Functional sequences and series.

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1 Pointwise and uniform convergence of functional sequences.

1.1 Definitions and examples.

Definition 1.1. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a sequence of functions defined on the same set* $f_n : E \to \mathbb{R}$ *(or* $f_n : E \to \mathbb{C}$ *). Assume that for every* $x \in E$ *a (numerical) sequence* $\{f_n(x)\}$ *has a limit* $f(x)$*. This defines on* E *a function*

$$f(x) = \lim_{n\to\infty} f_n(x)$$

which is called a ***pointwise limit*** *of a sequence* $\{f_n\}_{n=1}^{\infty}$.

Remark. Sometimes it is convenient consider a set of convergence

$$E_1 = \{x \in E : \{f_n(x)\} \text{ has a limit}\}.$$

Assume that $f_n, f : E \to \mathbb{R}$ and the sequence $\{f_n(x)\}$ converges pointwise to f on E. Assume that the functions f_n have some properties. It is a natural question whether these properties are preserved when we take the pointwise limit. E.g.:

- Let $f_n \in C(E)$ (all functions f_n are continuous on E). Is it true that f is continuous on E?

- Assume that $E = [a, b]$ is a segment and f_n are Riemann integrable on E. Is it true that f is Riemann integrable on E and

$$\int_a^b f_n \to \int_a^b f, \qquad n \to \infty?$$

- Let $E = [a, b]$ and $f_n \in C^1[a, b]$. Is it true that $f \in C^1[a, b]$ and $f_n' \to f'$?

Unfortunately, the answer to all these questions is in general NO. To obtain results of this type we need to introduce a different (a stronger) notion of convergence which will preserve good properties of functions.

Examples. Let $f_n : [0, 1] \to \mathbb{R}$.

1. If $f_n(x) = x^n$ then $f(x) = 0,\ x \in [0, 1)$; $f(1) = 1$. So the pointwise limit of continuous (even smooth) functions can be not continuous.

2. If $f_n(x) = \frac{1}{1+nx}$ then $f(x) = 0,\ x \in (0, 1]$; $f(0) = 1$.

3. If $f_n(x) = 2n^2xe^{-n^2x^2}$ then $f(x) = 0,\ x \in [0, 1]$. Here,

$$\int_0^1 f_n(x) = 2n^2 \int_0^1 e^{-n^2x^2}dx = [t = nx] = n \int_0^n e^{-t^2} \to +\infty,\ n \to \infty;$$

$$\int_0^1 f = 0.$$

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions on E with pointwise limit $f(x)$. This means that

$$\forall x \in E\ \forall \varepsilon > 0\ \exists N \in \mathbb{N} :\ \forall n > N\ \ |f_n(x) - f(x)| < \varepsilon.$$

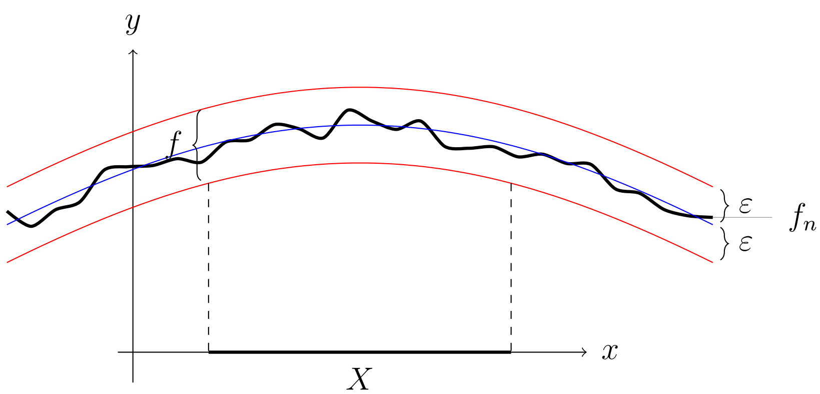
In other words, number N depends both on ε and $x \in E$.

Definition 1.2. *Let $f, f_n : E \to \mathbb{R}$ (or $f_n : E \to \mathbb{C}$), $n \in \mathbb{N}$. A function f is called an uniform limit of a (functional) sequence $\{f_n\}$ (a sequence $\{f_n\}$ converges uniformly to f on E) if*

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} : \; \forall n > N \; \forall x \in E \; |f_n(x) - f(x)| < \varepsilon.$$

In this case we write

$$f_n \rightrightarrows f \text{ on } E.$$



Remark. If a sequence $\{f_n\}$ converges uniformly to f on E then it converges to f pointwise (the opposite is not true).

Lemma 1.3. *Let $f, f_n : E \to \mathbb{R}$ (or $f_n : E \to \mathbb{C}$), $n \in \mathbb{N}$. The following assertion are equivalent:*

1. $f_n \rightrightarrows f$ *on* E;
2. $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} : \; \forall n > N \; \sup\limits_{x \in E} |f_n(x) - f(x)| < \varepsilon$;
3. $\lim\limits_{n \to \infty} \sup\limits_{x \in E} |f_n(x) - f(x)| = 0$;

4. There exists a sequence $\{\varepsilon_n\}$ of nonnegative numbers such that

$$|f_n(x) - f(x)| \le \varepsilon_n, \ \forall x \in E,$$

and $\varepsilon_n \to 0$.

Proof. **1** $\Rightarrow$ **2**. Assume that $f_n \rightrightarrows f$ on E. Then

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \ \forall n > N \ \forall x \in E \ \ |f_n(x) - f(x)| < \varepsilon/2.$$

Considering supremum in the last inequality we see that

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \ \forall n > N \ \sup_{x \in E} |f_n(x) - f(x)| \le \varepsilon/2 < \varepsilon.$$

2 $\Rightarrow$ **3** by definition of a limit.
3 $\Rightarrow$ **4**. It is enough to take $\varepsilon_n = \sup\limits_{x \in E} |f_n(x) - f(x)|$.
4 $\Rightarrow$ **1**. Let $\varepsilon > 0$. Then $\exists N \in \mathbb{N} : \ \forall n > N \ \varepsilon_n < \varepsilon$. Consequently,

$$\forall n > N \ \forall x \in E \ \ |f_n(x) - f(x)| < \varepsilon.$$

□

Example 1. Let $E = [0, 1]$, $f_n(x) = x^n$. Then f_n converge pointwise to

$$f(x) = \begin{cases} 0, & x \in [0, 1), \\ 1, & x = 1. \end{cases}$$

Is the convergence uniform? No. For any fixed n consider

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1)} x^n.$$

Since $x^n \to 1$ when $x \to 1-$, we see that

$$\sup_{x\in[0,1)} |f_n(x) - f(x)| = 1$$

and does not tend to 0. Thus, by Lemma 1.3 the sequence f_n does not converge to f uniformly.

Remark. To show that there is no uniform convergence it is not necessary to compute the supremum. It is sufficient to construct a sequence $x_n \in E$ such that $|f_n(x_n) - f(x_n)|$ does not tend to 0 as $n \to \infty$. In our case one can take $x_n = 1 - 1/n$ or $x_n = 1 - 1/n^2$.

Example 2. Let $E = [0, +\infty)$, $f_n(x) = \frac{1}{1+nx}$. Then $f(x) = 0$, $x \in (0, +\infty)$; $f(0) = 1$. Take $x_n = 1/n$. We have $f_n(x_n) = 1/2$, while $f(x_n) = 0$. Thus the convergence is not uniform.

Example 3. Let $E = [0, +\infty)$, $f_n(x) = \frac{x}{1+nx}$. Then $f(x) = \lim\limits_{n\to\infty} f_n(x) = 0\ \forall x \in E$. We have

$$|f_n(x) - f(x)| = \frac{x}{1+nx} \le \frac{x}{nx} = \frac{1}{n}, \quad \forall\, x > 0.$$

Hence, $f_n \rightrightarrows f$ on E.

Definition 1.4. *Let X be a set, $f : X \to \mathbb{R}$ (or to $\mathbb{C}$). A* ***uniform norm*** *of a function f is a value*

$$\|f\| = \sup_{x\in X} |f(x)|\,.$$

1.1.1 Examples of problems.

Problem 1. Find the pointwise limit function $f(x)$ of a sequence $\{f_n(x)\}$ on a set E, where

$$f_n(x) = \sqrt[n]{1 + x^n},\ E = [0, 2]$$

Solution: Consider two cases:
Case 1. If $x \in [0,1]$ then $1 \leq \sqrt[n]{1+x^n} \leq \sqrt[n]{2}$. And $\sqrt[n]{1+x^n} \to 1$ as $n \to \infty$ by Squeeze theorem since $\sqrt[n]{2} \to 1$.
Case 2. If $x \in (1,2)$ then

$$\sqrt[n]{1+x^n} = x\sqrt[n]{1+1/x^n} \to x$$

by previous case.

Consequently, $f_n(x) \to f(x) = \begin{cases} 1, & x \in [0,1]; \\ x, & x \in (1,2]. \end{cases}$

Problem 2. Prove that a sequence $\{f_n(x)\}$ converges uniformly on E if

$$f_n(x) = n(x^{1/n} - 1),\ E = [1,3].$$

Solution: Let's first find a pointwise limit. Let $x \in [1,3]$. Then

$$f_n(x) = n(x^{1/n} - 1) = n\left(e^{\ln(x)/n} - 1\right) \sim n\frac{\ln x}{n} = \ln x.$$

To prove the uniform convergence apply Taylor's-Lagrange's formula for e^t. For every $t >$ there exists $c \in (0,t)$ such that

maybe the most common many to estimate the difference between $f_n(x)$ and $f(x)$

$$e^t = 1 + t + e^c\frac{t^2}{2}.$$

Consequently, $|e^t - 1 - t| \leq \frac{e^2}{2}t^2$, for $t \in [0,2]$, and

$$|f_n(x) - \ln x| = \left|n\left(e^{\ln(x)/n} - 1\right) - \ln x\right| =$$

$$n\left|e^{\ln(x)/n} - 1 - \frac{\ln x}{n}\right| \leq \frac{e^2}{2}\frac{\ln^2 x}{n^2} \leq \frac{e^2\ln^2 3}{2n^2} \to 0.$$

Problem 3. Check the uniform convergence of a sequence $\{f_n(x)\}$ on E_1 and E_2 if

$$f_n(x) = n \operatorname{arctg} \frac{1}{nx}, \quad E_1 = (0, 2), \quad E_2 = (2, +\infty)$$

Solution: Let's find first a pointwise limit. Let $x \in (0, +\infty)$. Then

$$f_n(x) = n \operatorname{arctg} \frac{1}{nx} \sim n\frac{1}{nx} = \frac{1}{x}.$$

First, apply Taylor's-Lagrange's formula to see that

$$\operatorname{arctg} t = t + (\operatorname{arctg} t)''|_{t=c} \frac{t^2}{2}$$

for some c between 0 and t. Notice that

$$(\operatorname{arctg} t)'' = \left(\frac{1}{1+t^2}\right)' = \frac{-2t}{(1+t^2)^2}$$

and

$$|(\operatorname{arctg} t)''| \le \frac{1}{1+t^2} \le 1$$

since $|2t| \le 1 + t^2$. Consequently,

$$|\operatorname{arctg} t - t| \le t^2/2,\ t \in \mathbb{R}.$$

Case 1. Let $x \in E_2$. Then

$$\left|f_n(x) - \frac{1}{x}\right| = n\left|\operatorname{arctg} \frac{1}{nx} \sim n\frac{1}{nx} - \frac{1}{nx}\right| \le \frac{n}{2n^2x^2 \le \frac{1}{8n}} \to 0,$$

and $f_n \rightrightarrows 1/x$ on E_2.

Case 2. Let $x_n = 1/n \in E_1$. Then

$$f_n(1/n) = n \operatorname{arctg} 1 - n = n(\pi/4 - 1) \to +\infty$$

and $f_n \not\rightrightarrows 1/x$ on E_1.

