St. Petersburg University

Analytic Geometry. Axiomatic System for Geometry: First Look

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Course Details

- Your instructor: Victor Kats, PhD
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  - Feel free to write any questions: v.kats@spbu.ru
- Schedule
  - On Tuesdays 1:45PM-3:30PM
  - On Thursdays 8:00AM-9:45AM
- Plans for each class
  - 50+50 minutes for theoretical or mixed theoretical and practical lecture
  - Several questions on the material of classes for your self-study as a final of each topic It is recommended to write answers on the paper
- Some recommendations
  - Aim to have maximum of materials written by your own hand
  - Don't refrain from attending your homework assignments
  - Do not overdo with asking any questions you have

Geometry

- Geometry (from Ancient Greek γεωμετρία (geōmetría), "land measurement") is (with arithmetic) one of the oldest branches of mathematics
- Field of interest for Geometry is understanding of properties of space such as *the distance, shape, size, relative position of figures*, and their generalization
- Despite its age, Geometry remains in list of actual branches of mathematics
- Modern Geometry adopted concepts and methods from various actual branches of mathematics

Branches of Geometry

With Respect to Invariant Transformations

We are looking for some transformation of space which keeps properties of studied objects invariant

- **Euclidean geometry**. Studies properties of figures invariant with respect to motion (transition, rotation and scale)
  - **Plane geometry**
  - **Solid geometry**
- **Projective geometry**. Studies properties of figures invariant with respect to projective transformations
  - Original motivation to introduce is understanding the natural perspective
  - Projective transformation preserves straight lines
- **Affine geometry**. Studies properties of figures invariant with respect to affine transformations
  - Affine transformation or affinity (from the Latin, affinis, "connected with") is a geometric transformation that preserves lines and parallelism, but not necessarily Euclidean distances and angles.
- **Descriptive geometry**. Studies properties of presentation solid objects on plane

Branches of Geometry

Modern Branches

- Geometry of extra dimensions
- Non-Euclidean geometry. Studies alternatives to the parallel postulate
  - Spherical geometry
  - Elliptic geometry
  - Lobachevskian geometry (Hyperbolic geometry)
- Riemannian geometry
- Geometry of manifolds
- Topology

Branches of Geometry

Utilizing Various Mathematical Instruments

- **Analytic geometry** or Cartesian geometry. Study figures utilizing **coordinate method**. The core topic of our course
- Algebraic geometry (don't confuse with geometric algebra!). Utilizes general algebraic methods to study algebraic manifolds.
- Differential geometry. Studies smooth manifolds with techniques adopted from differential calculus, integral calculus, linear algebra and multilinear algebra

Axiomatic System

- An **axiom** (from Ancient Greek ἀξίωμα (axíōma), "that which is thought worthy or fit") (also **postulate**, or **assumption**) is a statement that is taken to be true, to serve as a premise or starting point for further reasoning and arguments
  - Axiom (more accurate non-logical axiom) is a formal logical expression used in deduction to build a mathematical theory, and might or might not be self-evident in nature.
- **Primitive notion** is a concept that is not defined in terms of previously-defined concepts
  - **Primitive terms**: point, line, plane, segment, etc.
  - **Primitive relations**: congruence, betweenness, incidence, containment, motion, etc.
- **Axiomatic system** is a set of primitive notions and axioms which establish fundament of some theory. Each statement of the underlined theory may be derived from these axioms
  - An axiomatic system is said to be **consistent** if it lacks contradiction
  - In an axiomatic system, an axiom is called **independent** if it cannot be proven or disproven from other axioms in the system
  - An axiomatic system is called **complete** if for every statement, either itself or its negation is derivable from the system's axioms
- For the same theory exists various axiomatic systems

Example of Axiom

- **Primitive notions**
  - Primitive terms: points, lines
  - Primitive relations: containment
- **Statement**: For every two points A and B there exists a line a that contains them both
- Equivalent forms of term "contains":
  - A and B lie upon a
  - A and B are points of a
  - a goes through A and through B
  - a joins A to B
  - etc...
- Notation: $AB = a$ or $BA = a$

Theorems and Proofs

- **Theorem** is a statement that has been proved, or can be proved
- The **proof** of a theorem is a logical argument [1] that uses the inference rules of a deductive system
- Proved theorem is a logical consequence of the axioms and previously proved theorems
- Process of proving usually begins from non-formal descriptive reasoning and must come to a valid argument as result

[1] An argument is a statement or group of statements called premises intended to determine the degree of truth or acceptability of another statement called a conclusion.

