11.10.24

2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

Algorithm

1) Find $b^2 - ac$, determine the type of equation.
2) We find the first integrals of the characteristic equations:

in the case when $a \neq 0$: $\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$,

in the case when $c \neq 0$: $\frac{dx}{dy} = \frac{b \pm \sqrt{b^2 - ac}}{c}$

3) The first integrals have the form:

in the case of hyperbolic type: $\varphi(x,y) = C, \ \psi(x,y) = C;$

in the case of elliptic type: $\alpha(x,y) \pm i\beta(x,y) = C$,

in the case of parabolic type: $\delta(x,y) = C$.

4) We replace variables:

in the case of hyperbolic type: $\begin{cases} \xi = \varphi(x,y); \\ \eta = \psi(x,y). \end{cases}$

in the case of elliptical type: $\begin{cases} \xi = \alpha(x,y); \\ \eta = \beta(x,y). \end{cases}$

in the case of parabolic type: $\begin{cases} \xi = \delta(x,y); \\ \eta = \varepsilon(x,y). \end{cases}$

where $\varepsilon(x,y)$ is any function of C^1 such that $\begin{vmatrix} \delta_x & \delta_y \\ \varepsilon_x & \varepsilon_y \end{vmatrix} \neq 0$.

The result of the replacement will be the canonical form of the equation.

Example 1

Bring it to a canonical form

$$u_{xx} + 2u_{xy} - 3u_{yy} + u_x + u_y = 0$$

Solution:

The type of equation:

$a = 1,\ \ b = 1,\ c = \text{-}3$

$b^2 - ac = 1 + 3 = 4\ \ > 0$

This equation has a hyperbolic type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$dy^2 - 2dydx - 3dx^2 = 0$

We solve this equation as a quadratic one.

$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$

$dy - \left(1 \pm \sqrt{4}\right)dx = 0$

$$\begin{cases} dy - 3dx = 0 \\ dy + dx = 0 \end{cases}$$

It is good that we have an equation with constant coefficients: a, b, c do not depend on x, y.

$$\begin{cases} y - 3x = C_1 \\ y + x = C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi = y - 3x \\ \eta = y + x \end{cases}$$

When moving to these variables, the equation (*) is greatly simplified.

$$u(x,y) \to U(\xi,\eta)$$

The derivative of the whole function:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

Take the second derivatives of $u_{x,}$ u_y.

$$u_x = U_\xi \cdot (\text{-}3) + U_\eta \cdot 1$$

$$u_y = U_\xi \cdot 1 + U_\eta \cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx} = -3\left(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x\right) + 1\left(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x\right) =$$

$$=-3\left(U_{\xi\xi}\cdot(-3)+U_{\xi\eta}\cdot 1\right)+1\left(U_{\eta\xi}\cdot(-3)+U_{\eta\eta}\cdot 1\right)=$$

$$=9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}$$

$$u_{xy}=-3\left(U_{\xi\xi}\cdot\xi_y+U_{\xi\eta}\cdot\eta_y\right)+1\left(U_{\eta\xi}\cdot\xi_y+U_{\eta\eta}\cdot\eta_y\right)=$$

$$=-3\left(U_{\xi\xi}\cdot 1+U_{\xi\eta}\cdot 1\right)+1\left(U_{\eta\xi}\cdot 1+U_{\eta\eta}\cdot 1\right)=$$

$$=-3U_{\xi\xi}-2U_{\xi\eta}+U_{\eta\eta}$$

$$u_{yy}=1\left(U_{\xi\xi}\cdot\xi_y+U_{\xi\eta}\cdot\eta_y\right)+1\left(U_{\eta\xi}\cdot\xi_y+U_{\eta\eta}\cdot\eta_y\right)=$$

$$=1\cdot U_{\xi\xi}+U_{\xi\eta}\cdot 1+U_{\eta\xi}\cdot 1+U_{\eta\eta}\cdot 1=$$

