Functional Analysis training Problems

$\boxed{1}$ Prove that in a Hilbert space, every bounded sequence has a weakly convergent subsequence.

Solution: In reflexive spaces (particularly Hilbert spaces), the closed unit ball is weakly compact by Banach-Alaoglu and Kakutani's theorem.

$\boxed{2}$ Find the spectrum of the multiplication operator $M_f : L^2[0,1] \to L^2[0,1]$, where $f(x) = \text{sign}(\sin(1/x))$ for $x > 0$ and $f(0) = 0$.

Solution: By the **Spectral Theorem for Multiplication Operators** (Theorem 4.2.22 in our course), for a multiplication operator M_f on $L^2(X, \mu)$, we have:

$$\sigma(M_f) = \operatorname{ess\,ran}(f) = \{\lambda \in \mathbb{C} : \mu(\{x : |f(x) - \lambda| < \varepsilon\}) > 0\ \forall \varepsilon > 0\}$$

The function $f(x) = \text{sign}(\sin(1/x))$ takes values ± 1 almost everywhere on $[0,1]$, since the set $\{x : \sin(1/x) = 0\}$ has measure zero. Therefore, the essential range is:

$$\operatorname{ess\,ran}(f) = \{-1, 1\}$$

Hence, $\sigma(M_f) = \{-1, 1\}$.

For $\lambda \notin \{-1, 1\}$, the function $(f(x) - \lambda)^{-1}$ is bounded almost everywhere, so $M_f - \lambda I$ is invertible with bounded inverse $M_{(f-\lambda)^{-1}}$.

For $\lambda = \pm 1$, the operator $M_f - \lambda I = M_{f-\lambda}$ is not invertible since $f(x) - \lambda$ vanishes on a set of positive measure.

$\boxed{3}$ Prove that a compact operator maps weakly convergent sequences to strongly convergent sequences.

Solution: Step 1. Since T is compact and $\{x_n\}$ is bounded (by the Uniform Boundedness Principle), the sequence $\{Tx_n\}$ has a strongly convergent subsequence. Let $\{Tx_{n_k}\}$ be such that $Tx_{n_k} \to y$ for some $y \in Y$.

Step 2. We claim that $y = Tx$. Indeed, for any $f \in Y^*$, we have:

$$f(Tx_{n_k}) = (T^* f)(x_{n_k}) \to (T^* f)(x) = f(Tx)$$

since $x_n \to x$ weakly and $T^*f \in X^*$. But also $f(Tx_{n_k}) \to f(y)$ by continuity of f. Therefore, $f(y) = f(Tx)$ for all $f \in Y^*$, so by Hahn-Banach theorem, $y = Tx$.

Step 3. Now we show that the entire sequence $\{Tx_n\}$ converges to Tx. Suppose not. Then there exists $\varepsilon > 0$ and a subsequence $\{Tx_{m_k}\}$ such that:

$$\|Tx_{m_k} - Tx\| \geq \varepsilon \quad \text{for all } k.$$

But $\{x_{m_k}\}$ also converges weakly to x, so by the same argument as in Steps 1-2, $\{Tx_{m_k}\}$ has a subsequence converging to Tx, which contradicts the inequality above.

Therefore, $Tx_n \to Tx$ strongly in Y.

$\boxed{\mathbf{4}}$ Show that the set $\{\sin(nt)\}_{n=1}^{\infty}$ in $L^2[0,\pi]$ converges weakly to 0 but not strongly.

Solution: By Riemann-Lebesgue, $\langle \sin(nt), f\rangle \to 0$ for all $f \in L^2$, but $\|\sin(nt)\|_2 = \sqrt{\pi/2} \not\to 0$.

$\boxed{\mathbf{5}}$ Find the norm of the functional $f(x) = x(0) - \int_0^1 tx(t)dt$ on $C[0,1]$ with $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$.

Solution:

1. Upper bound: For any x with $\|x\|_\infty \leq 1$:

$$|f(x)| = \left|x(0) - \int_0^1 tx(t)dt\right| \leq |x(0)| + \int_0^1 t|x(t)|dt \leq 1 + \frac{1}{2} = \frac{3}{2}.$$

Thus $\|f\| \leq \frac{3}{2}$.

$$x_n(t) = \begin{cases} 1 & \text{for } t = 0, \\ -1 & \text{for } t \geq \frac{1}{n}, \\ \text{linear from 1 to -1} & \text{for } 0 < t < \frac{1}{n}. \end{cases}$$

Explicitly: $x_n(t) = 1 - 2nt$ for $t \in [0, \frac{1}{n}]$, and $x_n(t) = -1$ for $t \geq \frac{1}{n}$.

