Proposition 1. Let $I = [a, b]$ be an interval and let $f : I \to I$ be continuous. Then f has at least one fixed point in I.

Pf: $g(x) = f(x) - x$. cont. since $f(x)$ cont.

$g(a) = f(a) - a \geqslant 0$ $\quad g(b) = f(b) - b \leqslant 0$.

by. mean value thm. $\exists\ g(\xi) = 0$. $\xi \in [a, b]$.

Proposition 2. Let $f : I \to I$ and assume that $|f'(x)| < 1$ for all x in I. Then there exists a unique fixed point for f in I. Moreover

$$|f(x) - f(y)| < |x - y|$$

for all $x, y \in I, x \neq y$.

Pf: $\exists\ x_1, x_2 \in I$. w.l.g. assume $x_1 < x_2$

$$\begin{cases} f(x_1) - x_1 = 0 \\ f(x_2) - x_2 = 0 \end{cases} \Rightarrow \frac{f(x_1) - f(x_2)}{x_1 - x_2} = 1.$$

by Laplace thm. $\exists\ \xi \in [x_1, x_2]$. $f'(\xi) = 1$. contradicts.

4. Show via an example that hyperbolic periodic points need not be isolated.

1. Find all periodic points for each of the following maps and classify them as attracting, repelling, or neither. Sketch the phase portraits.

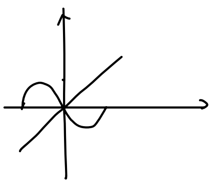
e. $S(x) = \frac{1}{2}\sin(x)$
f. $S(x) = \sin(x)$
g. $E(x) = e^{x-1}$
h. $E(x) = e^{x}$

f.

$S(x) = \sin x.$ $\quad x=0.$ non-hyperbolic

$x \to 0^+.$ $f(x) - x = \sin x - x = x - \frac{1}{6}x^3 + \cdots - x = -\frac{1}{6}x^3 + o(x^3).$

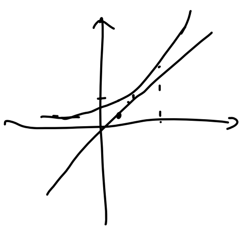
$f(x)$ more close to 0.



g. $E(x) = e^{x-1}$ $\quad$ fixed point $x = 1.$

$E'(x) = e^{x-1}.$ $\quad E'(1) = 1.$

consider $e^{x-1} - x = 1 + x - 1 + \frac{(x-1)^2}{2!} + o((x-1)^2) - x = \frac{(x-1)^2}{2!} + o((x-1)^2). > 0.$ $\quad$ repell.



$x < 1.$

$(1 - e^{x-1}) - (1 - x)$

$x > 1$

$(e^{x-1} - 1) - (x - 1)$

2. Sketch the graph of $F_4^n(x)$ on the unit interval, where $F_4(x) = 4x(1-x)$. Conclude that F_4 has at least 2^n periodic points of period n.

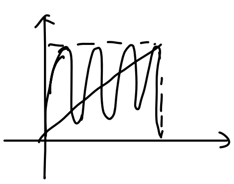
3. Sketch the graph of the tent map

$$T_2(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2 - 2x & \frac{1}{2} \le x \le 1 \end{cases}$$

on the unit interval. Use the graph of T_2^n to conclude that T_2 has exactly 2^n periodic points of period n.

2. $4x(1-x).$ $\quad 4(4x(1-x)(1 - 4x(1-x))).$

一定存在. $F^{n-1}(x) = \frac{1}{2}$ 的点. $F(\frac{1}{2}) = 1.$



有 2^n 个零点、 , 2^{n-1} 个 1.

$\Rightarrow$ 下次到 0.

p47 - 48.

1. Let $Q_c(x) = x^2 + c$. Prove that if $c < 1/4$, there is a unique $\mu > 1$ such that Q_c is topologically conjugate to $F_\mu(x) = \mu x(1-x)$ via a map of the form $h(x) = \alpha x + \beta$.

Sol: $h \circ Q_c = \alpha(x^2+c) + \beta = \alpha x^2 + \alpha c + \beta$.

