

22.10.2024

# Equations of mathematical physics

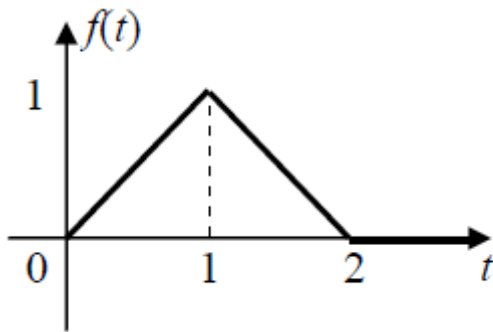
Examples of possible exam assignments

## FUNDAMENTALS OF OPERATIONAL CALCULUS

Ex 1. Using the definition, find images of the following functions

$$f(t) = e^{4t}$$

Ex 2. Find the image of the function given by the following graph:



Ex 3. Find images of the following functions:

$$f(t) = \int_0^t \cos(t - \tau) e^{2\tau} d\tau$$

Ex 4. Find the original corresponding to the image (using the Duhamel integral):

$$F(p) = \frac{2p^2}{(p^2 + 1)^2}$$

Ex 5. Solve the Cauchy problem:

$$x''' - x'' - 6x' = 0,$$

$$x(0) = 15, x'(0) = 2, x''(0) = 56.$$

## CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Ex 1. Find the general solution  $u = u(x, y, z)$  of the equation

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} + (x - y) \frac{\partial u}{\partial z} = 0.$$

Ex 2. Find a solution to the equation

$$(x + z)u'_x + (y + z)u'_y + (x + y)u'_z = 0.$$

Ex 3. Find solutions to the equation

$$(2y - u) u'_x + y u'_y = u.$$

Ex 4. Find a general solution to the partial differential equation

$$e^x \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = y e^x$$

Ex 5. To bring the differential equation to the canonical form and integrate it.

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + 3x \frac{\partial z}{\partial x} - 3y \frac{\partial z}{\partial y} = 0 \quad (x \neq 0, y \neq 0)$$

Ex 6. To bring the following differential equation to a canonical form:

$$\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} - 3 \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial x} + 6 \frac{\partial u}{\partial y} = 0$$

## HYPERBOLIC EQUATIONS

Ex1 Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

If

$$u|_{t=0} = x^2, \quad \frac{\partial u}{\partial t} \Big|_{t=0} = 0$$

Ex2 Find a solution to the equation:

$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = x, \quad \frac{\partial u(x, 0)}{\partial t} = \cos^2 x; \end{cases}$
<p>Ex3</p> <p>Find a solution to the equation:</p> $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x \sin t,$ <p>if</p> $u _{t=0} = \sin x, \quad \frac{\partial u}{\partial t} _{t=0} = \cos x, \quad x \in R.$
<p>Ex4 Find a solution to the equation:</p> $\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + e^x, \\ u(x, 0) = \sin x, \quad \frac{\partial u(x, 0)}{\partial t} = x + \cos x; \end{cases}$
<p>Ex5 Fourier method</p>
<p>PARABOLIC EQUATIONS</p>
<p>7.</p>
<p>8.</p>
<p>ELLIPTIC EQUATIONS</p>
<p>9.</p>
<p>10.</p>

Example 1.

Using the definition, find images of the following functions

$$f(t) = e^{4t}$$

Solution:

The function  $f(t) = e^{4t}$  is the original with a growth index of 4.

$$\begin{aligned} F(p) &= \int_0^{+\infty} e^{4t} \cdot e^{-pt} dt = \lim_{B \rightarrow +\infty} \int_0^B e^{-(p-4)t} dt = - \lim_{B \rightarrow +\infty} \left( \frac{1}{p-4} e^{-(p-4)t} \right) \Bigg|_0^B = \\ &= \lim_{B \rightarrow +\infty} \left( \frac{1}{p-4} - \frac{e^{-(p-4)B}}{p-4} \right) = \frac{1}{p-4}. \end{aligned}$$

Example 2.

$$f(t) = \begin{cases} 0, & t \leq 0, \quad t > 2, \\ t, & 0 < t \leq 1, \\ 2-t, & 1 < t \leq 2. \end{cases}$$

Solution:

We use the Laplace transform formula:

$$F(p) = \int_0^{+\infty} e^{-pt} f(t) dt = \int_0^1 t e^{-pt} dt + \int_1^2 (2-t) e^{-pt} dt.$$

$$\int_0^1 t e^{-pt} dt = \left[ \begin{array}{l} u = t, \quad dv = e^{-pt} dt, \\ du = dt, \quad v = \frac{e^{-pt}}{-p} \end{array} \right] = \frac{te^{-pt}}{-p} \Big|_0^1 + \frac{1}{p} \int_0^1 e^{-pt} dt =$$

$$= -\frac{e^{-p}}{p} - \frac{e^{-pt}}{p^2} \Big|_0^1 = -\frac{e^{-p}}{p} - \frac{e^{-p}}{p^2} + \frac{1}{p^2},$$

$$\int_1^2 (2-t) e^{-pt} dt = \left[ \begin{array}{l} u = 2-t, \quad dv = e^{-pt} dt, \\ du = -dt, \quad v = \frac{e^{-pt}}{-p} \end{array} \right] = \frac{(2-t)e^{-pt}}{-p} \Big|_1^2 - \frac{1}{p} \int_1^2 e^{-pt} dt =$$

$$= \frac{e^{-p}}{p} + \frac{e^{-pt}}{p^2} \Big|_1^2 = \frac{e^{-p}}{p} + \frac{e^{-2p}}{p^2} - \frac{e^{-p}}{p^2}.$$

$$F(p) = \frac{1}{p^2} (1 - 2e^{-p} + e^{-2p}) = \frac{(1 - e^{-p})^2}{p^2}.$$

Example 3.

$$f(t) = f_1(t) * f_2(t)$$

$$f_1(t) = \cos t, \quad f_2(t) = e^{2t}$$

$$\cos t \leftrightarrow \frac{p}{p^2 + 1}$$

$$e^{2t} \leftrightarrow \frac{1}{p - 2}$$

$$\int_0^t \cos(t - \tau) e^{2\tau} d\tau = \cos t * e^{2t} \leftrightarrow \frac{p}{p^2 + 1} \cdot \frac{1}{p - 2} = \frac{p}{(p^2 + 1)(p - 2)}.$$

#### Example 4

$$\frac{2p^2}{(p^2+1)^2} = 2p \cdot \frac{1}{p^2+1} \cdot \frac{p}{p^2+1}$$

$$\frac{1}{p^2+1} \quad \leftrightarrow \quad \sin t = f_1(t),$$

$$\frac{p}{p^2+1} \quad \leftrightarrow \quad \cos t = f_2(t)$$

$$f_1(0) = \sin 0 = 0, \quad f_1'(t) = \cos t,$$

$$2p \cdot \frac{1}{p^2+1} \cdot \frac{p}{p^2+1} \quad \leftrightarrow$$

$$\begin{aligned} 0 + 2 \int_0^t \cos \tau \cos(t-\tau) d\tau &= \\ &= 2 \int_0^t \frac{\cos t + \cos(t-2\tau)}{2} d\tau = \left( \tau \cos t - \frac{1}{2} \sin(t-2\tau) \right) \Big|_0^t = t \cos t + \sin t. \end{aligned}$$

#### Example 5

$$x(t) \quad \leftrightarrow \quad X(p),$$

$$x'(t) \quad \leftrightarrow \quad pX(p) - 15,$$

$$x''(t) \quad \leftrightarrow \quad p^2 X(p) - 15p - 2,$$

$$x'''(t) = p^3 X(p) - 15p^2 - 2p - 56.$$

$$(p^3 - p^2 - 6p)X(p) = 15p^2 - 13p - 36,$$

$$X(p) = \frac{15p^2 - 13p - 36}{p(p-3)(p+2)}.$$

$$p_1 = 0, p_2 = 3, p_3 = -2$$

$$P(p) = 15p^2 - 13p - 36, \quad Q(p) = p^3 - p^2 - 6p, \\ Q'(p) = 3p^2 - 2p - 6,$$

$$p_1 = 0$$

$$\left. \frac{P(p)}{Q'(p)} \right|_{p_1=0} = \frac{-36}{-6} = 6,$$

$$p_2 = 3$$

$$\left. \frac{P(p)}{Q'(p)} \right|_{p_2=3} = \frac{60}{15} = 4,$$

$$p_3 = -2$$

$$\left. \frac{P(p)}{Q'(p)} \right|_{p_3=-2} = \frac{50}{10} = 5,$$

and by the second decomposition theorem we get

$$x(t) = 6 + 5e^{-2t} + 4e^{3t}.$$

## CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Ex 1.

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x-y}$$

$$\left\{ \begin{array}{l} \frac{dy}{dx} = -\frac{y}{x}, \\ \frac{dz}{dx} = \frac{x-y}{x} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \frac{dy}{dx} = -\frac{y}{x}, \\ \frac{dz}{dx} = 1 - \frac{y}{x} = 1 + \frac{dy}{dx}, \end{array} \right.$$

$$xy = C_1$$

$$z - y - x = C_2$$

Then the general solution of the given equation has the form:

$$u(x, y, z) = \Phi(xy, z - x - y).$$

where  $\Phi(a,b)$  an arbitrary continuously differentiable function.

