Exercises (24 tasks, 12 points in total)

Each correctly solved task gives 0.5 points.

1. Show that for the feasible region defined by

$$(x_1-1)^2+(x_2-1)^2 \le 2, \qquad (x_1-1)^2+(x_2+1)^2 \le 2, \qquad x_1 \ge 0,$$

the MFCQ is satisfied at $x^* = (0,0)^T$ but the LICQ is not satisfied.

2. Consider the following modification of Example 6 (see page 4, LectureNotes 7.pdf), where t is a parameter to be fixed prior to solving the problem:

$$\min_x \ (x_1 - \tfrac{3}{2})^2 + (x_2 - t)^4$$

subject to

$$\begin{bmatrix} 1-x_1-x_2 \\ 1-x_1+x_2 \\ 1+x_1-x_2 \\ 1+x_1+x_2 \end{bmatrix} \ge 0.$$

(a) For what values of t does the point $x^* = (1,0)^T$ satisfy the KKT conditions?
(b) Show that when $t = 1$, only the first constraint is active at the solution, and find the solution.

3. Consider the half space defined by

$$H = \{x \in \mathbb{R}^n \mid a^T x + \alpha \ge 0\},$$

where $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ are given. Formulate and solve the optimization problem for finding the point in H that has the smallest Euclidean norm.

4. Solve the problem

$$\min_x \ x_1 + x_2 \quad \text{s.t. } x_1^2 + x_2^2 = 1$$

by eliminating the variable x_2. Show that the choice of sign for a square root operation during the elimination process is critical; the "wrong" choice leads to an incorrect answer.

5. Consider the problem of finding the point on the parabola

$$y = \tfrac{1}{5}(x-1)^2$$

that is closest to $(1,2)$ in the Euclidean norm sense. This can be written as

$$\min f(x,y) = (x-1)^2 + (y-2)^2 \quad \text{s.t. } (x-1)^2 = 5y.$$

(a) Find all the KKT points for this problem. Is the LICQ satisfied?
(b) Which of these points are solutions?
(c) By directly substituting the constraint into the objective function and eliminating x, show that the solutions of this problem cannot be solutions of the original problem.

6. Consider the problem
$$\min_{x\in\mathbb{R}^2} f(x) = -2x_1 + x_2$$
subject to
$$(1-x_1)^3 - x_2 \geq 0, \qquad x_2 + 0.25x_1^2 - 1 \geq 0.$$
The optimal solution is $x^* = (0,1)^T$, where both constraints are active.
(a) Does the LICQ hold at this point?
(b) Are the KKT conditions satisfied?
(c) Write down the sets $F(x^*)$ and $C(x^*, \lambda^*)$.
(d) Are the second-order necessary conditions satisfied? Are the second-order sufficient conditions satisfied?

7. Find the minima of
$$f(x) = x_1x_2$$
on the unit circle $x_1^2 + x_2^2 = 1$. Illustrate this problem geometrically.

8. Find the maxima of
$$f(x) = x_1x_2$$
over the unit disk defined by
$$1 - x_1^2 - x_2^2 \geq 0.$$

9. Convert the following linear program to standard form:
$$\max_{x,y}\ c^Tx + d^Ty \quad \text{s.t. } A_1x = b_1,\ A_2x + B_2y \leq b_2,\ l \leq y \leq u.$$

10. Verify that the dual
$$\max\ b^T\lambda \quad \text{s.t.} \quad A^T\lambda + s = c,\ s \geq 0$$
is the original primal problem
$$\min\ c^Tx \quad \text{s.t.} \quad Ax = b,\ x \geq 0.$$

11. Complete the proof of Theorem 31 (page 31, LectureNotes 8.pdf) by showing that if the dual
$$\max\ b^T\lambda \quad \text{s.t. } A^T\lambda \leq c$$
is unbounded above, the primal
$$\min\ c^Tx \quad \text{s.t.} \quad Ax = b,\ x \geq 0$$
must be infeasible.

12. Theorem 31 does not exclude the possibility that both primal and dual are infeasible. Give a simple linear program for which such is the case.

