Combinatorics

Lecture 2

Harbin, 2023

Polynomial formula

Question. Let's consider the word Combinatorics and will change the letters in this word. How many "words" will you get in this case? First, choose 2 places for the letter "c" – it can be done in C_{13}^2 ways. Then we choose 2 places for the letter "o" – this can be done in C_{11}^2 ways. Similarly, we get further $C_9^1, C_8^1, C_7^2, C_5^1, C_4^1, C_3^1, C_2^1, C_1^1$. In total, we have $C_{13}^2 C_{11}^2 C_9^1 C_8^1 C_7^2 C_5^1 C_4^1 C_3^1 C_2^1 C_1^1 = \frac{13!}{2!2!2!}$ possibilities.

Assume that we have n_1 objects of the form a_1, n_2 objects of the form a_2, ... , n_k objects of the form a_k. Let $n := n_1 + \ldots + n_k$. Denote by $P(n_1, \ldots, n_k)$ is the number of all possible permutations that can be obtained from these n objects. Arguing as in a problem with a word Combinatorics, we get that the following theorem is true:

Theorem 1.

$$P(n_1, \ldots, n_k) = C_n^{n_1} C_{n-n_1}^{n_2} \ldots C_{n-n_1-\ldots-n_{k-1}}^{n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}$$

Corollary 2 (Polynomial formula aka Multinomial Theorem).

$$(x_1 + \ldots + x_k)^n = \sum_{\substack{(n_1,\ldots,n_k) \\ n_1+\ldots+n_k=n}} P(n_1,\ldots,n_k) x_1^{n_1} \ldots x_k^{n_k}$$

Proof. Indeed,

$(x_1 + \ldots + x_k)^n = \underbrace{(x_1 + \ldots + x_k) \ldots (x_1 + \ldots + x_k)}_{n}$. Every

monomial of type $x_1^{n_1} \ldots x_k^{n_k}$ occurs exactly $P(n_1,\ldots,n_k)$ times. ∎

Corollary 3.

$$k^n = \sum_{\substack{(n_1,\ldots,n_k) \\ n_1+\ldots+n_k=n}} P(n_1,\ldots,n_k)$$

Proof. $k^n = (\underbrace{1 + \ldots + 1}_{k})^n$ ∎

<u>Example 1.</u> Let $n = 3, k = 3$, so $3 = n_1 + n_2 + n_3$. Then by the polynomial formula we have

$$
\begin{aligned}
(x_1 + x_2 + x_3)^3 &= P(3, 0, 0)x_1^3 + P(0, 3, 0)x_2^3 + P(0, 0, 3)x_3^3 + \\
&\quad + P(1, 1, 1)x_1 x_2 x_3 + P(1, 2, 0)x_1 x_2^2 + P(2, 1, 0)x_2 x_1^2 + \\
&\quad + P(1, 0, 2)x_1 x_3^2 + P(0, 1, 2)x_2 x_3^2 + P(2, 0, 1)x_1^2 x_3 + \\
&\quad + P(0, 2, 1)x_2^2 x_3
\end{aligned}
$$

$$(1)$$

$P(3, 0, 0) = 1, P(2, 1, 0) = 3, P(0, 3, 0) = 1, P(1, 0, 2) = 3,$
$P(0, 0, 3) = 1, P(0, 1, 2) = 3, \ P(1, 1, 1) = 6, P(2, 0, 1) = 3,$
$P(1, 2, 0) = 3, P(0, 2, 1) = 3.$
Finally we get: $(x_1 + x_2 + x_3)^3 =$
$x_1^3 + x_2^3 + x_3^3 + 6x_1 x_2 x_3 + 3x_1 x_2^2 + 3x_2 x_1^2 + 3x_1 x_3^2 + 3x_2 x_3^2 + 3x_1^2 x_3 + 3x_2^2 x_3$

Example 2. $(x_1 + x_2 + x_3)^4 = x_1^4 + 4x_1^3 x_3 + 6x_1^2 x_3^2 + 4x_1 x_3^3 + x_3^4 + 4x_1^3 x_2 + 12x_1^2 x_2 x_3 + 12x_1 x_2 x_3^2 + 4x_2 x_3^3 + 6x_1^2 x_2^2 + 12x_1 x_2^2 x_3 + 6x_2^2 x_3^2 + 4x_1 x_2^3 + 4x_2^3 x_3 + x_2^4$

Example 3. Find the coefficient of $x^3 y^2 z^5$ in $(x + y + z)^{10}$.