1.2 Properties of uniformly convergent sequences.

Lemma 1.5 (Arithmetical properties of the uniform convergence)**.** *1. Let $f_n \rightrightarrows f$, $g_n \rightrightarrows g$ on X, $\alpha, \beta \in \mathbb{R}$ (or in $\mathbb{C}$). Then $\alpha f_n + \beta g_n \rightrightarrows \alpha f + \beta g$.*

2. Let $f_n \rightrightarrows f$, g be bounded on X. Then $f_n g \rightrightarrows fg$ on X.

Proof. **1.** For every $x \in E$ we have

$$|(\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x)| \leq |\alpha| \, |f_n(x) - f(x)| + |\beta| \, |g_n(x) - g(x)| \, .$$

Consequently,

$$\sup_{x\in E} |(\alpha f_n + \beta g_n)(x) - (\alpha f + \beta g)(x)| \\ \leq |\alpha| \sup_{x\in E} |f_n(x) - f(x)| + |\beta| \sup_{x\in E} |g_n(x) - g(x)| \to 0, \ n \to \infty.$$

2. Assume that $\sup\limits_{x\in E} |g| \leq M$. Then

$$\sup_{x\in E} |f_n g(x) - fg(x)| \leq M \sup_{x\in E} |f_n(x) - f(x)| \to 0.$$

□

Theorem 1.6 (Bolzano–Cauchy criterion for uniform convergence of a sequence)**.** *Let $f_n : E \to \mathbb{R}$. The uniform convergence of a sequence $\{f_n\}$ on E is equivalent to the following condition*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \ \forall n, m > N \ \forall x \in E \ \ |f_n(x) - f_m(x)| < \varepsilon. \qquad (1)$$

Proof. Let $f_n \rightrightarrows f$ on E and let $\varepsilon > 0$. Then we can find N such that for any $n > N$ and any $x \in E$ $|f_n(x) - f(x)| < \varepsilon/2$. Then for any $m, n > N$ and any $x \in E$ we have

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Conversely, assume that (1) is satisfied. Then, for any fixed $x \in E$, the sequence $\{f_n(x)\}$ is a Cauchy sequence, and so it converges. Define $f(x) = \lim\limits_{n\to\infty} f_n(x)$. Now, for $\varepsilon > 0$ we choose N such that

$$\forall n, m > N \ \forall x \in E \ \ |f_n(x) - f_m(x)| < \varepsilon/2.$$

Passing to the limit over $m \to \infty$ (with a fixed $n > N$) we get

$$\forall n > N \ \forall x \in E \ \ |f_n(x) - f(x)| \leq \varepsilon/2 < \varepsilon,$$

which means that $f_n \rightrightarrows f$ on E. □

Theorem 1.7 (On interchange of limits)**.** *Let E be a subset of $\mathbb{R}$ or $\mathbb{C}$, p be a limit point of E, and $f_n, f : \mathbb{R} \to \mathbb{R}$ or $(\mathbb{C})$. Assume that*

1. *$f_n \rightrightarrows f$ on E;*
2. *for every $n \in \mathbb{N}$ there exists a limit $A_n = \lim\limits_{x\to p} f_n(x)$.*

Then limits $\lim\limits_{n\to\infty} A_n$ and $\lim\limits_{x\to p} f(x)$ exist and are equal:

$$\lim_{n\to\infty} \lim_{x\to p} f_n(x) = \lim_{x\to p} \lim_{n\to\infty} f_n(x).$$

Proof. Let $\varepsilon > 0$. Then, by Bolzano-Cauchy criteria there exists $N > 0$ such that

$$\forall \ n, m > N \ \forall x \in E \ |f_n(x) - f_m(x)| < \varepsilon.$$

Now letting $x \to p$ we see that

$$\forall\, n, m > N\ \forall x \in E\ \ |A_n - A_m| \le \varepsilon,$$

which implies that $\{A_n\}$ is a Cauchy sequence and has limit that will be denoted by A.

Now it is enough to show that $A = \lim\limits_{x\to p} f(x)$. We use uniform convergence $f_n \rightrightarrows f$ and convergence $A = \lim\limits_{n\to\infty} A_n$ to see that there exists $L > 0$ such that

$$\forall n > L,\ x \in E\ \ |f_n(x) - f(x)| < \varepsilon/3$$

and $K > 0$ such that

$$\forall k > K\ \ |A_k - A| < \varepsilon/3.$$

Let $M > \max(K, L)$ then $A_M = \lim\limits_{x\to p} f_M(x)$ and there exists $\delta > 0$ such that

$$|f_M(x) - A_M| < \varepsilon/3, \quad x \in E,\ 0 < |x - p| < \delta.$$

Finally for every $x \in E$ such that $0 < |x - p| < \delta$ we have

$$|f(x) - A| \le |f(x) - f_M(x)| + |f_M(x) - A_M| + |A_M - A| < \varepsilon$$

and $A = \lim\limits_{x\to p} f(x)$.

$\square$

Corollary 1.7.1 (Pointwise continuity of the uniform limit.)**.** *Let E be a subset of $\mathbb{R}$ or $\mathbb{C}$, $p \in E$, and $f_n, f : \mathbb{R} \to \mathbb{R}$ or $(\mathbb{C})$. Assume that*

1. $f_n \rightrightarrows f$ *on* E;

2. all functions f_n are continuous in p.

Then the function f is continuous in p.

Proof. If p is isolated the assertion is trivial. Assume that p is not isolated. Then the assumptions of Theorem 1.7 are satisfied with $A_n = f_n(p)$. Consequently,

$$\lim_{x\to p} f(x) = \lim_{n\to\infty} A_n = f(p)$$

and f is continuous at p. $\square$

Corollary 1.7.2 (Continuity of the uniform limit on a set.)**.** *Let E be a subset of $\mathbb{R}$ or $\mathbb{C}$, $p \in E$, and $f_n, f : \mathbb{R} \to \mathbb{R}$ or $(\mathbb{C})$. Assume that*

1. $f_n \rightrightarrows f$ on E;

2. all functions f_n are continuous on E.

Then the function f is continuous on E. Moreover, if every function f_n is uniform continuous then f is also uniform continuous.

Proof. Assume that functions f_n are uniform continuous. We partially repeat the proof of Theorem 1.7. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$ there exists $L > 0$ such that

$$\forall n > L,\ x \in E\ \ |f_n(x) - f(x)| < \varepsilon/3.$$

Let $M = L + 1$ then there exists $\delta > 0$ such that

$$|f_M(x) - f_M(y)| < \varepsilon/3, \quad x, y \in E,\ |x - y| < \delta.$$

Consequently, for every $x, y \in E$ such that $|x - y| < \delta$

$$|f(x) - f(y)| \leq |f(x) - f_M(x)| + |f_M(x) - f_M(y)| + |f_M(y) - f(y)| < \varepsilon.$$

This implies uniform continuity of f. $\square$

Example. We've already seen that pointwise limit of a sequence of continuous functions can be discontinuous. So pointwise continuity is not sufficient for permutation of limits:

$$\lim_{n\to\infty}\lim_{x\to 1-} x^n = 1 \neq \lim_{x\to 1-}\lim_{n\to\infty} x^n = 0.$$

At the same time uniform convergence is not necessary for continuity of a limit function. So, sequence $f_n = \sqrt{n}x(1-x^2)^n$ converges pointwise but not uniformly to 0 on $[0,1]$ since

$$f_n(1/\sqrt{n}) = \left(1 - \frac{1}{n}\right)^n \to e^{-1} \neq 0.$$

However, with some additional assumption we may deduce uniform convergence from continuity of a limit function.

Theorem 1.8 (Dini's theorem for a sequences). *Let K be compact subset of $\mathbb{R}$ or $\mathbb{C}$, and $f_n, f \in C(K)$. If f is a pointwise limit of a sequence $\{f_n\}$ and for every $x \in E$ a sequence $f_n(x)$ is increasing. Then $f_n \rightrightarrows f$ on E.*

Proof. Let $\varepsilon > 0$ and $g_n = f - f_n$. Let $E_n = \{x \in K : g_n(x) < \varepsilon\}$. Since f, f_n are continuous then E_n is open in K. The monotonicity of a sequence $f_n(x)$ implies that a sequence E_n is ascending, $E_n \subset E_{n+1}$. Finally, g_n converges pointwise to 0 and $K = \bigcup\limits_{n=1}^{\infty} E_n$ and by compactness of K there is a finite subcover. Since E_n is ascending this implies that there exists a number $N \in \mathbb{N}$ such that $K = E_N = E_n,\ n > N$. In fact this means that for $n > N$

$$|f(x) - f_n(x)| < \varepsilon,\ x \in K.$$

□

Lemma 1.9 (Squeeze theorem for Riemann integral)**.** *Let $f : [a, b] \to \mathbb{R}$. Then f is Riemann integrable if and only if for every $\varepsilon > 0$ there exist two functions $g_1, g_2 \in \mathcal{R}[a, b]$ such that*

$$g_1(x) \le f(x) \le g_2(x), \quad g_2(x) - g_1(x) < \varepsilon, \ x \in [a, b].$$

Proof. If $f \in \mathcal{R}[a, b]$ then we can take $g_1 = g_2 = f$.

Now, suppose the converse and let g_1 and g_2 satisfy the conditions of the theorem with $g_2 - g_1 \le \frac{\varepsilon}{3(b-a)}$. We will use criterion for integrability in terms of lower and upper Darboux's sums.

There exists $\delta > 0$ such that for every partition $\tau = \{x_k\}_{k=0}^n$ of a segment $[a, b]$ with mesh less than δ such that

$$S_\tau(g_1) - \frac{\varepsilon}{3} < \int_a^b g_1 < s_\tau(g_1) + \frac{\varepsilon}{3}$$

and

$$S_\tau(g_2) - \frac{\varepsilon}{3} < \int_a^b g_2 < s_\tau(g_2) + \frac{\varepsilon}{3}$$

Since $g_1 \le f \le g_2$ we see that

$$S_\tau(f) \le S_\tau(g_2) < \int_a^b g_2 + \frac{\varepsilon}{3}$$

and

$$s_\tau(f) \ge s_\tau(g_1) > \int_a^b g_1 - \frac{\varepsilon}{3}.$$

Consequently,

$$S_\tau(f) - s_\tau(f) < \int_a^b (g_2 - g_1) + \frac{2\varepsilon}{3} < \frac{(b-a)\varepsilon}{3(b-a)} + \frac{2\varepsilon}{3}\varepsilon$$

and function f is integrable. $\square$

Theorem 1.10 (Uniform Convergence and Integration)**.** *Let $\{f_n\}$ be a sequence of Riemann integrable functions on $[a,b]$. If $\{f_n\}$ converges uniformly to $\{f\}$ on $[a,b]$ then $f \in \mathcal{R}[a,b]$ and*

$$\int\limits_a^b f_n \to \int\limits_a^b f.$$

Proof. Let $\varepsilon > 0$ be arbitrary. By uniform convergence, there exists $K \in \mathbb{N}$ such that if $n \geq K$ then for all $x \in [a,b]$ we have

$$|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$$

or

$$f_n(x) - \frac{\varepsilon}{b-a} < f(x) < f_n(x) + \frac{\varepsilon}{b-a}.$$

By assumption, $f_n \pm \frac{\varepsilon}{4(b-a)}$ is Riemann integrable and thus by Lemma 1.9, f is Riemann integrable. Moreover, if $n \geq N$ then

$$-\frac{\varepsilon}{b-a} < f_n(x) \mid -f(x) < \frac{\varepsilon}{b-a}$$

implies (by monotonicity of integration)

$$-\varepsilon < \int_a^b f_n - \int_a^b f < \varepsilon$$

and thus

$$\left|\int_a^b f_n - \int_a^b f\right| < \frac{\varepsilon}{4}$$

This proves that the sequence $\int_a^b f_n$ converges to $\int_a^b f$. $\square$

Theorem 1.11 (Integral of the uniform limit)**.** *Let* $f_n \in C[a,b]$ *and* $f_n \rightrightarrows f$ *on* $[a,b]$*. Then*

Proof. By Theorem 1.7.2 $f \in C[a,b]$ and the integral $\int\limits_a^b f$ exists. Let $\varepsilon > 0$ then by uniform convergence there exists $N > 0$ such that for every $n > N$ and $x \in [a,b]$ $|f_n(x) - f(x)| < \frac{\varepsilon}{b-a}$. Then

$$\left| \int\limits_a^b f_n - \int\limits_a^b f \right| \leq \int\limits_a^b |f_n - f| < \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

□

Example 1. Consider a sequence

$$f_n(x) = n^2 x(1-x^2)^n.$$

This sequence converges pointwise to 0 on $[0,1]$. Nevertheless,

$$\int_0^1 f_n = n^2 \int_0^1 x(1-x^2)^n dx =$$
$$n^2 \left(-\frac{(1-x^2)^{n+1}}{2(n+1)} \right) \Bigg|_0^1 = \frac{n^2}{2(n+1)} \to +\infty,\ n \to \infty.$$

Example 2. A sequence

$$f_n(x) = n^2 x(1-x^2)^n.$$

This sequence converges pointwise to 0 on $[0,1]$. Nevertheless,

$$\int_0^1 f_n = \frac{n}{2(n+1)} \to \frac{1}{2},\ n \to \infty.$$

We now consider how the operation of differentiation behaves under uniform convergence. One would hope that if $f_n \rightrightarrows f$ uniformly and each f_n is differentiable then f' is also differentiable and maybe even that $f_n' \rightrightarrows f'$ at least pointwise and maybe even uniformly. Unfortunately, the property of differentiability is not generally inherited under uniform convergence. Example 1. A sequence $f_n(x) = \frac{\sin nx}{n}$ converges uniformly to 0 on $\mathbb{R}$ since $|f_n(x)| \leq \frac{1}{n}$. However a sequence of derivatives $f_n'(x) = \cos nx$ doesn't have limit when $x \neq 2\pi k,\ k \in \mathbb{Z}$.

