THEORY OF SECOND-ORDER SURFACES

1 Center of a second-order surface

Definition. *The center of a second-order surface is the center of symmetry of this surface.*

Theorem 1. *Let a second-order surface be given with respect to the general Cartesian coordinate system by the general equation*

$$\Phi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0. \quad (1.1)$$

In order for the origin of coordinate system to be the center of this surface, it is necessary and sufficient that its equation does not contain first order terms with x, y, and z, i.e., that

$$a_1 = a_2 = a_3 = 0,$$

so the equation (1.1) should be

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + a_0 = 0. \quad (1.2)$$

Proof of necessity. Let assume that the origin is the center of the surface (1.1) and take an arbitrary point $M(x, y, z)$ on this surface. Its coordinates will satisfy the equation (1.1), and since the origin is the center of symmetry of the surface (1.1), the coordinates of the point $M'(-x, -y, -z)$ will also satisfy the equation (1.1). $M'(-x, -y, -z)$ is symmetric to the point M with respect to the origin, i.e.

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 - \\ - 2a_1x - 2a_2y - 2a_3z + a_0 = 0. \quad (1.3)$$

From this relation and from the relation (1.1) we find

$$a_1x + a_2y + a_3z = 0.$$

This equation is satisfied by the coordinates of all points on the surface (1.1). Suppose that at least one of the numbers a_1, a_2, a_3 is not equal to zero. Then all points of the surface lie in the plane

$$a_1x + a_2y + a_3z = 0.$$

This can be if and only if the equation (1.1) defines two planes coinciding with the plane

$$a_1x + a_2y + a_3z = 0.$$

By analogy with second-order curves, one can prove that the left side of the equation (1.1) is decomposed into a product of two linear function of variables x, y, z, one of which is the form $a_1x + a_2y + a_3z$:

$$\Phi = (a_1x + a_2y + a_3z)(Ax + By + Cz + D).$$

The plane given by the equation

$$Ax + By + Cz + D = 0,$$

on the basis of the remark made above, must coincide with the plane $a_1x + a_2y + a_3z = 0$, so,

$$A : B : C = a_1 : a_2 : a_3, \qquad D = 0,$$

and that's why

$$\Phi = k(a_1x + a_2y + a_3z)^2.$$

We arrive at a contradiction with the fact that in the equation (1.1) at least one of the coefficients at x, y, or z is different from zero.

Theorem 2. *If relative to the general Cartesian coordinate system the second-order surface is given by the general equation*

$$\begin{aligned} a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0, \end{aligned} \tag{1.4}$$

then the coordinates x_0, y_0, z_0 of its center are determined from the system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z + a_1 = 0, \\ a_{21}x + a_{22}y + a_{23}z + a_2 = 0, \\ a_{31}x + a_{32}y + a_{33}z + a_3 = 0, \end{aligned} \tag{1.5}$$

moreover, in the case of inconsistency of this system, the surface has no center.

Proof. Let's perform a parallel translation of given coordinate system, in which the new origin will be the point $O'(x_0, y_0, z_0)$. Denoting the old coordinates of an arbitrary point M by x, y, z, and its new coordinates by x', y', z', we have

$$x = x' + x_0,$$
$$y = y' + y_0,$$
$$z = z' + z_0,$$

and the equation (1.1) becomes

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + 2a_{13}x'z' + 2a_{23}y'z' + a_{33}z'^2 + \\ + 2(a_{11}x_0 + a_{12}y_0 + a_{13}z_0 + a_1)x' + 2(a_{21}x_0 + a_{22}y_0 + a_{23}z_0 + a_2)y' + \\ + 2(a_{31}x_0 + a_{32}y_0 + a_{33}z_0 + a_3)z' + \Phi(x_0, y_0, z_0) = 0,$$

where $\Phi(x_0, y_0, z_0)$ is the result of substituting the coordinates x_0, y_0, z_0 of the point O' into the left side of the equation (1.1).

Based on Theorem 1, the point $O(x_0, y_0, z_0)$ will be the center of the surface (1.1) if and only if

$$a_{11}x_0 + a_{12}y_0 + a_{13}z_0 + a_1 = 0,$$
$$a_{21}x_0 + a_{22}y_0 + a_{23}z_0 + a_2 = 0,$$
$$a_{31}x_0 + a_{32}y_0 + a_{33}z_0 + a_3 = 0.$$

2 Classification of second-order surfaces by the nature of the locus of centers

Let the second-order surface be given by the general equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0 \quad (2.1)$$

with respect to a general Cartesian coordinate system.

