Chapter 1. & 2.

1.2.3 Exercise. Let X be the real l_1, $f : X \to \mathbb{R}$, $f(x) = \sum_{k=1}^{\infty} x_k$ if $x = (x_k)_{k=1}^{\infty}$, and $\operatorname{Ker} f = \{x \in l_1 \ : \ f(x) = 0\}$ is the ***kernel*** of f. Calculate codim(Ker f). What can be said about the case when l_1 is complex and $f : X \to \mathbb{C}$?

Sol: $\forall x, y \in l_1$. if $x \sim y$, that is $x - y \in \operatorname{Ker} f$. i.e. $\sum_{k=1}^{\infty} (x_k - y_k) = 0$

the sequence with equal sum. forms a equivalent class.

since $\forall x = (x_1, x_2, \dots)$, $\sum_{n=1}^{\infty} |x_k| < \infty$ $[x] = a \cdot [1]$. where $a = \sum_{n=1}^{\infty} x_k \in \mathbb{R}$ and $1 = \{1, 0, \dots\}$.

thus. $\operatorname{codim}(\operatorname{Ker} f) = \dim X/\operatorname{Ker} f = 1$.

1.7.8 Exercise. Prove that $\mathbb{R}_p^2 = (\mathbb{R}^2, \|\cdot\|_p)$ isometrically embedded in both L^p space and l^p space.

Pf: (1) $(\mathbb{R}^2, \|\cdot\|_p)$ is isometrically embedded in $L^p([0,1], \lambda')$

$\forall x, y \in \mathbb{R}^2$ then $\|x - y\|_p = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{\frac{1}{p}}$

denote mapping $x = (x_1, x_2) \xrightarrow{\varphi} f(t) = 2^{\frac{1}{p}} (x_1 \chi_{[0,\frac{1}{2}]}(t) + x_2 \chi_{(\frac{1}{2},1]}(t))$ $t \in [0,1]$

$$\|\varphi(x) - \varphi(y)\|_{L^p[0,1]} = \left(\int_0^1 |f(t)|^p du\right)^{\frac{1}{p}} = \left((2^{\frac{1}{p}})^p \cdot \frac{1}{2}(|x_1 - y_1|^p + |x_2 - y_2|^p)\right)^{\frac{1}{p}} = \|x - y\|_p.$$

(2) $(\mathbb{R}^2, \|\cdot\|_p)$ is isometrically embedded in l^2

$\forall x, y \in \mathbb{R}^2$ then $\|x - y\|_p = (|x_1 - y_1|^p + |x_2 - y_2|^p)^{\frac{1}{p}}$

denote mapping $x = (x_1, x_2) \xrightarrow{\varphi}$ sequence $(x_1, x_2, 0, 0, \dots)$.

$$\|\varphi(x) - \varphi(y)\|_{l^2} = \left(\sum_{i=1}^{\infty} (x_i - y_i)^p\right)^{\frac{1}{p}} = \left((x_1 - y_1)^p + (x_2 - y_2)^p\right)^{\frac{1}{p}} = \|x - y\|_p$$

2.1.18 Proposition (Compactness in l^p). *A subset $E \subseteq l^p$, $p \in [1, +\infty)$ is precompact if and only if E is bounded and has uniformly decaying tails, i.e.*

$$\sum_{k>n} |x_k|^p \le \varepsilon_n \to 0, \quad \textit{for all } x = (x_k) \in E,$$

where $\varepsilon_n \ge 0$ is some sequence of numbers (that does not depend on x).

2.1.19 Exercise. Prove compactness in l^p using theorem 2.1.14.

Pf: "$\Rightarrow$" l^∞ is Banach space. E is precompact implies E is totally bounded (by 2.1.13).

then by thm 2.1.14. E is bounded.

for any fixed $\varepsilon > 0$. denote $(y^{(1)}, y^{(2)}, \dots y^{(n)})$ a finite ε-net of E.

since $y^{(i)} \in l^p$, $\forall i \in [1:n]$. $\exists N_i \in \mathbb{N}$. $\sum_{k>N_i} |y_k^{(i)}|^p < \varepsilon$. denote $N = \max\{N_i\}$.