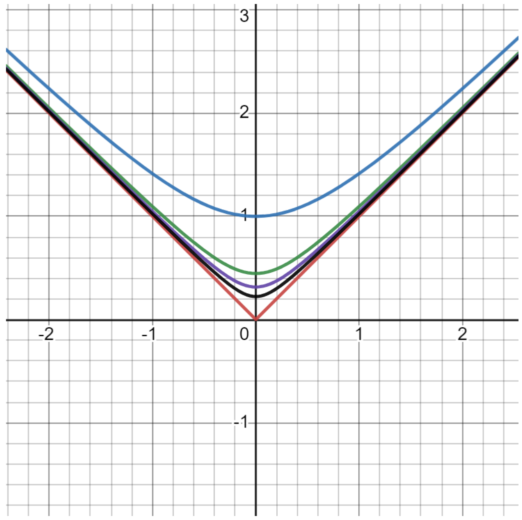
Example 2. A sequence $f_n(x) = \frac{x^{n+1}}{n+1}$ uniformly converges to 0 on $[0,1]$ since $|f_n(x)| \leq \frac{1}{n+1}$. However a sequence of derivatives $f_n'(x) = x^n$ converges on $[0,1]$ not uniformly and

$$\lim_{x\to 1-} f_n'(x) = 1 \neq \left(\lim_{x\to 1-} f_n(x)\right)' = 0.$$

Example 3. A sequence of differentiable functions $f_n(x) = \sqrt{\frac{nx^2+1}{n}}$ converges uniformly to function $f(x) = |x|$ on $[0,1]$ which is not differentiable at 0. To check this we can consider an estimate

$$|f_n(x) - f(x)| = \sqrt{x^2 + 1/n} - |x| = \frac{x^2 + 1/n - x^2}{\sqrt{x^2 + 1/n) + |x|}} \leq \frac{1}{\sqrt{n}}$$

or apply Dini's theorem 1.8 on a limit of monotone sequence.



These two examples show us that the uniform convergence doesn't imply neither convergence of derivatives nor the possibility of the passage to the limit under the sign of derivative. It turns out that the main assumption needed for all to be well is that the sequences $\{f_n'\}$ converges uniformly.

Theorem 1.12 (On differentiation of a limit function)**.** *Let $f_n : [a, b] \to \mathbb{R}$ be a differentiable functions. Assume that*

- *a sequence of derivatives $\{f_n'\}$ uniformly converges on $[a, b]$ to some function φ;*
- *$\exists c \in [a, b]$ such that $f_n(c)$ converges.*

Then

1. *a sequence $\{f_n\}$ uniformly converges on $[a, b]$ to some function f;*
2. *a function f is differentiable on $[a, b]$;*

3. $f' = \varphi$.

The last identity can be written as

$$\left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} f_n'.$$

Proof. Let's fix $p \in E$ and let

$$g_n(x) = \frac{f_n(x) - f_n(p)}{x - p}, \quad x \in E \setminus \{p\}.$$

We will prove that a sequence g_n uniformly converges on $E \setminus \{p\}$. By Lagrange's formula for every $n, m \in \mathbb{N}$ there exists a point ξ between p and x such that

$$|g_n(x) - g_m(x)| = \frac{(f_n - f_m)(x) - (f_n - f_m)(p)}{x - p} = (f_n - f_m)'(\xi).$$

Consequently,

$$\sup_{x\in E} |g_n(x) - g_m(x)| \sup_{x\in E} |f_n'(x) - f_m'(x)|$$

And this with uniform convergence of $\{f_n'\}$ and Bolzano-Cauchy criteria implies that sequence $g_n(x)$ uniformly converges on $E \setminus \{p\}$.

In particular we can let $p = c$. And since the multiplication by the bunded function doesn't interfere the uniform convergence this means that a sequence $\{f_n(x) - f_n(c)\}$ is also uniformly convergent on $E \setminus \{p\}$. Since all function of this sequence are zero at c we can say that this sequence converges on E. Recall that sequence $\{f_n(c)\}$ has limit. Then sequence $f_n = (f_n - f_n(c)) + f_n(c)$ has uniform limit on E that we'll denote as f. This proves the first assertion of the theorem.

Let

$$g(x) = \frac{f(x) - f(p)}{x - p}.$$

Then by previous reasoning $g_n \rightrightarrows g$ on $E \setminus \{p\}$ and by the definition of derivative $g_n(x) \to f_n'(p),\ x \to p$. Consequently, by Theorem 1.7 there exists a limit $\lim\limits_{x \to p} g(x)$ and

$$\lim_{x \to p} g(x) = \lim_{n \to \infty} f_n'(p) = \varphi(p).$$

So, by definition of derivative, $f'(p)$ exists and is equal to $\varphi(p)$, which prove the second and the third assertions of the theorem. □

2 Functional series.

Definition 2.1. *Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions defined on the same set $f_n : E \to \mathbb{R}$ (or $f_n : E \to \mathbb{C}$). A functional series*

$$\sum_{k=1}^{\infty} u_k(x)$$

is a sequence of partial sums $S_n(x) = \sum\limits_{k=1}^{n} u_k(x) = u_1(x) + u_2(x) + \ldots + u_n(x)$.

Assume that for every $x \in E$ this series converges. This defines a ***pointwise sum***

$$S(x) = \sum_{k=1}^{\infty} u_k(x).$$

We say that a functional series ***converges uniformly*** *on E if a sequence of partial sums converges uniformly on E, that is $S_n \rightrightarrows S$ on E.*

Remark 1. Any functional sequence $\{S_n\}_{n=1}^{\infty}$ can be considered as a series $\sum\limits_{k=1}^{\infty} u_k(x)$ with terms

$$u_1(x) = S_1, \quad u_k(x) = S_k(x) - S_{k-1}(x),\ k \geq 2.$$

Remark 2. uniform convergence of a series is equivalent to uniform convergence of its remainder to zero, that is

$$S_n \rightrightarrows S \iff S - S_n = \sum_{k=n+1}^{\infty} u_k(x) \rightrightarrows 0.$$

Theorem 2.2 (Bolzano-Cauchy criteria for uniform convergence of a series.)**.** *Let $u_n : E \to \mathbb{R}$ (or $\mathbb{C}$). The uniform convergence of a series $\sum\limits_{k=1}^{n} u_n(x)$ on E is equivalent to the following condition*

$$\forall \varepsilon > 0\ \exists N \in \mathbb{N} :\ \forall n > N\ \forall p \in \mathbb{N}\ \forall x \in E\ \left| \sum_{k=n+1}^{n+p} u_k(x) \right| < \varepsilon. \quad (2)$$

Proof. The proof follows from Bolzano-Cauchy theorem for uniform convergence of functional sequences applied to sequence $\{S_n\}$ of partial sums and the equation

$$S_{n+p} - S_n = \sum_{k=n+1}^{n+p} u_k(x).$$

□

Remark 1. Condition (2) is equivalent to

$$\forall \varepsilon > 0\, \exists N \in \mathbb{N} : \ \forall n > N\ \forall p \in \mathbb{N}\ \sup_{x \in E} \left| \sum_{k=n+1}^{n+p} u_k(x) \right| < \varepsilon. \qquad (3)$$

Remark 2. We often use the Bolzano-Cauchy sequence to prove that a series is not uniformly convergent. The following assertions are equivalent

1. A series $\sum\limits_{k=1}^{\infty} u_k(x)$ is not uniformly convergent on E;
2. There exists $\varepsilon^* > 0$ and an increasing sequence $\{n_j\}$ of natural numbers, sequence $\{p_j\}$ of natural numbers and a sequence $\{x_j\}$ of points of E such that

$$\left| \sum_{\mathrm{j}=n_j+1}^{n_j+p} u_{\mathrm{j}}(x) \right| > \varepsilon^*.$$

Corollary 2.2.1 (Necessary condition for uniform convergence of a series.)**.** *If a series* $\sum\limits_{k=1}^{n} u_n(x)$ *converges uniformly on* E *then* $u_n \rightrightarrows 0$ *on* E.

Example 1. A series $\sum\limits_{k=0}^{\infty} x^k$ converges to a sum $\frac{1}{1-x}$ on $[0,1)$ pointwise but not uniformly since $x^n \not\rightrightarrows 0$ on $[0,1)$.

2.1 Properties of uniformly convergent series

Proofs of all properties of functional series in this subsection directly follow from corresponding properties of functional sequences.

Theorem 2.3. *Let* E *be a subset of* $\mathbb{R}$ *or* $\mathbb{C}$, p *be a limit point of* E, *and* $u_k : \mathbb{R} \to \mathbb{R}$ *or* $(\mathbb{C})$. *Assume that*

1. *a series* $\sum_{k=1}^{\infty} u_k$ *uniformly converges to* S *on* E;

2. *for every* $k \in \mathbb{N}$ *there exists a limit* $u_k = \lim_{x \to p} a_k(x)$.

Then the series converges to sum A, *the limit* $\lim_{x \to p} S(x)$ *exists and is equal to* A,

$$\lim_{x \to p} \sum_{k=1}^{\infty} u_k(x) = \sum_{k=1}^{\infty} \lim_{x \to p} u_k(x).$$

Proof. Let $S_n = \sum_{k=1}^{n} u_k$ be a sequence of partial sums. Then $\lim_{x \to p} S_n(x) = \sum_{k=1}^{n} \lim_{x \to p} u_k$ and by theorem on interchange of limits

$$\lim_{x \to p} \sum_{k=1}^{\infty} u_k(x) = \lim_{x \to p} \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \lim_{x \to p} S_n(x) = \sum_{k=1}^{\infty} \lim_{x \to p} u_k(x).$$

□

Theorem 2.4. *Let* E *be a subset of* $\mathbb{R}$ *or* $\mathbb{C}$, $p \in E$, *and* $u_k : \mathbb{R} \to \mathbb{R}$ *or* $(\mathbb{C})$. *Assume that*

1. *a series* $\sum_{k=1}^{\infty} u_k$ *uniformly converges to* S *on* E;

2. *all functions* u_k *are continuous in* p *(are continuous on* E*).*

Then the function S *is continuous in* p *(are continuous on* E*).*

Proof. By assumption, the sequence of functions $S_n(x) = \sum_{k=1}^{n} u_k$ for $x \in E$ converges uniformly to f. Since each function f_n is continuous, and the sum of continuous functions is continuous, it follows that S_n is continuous. The result now follows by Theorem on continuity of the uniform limit. $\square$

Theorem 2.5 (Dini's theorem for a series)**.** *Let* $u_k \in C([0,1])$, $u_k \geq 0$. *If a series* $\sum\limits_{k=1}^{\infty} u_k$ *converges to a continuous sum then it converges uniformly on* $[0,1]$.

