SECOND-ORDER SURFACES

1 Types of second-order surfaces

Definition. *The* ***surface of the second order*** *(or the quadric surface) is defined as a locus of points in space whose Cartesian coordinates satisfy the equation of the form*

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0. \quad (1.1)$$

Obviously, this definition is invariant to the system of coordinates chosen. Indeed, the equation of the surface in any other system of coordinates $x'y'z'$ is obtained from the equation (1.1) by substituting x, y, and z by linear expressions with respect to x', y', z', and, consequently, in the coordinates $x'y'z'$ will also have the form (1.1).

Any plane intersects a quadric surface along a curve of the second order. Indeed, since the determination of surface is invariant with reference to the coordinate system chosen, we may regard the plane xy ($z = 0$) as a secant plane. And this plane obviously intersects the surface along the second-order curve

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0.$$

Theorem 1. *Any quadratic form can be reduced by a homogeneous orthogonal transformation to such a form (canonical) that the transformed form does not contain mixed terms.*

The roots of the characteristic equation become the coefficients of the transformed form.

Proof. Consider a quadratic form in three variables:

$$\varphi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2. \quad (1.2)$$

Let's introduce in space orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The variables x, y, z are considered as coordinates of the vector $\{x, y, z\}$ in this basis. Let us introduce a new orthonormal basis $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$. Coordinates x, y, z of an arbitrary vector $\mathbf{a}$ in the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ through the coordinates x', y', z' of that vector $\mathbf{a}$ in the basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ are expressed by the relations

$$\begin{aligned} x &= x' \cos\alpha_{11} + y' \cos\alpha_{12} + z' \cos\alpha_{13}, \\ y &= x' \cos\alpha_{21} + y' \cos\alpha_{22} + z' \cos\alpha_{23}, \\ z &= x' \cos\alpha_{31} + y' \cos\alpha_{32} + z' \cos\alpha_{33}, \end{aligned} \tag{1.3}$$

where α_{ij} is angle between vectors $\mathbf{e}_i$ and vectors $\mathbf{e}'_j$.

These relations express the orthogonal transformation Ω_1, since the matrix of this transformation is orthogonal.

Substituting into the expression (1.2) instead of x, y, z their values from the formulas (1.3), we obtain

$$\begin{aligned} \varphi' = a_{11}(x' \cos\alpha_{11} + y' \cos\alpha_{12} + z' \cos\alpha_{13})^2 + \\ + a_{22}(x' \cos\alpha_{21} + y' \cos\alpha_{22} + z' \cos\alpha_{23})^2 + \dots \end{aligned} \tag{1.4}$$

Let us first prove that the orthonormal basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ can be chosen so that in the expression (1.4 the coefficients at $x'z'$ and $y'z'$ will vanish. Writing from the relation (1.4) half of the coefficients at $x'z'$ and $y'z'$ and equating them to zero, we obtain equations that can be written as

$$\begin{aligned} &(a_{11} \cos\alpha_{13} + a_{12} \cos\alpha_{23} + a_{13} \cos\alpha_{33}) \cos\alpha_{11} + \\ &\quad (a_{21} \cos\alpha_{13} + a_{22} \cos\alpha_{23} + a_{23} \cos\alpha_{33}) \cos\alpha_{21} + \\ &\quad\quad (a_{31} \cos\alpha_{13} + a_{32} \cos\alpha_{23} + a_{33} \cos\alpha_{33}) \cos\alpha_{31} = 0, \end{aligned}$$

$$\begin{aligned} &(a_{11} \cos\alpha_{13} + a_{12} \cos\alpha_{23} + a_{13} \cos\alpha_{33}) \cos\alpha_{12} + \\ &\quad (a_{21} \cos\alpha_{13} + a_{22} \cos\alpha_{23} + a_{23} \cos\alpha_{33}) \cos\alpha_{22} + \\ &\quad\quad (a_{31} \cos\alpha_{13} + a_{32} \cos\alpha_{23} + a_{33} \cos\alpha_{33}) \cos\alpha_{32} = 0. \end{aligned}$$

These relations mean that the vector

$$\begin{aligned} \{a_{11} \cos\alpha_{13} + a_{12} \cos\alpha_{23} + a_{13} \cos\alpha_{33},\ a_{21} \cos\alpha_{13} + a_{22} \cos\alpha_{23} + a_{23} \cos\alpha_{33}, \\ a_{31} \cos\alpha_{13} + a_{32} \cos\alpha_{23} + a_{33} \cos\alpha_{33}\} \end{aligned}$$

must be orthogonal to the vectors

$$\mathbf{e}'_1 = \{\cos\alpha_{11},\ \cos\alpha_{21},\ \cos\alpha_{31}\} \text{ and } \mathbf{e}'_2 = \{\cos\alpha_{12},\ \cos\alpha_{22},\ \cos\alpha_{32}\}$$

in other words collinear to the vector

$$\mathbf{e}'_3 = \{\cos\alpha_{13},\ \cos\alpha_{23},\ \cos\alpha_{33}\},$$

i. e.

$$\begin{cases} a_{11}\cos\alpha_{13} + a_{12}\cos\alpha_{23} + a_{13}\cos\alpha_{33} = \lambda\cos\alpha_{13}, \\ a_{21}\cos\alpha_{13} + a_{22}\cos\alpha_{23} + a_{23}\cos\alpha_{33} = \lambda\cos\alpha_{23}, \\ a_{31}\cos\alpha_{13} + a_{32}\cos\alpha_{23} + a_{33}\cos\alpha_{33} = \lambda\cos\alpha_{33}, \end{cases} \tag{1.5}$$

or

$$\begin{cases} (a_{11} - \lambda)\cos\alpha_{13} + a_{12}\cos\alpha_{23} + a_{13}\cos\alpha_{33} = 0, \\ a_{21}\cos\alpha_{13} + (a_{22} - \lambda)\cos\alpha_{23} + a_{23}\cos\alpha_{33} = 0, \\ a_{31}\cos\alpha_{13} + a_{32}\cos\alpha_{23} + (a_{33} - \lambda)\cos\alpha_{33} = 0. \end{cases} \tag{1.6}$$

Since the vector $\mathbf{e}_3'$ must be unit ($\cos^2\alpha_{13} + \cos^2\alpha_{23} + \cos^2\alpha_{33} = 1$) and since the system (1.6) is linear homogeneous with respect to $\cos\alpha_{13}$, $\cos\alpha_{23}$, $\cos\alpha_{33}$, then it has non-zero solution if and only if the determinant of system equals zero:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0. \tag{1.7}$$

This equation is called the **characteristic equation** of the form φ. It is an equation of the third degree with respect to λ. But since we consider all numbers a_{ij} to be real, the equation (1.7) has at least one real root. Leaving aside for now the case when the equation (1.7) has a triple root, it can be asserted that the equation (1.7) has a simple real root, which we denote as follows: $\lambda = \lambda_3$. Substituting this value into the system (1.6), we get the system

$$\begin{cases} (a_{11} - \lambda_3)\cos\alpha_{13} + a_{12}\cos\alpha_{23} + a_{13}\cos\alpha_{33} = 0, \\ a_{21}\cos\alpha_{13} + (a_{22} - \lambda_3)\cos\alpha_{23} + a_{23}\cos\alpha_{33} = 0, \\ a_{31}\cos\alpha_{13} + a_{32}\cos\alpha_{23} + (a_{33} - \lambda_3)\cos\alpha_{33} = 0. \end{cases} \tag{1.8}$$

The system (1.8) cannot have two linearly independent solutions with respect to $\cos\alpha_{13}$, $\cos\alpha_{23}$, $\cos\alpha_{33}$. Indeed, otherwise the matrix

$$\begin{pmatrix} a_{11} - \lambda_3 & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda_3 & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda_3 \end{pmatrix}$$

would have rank less than 2, and hence the derivative of the characteristic determinant

$$\Delta(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix},$$

which is equal to

$$\Delta'(\lambda) = -\begin{vmatrix} a_{22}-\lambda & a_{23} \\ a_{32} & a_{33}-\lambda \end{vmatrix} - \begin{vmatrix} a_{11}-\lambda & a_{13} \\ a_{31} & a_{33}-\lambda \end{vmatrix} - \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix},$$

would vanish for $\lambda = \lambda_3$, i.e., the root $\lambda = \lambda_3$ of the characteristic equation would have multiplicity greater than 1, contrary to the assumption.

