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Analytic Geometry. Vectors and Operations with Vectors

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First Look on Vectors

- We call vector a segment with assigned to it direction, so one ts endpoint considered as initial point and second endpoint considered as terminal of vector
- Reminder. We postulated before that there is one and only one segment having some pair of points as endpoints, and vice-versa there is one and only one pair of endpoints of given segment
- Physics gives us alternative approach to understanding vectors. Here we assign direction to some physical measure (e.g. force, speed, etc.)
- In mus be noted that some "directed" physical measures are not invariant against mirror image of spaces and are called pseudovectors (or axial vectors)
- More deep generalization of term vector is tensor
- Here we start discussion on most general features of vectors

Directed Segments I

- We start with assigning direction to particular segments (we call them particular vectors)
- Lets assign a direction to arbitrary segment AB and A becomes its initial, and B becomes its terminal point
- We say that vector points from A to B and utilize notation $\overrightarrow{AB}$ or $\vec{a}$
- Alternative notations may fortune in literature: lines over ($\bar{a}$) and under ()$\underline{a}$) the vector denote, bold font ($\boldsymbol{F}$)
- Thus, we consider vector as ordered pair of its initial and terminal points
- Relations ("lies on", "parallel", "forms angle", etc.) of the vector with segments, lines and planes are inherited from relations of parent segment with these object
- We will not separate vectors on plane and in space in this discussion

Directed Segments II

- We say that vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are codirected and write $\overrightarrow{AB} \upuparrows \overrightarrow{CD}$ if for any arbitrary points M and N shaping segments AM overlapping AB and CN overlapping CD, and $AM = BM$, length of segment MN limited by constant finite value. In common words, points M and N follow each other on "parallel courses"
- There is no need to enforce this definition with demand of parallelism of AB and CD. If these segments lay on crossing line, distance between two following points may decrease until lines common points, but will grow without any limitation after following this point
- If vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ lay on parallel lines, but are not codirected, we say that they are anti-codirected and write $\overrightarrow{AB} \uparrow\downarrow \overrightarrow{CD}$

Directed Segments III

- Theorem 1: *there are two vectors, say $\overrightarrow{AB}$ and $\overrightarrow{CD}$, are codirected with third arbitrary vector, say $\overrightarrow{EF}$. Thus, given vectors are codirected*
- Proof: in given condition there are positive real numbers d_{AB} and d_{DC} and for any equal segments AM, CN, and EP containing B, D, and F respectively $MP < d_{AB}$ and $NP < d_{CD}$. Points M, N, and P shape triangle (or lay on a line as extreme case), thus $MN \leqslant MP + NP < d_{AB} + d_{CD}$. This is definition that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are codirected. $\square$
- We say vectors are equal if shaping them segments are equal (have equal length), and they are codirected. Notation: $\overrightarrow{AB} = \overrightarrow{CD}$
- Theorem 2: *there are two vectors, say $\overrightarrow{AB}$ and $\overrightarrow{CD}$, and for arbitrary vector $\overrightarrow{EF}$ $\overrightarrow{AB} = \overrightarrow{EF}$, and $\overrightarrow{CD} = \overrightarrow{EF}$. Thus, $\overrightarrow{AB} = \overrightarrow{CD}$*
- Proof: $\overrightarrow{AB}$ and $\overrightarrow{EF}$ are codirected, as well as $\overrightarrow{CD}$ and $\overrightarrow{EF}$. Thus, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are codirected. Their lengths are equal with the same value, thus are equal to each other. This is definition of vector equality as it is. $\square$

Directed Segments IV

- Consequence: relation of vectors equality is reflexive, symmetric, and transitive
  - $\overrightarrow{a} = \overrightarrow{a}$
  - $\overrightarrow{a} = \overrightarrow{b}$ and $\overrightarrow{b} = \overrightarrow{a}$
  - By Theorem 2: if $\overrightarrow{a} = \overrightarrow{b}$, and $\overrightarrow{b} = \overrightarrow{c}$, then $\overrightarrow{a} = \overrightarrow{c}$
- Theorem 3: Consider vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ laying on different lines. These vectors are codirected if and only if they lay on parallel lines, and they lay in the same half-plane with respect to line AC.
- Proof:
  - Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ lay on parallel lines, and they lay in the same half-plane with respect to line AC.
  - Consider arbitrary segments $AM = CN$, overlapping AB and CD respectively (thus, laying in the same half-plane with respect to AC).
  - Figure $AMNC$ resembles parallelogram for any AM and CN
  - Thus, $AC = MN$ and vectors are codirected. $\square$
  - Let $\overrightarrow{AB}$ and $\overrightarrow{CD}$ being codirected, and do not belong to the same line
  - Consider segment $CE||AB$, and laying in the same half-plane as CD with respect to line AC
  - $\overrightarrow{CD} \upuparrows \overrightarrow{AB}$ by definition

Directed Segments V

- Let angle $\angle DCE$ be ordinary
- Thus in the family of transverse segments $\{PM\}$, there $CP = CN$, distance between P and N grows without any limitation with growing CN.
- In the same time $MP = AC$ for any P(and N)
- Consider triangle MNP: $MN > NP - MP = NP - AC$, and NP grows without limitation with NP.
- Thus, $\overrightarrow{AB} \not\uparrow\uparrow \overrightarrow{CD}$ if angle $\angle CDE$ is ordinary.
- Therefore, CE overlaps CD, thus CD is parallel to AB and lies in the same half-pale with respect to AC. $\square$

- Particular case: laying on the same line vectors are codirected if and only if ray shaped by first vector contains second vector or vice-versa.

