Oct. 24th

Exercise 1.1. *Which of the following functions are metrics on* $\mathbb{R}^2$:

i) $d((x_1,x_2),(y_1,y_2)) = \min(|x_1-y_1|,|x_2-y_2|)$

ii) $d((x_1,x_2),(y_1,y_2)) = |x_1-y_1|^2+|x_2-y_2|^2$

iii) $d((x_1,x_2),(y_1,y_2)) = |x_1-y_1|+2|x_2-y_2|$

i). not metric

$d=0 \Leftrightarrow \min(|x_1-y_1|,|x_2-y_2|)=0$

$\Rightarrow$ we can let $|x_1-y_1|=0$. $|x_2-y_2|=1$.

$\not\Rightarrow (x_1,x_2)\neq(y_1,y_2)$. doesn't satisfy the identity axioms.

ii) not metric

Identity: $d=0 \Leftrightarrow |x_1-y_1|^2+|x_2-y_2|^2=0$

$\Leftrightarrow x_1=y_1$ and $x_2=y_2$

$\Leftrightarrow (x_1,x_2)=(y_1,y_2)$

symmetry: $d((x_1,x_2),(y_1,y_2)) \Leftrightarrow d = |x_1-y_1|^2+|x_2-y_2|^2$

$= |y_1-x_1|^2+|y_2-x_2|^2$

$= d((y_1,y_2),(x_1,x_2))$

Triangle: not satisfy.

counter-example: let $(x_1,x_2)=(1,1)$. $(y_1,y_2)=(3,3)$, $(z_1,z_2)=(2,2)$

$d((x_1,x_2),(y_1,y_2)) = |3-1|^2+|3-1|^2=8$

$d((x_1,x_2),(z_1,z_2))+d((z_1,z_2),(y_1,y_2)) = 2. < d((x_1,x_2),(y_1,y_2))$

iii) Identity: $d=0 \Leftrightarrow |x_1-y_1|+2|x_2-y_2|=0$, since $|x_1-y_1|;\ |x_2-y_2|\geq 0$

$\Leftrightarrow x_1=y_1$ and $x_2=y_2$

$\Leftrightarrow (x_1,x_2)=(y_1,y_2)$

symmetry: $d((x_1,x_2),(y_1,y_2)) \Leftrightarrow d = |x_1-y_1|+2|x_2-y_2|$

$= |y_1-x_1|+2|y_2-x_2|$

$= d((y_1,y_2),(x_1,x_2))$

Triangle. $d((x_1,x_2),(z_1,z_2))+d((y_1,y_2),(z_1,z_2)) = |x_1-z_1|+2|x_2-z_2|+|y_1-z_1|+2|y_2-z_2|$

$=(|x_1-z_1|+|z_1-y_1|+2(|x_2-z_2|+|z_2-y_2|) \underset{\text{triangle inequality of real numbers}}{\geq} |x_1-y_1|+2|x_2-y_2| = d((x_1,y_1),(x_2,y_2))$

Exercise 1.2. *Prove that if* d *and* d' *are two metrics on* X, *then* $d+d'$ *and* $\max(d,d')$ *also are metrics on* X.

Pf: $\forall x,y,z\in X$.

1) Identity. $(d+d')(x,y)=0 \xLeftrightarrow{d,d'\geq 0} d(x,y)=0$ and $d'(x,y)=0 \xLeftrightarrow{d,d' \text{ are metrics on } X} x=y$

$\max(d,d')(x,y)=0 \xLeftrightarrow[\text{def. of max.}]{d,d'\geq 0} d(x,y)=0$ and $d'(x,y)=0 \xLeftrightarrow{d,d' \text{ are metrics on } X} x=y$

2) symmetry $(d+d')(x,y) = d(x,y)+d'(x,y)$ (linearity of functions)

$= d(y,x)+d'(y,x)$ (symmetric of metrics d,d')

$= (d+d')(y,x)$

$\max(d,d')(x,y) = \begin{cases} d(x,y), & d(x,y)\geq d'(x,y) \\ d'(x,y) & d'(x,y)>d(x,y)\end{cases} \overset{\text{symmetry of metrics } d,d'}{=} \begin{cases} d(y,x), & d(y,x)\geq d'(y,x) \\ d'(y,x), & d'(y,x)>d(y,x)\end{cases} = \max(d,d')(y,x)$

3) triangular: $(d+d')(x,z)+(d+d')(z,y) = d(x,z)+d'(x,z)+d(y,z)+d'(y,z)$

$$= d(x,z)+d(y,z)+d'(x,z)+d'(z,y)$$
$$\geq d(x,y)+d'(x,y)$$
$$=(d+d')(x,y)$$

$\max(d,d')(x,z) + \max(d,d')(z,y)$

① $= d(x,z)+d(z,y) \geq \quad \geq d(x,y)$

$= d(x,z)+d(z,y) \geq d'(x,z)+d'(z,y) \geq d'(x,y)$ $\Big\} \Rightarrow \geq \max(d,d')(x,y)$

② $= d'(x,z)+d(z,y) \geq d(x,z)+d(z,y) \geq d(x,y)$

$= d'(x,z)+d(z,y) \geq d'(x,z)+d'(z,y) \geq d'(x,y)$ $\Big\} \Rightarrow \geq \max(d,d')(x,y)$

③ $= d'(x,z)+d'(z,y) \geq \quad \geq d'(x,y)$

$= d'(x,z)+d'(z,y) \geq d(x,z)+d(y,z) \geq d(x,y)$ $\Big\} \Rightarrow \geq \max(d,d')(x,y)$

④ $= d(x,z)+d'(z,y) \geq d(x,z)+d(y,z) \geq d(x,y)$

$= d(x,z)+d'(z,y) \geq d'(x,z)+d'(y,z) \geq d'(x,y)$ $\Big\} \Rightarrow \geq \max(d,d')(x,y)$

$\Downarrow$ def of max $\qquad$ $\Downarrow$ triangular of metric $d.d'$

Exercise 1.3. *Let d be a metric on X and d' be a metric on X'. Prove that*

$$D((x,x'),(y,y')) = \max(d(x,y), d'(x',y')) \quad \text{(product metric)}$$

is a metric on $X \times X'$.

Pf: 1) Identity. $D((x,x'),(y,y')) = \max(d(x,y).\, d'(x',y'))=0$. since $d.d' \geq 0$.

$d(x,y)=0$ and $d'(x',y')=0 \iff x=y$ and $x'=y' \iff (x,x')=(y,y')$

2) Symmetry. $D((x,x'),(y,y')) = \max(d(x,y), d'(x',y')) = \max(d(y,x), d'(y',x'))$
$$= D((y,y'),(x,x'))$$

3) Triangle $D((x,x'),(z,z')) + D((z,z'),(y,y'))$.
$$= \max(d(x,z), d(x',z')) + \max(d(z,y), d(z',y')).$$

① $\leq d(x,z)+d(z,y) \leq d(x,y)$
tri. inequality of d $\Rightarrow \leq \max(d(x,y), d(x',y'))$

② $\leq d(x',z')+d(z',y') \leq d(x',y')$
tri. inequality of d' $\quad = D((x,x'),(y,y'))$

Exercise 1.4. *Let (X,d) be a metric space and $x,y,z,w \in X$. Show that*

$$|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w).$$

Pf: since $d \geq 0$.

1). $d(x,y) \geq d(z,w)$

$d(x,y) - d(z,w) \leq d(x,z) + d(y,z) - (d(z,y) - d(y,w)) = d(x,z) + d(y,w)$

$\Downarrow$

(by triangle inequality of d. $d(x,y) \leq d(x,z)+d(z,y)$. $d(z,w) \geq d(z,y) - d(y,w)$)

2) $d(x,y) < d(z,w)$ $\qquad \Uparrow$

$|d(x,y) - d(z,w)| = d(z,w) - d(x,y) \leq d(y,w) + d(x,z) - (d(x,z) - d(y,z)) = d(x,z)+d(y,w)$.

Thus. $|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$.

Exercise 1.5. *Let (X,d) be a metric space and $x \in X$. Prove that $\cup_{r>0} B_r(x) = X$ and $\cap_{r>0} B_r(x) = \{x\}$.*

① " $\bigcup_{r>0} Br(x) \subset X$. " X is whole set. trivially.

" $\bigcup_{r>0} Br(x) \supset X$ " $\forall y \in X$. Let $r_0 = d(x,y)+1$.

$y \in B_{r_0}(x) \subset \bigcup_{r>0} Br(x)$

Since y is arbitrary. $X \subset \bigcup_{r>0} Br(x)$.

② " $\bigcap_{r>0} Br(x) \supset \{x\}$ ". For any $r>0$. $d(x,x) = 0 < r$.

$x \in Br(x)$ for any $r>0$.

Thus. $x \in \bigcap_{r>0} Br(x)$, i.e. $\{x\} \subset \bigcap_{r>0} Br(x)$

" $\bigcap_{r>0} Br(x) \subset \{x\}$ "

Assume $y \in \bigcap_{r>0} Br(x)$ and $y \neq x$.

Then let $r_1 = \frac{d(x,y)}{2}$, $y \notin B_{r_1}(x)$. contradicts to $y \in \bigcap_{r>0} Br(x)$

thus no elements other than x. contained in $\bigcap_{r>0} Br(x)$.

Exercise 1.6. *Let d be a metric on X and d' be a metric on X'. If $D((x,x'),(y,y')) = \max(d(x,y), d'(x',y'))$ is a metric on $X \times X'$, prove that $B_r((x,x')) = B_r(x) \times B_r(x')$.*

Pf: " $\subset$ ". $\forall (x_0, x_0') \in Br(x, x')$

$\Rightarrow D[(x_0, x_0'), (x, x')) < r.$ (def of ball)

$\Rightarrow \max(d((x_0, x), d'(x', x_0'))) < r.$ (def of metric D)

$\Rightarrow d(x_0, x) < r$ and $d'(x', x_0') < r.$ (def of maximum)

$\Rightarrow x_0 \in Br(x)$ and $x_0' \in Br(x'). \Rightarrow (x_0, x_0') \in Br(x) \times Br(x')$

" $\supset$ ". $\forall (y_0, y_0') \in Br(x) \times Br(x')$.

$\Rightarrow d(y_0, x) < r$ and $d'(y_0', x') < r$ (def of ball)

$\Rightarrow \max(d(y_0, x), d(y_0', x')) < r.$ (def of maximum)

$\Rightarrow D((y_0, y_0'), (x, x')) < r \Rightarrow (y_0, y_0') \in Br(x, x'))$

Exercise 2.1. *Show that the function* $f(x) = x^2$

i) $f\colon (\mathbb{Q}, |\cdot|) \to (\mathbb{Q}, |\cdot|_p)$ *is not continuous*

ii) $f\colon (\mathbb{Q}, |\cdot|_p) \to (\mathbb{Q}, |\cdot|)$ *is not continuous*

iii) $f\colon (\mathbb{Q}, |\cdot|_p) \to (\mathbb{Q}, |\cdot|_p)$ *is continuous*

10. Let p be a fixed prime and for $n \in \mathbb{N}$ define $\nu_p(n) = k$, where $p^k \mid n$ and $p^{k+1} \nmid n$. Extend ν_p to $\mathbb{Q}$ by setting

$$\nu_p(\tfrac{a}{b}) = \begin{cases} \nu_p(a) - \nu_p(b) & \text{if } \frac{a}{b} \neq 0, \\ \infty & \text{if } \frac{a}{b} = 0 \end{cases} \quad (p\text{-adic norm})$$

$X = \mathbb{Q}$, $d(x, y) = |x - y|_p$, where $|r|_p = p^{-\nu_p(r)}$ (*p-adic metric*)

i). let $x_1, x_2 \in \mathbb{Q}$. where $x_2 = \sqrt{x_1^2 + \frac{1}{p^n}}$.

since $|x_2 - x_1| = \left|\sqrt{x_1^2+\frac{1}{p^n}} - x_1\right| = \left|\dfrac{\frac{1}{p^n}}{x_1 + \sqrt{x_1^2+\frac{1}{p^n}}}\right| \leq \left|\dfrac{1}{2x_1 \cdot p^n}\right| \to 0$

for any $\delta > 0$, we can find $n \in \mathbb{N}$. s.t. $|x_2 - x_1| \leq \left|\frac{1}{2x_1 \cdot p^n}\right| < \delta$.

$|x_2^2 - x_1^2|_p = |\frac{1}{p^n}|_p = p^{-V_p(\frac{1}{p^n})} = p^n \geq 1$.

Thus the function is not continuous at x_1.

ii). let $x_1, x_2 \in \mathbb{Q}$. where $x_2 = x_1 + p^n$.

$|x_2 - x_1|_p = |p^n|_p = p^{-n}$.

for any $\delta > 0$, we can find $n \in \mathbb{N}$. s.t. $|x_2 - x_1|_p = p^{-n} < \delta$.

$|x_2^2 - x_1^2| = |x_1 + x_2|\,|x_1 - x_2| = |x_1 + x_2| \cdot p^n > |x_1| \cdot p^n := \varepsilon_0. \not\to 0.$

since x_1 is fixed. f not continuous at x_1

iii) let $x_1 = \frac{b_1}{a}$. $x_2 = \frac{b_2}{a}$. $b_1, b_2 \in \mathbb{Z}$. (if the dominator doesn't equal at first, use the least common multiple).

let $a = \alpha_0 p^m$. where m is nonnegative integer

$V_p(x_1 - x_2) = V_p(b_1 - b_2) - V_p(a) = V_p(b_1 - b_2) - m$

$V_p(x_1^2 - x_2^2) = V_p(b_1^2 - b_2^2). - V_p(a^2) = V_p(b_1^2 - b_2^2) - 2m.$

if. $p \nmid b_1 - b_2 \Rightarrow V_p(b_1 - b_2) = -m \Rightarrow |x_1 - x_2|_p = p^m \geq 1$.

we can always let $\delta < 1$ to omit the case.

thus. rewrite $b_1 - b_2 = \alpha p^k$. $|x_1 - x_2|_p = p^{m-k}$.

$|x_1^2 - x_2^2|_p = |(x_1 - x_2)(x_1 + x_2)|_p = p^{2m-(k+l)}$.

where $p^l \mid b_1 + b_2$, $p^{l+1} \nmid b_1 + b_2$, l is nonnegative integer

Thus. for any $\varepsilon > 0$. $\exists \delta = \min\{1, \frac{\sqrt{\varepsilon}}{p^{\frac{k}{2}}}\}$

for any $|x_1 - x_2|_p < \delta$.