- Now we build our first-look axiom system
- This system will be somehow redundant for more obviousness

Axiomatic System of Plane Geometry. Primitive Notions

- Primitive objects
  - Points. Descriptive meaning of point may be described as "exact location". Dot is a coarse model of point. To locate a point laying on some line or curve, we may use a stroke.
  - Segments. Descriptive meaning of segment may be described as "stretched thread"
- Primitive relations
  - Point is endpoint of segment
  - Point lies on segment
  - Two segments are equal

Notes

- We use uppercase Latin letters for points (A, B, C, M, ...)
- We use lowercase Latin letters for segments (a, b, c, m, ...)
- We say "**point lies on a segment**" for points lie **inside** segment
- We say "**point is contained in a segment**" and use notation $A \in a$ to label that
-
  - Point A lies on segment a or
  - Point A is endpoint of segment a

Axiomatic System of Plane Geometry. Additional Relations

- Segment b **contains** segment a ($a \subset b$) if all points $A \in a$ are contained in b
- Segments a and b **shape** segment c ($c = a \cup b$) if all points $C \in c$ are also points of a, or b or both. This may be generalized to n segments
- Segment c is **composed** of segments a and b if a and b shape c and the only common point of a and b is one of endpoints. This may be generalized to n segments
- Segment a **overlaps** b if they have single common endpoint and a lies on b
- For a par of segments there one overlaps another we may say that segments **overlap** without specification of the order in this relation
- Segments are crossing if there is only single common point laying on both segments

Axiomatic System of Plane Geometry. Skeleton

Usual sequence to build axiomatic system is following: (1) build axioms for a plane, (2) assign these axioms to any plane in space, and, finally, add axioms on non-planar objects.
Here is a skeleton of our axiomatic system on a plane.

- Linear axioms. Deal with segments and point which may lie on a single line. Notion of the plane is redundant here
  - Axioms of connections
  - Axioms of equivalence and measurement
- Planar axioms. Deal with figures on plane which do not lie on a single line
- Parallel postulate

Notes

- Plane itself is an "environment" to build plane geometry it is not necessary to include it into list of primitive notions
- Term "line" is more general in comparison to (line) "segment", but in most real problems we deal with finite parts of lines. We will derive term of line later.
- Term "figure" and its properties will be introduced as nested axiomatic system

Axiomatic System of Plane Geometry. Linear Axioms I

Axioms of shaping the segments

- $\mathbf{I}_1$. Axiom of existence.
  - There exists at least a single segment
  - Each segment has exactly two endpoints
  - There is at least one point laying on each existing segment
- $\mathbf{I}_2$. Axiom of building the segment
  - Pair of non-equal points are endpoints of single and only single segment
- $\mathbf{I}_3$. Axiom of splitting the segment
  - Each point laying on a segment splits it into two segments and original segment is composed of these two segments
- $\mathbf{I}_4$. Axiom of concatenation of the segments
  - If two segments contain at least two common points, then they shape a segment

Axiom $\mathbf{I}_2$ allows us use endpoints to denote a segment. For segment a with endpoints A and B we may denote a as AB

