e) (Continuation.) Show that Dedekind's theorem is equivalent to the axiom of completeness.

Proof. "Axiom of Completeness $\Rightarrow$ Dedekind's Theorem"

By Axiom of completeness, there exists $c \in \mathbb{R}$ such that $\forall x \in X \forall y \in Y (x \leqslant c \leqslant y)$. Since $X \cup Y = \mathbb{R}$, it follows that $(c \in X) \vee (c \in Y)$. If $c \in X$, then $c = \max X$; if $c \in Y$, then $c = \min Y$.

"Dedekind's Theorem $\Rightarrow$ Axiom of Completeness"

Let $X' = \{x \in \mathbb{R} | \forall y \in Y (x \leqslant y)\}$ and $Y' = \{y \in \mathbb{R} | \forall x \in X (x \leqslant y)\}$, then it is obvious that $(X \subset X') \wedge (Y \subset Y')$. Assume that there is no $c \in \mathbb{R}$ such that $\forall x \in X \forall y \in Y (x \leqslant c \leqslant y)$, which is equivalent to that $X' \cap Y' = \varnothing$. We claim that $\forall x \in X' \forall y \in Y' (x \leqslant y)$, otherwise there exist $x_0 \in X'$ and $y_0 \in Y'$ such that $y_0 < x_0$. Then, $\forall x \in X \forall y \in Y (x \leqslant y_0 < x_0 \leqslant y)$, which contradicts with the assumption that there is no $c \in \mathbb{R}$ such that $\forall x \in X \forall y \in Y (x \leqslant c \leqslant y)$.

For any $r \in \mathbb{R}$, if $\forall x \in X (x \leqslant r)$, then $r \in Y'$; otherwise, $\exists x_0 \in X (r < x_0)$, then $\forall y \in Y (r < x_0 \leqslant y)$, i.e., $r \in X'$. Hence $X' \cup Y' = \mathbb{R}$. By Dedekind's theorem, either X' has a maximal element or Y' has a minimal element. Without loss of generality, suppose that X' has a maximal element c, then $\forall x \in X \subset X' (x \leqslant c)$ and $c \in X' \Rightarrow \forall y \in Y (c \leqslant y)$, i.e., $\forall x \in X \forall y \in Y (x \leqslant c \leqslant y)$ which contradicts with the assumption. Hence, the assumption is false, i.e., there exist $c \in \mathbb{R}$ such that $\forall x \in X \forall y \in Y (x \leqslant c \leqslant y)$.

1. Bolzano-Weierstrass Principle $\Rightarrow$ Axiom of Completeness

Proof. Given two nonempty subset X, Y of $\mathbb{R}$ such that $\forall x \in X \forall y \in Y (x \leqslant y)$. Choose $a_1 \in X$, $b_1 \in Y$, then $\frac{a_1+b_1}{2}$ must satisfy one and only one of the following three conditions:

(a) $\forall x \in X \forall y \in Y \left(x \leqslant \frac{a_1+b_1}{2} \leqslant y\right)$;

(b) $\exists x_0 \in X \left(\frac{a_1+b_1}{2} < x_0\right)$;

(c) $\exists y_0 \in Y \left(y_0 < \frac{a_1+b_1}{2}\right)$.

If (a) holds, then we can let $c = \frac{a_1+b_1}{2}$, and we are done; if (b) holds, then let $a_2 := \frac{a_1+b_1}{2}$, $b_2 := b_1$; if (c) holds, then let $a_2 := a_1$, $b_2 = \frac{a_1+b_1}{2}$. We can do this procedure inductively, then either we obtain the real number c as required or obtain the nested closed interval $[a_1, b_1] \supset [a_2, b_2] \supset \cdots \supset [a_n, b_n] \supset \cdots$.

Since the set $\{a_n\}$ is infinite (In fact, if $X = \{x\}$ is a singleton, then $a_n = x$, $n = 1, 2, \cdots$. In this case, we can let $c = x$, which obviously satisfy the requirement.) and bounded, by Bolzano-Weierstrass Principle, there exists a limit point $c \in \mathbb{R}$ of $\{a_n\}$. W claim that $\forall y \in Y (c \leqslant y)$, otherwise there exists $y_0 \in Y$ such that $y_0 < c$. Since c is the limit point of $\{a_n\}$, there exists a_{n_0} such that $|a_{n_0} - c| < \frac{1}{2}(c - y_0)$, then

