Mathematical Logic

Lecture 3

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Semantics

Consider $\mathcal{S}_{Gr} = \{\circ, e\}$.
The formula $\forall x \exists y \; x \circ y \equiv e$ is a member of $\mathcal{F}_{\mathcal{S}_{Gr}}$
But what does this formula mean?

• If $\forall x \exists y$ quantify over the set $\mathbb{Z}$ of integers, if $\circ$ stands for addition in $\mathbb{Z}$, and if e denotes the integer 0, then the formula means:
"for every integer there is another integer such that the addition of both is equal to 0"
$\rightsquigarrow$ if interpreted in this way: the formula is true!

• If $\forall x \exists y$ quantify over $\mathbb{Z}$, if $\circ$ stands for multiplication in $\mathbb{Z}$, and if e denotes the integer 1, then the formula means:
"for every integer there is another integer such that the product of both is equal to 1"
$\rightsquigarrow$ if interpreted in this way: the formula is false!

Let us now make precise what we understand by such interpretations of symbol sets of first-order languages:

$\boxed{\textit{Definition } 1.}$ Let $\mathcal{S}$ be an arbitrary symbol set:
An $\mathcal{S}-$model is an ordered pair $\mathfrak{M} = (D, \mathfrak{J})$, such that:

1.) D is a set; $D \neq \varnothing$ (domain)

2.) $\mathfrak{J}$ is defined on $\mathcal{S}$ as follows (interpretation of $\mathcal{S}$):

- for $n-$ary predicates P in $\mathcal{S}$: $\mathfrak{J}(P) \subseteq D^n$
- for $n-$ary function signs f in $\mathcal{S}$: $\mathfrak{J}(f) : D^n \to D$
- for every constant c in $\mathcal{S}$: $\mathfrak{J}(c) \in D$

Examples.

(I) $\mathcal{S} = \mathcal{S}_{Gr} = \{\circ, e\}$, $D = \mathbb{Z}$, $\mathfrak{J}(\circ) = +$ on $\mathbb{Z}$, $\mathfrak{J}(e) = 0$
$\Longrightarrow \mathfrak{M} = (D, \mathfrak{J})$ is the model of the set of integers viewed as additive group.

(II) $\mathcal{S} = \mathcal{S}_{Fie} = \{+, \cdot, 0, 1\}$, $D = \mathbb{R}$, $\mathfrak{J}(+) = +$, $\mathfrak{J}(\cdot) = \cdot$, $\mathfrak{J}(0) = 0$, $\mathfrak{J}(1) = 1 \implies \mathfrak{M} = (D, \mathfrak{J})$ is the model of the real number field.

(III) Now we add $<$ to $\mathcal{S}_{Fie}$: $\mathcal{S}_{OrdFie} = \{+, \cdot, 0, 1, <\}$, $\mathfrak{J}(<) \subseteq \mathbb{R} \times \mathbb{R}$ such that $\Big((a, b) \in \mathfrak{J}(<) \Leftrightarrow a < b \Big) \implies \mathfrak{M} = (D, \mathfrak{J})$ is the model of the <u>ordered</u> real number field.

Just as an interpretation assigns meaning to predicates, function signs, and constants, we need a way of assigning values to variables:

$\boxed{\textit{Definition } 2.}$ A variable assignment over a model $\mathfrak{M} = (D, \mathfrak{J})$ is a function $s : \{v_0, v_1, \ldots\} \to D$.

Remark. As we will see, we need variable assignments in order to define the truth values of quantified formulas. Here is the idea : $\forall x \phi$ is true $\Leftrightarrow$ whatever $d \in D$ a variable assignment assigns to x, ϕ turns out to be true under this assignment.

Analogously for $\exists x \phi$ and the existence of an element $d \in D$. It is also useful to have a formal way of changing variable assignments: Let s be a variable assignment over $\mathfrak{M} = (D, \mathfrak{I})$, let $d \in D$:

We define

why

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有 $\forall, \exists$ 变量

$$s\frac{d}{x} : \{v_0, v_1, \ldots\} \to D$$

$$s\frac{d}{x}(y) = \begin{cases} d, & \text{if } y = x; \\ s(y), & \text{if } y \neq x. \end{cases}$$

(where x is some variable in $\{v_0, v_1, \ldots\}$).

