - y + 1 = 0.$$

Vertex of parabola:

$$\begin{cases} 4x^2 - 4xy + y^2 - 2x - 14y + 7 = 0 \\ 2x - y + 1 = 0 \end{cases} \quad \Rightarrow$$

$$-30x - 6 = 0 \quad \Rightarrow \quad x = -\frac{1}{5},\ y = \frac{3}{5}.$$

Formula of coordinate changing:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} -1/5 \\ 3/5 \end{pmatrix},$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x + 1/5 \\ y - 3/5 \end{pmatrix}.$$

Focus in canonical coordinate system:

$$F = \left(\frac{p}{2}, 0\right) = \left(\frac{3}{2\sqrt{5}}, 0\right).$$

Focus in the initial coordinate system:

$$F = \left(\frac{1}{10}, \frac{6}{5}\right).$$

Equations of directrix in canonical coordinate system:

$$x' = -\frac{p}{2} = -\frac{3}{2\sqrt{5}},$$

in the initial coordinate system:

$$2x + 4y + 1 = 0.$$

Tangents at the vertices in the initial coordinate system:

$$x + 2y - 1 = 0.$$

c. $12xy + 5y^2 - 12x - 22y - 19 = 0$

Invariants:

$$S = 5, \qquad \delta = -36, \qquad \Delta = 1296.$$

Characteristic equation:

$$\begin{vmatrix} 0-\lambda & 6 \\ 6 & 5-\lambda \end{vmatrix} = \lambda^2 - 5\lambda - 36 = 0.$$

Roots of characteristic equation:

$$\lambda_1 = 9, \qquad \lambda_2 = -4.$$

Reduced equation:

$$9x'^2 - 4y'^2 - 36 = 0.$$

Canonical equation:

$$\frac{x'^2}{4} - \frac{y'^2}{9} = 1.$$

Direction of canonical axes of ellipse:

$$\lambda_1 = 9: \quad \begin{pmatrix} -9 & 6 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_1, \beta_1\} = \{2, 3\},$$

$$\lambda_2 = -4: \quad \begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_2, \beta_2\} = \{3, -2\}.$$

Unit vectors of canonical coordinate system:

$$\mathbf{e}'_1 = \left\{ \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\}, \qquad \mathbf{e}'_2 = \left\{ \frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}} \right\}.$$

Center:

$$\begin{cases} y - 1 = 0, \\ 6x + 5y - 11 = 0, \end{cases} \quad \Rightarrow \quad (1, 1).$$

Formula of coordinate changing:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \dfrac{2}{\sqrt{13}} & \dfrac{3}{\sqrt{13}} \\ \dfrac{3}{\sqrt{13}} & -\dfrac{2}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \end{pmatrix}.$$

Vertices and foci in canonical coordinate system:

$$A(\pm 2, 0), \quad F(\pm\sqrt{13}, 0).$$

Vertices and foci relative to the initial coordinate system:

$$A\left(1 \pm \frac{4}{\sqrt{13}}, 1 \pm \frac{6}{\sqrt{13}}\right), \quad F_1(3, 4), \quad F_2(-1, -2).$$

Equations of directrices in canonical coordinate system:

$$x' = \pm\frac{a}{e} = \pm\frac{a^2}{c} = \pm\frac{4}{\sqrt{13}},$$

in the initial coordinate system:

$$2x + 3y - 1 = 0, \qquad 2x + 3y - 9 = 0.$$

Tangents at the vertices in canonical coordinate system:

$$x' = \pm 2,$$

in the initial coordinate system:

$$2x + 3y - 5 \pm 2\sqrt{13} = 0.$$

Problem 2. Find with using the invariants S, δ, Δ the necessary and sufficient conditions for the quadric $F(x, y) = 0$ to decompose into pair of perpendicular lines.

Solution.

For curve decomposing to pair intersecting lines the following condition should be satisfied: $\delta < 0$, $\Delta = 0$.

Let's consider the canonical equation of such curve:

$$a^2x^2 - b^2y^2 = 0 \quad \text{or} \quad (ax - by)(ax + by) = 0.$$

The lines would be orthogonal in case $a^2 = b^2$ or $a = \pm b$. Then the equation of curve takes the form:

$$x^2 - y^2 = 0,$$

hence additional condition is $S = 1 - 1 = 0$.

Problem 3. Find an equation of common tangents to ellipses

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \text{ and } \frac{x^2}{9} + \frac{y^2}{18} = 1.$$

Solution.

Let's write the equation of tangent line for two ellipses at two different tangent points (x_1, y_1) and (x_2, y_2):

$$\frac{xx_1}{45} + \frac{yy_1}{9} = 1 \text{ and } \frac{xx_2}{9} + \frac{yy_2}{18} = 1.$$

From here we have:

$$x_1 = 5x_2, \quad y_1 = \frac{y_2}{2}.$$

Substituting these expression in the equation of 1-st ellipse and taking into account that (x_2, y_2) belongs the 2-nd ellipse, we obtain the following system

$$\frac{25x_2^2}{45} + \frac{y_2}{36} = 1,$$
$$\frac{x_2^2}{9} + \frac{y_2^2}{18} = 1.$$

Solution of this system is

$$x_2^2 = 1, \qquad y_2^2 = 16.$$

Let consider all possible cases:

1. $x_2 = 1$, $y_2 = 4 \Rightarrow$ equation of tangent:

$$x + 2y - 9 = 0,$$

2. $x_2 = 1$, $y_2 = -4 \Rightarrow$ equation of tangent:

$$x - 2y - 9 = 0,$$

3. $x_2 = -1$, $y_2 = 4 \Rightarrow$ equation of tangent:

$$x - 2y + 9 = 0,$$

4. $x_2 = -1$, $y_2 = -4 \Rightarrow$ equation of tangent:

$$x + 2y + 9 = 0.$$