$y_0 < \frac{1}{2}c + \frac{1}{2}y_0 = -\frac{1}{2}(c - y_0) + c < a_{n_0}$, which contradict with the definition of a_{n_0}. Similarly, we claim that $\forall x \in X(x \leqslant c)$, otherwise there exists $x_0 \in X$ such that $c < x_0$. We can find a_{n_1} and b_{n_1} such that $|c - a_{n_1}| < \frac{1}{3}(x_0 - c)$ and $|a_{n_1} - b_{n_1}| < \frac{1}{3}(x_0 - c)$, then

$|b_{n_1} - c| = |b_{n_1} - a_{n_1} + a_{n_1} - c| \leqslant |b_{n_1} - a_{n_1}| + |a_{n_1} - c| < \frac{2}{3}(x_0 - c)$,

$b_{n_1} < c + \frac{2}{3}(x_0 - c) < \frac{2}{3}x_0 + \frac{1}{3}c < x_0$, which contradicts with the definition of b_{n_1}. Hence, $\forall x \in X \forall y \in Y(x \leqslant c \leqslant y)$, $\blacksquare$.

2. Borel–Lebesgue Principle ⇒ Axiom of Completeness

Proof. Suppose we obtain exactly the same nested intervals as in 1. Then $(a_1 - 1, b_1 + 1) \setminus [a_n, b_n] = (a_1 - 1, a_n) \cup (b_n, b_1 + 1)$ .Actually, we obtain a system of open intervals $\{(a_1 - 1, a_n), (b_n, b_1 + 1)\}$. We claim that $\bigcap_{n=1}^{\infty}[a_n, b_n]$ is nonempty, then $\{c\} = \bigcap_{n=1}^{\infty}[a_n, b_n]$ (the uniqueness follows from the fact that $b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \to 0\,(n \to \infty)$). Otherwise,

$\bigcap_{n=1}^{\infty}[a_n, b_n] = \varnothing$, then

$$\bigcup_{n=1}^{\infty}(a_1 - 1, b_1 + 1) \setminus [a_n, b_n] = (a_1 - 1, b_1 + 1) \setminus \left(\bigcap_{n=1}^{\infty}[a_n, b_n]\right) = (a_1 - 1, b_1 + 1) \setminus \varnothing$$
$$= (a_1 - 1, b_1 + 1) \supset [a_1, b_1]$$

i.e., the open intervals $\{(a_1 - 1, a_n), (b_n, b_1 + 1)\}$ cover closed interval

$[a_1, b_1]$. By Borel–Lebesgue Principle, there exists a finite subsystem of open intervals which can also cover $[a_1, b_1]$, which is a contradiction, since the union of these finite open intervals is included in certain $(a_1 - 1, a_n) \cup (b_n, b_1 + 1)$, which cannot cover $[a_1, b_1]$. As in 1, we can show that $\forall x \in X \forall y \in Y (x \leqslant c \leqslant y)$, $\blacksquare$.

3. Cauchy-Cantor Principle $\Rightarrow$ Axiom of Completeness

Proof. Suppose we obtain exactly the same nested intervals as in 1. Then, by Cauchy-Cantor Principle, there exists unique $c \in \mathbb{R}$ such that $\{c\} = \bigcap_{n=1}^{\infty} [a_n, b_n]$, and we can show that $\forall x \in X \forall y \in Y (x \leqslant c \leqslant y)$ as in 1, $\blacksquare$.

23. Show that if $\mathbb{R}$ and $\mathbb{R}'$ are two models of the set of real numbers and $f:\mathbb{R}\to\mathbb{R}'$ is a mapping such that $f(x+y)=f(x)+f(y)$ and $f(x\cdot y)=f(x)\cdot f(y)$ for any $x,y\in\mathbb{R}$, then

a) $f(0)=0'$;

b) $f(1)=1'$ if $f(x)\not\equiv 0'$, which we shall henceforth assume;

c) $f(m)=m'$ where $m\in\mathbb{Z}$ and $m'\in\mathbb{Z}'$, and the mapping $f:\mathbb{Z}\to\mathbb{Z}'$ is injective and preserves the order.

d) $f(\frac{m}{n})=\frac{m'}{n'}$, where $m,n\in\mathbb{Z}$, $n\neq 0$, $m',n'\in\mathbb{Z}'$, $n'\neq 0'$, $f(m)=m'$, $f(n)=n'$. Thus $f:\mathbb{Q}\to\mathbb{Q}'$ is a bijection that preserves order.

e) $f:\mathbb{R}\to\mathbb{R}'$ is a bijective mapping that preserves order.

