Numerical series

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Definition 1 *Let $\{a_k\}_{k=1}^{\infty}$ be a real or complex sequence. The symbol*

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \dots$$

*is called a **numeric series**, numbers a_k are called **terms**. The numbers $S_n = \sum_{k=1}^{n} a_k$ are referred to as **partial sums** of the series. If the sequence $\{S_n\}_{n=1}^{\infty}$ has the limit S (finite or infinite), then S is called the **sum** of the series and we write $\sum_{k=1}^{\infty} a_k = S$. Otherwise, the series has no a sum. If the sequence $\{S_n\}_{n=1}^{\infty}$ **converges**, that is S is a finite number, then it is said that the series converges, otherwise, the series is referred to as **divergent**.*

Thus,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k,$$

if the limit exists (finite or infinite).

Example. $\dfrac{1}{1\cdot 4} + \dfrac{1}{4\cdot 7} + \dots + \dfrac{1}{(3n-2)(3n+1)} + \dots$

$$S_n = \frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + \dots + \frac{1}{(3n-2)(3n+1)}$$

$$= \frac{1}{3}\left(\left(1-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \dots + \left(\frac{1}{3n-2} - \frac{1}{3n+1}\right)\right) = \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) \to \frac{1}{3}.$$

Example

$u = q\sin\alpha + q^2 \sin 2\alpha + \dots + q^n \sin n\alpha + \dots,$

$v = q\cos\alpha + q^2 \cos 2\alpha + \dots + q^n \cos n\alpha + \dots, |q| < 1.$ Let (u_n) and (v_n) be sequences of partial sums of u and v

$$u_n + iv_n = qe^{i\alpha} + q^2 e^{2i\alpha} + \dots + q^n e^{in\alpha} = \frac{qe^{i\alpha} - q^{n+1}e^{i(n+1)\alpha}}{1-qe^{i\alpha}} \to \frac{qe^{i\alpha}}{1-qe^{i\alpha}}$$

$$= q\left(\frac{\cos\alpha - q}{1-2q\cos\alpha + q^2} + i\frac{\sin\alpha}{1-2q\cos\alpha+q^2}\right).$$

So,

$$u = q\frac{\cos\alpha - q}{1-2q\cos\alpha+q^2}, \quad v = \frac{q\sin\alpha}{1-2q\cos\alpha+q^2}.$$

S1. If $\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{m} a_k$ converges, then for any $m\in\mathbb{N}$ the series $\sum_{k=m+1}^{\infty} a_k = \sum_{k=1}^{m} a_k$ converges and

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{\infty} a_k.$$

And conversely, if for some $m\in\mathbb{N}$ the series $\sum_{k=m+1}^{\infty} a_k$ converges, then the series $\sum_{k=1}^{\infty} a_k$ converges as well.

Definition 2 *The series* $\sum\limits_{k=m+1}^{\infty} a_k$ *is called the* m*-th* ***reminder*** *of the series* $\sum\limits_{k=1}^{\infty} a_k$.

S2. If $\sum\limits_{k=1}^{\infty} a_k$ converges, then $\sum\limits_{k=m+1}^{\infty} a_k \to 0$ as $m \to \infty$.

S3. Linearity. If $\sum\limits_{k=1}^{\infty} a_k$, $\sum\limits_{k=1}^{\infty} b_k$ converge, $\alpha, \beta \in \mathbb{R}$, or $\mathbb{C}$, then $\sum\limits_{k=1}^{\infty} (\alpha a_k + \beta b_k)$ converge and

$$\sum_{k=1}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^{\infty} a_k + \beta \sum_{k=1}^{\infty} b_k.$$

S4. If $(z_k)_{k=1}^{\infty}$ is a sequence of complex numbers, $x_k = \mathrm{Re} z_k$, $x_k = \mathrm{Im} z_k$, then the convergence of the series $\sum\limits_{k=1}^{\infty} z_k$ is equivalent to the simultaneous convergence of the series $\sum\limits_{k=1}^{\infty} x_k$ and $\sum\limits_{k=1}^{\infty} y_k$. Moreover,

$$\sum_{k=1}^{\infty} z_k = \sum_{k=1}^{\infty} x_k + i \sum_{k=1}^{\infty} y_k.$$

S4. Monotonicity. If $a_k, b_k \in \mathbb{R}$, $\sum\limits_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k \in \overline{\mathbb{R}}$, and $a_k \leq b_k$ for all $k \in \mathbb{N}$, then

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k.$$

Theorem 3 ***(Necessary condition for the convergence of the series.)*** *If the series* $\sum\limits_{k=1}^{\infty} a_k$ *converges, then* $a_n \longrightarrow 0$ *as* $n \to \infty$.

Proof. Let $\sum\limits_{k=1}^{\infty} a_k = S$, then $a_n = S_n - S_{n-1} \longrightarrow S - S = 0$ as $n \to \infty$.

Example. Let us prove that $\sum\limits_{k=1}^{\infty} \sin n\alpha$, $\alpha \neq \pi m$, $m \in \mathbb{Z}$ diverges.

Assume the converse. Then

$$\lim_{n \to \infty} \sin n\alpha = 0, \quad \lim_{n \to \infty} \sin(n+1)\alpha = 0,$$

that is

$$\lim_{n \to \infty} (\sin n\alpha \cos \alpha + \cos n\alpha \sin \alpha) = 0$$

so $\lim_{n \to \infty} \cos n\alpha = 0$. This contradicts to

$$\sin^2 n\alpha + \cos^2 n\alpha = 1.$$

Theorem 4 ***(Cauchy's criterion for the convergence of the series.)*** *The convergence of the series* $\sum_{k=1}^{\infty} a_k$ *is equivalent to*

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n > N \quad \forall p \in \mathbb{N} \quad \left| \sum_{k=n+1}^{n+p} a_k \right| < \varepsilon.$$

Example. $\dfrac{\cos x}{1^2}+\dfrac{\cos x^2}{2^2}+\ldots+\dfrac{\cos x^n}{n^2}+\ldots$

$$|S_{n+p}-S_n|=\left|\frac{\cos x^{n+1}}{(n+1)^2}+\frac{\cos x^{n+2}}{(n+2)^2}+\ldots+\frac{\cos x^{n+p}}{(n+p)^2}\right|\le\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(n+p)^2}$$

$$<\frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}\ldots+\frac{1}{(n+p-1)(n+p)}=\frac{1}{n}-\frac{1}{n+p}<\frac{1}{n}.$$

Example.

$$1-1+\frac{1}{2}+\frac{1}{2}-\frac{1}{2}-\frac{1}{2}+\cdots+\underbrace{\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}}_{n\text{ times}}$$

$$\underbrace{-\frac{1}{n}-\frac{1}{n}-\cdots-\frac{1}{n}}_{n\text{ times}}+\frac{1}{n+1}+\cdots$$

By Cauchy's criterion for any $N\in\mathbb{N}$ there exist $n\in\mathbb{N}$, $n>N$, $p\in\mathbb{N}$ such that

$$a_{n+1}=\frac{1}{p},\quad a_{n+2}=\frac{1}{p},\ldots,a_{n+p}=\frac{1}{p}.$$

So, $\sum\limits_{k=n+1}^{n+p}a_n=1$, therefore the series diverges.