So there are only two opposite unit vectors

$$\mathbf{e}'_3 = \{\cos\alpha_{13},\ \cos\alpha_{23},\ \cos\alpha_{33}\} \text{ and } \mathbf{e}''_3 = \{-\cos\alpha_{13},\ -\cos\alpha_{23},\ -\cos\alpha_{33}\},$$

whose coordinates satisfy the system (1.8). Choosing any of them

$$\mathbf{e}'_3 = \{\cos\alpha_{13},\ \cos\alpha_{23},\ \cos\alpha_{33}\},$$

positioning vectors

$$\mathbf{e}'_1 = \{\cos\alpha_{11},\ \cos\alpha_{21},\ \cos\alpha_{31}\} \text{ and } \mathbf{e}'_2 = \{\cos\alpha_{12},\ \cos\alpha_{22},\ \cos\alpha_{32}\}$$

perpendicular to the vector $\mathbf{e}'_3$ and to each other, we get that the form (1.4) takes the form

$$\varphi' = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + a'_{33}z'^2.$$

Now let's rotate the $Ox'y'z'$ axes around the Oz' axis; this rotation corresponds to the orthogonal transformation Ω_2

$$\begin{cases} x' = x''\cos\alpha - y''\sin\alpha, \\ y' = x''\sin\alpha + y''\cos\alpha, \\ z' = z'', \end{cases} \tag{1.9}$$

which can be chosen such that the form $a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2$ is transformed into the form $a''_{11}x''^2 + a''_{22}y''^2$.

Under orthogonal transformation $\Omega = \Omega_2\Omega_1$ the form φ is transformed into the form

$$\varphi'' = a''_{11}x''^2 + a''_{22}y''^2 + a''_{33}z''^2. \tag{1.10}$$

Theorem proven.

Note (and this is very important for what follows) that

$$\begin{aligned} a''_{33} = a'_{33} = (a_{11}\cos\alpha_{13} + a_{12}\cos\alpha_{23} + a_{13}\cos\alpha_{33})\cos\alpha_{13} + \\ + (a_{21}\cos\alpha_{13} + a_{22}\cos\alpha_{23} + a_{23}\cos\alpha_{33})\cos\alpha_{23} + \\ + (a_{31}\cos\alpha_{13} + a_{32}\cos\alpha_{23} + a_{33}\cos\alpha_{33})\cos\alpha_{33} = \\ = \lambda_3\cos^2\alpha_{13} + \lambda_3\cos^2\alpha_{23} + \lambda_3\cos^2\alpha_{33} = \lambda_3. \end{aligned}$$

Since the form φ'' does not contain mixed terms $x''y''$ and $x''z''$, then $a''_{11} = \lambda_1$, where λ_1 is also a root of the same characteristic equation (1.7) different from λ_3, and similarly $a''_{22} = \lambda_2$, where λ_2 is the root of the characteristic equation (1.7).

Thus, the canonical form of φ is

$$\varphi'' = \lambda_1 x''^2 + \lambda_2 y''^2 + \lambda_3 z''^2,$$

where λ_1, λ_2, λ_3 are the roots of the characteristic equation of the form φ. It also follows from what has been proved that:

1. all roots of the characteristic equation are real;

2. if all the roots λ_1, λ_2, λ_3 are simple, then by substituting into the system (1.6 λ_1 instead of λ, and replacing α_{13}, α_{23}, α_{33} on α_{11}, α_{21}, α_{31} we get a system from which one can find only two (mutually opposite) unit vectors of the new axis Ox', and for $\lambda = \lambda_2$ only two (also mutually opposite) unit vectors of the new axis Oy';

3. if $\lambda_1 = \lambda_2 \neq \lambda_3$, then the form φ'' is

   $$\varphi'' = \lambda_1 x''^2 + \lambda_1 y''^2 + \lambda_3 z''^2$$

   and under any orthogonal transformation of Ω_2 it will not change this form, since it follows from the formulas (1.9) that $x'^2 + y'^2 = x''^2 + y''^2$. Thus, already under the orthogonal transformation of Ω_1, the form φ will pass into the form

   $$\varphi' = \lambda_1 x'^2 + \lambda_1 y'^2 + \lambda_3 z'^2.$$

   Since the coordinates of the vector $\mathbf{e}'_1$ of the orthonormal basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$, in which the form φ does not contain terms $x'y'$ and $x'z'$, are always found from a system (1.6) in which λ must be equal to λ_1, then in case of $\lambda_1 = \lambda_2 \neq \lambda_3$ system (1.6) (in which λ is replaced by λ_1 and α_{13}, $alpha_{23}$, α_{33} on α_{11}, α_{21}, α_{31}) has two linearly independent solutions. As the vectors $\mathbf{e}'_1$ and $\mathbf{e}'_2$, as we have already indicated, in this case we can take any two vectors orthogonal to the vector $\mathbf{e}'_3$ and between yourself.

4. If, finally, $\lambda_1 = \lambda_2 = \lambda_3$, then the form φ' has the form

   $$\varphi' = \lambda_1 (x'^2 + y'^2 + z'^2).$$

   But from the relations (1.3) it follows that for any homogeneous orthogonal transformation

   $$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2,$$

hence the form φ from the very beginning has the canonical form

$$\varphi = \lambda_1(x^2 + y^2 + z^2).$$

Theorem 2. *General equation*

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0 \quad (1.11)$$

of second-order surface, given with respect to a general Cartesian coordinate system, by transforming the coordinate system to a rectangular system, can be reduced to one of the following five equations:

$$\begin{array}{lrl}
\text{(I)} & \lambda_1X^2 + \lambda_2Y^2 + \lambda_3Z^2 + D = 0, & \textit{where } \lambda_1 \neq 0,\ \lambda_2 \neq 0,\ \lambda_3 \neq 0, \\
\text{(II)} & \lambda_1X^2 + \lambda_2Y^2 + 2a_3'Z = 0, & \textit{where } \lambda_1 \neq 0,\ \lambda_2 \neq 0,\ a_3' \neq 0, \\
\text{(III)} & \lambda_1X^2 + \lambda_2Y^2 + D = 0, & \textit{where } \lambda_1 \neq 0,\ \lambda_2 \neq 0, \\
\text{(IV)} & \lambda_1X^2 + 2a_2'Y = 0, & \textit{where } \lambda_1 \neq 0,\ a_2' \neq 0, \\
\text{(V)} & \lambda_1X^2 + D = 0, & \textit{where } \lambda_1 \neq 0.
\end{array}$$

Proof. Let us first pass from the general Cartesian coordinate system to the Cartesian rectangular one. Since the order of the surface does not change during the transformation of the Cartesian coordinate system, the equation (1.11) will turn into an equation of the same form. Therefore, from the very beginning we assume that the equation (1.11) is given with respect to the Cartesian rectangular coordinate system $Oxyz$.

Theorem 1 of this section proves that it is possible to switch from the rectangular coordinate system $Oxyz$ to another rectangular system $Ox'y'z'$, such that in the system $Ox'y'z'$ the equation (1.11) is converted to the following:

$$\lambda_1x'^2 + \lambda_2y'^2 + \lambda_3z'^2 + 2a_1'x' + 2a_2'y' + 2a_3'z' + a_0 = 0, \tag{1.12}$$

where λ_1, λ_2, λ_3 are the roots of characteristic equation

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0. \tag{1.13}$$

The coordinates $\cos\alpha_{ij}$ of unit vectors $\mathbf{e}_1'$, $\mathbf{e}_2'$, $\mathbf{e}_3'$ of new axes Ox', Oy', Oz'

are found respectively from the systems:

$$\begin{cases} (a_{11}-\lambda_j)\cos\alpha_{1j}+a_{12}\cos\alpha_{2j}+a_{13}\cos\alpha_{3j}=0, \\ a_{21}\cos\alpha_{1j}+(a_{22}-\lambda_j)\cos\alpha_{2j}+a_{23}\cos\alpha_{3j}=0, \\ a_{31}\cos\alpha_{1j}+a_{32}\cos\alpha_{2j}+(a_{33}-\lambda_j)\cos\alpha_{3j}=0, \end{cases} \quad j=1,\,2,\,3. \tag{1.14}$$

If $\lambda_1\neq\lambda_2$, $\lambda_2\neq\lambda_3$, $\lambda_3\neq\lambda_1$ then we get the only three new axes Ox', Oy', Oz' (and positive direction of them). If $\lambda_1=\lambda_2\neq\lambda_3$, then from the system (1.14) with $j=3$ we find the only axis Oz' (and it's positive direction) , while the Ox' and Oy' axes are perpendicular to the Oz' axis and to each other. Finally, if $\lambda_1=\lambda_2=\lambda_3$, then already the initial equation (1.11) does not contain terms with xy, xz and yz and the quadratic form φ on the left side of the equation (1.11) has the form $\lambda(x^2+y^2+z^2)$.