Directed Segments VI

- Theorem 4: For arbitrary vector $\overrightarrow{AB}$ and any point C there is one and only one vector $\overrightarrow{CD} = \overrightarrow{AB}$
- Proof:
  - For point C not laying on the line shaped with AB there is plane containing all points A, B, and C
  - We take point D from this plane to shape segment $CD||AB$, $CD = AB$, and laying in the same half-plane as AB with respect to line AC
  - Theorem 3 establishes that $\overrightarrow{AB} = \overrightarrow{CD}$
  - Uniqueness of the line parallel to given and containing specified distant point grants Uniqueness of$\overrightarrow{CD}$
  - For point C laying on the line shaped with AB there are two cases: $\overrightarrow{AC} \upuparrows \overrightarrow{AB}$, thus CD is continuation of AC, int opposite case CD overlaps AC
  - Uniqueness of $\overrightarrow{CD} = \overrightarrow{AB}$ is consequence of the uniqueness of the segment of given length starting from endpoint point of arbitrary ray.

Zero Vector I

Consider statement:

- For arbitrary vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ necessary and sufficient condition for $\overrightarrow{AB} = \overrightarrow{CD}$ is $\overrightarrow{AC} = \overrightarrow{BD}$
- Proof:
  - For the case there AB and CD lay on different lines condition $\overrightarrow{AB} = \overrightarrow{CD}$ is equal to the statement that AB and CD are sides of parallelogram.
  - Opposite pair of sides, AC and BD are also equal and parallel to each other and lie in the same half-plane wit respect to AB (and as well CD), thus $\overrightarrow{AC} \upuparrows \overrightarrow{BD}$ and $\overrightarrow{AC} = \overrightarrow{BD}$.
  - For $\overrightarrow{AC} = \overrightarrow{BD}$ the same reasoning gives $\overrightarrow{AB} = \overrightarrow{CD}$. $\square$

For $A = C$ this statement is logically invalid, as we require now $\overrightarrow{AA} = \overrightarrow{BB}$.

Zero Vector II

Now we generalize our definition

- We call **vector** any ordered pair of points, distant or not
- For vector composed of a pair of distant pints we call first point int initial and second point terminal, and assign to the vector direction from initial to terminal point
- Vector composed of a pair of equal points we call **zero vector**
- We assign zero length to such vector and left its direction to be undefined

Vector as Abstract Object

- We call **(abstract) vector** the object assigned to the class of equal directed segments.
- This object keeps information on length and direction of any directed segment in specified class
- Vector in this sense resembles directed segment of specified length and direction established from any point
- We say that vector applied to point (or body)
- For example, vector $\boldsymbol{a}$ may be applied to point A, and shape directed segment $\overrightarrow{AB} = \boldsymbol{a}$. In the same time, application of this vector to distant point A'shapes new directed segment $\overrightarrow{A'B'} = \overrightarrow{AB} = \boldsymbol{a}$
- We will denote all zero vectors as $\mathbf{0}$
- We define vector with specification of its direction and length
- Direction may be specified with a ray "directing" vector
- Length has common sense with ordinary vectors and denoted as $|\boldsymbol{a}|$

Addition of Vectors I

- Consider translation of the point particle form point A to point B, and next to point C.
- Directed segments $\overrightarrow{AB}$ and $\overrightarrow{BC}$ represent these translations.
- As a final result we must consider translation from A to C and corresponding directed segment $\overrightarrow{AC}$
- It will be natural to consider $\overrightarrow{AC}$ as a sum of $\overrightarrow{AB}$ and $\overrightarrow{BC}$: $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$
- If these directed segments correspond to (abstract) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, we may define sum $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$ with $\boldsymbol{c}$ corresponding to AC by the procedure:
  - Select arbitrary point A
  - Apply $\boldsymbol{a}$ to A and obtain $\overrightarrow{AB} = \boldsymbol{a}$
  - Apply $\boldsymbol{b}$ to B and obtain $\overrightarrow{BC} = \boldsymbol{c}$
  - Construct $\boldsymbol{c} = \overrightarrow{AC}$
- Invariance of this definition against choose of A is a matter of proof

Addition of Vectors II

- Theorem: sum of vectors is invariant against point selected to establish directed segments.
- Proof:
  - Consider distant points A and A'. Definition of vector states $\boldsymbol{a} = \overrightarrow{AB} = \overrightarrow{A'B'}$ and $\boldsymbol{b} = \overrightarrow{BC} = \overrightarrow{B'C'}$
  - Thus, $\overrightarrow{AA'} = \overrightarrow{BB'}$ and $\overrightarrow{BB'} = \overrightarrow{CC'}$
  - Transition: $\overrightarrow{AC} = \overrightarrow{A'C'}$. $\square$
- We call described approach **The triangle low of vectors addition**
- Key disadvantage: symmetry is not obvious and requires proof.
- **The parallelogram low of vectors addition**
  - Select arbitrary point A, and apply both $\boldsymbol{a}$ and $\boldsymbol{b}$ to it: $\overrightarrow{AB} = \boldsymbol{a}$, $\overrightarrow{AC} = \boldsymbol{b}$
  - This triplet, A, B, C, allows shaping parallelogram $ABCD$
  - $\overrightarrow{AB} = \overrightarrow{CD} = \boldsymbol{a}$ and $\overrightarrow{AC} = \overrightarrow{BD} = \boldsymbol{b}$
  - $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{a} + \boldsymbol{b} = \overrightarrow{AC} + \overrightarrow{CD} = \boldsymbol{b} + \boldsymbol{a} = \boldsymbol{c}$