E.g., $s\frac{4}{v_0}(v_0) = 4$, $s\frac{4}{v_0}(v_1) = s(v_1)$

Given an $\mathcal{S}-$model together with a variable assignment over this model, we can define the semantic value of a term/formula:

$\boxed{\textit{Definition 3.}}$ Let $\mathfrak{M} = (D, \mathfrak{I})$ be an $\mathcal{S}-$model. Let s be a variable assignment over $\mathfrak{M}$:

$Val_{\mathfrak{M},s}$ ("semantic value function") is defined on $\mathcal{T}_{\mathcal{S}} \cup \mathcal{F}_{\mathcal{S}}$, such that:

(V1) $Val_{\mathfrak{M},s}(x) = s(x)$

(V2) $Val_{\mathfrak{M},s}(c) = \mathfrak{I}(c)$

(V3) $Val_{\mathfrak{M},s}(f(t_1, \ldots, t_n)) = \mathfrak{I}(f)(Val_{\mathfrak{M},s}(t_1), \ldots, Val_{\mathfrak{M},s}(t_n))$

(V4) $Val_{\mathfrak{M},s}(t_1 \equiv t_2) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(t_1) = Val_{\mathfrak{M},s}(t_2)$

(V5) $Val_{\mathfrak{M},s}(P(t_1, \ldots, t_n)) := 1 \Leftrightarrow (Val_{\mathfrak{M},s}(t_1), \ldots, Val_{\mathfrak{M},s}(t_n)) \in \mathfrak{I}(P)$

(V6) $Val_{\mathfrak{M},s}(\neg\phi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = 0$

(V7) $Val_{\mathfrak{M},s}(\phi \wedge \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = Val_{\mathfrak{M},s}(\psi) = 1$

(V8) $Val_{\mathfrak{M},s}(\phi \vee \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = 1$ or $Val_{\mathfrak{M},s}(\psi) = 1$

(V9) $Val_{\mathfrak{M},s}(\phi \rightarrow \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = 0$ or $Val_{\mathfrak{M},s}(\psi) = 1$

(V10) $Val_{\mathfrak{M},s}(\phi \leftrightarrow \psi) := 1 \Leftrightarrow Val_{\mathfrak{M},s}(\phi) = Val_{\mathfrak{M},s}(\psi)$

(V11) $Val_{\mathfrak{M},s}(\forall x \phi) := 1 \Leftrightarrow$ for all $d \in D$: $Val_{\mathfrak{M},s\frac{d}{x}}(\phi) = 1$

(V12) $Val_{\mathfrak{M},s}(\exists x \phi) := 1 \Leftrightarrow$ there is a $d \in D$, such that $Val_{\mathfrak{M},s\frac{d}{x}}(\phi) = 1$

For (V4) - (V12): in case the "iff" condition is not satisfied, the corresponding semantic value is defined to be 0.

Terminology:

$Val_{\mathfrak{M},s}(t)$ and $Val_{\mathfrak{M},s}(\phi)$ are the semantic values of t and ϕ respectively (relative to $\mathfrak{M}$), where

- $Val_{\mathfrak{M},s}(t) \in D$
- $Val_{\mathfrak{M},s}(\phi) \in \{1, 0\}$

Instead of writing that $Val_{\mathfrak{M},s}(\phi) = 1$, we may also say:

- ϕ is true at $\mathfrak{M}, s$
- $\mathfrak{M}, s$ make ϕ true
- $\mathfrak{M}, s$ satisfy ϕ
- briefly $\mathfrak{M}, s \models \phi$

We will also write for sets Φ of formulas:
$\mathfrak{M}, s \models \Phi \iff$ for all $\phi \in \Phi$ $\mathfrak{M}, s \models \phi$