$|x_1^2 - x_2^2|_p = \delta^2 \cdot p^{k-l} \leq \delta^2 \cdot p^k < \varepsilon$

Thus. f is continuous on $\mathbb{Q}$.

$d_p(x, y) < p^{-n}$

$x - y = p^n \cdot r$.

$x = y + p^n \cdot r$

$x^2 = y^2 + p^n \cdot s$

$d_p(x^2, y^2) < p^{-n}$.

Exercise 2.2.* *Let (X, d) and (Y, d') be metric spaces and $A \subset X, x \in A$. Prove that if $f\colon X \to Y$ is continuous at x then $f|_A\colon A \to Y$ is continuous at x, where the metric on A is the restriction of d.*

Pf. since $x \in A$. $\exists \delta_1$, for any $y \in \delta_1$. $d|_A(x,y) = d(x,y)$.

$f: X \to Y$ is cont. at x. $\forall \varepsilon > 0$. $\exists \delta_2 > 0$, For any $d(x,x') < \delta_2$ $d'(f(x), f(x')) < \varepsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$. For any $d|_A(x,x') = d(x,x') < \delta < \delta_2$.

we also have $d'(f(x), f(x')) < \varepsilon$, thus $f|_A$ is cont. at x

$f|_A^{-1}(B) = f^{-1}(B) \cap A$.

B is open in Y.

let $x \in A$. $f(x) \in Y$.

$\forall \varepsilon > 0$. $\exists \delta > 0$: $f(B_\delta(x)) \sqsubset B_\varepsilon(f(x))$

$f|_A(B^A_\delta(x)) = f(B_\delta(x) \cap A)$

$\sqsubset B_\varepsilon(f(x))$

$\{y \in A \mid d(y,x) < \delta\} = \{y \in X \mid d(y,x) < \delta\} \cap A$.

Exercise 2.3. *Show that every subsequence of a convergent sequence in a metric space converges to the same limit.*

Pf: Assume the converse.

Let sequence $\{x_n\}$ and its subsequence $\{x_{n_k}\}$. $\lim\limits_{n\to\infty} x_n = x$. $\lim\limits_{k\to\infty} x_{n_k} = y$.

let $\varepsilon_0 = \dfrac{d(x,y)}{4}$. $\exists N_1 \in \mathbb{N}$. $d(x_n, x) < \varepsilon_0$ for any $n > N_1$.

$\exists N_2 \in \mathbb{N}$. $d(x_{n_k}, y) < \varepsilon_0$ for any $n_k > N_2$.

$\exists N_3 \in \mathbb{N}$. $d(x_n, x_m) < \varepsilon_0$. for any $n, m > N_3$.

Let $N = \max\{N_1, N_2, N_3\}$ for any $n_k, n > N$.

$4\varepsilon_0 = d(x,y) \leq d(x_n, x) + d(x_{n_k}, y) + d(x_n, x_m) < 3\varepsilon_0$

since $\varepsilon_0 > 0$. this inequality causes a contradiction.

Exercise 2.4. *Prove that a sequence $\{x_n\}$ in a discrete metric space X converges if and only if there are $x \in X$ and $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.*

"$\Rightarrow$" $\{x_n\}$ conv. $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$. $d(x_n, x) < \varepsilon$. for any $n \in \mathbb{N}$.

if $\exists x_n \neq x$ for some $n > N$. $d(x_n, x) = 1$.

Let $\varepsilon_0 = \frac{1}{2}$. then $d(x, x_n) = 1 > \varepsilon_0$. $\{x_n\}$ not conv.

thus. $x_n = x$. for all $n > N$.

"$\Leftarrow$" $\forall \varepsilon > 0$. $\exists N$, for any $n > N$. $d(x_n, x) = 0 < \varepsilon$.

$\{x_n\}$ in X conv. to x.

Exercise 2.5. *Which of the following sequences converge in $\mathbb{Q}$ with respect to the p-adic metric:*

i) $\{p^n/(p^n-1)\}$

ii) $\{n!\}$

iii) $\{1/n\}$

10. Let p be a fixed prime and for $n \in \mathbb{N}$ define $\nu_p(n) = k$, where $p^k \mid n$ and $p^{k+1} \nmid n$. Extend ν_p to $\mathbb{Q}$ by setting

$$\nu_p(\tfrac{a}{b}) = \begin{cases} \nu_p(a) - \nu_p(b) & \text{if } \frac{a}{b} \neq 0, \\ \infty & \text{if } \frac{a}{b} = 0 \end{cases} \quad (p\text{-adic norm})$$

$X = \mathbb{Q}, d(x,y) = |x-y|_p$, where $|r|_p = p^{-\nu_p(r)}$ (*p-adic metric*)

i). $\left|\frac{p^n}{p^n-1} - 0\right|_p = \left|\frac{p^n}{p^n-1}\right|_p = p^{-n} \to 0$

$\forall \varepsilon > 0.$ $\exists N \in \mathbb{N}.$ for any $n > N.$ $p^{-n} < \varepsilon.$

thus. $\lim \left|\frac{p^n}{p^n-1}\right|_p = 0.$ $0 \in \mathbb{Q}$ conv.

ii) $|n! - 0|_p = p^{-j}$. where $p^j \mid n!$ $p^{j+1} \nmid n!$

$\forall \varepsilon > 0.$ $\exists N \in \mathbb{N}.$ for any $n > N.$ $p^{-j} < \varepsilon.$

$\lim |n!|_p = 0.$ $0 \in \mathbb{Q}.$ conv.

iii) not conv.

For any $N \in \mathbb{N}$, we can always find $p^{n_k} > N$.

$\left|\frac{1}{p^{n_k}}\right|_p = p^{n_k}$. $\left|\frac{1}{p^{n_k}+1}\right| = p^{-0} = 1.$

can't find a limit.

Exercise 2.6. *Let (X,d) be a metric space.*

i) *Let $y \in X$. Prove that $f\colon X \to \mathbb{R}$, $f(x) = d(x,y)$ is continuous.*

ii) *Prove that $d\colon X \times X \to \mathbb{R}$ is continuous, where $X \times X$ is equipped with the product metric (see Exercise 1.3)*

i). $\forall x \in X.$

$\forall \varepsilon > 0.$ $\exists \delta = \varepsilon.$ for any $x' \in X$ s.t. $d(x, x') < \delta$

$|f(x) - f(x')| = |d(x,y) - d(x',y)| \leq |d(x,x')| < \delta = \varepsilon.$

f is cont. at x. Since x is arbitrary. f is cont. on X.

ii). $\forall (x, x') \in X \times X$

$\forall \varepsilon > 0.$ $\exists \delta = \frac{\varepsilon}{3}$ for any (y, y') s.t. $D((x,x'),(y,y')) < \delta.$

i.e. $d(x,y) < \delta$ and $d(x',y') < \delta.$

$$|d(y,y') - d(x,x')| \underset{\substack{\Downarrow \\ \text{we have showed in Ex 1.4.}}}{\leq} d(x,y) + d(x',y') \leq 2\delta = \frac{2\varepsilon}{3} < \varepsilon$$

Exercise 3.1. *Prove that in a finite metric space, any subset is open.*

Pf: Let (X, d) is metric space.

let $A \subset X$.

$\forall x \in A$. since the space is finite. we can find $r_0 > 0$.

s.t. $B_{r_0}(x) \cap X = \{x\}$.

thus. in subset A. $B_{r_0}(x) = \{x\} \subset A$.

Exercise 3.2. *Let U be an open subset of a metric space (X, d). Show that $U \setminus \{y\}$ is also open for any $y \in U$.*

$\forall x \in U$, since U is open. then $\exists \delta > 0$. $B_\delta(x) \subset U$.

1). if $y \notin B_\delta(x)$. $B_\delta(x) \subset U \setminus \{y\}$

2) if $y \in B_\delta(x)$. Let $\delta' = \frac{d(x,y)}{2}$

$B_{\delta'}(x) \subset B_\delta(x) \setminus \{y\} \subset U \setminus \{y\}$.

Since x is arbitrary. $U \setminus \{y\}$ also open.

Exercise 3.3. *Let X, Y be two metric spaces and $f \colon X \to Y$. Let $X = A \cup B$ for open subsets $A, B \subset X$ such that $A \cap B = \varnothing$. Prove that if $f|_A$ and $f|_B$ are continuous then f itself is continuous.*

Let. d, d' be metrics of X, Y.

Pf: $f|_A$ cont. at x. $\forall \varepsilon > 0$. $\exists \delta_1 > 0$, for any $x_1, \cancel{x_2}\, x \in A$. s.t. $d|_A(x_1, \cancel{x_2}\, x) < \delta_1$

$d'(f|_A(x_1), f|_A(\cancel{x_2}\, x)) < \varepsilon$.

$f|_B$ cont. x' $\forall \varepsilon > 0$. $\exists \delta_2 > 0$, for any $\cancel{x_3, x_4}\, x_2 \in B$. s.t. $d|_B(\cancel{x_3}\, x', \cancel{x_4}\, x_2) < \delta_2$

$d'(f|_B(\cancel{x_1}\, x'), f|_B(x_2)) < \varepsilon$.

$f|_A : A \to Y \Rightarrow \forall \varepsilon > 0. \exists \delta > 0.$

$f(B_\delta(x)) \subset B_\varepsilon(f(x))$

Since A, B are open. and $X = A \cup B$.

$\forall x \in X$. w.l.g. $x \in A$. $\exists \delta_3$. $B_{\delta_3}(x) \subset A$.

A is open, $x \in A$. $\Rightarrow \exists r > 0 : B_r(X) \subset A$.

Let. $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. for any x'. s.t. $d(x, x') < \delta$. $\delta' = \min(r, \delta)$

we have $d'(f(x), f(x')) < \varepsilon$

Oct. 31st. Week 10.

Exercise 3.4. *Let X, Y be two metric spaces and let $X \times Y$ be equipped with the product metric (Exercise 1.3).*

i) *Prove that $A \times B$ is open in $X \times Y$ if $A \subset X, B \subset Y$ are open.*

ii) *Prove that the projection map $\pi_X : X \times Y \to X, \pi_X((x,y)) = x$ is continuous.*

Hint. Use Exercise 1.6.

Pf: i). $A \subset X$. $B \subset Y$ is open

$\forall x \in A. y \in B$ $\exists \delta_1, \delta_2$. $B_{\delta_1}(x) \subset A$. $B_{\delta_2}(y) \subset B$.

let $\delta = \min(\delta_1, \delta_2)$.

consider a open ball $B_\delta(x,y)$.

s.t. for any $(x', y') \in B_\delta(x,y)$. $D((x',y'),(x,y)) = \max\{d(x,x'), d(y,y')\} \leq \delta$.

$d(x', x) \leq \delta \leq \delta_1$ $\quad x' \in B_{\delta_1}(x) \subset A$. (or by Ex 1.6. $B_\delta(x,y) = B_\delta(x) \times B_\delta(y)$

$d(y', y) \leq \delta \leq \delta_2$. $\quad y' \in B_{\delta_2}(y) \subset B$. $\quad B_\delta(x) \subset B_{\delta_1}(x) \subset A$. $B_\delta(y) \subset B_{\delta_2}(y) \subset B$)

$(x', y') \subset A \times B$, since (x', y') is arbitrary. $B_\delta(x,y) \subset A \times B$.

Thus. $\forall (x,y) \in A \times B \cdot \exists \delta > 0 \cdot B_\delta(x,y) \subset A \times B$

ii). Consider arbitrary point $(x,y) \in X \times Y$.

$\forall \varepsilon > 0$. $\exists \delta = \varepsilon$. for any. (x', y') s.t. $D((x,y),(x',y')) < \delta$,

we have $d(x,x') < \delta$ and $d(y,y') < \delta$.

$$\left| d(\pi(x,y), \pi(x',y')) \right| = |d(x,x')| < \delta = \varepsilon$$

Exercise 3.5. *Prove that in the above example $[0,1]$ is an open subset of Y using the definition of open subset.* $Y = [0,1] \cup [2,3]$

Pf: for the inner point of $[0,1]$, that is point $a \in (0,1)$.

we can simply let $r = \min\left(\frac{d(0,a)}{2}, \frac{d(a,1)}{2}\right)$. $B_r(a) \subset Y$.

for the endpoint. w.l.g. consider $y = 0$.

$\forall \varepsilon \in (0,1)$. $\quad B^Y_\varepsilon(0) = \{y' \in Y \mid d(0,y') < \varepsilon\} = [0, \varepsilon)$

since $\varepsilon \in (0,1)$, $\quad B^Y_\varepsilon(0) \subset [0,1] \subset Y$.

$\therefore y = 0$ is the interior point.

$y = 1$. similarly. thus $[0,1]$ is open.

Exercise 3.6. *Let (X, d) be a metric space.*

i) *Prove that $d'(x, y) = \sqrt{d(x, y)}$ is a metric on X.*

ii) *Prove that d' is equivalent to d.*

i) identity. $d'(x,y)=0 \Leftrightarrow \sqrt{d(x,y)}=0 \Leftrightarrow d(x,y)=0 \Leftrightarrow x=y$.

symmetry. $d'(x,y)=\sqrt{d(x,y)} = \sqrt{d(y,x)} = d'(y,x)$.

triangle inequ. $d'(x,y) = \sqrt{d(x,y)} \leq \sqrt{d(x,z)+d(y,z)} \underset{①}{\leq} \sqrt{d(x,z)}+\sqrt{d(y,z)} \leq d'(x,z)+d'(y,z)$

(inequality ① holds because $0 \leq 2\sqrt{a}\sqrt{b} \Leftrightarrow a+b \leq a+2\sqrt{a}\sqrt{b}+b$

$a+b \leq (\sqrt{a}+\sqrt{b})^2 \Leftrightarrow \sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$. for an $a, b \geq 0$.).

ii). $\forall U \subset X$. U is open.

$\forall x \in U$, $\exists \delta_x > 0$. s.t. $B_{\delta_x}(x) \subset U$.

$B_{\delta_x}(x) = \{ y \in U \mid d(x,y) < \delta_x \}$.