Let's consider matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \qquad A^* = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_1 \\ a_{21} & a_{22} & a_{23} & a_2 \\ a_{31} & a_{32} & a_{33} & a_3 \end{pmatrix}.$$

The table below gives the necessary and sufficient attributes of the nature of the locus of the surface centers.

Rank A	Rank A^*	Locus of centers
3	3	point
2	3	no center
2	2	line
1	2	no center
1	1	plane

Theorem. *Classification of second-order surfaces according to types I-V (see Theorem 2, section 1 of the previous topic) coincides with the classification of second-order surfaces according to the nature of locus of centers, otherwise, according to the ranks of the matrices A and A^*.*

It is assumed that the second-order surface is given by a general equation with respect to a general Cartesian coordinate system.

Proof. Denote by B the matrix of the quadratic form included in the left side of each of reduced equations (I-V types) of the second-order surface:

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Then $B = C^T AC$, where C is the matrix of the coefficients of X, Y, Z in the transformation

$$\begin{aligned} x &= c_{11}X + c_{12}Y + c_{13}Z + c_1, \\ y &= c_{21}X + c_{22}Y + c_{23}Z + c_2, \\ z &= c_{31}X + c_{32}Y + c_{33}Z + c_3, \end{aligned}$$

expressing the coordinates x, y, z of a point in the given system $Oxyz$ in terms of the coordinates X, Y, Z of the same point in the system $O'XYZ$ (already rectangular), in which the surface equation of I-V types has reduced form.

Since

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

is a nondegenerate matrix, then the ranks of the matrices A and B are equal.

For surfaces of type I $rank(B) = 3$, hence $rank(A) = 3$. For surfaces of types II and III $rank(B) = 2$, so also $rank(A) = 2$. For surfaces of types IV and V $rank(B) = 1$, hence $rank(A) = 1$.

Further, if $rank(A) = 3$ (that is, I type surface), then $rank(A^*) = 3$.

If the surface belongs to III or V types, then, composing the system of equations (1.5), which determines the coordinates of the center of the surface, we can see that this system is consistent (moreover, for surfaces of type III, the locus of centers is the axis $O'Z$, and for surfaces of type V — the plane $O'ZY$). This means that for surfaces of type III and V the system (1.5) is also consistent, and therefore $rank(A) = rank(A^*)$ by the Rouché–Capelli theorem. Therefore, for surfaces of type III $rank(A) = rank(A^*) = 2$, and for surfaces of type V $rank(A) = rank(A^*) = 1$.

Finally, for surfaces of types II and IV, the system (1.5) compiled for their reduced equations is inconsistent, i.e., these surfaces do not have a center. This means that the system itself (1.5) is also inconsistent, hence $rank(A^*) > rank(A)$. But for surface of type II $rank(A) = 2$, so $rank(A^*) = 3$. For surface of type IV $rank(A) = 1$, hence $rank(A^*) = 2$.

A second-order surface with a single center is called a **central surface**. The central surfaces are ellipsoids (real and imaginary), hyperboloids (one-sheet and two-sheet) and cones (real and imaginary). In order for the surface given by the general equation with respect to the general Cartesian coordinate system to be central, it is necessary and sufficient that the determinant of the quadratic form included in the left side of the equation of this surface be different from zero:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0.$$

3 Intersection of a second-order surface with a line. Asymptotic directions, asymptotic cone and cone of asymptotic directions

Let's assume that, with respect to a general Cartesian coordinate system, a second-order surface is given by the general equation

$$F(x,y,z) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0. \quad (3.1)$$

We will investigate the intersection of this surface with a straight line, the equations of which we will take in parametric form:

$$x = x_0 + \alpha t,$$
$$y = y_0 + \beta t, \tag{3.2}$$
$$z = z_0 + \gamma t.$$

Here (x_0, y_0, z_0) is some point on the line, and $\{\alpha, \beta, \gamma\}$ is direction vector of line.

Substituting in the left side of the equation (3.1) instead of x, y and z their expressions from the formulas (3.2), we obtain the following equation for t:

$$At^2 + Bt + C = 0, \tag{3.3}$$

where

$$A = a_{11}\alpha^2 + a_{22}\beta^2 + a_{33}\gamma^2 + 2a_{12}\alpha\beta + 2a_{13}\alpha\gamma + 2a_{23}\beta\gamma, \tag{3.4}$$

$$B = \alpha(a_{11}x_0 + a_{12}y_0 + a_{13}z_0 + a_1) + \beta(a_{21}x_0 + a_{22}y_0 + a_{23}z_0 + a_2)+ \\ \gamma(a_{31}x_0 + a_{32}y_0 + a_{33}z_0 + a_3), \tag{3.5}$$

$$C = F(x_0, y_0, z_0). \tag{3.6}$$

The roots of the equation (3.3) will give us the coordinates of the intersection points of the line (3.2) with the surface (3.1).

