Dynamical System

our course - 1-dim discrete dynamical system.
- goal: understand the eventual or asymptotic behavior of iterative process.

1 Preliminary.

* Defaultly, function means C^∞ in our course.

Definitions:

Analysis

(1) homeomorphism. $f(x)$ - bijective. cont. $f^{-1}(x)$ cont.

C^r-diffeomorphism. $f(x)$ C^r-homeomorphism s.t. $f^{-1}(x)$ is C^r

Topology: (in $\mathbb{R}$. S is a subset of $\mathbb{R}$).

(1) limit point. x of S. $x \in \mathbb{R}$. $\exists \{x_n\} \subseteq S$. conv. to x.

(2) closed set S. if S contains all of its limit points.

(infinite union not closed.)

(3) open set S. $\forall x \in S$. $\exists \varepsilon > 0$. s.t. all t. in $x - \varepsilon < t < x + \varepsilon$. are contain in S.

(infinite intersection not open).

(4) dense. U is dense in S if $\bar{U} = S$.

△注意. 从语法上. map in some set / map on some set. 后者自动强调了 map 是满射.

Theorems.

(1). Mean Value Theorem.

Suppose $f: [a,b] \to \mathbb{R}$ is C^1. Then there exists $c \in [a,b]$.

s.t. $f(b) - f(a) = f'(c)(b-a)$.

(2). Intermediate Value Theorem.

Suppose. $f: [a,b] \to \mathbb{R}$ is continuous. Suppose that $f(a) = u$ and $f(b) = v$.

Then for any z between u and v. $\exists$ $c \in [a,b]$. s.t. $f(c) = z$.

(3). Implicit Function Theorem.

Suppose. $G: \mathbb{R}^2 \to \mathbb{R}$ is C^1-function (i.e. both p.d. of G exist. and cont.).

further that
$$\begin{cases} 1.\ G(x_0, y_0) = 0 \\ 2.\ \dfrac{\partial G}{\partial y}(x_0, y_0) \neq 0. \end{cases}$$

then there exist open intervals I about x_0 and J about y_0 and a C^1-func.

$p: I \to J$ satisfying
$$\begin{cases} 1.\ p(x_0) = y_0 \\ 2.\ G(x, p(x)) = 0 \text{ for all } x \in I. \end{cases}$$

§ fixed point.

Def. Fixed points for functions are point. satisfy $f(x) = x$.

Prop1 Let $I = [a,b]$. be an interval and let $f: I \to I$. be cont. Then f has at least one fixed point in I.

Prop2. Let $f: I \to I$. and assume. that $|f'(x)| < 1$. for all x in I. Then there exists. a unique fixed point for f in I. moreover. $|f(x) - f(y)| < |x-y|$ for all $x, y \in I$. $x \neq y$.

2. Mean Definitions.

Def1. (orbits). A set of points. forward orbits $O^+(x) = \{x, f(x), f^2(x) \dots\}$ if $f(x)$ is homeomorphism. backward orbits. $O^-(x) = \{x, f^{-1}(x), f^{-2}(x) \dots\}$. full orbits $O(x) = \{ \dots f^{-1}(x), x, f(x) \dots \}$.

Def2. (points). (1) x is fixed point ~~of~~ for f. if $f(x) = x$.

(2) x is periodic point of period n if $f^n(x) = x$. (the least positive n for $f^n(x) = x$ is prime period of x).

(3) $Per_n(f)$ - the set of periodic points of period n (not necessarily prime).

Def3. Let p be periodic of period n. A point x is forward asymptotic. to p if $\lim_{i \to \infty} f^{in}(x) = p$.

$W^s(p)$ - the stable set of p. all points forward asymptotic to p.

Remark: if p is nonperiodic. still define forward asymptotic. points. $|f^i(x) - f^i(p)| \xrightarrow[i \to \infty]{} 0$ If f is invertible. backward asymptotic points by $i \to -\infty$.

$W^u(p)$ - the unstable set of p. all point backward asymptotic to p.

Def4. (critical point). A point x is critical point of f if $f'(x) = 0$. The critical point is non-degenerate. if $f''(x) \neq 0$.

Def5. (eventually periodic point). A point is eventually periodic of period n. if x is not periodic but. $\exists m > 0$. s.t. $f^{n+i}(x) = f^i(x)$. for all $i \geq m$. (That. is $f^i(x)$ is periodic for $i \geq m$).