Addition of Vectors III

- On a single line (as well as on parallel lines) sum of vectors depends on their direction:
  - for $\boldsymbol{a} \upuparrows \boldsymbol{b}$ direction will be the same and legths will sum
  - for $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b}$ direction corresponds to greater (by length) vector and less length will be subtracted from greater
- Given definition of addtion of vectors corresponds to definition of vector as abstract object, but not as particular directed segment
- Constructive manipulations with particular directed segments is just application of this definition

Commutative Group

Key features of addition operation

1. For any pair $\boldsymbol{a}$ and $\boldsymbol{b}$ there is vector $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$
2. For any pair $\boldsymbol{a}$ and $\boldsymbol{b}$ $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$
3. $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$
4. $\boldsymbol{a} + \mathbf{0} = \boldsymbol{a}$
5. For any vector $\boldsymbol{a}$ there is vector $-\boldsymbol{a}$ and $\boldsymbol{a} + (-\boldsymbol{a}) = \mathbf{0}$
6. Operation of subtraction may be defined for any pair of vectors: if $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, we define $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$ if $\boldsymbol{a} = \boldsymbol{c} + \boldsymbol{b}$

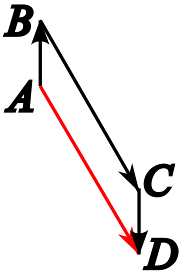
By this list of properties we may conclude that set of vectors form algebraic structure known as **commutative group** with operation of addition

Addition of Vectors. Problems Corner I

Problem 1

1. Arbitrary body moved from point A to point B, later to point C, and finally to point D
2. Distances are folloving $AB = 1\text{cm}$, $BC = 3\text{cm}$, and $CD = 1\text{cm}$
3. $\angle ABC = 60^\circ$, $\overrightarrow{AB} \upuparrows \overrightarrow{DC}$
4. Draw sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$ and find it's length.

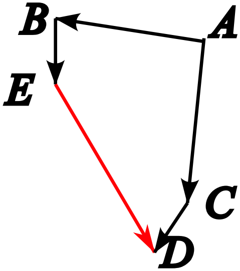
Addition of Vectors. Problems Corner II



Desired figure is parallelogram, $|\overrightarrow{AD}| = |\overrightarrow{BC}|$ =3cm

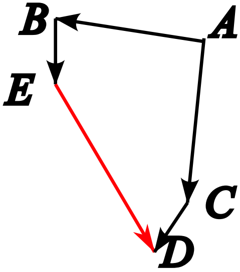
Problem 2

Addition of Vectors. Problems Corner III



Express $\overrightarrow{ED}$ as sum of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BE}$, and $\overrightarrow{CD}$

Addition of Vectors. Problems Corner IV



Express $\overrightarrow{ED}$ as sum of vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BE}$, and $\overrightarrow{CD}$

$-\overrightarrow{BE} + (-\overrightarrow{AB}) + \overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{ED}$

Home assignment

Problem 3

There is a triangle $\triangle ABC$ with equal sides. Plot and find length of vectors $\overrightarrow{AB}+\overrightarrow{AC}$, $\overrightarrow{AB}-\overrightarrow{AC}$. For calculations let length of side be $2\sqrt{3}$cm

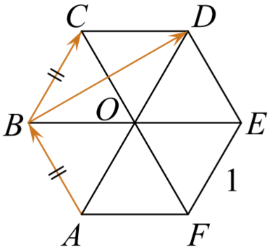
Problem 4

In triangle $\triangle ABC$ $\angle A = 120^\circ$ $AB = 1$cm, $AC = 2$cm. Calculate $\overrightarrow{AB}+\overrightarrow{AC}$, $\overrightarrow{BA}+\overrightarrow{AC}$

Addition of Vectors. Problems Corner VI

Problem 5

There is right hexagon $ABCDEF$ (each side is equal) showed on the plot. O is center of the figure. Find $|\overrightarrow{AB} + \overrightarrow{BC}|$, $|\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{ED}|$, and $|\overrightarrow{OD} + \overrightarrow{DB}|$. Length of each side is 1.



Collinearity and Coplanarity

- We call vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ **collinear** and write $\boldsymbol{a}||\boldsymbol{b}$ if there is arbitrary line and corresponding directed segments are parallel to it
- We call a vector **coplanar** with arbitrary plane if corrsponding directed segments are parallel with the plane
- Zero vector is **collinear** with any vector
- Zero vector is **coplanar** with any plane

Product of Vector and Real Number I

- For given vector $\boldsymbol{a}$ and real number x we call their product vector denoted as $x\boldsymbol{a}$ with features:
  - $|x\boldsymbol{a}| = |x||\boldsymbol{a}|$
  - If $\boldsymbol{a} = \mathbf{0}$ or $x = 0$ (or both), then $x\boldsymbol{a} = \mathbf{0}$
  - If $x > 0$, then $x\boldsymbol{a} \upuparrows \boldsymbol{a}$
  - If $x > 0$, then $x\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$
- Theorem 1: There are vectors $\boldsymbol{a} \neq \mathbf{0}$ and $\boldsymbol{b}$. Existence of real number x: $\boldsymbol{b} = x\boldsymbol{a}$ is necessary and sufficient condition for collinearity of $\boldsymbol{a}$ and $\boldsymbol{b}$. The number x is explicit for the pair of vectors.

