1.2 Sets and Elementary Operations on Them

1.2.1 The Concept of a Set

"We take a *set* to be an assemblage of definite, perfectly distinguishable objects of our intuition or our thought into a coherent whole." Thus did Georg Cantor,[7] the creator of set theory, describe the concept of a set.

Cantor's description cannot, of course, be considered a definition, since it appeals to concepts that may be more complicated than the concept of a set itself (and in any case, have not been defined previously). The purpose of this description is to explain the concept by connecting it with other concepts.

The basic assumptions of Cantorian (or, as it is generally called, "naive") set theory reduce to the following statements.

1^0. A set may consist of any distinguishable objects.
2^0. A set is unambiguously determined by the collection of objects that comprise it.
3^0. Any property defines the set of objects having that property.

And in fact the concept of the set of all sets, for example, is simply contradictory.

Proof Indeed, suppose that for a set M the notation $P(M)$ means that M is not an element of itself.

Consider the class $K = \{M \mid P(M)\}$ of sets having property P.

If K is a set either $P(K)$ or $\neg P(K)$ is true. However, this dichotomy does not apply to K. Indeed, $P(K)$ is impossible; for it would then follow from the definition of K that K contains K as an element, that is, that $\neg P(K)$ is true; on the other hand, $\neg P(K)$ is also impossible, since that means that K contains K as an element, which contradicts the definition of K as the class of sets that do not contain themselves as elements.

Consequently K is not a set. $\square$

In modern mathematical logic the concept of a set has been subjected to detailed analysis (with good reason, as we see). However, we shall not go into that analysis. We note only that in the current axiomatic set theories a set is defined as a mathematical object having a definite collection of properties.

1.2.2 The Inclusion Relation

The statement, “x is an element of the set X” is written briefly as

$$x \in X \quad (\text{or } X \ni x),$$

and its negation as

$$x \notin X \quad (\text{or } X \not\ni x).$$

When statements about sets are written, frequent use is made of the logical operators $\exists$ (“there exists” or “there are”) and $\forall$ (“every” or “for any”) which are called the *existence* and *generalization* quantifiers respectively.

Thus two sets are *equal* if they consist of the same elements.

The negation of equality is usually written as $A \neq B$.

If every element of A is an element of B, we write $A \subset B$ or $B \supset A$ and say that A is a *subset* of B or that B contains A or that B includes A. In this connection the relation $A \subset B$ between sets A and B is called the *inclusion relation* (Fig. 1.1).

Thus

$$(A \subset B) := \forall x \, \big((x \in A) \Rightarrow (x \in B)\big).$$

If $A \subset B$ and $A \neq B$, we shall say that the inclusion $A \subset B$ is *strict* or that A is a *proper* subset of B.

1.2.3 Elementary Operations on Sets

Let A and B be subsets of a set M.

a. The *union* of A and B is the set

$$A \cup B := \big\{x \in M \mid (x \in A) \vee (x \in B)\big\},$$

consisting of precisely the elements of M that belong to at least one of the sets A and B (Fig. 1.2).

b. The *intersection* of A and B is the set

$$A \cap B := \big\{x \in M \mid (x \in A) \wedge (x \in B)\big\},$$

formed by the elements of M that belong to both sets A and B (Fig. 1.3).

c. The *difference* between A and B is the set

$$A \backslash B := \big\{x \in M \mid (x \in A) \wedge (x \notin B)\big\},$$

consisting of the elements of A that do not belong to B (Fig. 1.4).

The difference between the set M and one of its subsets A is usually called the *complement* of A in M and denoted $C_M A$, or CA when the set in which the complement of A is being taken is clear from the context (Fig. 1.5).

Example As an illustration of the interaction of the concepts just introduced, let us verify the following relations (the so-called de Morgan[9] rules):

$$C_M(A \cup B) = C_M A \cap C_M B, \tag{1.1}$$

$$C_M(A \cap B) = C_M A \cup C_M B. \tag{1.2}$$

This set is called the *unordered pair* of sets A and B, to be distinguished from the *ordered* pair (A, B) in which the elements are endowed with additional properties to distinguish the first and second elements of the pair $\{A, B\}$. The equality

Now let X and Y be arbitrary sets. The set

$$X \times Y := \{(x, y) \mid (x \in X) \wedge (y \in Y)\},$$

formed by the ordered pairs (x, y) whose first element belongs to X and whose second element belongs to Y, is called the *direct* or *Cartesian product* of the sets X and Y (in that order!).

In the ordered pair $z = (x_1, x_2)$, which is an element of the direct product $Z = X_1 \times X_2$ of the sets X_1 and X_2, the element x_1 is called the *first projection* of the pair z and denoted $\mathrm{pr}_1 z$, while the element x_2 is the *second projection* of z and is denoted $\mathrm{pr}_2 z$.

By analogy with the terminology of analytic geometry, the projections of an ordered pair are often called the (first and second) *coordinates* of the pair.