Answer: $\frac{10!}{3!2!5!} = 7560$.

Corollary 4.

$$C_{n+m}^n = C_{n+m-1}^{n-1} + C_{n+m-2}^{n-1} + \ldots + C_{n-1}^{n-1}$$

Proof. Consider a $(n+1)$-element set A. Number of $m-$combinations with repetitions from its elements is $\tilde{C}_{n+1}^m = C_{n+m}^m = C_{n+m}^n$. On the other hand, in each combination with repetitions, element a_1 occurs from 0 to m times. The number of combinations with repetitions in which a_1 occurs exactly k times equals $\tilde{C}_{n-1}^{m-k} = C_{n+m-k-1}^{n-1}$. Summing over k from 0 to m we get the desired. ∎

Examples.

- $n = 2$: formula takes the form

$\frac{(m+1)(m+2)}{2} = C_{m+2}^2 = C_{m+1}^1 + C_m^1 \ldots + C_1^1 =$

$(m + 1) + m + \ldots + 2 + 1$

we have proved the formula for the sum of the first $m + 1$ natural numbers!

- $n = 3 : \quad C_{m+3}^3 = C_{m+2}^2 + \ldots + C_2^2$

$\frac{(m+1)(m+2)(m+3)}{6} = \frac{(m+1)(m+2)}{2} + \frac{m(m+1)}{2} + \ldots + \frac{1 \cdot 2}{2} =$

$= \frac{(m+1)^2}{2} + \frac{m+1}{2} + \frac{m^2}{2} + \frac{m}{2} + \ldots + \frac{1}{2} + \frac{1}{2} =$

$= \frac{1}{2} \left(1 + \ldots + (m+1) + 1^2 + 2^2 + \ldots + (m+1)^2 \right) =$

$= \frac{1}{2} \left(\frac{(m+1)(m+2)}{2} + 1^2 + 2^2 + \ldots + (m+1)^2 \right) \rightsquigarrow$

$1^2 + 2^2 + \ldots + (m+1)^2 = \frac{(m+1)(m+2)(m+3)}{3} - \frac{(m+1)(m+2)}{2} =$
$= \frac{(m+1)(m+2)(2m+3)}{6}$ — we found the sum of the squares of the first
$m + 1$ natural numbers.

With $n = 4$, you can get the formula for the sum of cubes

$$1^3 + 2^3 + \ldots + m^3 = \frac{m^2(m+1)^2}{4}(\textit{Exercise!}).$$

Lemma 5.

For all $n \geq 1, 0 \leq k \leq n$, we have $C_n^k \leq \frac{n^n}{k^k(n-k)^{n-k}}$

Proof. First note that for all $n \geq 1$, $C_n^0 = 1 \leq \frac{n^n}{(n)^n}$, i.e. for $k = 0$ the assertion of the lemma is true. Proof for $n \geq 1$ for all k, $1 \leq k \leq n$, by induction on n.

  Base case: n=1 − ok.

  Inductive step: Suppose that for some $n \geq 1$ for all k, $1 \leq k \leq n$, Lemma 5 is true. Consider $n + 1$. Then

$$C_{n+1}^k = \frac{n+1}{k}C_n^{k-1} \leq \frac{n+1}{k}\frac{n^n}{(k-1)^{k-1}(n-k+1)^{n-k+1}}\frac{(n+1)^n}{(n+1)^n}\frac{k^k}{k^k} =$$

$$= \frac{(n+1)^{n+1}}{k^k(n-k+1)^{n-k+1}}\frac{n^n}{(n+1)^n}\frac{k^{k-1}}{(k-1)^{k-1}} =$$

$$= \frac{(n+1)^{n+1}}{k^k(n-k+1)^{n-k+1}} \frac{(1+\frac{1}{k-1})^{k-1}}{(1+\frac{1}{n})^n} \leq \frac{(n+1)^{n+1}}{k^k(n-k+1)^{n-k+1}}$$

In the final inequality, we have used the fact that the sequence $a_n = (1+\frac{1}{n})^n$ is increasing. ∎

Proposition 6.