$\forall x \in E$. $\exists i \in [1:n]$. $\|x - y^{(i)}\|_p < \varepsilon$.

$$\sum_{k>N} |x_k|^p = \sum_{k>N} |x_k - y_k^{(i)} + y_k^{(i)}|^p \le \sum_{k>N} |x_k - y_k^{(i)}|^p + \sum_{k>N} |y_k^{(i)}|^p < 2\varepsilon.$$

since ε is arbitrary, we can denote it by ε_n and let $\varepsilon_n \to 0$.

"$\Leftarrow$" by thm. 2.1.14. it suffices to check that $\exists Y$ - finite dim. subspace. forms ε-net for any $\varepsilon > 0$.

$\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$. s.t. $\varepsilon_N < \varepsilon$.

denote the orthonormal basis $\{e_k\}$ of l^p.

$\forall a = (a_1, a_2, \dots) \in E$. $\|a - (a_1, a_2, \dots a_N, 0, \dots)\|_p = \sum_{k>N} |a_k|^p < \varepsilon_N < \varepsilon$.

thus. span$\{e_1, e_2, \dots e_N\}$. is finite and it's a ε-net of E.

Nov. 4th.

4.1.4 Exercise. Let X be a normed space. The kernel of a linear functional f on X is either closed or dense in X.

Pf: In prop.4.1.3 we have shown that a linear functional in normed space is continuous iff kernel is closed.

Thus, it's suffices to check that f is discontinuous iff $\ker f$ is dense.

$\forall$ point $x_0 \in X$. denote $f(x_0) = C_0$.

since f is discont. f is discont. at $x = 0$. by Prop. 1.9.3 discontinuity implies unboundedness.

$\exists \{x_n\} \to 0$ $\forall C \in \mathbb{R}$ we have $|f(x_n)| \geq C$ for some $n \in \mathbb{N}$.

denote $y_n := \dfrac{C_0 \cdot x_n}{f(x_n)}$. $\quad y_n \xrightarrow[n\to\infty]{} 0$. $\quad f(y_n) = \dfrac{C_0}{f(x_n)} \cdot f(x_n) \equiv C_0$.

since f is linear, $x_0 - y_n \in \operatorname{Ker} f$. thus we have $\{x_0 - y_n\} \subseteq \operatorname{Ker} f$ and $x_0 - y_n \to x_0$.

i.e. $\operatorname{Ker} f$ is dense in X.

3.2.4 Example. *Let $X = L^p(E, \mathcal{A}, \mu)$ with σ-finite measure μ, $p \in [1, +\infty]$, $\frac{1}{p} + \frac{1}{q} = 1$ (if $p = 1$, than $q = \infty$ and vice versa) and $w \in L^q(E, \mathcal{A}, \mu)$ is fixed and*

$$f(x) = \int_E x(t) \cdot w(t)\, d\mu(t).$$

Then f is linear bounded and $\|f\| = \|w\|_q \left(= \|w\|_{L^q(E,\mathcal{A},\mu)}\right)$.

Exercises. Prove the last statement for the «borderline case» $p = 1$.

Pf: 1) "$\leq$"

$$|f(x)| \leq \int_E |x(t)||w(t)|\,d\mu(t) \leq \|w\|_\infty \cdot \int_E |x(t)|\,d\mu(t) = \|w\|_\infty \cdot \|x\|_1$$

$\Rightarrow \|f\| \leq \|w\|_\infty$. (by Remark 1.9.10)

2) "$\geq$"

$\forall \varepsilon > 0$. $\exists E_1 \subseteq E$ s.t. for any $t \in E_1$, $\|w\|_\infty > w(t) > \|w\|_\infty - \varepsilon$

