Real Analysis 2024. Homework 8.

1. Find condition on p under which the integral is finite
$$\iint_E \frac{dxdy}{(x^2+y^2)^p}, \quad E = \{(x,y) : 0 < x < 1,\ 0 < y < x^2\}.$$

Solution.
$$I = \iint_E \frac{dxdy}{(x^2+y^2)^p} = \int_0^1 dx \int_0^{x^2} \frac{dy}{(x^2+y^2)^p} = [y = x^2 t] =$$
$$\int_0^1 dx \int_0^1 \frac{dy}{(x^2+t^2x^4)^p} = \int_0^1 \frac{dx}{x^{2p-2}} \int_0^1 \frac{dy}{(1+t^2x^2)^p}$$
Since
$$\frac{1}{2^p} \le \frac{1}{(1+t^2x^2)^p} \le 1, \quad 0 \le t, x \le 1,$$
then
$$\frac{1}{2^p} \int_0^1 \frac{dx}{x^{2p-2}} \le I \le \int_0^1 \frac{dx}{x^{2p-2}}$$
and I converges if and only if $p < 1/2$.

2. Find condition on p under which the integral is finite
$$\iint_{x+y>1} \frac{\sin x \sin y}{(x+y)^p} dxdy.$$
Caution: the integrability of f is equivalent to integrability of $|f|$!

Solution.

The integral converges if and only if
$$I = \iint_{x+y>1} \frac{|\sin x \sin y|}{(x+y)^p} dxdy$$
converges. Let $u = x$, $v = x + y$. The Jacobian of this change is equal to 1 and
$$I = \int_{v>1} \frac{|\sin u \sin(v-u)|}{v^p} dudv$$
The function
$$F(u) = \int_1^{+\infty} \frac{\sin(v-u)}{v^p} dv$$

is positive and 2π-periodic. Hence,

$$I = \int_{-\infty}^{+\infty} F(u)\,|\sin u|\,du = \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} F(u)\,|\sin u|\,du + \infty.$$

3. Find condition on p under which the integral is finite

$$I = \int_E \frac{dxdydz}{|x+y-z|^p}, \quad E = [-1,1]^3.$$

Solution. First, notice that the cube $[-1,1]^3$ is divided by a plane $z = x+y$ into two equal parts and

$$I = 2 \int\limits_{\substack{-1\le x,y\le 1,\\ x+y\le 1}} dxdy \int_{x+y}^{1} \frac{dz}{(z-x-y)^p}$$

The integral $\int_{x+y}^{1} \frac{dz}{(z-x-y)^p}$ converges if and only if $p<1$. In this case

$$\int_{x+y}^{1} \frac{dz}{(z-x-y)^p} = \frac{1}{1-p}\frac{1}{(1-x-y)^{p-1}}$$

and

$$I = \frac{2}{1-p} \int\limits_{\substack{-1\le x,y,\le 1\\ x+y\le 1}} (1-x-y)^{1-p} dxdy$$

This integral converges since $p<1$ and function $(1-x-y)^{1-p}$ is continuous in $\{-1\le x,y,\le 1; x+y\le 1\}$.