Axiomatic System of Plane Geometry. Derived Theorems I

1. Any point laying on a segment is never it's endpoint
   - Proof: if point C lies on segment AB it splits ($\mathbf{I}_3$) segment into segments AC and CB.
   - Thus, C must be different from A and B which are endpoints of given segment.
   - Thus, if we say that endpoint lies on the segment we may take A or B as C. Contradiction. $\square$
2. Consider segment AB, and points $C \in AB$ and $D \in AB$, and point M laying on CD. Then (1) $CD \subset AB$ and (2) M lies on AB
   - Proof: let's take a look on possible cases
     - Let C and D match A and B. Thus, by axiom $\mathbf{I}_2$ CD equals AB. Therefore, M lies on AB
     - Let only one point, C or D, match A or B. Let D match A (C lies on AB). Thus, CD equals AC and M lies on AC. Therefore, by axiom $\mathbf{I}_3$ C splits AB into AC and CB, and (1) $M \in AB$, (2) M differs form A and C (see above), (3) M differs from B as C is single common point of AC and CB. Finally, M lies on AB
     - Let both C and D lay on AB. Thus, C spits AB on AC and CB. Let D lay on AC, for D laying on CB situation is equivalent. Therefore, M lies on AC (see case above) and by the same reason M lies on AB.

   $\square$

Axiomatic System of Plane Geometry. Derived Theorems II

3. Consider segments a and b. If $a \subset b$ and $b \subset a$ then a equals b
    - Proof: Result for inner points seems trivial, but situation on endpoints must be studied.
    - Let A_1 and A_2 be endpoints of a, and B_1 and B_2 be endpoints of b.
    - $A_1 \in b$ and $A_2 \in b$ because $a \subset b$ and in respect to theorem 2 all points lay on a lay on b
    - $B_1 \in a$ and $B_2 \in a$ because $b \subset a$ and in respect to theorem 2 all points lay on b lay on a
    - Statement 1 establishes that any point laying on a segment can not be it's endpoint. Thus, endpoints of a correspond endpoints of b and they are the same
    - By axiom $\mathbf{I}_2$ segments a and b must be the same.$\square$
4. Segments with common endpoint overlap if there is any other common point of underlined segments
5. Proof: to be done as **home assignment**

Axiomatic System of Plane Geometry. Linear Axioms II

Axioms of equivalence and measurement

- **II**$_1$. Axiom of establishing the segment.
  - There is arbitrary segment a with selected endpoint A
  - For any given there is one and only one segment (1) equal with given, and (2) having common endpoint A and overlapping segment a
- **II**$_2$. Axiom of length
  - Real number 1 assigned to arbitrary segment e
  - Each segment may be assigned to real number which is denoted as its length with respect to scale e if following conditions fulfilled
    - Equal length assigned to equal segments
    - If C lies on AB length of AB is sum of length of AC and length of CB
- **II**$_3$. Axiom of existence of the segment of specified length
  - For any scale e and any positive real number l, there is a segment of length l

Note

- Length of the segment gives us understanding of the distance between points.
- Distance from point to itself may be generalized as zero length.

Axiomatic System of Plane Geometry. Derived Theorems I

Measured length in scale of e is a function $l(*) = |*|$ with set of segments as domain, and set of positive real numbers as codomain, and with properties:

1. $l(e) = 1$
2. if segment a equals segment b, then $l(a) = l(b)$
3. if segment c composed of segments a and b, then $l(c) = l(a) + l(b)$

Axiom $\mathbf{II}_2$ establishes existence of this function for any scale e.

1. For given scale e exists single and only single length function
   - Proof: to be done as **home assignment**. (Let e' be equal to e in terms of next theorem)
2. Let be e and e' be two scales and l and l' corresponding lengths. Transition from e to e' leads to transformation l to l' denoted as multiplication by a positive factor:

$$l(a) = cl'(a), \quad c = const > 0$$

   - Proof: TBD.□

Axiomatic System of Plane Geometry. Derived Theorems II

We can derive c:

$$1 = l(e) = cl(e'). \quad \Rightarrow \quad c = \frac{1}{l(e')}.$$

Thus,

$$l'(a) = \frac{l(a)}{l(e')}.$$

3. Consider segments AB and CD. If $|AB| = |CD|$, then AB equals CD
   - Proof: Result for inner points seems trivial, but situation on endpoints must be studied.
   - Axiom $\mathbf{II}_1$ grants a possibility to lay alongside AB single and only single segment AM equal with CD
   - $|AM| = |CD|$. Thus, $|AM| = |AB|$
   - Let AM not match AB. Thus, $AM \subset AB$ and M lies on AB or $AB \subset AM$ and B lies on AM
   - Continue this proof as **home assignment**. Check both possible cases, derive formulas for $|AB|$ and $|AM|$, and find contradictions

Axiomatic System of Plane Geometry. Derived Theorems III

Consequences

- Segments and their length (for selected constant scale) are equivalent in the same time
- As relation of equality for real numbers is reflexive, symmetric and transitive, relation of equality of segments is reflexive, symmetric and transitive too.