Then $\|x_n\|_\infty = 1$ and $f(x_n) \to \frac{3}{2}$.

Thus $\|f\| \geq \frac{3}{2}$.

$\boxed{6}$ Show that every finite-dimensional normed space is reflexive.

Solution: In finite dimensions, the canonical embedding $J : X \to X^{**}$ is linear and injective between spaces of equal dimension, hence surjective.

$\boxed{7}$ Prove that every compact operator is bounded.

Solution: If T is compact, image of unit ball is precompact, hence bounded, so $\|T\| < \infty$.

$\boxed{8}$ Show that ℓ^1 is not isomorphic to ℓ^2.

Solution: ℓ^2 is reflexive while ℓ^1 is not. Alternatively, ℓ^2 has Hilbert space structure while ℓ^1 doesn't satisfy parallelogram law.

$\boxed{9}$ Show that the range of a compact operator is separable.

Solution: $T(B_X)$ is precompact, hence separable. Since $T(X) = \cup_n nT(B_X)$, the range is separable.

$\boxed{10}$ Prove that a closed subspace of a Banach space is complete.

Solution: If $Y \subset X$ is closed and $\{y_n\} \subset Y$ is Cauchy, then $y_n \to y \in X$. Since Y closed, $y \in Y$.

$\boxed{11}$ Show that the weak limit of a sequence is unique.

Solution: Suppose $x_n \to x$ weakly and $x_n \to y$ weakly in a normed space X. Then for every $f \in X^*$:

$$\lim_{n\to\infty} f(x_n) = f(x) \quad \text{and} \quad \lim_{n\to\infty} f(x_n) = f(y).$$

Hence $f(x) = f(y)$ for all $f \in X^*$, i.e., $f(x - y) = 0$ for all $f \in X^*$.

Now, if $x \neq y$, then by the **corollary to Hahn-Banach theorem** (that X^* separates points of X), there exists $f \in X^*$ such that $f(x - y) \neq 0$, which contradicts the above.

Therefore, $x = y$.

$\boxed{\textbf{12}}$ Find the spectrum of the operator $S : \ell^2 \to \ell^2$ defined by:

$$S(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, \dots) = (x_2, x_3, x_1, x_5, x_6, x_4, x_8, x_9, x_7, \dots)$$

Solution:

1. **Operator structure:** $\ell^2 = \bigoplus_{k=0}^{\infty} H_k$, where $H_k \cong \mathbb{C}^3$ are 3-dimensional blocks. On each H_k, S acts as the cyclic permutation matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

2. **Spectrum of M:**

$$\det(M - \lambda I) = -\lambda^3 + 1 = 0 \Rightarrow \lambda^3 = 1$$

   Eigenvalues: $1, e^{2\pi i/3}, e^{4\pi i/3}$

3. **Spectrum of S:** Since S is an orthogonal direct sum of copies of M:

$$\sigma(S) = \sigma(M) = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$$

   All are eigenvalues of infinite multiplicity.

4. **No other points** belong to $\sigma(S)$ because for $\lambda \notin \sigma(M)$, the resolvent $(S - \lambda I)^{-1}$ exists and is bounded, constructed as the direct sum of $(M - \lambda I)^{-1}$ on each block H_k.

$\boxed{\textbf{13}}$ Let $T : L^2[0,1] \to L^2[0,1]$ be the integral operator with kernel $K(t,s) = \min(t,s)$:

$$(Tf)(t) = \int_0^1 \min(t,s) f(s)\, ds.$$

Prove that the adjoint operator T^* is compact.

Solution:

1. Adjoint operator: Since the kernel is real and symmetric, $K(t,s) = K(s,t)$, we have:

$$(T^* f)(t) = \int_0^1 \min(t,s) f(s)\, ds = Tf.$$

Thus $T^* = T$.

2. Compactness: The kernel $K(t,s) = \min(t,s)$ is continuous on $[0,1] \times [0,1]$, hence in $L^2([0,1] \times [0,1])$. Therefore, $T^* = T$ is a Hilbert-Schmidt operator, and thus compact.

$\boxed{\mathbf{14}}$ Show that $L^2[0,1]$ is separable.

Solution: Polynomials with rational coefficients are dense in $C[0,1]$, which is dense in $L^2[0,1]$.

$\boxed{\mathbf{15}}$

Find the norm of the functional $f : C[0,1] \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} x(t_k),$$

where $t_0 = 0.5$ and $t_{k+1} = t_k^2$ for $k \geq 0$.

