CA HW 8

1. $f(z) = \frac{\cos z}{z}, 0 < |z|$

2. $f(z) = e^{-1/z^2}, 0 < |z|$

3. $f(z) = \frac{e^z}{z-1}, 0 < |z-1|$

1. $f(z) = \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-1}}{(2n)!}$ valid for $|z|$

2. $f(z) = \sum_{n=0}^{\infty} \frac{(z^{-2})^n}{n!} = \sum_{n=0}^{\infty} \frac{1\,(-1)^n}{z^{2n}\,n!}$ valid for $|z| > 0$.

3. $f(z) = \frac{e \cdot e^{z-1}}{z-1} = e \cdot \sum_{n=0}^{\infty} \frac{(z-1)^{n-1}}{n!}$ valid for $|z-1| > 0$

$f(z) = \frac{1}{z(z-3)} = \frac{1}{3}\left(-\frac{1}{z} + \frac{1}{z-3}\right) = -\frac{1}{z} - \frac{1}{3(1-\frac{z}{3})} = -\frac{1}{z} - \frac{1}{3}\left[\frac{1}{1-\frac{z}{3}}\right] = \left(-\frac{1}{z} - \frac{1}{3}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n\right) \times \frac{1}{3} = -\frac{1}{3z} - \frac{1}{3^2} - \frac{z}{3^3}$

4. $0 < |z| < 3$

5. $0 < |z-3| < 3$

6. $1 < |z-4| < 4$

4. $f(z) = \frac{1}{z^2}\left(\frac{1}{1 - \frac{3}{z}}\right) = \frac{1}{z^2}\left(\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n\right) = \sum_{n=0}^{\infty} \frac{3^n}{z^{n+2}}$

5. $f(z) = \frac{1}{3}\left(\frac{1}{z} - \frac{1}{z-3}\right) = \frac{1}{3}\cdot\left(\frac{1}{(z-3)+3} - \frac{1}{z-3}\right) = \frac{1}{9}\cdot\frac{1}{1+\frac{(z-3)}{3}} - \frac{1}{z-3}$

$= \sum_{n=0}^{\infty}(-1)^n\frac{(z-3)^n}{3^{n+2}} - \frac{1}{3}\,\frac{1}{z-3} = \sum_{n=-1}^{\infty}(-1)^n\frac{(z-3)^n}{3^{n+2}}$

6. $f(z) = \frac{1}{3}\left(\frac{1}{4+(z-4)} - \frac{1}{1+(z-4)}\right) = \frac{1}{3}\left(\frac{1}{4}\cdot\frac{1}{1+\frac{z-4}{4}} - \frac{1}{z-4}\cdot\frac{1}{1+\frac{1}{z-4}}\right)$

$= \frac{1}{3}\left[\sum_{n=0}^{\infty}\frac{(-1)^n(z-4)^n}{4^{n+1}} - \sum_{n=0}^{\infty}(-1)^n\cdot\frac{1}{(z-4)^{n+1}}\right]$

$= \frac{1}{3}\left[\cdots - \frac{1}{(z-4)^3} + \frac{1}{(z-4)^2} - \frac{1}{z-4} + \frac{1}{4} - \frac{z-4}{4^2} + \frac{(z-4)^2}{4^3} - \frac{(z-4)^3}{4^4} + \cdots\right]$

$f(z) = \frac{1}{(z-1)(z-2)}$

7. $1 < |z| < 2$

8. $0 < |z-1| < 1$

7. $f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} + \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}$

$= \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$

$= \cdots + z^{-3} + z^{-2} + z^{-1} + \frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \frac{z^3}{2^4} + \cdots$

8. $f(z) = \frac{1}{z-1}\left(\frac{1}{(z-1)-1}\right) = -\frac{1}{z-1}\cdot\sum_{n=0}^{\infty}(z-1)^n = -\sum_{n=0}^{\infty}(z-1)^{n-1}$

