Taylors series

TASKS

Using the definition of the functions of the complex variable $\cos z$ and $\sin z$, prove that:

1. $\sin z \cdot \cos z = \frac{1}{2} \sin 2z$
2. $\sin^2 z + \cos^2 z = 1$.

Using the definition of the functions of the complex variable $\operatorname{sh} z$ and $\operatorname{ch} z$, prove that:

3. $\operatorname{ch}^2 z - \operatorname{sh}^2 z = 1$
4. $\operatorname{ch} 2z = \operatorname{ch}^2 z + \operatorname{sh}^2 z$.

Decompose the z function in a series of degrees:

5. $e^z \sin z = \sum_{n=0}^{\infty} \frac{2^{n/2} sin(\pi n/4)}{n!} z^n$
6. $\operatorname{ch} z \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(4n)!} z^{4n}$
7. $e^{z \operatorname{ctg} \alpha} \cos z, \quad \sin x \neq 0 \quad e^{z \operatorname{ctg} \alpha} \cos z = \sum_{n=0}^{\infty} \frac{cos(n\alpha)}{sin^n \alpha} \frac{z^n}{n!}$
8. $e^{z \cos \alpha} \cos(z \sin \alpha) = \sum_{n=0}^{\infty} \frac{cos(n\alpha)}{n!} z^n$.

Lorans series

TASKS

In Problems 1-3, expand the given function in a Laurent series valid for the given annular domain.

1. $f(z) = \frac{\cos z}{z}, 0 < |z|$

   Answer: $\frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \frac{z^5}{6!}$

2. $f(z) = e^{-1/z^2}, 0 < |z|$

   Answer: $1 - \frac{1}{1!z^2} + \frac{1}{2!z^4} - \frac{1}{3!z^4}$

3. $f(z) = \frac{e^z}{z-1}, 0 < |z-1|$

   Answer: $\frac{e}{z-1} + e + \frac{e(z-1)}{2!} + \frac{e(z-1)^2}{3!}$

In Problems $4-6$, expand $f(z) = \frac{1}{z(z-3)}$ in a Laurent series valid for the indicated annular domain.

4. $0 < |z| < 3$

Answer: $-\frac{1}{3z} - \frac{1}{3^2} - \frac{z}{3^3} - \frac{z^2}{3^4}$

5. $0 < |z-3| < 3$

Answer: $\frac{1}{3(z-3)} - \frac{1}{3^!} + \frac{z-3}{3^2} - \frac{(z-3)^2}{3^4}$

6. $1 < |z-4| < 4$

Answer: $.. - \frac{1}{3(z-4)^2} + \frac{1}{3(z-4)} - \frac{1}{12} + \frac{z-4}{3 \cdot 4^2} - \frac{(z-4)^2}{3 \cdot 4^3}$

In Problems 7,8, expand $f(z) = \frac{1}{(z-1)(z-2)}$ in a Laurent series valid for the given annular domain.

7. $1 < |z| < 2$

Answer: $.. - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{2^2} - \frac{z^2}{2^3}$

8. $0 < |z-1| < 1$

Answer: $\frac{-11}{z-1} - 1 - (z-1) - (z-1)^2$

9.Expand $f(z) = \frac{z}{(z+1)(z-2)}$ in a Laurent series valid for the given annular domain. $0 < |z+1| < 3$

Answer: $\frac{1}{3(z+1)} - \frac{2}{3^2} - \frac{2(z+1)}{3^3} - \frac{2(z+1)^2}{3^4}$

Residues

TASKS

Use an appropriate Laurent series to find the indicated residue.

1. $f(z) = \frac{2}{(z-1)(z+4)}; \mathrm{Res}(f(z), 1)$

   Answer: 2/5

2. $f(z) = \frac{4z-6}{z(2-z)}; \mathrm{Res}(f(z), 0)$

   Answer: -3

Find the residue at each pole of the given function.

3. $f(z) = \frac{z}{z^2+16}$

Answer: Res(f(z),-4i)=1/2,Res(f(z),4i)=1/2
4. $f(z) = \frac{5z^2-4z+3}{(z+1)(z+2)(z+3)}$

Answer: Res(f(z),-1)=6,Res(f(z),-2)=-31,Res(f(z),-3)=30
5. $f(z) = \sec z$

Answer: $Res(f(z), \frac{(2n+1)\pi}{2}) = (-1)^{n+1}, n \in Z$

Use Cauchy's residue theorem, where appropriate, to evaluate the given integral along the indicated contours.

6. $\oint_C \frac{1}{(z-1)(z+2)^2} dz$ (a) $|z| = \frac{1}{2}$

Answer: 0
(b) $|z| = \frac{3}{2}$

Answer: $2\pi i/9$
(c) $|z| = 3$

Answer: 0

Use Cauchy's residue theorem to evaluate the given integral along the indicated contour.

7. $\oint_C \frac{1}{z^2+4z+13} dz, C : |z-3i| = 3$

Answer: $\pi/3$
8. $\oint_C \frac{\tan z}{z} dz, C : |z-1| = 2$

Answer: -4i

Evaluation of Real Trigonometric and Improper Integrals

TASKS

Evaluate the given trigonometric integral.

1. $\int_0^{2\pi} \frac{1}{1+0.5\sin\theta} d\theta$

   Answer: $4\pi/\sqrt{3}$

2. $\int_0^{\pi} \frac{1}{2-\cos\theta} d\theta$[Hint: Let $t = 2\pi - \theta$.]

   Answer: $\pi/\sqrt{3}$

3. $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$

   Answer: $\pi/6$

Establish the given general result.

4. $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} d\theta = \frac{a\pi}{\left(\sqrt{a^2-1}\right)^3}, a > 1$

Evaluate the Cauchy principal value of the given improper integral.

5. $\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2} dx$

Answer: π

6. $\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4} dx$

Answer: $\pi/2$

7. $\int_0^{\infty} \frac{x^2}{x^6+1} dx$

Answer: $\pi/6$

8. $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1} dx$

Answer: πe^{-1}

9. $\int_0^{\infty} \frac{\cos 2x}{x^4+1} dx$

Answer: $\frac{\pi e^{-\sqrt{2}}}{2\sqrt{2}}(cos\sqrt{2} + sin\sqrt{2})$