<u>Example.</u> Let $\mathfrak{M} = (D, \mathfrak{I})$ be the model of the ordered real number field. Let s be a variable assignment over $\mathfrak{M}$, such that $s(v_1) = 3$:

$Val_{\mathfrak{M},s}(\exists v_0 \; v_1 < v_0 + 1) = 1 \Leftrightarrow$ there is a $d \in D = \mathbb{R}$, such that:
$Val_{\mathfrak{M},s\frac{d}{v_0}}(v_1 < v_0 + 1) = 1 \quad$ by (V12)
$\Leftrightarrow$ there is a $d \in D = \mathbb{R}$, such that:
$(Val_{\mathfrak{M},s\frac{d}{v_0}}(v_1), Val_{\mathfrak{M},s\frac{d}{v_0}}(v_0 + 1)) \in \mathfrak{I}(<)$
$\Leftrightarrow$ there is a $d \in D = \mathbb{R}$, such that:
$(s\frac{d}{v_0}(v_1), \mathfrak{I}(+)(Val_{\mathfrak{M},s\frac{d}{v_0}}(v_0), Val_{\mathfrak{M},s\frac{d}{v_0}}(1))) \in \mathfrak{I}(<)$ by (V1), (V3)
$\Leftrightarrow$ there is a $d \in D = \mathbb{R}$, such that:
$(s\frac{d}{v_0}(v_1), \mathfrak{I}(+)(s\frac{d}{v_0}(v_0), \mathfrak{I}(1))) \in \mathfrak{I}(<)$ by (V1), (V2)
$\Leftrightarrow$ there is a $d \in D = \mathbb{R}$, such that:
$(s(v_1), \mathfrak{I}(+)(d, 1)) \in \mathfrak{I}(<)$

$\Leftrightarrow$ there is a $d \in D = \mathbb{R}$, such that:
$(3, d+1) \in \mathfrak{I}(<)$
$\Leftrightarrow$ there is a $d \in D = \mathbb{R}$, such that:
$3 < d + 1$
Such a $d \in \mathbb{R} = D$ exists:
$\Rightarrow Val_{\mathfrak{M},s}(\exists v_0 \; v_1 < v_0 + 1) = 1$

Examples like these tell us:

Remark.

1.) The semantic value of a term t only depends (i) on the interpretation of the constants and functions signs that occur in t and (ii) on the values the assignment function assigns to the variables that occur in t.

2.) The semantic value of a formula ϕ only depends (i) on the interpretation of the constants, functions signs, and predicates that occur in ϕ and (ii) on the values the assignment function assigns to the variables that occur in ϕ freely (the assignment of values to bound occurrences of variables are "erased" by the quantifiers which bind these occurrences).

Lemma 1 (Coincidence Lemma).

Let $\mathcal{S}_1, \mathcal{S}_2$ be two symbol sets. Let $\mathfrak{M}_1 = (D, \mathfrak{I}_1)$ be an $\mathcal{S}_1-$model , $\mathfrak{M}_2 = (D, \mathfrak{I}_2)$ be an $\mathcal{S}_2-$model. Let s_1 be a variable assignment over $\mathfrak{M}_1$, s_2 a variable assignment over $\mathfrak{M}_2$. Finally, let $\mathcal{S} = \mathcal{S}_1 \cap \mathcal{S}_2$.

1.) For all terms $t \in \mathcal{T}_{\mathcal{S}}$:

If $\mathfrak{I}_1(c) = \mathfrak{I}_2(c)$ for all c in t

$\mathfrak{I}_1(f) = \mathfrak{I}_2(f)$ for all f in t

$s_1(x) = s_2(x)$ for all x in t

then $Val_{\mathfrak{M}_1,s_1}(t) = Val_{\mathfrak{M}_2,s_2}(t)$

2.) For all formulas $\phi \in \mathcal{F}_{\mathcal{S}}$:

If $\mathfrak{I}_1(c) = \mathfrak{I}_2(c)$ for all c in ϕ

$\mathfrak{I}_1(f) = \mathfrak{I}_2(f)$ for all f in ϕ

$\mathfrak{I}_1(P) = \mathfrak{I}_2(P)$ for all P in ϕ

$s_1(x) = s_2(x)$ for all x in $free(\phi)$

then $Val_{\mathfrak{M}_1,s_1}(\phi) = Val_{\mathfrak{M}_2,s_2}(\phi)$

Corollary 2.

