17.10.24

3.2.THE EQUATION OF STRING VIBRATION AND ITS SOLUTION BY THE METHOD OF SEPARATION OF VARIABLES (FOURIER METHOD)

The method of separation of variables, or the Fourier method, is one of the most common methods for solving partial differential equations. We will present this method for the problem of vibrations of a string fixed at the ends.

3.2.1. THE EQUATION OF FREE VIBRATIONS OF THE STRING

The following mixed problem is considered.

Task 1.

Let it be required to find a solution:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2},\ 0 < x < l,\ t > 0,$$

satisfying the initial and boundary conditions:

$$u(x,0) = \varphi(x),$$

$$\frac{\partial u(x,0)}{\partial t} = \psi(x),$$

$$u(0,t) = 0,$$

$$u(l,t) = 0,\ t \ge 0.$$

We are looking for a solution to this problem in the form of a product:

$$u(x,t) = X(x)T(t),$$

substituting which into this equation, we have

$$X(x)T''(t) = a^2 X''(x)T(t).$$

Dividing both parts of this equation by $a^2 X(x)T(t)$, we obtain

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)}. \qquad (3.22)$$

The right side of equality (3.22) is a function of only variable x, and the left side is only t, so the right and left sides of equality (3.22), when changing their arguments. retain a constant value. It is convenient to denote this value by $-\lambda$, that is

$$\frac{T''(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

$$X''(x) + \lambda X(x) = 0,$$

$$T''(t) + \lambda a^2 T(t) = 0.$$

The general solutions of these equations have the form

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

$$T(t) = C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t,$$

where $A,B,C,D-$ are arbitrary constants, and the function $u(x,t)$ is

$$u(x,t) = \left(A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x\right)\left(C\cos a\sqrt{\lambda}t + D\sin a\sqrt{\lambda}t\right).$$

Constants A and B can be found using the boundary conditions of Task 1. Since

$$T(t) \not\equiv 0,$$

then

$$X(0)=0,\ X(l)=0.$$

$$X(0)=A=0,$$

$$X(l)=A\cos\sqrt{\lambda}l+B\sin\sqrt{\lambda}l=0,$$

that is,

$$A=0,$$

$$B\sin\sqrt{\lambda}l=0.$$

From where

$$\sqrt{\lambda}=\frac{k\pi}{l},\ k=1,2,\dots.$$

So,

$$X(x)=B\sin\frac{k\pi}{l}x.$$

The values $\lambda=\frac{k^2\pi^2}{l^2}$ found are called *eigenvalues* for a given boundary value Task 1, and the functions $X(x)=B\sin\frac{k\pi}{l}x$ are called *eigenfunctions.*

With the values of λ found, we get

$$T(t)=C\cos\frac{ak\pi}{l}t+D\sin\frac{ak\pi}{l}t,$$

$$u_k(x,t)=\sin\frac{k\pi}{l}x\left(a_k\cos\frac{ak\pi}{l}t+b_k\sin\frac{ak\pi}{l}t\right),\ k=1,2,\dots .$$

Since the equation is linear and homogeneous, the sum of the solutions is also a solution that can be represented as a series:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{ak\pi}{l}t+b_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x .$$

In this case, the solution must satisfy the initial condition:

$$u(x,0)=\sum_{k=1}^{\infty}a_k\sin\frac{k\pi}{l}x=\varphi(x) .$$

If the function $\varphi(x)$ decomposes into a Fourier series in the interval $(0,l)$ in terms of sines, then

$$a_k=\frac{2}{l}\int_0^l\varphi(x)\sin\frac{k\pi}{l}x\,dx . \qquad (3.23)$$

From the initial condition

$$\frac{\partial u(x,0)}{\partial t}=\psi(x)$$

we have

$$\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=\sum_{k=1}^{\infty}\frac{ak\pi}{l}b_k\sin\frac{k\pi}{l}x=\psi(x) .$$

We determine the Fourier coefficients of this series:

$$\frac{ak\pi}{l}b_k=\frac{2}{l}\int_0^l\psi(x)\sin\frac{k\pi}{l}x\,dx ,$$

from where

$$b_k = \frac{2}{ak\pi}\int_0^l \psi(x)\sin\frac{k\pi}{l}xdx\,. \qquad (3.24)$$

Thus, the solution of the string oscillation equation can be represented as the sum of an infinite series:

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty}\left(a_k\cos\frac{ak\pi}{l}t + b_k\sin\frac{ak\pi}{l}t\right)\sin\frac{k\pi}{l}x\,, \qquad (3.25)$$

where a_k, b_k are determined by formulas (3.23) and (3.24).

