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2.3.1. APPLICATION OF THE LAPLACE TRANSFORM TO SOLVING FIRST-ORDER LINEAR EQUATIONS

An unknown function satisfying a first-order linear partial differential equation and given conditions can be found using a one-time or two-time Laplace transform, depending on the type of conditions.

In the first case, the transformation is applied to a partial differential equation for one of the independent variables, assuming that the other remains unchanged.

The result is an operator equation with respect to the image, which is an ordinary differential equation with a parameter.

After integrating the operator equation from the image found from it, the original is found as a solution to the original equation.

In the second case, the Laplace transform is applied sequentially, resulting in an equation from which a two-fold image of the desired function is found.

Using inverse transformations, the original function is restored.

The solution of the partial differential equation found using the two-fold Laplace transform does not depend on the sequence in which the forward and reverse transformations were applied.

Example 1

In the region $x > 0$, $y > 0$, using the Laplace transform to find a solution to the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x + y \text{ ,}$$

satisfying the conditions: $u(0, y) = u(x, 0) = 1$.

<u>Solution:</u>

We apply the Laplace transform with respect to the variable x to the given equation , assuming $u(x,y) \leftrightarrow U(p,y)$. Since

$$\frac{\partial u}{\partial x} \leftrightarrow pU(p,y) - u(0,y) = pU(p,y) - 1 \text{ ,}$$

$$\frac{\partial u}{\partial y} \leftrightarrow \frac{\partial U(p,y)}{\partial y} \text{ ,}$$

$$u(x,0) = 1 \leftrightarrow U(p,0) = \frac{1}{p} \text{ ,}$$

the specified transformation gives the operator equation:

$$pU(p,y) - 1 + \frac{\partial U(p,y)}{\partial y} = \frac{1}{p^2} + \frac{y}{p} \text{ ,}$$

to which the condition

$$U(p,0) = \frac{1}{p}$$

should be added.

Thus, a single Laplace transform with respect to the variable x gives the problem

$$\begin{cases} pU(p,y) + \dfrac{\partial U(p,y)}{\partial y} = \dfrac{1}{p^2} + \dfrac{y}{p} + 1, \quad y > 0, \\ U(p,0) = \dfrac{1}{p}. \end{cases} \tag{*}$$

The resulting equation can be considered as an ordinary first - order differential equation with constant coefficients for the function U, with an independent variable y and a parameter p. Let's solve the Cauchy problem (*) in two ways.

First, by solving a differential equation, it is possible to construct its general solution:

$$U(p,y) = Ce^{-py} + \frac{y}{p^2} + \frac{1}{p},$$

and select a solution that satisfies the given initial condition:

$$U(p,y) = \frac{y}{p^2} + \frac{1}{p}.$$

It is easy to build a corresponding original for the found image:

$$u(x,y) = yx + 1.$$

The second method involves solving the problem (*) using the Laplace transform with respect to the variable y.

Assuming $U(p,y) \leftrightarrow V(p,q)$, we construct the operator equation

$$qV(p,q) - \frac{1}{p} + pV = \frac{1}{p^2 q} + \frac{1}{pq^2} + \frac{1}{q},$$

from where we find

$$V(p,q)=\frac{1}{p^2q^2}+\frac{1}{pq}.$$

By performing inverse transformations, we find a solution to the problem formulated in the example condition.

Both the first method, the one—time Laplace transform, and the second, the two-time Laplace transform, give the same result.

2.4. CLASSIFICATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

A linear partial differential equation of the second order is called the equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f(x)\,,\ x \in D\,, \qquad (2.3)$$

where the coefficients are real functions of the point x in the region D:

$$a_{ij} = a_{ij}(x)\,,\ b_i = b_i(x)\,,\ c = c(x)\,.$$

Equation (2.3) corresponds to the characteristic form:

$$Q(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{i,j=1}^{n} a_{ij}\lambda_i\lambda_j\,,$$

which is quadratic.

At each fixed point $x \in D$, using a non-special affine transformation of variables:

$$\lambda_i = \lambda_i(\mu_1, \mu_2, \ldots, \mu_n)\,,\ i = 1, 2, \ldots, n$$

the quadratic form Q can be reduced to the canonical form

$$\tilde{Q}(\mu_1, \mu_2, \ldots, \mu_n) = \sum_{i=1}^{n} \alpha_i \mu_i^2\,, \qquad (2.4)$$

where $\alpha_i \in \{-1, 0, 1\}$.

The canonical form of the quadratic form determines the type of equation (2.3).

The linear equation (2.3) will be called *elliptical* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, all $\alpha_i \neq 0$ and one sign.

Equation (2.3) will be called *hyperbolic* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, all $\alpha_i \neq 0$, but not all of the same sign.

Equation (2.3) will be called *parabolic* at point $x \in D$, if, in the canonical form of the quadratic form (2.4) with coefficients calculated at point $x \in D$, at least one of the coefficients $\alpha_k = 0$.