Proof. Sequence of partial sums is increasing since terms of a series are nonnegative. Thus we can apply Dini's therem for sequences of functions to obtain the proof. $\square$

Theorem 2.6 (Term-by-Term Integration)**.** *Let* $u_k \in \mathcal{R}[a,b]$. *Assume that a series* $\sum\limits_{k=1}^{\infty} u_k$ *uniformly converges on* $[a,b]$. *Then the sum is Riemann integrable and*

$$\int\limits_a^b \sum_{k=1}^{\infty} u_k = \sum_{k=1}^{\infty} \int\limits_a^b u_k.$$

Theorem 2.7 (Term-by-Term Differentiation)**.** *Let* $u_k : [a,b] \to \mathbb{R}$ *be a differentiable functions. Assume that*

- *a series* $\sum\limits_{k=1}^{\infty} u_k'$ *uniformly converges on* $[a,b]$;
- $\exists c \in [a,b]$ *such that* $\sum\limits_{k=1}^{\infty} u_k(c)$ *converges.*

Then

1. *a series* $\sum\limits_{k=1}^{\infty} u_k$ *uniformly converges on* $[a, b]$;

2. *its sum is differentiable on* $[a, b]$;

3. $\left(\sum\limits_{k=1}^{\infty} u_k(x)\right)' = \sum\limits_{k=1}^{\infty} u_k'(x).$

2.2 Sufficient conditions for uniform convergence.

Theorem 2.8 (Weierstrass M-test)**.** *Let $\{u_k\}$ be a sequence of real or-complex valued functions on E and $\{M_k\}$ be a sequence of nonnegative numbers such that*

$$|u_k(x)| \leq M_k,\ x \in E,$$

and

$$\sum_{k=1}^{n} M_k < +\infty.$$

Then the series

$$\sum_{k=1}^{n} u_k(x)$$

converges uniformly on E.

Proof. Let $\varepsilon > 0$. Then, by Bolzano-Cauchy criteria for convergence of numerical series

$$\exists N \in \mathbb{N}:\ \forall n > N\ \forall p \in \mathbb{N}\ \left|\sum_{k=n+1}^{n+p} M_k\right| < \varepsilon.$$

Consequently, for every $n > N$ and $p \in \mathbb{N}$

$$\sup_{x\in E}\left|\sum_{k=n+1}^{n+p} u_k(x)\right| \leq \sup_{x\in E}\sum_{k=n+1}^{n+p} |u_k(x)| \leq \sum_{k=n+1}^{n+p} M_k < \varepsilon$$

and series $\sum\limits_{k=1}^{n} u_n(x)$ uniformly converges on E. $\square$

Example 1. Series $\sum\limits_{k=1}^{n} \frac{\sin nx}{n^p}$ and $\sum\limits_{k=1}^{n} \frac{\cos nx}{n^p}$ converge uniformly on $\mathbb{R}$ for $p > 1$ since

$$\frac{|\sin nx|}{n^p}, \ \frac{|\cos nx|}{n^p} \leq \frac{1}{n^p}$$

and a series $\sum\limits_{k=1}^{n} \frac{1}{n^p}$ converges if $p > 1$.

Example 2. Prove that

$$\int_0^{\pi} \left(\sum_{n=1}^{\infty} \frac{n \sin(nx)}{e^n} \right) = \frac{2e}{e^2 - 1}$$

Proof. For any $x \in \mathbb{R}$ it holds that

$$\left| \frac{n \sin(nx)}{e^n} \right| \leq \frac{n}{e^n}$$

A straightforward application of the Ratio test shows that $\sum_{n=1}^{\infty} \frac{n}{e^n}$ is a convergent series. Hence, by the M-Test, the given series converges uniformly on $A = \mathbb{R}$, and in particular on $[0, \pi]$. By theorem on term-by-term integration of a uniformly convergent series we see that

$$\begin{aligned}
\int_0^{\pi} \sum_{n=1}^{\infty} \frac{n \sin(nx)}{e^n} dx &= \sum_{n=1}^{\infty} \int_0^{\pi} \frac{n \sin(nx)}{e^n} dx \\
&= \sum_{n=1}^{\infty} - \left. \frac{\cos(nx)}{e^n} \right|_0^{\pi} = \sum_{n=0}^{\infty} \left[\left(\frac{1}{e} \right)^n - \left(\frac{-1}{e} \right)^n \right] \\
&= \left(\frac{1}{1 - 1/e} - 1 \right) - \left(\frac{1}{1 + 1/e} - 1 \right) = \frac{2e}{e^2 - 1}
\end{aligned}$$

$\square$

Remark. If the series satisfies conditions of Weierstrass M-test it converges not only uniformly but also absolutely ($\sum_{k=1}^{\infty} |u_k(x)|$ converges uniformly). At the same time there exist series that converge absolutely but not uniformly (for example, $\sum_{k=1}^{n} x^n$ on $(-1, 1)$) and series that converge uniformly but not absolutely (as an example you can consider any conditionally convergent numerical series as $\sum_{k=1}^{n} \frac{(-1)^n}{n}$.)

Theorem 2.9 (Dirichlet's tests for uniform convergence of a series.)**.** *Let $\{u_k\}$ be a sequence of real or complex valued functions on E, $\{v_k\}$ be a sequence of real-valued functions on E.. Assume that*

- *a sequence $U_n(x) = \sum_{k=1}^{n} u_k(x)$ is uniformly bounded, that is there exists a constant $M > 0$ such that*

$$\left| \sum_{k=1}^{n} u_k(x) \right| \leq M, \ \forall x \in E, \ n \in \mathbb{N};$$

- *For every $x \in E$ a sequence $\{v_n(x)\}$ is monotone;*

- *$v_n \rightrightarrows 0$ on E.*

Then a series $\sum_{k=1}^{\infty} u_k(x)v_k(x)$ converges uniformly on E.

Lemma 2.10 (Abel's lemma or summation by parts formula)**.** *Let* $\{a_k\}, \{b_k\}$

be two numerical sequences, $A_n = \sum_{k=1}^{n} a_k$. *Then*

$$\sum_{k=n+1}^{m} a_k b_k = A_m b_m - A_n b_{n+1} - \sum_{k=n+1}^{m-1} A_k(b_{k+1} - b_k) =$$
$$(A_m - A_n)b_{n+1} - \sum_{k=n+1}^{m} (A_k - A_m)(b_{k+1} - b_k). \quad (4)$$

Proof of the Lemma. The first identity follows from the following transformations

$$\sum_{k=n+1}^{m} a_k b_k = \sum_{k=n+1}^{m} (A_k - A_{k-1})b_k = \sum_{k=n+1}^{m} A_k b_k - \sum_{k=n+1}^{m} A_{k-1} b_k =$$
$$\sum_{k=n+1}^{m} A_k b_k - \sum_{k=n}^{m-1} A_k b_{k+1} = A_m b_m - A_n b_{n+1} - \sum_{k=n+1}^{m-1} A_k(b_{k+1} - b_k).$$

The second part can be verified analogously:

$$\sum_{k=n+1}^{m} a_k b_k = \sum_{k=n+1}^{m} ((A_k - A_m) - (A_{k-1} - A_m))b_k =$$
$$\sum_{k=n+1}^{m} (A_k - A_m)b_k - \sum_{k=n+1}^{m} (A_{k-1} - A_m)b_k =$$
$$\sum_{k=n+1}^{m} (A_k - A_m)b_k - \sum_{k=n}^{m-1}(A_k - A_m)b_{k+1} =$$
$$(A_m - A_n)b_{n+1} - \sum_{k=n+1}^{m-1} (A_k - A_m)(b_{k+1} - b_k).$$

$\square$

Proof of Theorem 2.9. Let $S_n = \sum_{k=1}^{n} u_k v_k$. Then

$$|S_n - S_m| \leq |U_m|\,|v_m| + |U_n|\,|v_{n+1}| + \sum_{k=n+1}^{m-1} |U_k|\,|v_{k+1} - v_k| \leq$$
$$M\left(|v_m| + |v_{n+1}| + \sum_{k=n+1}^{m-1} |v_{k+1} - v_k|\right). \quad (5)$$

Notice that a sequence $\{v_n(x)\}$ is monotonic for every $x \in E$ and the sign of the difference $v_{k+1}(x) - v_k(x)$ doesn't depend on k. Consequently,

$$\sum_{k=n+1}^{m-1} |v_{k+1} - v_k| = \left|\sum_{k=n+1}^{m-1} (v_{k+1}(x) - v_k(x))\right| =$$
$$|v_m(x) - v_{n+1}(x)| \leq 2\max|v_m(x)|\,, |v_n(x)|.$$

Applying this to estimate (5) we see that

$$|S_n - S_m| \leq 4M\max|v_m(x)|\,, |v_n(x)|$$

and since $v_n \rightrightarrows 0$ this implies that sequence S_n satisfies Bolzano-Cauchy criteria for uniform convergence. □

Corollary 2.10.1 (Leibniz's test for alternating series.)**.** *Let $\{v_n\}$ be a sequence of real-valued functions such that for every $x \in E$ a sequence $\{v_n(x)\}$ is monotone. An alternating series*

$$\sum_{k=1}^{\infty} (-1)^{k-1} v_k(x) \quad (6)$$

converges uniformly if and only if $v_n \rightrightarrows 0$ on E.

Proof. Let $U_n = \sum_{k=1}^{n}(-1)^{k-1}$. Then $U_n = 1$ if n is odd and $U_n = 0$ if n is even. So $|U_n| \le 1$ for every $n \in \mathbb{N}$. If $v_n \rightrightarrows 0$ on E and $\{v_n(x)\}$ is monotone for every fixed x then the series (6) satisfies all assumptions of Dirichlet's test and converges uniformly on E.

If $v_n \not\rightrightarrows 0$ on E then the necessary condition for uniform convergence is not satisfied and the series (6) diverges. □

Theorem 2.11 (Abel's test for uniform convergence of a series.)**.** *Let E be compact, $\{u_n\}$ be a sequence of real or complex valued functions on E, $\{v_n\}$ be a sequence of real-valued functions on E.*

Assume that

- *a series $\sum_{n=1}^{\infty} u_n$ converges uniformly on E;*
- *For every $x \in E$ a sequence $\{v_n(x)\}$ is monotone;*
- *a sequence $\{v_n(x)\}$ is uniformly bounded, that is there exists a constant $M > 0$ such that*

$$|v_n(x)| \le M, \ \forall x \in E, \ n \in \mathbb{N};$$

then a series $\sum_{n=1}^{\infty} u_n(x)v_n(x)$ converges uniformly on E.

Proof. Let $S_n = \sum_{k=1}^{n} u_k v_k$. Let $M > 0$ be such that $|v_n(x)| \le M$ for every $x \in E$ and $n \in \mathbb{N}$.

Fix $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for every $k, j > N$ and $x \in E$

$$|U_j(x) - U_k(x)| < \frac{\varepsilon}{3M}.$$

Applying second part of Abel's lemma on summation by parts 2.10 with $n = 0$ we see that for $m, n > N$

$$|S_m - S_n| \le |U_m - U_n|\,|v_{n+1}| + \sum_{k=n+1}^{m} |U_m - U_k|\,|v_{k+1} - v_k|$$

$$< \frac{\varepsilon}{3M}\left(|v_{n+1}| + \sum_{k=n+1}^{m} |v_{k+1} - v_k|\right) = \frac{\varepsilon}{3M}(M + |v_{m+1} - v_{n+1}|) \le \varepsilon.$$

□

2.3 Investigation of series $\sum_{k=1}^{\infty} b_k \sin kx$, $\sum_{k=1}^{\infty} b_k \cos kx$, $\sum_{k=1}^{\infty} b_k e^{ikx}$.

