Equations in Analytic Geometry

1 Equation and locus

Locus or **graph** of an equation in two variables is the curve or straight line containing all the points, and only the points, whose coordinates satisfy the equation. Before plotting the graph of an equation, it is most often very help¬ ful to determine from the form of the equation certain properties of the curve. Such properties are: *intercepts*, *symmetry*, *extent*.

Intercepts are the directed (positive or negative) distances from the origin to the points where the curve intersects the coordinate axes.

To determine intercepts we are letting firs or second coordinates to zero

Example. Observe equation $y^2 + 2x = 16$. Letting $y = 0$ yields intercept in point $(8, 0)$. Letting $x = 0$ yields intercepts in points $(0, +4)$, and $(0, -4)$. x-intercept is 8, and y-intercepts are $+4$ and -4

Two points are **symmetric with respect to a line** if that line is the perpendicular bisector of the line connecting the two points. Two points are **symmetric about a point** if that point is the midpoint of the line connecting the two given points.

- If an equation remains unchanged when x is replaced by $-x$, the graph is symmetric with respect to the y-axis. Example: $x^2 - 6y + 12 = 0$, or $x = \pm\sqrt{6y - 12}$.
- If an equation remains unchanged when y is replaced by $-y$, the graph is symmetric with respect to the x-axis. Example: $y^2 - 4x - 7 = 0$, or $y = \pm\sqrt{4x + 7}$.
- If an equation remains unchanged when x is replaced by $-x$ and y is replaced by $-y$, the graph is symmetric with respect to the origin. Example: $x^3 + x + y^3 = 0$.

If certain values of one variable cause the other variable to become *imaginary*, **such values must be excluded** (extent).

Example: $x = \pm\sqrt{6y - 12}$ for $y < 2$ value under the root is negative, therefore the curve lies to the right from line $x = 2$

2 Forms of equations

If the radius vector of a point enters an equa- tion as a whole without subdividing it into separate coordinates, such an equation is called a **vectorial equation**.

Example: $\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{0}$

If the radius vector of a point enters an equation through its coordinates, such an equation is called a **coordinate equation**

Example: $x^2 + y^2 + z^2 = 1$

One or two degrees of freedom can be implemented in an equation explicitly when the radius of a point is given as a function of one or two variables, which are called parameters. In this case the equation is called **parametric**.

Example: $\begin{cases} x = at; \\ y = bt; \\ z = L \end{cases}$

Non-parametric equations behave as obstructions decreasing the number of degrees of freedom from the initial three to one or two.

3 Example. Equation of circle on plane. Equation of sphere in space

Circle means figure shaped of points laying in an equal distance from the center of it.

Thus, if center of circle is point $C(x_0, y_0)$, equal distance R from eat means

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

In parametric form:

$$\begin{cases} x = x_0 + R\cos\varphi \\ y = y_0 + R\sin\varphi \end{cases}$$

Vectorial equation:

$$(\boldsymbol{r} - \boldsymbol{c}) \cdot (\boldsymbol{r} - \boldsymbol{c}) = R^2$$

Here $\boldsymbol{c}$ is radius vector for C

For the sphere as a locus of point laying in equal distance R form center $C(x_0, y_0, z_0)$ we have

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

In parametric form:

$$\begin{cases} x = x_0 + R\sin\theta\cos\varphi \\ y = y_0 + R\sin\theta\sin\varphi \\ z = z_0 + R\cos\theta \end{cases}$$

Vectorial equation matches plane case of circle

4 Degrees of freedom

Coordinates of a single fixed point P in the space $\mathbb{E}$ are three *fixed numbers.* If these coordinates are changing, we get a moving point that runs over some set within the space $\mathbb{E}$.

The case of moving along some line (not necessary straight one, we call it **curve**) differs from the case of a **surface** by its dimension or, in other words, by the number of **degrees of freedom**.

Point in space has *three degrees of freedom.*

Each surface is two-dimensional – a point on a surface has *two degrees of freedom.*

Each curve (including straight line) is one-dimensional — a point on a line has *one degree of freedom.*

5 Correspondence between equation and locus

Any correct valid equation explicitly define one and only one locus.