4. For arbitrary segment AB there is bijection with domain of all points $M \in AB$ and codomain $[0, |AB|]$ with following condition.
   Let $M_1, M_2 \in AB$ and their correspondences $M_1 \mapsto x_1$ and $M_2 \mapsto x_2$ be ordered:
   $0 < x_1 < x_2 < |AB|$, therefore, $AM_1 \subset AM_2$ and $|M_1M_2| = x_2 - x_1$.
   - Proof: For any $M \in AB$ by axiom $\mathbf{II}_2$ exists positive length $|AM|$ and $|AB| = |AM| + |MB|$. Thus, $|AM| < |AB|$
   - By axiom $\mathbf{II}_3$ for any $x > 0$ exist segment of length x and by axiom $\mathbf{II}_1$ one and only one segment of underlined length may be laid alongside AB from endpoint A
   - for $x < |AB|$ M lies on AB, $AM \subset AB$. If we let $AB \subset AM$ or their equality then axiom $\mathbf{II}_2$ grants contradiction: $|AB| \leqslant |AM|$.
   - Thus, we established bijection between $0 < x < |AB|$ and M laying on AB. Values 0 and $|AB|$ correspond endpoints of the segment.

Axiomatic System of Plane Geometry. Derived Theorems IV

- If M_1 lies on AM_2, then by axiom $\mathbf{II}_2$ $|AM_2| = |AM_1| + |M_1M_2|$ and $|M_1M_2| = x_2 - x_1$
- Let $0 < x_1 < x_2 < |AB|$ but M_1 does not lay on AM_2
- M_2 splits AB into AM_2 and M_2B, thus M_1 in this case lies on M_2B and $|M_2B| > |M_1B|$
- $|BM_2| + |M_2A| = |AB| = |BM_1| + |M_1A| \Rightarrow x_2 = |AM_2| < |AM_1| = x_1$. Contradiction.$\square$

Mapping points of the segment into the real number axis using length as function preserves order of the points

Axiomatic System of Plane Geometry. Derived Theorems V

Now we revisit theorem 2

Let be e and e' be two scales and l and l' corresponding lengths. Transition from e to e' leads to transformation l to l' denoted as multiplication by a positive factor:

$$l(a) = cl'(a), \quad c = const > 0$$

- Proof: Consequences of theorem 3 and axiom $\mathbf{II}_3$ establish existence of the segments for any given positive length and any given scale.
- This statement establishes family of bijective functions with domain of pozitive real numbers and codomain formed with classes of equal segments, which are lengths of specified scale.
- Change of scale now means mapping from one bijection to another and depends only on length value for first scale

$$l'(a) = f(l(a))$$

Axiomatic System of Plane Geometry. Derived Theorems VI

- By axiom $\mathbf{II}_2$ function f must be additive (c composed with segments a and b):

$$l(c) = l(a) + l(b) \quad \text{and} \quad l'(c) = l'(a) + l'(b).$$
$$f(l(a) + l(b)) = f(l(a)) + f(l(b))$$
$$f(x + y) = f(x) + f(y)$$

- So, f defined on both domain and codomain of positive real number and is additive.

Axiomatic System of Plane Geometry. Derived Theorems VII

- Let's proof $f(x) = cx$, $c = const > 0$
  - $f(nx) = f\left(\sum_1^n x\right) = \sum_1^n f(x) = nf(x)$
  - Let $r = m/n$, m and n are positive integers and subsititute $x = ry$
  - $f(my) = f\left(n\frac{m}{n}y\right) = nf\left(\frac{m}{n}y\right) = nf(ry) = nf(x)$
  - $f\left(\frac{m}{n}y\right) = \frac{1}{n}f(my) = \frac{m}{n}f(y)$
  - Let $y = 1$, thus for any rational r: $f(r) = f(r \cdot 1) = rf(1) = cr$, $c = f(1) = const > 0$
  - f is monotonic: if $x_1 > x_2 > 0$, then $x_1 = x_2 + x$, $x > 0$ and $f(x_1) = f(x_2 + x) = f(x_2) + f(x) > f(x_2)$