$F_\mu \circ h = \mu(\alpha x+\beta)(1-(\alpha x+\beta)) = -\mu(\alpha x+\beta)^2 + \mu(\alpha x+\beta)$

$= -\mu\alpha^2 x^2 + (\alpha\mu - 2\alpha\beta\mu)x + \beta(\mu - \beta\mu)$

Let $\begin{cases} \alpha = -\mu\alpha^2 & ① \\ \alpha\mu - 2\alpha\beta\mu = 0 & ② \\ \beta(\mu-\beta\mu) = \alpha c + \beta. & ③ \end{cases} \Rightarrow \begin{cases} \alpha = -\frac{1}{\mu} \\ \beta = \frac{1}{2}. \end{cases}$ by ① ②

the ③ implies. $\frac{1}{2}\mu - \frac{1}{4}\mu = -\frac{c}{\mu} + \frac{1}{2}. \Rightarrow \mu^2 - 2\mu + 4c = 0.$ $\quad \Delta = \sqrt{4+16c}$

$\mu = \pm\sqrt{1-4c} + 1.$ when $c < \frac{1}{4}$. ③ always have real root and $\mu > 1$.

2. A point p is a *non-wandering* point for f, if, for any open interval J containing p, there exists $x \in J$ and $n > 0$ such that $f^n(x) \in J$. Note that we do not require that p itself return to J. Let $\Omega(f)$ denote the set of non-wandering points for f.

a. Prove that $\Omega(f)$ is a closed set.
b. If F_μ is the quadratic map with $\mu > 2 + \sqrt{5}$, show that $\Omega(F_\mu) = \Lambda$.
c. Identify $\Omega(F_\mu)$ for each μ satisfying $0 < \mu \leq 3$.

a). $\forall q \in (\Omega(f))^c$. $\exists V_q$. s.t. $\forall x \in V_q$. $\forall n > 0$. $f^n(x) \notin V_q$.

$\exists B_\varepsilon(q) \subseteq V_q$. since V_q is open. for any. $q' \in B_\varepsilon(q)$. $\exists V_{q'} = V_q$. ...

thus. $B_\varepsilon(q) \subseteq (\Omega(f))^c$ thus. $(\Omega(f))^c$ is open.

b). F_μ has dense orbit.

$\exists$ dense trajectory for Λ

$\forall x \in \Lambda$. $\exists F_\mu^n(x) = p_n \to x$. i.e. $F_\mu^n(x) \subseteq J$.

c).

3. A point p is *recurrent* for f if, for any open interval J about p, there exists $n > 0$ such that $f^n(p) \in J$. Clearly, all periodic points are recurrent.

a. Give an example of a non-periodic recurrent point for F_μ when $\mu > 2 + \sqrt{5}$.
b. Give an example of a non-wandering point for F_μ which is not recurrent.

p43.

4. Let Σ' consist of all sequences in Σ_2 satisfying: if $s_j = 0$ then $s_{j+1} = 1$. In other words, Σ' consists of only those sequences in Σ_2 which never have two consecutive zeros.

a. Show that σ preserves Σ' and that Σ' is a closed subset of Σ.
b. Show that periodic points of σ are dense in Σ'.
c. Show that there is a dense orbit in Σ'.
d. How many fixed points are there for $\sigma, \sigma^2, \sigma^3$ in Σ' ?
e. Find a recursive formula for the number of fixed points of σ^n in terms of the number of fixed points of σ^{n-1} and σ^{n-2}.

a). $\forall s \in \Sigma'$. $\sigma(s)$ erase the first number.

$\sigma(s)$. satisfy that. if $s_{j-1} = 0$ then $s_j = 1$. denote $i = j-1$. we have $\sigma(s) \in \Sigma'$.

$\forall \{s_n\} \subseteq \Sigma'$. and $s_n \to s$. show that $s \in \Sigma'$

assume $s \notin \Sigma'$. then $\exists i \in \mathbb{N}$ s.t. $s_i = 0$ and $s_i = 0$

but. for s_n. if $s_{ni} = 1$. then $d[s, s_n] > \frac{1}{2^i}$

if $s_{ni} = 0 \Rightarrow s_{n,i+1} = 1$. then $d[s, s_n] > \frac{1}{2^{i+1}}$. which contradicts to $s_n \to s$.

b). denote set Σ_p as all periodic points of σ.