But any locus may be expressed with a variety of equal equations. One section upper we demonstrated as example various equations expressing circle and sphere.

And if any pair of equation express exactly the same locus, that equations are equal.

Let us refer an example.

Suppose functions $f_1(x, y)$, and $f_2(x, y)$ have a domain of $|x| \leqslant 1$, $|y| \leqslant 1$. Form of that function is

$$f_1(x, y) = x^2 + y^2 - 1,$$
$$f_2(x, y) = \begin{cases} x - \sqrt{1 - y^2}, & \text{if } x \geqslant 0 \\ x + \sqrt{1 - y^2}, & \text{if } x < 0 \end{cases}$$

Both equations $f_1(x, y) = 0$, and $f_2(x, y) = 0$ express circle with radius 1 and center in the origin.

Equation of the locus may be transformed from one equal form to another equal form

Problem 1

There is ellipse expressed with its equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Express it with polar coordinates

Solution

$$\begin{cases} x = a \cos \varphi \\ y = a \sin \varphi \end{cases}$$

$\varphi \in [0, 2\pi]$

Problem 2
Express a branch of hyperbola for positive x in parametric form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad x \geqslant 0$$

Solution
Hyperbola has parametric expression with *hyperbolic functions*:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\begin{cases} x = a\cosh t \\ y = b\sinh t \end{cases}$$

$t \in (-\infty, +\infty)$

Problem 3
Find equal parametric equations for curve $x^4 - 2ax^3 + a^2y^2 = 0$

Solution
We divide equation by x^2a^2 and combine squares to the left:

$$\frac{x^2}{a^2} + \frac{y^2}{x^2} = 2\frac{x}{a}$$

Let $\frac{x}{a} = r\cos\varphi$, $\frac{y}{x} = r\sin\varphi$.
Sum of the squares in the left successfully sterilizes sines and cosines:

$$r^2 = 2r\cos\varphi$$
$$r = 2cos\varphi$$

Thus, parametric expression for this curve is

$$x = 2a\cos^2\varphi$$
$$y = 4a\cos^3\varphi\sin\varphi$$

Letting $psi = 2\varphi$ yields more simple form:

$$x = a(1 + \cos\psi)$$
$$y = a(\sin\psi + \frac{\sin 2\psi}{2})$$

$\psi \in [0, 2\pi]$

6 Plotting the locus. Examples

Problem 1

Discuss and plot the locus of the ellipse $9x^2 + 16y^2 = 144$

Solution

Intercepts. When $y = 0$ $x = \pm 4$. When $x = 0$ $y = \pm 3$. Hence, the x-intercepts are ± 4, and the y-intercepts are ± 3.

Symmetry. Since the equation contains only even powers of x and y, the curve is symmetric about both axes and therefore about the origin. Hence, it is sufficient to plot only that part of the curve which lies in the first quadrant and determine the rest of the curve by symmetry.