3 Hyperbolicity.

Def 1. Let p be a periodic point of prime period n. The point p is hyperbolic. if $|(f^n)'(p)| \neq 1$. ($(f^n)'(p)$ is multiplier of periodic point).

Prop. 1. Let p be a hyperbolic fixed point. with $|f'(p)| < 1$.
Then there exist an open interval U. about p. s.t.
if $x \in U$ then $\lim_{n\to\infty} f^n(x) = p$

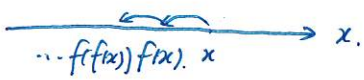
Remark: similarly result. when p is i period and. $|(f^i)'(p)| < 1$.
then $\lim_{n\to\infty} (f^i)^n(x) = p$. for some U and $x \in U$. $\rightarrow$ local stable set. W^s_{loc}

Prop 2. Let p be a hyperbolic fixed point s.t. $|f'(p)| > 1$. Then there exists an open interval U of p. s.t. if $x \in U$. $x \neq p$. then $\exists k > 0$. s.t. $f^k(x) \notin U$.
local unstable set W^u_{loc}.

Def 2. Let p be a hyperbolic point of period n.

1) $|(f^n)'(p)| < 1$. – p attracting period point / a sink.
$\rightarrow$ enough to consider $n=1$.
2). $|(f\ )'(p)| > 1$. – p repelling fixed point / a source.

△ phase portrait.



bifurcation.

alternative def.
attracting: $\exists U(x)$: $f^n(y) \to x$. $\forall y \in U(x)$
repelling: $\exists U(x)$: $\forall y \in U(x)$. $\exists n$. $f^n(y) \notin U(x)$

△找 periodic point 时. $\lim_{n\to\infty} f^n(x) = \pm\infty$ 可以说明 x 不是 periodic / eventually periodic.

△ For non-hyperbolic point. 也可能是 attract / repell 的，或一侧 attract，一侧 repell
此时要对 $f^n(x)$ 和 x. 作差. 看. $f^n(x)$ 是更接近 / 更远离该点.
(1) $|f^n(x) - x_0|$, $|x - x_0|$. (可能要分左右. 注意 $+-$).

(2) 也可以求导. 考查 $U(x_0)$ 中 $(f^n(x))'$

4 Quadratic Family.

quadratic family $F_\mu(x) = \mu x(1-x)$.

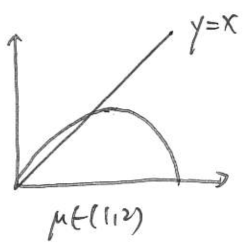
$F_\mu(0) = F_\mu(1) = 0$. $F_\mu(p_\mu) = p_\mu$. $p_\mu = \frac{\mu-1}{\mu}$ (if $\mu>1$. $0<p_\mu<1$).

Prop1. Suppose $\mu>1$. If $x<0$. then $F_\mu^n(x) \xrightarrow[n\to\infty]{} -\infty$ if $x>1$. then $F_\mu^n(x) \xrightarrow[n\to\infty]{} -\infty$

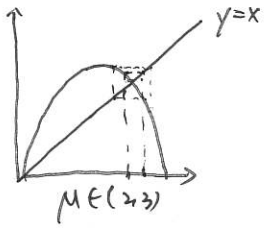
Prop2. (1) $\mu\in(1,3)$. F_μ has an attracting fixed point at $p_\mu = \frac{\mu-1}{\mu}$

a repelling fixed point at 0.

$x\in(0,1)$. $\lim_{n\to\infty} F_\mu^n(x) = p_\mu$.



$\mu\in(1,2)$



$\mu\in(2,3)$

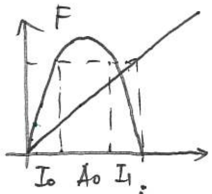
pf: consider $F_\mu(\hat{p}_\mu) = p_\mu$

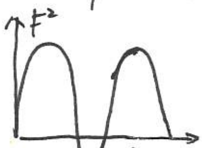
1st. $[\frac{1}{2}, \hat{p}_\mu]$

2nd $[\hat{p}_\mu, p_\mu]$

3rd. $(0, \hat{p}_\mu)$, $(p_\mu, 1)$

(2) $\mu>4$





A_0 - open. $f(A_0)>1$. $f^2(A_0)<0$ ... $\to -\infty$

△对 F^n. 稳定的部分将会呈现类似 Cantor set 的 disconnect set. ("中间"部分不断被去除)

$A_n = \{x\in I. F^i(x)\in I$ for $i\le n$ but $F^{i+1}(x)\notin I)$. $\Delta = I - (\bigcup_{n=0}^{\infty} A_n)$. 、 A_i 中的点经 $i+1$ 次运算 <0. 最终会 $\to -\infty$.