Let ϕ be an $\mathcal{S}-$sentence, let s_1, s_2 be variable assignments over an $\mathcal{S}-$model $\mathfrak{M}$:
It follows that $Val_{\mathfrak{M},s_1}(\phi) = Val_{\mathfrak{M},s_2}(\phi)$.

Proof. Since ϕ is assumed to be a sentence, $free(\phi) = \varnothing$. Therefore, trivially, $s_1(x) = s_2(x)$ for all $x \in free(\phi)$. So we can apply the coincidence lemma, where in this case $\mathfrak{M}_1 = \mathfrak{M}_2 = \mathfrak{M}$, and we are done.$\blacksquare$
Remark. We see that as far as sentences are concerned, it is irrelevant which variable assignment we choose in order to evaluate them: a sentence ϕ is true in a model $\mathfrak{M}$ relative to some variable assignment over $\mathfrak{M}$ iff ϕ is true in $\mathfrak{M}$ relative to all variable assignments over $\mathfrak{M}$.
Therefore we are entitled to write for sentences ϕ and sets Φ of sentences:

$$\mathfrak{M} \models \phi \text{ and } \mathfrak{M} \models \Phi$$

without mentioning a variable assignment s at all.

Example. (We use again $\mathcal{S}_{OrdFie} = \{+, \cdot, 0, 1, <\}$)
Let Φ be the following set of sentences:

- $\forall x \forall y \forall z \quad (x + y) + z \equiv x + (y + z)$
- $\forall x \quad x + 0 \equiv x$
- $\forall x \exists y \quad x + y \equiv 0$
- $\forall x \forall y \quad x + y \equiv y + x$

$\rightsquigarrow$ These axioms describe the Abelian group $(\mathbb{R}, +)$

- $\forall x \forall y \forall z \quad (x \cdot y) \cdot z \equiv x \cdot (y \cdot z)$
- $\forall x \quad x \cdot 1 \equiv x$
- $\forall x \forall y \quad x \cdot y \equiv y \cdot x$
- $(\neg x = 0 \Rightarrow \exists y, x \cdot y = 1)$

$\rightsquigarrow$ These axioms describe the Abelian group $(\mathbb{R} \setminus \{0\}, \cdot)$

- $\forall x \forall y \forall z \quad x \cdot (y + z) \equiv x \cdot y + x \cdot z$
- $\neg 0 = 1$

$\rightsquigarrow$ All axioms up to here taken together describe the real field $(\mathbb{R}, +, \cdot)$

- $\forall x \, \neg x < x$
- $\forall x \forall y \forall z \quad ((x < y) \wedge (y < z) \rightarrow (x < z))$
- $\forall x \forall y \quad (x < y \vee x \equiv y \vee y < x)$
- $\forall x \forall y \forall z \quad (x < y \rightarrow x + z < y + z)$
- $\forall x \forall y \forall z \quad (x < y \wedge 0 < z \rightarrow x \cdot z < y \cdot z)$

$\rightsquigarrow$ All axioms taken together describe the real ordered field $(\mathbb{R}, +, \cdot)$

Now let $\mathfrak{M}$ be the model of the real ordered field: then $\mathfrak{M} \models \Phi$

$\boxed{\textit{Definition 4.}}$ For all $\phi \in \mathcal{F}_{\mathcal{S}}$, $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$:
ϕ follows logically from Φ, briefly: $\Phi \models \phi$ iff for all $\mathcal{S}-$models $\mathfrak{M}$, for all variable assignments s over $\mathfrak{M}$:

$$\text{if } \mathfrak{M}, s \models \Phi, \text{then } \mathfrak{M}, s \models \phi$$