Axiomatic System of Plane Geometry. Derived Theorems VIII

  - For any positive real x: $r_1 < x < r_2$, r_1, r_2 are positive rational numbers

$$f(r_1) < f(x) < f(r_2)$$
$$cr_1 < f(x) < cr_2$$

  - With replacing r_1 and r_2 with rational sequences growing and descending to x we obtain in limit $f(x) = cx$
- Thus, $l'(a) = f(l(a)) = cl(a)$ $\square$

Axiomatic System of Plane Geometry. Concept of Line I

- In our axiomatic system we introduced segment as a primitive term is redundant because it is quite more demonstrative and easy-to-presentation
- Now we derive definition and properties of line
- Let A and B be a pair of points. We call **straight line** (AB) (or just line) union of all segments containing both points A and B
- Alternative
  - **Locus** (Latin word for "place", "location") is a set of all points, whose location satisfies or is determined by one or more specified conditions.
  - Line is locus of points contained in all possible segments containing two specified points

Axiomatic System of Plane Geometry. Concept of Line II

- Theorem: For every two points A and B there exists single and only single line (AB) that contains them both
- Proof: by axiom $\mathbf{I}_1$ exists line containing two point, because each pair of points may be connected with segment
- Let C and D be points lay on the line (AB). Proof that lines (AB) and (CD) are equivalent
- Triplets A, B, C and A, C, D shape two segments with two common points
- Axiom $\mathbf{I}_4$ grant shaping the segment a containing all four underlined points
- Let b be one of segments shaping (AB). A and B are common points for a and b
- Thus, exists c shaped from a and b by axiom $\mathbf{I}_4$ and (CD) contains c
- And each segment shaping (AB) contained in (CD): $(AB) \subset (CD)$
- In the same time with taking p from segments shaping (CD) we may demonstrate in the same manner it contained in (AB), thus $(AB) \supset (CD)$
- Thus, (AB) and (CD) are equivalent. $\square$

Axiomatic System of Plane Geometry. Concept of Ray

- **Ray** OB or half-line is union of all segments with just one common endpoint O and common point different from this endpoint B
- Alternative: ray is a locus of points contained in all segments with common endpoint and containing given point as common
- Each point on a line shapes two rays
- Each point on a ray shapes equivalent ray

Axiomatic System of Figures

- Primitive objects
  - Points. Descriptive meaning of point may be described as "exact location"
  - Figures. Descriptive meaning of figure may be described as "locus for collection of points"
- Primitive relations
  - Point lies on a figure
  - Figure contains a point

1. Figure unambiguously determined by all points, which it contains.
   - Thus, if figures F and F' contain exactly the same collection of points they are equivalent
2. Point is a figure. It contains only and only itself
3. For any explicit (and verifiable) low acting on a collection of points exists a figure which contains all points satisfying underlined low and only these points

Note Bene! Set of points satisfying low introduced in figure axiom 3 may be potentially empty.

Axiomatic System of Plane Geometry. Planar Axioms

- **III**$_1$. Axiom of plain separation
  - Each segment and containing it line establish along with points laying on that line two classes of points:
    - Points laying on one side of that line
    - Points laying on other side of that line

  Union of locus of one of underlined classes and that line shape **half-plane**

  Alternative:
- **III**$_1$. Axiom of plain separation
  - Single straight line established on a plane establishes exactly two halp-planes

Axiomatic System of Plane Geometry. Concept of Angle I

- A pair of segments with common endpoint establish **angle**
- Each segment may be replaced with any segment overlapping it and established from underlined endpoint
- Thus, we can say that angle is pair of rays with common endpoint
- We call that common endpoint the **edge** of the angle
- Theorem: only and only tree cases are possible:
  1. One segment lays in a single half-plane against second (ordinary angle)
  2. Segments shape new segment with single common point (straight angle)
  3. Segments overlap (zero angle)
- Proof: Let segments AB and AC shape angle with edge A
- We obtain case 1 just letting A be the single common point of AB, and AC, and any segment containing AB, or AC
- Let M be common point of segment AB and segment $a \supset AC$
- Segments AB and a shape segment MN by axiom $\mathbf{I}_4$