Extents. Solving explicit expressions for x and y yields

$$y = \pm\frac{3}{4}\sqrt{16 - x^2} \quad x = \pm\frac{4}{3}\sqrt{9 - y^2}$$

Thus, $|x|$ is not greater than 4, and $|y| \leqslant 3$

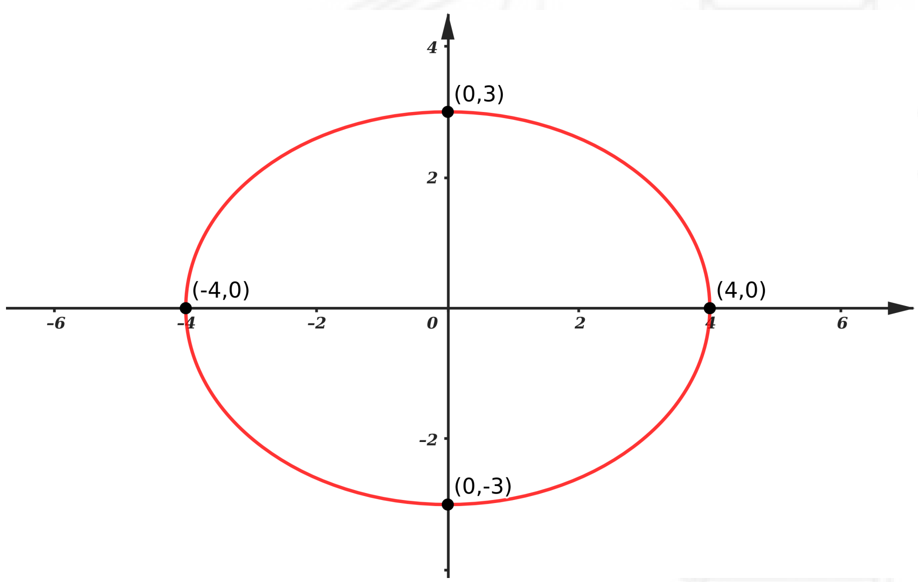

Figure 1: Plot of ellipse

x	0	± 1	± 2	± 3	± 3.5	± 4
y	± 3	± 2.9	± 2.6	± 2.0	± 1.5	0

To plot the locus we calculate several values of y for $0 \leqslant x \leqslant 4$, and propagate values to three other quadrants with symmetry

Problem 2

Discuss and plot the graph of the parabola $y^2 - 2y - 4x + 9 = 0$.

Solution

As first step, we explicitly express y with x or x with y:

$$y = 1 \pm 2\sqrt{x-2}$$
$$x = \frac{y^2 - 2y + 9}{4}$$

Intercepts. When $y = 0$, $x = \dfrac{9}{4}$.

When $x = 0$, $y = 1 \pm 2i\sqrt{2}$, and y is complex.

Hence, x-intercept is $\dfrac{9}{4}$ and *there is no y-intercept.*

Symmetry. The curve is not symmetric about either of the coordinate axes nor about the origin.

The only symmetry is about line $y = 1$, as each value of x gives value being as much greater than 1 as the other is less than 1.

Extents.

From expression for y: $x \geqslant 2$.

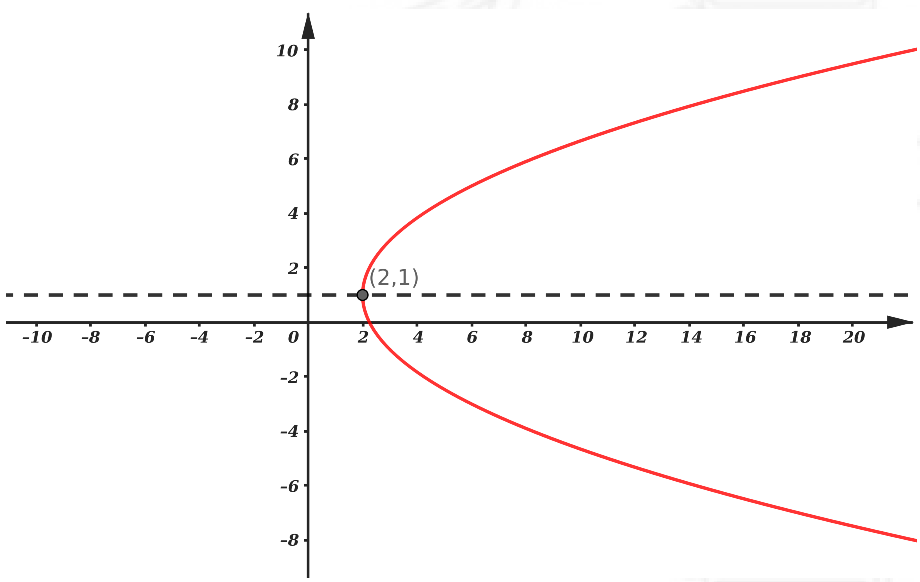

Figure 2: Plot of parabola

From expression for x: there are no limitations for y.

