1.

(a) Give the definition of the matrix exponent.

(b) Let $A_{[n*n]}$, $B_{[n*n]}$ be square matrices. Whether it true, that $e^{A+B} = e^A e^B$?

2. Give Poincare classification of singular points of a second-order linear homogeneous system with constant coefficients. You have to give all possible types of singular points, draw the pictures and specify the conditions.

3. Formulate the theorem on the stability of the zero solution of the first approximation system.

4. Solve the equation

$$y'' + 4y' + 4y = xe^{2x}.$$

5. Solve the equation

$$y'' + y = 2(\frac{1}{\cos x})^3.$$

6. Solve the equation

$$(e^x + 1)y'' - 2y' - e^x y = 0.$$

The one particular solution $y_1 = e^x - 1$ is given.

7. Solve the equation

$$y''^3 + xy'' = 2y'.$$

8. Solve the equation

$$\begin{cases} \dot{x} = 2x - y \\ \dot{y} = x + 2e^t \end{cases}$$

9. Find all equilibria and study the stability

$$\begin{cases} \dot{x} = e^y - e^x \\ \dot{y} = \sqrt{3x + y^2} - 2 \end{cases}$$

1.
(a) Give the definition of the matrix exponent.
(b) Let $A_{[n*n]}$, $B_{[n*n]}$ be square matrices. Whether it true, that $e^{A+B} = e^A e^B$?

(a) $e^A = \sum\limits_{k=1}^{\infty} \frac{1}{k!} A^k$

(b). it's true if $AB = BA$.

$$e^{A+B} = \sum\limits_{k=0}^{\infty} \frac{1}{k!} (A+B)^k$$

$$e^A e^B = \sum\limits_{k=0}^{\infty} \frac{1}{k!} A^k \cdot \sum\limits_{m=0}^{\infty} \frac{1}{m!} B^m$$

$$= \sum\limits_{k=0}^{+\infty} \left(\sum\limits_{m=0}^{k} \frac{1}{m!(k-m)!} A^m B^{k-m} \right)$$

1) $k=0$ $E = E$.

2) $k=1$ $A+B = A+B$

3) $k=n$. $RHS = \sum\limits_{m=0}^{n} \frac{1}{m!(n-m)!} A^m B^{n-m} = \sum\limits_{m=0}^{n} \frac{C_n^m}{n!} A^m B^{n-m}$

$$= \frac{1}{n!} (A+B)^n \cdot = LHS$$

2. Give Poincare classification of singular points of a second-order linear homogeneous system with constant coefficients. You have to give all possible types of singular points, draw the pictures and specify the conditions.

3. Formulate the theorem on the stability of the zero solution of the first approximation system.

$\exists K \geqslant 1. \ \sigma > 0. \quad c \in (0, \frac{\sigma}{K}). \quad$ s.t. $\ \|\Phi(t,\sigma)\| \leqslant K e^{-\sigma(t-\tau)}$

for any $t, \sigma \in IR, \ t_0 \leqslant \sigma < t < +\infty.$ and $\|g(t,x)\| < c\|x\|.$ for any $t \geqslant t_0. \ \|x\| < \rho_0$

Then $x \equiv 0.$ is asymptotically stable

4. Solve the equation

$$y'' + 4y' + 4y = xe^{2x}.$$

(1) Consider $y'' + 4y' + 4y = 0. \qquad \lambda_{1,2} = -2.$

Let $y_1 = e^{-2x} \qquad y_2 = x e^{-2x}$

denote. $\psi(x) = (\alpha x + \beta) e^{2x}$

$(4\alpha x + 4\alpha + 4\beta) e^{2x} + 4(2\alpha x + \alpha + 2\beta) e^{2x} + 4(\alpha x + \beta) e^{2x} = x e^{2x}$

$\begin{cases} 4\alpha + 8\alpha + 4\alpha = 1. \\ 4\alpha + 4\beta + 4\alpha + 8\beta + 4\beta = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \frac{1}{16} \\ \beta = -\frac{1}{32} \end{cases}$

$\psi = C_1 \bar{e}^{2x} + C_2 x e^{-2x} + (\frac{1}{16} x - \frac{1}{32}) e^{2x}.$

5. Solve the equation

$$y'' + y = 2(\frac{1}{\cos x})^3.$$

$\lambda^2 + 1 = 0. \qquad \lambda = \pm i.$

$\psi(x) = C_1(x) \cos x + C_2(x) \sin x$

$\begin{cases} \dot{C_1} \cos x + \dot{C_2} \sin x = 0 \\ -\dot{C_1} \sin x + \dot{C_2} \cos x = \frac{2}{\cos^3 x} \end{cases} \Rightarrow \begin{cases} \dot{C_1} = -\dot{C_2} \tan x \\ \dot{C_2} \frac{\sin^2 x}{\cos x} + \dot{C_2} \cos x = \frac{2}{\cos^3 x} \end{cases}$

$\dot{C_2} = \frac{2}{\cos^2 x} \Rightarrow C_2 = 2\tan x + C_2.$

$\dot{C_1} = -\frac{2\tan x}{\cos^2 x} \Rightarrow C_1 = -\int \frac{2\tan x}{\cos^2 x} dx = -\int 2\tan x \, d(\tan x) = -\tan^2 x + C_1$