Axiomatic System of Plane Geometry. Concept of Angle II

- MN may be extended to satisfy condition $MN \supset AB$, $MN \supset AC$
- A splits MN into MA and AN
- Two cases are possible:AB and AC contained in the same segment MA or AN and overlap because the long one of them contains short one and series of their common points may be shown
- Or the points are contained in different segments with single common point. Let $AB \subset MA$ and $AC \subset AN$. Thus, $|MB| < |MA| < |MC|$ and A lies on BC (axiom $\mathbf{I}_3$). Therefore, A splits BC into AB and AC. $\square$

Axiomatic System of Plane Geometry. Angles Comparison

- Let segment a be a side of angle, and it's endpoint A be angle's edge
- We say that ordinary angle **established** form side a or from edge A
- Notation for angles by edge O: $\angle O$, by sides a and b: $\angle ab$, by sites AB and AC: $\angle BAC$
- We call segment with endpoints on opposite sides of angle a **transverse segment**
- Let's consider two angles $\angle BAC$ and $\angle B'A'C'$ and respective **transverse segment** AB and $A'B'$
- We call transverse segments for two angles **corresponding** if pairs of two angles sides appear to be equal: $AB \cong A'B'$ and $AC \cong A'C'$
- We call **angles equal** if in this case transverse segments are equal too

Axiomatic System of Plane Geometry. Planar Axioms I

- **III**$_2$. Axiom of angle establishing and comparison
  - For a pair of equal ordinary angles all corresponding transverse segments are equal
  - An ordinary angle may be established for any given ordinary angle and segment with specified half-plane and endpoint of it. Established angle will be equal with given, have edge in specified endpoint and lay in specified half-plane.

This axiom describes process of plotting angle equal to given

- Segment lies inside the angle if one of its endpoints is angle's edge, and there is a transverse segment having cross point with this segment
- Ray lies inside the angle if its endpoint is angle's edge and each segment contained in the ray lies inside the angle
- Each segment or ray established from the edge of straight angle lies inside it

Axiomatic System of Plane Geometry. Planar Axioms II

- **III**$_3$. Axiom of angle measure
  - Arbitrary angle ε assigned to integer 1.
  - Each angle may be assigned with positive real number called **measure** wit respect to scale ε with properties:
    - Measures of equal angles are equal
    - If segment c lies inside angle $\angle ab$, measure of $\angle ab$ is sum of measures of $\angle ac$ and $\angle cb$

Axiomatic System of Plane Geometry. Derived Theorems I

1. All straight angles are equal.
   - Proof: Let $\angle AOB$ be straight. Any transvers segment, say AB contained in the segment formed by this angle. Thus, edge O contained in AB and splits it into two segments AO and OB
   - Therefore, comparison of straight angles AOB and $A'O'B'$ leads to comparison corresponding transversal segments AB and $A'B'$. Condition of correspondence is $AO \cong A'O'$ and $OB \cong O'B'$ thus $AB \cong A'B'$ and angles are equal. $\square$
2. Consider angle $\angle AOB$, it's transversal segment AB and segment OC laying inside the angle and crossing AB. Segment OC or at least ray with endpoint O established by OC (continuation of OC) will cross any transverse segment of the angle
   - Proof: to be done as **home assignment**. (Prove that endpoints of each transversal segments lay in different half-planes with respect to OC)

For three points ABC not-laying on the same line we call aggregate of segments AB, AC and BC the **triangle** and write $\triangle ABC$.

