22.10.2024

Equations of mathematical physics
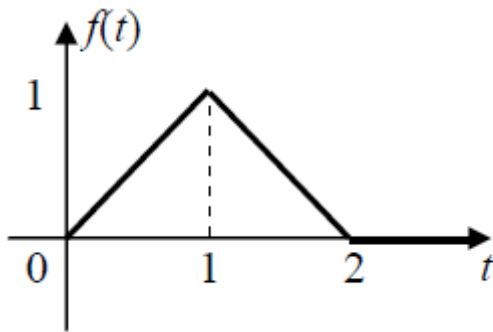
Examples of possible exam assignments

FUNDAMENTALS OF OPERATIONAL CALCULUS
Ex 1. Using the definition, find images of the following functions $$f(t)=e^{4t}$$
Ex 2. Find the image of the function given by the following graph:
Ex 3. Find images of the following functions: $$f(t)=\int_{0}^{t}\cos(t-\tau)e^{2\tau}d\tau$$
Ex 4. Find the original corresponding to the image (using the Duhamel integral): $$F(p)=\frac{2p^2}{\left(p^2+1\right)^2}$$
Ex 5. Solve the Cauchy problem: $$x'''-x''-6x'=0,$$ $$x(0)=15,\ x'(0)=2,\ x''(0)=56.$$

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Ex 1. Find the general solution $u = u(x, y, z)$ of the equation

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} + (x - y)\frac{\partial u}{\partial z} = 0.$$

Ex 2. Find a solution to the equation

$$(x + z)u'_x + (y + z)u'_y + (x + y)u'_z = 0.$$

Ex 3. Find solutions to the equation

$$(2y - u)\, u'_x + y\, u'_y = u.$$

Ex 4. Find a general solution to the partial differential equation

$$e^x \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = ye^x$$

Ex 5. To bring the differential equation to the canonical form and integrate it.

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + 3x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0 \quad (x \neq 0, \ y \neq 0)$$

Ex 6. To bring the following differential equation to a canonical form:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 0$$

HYPERBOLIC EQUATIONS

Ex1 Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

If

$$u\big|_{t=0} = x^2, \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0$$

Ex2 Find a solution to the equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = x, \quad \dfrac{\partial u(x,0)}{\partial t} = \cos^2 x; \end{cases}$$

Ex3

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x \sin t ,$$

if

$$u\big|_{t=0} = \sin x , \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = \cos x , \quad x \in R .$$

Ex4 Find a solution to the equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + e^x, \\ u(x,0) = \sin x, \quad \dfrac{\partial u(x,0)}{\partial t} = x + \cos x; \end{cases}$$

Ex5 Fourier method

PARABOLIC EQUATIONS

7.

8.

FUNDAMENTALS OF OPERATIONAL CALCULUS

Examples of possible exam assignments

3. HYPERBOLIC EQUATIONS (FOURIER METHOD)

$$\begin{cases} u_{tt} = a^2 u_{xx} & (*) \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \dfrac{1}{8}\sin\dfrac{3\pi x}{l} \\ u_t(x,0) = 0 \end{cases}$$

4. PARABOLIC EQUATIONS

$$\begin{cases} u_t = a^2 u_{xx} \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = \dfrac{Ax(l-x)}{l^2} \end{cases}$$

Example 1.

Using the definition, find images of the following functions

$$f(t) = e^{4t}$$

Solution:

The function $f(t) = e^{4t}$ is the original with a growth index of 4.

$$F(p) = \int\limits_{0}^{+\infty} e^{4t} \cdot e^{-pt}\,dt = \lim_{B \to +\infty} \int\limits_{0}^{B} e^{-(p-4)t}\,dt = -\lim_{B \to +\infty}\left(\frac{1}{p-4} e^{-(p-4)t}\Bigg|_{0}^{B}\right) =$$

$$= \lim_{B \to +\infty}\left(\frac{1}{p-4} - \frac{e^{-(p-4)B}}{p-4}\right) = \frac{1}{p-4}.$$

Example 2.

$$f(t) = \begin{cases} 0, & t \le 0, \quad t > 2, \\ t, & 0 < t \le 1, \\ 2-t, & 1 < t \le 2. \end{cases}$$