Example 1

Determine the type of equation for $u = u(x, y)$:

$$u_{xx} - 4u_{xy} + 8u_{yy} + u_x - 6u + y = 0 .$$

Solution:

The given equation corresponds to the quadratic form

$$Q(\lambda_1, \lambda_2) = \lambda_1^2 - 4\lambda_1\lambda_2 + 8\lambda_2^2,$$

which we bring to the canonical form by sequentially highlighting the complete squares:

$$Q(\lambda_1, \lambda_2) = \lambda_1^2 - 4\lambda_1\lambda_2 + 8\lambda_2^2 = \lambda_1^2 - 4\lambda_1\lambda_2 + 4\lambda_2^2 + 4\lambda_2^2 =$$

$$= (\lambda_1 - 2\lambda_2)^2 + (2\lambda_2)^2 = \mu_1^2 + \mu_2^2 = \tilde{Q}(\mu_1, \mu_2) .$$

Since both coefficients in the canonical form of a quadratic form have the same sign, the given equation has an elliptical type in the entire domain of setting the variables x, y.

Example 2

Determine the type of equation for $u = u(x, y, z)$:

$$u_{xx} - 4u_{yy} + 2u_{xz} + 4u_{yz} + 2u_x - u_y = xyz^2.$$

<u>Solution:</u>

The given equation corresponds to the quadratic form

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 - 4\lambda_2^2 + 2\lambda_1\lambda_3 + 4\lambda_2\lambda_3.$$

Let's bring it to a canonical form, sequentially highlighting the full squares:

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + 2\lambda_1\lambda_3 + \lambda_3^2 - \lambda_3^2 - 4\lambda_2^2 + 4\lambda_2\lambda_3 =$$
$$= (\lambda_1 + \lambda_3)^2 - (2\lambda_2 - \lambda_3)^2 = \mu_1^2 - \mu_2^2 = \tilde{Q}(\mu_1, \mu_2, \mu_3).$$

Since one of the coefficients in the canonical form of the quadratic form is 0 (for μ_3^2), the given equation has a parabolic type in the entire domain of setting the variables x, y, z.

2.5. CLASSIFICATION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH TWO VARIABLES

Consider a second-order equation, linear with respect to the higher derivatives, for an unknown function $u(x,y)$ of two independent variables x and y:

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + F(x,y,u,u_x,u_y) = 0, \qquad (2.5)$$

where the real functions $a(x,y), b(x,y), c(x,y)$ are defined in the domain D.

2.5.1. REPLACING INDEPENDENT VARIABLES

Let's introduce independent variables

$$\xi = \xi(x,y), \ \eta = \eta(x,y), \qquad (2.6)$$

where ξ, η are twice continuously differentiable functions in the domain D.

We require that the Jacobian of the transformation be nonzero:

$$\frac{D(\xi,\eta)}{D(x,y)} \neq 0 .$$

Let's try to choose the transformation (2.6) in such a way that equation (2.5) has the simplest form in the new variables. We transform equation (2.5) to new variables, assuming

$$U(\xi,\eta) = u(x(\xi,\eta), y(\xi,\eta)) .$$

Then we get

$$u_x = U_\xi \xi_x + U_\eta \eta_x ,$$

$$u_y = U_\xi \xi_y + U_\eta \eta_y ,$$

$$u_{xx} = U_{\xi\xi}\xi_x^2 + 2U_{\xi\eta}\xi_x\eta_x + U_{\eta\eta}\eta_x^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx} ,$$

$$u_{yy} = U_{\xi\xi}\xi_y^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}\eta_y^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy} ,$$

$$u_{xy} = U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}\left(\xi_x\eta_y + \xi_y\eta_x\right) + U_{\eta\eta}\eta_x\eta_y + U_\xi \xi_{xy} + U_\eta \eta_{xy} .$$

In the new variables, equation (2.5) will take the form

$$\bar{a}U_{\xi\xi} + 2\bar{b}U_{\xi\eta} + \bar{c}U_{\eta\eta} + \bar{F} = 0 , \tag{2.7}$$

where

$$\bar{a} = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 , \tag{2.8}$$

$$\bar{c} = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 ,$$

$$\bar{b} = a\xi_x\eta_x + 2b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y ,$$

$\bar{F} = \bar{F}\left(\xi, \eta, U, U_\xi, U_\eta\right) -$ a function that does not depend on the higher derivatives.

Definition 1. Equation (2.5) has at the point (x,y):

– *hyperbolic* type if $b^2 - ac > 0$ at point (x,y),

– *elliptical* type if $b^2 - ac < 0$ at point (x,y),

– *parabolic* type if $b^2 - ac = 0$ at point (x,y).

If the type of equation is preserved at all points of the domain D, then the equation is called an equation of this type in the entire domain D.

If an equation belongs to different types at different points in the domain, then it is called a *mixed type* equation in the domain D.

2.5.2.THE EQUATION OF CHARACTERISTICS

Now let's figure out how to introduce new variables x and h so that equation (2.5) takes the simplest form.

Assumption. Equation (2.5) belongs to a certain type in the entire domain D and $a(x,y)$ and $c(x,y)$ not equal to zero at the same time.