9. $f(z) = \frac{z}{(z+1)(z-2)}$

$0 < |z+1| < 3$

9. $f(z) = \frac{1}{3}\cdot\frac{1}{z+1} + \frac{2}{3}\cdot\frac{1}{z-2} = \frac{1}{3}\cdot\frac{1}{z+1} - \frac{2}{3^2}\cdot\frac{1}{1-\frac{z+1}{3}}$

$= \frac{1}{3}\cdot\frac{1}{z+1} - 2\sum_{n=0}^{\infty}\frac{(z+1)^n}{3^{n+2}}$

CA HW9.

1. $f(z) = \frac{2}{(z-1)(z+4)}$; $\mathrm{Res}(f(z), 1)$

2. $f(z) = \frac{4z-6}{z(2-z)}$; $\mathrm{Res}(f(z), 0)$

1. $z=1$ is simple pole.

$$\mathrm{res}(f(z),1) = \frac{\varphi(1)}{\psi'(1)} = \frac{2}{2z+3\big|_{z=1}} = \frac{2}{5}$$

2. $z=0$ is simple pole

$$\mathrm{res}(f(z),0) = \frac{\varphi(0)}{\psi'(0)} = \frac{4z-6\big|_{z=0}}{2} = -3$$

3. $f(z) = \frac{z}{z^2+16}$

4. $f(z) = \frac{5z^2-4z+3}{(z+1)(z+2)(z+3)}$

5. $f(z) = \sec z$

3. pole: $z = \pm 4i$

$$\mathrm{Res}(f, 4i) = \frac{4i}{2\cdot 4i} = \frac{1}{2}$$

$$\mathrm{Res}(f, -4i) = \frac{-4i}{2\cdot(-4i)} = \frac{1}{2}$$

4. pole $z=-1,\ z=-2,\ z=-3$

$5z^2-4z+3$

$f(z) = \frac{\varphi(z)}{\psi(z)}$. $\quad \varphi(z) = \cancel{10z-4}$ $\quad \psi(z) = z^3+6z^2+11z+6$ $\quad \psi'(z) = 3z^2+12z+11$

$$\mathrm{res}(f,-1) = \frac{\cancel{-10-4}\ 12}{3-12+11} = \cancel{-7}\ 6$$

$$\mathrm{res}(f,-2) = \frac{\cancel{-20-4}\ 31}{-1} = \cancel{24}\ -31$$

$48+12$

$$\mathrm{res}(f,-3) = \frac{\cancel{-30-4}\ 60}{\cancel{27-36+11}\ 2} = \cancel{-17}\ 30$$

5. $f(z) = \sec z = \frac{1}{\cos z}$ $\qquad z = \frac{\pi}{2}+k\pi.\ k\in\mathbb{Z}.$

each pole is simple pole.

$$\mathrm{res}(f, \tfrac{\pi}{2}+k\pi) = \frac{1}{-\sin z} = (-1)^{k+1}.\quad k\in\mathbb{Z}.$$

6. $\oint_C \frac{1}{(z-1)(z+2)^2}dz$

(a) $|z| = \frac{1}{2}$

(b) $|z| = \frac{3}{2}$

(c) $|z| = 3$

Δ: firstly find the residue.

$$\mathrm{res}(f,1) = \frac{1}{3z^2+6z\big|_{z=1}} = \frac{1}{9}$$

$$\mathrm{res}(f,-2) = \lim_{z\to-2}\frac{d((z+2)^2 f(z))}{dz} = -\frac{1}{9}.$$

(a). no singular points with in the contour.

$$\oint_{|z|=\frac{1}{2}} \frac{1}{(z-1)(z+2)^2}dz = 0$$

(b) $z=1$ as the simple pole in the circle $|z|=\frac{3}{2}$

$$\oint_{|z|=\frac{3}{2}} \frac{1}{(z-1)(z+2)^2}dz = 2\pi i\cdot\left(\frac{1}{9}\right) = \frac{2}{9}\pi i$$