Solution

1. Upper bound: For any $x \in C[0,1]$ with $\|x\|_\infty \leq 1$, we have $|x(t_k)| \leq 1$ for all k. Therefore:

$$|f(x)| = \left| \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} x(t_k) \right| \leq \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |x(t_k)| \leq 1.$$

Thus $\|f\| \leq 1$.

2. Lower bound: Take the constant function $x_0(t) \equiv 1$. Then $\|x_0\|_\infty = 1$ and:

$$f(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1 = 1.$$

Thus $\|f\| \geq 1$.

3. Conclusion: $\|f\| = 1$.

Remark: The sequence $\{t_k\}$ converges to 0, so the functional computes the value at this fixed point.

$\boxed{\textbf{16}}$ Let $A : \ell^2 \to \ell^2$ be the diagonal operator defined by

$$A(x_1, x_2, x_3, \dots) = (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots).$$

Prove that the set

$$\Lambda = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}$$

is compact.

Solution

The set Λ is exactly the spectrum $\sigma(A)$ of the operator A, which is compact by corollary of Theorem on the Resolvent.

$\boxed{\textbf{17}}$ Let X be a Banach space and $\{T_n\}$ a sequence of bounded linear operators on X such that for every $x \in X$, the sequence $\{T_n x\}$ converges. Prove that:

1. The operators T_n are uniformly bounded: $\sup_n \|T_n\| < \infty$
2. The limit operator $Tx = \lim_{n\to\infty} T_n x$ is linear and bounded
3. $\|T\| \leq \liminf_{n\to\infty} \|T_n\|$

Solution:

1. By the **Uniform Boundedness Principle (Banach-Steinhaus)**, since for each $x \in X$ the sequence $\{T_n x\}$ converges (and hence is bounded), the operators $\{T_n\}$ are uniformly bounded.

2. Linearity follows from the linearity of limits:

$$T(\alpha x+\beta y) = \lim_{n\to\infty} T_n(\alpha x+\beta y) = \alpha \lim_{n\to\infty} T_n x+\beta \lim_{n\to\infty} T_n y = \alpha Tx+\beta Ty.$$

Boundedness follows from the uniform boundedness of $\{T_n\}$.

3. For any $x \in X$ with $\|x\| \leq 1$:

$$\|Tx\| = \lim_{n\to\infty} \|T_n x\| \leq \liminf_{n\to\infty} \|T_n\| \cdot \|x\| \leq \liminf_{n\to\infty} \|T_n\|.$$

Taking supremum over $\|x\| \leq 1$ gives the result.

$\boxed{\mathbf{18}}$ Let X, Y be Banach spaces and $T : X \to Y$ a surjective bounded linear operator. Prove that:

1. T is an open mapping

2. If T is also injective, then T^{-1} is bounded

3. There exists a constant $C > 0$ such that for every $y \in Y$, there is $x \in X$ with $Tx = y$ and $\|x\| \leq C\|y\|$

Solution:

1. By the **Open Mapping Theorem**, any surjective bounded linear operator between Banach spaces is open.

2. If T is bijective, then by the **Bounded Inverse Theorem**, T^{-1} is bounded.

3. Take $C = \|T^{-1}\|$. For any $y \in Y$, let $x = T^{-1}y$. Then:

$$\|x\| = \|T^{-1}y\| \leq \|T^{-1}\|\|y\| = C\|y\|.$$

$\boxed{\mathbf{19}}$ Let X be a normed space and $A, B \subset X$ disjoint convex sets with A open.

1. Prove that there exists $f \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re} f(a) < \alpha \leq \operatorname{Re} f(b) \quad \text{for all } a \in A, b \in B$$

2. Show that if B is also open, then the inequality can be made strict

3. Give an example where the separation is not strict when B is not open

Solution:

1. This is the direct consequence of the **Hahn-Banach Separation Theorem** for disjoint convex sets when one is open.

2. If B is also open, then both A and B are open convex disjoint sets, and the separation can be made strict on both sides:

$$\operatorname{Re} f(a) < \alpha < \operatorname{Re} f(b) \quad \text{for all } a \in A, b \in B.$$

3. In $\mathbb{R}^2$, take:

$$A = \{(x, y) : x > 0\}, \quad B = \{(0, y) : y \in \mathbb{R}\}.$$

The functional $f(x, y) = x$ separates them with:

$$f(a) > 0 = f(b) \quad \text{for all } a \in A, b \in B,$$

but the separation is not strict for B since $f(b) = 0$ for all $b \in B$.