$\forall y \in U$, $d(x,y) < \delta_x$ since $d(x,y) \geq 0$, $\delta_x > 0$.

$d(x,y) < \delta_x \Rightarrow \sqrt{d(x,y)} < \sqrt{\delta_x} \Rightarrow d'(x,y) < \sqrt{\delta_x}$. $\Rightarrow y \in B'_{\sqrt{\delta_x}}(x)$

$B_{\delta_x}(x) = B'_{\sqrt{\delta_x}}(x)$, which shows the equivalence.

Exercise 3.7. *Let $\mathcal{B}$ be a base of a metric space X. Prove that any open subset of X is the union of elements of $\mathcal{B}$.*

$\forall x \in V$. $\exists \omega_x$: $x \in U_{\omega_x} \subset V$.

$\bigcup_{x \in V} U_{\omega_x} = V$

Pf. $\mathcal{B} = \{U_\omega\}_{\omega \in \Omega}$ $\forall V \subset X$. V is open

By Pro 3.2. $V = \bigcup_{x \in V} B_{\delta_x}$

1). $V \subset \bigcup_{x \in V} U_{\omega_x}$

$\forall x \in V \subset U$. $\exists U_{\omega_x} \in \mathcal{B}$ s.t. $x \in U_{\omega_x} \subset \bigcup U_{\omega_x}$.

since x is arbitrary. $V \subset \bigcup_{x \in V} U_{\omega_x}$.

2). $\bigcup U_{\omega_x} \subset V$.

$\forall U_{\omega_x}$, $U_{\omega_x} = \bigcup_{x \in U_{\omega_x}} B_{\delta_x}$ by Pro 3.1.

since $U_{\omega_x} \subset V$. $\Rightarrow \bigcup_{x \in U_{\omega_x}} B_{\delta_x} \subset \bigcup_{x \in V} B_{\delta_x}$

thus $U_{\omega_x} \subset \bigcup_{x \in V} B_{\delta_x}$ for every U_{ω_x}. i.e. $\bigcup U_{\omega_x} \subset V$

Thus, $V = \bigcup U_{\omega_x}$ where U_{ω_x} is element from $\mathcal{B}$

Nov. 2nd.

Exercise 4.1. *The closed ball with center $x \in X$ and radius $r > 0$ is defined as*

$$\overline{B}_r(x) = \{y \in X \mid d(x,y) \le r\}.$$

Prove that the closed ball is closed.

Pf: $X \setminus \overline{B_r}(x) = \{ y \in X \mid d(x,y) > r \}$

$\forall y \in X \setminus \overline{B_r}(x)$. let $\varepsilon = \frac{d(x,y)-r}{2} > 0$, $B_\varepsilon(y) \subset X \setminus \overline{B_r}(x)$.

since $\forall y_0 \in B_\varepsilon(y)$. $d(y_0, x) \ge d(x,y) - d(y_0,y)$

$$\ge d(x,y) - \frac{d(x,y)-r}{2} = \frac{d(x,y)+r}{2} > r.$$

$X \setminus \overline{B_r}(x)$ is open.

Exercise 4.2. *Let (X, d) be a metric space.*

i) *Prove that any open subset of X can be expressed as a union of closed balls.*

ii) *Prove that any closed subset of X can be expressed as an intersection of open sets.*

i) Pf: $\forall U \subset X$, U is open.

$\forall x \in U$, $\exists \delta > 0$, $B_\delta(x) \subset U$.

we can claim that $\overline{B_{\frac{\delta}{2}}}(x) \subset B_\delta(x) \subset U$.

thus. for every x, we can find a closed ball belongs to U.

$U = \bigcup_{x \in U} \overline{B_{\frac{\delta_x}{2}}}(x)$

ii). $\forall V \subset X$. V is closed.

$X \setminus V$ is open. $\forall y \in X \setminus V$, $\exists \delta > 0$ $B_\delta(y) \subset X \setminus V$

we have $\overline{B_{\frac{\delta}{2}}}(y) \subset B_\delta(y) \subset X \setminus V$, $X \setminus \overline{B_{\frac{\delta}{2}}}(y)$ is open.

for every $y \in X \setminus V$, we can find $\overline{B_{\frac{\delta}{2}}}(y) \subset X \setminus V$. thus $V \subset X \setminus \overline{B_{\frac{\delta}{2}}}(y)$.

Thus $V \subset \bigcap_{y \in X \setminus V} X \setminus \overline{B_{\frac{\delta_y}{2}}}(y)$

And $\forall y' \in \bigcap X \setminus \overline{B_{\frac{\delta_y}{2}}}(y)$ for any y, $y' \notin \overline{B_{\frac{\delta_y}{2}}}(y)$ for any y.

$y' \notin X \setminus V$. $\Rightarrow y' \in V$. $\Rightarrow V \supset \bigcap_{y \in X \setminus V} X \setminus \overline{B_{\frac{\delta_y}{2}}}(y)$.

$$V = \bigcap_{y \in X \setminus V} X \setminus \overline{B_{\frac{\delta_y}{2}}}(y).$$

Exercise 4.3. *Let X, Y be two metric spaces and $A \subset X, B \subset Y$ be closed. Prove that $A \times B$ is closed in $X \times Y$ equipped with the product metric (Exercise 1.3).*

Pf: show $X \times Y \setminus A \times B$ is open.

$X \times Y \setminus A \times B = \{ (x,y) \mid (x,y) \notin A \times B \} = \{ (x,y) \mid x \notin A \text{ and } y \notin B \}$

$= \{ (x,y) \mid x \in X \setminus A,\ y \in Y \setminus B \}. = \{ (x,y) \mid (x,y) \in (X \setminus A) \times (Y \setminus B) \}.$

since $X \setminus A$, $Y \setminus B$ are open.

$\forall x_1 \in X \setminus A,\ y_1 \in Y \setminus B,\ \exists \delta_1, \delta_2 > 0.\quad B_{\delta_1}(x_1) \subset X \setminus A,\ B_{\delta_2}(y_1) \subset Y \times B.$

Let $\delta = \min(\delta_1, \delta_2)$. $\quad B_\delta(x_1) \subset X \setminus A,\ B_\delta(y_1) \subset Y \setminus B.$

$B_\delta(x_1, y_1) = B_\delta(x_1) \times B_\delta(y_1) \subset X \setminus A \times Y \setminus B \quad$ (by Ex 1.6.)

Thus $X \times Y \setminus A \times B$ is open

prove:

$(X \times Y) \setminus (A \times B) = (X \setminus A) \times Y \cup X \times (Y \setminus B)$

$X \setminus A$, Y are open.

Nov. 7th.

Exercise 4.4. *Prove that if two metrics d and d' on X are equivalent then the closures of any subset A with respect to them are equal.*

d and d' are equivalent.

the family of all closed set coincide

$\bar{A}$ = intersection of all closed subset contains A.

$\bar{A}_d = \bigcap_{\substack{F \text{ is closed in } d' \\ A \subset F}} F = \bigcap_{\substack{F \text{ is closed in } d \\ A \subset F}} F = \bar{A}_{d'}$

Pf: denote $\bar{A}$ closure w.r.t. d. $\bar{A}'$ w.r.t. d'.

it suffices to check $X \setminus \bar{A} = X \setminus \bar{A}'$

since $X \setminus \bar{A}$. $X \setminus \bar{A}'$ are open.

by pro 3.2. $X \setminus \bar{A} = \bigcup B_{\delta_x}(x)$

For any $B_{\delta_x}(x)$, it's open in (X, d), the equivalence shows. it's open in (X, d')

thus for the center point. x $\exists \delta_x'$. $B'_{\delta_x'}(x) \subset B_\delta(x)$.

$X \setminus \bar{A}' = \bigcup B'_{\delta_x'}(x) \subset \bigcup B_{\delta_x}(x) = X \setminus \bar{A}$.

"$\supset$" $\forall y \in \bigcup B_{\delta_x}(x)$, $y \in B_{\delta_y}(y)$. $B_{\delta_y}(y)$ is open in $X \setminus \bar{A}'$ $\exists \delta_y'$, $y \in B'_{\delta_y'}(y) \subset B_{\delta_y}(y)$.

thus $y \in \bigcup B'_{\delta_x'}(x)$ Since y is arbitrary. $\bigcup B_{\delta_x}(x) \subset \bigcup B'_{\delta_x'}(x)$

$X \setminus \bar{A}' = \bigcup B'_{\delta_x'}(x) = \bigcup B_{\delta_x}(x) = X \setminus \bar{A} \Rightarrow \bar{A} = \bar{A}'$

Exercise 4.5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that its graph $\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}\}$ is a closed subset of $\mathbb{R}^2$.*

1) $\mathbb{R}^2 \setminus \Gamma_f$ is open

2) $(x_n, f(x_n)) \to (a, b) \overset{?}{\Rightarrow} (a,b) \in \Gamma_f$

$\Updownarrow$

$f(x_n) \to b$ when $x_n \to a$ $\quad$ $b = f(a)$

since $x_n \to a$. cont. $f(x_n) \to f(a) = b$.

Pf: let $\{(x_n, f(x_n))\} \subset \Gamma_f$. and $\lim (x_n, f(x_n)) = (x, f(x))$.

its suffices to check. $(x, f(x)) \in \Gamma_f$.

$\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$. for any $n > N$, $d((x_n, f(x_n)), (x, f(x))) < \varepsilon$.

$\Rightarrow d(x_n, x) < \varepsilon$ and $d(f(x), f(x_n)) < \varepsilon$.

$\Rightarrow$ thus we can claim that. $\lim f(x_n) = f(x)$ for any $\{x_n\}$ with $\lim x_n = x$.

By Pro 2.3. and f is cont. $(x, f(x)) \in \Gamma_f$.

Exercise 4.6. *Prove that $\overline{A \cup B} = \bar{A} \cup \bar{B}$ for any subsets A, B.*

$x \in \overline{A \cup B} \Rightarrow \{C_n\} \to x$. $\Rightarrow C_{n_i} \in A$. $\Rightarrow x \in \bar{A}$

$C_n \in A \cup B$. $\quad C_{n_i} \to x$

"$\subset$" $\forall x \in \overline{A \cup B}$.

任取但已给定

$\forall \varepsilon > 0$. $\exists x_1 \in B_\varepsilon(x) \cap (A \cup B)$.

$\Rightarrow x_1 \in B_\varepsilon(x)$ and $x_1 \in (A \cup B)$

w.l.g. $x_1 \in A$. thus $x_1 \in B_\varepsilon(x) \cap A$. for arbitrary $\varepsilon > 0$.

仅说明了 some

thus $x \in \bar{A}$. $\Rightarrow \overline{A \cup B} \subset \bar{A} \cup \bar{B}$.

"$\supset$". $\forall x' \in \bar{A} \cup \bar{B}$. w.l.g. let $x' \in \bar{A}$.

$\Rightarrow \forall \varepsilon' > 0$, $\exists x_1' \in B_{\varepsilon'}(x') \cap A$

$x_1' \in B_{\varepsilon'}(x')$ and $x_1' \in A \subset A \cup B$.

thus $x_1' \in B_{\varepsilon'}(x') \cap (A \cup B)$. $\Rightarrow x_1' \in \overline{A \cup B}$ $\Rightarrow \bar{A} \cup \bar{B} \subset \overline{A \cup B}$

Exercise 4.7. *Find two $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.*

Let $A = (0,1)$. $B = (1,2)$.

$\overline{A \cap B} = \overline{\phi} = \phi$.

$\bar{A} \cap \bar{B} = \{1\}$.

Exercise 5.1. *If $\{x_n\}, \{y_n\}$ are Cauchy sequences in a metric space (X, d), show that the sequence $a_n = d(x_n, y_n)$ converges in $\mathbb{R}$.*

Pf: $\forall \varepsilon > 0$. $\exists N_1$. for any $m_1, n_1 > N_1$ $d(x_{m_1}, x_{n_1}) < \frac{\varepsilon}{2}$.

$\exists N_2$, for any $m_2, n_2 > N_2$ $d(x_{m_2}, x_{n_2}) < \frac{\varepsilon}{2}$.

Let $N = \max(N_1, N_2)$. for any $m, n > N$

$\left| d(x_m, y_m) - d(x_n, y_n) \right| \leq d(x_m, x_n) + d(y_m, y_n) < \varepsilon$. (by Ex. 1.4.)

thus the sequence $\{a_n\}$ is Cauchy's. $\exists a = \lim a_n$.

the space $\mathbb{R}$ is complete trivially. $a \in \mathbb{R}$. $\{a_n\}$ conv.

Exercise* 5.2. *Let $X = \mathbb{N}, d(m, n) = |1/m - 1/n|$. Show that (X, d) is not complete.*

Pf: consider sequence $x_n = n$.

$\forall \varepsilon > 0$. $\exists N = [\frac{1}{\varepsilon}]$, for any $n, m > N$.

w.l.g. let $n > m$. $d(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right| = \left|\frac{n-m}{mn}\right| \leq \left|\frac{n-m}{m}\right| \cdot \left|\frac{1}{n}\right| < \left|\frac{1}{n}\right| < \varepsilon$.

thus $\{x_n\}$ is Cauchy's.

But. $\lim x_n$ not exist

$\forall x \in X$. $d(x, x_m) = \left|\frac{m-x}{mx}\right| \leq \left|\frac{1}{x}\right| \nrightarrow 0$.

not the limit.

Exercise 5.3. *Show that a metric space with a finite number of elements is complete.*

Let $X = \{x_1, \dots x_n\}$. d be it's metric.

denote $\frac{\min d(x_i, x_j)}{2} = \varepsilon_0$. $i, j = 1, 2, \dots n$.

Let $\varepsilon = \varepsilon_0$. for any sequence $\{x_{n_k}\}$ in the X. we have. $d(x_m, x_n) \geq \varepsilon_0$.

thus there is no Cauchy sequence.

Which satisfy the def. of completeness automatically.

Exercise 5.4. *Show that $\mathbb{Z}$ with p-adic metric is not complete.*

Hint. $1 + p + p^2 + \cdots = \frac{1}{1-p}$?

$\frac{1-p^{n+1}}{1-p}$

Consider the sequence. $x_n = \sum_{k=0}^{n} p^k$.

