Equations of Line in Space

Assume that some straight line a in the space $\mathbb{E}$ is chosen and fixed. In order to study various equations determining this line we choose some coordinate system O, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in the space.

Then we can describe the points of the line by their radius vectors relative to the origin O.

We start with vectorial forms of equation of the line as they are more general and more close to definition of the line.

1 Equation of line in vectorial parametric form

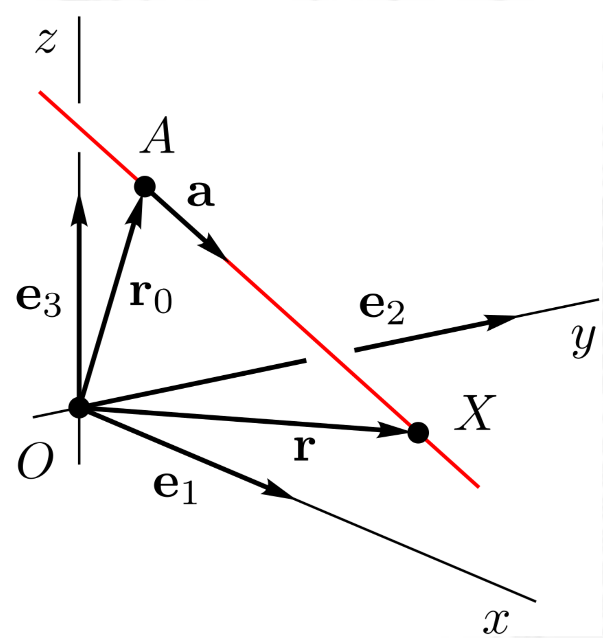

Figure 1: Derivation of vectorial parametric equation

Let's consider a line a in space.

Let A be some fixed point on this line, and X be arbitrary point of the line distant form A.

The position of the point X relative to the point A is marked by the vector $\overrightarrow{AX}$, while the position of the point A itself is determined by its radius vector $\boldsymbol{r}_0 = \overrightarrow{OA}$.

Therefore, position of X with respect to origin may be expressed with radius vector:

$$\boldsymbol{r} = \overrightarrow{OX} = \boldsymbol{r}_0 + \overrightarrow{AX}$$

By selecting and fixing arbitrary non-zero vector $\boldsymbol{a}||\overrightarrow{AX}$, $\boldsymbol{a} \neq \boldsymbol{0}$, we express $\overrightarrow{AX}$ as

$$\overrightarrow{AX} = \boldsymbol{a} \cdot t$$

t here is arbitrary number (positive, negative, or zero) explicitly defined with choice of $\boldsymbol{a}$. Modulus of t resembles length AX with respect to scale equal with $|\boldsymbol{a}|$, and sign means direction of displacement Thus, we established local coordinate system on the line a with origin in point A and basis vector $\boldsymbol{a}$.

From this two formulas we immediately derive

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a} \cdot t \tag{1}$$

This expression (1) is called **vectorial parametric equation of a line**.

Vector $\boldsymbol{a}$ is called a **directional vector of the line**, and point A called the **initial point**.

Real number t is **parameter** of this equation. Each particular value of the parameter t corresponds to some definite point on the line. The initial point A with the radius vector $\boldsymbol{r}_0$ is associated with the value $t = 0$.

This equation exactly corresponds with equation of uniform motion of the body[1].

Parameter t corresponds with time of motion, and vector $\boldsymbol{a}$ with constant velocity vector of the body. Vector $\boldsymbol{r}_0$ means initial position of the body. Straight line is **trajectory** of the body.

Let us discuss typical problems involving this type of straight line equation.

Remark. Form of the vectorial parametric equation of the line on plane and in space is the same

2 Problems involving vectorial parametric form of straight line equation

We further assume that our coordinate system has right orthonormal basis.

2.1 Constructing the equation

Problem 1

Express vectorial parametric equation of a line crossing points $A(4, 12, 1)$, and $B(6, 6, 5)$

Solution

Suppose A be initial point. Vector $\overrightarrow{AB}$ has coordinates $\begin{pmatrix} 6-4 \\ 6-12 \\ 5-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$. Let it be directional vector.

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$$

We may let point B be the initial point of our line. Equation in this case took form

$$\boldsymbol{r} = \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$$

[1] This interpretation contains strong idealization, as we overlook all possible resistance factors.