(c) $z=1,\ z=-2$ in the circle. $|z|=3$

$$\oint_{|z|=3} \frac{1}{(z-1)(z+2)^2}dz = 2\pi i\left(\frac{1}{9}-\frac{1}{9}\right) = 0$$

7. $\oint_C \frac{1}{z^2+4z+13}dz, C : |z-3i| = 3$

8. $\oint_C \frac{\tan z}{z}dz, C : |z-1| = 2$

7. pole : $z = -2 \pm 3i$. only $z = -2+3i$ in the circle.

$$\operatorname{res}(f, -2+3i) = \frac{1}{2z+4 \,|_{z=-2+3i}} = -\frac{i}{6}$$

$$\oint_C \frac{1}{z^2+4z+13}dz = 2\pi i \cdot -\frac{i}{6} = \frac{\pi}{3}$$

8. pole : $z = 0, \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$. 只有 $z = \frac{\pi}{2}$ 在内.

$$\operatorname{res}(f, 0) = \frac{\sin z \,|_{z=0}}{\cos z - z\sin z \,|_{z=0}} = 0.$$

$$\operatorname{res}(f, \tfrac{\pi}{2}+k\pi) = \frac{\sin z \,|_{z=\frac{\pi}{2}+k\pi}}{\cos z - z\sin z \,|_{z=\frac{\pi}{2}+k\pi}} = \frac{(-1)^k}{-(\frac{\pi}{2}+k\pi)(-1)^k} = -\frac{1}{\frac{\pi}{2}+k\pi} \quad -\frac{1}{\frac{\pi}{2}}$$

$$\oint_C \frac{\tan z}{z}dz = \cancel{-}2\pi i \sum_{k\in\mathbb{Z}} \frac{1}{\frac{\pi}{2}+k\pi} = -4i \sum_{k\in\mathbb{Z}} \cancel{\frac{1}{2k+1}} \quad -4i$$

$$-\frac{\pi}{2}$$

CA HW10

1. $\int_0^{2\pi} \frac{1}{1+0.5\sin\theta} d\theta$

2. $\int_0^{\pi} \frac{1}{2-\cos\theta} d\theta$[Hint: Let $t = 2\pi - \theta$.]

3. $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta$

1. Let $z = e^{i\theta}$. $d\theta = \frac{dz}{iz}$ $\sin\theta = \frac{1}{2i}(z - z^{-1})$ $C: |z| = 1$.

$$\int_0^{2\pi} \frac{1}{1+\frac{1}{2}\sin\theta} d\theta = \oint_C \frac{1}{1+\frac{1}{4i}(z-z^{-1})} \frac{dz}{iz} = \oint_C \frac{dz}{zi+\frac{1}{4}(z^2-1)} = 4 \cdot \oint_C \frac{dz}{[z+(2+\sqrt{3})i][z+(2-\sqrt{3})i]} = 4 \cdot 2\pi i \cdot \left(\frac{1}{2\sqrt{3}i}\right) = \frac{4\pi}{\sqrt{3}}$$

2. $$\int_0^{\pi} \frac{1}{2-\cos\theta} d\theta \overset{t=2\pi-\theta}{=\!=} \int_{\pi}^{2\pi} \frac{1}{2-\cos t} dt$$

thus $$\int_0^{\pi} \frac{1}{2-\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{2-\cos\theta} d\theta = \frac{1}{2}\oint_C \frac{1}{2-\frac{1}{2}(z+z^{-1})} \frac{dz}{iz}$$

$$= \oint_C \frac{1}{4iz - i(z^2+1)} dz = \frac{-1}{i}\oint_C \frac{dz}{z^2-4z+1} = \frac{-1}{i} \cdot 2\pi i \ \text{res}_{z=2-\sqrt{3}} = \frac{-1}{i} \cdot 2\pi i \cdot \frac{1}{2(2-\sqrt{3})-4} = \frac{\pi}{\sqrt{3}}.$$