$\forall s \in \Sigma'$. and $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$.

then there exists $s_p \in \Sigma_p$ of period N. s.t. $s_i = s_{pi}$ for $i \in [1:N]$.

thus we have $d(s_p, s) < \varepsilon$. by the arbitariness of s, Σ_p is dense in Σ'.

c) consider $s_0 = \{ 01 \mid 01\ 10\ 11 \mid 011\ 010\ 110\ 101\ 111 \mid \cdots$
1block 2block 3 block.

s_0 contains all blocks of 1's and 0's with non-consecutive 0.

$\forall s \in \Sigma'$. $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$. $\exists k$, s.t $\sigma^k(s_0)_i = s_i$ for all $i \in [1:N]$.

thus. $d[s, \sigma^k(s_0)]$. the orbit $O^+(s_0)$ is dense in Σ'.

d). for σ. 1 fixed point (1111...)

for σ^2 3 fixed point (101010...) (0101...) (1111...)

for σ^3. 4 fixed point. (1111...) (110110...) (011011...) (101101...).

e). the fixed point of σ^n is the periodic point of σ with period n. denote the required number is $S(n)$

consider the n digit sequence with the limitation of non - consecutive 0.

if the sequence end with 1 : then the former $n-1$ digits can be any acceptable sequence $- S(n-1)$
if the sequence end with 0 : then the penultimate number is 1. then former $n-2$ number can be any acceptable sequence $- S(n-2)$.
Thus we conclude that $S(n) = S(n-1) + S(n-2)$. ($n \geq 3$. $S(1) = 1$. $S(2) = 3$).

P43.

5. Let Σ_N consist of all sequences of natural numbers $1, 2, \ldots, N$. There is a natural shift on Σ_N.

a. How many periodic points does σ have in Σ_N ?

b. Show that σ has a dense orbit in Σ_N.

a) fixed point N

periodic point. of period 2 N^2 (not necessarily to be prime, the same below).

periodic point. of period 3 N^3

periodic point. of period n N^n

b) consider $s^* = (1 2 3 \ldots (N-1) N \mid 11\ 12\ 13 \cdots (N-1)N\ NN \mid 111 \cdots NNN \mid \cdots)$ listing all block's of 1's to N's of leng n, then length n+

$\qquad\qquad\qquad$ 1 block $\qquad$ 2 block $\qquad$ 3 block

same iterates of σ applied to s^* yields a sequence which agrees with any given sequence in an arbitrary large number of place.

For example $\forall s \in \Sigma_N$. and $\forall \varepsilon > 0$. $\exists n$. s.t. $\frac{1}{2^n} < \varepsilon$. and $\exists m \in \mathbb{N}$. $\sigma^m(s^*) \in O^+(s^*)$. s.t. $s_i = \sigma^m(s)_i$ for $i \in [1:n]$.

thus $d[s, \sigma^m(s^*)] < \varepsilon$. The orbit $O^+(s^*)$ w.r.t. σ meets the requirement.

6. Let $\mathbf{s} \in \Sigma_2$. Define the stable set of $\mathbf{s}$, $W^s(\mathbf{s})$, to be the set of sequences $\mathbf{t}$ such that $d[\sigma^i(\mathbf{s}), \sigma^i(\mathbf{t})] \to 0$ as $i \to \infty$. Identify all of the sequences in $W^s(\mathbf{s})$.

p48.

recurrent $\not\Rightarrow$ non-wandering.

3. A point p is *recurrent* for f if, for any open interval J about p, there exists $n > 0$ such that $f^n(p) \in J$. Clearly, all periodic points are recurrent. (无限次回到小邻域中)

a. Give an example of a non-periodic recurrent point for F_μ when $\mu > 2 + \sqrt{5}$.

b. Give an example of a non-wandering point for F_μ which is not recurrent.

a). $s^* = \{01 | 00011011 | \cdots\}$, the construction implies $s^* \notin \Lambda$

the orbit $O^+(s^*)$ is dense. $\exists n.\ f^n(s^*) \in J$. for any J.

b) eventually fixed point

For example $x = 1$. $\forall J \ni 1$. $\exists x_0 \in Per(F_\mu)$ and $x_0 \in J$ since periodic point is dense in Λ.

so $x = 1$ is non-wandering. But $f^n(1) = 0$ for any $n \geq 1$. thus it's not. recurrent.

Apr. 2nd.

$f: [0,1] \to [0,1]$. cont. $f([0,1]) = [0,1]$.

Prove: f^2 at least 2 fixed point.

Prove: if 0 and 1 are not fixed. then f^2 has at least 3 fixed point