Changing of the directional vector with it anticodrected "twin" yields two more equations:

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} 6 \\ 6 \\ 5 \end{pmatrix} + t \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$$

Definition. Vectorial parametric equation of the line containong distant points $P_1(x_1, y_1, z_1)$, and $P_2(x_2, y_2, z_2)$ has form

$$\boldsymbol{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \pm t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

or

$$\boldsymbol{r} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \pm t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

or

$$\boldsymbol{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \pm t \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

or

$$\boldsymbol{r} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \pm t \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \\ z_1 - z_2 \end{pmatrix}$$

Problem 2

Remark: in physics and mechanics **point mass** is the concept of a physical object (typically matter) that has nonzero mass, and yet explicitly and specifically is (or is being thought of or modeled as) *infinitesimal* (infinitely small) in its volume or linear dimensions. Geometrical point with assigned mass is a good model for this concept.

Express as vectorial parametric equation a trajectory of point mass which started from the point $A(7, 7, 5)$ and moves uniformly shaping angles $\pi/3$ with positive direction of first and second coordinate axis, and angle $\pi/4$ with positive direction of third coordinate axis.

Solution

Direction cosines of this line are $(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2})$

Initial point of the line is $A(7, 7, 5)$, thus $\boldsymbol{r}_0 = \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix}$

$$\boldsymbol{r} = \begin{pmatrix} 7 \\ 7 \\ 5 \end{pmatrix} \pm t \begin{pmatrix} \frac{\sqrt{1}}{2} \\ \frac{1}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$$

2.2 Investigation of line features

Problem 3

Find direction cosines of a line expressed with vectorial parametric equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a}t$$

Solution

Vector $\boldsymbol{a}$ is collinear with line in question, hence its direction cosines coincide with direction cosines of the line.

$$|\boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \tag{2}$$

$$\cos\alpha = \frac{a_x}{|\boldsymbol{a}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{e}_1}{|\boldsymbol{a}|}$$

$$\cos\beta = \frac{a_x}{|\boldsymbol{a}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{e}_2}{|\boldsymbol{a}|}$$

$$\cos\gamma = \frac{a_x}{|\boldsymbol{a}|} = \frac{\boldsymbol{a} \cdot \boldsymbol{e}_3}{|\boldsymbol{a}|}$$

Recap expression for the slope of a line on a plane expressed with vectorial parametric equation:

$$m = \frac{y_a}{x_a} = \frac{\boldsymbol{a} \cdot \boldsymbol{e}_2}{\boldsymbol{a} \cdot \boldsymbol{e}_1}$$

Problem 4

Using the vectorial parametric equation of a line, identify whether the points $A(-7, 5, 1)$, $B(-1, 2, 2)$, and $C(5, -1, 3)$ lie on the same line (are collinear).

Solution

There are a few ways of checking whether the three given points are collinear, one of which is to find the equation between one pair of points and then check if the third point satisfies the equation.

Let's do this by finding the vectorial parametric form of the equation of the line between $A(-7, 5, 1)$ and $B(-1, 2, 2)$.

We will take the equation in form

$$\boldsymbol{r} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 - (-7) \\ 2 - 5 \\ 2 - 1 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} + t \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix}$$

Substitution of coordinates of C into equation yields for x:

$$5 = -7 + 6t$$
$$6t = 12$$
$$t = 2$$

If now we subsitute $t = 2$ into equation of line connecting A and B, we obtain:

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 6 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -7 + 12 \\ 5 - 6 \\ 1 + 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix},$$

And $\boldsymbol{r}$ is radius vector for C. Thus, C lies on the line connecting A and B.

Suppose, $\tau = 2t$. This change of variable yields equivalent equation:

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} + \tau \begin{pmatrix} 12 \\ -6 \\ 2 \end{pmatrix}$$

Direction vector for this equivalent form of equation is $\overrightarrow{AC} = \begin{pmatrix} 5-(-7) \\ -1-5 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \\ 2 \end{pmatrix}$.

Proposition. *Three points lay on the same line if each pair of them yields equivalent equation of line. From the vectorial parametric equation of line it means that all vectors connecting these two points are collinear.*

Note: this also makes legal to use term "collinear" for such triplet of points.

