1.5. APPLICATION OF THE LAPLACE TRANSFORM TO THE SOLUTION OF INTEGRAL EQUATIONS AND SYSTEMS

Example 1

Solve the integral equation

$$\int_0^t e^{2(t-\tau)} y(\tau)\, d\tau = t^2 e^t$$

<u>Solution:</u>

In this equation $K(t-\tau) = e^{2(t-\tau)}$, therefore $K(t) = e^{2t}$.

Let's find an image of this function $\frac{1}{p-2} \leftrightarrow e^{2t}$.

Let's find the image of the right side of the equation, that is, the function $t^2 e^t$:

$$\frac{2}{(p-1)^3} \leftrightarrow t^2 e^t$$

Let's write down the equation

$$Y(p)\frac{1}{p-2} = \frac{2}{(p-1)^3}$$

From here

$$Y(p) = \frac{2p-4}{(p-1)^3}$$

Using the method of undetermined coefficients, we will find the decomposition of a fraction into the simplest fractions:

$$Y(p) = \frac{2p-4}{(p-1)^3} = \frac{A}{(p-1)^3} + \frac{B}{(p-1)^2} + \frac{C}{p-1}$$

Let's bring the right part to the common denominator and equate the numerators of the resulting and the original fraction:

$$A+B(p-1)+C(p-1)^2 \equiv 2p-4$$

Let's find the coefficients A, B, C.

$$p=1 \qquad A=-2$$

$$p=2 \qquad A+B+C=0$$

$$p=0 \qquad A-B+C=-4$$

$$2B=4 \quad \Rightarrow \quad B=2;$$

$$2A+2C=-4 \quad \Rightarrow \quad A+C=-2 \quad \Rightarrow \quad C=0$$

In that way

$$Y(p)=\frac{2p-4}{(p-1)^3}=-\frac{2}{(p-1)^3}+2\frac{1}{(p-1)^2}$$

We will find the original corresponding to the image:

$$y(t)=-t^2e^t+2te^t=te^t(2-t)$$

So, the solution of this integral equation is the function

$$y(t)=te^t(2-t)$$

Example 2

Solve the integral equation

$$y(t)=1+t+\int_0^t cos(t-\tau)y(\tau)d\tau$$

<u>Solution:</u>

In this case $\qquad f(t)=1+t \quad \Rightarrow \quad F(p)=\frac{1}{p}+\frac{1}{p^2}$;

$$K(t)=cos\,t \quad \Rightarrow \quad K^*(p)=\frac{p}{p^2+1}$$

The integral $\int_0^t cos(t-\tau)y(\tau)d\tau$ is a convolution of the function $cos\,t$ and $y(t)$.

Image of the equation:

$$Y(p)=\frac{1}{p}+\frac{1}{p^2}+Y(p)\frac{p}{p^2+1}$$

We'll find $Y(p)$.

$$Y(p)\left(1-\frac{p}{p^2+1}\right)=\frac{1}{p}+\frac{1}{p^2} \quad \Rightarrow$$

$$Y(p)\frac{p^2-p+1}{p^2+1}=\frac{p+1}{p^2} \quad \Rightarrow$$

$$Y(p)=\frac{(p^2+1)(p+1)}{p^2(p^2-p+1)}$$

Let's imagine the image of the solution as the sum of the simplest fractions:

$$Y(p)=\frac{(p^2+1)(p+1)}{p^2(p^2-p+1)}=\frac{A}{p^2}+\frac{B}{p}+\frac{Cp+D}{p^2-p+1}$$

Let's find the decomposition coefficients.

$$A(p^2-p+1)+Bp(p^2-p+1)+Cp^3+Dp^2 \equiv p^3+p^2+p+1$$

We equate the coefficients with the same degrees p in the right and left parts of the identity:

p^3	$B+C=1$
p^2	$A-B+D=1$
p	$-A+B=1$
p^0	$A=1$

We will get $A=1, B=2, C=-1, D=2.$

Therefore, the decomposition has the form:

$$Y(p)=\frac{\left(p^2+1\right)(p+1)}{p^2\left(p^2-p+1\right)}=\frac{1}{p^2}+\frac{2}{p}+\frac{2-p}{p^2-p+1}\ .$$