If in the equation (3.3) the coefficient at t^2 is non-zero, then the equation (3.3) has two roots, so the line (3.2) intersects the surface (3.1) at two points.

If

$$a_{11}\alpha^2 + a_{22}\beta^2 + a_{33}\gamma^2 + 2a_{12}\alpha\beta + 2a_{13}\alpha\gamma + 2a_{23}\beta\gamma = 0, \tag{3.7}$$

then the line with direction vector $\{\alpha, \beta, \gamma\}$ either intersects the surface (3.1) at only one point, or does not intersect it, or is part of the surface (3.1). In this case, we say that the direction vector $\{\alpha, \beta, \gamma\}$ has **asymptotic direction** with respect to the surface (3.1). So, the coordinates of vectors with an asymptotic direction are determined from the relation (3.7).

The equation

$$\varphi = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 = 0 \tag{3.8}$$

as a homogeneous equation of x, y, z defines a conical surface (real or imaginary) formed by lines passing through the origin. The coordinates x, y, z of the point M lying on the generatrix of the cone are the coordinates of the vector $\overrightarrow{OM}$. Thus, the generatrices of the cone (3.8) are lines having asymptotic directions of the surface (3.1) and vice versa: any asymptotic direction is the direction of one of the generatrices of the cone (3.8). The cone defined by the equation (3.8), as well as any cone obtained by translating this cone, is called the **cone of asymptotic directions** of the surface (3.1) (the vertex of the cone, given by the equation (3.8) is at the origin).

The asymptotic cone of second-order surfaces with a center is the cone of asymptotic directions whose vertex lies at the center of the surface.

Thus, the asymptotic cone of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

imaginary ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1,$$

imaginary cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0,$$

i.e., is imaginary.

Asymptotic cone of one-sheet and two-sheet hyperboloids

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \pm 1$$

has the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0,$$

and the asymptotic cone of a second-order cone coincides with itself.

Further, since the hyperbolic paraboloid

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z \, (p > 0, \ q > 0)$$

has no center, then, according to the definition given above, it has no asymptotic cone. One of the cones of asymptotic directions is expressed by the equation

$$\frac{x^2}{p} - \frac{y^2}{q} = 0,$$

and it decomposes into pair of intersecting planes

$$\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 0 \text{ and } \frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 0.$$

All cones of asymptotic directions are obtained by all translations of this pair of planes.

Elliptic paraboloid

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z \, (p > 0,\ q > 0)$$

also has no asymptotic cone, and all its cones of asymptotic directions are obtained by all translations of pair of imaginary intersecting planes

$$\frac{x^2}{p} + \frac{y^2}{q} = 0$$

or

$$\frac{x}{\sqrt{p}} + i\frac{y}{\sqrt{q}} = 0, \quad \frac{x}{\sqrt{p}} - i\frac{y}{\sqrt{q}} = 0.$$

For an elliptical cylinder (real or imaginary) and for a pair of imaginary intersecting planes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$$

an asymptotic cone consists of pair of imaginary intersecting planes

$$\frac{x}{a} \pm i\frac{y}{b} = 0.$$

For a hyperbolic cylinder and a pair of intersecting planes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

the asymptotic cone decomposes into pair of real intersecting planes

$$\frac{x}{a} + \frac{y}{b} = 0 \text{ and } \frac{x}{a} - \frac{y}{b} = 0.$$

For a parabolic cylinder $y^2 = 2px$, the cone of asymptotic directions is the plane $y = 0$, which is the plane of symmetry of this surface passing through the axis of the directing curve: $y^2 = 2px$, $z = 0$, and any of planes parallel to it.

Finally, for a pair of parallel or coinciding planes, the asymptotic cone coincides with the plane of surface centers.

The cone of asymptotic directions (for surfaces having a center) is obtained from the asymptotic cone by any translation.

Let the surface defined by the general equation (3.1) with respect to the general Cartesian coordinate system have a single center. Noting that δ and Δ are translation invariants of a general Cartesian coordinate system, we conclude that the equation of the given surface after the parallel translation of the coordinate axes, in which the new origin of coordinates is the center of the surface, will take the form

$$a_{11}x'^2 + 2a_{12}x'y' + a_{22}y'^2 + 2a_{13}x'z' + 2a_{23}y'z' + a_{33}z'^2 + \frac{\Delta}{\delta} = 0.$$

So the equation

$$F(x, y, z) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = \frac{\Delta}{\delta}$$

is the equation of the asymptotic cone of the central surface given by the general equation (3.1) with respect to the general Cartesian coordinate system.