Ex.2

$$\frac{dx}{x+z} = \frac{dy}{y+z} = \frac{dz}{x+y}$$

By the property of equal fractions we have

$$\frac{dx - dz}{z - y} = \frac{dy - dz}{z - x} \Rightarrow (x - z)d(x - z) = (y - z)d(y - z).$$

Integrating the last equality, we get the first integral

$$\psi_1(x, y, z) = (x - z)^2 - (y - z)^2 = (x - y)(x + y - 2z).$$

By the property of equal fractions we have

$$\frac{dx + dy + dz}{2(x + y + z)} = \frac{dx - dy}{x - y} \Leftrightarrow \frac{d(x + y + z)}{x + y + z} = \frac{2d(x - y)}{x - y},$$

Integrating

$$\psi_2(x, y, z) = \frac{x + y + z}{(x - y)^2}$$

Then the general solution of the given equation has the form

$$u(x, y, z) = \Phi \left( (x - y)(x + y - 2z), \frac{x + y + z}{(x - y)^2} \right).$$

where  $\Phi(a, b)$  an arbitrary continuously differentiable function.

Ex3.

$$\frac{dx}{2y - u} = \frac{dy}{y} = \frac{du}{u}.$$

$$\frac{dy}{y} = \frac{du}{u} \quad \Rightarrow \quad \frac{u}{y} = C_1,$$

$$\psi_1(x, y, u) = \frac{u}{y}.$$

$$\frac{dx}{2y - u} = \frac{2dy - du}{2y - u} \quad \Rightarrow \quad dx = 2dy - du \quad \Rightarrow \quad x - 2y + u = C_2.$$

$$\psi_2(x, y, u) = x - 2y + u.$$

$$\Phi\left(\frac{u}{y}, x - 2y + u\right) = 0.$$

where  $\Phi(a, b)$  an arbitrary continuously differentiable function.

Ex4

Creating a system

$$\frac{dx}{e^x} = \frac{dy}{y^2} = \frac{du}{ye^x}$$

From the first equation, we find one first integral

$$\frac{1}{y} - e^{-x} = C_1$$

and from the second, taking into account the equality

$$e^x = \frac{y}{1 - yC_1}$$

another first integral

$$u - \frac{\ln|y| - x}{e^{-x} - y^{-1}} = C_2$$

follows.

Thus, the general integral of this equation will be

$$\Phi\left(\frac{1}{y} - e^{-x}, \frac{\ln|y| - x}{e^{-x} - y^{-1}} - u\right) = 0$$

The general solution has the form

$$u = \frac{\ln|y| - x}{e^{-x} - y^{-1}} + \varphi\left(\frac{1}{y} - e^{-x}\right).$$

Ex 5

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} + 3x \frac{\partial z}{\partial x} - 3y \frac{\partial z}{\partial y} = 0 \quad (x \neq 0, y \neq 0)$$

$$A = x^2, B = 0, C = -y^2,$$

$$B^2 - AC = -x^2(-y^2) = x^2 y^2 > 0 \quad \text{if} \quad x \neq 0, y \neq 0.$$

Therefore, this equation is of a hyperbolic type. The characteristic equation (2.28) takes the form:

$$x^2 dy^2 - y^2 dx^2 = 0$$

or

$$y^2 dx^2 - x^2 dy^2 = 0$$

$$\begin{cases} ydx + xdy = 0, \\ ydx - xdy = 0. \end{cases}$$

Integrating these equations, we obtain

$$xy = C_1,$$

$$y/x = C_2.$$

Let's introduce new variables

$$\xi = xy, \quad \eta = y/x.$$

We find partial derivatives

$$\frac{\partial z}{\partial x} = y \frac{\partial z}{\partial \xi} - \frac{y}{x^2} \frac{\partial z}{\partial \eta},$$

$$\frac{\partial z}{\partial y} = x \frac{\partial z}{\partial \xi} + \frac{1}{x} \frac{\partial z}{\partial \eta},$$

$$\frac{\partial^2 z}{\partial x^2} = y^2 \frac{\partial^2 z}{\partial \xi^2} - 2 \frac{y^2}{x^2} \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{y^2}{x^4} \frac{\partial^2 z}{\partial \eta^2} + 2 \frac{y}{x^3} \frac{\partial z}{\partial \eta},$$

$$\frac{\partial^2 z}{\partial y^2} = x^2 \frac{\partial^2 z}{\partial \xi^2} + 2 \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{1}{x^2} \frac{\partial^2 z}{\partial \eta^2}.$$