$\forall \varepsilon > 0$. $\exists N = [-\frac{\varepsilon}{\ln p}]$. for any $m, n > N$. w.l.g. $n > m$

$d(x_m, x_n)_p = |p^m(p^{n-m} - 1)|_p = p^{-m} < \varepsilon$. thus it's Cauchy's.

but $\lim_{n\to\infty} x_n = \frac{1-p^{n+1}}{1-p} \to +\infty$, limit not exist, the space is not complete.

$\forall z \in \mathbb{Z}$, $d(z, x_n) = \left| z - \frac{1-p^{n+1}}{1-p} \right| = \left| \frac{z - zp - 1 + p^{n+1}}{1-p} \right|$

Exercise 5.5. *Prove that the intersection of any collection of complete subsets of a metric space is complete.*

Pf: Let (X, d) be metric space. $\{U_i\}$, $U_i \subset X$. U_i is complete subset.

Consider $\{x_n\} \subset \cap U_i$ and $\{x_n\}$ is Cauchy Sequence.

thus $\{x_n\}$ is Cauchy's in every U_i.

Since U_i is complete. $\exists x \in U_i$. s.t. $\lim x_n = x$.

$x_n \in A_\omega$, $\forall \omega \in \Omega$. $x_n \to x$ in A_ω
$x_n \in A_{\omega'}$ $x_n \to x'$ in $A_{\omega'}$ $\Big| \Rightarrow x = x'$

x belong to every U_i. i.e. $x \in \cap U_i$

uniqueness of limit needs to be formulated

Thus $\cap U_i$ is complete.

Nov. 14th

Exercise 6.1. *Prove that φ is bijection if X is complete.*

Pf: $\forall x \in X$. $\varphi(X) = [x]$. it suffices to show φ is surjective.

$\forall [S] \in \hat{X}$. let $s = \{x_i\} \in S$.

$\{x_i\}$ is one of Cauchy sequence in this equivalent class.

X is complete. $\exists x = \lim\limits_{i\to\infty} x_i$, $x \in X$. thus $[s] \sim [x]$.

i.e. $\forall [S] \in \hat{X}$, $\exists x \in \varphi(X)$. s.t. $\varphi(x) = [S] \Rightarrow \varphi$ is surjective.

Exercise 6.2. *Prove that any isometry is bijective.*

Pf: (X, d). (Y, d') be metric space. $f: X \to Y$. is isometry.

it suffices to show it's injective

let $f(x_1) = f(x_2)$ $\Rightarrow d'(f(x_1), f(x_2)) = 0$ (identity of metric)

$\Rightarrow d(x_1, x_2) = 0$ (def. of isometry).

$\Rightarrow x_1 = x_2$ (identity of metric)

which implies the injectivity.

Exercise 6.3. *Let (Y, d) be a metric space and $X \subset Y$. Prove that there is an isometry $\psi : \overline{X} \to Z$ for a certain $Z \subset \hat{X}$.*

Denote $\psi(x) = \overline{\varphi}(X) = [x]$. for any $x \in \overline{X}$.

$\forall x' \in \overline{X}$. x' is adherent. $\forall \frac{1}{n} > 0$ $\exists x_n' \in X \cap B_\varepsilon(x')$. i.e. $d(x', x_n') < \frac{1}{n}$.

$\{x_n'\}$ forms Cauchy sequence and $\{x_n'\} \to x'$.

$[x'] \sim [x_n']$. where. $[x']$ is equivalent to $x', x', \dots$

$\psi(x') = [x_n']$. $\in \hat{X}$. since $\{x_n'\} \in S$.

Let $Z = \{z \mid z = \psi(x),\ x \in \overline{X}\}$. $Z \subset \hat{X}$. by def. of Z. ψ is surjective.

$\forall x_1, x_2 \in \overline{X}$. $D(\psi(x_1), \psi(x_2)) = D([x_{n_1}'], [x_{n_2}']) = \lim d(x_{n_1}', x_{n_2}')$

construct $\{x_{n_1}'\}$. $\{x_{n_2}'\}$ as above. $= d(x_1, x_2)$., since $\{x_{n_1}'\} \to x_1$. $\{x_{n_2}'\} \to x_2$.

Now we prove ψ is isometry

Exercise 7.1. *Let A, B be subsets of a metric space (X, d).*

i) *Prove that $A \cup B$ is bounded if A and B are bounded.*

ii) *Prove that $A \cup B$ is totally bounded if A and B are totally bounded.*

i). fixed $x_1 \in A$ and $x_2 \in B$. denote $d(x_1, x_2) = R_1$

A. B are bounded. $\exists R_2, R_3 > 0$. $d(x_1, x_1') < R_2$ for any $x_1' \in A$.

$d(x_2, x_2') < R_3$ for any $x_2' \in B$.

$\forall x, y \in A \cup B$.

1) if $x, y \in A$ or $x, y \in B$. denote $R = R_1$ or R_2. $A \cup B$ is bounded.

2) if x, y not belongs to the same set. w.l.g let $x \in A$. $y \in B$

$$d(x,y) \leq d(x, x_1) + d(x_1, x_2) + d(y, x_2) < R_1 + R_2 + R_3$$

denote $R = R_1 + R_2 + R_3$. $A \cup B$ is bounded.

ii). Pf: $\forall \varepsilon>0$. $\exists m_1, m_2 \in \mathbb{N}$. $A = \bigcup_{1\le k_1\le m_1} B_\varepsilon(x^1_{k_1})$ $B = \bigcup_{1\le k_2\le m_2} B_\varepsilon(x^2_{k_2})$

$$A \cup B = \bigcup_{\substack{i=1 \quad 1\le k_1\le m_1 \\ i=2 \quad 1\le k_2\le m_2}} B_\varepsilon(x^i_k).$$

Exercise 7.2. *Let X be the set of binary sequences $\{x_n\}$ with each $x_n \in \{0,1\}$. Define a metric on X by $d(\{x_n\},\{y_n\}) = \sum_{n\ge1} |x_n - y_n|/2^n$. Prove that (X,d) is totally bounded.*

Hint. For $\varepsilon > 0$, choose $m \in \mathbb{N}$ such that $\varepsilon > 1/2^m$ and consider the set of the sequences $\{y_n\} \in X$ such that $y_n = 0$ for all $n > m$.

Pf: $\forall \varepsilon>0$ $\exists m \in \mathbb{N}$. s.t. $\frac{1}{2^m} < \varepsilon$

consider the sequences $\{y_n\}$ such that $y_n = 0$ for all $n > m$.

the set $\{\{y_n\}\}$ has 2^m elements. ($y_1, \dots y_m$ can be 0 or 1)

Let these sequences be the center of balls.

$\forall \{x_n\} \in X$. $\exists \{y_n\}$ s.t. for $1 \le n \le m$. $x_n = y_n$.

$$d(\{x_n\}, \{y_n\}) = \sum_{n\ge1} |x_n - y_n|/2^n = \sum_{n=m+1}^{\infty} |x_n|/2^n \le \sum_{n=m+1}^{\infty} \frac{1}{2^n} < \frac{1}{2^m} < \varepsilon.$$

thus $\{x_n\} \in B_\varepsilon(\{y_n\})$. i.e. $X = \bigcup B_\varepsilon(\{y_n\})$. $\to$ 2^m balls, finite.

Exercise 7.3. *Prove that the completion of a totally bounded metric space X is totally bounded.*

Hint. For $\varepsilon > 0$, choose $x_1, \dots, x_n \in X$ such that $X = \cup_{j=1}^n B_{\varepsilon/2}(x_j)$ and show that $\hat{X} = \cup_{j=1}^n B_\varepsilon(\varphi(x_j))$ using the fact that $\varphi(X)$ is dense in $\hat{X}$ (see Proposition 6.1).

Pf: $\forall \varepsilon>0$. $\exists x_1, \dots x_n \in X$. s.t. $X = \bigcup_{j=1}^n B_{\frac{\varepsilon}{2}}(x_j)$

since $\varphi(X)$ is dense in $\hat{X}$ i.e. for any $[S] \in \hat{X}$ and this $\varepsilon>0$.

$\exists x_i \in X$, $D(\varphi(x_i), [S]) < \frac{\varepsilon}{2}$.

and there exists some ball $B_{\frac{\varepsilon}{2}}(x_j)$ contains x_i

$$D([x_i], [x_j]) = \lim d(x_i, x_j) \underset{\text{for constant sequence}}{=\!=\!=} d(x_i, x_j) < \frac{\varepsilon}{2}.$$

$D(\varphi(x_j), [S]) \le D(\varphi(x_i), [S]) + D(\varphi(x_i), \varphi(x_j)) < \varepsilon$.

$[S] \in B_\varepsilon(\varphi(x_j))$

i.e. $\hat{X} = \bigcup_{j=1}^n B_\varepsilon(\varphi(x_j))$

Nov. 22nd

Exercise 7.4. *Let X be the set of sequences in $[0,1]$. Define a metric on X by $d(\{x_i\},\{y_i\}) = \sup|x_i - y_i|$. Prove that (X,d) is not separable.*

Hint. If $\{y^{(k)}\}$ is a dense subset of X, where $y^{(k)} = \{y_i^{(k)}\}$, consider $\{x_i\} \in X$ defined by

$$x_i = \begin{cases} y_i^{(i)} + \frac{1}{2}, & \text{if } y_i^{(i)} \le \frac{1}{2} \\ y_i^{(i)} - \frac{1}{2}, & \text{if } y_i^{(i)} > \frac{1}{2} \end{cases}$$

Pf: $\forall \{y^{(k)}\}$ is a countable subset of X. $k \in \mathbb{N}$.

consider $\{x_i\} \in X$. $x_i = \begin{cases} y_i^{(i)} + \frac{1}{2}. & \text{if } y_i^{(i)} \le \frac{1}{2} \\ y_i^{(i)} - \frac{1}{2} & \text{if } y_i^{(i)} > \frac{1}{2}. \end{cases}$

$\forall \{y^{(j)}\} \in \{y^{(k)}\}_{k\in\mathbb{N}}$. $d(\{y_i^{(j)}\}, \{x_i\}) = \sup|x_i - y_i^{(j)}| \geq |x_j - x_j^{(j)}| = \frac{1}{2}$.

which means. $\{y^{(k)}\}$ is not a dense of X.

i.e. X has no countable dense.

Exercise 7.5. *Show that if a discrete metric space is separable then it is countable.*

Hint. The only dense subset of a discrete metric space X is X.

Pf: let (X,d) be discrete metric space

let $Y \subset X$. $X \setminus Y \neq \phi$. i.e. $\exists\, a \in X\setminus Y$.

we can claim that Y is not a dense of X.

since let $\varepsilon < 1$. $\forall y \in Y$. $d(a,y) = 1 > \varepsilon$.

thus the only dense subset of a discrete metric space X is X.

i.e. X is separable $\Rightarrow$ it has countable dense set $\Rightarrow$ X is countable.

Exercise 7.6. *Let $(X,d), (Y,d')$ be metric spaces and $f\colon X \to Y$ be a continuous surjective map. Prove that Y is separable if X is separable.*

Hint. If D is a countable dense subset of X, show that $f(D)$ is countable and dense in Y.

Pf: let $\{x_n\}$ be a dense set of X. $\exists \{y_n\} \subset Y$ s.t. $f(x_n) = y_n$ for all $n \in \mathbb{N}$.

it suffices to show $\{y_n\}$ is dense in Y.

$\forall y \in Y$. by the surjectivity. $\exists x \in X$, $f(x) = y$.

since $\{x_n\}$ is dense in X. $\forall r > 0$. $\exists x_i \in \{x_n\}$ s.t. $d(x_i, x) < r$.

since f is continuous, $\forall x_n \in X$ $\forall \varepsilon > 0$. $\exists \delta$. for any. $d(x_n, x_m) < \delta$. $d'(f(x_n), f(x_m)) < \varepsilon$.

Let $r = \delta$. we have $d'(f(x_i), f(x)) = d'(y_i, y) < \varepsilon$. Thus, $\{y_i\}$ is dense in Y.

Exercise 7.7. *Show that every open subset of $\mathbb{R}$ is a countable union of open intervals.*

Hint. Use Proposition 7.3.

Pf: it's obvious to show $\overline{\mathbb{Q}} = \mathbb{R}$. $\mathbb{Q}$ is countable. Thus, $\mathbb{R}$ is separable.

$\forall V \subset \mathbb{R}$, V is open.

consider an open cover of V. denote by $\{U_\omega\}$, $V = \bigcup_{\omega \in \Omega} U_\omega$

for every U_ω, since U_ω is open, it's a union of open balls. $U_\omega = \bigcup B_{\delta x}$

In $\mathbb{R}$, "open ball" is a synonym for "open interval".

$V = \bigcup_{\omega \in \Omega} \bigcup B_{\delta x}$. by Pro 7.3. Since $\mathbb{R}$ is separable,

$\exists \{B_i\}_{i \in \mathbb{N}}$. as a subcover of V, which is countable. $\square$.

Exercise 8.1. *Prove that a closed subset of a compact metric space is compact.*

Pf: Let (X, d) be compact metric space. $Y \subset X$. Y is closed.

$\forall \{y_n\} \subset Y \subset X$. since X is compact. $\exists \{y_{n_i}\} \subset \{y_n\}$ which

is convergent in X. denote $y \in X$. $y = \lim y_{n_i}$.

$\{y_{n_i}\} \subset Y$. Y is closed. $\lim y_{n_i} = y$.

by Pro 4.5. $y \in Y$. thus Y is compact.

Exercise 8.2. *Prove that a discrete metric space is compact only if it is finite using each of the three definitions of a compact metric space.*

(X, d). metric space. discrete.

Pf: ① If the space infite. consider the cover $X = \bigcup_{x \in X} B_{\frac{1}{2}}(x)$

each open set $B_{\frac{1}{2}}(x)$ only contains one element.

the finite union of $B_{\frac{1}{2}}(x)$ can't contain infite number of elements.

which contradicts to X is compact.

② $\forall \{x_n\} \subset X$. $\exists \{x_{n_i}\}$ s.t. $\{x_{n_i}\}$ conv. to some $x \in X$. assume. $x_{n_i} \neq x_{n_j}$ when $n_i \neq n_j$

thus $\forall \varepsilon \in (0,1)$. $\exists N \in \mathbb{N}$. for $n_i, n_j > N$, $d(x_{n_i}, x_{n_j}) < \varepsilon$.

which is impossible in discrete metric when $n_i \neq n_j$.

which means no sequence in X. i.e. X is finite.