3. $$\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta = \oint_C \frac{\frac{1}{2}(z^2+z^{-2})}{5-2(z+z^{-1})} \frac{dz}{iz} = \frac{1}{2i}\oint_C \frac{z^4+1}{5z^3-2(z^4+z^2)} dz = \frac{1}{2i} \cdot 2\pi i \sum \text{res}.$$

$$\text{res}_{z=\frac{1}{2}} = \frac{z^4+1 \,\big|_{z=\frac{1}{2}}}{-8z^3+15z^2-4z\,\big|_{z=\frac{1}{2}}} = \frac{\frac{17}{16}}{\frac{3}{4}} = \frac{17}{12}$$

$$\text{res}_{z=0} = \lim_{z\to 0} \frac{d(f(z)\cdot z^2)}{dz} = -\frac{5}{4}$$

$$\Rightarrow \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta} d\theta = \frac{\pi}{6}$$

4. $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} d\theta = \frac{a\pi}{(\sqrt{a^2-1})^3}, a > 1$

Pf: similar as 2. $$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\oint \frac{dz}{a^2+(z+z^{-1})a+\frac{2+z^2+z^{-2}}{4}} \cdot \frac{1}{iz} = \frac{2}{i} \cdot \oint \frac{z}{4a^2z^2+4(z^3+z)a+2z^2+z^4+1} dz$$

$$= \frac{2}{i}\oint_C \frac{z\,dz}{z^4+4az^3+(4a^2+2)z^2+4az+1} = \frac{2}{i}\oint_C \frac{z}{(z^2+2az+1)^2} dz$$

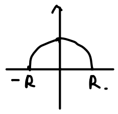
pole: $z = \sqrt{a^2-1} - a$. $z = -\sqrt{a^2-1} - a$.

$$\text{res}(f, \sqrt{a^2-1}-a) = \lim_{z\to a} \frac{d((z+a-\sqrt{a^2-1})^2 \cdot f(z))}{dz} = \lim_{z\to\sqrt{a^2-1}-a} \left[\frac{z}{(z+a+\sqrt{a^2-1})^2}\right]' = \frac{4a\sqrt{a^2-1}}{16(a^2-1)^2} = \frac{a(a^2-1)^{-\frac{3}{2}}}{4}$$

$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{2}{i} \cdot 2\pi i \cdot \frac{a(a^2-1)^{-\frac{3}{2}}}{4} = \frac{\pi a}{(\sqrt{a^2-1})^3}$$

Calculate. P.V.

5. $\int_{-\infty}^{\infty} \frac{1}{x^2-2x+2}dx$ C :



denote $f(z) = \frac{1}{z^2-2z+2}$. pole: $z = 1 \pm i$. upper half pole: $z = 1+i$.

$$\int_{C_R} \frac{1}{z^2-2z+2} dz + \int_{-R}^{R} \frac{1}{x^2-2x+2} dx = \oint_C \frac{dz}{z^2-2z+2} = 2\pi i \cdot res(f, 1+i) = 2\pi i \cdot \frac{1}{2(1+i)-2} = \pi.$$

since $deg(z^2-2z+2) = 2$. $deg(1) = 0$. $\int_{C_R} \frac{1}{z^2-2z+2} dz \xrightarrow[R\to\infty]{} 0$

thus. $P.V. \int_{-\infty}^{\infty} \frac{1}{x^2-2x+2} dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2-2x+2} dx = \pi.$

6. $\int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx$

denote $f(z) = \frac{2z^2-1}{z^4+5z^2+4}$ pole: $z = \pm i, \pm 2i$. upper half: $z = i, 2i$.