Let's convert all fractions into table fractions:

$$Y(p)=\frac{1}{p^2}+\frac{2}{p}+\frac{2-p}{p^2-p+1}=\frac{1}{p^2}+2\frac{1}{p}+\frac{\frac{3}{2}+\frac{1}{2}-p}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$=\frac{1}{p^2}+2\frac{1}{p}+\sqrt{3}\frac{\frac{\sqrt{3}}{2}}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{p-\frac{1}{2}}{\left(p-\frac{1}{2}\right)^2+\frac{3}{4}}$$

We will find the corresponding original:

$$y(t)=t+2+e^{\frac{t}{2}}\left(\sqrt{3}\sin\frac{\sqrt{3}}{2}t-\cos\frac{\sqrt{3}}{2}t\right)$$

This function is the solution of a given integral equation.

Example 3

Solve the integral equation

$$y(t)=cos\,t-\int_0^t e^{t-\tau}y(\tau)\,d\tau$$

Solution:

Here $f(t)=\cos t;\quad K(t)=e^t$.

Since $\int_0^t e^{t-\tau}y(\tau)\,d\tau$ is a convolution of functions e^t and $y(t)$, that is,

$$e^t * y(t) = \int_0^t e^{t-\tau} y(\tau)\,d\tau \leftrightarrow K^*(p)Y(p)$$, where

$$K(t) \leftrightarrow K^*(p) = \frac{1}{p-1},\ \ f(t) \leftrightarrow F(p) = \frac{p}{p^2+1};$$

$$y(t) \leftrightarrow Y(p).$$

The image of the integral equation takes the form

$$Y(p) = \frac{p}{p^2+1} - \frac{1}{p-1}Y(p)$$

$$Y(p) + \frac{1}{p-1}Y(p) = \frac{p}{p^2+1} \quad \Rightarrow$$

$$Y(p)\frac{p}{p-1} = \frac{p}{p^2+1} \quad \Rightarrow$$

$$Y(p) = \frac{p-1}{p^2+1}$$

Based on the image of the solution, we will find its original:

$$Y(p) = \frac{p-1}{p^2+1} = \frac{p}{p^2+1} - \frac{1}{p^2+1} \quad \leftrightarrow \quad \cos t - \sin t = y(t)$$

So, the solution of the integral equation is the function:

$$y(t) = \cos t - \sin t$$

Example 4 (HOMEWORK 5, the deadline is September 24th)

№1	№2
$y(x) = x + \int_0^x \sin(x-t)y(t)dt$;	$y''(x) - 2y'(x) + y(x) + 2\int_0^x \cos(x-t)y''(t)dt +$ $+2\int_0^x \sin(x-t)y'(t)dt = \sin x,\ y(0) = y'(0) = 0$;

2. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

2.1. PARTIAL DIFFERENTIAL EQUATIONS

Denote by D the region of the n -dimensional space R^n of points $x=(x_1,x_2,...,x_n)$, $x_1,x_2,...,x_n$, $n\geq 2$ — Cartesian coordinates of point x.

An equation of the form

$$F\left(x,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},...,\frac{\partial^k u}{\partial x_1^{i_1}...\partial x_n^{i_n}},\frac{\partial u}{\partial x_n^m}\right)=0,\ x\in D, \qquad (2.1)$$

$$\sum_{j=1}^{n} i_j = k,\ k=0,1,...,m,\ m\geq 1$$

is called a *partial differential equation of the order* m with respect to an unknown function $u=u(x)$, where $F=F\left(x,u,\frac{\partial u}{\partial x_1},...\right)-$ is a given real function of points $x\in D$, an unknown function u and its partial derivatives. The left side of equality (2.1) is called a *partial differential operator of order* m.

The real function $u=u(x_1,x_2,...,x_n)$, defined in the domain D of the assignment of equation (2.1), continuous together with its partial derivatives included in this equation and converting it into an identity, is called the *classical (regular) solution* of equation (2.1).