To plot the locus we calculate several values of y for $2 \leqslant x \leqslant 6$.

x	2	$\frac{9}{4}$	3	4	5	6
y	0	0, 2	3, -1	3.8, -1.8	4, 5, -2.5	5, -3

Problem 3
Plot the locus of $x^2 - x + xy + y - 2y^2 = 0$

Solution
Sometimes an equation will factor, in which case the locus of the equation will consist of the loci of the several factors. Since the equation factors into

$$(x - y)(x + 2y - 1) = 0,$$

its locus is the two intersecting lines

$$x = y$$
$$x = -2y + 1$$

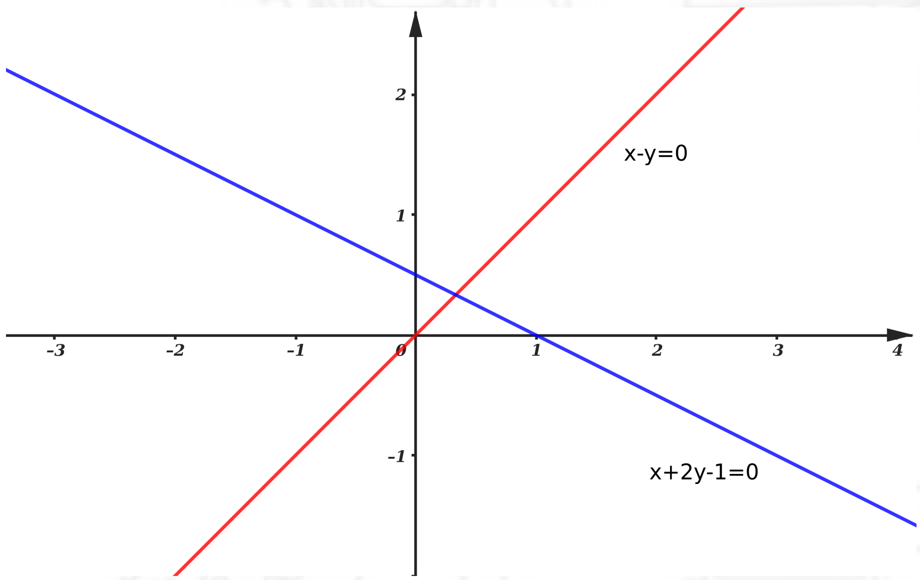

Figure 3: Plot of two crossing lines

Problem 4
Discuss and plot the graph of $x^2y - 4y + x = 0$.
Intercepts.
When $x = 0$, $y = 0$.
When $y = 0$, $x = 0$
Symmetry. With letting $-x$ for x and $-y$ for y, the equation becomes

$$-x^2y + 4y - x = 0,$$

and restores original form with multiply by -1.

Hence, the curve is symmetric by the origin. There is no symmetry line.

Extents.

Expression for y: $y = \dfrac{x}{4-x^2} = \dfrac{x}{(2-x)(2+x)}$.

This yields two *vertical asymptotes* $x = \pm 2$.

Expression for x: $x = \dfrac{-1 \pm \sqrt{16y^2+1}}{2y}$.

This yields *horizontal asymptote* $y = 0$.

There are no limitations for the values of x and y.

To plot the locus we calculate several values of y for $-4 \leqslant x \leqslant 4$

x	-4	-3	-2.5	-2	-1.5	-1	0	1	1.5	2	2.5	3	4
y	0.3	0.6	1.1	∞	-0.9	-0.3	0	0.3	0.9	∞	-1.1	-0.6	-0.3