Solution:

We use the Laplace transform formula:

$$F(p) = \int\limits_{0}^{+\infty} e^{-pt} f(t)\,dt = \int\limits_{0}^{1} t e^{-pt}\,dt + \int\limits_{1}^{2}(2-t)e^{-pt}\,dt.$$

$$\int_0^1 te^{-pt}\,dt = \begin{bmatrix} u=t, & dv=e^{-pt}\,dt, \\[2mm] du=dt, & v=\dfrac{e^{-pt}}{-p} \end{bmatrix} = \left. \frac{te^{-pt}}{-p}\right|_0^1 + \frac{1}{p}\int_0^1 e^{-pt}\,dt =$$

$$= -\frac{e^{-p}}{p} - \left.\frac{e^{-pt}}{p^2}\right|_0^1 = -\frac{e^{-p}}{p} - \frac{e^{-p}}{p^2} + \frac{1}{p^2},$$

$$\int_1^2 (2-t)e^{-pt}\,dt = \begin{bmatrix} u=2-t, & dv=e^{-pt}\,dt, \\[2mm] du=-dt, & v=\dfrac{e^{-pt}}{-p} \end{bmatrix} = \left.\frac{(2-t)e^{-pt}}{-p}\right|_1^2 - \frac{1}{p}\int_1^2 e^{-pt}\,dt =$$

$$= \frac{e^{-p}}{p} + \left.\frac{e^{-pt}}{p^2}\right|_1^2 = \frac{e^{-p}}{p} + \frac{e^{-2p}}{p^2} - \frac{e^{-p}}{p^2}.$$

$$F(p) = \frac{1}{p^2}\left(1-2e^{-p}+e^{-2p}\right) = \frac{\left(1-e^{-p}\right)^2}{p^2}.$$

Example 3.

$$f(t) = f_1(t) * f_2(t)$$

$$f_1(t) = \cos t, \quad f_2(t) = e^{2t}$$

$$\cos t \leftrightarrow \frac{p}{p^2+1}$$

$$e^{2t} \leftrightarrow \frac{1}{p-2}$$

$$\int_0^t \cos(t-\tau)e^{2\tau}\,d\tau = \cos t * e^{2t}$$

$$\leftrightarrow \quad \frac{p}{p^2+1}\cdot\frac{1}{p-2} = \frac{p}{\left(p^2+1\right)(p-2)}.$$

Example 4

$$\frac{2p^2}{\left(p^2+1\right)^2} = 2p \cdot \frac{1}{p^2+1} \cdot \frac{p}{p^2+1}$$

$$\frac{1}{p^2+1} \qquad \sin t = f_1(t).$$
$$\leftrightarrow$$

$$\frac{p}{p^2+1} \qquad \cos t = f_2(t)$$
$$\leftrightarrow$$

$$f_1(0) = \sin 0 = 0, \quad f_1'(t) = \cos t,$$

$$2p \cdot \frac{1}{p^2+1} \cdot \frac{p}{p^2+1}$$
$$\leftrightarrow$$

$$0 + 2\int_0^t \cos\tau \cos(t-\tau)\,d\tau =$$

$$= 2\int_0^t \frac{\cos t + \cos(t-2\tau)}{2}\,d\tau = \left(\tau\cos t - \frac{1}{2}\sin(t-2\tau)\right)\Big|_0^t = t\cos t + \sin t.$$

Example 5

$$x(t) \quad\leftrightarrow\quad X(p),$$

$$x'(t) \qquad pX(p) - 15,$$

$$x''(t) \qquad p^2 X(p) - 15p - 2,$$

$$x'''(t) \qquad p^3 X(p) - 15p^2 - 2p - 56.$$

$$\left(p^3 - p^2 - 6p\right)X(p) = 15p^2 - 13p - 36,$$

$$X(p) = \frac{15p^2 - 13p - 36}{p(p-3)(p+2)}.$$

$$p_1 = 0, \ p_2 = 3, \ p_2 = -2$$

$$P(p) = 15p^2 - 13p - 36, \quad Q(p) = p^3 - p^2 - 6p,$$
$$Q'(p) = 3p^2 - 2p - 6,$$

$$p_1 = 0$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_1=0} = \frac{-36}{-6} = 6,$$

$$p_2 = 3$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_2=3} = \frac{60}{15} = 4,$$

$$p_3 = -2$$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_3=-2} = \frac{50}{10} = 5,$$

and by the second decomposition theorem we get

$$x(t) = 6 + 5e^{-2t} + 4e^{3t}.$$

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Ex 1.