Δ is the set of point never escape from $I=[0,1]$

why $\mu=3$ is bifurcation !!

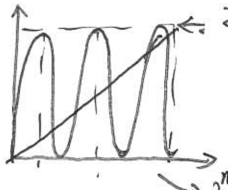
Def. (Cantor set). A set Λ. is a Cantor set. if it is a closed, totally disconnected. and perfect subset of I. (A set is totally disconnected if it contains no intervals; A set is perfect if every point in it is a limit point of other point in the set). (Λ is perfectly. no isolated point)

Thm1. If $\mu > 2+\sqrt{5}$. then Λ is a Cantor set.

Def A set $T\subseteq\mathbb{R}$ is repelling (resp. attracting) hyperbolic set for f if T is closed, bounded and invariant under f and there exists an $N>0$ s.t. $|(f^n)'(x)|>1$ (resp. <1) for all $n\ge N$. and $x\in T$.

$\underline{\underline{Q}}$: For $\mu = 4$. $F_4 = 4x(1-x)$ has at least 2^n periodic points of period n.

F_4^n. has. graph:



(why exist: $F^{n-1}(x) = \frac{1}{2} \Rightarrow F^n(x) = 1$).

($F^{n-1}(x) = 1 \Rightarrow F^n(x) = 0$).

each $\cap$. two intersection point. $\rightarrow$ 2^n periodic point of period n.

<u>Lemma.</u> If $f(I) \supset I$. then ~~there~~ exist fixed point in $I \subseteq \mathbb{R}$

*<u>Def.</u> (stability).

x is a stable fixed point. if. $\forall U(x). \exists V(x) \subseteq U(x)$. s.t. $\forall y \in V(x)$. $\forall n$. $f^n(y) \in U(x)$

5 Symbolic dynamics.

Goal: give a model for quadratic map on Cantor set.

<u>Def 1.</u> (sequence space). $\Sigma_2 = \{ s = (s_0 s_1 s_2 \dots) \mid s_j = 0 \text{ or } 1 \}$ - sequence space on the two symbols 0 and 1. (space Σ_n can be considered similarly).

the distance between sequence: $d[s,t] = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} < 2$ - metric on Σ_2.

(also make Σ_2 into a metric space)

<u>Prop 1.</u> $\forall s,t \in \Sigma_2$. s.t. $s_i = t_i$ for all $i \in [0:n]$ then $d[s,t] \le \frac{1}{2^n}$

conversely, if $d[s,t] < 1/2^n$. then $s_i = t_i$ for $i \le n$.

<u>Def 2.</u> (shift map). $\sigma : \Sigma_2 \to \Sigma_2$. is given by $\sigma(s_0 s_1 s_2 \dots) = (s_1 s_2 s_3 \dots)$

<u>Prop 2.</u> Map $\sigma : \Sigma_2 \to \Sigma_2$ is cont. (2 to 1 map).

Pf: $\forall \varepsilon > 0$. $\exists n$. s.t. $\frac{1}{2^n} < \varepsilon$. and let $\delta = \frac{1}{2^{n+1}}$

if $d[s,t] < \delta$. i.e. $s_i = t_i$ for all $i \in [0:n+1]$.

$\Rightarrow \sigma(s)_i = \sigma(t)_i$ for all $i \in [0:n]$. $\Rightarrow d[\sigma(s), \sigma(t)] \le \frac{1}{2^n} < \varepsilon$

<u>For Quadratic Map: F_μ.</u>

(1) periodic point: correspond to repeating sequence. (like $s = (s_0 \dots s_{n-1}, s_0, \dots s_{n-1}, \dots)$)

(2). $\text{Card}(\text{Per}_n(\sigma)) = 2^n$

(3) $\text{Per}(\sigma)$ is dense in Σ_2.

(4) There exists a dense orbit for σ in Σ_2.