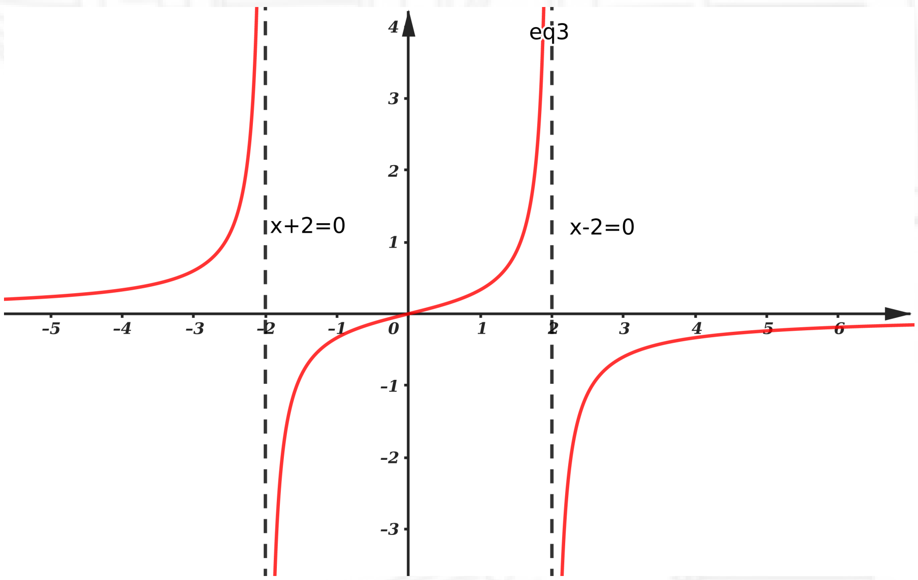

Figure 4: Plot of the curve with asymptotes

7 Identification properties of the locus with equation

Suppose locus of some figure expressed with equation

$$x^2 + y^2 + 2ax + 2by + c = 0(a^2 + b^2 - c > 0)$$

For this equation we can restore full squares:

$$x^2 + 2ax + a^2 + y^2 + 2by + b^2 - a^2 - b^2 + c = 0$$
$$(x+a)^2 + (y+b)^2 - \left(\sqrt{a^2+b^2-c}\right)^2 = 0$$

This equation expresses points laying on an equal distance $\sqrt{a^2+b^2-c}$ from the point $C(-a,-b)$.

Thus, figure (curve) expressed with this equation is a circle with center $C(-a,-b)$ and radius $\sqrt{a^2+b^2-c}$.

Problem 1

Find the coordinates of the center and radius of the sphere $x^2+y^2+z^2-6x+4y-8z=7$

Solution

Completing the squares yields

$$x^2-6x+9+y^2+4y+4+z^2-8z+16=36$$
$$(x-3)^2+(y+2)^2+(z-4)^2=36$$

Thus center is $C(3,-2,4)$, and radius is 6.

8 Restore equation by locus description

The law describing figure may be expressed with some non-formal words.

Problem 1

Find the equation of the straight lines which is 10 units to the right of the line $x+4=0$

Solution

$x=-4+10$, or $x=6$. This is the equation of the line which is 10 units to the right of the line $x+4=0$. It is parallel to the y-axis and 6 units to the right of the y-axis.

Problem 2

Determine the equation of the line parallel to the x-axis and 5 units from the point $(3,-4)$.

Solution

Let (x,y) be any point on the required line. $y=-4\pm5$,or $y=1$ and $y=-9$.

Problem 3

Derive the equation of the locus of a point $P(x,y)$ which moves in such a way that it is always equidistant from the points $A(-2,3)$ and $B(3,-1)$

Solution

Described condition means

$$
\begin{aligned}
&AP = BP\\
&\sqrt{(x+2)^2+(y-3)^2} = \sqrt{(x-3)^2+(y+1)^2}\\
&(x+2)^2+(y-3)^2 = (x-3)^2+(y+1)^2\\
&x^2+4x+4+y^2-6y+9 = x^2-6x+9+y^2+2y+1\\
&10x-8y+3=0
\end{aligned}
$$

Points $\left(0, \dfrac{8}{3}\right)$ and $\left(-\dfrac{3}{10}, 0\right)$ lay on the line expressed by this equation.

Its slope is $m' = \dfrac{3}{8} / \dfrac{3}{10} = \dfrac{10}{8} = \dfrac{5}{4}$

Slope of line AB is $m = \dfrac{-1-3}{3+2} = -\dfrac{4}{5}$.

Therefore, derived equation is the equation of the perpendicular bisector of the segment of the line connecting the two points.

Problem 4

A point $P(x,y)$ moves so that the sum of the squares of its distances from points $A(0,0)$ and $B(2,-4)$ is always 20.

Derive the equation of its locus.

Solution

$$
\begin{aligned}
&(AP)^2+(BP)^2 = 20\\
&x^2+y^2+(x-2)^2+(y+4)^2 = 20\\
&x^2+y^2+x^2-4x+4+y^2+8y+16 = 20\\
&x^2+y^2-2x+4y=0\\
&x^2-2x+1+y^2+4y+4-1-4=0\\
&(x-1)^2+(y+2)^2=5
\end{aligned}
$$

This is equation of circle with center in point $(1,-2)$. Segment AB is diameter of this circle

Problem 5

Given two points $P_1(2,4)$ and $P_2(5,-3)$. Determine the equation of the locus of the point $P(x,y)$ if the slope of PP_1 is 1 more than the slope of PP_2.