Problem 5

Find x- y- and z-intercept of line expressed with vectorial parametric equation:

$$\boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix}$$

Solution

Let $\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

Putting $x = 0$ now yields for first coordinate:

$$\begin{aligned} x_0 + t_x x_a &= 0 \\ t_x x_a &= -x_0 \\ t_x &= -\frac{x_0}{x_a} \end{aligned}$$

Also, putting $y = 0$ yields

$$\begin{aligned} y_0 + t_y y_a &= 0 \\ t_y y_a &= -y_0 \\ t_y &= -\frac{y_0}{y_a}, \end{aligned}$$

and putting $z = 0$ yields

$$\begin{aligned} z_0 + t_z z_a &= 0 \\ t_z z_a &= -z_0 \\ t_z &= -\frac{z_0}{z_a}, \end{aligned}$$

Suppose in the moment of $t = t_x = t_y$ $x = 0$ and $y = 0$. It means that line crosses axis oZ, and

$$z = z_0 - \frac{x_0 z_a}{x_a} = z_0 - \frac{x_0 z_a}{x_a}$$

z-intercept it

$$\begin{pmatrix} 0 \\ 0 \\ z_0 - \dfrac{x_0 z_a}{x_a} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 0 \\ z_0 - \dfrac{y_0 z_a}{y_a} \end{pmatrix}$$

If $t_x \neq t_y$, line hasn't z-intercept.
If in some moment of $t = t_x = t_z$ $x = 0$ and $z = 0$, y-intercept is

$$\begin{pmatrix} 0 \\ y_0 - \dfrac{x_0 y_a}{x_a} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ y_0 - \dfrac{z_0 y_a}{z_a} \\ 0 \end{pmatrix}.$$

And, if in the moment of $t = t_y = t_z$ $y = 0$ and $z = 0$, x-intercept is

$$\begin{pmatrix} x_0 - \dfrac{y_0 x_a}{y_a} \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} x_0 - \dfrac{z_0 x_a}{z_a} \\ 0 \\ 0 \end{pmatrix}.$$

Definition. We say that γ is angle two any lines in space if γ is angle between their direction vectors.

Problem 6
Find angle between two lines expressed with vectorial parametric equations:

$$\boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$
$$\boldsymbol{r}' = \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} + \tau \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Solution
Cosines of angle between these two lines is defined in terms of dot product:

$$\cos\gamma = \frac{xx' + yy' + zz'}{\sqrt{x^2 + y^2 + z^2}\sqrt{x'^2 + y'^2 + z'^2}}$$

Problem 7

Find intersection point of two lines expressed with vectorial parametric equations:

$$
\boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} x \\ y \\ z \end{pmatrix};
$$

$$
\boldsymbol{r}' = \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} + \tau \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}
$$

Solution

Suppose lines have different initial points. In opposite case solution is trivial.
Equality $\boldsymbol{r} = \boldsymbol{r}'$ may be expanded as

$$
\begin{cases} x_0 + tx = x_0' + \tau x' \\ y_0 + ty = y_0' + \tau y' \\ z_0 + tz = z_0' + \tau z' \end{cases}
$$

x_0, y_0, x_0', y_0', x, y, x', y' are here known parameters, and we are looking for t and τ.
Reordering yields:

$$
\begin{cases} tx - \tau x' - (x_0' - x_0) = 0 \\ ty - \tau y' - (y_0' - y_0) = 0 \\ tz - \tau z' - (z_0' - z_0) = 0 \end{cases}
$$

This system has one not-trivial solution if

$$
\begin{vmatrix} x & x' & (x_0' - x_0) \\ y & y' & (y_0' - y_0) \\ z & z' & (z_0' - z_0) \end{vmatrix} = 0
$$

If we checked this condition, we can solve system of firs and second equations:

$$
\begin{cases} tx - \tau x' = (x_0' - x_0) = \Delta_x \\ ty - \tau y' = (y_0' - y_0) = \Delta_y \end{cases}
$$

Application of Cramer's rule yields:

$$
\Delta = \begin{vmatrix} x & x' \\ y & y' \end{vmatrix} = xy' - x'y
$$

$$\Delta_1 = \begin{vmatrix} \Delta x & x' \\ \Delta y & y' \end{vmatrix} = \Delta x y' - x' \Delta y$$

$$\Delta_2 = \begin{vmatrix} x & \Delta x \\ y & \Delta y \end{vmatrix} = x \Delta y - \Delta x y$$

Therefore,

$$t = \frac{\Delta_1}{\Delta} = \frac{\Delta x y' - x' \Delta y}{x y' - x' y}$$
$$\tau = \frac{\Delta_2}{\Delta} = \frac{x \Delta y - \Delta x y}{x y' - x' y}$$

Solution of first and third equation, as well second and third must yield the same result.

2.3 Special cases of the lines

Problem 8

Derive vectorial parametric equation of lines parallel with one of coordinate axes and passing through point $A(x, y, z)$.

Solution

For line parallel with Ox we choose direction vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

For line parallel with Oy we choose direction vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

For line parallel with Oz we choose direction vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Lies are:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

It is important to underline that these direction vectors may be replaced with any non-zero multiply.