Pf(4). consider $s^* = (\underbrace{0\ 1}_{1\text{ block}} \mid \underbrace{00\ 01\ 10\ 11}_{2\text{ block}} \mid \underbrace{000\ 001 \dots}_{3\text{ block}} \mid \dots)$.

s^* listing all 0's and 1's of length n, then length $n+1$.

$\forall s \in \Sigma_2$. $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$.

then iterate s^* to N's block. find $\sigma^k(s^*)$ s.t. $\sigma^k(s^*)_i = s_i$ $i \in [1:N]$.

thus. $d[s, \sigma^k(s^*)] < \varepsilon$. we have $\overline{O^+(s^*)} = \Sigma_2$.

6 Topological Conjugacy

Goal: show "σ" on Σ_2 and f on Λ is essentially same.

Def 6.1 (itinerary). The itinerary of x is a sequence $S(x) = s_0 s_1 s_2 \ldots$
where $s_j = 0$ if $F_\mu^j(x) \in I_0$, $s_j = 1$ if $F_\mu^j(x) \in I_1$.
(Recall: $\Lambda \subset I_0 \cup I_1$.)

Thm 6.1 If $\mu > 2+\sqrt{5}$, then $S: \Lambda \to \Sigma_2$ is a homoemorphism.
△ 定理给出了对Δ"编码"的方法，证明了Δ，Σ₂是"等价"的.

Idea: $s = s_0 s_1 s_2 \ldots$ $x \in \Lambda$, $S(x) = s$

$$I_{s_0 s_1 \ldots s_n} = \{x \in I \mid x \in I_{s_0}, F_\mu(x) \in I_{s_1}, \ldots, F_\mu^n(x) \in I_{s_n}\}$$
$$= I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap \ldots \cap F_\mu^{-n}(I_{s_n}).$$

$I_{s_0 s_1 \ldots s_n}$ form a nested sequence of nonempty closed interval as $n \to \infty$.

$$I_{s_0 s_1 \ldots s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1 \ldots s_n}).$$

these interval $\{I_{s_0 s_1 \ldots s_n}\}_n$ are nested. $\bigcap_{n\geq 0} I_{s_0 s_1 \ldots s_n}$ is nonempty.
if $x \in \bigcap_{n\geq 0} I_{s_0 s_1 \ldots s_n}$ then $x \in I_{s_0}$, $F_\mu(x) \in I_{s_1}, \ldots$ $S(x) = (s_0 s_1 \ldots)$.

Remark: $\operatorname{diam}(I_{s_0 s_1 \ldots s_n}) \xrightarrow[n\to\infty]{} 0$. $\bigcap_{n\geq 0} I_{s_0 s_1 \ldots s_n}$ consists of a unique point.

Thm 6.2. $S \circ F_\mu = \sigma \circ S$

Pf: A point x in Λ may be defined uniquely by the nested sequence $\bigcap_{n\geq 0} I_{s_0 s_1 \ldots s_n}$
(determine by the itinerary $S(x)$).

$$I_{s_0 \ldots s_n} = I_{s_0} \cap F_\mu^{-1}(I_{s_1}) \cap \cdots \cap F_\mu^{-n}(I_{s_n}).$$

$$F_\mu(I_{s_0 \ldots s_n}) = I_{s_1} \cap F_\mu^{-1}(I_{s_2}) \cap \cdots \cap F_\mu^{-n+1}(I_{s_n}) = I_{s_1 \ldots s_n}. \quad (F_\mu(I_{s_0}) = I)$$

Thus. $S \circ F_\mu(x) = S \circ F_\mu\left(\bigcap_{n\geq 0}^{\infty} I_{s_0 s_1 \ldots s_n}\right) = S\left(\bigcap_{n=1}^{\infty} I_{s_1 \ldots s_n}\right) = s_1 s_2 \ldots = \sigma S(x)$

Def 6.2. Let $f: A \to A$ and $g: B \to B$ be two maps. f and g are topological conjugate if $\exists h: A \to B$, h is homoeomorphism. s.t. $h \circ f = g \circ h$.
h is called a topological conjugacy.

Thm 6.3. Let $F_\mu(x) = \mu x(1-x)$ with $\mu < 2+\sqrt{5}$. Then:

(1). The Cardinality of $Per_n(F_\mu)$ is 2^n

(2). $Per(F_\mu)$ is dense in Λ

(3). F_μ has a dense orbit in Λ

△都是同胚"保"的性质.