Solution

Slope of the line has expression through two points laying on it:

$$m = \frac{y_2-y_1}{x_2-x_1}$$

Condition that slope PP_1 is PP_2 plus 1 may be expressed as

$$\frac{y-4}{x-2} = \frac{y+3}{x-5} + 1$$
$$\frac{y-4}{x-2} = \frac{y+3+x-5}{x-5}$$
$$(y-4)(x-5) = (x-2)(x+y-2)$$
$$xy - 5y - 4x + 20 = x^2 + xy - 2x - 2x - 2y + 4$$
$$x^2 + xy - 2x - 2x - 2y + 4 - xy + 5y + 4x - 20 = 0$$
$$x^2 + 3y - 16 = 0$$

This curve is parabola.

Problem 6

A point $P(x, y)$ moves in such a way that the difference of its distances from $F_1(1, 4)$ to $F_2(1, -4)$ is always equal to 6.

Derive the equation of its locus.

Solution

$$PF_1 - PF_2 = 6.$$
$$\sqrt{(x-1)^2 + (y-4)^2} - \sqrt{(x-1)^2 + (y+4)^2} = 6$$
$$\sqrt{(x-1)^2 + (y-4)^2} = 6 + \sqrt{(x-1)^2 + (y+4)^2}$$

squaring

$$(x-1)^2 + (y-4)^2 = 36 + (x-1)^2 + (y+4)^2 + 12\sqrt{(x-1)^2 + (y+4)^2}$$
$$y^2 - 8y + 16 = 36 + y^2 + 8y + 16 + 12\sqrt{(x-1)^2 + (y+4)^2}$$
$$-16y - 36 = 12\sqrt{(x-1)^2 + (y+4)^2}$$
$$4y + 9 = -3\sqrt{(x-1)^2 + (y+4)^2}$$

squaring again

$$16y^2 + 72y + 81 = 9x^2 - 18x + 9 + 9y^2 + 72y + 144$$
$$9x^2 - 7y^2 - 18x + 72 = 0$$

This is equation of hyperbola

Problem 7

Find the equation of the perpendicular bisector of the line joining the points $A(2, -1, 3)$ and $B(-4, 2, 2)$.

Solution

This description means that desired figure:

1. Contains midpoint P_0 of the segment AB.

   Coordinates of this midpoint are $x = \dfrac{2-4}{2} = -1$, $y = \dfrac{-1+2}{2} = \dfrac{1}{2}$, $z = \dfrac{3+2}{2} = \dfrac{5}{2}$

2. Any segment P_0P established from this midpoint P_0 is perpendicular for AB.

Point (2) yields equal triangles $\triangle AP_0P$ and $\triangle BP_0P$: $AP_0 = BP_0$ (midpoint), P_0P is common, $\angle AP_0P = \angle AP_0P = \pi/2$ (perpendicular).

Thus, condition on the points $P(xy)$ is

$$
\begin{aligned}
&AP = BP\\
&\sqrt{x-2)^2+(y+1)^2+(z-3)^2} = \sqrt{(x+4)^2+(y-2)^2+(z-2)^2}\\
&x^2-4x+4+y^2+2y+1+z^2-6z+9 = x^2+8x+16+y^2-4y+4+z^1-4z+4\\
&12x-6y+2z+10=0\\
&6x-3y+z+5=0
\end{aligned}
$$

This is the equation of a plane every point of which is equidistant from the two given points.
1. Calculated coordinate of segment midpoint satisfy this equation:

$$6\cdot(-1)-3\cdot\frac{1}{2}+\frac{5}{1}+5 = 5-6+\frac{5-3}{2} = 0$$

2. Intercepts of this plane with coordinate axes are $(-5/6, 0, 0)$, $(0, 5/3, 0)$ and $(0, 0, -5)$, and cuts the line at $(-1, 1/2, 5/2)$.

Problem 8

Find the equation of the locus of a point $P(x, y, z)$ the difference of whose distances from $A(4, 0, 0)$ and $B(-4, 0, 0)$ is 6.