$$\int_{C_R} f(z) dz + \int_{-R}^{R} f(x) dx = \oint_C f(z) dz = 2\pi i \cdot [res(f, i) + res(f, 2i)] = 2\pi i \cdot -\frac{i}{4} = \frac{\pi}{2}.$$

$\int_{C_R} f(z) dz \xrightarrow[R\to\infty]{} 0$ similarly. thus, $P.V. \int_{-\infty}^{\infty} f(x) dx = \lim_{R\to+\infty} \int_{-R}^{R} f(x) dx = \frac{\pi}{2}$

7. $\int_0^{\infty} \frac{x^2}{x^6+1}dx$

denote. $f(z) = \frac{z^2}{z^6+1}$ pole on upper half: $z_1 = i$, $z_2 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$, $z_3 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$$P.V. \int_0^{\infty} \frac{x^2}{x^6+1} dx = \frac{1}{2} \lim_{R\to 0} \int_{-R}^{R} \frac{x^2}{x^6+1} dx.$$

$$\int_{C_R} f(z) dz + \int_{-R}^{R} f(x) dx = \oint_C f(z) dz = 2\pi i \sum_{k=1}^{3} res(f, z_k) = 2\pi i \cdot \left(-\frac{1}{6i} + \frac{1}{6i} + \frac{1}{6i}\right) = \frac{\pi}{3}$$

$$\int_0^{\infty} \frac{x^2}{x^6+1} dx = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} f(x) dx = \frac{1}{2} \cdot \frac{\pi}{3} = \frac{\pi}{6}$$

8. $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}dx$

denote $f(x) = \frac{x}{x^2+1}$ $\int_{-\infty}^{\infty} \frac{x}{x^2+1} \cdot e^{ix} dx = \int_{-\infty}^{\infty} \frac{x}{x^2+1} \cos x \, dx + i \int_{-\infty}^{\infty} \frac{x}{x^2+1} \sin x \, dx.$

$$\int_{C_R} f(z) \cdot e^{iz} dz + \int_{-R}^{R} f(x) \cdot e^{ix} dx = \oint_C \frac{z}{z^2+1} e^{iz} dz. = 2\pi i \, Res(f(z) e^{iz}, i) = 2\pi i \left. \frac{z \cdot e^{iz}}{2z} \right|_{z=i} = \frac{\pi}{e} i$$

$\int_{C_R} f(z) \cdot e^{iz} dz \to 0$ as $R \to \infty$.

$$P.V. \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} dx = Im \cdot \lim_{R\to\infty} \int_{-R}^{R} \frac{x}{x^2+1} e^{ix} dx = \frac{\pi}{e}.$$

9. $\int_0^{\infty} \frac{\cos 2x}{x^4+1}dx$

$$\int_0^{\infty} \frac{\cos 2x}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos 2x}{x^4+1} dx$$

denote $f(z) = \frac{1}{z^4+1}$ pole: $z = \pm\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$.

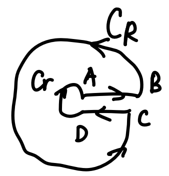
$$\int_{C_R} f(z) \cdot e^{i2z} dz + \int_{-R}^{R} f(x) \cdot e^{i(2x)} dx = \oint_C \frac{e^{i(2z)}}{z^4+1} dz = 2\pi i \cdot \left(res\left(f(z) \cdot e^{i(2z)}, \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) + res\left(f(z) \cdot e^{i(2z)}, -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) \right)$$

$$= 2\pi i \cdot \left(\frac{e^{-\sqrt{2}} \cdot e^{\sqrt{2}i}}{4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)} + \frac{e^{-\sqrt{2}} \cdot e^{-\sqrt{2}i}}{4\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)} \right) = \frac{\pi i}{2e^{\sqrt{2}}} \cdot \left(e^{(\sqrt{2} - \frac{3\pi}{4})i} + e^{(-\sqrt{2} - \frac{\pi}{4})i} \right).$$