We also say equivalently:
Φ logically implies ϕ ; or ϕ is a logical consequence of Φ.
Example. Let $\mathcal{S} = \mathcal{S}_{Gr} = \{e, \circ\}$:
Let Φ be the set that has the group axioms as its only members. It follows that $\Phi \models \forall x \exists y \quad y \circ x \equiv e$
This is because if $\mathfrak{M}$ is a model of Φ, i.e., $\mathfrak{M} \models \Phi$, then $(D, \mathfrak{I}(\circ), \mathfrak{I}(e))$ is a group.
Furthermore:
$\Phi \not\models \forall x \forall y \ x \circ y = y \circ x$
$\Phi \not\models \neg\forall x \forall y \ x \circ y = y \circ x$

We see that it is not generally the case that: $\Phi \not\models \phi \implies \Phi \models \neg\phi$
But : $\mathfrak{M}, s \not\models \phi \implies \mathfrak{M}, s \models \neg\phi$
Now we single out important semantic concepts that apply to formulas of a particular type. Some formulas have the property of being true under all interpretations:

$\boxed{\textit{Definition } 5.}$ For all $\phi \in \mathcal{F}_{\mathcal{S}}$: ϕ is logically true iff for all $\mathcal{S}-$models $\mathfrak{D}$, for all variable assignments s over $\mathfrak{D}$,

$$\mathfrak{M}, s \models \phi$$

Example. (i) $\phi \vee \neg\phi$
(ii) $\forall x \exists y \; x \equiv y$ are logically true.
(iii) $P(c)$
(iv) $\exists x P(x)$ are not logically true.

Some formulas are true under some interpretation:

$\boxed{\textit{Definition 6.}}$ For all $\phi \in \mathcal{F}_{\mathcal{S}}$, $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$:
ϕ is satisfiable iff there is an $\mathcal{S}-$model $\mathfrak{M}$ and a variable assignment s over $\mathfrak{M}$, such that: $\mathfrak{M}, s \models \phi$.
$\Phi \subseteq \mathcal{F}_{\mathcal{S}}$ is (simultaneously) satisfiable iff there are $\mathfrak{M}, s$ such that $\mathfrak{M}, s \models \phi$.

Example. (i) $\phi \vee \neg\phi$, (ii) $P(c)$, (iii) $\neg P(c)$ are satisfiable.
(iv) $\phi \wedge \neg\phi$, (v) $\neg\forall x\ x \equiv x$ are not satisfiable.
(vi) $\{P(c), \exists x\ Q(x,x)\}$ is satisfiable.
(vii) $\{P(c), \exists x\ Q(x,x), P(c) \rightarrow \forall x \neg Q(x,x)\}$ is not simultaneously satisfiable.

Logical consequence, logical truth, and satisfiability are themselves logically related to each other:

Lemma 3.

For all $\phi \in \mathcal{F}_{\mathcal{S}}$, $\Phi \subseteq \mathcal{F}_{\mathcal{S}}$:
1. ϕ is logically true iff $\varnothing \models \phi$.
2. $\Phi \models \phi$ iff $\Phi \cup \{\neg\phi\}$ is not satisfiable.
3. ϕ is logically true iff $\neg\phi$ is not satisfiable.

Proof. 1.) $\varnothing \models \phi \Leftrightarrow$ for all $\mathfrak{M}, s$: if $\mathfrak{M}, s \models \varnothing$, then $\mathfrak{M}, s \models \phi \Leftrightarrow$ for all $\mathfrak{M}, s$: $\mathfrak{M}, s \models \phi \Leftrightarrow \phi$ is logically true.
2.) $\Phi \models \phi \Leftrightarrow$ for all $\mathfrak{M}, s$: if $\mathfrak{M}, s \models \Phi$, then $\mathfrak{M}, s \models \phi \Leftrightarrow$ not there are $\mathfrak{M}, s$, such that : $\mathfrak{M}, s \models \Phi$, $\mathfrak{M}, s \not\models \phi \Leftrightarrow$ not there are $\mathfrak{M}, s$, such that : $\mathfrak{M}, s \models \Phi$, $\mathfrak{M}, s \models \neg\phi \Leftrightarrow$ not there are $\mathfrak{M}, s$, such that : $\mathfrak{M}, s \models \Phi \cup \{\neg\phi\} \Leftrightarrow \Phi \cup \{\neg\phi\}$ not satisfiable.
3.) ϕ is logically true $\Leftrightarrow \varnothing \models \phi \Leftrightarrow \{\neg\phi\}$ is not satisfiable $\Leftrightarrow \neg\phi$ is not satisfiable. ■