Solution

$$
\begin{aligned}
&AP - BP\\
&\sqrt{(x-4)^2+y^2+z^2} - \sqrt{(x+4)^2+y^2+z^2} = 6\\
&\text{squaring and simplifying}\\
&4x+9 = -3\sqrt{(x+4)^2+y^2+z^2}\\
&\text{and again squaring and simplifying}\\
&7x^2-9y^2-9z^2 = 63
\end{aligned}
$$

This surface is a hyperboloid of revolution about the x-axis.

9 Intersection of curves

Suppose we expressed two curves, say γ_1 and γ_2 wit equations

$$f_1(x, y) = 0;$$
$$f_2(x, y) = 0.$$

Let $A(x_0, y_0)$ be a point of intersection of γ_1 and γ_2.
Thus, A lies on both γ_1 and γ_2.
The fact that A lies on γ_1 we are expressing with $f_1(x_0, y_0) = 0$.
The fact that A lies on γ_2 we are expressing with $f_2(x_0, y_0) = 0$.
Therefore, coordinates of A are solution of the system of equations

$$\left\{ f_1(x, y) = 0 f_2(x, y) = 0, \right.$$

and any real solution of this system expresses in intersection point of these two curves.

Suppose γ_2 expressed in parametric form:

$$x = \varphi_2(\tau)$$
$$y = \psi_2(\tau).$$

For this case we are looking for solution of system

$$\left\{ f_1(x, y) = 0 x = \varphi_2(\tau), \quad y = \psi_2(\tau). \right.$$

And, finally, if γ_1 also expressed in parametric form

$$x = \varphi_1(t)$$
$$y = \psi_1(t),$$

our system obtains form

$$\left\{ x = \varphi_1(t), \quad y = \psi_1(t) x = \varphi_2(\tau), \quad y = \psi_2(\tau). \right.$$

10 Intersection of curves. Example

We need to find intersection points for circles

$$x^2 + y^2 = 2ax, \quad x^2 + y^2 = 2by$$

System to be solved:

$$\begin{cases} {}^2 + y^2 = 2ax, \\ x^2 + y^2 = 2by. \end{cases}$$

Subtraction of the first equation from second yields dependence: $y = \frac{a}{b}x$.

While we substitute it into first equation, we obtain:

$$\left(1 + \frac{a^2}{b^2}\right) x^2 - 2ax = 0;$$

Solutions for this equation are $x_1 = 0$ and $x_2 = \frac{2ab^2}{a^2+b^2}$.

Hence, $y_1 = 0$ and $y_2 = \frac{2a^2b}{a^2+b^2}$.

Points $(0,0)$ and $\left(\frac{2ab^2}{a^2+b^2}, \frac{2a^2b}{a^2+b^2}\right)$ are intersection points for these circles.

This gives a receipt how to solve systems of equations which have no well-expressed roots or calculation of these roots is technically difficult cases.

Solution for such system of equations may be found by plotting curves corresponding with both equations using the same coordinate system.

Intersections of the locuses yield solution of the system of equations.

Problem 1

Solve the following pair of simultaneous equations graphically, then check results by solving them algebraically.

$$x - y + 2 = 0$$
$$xy = 8$$

First equation represents a straight line whose intercepts are (-2.0) and $(0,2)$.

Let us plot graph for second curve.

Second equation yields:

$$x = \frac{8}{y},$$
$$y = \frac{8}{x}.$$

Then $x = 0$ y is infinite. Then $y = 0$ x is infinite.

Thus, $y = 0$ and $x = 0$ are horizontal and vertical asymptotes.

Replacement x to $-x$ and y to $-y$ preserves the equation, thus it is symmetric with respect to origin

x	0	1	2	3	4	-1	-2	-3	-4
y	∞	8	4	$\frac{8}{3}$	2	-8	-4	$-\frac{8}{3}$	-2

From the graph the solutions read (-4,-2) and (2,4).