$$P.V. \int_0^{\infty} \frac{\cos 2x}{x^4+1} dx = \frac{1}{2} \lim_{R\to\infty} \int_{-R}^{R} f(x) \cdot e^{i(2x)} dx = \frac{1}{2} Re \lim_{R\to 0} \int_{-R}^{R} f(z) \cdot e^{i(2z)} dz$$

$$= \frac{\pi}{4e^{\sqrt{2}}} \left[\sin\left(\frac{3\pi}{4} - \sqrt{2}\right) + \sin\left(\frac{\pi}{4} + \sqrt{2}\right) \right] = \frac{\pi}{2e^{\sqrt{2}}} \cos\left(\sqrt{2} - \frac{\pi}{4}\right)$$

CA HW11



1. $\int_0^\infty \frac{1}{\sqrt{x}(x^2+1)}dx = \frac{\pi}{\sqrt{2}}$
2. $\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)}dx = \frac{\pi}{3}$

1. denote the function $f(z) = \frac{1}{\sqrt{z}\,(z^2+1)}$

pole : $z = \pm i$. discontinuity point. $z=0$.

$$\oint_C \frac{1}{\sqrt{z}(z^2+1)}dz = 2\pi i \, Res\,(f(z), \pm i) = 2\pi i \frac{2z^{\frac{1}{2}}}{5z^2+1} = 2\pi i \cdot \left(\frac{\sqrt{2}+\sqrt{2}i}{-4} + \frac{-\sqrt{2}+\sqrt{2}i}{-4} \right) = \sqrt{2}\pi$$

$$\oint_C = \oint_{C_R} - \oint_{Cr} + \int_{AB} + \int_{DC} = \sqrt{2}\pi.$$

on AB. $z = x \cdot e^{0i} = x.$

on DC $z = x \cdot e^{2\pi i}.$

$$\int_{CD} = \int_R^r \frac{d(x \cdot e^{2\pi i})}{(x\cdot e^{2\pi i})^{\frac{1}{2}} \cdot ((x\cdot e^{2\pi i})^2+1)} = -\int_r^R \frac{dx}{e^{\pi i}\, x^{\frac{1}{2}}(x^2+1)} = \int_r^R \frac{dx}{x^{\frac{1}{2}}(x^2+1)}$$

$\int_{C_R} \xrightarrow[R\to+\infty]{} 0$ $\quad \int_{Cr} \xrightarrow[r\to 0]{} 0$ $\quad$ Thus $\oint_C = 2\int_{AB} = \sqrt{2}\pi.$

$$P.V. \int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \lim_{\substack{R\to\infty \\ r\to 0}} \int_r^R \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

2. $f(z) = \frac{1}{\sqrt{z}\,(z+1)(z+4)}$

pole : $z=-1$ or $z=-4$.

$$\oint_C \frac{dz}{\sqrt{z}(z+1)(z+4)} = 2\pi i \left(Res\,(f,-1) + Res\,(f,-4) \right) = 2\pi i \left[\left. \frac{1}{\frac{5}{2}z^{\frac{3}{2}} + \frac{15}{2}z^{\frac{1}{2}} + 2z^{-\frac{1}{2}}} \right|_{z=-1,-4} \right] = \frac{\pi}{3}$$

since similar as 1. $\int_{AB} = \frac{1}{2}\oint_C = \frac{\pi}{6}.$

$$P.V. \int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)}dx = \lim_{\substack{r\to 0 \\ R\to+\infty}} \int_r^R \frac{1}{\sqrt{x}(x+1)(x+4)}dx = \frac{\pi}{6}.$$