We assume that $a(x,y)\neq 0$.

It can be seen from the relation (2.8) that in order for $\bar{a}=0$, it is necessary as a function of $\xi(x,y)$ to take the solution of the equation:

$$az_x^2+2bz_xz_y+cz_y^2=0. \qquad (2.9)$$

Definition 2. Equation (2.9) is called the characteristic equation for equation (2.5).

Lemma. Let the function $z(x,y)$ be continuously differentiable in the domain D and such that $z_y\neq 0$. In order for the family of curves $z(x,y)=C$ to represent the characteristics of equation (2.5), it is necessary and sufficient that the expression $z(x,y)=C$ be the general integral of the ordinary differential equation

$$a(x,y)(dy)^2-2b(x,y)dxdy+c(x,y)(dx)^2=0. \qquad (2.10)$$

Definition 3.

Equation (2.10) is called the *equation of characteristics* for equation (2.5).

Assuming $\xi=\varphi(x,y)$, where $\varphi(x,y)=C$ is the integral of equation (2.10), we nullify the coefficient at $U_{\xi\xi}$ in equation (2.7).

If $\psi(x,y)=C$ is another integral of equation (2.10), independent of $\varphi(x,y)$, then assuming $\eta=\psi(x,y)$, we also zero the coefficient at $U_{\eta\eta}$.

Equation (2.10) splits into two equations:

$$\frac{dy}{dx}=\frac{b+\sqrt{b^2-ac}}{a}, \qquad (2.11)$$

$$\frac{dy}{dx}=\frac{b-\sqrt{b^2-ac}}{a}. \qquad (2.12)$$

Definition 4. Solutions of equations (2.11), (2.12) are called *characteristics* for equation (2.5).

2.5.3. CANONICAL FORMS OF EQUATIONS

Consider the region D, at all points of which equation (2.5) has the same type.

1. For an equation of hyperbolic type $b^2 - ac > 0$, the right-hand sides of equations (2.11) and (2.12) are valid and different.

Their general integrals, $\varphi(x,y) = C_1$ and $\psi(x,y) = C_2$, define families of characteristics that do not touch each other.

Choosing $\xi = \varphi(x,y), \eta = \psi(x, y)$, we get $\bar{a} = 0$, $\bar{c} = 0$.

Therefore, equation (2.7), after division by $\bar{b} \neq 0$, takes the form

$$U_{\xi\eta} = \bar{F}(\xi, \eta, U, U_\xi, U_\eta). \qquad (2.13)$$

Definition 5. The form of equation (2.13) is called *the first canonical form* of the hyperbolic type equation.

Another canonical form is often used, which can be obtained by replacing:

$$\alpha = \frac{1}{2}(\xi - \eta), \ \beta = \frac{1}{2}(\xi + \eta).$$

In this case, the equation takes the form

$$U_{\alpha\alpha} - U_{\beta\beta} = \bar{F}_1(\xi, \eta, U, U_\xi, U_\eta).$$

2. Let the equation (2.5) be of elliptical type in the domain D, that is, $b^2 - ac < 0$.

Then the equations of characteristics (2.11) and (2.12) with real coefficients a,b,c have complex conjugate right-hand sides. All the characteristics will be complex.

Assuming that the coefficients a,b,c are defined in the complex domain, and making a formal substitution:

$$\xi = \xi(x,y)\,,\ \eta = \xi^*(x,y)\,,$$

where $\xi(x,y) = C_1$ and $\xi^*(x,y) = C_2$ - complex conjugate integrals (2.11) and (2.12), we obtain the equation

$$U_{\xi\eta} = \bar{F}_2(\xi,\eta,U,U_\xi,U_\eta) \qquad (2.14)$$

in the complex domain.

If we make another replacement:

$$\alpha = \frac{1}{2}(\xi+\eta) = \mathrm{Re}\,\xi\,,\ \beta = -\frac{i}{2}(\xi-\eta) = \mathrm{Im}\,\xi\,,$$

then equation (2.14) will take the form

$$U_{\alpha\alpha} + U_{\beta\beta} = \bar{F}_3(\xi,\eta,U,U_\xi,U_\eta) \qquad (2.15)$$

already in the real domain.

Definition 6. The form (2.15) of the transformed equation (2.5) is the canonical form of an elliptic equation.

3. Finally, let us consider a parabolic equation in the region D:

$$b^2 - ac = 0 .$$

In this case, there is only one equation of characteristics

$$\frac{dy}{dx} = \frac{b}{a} .$$

Let $\xi(x,y) = C$ be its integral. Let's take an arbitrary twice differentiable function $\eta(x,y)$ such that the condition

$$\frac{D(\xi,\eta)}{D(x,y)} \neq 0$$

is satisfied.

Then, when replacing $\xi = \xi(x,y), \eta = \eta(x,y)$, equation (2.7) takes the form

$$U_{\eta\eta} = \overline{F}_4(\xi,\eta,U,U_\xi,U_\eta) . \qquad (2.16)$$

Definition 7.

The form (2.16) of the transformed equation (2.5) is the *canonical form* of a parabolic equation.