Def 6.4 A point p is a non-wandering point for f, if $\forall J$ - open interval containing p, there exists $x \in J$ and $n > 0$ s.t. $f^n(x) \in J$. △不要求 $f^n(p) \in J$.
denote $\Omega(f)$ - the set of non-wandering points for f.

Def 6.5. A point p is a recurrent for f, if $\forall J$ - open interval of p. $\exists n > 0$ s.t. $f^n(p) \in J$.

Remark: periodic point $\underset{(1)}{\overset{\Rightarrow}{\not\Leftarrow}}$ recurrent $\underset{(2)}{\overset{\Rightarrow}{\not\Leftarrow}}$ non-wandering

(1). $s^* = \{01|00011011|\cdots\}$ point s^* has dense orbit.
$\exists n$. $f^n(s^*) \in J$ for any J. but non-periodic.

(2) eventually fixed point (邻域内一定有个周期点，但回不去)
$x = 1$. $\forall J \ni 1$. $\exists x_0 \in Per(F_\mu)$ and $x_0 \in J$. ($Per(F_\mu)$ dense in Λ).
and $f^n(1) = 0$ for any $n \geq 1$. not recurrent.

7. Chaos.

Def 7.1. $f: J \to J$ is topologically transitive if $\forall\, U, V \subseteq J$, there exists $k > 0$ s.t. $f^k(U) \cap V \neq \emptyset$.

△"拓扑传递". 从一个开集出发的点能传到另一个开集.

Def 7.2. $f: J \to J$ has sensitive dependence on initial conditions, if $\exists\, \delta > 0$. s.t. $\forall x \in J$ $\forall N$-neighborhood of x. $\exists\, y \in N$. $n \geq 0$. s.t. $|f^n(x) - f^n(y)| > \delta$.

Remark: F_μ. $\mu > 2+\sqrt{5}$. is sensitive ~~de~~ on Λ (△取 $\delta = \Lambda$ 中被截的中间部分).

Def 7.3. Let V be a set. $f: V \to V$ is chaotic on V. if.

(1) f has sensitive dependence on initial conditions. 初始状态微小差异指数级放大

(2). f is t.p. transitive. 轨迹会无限接近任何点，无法被限制在某个子区间内.

(3). periodic points are dense in V.

△ if $\exists$ dense trajectory $\Rightarrow$ t.p. transitive (对连续映射两者等价).

Thm 7.1. Let $f: I \to I$. $I \subseteq \mathbb{R}$. If $\forall J \subseteq I$. J is segment, $\exists\, n$, $f^n(J) = I$.
Then f is chaotic.

Example: of chaotic.

(1) $f: S^1 \to S^1$ by $f(\theta) = 2\theta$.

(2) $F_\mu(x) = \mu x(1-x)$. on Λ, $\mu > 2+\sqrt{5}$.

(3). $F_4(x) = 4x(1-x)$ on $I = [0,1]$.

Pf: denote $\underline{g(\theta) = 2\theta}$. $h_1: S^1 \to [-1,1]$. $\underline{h_1(\theta) = \cos\theta}$. $\underline{q(x) = 2x^2 - 1}$.

$h_1 \circ g\,(\theta) = \cos 2\theta = 2\cos^2\theta - 1. = q \circ h_1(\theta)$.

q is t.p. conjugate to F_4. (i.e. $\exists$ $\underline{h_2(t) = \frac{1-t}{2}}$. $F_4 \circ h_2 = h_2 \circ q$).

(we say in such case. F_4 and g are semi-conjugate

$$\begin{tikzcd}
S^1 \arrow[r, "g"] \arrow[d, "h_1"'] & S^1 \arrow[d, "h_1"] \arrow[dd, bend left=60, "h=1-x^2"] \\
{[-1,1]} \arrow[r, "q"] \arrow[d, "h_2"'] & {[-1,1]} \arrow[d, "h_2"] \\
{[0,1]} \arrow[r, "F_4"] & {[0,1]}
\end{tikzcd}$$

not in. but sur.

Def 7.4. $f: J \to J$ is expansive if $\exists\, \nu > 0$ s.t. $\forall x, y \in J$ $x \neq y$ $\exists\, n$. s.t $|f^n(x) - f^n(y)| > \nu$. (与 sensitive dependent 的差异是 $\forall x, y$ 均在迭代后可分).