3. $\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx = \frac{\pi}{\sin\alpha\pi}, 0<\alpha<1.$

$$f(z) = \frac{1}{z^{1-\alpha}(1+z)} \qquad \text{pole } z=-1.$$

$$\oint_C \frac{dz}{z^{1-\alpha}(1+z)} = 2\pi i \operatorname{Res}(f,-1) = \frac{2\pi i}{(-1)^{1-\alpha}}$$

$$\int_{CD} = -\int_r^R \frac{dx}{e^{2\pi(1-\alpha)i}(1+x)x^{1-\alpha}} = [-\cos(2\pi\alpha) - i\sin(2\pi\alpha)]\int_{AB}.$$

$$[(1-\cos 2\pi\alpha) - i\sin 2\pi\alpha]\int_{AB} = \frac{2\pi i}{(-1)^{1-\alpha}} \Rightarrow [\sin^2\pi\alpha - i\cos\pi\alpha\sin\pi\alpha]\int_{AB} = \frac{\pi i}{(-1)^{1-\alpha}}.$$

$$\int_{AB} = \frac{\pi}{\sin\pi\alpha\cdot e^{-\pi\alpha i}(-1)^{1-\alpha}} = \frac{\pi}{\sin\pi\alpha\cdot e^{\pi\alpha i}\cdot(e^{\pi i})^{2-\alpha}} = \frac{\pi}{\sin\pi\alpha}.$$

$$\int_0^\infty \frac{x^{\alpha-1}}{x+1}dx = \lim_{\substack{r\to 0\\ R\to\infty}}\int_r^R \frac{x^{\alpha-1}}{x+1}dx = \frac{\pi}{\sin\pi\alpha}.$$

CA HW12

1. $f(z) = z^6 - 2iz^4 + (5-i)z^2 + 10, C$ encloses all the zeros of f

1. by D'Alembert's thm. $f(z)$ has exact 6 zero (counting the order)

$$\oint_C \frac{f'(z)}{f(z)} = 2\pi i\,(N_0 - N_p) = 12\pi i.$$

Use the argument principle in (28) of Theorem 6.20 to evaluate the given integral on the indicated closed contour C. You will have to identify $f(z)$ and $f'(z)$.

2. $\oint_C \frac{2z+1}{z^2+z} dz, C$ is $|z| = 2$
3. $\oint_C \frac{z}{z^2+4} dz, C$ is $|z| = 3$

2. $f(z) = z^2 + z \quad f'(z) = 2z+1.$

zero : $z=0$ and $z=-1$. (within C).

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i\,(N_0) = 4\pi i$$

3. $$\frac{1}{2}\oint_C \frac{2z}{z^2+4} dz = \frac{1}{2}\oint \frac{f'(z)}{f(z)} dz = \pi i \cdot N_0 = 2\pi i.$$

where $f(z) = z^2+4$. $f'(z) = 2z$. Zero $z = \pm 2i$ within C

CA HW13.

1. Consider primary branch of logarithm in $U\{z \in \mathbb{C} : |z-1| < 1\}$ defined by the series
$$\ln z = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(z-1)^n.$$
Let $a \in U$. Prove that Taylor series centered at point a converges on the disk of radius $r = \cancel{|1-a|}\ |a|$

Pf: $\ln z = \ln|a| + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n a^n}(z-a)^n$

by Cauchy-Hadamart's formula.

$$R = \frac{1}{\lim_{n\to\infty}\left(\left|\frac{(-1)^{n-1}}{n a^n}\right|\right)^{\frac{1}{n}}} = |a| \cdot \left(\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}\right)^{-1} = |a|$$

2. Prove that the relation "to be analytic continuation" is equivalence relation on a family of all canonical elements.

Pf: denote the element $F_k = (U_k, f_k)$. $k = 1, 2, 3$. $U_i \cap U_j \neq \phi$. $i, j = 1, 2, 3$.

$f_i = f_j$ on $U_i \cap U_j$

1). reflexivity: F_1 is AC of F_1. $f_1 = f_1$ on $U_1 \cap U_1 = U_1$ $f_1 \in H(U_1)$.

