Functional Analysis Set of topics for exam

Star symbol (*) at the beginning of the line and **bold** font means that this question can be asked with proof

1. Linear spaces and linear operators Definition of quotient space

   Definition of metric space

   Definition of a normed vector space

   Definition of equivalent norms

   Definition of Banach space

   Definition of absolutely converges series

   Theorem (completeness criterion)

   Definition of inner product

   Definition of inner product space

   Theorem (Elementary inequalities for the norm generated by the inner product)

   Proposition (Canonical inner product on L^2)

   Cauchy-Bunyakovsky-Schwarz inequality in L^2 and l^2

   ***Parallelogram law**

   ***Proposition (Relations between L^p spaces)**

   Definition of orthogonal complement

   Definition of a Hilbert space

   Theorem (Orthogonality principle)

   Definition of orthogonal and orthonormal systems

   ***Theorem (Convergence of orthogonal series)**

   Definition of linear functional and linear operators

   Definition of bounded linear mapping

   Definition of space of bounded linear operators and operator norm

   ***Theorem (Alternative expressions for the operator norm)**

2. Compactness and separability.

   Definition of compact and relative compact spaces

   Basic properties of compact sets

   Definition of epsilon-net

   Definition of totally bounded

   Theorem (Heine-Borel)

   ***Lemma («the almost perpendicular» lemma, Riesz Lemma)**

Theorem (Riesz).

Theorem (Hausdorff compactness criterion)

***Theorem (Compactness criterion for infinite-dimensional spaces, approximation by finite dimensional subspaces).**

Definition of equi-continuous

Theorem (Arzela-Ascoli, a compactness criteria for C(K))

Proposition (Compactness in l^p)

Proposition (Compactness in L^1)

Definition of Space separates points and strong separates points

***Theorem (The Stone–Weierstrass Theorem, R-case)**

***Theorem (The Stone–Weierstrass Theorem, C-case)**

Definition of integrable step function

3. Applications of Baire theorem

   Definition (Baire Category)

   ***Theorem (Baire Category Theorem)**

   ***Theorem (Completeness of the operator space)**

   Definition of Dual space

   ***Theorem (Banach-Steinhaus uniform boundedness principle)**

   Definition of Fundamental set

   ***Corollary (Banach-Steinhaus theorem)**

   Definition (Perfectly convex set)

   ***Theorem (Open mapping theorem by S. Banach)**

   Inverse Operator Theorem

   Definition of quotient map

   Corollary (Surjective operators are essentially quotient maps)

   Definition of graph.

   ***Closed graph theorem**

   Theorem (Hellinger-Toeplitz)

4. Bounded linear functionals

   ***Proposition (Linear functionals and hyperplanes)**

   ***Proposition. A linear functional on a normed space is continuous if and only if its kernel is closed**

   ***Theorem (Riesz representation theorem)**

   Theorem (Radon-Nikodym theorem for finite measures)

   ***Radon-Nikodym theorem for σ-finite measures**

Definition of charge

Proposition (Continuity of a charge)

***Theorem (Radon-Nikodym theorem for charges)**

Theorem The dual of L^p

Theorem ((C(K))*, Riesz-Markov theorem)

The Axiom of Choice

Definition of quasi-seminorm and seminorm

Theorem (Hahn–Banach)

***Proposition (Supporting functional)**

Theorem (Second dual space)

Definition of absorbing set. Minkowski functional

Theorem (Separating a point from a convex set)

Theorem (Separation of open convex sets)

Corollary (Separation of closed convex sets)

5. The Weak and Weak* Topologies

Definition of topological vector space

***Proposition (Weak and strong boundedness)**

Lemma (Mazur)

***Lemma (Testing weak convergence on a dense set)**

***Theorem (Weak convergence in c^0 and l^p)**

***Theorem (Weak convergence in C(K))**

***Theorem (Weak convergence in L^p)**

Lemma (Weak Closure of the Unit Sphere)

***Theorem (Banach-Alaoglu)**

***Theorem (Universality of C(K))**

Definition of face and extremal point

Theorem (Krein–Milman)

***Theorem (Banach–Alaoglu: The Separable Case)**

Definition of Φ-invariant measure

Definition Φ -ergodic and μ -ergodic

***Theorem (Ergodic Measures are Extremal)**

Theorem (Von Neumann's Mean Ergodic Theorem)

Definition of Projection

Theorem (Ergodic Theorem)

6. Compact operators. Elements of spectral theory

   Definition of compact operator

   ***Proposition (Compactness of the operator with a continuous kernel)**

   ***Proposition (Properties of $\mathbf{K(X \to Y)}$)**

   Definition of Hilbert-Schmidt operators

   ***Proposition (Hilbert-Schmidt integral operators)**

   Definition of Adjoint operators

   Definition of annihilator

   ***Proposition (Duality of kernel and image)**

   ***Theorem (Schauder)**

   ***Proposition (Isomorphic embeddings)**

   Theorem (Fredholm alternative)

   Definition of Spectrum of linear operator

   Definition of Resolvent operator

   Theorem (Resolvent set and resolvent operators properties)

   Definition of spectral radius

   Theorem (Gelfand's formula)

   ***Theorem (Point spectrum of compact operators)**

   ***Corollary (Classification of spectrum of compact operators)**

   ***Theorem (Spectrum of unitary operators)**

7. Self-adjoint operators on Hilbert space

   Definition of Self-adjoint operators

   ***Proposition (Norm of a self-adjoint operator)**

   ***Theorem about spectrum interval**

   ***Theorem (Spectral theorem for compact self-adjoint operators, Hilbert-Schmidt theorem)**

   ***Theorem (Separation of variables)**

   Definition of positive operators

   Definition of partial order

   Definition of Polynomials of an operator

   ***Theorem about spectrum of polynomials of an operator**