Case I. Suppose that in the equation (1.12)

$$\lambda_1\neq 0,\quad \lambda_2\neq 0,\quad \lambda_3\neq 0.$$

Then the equation (1.12) can be rewritten as

$$\lambda_1\left(x'+\frac{a_1'}{\lambda_1}\right)^2+\lambda_2\left(y'+\frac{a_2'}{\lambda_2}\right)^2+\lambda_3\left(z'+\frac{a_3'}{\lambda_3}\right)^2+D=0, \tag{1.15}$$

where

$$D=a_0-\frac{a'^2_1}{\lambda_1}-\frac{a'^2_2}{\lambda_2}-\frac{a'^2_3}{\lambda_3}.$$

Carrying out the translation of the system $Ox'y'z'$ so that the new origin is the point

$$O'\left(-\frac{a'_1}{\lambda_1},\,-\frac{a'_2}{\lambda_2},\,-\frac{a'_3}{\lambda_3}\right),$$

where the coordinates of the point O' are given with respect to the system $Ox'y'z'$, and denoting the coordinates of the points in the new system by X, Y, Z, we have

$$X=x'+\frac{a'_1}{\lambda_1},\quad Y=y'+\frac{a'_2}{\lambda_2},\quad Z=z'+\frac{a'_3}{\lambda_3},$$

and the equation (1.15) becomes

$$\boxed{\lambda_1X^2+\lambda_2Y^2+\lambda_3Z^2+D=0.} \tag{1.16}$$

Case II. Let suppose that in the equation (1.12)

$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \lambda_3 = 0, \quad a'_3 \neq 0.$$

Then the equation (1.12) is transformed as follows:

$$\lambda_1 \left(x' + \frac{a'_1}{\lambda_1} \right)^2 + \lambda_2 \left(y' + \frac{a'_2}{\lambda_2} \right)^2 + 2a'_3 \left(z' + \frac{D}{2a'_3} \right) = 0,$$

where

$$D = a_0 - \frac{a'^2_1}{\lambda_1} - \frac{a'^2_2}{\lambda_2}.$$

Translating the system $Ox'y'z'$ so that the point

$$O' \left(-\frac{a'_1}{\lambda_1}, -\frac{a'_2}{\lambda_2}, -\frac{D}{2a'_3} \right)$$

becomes the new origin (the coordinates of the point O' are given with respect to the system $Ox'y'z'$), i.e., assuming

$$X = x' + \frac{a'_1}{\lambda_1}, \quad Y = y' + \frac{a'_2}{\lambda_2}, \quad Z = z' + \frac{D}{2a'_3},$$

we reduce the equation (1.12) to the form

$$\boxed{\lambda_1 X^2 + \lambda_2 Y^2 + 2a'_3 Z = 0.} \tag{1.17}$$

Case III. Let in the equation (1.12)

$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \lambda_3 = 0, \quad a'_3 = 0.$$

Then the equation (1.12) has the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + 2a'_1 x' + 2a'_2 y' + a_0 = 0$$

or

$$\lambda_1 \left(x' + \frac{a'_1}{\lambda_1} \right)^2 + \lambda_2 \left(y' + \frac{a'_2}{\lambda_2} \right)^2 + D = 0, \tag{1.18}$$

where

$$D = a_0 - \frac{a'^2_1}{\lambda_1} - \frac{a'^2_2}{\lambda_2}.$$

Translating the coordinate system $Ox'y'z'$ so that the new origin is the point

$$O'\left(-\frac{a'_1}{\lambda_1}, -\frac{a'_2}{\lambda_2}, 0\right),$$

i.e. assuming

$$X = x' + \frac{a'_1}{\lambda_1}, \quad Y = y' + \frac{a'_2}{\lambda_2}, \quad Z = z',$$

we reduce the equation (1.18) to the form

$$\boxed{\lambda_1 X^2 + \lambda_2 Y^2 + D = 0.} \tag{1.19}$$

Case IV. In the equation (1.12) $\lambda_2 = \lambda_3 = 0$, but at least one of the coefficients a_2' or a_3' is non-zero .

If $\lambda_2 = \lambda_3 = 0$, $a_2' \neq 0$, $a_3' = 0$ or $\lambda_2 = \lambda_3 = 0$, $a_2' = 0$, $a_3' \neq 0$, then only by translating the axes, the equation (1.12) can be transformed into

$$\lambda_1 X^2 + 2a_2' Y = 0$$

or (if $a_2' = 0$, $a_3' \neq 0$) to the form

$$\lambda_1 X^2 + 2a_3' Z = 0.$$

If $a_2' \neq 0$ and $a_3' \neq 0$, then by translating the axes $Ox'y'z'$ we get rid of the first power of x' in the equation (1.12) and in some system $O'x''y''z''$ it will take the form

$$\lambda_1 x''^2 + 2a_2' y'' + 2a_2' z'' = 0.$$

Then rotating the axes $O'x''y''z''$ around the new axis $O'x''$

$$X = x'', \quad Y = \frac{a_2' y'' + a_3' z''}{\sqrt{a'^2_2 + a'^2_3}}, \quad Z = \frac{a_3' y'' - a_2' z''}{\sqrt{a'^2_2 + a'^2_3}},$$

we reduce the last equation to the form

$$\lambda_1 X^2 + 2\sqrt{a'^2_2 + a'^2_3}\, Y = 0$$

or, denoting half the coefficient of Y as before by a_2'':

$$\boxed{\lambda_1 X^2 + 2a_2'' Y = 0.} \tag{1.20}$$

Case V. $\lambda_2 = \lambda_3 = a_2 = a_3 = 0$,
The equation (1.12) is

$$\lambda_1 x'^2 + 2a_1' x' + a_0 = 0$$

and by translating the coordinate axes is reduced to the form

$$\boxed{\lambda_1 X^2 + D = 0,} \tag{1.21}$$

where

$$D = a_0 - \frac{a_1'^2}{\lambda_1}.$$

Theorem 3. *General equation*

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0$$

of second-order surface, defined with respect to a general Cartesian coordinate system, expresses one of the following seventeen surfaces.

1°. *ellipsoid* $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1;$

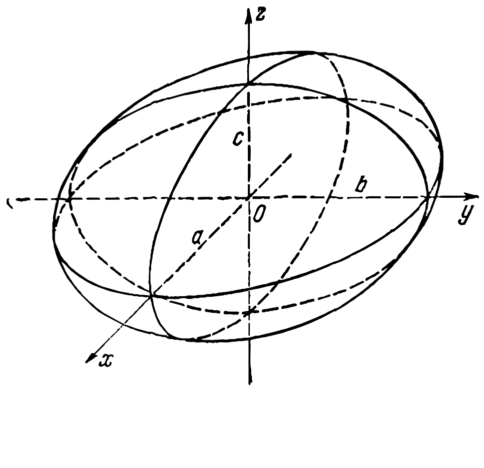

Figure 1. Ellipsoid.

