Tasks 2 (29.10). Mixture

Model the following random variables with N=10000, which from a mixture of some distributions (using both methods), and compare the theoretical moments (math.expectation and variance) with empirical ones. And compare their plots for distribution functions and density functions. And calculate the difficulty of the algorithm for both cases.

1. Simulate a mixture of the two following distributions. One is normal with parameters mean = 1, variance = 9; the other with the following density f(x) with $\alpha = 3$ and $\beta = 2$. The mixture weight p is 0.4.

$$f(x) = \frac{\alpha}{2} e^{-\alpha|x-\beta|}$$

2. The random variable X has the following probability density function:

$$f(x) = \begin{cases} a\,(1 - x^2), & -1 \le x \le 1, \\ \dfrac{1}{2}\, e^{-(x-2)}, & x \ge 2, \\ 0, & \text{otherwise.} \end{cases}$$

3. Consider a function $f(x)$ consisting of three components, one of which is constant and equal to $\frac{1}{4}$ on the interval $[1, 2)$.

$$f(x) = \begin{cases} \dfrac{1}{8}(x+1), & -1 \le x < 1, \\ \dfrac{1}{4}, & 1 \le x < 2, \\ \dfrac{1}{2} e^{-(x-2)}, & x \ge 2, \\ 0, & \text{otherwise.} \end{cases}$$

Task 3 (11.05). Rejecting sample.

For tasks 2 and 3, additionally perform a simulation using the rejection sampling method. Explain the choice of the majorizing function.

Problem. Levy Distribution.
Perform a simulation for Levy distribution with parameters $\mu = 1$ and $c = 4$.

Definition. Lets some random variable has the probability density function with parameters μ (location) and $c > 0$ (scale):

$$f(x;\mu,c) = \begin{cases} \sqrt{\dfrac{c}{2\pi}}\, \dfrac{\exp\left(-\dfrac{c}{2(x-\mu)}\right)}{(x-\mu)^{3/2}}, & x > \mu, \\ 0, & x \le \mu. \end{cases}$$

Sampling via the Normal Distribution

A direct and exact method for sampling from the Lévy distribution is based on a standard normal variable $Z \sim \mathcal{N}(0,1)$:

$$X = \mu + \frac{c}{Z^2}.$$

Then the random variable X follows a Lévy distribution with parameters (μ, c).
Explanation: Let $Y = c/Z^2$. The probability density of Y is

$$f_Y(y) = \sqrt{\frac{c}{2\pi}}\, y^{-3/2} e^{-c/(2y)}, \quad y > 0,$$

which matches the Lévy density for $\mu = 0$.

Rejection Sampling Method

For rejection sampling convenient proposal distributions are (choose the best parameters):
1. The Pareto distribution:

$$g(x) = \frac{a\, x_m^a}{x^{a+1}}, \quad x \geq x_m,$$

2. Unif[a,b]
3. Shifted exponential distribution

$$f(x, a, b) = b\exp(-b(x-a)), x > a.$$

Choose 2 proposal distribution and explain it. 4. Choose another majorizing distribution by your own sight.