Theorem. Let $\varphi(x) \in C^2([0,l])$, $\psi(x) \in C^1([0,l])$, in addition, $\varphi(x)$ has a third, and $\psi(x)$ has a second piecewise continuous derivative and the relations are fulfilled: $\varphi(0) = \varphi(l) = 0,\ \varphi''(0) = \varphi''(l) = 0,\ \psi(0) = \psi(l) = 0$. Then the sum of the series (3.25) with coefficients defined by formulas (3.23), (3.24) is the solution to Task 1.

Example 1

Find the deviation $u(x;t)$ from the equilibrium position of a homogeneous horizontal string fixed at the ends $x = 0$ and $x = l$, if at the initial moment the string had the shape $\frac{1}{8}\sin\frac{3\pi x}{l}$, and the initial velocities were absent.

Solution:

$$\begin{cases} u_{tt} = a^2 u_{xx} & (*) \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \frac{1}{8}\sin\frac{3\pi x}{l} \\ u_t(x,0) = 0 \end{cases}$$

The method of separation of variables, the Fourier method:

we will look for a solution in the form

$$u(x,t) = T(t)\cdot X(x)$$

$$u_{tt}(x,t) = T''(t)\cdot X(x)$$

$$u_{xx}(x,t) = T(t)\cdot X''(x)$$

We substitute it into equation (*):

$$T''(t)X(x) = a^2 T(t)X''(x)$$

Divide by $a^2T(t)X(x)$, we get

$$\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda\ .$$

Consider

$$\frac{X''(x)}{X(x)} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0 \\ X(l) = 0 \end{cases}$$

We obtain an ordinary differential equation of the second order. This is the task of Sturm-Liouville theory.

We are solving this problem:

1) $\lambda = 0 \quad \Rightarrow \quad X''(x) = 0$

$$X(x) = C_1 x + C_2$$

Substituting into the boundary conditions, we get:

$$C_1 = 0$$
$$C_2 = 0$$

That is, for $\lambda = 0$, the only solution is: $X(x) \equiv 0$. This solution does not suit us.

2) $\lambda < 0 \quad \Rightarrow \quad X(x) \equiv 0$

3) $\lambda > 0 \quad \Rightarrow \quad X(x) = A\sin\left(\sqrt{\lambda}\, x\right) + B\cos\left(\sqrt{\lambda}\, x\right)$

Substitute the boundary conditions, we get:

$$X(0) = B = 0$$
$$X(l) = A sin\left(\sqrt{\lambda}\, l\right) = 0$$

There are two options:

If $A = 0$, then $X(x) \equiv 0$. It doesn't suit us.

If $A \neq 0$, then

$$sin\left(\sqrt{\lambda}\, l\right) = 0$$

$$\sqrt{\lambda}\, l = \pi n, \quad n \in \mathbb{Z}$$

$$\sqrt{\lambda} = \frac{\pi n}{l},$$

$$\lambda = \left(\frac{\pi n}{l}\right)^2$$

$$X(x) = A\sin\left(\frac{\pi n}{l}x\right)$$

The solution is definitely ambiguous. $A-$ can be any, $n-$ can be different numbers.

So

$$X_n(x) = A_n \sin\left(\frac{\pi n}{l}x\right)$$

We have found Sturm-Liouville's *eigenfunctions.*

$$T''(t) + a^2\lambda T(t) = 0$$

$$T''(t) + \left(\frac{a\pi n}{l}\right)^2 T(t) = 0$$

$$T(t) = C\,sin\left(\frac{a\pi n}{l}t\right) + D\,cos\left(\frac{a\pi n}{l}t\right)$$

$$T_n(t) = C_n \sin\left(\frac{a\pi n}{l}t\right) + D_n \cos\left(\frac{a\pi n}{l}t\right)$$

$$u_n(x,t) = \left(C_n \sin\left(\frac{a\pi n}{l}t\right) + D_n \cos\left(\frac{a\pi n}{l}t\right)\right) \cdot A_n \sin\left(\frac{\pi n}{l}x\right) =$$

$$= \left(a_n \sin\left(\frac{a\pi n}{l}t\right) + b_n \cos\left(\frac{a\pi n}{l}t\right)\right) \cdot \sin\left(\frac{\pi n}{l}x\right)$$

$$u_n(x,t) = \left(a_n \sin\left(\frac{a\pi n}{l}t\right) + b_n \cos\left(\frac{a\pi n}{l}t\right)\right) \cdot \sin\left(\frac{\pi n}{l}x\right)$$

We don't know these constants a_n and b_n, there are an infinite number of them.