For all $k \geq 2$ and $n_1, \ldots, n_k \geq 0$, such that $n_1 + \ldots + n_k = n$ we have $P(n_1, \ldots, n_k) \leq \frac{n^n}{n_1^{n_1} \ldots n_k^{n_k}}$

Proof. Induction on n. Base case − Lemma 5. ∎

From the functional equation $C_n^k = \frac{n-k+1}{k} C_n^{k-1}$ one easily finds that for every n the binomial coefficients C_n^k form a sequence that is symmetric and unimodal : it increases towards the middle, so that the middle binomial coefficients are the largest ones in the sequence:

$$1 = C_n^0 < C_n^1 < \ldots C_n^{\lfloor \frac{n}{2} \rfloor} = C_n^{\lceil \frac{n}{2} \rceil} > \ldots > C_n^{n-1} > C_n^n = 1$$

Here $\lfloor x \rfloor$ (resp. $\lceil x \rceil$) denotes the number x rounded down (resp. rounded up) to the nearest integer.

In 1928 Emanuel Sperner asked and answered the following question: Suppose we are given the set $N = \{1, 2, \ldots, n\}$. Call a family $\mathcal{F}$ of subsets of N an antichain if no set of $\mathcal{F}$ contains another set of the family $\mathcal{F}$. What is the size of a largest antichain?

Clearly, the family $\mathcal{F}_k$ of all $k-$sets satisfies the antichain property with $|\mathcal{F}_k| = C_n^k$. Looking at the maximum of the binomial coefficients as above we conclude that there is an antichain of size $C_n^{\lfloor \frac{n}{2} \rfloor} = max_k C_n^k$.

Sperner's theorem now asserts that there are no larger ones.

Theorem 7.

The size of a largest antichain of an $n-$set is $C_n^{\lfloor \frac{n}{2} \rfloor}$.

Proof. Let $\mathcal{F}$ be an arbitrary antichain. Then we have to show $|\mathcal{F}| \leq C_n^{\lfloor \frac{n}{2} \rfloor}$. The key to the proof is that we consider chains of subsets $\varnothing = C_0 \subseteq C_1 \ldots \subseteq C_n = N$, where $|C_i| = i$ for $i = 0, \ldots, n$. How many chains are there? Clearly, we obtain a chain by adding one by one the elements of N, so there are just as many chains as there are permutations of N, namely $n!$.

Next, for a set $A \in \mathcal{F}$ we ask how many of these chains contain A. Again this is easy. To get from $\varnothing$ to A we have to add the elements of A one by one, and then to pass from A to N we have to add the remaining elements.

Thus if A contains k elements, then by considering all these pairs of chains linked together we see that there are precisely $k!(n-k)!$ such chains. Note that no chain can pass through two different sets A and B of $\mathcal{F}$, since $\mathcal{F}$ is an antichain.

To complete the proof, let m_k be the number of $k-$sets in $\mathcal{F}$. Thus $|\mathcal{F}| = \sum_{k=0}^{n} m_k$. Then the number of chains passing through some member of $\mathcal{F}$ is

$$\sum_{k=0}^{n} m_k k!(n-k)!$$

and this expression cannot exceed the number $n!$ of all chains. Hence we conclude

$$\sum_{k=0}^{n} m_k \frac{k!(n-k)!}{n!} = \sum_{k=0}^{n} \frac{m_k}{C_n^k} \leq 1$$

Replacing the denominators by the largest binomial coefficient, we therefore obtain

$$\frac{1}{C_n^{\lfloor \frac{n}{2} \rfloor}} \sum_{k=0}^{n} m_k \leq 1 \quad \text{that is} \quad |\mathcal{F}| = \sum_{k=0}^{n} m_k \leq C_n^{\lfloor \frac{n}{2} \rfloor} \blacksquare$$

Exercises

Exercise 1. Determine x such that $\sum_{i=0}^{n} C_n^i 8^i = x^n$

Exercise 2. Find the coefficient of $x_1^3 x_2 x_3^2$ in the expansion of $(2x_1 - 3x_2 + 5x_3)^6$.