$$=U_{\xi\xi}+2U_{\xi\eta}+U_{\eta\eta}$$

Substituting everything into our equation:

$$u_{xx}+2u_{xy}-3u_{yy}+u_x+u_y=0$$

$$9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}+2\left(-3U_{\xi\xi}-2U_{\xi\eta}+U_{\eta\eta}\right)-$$
$$-3\left(U_{\xi\xi}+2U_{\xi\eta}+U_{\eta\eta}\right)+U_{\xi}\cdot(-3)+U_{\eta}\cdot 1+U_{\xi}\cdot 1+U_{\eta}\cdot 1=0$$

$$9U_{\xi\xi}-6U_{\eta\xi}+U_{\eta\eta}-6U_{\xi\xi}-4U_{\xi\eta}+2U_{\eta\eta}-$$
$$-3U_{\xi\xi}-6U_{\xi\eta}-3U_{\eta\eta}-3U_{\xi}+U_{\eta}+U_{\xi}+U_{\eta}=0$$

$$-16U_{\xi\eta}+2U_{\eta}-2U_{\xi}=0$$

$$U_{\xi\eta} = \frac{2U_{\xi} - 2U_{\eta}}{-16}$$

Example 2 EQUATION WITH VARIABLE COEFFICIENTS

To bring to a canonical form in each area where the type is preserved, the equation

$$yu_{xx} + u_{yy} = 0$$

Solution:

Step 1. The type of equation:

$a = y, \;\; b = 0, \; c = 1$

$\Delta = b^2 - ac = 0 - y \cdot 1 = -y$

Therefore,

1) in the half-plane $y < 0$, $\Delta > 0 \;\Rightarrow$ means hyperbolic type,

2) in the half-plane $y > 0$, $\Delta < 0 \;\Rightarrow$ means elliptical type,

3) on the straight $y = 0$ discriminant $\Delta = 0 \;\Rightarrow$ means parabolic type.

Step 2.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$y dy^2 + dx^2 = 0$$

We solve this equation as a quadratic one.

Since $c = 1 \neq 0$, the characteristic equations have the form:

$$\frac{dx}{dy} = \frac{b \pm \sqrt{\Delta}}{c}, \quad \text{that is,} \quad \frac{dx}{dy} = \pm\sqrt{-y}$$

This is an equation with separable variables. We solve them:

1. in the half-plane $y < 0$

$$dx = \pm\sqrt{-y}dy \qquad \Rightarrow \qquad x + c = \mp\frac{2}{3}(-y)^{\frac{3}{2}}.$$

Therefore, the first integrals have the form:

$$\boxed{\varphi(x,\, y) = x + \tfrac{2}{3}(-y)^{\frac{3}{2}}} = c, \qquad \boxed{\psi(x,\, y) = x - \tfrac{2}{3}(-y)^{\frac{3}{2}}} = c$$

2. in the half-plane $y > 0$

$$dx = \pm i\sqrt{y}dy \qquad \Rightarrow \qquad x + c = \pm i\frac{2}{3}y^{\frac{3}{2}}.$$

Therefore, the first integrals have the form:

$$\alpha(x,\, y) \pm i\beta(x,\, y) = c,$$

where

$$\boxed{\alpha(x, y) = x}, \qquad \boxed{\beta(x, y) = \tfrac{2}{3}\, y^{\frac{3}{2}}}$$

3. on the straight $y = 0$

$$dx = 0 \cdot dy \qquad \Rightarrow \qquad x = c$$

Therefore, the first integral (the only linearly independent one) has the form:

$$\boxed{\delta(x, y) = x.}$$

Step 3.

Replacing variables.

According to the algorithm, it is necessary to carry out a replacement.