2°. *imaginary ellipsoid* $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = -1;$

3°. *one-sheet hyperboloid* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1;$

4°. *two-sheet hyperboloid* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1;$

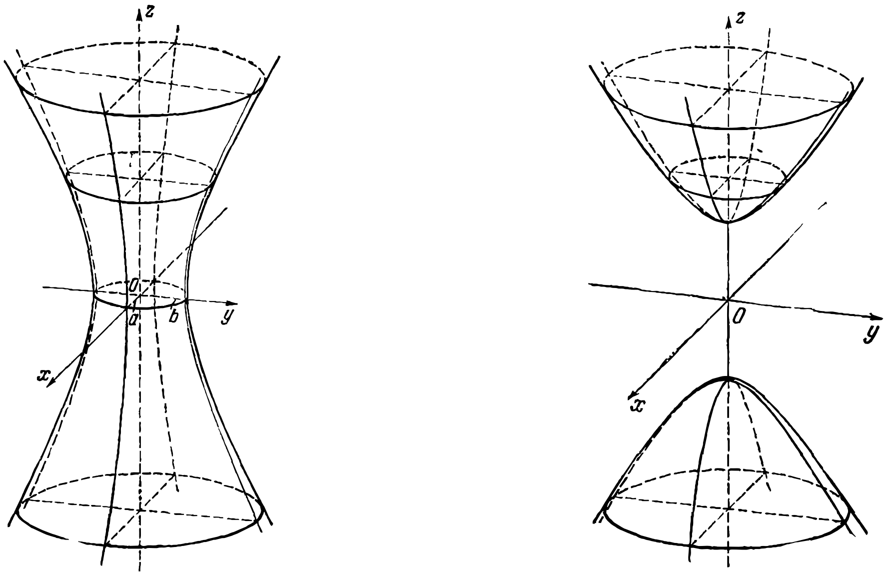

Figure 2. One- and two-sheet hyperboloids.

5°. *cone* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0;$

6°. *imaginary cone* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 0;$

7°. *elliptic paraboloid* $\dfrac{x^2}{p} + \dfrac{y^2}{q} = 2z,\ p > 0,\ q > 0;$

8°. *hyperbolic paraboloid* $\dfrac{x^2}{p} - \dfrac{y^2}{q} = 2z,\ p > 0,\ q > 0;$

9°. *elliptical cylinder* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1;$

10°. *imaginary elliptical cylinder* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1;$

11°. *pair of imaginary intersecting planes* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0;$

12°. *hyperbolic cylinder* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1;$

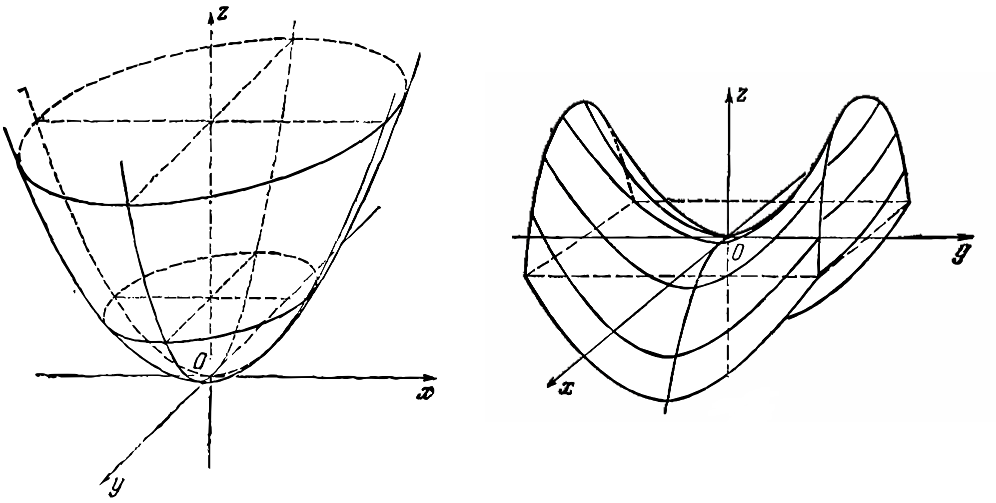

Figure 3. Elliptic and hyperbolic paraboloids.

13°. *pair of intersecting planes* $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$;

14°. *parabolic cylinder* $y^2 = 2px,\ p > 0$;

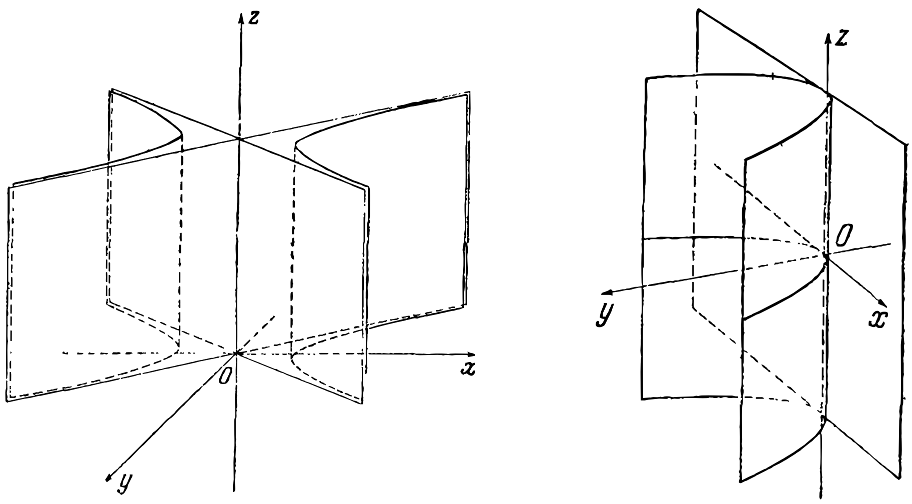

Figure 4. Hyperbolic and parabolic cylinders.

15°. *pair of parallel planes* $x^2 = a^2,\ a \neq 0$;

16°. *pair of imaginary parallel planes* $x^2 = -a^2,\ a \neq 0$;

17°. *pair of coincident planes* $x^2 = 0$.

Proof. According to the two previous theorems, we need to consider the following cases[1].

Case I.

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + D = 0,$$
$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \lambda_3 \neq 0.$$

1°. λ_1, λ_2, λ_3 have the same sign, D has the opposite sign. In this case, the equation (1.16) can be rewritten like this:

$$\frac{x^2}{-\dfrac{D}{\lambda_1}} + \frac{y^2}{-\dfrac{D}{\lambda_2}} + \frac{z^2}{-\dfrac{D}{\lambda_3}} = 1.$$

and since $-\dfrac{D}{\lambda_1} > 0$, $-\dfrac{D}{\lambda_2} > 0$, $-\dfrac{D}{\lambda_3} > 0$, then we can put

$$-\frac{D}{\lambda_1} = a^2, \quad -\frac{D}{\lambda_2} = b^2, \quad -\frac{D}{\lambda_3} = c^2,$$

and get the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{1.22}$$

2°. λ_1, λ_2, λ_3 and D have the same sign. In this case, we rewrite the equation (1.16) as

$$\frac{x^2}{\dfrac{D}{\lambda_1}} + \frac{y^2}{\dfrac{D}{\lambda_2}} + \frac{z^2}{\dfrac{D}{\lambda_3}} = -1.$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1, \tag{1.23}$$

where

$$a^2 = \frac{D}{\lambda_1}, \quad b^2 = \frac{D}{\lambda_2}, \quad c^2 = \frac{D}{\lambda_3}.$$

3°. λ_1 and λ_2 have the same sign, while λ_3 and D have the opposite sign. Let's rewrite the equation (1.16) as

$$\frac{x^2}{-\dfrac{D}{\lambda_1}} + \frac{y^2}{-\dfrac{D}{\lambda_2}} - \frac{z^2}{\dfrac{D}{\lambda_3}} = 1.$$

[1] In the reduced equations of second-order surfaces with respect to the Cartesian rectangular coordinate system, we will again denote the coordinates by the letters x, y, z.

