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Homework

Problem 1

Find all stationary curves of the following variational problem:

$$I(x,y) = \int_1^2 \frac{1}{y}\sqrt{1+x'^2+y'^2}\,\mathrm{d}t,$$
$$x(1)=1,\quad y(1)=1,\quad x(2)=3,\quad y(2)=4.$$

Sol: $\mathcal{L}(t;x,y;x',y') = \frac{1}{y}\sqrt{1+x'^2+y'^2}$ - independent with t.

by the integral of motion, we have $x'\mathcal{L}_{x'} + y'\mathcal{L}_{y'} - \mathcal{L} = \text{const}$

That is. $\frac{1}{y}\frac{x'^2}{\sqrt{1+x'^2+y'^2}} + \frac{1}{y}\cdot\frac{y'^2}{\sqrt{1+x'^2+y'^2}} - \frac{1}{y}\sqrt{1+x'^2+y'^2} = \text{const}$

$\Rightarrow \frac{1}{y}\left(\sqrt{1+x'^2+y'^2} - \frac{1}{\sqrt{1+x'^2+y'^2}} - \sqrt{1+x'^2+y'^2}\right) = \text{const} \Rightarrow \frac{1}{y}\sqrt{1+x'^2+y'^2}^{\,-1} = \text{const}$ (1)

by E-L equation $\mathcal{L}_x - \frac{d}{dt}\mathcal{L}_{x'} = 0 \Rightarrow \frac{d}{dt}\left(\frac{1}{y}\cdot\frac{x'}{\sqrt{1+x'^2+y'^2}}\right) \Rightarrow \frac{d}{dt}(\text{const}\cdot x') = 0 \Rightarrow x'' = 0$.

thus $x(t) = at+b$. since $\begin{cases} x(1)=1 \\ x(2)=3 \end{cases} \Rightarrow x(t) = 2t-1$

(1) $\Rightarrow \frac{1}{y}\sqrt{5+y'^2}^{\,-1} = \text{const} \Rightarrow \frac{c}{y^2} = 5+(y')^2 \Rightarrow y' = \pm\sqrt{\frac{c_1}{y^2}-5}$

$\xRightarrow{y>0} \frac{dy}{dt} = \pm\frac{1}{y}\sqrt{c_1-5y^2} \Rightarrow \frac{y\,dy}{\sqrt{c_1-5y^2}} = \pm dt \Rightarrow t = \pm\frac{1}{5}\sqrt{c_1-5y^2} + c_2$

since: $\begin{cases} y(1)=1 \\ y(2)=4 \end{cases}$ no sol for $t = \frac{1}{5}\sqrt{c_1-5y^2}+c_2$

$c_1 = 105$ $c_2 = 3$ for $t = -\frac{1}{5}\sqrt{c_1-5y^2}+c_2$

$\Rightarrow t = -\frac{1}{5}\sqrt{105-5y^2}+3 \Rightarrow y = \sqrt{-5(t-3)^2+21}$ $t\in[1,2]$

Stationary curve $\begin{cases} x(t) = 2t-1 \\ y(t) = \sqrt{-5(t-3)^2+21} \end{cases}$

Problem 2

Find all stationary curves of the following variational problem and check whether they furnish an extremum:

$$I(x) = \int_0^{\frac{25\pi}{12}} (x'^2 + 2xx' - 9x^2)\,\mathrm{d}t, \quad x(0) = 0, \quad x\left(\frac{25\pi}{12}\right) = \sqrt{2}.$$

Sol: Lagrangian $\mathcal{L}(t,x,x') = (x')^2 + 2xx' - 9x^2$

E-L equation $\mathcal{L}_x - \frac{d}{dt}(\mathcal{L}_{x'}) = 0 \implies 2x' - 18x - \frac{d}{dt}(2x' + 2x) \implies x'' + 9x = 0$.

homogenous equation. general sol is $x(t) = A\cos 3t + B\sin 3t$. $\xRightarrow{\begin{cases} x(0)=0 \\ x(\frac{25}{12}\pi)=\sqrt{2}\end{cases}} \begin{cases} A = 0 \\ B = 2 \end{cases}$

stationary curve $x(t) = 2\sin 3t$.

$Q(t) = \frac{1}{2}\left(\mathcal{L}_{xx} - \frac{d}{dt}\mathcal{L}_{xv}\right) = -9$

$P(t) = \frac{1}{2}\mathcal{L}_{vv} = 1$.

Jacobi equation. $Qh - \frac{d}{dt}(Ph') = 0 \implies h'' + 9h = 0 \implies h(t) = A_1\sin 3t + B_1\cos 3t$.