Proof: a) $f(0)=f(0+0)=f(0)+f(0)\Rightarrow f(0)=0'$.

b) Suppose $x_0\in\mathbf{R}$ such that $f(x_0)\neq 0'$, then

$$f(x_0)=f(x_0\cdot 1)=f(x_0)\cdot f(1)\Rightarrow [f(x_0)]^{-1}f(x_0)=[f(x_0)]^{-1}f(x_0)\cdot f(1)$$
$$\Rightarrow 1'=1'\cdot f(1)\Rightarrow f(1)=1'$$

c) First of all, we show that $f(\mathbf{N})=\mathbf{N}'$.

$$(f(1)=1'\in\mathbf{N}')\wedge(n\in\mathbf{N}\wedge f(n)\in\mathbf{N}'\Rightarrow f(n+1)=f(n)+f(1)=f(n)+1'\in\mathbf{N}')$$
$$\Rightarrow f(\mathbf{N})\subset\mathbf{N}'$$

$$(f(\mathbf{N})\subset\mathbf{N}')\wedge(1'=f(1)\in f(\mathbf{N}))$$
$$\wedge(n\in\mathbf{N}\Rightarrow f(n)+1'=f(n)+f(1)=f(n+1)\in f(\mathbf{N}))\Rightarrow f(\mathbf{N})=\mathbf{N}'$$

$\forall x\in\mathbf{R}$,

$$0'=f(0)=f(x+(-x))=f(x)+f(-x)\Rightarrow f(-x)=-f(x).$$

$\forall x,y\in\mathbf{R}$, $f(x-y)=f(x+(-y))=f(x)+f(-y)=f(x)-f(y)$.

Hence $f(-\mathbf{N})=-f(\mathbf{N})=-\mathbf{N}'$.

$$(f(\mathbf{N})=\mathbf{N}')\wedge(f(-\mathbf{N})=-\mathbf{N}')\wedge(f(0)=0')\Rightarrow f(\mathbf{Z})=\mathbf{Z}'.$$

Next, we show that $f: \mathbb{R} \to \mathbb{R}'$ is injective $\Leftrightarrow (f(x)=0' \Leftrightarrow x=0)$.

" $\Rightarrow$ " is trivial. Conversely, if $f(r_1)=f(r_2)$, then

$f(r_1)-f(r_2)=0' \Rightarrow f(r_1-r_2)=0' \Rightarrow r_1-r_2=0 \Rightarrow r_1=r_2$, i.e.,

$f: \mathbb{R} \to \mathbb{R}'$ is injective.

So, we just need to show that $f(x)=0' \Leftrightarrow x=0$. " $\Leftarrow$ " is proved in a). Conversely, assume that $x \neq 0$, then

$1'=f(x \cdot x^{-1})=f(x) \cdot f(x^{-1})=0' \cdot f(x^{-1})=0'$, which contradicts with the fact that $0'<1'$. Hence, the assumption is false, i.e.,

$f(x)=0' \Rightarrow x=0$.

Thus, we have shown that $f: \mathbb{R} \to \mathbb{R}'$ is injective, not only restricted on $\mathbb{Z}$.

Suppose $m,n \in \mathbb{Z}$ such that $m<n$ (i.e., $n-m \in \mathbb{N}$), then

$f(n)-f(m)=f(n-m) \in \mathbb{N}'$. Hence, $f(n)-f(m)>0$, i.e.,

$f(m)<f(n)$.

d) $\forall x \in \mathbb{R} \backslash \{0\}$,

$1'=f(1)=f(x \cdot x^{-1})=f(x) \cdot f(x^{-1}) \Rightarrow f(x^{-1})=[f(x)]^{-1}$.

$\forall x \in \mathbb{R}\, \forall y \in \mathbb{R} \backslash \{0\}$, $f(x \cdot y^{-1})=f(x) \cdot f(y^{-1})=f(x) \cdot [f(y)]^{-1}$.