1. in the half-plane $y < 0$

$$\begin{cases} \xi = x + \frac{2}{3}(-y)^{\frac{3}{2}}; \\ \eta = x - \frac{2}{3}(-y)^{\frac{3}{2}}. \end{cases}$$

$$\begin{aligned} &\xi_x = 1, \\ &\xi_y = -\sqrt{-y}, \\ &\eta_x = 1, \\ &\eta_y = \sqrt{-y} \end{aligned}$$

Then by entering the function $U(\xi, \eta)$, we get:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

Take the second derivatives of $u_{x,}\ u_y$.

$$u_x = U_\xi \cdot 1 + U_\eta \cdot 1 = U_\xi + U_\eta$$

$$u_y = U_\xi \cdot \left(-\sqrt{-y}\right) + U_\eta \cdot \left(\sqrt{-y}\right) = \sqrt{-y}\left(-U_\xi + U_\eta\right)$$

From these derivatives we take the second derivatives:

$$u_{xx} = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = -y\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - \frac{1}{2\sqrt{-y}}\left(-U_\xi + U_\eta\right)$$

Substituting the found derivatives into the original equation, we get:

$$yu_{xx} + u_{yy} = 0$$

$$y\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) - y\left(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - \frac{1}{2\sqrt{-y}}\left(-U_\xi + U_\eta\right) =$$

$$= y\left[4U_{\xi\eta} - \frac{1}{2(-y)^{\frac{3}{2}}}\left(-U_\xi + U_\eta\right)\right] = 0$$

Dividing by $4y$ and expressing $2(-y)^{\frac{3}{2}} = \frac{3}{2}(\xi - \eta)$, we get the canonical form:

$$U_{\xi\eta} - \frac{1}{6(\xi - \eta)}\left(-U_\xi + U_\eta\right) = 0$$

2. in the half-plane $y>0$

$$\begin{cases} \xi = x; \\ \eta = \frac{2}{3} y^{\frac{3}{2}}. \end{cases}$$

$$\xi_x = 1,$$
$$\xi_y = 0,$$
$$\eta_x = 0,$$
$$\eta_y = \sqrt{y}$$

Then by entering the function $U(\xi,\eta)$, we get:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

$$u_x = U_\xi + U_\eta \cdot 0 = U_\xi$$
$$u_y = U_\xi \cdot 0 + U_\eta \sqrt{y} = U_\eta \sqrt{y}$$

$$u_{xx} = U_{\xi\xi}$$
$$u_{yy} = U_{\eta\eta} y + \frac{1}{2\sqrt{y}} U_\eta$$

Substituting the found derivatives into the original equation, we get:

$$yu_{xx} + u_{yy} = 0$$

$$yU_{\xi\xi}+U_{\eta\eta}y+\frac{1}{2\sqrt{y}}U_{\eta}=y\left(U_{\xi\xi}+U_{\eta\eta}\right)+\frac{1}{2\sqrt{y}}U_{\eta}=$$

$$=y\left(U_{\xi\xi}+U_{\eta\eta}+\frac{1}{2y^{\frac{3}{2}}}U_{\eta}\right)=\left[2y^{\frac{3}{2}}=3\eta\right]=$$

$$=y\left(U_{\xi\xi}+U_{\eta\eta}+\frac{1}{3\eta}U_{\eta}\right)=0$$

Dividing by y, we get the canonical form:

$$U_{\xi\xi}+U_{\eta\eta}+\frac{1}{3\eta}U_{\eta}=0$$

3. on the straight $y=0$

$$\begin{cases} \xi = x; \\ \eta = y. \end{cases}$$

We need to arbitrarily choose $\eta(x,y)$ so that the functions ξ,η form a linearly independent pair.

$\xi_x=1,$
$\xi_y=0,$
$\eta_x=0,$
$\eta_y=1$

Then by entering the function $U(\xi,\eta)$, we get:

$u_x=U_{\xi}\xi_x+U_{\eta}\eta_x$
$u_y=U_{\xi}\xi_y+U_{\eta}\eta_y$

$$u_x = U_\xi$$
$$u_y = U_\eta$$

$$u_{xx} = U_{\xi\xi}$$
$$u_{yy} = U_{\eta\eta}$$

Substituting the found derivatives into the original equation for $y=0$, we get:

$$u_{yy} = U_{\eta\eta} = 0$$

So, the canonical form of the original equation on the line $y=0$:

$U_{\eta\eta} = 0$ or, what is the same, $u_{yy} = 0$.