Note coordinate axes may be expresses with vectorial parametric equations as:

$$\begin{aligned} Ox \text{ axis: } \boldsymbol{r} &= t\boldsymbol{e}_1 \\ Oy \text{ axis: } \boldsymbol{r} &= t\boldsymbol{e}_2 \\ Oz \text{ axis: } \boldsymbol{r} &= t\boldsymbol{e}_3 \end{aligned}$$

3 Coordinate parametric equations of a line

Let's determine the vectors $\boldsymbol{r}$, $\boldsymbol{r}_0$, and $\boldsymbol{a}$ in the vectorial parametric equation (1) through their coordinates:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \quad \boldsymbol{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

Separated writing of the equation for first and second coordinate form (1) yields us **coordinate parametric equations of a line**:

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \\ z = z_0 + a_z t \end{cases} \tag{3}$$

Condition on non-zero $\boldsymbol{a}$ now takes form that *a_x, a_y, and a_z must not vanish simultaneously.*

Mechanical explanation for this system of equation is following.

We assume displacement of our point mass being expanded on coordinate axes. Thus, velocity vector is also expanded. For each component we need to write equation for varying coordinate.

Hence, we have three scalar motion equations for each coordinate. Their recombination restores vectorial displacement.

This direct connection of these two forms automatically yields us analogs for all dependencies and formulas we investigated for vectorial parametric equation.

Let us recap and modify that dependencies for coordinates with right orthonormal basis.

1. Direction cosines of the line are

$$\cos\alpha = \frac{a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}}, \quad \cos\beta = \frac{a_y}{\sqrt{a_x^2 + a_y^2 + a_z^2}}, \quad \cos\gamma = \frac{a_z}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

2. z-intercept has expression

$$\begin{cases} x = 0 \\ y = 0 \\ z = z_0 - a_z \dfrac{x_0}{a_x} = z - a_z \dfrac{y_0}{a_y}, \end{cases}$$

if $\dfrac{x_0}{a_x} \neq \dfrac{y_0}{a_y}$, line has no z-intercept.

3. y-intercept has expression

$$\begin{cases} x = 0 \\ y = y_0 - a_y \dfrac{x_0}{a_x} = y - a_y \dfrac{z_0}{a_z} \\ z = 0, \end{cases}$$

if $\dfrac{x_0}{a_x} \neq \dfrac{z_0}{a_z}$, line has no y-intercept.

4. x-intercept has expression

$$\begin{cases} x = x_0 - a_x \dfrac{y_0}{a_y} = x_0 - a_x \dfrac{z_0}{a_z} \\ y = 0 \\ z = 0, \end{cases}$$

if $\dfrac{y_0}{a_y} \neq \dfrac{z_0}{a_z}$, line has no x-intercept.

5. Let a and a' be two lines expressed with parametric equations

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \\ z = z_0 + a_z t \end{cases}$$

and

$$\begin{cases} x' = x'_0 + a'_x \tau \\ y' = y'_0 + a'_y t\tau \\ z' = z'_0 + a'_z t\tau \end{cases}$$

Cosines of the angle between this two lines is

$$\cos\theta = \frac{a_x a'_x + b_x b'_x + c_x c'_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{a'^2_x + a'^2_y + a'^2_z}}$$

Problem 1

Express coordinate parametric equation of a line crossing points $A(4, 12, 1)$, and $B(6, 6, 5)$.

Solution

Suppose A be initial point. Thus, $x_0 = 4$, $y_0 = 12$, $z_0 = 1$.
Suppose point will be moved from A to B with value of parameter $t = 1$

$$x_b = x_a + a_x$$
$$y_b = y_a + a_y$$
$$y_b = y_a + a_z$$

Thus, $a_x = x_b - x_a = 6 - 4 = 2$, $a_y = y_b - y_a = 6 - 12 = -6$, $a_z = z_b - z_a = 5 - 1 = 4$
The system of equations is

$$\begin{cases} x = 4 + 2t \\ y = 12 - 6t \\ z = 1 + 4t \end{cases}$$

Alternatively, we can take point B as initial point:

$$\begin{cases} x = 6 + 2t \\ y = 6 - 6t \\ z = 5 + 4t, \end{cases}$$

or reverse direction of shift:

$$\begin{cases} x = 4 - 2t \\ y = 12 + 6t \\ z = 1 - 4t, \end{cases}$$
$$\begin{cases} x = 6 - 2t \\ y = 6 + 6t \\ z = 5 - 4t \end{cases}$$