Recall that

$$e^{ix} = \cos x + i \sin x, \ \ \cos x = \frac{e^{inx} + e^{-inx}}{2}, \ \sin x = \frac{e^{inx} - e^{-inx}}{2i}.$$

Lemma 2.12. *Assume that $\frac{x}{2\pi} \notin \mathbb{Z}$. Then*

$$U_n(x) = \sum_{k=0}^{n} e^{ikx} \quad = \tfrac{1-e^{i(n+1)x}}{1-e^{ix}}, \tag{7}$$

$$V_n(x) = \sum_{k=1}^{n} \sin kx \ = \tfrac{\sin \frac{nx}{2} \sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}}, \tag{8}$$

$$W_n(x) = \sum_{k=0}^{n} \cos kx \ = \tfrac{\cos \frac{nx}{2} \sin \frac{(n+1)x}{2}}{\sin \frac{x}{2}}; \tag{9}$$

and
$$|U_n(x)|\,,\ |V_n(x)|\,,\ |W_n(x)| \le \frac{1}{|\sin(x/2)|} \tag{10}$$

Proof. The first formula (7) follows from the formula for partial sums of geometric series
$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q},$$
with $q = e^{ix}$.

To prove (8,9) notice that $V_n = \operatorname{Im} U_n$ and $W_n = \operatorname{Re} W_n$ and transform the formula for a sum in (7).
$$U_n = \frac{1-e^{i(n+1)x}}{1-e^{ix}} = e^{i\frac{n}{2}x}\frac{e^{i\frac{n+1}{2}} - e^{-i\frac{n+1}{2}x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} = e^{i\frac{n}{2}x}\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}.$$
Now the estimate (10) is obvious. □

Theorem 2.13. *Let $[a,b] \subset (2\pi m, 2\pi(m+1)]$ and b_k be a monotone numerical sequence that tends to 0. Then series $\sum_{k=1}^{\infty} b_k \sin kx$, $\sum_{k=0}^{\infty} b_k \cos kx$, $\sum_{k=0}^{\infty} b_k e^{ikx}$ converge uniformly on $[a,b]$.*

Proof. The result of this theorem follows from Dirichlet's test (Theorem 2.9). Indeed, the estimate (10) implies that
$$\min_{x\in[a,b]} \left|\sin\frac{x}{2}\right| = \min\left(\left|\sin\frac{a}{2}\right|, \left|\sin\frac{b}{2}\right|\right) = \delta > 0$$
and, in notations of previous lemma, $|U_n(x)|\,,\ |V_n(x)|\,,\ |W_n(x)| \le \delta^{-1}$. □

Theorem 2.14. *Let $p \in (0,1]$. Then series* $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$, $\sum_{k=1}^{\infty} \frac{\cos kx}{k^p}$, $\sum_{k=1}^{\infty} \frac{e^{ikx}}{k^p}$ *do not converge uniformly on* $(0, 2\pi)$.

Proof. We will prove this by contradiction of Bolzano-Cauchy criteria for uniform convergence of a series (2). Let $n \in \mathbb{N}$, $x_n = \frac{1}{2n}$. Then

$$\sin kx_n \geq \sin \frac{1}{2}, \ n \leq k \leq 2n$$

and, since $p \in (0,1]$,

$$\sum_{k=n+1}^{2n} \frac{\sin kx_n}{k^p} \geq n\frac{\sin \frac{1}{2}}{(2n)^p} \geq n^{1-p}2^{-p} \sin \frac{1}{2} \geq 2^{-p} \sin \frac{1}{2}.$$

Analogously,

$$\cos kx_n \geq \cos 1, \ n \leq k \leq 2n$$

and, since $p \in (0,1]$,

$$\sum_{k=n+1}^{2n} \frac{\cos kx_n}{k^p} \geq n\frac{\cos 1}{(2n)^p} \geq n^{1-p}2^{-p} \cos 1 \geq 2^{-p} \cos 1.$$

Consequently, series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$, $\sum_{k=1}^{\infty} \frac{\cos kx}{k^p}$ and $\sum_{k=1}^{\infty} \frac{e^{ikx}}{k^p}$, as combination of two previous, do not converge uniformly on $[0, 2\pi]$. □

2.4 Examples of problems.

In all following problem $u_k(x)$ will denote a term of a series.

Problem 1. Check the uniform convergence of

$$\sum_{k=1}^{\infty} \frac{x}{(k-1)x+1)(kx+1)},$$

on $(0, +\infty)$.

Solution. Let's find a partial sum:

$$S_n(x) = \sum_{k=1}^{n} \frac{x}{((k-1)x+1)(kx+1)} =$$

$$\sum_{k=1}^{n} \left(\frac{1}{(k-1)x+1} - \frac{1}{kx+1} \right) = 1 - \frac{1}{nx+1}.$$

Consequenly, $S(x) = \lim_{n\to\infty} S_n(x) = 1$ for every $x \in (0, +\infty)$. However,

$$\sup_{x\in(0,+\infty)} |S_n(x) - S(x)| = \sup_{x\in(0,+\infty)} \frac{1}{1+nx} = 1$$

and the convergence is not uniform.

Problem 2. Check the uniform convergence of

$$\sum_{k=1}^{\infty} \frac{kx}{1+k^5x^2},$$

on $\mathbb{R}$. **Solution.** Let's find supremum of absolute value of every term of this series

$$u_k(x) = \frac{k\,|x|}{1+k^5x^2}.$$

Since u_k is differentiable it is enough to check values at stationary points (i.e. points at which $u_k' = 0$) and limits at $\pm\infty$. First, $\lim_{x\to\infty} |u_k(x)| = 0$.

Now ($u_k(x)$ is smooth. consider $u_n'(x)$ to find supremum)

$$u_n'(x) = \frac{k(1+k^5x^2) - 2k^6x^2}{(1+k^5x^2)^2} = \frac{k(1-k^5x^2}{(1+k^5x^2)^2} = 0.$$

iff $x = k^{-5/2}$. And

$$\sup_{x\in\mathbb{R}} |u_k(x)| = \frac{1}{2k^{3/2}}.$$

or we can use $\frac{1+k^5x^2}{2} \leqslant \sqrt{1\cdot k^5x^2}$.

$\Rightarrow |u_k(x)| \leqslant \frac{1}{2k^{\frac{3}{2}}}$

So $|u_k(x)| \leq \frac{1}{2k^{3/2}}$ and since $\sum\limits_{k=1}^{\infty} \frac{1}{k^{3/2}}$ is convergent the initial series converges uniformly on $\mathbb{R}$.

Problem 3. Check the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}} \left(x^n + x^{-n}\right),$$

on $E = \left\{x \in \mathbb{R} : \frac{1}{2} \leqslant |x| \leqslant 2\right\}$.

Solution. First, consider an estimate

$$\sup_{\frac{1}{2}\leqslant|x|<2} \left(x^n + x^{-n}\right) = 2^n + 2^n < 2^{n+1}.$$

Since the series $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}} 2^{n+1}$ is convergent by D'Alambert's ratio test the initial series converges uniformly on E.

Problem 4. Check the uniform convergence of the series

$$\sum_{k=1}^{\infty} 2^k \sin\frac{1}{3^k x}$$

on $(0, +\infty)$.

Solution. First, $2^k \sin\frac{1}{3^kx} \sim \left(\frac{2}{3}\right)^k \frac{1}{x}$ as $n \to \infty$, and this implies pointwise convergence.

just investigate the supremum $\sup\limits_{x>0} 2^k \left|\sin\frac{1}{3^k x}\right| = 2^k$

We will prove that a series is not uniformly convergent by the following negation of Bolzano-Cauchy criteria:

$$\left|S_{2n}(3^{-n}) - S_n(3^{-n})\right| = \left|\sum_{k=n+1}^{2n} 2^k \sin\frac{1}{3^{k-n}}\right| \overset{<}{>} 2^{n+1}\sin\frac{1}{3} > 1, \quad n > 1.$$

Problem 5. Check the uniform convergence of the series

$$\sum_{k=1}^{\infty} \frac{\sin x \sin kx}{\sqrt{k+x}}$$

on $[0, +\infty)$.

Solution. We will consider this series by Dirichlet's test. Let

$$u_k(x) = \sin x \sin kx, \quad v_k(x) = \frac{1}{\sqrt{k+x}}.$$

Then partial sums $U_n = \sum\limits_{k=1}^{n} \sin x \sin kx$ are bounded since

$$\left|\sum_{k=1}^{n} \sin x \sin kx\right| = \left|\frac{\sin x}{\sin\frac{x}{2}}\right| \left|\sin\frac{kx}{2}\sin\frac{k+1}{2}x\right| = 2\left|\cos\frac{x}{2}\right| \left|\sin\frac{kx}{2}\sin\frac{k+1}{2}x\right| \leq 2.$$

The sequence $\{v_k(x)\}$ is monotonic in k for every fixed x. Also $v_k \rightrightarrows 0$ since $|v_k| \leq 1/\sqrt{k}$. Consequently, the series $\sum\limits_{k=1}^{\infty} u_k v_k$ converges uniformly on $[0, +\infty)$ by Dirichlet's test.

Problem 6. Prove that the sum of the series

$$S(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^3}$$

is continuously differentiable on $\mathbb{R}$.

Solution. By Weierstrass M-test it is easy to see that a series

$$\sum_{k=1}^{\infty}\left(\frac{\sin kx}{k^3}\right)' = \sum_{k=1}^{\infty}\frac{\sin kx}{k^2}$$

is uniformly convergent on $\mathbb{R}$. Consequently, by the theorem of term-by-term differentiation of a series and theorem on continuity of the uniformly convergent series $S(x)$ and

$$S'(x) = \sum_{k=1}^{\infty}\frac{\sin kx}{k^2}$$

are continuous on $\mathbb{R}$.

Problem 6. Find the set of convergence of a series and investigate the continuity of a sum

$$S(x) = \sum_{k=1}^{\infty}\left(x + \frac{1}{k}\right)^k.$$

Solution. By Cauchy radical test the series is convergent when

$$\lim_{k\to\infty}\sqrt[k]{|u_k(x)|} = \lim_{k\to\infty}\left|x + \frac{1}{k}\right| = |x| < 1.$$

Also if $|x| \geq 1$ then $|u_k(x)| \not\to 0$ and the series is divergent. So $S(x)$ exists on $(-1, 1)$.

Notice that the convergence on $(-1, 1)$ is not uniform since

$$\sup_{x\in(-1,1)}|u_k(x)| = \left(1 + \frac{1}{k}\right)^k \to e \neq 0.$$

To investigate continuity let $0 < r < 1$ and consider $x \in (-r, r)$. In this case

$$|u_k(x)| \leq u_k(r) = \left(r + \frac{1}{k}\right)^k$$

and the Weierstrass M-test implies the uniform convergence of $\sum\limits_{k=1}^{\infty} u_k$ on $(-r, r)$. Finally, by continuity of the sum of uniformly convergent series the sum $S(x)$ is continuous on $(-r, r)$ and, since $r \in (0, 1)$ is arbitrary, on $(-1, 1)$.

Remark. This example shows that for continuity of a sum $S(x) = \sum\limits_{k=1}^{\infty} u_k(x)$ on a set E it is enough to check that every point $p \in E$ has a neighborhood V_p such that the series converges absolutely on $E \cap V_p$.

Problem 5. Check the uniform convergence of the series

$$\sum_{k=1}^{\infty} \left(1 - \cos \sqrt[3]{\frac{x}{k^2}}\right)$$

on $E_1 = (0, 1)$, and $E_2 = (1, +\infty)$.

First, notice that $u_k(x) \sim \frac{x^{2/3}}{2k^{4/3}},\ k \to +\infty$, and this series is pointwise convergent on E_1 and E_2.

Consider, first the second set E_2. Then $u_k(k^2) = 1 - \cos 1$ and the sires doesn't satisfy the necessary condition for uniform convergence, $u_k \not\rightrightarrows 0$ on E_2.

Let $x \in E_2$. Then $|u_k(x)| \leq u_k(1)$ and, since $\sum\limits_{k=1}^{\infty} u_k(1)$ is convergent, the series $\sum\limits_{k=1}^{\infty} u_k$ satisfies conditions of Weierstrass M-test and is uniformly convergent on E.