Product of Vector and Real Number II

- Proof:
  - If $\boldsymbol{b} = x\boldsymbol{a}$, then $\boldsymbol{a}||\boldsymbol{b}$ by given definitions of vectors collinearity and product of vector and number
  - Thus, $\boldsymbol{a} \upuparrows \boldsymbol{b}$, or $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b}$ or $\boldsymbol{b} = \mathbf{0}$
  - Consider $\boldsymbol{a}||\boldsymbol{b}$ and construct x
    - Let $\boldsymbol{b} = \mathbf{0} \Rightarrow x = 0$: $\mathbf{0} = 0\boldsymbol{a}$
    - Let $\boldsymbol{a} \upuparrows \boldsymbol{b} \Rightarrow x = \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|} > 0$: $|x\boldsymbol{a}| = |x||\boldsymbol{a}| = \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}|\boldsymbol{a}| = |\boldsymbol{b}|$ and $x\boldsymbol{a} \upuparrows \boldsymbol{a} \upuparrows \boldsymbol{b} \Rightarrow x\boldsymbol{a} \upuparrows \boldsymbol{b}$, thus $\boldsymbol{b} = x\boldsymbol{a}$
    - Let $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b} \Rightarrow x = -\frac{|\boldsymbol{b}|}{|\boldsymbol{a}|} < 0$: $|x\boldsymbol{a}| = |x||\boldsymbol{a}| = \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}|\boldsymbol{a}| = |\boldsymbol{b}|$ and $x\boldsymbol{a} \uparrow\downarrow \boldsymbol{a} \uparrow\downarrow \boldsymbol{b} \Rightarrow x\boldsymbol{a} \upuparrows \boldsymbol{b}$, thus $\boldsymbol{b} = x\boldsymbol{a}$ $\square$

Note: build of anti-codirected counterpart of equal length for given vector resembles product of -1 and vector.

Product of Vector and Real Number III

Consider features of number-vector product

1. If $\boldsymbol{a} = \mathbf{0}$ or $x = 0$, then $x\boldsymbol{a} = \mathbf{0}$
2. $1 \cdot \boldsymbol{a} = \boldsymbol{a}$
3. $(-1) \cdot \boldsymbol{a} = -\boldsymbol{a}$
4. For arbitrary vector $\boldsymbol{a}$, and arbitrary real numbers x and y $x(y\boldsymbol{a}) = (xy)\boldsymbol{a}$
5. For arbitrary vector $\boldsymbol{a}$, and arbitrary real numbers x and y $(x + y)\boldsymbol{a} = x\boldsymbol{a} + y\boldsymbol{a}$
6. For arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and arbitrary real number x $x(\boldsymbol{a} + \boldsymbol{b}) = x\boldsymbol{a} + x\boldsymbol{b}$

Feature 1, 2, and 3 are direct consequence form definition, but features 4, 5, and 5 require some proof.
General steps for proof:

- Check zero-cases
- Check length of left and right sides
- Monitor redirection of vectors in left and right side with respect to original vectors

Product of Vector and Real Number IV

Proof for feature 4 ($x(y\boldsymbol{a}) = (xy)\boldsymbol{a}$):

- Let $\boldsymbol{a} = \mathbf{0}$. $x(y\mathbf{0}) = x\mathbf{0} = \mathbf{0}$, and $(xy)\boldsymbol{a} = (xy)\mathbf{0} = \mathbf{0}$
- Let $x = 0$ or $y = 0$ (or both are zero), thus $xy = 0$ and $(xy)\boldsymbol{a} = \mathbf{0}$. $0 \cdot (y\boldsymbol{a}) = 0 \cdot \boldsymbol{b} = \mathbf{0}$, or $x(0 \cdot \boldsymbol{a}) = x\mathbf{0} = \mathbf{0}$.
- Consider all-non-zero case
  - $|x(y\boldsymbol{a})| = |x||y\boldsymbol{a}| = |x||y||\boldsymbol{a}|$ and $|(xy)\boldsymbol{a}| = |xy||\boldsymbol{a}| = |x||y||\boldsymbol{a}|$
  - $xy > 0 \Rightarrow x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. Thus, $(xy)\boldsymbol{a} \upuparrows \boldsymbol{a}$
    - $x > 0$ and $y > 0$. Thus, $(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$, and $x\boldsymbol{b} \upuparrows \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \upuparrows y\boldsymbol{a} \upuparrows \boldsymbol{a}$, and $x(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$
    - $x < 0$ and $y < 0$. Thus, $(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$, and $x\boldsymbol{b} \uparrow\downarrow \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \uparrow\downarrow y\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$, and $x(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$
  - $xy < 0 \Rightarrow x < 0$ and $y > 0$, or $x > 0$ and $y < 0$. Thus, $(xy)\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$
    - $x < 0$ and $y > 0$. Thus, $(y\boldsymbol{a}) \upuparrows \boldsymbol{a}$, and $x\boldsymbol{b} \uparrow\downarrow \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \uparrow\downarrow y\boldsymbol{a} \upuparrows \boldsymbol{a}$, and $x(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$
    - $x > 0$ and $y < 0$. Thus, $(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$, and $x\boldsymbol{b} \upuparrows \boldsymbol{b}$ for any $\boldsymbol{b}$, and, as particular case, $x(y\boldsymbol{a}) \upuparrows y\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$, and $x(y\boldsymbol{a}) \uparrow\downarrow \boldsymbol{a}$
  - Left and right operations preserve length and direction of result vector. $\square$

Product of Vector and Real Number V

Proof for feature 5 ($(x+y)\boldsymbol{a} = x\boldsymbol{a} + y\boldsymbol{a}$):