3. Let line a cross one side, say AB of $\triangle ABC$. If a contains none of points A, B, and C, then it crosses one other side of triangle, AC or BC.
   - Proof: Let's observe point C, as A and B lay in different half-planes with respect to a by condition.
   - C must lay in one and only one of half-planes constructed with respect to a, say it the same half-plane with A. Thus, a crosses BC, but not AC
   - Reverse is correct too. $\square$

Axiomatic System of Plane Geometry. Derived Theorems III

4. Observe angles $\angle ab$ and $\angle ac$. If these angles are established in the same half-plane with respect to segment a, then c lies in $\angle ab$ or b lies $\angle ac$
   - Proof: Denote edge of the angles as O and select point A on segment a and point B on segment B
   - We also continue segment a behind the edge and select arbitrary point D on this continuation
   - Consider triangle $\triangle ABD$. Segment c crosses its side AD, thus line established by c also crosses AB or BD
   - Crossing AB which is transverse segment of $\angle ab$ means that c lies inside $\angle ab$. $\square$
   - If c crosses BD lets denote crosspoint as C' and consider triangle $AC'D$
   - line established with b crosses AD and continuation of $C'D$ in point B, thus it crosses AC'
   - Thus b lies inside $\angle ac$. $\square$

Axiomatic System of Plane Geometry. Derived Theorems IV

We call angles **alternate** if they have a common side and two other sides shape the straight angle.
We call angle which equals to its alternate angle the **right angle**.

5. Equal angles have equal measure.
6. Angle measure is reflexive, symmetric and transitive
7. All right angles are equal. Measure of right angle is $1/2$ of the measure of straight angle for any scale

- Prove these theorems as **home assignment**. (The proof based on analogous statements for segments)

Axiomatic System of Plane Geometry. Measure Units for Angle

- There are three most often used unit systems for angle measurement
- (1) We assign to right angle measure 90°
- (3) We assign to right angle measure 100^g (grad)
- (2) We assign to right angle measure $\pi/2$ (radians)

Radian measure of angle has a wide application in physics. Canonical forms of all physical lows involving angles measure are usually written with respect to this measure

Axiomatic System of Plane Geometry. Parallel Postulate

- **IV**$_1$. Parallel postulate for segments
  - Consider equal segments AC and BD forming right angles with segment AB and laying in the same halp-plane in respect to AB. Therefore, segments AB and CD are equal

Notes

- Points A, B, C, D shape figure called **rectangle**
- Any pair of equal segments overlapping AC abd BD possess the same property
- The segments may be continued to lines called **parallel**. We use notation $a||b$ for such lines
- Classic parallel postulate:
  - For given line a and point A not laying on the line exists one and only one line parallel to the line a
- Euclid's parallel postulate:
  - If a straight line falling on two straight lines make the alternate angles equal to one another, the straight lines will be parallel to one another
- Modern geometry constructs theories with excluded parallel postulate

Axiomatic System of Solid Geometry. Axioms for Arbitrary Plane I

Generalization of Planar Axiom for any Plane in Space

Additional primitive notions

- Primitive object: **plane**
- Primitive relation: point lies on plane

Additional derived definition:

- Segment lies on the plane if each point contained in the segment lies on the plane

Linear axioms stay "as is":

- $\mathbf{I}_1$. Axiom of existence.
- $\mathbf{I}_2$. Axiom of building the segment
- $\mathbf{I}_3$. Axiom of splitting the segment
- $\mathbf{I}_4$. Axiom of concatenation of the segments
- $\mathbf{II}_1$. Axiom of laying the segment.
- $\mathbf{II}_2$. Axiom of length
- $\mathbf{II}_3$. Axiom of existence of the segment of specified length

Axiomatic System of Solid Geometry. Axioms for Arbitrary Plane II

Generalization of Planar Axiom for any Plane in Space

Planar axioms need clarification with specific plane

- **III$_1$**. Axiom of plain separation
  - On each plane each segment contained in it establishes two sides
- **III$_2$**. Axiom of angle establishing and comparison
  - On each plane an ordinary angle may be established for any given ordinary angle and segment laying in that plane with specified half-plane in respect to given plane and endpoint of it. Established angle will be equal with given, have edge in specified endpoint and lay in specified half-plane.