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x-y}$$

$$\begin{cases} \dfrac{dy}{dx} = -\dfrac{y}{x}, \\ \dfrac{dz}{dx} = \dfrac{x-y}{x} \end{cases} \Leftrightarrow \begin{cases} \dfrac{dy}{dx} = -\dfrac{y}{x}, \\ \dfrac{dz}{dx} = 1 - \dfrac{y}{x} = 1 + \dfrac{dy}{dx}, \end{cases}$$

$$xy = C_1$$

$$z - y - x = C_2$$

Then the general solution of the given equation has the form:

$$u(x, y, z) = \Phi(xy, z - x - y).$$

where $\Phi(a,b)$ an arbitrary continuously differentiable function.

Ex.2

$$\frac{dx}{x+z} = \frac{dy}{y+z} = \frac{dz}{x+y}$$

By the property of equal fractions we have

$$\frac{dx - dz}{z - y} = \frac{dy - dz}{z - x} \Rightarrow (x - z)d(x - z) = (y - z)d(y - z).$$

Integrating the last equality, we get the first integral

$$\psi_1(x, y, z) = (x - z)^2 - (y - z)^2 = (x - y)(x + y - 2z).$$

By the property of equal fractions we have

$$\frac{dx + dy + dz}{2(x + y + z)} = \frac{dx - dy}{x - y} \quad \Leftrightarrow \quad \frac{d(x + y + z)}{x + y + z} = \frac{2d(x - y)}{x - y},$$

Integrating

$$\psi_2(x, y, z) = \frac{x + y + z}{(x - y)^2}$$

Then the general solution of the given equation has the form

$$u(x, y, z) = \Phi\left((x - y)(x + y - 2z), \frac{x + y + z}{(x - y)^2}\right).$$

where $\Phi(a,b)$ an arbitrary continuously differentiable function.

Ex3.

$$\frac{dx}{2y - u} = \frac{dy}{y} = \frac{du}{u}.$$

$$\frac{dy}{y} = \frac{du}{u} \quad \Rightarrow \quad \frac{u}{y} = C_1,$$

$$\psi_1(x, y, u) = \frac{u}{y}.$$

$$\frac{dx}{2y - u} = \frac{2dy - du}{2y - u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

$$\psi_2(x, y, u) = x - 2y + u.$$

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0.$$

where $\Phi(a,b)$ an arbitrary continuously differentiable function.

Ex4

Creating a system

$$\frac{dx}{e^x} = \frac{dy}{y^2} = \frac{du}{ye^x}$$

From the first equation, we find one first integral

$$\frac{1}{y} - e^{-x} = C_1$$

and from the second, taking into account the equality

$$e^x = \frac{y}{1 - yC_1}$$

another first integral

$$u - \frac{\ln|y| - x}{e^{-x} - y^{-1}} = C_2$$

follows.

Thus, the general integral of this equation will be

$$\Phi\left(\frac{1}{y} - e^{-x}, \frac{\ln|y| - x}{e^{-x} - y^{-1}} - u\right) = 0$$

The general solution has the form

$$u = \frac{\ln|y| - x}{e^{-x} - y^{-1}} + \varphi\left(\frac{1}{y} - e^{-x}\right).$$

Ex 5

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + 3x \frac{\partial z}{\partial x} - 3y \frac{\partial z}{\partial y} = 0 \quad (x \neq 0, \ y \neq 0)$$

$A = x^2, \ B = 0, \ C = -y^2$

$B^2 - AC = -x^2(-y^2) = x^2 y^2 > 0 \qquad \text{if} \qquad x \neq 0, y \neq 0.$