Consider the initial conditions (initial amplitude and initial velocities).

$$u(x;0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n}{l}x\right) = \frac{1}{8}\sin\left(\frac{3\pi}{l}x\right)$$

$$u_t(x;0) = \sum_{n=1}^{\infty} a_n \cdot \frac{a\pi n}{l} \cdot 1 \cdot \sin\left(\frac{\pi n}{l}x\right) = 0$$

We get that $a_n = 0 \quad \forall n$.

$$b_1 \sin\left(\frac{\pi}{l}x\right) + b_2 \sin\left(\frac{2\pi}{l}x\right) + b_3 \sin\left(\frac{3\pi}{l}x\right) + b_4 \sin\left(\frac{4\pi}{l}x\right) + \ldots = \frac{1}{8}\sin\left(\frac{3\pi}{l}x\right).$$

We get that $b_3 = \frac{1}{8}, \quad b_n = 0 \quad n \neq 3$.

<u>Answer:</u> $u(x,t) = \frac{1}{8}\cos\left(\frac{3a\pi}{l}t\right)\sin\left(\frac{3\pi}{l}x\right)$.

Example 2

Let the initial rejection of the string fixed at points $x=0$ and $x=l$ be zero, and the initial velocity

$$\frac{\partial u}{\partial t}=\begin{cases} v_0, & \left|x-\frac{l}{2}\right|<\frac{h}{2}, \\ 0, & \left|x-\frac{l}{2}\right|>\frac{h}{2}. \end{cases}$$

Determine the shape of the string for any time t.

<u>Solution:</u>

Here $\varphi(x)=0$, and $\psi(x)=v_0$ in the interval $\left(\frac{l-h}{2},\frac{l+h}{2}\right)$, and $\psi(x)=0$ outside this interval.

Therefore,

$$a_k=0,$$

$$b_k=\frac{2}{ak\pi}\int_{(l-h)/2}^{(l+h)/2} v_0\sin\frac{k\pi}{l}x\,dx=-\frac{2v_0}{ak\pi}\frac{l}{k\pi}\cos\frac{k\pi}{l}x\Bigg|_{(l-h)/2}^{(l+h)/2}=$$

$$=\frac{2v_0 l}{ak^2\pi^2}\left[\cos\frac{k\pi(l-h)}{2l}-\cos\frac{k\pi(l+h)}{2l}\right]=\frac{4v_0 l}{ak^2\pi^2}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2l}.$$

Hence

$$u(x,t)=\frac{4v_0 l}{a\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{2}\sin\frac{k\pi h}{2l}\sin\frac{ak\pi t}{l}\sin\frac{k\pi x}{l}.$$

Example 3

A string is given, fixed at the ends $x=0$ and $x=l$. Let's assume that at the initial moment the shape of the string has the form of a polyline OAB, shown in Fig. Find the shape of the string for any time t if there are no initial velocities.

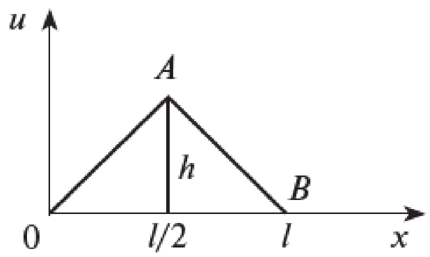

Fig. The shape of the string at the initial moment of time

Solution:

