Real Analysis 2024. Homework 2.

1. Let (f_n) be a sequence of real-valued measurable functions on X. Then the set
$$E = \{x \in X \mid f_n(x) \text{ converges to some limit } \}$$
is measurable. (Hint: use Cauchy criterion).

Proof. By translating the Cauchy criterion in terms of set operations, one can write
$$E = \bigcap_{k \geqslant 1} \bigcup_{N \geqslant 1} \{x || f_n(x) - f_m(x) |< 1/k\}$$
and the result follows because $|f_n - f_m|$ is measurable. □

2. Assume that μ is a measure on $(X, \mathcal{A})$. Prove that
$$\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$$
for every $E, F \in \mathcal{A}$.

Proof. Notice that
$$E = (E \backslash (E \cap F)) \cup (E \cap F); \quad F = (F \backslash (E \cap F)) \cup (E \cap F);$$
$$E \cup F = (E \backslash (E \cap F)) \cup (E \cap F) \cup (E \cap F).$$
Sets in the unions above are disjoint and measurable. Hence,
$$\mu E = \mu(E \backslash (E \cap F)) + \mu(E \cap F);$$
$$\mu F = \mu(F \backslash (E \cap F)) + \mu(E \cap F);$$
$$\mu(E \cup F) = \mu(E \backslash (E \cap F)) + \mu(E \cap F) + \mu(E \cap F).$$
and
$$\mu E + \mu F = \mu(E \backslash (E \cap F)) + \mu(F \backslash (E \cap F)) + 2\mu(E \cap F) = \mu(E \cup F) + \mu(E \cap F).$$
□

3. Prove that Borel $\sigma-$algebra $\mathcal{B}_{\mathbb{R}^2}$ is the same as product $\sigma-$algebra $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}}$.

   Proof. First of all notice that the product of two Borel sets is always Borel, consequently, $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\mathbb{R}^2}$.

   Assume that $G \in \mathcal{B}_{\mathbb{R}^2}$. We will

   □

4. Prove that a Cantor's set is a Borel set (see MA(3)).

   Proof. By the construction, Cantor's set is obtained as the intersection of a nested sequence of closed sets. Consequently, it is a set of type F_σ and is a Borel set. □

5. Consider a sequence of measures μ_i on a $\sigma-$algebra $\mathcal{A}$ and a sequence of nonnegative numbers $\alpha_i \in \mathcal{A}$. For a set $E \in \mathcal{A}$ let

$$\mu(E) = \sum_{i=1}^{\infty} \alpha_i \mu_i(E).$$

   Prove that μ is a measure on $\mathcal{A}$.

   Proof. First, $\mu(\varnothing) = 0$ since $\mu_i(\varnothing) = 0$. To prove countable additivity consider a sequence of disjoint sets E_k and $E = \bigcup\limits_{k=1}^{\infty} E_k$. Hence,

$$\mu(E) = \sum_{i=1}^{\infty} \alpha_i \mu_i(E) = \sum_{i=1}^{\infty} \left(\sum_{k=1}^{\infty} \alpha_i \mu_i(E_k) \right) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_i \mu_i(E_k) \right) = \sum_{k=1}^{\infty} \mu(E_k). \quad (1)$$

   □

The change of the order of summation in the proof can be explained in the following lemma.

Lemma 0.1. *Assume that $a_{k,i} \geq 0$, $k, i \in \mathbb{N}$. Then*

$$S_1 = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} a_{k,i} = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{k,i} = S_2.$$

Proof. Assume first that

$$S_1 = +\infty.$$

Then for every $M > 0$ there exists $N > 0$ such that

$$\sum_{k=1}^{N} \sum_{i=1}^{\infty} a_{k,i} > M$$

Then

$$S_2 = \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{k,i} \geq \sum_{i=1}^{\infty} \sum_{k=1}^{N} a_{k,i} = \sum_{k=1}^{N} \sum_{i=1}^{\infty} a_{k,i} > M.$$

Since $M > 0$ is arbitrary this implies that $S_2 = +\infty = S_1$. (And finitenss of one of the sums implies finitness of the other).

Assume that $S_1 < +\infty$. Then for every $\varepsilon > 0$ there exists $N > 0$ such that

$$S_1 - \varepsilon \leq \sum_{k=1}^{N} \sum_{i=1}^{\infty} a_{k,i} = \sum_{i=1}^{\infty} \sum_{k=1}^{N} a_{k,i} \leq S_1.$$

Since $\sum\limits_{k=1}^{N} a_{k,i} \leq \sum\limits_{k=1}^{\infty} a_{k,i}$ then

$$S_1 - \varepsilon \leq \sum_{i=1}^{\infty} \sum_{k=1}^{N} a_{k,i} \leq \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{k,i} = S_2.$$

Since $\varepsilon > 0$ is arbitrary this implies that $S_1 \leq S_2$ and, by symmetry, $S_2 \leq S_1$ and, hence, $S_1 = S_2$. $\square$