2) symmetry: F_1 is AC of F_2: $f_1 = f_2$ on $U_1 \cap U_2$. $f_1 \in H(U_1)$. $f_2 \in H(U_2)$.

i.e. $f_2 = f_1$ on $U_2 \cap U_1$. F_2 is AC of F_1.

3) transitivity: F_1 is AC of F_2, F_2 is AC of F_3.

we have $f_1 = f_2 = f_3$ on $U_1 \cap U_2 \cap U_3 \neq \phi$

i.e. $f_1 = f_3$ on $U_1 \cap U_2 \cap U_3 \subset U_1 \cap U_3$. , by uniqueness thm. $f_1 = f_3$ on $U_1 \cap U_3$.

i.e. F_1 is AC of F_3.

3. *Prove that on the boundary of disk of convergence of a power series the sum
$$f(z) = \sum_{k=0}^{\infty} c_k z^k$$
has a singular point.

Definition 0.1. *Let $a \in \mathbb{C}$, $R \in (0, +\infty)$, $U = \{z \in \mathbb{C} : |z - R|$, $F = (U, f)$ be a canonical element*

$$f(z) = \sum_{k=0}^{\infty}, \quad |z - z_0| < R.$$

A point $z_1 \in \overline{U}$ is called a regular point of element (U, f) if there exists DAC of F with center at z_1. Otherwise z_1 is a simgular point.

Pf: denote the disk of convergence $U = \{|z| < R\}$.

Assume the converse. if $\forall z \in \partial U$. z is a regular point.

i.e. $\exists$ DAC of F centered at z. i.e. $\exists\ G = (U_\varepsilon(z), g)$. s.t. $g(z) = f(z)$ for any $z \in U_\varepsilon(z) \cap U$

then we can make the analytic continuation at every $z \in \partial U$. which is contradicts with the definition of boundary of disk of conv.

Complex Analysis 2024. Homework 14.

1. Provide an example of a function with a nonisolated singular point.

2. Let $D \subset \mathbb{C}$ be simply connected domain, $f \in H(D)$ and $f(D) \subset \mathbb{C}\backslash\{0\}$. Hence there exist functions f and g holomorphic in domain D such that
$$f = g^2, \quad f = e^h, \quad \text{in } D.$$

3. Let $D \subset \mathbb{C}$ be simply connected domain, $f \in H(D)$, $f \not\equiv 0$. Prove that $f = g^2$ for some $g \in H(D)$ if and only if orders of all zeros of f are even.

4. Describe all values z^z, $z \neq 0$. Consider a holomorphic branch f of a function Lnz such that $f(1) = 0$. Let $x \in (-\infty, 0)$. Calculate
$$\lim_{t \to 0+} (f(x + it) - f(x - it)).$$

5. Let f_0 be holomorphic branch of $z^{1/3}$ such that $f(1) = e^{2\pi i/3}$. Consider a path $\gamma(t) = (1 + t)e^{7\pi i t}$, $0 \leq t \leq 1$. Calculate value of analytic continuation of f_0 at point -2.

6. Suppose that analytic functions $\mathcal{F}$ and $\mathcal{G}$ have single-valued branches in domain D. Prove that $\mathcal{F} + \mathcal{G}$ and $\mathcal{F}\mathcal{G}$ have single-valued branches in domain D. Use this property to prove that $\sqrt{1 - z^2}$ has single-valued branch in $\mathbb{C}\backslash\left((-\infty, -1] \cup [1, +\infty)\right)$. Suppose that $f(z)$ is a branch such that $f(0) = 1$. Calculate $f(i)$.

7. Let $a, b \in \mathbb{C}$. Prove that function $z^a(1 - z)^b$ has single-valued branch in domain $\mathbb{C}\backslash[0, 1]$ if and only if $a + b \in \mathbb{Z}$.

8. Prove that $\ln(z)\ln(1 + z^2)$ has single-valued branch in $\mathbb{C}\backslash(-i\infty, i]$.