Since $-\dfrac{D}{\lambda_1} > 0$, $-\dfrac{D}{\lambda_2} > 0$, $\dfrac{D}{\lambda_3} > 0$, then we can put

$$-\frac{D}{\lambda_1} = a^2, \quad -\frac{D}{\lambda_2} = b^2, \quad \frac{D}{\lambda_3} = c^2$$

and obtain the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \tag{1.24}$$

4°. λ_1, λ_2 and D have the same sign, λ_3 have the opposite sign. Let's rewrite the equation (1.16) as

$$\frac{x^2}{\dfrac{D}{\lambda_1}} + \frac{y^2}{\dfrac{D}{\lambda_2}} - \frac{z^2}{-\dfrac{D}{\lambda_3}} = -1.$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \tag{1.25}$$

where

$$a^2 = \frac{D}{\lambda_1}, \quad b^2 = \frac{D}{\lambda_2}, \quad c^2 = -\frac{D}{\lambda_3}.$$

5°. λ_1 and λ_2 have the same sign, λ_3 has the opposite sign, $D = 0$. In this case, the equation (1.16) can be rewritten like this:

$$|\lambda_1|x^2 + |\lambda_2|y^2 - |\lambda_3|z^2 = 0$$

or

$$\frac{x^2}{\dfrac{1}{|\lambda_1|}} + \frac{y^2}{\dfrac{1}{|\lambda_2|}} - \frac{z^2}{\dfrac{1}{|\lambda_3|}} = 0,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0, \tag{1.26}$$

where

$$a^2 = \frac{1}{|\lambda_1|}, \quad b^2 = \frac{1}{|\lambda_2|}, \quad c^2 = \frac{1}{|\lambda_3|}.$$

6°. λ_1, λ_2, λ_3 have the same sign, and $D = 0$. In this case, the equation (1.16) can be rewritten like this:

$$|\lambda_1|x^2 + |\lambda_2|y^2 + |\lambda_3|z^2 = 0$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0. \tag{1.27}$$

Case II.

$$\lambda_1 x^2 + \lambda_2 y^2 + 2a_3' z = 0,$$
$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad a_3' \neq 0.$$

7°. λ_1 and λ_2 are like-sign.

By choosing the positive direction of the Oz axis, we can obtain that the coefficient at z in the equation (1.17) will have the opposite sign of λ_1 and λ_2. In this case, the equation (1.17) can be rewritten like this:

$$\frac{x^2}{-\dfrac{a_3'}{\lambda_1}} + \frac{y^2}{-\dfrac{a_3'}{\lambda_2}} = 2z,$$

or, assuming

$$-\frac{a_3'}{\lambda_1} = p, \quad -\frac{a_3'}{\lambda_2} = q$$

($p > 0$ and $q > 0$, since a_3' has the opposite sign of λ_1 and λ_2), we have:

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z. \tag{1.28}$$

8°. λ_1 and λ_2 have different signs.

The positive direction of the Oz axis can always be chosen so that the sign of a_3' is opposite to the sign of λ_1. Rewriting then the equation (1.17) as

$$\frac{x^2}{-\dfrac{a_3'}{\lambda_1}} - \frac{y^2}{\dfrac{a_3'}{\lambda_2}} = 2z,$$

and noticing that

$$-\frac{a_3'}{\lambda_1} = p > 0, \quad \frac{a_3'}{\lambda_2} = q > 0,$$

we get the equation:

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z. \tag{1.29}$$

Cases III, IV, V can be investigated in a similar way, leading to equations 9°-17°.

2 Invariant theory

In this section, we will use the theorem formulated earlier on the transformation of the quadratic part of a function, but for the case of three and four variables.

Namely, we perform the linear inhomogeneous transformation:

$$\begin{aligned} x &= c_{11}x' + c_{12}y' + c_{13}z' + c_1, \\ y &= c_{21}x' + c_{22}y' + c_{23}z' + c_2, \\ z &= c_{31}x' + c_{32}y' + c_{33}z' + c_3 \end{aligned} \tag{2.1}$$

on the second-order function F

$$\begin{aligned} F = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0. \end{aligned} \tag{2.2}$$

Let

$$\begin{aligned} F' = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_{13}x'z' + 2a'_{23}y'z' + a'_{33}z'^2 + \\ + 2a'_1x' + 2a'_2y' + 2a'_3z' + a'_0 = 0 \end{aligned} \tag{2.3}$$

is the function into which the function F is transformed. Then we have the relations

$$\begin{pmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{pmatrix} = \\ = \begin{pmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \tag{2.4}$$

and

$$\begin{pmatrix} a'_{11} & a'_{12} & a'_{13} & a'_1 \\ a'_{21} & a'_{22} & a'_{23} & a'_2 \\ a'_{31} & a'_{32} & a'_{33} & a'_3 \\ a'_1 & a'_2 & a'_3 & a'_0 \end{pmatrix} =$$

$$= \begin{pmatrix} c_{11} & c_{21} & c_{31} & 0 \\ c_{12} & c_{22} & c_{32} & 0 \\ c_{13} & c_{23} & c_{33} & 0 \\ c_1 & c_2 & c_3 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_1 \\ a_{21} & a_{22} & a_{23} & a_2 \\ a_{31} & a_{32} & a_{33} & a_3 \\ a_1 & a_2 & a_3 & a_0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_1 \\ c_{21} & c_{22} & c_{23} & c_2 \\ c_{31} & c_{32} & c_{33} & c_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.5)$$

Indeed, the quadratic form included in the function F is transformed into the quadratic form included in the function F' under the *homogeneous* transformation

$$\begin{aligned} x &= c_{11}x' + c_{12}y' + c_{13}z', \\ y &= c_{21}x' + c_{22}y' + c_{23}z', \\ z &= c_{31}x' + c_{32}y' + c_{33}z'. \end{aligned} \quad (2.6)$$

From here the formula (2.4) follows.

Further, the function F can be obtained from the quadratic form:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1xt + 2a_2yt + 2a_3zt + a_0t^2$$

for $t = 1$, and the inhomogeneous transformation (2.1) is obtained from the homogeneous one:

$$\begin{aligned} x &= c_{11}x' + c_{12}y' + c_{13}z' + c_1t', \\ y &= c_{21}x' + c_{22}y' + c_{23}z' + c_2t', \\ z &= c_{31}x' + c_{32}y' + c_{33}z' + c_3t', \\ t &= t' \end{aligned}$$

for $t' = 1$.

From these considerations, the formula (2.5) is obtained. From the relations (2.4) and (2.5) it follows that under a linear transformation (2.1) on the variables x, y, z of the function F, at which it transforms into the function F', the following relations hold:

$$\begin{vmatrix} a'_{11} & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}^2, \quad (2.7)$$

$$\begin{vmatrix} a'_{11} & a'_{12} & a'_{13} & a'_1 \\ a'_{21} & a'_{22} & a'_{23} & a'_2 \\ a'_{31} & a'_{32} & a'_{33} & a'_3 \\ a'_1 & a'_2 & a'_3 & a'_0 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_1 \\ a_{21} & a_{22} & a_{23} & a_2 \\ a_{31} & a_{32} & a_{33} & a_3 \\ a_1 & a_2 & a_3 & a_0 \end{vmatrix} \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix}^2 . \tag{2.8}$$

Definition. *A polynomial function of the coefficients of a second-order surface is called the* ***orthogonal invariant*** *of this polynomial under an orthogonal transformation if it conserves its value under inhomogeneous orthogonal transformations of the variables.*

Theorem 1. *Functions*[a]

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \tag{2.9}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_1 \\ a_{21} & a_{22} & a_{23} & a_2 \\ a_{31} & a_{32} & a_{33} & a_3 \\ a_1 & a_2 & a_3 & a_0 \end{vmatrix}, \tag{2.10}$$

$$K_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \tag{2.11}$$

$$S = a_{11} + a_{22} + a_{33} \tag{2.12}$$

are orthogonal invariants of the second-order function:

$$F = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0.$$

[a] δ is called the discriminant of the form φ.

Proof. Since the determinant of an orthogonal transformation is equal to ± 1, its square is equal to 1 and the invariance of δ and Δ follows from the relations (2.7) and (2.8).

To prove that K_1 and S are also orthogonal invariants, we first note that the coefficients a_{11}, a_{22}, a_{33}, a_{12} , a_{13}, a_{23} are invariants of translation:

$$\begin{aligned} x &= x' + c_1, \\ y &= y' + c_2, \\ z &= z' + c_3. \end{aligned} \tag{2.13}$$

This is proved in the same way as for second-order curves. Therefore, it suffices to prove that K_1 and S are invariants of a homogeneous orthogonal transformation:

$$\begin{aligned} x &= c_{11}x' + c_{12}y' + c_{13}z', \\ y &= c_{21}x' + c_{22}y' + c_{23}z', \\ z &= c_{31}x' + c_{32}y' + c_{33}z'. \end{aligned} \tag{2.14}$$

This transformation satisfies the relation

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$

(this is proved in the same way as the corresponding equality $x^2 + y^2 = x'^2 + y'^2$ was proved in section of semi-invariant theorem of the second-order curve).