Equation (2.1) is called linear if F depends linearly on all variables of the form

$$\frac{\partial^k u}{\partial x_1^{i_1}...\partial x_n^{i_n}},\ 0\leq k\leq m.$$

The linear equation can be written as

$$\sum_{k=0}^{m}\sum_{i_1,\dots,i_n} a_{i_1\dots i_n}(x)\frac{\partial^k u}{\partial x_1^{i_1}\dots\partial x_n^{i_n}} = f(x)\,,\ \sum_{j=1}^{n} i_j = k\,,\ x\in D$$

or in the form of

$$Lu = f(x)\,,\ x\in D\,,$$

where $L-$ linear differential operator of order m:

$$L \equiv \sum_{k=0}^{m}\sum_{i_1,\dots,i_n} a_{i_1\dots i_n}(x)\frac{\partial^k}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}\,,\ \sum_{j=1}^{n} i_j = k\,.$$

A linear equation is called homogeneous if $f(x)\equiv 0$, inhomogeneous if $f(x)\neq 0$.

Equation (2.1) of order m is called quasilinear if F linearly depends only on partial derivatives of order m:

$$\frac{\partial^m u}{\partial x_1^{i_1}\dots\partial x_n^{i_n}}\,,\ \sum_{j=1}^{n} i_j = m\,.$$

<u>Control tasks</u>

Example 1

Find out if the following equalities are partial differential equations:

1. $\cos(u_x+u_y)-\cos u_x\cos u_y+\sin u_x\sin u_y=0$;

   <u>Solution:</u>

1. By converting the cosine of the sum into the product of cosines and sines, we obtain the identity

$$\cos(u_x+u_y)=\cos u_x\cdot\cos u_y-\sin u_x\cdot\sin u_y$$

$$\cos u_x \cdot \cos u_y - \sin u_x \cdot \sin u_y - \cos u_x \cdot \cos u_y + \sin u_x \cdot \sin u_y = 0$$

$$0 = 0$$

This is not a differential equation.

2. $u_{xx}^2 + u_{yy}^2 - \left(u_{xx} - u_{yy}\right)^2 = 0$;

<u>Solution:</u>

We open the brackets, give similar terms, and get

$$u_{xx}^2 + u_{yy}^2 - \left(u_{xx}^2 - 2u_{xx}u_{yy} + u_{yy}^2\right) = 0$$

$$u_{xx}^2 + u_{yy}^2 - u_{xx}^2 + 2u_{xx}u_{yy} - u_{yy}^2 = 0$$

$$2u_{xx}u_{yy} = 0$$

Equation (2) is a differential equation.

3. $\sin^2\left(u_{xx} + u_{xy}\right) + \cos^2\left(u_{xx} + u_{xy}\right) - u = 1$;

<u>Solution:</u>

Using the basic trigonometric identity, we obtain

$$\sin^2\left(u_{xx} + u_{xy}\right) + \cos^2\left(u_{xx} + u_{xy}\right) = 1$$

$$1 - u = 1$$

$$-u = 0$$

Equation (3) is not a differential equation.

Example 2

Determine the order of the equations:

1. $\ln\left|u_{xx}u_{yy}\right|-\ln\left|u_{xx}\right|-\ln\left|u_{yy}\right|+u_x+u_y=0$

Solution:

1. Converting the sum of the logarithms into the logarithm of the product and giving similar terms, we get

$$\ln\left|u_{xx}\right|+\ln\left|u_{yy}\right|=\ln\left|u_{xx}\,u_{yy}\right|$$

$$\ln\left|u_{xx}u_{yy}\right|-\ln\left|u_{xx}u_{yy}\right|+u_x+u_y=0$$

$$u_x+u_y=0$$

The order of the differential equation (1) is the first.