Lemma 1 *Let $a_k\ge 0$ for all $k\in\mathbb{N}$. Then the convergence of the series $\sum\limits_{k=1}^{\infty}a_k$ is equavalent to the boundedness from above of the sequence $\{S_n\}$.*

Example. Prove that if $\sum\limits_{n=1}^{\infty}a_n, a_n\ge 0$ converges, then $\sum\limits_{n=1}^{\infty}a_n^2$ converges as well.

Let S_n and C_n be partial sums of the first and the second series. The sequence (C_n) is nondecreasing. Since $a_n\ge 0$

$$C_n=a_1^2+a_2^2+\ldots+a_n^2<(a_1+a_2+\ldots+a_n)^2=S_n^2\le\text{const}$$

By Lemma, the second series converges.

Theorem 5 ***(Comparison test for convergence of positive series.)*** *Let $a_k,b_k\ge 0$ for all $k\in\mathbb{N}, a_k=O(b_k)$ as $k\to\infty$.*

1. *If the series $\sum\limits_{k=1}^{\infty}b_k$ converges, then the series $\sum\limits_{k=1}^{\infty}a_k$ converges as well.*
2. *If the series $\sum\limits_{k=1}^{\infty}a_k$ diverges, then the series $\sum\limits_{k=1}^{\infty}b_k$ diverges as well.*

Corollary 1 ***(Comparison test in a limit form.)*** *Let $a_k\ge 0$, $b_k>0$ for all $k\in\mathbb{N}$ and there exists $\lim\limits_{k\to\infty}\dfrac{a_k}{b_k}=\ell\in[0,+\infty]$.*

1. *If $\ell\in[0,+\infty)$, and the series $\sum\limits_{k=1}^{\infty}b_k$ converges, then the series $\sum\limits_{k=1}^{\infty}a_k$ converges as well.*
2. *If $\ell\in(0,+\infty]$, and the series $\sum\limits_{k=1}^{\infty}a_k$ converges, then the series $\sum\limits_{k=1}^{\infty}b_k$ converges as well.*
3. *If $\ell\in(0,+\infty)$, then the series $\sum\limits_{k=1}^{\infty}a_k u\sum\limits_{k=1}^{\infty}b_k$ converge or diverge simultaneously.*

Example. $\sqrt{2}+\sqrt{2-\sqrt{2}}+\sqrt{2-\sqrt{2+\sqrt{2}}}+\ldots$

$$a_n=\sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}},\quad n\in\mathbb{N},$$

$\sqrt{2}=2\cos\frac{\pi}{4}$, so $a_n=\sqrt{2-2\cos\frac{\pi}{2^n}}=2\sin\frac{\pi}{2^{n+1}}<\frac{\pi}{2^n}$. The series $\sum\limits_{n=1}^{\infty}\frac{\pi}{2^n}$ converges, so by comparison test the initial series converges.

Theorem 6 ***(Cauchy's root test.)*** *Let $a_k\geq 0$ for all $k\in\mathbb{N}$, $\mathcal{K}=\overline{\lim}_{n\to\infty}\sqrt[n]{a_n}$.*

1. If $\mathcal{K}>1$, then the series $\sum\limits_{k=1}^{\infty}a_k$ diverges.

2. If $\mathcal{K}<1$, then the series $\sum\limits_{k=1}^{\infty}a_k$ converges.

Example. $\sum\limits_{n=1}^{\infty}\left(\frac{1+\cos n}{2+\cos n}\right)^{2n-1nn}$. Since

$$\overline{\lim_{n\to\infty}}\left(\frac{1+\cos n}{2+\cos n}\right)^{2-\frac{\ln n}{n}}\leqslant\lim_{n\to\infty}\left(\frac{2}{3}\right)^{2-\frac{\ln n}{n}}=\frac{4}{9}<1,$$

by Cauchy's root test it follows that the series converges.

Example. Whether the series $\sum\limits_{n=1}^{\infty}\frac{n!}{n\sqrt{n}}$ converges or diverges.

By Stirling's formula

$$n!\sim\left(\frac{n}{e}\right)^n\sqrt{2\pi n},\qquad n\to\infty,$$

we obtain

$$\sqrt[n]{a_n}\sim e^{-1}(2\pi)^{1/(2n)}n^{1/(2n)-1/\sqrt{n}}n\sim\frac{n}{e},\quad n\to\infty$$

The series diverges.

Theorem 7 ***(d'Alembert's ratio test.)*** *Let $a_k>0$ for all $k\in\mathbb{N}$ and there exists $\mathcal{D}=\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}\in[0,+\infty]$.*

1. If $\mathcal{D}>1$, then series $\sum\limits_{k=1}^{\infty}a_k$ diverges.

2. If $\mathcal{D}<1$, then series $\sum\limits_{k=1}^{\infty}a_k$ converges.

Example. $\sum\limits_{n=1}^{\infty}\frac{n!}{(i+2)(i+4)\ldots(i+2n)}$.

By $x+iy=\sqrt{x^2+y^2}(\cos\varphi+i\sin\varphi)=\sqrt{x^2+y^2}e^{i\varphi}$, we get

$$\frac{1}{(i+2)(i+4)\ldots(i+2n)}=\frac{\cos\varphi_n-t\sin\varphi_n}{\sqrt{5}\sqrt{17}\ldots\sqrt{4n^2+1}},$$

where $\varphi_n=\sum\limits_{k=1}^{n}\arctan\frac{1}{2k}$. Since

$$\frac{n!\,|\cos\varphi_n|}{\sqrt{5}\sqrt{17}\ldots\sqrt{4n^2+1}}\leq\frac{n!}{\sqrt{5}\sqrt{17}\ldots\sqrt{4n^2+1}},$$

$$\frac{n!\,|\sin\varphi_n|}{\sqrt{5}\sqrt{17}\ldots\sqrt{4n^2+1}} \le \frac{n!}{\sqrt{5}\sqrt{17}\ldots\sqrt{4n^2+1}}$$

and the series

$$\sum_{n=1}^{\infty} \frac{n!}{\sqrt{5}\sqrt{17}\ldots\sqrt{4n^2+1}}$$

converges by d'Alembert's ratio test, it follows that the initial series converges.