- Let $\boldsymbol{a} = \mathbf{0}$. $(x+y)\mathbf{0} = \mathbf{0}$, and $x\mathbf{0} + y\mathbf{0} = \mathbf{0}$
- Let $x + y = 0$, thus $x = (-1) \cdot y$. $0 \cdot \boldsymbol{a} = \mathbf{0}$, and $x\boldsymbol{a} + y\boldsymbol{a} = x\boldsymbol{a} + (-1)x\boldsymbol{a} = x\boldsymbol{a} - x\boldsymbol{a} = \mathbf{0}$
- Consider all-non-zero case and $xy > 0$.
  - $|x+y| = |x| + |y| \Rightarrow |(x+y)\boldsymbol{a}| = |(x+y)||\boldsymbol{a}| = (|x| + |y|)|\boldsymbol{a}| = |x||\boldsymbol{a}| + |y||\boldsymbol{a}|$
  - $x\boldsymbol{a} \upuparrows y\boldsymbol{a} \Rightarrow |x\boldsymbol{a} + y\boldsymbol{a}| = |x\boldsymbol{a}| + |y\boldsymbol{a}| = |x||\boldsymbol{a}| + |y||\boldsymbol{a}|$
  - $\operatorname{sign}(x) = \operatorname{sign}(y) = \operatorname{sign}(x+y) = s$
  - $(x+y)\boldsymbol{a} = s|x+y|\boldsymbol{a}$, and $x\boldsymbol{a} + y\boldsymbol{a} = s|x|\boldsymbol{a} + s|y|\boldsymbol{a}$, thus all $(x+y)\boldsymbol{a}$, $x\boldsymbol{a}$ and $y\boldsymbol{a}$ are (anti-)codirected with $\boldsymbol{a}$ in the same time and have the same direction.
- Consider all-non-zero case and $xy < 0$.
  - Let $\operatorname{sign}(-y) = \operatorname{sign}(x+y)$ without leak of generalization
  - $(x+y)\boldsymbol{a} - y\boldsymbol{a} = (x+y-y)\boldsymbol{a} = x\boldsymbol{a} \Rightarrow (x+y)\boldsymbol{a} = x\boldsymbol{a} + y\boldsymbol{a}$

Product of Vector and Real Number VI

Proof for feature 6 ($x(\boldsymbol{a}+\boldsymbol{b}) = x\boldsymbol{a} + x\boldsymbol{b}$)

- Let $x = 0$. $0 \cdot (\boldsymbol{a}+\boldsymbol{b}) = \mathbf{0}$, and $0 \cdot \boldsymbol{a} + 0 \cdot \boldsymbol{b} = \mathbf{0} + \mathbf{0} = \mathbf{0}$
- Let $\boldsymbol{a} \upuparrows \boldsymbol{b}$.
  - $\boldsymbol{a} \upuparrows \boldsymbol{b} \upuparrows \boldsymbol{a}+\boldsymbol{b}$, and $x\boldsymbol{a} \upuparrows x\boldsymbol{b} \Rightarrow x\boldsymbol{a} \upuparrows x\boldsymbol{b} \upuparrows x(\boldsymbol{a}+\boldsymbol{b})$
  - $|x(\boldsymbol{a}+\boldsymbol{b})| = |x||\boldsymbol{a}+\boldsymbol{b}| = |x|(|\boldsymbol{a}|+|\boldsymbol{b}|) = |x||\boldsymbol{a}| + |x||\boldsymbol{b}|$
  - $|x\boldsymbol{a}+x\boldsymbol{b}| = |x\boldsymbol{a}| + |x\boldsymbol{b}| = |x||\boldsymbol{a}| + |x||\boldsymbol{b}|$
- Let $\boldsymbol{a} \uparrow\downarrow \boldsymbol{b}$.
  - Let $(\boldsymbol{a}+\boldsymbol{b}) \upuparrows -\boldsymbol{b}$ without leak of generalization
  - $x(\boldsymbol{a}+\boldsymbol{b}) \upuparrows -x\boldsymbol{b}$
  - $x(\boldsymbol{a}+\boldsymbol{b}) + (-x\boldsymbol{b}) = x(\boldsymbol{a}+\boldsymbol{b}+(-\boldsymbol{b})) = x\boldsymbol{a} \Rightarrow x(\boldsymbol{a}+\boldsymbol{b}) = x\boldsymbol{a} + x\boldsymbol{b}$

Product of Vector and Real Number VII

- General case $\boldsymbol{a} \nparallel \boldsymbol{b}$
  - Consider directed segments: $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$
  - Sum $\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{OB} = \boldsymbol{a} + \boldsymbol{b}$ is diagonal of corresponding parallelogram $OADB$
  - Let $x > 0$. Directed segments $\overrightarrow{OA'} = x\boldsymbol{a}$ and $\overrightarrow{OB'} = x\boldsymbol{b}$ overlap $\overrightarrow{OA}$ and $\overrightarrow{OB}$ Sum $\overrightarrow{OD'} = \overrightarrow{OA'} + \overrightarrow{OB'} = x\boldsymbol{a} + x\boldsymbol{b}$ is diagonal of corresponding parallelogram $OA'D'B'$ similar with $OADB$
  - $\overrightarrow{OD} \upuparrows \overrightarrow{OD'}$, as opposite sides of parallelograms overlap
  - $OD' = xOD \Rightarrow |\overrightarrow{OD'}| = |x\overrightarrow{OD}| = |x(\boldsymbol{a} + \boldsymbol{b})| = |x\boldsymbol{a} + x\boldsymbol{b}|$ as parallelograms are similar
  - $x\overrightarrow{OA} + x\overrightarrow{OB} = x(\overrightarrow{OA} + \overrightarrow{OB}) \Rightarrow x\boldsymbol{a} + x\boldsymbol{b} = x(\boldsymbol{a} + \boldsymbol{b})$