2. $u_x u_{xy}^2+\left(u_{xx}^2-2u_{xy}^2+u_y\right)^2-2xy=0$

Solution:

We open the brackets, give similar terms, and get

$$\left(u_{xx}^2-2u_{xy}^2+u_y\right)^2=u_{xx}^4-4u_{xx}^2u_{xy}^2+4u_{xy}^4+2u_{xx}^2u_y-4u_{xy}^2u_y+u_y^2$$

$$u_x u_{xy}^2+u_{xx}^4-4u_{xx}^2u_{xy}^2+4u_{xy}^4+2u_{xx}^2u_y-4u_{xy}^2u_y+u_y^2-2xy=0$$

The order of the differential equation (2) is second.

Example 3

Find out which of the following equations are linear and which are nonlinear (quasilinear):

1. $2\sin(x+y)u_{xx} - x\cos y u_{xy} + xyu_x - 3u + 1 = 0$;

<u>Solution:</u>

In this equation, the coefficients before the second and first derivatives are functions of x and y, so the equation is linear. The function $f(x,y) = 1$, that is, the equation is inhomogeneous.

Equation (1) is linear and inhomogeneous.

2. $x^2 yu_{xxy} + 2e^x y^2 u_{xy} - (x^2 y^2 + 1)u_{xx} - 2u = 0$;

<u>Solution:</u>

In this equation, the coefficients before the second and third derivatives are functions of x and y, so the equation is linear, of the third order.

The function $f(x,y) = 0$, that is, the equation is homogeneous.

Equation (2) is linear and homogeneous.

3. $3u_{xy} - 6u_{xx} + 7u_y - u_x + 8x = 0$;

<u>Solution:</u>

In this equation, the coefficients before the second and first derivatives are constant values, so the equation is linear with constant coefficients. The function $f(x,y)=8x$, that is, the equation is inhomogeneous.

Equation (3) is linear with constant coefficients and inhomogeneous.

4. $u_x u_{xy}^2 + 2xuu_{yy} - 3xyu_y - u = 0$;

<u>Solution:</u>

The first term can be represented as follows: $u_x u_{xy}^2 = u_x u_{xy} u_{xy}$. It was found that the coefficient before the highest second derivative also depends on the second derivative u_{xy}, that is, the equation is nonlinear.

Equation (4) is nonlinear.

2.2. EXAMPLES OF THE SIMPLEST PARTIAL DIFFERENTIAL EQUATIONS

Let's look at some examples of partial differential equations.

Example 1

Find the function $u=u(x,y)$ satisfying the differential equation:

$$\frac{\partial u}{\partial x}=1$$

Solution:

Integrating, we get

$$u=x+\varphi(y),$$

where $\varphi(y)$ - an arbitrary function. This is the general solution of this differential equation.

Example 2

Solve the equation

$$\frac{\partial^2 u}{\partial y^2}=6y,$$

where $u=u(x,y)$.

Solution:

Integrating twice by y, we get

$$\frac{\partial u}{\partial y} = 3y^2 + \varphi(x),$$

$$u = y^3 + y\varphi(x) + \psi(x),$$

where $\varphi(x)$ and $\psi(x)$ an arbitrary functions.

Example 3

Solve the equation

$$\frac{\partial^2 u}{\partial x \partial y} = 0.$$

Solution:

Integrating the equation with respect to x, we have

$$\frac{\partial u}{\partial y} = f(y).$$

Integrating obtained result by y, we find

$$u = \varphi(x) + \psi(y),$$

where $\psi(y) = \int f(y)dy$, $\varphi(x)$ and $\psi(y)$ - arbitrary functions.

Example 4

Solve the equation

$$x^2 \frac{\partial^2 u}{\partial x \partial y} + 2x \frac{\partial u}{\partial y} = 0, \; x \neq 0.$$

Solution:

This equation can be reduced to the form

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial y}\right)=0\,.$$

Integrating the equation with respect to the variable x, we obtain

$$x^2\frac{\partial u}{\partial y}=f(y)\,,$$

where $f(y)$ - arbitrary function.

Integrating the result obtained with respect to the variable y, we find

$$u=\varphi(x)+\psi(y)\,,$$

where $\psi(y)=\frac{1}{x^2}\int f(y)dy$, $\varphi(x)$ and $\psi(y)$ - are arbitrary functions.