③ let $\varepsilon_0 = \frac{1}{2}$. since X is totally bound. $\exists \{x_1, \ldots x_n\}$. $\bigcup B_{\frac{1}{2}}(x_i) = X$.

In discrete space. $B_{\frac{1}{2}}(x)$ only contains the center element.

thus. X is finite.

Exercise 8.3. *Let $(X, d), (Y, d')$ be metric spaces. Prove that $X \times Y$ equipped with the product metric (Exercise 1.3) is compact if and only if both X and Y are compact.*

Hint. If $\{(x_n, y_n)\}$ is a sequence in $X \times Y$, choose a convergent subsequence $\{x_{n_k}\}$ and then a convergent subsequence $\{y_{n_{k_i}}\}$. If $\{(x_n)\}$ is a sequence in X, consider a sequence $\{(x_n, y)\}$ for some $y \in Y$.

Pf: $\Rightarrow$ $\forall \{x_n\} \subset X$. $y \in Y$. consider sequence $\{(x_n, y)\} \subset X \times Y$.

since $X \times Y$ is compact, $\exists \{(x_{n_k}, y)\}. \subset \{(x_n, y)\}$ conv. to some $(x, y') \in X \times Y$.

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ for any $n_k > N$, $D((x_{n_k}, y), (x, y')) < \varepsilon$.

i.e. $\max(d(x_{n_k}, x), d(y', y)) < \varepsilon$.

$d(y', y)$ is fixed, we can let $y' = y$. $d(y', y) = 0$

$\Rightarrow$ $d(x_{n_k}, x) < \varepsilon$. and $x \in X$ which means $\{x_{n_k}\}$ conv in X.

X is compact. Y is compact can be showed similarly.

"$\Leftarrow$" $\forall \{(x_n, y_n)\} \subset X \times Y$. we have $\{x_n\} \subset X$

X is compact. $\exists \{x_{n_k}\}$ conv. to $x \in X$.

then consider $\{y_{n_k}\}$. Y is compact. $\exists \{y_{n_{k_i}}\} \subset \{y_{n_k}\}$, which conv. to $y \in Y$.

$\forall \varepsilon > 0$. $\exists N_1 \in \mathbb{N}$. for any $n_{k_i} > N_1$ $d'(y_{n_{k_i}}, y) < \varepsilon$.

$\exists N_2 \in \mathbb{N}$ for any $n_{k_i} > N_2$ $d(x_{n_{k_i}}, x) < \varepsilon$.

Let $N = \max\{N_1, N_2\}$ for any $n_{k_i} > N$. $D((y_{n_{k_i}}, x_{n_{k_i}}), (x, y)) = \max(d(x_{n_{k_i}}, x), d'(y_{n_{k_i}}, y)) < \varepsilon$.

thus. $\{(x_{n_{k_i}}, y_{n_{k_i}})\} \to (x, y) \in X \times Y$ i.e. $\{(x_n, y_n)\}$ has a conv. subsequence.

Exercise 8.4. *Let $(X, d), (Y, d')$ be metric spaces and $f : X \to Y$ be continuous. Show that if X is compact, then the image of ~~a closed~~ subset $A \subset X$ under f is closed.*

by Pro 8.5 $\Rightarrow$ image $f(A)$ is compacted. $\Rightarrow$ by Coro 8.3. is closed.

Pf: by Ex 8.1. A is compact. $\forall \{x_n\} \subset A$. $\exists \{x_{n_k}\}. \subset \{x_{n_k}\}$ conv. to some $x \in A$.

denote $f(A) = B \subset Y$. $f|_A : A \to B$. also continuous, denote $f|_A(x) = y$

$f|_A$ is cont. $\Rightarrow$ $\lim f|_A(x_{n_k}) = f|_A(x) = y \in B$. for any $\{x_{n_k}\}$ $\lim x_{n_k} = x$ (Pro 2.3).

//一般性结论.

thus. for any $\{y_{n_k}\} \subset B$. $\exists \{x_{n_k}\} \subset A$. $f(x_{n_k}) = y_{n_k}$. and $\lim y_{n_k} = y \in B$.

Thus. B is closed. (by. Pro 4.5).

与之间 $\forall \{x_n\}$ 中的$\{x_n\}$不是同一序列，可以换个符号.

Exercise 8.5. *Let (X, d) be metric space and A be its compact subset.*

i) *Prove that for any $x \in X$ there exists $a \in A$ such that $d(x, a) \leq d(x, a')$ for any $a' \in A$.*

ii) *Prove that there are $a, b \in A$ such that $d(a, b) \geq d(a', b')$ for any $a', b' \in A$.*

Hint. Use Exercise 2.6. For ii) use also Exercise 8.3.

Pf: i). $\forall x \in X$. Let $f: A \to \mathbb{R}$. $f(x) = d(x, a)$. is continutous (by Ex 2.b).

by Pro 8.7. . f has its minimum value. Let it be $y \in \mathbb{R}$.

by continuity and compactness $\exists a \in A$, s.t. $f(a) = y$.

$\forall a' \in A$. $f(a') \geqslant f(a) \Rightarrow d(x, a') \geqslant d(x, a)$

ii). since A is compact. $A \times A$ is compact. (equipped with product metric).

$d: A \times A \to \mathbb{R}$. Exercise 2.b.2. shows it's continuous.

similarly, by Pro 8.7. d also has its maximal value. $M \in \mathbb{R}$.

by continuity and compactness of d. $\exists (a, b)$. $d(a, b) = M$.

$\forall a' \in A$ $\forall b' \in A$. , $(a', b') \in A \times A$ $d(a, b) \geqslant d(a', b')$

Exercise 8.6. *Let (X, d) be a compact metric space. Prove that for each open cover $\{U_\omega\}_{\omega \in \Omega}$ of X there is $\varepsilon > 0$ such that each open ball $B_\varepsilon(x)$ is contained in one of U_ω.*

Hint. Suppose the result is false. Then for every $n \in \mathbb{N}$ there is $B_{1/n}(x_n)$ which is not contained in any U_ω. Choose a convergent subsequence of $\{x_n\}$ and consider U_ω containing its limit.

Pf: Assume the converse.

Let $\varepsilon = \frac{1}{n}$. for any $n \in \mathbb{N}$. $\exists B_{\frac{1}{n}}(x_n)$ which is not contained in any U_ω.

the procedure construct a sequence $\{x_n\}$. s.t. $B_{\frac{1}{n}}(x_n)$ not contained in any U_ω.

X is compact. $\exists \{x_{n_k}\} \sqsubset \{x_n\}$ which is conv. to some $x \in X$.

$x \in X$. x must belongs to some U_ω. $\omega \in \Omega$; w.l.g. let $x \in U_\omega$.

since U_ω is open. $\exists \delta > 0$. $B_\delta(x) \sqsubset U_\omega$.

since $\{x_{n_k}\} \to x$. $\forall \varepsilon = \frac{\delta}{2}$. $\exists N \in \mathbb{N}$. for any $n_k > N$. $d(x, x_{n_k}) < \varepsilon$.

consider some $n_k > \max\{N, \frac{2}{\delta}\}$. $\forall x' \in B_{\frac{1}{n_k}}(x_{n_k})$.

$d(x', x) \leqslant d(x', x_{n_k}) + d(x, x_{n_k}) < \frac{\delta}{2} + \frac{\delta}{2} = \varepsilon$. $x' \in B_\delta(x)$

i.e. $B_{\frac{1}{n_k}}(x_{n_k}) \sqsubset B_\delta(x) \sqsubset U_\omega$. which contradicts to the assumption.

Exercise 9.1. *Show that a metric space X is disconnected iff there exist closed subsets $U, V \subset X$ such that $U \cup V = X$ and $U \cap V = \varnothing$.*

Pf: X is disconnected $\Leftrightarrow$ $\exists$ open and nonempty $U. V \subset X$. $U \cup V = X$. $U \cap V = \phi$

$\Leftrightarrow$ $\exists$ closed and nonempty set. $U' = X \setminus U$. $V' = X \setminus V$.

s.t. $U' \cup V' = (X \setminus U) \cup (X \setminus V) = V \cup U = X$.

$U' \cap V' = (X \setminus U) \cap (X \setminus V) = V \cap U = \phi$. $\square$

Exercise 9.2. *Prove that $\mathbb{Z}$ in p-adic metric is disconnected.*

Hint. Show that $\mathbb{Z} = \cup_{n=0}^{p-1} B_1(n)$ and these open balls do not overlap.

Pf: fix $p \in \mathbb{N}$ for any $z \in \mathbb{Z}$. $z = p^k \cdot \alpha + n$,

where k is the multiplicity. $k \geq 1$. $p \nmid \alpha$. n is remainder $n = 0, 1, \ldots p-1$.

$$d_p(z, n) = |p^k \cdot \alpha| = \begin{cases} p^{-\infty} & \alpha = 0. \\ p^{-k} & \alpha \neq 0 \end{cases} \qquad d_p(z, n) < 1.$$

thus. $z \in B_1(n)$. where $z = p^k \cdot \alpha + n$. $\Rightarrow$ $z = \bigcup_{n=0}^{p-1} B_1(n)$.

By the construction, balls $\{B_1(n)\}$ not overlap. since for fixed $z \in \mathbb{Z}$. the remainder n is unique.

consider $B_1(0)$ $\mathbb{Z} \setminus B_1(0) = \bigcup_{n=1}^{p-1} B_1(n)$. union of open balls, which is open.

thus $B_1(0)$ is closed and open. clearly $B_1(0) \neq \phi$ or $\mathbb{Z}$. $\mathbb{Z}$ is disconnected.

Exercise 9.3. *Let X be a metric space, and let $\{A_\omega\}_{\omega \in \Omega}$ be a collection of connected subsets of X such that for any $\omega, \omega' \in \Omega$ there are $\omega_1 = \omega, \omega_2, \ldots, \omega_n = \omega' \in \Omega$ such that $A_{\omega_i} \cap A_{\omega_{i+1}} \neq \varnothing$ for any $1 \leq i \leq n-1$. Prove that $A = \cup_\Omega A_\omega$ is connected.*

Pf: Let $f: A \to \{0, 1\}$ be continuous. $A\omega$ is connected.

$f(A\omega) = 0$ or 1. w.l.g. $f(A\omega) = 0$.

for any $\omega' \in \Omega$. $A\omega_1 \cap A\omega_2 \neq \phi \Rightarrow f(A\omega) = f(A\omega_2) = 0$.

similarly. $A\omega_2 \cap A\omega_3 \neq \phi \ldots A\omega_{n-1} \cap A\omega_n \neq \phi$

$\Rightarrow f(A\omega') = f(A\omega_{n-1}) = \cdots = f(A\omega_2) = f(A\omega_1) = f(A\omega) = 0$.

i.e. $f(A\omega') = 0$ for any $\omega' \in \Omega$. f is not surjective.

thus A is connected.

Exercise 9.4. *Prove that a circle in $\mathbb{R}^2$ is connected.*

Pf: $f: \mathbb{R} \to \mathbb{R}$. $f \equiv r \in \mathbb{R}$ constant map. continuous.

By Pro 9.8 the graph $\Gamma_f = \{ (\varphi, f(\varphi)) \mid \varphi \in \mathbb{R} \}$. is connected in $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

any circle in $\mathbb{R}^2$ can be expressed as the graph Γ_f by put the initial point at the center of the circle. Thus a circle in $\mathbb{R}^2$ is connected.

Next time/

half.

disjoint. / connect

or

Nov. 28th.

Exercise 9.5. *Let X be a metric space. For $x, y \in X$, define $x \sim y$ if $x, y \in V$ for some connected $V \subset X$. Prove that $\sim$ is an equivalence relation on X and the equivalence class of $x \in X$ coincides with its connected component.*

Pf: 1. Equivalent relation:

i) "$x \sim x$". $x \in V$. V is connected. $\Rightarrow$ $x \in V$. V is connected.

ii) "$x \sim y$" $\Rightarrow$ "$y \sim x$" $x, y \in V \Rightarrow y, x \in V$.

iii) "$x \sim y$" $\wedge$ "$y \sim z$" $\Rightarrow$ "$x \sim z$". if $x, y \in U$. $y, z \in V$.

U. V are connected subset of X. since $y \in U \cap V$. i.e. $U \cap V \neq \phi$.

By Lemma 9.5. $U \cup V$ is connected. and $x, z \in U \cup V$.

2. Show $[x] = U$. U is the connected component. of x.

"$\subset$" $\forall\ y \in [x]$. $\exists V$, s.t. $x, y \in V$. V is connected and contains $x. \Rightarrow y \in V \subset U$.

"$\supset$" $\forall\ z \in U$, let $V' = U$. s.t. $x, y \in V'$ for some connected V'. i.e. $z \in [x]$

Exercise 9.6. *Prove that a non-empty connected subset of a metrical space that is both open and closed is a connected component.*

Hint. Let A is connected, open and closed, $x \in A$. For a connected subset V show that $A \cap V$ and $V \setminus A$ are open in V.

Pf: Let A is connected. open and closed Let $x \in A$.

By Pro 9.9. there exist connected component V. of x.

thus we have $A \subset V$. thus $A = V \cap A$.

Since A is open in X, By Pro3.4. A is open in V.

Since. $A \subset V \subset X$. $V \setminus A \subseteq X \setminus A$. $V \setminus A = (X \setminus A) \cap V$

A is closed in X. $X \setminus A$ is open in X. $V \setminus A$ is open in V.

Thus A is open and closed in V. V is connected. i.e. $A = V$ or $A = \phi$.

By condition, $A \neq \phi$. thus $A = V$.

Exercise 9.7. i) *Prove that a circle in $\mathbb{R}^2$ is path-connected.*

ii) *Prove that a sphere in $\mathbb{R}^3$ is path-connected.*

Pf: i). Let V be a circle in $\mathbb{R}^2$. with radius r.

$\forall x, y \in V$. consider x, y in polar coordinates.

Let $x = (r, \theta_1)$. $y = (r, \theta_2)$. w.l.g. $\theta_1 > \theta_2$.

$\varphi(t) = (r, \ t(\theta_1 - \theta_2) + \theta_2)$. $\varphi(0) = (r, \theta_2)$ $\varphi_1(1) = (r, \theta_1)$

ii). Let S be a sphere in $\mathbb{R}^3$ with center p and radius r.

in Cartesian coordinates. Let $p = (a, b, c)$.