Product of Vector and Real Number VIII

- Let $x = -1$. In terms of previous points $\overrightarrow{OA'} = -\overrightarrow{OA}$, $\overrightarrow{OB} = -\ \overrightarrow{OB'}$,
- AA' and BB' shape crossing lines (O is cross point)
- $\angle AOB = \angle A'OB'$, thus $OADB$ and $OA'D'B'$ are equal parallelograms
- Segments OD and OD' lay on the single line and do not overlap, thus $\overrightarrow{OD} \uparrow\downarrow \overrightarrow{OD'}$, and $OD = OD'$ $\overrightarrow{OD'} = \overrightarrow{OA'} + \overrightarrow{OB'} = (-1) \cdot \overrightarrow{OA} + (-1) \cdot \overrightarrow{OB} = -\overrightarrow{OD} = (-1) \cdot (\overrightarrow{OA} + \overrightarrow{OB}) \Rightarrow$ $(-1) \cdot \overrightarrow{OA} + (-1) \cdot \overrightarrow{OB} = (-1) \cdot (\overrightarrow{OA} + \overrightarrow{OB})$
- $(-1) \cdot \boldsymbol{a} + (-1) \cdot \boldsymbol{b} = (-1) \cdot (\boldsymbol{a} + \boldsymbol{b})$
- Let $x < 0$. $x = -1 \cdot |x|$
- $x(\boldsymbol{a} + \boldsymbol{b}) = -1 \cdot |x|(\boldsymbol{a} + \boldsymbol{b}) = -1 \cdot (|x|\boldsymbol{a} + |x|\boldsymbol{b}) = -1 \cdot |x|\boldsymbol{a} + (-1) \cdot |x|\boldsymbol{b} = x\boldsymbol{a} + x\boldsymbol{b}$

Angle Between Vectors

- Consider vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. Establish directed segments from the same point O: $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. We distinguish angle $\angle AOB$ as angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Consider point O' distant form O and directed segments $\overrightarrow{O'A'} = \boldsymbol{a}$ and $\overrightarrow{O'B'} = \boldsymbol{b}$. $\overrightarrow{OA} = \overrightarrow{O'A'}$ and $\overrightarrow{OB} = \overrightarrow{O'B'}$. $\overrightarrow{AB} = \overrightarrow{A'B'} = \boldsymbol{b} - \boldsymbol{a}$. Thus $\triangle AOB = \triangle A'O'B'$, and corresponding angles are equal
- Thus angle between vectors depends on only their relative direction
- We will write $\angle(\boldsymbol{a}, \boldsymbol{b})$
- We call vectors shaping right angle **orthogonal**
- How we can describe our vectors?

Dot Product I

- Consider vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
- We call number $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos \angle(\boldsymbol{a}, \boldsymbol{b})$ the **dot product** (or scalar product) of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Sometimes we overlook dot: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}\boldsymbol{b}$
- If $\boldsymbol{a} = \mathbf{0}$, or $\boldsymbol{b} = \mathbf{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- If $\boldsymbol{b} = \boldsymbol{a}$, then we write $\boldsymbol{a} \cdot \boldsymbol{a} = \boldsymbol{a}^2 = |\boldsymbol{a}|^2$
- For orthogonal not zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, $\cos \angle(\boldsymbol{a}, \boldsymbol{b}) = 0$, $\boldsymbol{a} \cdot \boldsymbol{b} = 0$
- Key features of dot product
  1. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
  2. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and number x: $(x\boldsymbol{a}) \cdot \boldsymbol{b} = x(\boldsymbol{a} \cdot \boldsymbol{b})$
     - Particular case: $(-\boldsymbol{a}) \cdot \boldsymbol{b} = -(\boldsymbol{a} \cdot \boldsymbol{b})$
  3. For vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$: $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$

Dot Product II

- Proof:
  1. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$: $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
     - This is a simple derivative from definition
  2. For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and number x: $(x\boldsymbol{a}) \cdot \boldsymbol{b} = x(\boldsymbol{a} \cdot \boldsymbol{b})$
     - Case if $x = 0$, or (and) $\boldsymbol{a} = \mathbf{0}$, or (and) $\boldsymbol{b} = \mathbf{0}$ appears be trivial
     - Let $x > 0$, thus $x\boldsymbol{a} \upuparrows \boldsymbol{a}$, and $|x| = x$. Therefore, $\angle(\boldsymbol{a}, \boldsymbol{b}) = \angle(x\boldsymbol{a}, \boldsymbol{b})$, and $(x\boldsymbol{a}) \cdot \boldsymbol{b} = |x\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(x\boldsymbol{a}, \boldsymbol{b}) = x|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(\boldsymbol{a}, \boldsymbol{b}) = x(\boldsymbol{a} \cdot \boldsymbol{b})$
     - Let $x < 0$, thus $x\boldsymbol{a} \uparrow\downarrow \boldsymbol{a}$, and $|x| = -x$. Therefore, $\angle(x\boldsymbol{a}, \boldsymbol{b}) = \pi - \angle(\boldsymbol{a}, \boldsymbol{b})$, and $\cos\angle(x\boldsymbol{a}, \boldsymbol{b}) = -\cos\angle(\boldsymbol{a}, \boldsymbol{b})$, and $(x\boldsymbol{a}) \cdot \boldsymbol{b} = |x\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(x\boldsymbol{a}, \boldsymbol{b}) = -x|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot (-\cos\angle(\boldsymbol{a}, \boldsymbol{b})) = x(|\boldsymbol{a}| \cdot |\boldsymbol{b}| \cdot \cos\angle(\boldsymbol{a}, \boldsymbol{b})) = x(\boldsymbol{a} \cdot \boldsymbol{b})$

Dot Product III

3. For vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$: $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
   - We star with deriving two supplementary equations:

$$(\boldsymbol{a} + \boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a} \cdot \boldsymbol{b}$$
$$(\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{a} \cdot \boldsymbol{b}$$
$$(\boldsymbol{a} + \boldsymbol{b})^2 + (\boldsymbol{a} - \boldsymbol{b})^2 = 2(\boldsymbol{a}^2 + \boldsymbol{b}^2)$$