$\begin{cases} h(0) = 0 \\ h'(0) = 1 \end{cases} \implies h(t) = \frac{1}{3}\sin 3t$

on $[0, \frac{25}{12}\pi]$. $\exists\, t_0 = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

s.t. $h(t_0) = 0$. i.e. we have conjugate points.

The Jacobi thm don't work.

$$\Delta I = (x_0 + h, x_0) = \int_0^{\frac{25}{12}\pi} [(-9)h^2 + 2hh' + h'^2]\,dt + O(h^2)$$

$$= \int_0^{\frac{25}{12}\pi} [(h')^2 - 9h^2]\,dt + O(h^2)$$

let $h = \sin\frac{12t}{25}$ s.t. $h(0) = h(\frac{25}{12}\pi) = 0$. $\Delta I = \left(\frac{144}{625} - 9\right)\int_0^{\frac{25}{12}\pi} h^2\,dt + O(h^2) < 0$.

let $h = \sin\frac{120}{25}t$ s.t. $h(0) = h(\frac{25}{12}\pi) = 0$ $\Delta I = \left(\frac{14400}{625} - 9\right)\int_0^{\frac{25}{12}\pi} h^2\,dt + O(h^2) > 0$.

ΔI not definite. thus stationary curve $x(t) = 2\sin 3t$ is not extremum.

Problem 3

Find all stationary points of the following isoperimetric problem:

$$I(x,y) = \int_0^1 \left(x'^2 - x'y' - y'^2(t)\right) \mathrm{d}t,$$

$$x(0) = 1, \quad y(0) = 2, \quad x(1) = 3, \quad y(1) = 2,$$

$$\int_0^1 \left(x' + 2ty + \frac{1}{3}y'^2\right) \mathrm{d}t = \frac{20}{3}.$$

Sol: define the new Lagrangian: $\mathcal{M} = \mathcal{L}_1 + \lambda \mathcal{L}_2 = x'^2 - x'y' - y'^2 + \lambda\,(x' + 2ty + \frac{1}{3}y'^2)$.

E-L equation:

$$\begin{cases} \dfrac{d}{dt}\left(2x' - y' + \lambda\right) = 0 \\ 2\lambda t - \dfrac{d}{dt}\left(-x' - 2y' + \dfrac{2\lambda}{3}y'\right) = 0 \end{cases} \Rightarrow \begin{cases} 2x'' = y'' \\ x'' + (2 - \dfrac{2\lambda}{3})y'' + 2\lambda t = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = \dfrac{\lambda}{4\lambda - 15}t^3 + B_1 t + C_1 \\ y = \dfrac{2\lambda}{4\lambda - 15}t^3 + B_2 t + C_2 \end{cases} \qquad \begin{cases} x(0) = 1 \\ y(0) = 2 \end{cases} \qquad \begin{cases} x(1) = 3 \\ y(1) = 2 \end{cases}$$

$$\Rightarrow \begin{cases} B_1 + \dfrac{\lambda}{4\lambda - 15} = 2 \quad \Rightarrow B_1 = 2 + \dfrac{B_2}{2} \\ B_2 + \dfrac{2\lambda}{4\lambda - 15} = 0 \quad \Rightarrow \dfrac{\lambda}{4\lambda - 15} = -\dfrac{B_2}{2} \\ \displaystyle\int_0^1 \frac{3\lambda}{4\lambda - 15}t^2 + B_1 + \frac{4\lambda}{4\lambda - 15}t^4 + 2B_2 t^2 + 4t + \frac{1}{3}\left(\frac{6\lambda}{4\lambda - 15}t^2 + B_2\right)^2 dt \end{cases}$$

$$= \int_0^1 [3B_2^2 - 2B_2]\,t^4 + \left[\frac{4B_2\lambda + 3\lambda}{4\lambda - 15} + 2B_2\right]t^2 + 4t + B_1 + \frac{1}{3}B_2^2 \, dt$$

$$= \int_0^1 [3B_2^2 - 2B_2]t^4 + \left[4B_2 \cdot -\frac{B_2}{2} + 3 \cdot -\frac{B_2}{2} + 2B_2\right]t^2 + 4t + B_1 + \frac{1}{3}B_2^2 \, dt.$$

$$= \frac{1}{5}(3B_2^2 - 2B_2]\,t^5 + \frac{1}{3}[-2B_2^2 + \frac{1}{2}B_2]\,t^3 + 2t^2 + (B_1 + \frac{1}{3}B_2^2)t \,\Big|_0^1$$