Theorem 8 *(Cauchy's integral test.) Let the function f be monotone on $[1,+\infty)$. Then the series $\sum\limits_{k=1}^{\infty} f(k)$ and the integral $\int_1^{+\infty} f$ converge or diverge simultaneously.*

Proof. Suppose f decreases. If $f(x_0) < 0$ for some x_0, then $\lim\limits_{x\to+\infty} f(x) \le f(x_0) < 0$ and the series and the integral diverge to $-\infty$. So, we consider $f \ge 0$. In this case the sum and the integral exist and belong to $[0,+\infty]$.

By decreasing f for all $k \in \mathbb{N}$ we get

$$f(k+1) \le \int_k^{k+1} f \le f(k).$$

Fix $n \in \mathbb{N}$ and add the inequalities k from 1 to n :

$$\sum_{k=1}^{n} f(k+1) \le \int_1^{n+1} f \le \sum_{k=1}^{n} f(k).$$

Changing the index in the LHS and passing the limit n к ∞, we obtain

$$\sum_{k=2}^{\infty} f(k) \le \int_1^{+\infty} f \le \sum_{k=1}^{\infty} f(k).$$

Example. $\sum\limits_{k=1}^{\infty} \frac{1}{k^\alpha}$ converges for $\alpha > 1$ and diverges for $\alpha \le 1$ by Cauchy's integral test. We compare the series and the integral $\int_1^{+\infty} \frac{\mathrm{d}x}{x^\alpha}$.

Example. $\sum\limits_{n=1}^{\infty} \frac{1}{\log^2\left(\sin\frac{1}{n}\right)}$.

$\sin\frac{1}{n} > \frac{2}{\pi n}, n \in \mathbb{N}, \Rightarrow \log^2\left(\sin\frac{1}{n}\right) < \log^2\left(\frac{\pi n}{2}\right)$. So,

$$\frac{1}{\log^2\left(\sin\frac{1}{n}\right)} > \frac{1}{\log^2\left(\frac{\pi n}{2}\right)} > \frac{2}{\pi n \log\frac{\pi n}{2}} = O\left(\frac{1}{n\log n}\right), n \to \infty.$$

By Cauchy's integral test and comparison test, the series diverges.

Example. $\sum\limits_{n=1}^{\infty}\left(n^{n^\alpha}-1\right)$ Since $\lim\limits_{n\to\infty} a_n \ne 0$ for $\alpha \ge 0$ it follows that the series diverges. Consider $\alpha < 0$. By Taylor's formula, we obtain

$$n^{n^\alpha} - 1 = \exp\left(n^\alpha \log n\right) - 1 = \frac{\log n}{n^{-\alpha}} + o\left(\frac{\log n}{n^{-\alpha}}\right) = O\left(\frac{\log n}{n^{-\alpha}}\right),\ n \to \infty.$$

By Cauchy's integral test and comparison test, the series converges for $\alpha < -1$.

Example. Whether the series $\sum_{n=1}^{\infty} a_n$ converges or diverges, $a_n = (\sqrt{n+1} - \sqrt{n})^p \log \frac{n-1}{n+1}$, $n > 1$. By Taylor's formula, we get

$$a_n = \frac{1}{(\sqrt{n+1}+\sqrt{n})^p} \ln\left(1 - \frac{2}{n+1}\right) = n^{-\frac{p}{2}} \left(2 + o\left(\frac{1}{n}\right)\right)^{-p} \left(-\frac{2}{n+1} + o\left(\frac{1}{n}\right)\right) =$$

$$n^{-\frac{p}{2}} 2^{-p} \left(1 + o\left(\frac{1}{n}\right)\right)\left(-\frac{2}{n+1} + o\left(\frac{1}{n}\right)\right) = O\left(\frac{1}{n^{1+\frac{p}{2}}}\right), \quad n \to \infty.$$

By the previous example the series converges for $p > 0$.

Example. Let $0 \le \lambda_1 < \lambda_2 < \ldots, n \in \mathbb{N}$ be roots of the equation $\tan x = x$. Whether the series $\sum_{n=1}^{\infty} \lambda_n^{-2}$ converges or diverges.

Since $n\pi < \lambda_n < n\pi + \frac{\pi}{2}$, it follows that

$$\frac{1}{\left(n\pi + \frac{\pi}{2}\right)^2} < \frac{1}{\lambda_n^2} < \frac{1}{n^2\pi^2}.$$

By comparison test the series converges.

Example. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(n+a)^{n+b}(n+b)^{n+a}}, a > 0, b > 0$

$$a_n = \frac{n^{2n}}{(n+a)^{n+b}(n+b)^{n+a}} = \frac{1}{n^{a+b}\left(1+\frac{a}{n}\right)^{n+b}\left(1+\frac{b}{n}\right)^{n+a}}.$$

$\left(\left(1+\frac{a}{n}\right)^{b+n}\right) \to e^a$ and $\left(\left(1+\frac{b}{n}\right)^{a+n}\right) \to e^b$ as $n \to \infty$, so $a_n \sim \frac{e^{-a-b}}{n^{a+b}}$ as $n \to \infty$. By cmparison test the series converges for $a + b > 1$.

Example. $\sum_{k=2}^{\infty} \frac{1}{k^\alpha \log^\beta k}$ converges for $\alpha > 1, \beta$ is arbitrary, or $\alpha = 1, \beta > 1$. It diverges in other cases. We compare the series and the integral $\int_2^{+\infty} \frac{dx}{x^\alpha \log^\beta x}$.

Remark. Suppose f decreases on $[1, +\infty), f \ge 0$.

$$A_n := \sum_{k=1}^{n} f(k) - \int_1^{n+1} f.$$

$\{A_n\}$ is increasing.

$$A_{n+1} - A_n = f(n+1) - \int_{n+1}^{n+2} f \ge 0$$

Moreover,

$$0 \le A_n = f(1) - f(n+1) + \sum_{k=2}^{n+1} f(k) - \int_1^{n+1} f \le f(1) - f(n+1) \le f(1),$$

So, $\{A_n\}$ is bounded. Therefore, there exists a finit limit $\lim_{n\to\infty} A_n = c \ge 0$. In other words, $A_n = c + \varepsilon_n$, where $\varepsilon_n \to 0$. Thus, we obtain the following asymptotic formula

$$\sum_{k=1}^{n} f(k) = \int_1^{n+1} f + c + \varepsilon_n, \quad \varepsilon_n \to 0$$