Let consider the auxiliary quadratic form

$$\begin{aligned} \psi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 - \\ - \lambda(x^2 + y^2 + z^2). \end{aligned} \tag{2.15}$$

With orthogonal transformation (2.14), it will take the form

$$\begin{aligned} \psi' = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_{13}x'z' + 2a'_{23}y'z' + a'_{33}z'^2 - \\ - \lambda(x'^2 + y'^2 + z'^2). \end{aligned} \tag{2.16}$$

By what has been proved, the discriminant of a quadratic form is an orthogonal invariant, which means that

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = \begin{vmatrix} a'_{11} - \lambda & a'_{12} & a'_{13} \\ a'_{21} & a'_{22} - \lambda & a'_{23} \\ a'_{31} & a'_{32} & a'_{33} - \lambda \end{vmatrix}.$$

This equality is true for all values of λ, therefore, the corresponding coefficients for λ^2 and λ are equal on the left and right sides, i.e.

$$a_{11} + a_{22} + a_{33} = a'_{11} + a'_{22} + a'_{33},$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{vmatrix} + \begin{vmatrix} a'_{11} & a'_{13} \\ a'_{31} & a'_{33} \end{vmatrix} + \begin{vmatrix} a'_{22} & a'_{23} \\ a'_{32} & a'_{33} \end{vmatrix}.$$

Theorem 2. *Functions*

$$K_2 = \begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{33} & a_3 \\ a_3 & a_0 \end{vmatrix}, \tag{2.17}$$

$$K_3 = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} & a_1 \\ a_{31} & a_{33} & a_3 \\ a_1 & a_3 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} & a_2 \\ a_{32} & a_{33} & a_3 \\ a_3 & a_3 & a_0 \end{vmatrix} \tag{2.18}$$

are invariants of the homogeneous orthogonal transformation. These functions K_2 and K_3 are called "semi-invariants".
If the function

$$F = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0$$

by the homogeneous orthogonal transformation can be reduced to the form

$$F' = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_1x' + 2a'_2y' + a_0, \tag{2.19}$$

then K_3 is an orthogonal invariant, and if F can be reduced to the form

$$F' = a'_{11}x'^2 + 2a'_1x' + a_0, \tag{2.20}$$

then K_2 (and K_3) is an orthogonal invariant.

Proof. Let consider the auxiliary function

$$\Phi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 - \lambda(x^2 + y^2 + z^2).$$

Performing a homogeneous orthogonal transformation (2.14), we obtain the function

$$\Phi' = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_{13}x'z' + 2a'_{23}y'z' + a'_{33}z'^2 + \\ + 2a'_1x' + 2a'_2y' + 2a'_3z' + a'_0 - \lambda(x'^2 + y'^2 + z'^2).$$

where $a'_0 = a_0$.

As proved, Δ is an orthogonal invariant. Using this with respect to the function Φ, we get

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & a_1 \\ a_{21} & a_{22}-\lambda & a_{23} & a_2 \\ a_{31} & a_{32} & a_{33}-\lambda & a_3 \\ a_1 & a_2 & a_3 & a_0 \end{vmatrix} = \begin{vmatrix} a'_{11}-\lambda & a'_{12} & a'_{13} & a'_1 \\ a'_{21} & a'_{22}-\lambda & a'_{23} & a'_2 \\ a'_{31} & a'_{32} & a'_{33}-\lambda & a'_3 \\ a'_1 & a'_2 & a'_3 & a'_0 \end{vmatrix}$$

(identity with respect to λ). Equating the coefficients of λ and λ^2 on the left and right sides, we get

$$\begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{33} & a_3 \\ a_3 & a_0 \end{vmatrix} = \begin{vmatrix} a'_{11} & a'_1 \\ a'_1 & a'_0 \end{vmatrix} + \begin{vmatrix} a'_{22} & a'_2 \\ a'_2 & a'_0 \end{vmatrix} + \begin{vmatrix} a'_{33} & a'_3 \\ a'_3 & a'_0 \end{vmatrix},$$

$$\begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} & a_1 \\ a_{31} & a_{33} & a_3 \\ a_1 & a_3 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} & a_2 \\ a_{32} & a_{33} & a_3 \\ a_3 & a_3 & a_0 \end{vmatrix} =$$
$$= \begin{vmatrix} a'_{11} & a'_{12} & a'_1 \\ a'_{21} & a'_{22} & a'_2 \\ a'_1 & a'_2 & a'_0 \end{vmatrix} + \begin{vmatrix} a'_{11} & a'_{13} & a'_1 \\ a'_{31} & a'_{33} & a'_3 \\ a'_1 & a'_3 & a'_0 \end{vmatrix} + \begin{vmatrix} a'_{22} & a'_{23} & a'_2 \\ a'_{32} & a'_{33} & a'_3 \\ a'_3 & a'_3 & a'_0 \end{vmatrix}.$$

Let suppose now that there is such a homogeneous orthogonal transformation ω_1 that transforms the function F into the function (2.19). For the function (2.19), the semi-invariant K_3 has the value

$$K_3 = K'_3 = \begin{vmatrix} a'_{11} & a'_{12} & a'_1 \\ a'_{21} & a'_{22} & a'_2 \\ a'_1 & a'_2 & a'_0 \end{vmatrix}, \tag{2.21}$$

equal to its value calculated by the formula (2.17). Determinant

$$\begin{vmatrix} a'_{11} & a'_{12} & a'_1 \\ a'_{21} & a'_{22} & a'_2 \\ a'_1 & a'_2 & a'_0 \end{vmatrix}$$

does not change if we perform the transformation ω_2 on the function (2.19) as follows:

$$x' = x'' + c_1,$$
$$y' = y'' + c_2$$

(this comes from the fact that $c_{11} = c_{22} = 1$, $c_{12} = c_{21} = 0$).

Let ω be an arbitrary orthogonal transformation. Consider the orthogonal transformation $\omega' = \omega\omega_1^{-1}$; then $\omega = \omega'\omega_1$. Next, we represent the orthogonal transformation ω' as the product of the homogeneous orthogonal transformation ω_3 and the translation ω_2; then

$$\omega = \omega_3\omega_2\omega_1.$$

After a homogeneous orthogonal transformation ω_1, the function F will transform into the function (2.19) and, according to what has been proved, K_3 will not change and will be equal to the value calculated by the formula (2.21).

With the translation ω_2, the function F' will transform into the function

$$F'' = a''_{11}x''^2 + 2a''_{12}x''y'' + a''_{22}y''^2 + 2a''_1x'' + 2a''_2y'' + a''_0$$

and according to what has been proved

$$K_3 = K'_3 = K''_3 = \begin{vmatrix} a''_{11} & a''_{12} & a''_1 \\ a''_{21} & a''_{22} & a''_2 \\ a''_1 & a''_2 & a''_0 \end{vmatrix},$$

finally, after a homogeneous orthogonal transformation ω_3, the function F'' becomes the function

$$F''' = a'''_{11}x'''^2 + 2a'''_{12}x'''y''' + a'''_{22}y'''^2 + a'''_{33}z'''^2 + \dots$$

and hence

$$K_3 = K'_3 = K''_3 = K'''_3 = \begin{vmatrix} a'''_{11} & a'''_{12} & a'''_1 \\ a'''_{21} & a'''_{22} & a'''_2 \\ a'''_1 & a'''_2 & a'''_0 \end{vmatrix} + \begin{vmatrix} a'''_{11} & a'''_{13} & a'''_1 \\ a'''_{31} & a'''_{33} & a'''_3 \\ a'''_1 & a'''_3 & a'''_0 \end{vmatrix} + \begin{vmatrix} a'''_{22} & a'''_{23} & a'''_2 \\ a'''_{32} & a'''_{33} & a'''_3 \\ a'''_2 & a'''_3 & a'''_0 \end{vmatrix}.$$

It can be proved similarly that K_2 is an orthogonal invariant if the function F can be reduced by a homogeneous orthogonal transformation to the form

$$F' = a'_{11}x'^2 + 2a'_2x' + a'.$$

3 Definition of the canonical equation of a second-order surface using an invariants

The Table 1 indicates the necessary and sufficient attributes that a second-order surface is a surface of type I, II, III, IV or V:

Table 1

Type number	Type attribute
I	$\delta \neq 0$
II	$\delta = 0,\ \Delta \neq 0$
III	$\delta = 0,\ \Delta = 0,\ K_1 \neq 0$
IV	$\delta = 0,\ \Delta = 0,\ K_1 = 0,\ K_3 \neq 0$
V	$\delta = 0,\ \Delta = 0,\ K_1 = 0,\ K_3 = 0,\ S \neq 0$

Proof. If a second-order surface is given by a general equation with respect to a Cartesian rectangular coordinate system, then, as was also shown in section 1, when the given Cartesian rectangular coordinate system $Oxyz$ is transformed into another Cartesian rectangular system $Ox'y'z'$ the equation of surface can be reduced to the form

$$\lambda_1 {x'}^2 + \lambda_2 {y'}^2 + \lambda_3 {z'}^2 + 2a_1'x' + 2a_2'y' + 2a_3'z' + a_0 = 0,$$

where λ_1, λ_2, λ_3 are the roots of the characteristic equation.