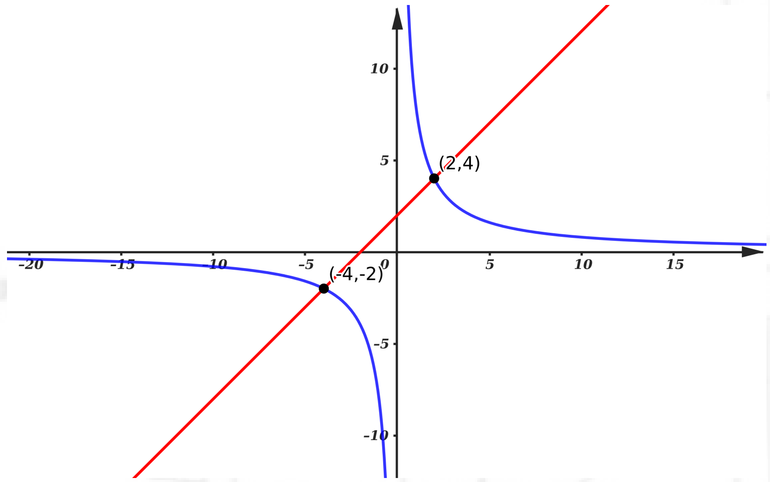

Figure 5: Graphical solution of system

Algebraic solution may be following:
From first equation $y = x + 2$. Substitution into second equation yields

$$x(x+2) = 8$$
$$x^2 + 2x - 8 = 0$$

Solutions for this equation are $x = -4$, and $x = 2$, corresponding y-s are -2 and 4. $\square$

Problem 2
Solve the following pair of simultaneous equations graphically, then check your results analytically.

$$4x^2 + y^2 = 100$$
$$9x^2 - y^2 = 108$$

Solution
Solving for y yields:

$$y = \pm\sqrt{100 - 4x^2}$$
$$y = \pm 3\sqrt{x^2 - 12}$$

Solving by x yields

$$x = \pm\frac{1}{2}\sqrt{100 - y^2}$$
$$x = \pm\frac{1}{3}\sqrt{y^2 + 108}$$

Therefore, for *first curve* x lies in $-5 \leqslant x \leqslant 5$, and y in $-10 \leqslant x \leqslant 10$.

For second curve $x \leqslant -\sqrt{12}$, and $x \geqslant \sqrt{12}$, and there is no limitation for y.
Each of the curves is symmetric about both axes and the origin.
Tables of values for the curves are

x	0	± 1	± 2	± 3	± 4	± 5
y	± 10	± 9.8	± 9.2	± 8	± 6	0

and

x	$\pm\sqrt{12}$	± 4	± 5	± 6
y	0	± 6	± 10.8	± 14.7

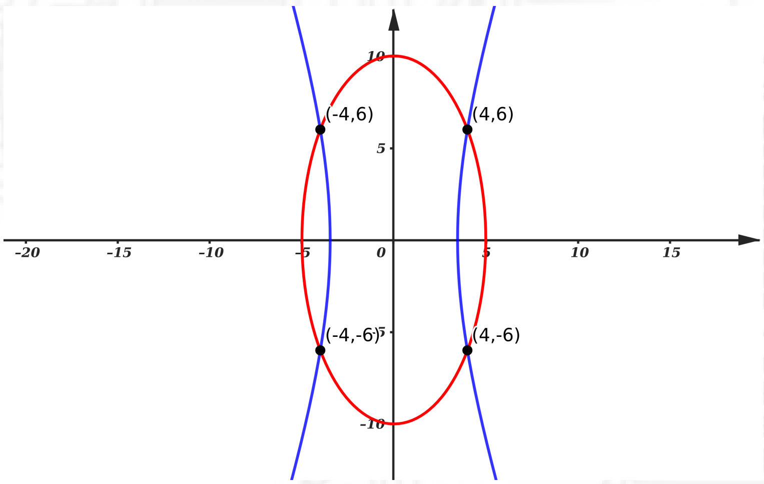

Figure 6: Graphical solution of system

Reading from the graph the solutions are $(4, \pm 6)$, $(4, \pm 6)$
Algebraic solution is following:

$$4x^2 + y^2 = 100$$
$$9x^2 - y^2 = 108$$

$$13x^2 = 208$$
$$x^2 = 16$$
$$x = \pm 4$$

$$y^2 = 9x^2 - 108 = 144 - 108 = 36$$
$$y = \pm 6$$