Chapter 1. Polynomials

§1. Fields

A. Notations

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ $\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$ $\mathbb{R} = \{\text{real number}\}$, $\mathbb{C} = \{\text{complex number}\}$

B. Fields

Def 1. Let $\mathbb{F}$ be a subset of $\mathbb{C}$. If $0, 1 \in \mathbb{F}$, and $\mathbb{F}$ is closed under addition, subtraction, ~~sub~~ multiplication and division, namely, for any $a, b \in \mathbb{F}$, there hold

$$a+b \in \mathbb{F}, \quad a-b \in \mathbb{F}, \quad ab \in \mathbb{F}, \text{ and } \frac{a}{b} \in \mathbb{F} \text{ with } b \neq 0,$$

then $\mathbb{F}$ is said to be a (number) field.

Examples.

(1) $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields, called the rational field, real field and complex field, respectively.

(2) $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a field.

(3) $\mathbb{Z}$ and $\mathbb{N}$ are not fields. The set of all odd (resp. even) integers is not a field.

Prop 1. Let $\mathbb{F} \subseteq \mathbb{C}$ be a field. Then $\mathbb{Q} \subseteq \mathbb{F}$.

Proof. Since $\mathbb{F}$ be a field, $0, 1 \in \mathbb{F}$. Then $n = \overbrace{1 + \cdots + 1}^{n \text{ times}} \in \mathbb{F}$, and thus $-n = 0 - n \in \mathbb{F}$. This means that $\mathbb{Z} \in \mathbb{F}$. Finally, every rational number has the form $\frac{a}{b}$, with $a, b \in \mathbb{Z}$ and $b \neq 0$. As $\mathbb{F}$ is closed under division, we have $\frac{a}{b} \in \mathbb{F}$. Hence $\mathbb{Q} \subseteq \mathbb{F}$. $\square$.

§2. Polynomials in one indeterminate

In this section, we always assume that $\mathbb{F}$ is a fixed field, and we refer to x as an indeterminate.

A. Concepts of polynomials.

Def 2. We call an expression of the form

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

a polynomial over $\mathbb{F}$, where $a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{F}$ are called its coefficients, a_kx^k is called the kth term and a_k is called the coefficient of the kth term. In particular, a_0 is called the constant term.

Denote by $\mathbb{F}[x]$ the set of all polynomials over $\mathbb{F}$.

Examples. ($\mathbb{F}=\mathbb{R}$)

$f(x)=x^2-3x+2$

$f(x)=-\frac{1}{3}x^3+x$

$f(x)=\pi$ (here $a_0=\pi$, and $a_1=a_2=\cdots=0$)

$f(x)=0$ (here all the coefficients are 0)

If

$$f(x)=a_nx^n+\cdots+a_1x+a_0$$

is a polynomial over $\mathbb{F}$ and n is the largest integer such that $a_n\neq 0$, then we say that n is the degree of f, and write $n=\deg(f)$. In this case, we refer a_nx^n and a_n as the leading term and the leading coefficient respectively.

Special polynomials:

(a) Monic ploynomial : leading coefficient is equal to 1

$f(x)=x^2+1$, $f(x)=x^3+ax+b$

(b) zero ploynomial : all coefficients are 0. Denote by 0.

※ Note that we do not assign a degree to a zero polynomial.

(c) Constant or scalar polynomials : polynomials of degree 0 and the zero polynomial.

$$f(x) = a, \quad a \in \mathbb{F}.$$

(d) Linear polynomial : one of degree 1.

$$f(x) = x + a, \quad f(x) = ax + b.$$

Example 2. Let $f(x) = 7x^5 - 8x^3 + 4x - \sqrt{2}$. Then f has degree 5. The leading coefficient is 7, and the constant term is $-\sqrt{2}$.

B. operations

Let

$$f(x) = a_n x^n + \cdots + a_0, \quad g(x) = b_m x^m + \cdots + b_0,$$

be two polynomials over $\mathbb{F}$ of degree n and m, respectively.

If, say, $n > m$, we let $b_j = 0$ if $j > m$ and we also write

$$g(x) = 0x^n + \cdots + b_m x^m + \cdots + b_0.$$

Then we define

$$f(x) \pm g(x) = (a_n \pm b_n) x^n + \cdots + (a_1 \pm b_1) x + (a_0 \pm b_0).$$

If $c \in \mathbb{F}$, then we define

$$c f(x) = c a_n x^n + c a_{n-1} x^{n-1} + \cdots + c a_1 x^1 + c a_0.$$

We also take the product which we write as

$$f(x) g(x) = a_n b_m x^{m+n} + (a_{n-1} b_m + a_n b_{m-1}) x^{m+n-1} + \cdots + a_0 b_0.$$

In fact, if we write $f(x) g(x) = c_{m+n} x^{m+n} + \cdots + c_1 x + c_0$, then

$$c_i = a_0 b_i + a_1 b_{i-1} + \cdots + a_i b_0 = \sum_{\substack{j+k=i \\ 0 \le j,k \le i}} a_j b_k = \sum_{j=0}^{i} a_j b_{i-j}.$$

In particular, we have

$$0 + f(x) = f(x), \quad 1 f(x) = f(x), \quad 0 f(x) = 0, \quad f(x) - f(x) = 0.$$

$$\deg(f(x) \pm g(x)) \le \max(\deg f(x), \deg(g(x))), \text{ if } f(x) \pm g(x) \ne 0.$$

$$\deg(f(x) g(x)) = \deg(f(x)) + \deg(g(x)). \text{ (prove)}, \ f(x) \ne 0 \text{ and } g(x) \ne 0.$$