3 Power series

3.1 Definition

Definition 3.1. *Let $\{c_n\}$ be a complex sequence, $w_0 \in \mathbb{C}$ complex number. A functional series*

$$\sum_{n=0}^{+\infty} c_n (w - w_0)^n, \ w \in \mathbb{C}, \tag{11}$$

is called a ***(complex) power series****.*

If $\{a_n\}$ is a real sequence, $x_0 \in \mathbb{R}$ then a functional series

$$\sum_{n=0}^{+\infty} a_n (x - x_0)^n, \ x \in \mathbb{R}, \tag{12}$$

is called a ***(real) power series****. Numbers c_n, a_n are coefficients of the power series (11).*

Remark.

- Partial sums $S_n(t)$ of the power series are polynomials of degree not greater than n.
- The change $z = w - w_0$ transforms the series (11) to the following form
$$\sum_{n=0}^{+\infty} c_n z^n \tag{13}$$
For compactness of reasoning we will further consider this form of the power series.

The definition of a power series naturally raises the following question:

– What is the set of convergence of a power series?

– What properties has a sum of a power series?

– What functions can be represented by a power series and how?

Lemma 3.2. *Assume that the power series (13) converges at z_0. Then it converges at every point z such that $|z| < |z_0|$.*

Proof. Assume that a series $\sum\limits_{n=0}^{+\infty} a_n z_0^n$ converges. Then terms of the series are uniformly bounded and there exists $M > 0$ such that

$$|a_n z_0^n| \leq M.$$

Let $|z| < |z_0|$ and $0 \leq q = \left|\frac{z}{z_0}\right| < 1$. Then

$$|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n \leq M q^n$$

and the series $\sum\limits_{n=0}^{+\infty} a_n z^n$ converges by simple comparison test. □

Remark. This lemma, in fact, tells us that the power series converges on some interval in real-variable case and on some disc (with some points on the boundary) in the complex-variable case.

Definition 3.3. *We say that $0 \leq R \leq \infty$ is a **radius of convergence** of the series (12) if it converges when $|z| < R$ and diverges for $|z| > R$.*

- *When $R = 0$ then the power series (13) converges only at $z = 0$, and is called* ***nowhere convergent****.*

- *When $R = +\infty$ we say that the series is* ***everywhere convergent****. In complex-variable case this means that the series is convergent on $\mathbb{C}$ in real-variable case the series converges on $\mathbb{R}$*

3.2 Formulas for radius of convergence.

By the definition of radius convergence we have

$$R = \sup\{|x| : \text{series (13) converges at } x\}.$$

Recall the definition of upper limit.

Definition 3.4.
- *A number $A \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is a partial limit of real sequence $\{x_n\}_{n=1}^{\infty}$ if there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to A$ as $k \to \infty$.*

- *The upper limit of sequence $\{x_n\}$ is defined as maximal partial limit*

$$\limsup\{x_n\} = \overline{\lim} x_n = \max\{A : A \textit{ is a partial limit}\}$$

- *The lower limit of sequence $\{x_n\}$ is defined as minimal partial limit*

$$\liminf\{x_n\} = \underline{\lim} x_n = \min\{A : A \textit{ is a partial limit}\}.$$

Lemma 3.5. *Let $\{x_n\}$ be a real sequence. Then*

$$\overline{\lim} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k;$$

$$\underline{\lim} x_n = \lim_{n\to\infty} \inf_{k\geq n} x_k.$$

Theorem 3.6 (Cauchy-Adamar formula.)**.** *Every power series has a radius convergence and*

$$\frac{1}{R} = \overline{\lim} \sqrt[n]{|c_n|}. \tag{14}$$

Here we assume $1/0 = +\infty$ *and* $1/+\infty = 0$.

Lemma 3.7. *Let* $\{x_n\}$, $\{y_n\}$ *be two real sequences. Assume that there exists a limit* $\lim x_n \in (0, \infty)$. *Then*

$$\overline{\lim} x_n y_n = \lim x_n \overline{\lim} y_n. \tag{15}$$

Proof of Lemma 3.7. Let $A = \lim x_n$, $B = \overline{\lim} y_n$, $C = \overline{\lim} x_n y_n$. We will prove that $C = AB$.

Let n_k be an increasing sequence such that $x_{n_k} y_{n_k} \to C$. Then $x_{n_k} \to A$ and $y_{n_k} = x_{n_k} y_{n_k} / x_{n_k} \to C/A$. Consequently, C/A is a partial limit of a sequence $\{y_n\}$ and $C/A \leq B$, that is $C \leq AB$.

To prove the inverse inequality consider an increasing sequence m_l such that $y_{m_l} \to B$. Then $x_{m_l} y_{m_l} \to AB$. This means that AB is a partial limit of a sequence $\{x_n y_n\}$ and $AB \leq C$.

□

Proof of Theorem 3.6. Let's prove that value R defined by formula (14) satisfies the definition of radius of convergence. Let $z \neq 0$ and apply the radical test (Cauchy's root test) for convergence of numerical series. Then

$$q = \overline{\lim} \sqrt[n]{|c_n|\, z^n} = \overline{\lim}\, |z| \sqrt[n]{|c_n|} = |z|\, \overline{\lim} \sqrt[n]{|c_n|} = |z|/R.$$

If $|z| < R$ then $q < 1$ and the series is convergent. If $|z| > R$ then $q > 1$ and the series is divergent. □

Lemma 3.8. *If there exists a finite or infinite limit* $\lim_{n\to+\infty} |c_n/c_{n+1}|$ *then it is equal to the radius of convergence.*

Proof. Let $R = \lim_{n\to\infty} |c_n/c_{n+1}|$. Let $z \neq 0$ and apply the d'Alambert's ratio test for convergence of the series $\sum_{k=0}^{\infty} c_n z^n$

$$D = \lim_{n\to\infty} \frac{c_{n+1}z^{n+1}}{c_n z^n} = |z|/R.$$

If $|z| < R$ then $D < 1$ and the series is convergent. If $|z| > R$ then $D > 1$ and the series is divergent. □

Definition 3.9. *Assume that* R *is radius of convergence of power series (11). An open disc*

$$B(w_0, R) = \{z \in \mathbb{C} : |w - w_0| < R\}$$

is called a ***disc of convergence*** *of series (11). Here we assume* $B(w_0, +\infty) = \mathbb{C}$ *and* $B(w_0, 0) = \emptyset$.

Remark. By definition a disc of convergence is open. By the radical test the power series converges absolutely on the disc of convergence. If $R \in (0, \infty)$ the set of convergence contains disc of convergence and may contain dome points of the boundary. The behavior of a series on the boundary $|w - w_0| = R$ can be various.

Definition 3.10. *Assume that* R *is radius of convergence of a real power series (12). An open interval* $(x_0 - R, x_0 + R)$ *is called an* ***interval of convergence*** *of series (12).*

Remark. If $R \in (0, \infty)$ the set of convergence of real power series is on of the intervals $(x_0 - R, x_0 + R)$, $[x_0 - R, x_0 + R)$, $(x_0 - R, x_0 + R]$, $[x_0 - R, x_0 + R]$.

Example 1. Consider a power series defined by the geometric series

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad |z| < 1.$$

If $|z| \geq 1$ the series is divergent. Consequently, the radius of convergence is equal to 1 and a set of convergence coincides with the disc of convergence.

Example 2. Radius of convergence of a series $\sum\limits_{k=1}^{\infty} \frac{z^k}{k^2}$ is equal to 1 since $\lim\limits_{k\to\infty} \sqrt[k]{1/k^2}$. If $|z| = 1$ then $\left|\frac{z^k}{k^2}\right| \leq \frac{1}{k^2}$ and a series is convergent uniformly and absolutely on the circle $|z| = 1$.

Example 3. Radius of convergence of a series $\sum\limits_{k=1}^{\infty} \frac{z^k}{k}$ is equal to 1. If $z = 1$ this series diverges as harmonic series. Let $|z| = 1$ and $z \neq 1$. Then

$$\left|\sum_{k=1}^{n} z^k\right| = \left|\frac{z - z^{n+1}}{1-z}\right| \leq \frac{1}{|1-z|}$$

and a sequence $1/k$ tends monotonically to 0. Consequently, by Dirichlet's test the series converges. So, the set of convergence of this series is closed disc without point 1.

Example 4. The radius of convergence of a series $\sum\limits_{k=0}^{\infty} \frac{z^k}{k!}$ is

$$R = \lim_{k\to\infty} \frac{1/k!}{1/(k+1)!} = \lim_{k\to\infty} (k+1) = +\infty.$$

Similarly the radius of convergence of a series $\sum\limits_{k=0}^{\infty} k! z^k$ is $R = 0$.

3.3 Properties of a sum of a power series.

Theorem 3.11 (Uniform convergence of a power series.). *Let $R \in (0, +\infty]$ be the radius of convergence of a power series (13). Then for every $r \in (0, R)$ this series converges uniformly on $\overline{B}(0, r)$.*

Proof. Let $|z| \leq r < R$. Then $\left|c_k z^k\right| \leq |c_k| r^k$. Since the power series converges absolutely on the disc of convergence we see that

$$\sum_{k=0}^{\infty} |c_k| r^k < +infty.$$

Then by Weierstrass M-test the series is convergent uniformly on the closed disc $\overline{B}(0, r)$. □

Corollary 3.11.1. *A sum of power series is continuous on the disc of convergence.*

Proof. Let

$$f(z) = \sum_{k=0}^{\infty} c_k z^k, \ |z| < R.$$

Let $z_1 \in \mathbb{C}$ be such that $|z| < R$ and let $r \in (|z|, R)$. Then by Theorem 3.11 the series is uniformly convergent in $\overline{B}(0, r)$ and since terms of power series are continuous the sum f is continuous in $\overline{B}(0, r)$ and, in particular, in z_1. Since the choice of z_1 is arbitrary the function f is continuous in the disc of convergence. □

Theorem 3.12 (Abel's theorem)**.** *Let $R \in (0, +\infty]$ be the radius of convergence of real power series. If the series*

$$S(R) = \sum_{n=0}^{+\infty} a_n R^n$$

converges then it converges uniformly on $[0, R]$ and

$$S(R) = \lim_{x \to R-} S(x).$$

Analogously, if the series

$$S(-R) = \sum_{n=0}^{+\infty} a_n (-R)^n$$

converges then it converges uniformly on $[-R, 0]$ and

$$S(-R) = \lim_{x \to -R+} S(x).$$

Proof. Consider terms of power series as following

$$a_k x^k = a_k R^k \left(\frac{x}{R}\right)^k.$$

The numerical series $\sum\limits_{k=0}^{\infty} a_k R^k$ converges (uniformly on $[0, R]$ since it's terms are constant). A sequence $\left\{\left(\frac{x}{R}\right)^k\right\}$ is uniformly bounded on $[0, R]$ and decreasing. Consequently, by Abel's test, the series $\sum\limits_{k=0}^{\infty} a_k x^k$ converges uniformly and it's sum is continuous on $[0, R]$. $\square$

Theorem 3.13 (On derivative and integral of a power series.)**.** *Let $R > 0$ be a radius of convergence of real power series*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Then

1. *a function f is smooth in the interval $(-R, R)$ and*

$$f^{(m)}(x) = \sum_{k=m}^{+\infty} \frac{k!}{(k-m)!} a_k x^{k-m}.$$

2. *For every $x \in (-R, R)$*

$$\int_0^x f = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}.$$

3. *If a series $\sum\limits_{k=0}^{\infty} \frac{a_k}{k+1} R^{k+1}$ converges then the integral (that can be improper)*

$$\int_0^R f = \sum_{k=0}^{\infty} \frac{a_k}{k+1} R^{k+1}.$$

First, let's introduce the following Lemma that follows from Lemma 3.7 and the limit $\lim \sqrt[n]{n} = 1$.

Lemma 3.14. *Radiuses of convergence of series*

$$\sum_{k=0}^{\infty} c_k (z - z_0)^k, \quad \sum_{k=1}^{\infty} k c_k (z - z_0)^k, \quad \sum_{k=0}^{\infty} \frac{c_k}{k+1} (z - z_0)^k$$

are equal.