Since E is σ-finite, we can find $\mu E_1 < \infty$

denote $x(t) = \begin{cases} \dfrac{\|w\|_\infty}{w(t) \cdot \mu E_1} & t \in E_1 \\ 0 & t \notin E_1 \end{cases}$

$$\|x\|_1 = \int_E |x(t)|\,d\mu(t) = \int_{E_1} |x(t)|\,d\mu(t) \leq \frac{\|w\|_\infty}{\|w\|_\infty - \varepsilon}$$

$$f(x) = \int_E w(t)\,x(t)\,d\mu = \int_{E_1} w(t)\,x(t)\,d\mu = \int_{E_1} \frac{\|w\|_\infty}{\mu E_1}\,d\mu = \|w\|_\infty.$$

thus $\|f\| \geq \dfrac{|f(x)|}{\|x\|_1} \geq \|w\|_\infty - \varepsilon \xrightarrow[\varepsilon \to 0]{} \|w\|_\infty$. i.e. $\|f\| \geq \|w\|_\infty$

thus. $\|f\| = \|w\|_\infty$ holds for $p = 1$ $q = \infty$.

4.3.11 Exercises. 1) Consider a normed space $X = \mathbb{R}^2$ and a unit vector $x_0 \in X$. Let f be a supporting functional of x_0. Interpret geometrically the level set $\{x : f(x) = 1\}$ as a tangent hyperplane for the unit ball $B_1(\mathbf{0})$ at point x_0. Construct an example of a normed space for which the supporting functional of x is not unique.

2) (a variation of the exercise 1.9.11) Construct a bounded linear functional on $C[0,1]$ which does not attain its norm.

(1) $f(x) \equiv 1$. a support functional of x_0.

$\|f\| = 1$. $f(x_0) = 1 = \|x_0\|$

HW3. Functional Analysis

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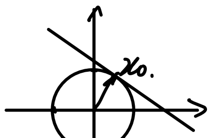
4.3.11 Exercises. 1) Consider a normed space $X = \mathbb{R}^2$ and a unit vector $x_0 \in X$. Let f be a supporting functional of x_0. Interpret geometrically the level set $\{x : f(x) = 1\}$ as a tangent hyperplane for the unit ball $B_1(\mathbf{0})$ at point x_0. Construct an example of a normed space for which the supporting functional of x is not unique.

Sol: (i) interpretion:

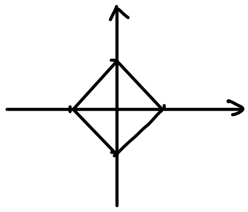
$x_0 \in X$. there exists $f \in X^*$ s.t. $\|f\| = 1$. $f(x_0) = \|x_0\|$

in $\mathbb{R}^2$. the level set is a line; the tangent hyperplane for the unit ball $B_1(0)$ is the tangent line of unit circle.



(ii). let the norm be ℓ^1-norm. $\|x\| = \|x_1\| + \|x_2\|$

the unit ball $B_1(0)$ is ◇ norm.



let $x_0 = (1,0)$. let $f_1(x) = \|x_0\|$.

$f_2(x) = x_1 + x_2$. where $(x_1, x_2) = x$.

f_1, f_2 are both support functional of x_0.

2) (a variation of the exercise 1.9.11) Construct a bounded linear functional on $C[0,1]$ which does not attain its norm.

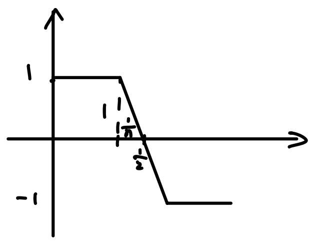
Sol: denote linear function $f: C[0,1] \to \mathbb{R}$.

f: $f(g) = \int_0^1 hg\,dx$. $h(x) = \begin{cases} 1 & x \in [0, \frac{1}{2}] \\ -1 & x \in (\frac{1}{2}, 1] \end{cases}$ norm in $C[0,1]$ is $\|\cdot\|_\infty$

$g_n = \begin{cases} 1 & [0, \frac{1}{2} - \frac{1}{n}] \\ -nx + \frac{n}{2} & (\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}] \\ -1 & (\frac{1}{n} + \frac{1}{2}, 1] \end{cases}$ $n \neq 1$.

$\|g_n\| = 1$.