Therefore, this equation is of a hyperbolic type. The characteristic equation (2.28) takes the form:

$$x^2 dy^2 - y^2 dx^2 = 0$$

or

$$y^2 dx^2 - x^2 dy^2 = 0$$

$$\begin{cases} y dx + x dy = 0, \\ y dx - x dy = 0. \end{cases}$$

Integrating these equations, we obtain

$$xy = C_1,$$

$$y / x = C_2.$$

Let's introduce new variables

$$\xi = xy, \quad \eta = y / x.$$

We find partial derivatives

$$\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial \xi} - \frac{y}{x^2}\frac{\partial z}{\partial \eta},$$

$$\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial \xi} + \frac{1}{x}\frac{\partial z}{\partial \eta},$$

$$\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial \xi^2} - 2\frac{y^2}{x^2}\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial \eta^2} + 2\frac{y}{x^3}\frac{\partial z}{\partial \eta},$$

$$\frac{\partial^2 z}{\partial y^2} = x^2\frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{1}{x^2}\frac{\partial^2 z}{\partial \eta^2}.$$

Substituting these expressions into the original equation, we obtain its canonical form:

$$\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{1}{xy}\frac{\partial z}{\partial \eta} = 0,$$

or

$$\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{1}{\xi}\frac{\partial z}{\partial \eta} = 0.$$

Let's integrate the last equation. Let's denote

$$\frac{\partial z}{\partial \eta} = w.$$

Then we have

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{\partial w}{\partial \xi}, \quad \frac{\partial w}{\partial \xi} + \frac{1}{\xi}w = 0, \quad \frac{1}{w}\frac{\partial w}{\partial \xi} + \frac{1}{\xi} = 0.$$

$$\ln|w| + \ln|\xi| = \ln|\mu(\eta)|, \quad w\xi = \mu(\eta).$$

Returning to the z function, we will have the differential equation

$$\frac{\partial z}{\partial \eta}\xi = \mu(\eta),$$

integrating, we find:

$$\xi z = \int \mu(\eta)d\eta + v(\xi).$$

Therefore,

$$z = \frac{1}{\xi}\mu(\eta) + v(\xi),$$

where μ and v are arbitrary doubly continuously differentiable functions of their arguments. Returning to the variables x and y, we finally get:

$$z = \frac{1}{xy}\mu(y/x) + v(xy).$$

Ex 6

To bring the following differential equation to a canonical form:

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial y} = 0$$

Solution:

The type of equation:

$$a = 1, \quad b = 1, \quad c = -3$$

$$b^2 - ac = 1 + 3 = 4 \ > 0$$

<u>This equation has a hyperbolic type.</u>

Next, we need to bring it to a canonical form.

We write the <u>characteristic equation:</u>

$$dy^2 - 2dydx - 3dx^2 = 0$$

We solve this equation as a quadratic one.

$$a\,dy - \left(b \pm \sqrt{b^2 - ac}\right)dx = 0$$

$$dy - \left(1 \pm \sqrt{4}\right)dx = 0$$

$$\begin{cases} dy - 3dx = 0 \\ dy + dx = 0 \end{cases}$$

It is good that we have an equation with constant coefficients: a, b, c do not depend on x, y.

$$\begin{cases} y - 3x = C_1 \\ y + x = C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi = y - 3x \\ \eta = y + x \end{cases}$$

When moving to these variables, the equation (*) is greatly simplified.

$$u(x, y) \rightarrow U(\xi, \eta)$$

The derivative of the whole function:

$$u_x = U_\xi \xi_x + U_\eta \eta_x$$
$$u_y = U_\xi \xi_y + U_\eta \eta_y$$

Take the second derivatives of $u_x,\ u_y$.

$$u_x = U_\xi \cdot (-3) + U_\eta \cdot 1$$

$$u_y = U_\xi \cdot 1 + U_\eta \cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx} = -3\left(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x\right) + 1\left(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x\right) =$$

$$= -3\left(U_{\xi\xi} \cdot (-3) + U_{\xi\eta} \cdot 1\right) + 1\left(U_{\eta\xi} \cdot (-3) + U_{\eta\eta} \cdot 1\right) =$$

$$= 9U_{\xi\xi} - 6U_{\eta\xi} + U_{\eta\eta}$$

$$u_{xy} = -3\left(U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y\right) + 1\left(U_{\eta\xi} \cdot \xi_y + U_{\eta\eta} \cdot \eta_y\right) =$$

$$= -3\left(U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 1\right) + 1\left(U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 1\right) =$$

$$= -3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = 1\left(U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y\right) + 1\left(U_{\eta\xi} \cdot \xi_y + U_{\eta\eta} \cdot \eta_y\right) =$$

$$= 1 \cdot U_{\xi\xi} + U_{\xi\eta} \cdot 1 + U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 1 =$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Substituting everything into our equation:

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} + 2\left(-3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\right) - 3\left(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}\right) +$$

$$+2\left(U_{\xi} \cdot (-3) + U_{\eta}\right) + 6\left(U_{\xi} + U_{\eta}\right) = 0$$

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} - 6U_{\xi\xi} - 4U_{\xi\eta} + 2U_{\eta\eta} - 3U_{\xi\xi} - 6U_{\xi\eta} - 3U_{\eta\eta} -$$

$$-6U_{\xi} + 2U_{\eta} + 6U_{\xi} + 6U_{\eta} = 0$$

$$-16U_{\xi\eta} + 8U_{\eta} = 0$$

$$-U_{\xi\eta} = \frac{1}{2}U_{\eta}$$

This expression is already the canonical form of a hyperbolic equation.

HYPERBOLIC EQUATIONS

Ex1

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

$$u\big|_{t=0} = x^2, \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = 0$$

$$\psi(x) = 0$$

$$u(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2}$$

$$a = 1, \quad \varphi(x) = x^2$$

$$u(x,t) = \frac{(x+t)^2 + (x-t)^2}{2}$$

or

$$u(x,t) = x^2 + t^2$$

Ex. 2 (on your own for training).

Ex.3.

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x \sin t ,$$

if

$$u\big|_{t=0} = \sin x , \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = \cos x , \quad x \in R .$$

Solution:

$$u_{tt} - a u_{xx} = f(x;t)$$

$$a = 1$$

$$f(x;t) = x \cdot \sin t$$

$$u\big|_{t=0} = \sin x : u(x;0) = \sin x, \quad \text{so} \quad \varphi(x) = \sin x$$

$$u_t'\big|_{t=0} = \cos x : u_t'(x;0) = \cos x, \quad \text{so} \quad \psi(x) = \cos x$$

Since $u(x;t) = \upsilon(x;t) + \omega(x;t)$, we will find $\upsilon(x;t)$ and $\omega(x;t)$.

Let $\upsilon(x;t) = \upsilon_1 + \upsilon_2$.

$$\upsilon_1(x,t) = \frac{\varphi(x+at) + \varphi(x-at)}{2}$$

$$\upsilon_1 = \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) = \frac{1}{2}(\sin(x+t) + \sin(x-t))$$

$$\upsilon_2 = \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)\,dy = \frac{1}{2}\int_{x-t}^{x+t} \cos y\,dy = \frac{1}{2}\sin y\Big|_{x-t}^{x+t} =$$

$$= \frac{1}{2}(\sin(x+t) - \sin(x-t))$$

$$\upsilon(x;t) = \frac{1}{2}(\sin(x+t) + \sin(x-t)) + \frac{1}{2}(\sin(x+t) - \sin(x-t)) =$$

$$= \sin(x+t)$$

We have $\upsilon(x;t) = \sin(x+t)$.

Now let's find $\omega(x;t)$.

Since $f(x;t) = x \cdot \sin t$, then

$$\omega = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi;\tau)\,d\xi\,d\tau = \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} \xi\cdot\sin\tau\,d\xi\,d\tau = \frac{1}{2}\int_0^t \sin\tau\,d\tau\,\frac{\xi^2}{2}\Big|_{x-t+\tau}^{x+t-\tau} =$$

$$= \frac{1}{4}\int_0^t \sin\tau\left[(x+t-\tau)^2 - (x-t+\tau)^2\right]d\tau =$$

$$= \frac{1}{4}\int_0^t \sin\tau\left[(x+t-\tau+x-t+\tau)(x+t-\tau-x+t-\tau)\right]d\tau =$$

$$= x\int_0^t \sin\tau\cdot(t-\tau)\,d\tau = x(t-\sin t)$$

We get

$$u = \sin(x+t) + xt - x\sin t$$

Ex 4 (on your own for training).