$$= \frac{3}{5}B_2^2 - \frac{2}{5}B_2 + -\frac{2}{3}B_2^2 + \frac{1}{6}B_2 + 4 + \frac{1}{2}B_2 + \frac{1}{3}B_2^2$$

$$= \frac{4}{15}B_2^2 + \frac{4}{15}B_2 + 4 = \frac{20}{3} \Rightarrow B_2 = \frac{\pm\sqrt{41} - 1}{2} \qquad B_1 = \frac{\pm\sqrt{41} + 7}{4}$$

$$\Rightarrow \begin{cases} x(t) = \dfrac{\mp\sqrt{41} + 1}{4}t^3 + \dfrac{\pm\sqrt{41} + 7}{4}t + 1 \\ y(t) = \dfrac{\mp\sqrt{41} + 1}{2}t^3 + \dfrac{\pm\sqrt{41} - 1}{2}t + 2 \end{cases}$$

Problem 4

Find the distance between two curves:

$$t^2 + x^2 = 1, \quad x^3 = \left(t - \frac{35}{8}\right)^2.$$

Sol: write the problem as the form of variation problem with moving ends.

$$I(x) = \int_{(t_A,x_A)}^{(t_B,x_B)} \sqrt{1+(x')^2}\, dt. \qquad \varphi_A(t,x) = x^2+t^2-1. \qquad \varphi_B(t,x) = \left(t-\frac{35}{8}\right)^2 - x^3$$

E-L equation $\frac{d}{dt}\left(\frac{x'}{\sqrt{1+(x')^2}}\right) \Rightarrow x'' = 0 \quad \Rightarrow \quad x(t) = at+b.$

$$\mathcal{L} - x'\mathcal{L}_{x'} = \sqrt{1+x'^2} - \frac{x'^2}{\sqrt{1+x'^2}} = \frac{1}{\sqrt{1+x'^2}} = \frac{1}{\sqrt{1+a^2}} \qquad \begin{cases} \varphi_{A,t} = 2t & \varphi_{A,x} = 2x \\ \varphi_{B,t} = 2\left(t-\frac{35}{8}\right) & \varphi_{B,x} = -3x^2 \end{cases}$$

the transversality condition

$$\begin{cases} \left(\frac{x'}{\sqrt{1+x'^2}} \cdot 2t - \frac{1}{\sqrt{1+a^2}} \cdot 2x\right)\Big|_{t=t_A} = 0 \\ \left(\frac{x'}{\sqrt{1+x'^2}} \cdot 2\left(t-\frac{35}{8}\right) - \frac{1}{\sqrt{1+a^2}} \cdot (-3x^2)\right)\Big|_{t=t_B} = 0 \end{cases} \Rightarrow \begin{cases} 2ta - 2(ta+b)\Big|_{t=t_A} = 0 \\ 2a\left(t-\frac{35}{8}\right) + 3(at+b)^2\Big|_{t=t_B} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} b = 0 \\ a = \dfrac{\frac{35}{4} - 2t_B}{3t_B^2} \end{cases} \Rightarrow 9a^2 t_B^4 = 4\left(t_B - \frac{35}{8}\right)^2 \quad (1)$$

$$\begin{cases} \varphi_A(t_A,x_A) = 0 \Rightarrow (a^2+1)t_A^2 - 1 = 0 \quad (2) \\ \varphi_B(t_B,x_B) = 0 \Rightarrow a^3t_B^3 - \left(t_B - \frac{35}{8}\right)^2 = 0 \xrightarrow[\text{by }(1)]{\text{substitute } (t_B-\frac{35}{8})^2} a^3t_B^3 = \frac{9}{4}a^2t_B^4 \Rightarrow a = \frac{9}{4}t_B \quad (3). \end{cases}$$

$$\Rightarrow 27t_B^3 + 8t_B - 35 = 0 \Rightarrow (t_B-1)(27t_B^2 - 27t_B + 35) = 0 \Rightarrow t_B = 1 \xrightarrow[\text{by }(2)\&(3).]{} \begin{cases} a = \frac{9}{4} \\ t_A = \frac{4}{\sqrt{97}} \end{cases}$$

$\Delta = 27^2 - 4 \cdot 27 \cdot 35 < 0.$

single real root $t_B = 1.$

thus.

$$I\left(\frac{9}{4}t\right) = \int_{\frac{4}{\sqrt{97}}}^{1} \sqrt{1+\left(\frac{9}{4}\right)^2}\, dt = \left(1 - \frac{4}{\sqrt{97}}\right) \cdot \frac{\sqrt{97}}{4} = \frac{\sqrt{97}}{4} - 1$$