Proof of Theorem 3.13. **1.** Since by previous lemma the radius of convergence doesn't change after differentiation it is enough to prove that the statement for $m = 1$. Let $|x| < R$ and $r \in (|x|, R)$. Then the power series $\sum\limits_{k=1}^{\infty} k a_k x^{k-1}$ converges uniformly on $[-r, r]$ and by theorem on differentiation of series $f'(x) = \sum\limits_{k=1}^{\infty} k a_k x^{k-1}$.

2. Analogously to previous we can apply the formula for the integral of the uniformly convergent series.

3. To prove this part we apply Abel's theorem to see that the sum $F(x)$ of a power series $\sum\limits_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$ exists and is continuous on $[0, R]$ and, consequently,

$$\int\limits_0^R f = \lim_{x \to R-} \int\limits_0^x f = \lim_{x \to R-} F(x) = F(R) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} R^{k+1}.$$

□

3.4 Taylor series. Analytic functions.

Theorem 3.15 (Uniqueness of a decomposition of a Taylor series into a power series)**.** *Let* $R \in (0, +\infty]$ *and*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-p)^k, \quad |x-p| < R.$$

Then coefficients a_k are defined uniquely by the formula

$$a_k = \frac{f^{(k)}(p)}{k!}.$$

Proof. By Theorem 3.13 for every $m \in \mathbb{N}_0$

$$f^{(m)}(x) = \sum_{k=m}^{\infty} \frac{k!}{(k-m)!} a_k (x-p)^{k-m}, \ |x-p| < R.$$

If $x = p$ then all terms with numbers $k \geq m+1$ vanish. Consequently, $f^{(m)}(p) = c_m m!$. □

Definition 3.16. *Assume that f has at point p derivatives of arbitrary order. The power series*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(p)}{k!} (x-p)^k \tag{16}$$

*is called **Taylor series** of function f at point p.*
*If $p = 0$ then the series (16) is sometimes called **McLaurin series**.*
*Numbers $a_k = \frac{f^{(k)}(p)}{k!}$ are called **Taylors coefficients**.*

Remark 1. Partial sums of Taylor series are Taylor polynomials $T_{n,p}(x)$ and

$$f(x) = T_{n,a}(x) + O(x-p)^{n+1}, \ x \to a.$$

Remark 2. Taylor series (16) may converge or not converge to a function f.

Further we will consider the question on expression of a function by power series. We know that if a function can be decomposed into a power series in neighborhood of a point a then it is infinitely differentiable in this neighborhood. Assume now that $f \in C^{(\infty)}(a-R, a+R)$. Then we can construct a Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(p)}{k!}(x-p)^k$$

for function f at p. What is the behavior of this Taylor series. If $x = p$ then this series converges to $f(p)$. For $x \neq p$ there are three cases that will be represented by the following examples.

Case 1. Series converges to $f(x)$. As example we can consider the equality

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}, \quad |x| < 1. \tag{17}$$

Case 2. Series diverges. As example we can consider the same series (17) as in the previous case but for $|x| \geq 1$.

Case 3. Series converges but not to $f(x)$. As example consider a function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then for every $k \in \mathbb{N} \cup \{0\}$

$$f^{(k)}(x) = \begin{cases} P_{3k}\left(1/x\right) e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases},$$

where P_{3k} is a polynomial of degree not greater than $3k$ and $f^{(k)}(0) = 0$.

This implies that Taylor series converges to 0 everywhere while

$$f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0, \quad x \neq 0.$$

Proof. **Induction.** For $k = 0$ the assertion is true by the definition of function f. Let's verify the induction step

$$f^{(k+1)}(x) = \left(P_{3k}(1/x)e^{-1/x^2}\right)' = \left(-\frac{P'_{3k}(1/x)}{x^2} + \frac{2P_{3k}(1/x)}{x^3}\right) e^{-1/x^2} = P_{3(k+1)}(1/x)e^{-1/x^2}.$$

$$f^{(k+1)}(0) = \lim_{h\to 0} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = \lim_{h\to 0} h^{-1} P_{3k}\left(1/h\right) e^{-1/h^2} = 0.$$

□

Definition 3.17. *We say that a function f is* ***analytic*** *at a if there exists a neighborhood of a point a in which this function can be decomposed as a power series, that is there exist $r > 0$ and numbers $a_1, a_2, \ldots$ such that*

$$f(x) = \sum_{k=0}^{\infty} a_n(x-p)^k,\ \ |x-p| < r,\ \ r > 0. \tag{18}$$

A function $f : (\alpha, \beta) \to \mathbb{R}$ is analytic (real-analytic) on (α, β) if it is analytic at every point of (α, β). A set of all real-analytic functions on (α, β) is denoted by $\mathcal{A}(\alpha, \beta)$.

The previous example shows us that

$$\mathcal{A}(\alpha, \beta) \subsetneq C^{(\infty)}(\alpha, \beta)$$

and in the following theorem we will consider a simple sufficient condition for analiticity of smooth function.

Theorem 3.18 (Sufficient condition for analyticity). *Let $f \in C^\infty(p-r, p+r)$ be such that there exists $M > 0$ such that*

$$\left|f^{(k)}(x)\right| \le M, \quad \forall k \in \mathbb{N},\ x \in (p-r, p+r).$$

Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(p)}{k!}(x-p)^k,\ |x-p| < r.$$

Proof. Let $n \in \mathbb{N}$, $x \in (p-r, p+r)$. Then by Taylor-Lagrange's theorem there exists a point between x and p such that

$$f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(p)}{k!}(x-p)^k + \frac{f^{(n+1)}(c)}{k!}(x-p)^{n+1},$$

where

$$\left|\frac{f^{(n+1)}(c)}{k!}(x-a)^{n+1}\right| \le M\frac{r^{n+1}}{(n+1)!} \to 0,\ n \to +\infty.$$

Consequently,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(p)}{k!}(x-p)^k, \quad |x-p| < r.$$

□

Example 1. Functions $\cos x$, $\sin x$, e^x, $\cosh x$, $\sinh x$ are analytic on $\mathbb{R}$ since derivatives of these functions are uniformly bounded on every bounded segment.

Example 2. Taylor series for logarithm. Consider a power series

$$f(x)=\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n}x^n \tag{19}$$

and Leibniz's series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}. \tag{20}$$

Then the radius of convergence of a series (19) is equal

$$R=\lim_{n\to\infty}\left|\frac{n}{n+1}\right|=1.$$

We want to find the sum $f(x)$. By Theorem 3.13 on derivative and integral of power series

$$f'(x)=\sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}=\frac{1}{1+x},\quad |x|<1.$$

Finally

$$f(x)=f(0)+\int_0^x\frac{dt}{1+t}=\ln(1+x),\quad |x|<1.$$

A numerical series (20) is convergent by Leibniz test and by Abel's theorem

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\lim_{x\to 1-}S(x)=\ln(2).$$

Theorem 3.19 (Example 4. Binomial series)**.** *Let* $\alpha\in\mathbb{R}$*. Then*

$$(1+x)^\alpha=\sum_{k=0}^{\infty}\binom{\alpha}{k}x^k,\quad x\in(-1,1), \tag{21}$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!},\ k \in \mathbb{N}, \quad \binom{\alpha}{0} = 0,$$

are generalized binomial coefficients.

Proof. Let's find the radius of convergence of the series (21):

$$R = \lim_{n\to\infty} \frac{\binom{\alpha}{n}}{\binom{\alpha}{n+1}} = \lim \frac{n+1}{n+\alpha} = 1.$$

Let $S(x) = \sum\limits_{k=0}^{\infty} \binom{\alpha}{k} x^k$. Then

$$S'(x) = \sum_{k=1}^{\infty} k\binom{\alpha}{k} x^{k-1} = \sum_{k=0}^{\infty}(k+1)\binom{\alpha}{k+1} x^k =$$

$$\sum_{k=0}^{\infty}(\alpha-k)\binom{\alpha}{k} x^k,\ |x|<1,$$

and

$$xS'(x) = \sum_{k=0}^{\infty} k\binom{\alpha}{k} x^k, \quad (1+x)S'(x) = \sum_{k=0}^{\infty} \alpha\binom{\alpha}{k} x^k = \alpha S(x).$$

Consequently

$$\frac{S'(x)}{S(x)} = \frac{1}{1+x} \quad \text{and} \quad (\ln S(x))' = \frac{\alpha}{1+x}.$$

Finally,

$$\ln S(x) = S(0)\alpha\ln(1+x) = \alpha\ln(1+x) = \ln(1+x)^\alpha \quad \text{and} \quad S(x) = (1+x)^\alpha.$$

□

Example 5. Taylor series for arcsin . Consider binomial series with $\alpha = -1/2$. Then

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^{2n} \tag{22}$$

since

$$\binom{-1/2}{n} = \frac{-\frac{1}{2}\left(-\frac{1}{2}-1\right)\dots\left(-\frac{1}{2}-n+1\right)}{n!} =$$
$$(-1)^n \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n n!} = (-1)^n \frac{(2n-1)!!}{(2n)!!} = (-1)^n \frac{(2n-1)!!}{(2n)!}.$$

Consequently,

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!(2n+1)} x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!} x^{2n+1}, \quad |x| < 1. \tag{23}$$

Exercise. Check that the series (23) converges for $x = \pm 1$.

3.5 Applications of power series.

3.5.1 Stirling's formula.

Lemma 3.20 (Wallis's formula)**.**

$$\pi = \lim_{n\to\infty} \frac{1}{n} \left(\frac{(2n)!!}{(2n-1)!!} \right)^2$$

Lemma 3.21.

$$e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}. \tag{24}$$

Proof. Consider power series for logarithm,

$$\ln(1+x) = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} x^k, \quad x \in (-1,1).$$

Then

$$\ln(1-x) = -\sum_{k=1}^{+\infty} \frac{x^k}{k}, \quad x \in (-1,1)$$

and

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}(\ln(1+x) - \ln(1-x)) = \sum_{k=1}^{+\infty} \frac{x^{2k+1}}{2k+1}, \quad x \in (-1,1).$$

Let $x = \frac{1}{2n+1} \in (0,1)$. Then $\frac{1+x}{1-x} = 1 + \frac{1}{n}$ and

$$\frac{1}{2}\ln\left(1+\frac{1}{n}\right) = \sum_{k=0}^{+\infty} \frac{1}{(2k+1)(2n+1)^{2k+1}}.$$

Multiplying by $2n+1$ we see that

$$\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) = 1 + \sum_{k=1}^{+\infty} \frac{1}{(2k+1)(2n+1)^{2k}}.$$

Estimating $\frac{1}{2k+1} < \frac{1}{3}$, $k \geq 2$, we see that

$$1 < \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) < 1 + \frac{1}{3}\sum_{k=1}^{+\infty} \frac{1}{(2n+1)^{2k}} =$$

$$1 + \frac{1}{3}\frac{1}{(2n+1)^2}\frac{1}{1-\frac{1}{(2n+1)^2}} = 1 + \frac{1}{12n(n+1).}$$

Applying exponent to this estimate we prove the lemma. $\square$

Theorem 3.22 (Stirling's formula.)**.** *Let* $n \in \mathbb{N}$*. Then there exists* $\theta_n \in (0,1)$ *such that*

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}}.$$

Proof. Let $a_n = \frac{n!e^n}{n^{n+1/2}}$. Then by the left hand side estimate from (24)

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)^{n+3/2}}{(n+1)en^{n+1/2}} = e^{-1}\left(1+\frac{1}{n}\right)^{n+1/2} > 1$$

and a_n is strictly decreasing, positive, and, since, convergent. Let $a = \lim a_n$.

If $b_n = a_n e^{-1/12n}$ then by right hand side estimate from (24)

$$\frac{b_n}{b_{n+1}} = e^{-1-\frac{1}{12n(n+1)}}\left(1+\frac{1}{n}\right)^{n+1/2} < 1.$$

So b_n is decreasing, $b_n \le a_n$ and $\lim b_n = a$.

Consequently,

$$a_n e^{-1/12n} \le a \le a_n$$

and $a_n = ae^{\frac{\theta_n}{12n}}$, where $\theta_n \in (0,1)$. Now it is enough to find a.