$S = \{(x, y, z) \mid x = a + r\cos\beta\cos\alpha \quad y = b + r\cos\alpha\sin\beta \quad z = c + r\sin\alpha.\}$

why $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2})$. $\beta \in [0, 2\pi)$.

construct a map $f(\alpha, \beta) \equiv r$. cont. and surjective.

the set $U = \{(\alpha, \beta) \mid \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}). \ \beta \in [0, 2\pi)\}$. is path-connected.

since $\forall x, y \in U$. w.l.g. $\|y\| > \|x\|$ $\varphi(t) = x + \|y - x\| t$. $\varphi(0) = x$

$\varphi(1) = y$

By Pro. 9.12. S is path-connected.

Exercise 9.8. *Let X be a metric space, and let $\{A_\omega\}_{\omega\in\Omega}$ be a collection of path-connected subsets of X such that $A_\omega \cap A_{\omega'} \neq \varnothing$ for all $\omega, \omega' \in \Omega$. Then, $A = \cup_\Omega A_\omega$ is path-connected.*

Pf: $\forall \omega, \omega' \in \Omega$. since $A_\omega \cap A_{\omega'} \neq \phi$ let $x \in A_\omega \cap A_{\omega'}$.

$\forall y \in A_\omega$, $z \in A_{\omega'}$ since A_ω and $A_{\omega'}$ are path-connected.

by equivalence. $x \sim y$ and $z \sim x \Rightarrow z \sim y$, thus $A_\omega \cup A_{\omega'}$ is path-connected.

Thus $A = \bigcup_{\omega\in\Omega} A_\omega$ is path-connected.

Exercise 10.1. *Let $(X, \mathcal{T})$ be the line with two origins.*

1. *Check that $(X, \mathcal{T})$ is a topological space.*
2. *Prove there is no open sets $A, B \subset X$ such that $p \in A, q \in B$ and $A \cap B = \varnothing$.*

Pf: Actually, when U is open subset of $\mathbb{R}$ s.t. $0 \notin U$, we have $U \setminus \{0\} = U$.

$\forall U' \in \mathcal{T}$, $U' = U \setminus \{0\} \cup V$. where U is any open subset of $\mathbb{R}$.

V is one of these four set $\{q, p\}, \{p\}, \{q\}, \phi$. problem. can't generalize. classify

1. I) ϕ is an open set s.t. $0 \notin \phi$. $\phi \in \mathcal{T}$.

$X = (\mathbb{R} \setminus \{0\}) \cup \{p, q\}$. $\mathbb{R}$ is an open subset of $\mathbb{R}$ s.t. $0 \in \mathbb{R}$.

II) If $U_1', U_2', \dots U_k' \in \mathcal{T}$,

$$\bigcap_{1 \le i \le k} U_i' = \bigcap_{1 \le i \le k} ((U_i \setminus \{0\}) \cup V_i) = \left(\bigcap_{1 \le i \le k} U_i\right) \setminus \{0\} \cup \left(\bigcap V_i\right)$$

Since $\bigcap_{1 \le i \le k} U_i$ is a open subset of $\mathbb{R}$. (by Pro 3.1).

$\bigcap_{1 \le i \le k} V_i = \{q, p\}$ or $\{p\}$ or $\{q\}$ or ϕ. thus $\bigcap_{1 \le i \le k} U_i' \in \mathcal{T}$.

III) If $\{U_\omega'\}_{\omega \in \Omega}$ is a collection of sets in $\mathcal{T}$.

$$\bigcup_{\omega \in \Omega} U_\omega' = \bigcup_{\omega \in \Omega} ((U_\omega \setminus \{0\}) \cup V_\omega) = \left(\bigcup_{\omega \in \Omega} U_\omega\right) \setminus \{0\} \cup \left(\bigcup_{\omega \in \Omega} V_\omega\right)$$

since $\bigcup_{\omega \in \Omega} U_\omega$ is a open subset of $\mathbb{R}$. (by Pro 3.1).

$\bigcup_{\omega \in \Omega} V_\omega = \{q, p\}$ or $\{p\}$ or $\{q\}$ or ϕ. thus, $\bigcup_{\omega \in \Omega} U_\omega' \in \mathcal{T}$

2. Assume the converse. $\exists A, B \subset X$ s.t. $p \in A$, $q \in B$. $A \cap B = \phi$.

thus we can let $A = (U_1 \setminus \{0\}) \cup \{p\}$ $B = (U_2 \setminus \{0\}) \cup \{q\}$.

U_1, U_2 are open subset of $\mathbb{R}$ s.t. $0 \in U_1$ and $0 \in U_2$; $A \cap B = \phi \Leftrightarrow U_1 \cap U_2 = \phi$

U_1 is open. $0 \in U_1$, $\exists \delta_1$, s.t. $B_{\delta_1}(0) \subset U_1$. Similarly. $\exists \delta_2$, s.t. $B_{\delta_2}(0) \subset U_2$.

Let $\delta = \min\{\delta_1, \delta_2\}$. $B_\delta(0) \subset B_{\delta_1}(0) \subset U_1$, $B_\delta(0) \subset B_{\delta_2}(0) \subset U_2$

thus $B_\delta(0) \subset U_1 \cap U_2$. contradicts to $U_1 \cap U_2 = \phi$. $\square$

Exercise 10.2. *Let $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topologies on X?*

1. $\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}$;
2. $\varnothing, X, \{a\}, \{b\}, \{a, b\}, \{b, d\}$;
3. $\varnothing, X, \{a, c, d\}, \{b, c, d\}$.

Pf: 1. Yes.

2. not. $\{a, b\} \cup \{b, d\} = \{a, b, d\} \notin \mathcal{T}$.

3. not. $\{a, c, d\} \cap \{b, c, d\} = \{c, d\} \notin \mathcal{T}$

Exercise 10.3. *Let $(X, \mathcal{T})$ be a topological space and $A \subset X$ be equipped with the relative topology $\mathcal{T}'$ induced by $\mathcal{T}$. Show that on $B \subset A$ the relative topology induced by $\mathcal{T}'$ coincides with the relative topology induced by $\mathcal{T}$.*

Pf: Denote $\mathcal{T}'' = \{U' \cap B \mid U' \in \mathcal{T}'\}$ as the relative topology induced by $\mathcal{T}'$

for any $U' \in \mathcal{T}'$, there exist some $U \in \mathcal{T}$ s.t. $U' = A \cap U$.

for any $U'' \in \mathcal{T}'$, $U'' = U' \cap B = (A \cap U) \cap B = B \cap U$ (since $B \subset A$).

thus $\mathcal{T}'' = \{U \cap B \mid U \in \mathcal{T}\}$, which means the topologies coincide.

Exercise 10.4. *Let X, Y be discrete topological spaces. Is the product topology on $X \times Y$ discrete?*

Check: Let $\mathcal{T}$ is collection of all subset of X; S is collection of all subset of Y

$R = \{\bigcup_{\omega \in \Omega} (U_\omega \times V_\omega) \mid U_\omega \in \mathcal{T}, V_\omega \in S\}$.

Take arbitrary subset $A \times B \subset X \times Y$. problem. $\forall A \in R$

$A = \bigcup (U_\omega \times V_\omega)$

since $A \subset X$, $B \subset Y$. $A \in \mathcal{T}$; $B \in S$. thus $A \times B \in R$.

i.e. R is discrete.

$\forall V \subset X \times Y$.

$V = \bigcup \{(x_v, y_v)\}$

$\{(x, y)\} = \{x\} \times \{y\}$

$\Downarrow$ open. $\quad \Downarrow$ open $\quad \downarrow$ open

Dec. 13th

Exercise 11.1. *Let X be a topological space and $Y \subset X$. Prove that $F' \subset Y$ is closed in the relative topology iff $F' = F \cap Y$ for some closed $F \subset X$.*

Pf: Denote the relative topology $\mathcal{T}' = \{A \cap Y \mid A \in \mathcal{T}\}$.

"$\Rightarrow$" F' is closed in the relative topology, $Y \setminus F' \in \mathcal{T}'$

there exists $A \subset X$. s.t. $A \in \mathcal{T}$ and $A \cap Y = Y \setminus F'$

denote $F = X \setminus A$. F is closed and $F \subset X$.

$$F \cap Y = (X \setminus A) \cap Y \underset{\text{since } X \supset Y}{=} Y \setminus (A \cap Y) = Y \setminus (Y \setminus F') = F'$$

"$\Leftarrow$" F is closed, $X \setminus F \in \mathcal{T}$ $\exists B = (X \setminus F) \cap Y$ and $B \in \mathcal{T}'$.

$Y \setminus B$ is closed in $\mathcal{T}'$.

$$Y \setminus B = Y \setminus ((X \setminus F) \cap Y) = Y \setminus (Y \setminus (F \cap Y)) = F \cap Y = F'$$

thus F' is closed in the relative space.

Exercise 11.2. *Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, c\}, \{a, b, c\}, \{a, b\}\}$. Find the closure of $\{a\}$ in $(X, \mathcal{T})$.*

Pf: denote $\{a\} = A$. $a \in \bar{A}$, trivially. check if $b.c.d$ are adherent points

. b. $\{b\} \cap A = \phi$. $\Rightarrow b \notin \bar{A}$.

c $X \cap A = \{a\}$. $\{a, c\} \cap A = \{a\}$. $\{a, b, c\} \cap A = \{a\}$. $\Rightarrow c \in \bar{A}$

d. $X \cap A = \{a\}$. $\Rightarrow d \in \bar{A}$

$\bar{A} = \{a, b, d\}$.

Exercise 11.3. *Let $X = \mathbb{R}$ and $\mathcal{T}$ be the collection of subsets of $\mathbb{R}$ which, in addition to $\varnothing$ and $\mathbb{R}$, contains all the subsets of the form (x, ∞) for $x \in \mathbb{R}$. Which of the following maps from $(\mathbb{R}, \mathcal{T})$ to $(\mathbb{R}, \mathcal{T})$ are continuous:*

i) $f(x) = x^2$;

ii) $f(x) = x^3$;

iii) $f(x) = -x^3$.

i). not continuous.

counter example: image: $(-1, +\infty)$. pre-image: $[0, +\infty)$. not open.

ii) continuous.

$\forall$ image $(x, +\infty) \subset \mathbb{R}$. pre-image $(\sqrt[3]{x}, +\infty)$. $\sqrt[3]{x} \in \mathbb{R}$, and $(\sqrt[3]{x}, +\infty) \in \mathcal{T}$

iii) not continuous.

image $(x, +\infty) \in \mathcal{T}$. pre-image $(-\infty, -\sqrt[3]{x})$. $\notin \mathcal{T}$.

Exercise 11.4. (cf. Exercise 2.2) *Let X, Y be topological spaces and $f: X \to Y$ be a continuous map.*

1. *For $A \subset X$, prove that $f|_A: A \to Y$ is continuous, where A is equipped with the relative topology.*
2. *If $B \subset Y$ and $\operatorname{Im} f \subset B$, prove that $f: X \to B$ is continuous, where B is equipped with the relative topology.*

Pf: 1. $\forall U \subset Y$, U is open. $f^{-1}(U)$ is open in X.

$f^{-1}(U) \cap A$ is open in relative topology.

and $f|_A^{-1}(U) = f^{-1}(U) \cap A$.

2. $\forall U' \subset B$, U' is open in B. $\exists U$ is open in Y. s.t. $U' = U \cap B$

and $U = (U \cap B) \cup (U \cap (Y \setminus B))$.

since $\operatorname{Im} f \subset B$, $f^{-1}(U) = f^{-1}(U \cap B) = f^{-1}(U')$.

$f^{-1}(U)$ is open in X implies $f^{-1}(U')$ is open in X.

Exercise 11.5. (cf. Exercise 3.4) *Let X, Y be two topological spaces and let $X \times Y$ be equipped with the product topology. Prove that the projection map $\pi_X: X \times Y \to X, \pi_X((x, y)) = x$ is continuous.*

Pf: denote the topology, $(X, \mathcal{T})$, (Y, S), $(X \times Y, R)$.

$\forall U \in \mathcal{T}$. the pre-image $\pi_X^{-1}(U) = U \times Y$.

since $U \in \mathcal{T}$, $Y \in S$, $U \times Y \subset R$. thus $\pi_X^{-1}(U)$ is open in $X \times Y$.

Exercise 12.1. *Let $X = [-1, 1]$ and $\mathcal{T}$ be the collection of all subsets of $[-1, 1]$ which either do not contain 0 or contain $(-1, 1)$.*

1. *Show that $(X, \mathcal{T})$ is a topological space.*
2. *Show that $(X, \mathcal{T})$ is compact.*

Pf: 1. I) $0 \notin \phi$, $\phi \in \mathcal{T}$; $(-1,1) \subset X$, $X \in \mathcal{T}$

II) if $U_1, \dots U_k \in \mathcal{T}$. ① at least one of the not contain 0. $\Rightarrow 0 \notin \bigcap_{1 \le i \le k} U_i$, $\bigcap_{1 \le i \le k} U_i \in \mathcal{T}$

② none of them not contain 0. $\Rightarrow$ all of them contains $(-1,1)$

$\Rightarrow (-1,1) \subset \bigcap U_i \Rightarrow \bigcap U_i \in \mathcal{T}$.

III). if $\{U_w\}_{w \in \Omega}$. $U_w \in \mathcal{T}$. ① at least one of them contain $(-1,1)$, $(-1,1) \subset \bigcup U_w$.

② one of them contain $(-1,1) \Rightarrow$ all of them not contain 0.

$\Rightarrow 0 \notin \bigcup_{w \in \Omega} U_w$

2. $\forall \{U_w\}_{w \in \Omega}$ which covers X. at least one of them contain $(-1,1)$. denote by U_{w_1}.

Similarly. $\exists w_2, w_3 \in \Omega$. s.t. $-1 \in U_{w_2}$, $1 \in U_{w_3}$.

thus $X = U_{w_1} \cup U_{w_2} \cup U_{w_3}$. $U_{w_{1,2,3}} \in \{U_w\}_{w \in \Omega}$, which is finite subcover.