<u>Answer:</u>

$$\begin{cases} U_{\xi\eta} - \dfrac{1}{6(\xi-\eta)}\left(-U_\xi + U_\eta\right) = 0 \quad \textit{in the area } \ y<0, \textit{ hyperbolic type} \\ U_{\xi\xi} + U_{\eta\eta} + \dfrac{1}{3\eta}U_\eta = 0 \quad \textit{in the area } y>0, \textit{ elliptical type} \\ U_{\eta\eta} = 0 \quad \textit{in the area } y=0, \textit{ parabolic type.} \end{cases}$$

REDUCTION TO THE CANONICAL FORM OF PARTIAL DIFFERENTIAL EQUATIONS OF THE 2ND ORDER WITH CONSTANT COEFFICIENTS

In this section, we will consider second-order partial differential equations with constant coefficients and n independent variables:

$$\sum_{i,j=1}^{n} a_{ij}\, u_{x_i x_j} + f(x_1,\, \dots,\, x_n;\; u,\, u_{x_1},\, \dots,\, u_{x_n}) = 0, \tag{1}$$

$a_{ij} = const \in \mathbb{R},\ i, j = \overline{1,n}$.

Definition 1

The characteristic quadratic form of equation (1) is the expression:

$$Q(\lambda_1,\, \dots,\, \lambda_n) = \sum_{i,j=1}^{n} a_{ij}\, \lambda_i \lambda_j. \tag{2}$$

The normal form of the quadratic form (2) is its form:

$$\tilde{Q}(\mu_1,\, \dots,\, \mu_n) = \sum_{k=1}^{n} \beta_k\, \mu_k^2, \qquad \beta_k \in \{-1,\ 0,\ 1\}. \tag{3}$$

The canonical form of equation (1) is the form in which its characteristic quadratic form takes the normal (or canonical) form:

$$\sum_{k=1}^{n} \beta_k \, u_{x_k x_k} + g(x_1, \ldots, x_n; \; u, u_{x_1}, \ldots, u_{x_n}) = 0. \qquad \textbf{(4)}$$

Definition 2

Equation (1) refers to

1) **hyperbolic type**, if the coefficients β_k are different from zero and not all of the same sign;

2) **elliptical type**, if all coefficients β_k are nonzero and all of the same sign;

3) **parabolic type,** if at least one of the coefficients β_k is zero.

<u>Algorithm</u>

1) We reduce the characteristic quadratic form to the canonical (normal) form (3) (by the method of selecting complete squares). We write out the transformation matrix that performs this process:

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \cdots \\ \mu_n \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_{11} & \alpha_{12} & \vdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \vdots & \alpha_{2n} \\ \cdots & \cdots & \ddots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \vdots & \alpha_{nn} \end{pmatrix}}_{A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \cdots \\ \lambda_n \end{pmatrix}, \qquad \det A \neq 0.$$

2) We find the matrix Γ of the substitution of variables according to the law:

$$\Gamma = \left(A^T\right)^{-1}.$$

3) We replace variables:

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \dots \\ \xi_n \end{pmatrix} = \underbrace{\begin{pmatrix} \gamma_{11} & \gamma_{12} & \vdots & \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \vdots & \gamma_{2n} \\ \dots & \dots & \ddots & \dots \\ \gamma_{n1} & \gamma_{n2} & \vdots & \gamma_{nn} \end{pmatrix}}_{\Gamma} \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$$

The result of the replacement will be the canonical form (4) of equation (1).