   - In the case $\boldsymbol{a} = \boldsymbol{0}$, or $\boldsymbol{b} = \boldsymbol{0}$ the equations are successfully valid.
   - Consider directed segments $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OB} = \boldsymbol{b}$. Thus, $\overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{AB}$
   - Now we can apply theorem of cosines for this triangle and definition of the dot product:

$$AB^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cdot \cos \angle O$$
$$(\boldsymbol{ab})^2 = |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 - 2 \cdot |\boldsymbol{a}| \cdot |\boldsymbol{b}| \cos \angle(\boldsymbol{ab})$$
$$(\boldsymbol{a} - \boldsymbol{b})^2 = \boldsymbol{a}^2 + \boldsymbol{b}^2 - 2\boldsymbol{a} \cdot \boldsymbol{b}$$

   - Consider replacement $\boldsymbol{b'} = -\boldsymbol{b}$.
   - $\boldsymbol{a} \cdot \boldsymbol{b'} = \boldsymbol{a} \cdot (-\boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b}$. $\boldsymbol{b'}^2 = -\boldsymbol{b}^2$
   - $(\boldsymbol{a} + \boldsymbol{b})^2 = (\boldsymbol{a} - \boldsymbol{b'})^2 = \boldsymbol{a}^2 + \boldsymbol{b'}^2 - 2\boldsymbol{a} \cdot \boldsymbol{b'} = \boldsymbol{a}^2 + \boldsymbol{b}^2 + 2\boldsymbol{a} \cdot \boldsymbol{b}$
   - Third formula we obtain with summarizing two proved

Dot Product IV

- Now we proceed with desired feature $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- For vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ consider combinations:
- $\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b}$
- $\boldsymbol{q} = \boldsymbol{a} + \boldsymbol{c}$
- $(\boldsymbol{p} + \boldsymbol{q})^2 + (\boldsymbol{p} - \boldsymbol{q})^2 = 2(\boldsymbol{p}^2 + \boldsymbol{q}^2)$
- $(\boldsymbol{p} + \boldsymbol{q})^2 = (2\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}))^2 = 4\boldsymbol{a}^2 + (\boldsymbol{b} + \boldsymbol{c})^2 + 4\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c})$
- $(\boldsymbol{p} - \boldsymbol{q})^2 = (\boldsymbol{b} - \boldsymbol{c})^2 = 2(\boldsymbol{b}^2 + \boldsymbol{c}^2) - (\boldsymbol{b} + \boldsymbol{c})^2$
- $(\boldsymbol{p} + \boldsymbol{q})^2 + (\boldsymbol{p} - \boldsymbol{q})^2 = 4\boldsymbol{a}^2 + 4\boldsymbol{a}(\boldsymbol{b} + \boldsymbol{c}) + 2(\boldsymbol{b}^2 + \boldsymbol{c}^2)$
- $2(\boldsymbol{p}^2 + \boldsymbol{q}^2) = 2(\boldsymbol{a} + \boldsymbol{b})^2 + 2(\boldsymbol{a} + \boldsymbol{c})^2 = 4\boldsymbol{a}^2 + 2\boldsymbol{b}^2 + 4\boldsymbol{ab} + 2\boldsymbol{c}^2 + 4\boldsymbol{ac}$
- Comparing this two equation we obtain described

$$\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$$

$\square$

Some Applications of Dot Product I

- **Work** in mechanics
- Let material point is affected with force $\boldsymbol{F}$ which caused arbitrary displacement described with $\boldsymbol{s}$
- $W = \boldsymbol{F} \cdot \boldsymbol{s} = Fs \cos \angle(\boldsymbol{F}, \boldsymbol{s})$
- Let material point be affected with a pair of forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$
- $W = (\boldsymbol{F}_1 + \boldsymbol{F}_2) \cdot \boldsymbol{s} = \boldsymbol{F}_1 \cdot \boldsymbol{s} + \boldsymbol{F}_2 \cdot \boldsymbol{s}$
- We call a **parallelepiped** a three-dimensional figure formed by six opposed parallelograms
- Consider parallelepiped shaped with triplet of non-coplanar directed segments established from arbitrary point O: $\overrightarrow{OA} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$, $\overrightarrow{OC} = \boldsymbol{c}$
- It's diagonal resembles sum of these vectors $\boldsymbol{d} = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}$
- Length of diagonal:$d^2 = \boldsymbol{d}^2 = (\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})^2$

Some Applications of Dot Product II

- We call an **axis** a line with predicted direction. In other words it is aggregate of all vectors parallel with arbitrary line and codirected
- We usually associate axis with vector $\boldsymbol{e}$, $|\boldsymbol{e}| = 1$, and call it **unit vector**
- Consider vector a and axis directed with vector $\boldsymbol{e}$
- $\overrightarrow{AB} = \boldsymbol{a}$, and $A'B'$ is a projection of segment AB on a line corresponding to given axis
- We will call signed length of segment $A'B'$ the **projection of vector on axis**.
- We take sign '+' if $\overrightarrow{A'B'} \upuparrows \boldsymbol{e}$, and sign '-' in opposite case
- $p_{\boldsymbol{e}}(\boldsymbol{a}) = \boldsymbol{a} \cdot \boldsymbol{e} = |\boldsymbol{a}| \cos(\boldsymbol{a}, \boldsymbol{e})$

Dot Product. Problems Corner I

Problem 1
There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ shaping angle $\pi/6$ radians. $\boldsymbol{a} = 6$ and $\boldsymbol{b} = 8$.
Find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Dot Product. Problems Corner II

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ shaping angle $\pi/6$ radians. $\boldsymbol{a} = 6$ and $\boldsymbol{b} = 8$.
Find $\boldsymbol{a} \cdot \boldsymbol{b}$.
$\cos \angle(\boldsymbol{a}, \boldsymbol{b}) = \dfrac{\sqrt{3}}{2}$. $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos \angle(\boldsymbol{a}, \boldsymbol{b}) = 6 \cdot 8 \cdot \dfrac{\sqrt{3}}{2} = 24\sqrt{3}$

Dot Product. Problems Corner III

Problem 2
There are two vectors of equal length, say 12cm and anti-codirected. Find their dot product.