$\psi(x) = C_1 \cos x + C_2 \sin x + 2\tan x \sin x - \tan^2 x \cos x.$

6. Solve the equation

$$(e^x + 1)y'' - 2y' - e^x y = 0.$$

The one particular solution $y_1 = e^x - 1$ is given.

$$y'' - \frac{2}{e^x+1} y' - \frac{e^x}{e^x+1} y = 0$$

$$W(x) = \tilde{C} \cdot e^{\int \frac{2}{e^x+1} dx} = C \cdot \left(\frac{e^x}{e^x+1}\right)^2$$

$$\int \frac{2}{e^x+1} dx = \int \frac{2\, d(e^x)}{e^x(e^x+1)} = 2 \ln \left|\frac{e^x}{e^x+1}\right| + \hat{C}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = (e^x-1)\, y_2' - e^x y_2 = C \cdot \left(\frac{e^x}{e^x+1}\right)^2$$

$$y_2' - \frac{e^x}{e^x-1} y_2 = \frac{C \cdot e^x}{(e^x+1)^2(e^x-1)}$$

$$\frac{dy_2}{dx} = \frac{e^x}{e^x-1} y_2 \iff \frac{dy_2}{y_2} = \frac{d(e^x)}{e^x-1} \iff y_2 = C(x)(e^x-1)$$

$$C'(x)(e^x-1) = \frac{C \cdot e^{2x}}{(e^x+1)^2(e^x-1)} \qquad C'(x) = \frac{C\, e^{2x}}{(e^{2x}-1)^2}$$

$$C(x) = \frac{C}{2} \int \frac{d e^{2x}}{(e^{2x}-1)^2} = -\frac{C}{2} \cdot \frac{1}{e^{2x}-1} + \tilde{C}$$

$$y = C_1 \cdot \frac{1}{e^x+1} + C_2 \cdot (e^x-1)$$

7. Solve the equation

$$y''' + xy'' = 2y'.$$

denote. $y' = z$, $z'' + xz' = 2z \Rightarrow z'' + xz' - 2z = 0$

a particular solution : $z_1 = x^2 + 1$.

$$W(x) = \tilde{C} \cdot e^{-\int x\, dx} = C \cdot e^{-\frac{x^2}{2}}$$

$$\begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} = (x^2+1) \cdot z_2' - 2x z_2 = C \cdot e^{-\frac{x^2}{2}}$$

consider $(x^2+1) \cdot z_2' - 2x z_2 = 0$

$$\Rightarrow (x^2+1)\frac{dz_2}{dx} = 2x\,z_2 \Rightarrow \frac{dz_2}{z_2} = \frac{2x\,dx}{x^2+1} \Rightarrow z_2 = c(x)\,(x^2+1).$$

$$(x^2+1)^2 c'(x) = c \cdot e^{-\frac{x^2}{2}}$$

$$\Rightarrow c(x) = c \int \frac{e^{-\frac{x^2}{2}}\,dx}{(x^2+1)^2}$$

8. Solve the equation

$$\begin{cases} \dot{x} = 2x - y \\ \dot{y} = x + 2e^t \end{cases}$$

$$\begin{cases} \dot{x} = 2x - y \\ \dot{y} = x \end{cases} \qquad \begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda(\lambda-2) + 1 \qquad \lambda_{1,2} = 1.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha_1 + \beta_1 t \\ \alpha_2 + \beta_2 t \end{pmatrix} e^t$$

$$\begin{cases} \alpha_1 + \beta_1 t + \beta_1 = 2\alpha_1 + 2\beta_1 t - \alpha_2 - \beta_2 t \\ \alpha_2 + \beta_2 t + \beta_2 = \alpha_1 + \beta_1 t \end{cases} \Rightarrow \begin{cases} \beta_1 = \alpha_1 - \alpha_2 \\ \beta_1 = 2\beta_1 - \beta_2 \\ \alpha_2 + \beta_2 = \alpha_1 \\ \beta_2 = \beta_1 \end{cases}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_1 + c_2 t \end{pmatrix} e^t$$

$$\lambda_1 = \lambda_2 = \lambda_0.$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A_1 t^2 + B_1 t + C_1 \\ A_2 t^2 + B_2 t + C_2 \end{pmatrix} e^t$$