$\forall \dfrac{m}{n} \in \mathbb{Q}$ (where $m,n \in \mathbb{Z}$ and $n \neq 0$),

$f\left(\dfrac{m}{n}\right)=f(m \cdot n^{-1})=f(m) \cdot [f(n)]^{-1}=\dfrac{f(m)}{f(n)}$, where

$f(m), f(n) \in \mathbb{Z}'$ and $f(n) \neq 0'$. Hence, $f(\mathbb{Q}) \subset \mathbb{Q}'$. Since

$f|_{\mathbb{Z}}: \mathbb{Z} \to \mathbb{Z}'$ is bijective, $\forall \frac{m'}{n'} \in \mathbb{Q}'$ (where $m', n' \in \mathbb{Z}'$ and $n' \neq 0'$), we can find $m, n \in \mathbb{Z}$ ($n \neq 0$) such that $f(m) = m'$, $f(n) = n'$ and $f\left(\frac{m}{n}\right) = \frac{f(m)}{f(n)} = \frac{m'}{n'}$. Hence, $f(\mathbb{Q}) = \mathbb{Q}'$ and $f|_{\mathbb{Q}}: \mathbb{Q} \to \mathbb{Q}'$ is bijective. Moreover, it is easy to see that $f(\mathbb{Q}^+) = \mathbb{Q}'^+$ (from the facts that $f\left(\frac{m}{n}\right) = \frac{f(m)}{f(n)}$ and $f(\mathbb{N}) = \mathbb{N}'$), hence $f|_{\mathbb{Q}}: \mathbb{Q} \to \mathbb{Q}'$ preserves the order.

e) We just need to show that $f|_{\mathbb{R}\setminus\mathbb{Q}}: \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}' \setminus \mathbb{Q}'$ is bijective and preserves the order.

$\forall x > 0$, $f(x) = f\left(x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}\right) = f\left(x^{\frac{1}{2}}\right) \cdot f\left(x^{\frac{1}{2}}\right) \geqslant 0$. By injectivity of $f: \mathbb{R} \to \mathbb{R}'$ and $x > 0$ (thus $x \neq 0$), it follows that $f(x) \neq 0$, hence $f(x) > 0$, i.e., $f(\mathbb{R}^+) \subset \mathbb{R}'^+$. It is easy to see that $f(\mathbb{R}^+) \subset \mathbb{R}'^+$ implies that $f: \mathbb{R} \to \mathbb{R}'$ preserves the order.

Since $f: \mathbb{R} \to \mathbb{R}'$ is injective and $f|_{\mathbb{Q}}: \mathbb{Q} \to \mathbb{Q}'$ is surjective, it follows that $f(\mathbb{R} \setminus \mathbb{Q}) \subset \mathbb{R}' \setminus \mathbb{Q}'$.

$\forall r' \in \mathbb{R}' \setminus \mathbb{Q}'$, let $X' = \{x' \in \mathbb{Q}' | x' < r'\}$ and $Y' = \{y' \in \mathbb{Q}' | y' > r'\}$, $X = f^{-1}(X')$ and $Y = f^{-1}(Y')$. Since $f|_{\mathbb{Q}}: \mathbb{Q} \to \mathbb{Q}'$ is bijective and preserves the order, it follows that

$(X \cup Y = \mathbb{Q}) \wedge (\forall x \in X \forall y \in Y (x < y))$. By the axiom of completeness, there exists $r \in \mathbb{R}$ such that

$\forall x\in X\forall y\in Y(x\leqslant r\leqslant y)$. We claim that r is unique and $r\in\mathbb{R}\setminus\mathbb{Q}$. If there exist two distinct such numbers $r_1, r_2\in\mathbb{R}$ (suppose $r_1<r_2$), then there exists $q\in\mathbb{Q}$ such that $r_1<q<r_2$. Thus $\forall x\in X\forall y\in Y(x\leqslant r_1<q<r_2\leqslant y)$, which contradicts with the facts $(q\in\mathbb{Q})\wedge(X\cup Y=\mathbb{Q})$. Hence, r is unique. Assume that $r\in\mathbb{Q}$, without loss of generality, $r\in X$, then $f(r)\in X'$, hence $f(r)<r'$. There exists $q'\in X'$ such that $f(r)<q'<r'$. Hence, $r<\left(f|_{\mathbb{Q}}\right)^{-1}(q')\in X$, which is a contradiction. Hence, r is irrational. Finally, we show that $f(r)=r'$. Without loss of generality, assume that $f(r)>r'$, then there exists $q'\in\mathbb{Q}'$ such that $r'<q'<f(r)$. Hence, $q=\left(f|_{\mathbb{Q}}\right)^{-1}(q')\in Y$, thus $r\leqslant q$, which contradicts with $q'<f(r)\Rightarrow f(q)<f(r)\Rightarrow f(r-q)>0\Rightarrow r-q>0\Rightarrow r>q$. Hence, we must have $f(r)=r'$. In conclusion, $f\colon \mathbb{R}\to\mathbb{R}'$ is bijective.