If the integral and the series converge, than $c = \sum_{k=1}^{\infty} f(k) - \int_1^{+\infty} f$. The result is most applicable when the integral and the series diverge. In this case

$$\sum_{k=1}^{n} f(k) \sim \int_1^{n+1} f.$$

Example. For the harmonic series, we obtain

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \int_1^{n+1} \frac{\mathrm{d}x}{x} + \gamma + \varepsilon_n = \log(n+1) + \gamma + \varepsilon_n.$$

The constant γ is called Euler's constant. Taking into account $\log(n+1) - \log n = \log\left(1 + \frac{1}{n}\right) \to 0$, we get

$$H_n = \log n + \gamma + \delta_n, \qquad \delta_n \to 0.$$

In praticular,

$$H_n \sim \log n$$

Since

$$\log(n+1) = \sum_{k=1}^{n} (\log(k+1) - \log k) = \sum_{k=1}^{n} \log\left(1 + \frac{1}{k}\right)$$

it follows that

$$\gamma = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \log\left(1 + \frac{1}{k}\right)\right).$$

It could be calculated that

$$\gamma = 0,5772156649\ldots$$

Let us prove the estimate for the error δ_n

$$0 < \delta_n < \frac{1}{2n}.$$

We have

$$\delta_n = \frac{1}{n} + \sum_{k=n}^{\infty} \left(\log\left(1 + \frac{1}{k}\right) - \frac{1}{k}\right).$$

We claim that

$$\frac{1}{k+1} < \log\left(1 + \frac{1}{k}\right) < \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right), \qquad k \in \mathbb{N}.$$

By $\log(1+x) < x (x > -1, x \neq 0)$, we get

$$\log\left(1 + \frac{1}{k}\right) = -\log\frac{k}{k+1} = -\log\left(1 - \frac{1}{k+1}\right) > \frac{1}{k+1}.$$

To check the right ineqiality rewrithe it in the form

$$\log u < \frac{1}{2}\left(u - \frac{1}{u}\right), \qquad u = 1 + \frac{1}{k} > 1$$

We denote by $f(u)$ the LHS, and by $g(u)$ the RHS. Since $f(1) = g(1)$ and $u > 1$

$$f'(u) = \frac{1}{u} < \frac{1}{2}\left(1 + \frac{1}{u^2}\right) = g'(u)$$

we conclude that $f(u) < g(u)$ for all $u > 1$.

So, on the one hand,

$$\delta_n > \frac{1}{n} + \sum_{k=n}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k}\right) = 0.$$

On the other hand,

$$\delta_n < \frac{1}{n} + \frac{1}{2}\sum_{k=n}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k}\right) = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}.$$

Example. Let $\alpha \in (0,1)$, $f(x) = \dfrac{1}{x^\alpha}$. Then we get

$$\sum_{k=1}^{n}\frac{1}{k^\alpha} = \int_1^{n+1}\frac{dx}{x^\alpha} + c_\alpha + \varepsilon_n, \quad \varepsilon_n \to 0.$$

Taking into account

$$\int_1^{n+1}\frac{dx}{x^\alpha} = \frac{(n+1)^{1-\alpha} - 1}{1-\alpha} = \frac{n^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha} + o(1),$$

we have

$$\sum_{k=1}^{n}\frac{1}{k^\alpha} = \frac{n^{1-\alpha}}{1-\alpha} + d_\alpha + \delta_n, \quad \delta_n \to 0,$$

where $d_\alpha = c_\alpha - \dfrac{1}{1-\alpha}$. So,

$$\sum_{k=1}^{n}\frac{1}{k^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}, \quad \alpha \in (0,1)$$

Remark. Suppose f decreases on $[1,+\infty)$, $f \geq 0$, the integral and the series converge. Then

$$\int_{n+1}^{+\infty} f \leq \sum_{k=n+1}^{\infty} f(k) \leq \int_{n}^{+\infty} f$$

Example. Consider $\alpha > 1$, $f(x) = \dfrac{1}{x^\alpha}$. By

$$\int_m^{\infty}\frac{dx}{x^\alpha} = \frac{1}{(\alpha-1)m^{\alpha-1}}$$

we obtain

$$\frac{1}{(\alpha-1)(n+1)^{\alpha-1}} \leq \sum_{k=n+1}^{\infty}\frac{1}{k^\alpha} \leq \frac{1}{(\alpha-1)n^{\alpha-1}}.$$

So,

$$\sum_{k=n+1}^{\infty}\frac{1}{k^\alpha} \sim \frac{1}{(\alpha-1)n^{\alpha-1}}, \quad \alpha > 1$$

Definition 9 *It is said that series* $\sum_{k=1}^{\infty} a_k$ *absolutely converges, if the series* $\sum_{k=1}^{\infty} |a_k|$ *converges.*

Remark. If the series $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ absolutely converge, $\alpha, \beta \in \mathbb{R}$ (or $\mathbb{C}$), then the series $\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k)$ absolutely converges. It follows from the equality

$$|\alpha a_k + \beta b_k| \leq |\alpha| \cdot |a_k| + |\beta| \cdot |b_k|$$

and the comparison test.

Remark. If (z_k) is a sequence of complex numbers,$x_k = \mathrm{Re} z_k$, $y_k = \mathrm{Im} z_k$, then absolute convergence of the series $\sum_{k=1}^{\infty} z_k$ is equivalent to simultaneous absolute convergence of the series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$. It follows from the inequality

$$|x_k|, |y_k| \le |z_k| \le |x_k| + |y_k|$$

and the comparison test.

Remark. If the series $\sum_{k=1}^{\infty} a_k$ has a sum, then

$$\left|\sum_{k=1}^{\infty} a_k\right| \le \sum_{k=1}^{\infty} |a_k|.$$

It is sufficient to pass to the limit in the inequality for the partial sums.

Lemma 2 *If the series absolutely converges, then it converges.*

We gige two proofs.

The first proof. Let $\varepsilon > 0$ be given. By Cauchy's criterion for the convergence of the series $\sum_{k=1}^{\infty} |a_k|$ we find $N \in \mathbb{N}$ such that for any $n > N, p \in \mathbb{N}$ we get $\sum_{k=n+1}^{n+p} |a_k| < \varepsilon$. Then

$$\left|\sum_{k=n+1}^{n+p} a_k\right| \le \sum_{k=n+1}^{n+p} |a_k| < \varepsilon.$$

So by Cauchy;s criterion the series $\sum_{k=1}^{\infty} a_k$ converges.

The positive and the negative parts of the number $x \in \mathbb{R}$ are defined by

$$x_+ = \max\{x, 0\}, \quad x_- = \max\{-x, 0\}.$$

$x_+ - x_- = x, x_+ + x_- = |x|, 0 \le x_\pm \le |x|$.