Exercise 12.2. *Let X be a topological space. Show that if $A, B \subset X$ are compact then $A \cup B$ is compact.*

Denote the topology space $(X, \mathcal{T})$.

Pf: $\forall \{U_\omega\}_{\omega \in \Omega}$ which covers $A \cup B$ and $U_\omega \in \mathcal{T}$.

each $U_\omega = (U_\omega \cap A) \cup (U_\omega \cap B)$.

denote $U_\omega^A = U_\omega \cap A$. $U_\omega^B = U_\omega \cap B$.

$\{U_\omega^A\}$ $\{U_\omega^B\}$ covers $A \cdot B$ relatively. $\Rightarrow \exists \{U_{\omega_i}^A\}_{1\le i\le k}$, $\{U_{\omega_j}^B\}_{1\le j\le m}$ covers $A \cdot B$.

Denote $U_{\omega_{1i}}$, $\omega_{1i} \in \Omega$, $1\le i\le k$. where $U_{\omega_{1i}} = U_{\omega_i}^A \cup U_{\omega_i}^B$, $\bigcup_{1\le i\le k} U_{\omega_{1i}} \sqsupset \bigcup_{1\le i\le k} U_{\omega_i}^A = A$.

$U_{\omega_{2j}}$, $\omega_{2j} \in \Omega$, $1\le j\le m$. where $U_{\omega_{2j}} = U_{\omega_j}^A \cup U_{\omega_j}^B$. $\bigcup_{1\le j\le m} U_{\omega_{2j}} \sqsupset \bigcup_{1\le j\le m} U_{\omega_j}^B = B$.

thus. the set $\{U_{\omega_{11}}, U_{\omega_{12}}, \dots U_{\omega_{1k}}, U_{\omega_{21}}, U_{\omega_{22}}, \dots U_{\omega_{2m}}\}$. covers $A \cup B$. and the number of the elements in this set not exceed $k+m$. which is finite.

Exercise 12.3. *Let X be a topological space and $B \subset X$ be equipped with the relative topology. Show that if $A \subset B$ is compact as a subset of B then A is compact as a subset of X.*

Pf: $\forall \{U_\omega\}_{\omega\in\Omega}$ s.t. $U_\omega \in \mathcal{T}$ and. $\{U_\omega\}_\Omega$ covers A.

For each U_ω, $\exists U_\omega' = U_\omega \cap B$ and U_ω' is open in B.

$\bigcup(U_\omega \cap B) = (\bigcup U_\omega) \cap B$. since $A \sqsubset B$. $\{U_\omega'\}_\Omega$ covers A.

there exist $\{U_{\omega_1}', U_{\omega_2}', \dots U_{\omega_k}'\}$ where $\omega_1, \dots \omega_k \in \Omega$, covers A.

then we have the corresponding $\{U_{\omega_i}\}_{1\le i\le k}$, s.t. $U_{\omega_i} \cap B = U_{\omega_i}'$. covers A.

Exercise 12.4. *Let X be the real line with two origins. Show that $(0, 1] \cup \{p\}$ and $(0, 1] \cup \{q\}$ are compact, but their intersection $(0, 1]$ is not.*

Hint. The topology induced from X to $(0, 1]$ coincides with the Euclidean topology.

Pf: ① $\forall \{U_\omega\}_{\omega\in\Omega}$ as a cover of $(0,1] \cup \{p\}$.

at least one of them contains p, denote by U_ω. $U_\omega = (U \setminus \{0\}) \cup \{p\} \cup (0,1] \cup \{p\}$.

U is open in $\mathbb{R}$ and $0 \in U$. thus, $U_\omega = (0,a) \cup \{p\}$. $0<a\le 1$

denote $A = [a,1]$. $(0,1] \cup \{p\} = U_\omega \cup A$.

since A is close, we can always find finite subcover of A. denote by $\{U_{\omega_i}\}_{\omega_i\in\Omega, 1\le i\le k}$.

and $\bigcup_{1\le i\le k} U_{\omega_i} \cup U_\omega$ is the finite subcover of $(0,1] \cup \{p\}$ $\square$.

similarly, $(0,1] \cup \{q\}$. is compact.

② $\{U_\omega\}_{\omega\in\Omega}$ as a cover of $(0,1]$. For each U_ω, $U_\omega = (0,1] \cap U_\omega'$.

where U_ω' is an open subset of $\mathbb{R}$ s.t. $0 \in U$.

For example, $U_n' = (\frac{1}{n}, +\infty)$. $U_n = (\frac{1}{n}, 1]$ $n \in \mathbb{N}$.

$\bigcup_{n=1}^{\infty} U_n = [0,1]$, but no finite subcover of $\{U_n\}$ covers $(0,1]$.

Since for any finite $\{U_i\}_{1\le i\le k}$. Let $N = \max\{1,2,\dots k\}$. $\frac{1}{N+1} \in (0,1]$ but $\notin \bigcup_{1\le i\le k} U_i$

Exercise 13.1. *Show that the line with two origins is connected.*

Pf: Denote $\mathbb{R}\setminus\{0\}\cup\{p,q\}=X$. and the topological space is $(X,\mathcal{T})$.

Assume the converse. i.e. $\exists$ non-empty $A, B\in\mathcal{T}$, $A\cap B=\phi$, $A\cup B=X$.

By Ex 10.1. the situation $p\in A$, $q\in B$ or $q\in A$, $p\in B$ is impossible.

thus, w.l.g. Let $A=U_1\setminus\{0\}\cup\{p,q\}$. U_1, U_2 are open and nonempty in $\mathbb{R}$.
$B=U_2$ and $0\in U_1$, $0\notin U_2$.

since $A\cup B=X$. $\Rightarrow$ $U_1\cup U_2=\mathbb{R}$.

$A\cap B=\phi$. $U_1\subset A$, $U_2\subset B$, $\Rightarrow U_1\cap U_2=\phi$

which contradicts to $\mathbb{R}$ is connected (w.r.t. Euclidean topology)

Exercise 13.2. *Let X, Y be topological spaces and $f\colon X\to Y$ be a bijection such that both f and f^{-1} are continuous. Show that if C is the connected component of $x\in X$ then $f(C)$ is the connected component of $f(x)$.*

Pf: $f|_C : C\to f(C)$ is cont. and surjective. C is connected $\Rightarrow f(C)$ is connected.

by Prop 13.7. the connected component of $f(x)$ must exists. let it be $B\subset Y$.

and we have $f(C)\subset B$.

since $f^{-1}|_B : B\to f^{-1}(B)$. is cont. and sur. $f^{-1}(B)$ is connected.

and. $f(x)\in B$, $x\in f^{-1}(B)$. thus $f^{-1}(B)\subset C$ (by def. connected component).

$f^{-1}(B)\subset C \Rightarrow B\subset f(C) \Rightarrow B=f(C)$.

i.e. $f(C)$ is Connected. component of $f(x)$

Exercise 13.3. *Let $X=\{a,b,c,d\}$ and $\mathcal{T}=\{\varnothing, X, \{a\}, \{b\}, \{a,c\}, \{a,b,c\}, \{a,b\}\}$. Show that $(X,\mathcal{T})$ is path-connected.*

Pf: find the path

$a\to d$ $\varphi=\begin{cases} a & t\in[0,1) \\ d & t=1\end{cases}$ $b\to d$ $\varphi=\begin{cases} b & t\in[0,1) \\ d & t=1\end{cases}$ $c\to d$ $\varphi=\begin{cases} c & t\in[0,1) \\ d & t=1\end{cases}$

the pre-image can be $[0,1)$, $[0,1]$, ϕ all open in $[0,1]$. φ are cont.

$a\to b$ $\varphi=\begin{cases} a & t\in[0,\frac{1}{3}) \\ c & t\in[\frac{1}{3},\frac{2}{3}) \\ d & t=\frac{2}{3} \\ b & t\in(\frac{2}{3},1]\end{cases}$. the pre-image can be. ϕ, $[0,1]$, $[0,\frac{1}{3})$, $(\frac{2}{3},1]$, $[0,\frac{2}{3})$, $[0,1]\setminus\{\frac{2}{3}\}$, $[0,\frac{1}{3})\cup(\frac{2}{3},1]$. open in $[0,1]$, φ is cont.

$a \to c$ $\quad \varphi = \begin{cases} a & t \in [0, \frac{1}{3}) \\ b & t \in (\frac{1}{3}, \frac{2}{3}) \\ d & t = \frac{1}{3} \text{ or } \frac{2}{3} \\ c & t \in (\frac{2}{3}, 1]. \end{cases}$ $\qquad b \to c$ $\quad \varphi = \begin{cases} b & t \in [0, \frac{1}{3}) \\ d & t = \frac{1}{3} \\ a & t \in (\frac{1}{3}, \frac{2}{3}) \\ c & t \in [\frac{2}{3}, 1] \end{cases}$

thus each two point in X has a path.

Exercise 13.4. *Show that the line with two origins is path-connected.*

Hint. Let $X = \mathbb{R} \setminus \{0\}) \cup \{p, q\}$ be the line with two origins. Then $\varphi \colon [0,1] \to X$,

$$\varphi(t) = \begin{cases} 2t - 1 & \text{if } t \neq \cancel{2}\ 1, \\ p & \text{if } t = \cancel{2}\ 1. \end{cases}$$

is a path connecting -1 and 1.

Pf: find the path. $\forall\, a, b \in X$.

1) $a = p$, $b = q$. $\quad \varphi(t) = \begin{cases} p & t = 1 \\ t(1-t) & t \in (0,1) \\ q & t = 0. \end{cases}$

2) $a = p$, $b \in \mathbb{R} \setminus \{0\}$ ($a = q$ similarly).

$$\varphi(t) = \begin{cases} p & t = 0 \\ bt & t \in (0,1] \end{cases}$$

3). $a, b \in \mathbb{R} \setminus \{0\}$. $\quad \varphi(t) = \begin{cases} (b-a)t + a. & t \neq \frac{a}{a-b} \\ p & t = \frac{a}{a-b}. \end{cases}$

Exercise 13.5. *Prove that the product of two path-connected topological spaces with product topology is path-connected. Do not forget to prove that the path you choose is continuous.*

Pf: let the path-connected t.s. be X_1, X_2

$\forall\, (a_1, b_1), (a_2, b_2) \in X_1, X_2$

since X is path-connected, $\exists\ \varphi_1(t) : [0,1] \to X_1$ cont. and $\varphi_1(0) = a_1$. $\varphi_1(1) = a_2$.

similarly. $\varphi_2(t) : [0,1] \to X_2$ and $\varphi_2(0) = b_1$ $\varphi_2(1) = b_2$.

Denote $\Phi(t) : [0,1] \to X \times Y$ where $\Phi(t) = (\varphi_1(t), \varphi_2(t))$. $\Phi(0) = (a_1, b_1)$, $\Phi(1) = (a_2, b_2)$

It remains to check Φ is cont.

$\forall\ A = \bigcup (U_\omega \times V_\omega)$, U_ω is open in X, V_ω is open in Y.

$\Phi^{-1}(A) = \bigcup_{\omega \in \Lambda} \Phi^{-1}(U_\omega \times V_\omega) = \bigcup_{\omega \in \Lambda} (\varphi_1^{-1}(U_\omega) \cap \varphi_2^{-1}(V_\omega))$ (Consider the proof Pro 13.6).

since φ_1, φ_2 cont. $\varphi_1^{-1}(U_\omega)$, $\varphi_2^{-1}(V_\omega)$ are open. the intersection set open as well.

$\Phi^{-1}(A)$ is open in $[0,1]$. thus Φ is continuous

Dec 12th.

Exercise 14.1. *Let* $X = [0,1]$ *and* $x \sim y$ *if* $x, y \in [0, 1/3]$ *or* $x, y \in (1/3, 2/3]$ *or* $x, y \in (2/3, 1]$. *Describe explicitly the topology on the quotient set of* X *by* $\sim$.

Define the topology on X, $\mathcal{T} = \{X, \phi, [0,\frac{1}{3}], (\frac{1}{3},\frac{2}{3}], (\frac{2}{3},1], [0,\frac{2}{3}], (\frac{1}{3},1], [0,\frac{1}{3}] \cup (\frac{2}{3},1]\}$

(The three axioms of topology can be checked easily).

And the quotient topology $\overline{\mathcal{T}} = \{[\frac{1}{3}], [\frac{2}{3}], [1]\}$.

Check: $\pi^{-1}([\frac{1}{3}]) = [0,\frac{1}{3}] \in \mathcal{T}$

$\pi^{-1}([\frac{2}{3}]) = (\frac{1}{3}, \frac{2}{3}] \in \mathcal{T}$

$\pi^{-1}([1]) = (\frac{2}{3}, 1] \in \mathcal{T}$.

Exercise 14.2. *Let* X, Y *be topological spaces and* $\sim$ *be an equivalence relation on* X. *Let* $f : X \to Y$ *be a map such that* $f(x) = f(x')$ *for any* $x \sim x', x, x' \in X$. *Then*

i) *There exists a unique map* $\overline{f} : \overline{X} \to Y$ *such that* $f = \overline{f} \circ \pi$ *where* $\overline{X}$ *is the quotient space of* X *with respect to* $\sim$

ii) *If* f *is continuous then* $\overline{f}$ *is continuous* (Pro 14.2).

iii) *If* f *is open (i.e. the image of every open subset of* X *is open in* Y*) then* $\overline{f}$ *is open*

$a \in \overline{X}$
$[x] = a$
$\overline{f}(a) = f(x)$

Pf: i) existence: Denote: $\overline{f}([x]) = f(x)$. for any $x \in X$.