If all the roots are different from zero, then the equation of the surface by translation of the axes can be reduced to the equation of the first group. If one of the roots of the characteristic equation, for example λ_3, is equal to zero, but $a_3' \neq 0$, then we can get the reduced equation of the second group.

If one of the roots is equal to 0, for example $\lambda_3 = 0$ and $a_3' = 0$, then we can get the reduced equation of the third group, etc.

1. Let a second-order surface be a surface of type I. Then, as was shown in section 1, the equation of this surface can be reduced to the form

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + D = 0,$$

where $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$. In this case

$$\delta = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} = \lambda_1\lambda_2\lambda_3 \neq 0.$$

2. Let a second-order surface be a surface of type II. Then (section 1) its equation can be reduced to the form

$$\lambda_1 X^2 + \lambda_2 Y^2 + 2a_3' Z = 0,$$

where λ_1 and λ_2 are non-zero roots of the characteristic equation ($\lambda_3 = 0$) and $a'_3 \neq 0$. We find

$$\delta = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & a'_3 \\ 0 & 0 & a'_3 & 0 \end{vmatrix} = -\lambda_1\lambda_2 a'^2_3 \neq 0.$$

3. Let a second-order surface be a surface of type III. Then (section 1) its equation can be reduced to the form

$$\lambda_1 X^2 + \lambda_2 Y^2 + D = 0,$$

where λ_1 and λ_2 are non-zero roots of the characteristic equation. From here we find

$$\delta = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D \end{vmatrix} = 0,$$

$$K_1 = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \lambda_2 & 0 \\ 0 & 0 \end{vmatrix} = \lambda_1\lambda_2 \neq 0.$$

4. Let a second-order surface be a surface of type IV. Then its equation can be reduced to the form

$$\lambda_1 X^2 + 2a'_2 Y^2 = 0,$$

where λ_1 is a non-zero root of the characteristic equation ($\lambda_2 = \lambda_3 = 0$), and $a'_2 \neq 0$. From here we find

$$\delta = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2' \\ 0 & 0 & 0 & 0 \\ 0 & a_2' & 0 & 0 \end{vmatrix} = 0,$$

$$K_1 = \begin{vmatrix} \lambda_1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

$$K_3 = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & a_2' \\ 0 & a_2' & 0 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_2' \\ 0 & 0 & 0 \\ a_2' & 0 & 0 \end{vmatrix} = -\lambda_1 a_2'^2 \neq 0.$$

5. Let, finally, a second-order surface be a surface of type V. Then its equation can be reduced to the form

$$\lambda_1 X^2 + D = 0,$$

where λ_1 is a non-zero root of the characteristic equation ($\lambda_2 = \lambda_3 = 0$). From here

$$\delta = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D \end{vmatrix} = 0,$$

$$K_1 = \begin{vmatrix} \lambda_1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

$$K_3 = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D \end{vmatrix} + \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D \end{vmatrix} + \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D \end{vmatrix} = 0,$$

$$S = \lambda_1 \neq 0.$$

The necessity of attributes is proved. Since these attributes are pairwise incompatible, they are sufficient.

Theorem 2.

I. *If a second-order surface given by a general equation with respect to a rectangular coordinate system is a I-st type surface, then its reduced equation has the form*

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + \frac{\Delta}{\delta} = 0, \tag{3.1}$$

where λ_1, λ_2, λ_3 *are the roots of the characteristic equation*

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix} = 0,$$

or

$$\lambda^3 - S\lambda^2 + K_1\lambda - \delta = 0.$$

II. *If a second-order surface is a II-nd type surface, then its reduced equation is*

$$\lambda_1 X^2 + \lambda_2 Y^2 \pm \sqrt{-\frac{\Delta}{K_1}} Z = 0, \tag{3.2}$$

where λ_1 *and* λ_2 *are non-zero roots of the characteristic equation.*

III. *If a second-order surface is a III-rd type surface, then its reduced equation is*

$$\lambda_1 X^2 + \lambda_2 Y^2 + \frac{K_3}{K_1} = 0, \tag{3.3}$$

where λ_1 *and* λ_2 *are non-zero roots of the characteristic equation.*

IV. *If a second-order surface is a IV-th type surface, then its reduced equation is*

$$SX^2 \pm \sqrt{-\frac{K_3}{S}} Y = 0. \tag{3.4}$$

V. *Finally, if a second-order surface is a V-th type surface, then its reduced equation has the form*[a]

$$SX^2 + \frac{K_2}{S} = 0. \tag{3.5}$$

[a] In cases IV and V, the invariant S is equal to a non-zero root of the characteristic equation.

Proof.

I. If a second-order surface is a surface of type I, then its canonical equation has the form

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + D = 0,$$

where

$$\lambda_1 \neq 0, \quad \lambda_2 \neq 0, \quad \lambda_3 \neq 0.$$

We find

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & D \end{vmatrix} = \lambda_1\lambda_2\lambda_3 D = \delta D,$$

hence $D = \dfrac{\Delta}{\delta}$.

II. If a second-order surface is a surface of type II, then its canonical equation has the form

$$\lambda_1 X^2 + \lambda_2 Y^2 + 2a_3' Z = 0,$$

where λ_1 and λ_2 are non-zero roots of the characteristic equation and $a_3' \neq 0$. From here we find

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & a_3' \\ 0 & 0 & a_3' & 0 \end{vmatrix} = -\lambda_1\lambda_2 {a_3'}^2 = -K_1 {a_3'}^2,$$

hence $a_3' = \pm\sqrt{-\dfrac{\Delta}{K_1}}$.

The other assertions of the theorem can be proved similarly.

Theorem 3. *The table 2 gives necessary and sufficient attributes for each of the seventeen classes of second-order surfaces.*

Proof of the necessity. In this section, it was proved that if, with respect to the Cartesian rectangular coordinate system $Oxyz$, a second-order surface is given by a general equation, then by transforming the given coordinate system $Oxyz$ into the Cartesian rectangular one $O'x'y'z'$ this equation can be transformed to one of the reduced equations (3.1)-(3.5). In all these equations λ_i are non-zero roots of the characteristic equation, and the invariants δ, Δ, etc. are calculated according to the formulas specified in the section 2.