8. Structural Stability. —系统在小扰动下的稳定性.

△ 如果"f"附近每个"nearby"映射都 t.p. conjugate to f, 则f结构稳定

Def 8.1. Let. f, g two maps. The C^0-distance between f, g. written $d_0(f,g)$ given by. $d_0(f,g) = \sup_{x\in\mathbb{R}} |f(x)-g(x)|$ → 用于定义"邻近".

The C^r-distance $d_r(f,g)$ is given by $d_r(f,g) = \sup_{x\in\mathbb{R}} (|f(x)-g(x)|, \dots |f^{(r)}(x)-g^{(r)}(x)|)$

(d_r is not a metric for functional space; d_r possibly be $\pm\infty$).

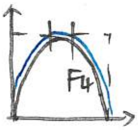
△ 从直觉上, 两映射是"C^r-close"的表明其前 r 阶导相差小.

Def 8.2. Let $f: J\to J$. f is C^r-structurally stable. on J. if there exists $\varepsilon>0$. s.t. if $d_r(f,g)<\varepsilon$ for $g: J\to J$, then f is t.p. conjugate to g.

△ 条件上 r 越大 稳定性越强.

Type of problems: check the structural stability. ↗ 证明 t.p. conjugate 在邻近处不存在. 找"突变"的动力学性质.

(1) $F_4 = 4x(1-x)$ not structually stability.



for some C^0-closed g. ∃ interval. → $-\infty$. (always have different dynamics).

(2) $L(x) = \frac{1}{2}x$. C^1-structurally stability on $\mathbb{R}$.

Thm 8.3. The quadratic map $F_\mu(x) = \mu x(1-x)$ is C^2 structurally stable if $\mu > 2+\sqrt{5}$.

Prop. 8.4. A hyperbolic fixed point for f is C^1 structurally stable locally.

→ 1 dim Sternberg's Thm.

Thm 8.5. Let p be a hyperbolic fixed point for f and suppose $f'(p) = \lambda$ with $|\lambda| \neq 0, 1$. Then there are neighborhood U of p and V of $0 \in \mathbb{R}$ and a homeomorphism $h: U \to \mathbb{R}$ which conjugates f on U to the linear map $L(x) = \lambda x$ on V. (i.e. $L \circ h = h \circ f$).

§ 9. Sarkovsky Thm.

Thm 9.1.

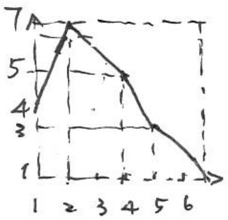
Let f $\mathbb{R} \to \mathbb{R}$ be continuous. Suppose f has a periodic point of period three. Then f has periodic points of all other periods.

Def 9.1. A new order of natural number:

$$\underbrace{3 \rhd 5 \rhd 7}_{\text{the odd number}} \rhd \cdots \rhd \underbrace{2\cdot 3 \rhd 2\cdot 5}_{2\times\text{odd}} \rhd \cdots \rhd \underbrace{2^2\cdot 3}_{2^2\times\text{odd}} \rhd 2^2\cdot 5 \rhd \cdots \rhd \cdots \rhd \underbrace{2^n}_{2^n\text{ power}} \rhd \cdots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd \underbrace{1}_{1}.$$

Thm 9.2. (Sarkovsky thm. only for 1-dim. no extension for higher dim)
Let $f: \mathbb{R} \to \mathbb{R}$ be cont. Suppose f has a periodic point of prime period k. If $k \rhd l$ in the above order. f also has a periodic point of order l.

Ex: construct the function with period n and no period $n-2$.



if n is odd. $f(1) = \frac{n+1}{2}$.

$f(2) = n$.

$f(3) = n-1$

...

$f(\frac{n+1}{2}) = \frac{n+3}{2}$.

$f(\frac{n+3}{2}) = \frac{n-1}{2}$.

...

$f(n) = 1$. $\triangle$ $x=1$ is periodic point of period n.

e.g. $n=7$. $f^5[1,2] = [2,7]$

...

$f^5[4,5]$: $[4,5] \to [3,5] \to [3,6] \to [2,6]$

$\to [2,7] \to [1,7]$. has 1 fixed point.

and only one (because in $[2,6]$. decreasing).