The angular coefficient of the straight line OA is equal to $\frac{h}{l/2}$, the equation of this straight line is $u=\frac{2h}{l}x$. The straight line AB cuts off the segments: l and $2h$ on the coordinate axes, which means that the equation of the straight line AB: $u=\frac{2h}{l}(l-x)$. So,

$$\varphi(x)=\begin{cases}\frac{2h}{l}x, & 0\le x\le\frac{l}{2}\\ \frac{2h}{l}(l-x), & \frac{l}{2}\le x\le l,\end{cases}$$

$$\psi(x)=0.$$

We find

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{k\pi x}{l}dx =$$

$$= \frac{4h}{l^2}\int_0^{l/2} x\sin\frac{k\pi x}{l}dx + \frac{4h}{l^2}\int_{l/2}^{l}(l-x)\sin\frac{k\pi x}{l}dx\ ,$$

$$b_k = 0\ .$$

Integrating by parts, we get

$$a_k = -\frac{4h}{k\pi l}x\cos\frac{k\pi x}{l}\Big|_0^{l/2} + \frac{4h}{k\pi l}\int_0^{l/2}\cos\frac{k\pi x}{l}dx -$$

$$-\frac{4h}{k\pi l}(l-x)\cos\frac{k\pi x}{l}\Big|_{l/2}^{l} - \frac{4h}{k\pi l}\int_{l/2}^{l}\cos\frac{k\pi x}{l}dx =$$

$$= -\frac{2h}{k\pi}\cos\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\sin\frac{k\pi x}{l}\Big|_0^{l/2} + \frac{2h}{k\pi}\cos\frac{k\pi}{2} - \frac{4h}{k^2\pi^2}\sin\frac{k\pi x}{l}\Big|_{l/2}^{l} =$$

$$= \frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2} + \frac{4h}{k^2\pi^2}\sin\frac{k\pi}{2} = \frac{8h}{k^2\pi^2}\sin\frac{k\pi}{2}\ .$$

Therefore,

$$u(x,t) = \frac{8h}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\sin\frac{k\pi}{2}\sin\frac{k\pi x}{l}\cos\frac{ak\pi t}{l}\ .$$

Example 4

The string fixed at the ends $x = 0$ and $x = l$ has the shape of a parabola

$$u(x,0) = \frac{4}{l^2}x(l-x)\ .$$

at the initial moment of time. Find the shape of the string at any given time if there are no initial velocities.

<u>Solution:</u>

Here

$$\varphi(x) = \frac{4}{l^2} x\left(l - x\right),$$

$$\psi(x) = 0\,.$$

Therefore, we have

$$a_k = \frac{2}{l}\int_0^l \frac{4}{l^2} x(l-x)\sin\frac{k\pi x}{l}dx,$$

Applying the integration by parts method twice, we get

$$a_k = \frac{8}{l^3}\int_0^l (lx - x^2)\sin\frac{k\pi x}{l}dx =$$

$$= \frac{8}{l^3}\left(-\frac{l\left(lx - x^2\right)}{k\pi}\cos\frac{k\pi x}{l}\Bigg|_0^l + \frac{l}{k\pi}\int_0^l \left(l - 2x\right)\cos\frac{k\pi x}{l}dx \right) =$$

$$= \frac{8}{l^2 k\pi}\int_0^l \left(l - 2x\right)\cos\frac{k\pi x}{l}dx =$$

$$= \frac{8}{l^2 k\pi}\left(\frac{l\left(l - 2x\right)}{k\pi}\sin\frac{k\pi x}{l}\Bigg|_0^l + \frac{2l}{k\pi}\int_0^l \sin\frac{k\pi x}{l}dx \right) =$$

$$= \frac{16}{lk^2\pi^2}\int_0^l \sin\frac{k\pi x}{l}dx = \frac{16}{lk^2\pi^2}\left(-\frac{l}{k\pi}\cos\frac{k\pi x}{l} \right)\Bigg|_0^l =$$

$$= -\frac{16}{k^3\pi^3}\left(\cos k\pi - 1\right) = \frac{16}{k^3\pi^3}\left(1 - (-1)^k\right) = \left[\begin{array}{ll} \dfrac{32}{k^3\pi^3}, & k = 2n+1, \\ 0, & k = 2n. \end{array}\right.$$

Then the solution of the problem will take the following form:

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{\left(2n+1\right)^3}\sin\frac{\left(2n+1\right)\pi x}{l}\cos\frac{a\left(2n+1\right)\pi t}{l}.$$

Task 1	$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = x^2, \quad \dfrac{\partial u(x,0)}{\partial t} = 1; \end{cases}$$
Task 2	$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = 1, \quad \dfrac{\partial u(x,0)}{\partial t} = 0, \\ u(0,t) = 0, \quad u\left(\dfrac{\pi}{2}, t\right) = 0; \end{cases}$$

Do it before October 20th. Homework is not accepted after October 20th.