Sometimes two formulas "say the same" :

$\boxed{\textit{Definition 7.}}$ For all $\phi, \psi \in \mathcal{F}_{\mathcal{S}}$: ϕ is logically equivalent to ψ iff $\phi \models \psi$ and $\psi \models \phi$.

Example. (i) $\phi \wedge \psi$ is logically equivalent to $\neg(\neg\phi \vee \neg\psi)$
(ii) $\phi \rightarrow \psi$ is logically equivalent to $\neg\phi \vee \psi$
(iii) $\phi \leftrightarrow \psi$ is logically equivalent to $\neg\exists x \neg\phi$

Proposition 4.

Let $\mathcal{S}, \mathcal{S}'$ be symbol sets, such that $\mathcal{S} \subseteq \mathcal{S}'$. Let ϕ be an $\mathcal{S}-$formula ($\Rightarrow \phi$ is also an $\mathcal{S}'$-formula). Then:
ϕ is $\mathcal{S}-$satisfiable iff ϕ is $\mathcal{S}'$-satisfiable.

Proof. Assume that ϕ is $\mathcal{S}-$satisfiable. By definition, there is an $\mathcal{S}-$model $\mathfrak{M}$ and there is a variable assignment s over $\mathfrak{M}$, such that:

$$\mathfrak{M}, s \models \phi$$

Now we define an $\mathcal{S}'$-model $\mathfrak{M}'$: let $D' = D$, $\mathfrak{I}'|_{\mathcal{S}} = \mathfrak{I}$ (i.e., $\mathfrak{I}$ and $\mathfrak{I}'$ are identical on $\mathcal{S}$), $\mathfrak{I}'$ on $\mathcal{S}' \setminus \mathcal{S}$ is chosen arbitrarily. Furthermore, let $s' := s$. By the coincidence lemma it follows that $\mathfrak{M}', s' \models \phi$ (since ϕ is an $\mathfrak{S}-$formula, the symbols in $\mathfrak{S}$ are interpreted in the same way by $\mathfrak{I}'$ and $\mathfrak{I}$, and the two models $\mathfrak{M}'$ and $\mathfrak{M}$ have the same domain). Hence, ϕ is $\mathcal{S}'$-satisfiable ($\mathfrak{M}'$ is called an expansion of $\mathfrak{M}$).

The reverse implication is similar (in this case one simply "forgets" about the interpretation of symbols in $\mathcal{S}' \setminus \mathcal{S}$: this yields a so-called reduct of $\mathfrak{M}'$).■

By Lemma 3: analogously for logical consequence, logical truth, and so forth.

Exercises

Exercise 1. Let D be finite and non-empty, let $\mathcal{S}$ be finite. Show that there are only finitely many $\mathcal{S}-$models with domain D.

Exercise 2. The convergence of a real-valued sequence $(x_n)_{n\in\mathbb{N}}$ to a limit x is usually defined as follows:
(Conv) For all $\epsilon > 0$ there is a natural number n, such that for all natural numbers $m > n$ it holds that : $|x_m - x| < \epsilon$.
Represent (Conv) in a first-order language by choosing an appropriate symbol set $\mathcal{S}$ and define the corresponding $\mathcal{S}-$model.

Exercise 3. Show that for arbitrary $\mathcal{S}-$formulas ϕ, ψ, ρ :

$(\phi \vee \psi) \models \rho \iff \phi \models \rho$ and $\psi \models \rho$.