Example 3

To bring the equation to a canonical form:

$$u_{xx} + 2u_{xy} + 5u_{yy} - 32u = 0.$$

<u>Solution:</u>

Step 1

The characteristic quadratic form of this equation has the form

$$Q(\lambda_1,\ \lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + 5\lambda_2^2.$$

Let's bring it to the canonical form:

$$Q(\lambda_1,\ \lambda_2) = \lambda_1^2 + 2\lambda_1\lambda_2 + 5\lambda_2^2 = (\lambda_1 + \lambda_2)^2 + (2\lambda_2)^2 = \mu_1^2 + \mu_2^2,$$

where

$$\begin{cases} \mu_1 = \lambda_1 + \lambda_2; \\ \mu_2 = 2\lambda_2 \end{cases}$$

that is

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}}_{A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Step 2

Let's find the matrix of substitution of variables Γ:

$$\Gamma = \left(A^T\right)^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$$

Step 3

We replace the variables:

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

that is,

$$\begin{cases} \xi = x; \\ \eta = \frac{1}{2}\left(-x + y\right). \end{cases}$$

To put new variables in the original equation, let's put

$$U(\xi,\eta) = u(x,y)$$

and find $u_x, u_y, u_{xx}, u_{xy}, u_{yy}$ as derivatives of a complicated function $U(\xi(x,y),\eta(x,y))$:

$$u_x = U_\xi - \frac{1}{2}U_\eta$$

$$u_y = \frac{1}{2}U_\eta$$

$$u_{xx} = U_{\xi\xi} - U_{\xi\eta} + \frac{1}{4}U_{\eta\eta}$$

$$u_{xy} = \frac{1}{2}U_{\xi\eta} - \frac{1}{4}U_{\eta\eta}$$

$$u_{yy} = \frac{1}{4}U_{\eta\eta}$$

Substituting the found derivatives into the left side of the original equation and giving similar ones, we get:

$$u_{xx} + 2u_{xy} + 5u_{yy} - 32u = \left(U_{\xi\xi} - U_{\xi\eta} + \frac{1}{4}U_{\eta\eta}\right) + 2\left(\frac{1}{2}U_{\xi\eta} - \frac{1}{4}U_{\eta\eta}\right) +$$

$$+5\left(\frac{1}{4}U_{\eta\eta}\right) - 32U = U_{\xi\xi} + U_{\eta\eta} - 32U$$

<u>Answer:</u>

the equation has an elliptical type,

$U_{\xi\xi} + U_{\eta\eta} - 32U = 0$, where $\xi = x,\ \ \eta = \frac{1}{2}(-x + y)$.

HOMEWORK 7 (The deadline is October 14, 2024)

Version 1	**Version 2**
$u_{xx} - yu_{yy} = 0.$	$xu_{xx} - 2\sqrt{xy}u_{xy} + yu_{yy} + \frac{1}{2}u_y = 0.$
Wang Jiahe	Yan Shukun
Guan Haochen	Liu Yudong
Liu Tianxing	Wang Youshen
Ma Yueyang	Lu Qibo
Wang Changzhi	Chen Langbo
Yang Zihao	An Junhao
Zhong Yuhao	Yu Hang
Wu Haonan	Ni Zhongshuo
Yan Sensen	Li Kangjian
Wang Yudong	Lin Enbei
Li Sicheng	Xia Xinglin
Li Kaiyan	Huang Yifan
Yang Guowei	Shen Xingye
Kong Xiangning	Wang Haojun
Liu Jiashan	Li Jiashen
Zhao Yixiao	Chen Shiwen
Li Xinyi	Wang Leihan
Zhao Xiaohui	Yang Yuhao
Qu Linfeng	Liu Xingyu
Zhou Zixin	Qian Keqing
Yu Rongyi	Wu Jiaxin
Mei Mingzhe	Lu Mingyu
Zhang Hongbo	
Yan Shukun	

14.10.2024

2.6. BASIC EQUATIONS OF MATHEMATICAL PHYSICS