$f(g_n) = 1 - \frac{1}{2n} + 0 = 1 - \frac{1}{2n}$.



$$\frac{|f(g_n)|}{\|g_n\|} = \frac{|1 - \frac{1}{2n}|}{1} \xrightarrow[n\to\infty]{} 1. \quad \text{thus. } \|f\| = \sup_{\|g\| \neq 0} \frac{|f(g)|}{\|g\|} \geq 1.$$

Assume there exists $g \in C[0,1]$ s.t. $\|f\| = \frac{|f(g)|}{\|g\|} = 1$.

w.l.g. let $\|g\|_\infty = 1$.

Since $|f(g)| = \left|\int_0^{1/2} g\,dx - \int_{1/2}^1 g\,dx\right| = 1$.

$\left|\int_0^{1/2} g\,dx\right| \leq \int_0^{1/2} |g|\,dx \leq \int_0^{1/2} \|g\|_\infty\,dx = \frac{1}{2}$. $\left|\int_{1/2}^1 g\,dx\right| \leq \frac{1}{2}$. similarly.

to get $|f(g)| = 1$. we need every inequality above take equal signs.

that is $g = \begin{cases} 1 & x \in [0, \frac{1}{2}) \\ -1 & x \in [\frac{1}{2}, 1] \end{cases}$ or $g = \begin{cases} 1 & x \in [0, \frac{1}{2}] \\ -1 & x \in (\frac{1}{2}, 1] \end{cases}$ which contradicts with $g \in C[0,1]$

Thus. the norm can't be attained.

4.3.19 Exercise (easy). Let E be a subset of a normed space X such that $0 \in \operatorname{Int} E$. Then E is an absorbing set.

Pf: $0 \in \operatorname{Int} E$. $\exists r > 0$. $\overline{B_r(0)} \subseteq E$.

$\forall x \in X$. let $t = \frac{\|x\|}{r} > 0$.

$\|t^{-1}x\| = \frac{\|x\|}{t} = r$. whence $t^{-1}x \in E$.

$x = t(t^{-1}x) \in tE$. E is an absorbing set.

4.3.20 Proposition (Minkowski functional).

(I) *Let E be a absorbing convex subset of a linear vector space X such that $\mathbf{0} \in E$. Then Minkowski functional generated by E is a quasi-seminorm* (see definition 4.3.6).

(II) *For any quasi-seminorm p on a linear vector space X the sub-level set*

$$E = \{x \in X \ : \ p(x) \leq 1\}$$

is an absorbing convex set, and $\mathbf{0} \in E$.

Pf: (I). $p_E(x) = \inf\{t > 0 : x \in tE\}$ $x \ni X$.

① $p_E(x) \geqslant 0$ by def.

② Let $a = p_E(x)$. $x \in (a+\varepsilon)E$ for any $\varepsilon > 0$. thus. $\lambda x \in |\lambda|(a+\varepsilon)E$ for any $\varepsilon > 0$

when. $|\lambda| a = \inf\{t > 0 : \lambda x \in tE\}$. i.e. $p_E(\lambda x) = |\lambda| p_E(x)$.

③ let $a = p_E(x)$ $b = p_E(y)$. $x \in (a+\varepsilon_a)E$ and $y \in (b+\varepsilon_b)E$. for any $\varepsilon_a, \varepsilon_b > 0$.

Since X is convex, we have $x+y \in (a+\varepsilon_a)E + (b+\varepsilon_b)E \subseteq (a+b+\varepsilon_a+\varepsilon_b)E$

i.e. $p_E(x+y) \leq (a+\varepsilon_a)+(b+\varepsilon_b) = a+b = p_E(x) + p_E(y)$.

(II) ① let $x, y \in E$. $p(x) \leq 1$. $p(y) \leq 1$.

$\forall \varepsilon \in [0,1]$. $p(\varepsilon x + (1-\varepsilon)y) \leq p(\varepsilon x) + p((1-\varepsilon)y)$ (by prop. semi-norm)

$= \varepsilon p(x) + (1-\varepsilon)p(y)$. ≤ 1.

thus. $\varepsilon x + (1-\varepsilon)y \in E$. E is convex.

② $p(0) = 0 \leq 1$. $0 \in E$.

③ $\forall x \in X$. $\exists t > 0$. $p(tx) = t\,p(x) \leq 1$.

i.e. $x \in tE$. thus E is absorbing.