$$\begin{cases} A_1 t^2 + B_1 t + C_1 + 2A_1 t + B_1 = 2(A_1 t^2 + B_1 t + C_1) - (A_2 t^2 + B_2 t + C_2). \\ A_2 t^2 + B_2 t + C_2 + 2A_2 t + B_2 = A_1 t^2 + B_1 t + C_1 + 2 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = 2A_1 - A_2 \\ B_1 + 2A_1 = 2B_1 - B_2 \\ C_1 + B_1 = 2C_1 - C_2 \\ A_2 = A_1 \\ B_2 + 2A_2 = B_1 \\ C_2 + B_2 = C_1 + 2. \end{cases} \Rightarrow \begin{cases} A_1 = A_2 \\ B_1 - B_2 = 2A_1 \\ C_1 - C_2 = B_1 \\ C_1 - C_2 = B_2 - 2 \end{cases} \Rightarrow \begin{cases} A_1 = A_2 = -1 \\ B_1 = B_2 - 2. \\ C_1 = B_2 - 2 + C_2. \end{cases} \Rightarrow \begin{cases} A_1 = A_2 = -1 \\ B_2 = 0 \\ C_2 = 0 \\ B_1 = -2. \quad C_1 = -2. \end{cases}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 + c_2 + c_2 t \\ c_1 + c_2 t \end{pmatrix} e^t + \begin{pmatrix} -t^2 - 2t - 2 \\ -t^2 \end{pmatrix} e^t$$

9. Find all equilibria and study the stability

$$\begin{cases} \dot{x} = e^y - e^x \\ \dot{y} = \sqrt{3x + y^2} - 2 \end{cases}$$

$$\begin{cases} e^y = e^x \\ \sqrt{3x+y^2} = 2 \end{cases} \Rightarrow \begin{cases} x = -4 \\ y = -4 \end{cases} \quad \begin{cases} x = 1 \\ y = 1 \end{cases}$$

1). $x = -4. \quad y = -4$ $\begin{cases} x_1 = x+4 \\ y_1 = y+4 \end{cases}$

$$\begin{cases} \dot{x}_1 = e^{-4}(-x_1 + y_1) + o(\rho) \\ \dot{y}_1 = \frac{1}{8}(3x_1 - 4y_1) + o(\rho) \end{cases}$$

$$\begin{vmatrix} -\frac{1}{e^4} - \lambda & \frac{1}{e^4} \\ \frac{3}{8} & -\frac{1}{2} - \lambda \end{vmatrix} = (\lambda + \frac{1}{2})(\lambda + \frac{1}{e^4}) - \frac{3}{8e^4} = \lambda^2 + \left(\frac{1}{e^4} + \frac{1}{2}\right)\lambda + \frac{1}{8e^4}$$

$$\Delta = \left(\frac{1}{e^4} + \frac{1}{2}\right)^2 - 4 \cdot \frac{1}{8e^4} > 0. \qquad \lambda_1 \lambda_2 > 0 . \quad \lambda_1 + \lambda_2 < 0$$

thus $Re \, \lambda_1, \lambda_2 = \lambda_1, \lambda_2 < 0$.   asymptotically stable.

2) $x = 1 \quad y = 1.$ $\begin{cases} x_2 = x-1 \\ y_2 = y-1 \end{cases}$

$$\begin{cases} \dot{x}_2 = e(-x_2 + y_2) + o(\rho) \\ \dot{y}_2 = \frac{1}{8}(3x_2 + 2y_2) + o(\rho) \end{cases} \qquad \begin{vmatrix} -e - \lambda & e \\ \frac{3}{8} & \frac{1}{4} - \lambda \end{vmatrix} = (\lambda + e)(\lambda - \frac{1}{4}) - \frac{3}{8}e$$

$$= \lambda^2 + (e - \frac{1}{4})\lambda - \frac{5}{8}e.$$

$\Delta > 0.$ $\lambda_1 \lambda_2 < 0.$ $\lambda_1 + \lambda_2 < 0.$

$\exists \, Re \, \lambda > 0.$ unstable.

7. Solve the equation

$$y''^3 + xy'' = 2y'.$$

4. 如果方程对 x 和 y 是广义齐次的，即以 kx 代替 x，以 $k^m y$ 代替 y(同时以 $k^{m-1}y'$ 代替 y'，以 $k^{m-2}y''$ 代替 y'' 等等) 方程不变，则方程可以降阶．为了知道方程是不是齐次的，并且求出数 m，应当使经上述代换后，方程的每项里出现数 k 的幂指数彼此相等．例如经这一代换后在方程 $2x^4y'' - 3y^2 = x^4$ 的第一项里数 k 将出现 $4+(m-2)$ 次幂，在第二项里将出现 $2m$ 次幂，在第三项里将出现 4 次幂．因此 m 应当满足方程

$$4+(m-2)=2m=4.$$

$$k^{m-2}y''$$

$$3m-6 = m-2+m = m-4.$$

由此 $m=2$．如果所得到的关于 m 的方程是不相容的，则微分方程在上说的意义下就不是齐次的．

求出数 m 后，要作代换 $x=e^t$，$y=ze^{mt}$，其中 $z=z(t)$ 是新的未知函数，而 t 是新的自变量．我们将得到不出现自变量 t 的方程．这个方程可以用前面讨论过的方法降阶．

431. $y=C_1[1\pm\text{ch}(x+C_2)]$; $y=Ce^{\pm x}$. **432.** $x=C_1p+3p^2$; $y=\frac{12}{5}p^5+\frac{5}{4}$ $\cdot C_1 p^4 + C_1^2\frac{p^3}{6}+C_2$; $y=C$. **433.** $y=C_1\frac{x^2}{2}-C_1^2x+C_2$; $y=(x^3/12)+C$.