The second proof. Let $a_k \in \mathbb{R}$ for all $k \in \mathbb{N}$. Since the series $\sum_{k=1}^{\infty} |a_k|$ converges, by comparison test it follows that the series $\sum_{k=1}^{\infty} (a_k)_\pm$ converge, so the series $\sum_{k=1}^{\infty} a_k$ converges as the difference.

If $a_k \in \mathbb{C}, x_k = \mathrm{Re} a_k, y_k = \mathrm{Im} a_k$, than by remark the series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ absolutely converge. So, they converge, then by linearity the series $\sum_{k=1}^{\infty} a_k$ converges.

Remark. The statement of the Lemma is not invertible. There exist convergent series such that they are not absolutely converge. Such series are called **conditionally convergent**.

Remark. If the series $\sum_{k=1}^{\infty} a_k$ is conditionally convergent, and the series $\sum_{k=1}^{\infty} b_k$ absolutely converges, then the series $\sum^{\infty} (a_k + b_k)$ is conditionally convergent.

Theorem 10 ***(Dirichlet's and Abel's tests for convergence of series.)*** *Let (a_k) be a real or complex sequaence, (b_k) be a monotone sequence.*

1. ***Dirichlet's test.*** *If the sequence $A_n = \sum_{k=1}^{n} a_k$ is bounded, a $b_n \to 0$, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.*

2. ***Abel's test.*** *If the series $\sum_{k=1}^{\infty} a_k$ converges, and $\{b_k\}$ is bounded, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.*

We will prove these tests later for more general case.

The series $\sum_{k=1}^{\infty}(-1)^{k-1} b_k$ or $\sum_{k=1}^{\infty}(-1)^{k} b_k$, where $b_k \geq 0$ for all k, are called **alternating series**.

Theorem 11 ***(Leibniz's test for convergence of series.)*** *Let $\{b_n\}$ be monotone, $b_n \to 0$. Then the series $\sum_{k=1}^{\infty}(-1)^{k-1} b_k$ converges.*

Dirichlet's test implies Leibniz's test for $a_k = (-1)^{k-1}$. Nevertheless, it is useful to give independent proof to have the estimate for the reminder.

Proof. Let $\{b_n\}$ decreases. So, $b_n \geq 0$. Consider the sequance $\{S_{2m}\}$. It increases, indeed,

$$S_{2m} - S_{2(m-1)} = b_{2m-1} - b_{2m} \geq 0.$$

It is bounded from above, indeed,

$$S_{2m} = b_1 + (-b_2 + b_3) + \ldots + (-b_{2m-2} + b_{2m-1}) - b_{2m} \leq b_1.$$

So, $\{S_{2m}\}$ converges to a limit S. Therefore, by $b_{2m+1} \to 0$ we get

$$S_{2m+1} = S_{2m} + b_{2m+1} \longrightarrow S,$$

thus $S_n \to S$.

Remark. Since

$$S_{2m} = (b_1 - b_2) + \ldots + (b_{2m-1} - b_{2m}) \geq 0 \quad S_{2m} \leq b_1,$$

it follows that $0 \leq S \leq b_1$. The series satisfying Leibniz's test are often called Leibniz's series.

Remark. The reminder of Leibniz's series does not exceed its first term in modulus and and the signs of the reminder and its first term coincide

$$0 \leq (-1)^n (S - S_n) \leq b_{n+1}.$$

Example. $\sum_{n=1}^{\infty} \frac{\log^{100} n}{n} \sin \frac{n\pi}{4}$.

Since

$$\left|\sum_{k=1}^{n} \sin \frac{k\pi}{4}\right| = \left(\sin \frac{\pi}{8}\right)^{-1} \left|\sin \frac{n\pi}{8} \sin \frac{n+1}{8}\pi\right| < \frac{1}{\sin \frac{\pi}{8}}$$

and $\left(n^{-1} \log^{100} n\right)_{n\in\mathbb{N}}$ monotonically tends to 0 for $n > \lfloor e^{100} \rfloor + 1$, it follows that by Dirichlet's test the series converges.

Example. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n} + (-1)^n}$.

$$\frac{(-1)^n}{\sqrt{n}+(-1)^n} = (-1)^n\frac{\sqrt{n}-(-1)^n}{n-1} = (-1)^n\frac{\sqrt{n}}{n-1} - \frac{1}{n-1},$$

by Leibniz's test $\sum\limits_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{n-1}$ converges, $\sum\limits_{n=2}^{\infty}\frac{1}{n-1}$ diverges to $+\infty$, so, the initial series diverges to $+\infty$.

Example. $\sum\limits_{n=1}^{\infty}\sin\left(\pi\sqrt{n^2+k^2}\right)$.

$$\sin\left(\pi\sqrt{n^2+k^2}\right) = (-1)^n\sin\pi\left(\sqrt{n^2+k^2}-n\right) \equiv (-1)^n b_n,$$

where $b_n = \sin\frac{\pi k^2}{\sqrt{n^2+k^2}+n}$ monotonically tends to 0 as $n\to\infty$ (for $n > n_0$), so by Leibniz's test the series converges.

If we group terms in the divergent series

$$1-1+1-1+\ldots$$

by two different ways, we get series convergent to different sums

$$(1-1)+(1-1)+\ldots = 0+0+\ldots = 0,$$

$$1+(-1+1)+(-1+1)+\ldots = 1+0+0+\ldots = 1.$$

Let

$$\sum_{k=1}^{\infty} a_k$$

be given, let $\{n_j\}_{j=0}^{\infty}, n_0 = 0$ be a strictly increasing sequence of integers. We denote

$$A_j = \sum_{k=n_j+1}^{n_{j+1}} a_k, \quad j\in\mathbb{Z}_+.$$

Then it is said that the series

$$\sum_{j=0}^{\infty} A_j$$

is got from the series $\sum\limits_{k=1}^{\infty} a_k$ via grouping the terms (introducing brackets).

Theorem 12 ***(Grouping terms of the series.)*** *1. If $\sum\limits_{k=1}^{\infty} a_k = S (S\in\overline{\mathbb{R}}$ or $\mathbb{C}\cup\{\infty\})$, then $\sum\limits_{j=0}^{\infty} A_j = S$.*

2. If $\sum\limits_{j=0}^{\infty} A_j = S (S\in\overline{\mathbb{R}}$ or $\mathbb{C}\cup\{\infty\})$, $a_n\to 0$, and there exists $L\in\mathbb{N}$ such that each bracket contains no more than L terms, then $\sum\limits_{k=1}^{\infty} a_k = S$.

3. If $a_k\in\mathbb{R}$, $\sum\limits_{j=0}^{\infty} A_j = S\in\overline{\mathbb{R}}$, and all terms in each group have the same sign, then $\sum\limits_{k=1}^{\infty} a_k = S$.