$\forall x \in X$, $\overline{f} \circ \pi(x) = \overline{f}([x]) = f(x)$. $\Rightarrow f = \overline{f} \circ \pi$.

uniqueness: $\exists \overline{f_1}, \overline{f_2}$ s.t. $\exists [x_0] \in \overline{X}$, $\overline{f_1}([x_0]) \neq \overline{f_2}([x_0])$.

i.e. for this $x_0 \in X$. $\overline{f_1}(\pi(x_0)) \neq \overline{f_2}(\pi(x_0)) \Rightarrow f(x_0) \neq f(x_0)$ contradicts.

ii) Denote the topological space. $(X, \mathcal{T})$, $(\overline{X}, \overline{\mathcal{T}})$, (Y, S)

$\forall V \in S$, since f is cont, $(\overline{f} \circ \pi)^{-1}(V) \in \mathcal{T}$.

i.e. $\pi^{-1}(\overline{f}^{-1}(V)) \in \mathcal{T}$.

by def of π. $\overline{f}^{-1}(V) \in \overline{\mathcal{T}}$, thus $\overline{f}$ is cont.

iii) $\forall U \in \overline{\mathcal{T}}$, Check $\overline{f}(U) = f(\pi^{-1}(U))$ is open

$\pi^{-1}(U) \in \mathcal{T}$, by def of $\mathcal{T}$; since f is open $f(\pi^{-1}(U)) \in S$

Exercise 14.3. *The relation $x \sim x'$ if $x - x' = 2\pi k, k \in \mathbb{Z}$ is an equivalence relation on $\mathbb{R}$. Let $\overline{\mathbb{R}}$ be the quotient space of $\mathbb{R}$ with respect to $\sim$ and $S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the circle of radius 1 with center at 0. Let $f : \mathbb{R} \to S$ be defined by $f(t) = (\cos t, \sin t)$.*

i) *Check that f satisfies the condition of Exercise 14.2 and thus defines the map $\overline{f} : \overline{\mathbb{R}} \to S$ satisfying $f = \overline{f} \circ \pi$.*

ii) *Check that f is continuous and open*

iii) *Prove that $\overline{f}$ is a bijection*

iv) *Prove that $\overline{f}$ is continuous and its inverse is also continuous.*

Hint. Use Exercise 14.2. The continuity of $\overline{f}^{-1}$ follows from the fact that $\overline{f}$ is an open map.

Denote the topology. $(\mathbb{R}, \mathcal{T})$, $(\overline{\mathbb{R}}, \overline{\mathcal{T}})$. $(S, \mathcal{T}')$

i). $\forall t_1 \sim t_2$, $t_1, t_2 \in \mathbb{R}$. denote $t_1 = t_2 + 2\pi k$.

$f(t_1) = (\cos t_1, \sin t_1) = (\cos(t_2 + 2\pi k), \sin(t_2 + 2\pi k)) = (\cos t_2, \sin t_2) = f(t_2)$.

map $\overline{f}([t]) = (\cos t, \sin t)$

ii). let the topology of $\mathbb{R}$ and S be default Euclidean topology.

then $f(t) = f_1(t) \times f_2(t)$. $f_1(t) = \cos t$ $f_2(t) = \sin t$.

$f(t)$ is cont. since f_1, f_2 are cont. (in Euclidean metric, as well as Euclidean t.p.)

f is open since $f^{-1}(t) = (\arccos t, \arcsin t)$ is cont.

iii) $\overline{f}([t]) = (\cos t, \sin t)$.

surjective: $\forall (x, y) \in S$, since $x^2 + y^2 = 1$. $\exists t \in [0, 2\pi)$. $\begin{cases} x = \cos t \\ y = \sin t. \end{cases}$

and $[t] = \{t + 2\pi k \mid k \in \mathbb{Z}\}$. i.e. $\exists [t] \in \overline{\mathbb{R}}$, s.t. $\overline{f}([t]) = (\cos t, \sin t)$

injective: if $\overline{f}([t_1]) = \overline{f}([t_2])$ then $(\cos t_1, \sin t_1) = (\cos t_2, \sin t_2)$.

i.e. $t_1 = t_2 + 2\pi k$, $k \in \mathbb{Z}$. $t_1, t_2 \in [t_1]$ and $t_1, t_2 \in [t_2] \Rightarrow [t_1] = [t_2]$

iv) $\overline{f}$ is cont. by ii) and Ex 14.2. ii).

$\forall U \in \overline{\mathcal{T}}$, the pre-image of $\overline{f}^{-1}$ is $(\overline{f}^{-1})^{-1}(U) = \overline{f}(U)$

$\overline{f}(U) \in \mathcal{T}'$, since $\overline{f}$ is open (implied by f is open and Ex. 14.2).

thus. $\overline{f}^{-1}$ is cont.

Exercise 14.4. *The relation $(x,y) \simeq (x',y')$ if $x^2+y^2 = x'^2+y'^2 = 1$ and $x=-x', y=-y'$ is an equivalence relation on the closed disc $D = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 1\}$ which defines the quotient space $\overline{D}$. Let $S = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2=1\}$ be the unit sphere and $\mathbb{P}^2$ be its quotient space by the equivalence relation $(x,y,z)\sim(x',y',z')$ if $(x,y,z) = -(x',y',z')$. Let f be the orthogonal projection from S to $\overline{D}$ defined by*

$\pi_1 : D \to \overline{D}$

$f : S \to \overline{D}$

$\pi_2 : S \to P^2$

$$f((x,y,z)) = \begin{cases} \{(x,y)\} & \text{if } z>0,\\ \{(-x,-y)\} & \text{if } z<0,\\ \{(x,y),(-x,-y)\} & \text{if } z=0 \end{cases}$$

i) *Check that f is correctly defined, satisfies the condition of Exercise 14.2, and thus defines the map $\overline{f}\colon \mathbb{P}^2 \to \overline{D}$ satisfying $f = \overline{f}\circ\pi$.*

ii) *Check that f is continuous and open*

iii) *Prove that $\overline{f}$ is a bijection*

iv) *Prove that $\overline{f}$ is continuous and its inverse is also continuous.*

Pf: i) $\forall (x_1,y_1,z_1) \sim (x_2,y_2,z_2) \in S$. $\Rightarrow (x_1,y_1,z_1) = -(x_2,y_2,z_2)$.

① $z_1>0$. $z_2<0$

$f(x_1,y_1,z_1) = \{(x_1,y_1)\} = \{(-x_2,-y_2)\} = f(x_2,y_2,z_2)$

② $z_1<0$. $z_2>0$

$f(x_1,y_1,z_1) = \{-(x_1,y_1)\} = \{(x_2,y_2)\} = f(x_2,y_2,z_2)$

③ $z_1=z_2=0$

$f(x_1,y_1,z_1) = \{(x_1,y_1), -(x_1,y_1)\} = \{-(x_2,y_2), (x_2,y_2)\} = f(x_2,y_2,z_2)$

$\Rightarrow f(x_1,y_1,z_1) = f(x_2,y_2,z_2)$

(ii) $\forall$ open set A in $\bar{D}$ $\quad A = \bigcup (x_w \times y_w)$ $\quad f^{-1}(A) = \bigcup_{\substack{w\in n\\ t\in\mathbb{R}}} (x_w\times y_w\times z_t)$. $f^{-1}(A)$ is open.

$\forall$ open set B in S $\quad B = \bigcup(x_w\times y_w\times z_w)$ $\quad f(B) = \bigcup(x_w\times y_w)$. which is open

iii) $\bar{f}([x,y,z]) = [x,y]$

$\forall [x,y]\in\bar{D}$. $\exists z\in\mathbb{R}$. $\bar{f}([x,y,z]) = [x,y]$ $\Rightarrow$ surjective.

if $[x_1,y_1] = [x_2,y_2]$, two case: $(x_1,y_1) = (x_2,y_2)$ or $(x_1,y_1) = -(x_2,y_2)$

$\forall z\in\mathbb{R}$. $[x_1,y_1,z] = \{(x_1,y_1,z), -(x_1,y_1,z)\} = \{-(x_2,y_2,z), (x_2,y_2,z)\}$

$= [x_1,y_1,z]$. $\Rightarrow$ injective.

iv). $f: S\to\bar{D}$. $\pi_2 : S\to P^2$. $\bar{f}: P^2\to\bar{D}$. $f = \bar{f}\circ\pi$.

$\bar{f}$ is cont. since f is cont.

$\bar{f}^{-1}$ is cont since f is open, f^{-1} is cont.

Exercise 15.1. *Give a formal proof that* $[0,1)$ *and* $[0,\infty)$ *are homeomorphic*

Pf: Define $f: [0,1) \to [0,\infty)$. $f(x) = \tan\frac{\pi x}{2}$. it suffices to check f is homeomorphism.

surjective: $\forall\, a \in [0,\infty)$, $\exists\, x_a = \frac{2}{\pi}\arctan a$ s.t. $f(x_a) = a$ and $x_a \in [0,1)$.

injective: $f(x_1) = f(x_2)$, $x_1, x_2 \in [0,1) \Rightarrow \tan\frac{\pi x_1}{2} = \tan\frac{\pi x_2}{2}$, $\frac{\pi x_1}{2}, \frac{\pi x_2}{2} \in [0, \frac{\pi}{2})$.

$\Rightarrow \frac{\pi x_1}{2} = \frac{\pi x_2}{2}$. $\Rightarrow x_1 = x_2$.

continuity of f: $f(x) = \tan\frac{\pi x}{2}$. is cont. on $[0,1)$ w.r.t. Euclidean metric

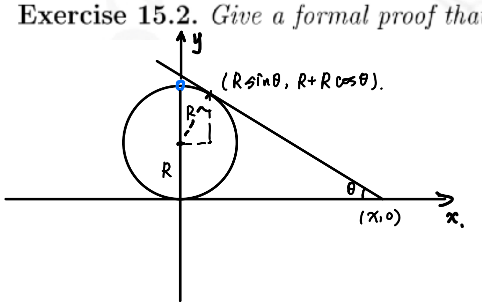
since $y_1 = \sin\frac{\pi x}{2}$ and $y_2 = \cos\frac{\pi x}{2}$ cont. on $[0,1)$ and $y_2 \neq 0$ on $[0,1)$.

continuity of f^{-1}: $\forall\, y \in [0,\infty)$ w.r.t. Euclidean Metric, $\forall \varepsilon > 0$, $\exists\, \delta = \frac{2}{\pi\varepsilon}$ for any $d(y,y') < \delta$

$$d(f^{-1}(y), f^{-1}(y')) = \left| \arctan \frac{\frac{\pi}{2}(y-y')}{1+\frac{\pi^2}{4}yy'} \right| \leq \left| \frac{\frac{\pi}{2}(y-y')}{1+\frac{\pi}{2}yy'} \right| \leq \frac{\pi}{2} \cdot |y-y'| < \varepsilon.$$

f^{-1} is cont. on $[0,\infty)$. w.r.t. Euclidean Metric. Thus f^{-1} is cont.

Exercise 15.2. *Give a formal proof that a punctured circle and* $\mathbb{R}$ *are homeomorphic*



denote the set of punctured circle by P.

$\theta = 2\arctan\frac{R}{x}$

Pf: $\forall x \in \mathbb{R}$. put it on the x-axis on the Cartesian coordinates. systen w.r.t. plane.

Let it's coordinates be $(x, 0)$.

put to punctured circle on the plane. radius R, center $(0,R)$. punctured point $(0,2R)$.

take the tangent of the circle through $(x,0)$. let the left angle between the tangent line and x-axis be. θ (particularly, the line parallel to x-axis $\theta = 0$), thus $\theta \in [0, 2\pi)$

$f: \mathbb{R} \to [0,2\pi)$ $f(x) = 2\arctan\frac{R}{x}$. the procedure ensures that x and θ has 1-to-1 corresponding. thus f is bijective. f and f^{-1} is cont. (We have showed in Ex 15.1),

Thus $\mathbb{R} \cong [0,2\pi)$

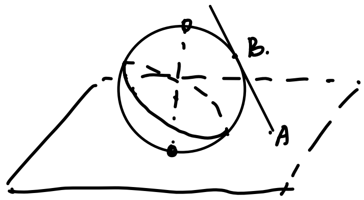
$g: [0,2\pi) \to P$. $g(\theta) = (R\sin\theta, R\cos\theta + R)$. denote $g_1(\theta) = R\sin\theta$, $g_2(\theta) = R\cos\theta + R$.

g is bijective by def by sin, cos. g_1, g_2 is cont. g is cont. g^{-1} is cont. similarly.

thus $[0,2\pi) \cong P$. $\Rightarrow$ $\mathbb{R} \cong P$ (transitivity of equivalence relation)

Exercise 15.3. *Explain why a twice-punctured sphere is homeomorphic to an open cylinder*

Pf: by Example 9. open cylinder $\cong$ punctured plane.



Let the line connecting the punctured line through the center of sphere.

Let the puncture of plane and one of the puncture of sphere coincide.

For arbitrary point in the plane. take the tangent line of the sphere. through the point. this construct the homeomorphism.

thus punctured plane $\cong$ twice punctured sphere.

Exercise 15.4. *Prove that the image of a cut-point under a homeomorphism is a cut-point*

Pf: Denote $X \cong Y$. $f: X \to Y$ is a homeomorphism. $a \in X$ is a cut point.

since $X \cong Y$ denote the number of connected component by n.

Denote $A = X \setminus \{a\}$. and $B = Y \setminus \{f(a)\}$.

by prop. 15.4. $f|_A : A \to B$ is also homeomorphism.

since a is a cut-point the number of connected component of A is $n+k$. $k \in \mathbb{N}$.

By Prop 15.3. the number of connected component of B. is $n+k$. as well

compare $B = Y \setminus \{f(a)\}$ and Y. $f(a)$ must be cut-point.

Exercise 15.5. *Give a formal proof that the letters X and Y are not homeomorphic*

Pf: Assume the converse. $X \cong Y$. $\exists f: X \to Y$ is homeomorphism.

$X \to a$ Y.

Consider the point a. $A \triangleq X \setminus \{a\}$. A has 4 connected components.

since f is bijective. $\exists ! f(a) \in Y$. $f(A) = Y \setminus \{f(a)\}$.

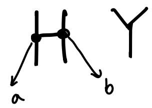
by Pro 15.4. $f|_A : A \to f(A)$ is homeomorphism.

$f(A)$ has 4 connected components.

But it's impossible since the once punctured Y has at most 3. connected components.

Exercise 15.6. *Give a formal proof that the letters* H *and* Y *are not homeomorphic*

Pf: Assume the converse. $H \cong Y$. $\exists f: H \to Y$ is homeomorphism.



consider points a, b. $A \triangleq H \setminus \{a, b\}$ A has 5 connected components.

since f is bijective. $\exists! f(a) \in Y$. $\exists! f(b) \in Y$ $f(A) = Y \setminus \{f(a), f(b)\}$.

by Pro 15.4. $f|_A : A \to f(A)$ is homeomorphism.

$f(A)$ has 5 connected components.

But it's impossible since the twice punctured Y has at most 4. connected components.

Ex 15.2