Table 2

	Surface	Attribute
1	ellipsoid	$K_1 > 0$, $S\delta > 0$, $\Delta < 0$
2	imaginary ellipsoid	$K_1 > 0$, $S\delta > 0$, $\Delta > 0$
3	imaginary cone	$K_1 > 0$, $S\delta > 0$, $\Delta = 0$
4	one-sheet hyperboloid	$\delta \neq 0$, $\Delta > 0$ or $K_1 \leqslant 0$ or $S\delta \leqslant 0$
5	two-sheet hyperboloid	$\delta \neq 0$, $\Delta < 0$ or $K_1 \leqslant 0$ or $S\delta \leqslant 0$
6	second-order cone	$\delta \neq 0$, $\Delta = 0$ or $K_1 \leqslant 0$ or $S\delta \leqslant 0$
7	elliptic paraboloid	$\delta = 0$, $\Delta < 0$
8	hyperbolic paraboloid	$\delta = 0$, $\Delta > 0$
9	elliptical cylinder	$\delta = 0$, $\Delta = 0$, $K_1 > 0$, $SK_3 < 0$
10	imaginary elliptical cylinder	$\delta = 0$, $\Delta = 0$, $K_1 > 0$, $SK_3 > 0$
11	pair of imaginary intersecting planes	$\delta = 0$, $\Delta = 0$, $K_1 > 0$, $K_3 = 0$
12	hyperbolic cylinder	$\delta = 0$, $\Delta = 0$, $K_1 < 0$, $K_3 \neq 0$
13	pair of intersecting planes	$\delta = 0$, $\Delta = 0$, $K_1 < 0$, $K_3 = 0$
14	parabolic cylinder	$\delta = 0$, $\Delta = 0$, $K_1 = 0$, $K_3 \neq 0$
15	pair of parallel planes	$\delta = 0$, $\Delta = 0$, $K_1 = 0$, $K_3 = 0$, $K_2 < 0$
16	pair of imaginary parallel planes	$\delta = 0$, $\Delta = 0$, $K_1 = 0$, $K_3 = 0$, $K_2 > 0$
17	pair of coincident planes	$\delta = 0$, $\Delta = 0$, $K_1 = 0$, $K_3 = 0$, $K_2 = 0$

1°. If the equation (3.1) is an ellipsoid equation, then the roots λ_1, λ_2, and λ_3 have the same sign, and the value $\dfrac{\Delta}{\delta}$ has the opposite sign. But since $\delta = \lambda_1\lambda_2\lambda_3$, then $\Delta < 0$ and so on,

$$K_1 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 > 0, \quad S\delta = (\lambda_1 + \lambda_2 + \lambda_3)\lambda_1\lambda_2\lambda_3 > 0.$$

2°. If the equation (3.1) is the equation of an imaginary ellipsoid, then all values λ_1, λ_2, λ_3, $\dfrac{\Delta}{\delta}$ of the same sign: since $\delta = \lambda_1\lambda_2\lambda_3$, then $\Delta > 0$. The relations $K_1 > 0$ and $S\delta > 0$ are proved in the same way as in item 1.

3°. If the equation (3.1) is an imaginary cone equation, then λ_1, λ_2, λ_3 have the same sign, and $\dfrac{\Delta}{\delta} = 0$, whence $\Delta = 0$; the inequalities $K_1 > 0$, $S\delta > 0$ are obtained in the same way as in item 1.

4°. If the equation (3.1) is the equation of a one-sheet hyperboloid, then from the values λ_1, λ_2, λ_3, $\dfrac{\Delta}{\delta}$ two are positive and two are negative: if, for example, $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 < 0$, $\dfrac{\Delta}{\delta} < 0$, then $\delta < 0$, $\Delta > 0$ and if, for example,

$K_1 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 > 0$, then $S\delta = (\lambda_1 + \lambda_2 + \lambda_3)\lambda_1\lambda_2\lambda_3$ has the opposite sign to $\lambda_1 + \lambda_2 + \lambda_3$.

Let us prove that $\lambda_1 + \lambda_2 + \lambda_3 \geqslant 0$ (then $S\delta \leqslant 0$). Indeed, if we had $\lambda_1 + \lambda_2 + \lambda_3 < 0$, then

$$-\lambda_3 > \lambda_1 + \lambda_2,$$
$$-\lambda_3\lambda_1 > \lambda_1^2 + \lambda_1\lambda_2,$$
$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1^2 < 0,$$
$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 < 0$$

contrary to the assumption. The same result ($\delta \neq 0$, $\Delta > 0$, or $K_1 \leqslant 0$, or $S\delta \leqslant 0$) can be obtained by assuming that $\lambda_1 < 0$, $\lambda_2 < 0$, $\lambda_3 > 0$, $\dfrac{\Delta}{\delta} > 0$.

5°. If the equation (3.1) is the equation of a two-sheet hyperboloid, then two of the roots λ_1, λ_2, λ_3 have the same sign as $\dfrac{\Delta}{\delta}$, and the third root has the opposite sign.

Let, for example, $\lambda_1 > 0$, $\lambda_2 < 0$, $\dfrac{\Delta}{\delta} > 0$, $\lambda_3 < 0$. Then $\Delta < 0$, $\delta < 0$, and the fact that either $K_1 \leqslant 0$ or $S\delta \leqslant 0$ is proved in the same way as in item 4.

6°. If the equation (3.1) is a equation of second-order cone, then $\dfrac{\Delta}{\delta} = 0$ whence $\Delta = 0$, $\delta \neq 0$, and further, two of the roots λ_1, λ_2, λ_3 have the same sign, and the third has the opposite sign. Hence, as in item 4, it is proved that either $K_1 \leqslant 0$ or $S\delta \leqslant 0$.

Let us pass to the study of the second-order surfaces of type II.

7°. If the equation (3.2) is an equation of an elliptic paraboloid, then λ_1 and λ_2 in the equation (3.2) are numbers of the same sign, and, therefore, $K_1 = \lambda_1\lambda_2 > 0$ and from the equation (3.2) it follows that $\Delta < 0$ (value $-\dfrac{\Delta}{\delta}$ under the radical in the equation (3.2) is positive).

8°. If the equation (3.2) is an equation of a hyperbolic paraboloid, then the numbers λ_1 and λ_2 have different signs, so $K_1 = \lambda_1\lambda_2 < 0$, and from conditions $-\dfrac{\Delta}{\delta} > 0$ we find $\Delta > 0$.

The consideration of other cases is essentially similar to the derivation for curves of the second order.

The sufficiency of all attributes is proved by contradiction (these attributes mutually exclude each other).

4 Table of second-order surface classification

Equation:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + 2a_1x + 2a_2y + 2a_3z + a_0 = 0$$

Characteristic equation:

$$\lambda^3 - S\lambda^2 + K_1\lambda - \delta = 0$$

Invariants:

$$S = a_{11} + a_{22} + a_{33} \qquad K_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_1 \\ a_{21} & a_{22} & a_{23} & a_2 \\ a_{31} & a_{32} & a_{33} & a_3 \\ a_1 & a_2 & a_3 & a_0 \end{vmatrix}$$

Semi-invariants:

$$K_2 = \begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{33} & a_3 \\ a_3 & a_0 \end{vmatrix}$$

$$K_3 = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} & a_1 \\ a_{31} & a_{33} & a_3 \\ a_1 & a_3 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} & a_2 \\ a_{32} & a_{33} & a_3 \\ a_3 & a_3 & a_0 \end{vmatrix}$$

Locus of centers	Type attribute	Class	Surface	Reduced Equation	Canonical Equation
point	$\delta \neq 0$	1	ellipsoid	$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \dfrac{\Delta}{\delta} = 0$	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
		2	imaginary ellipsoid		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = -1$
		3	imaginary cone		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 0$
		4	one-sheet hyperboloid		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
		5	two-sheet hyperboloid		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$
		6	second-order cone		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0$
no center	$\delta = 0,\ \Delta \neq 0$	7	elliptic paraboloid	$\lambda_1 x^2 + \lambda_2 y^2 \pm \sqrt{-\dfrac{\Delta}{K_1}} z = 0$	$\dfrac{x^2}{p} + \dfrac{y^2}{q} = 2z,\ p > 0,\ q > 0$
		8	hyperbolic paraboloid		$\dfrac{x^2}{p} - \dfrac{y^2}{q} = 2z,\ p > 0,\ q > 0$
line	$\delta = 0,\ \Delta = 0,$ $K_1 \neq 0$	9	elliptical cylinder	$\lambda_1 x^2 + \lambda_2 y^2 + \dfrac{K_3}{K_1} = 0$	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
		10	imaginary elliptical cylinder		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = -1$
		11	pair of imaginary intersecting planes		$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 0$
		12	hyperbolic cylinder		$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$
		13	pair of intersecting planes		$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$
no center	$\delta = 0,\ \Delta = 0,$ $K_1 = 0,\ K_3 \neq 0$	14	parabolic cylinder	$Sx^2 \pm \sqrt{-\dfrac{K_3}{S}} y = 0$	$y^2 = 2px,\ p > 0$
plane	$\delta = 0,\ \Delta = 0,$ $K_1 = 0,\ K_3 = 0$	15	pair of parallel planes	$Sx^2 + \dfrac{K_2}{S} = 0$	$x^2 = a^2,\ a \neq 0$
		16	pair of imaginary parallel planes		$x^2 = -a^2,\ a \neq 0$
		17	pair of coincident planes		$x^2 = 0$