In item 3. non-stricte sign is meant, that is for any $j \in \mathbb{Z}_+$ for any $\mu, \nu = n_j + 1, \dots, n_{j+1}$ we have $a_\mu a_\nu \ge 0$.

Proof. We denote

$$S_n = \sum_{k=1}^{n} a_k, \quad T_m = \sum_{j=0}^{m} A_j.$$

1. By definition $T_m = S_{n_{m+1}}$, that is $\{T_m\}$ is a consequence of $\{S_n\}$. Therefore, if $S_n \to S$, than $T_m \to S$.

We prove items 2. and 3. for finite sum S. Let series $\sum\limits_{j=0}^{\infty} A_j$ converges to S, that is $S_{n_m} \to S$. Let us prove that $S_n \to S$.

2. Let $\varepsilon > 0$ be given, find $M, K \in \mathbb{N}$ such that

$$\begin{array}{ll} |S_{n_m} - S| < \frac{\varepsilon}{2} & m > M, \\ |a_k| < \frac{\varepsilon}{2L} & k > K. \end{array}$$

Set $N = \max\{n_{M+1}, K\}$. Let $n > N$. Let m be the number such that $n_m \le n < n_{m+1}$, then $m > M$. So

$$|S_n - S| \le |S_n - S_{n_m}| + |S_{n_m} - S| \le \sum_{k=n_m+1}^{n} |a_k| + |S_{n_m} - S| < \frac{\varepsilon}{2L} \cdot L + \frac{\varepsilon}{2} = \varepsilon.$$

3. Let $\varepsilon > 0$ be given, find $M \in \mathbb{N}$ such that for all $m > M$ we get $|S_{n_m} - S| < \varepsilon$. Set $N = n_{M+1}$. Let $n > N$. Let m be the number such that $n_m \le n < n_{m+1}$, then $m > M$. If $a_{n_m+1}, \dots, a_{n_{m+1}} \ge 0$, than $S_{n_m} \le S_n \le S_{n_{m+1}}$, if $a_{n_m+1}, \dots, a_{n_{m+1}} \le 0$, then $S_{n_{m+1}} \le S_n \le S_{n_m}$. In both cases

$$|S_n - S| \le \max\{|S_{n_{m+1}} - S|, |S_{n_m} - S|\} < \varepsilon. \qquad \square$$

Example. $\sum\limits_{n=1}^{\infty} \dfrac{i^n}{n}$. By

$$\frac{(i)^n}{n} = \begin{cases} \dfrac{1}{n}, \ n = 4k, \\ \dfrac{i}{n}, \ n = 4k+1, \\ -\dfrac{1}{n}, \ n = 4k+2, \\ -\dfrac{i}{n}, \ n = 4k+3, k \in \mathbb{N}, \end{cases}$$

convergence of the series is equivalent to simultaneous convergence of the series

$$\sum_{n=1}^{\infty} a_n, \quad a_n = \begin{cases} \dfrac{1}{n}, \ n = 4k \\ -\dfrac{1}{n}, \ n = 4k+2 \\ 0, \ n = 2k+1 \end{cases}$$

and

$$\sum_{n=1}^{\infty} b_n, \quad b_n = \begin{cases} \dfrac{1}{n}, \ n = 4k+1 \\ -\dfrac{1}{n}, \ n = 4k+3 \\ 0, \ n = 2k \end{cases}$$

By the Theorem on grouping terms of the series, item 2. or 3. the last is equivalent to convergence of the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{2k}$, $\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}$, respectively. It remains to apply Leibniz's test.

Example. $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^\alpha}$ converges for $\alpha\in(0,1]$ by Leibniz's test. We know that $\sum_{k=1}^{\infty}\frac{1}{k^\alpha}$ diverges for $\alpha\in(0,1]$. Therefore, $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^\alpha}$ is an example of conditionally convergent series.

Example. Let us find the sum of the series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots$$

By the asymptotic formula

$$H_n=\log n+\gamma+\delta_n,\quad \delta_n\to 0.$$

we obtain

$$S_{2n}=1-\frac{1}{2}+\dots+\frac{1}{2n-1}-\frac{1}{2n}=1+\frac{1}{2}+\dots+\frac{1}{2n-1}+\frac{1}{2n}-2\left(\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2n}\right)$$
$$=H_{2n}-H_n=\log 2n+\gamma+\delta_{2n}-(\log n+\gamma+\delta_n)=\log 2+\delta_{2n}-\delta_n\longrightarrow\log 2 \text{ as } n\to\infty.$$

So,

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}=\log 2$$

Remark. Later we will prove the formula (pass the limit)

$$\log(1+x)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^k,\quad -1<x\le 1.$$

It is sufficient to put $x=1$.

Example. The series

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\dots$$

is the rearrangement of the series $\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}$. Let us find the sum of the rearrangement We denote the partial sums by T_n and S_n respectively. Then

$$T_{3m}=\sum_{k=1}^{m}\left(\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\right)=\frac{1}{2}\sum_{k=1}^{m}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\frac{1}{2}S_{2m}\underset{m\to\infty}{\longrightarrow}\frac{1}{2}\log 2,$$

$$T_{3m+1}=T_{3m}+\frac{1}{2m+1}\underset{m\to\infty}{\longrightarrow}\frac{1}{2}\log 2,$$
$$T_{3m+2}=T_{3m+1}-\frac{1}{4m+2}\underset{m\to\infty}{\longrightarrow}\frac{1}{2}\log 2.$$

Thus, $T_n\to\frac{1}{2}\log 2$. So, the rearrangement change the sum.

Consider the bijection $\varphi:\mathbb{N}\to\mathbb{N}$ (It is called the rearrangement of $\mathbb{N}$). Then it is said that the series

$$\sum_{k=1}^{\infty}a_{\varphi(k)}$$

is the rearrangement of the series $\sum_{k=1}^{\infty}a_k$.

Theorem 13 *(Rearrangement of the absolutely convergent series.) Let the series* $\sum_{k=1}^{\infty} a_k$ *be absolutely convergent. Let its sum be equal to* S, $\varphi : \mathbb{N} \to \mathbb{N}$ *be e bijection. Then the series* $\sum_{k=1}^{\infty} a_{\varphi(k)}$ *absolutely converges to* S.