Dot Product. Problems Corner IV

There are two vectors of equal length, say 12 and anti-codirected. Find their dot product.
If we establish directed segments from the same initial point, we notice that out vectors share straight angle
To calculate cosines for supplementary angle greater than right, we took cosines of corresponding angle with minus sign. Thus, for straight angle cosines is -1. (We postulated that cosines of zero angle is 1).
Our dot product is $-12^2 = -144$

Dot Product. Problems Corner V

Problem 3

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $\boldsymbol{a} = 4\sqrt{2}$ and $\boldsymbol{b} = 8$. $\angle(\boldsymbol{ab}) = 45^\circ$

Find dot product of vectors $\boldsymbol{c}$ and $\boldsymbol{d}$, which resemble combinations of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\begin{aligned} \boldsymbol{c} &= -2\boldsymbol{a} + \boldsymbol{b} \\ \boldsymbol{d} &= \boldsymbol{a} - \boldsymbol{b} \\ \boldsymbol{c} \cdot \boldsymbol{d} &= ? \end{aligned}$$

Dot Product. Problems Corner VI

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $\boldsymbol{a} = 4\sqrt{2}$ and $\boldsymbol{b} = 8$. $\angle(\boldsymbol{ab}) = 45^\circ$
Find dot product of vectors $\boldsymbol{c}$ and $\boldsymbol{d}$, which resemble combinations of $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\begin{aligned}
\boldsymbol{c} &= -2\boldsymbol{a} + \boldsymbol{b} \\
\boldsymbol{d} &= \boldsymbol{a} - \boldsymbol{b} \\
\boldsymbol{c} \cdot \boldsymbol{d} = (-2\boldsymbol{a} + \boldsymbol{b})(\boldsymbol{a} - \boldsymbol{b}) &= \\
= (-2\boldsymbol{a} + \boldsymbol{b})\boldsymbol{a} + (-2\boldsymbol{a} + \boldsymbol{b})(-\boldsymbol{b}) &= \\
= -2\boldsymbol{a}^2 + \boldsymbol{ba} + 2\boldsymbol{ba} - \boldsymbol{b}^2 &= \\
= -2\boldsymbol{a}^2 + 3\boldsymbol{ba} - \boldsymbol{b}^2 &= \\
= -2|\boldsymbol{a}|^2 + 3|\boldsymbol{b}||\boldsymbol{a}| \cos\angle(\boldsymbol{b}, \boldsymbol{a}) - |\boldsymbol{b}|^2 &= \\
= -2(4\sqrt{2})^2 + 38 \cdot 4\sqrt{2} \cos 45^\circ - 8^2 &= \\
= -64 + 96\sqrt{2}\frac{\sqrt{2}}{2} - 64 = -32 &
\end{aligned}$$

Dot Product. Problems Corner VII

Problem 4
There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $\boldsymbol{a} = 3$ and $\boldsymbol{b} = 2$. $\angle(\boldsymbol{a}, \boldsymbol{b}) = \pi/3$
Find length of vector $\boldsymbol{x} = -\boldsymbol{a} + \boldsymbol{b}$

Dot Product. Problems Corner VIII

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $\boldsymbol{a} = 3$ and $\boldsymbol{b} = 2$. $\angle(\boldsymbol{a}, \boldsymbol{b}) = \pi/3$
Find length of vector $\boldsymbol{x} = -\boldsymbol{a} + 3\boldsymbol{b}$
We know that $\boldsymbol{x}^2 = |\boldsymbol{x}|^2$
$\boldsymbol{x}^2 = (-\boldsymbol{a} + 3\boldsymbol{b})^2 = \boldsymbol{a}^2 - 6\boldsymbol{ab} + 9\boldsymbol{b}^2 = |\boldsymbol{a}|^2 - 6|\boldsymbol{a}||\boldsymbol{b}| \cos\angle(\boldsymbol{ab}) + 9|\boldsymbol{b}|^2 =$
$3^2 - 6 \cdot 3 \cdot 2 \cos\frac{\pi}{3} + 9 \cdot 3^2 = 3 - \frac{36}{2} + 36 = 27$
$|\boldsymbol{x}| = \sqrt{27}$

Dot Product. Problems Corner IX

Problem 5

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $\boldsymbol{a} = 4$ and $\boldsymbol{b} = 2\sqrt{2}$. $\boldsymbol{ab} = 8$

Find angle between vectors

Dot Product. Problems Corner X

There are vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. $\boldsymbol{a} = 4$ and $\boldsymbol{b} = 2\sqrt{2}$. $\boldsymbol{ab} = 8$
Find angle between vectors
From the definition of dot product we can derive:

$$\boldsymbol{ab} = |\boldsymbol{a}||\boldsymbol{b}| \cos \angle(\boldsymbol{a}, \boldsymbol{b})$$

$$\cos \angle(\boldsymbol{a}, \boldsymbol{b}) = \frac{\boldsymbol{ab}}{|\boldsymbol{a}||\boldsymbol{b}|}$$

$$\cos \angle(\boldsymbol{a}, \boldsymbol{b}) = \frac{8}{4 \cdot 2\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\angle(\boldsymbol{a}, \boldsymbol{b}) = \arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$