13. Show that the dual of the linear program

$$\min c^T x \quad \text{s.t. } Ax \geq b, \; x \geq 0$$

is

$$\max b^T \lambda \quad \text{s.t. } A^T \lambda \leq c, \; \lambda \geq 0.$$

14. Consider the following linear program:

$$\min -5x_1 - x_2$$

subject to

$$x_1 + x_2 \leq 5, \quad 2x_1 + \tfrac{1}{2}x_2 \leq 8, \quad x \geq 0.$$

(a) Add slack variables x_3 and x_4 to convert this problem to standard form.
(b) Using the simplex procedure (page 16, LectureNotes 9.pdf), solve this problem showing at each step the basis and the vectors $\lambda, s_{\mathcal{N}}, x_{\mathcal{B}}$, and the value of the objective function. (The initial choice of $\mathcal{B}$ for which $x_{\mathcal{B}} \geq 0$ should be obvious once you have added the slacks in part (a).)

15. (a) Solve the quadratic program

$$\min f(x) = 2x_1 + 3x_2 + 4x_1^2 + 2x_1x_2 + x_2^2$$

subject to $x_1 - x_2 \geq 0$, $x_1 + x_2 \leq 4$, $x_1 \leq 3$. Illustrate it geometrically.
(b) If the objective function is redefined as $q(x) = -f(x)$, does the problem have a finite minimum? Are there local minimizers?

16. The problem of finding the shortest distance from a point x_0 to the hyperplane $\{x \mid Ax = b\}$, where A has full row rank, can be formulated as the quadratic program

$$\min \tfrac{1}{2}(x - x_0)^T(x - x_0) \quad \text{s.t. } Ax = b.$$

Show that the optimal multiplier is

$$\lambda^* = (AA^T)^{-1}(b - Ax_0),$$

and that the solution is

$$x^* = x_0 + A^T(AA^T)^{-1}(b - Ax_0).$$

Show that in the special case in which A is a row vector, the shortest distance from x_0 to the solution set of $Ax = b$ is $|b - Ax_0|/\|A\|^2$.

17. Use Theorem 21 (page 2, LectureNotes 7.pdf) to verify that the first-order necessary conditions for equality-constrained QPs (page 18, LectureNotes 10.pdf) are given by the KKT system (page 18, LectureNotes 10.pdf).

18. Verify that the inverse of the KKT matrix (page 25, LectureNotes 10.pdf) is given by

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}^{-1} = \begin{bmatrix} C & E \\ E^T & F \end{bmatrix},$$

with

$$C = G^{-1} - G^{-1}A^T(AG^{-1}A^T)^{-1}AG^{-1}, \quad E = G^{-1}A^T(AG^{-1}A^T)^{-1}, \quad F = -(AG^{-1}A^T)^{-1}.$$

19. For each of the alternative choices of initial working set W_0 in the example (page 21, LectureNotes 11.pdf), $W_0 = \{3\}$, $W_0 = \{5\}$, and $W_0 = \emptyset$, work through the first two iterations of the Active-Set Method for Convex QP (page 18, LectureNotes 11.pdf).

20. Program the Active-Set Method for Convex QP (page 18, LectureNotes 11.pdf) and use it to solve the problem

$$\min x_1^2 + 2x_2^2 - 2x_1 - 6x_2 - 2x_1x_2$$

subject to

$$\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 \leq 1, \quad -x_1 + 2x_2 \leq 2, \quad x_1, x_2 \geq 0.$$

Choose three starting points: one in the interior, one at a vertex, and one on a non-vertex boundary point.

21. Consider equality-constrained QPs (page 18, LectureNotes 10.pdf), and assume that A has full row rank and that Z is a basis for the null space of A. Prove that there are no finite solutions if Z^TGZ has negative eigenvalues.

22. (a) Assume that $A \neq 0$. Show that the KKT matrix

$$K = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \quad \text{is indefinite.}$$

(b) Prove that if the KKT matrix is nonsingular, then A must have full rank.

23. Consider the quadratic program

$$\max\ 6x_1 + 4x_2 - 13 - x_1^2 - x_2^2 \quad s.t. \quad x_1 + x_2 \leq 3, \quad x_1 \geq 0, \quad x_2 \geq 0.$$

First solve it graphically, and then use your implementation of the Active-Set Method for Convex QP.

24. Let W be an $n \times n$ symmetric matrix, and suppose that Z is of dimension $n \times t$. Suppose that Z^TWZ is positive definite and that $\bar{Z}$ is obtained by removing a column from Z. Show that $\bar{Z}^TW\bar{Z}$ is positive definite.