Proof. 1. Let $a_k \geq 0$ for all $k \in \mathbb{N}$. We denote

$$S_n = \sum_{k=1}^{n} a_k, \quad T_n = \sum_{k=1}^{n} a_{\varphi(k)}.$$

For all n we have

$$T_n \leq S_m \leq S,$$

where $m = \max\{\varphi(1), \ldots, \varphi(n)\}$. So, $\sum_{k=1}^{\infty} a_{\varphi(k)}$ converges, and $T \leq S$. We have proved that the sum of the rearrangement positive series is no greater then the sum of the initial series. Applying this fact to the rearrangement φ^{-1}, we obtain $S \leq T$.

2. $a_k \in \mathbb{R}$. By comparison test positive series with terms $(a_k)_{\pm}$ converge. By item 1. the series with terms $\left(a_{\varphi(k)}\right)_{\pm}$ converge to the same sums. Therefore, the series $\sum_{k=1}^{\infty} a_{\varphi(k)}$ converges as the difference and

$$\sum_{k=1}^{\infty} a_{\varphi(k)} = \sum_{k=1}^{\infty} \left(a_{\varphi(k)}\right)_{+} - \sum_{k=1}^{\infty} \left(a_{\varphi(k)}\right)_{-} = \sum_{k=1}^{\infty} (a_k)_{+} - \sum_{k=1}^{\infty} (a_k)_{-} = \sum_{k=1}^{\infty} a_k.$$

3. $a_k \in \mathbb{C}$, $x_k = \mathrm{Re} a_k, y_k = \mathrm{Im} a_k$. By the remark on absolute convergence of the series with complex terms, the series with real terms x_k and y_k absolutely converge. It remains to apply item 2., we get

$$\sum_{k=1}^{\infty} a_{\varphi(k)} = \sum_{k=1}^{\infty} x_{\varphi(k)} + i \sum_{k=1}^{\infty} y_{\varphi(k)} = \sum_{k=1}^{\infty} x_k + i \sum_{k=1}^{\infty} y_k = \sum_{k=1}^{\infty} a_k. \qquad \square$$

Question. Can the arrangement of the divergent positive series give convergent seires?

Lemma 3 . *If the series* $\sum_{k=1}^{\infty} a_k$ *with real terms are conditionally convergent, then both series* $\sum_{k=1}^{\infty} (a_k)_{+}$ *and* $\sum_{k=1}^{\infty} (a_k)_{-}$ *are divergent.*

Proof. Suppose that both series are convergent, then the series $\sum_{k=1}^{\infty} |a_k|$ converges as a sum. Suppose that one series converges and another diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges as the difference of convergent and divergent series. $\square$

Theorem 14 ***(B. Riemann, The arrangement of the conditionally convergent series.)*** *Let the series* $\sum_{k=1}^{\infty} a_k$ *with real terms is conditionally convergent. Then for any* $S \in \overline{\mathbb{R}}$ *there exists the rearrangement* $\varphi(k)$ *such that* $\sum_{k=1}^{\infty} a_{\varphi(k)} = S$. *There exists the rearrangement* $\varphi(k)$ *such that* $\sum_{k=1}^{\infty} a_{\varphi(k)}$ *has no sum.*

Proof. Let us prove the case $S \in [0, +\infty)$. Other cases you can prove by yourself. Let $\{b_p\}$ и $\{c_q\}$ be sequences of all nonnegetive and negative terms of the series $b_p = a_{n_p}$, $c_q = a_{m_q}$. By Lemma both series $\sum\limits_{p=1}^{\infty} b_p$ and $\sum\limits_{q=1}^{\infty} c_q$ diverge. Set $p_0 = q_0 = 0$. Denote by p_1 the least natural number such that

$$\sum_{p=1}^{p_1} b_p > S,$$

that is

$$\sum_{p=1}^{p_1-1} b_p \le S < \sum_{p=1}^{p_1} b_p.$$

Then denote by q_1 the least natural number such that

$$\sum_{q=1}^{q_1} c_q < S - \sum_{p=1}^{p_1} b_p,$$

that is

$$\sum_{p=1}^{p_1} b_p + \sum_{q=1}^{q_1} c_q < S \le \sum_{p=1}^{p_1} b_p + \sum_{q=1}^{q_1-1} c_q.$$

The existence of p_1 and q_1 follows from divergence of the series $\sum\limits_{p=1}^{\infty} b_p$ and $\sum\limits_{q=1}^{\infty} c_q$. We continue the procedure. Let $p_1, \dots, p_{s-1}, q_1, \dots, q_{s-1}$ be chosen. We denote by p_s the least natural number such that

$$\sum_{p=1}^{p_s} b_p > S - \sum_{q=1}^{q_{s-1}} c_q,$$

that is

$$\sum_{p=1}^{p_s-1} b_p + \sum_{q=1}^{q_{s-1}} c_q \le S < \sum_{p=1}^{p_s} b_p + \sum_{q=1}^{q_{s-1}} c_q.$$

Then we denote by q_s the least natural number such that

$$\sum_{q=1}^{q_s} c_q < S - \sum_{p=1}^{p_s} b_p$$

that is

$$\sum_{p=1}^{p_s} b_p + \sum_{q=1}^{q_s} c_q < S \le \sum_{p=1}^{p_s} b_p + \sum_{q=1}^{q_s-1} c_q.$$

The existence of p_s and q_s follows from divergence of the series $\sum\limits_{p=1}^{\infty} b_p$ and $\sum\limits_{q=1}^{\infty} c_q$. The series

$$b_1 + \ldots + b_{p_1} + c_1 + \ldots + c_{q_1} + \ldots + b_{p_{s-1}+1} + \ldots + b_{p_s} + c_{q_{s-1}} + \ldots + c_{q_s} + \ldots$$

is the rearrangement of the initial series. Let us prove that it converges to S. Grouping the terms of the same sign we get

$$B_1 + C_1 + \ldots + B_s + C_s + \ldots$$

where $B_s = \sum\limits_{p=p_{s-1}+1}^{p_s} b_p, C_s = \sum\limits_{q=q_{s-1}+1}^{q_s} c_q$. Denote its partial sums by T_n. By the procedure $0 < T_{2s-1} - S \le b_{p_s}, c_{q_s} \le T_{2s} - S < 0$. Since the series $\sum\limits_{k=1}^{\infty} a_k$ converges, it follows that $b_s, c_s \to 0$. So, $T_n \to S$. It remains to apply the Theorem on grouping terms. □

By the commutative and distributive laws

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{m} b_j\right) = \sum_{k=1}^{n}\sum_{j=1}^{m} a_k b_j.$$

For infinite sums the following questions appear. Whether the series of various products $a_k b_j$ converges and what order we need to prefer.