Now $n! \sim a\sqrt{n}\left(\frac{n}{2}\right)^n$ and, by Wallis formula,

$$\pi = \frac{1}{n}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 = \frac{1}{n}\left(\frac{((2n)!!)^2}{(2n)!}\right)^2 = \\ \frac{1}{n}\left(\frac{2^{2n}(n!)^2}{(2n)!}\right)^2 \sim \frac{1}{n}\left(\frac{a\sqrt{n}}{2}\right)^2 = \frac{a^2}{2},$$

since $(2n-1)!! = (2n)!/(2n)!!$ and $(2n)!! = 2^n n!$. $\square$

3.5.2 Sum of inverse squares.

Recall that formula (23)

$$\arcsin x = x + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^n n!(2n+1)} x^{2n+1} = x + \sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!} x^{2n+1}$$

and let $x = \sin(t)$ in this formula. Consequently, by Theorem 3.13

$$\frac{\pi^2}{8} = \int\limits_0^{\pi/2} t dt = \int\limits_0^{\pi/2} \arcsin(\sin t) dt = I_0 + \sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!} I_{2n+1}, \quad (25)$$

where

$$I_{2n+1} = \int\limits_0^{\pi/2} \sin^{2n+1}(t) dt = \frac{(2n)!!}{(2n+1)!!} = \frac{2^n n!}{(2n+1)!!}.$$

Hence identity (25) implies

$$\frac{\pi^2}{8} = 1 + \sum_{n=1}^{\infty} \frac{((2n-1)!!)^2}{(2n+1)!} \frac{(2n)!!}{(2n+1)!!} =$$

$$1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!(2n)!!}{(2n+1)!} \frac{(2n-1)!!}{(2n+1)!!} =$$

$$1 + \sum_{n=1}^{\infty} \frac{(2n)!}{(2n+1)!} \frac{1}{2n+1} =$$

$$1 + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \quad (26)$$

Also this formula allows us to calculate the sum of inverse squares. So

$$S = \sum_{n=0}^{\infty} \frac{1}{n^2} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4}S,$$

and

$$S = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}. \tag{27}$$

3.6 Examples of problems on power series.

Types of problems.

1. Calculation of radius of convergence of power series.
2. Summation of power series (direct calculation of the sum, application of theorems on derivative and integral of a power series, application of Taylor's series).
3. Summation of numerical series.
4. Calculation of integrals by power series.
5. Solution of differential equations by power series.

3.6.1 Calculation of the radius of convergence.

1. $\sum\limits_{n=0}^{\infty} \left(\frac{n+1}{2n+3}\right)^n x^n$

$$R^{-1} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}, \; R = 2.$$

2. $\sum_{n=0}^{\infty} \left(\frac{n+1}{n+5}\right)^{n^2} x^n$

$$R^{-1} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n+1}{n+5}\right)^n = \lim_{n\to\infty} \left(1 - \frac{4}{n+5}\right)^n = e^{-4},$$

$$R = e^4.$$

3. $\sum_{n=0}^{\infty} n! x^{n!}$

$a_k = \begin{cases} 0, & k \neq n! \\ k, & k = n! \end{cases}$

$$R^{-1} = \lim_{n\to\infty} \sqrt[n!]{n!} = 1.$$

$\overline{\lim_{n\to\infty}} = \max\left(0, \lim_{n\to\infty} \sqrt[n!]{n!}\right).$

4. $\sum_{n=0}^{\infty} \ln\left(\cos \frac{1}{3^n}\right) z^n$

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\ln\left(\cos\frac{1}{3^n}\right)}{\ln\left(\cos\frac{1}{3^{n+1}}\right)} =$$

$$\lim_{n\to\infty} \frac{1-\cos\frac{1}{3^n}}{1-\cos\frac{1}{3^{n+1}}} = \lim_{n\to\infty} \frac{\frac{1}{3^{2n}}}{\frac{1}{3^{2n+2}}} = 9.$$

3.6.2 Calculation of Taylor series.

Example 1. Find a Taylor series of a function

$$f(x) = \ln(4 + 3x - x^2)$$

at $p = 2$ and find radius of analyticity (radius of convergence of Taylor series at p to function f).

Solution.

$$f(x) = \ln((4-x)(1+x)) = \ln(4-x) + \ln(1+x) = [t = x-2] =$$
$$\ln(2-t) + \ln(3+t) = \ln 6 + \ln(1-t/2) + \ln(1+t/3) =$$
$$\ln(6) - \sum_{n=1}^{\infty} \frac{t^n}{n2^n} + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n3^n} = \ln(6) + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{3^n} - \frac{1}{2^n} \right) \frac{(x-2)^n}{n}.$$

The radius of analyticity R is equal to the minimum of the radii of convergence of these series. So $R = 2$.

Example 2.

Solution. Find Taylor series at 0 of function

$$f(x) = \operatorname{arctg}(x + \sqrt{1+x^2}).$$

$$f'(x) = \frac{1 + \frac{x}{1+x^2}}{1 + (x + \sqrt{1+x^2})^2} = \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}(2 + 2x^2 + 2x\sqrt{1+x^2})} =$$
$$\frac{x + \sqrt{1+x^2}}{2(1+x^2)(x + \sqrt{1+x^2})} = \frac{1}{2(1+x^2)} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n x^{2n}, \ x \in (-1, 1).$$

Hence,

$$f(x) = f(0) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \frac{\pi}{4} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}.$$

Example 3. Find Taylor series at 0 of function $f(x) = x \ln(x + \sqrt{1+x^2}) - \sqrt{1+x^2}$.

Solution.

$$f'(x) = \ln(x + \sqrt{1 + x^2}), \;\; f'(0) = 0;$$
$$f''(x) = \frac{1}{\sqrt{1+x^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} x^{2n}$$

Hence,

$$f(x) = f(0) + f'(0)x + \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!} \frac{x^{2n+2}}{(2n+1)(2n+2)} =$$
$$-1 + \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n+2)!!} \frac{x^{2n+2}}{(2n+1)}.$$

3.6.3 Summation of power series.

Example 1. Find a sum
$$\sum_{n=0}^{\infty} \frac{3n+1}{n!} x^{3n}.$$

Solution.

$$\sum_{n=0}^{\infty} \frac{3n+1}{n!} x^{3n} = \sum_{n=1}^{\infty} \frac{3}{(n-1)!} x^{3n} + \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} =$$
$$\sum_{n=0}^{\infty} \frac{3}{n!} x^{3(n+1)} + e^{x^3} = 3x^3 e^{x^3} + e^{x^3}, \;\; x \in \mathbb{R}.$$

Example 2. Find a sum
$$\sum_{n=0}^{\infty} \frac{1+n^2}{2^n n!} x^n.$$

Solution.

$$\sum_{n=0}^{\infty}\frac{1+n^2}{2^n n!}x^n=\sum_{n=0}^{\infty}\frac{1+n^2}{n!}\left(\frac{x}{2}\right)^n=e^{x/2}+\sum_{n=1}^{\infty}\frac{n}{(n-1)!}\left(\frac{x}{2}\right)^n=$$

$$e^{x/2}+\sum_{n=1}^{\infty}\frac{n-1+1}{(n-1)!}\left(\frac{x}{2}\right)^n=e^{x/2}+\sum_{n=2}^{\infty}\frac{1}{(n-2)!}\left(\frac{x}{2}\right)^n+\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left(\frac{x}{2}\right)^n=$$

$$e^{x/2}+\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2}\right)^{n+2}+\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2}\right)^{n+1}=$$

$$e^{x/2}+\left(\frac{x}{2}\right)^2e^{x/2}+\frac{x}{2}e^{x/2}=\left(1+\frac{x}{2}+\left(\frac{x}{2}\right)^2\right)e^{x/2}.$$

Example 3. Find a sum

$$f(x)=\sum_{n=1}^{\infty}\frac{x^n}{n(n+1)}.$$

Solution.

$$f(x)=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)x^n=\sum_{n=1}^{\infty}\frac{x^n}{n}-\frac{1}{x}\sum_{n=1}^{\infty}\frac{x^{n+1}}{n+1}=$$

$$=-\ln(1-x)-\frac{1}{x}\left(-x+\sum_{n=1}^{\infty}\frac{x^n}{n}\right)=-\ln(1-x)+\frac{x+\ln(1-x)}{x}=$$

$$1+(1-x)\frac{\ln(1-x)}{x},\ \ |x|<1.$$

Moreover considering the limit $x\to 1-$ and applying Abel's theorem we see that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1.$$

Analogously with $x \to -1+$ we get

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} = 1 - 2\ln(2).$$

3.6.4 Summation of numerical series.

Example 1. Find a sum

$$S = \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)}.$$

Solution. Consider a power series $f(x) = \sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(n+1)(2n+1)}$, that is convergent on $[-1, 1]$. Then

$$f'(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{n+1} = \frac{1}{x^2} \sum_{n=0}^{\infty} \frac{x^{2(n+1)}}{n+1} = -\frac{\ln(1-x^2)}{x^2}.$$

$$f(x) = f(0) - \int_0^x \frac{\ln(1-t^2)}{t^2} dt = \ln(1+x) + \frac{x-1}{x} \ln(1-x) + \frac{\ln(1+x)}{x}.$$

Applying Abel's theorem we obtain

$$S = \lim_{x \to 1-} f(x) = 2\ln 2.$$

Example 2. Find a sum

$$S = \sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n+1)!}.$$

Solution. Consider a power series $f(x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} n x^{2n-1}$ that is convergent on $\mathbb{R}$. Then

$$\int f(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n} = \frac{1}{2x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} - \frac{1}{2} = \frac{\sin x}{2x} - \frac{1}{2}.$$

$$f(x) = \left(\frac{\sin x}{2x} - \frac{1}{2} \right)' = \frac{x \cos x - \sin x}{2x^2}.$$

$$S = f(1) = \frac{\cos 1 - \sin 1}{2}.$$

Definition 3.23. *The* ***ordinary generating function*** *of a sequence a_n is a function*

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n.$$

When the term generating function is used without qualification, it is usually taken to mean an ordinary generating function.

3.6.5 Calculation of an integral.

Example. Calculate the integral

$$I = \int\limits_0^1 \frac{\ln(1+x)}{x} dx.$$

Solution. Consider expansion

$$\frac{\ln(1+x)}{x} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^{n-1}.$$

Then by Theorem 3.13

$$\int_0^1 \frac{\ln(1+x)}{x}dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

The sum of this numerical series is calculated below by application of formulas (26) and (27) for sums of inverse squares

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}.$$

3.6.6 Solution of differential equation.

Example. Assume that analytic function $y(x)$ satisfies differential equation and initial condition

$$y'' - xy = 0, \quad y(0) = 1,\ y'(0) = 0.$$

Find it's Taylor series at 0.
Solution. Let

$$y = \sum_{k=0}^{\infty} a_k z^k.$$

Then

$$y'' = \sum n(n-1)a_n x^{n-2} = 2a_2 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k,$$

$$xy = \sum_{k=1}^{\infty} a_{k-1}x^k,$$

and the equation will have the following form

$$2a_2 + \sum_{k=1}^{\infty}(k+2)(k+1)a_{k+2}x^k = \sum_{k=1}^{\infty} a_{k-1}x^k.$$

Consequently,

$$a_2 = 0, \quad (k+1)(k+2)a_{k+2} = a_{k-1},\ k \in \mathbb{N}.$$

Since $a_2 = 0$ we see that $a_{3k-1} = 0,\ k \in \mathbb{N}$. Also from this formula we see that

$$a_{3k} = \frac{a_0}{(2\cdot 3)(5\cdot 6)\ldots((3k-1)\cdot 3k)}, \quad a_{3k+1} = \frac{a_1}{(3\cdot 4)(6\cdot 7)\ldots(3k\cdot(3k+1))};$$

Notice that $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$. Finally,

$$y = 1 + \frac{x^3}{2\cdot 3} + \frac{x^6}{(2\cdot 3)(5\cdot 6)} + \ldots + \frac{x^{3k}}{(2\cdot 3)(5\cdot 6)\ldots((3k-1)\cdot 3k)} + \ldots.$$