Next, we clarify definition of "segment laying in the angle" with condition that them both lay in the same plane. After this remaning angle measure axiom stays "as is"

- **III$_3$**. Axiom of angle measure
  - Arbitrary angle ε assigned to integer 1.
  - Each angle may be assigned to positive real number called measure wit respect to scale ε with properties:
    - Measures of equal angles are equal
    - If segment c lies inside angle $\angle ab$, measure of $\angle ab$ is sum of measures of $\angle ac$ and $\angle cb$

Axiomatic System of Solid Geometry. Axioms for Arbitrary Plane III

Generalization of Planar Axiom for any Plane in Space

- **IV**$_1$. Parallel postulate for segments
  - Consider equal segments AC and BD laying in the same plane and forming right angles with segment AB and laying in the same half-plane in respect to AB. Therefore, segments AB and CD are equal

Axiomatic System of Solid Geometry. Axioms of Space

- **S**$_1$. Each triplet of points there is plane containing them
- **S**$_2$. For each triplet of points may be demonstrated point not laying in the plane containing them
- **S**$_2$. Locus of intersection of two planes possessing a common point is a line

Axiomatic System of Solid Geometry. Derived Theorems I

1. There is one and only one line containing given a pair of points
   - Proof repeats analogous for plane geometry. $\square$
2. If line, say a, and plane, say α contain two common points, say, A and B, then the line lies in the plane
   - Proof: By adding point C non-laying on a to pair A and B we will form triplet and thus establish plane β containing A, B, and C.
   - This plane intersects with specified plane α shaping line containing A and B.
   - Exists one and only one line containing A and B and this line is a
   - Thus, a lies in α $\square$
3. On each plane all planar axioms and derived from these axioms terms and theorems are valid and identical without any respect to selected plane

Axiomatic System of Solid Geometry. Derived Theorems II

4. Each triplet of points, say A, B and C which do not lay on the same line shapes single and only single plane
   - Proof: Axiom $\mathbf{S}_1$ grans existence of such plane
   - Let exist two such planes. Then any two points of the triplet, say A and B are contained in both planes.
   - By $\mathbf{S}_3$ intersection of such planes shape a line (AB)
   - But C is not laying on that line by condition.
   - Thus, C may not be possessed by both constructed planes in the same time. Contradiction. $\square$
5. Each plane, say α, separates all points non-laying on it int two classes, say F_1 and F_2 by condition "for two distant points segment containing underlined points as endpoints has no common points with plane α"
   - Proof: Consider point A laying outside plane α and containing in class F_1 and distant from A and each other points M and N also laying outside plane α.
   - This triplet shapes plane β not equal with α
   - Thus α and β have no intersection or such intersection is a line, say a
   - Segments AM, AN, and MN lay in the plane β

Axiomatic System of Solid Geometry. Derived Theorems III

- If α and β have no intersection underlined segments do not intersect α, and $M \in F_1$, $N \in F_1$
- If α and β have intersection (a line), this line a splits β into two half-planes
- Points M and N may lay in any half-plane, but not on the line a
- Thus, segments AM, AN and MN intersect a or lie by a single side of it and newer have endpoint on it
- Therefore, M and N are contained in one of classes F_1 or F_2, depending on intersections picture. $\square$

Axiomatic System of Solid Geometry. Derived Theorems IV

Conclusions from derived theorems

1. Line and point not laying on the line shape single and only single plane.
2. Two crossing line shape single and only single plane.

Conclusions

- We build consistent and complete axiomatic system for Euclidean geometry on plane in 3-dimensions space
- Our axiomatic system based on well-demonstartive primitive notions of point, segment, and plane
- Replacing of notion "plane" with arbitrary "finite" notion is possible, but such replacement will be unintuitive and difficult explain and to demonstrate
- Significant disadvantage of our approach to building this axiomatic system is strong connection between axioms of measure with properties of positive real numbers which we must postulate or proof
- As next step we exclude this connection

Questions for Self-Study

- What is axiom and theorem?
- What is point?
- Why we decided to apply segment instead of line as primitive notion?
- What collection of axioms makes up axiomatic system of Euclidean planar geometry?
- What clarification of "planar" axioms is required then we build axiomatic system for space?
- What are additional axioms for Euclidean space geometry?
- What is figure? List figures we introduced in this topic and try to postulate lows shaping them
- What is angle? How we measure angles?