Definition 15 *Let* $\sum\limits_{k=1}^{\infty} a_k$ *and* $\sum\limits_{j=1}^{\infty} b_j$ *be numerical series,* $\gamma = (\varphi, \psi) : \mathbb{N} \to \mathbb{N}^2$ *be a bijection. Then the series*

$$\sum_{l=1}^{\infty} a_{\varphi(l)} b_{\psi(l)}$$

is called the product of series $\sum\limits_{k=1}^{\infty} a_k$ *and* $\sum\limits_{j=1}^{\infty} b_j$*, corresponding to the numeration* γ*.*

Theorem 16 ***(O. Cauchy, The product of series.)*** *If the series* $\sum\limits_{k=1}^{\infty} a_k$, $\sum\limits_{j=1}^{\infty} b_j$ *absolutely converge to the sums* A *and* B*, then for any numeration their product absolutely converges to* $\leftarrow AB$*.*

Proof. Let $\gamma = (\varphi, \psi) : \mathbb{N} \to \mathbb{N}^2$ be a bijection. We denote

$$\sum_{k=1}^{\infty} |a_k| = A^*, \quad \sum_{j=1}^{\infty} |b_j| = B^*.$$

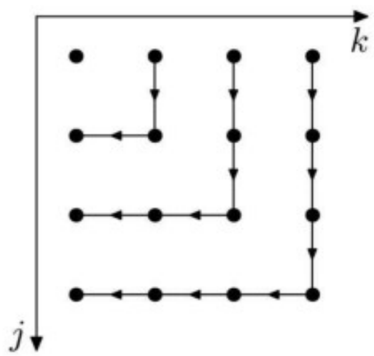
For all $\nu \in \mathbb{N}$

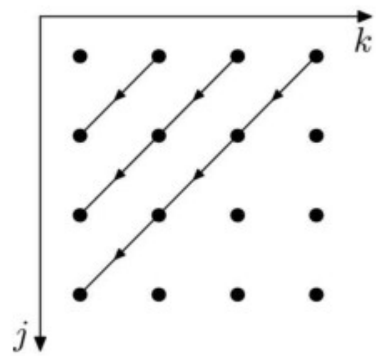
$$\sum_{l=1}^{\nu} \left|a_{\varphi(l)} b_{\psi(l)}\right| \le \left(\sum_{k=1}^{n} |a_k|\right)\left(\sum_{j=1}^{m} |b_j|\right) \le A^* B^*,$$

where $n = \max\limits_{1 \le l \le \nu} \varphi(l), m = \max\limits_{1 \le l \le \nu} \psi(l)$. So the partial sums of the series $\sum\limits_{l=1}^{\infty} \left|a_{\varphi(l)} b_{\psi(l)}\right|$ are bounded from above. So, the series $\sum\limits_{l=1}^{\infty} a_{\varphi(l)} b_{\psi(l)}$ absolutely converges. By the Theorem on rearrangement of the absolutely convergent series its sum does not depend on a rearrangement. That is why is $\widetilde{\gamma} = (\widetilde{\varphi}, \widetilde{\psi})$ is another numeration $\mathbb{N}^2$, then the series $\sum\limits_{l=1}^{\infty} a_{\tilde{\varphi}(l)} b_{\tilde{\psi}(l)}$, which us a result of the rearrangement $\gamma^{-1} \circ \widetilde{\gamma}$ for the series $\sum\limits_{l=1}^{\infty} a_{\varphi(l)} b_{\psi(l)}$ absolutely converges and has the same sum. To calculate the sum we consider the numeration "by squares" and the partial sums S_{n^2}, then

$$S_{n^2} = \sum_{k,j=1}^{n} a_k b_j = \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{j=1}^{n} b_j\right) \xrightarrow[n\to\infty]{} AB. \qquad \square$$

Lemma 4 *Is the series* $\sum\limits_{k=1}^{\infty} a_k$, $\sum\limits_{j=1}^{\infty} b_j$ *converge to the sums* A *and* B*, then their product "by squares" converges to* AB*. We emphasize that we do not need absolute convergence here.*





Proof. Let A_n and B_n be the partial sums of the initial series, S_n be the partial sum of the series "by squares". By the last Theorem $S_{n^2} \to AB$. For $n \in \mathbb{N}$ we define $m_n := [\sqrt{n}]$. Then $S_n = S_{m_n^2} + \theta_n$, where θ_n has the form $a_{m_n+1}B_J + b_{m_n+1}(A_K - A_M)$ $(J, K, M \in \mathbb{Z}_+)$. The partial sums of the convergent series are bounded, and terms tends to 0.so $\theta_n \to 0$, that is $S_n \to AB$. □

The most popular numeration is the numeration "by diagonals"

Definition 17 *The series* $\sum\limits_{k=1}^{\infty} c_k$*, where*

$$c_k = \sum_{j=1}^{k} a_j b_{k+1-j},$$

is called the product of the series $\sum\limits_{k=1}^{\infty} a_k$ *and* $\sum\limits_{j=1}^{\infty} b_j$ ***"by diagonals"*** *or* ***Cauchy's product.***

Remark. It is more convenient to start summation in Cauchy's product with zero. Cauchy's product of the series $\sum\limits_{k=0}^{\infty} a_k$ and $\sum\limits_{j=0}^{\infty} b_j$ is the series $\sum\limits_{k=0}^{\infty} c_k$, where

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}$$

Example. The series $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{\sqrt{k}}$ conditionally converges. Let us find its Cauchy's square, that is the series with terms

$$c_k = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{\sqrt{j}} \cdot \frac{(-1)^{k-j}}{\sqrt{k+1-j}} = (-1)^{k-1} \sum_{j=1}^{k} \frac{1}{\sqrt{j(k+1-j)}}.$$

By

$$|c_k| \geq \sum_{j=1}^{k} \frac{1}{\sqrt{k}\sqrt{k}} = 1$$

we have $c_k \not\to 0$, so the series $\sum\limits_{k=1}^{\infty} c_k$ diverges. At the same time the square of the initial series "by squares" converges.

Exercise. If two series converge, and at least one om them absolutely converges, then their Cauchy's product converges.

Exercise. If the series $\sum_{k=1}^{\infty} a_k, \sum_{k=1}^{\infty} b_k$ converge to A and B, and their Cauchy's product converges to C, then $C = AB$.

Remark. Cauchy's product of two divergent series might be convergent. For example,

$$a_k = \begin{cases} 1, & k = 0, \\ 2^{k-1}, & k \in \mathbb{N}, \end{cases} \qquad b_j = \begin{cases} 1, & j = 0 \\ -1, & j \in \mathbb{N} \end{cases}$$

Since $a_k, b_j \not\to 0$, it follows that series-multipliers diverge, while $c_0 = 1$, and for $k \in \mathbb{N}$

$$c_k = -1 - \sum_{j=1}^{k-1} \left(2^{j-1} + 2^{k-1}\right) = 0.$$