Substituting these expressions into the original equation, we obtain its canonical form:

$$\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{1}{xy} \frac{\partial z}{\partial \eta} = 0,$$

or

$$\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{1}{\xi} \frac{\partial z}{\partial \eta} = 0.$$

Let's integrate the last equation. Let's denote

$$\frac{\partial z}{\partial \eta} = w.$$

Then we have

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{\partial w}{\partial \xi}, \quad \frac{\partial w}{\partial \xi} + \frac{1}{\xi} w = 0, \quad \frac{1}{w} \frac{\partial w}{\partial \xi} + \frac{1}{\xi} = 0.$$

$$\ln|w| + \ln|\xi| = \ln|\mu(\eta)|, \quad w\xi = \mu(\eta).$$

Returning to the  $z$  function, we will have the differential equation

$$\frac{\partial z}{\partial \eta} \xi = \mu(\eta),$$

integrating, we find:

$$\xi z = \int \mu(\eta) d\eta + \nu(\xi).$$

Therefore,

$$z = \frac{1}{\xi} \mu(\eta) + \nu(\xi),$$

where  $\mu$  and  $\nu$  are arbitrary doubly continuously differentiable functions of their arguments. Returning to the variables  $x$  and  $y$ , we finally get:

$$z = \frac{1}{xy} \mu(y/x) + \nu(xy).$$

Ex 6

To bring the following differential equation to a canonical form:

$$\frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} - 3 \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial u}{\partial x} + 6 \frac{\partial u}{\partial y} = 0$$

Solution:

The type of equation:

$$a = 1, \quad b = 1, \quad c = -3$$

$$b^2 - ac = 1 + 3 = 4 > 0$$

This equation has a hyperbolic type.

Next, we need to bring it to a canonical form.

We write the characteristic equation:

$$dy^2 - 2dydx - 3dx^2 = 0$$

We solve this equation as a quadratic one.

$$a dy - \left( b \pm \sqrt{b^2 - ac} \right) dx = 0$$

$$dy - \left( 1 \pm \sqrt{4} \right) dx = 0$$

$$\begin{cases} dy - 3dx = 0 \\ dy + dx = 0 \end{cases}$$

It is good that we have an equation with constant coefficients:  $a, b, c$  do not depend on  $x, y$ .

$$\begin{cases} y - 3x = C_1 \\ y + x = C_2 \end{cases}$$

Let's make a substitution:

$$\begin{cases} \xi = y - 3x \\ \eta = y + x \end{cases}$$

When moving to these variables, the equation (\*) is greatly simplified.

$$u(x, y) \rightarrow U(\xi, \eta)$$

The derivative of the whole function:

$$u_x = U_{\xi} \xi_x + U_{\eta} \eta_x$$

$$u_y = U_{\xi} \xi_y + U_{\eta} \eta_y$$

Take the second derivatives of  $u_x, u_y$ .

$$u_x = U_{\xi} \cdot (-3) + U_{\eta} \cdot 1$$

$$u_y = U_{\xi} \cdot 1 + U_{\eta} \cdot 1$$

From these derivatives we take the second derivatives:

$$u_{xx} = -3(U_{\xi\xi} \cdot \xi_x + U_{\xi\eta} \cdot \eta_x) + 1(U_{\eta\xi} \cdot \xi_x + U_{\eta\eta} \cdot \eta_x) =$$

$$= -3(U_{\xi\xi} \cdot (-3) + U_{\xi\eta} \cdot 1) + 1(U_{\eta\xi} \cdot (-3) + U_{\eta\eta} \cdot 1) =$$

$$= 9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta}$$

$$u_{xy} = -3(U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y) + 1(U_{\eta\xi} \cdot \xi_y + U_{\eta\eta} \cdot \eta_y) =$$

$$= -3(U_{\xi\xi} \cdot 1 + U_{\xi\eta} \cdot 1) + 1(U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 1) =$$

$$= -3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

$$u_{yy} = 1(U_{\xi\xi} \cdot \xi_y + U_{\xi\eta} \cdot \eta_y) + 1(U_{\eta\xi} \cdot \xi_y + U_{\eta\eta} \cdot \eta_y) =$$

$$= 1 \cdot U_{\xi\xi} + U_{\xi\eta} \cdot 1 + U_{\eta\xi} \cdot 1 + U_{\eta\eta} \cdot 1 =$$

$$= U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

Substituting everything into our equation:

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} + 2(-3U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}) - 3(U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}) + \\ + 2(U_{\xi} \cdot (-3) + U_{\eta}) + 6(U_{\xi} + U_{\eta}) = 0$$

$$9U_{\xi\xi} - 6U_{\xi\eta} + U_{\eta\eta} - 6U_{\xi\xi} - 4U_{\xi\eta} + 2U_{\eta\eta} - 3U_{\xi\xi} - 6U_{\xi\eta} - 3U_{\eta\eta} - \\ - 6U_{\xi} + 2U_{\eta} + 6U_{\xi} + 6U_{\eta} = 0$$

$$-16U_{\xi\eta} + 8U_{\eta} = 0$$

$$-U_{\xi\eta} = \frac{1}{2}U_{\eta}$$

This expression is already the canonical form of a hyperbolic equation.

## HYPERBOLIC EQUATIONS

Ex1

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2},$$

$$u|_{t=0} = x^2, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0$$

$$\psi(x) = 0$$

$$u(x, t) = \frac{\varphi(x + at) + \varphi(x - at)}{2}$$

$$a = 1, \quad \varphi(x) = x^2$$

$$u(x,t) = \frac{(x+t)^2 + (x-t)^2}{2}$$

or

$$u(x,t) = x^2 + t^2$$

Ex. 2 (on your own for training).

Ex.3.

Find a solution to the equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + x \sin t,$$

if

$$u|_{t=0} = \sin x, \quad \left. \frac{\partial u}{\partial t} \right|_{t=0} = \cos x, \quad x \in \mathbb{R}.$$

Solution:

$$u_{tt} - a u_{xx} = f(x;t)$$

$$a = 1$$

$$f(x;t) = x \cdot \sin t$$

$$u|_{t=0} = \sin x : u(x;0) = \sin x, \quad \text{so} \quad \varphi(x) = \sin x$$

$$u'_t|_{t=0} = \cos x : u'_t(x;0) = \cos x, \quad \text{so} \quad \psi(x) = \cos x$$

Since  $u(x;t) = v(x;t) + w(x;t)$ , we will find  $v(x;t)$  and  $w(x;t)$ .

Let  $v(x; t) = v_1 + v_2$ .

$$v_1(x, t) = \frac{\varphi(x + at) + \varphi(x - at)}{2}$$

$$v_1 = \frac{1}{2}(\varphi(x + t) + \varphi(x - t)) = \frac{1}{2}(\sin(x + t) + \sin(x - t))$$

$$\begin{aligned} v_2 &= \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy = \frac{1}{2} \int_{x-t}^{x+t} \cos y dy = \frac{1}{2} \sin y \Big|_{x-t}^{x+t} = \\ &= \frac{1}{2}(\sin(x + t) - \sin(x - t)) \end{aligned}$$

$$\begin{aligned} v(x; t) &= \frac{1}{2}(\sin(x + t) + \sin(x - t)) + \frac{1}{2}(\sin(x + t) - \sin(x - t)) = \\ &= \sin(x + t) \end{aligned}$$

We have  $v(x; t) = \sin(x + t)$ .

Now let's find  $\omega(x; t)$ .

Since  $f(x; t) = x \cdot \sin t$ , then

$$\begin{aligned} \omega &= \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi; \tau) d\xi d\tau = \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} \xi \cdot \sin \tau d\xi d\tau = \frac{1}{2} \int_0^t \sin \tau d\tau \frac{\xi^2}{2} \Big|_{x-t+\tau}^{x+t-\tau} = \\ &= \frac{1}{4} \int_0^t \sin \tau [(x + t - \tau)^2 - (x - t + \tau)^2] d\tau = \\ &= \frac{1}{4} \int_0^t \sin \tau [(x + t - \tau + x - t + \tau)(x + t - \tau - x + t - \tau)] d\tau = \\ &= x \int_0^t \sin \tau \cdot (t - \tau) d\tau = x(t - \sin t) \end{aligned}$$

We get

$$u = \sin(x + t) + xt - x \sin t$$

Ex 4 (on your own for training).