4 Diametral plane conjugate to a non-asymptotic direction. Special directions relative to a second-order surface

Theorem 1. *The locus of midpoints of parallel chords of a second-order surface is a plane. This plane is called the* ***diametral plane*** *of the given surface conjugate to the chords of the given direction. If a second-order surface is given with respect to a general Cartesian coordinate system by the general equation*

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0, \quad (4.1)$$

and its chords are collinear to the vector $\{\alpha, \beta, \gamma\}$ *(which has no asymptotic direction), then the equation of the diametral plane conjugate to these chords has the form*

$$\alpha(a_{11}x + a_{12}y + a_{13}z + a_1) + \beta(a_{21}x + a_{22}y + a_{23}z + a_2) + \\ \gamma(a_{31}x + a_{32}y + a_{33}z + a_3) = 0. \quad (4.2)$$

Proof. The proof of this theorem is the same as the theorem on the second-order curve diameter equation.

It follows from the equation (4.2) that all diametral planes of second-order surfaces contain the locus of its centers if it is not empty, since the equation (4.2) is satisfied if

$$a_{11}x + a_{12}y + a_{13}z + a_1 = 0,$$
$$a_{21}x + a_{22}y + a_{23}z + a_2 = 0,$$
$$a_{31}x + a_{32}y + a_{33}z + a_3 = 0.$$

The vector $\mathbf{a} = \{\alpha, \beta, \gamma\}$ is non-coplanar with the diametrical plane that conjugate its direction, since normal vector to plane (4.2) has coordinates

$$\mathbf{a}^* = \{a_{11}\alpha + a_{12}\beta + a_{13}\gamma, a_{21}\alpha + a_{22}\beta + a_{23}\gamma, a_{31}\alpha + a_{32}\beta + a_{33}\gamma\},$$

where

$$\alpha(a_{11}\alpha + a_{12}\beta + a_{13}\gamma) + \beta(a_{21}\alpha + a_{22}\beta + a_{23}\gamma) + \gamma(a_{31}\alpha + a_{32}\beta + a_{33}\gamma) =$$
$$= a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 + 2a_{13}\alpha\gamma + 2a_{23}\beta\gamma + a_{33}\gamma^2 \neq 0.$$

Theorem 2. *Let there be a vector* $\mathbf{b}$ *of non-asymptotic direction, coplanar with the diametral plane, which is conjugate to the chords of the second-order surface. The vector* $\mathbf{a}$ *is collinear to the given chords. Then it can be asserted that the vector* $\mathbf{a}$ *will be coplanar with the diametral plane conjugate to the surface chords, which are collinear with the vector* $\mathbf{b}$.

Proof. Let the vector $\mathbf{b} = \{\alpha_1, \beta_1, \gamma_1\}$ be coplanar with the plane (4.2) and have no asymptotic direction of the surface (4.1). Then

$$\alpha_1(a_{11}\alpha + a_{12}\beta + a_{13}\gamma) + \beta_1(a_{21}\alpha + a_{22}\beta + a_{23}\gamma) + \gamma_1(a_{31}\alpha + a_{32}\beta + a_{33}\gamma) = 0$$

or

$$\alpha(a_{11}\alpha_1 + a_{12}\beta_1 + a_{13}\gamma_1) + \beta(a_{21}\alpha_1 + a_{22}\beta_1 + a_{23}\gamma_1) + \gamma(a_{31}\alpha_1 + a_{32}\beta_1 + a_{33}\gamma_1) = 0,$$

which means that the vector $\{\alpha, \beta, \gamma\}$ is coplanar with the diametral plane conjugate to chords, which collinear to the vector $\{\alpha_1, \beta_1, \gamma_1\}$.

It follows from the proved theorem that if the diametral planes δ_1 and δ_2 conjugate to the directions of two noncollinear vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ (having no asymptotic direction) intersect, and the line of their intersection does not have an asymptotic direction relative to the second-order surface, then the diametral plane conjugate to the direction of the chords parallel to this line is coplanar with the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. In this case, we also get three diametral planes, which have the property that any of them is conjugate to the direction of the line along which the other two intersect.

Definition. *A special direction with respect to a second-order surface is the direction of a line parallel to all diametral planes of this surface.*

Theorem 3. *Let a second-order surface be given with respect to the general Cartesian coordinate system by the general equation*

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0. \quad (4.3)$$

If this equation is the equation of a central surface (a surface has a single center), then it has no special directions. All other surfaces have special directions. For non-zero vector $\{\alpha_0, \beta_0, \gamma_0\}$ to have a special direction relative to the surface (4.3) it is necessary and sufficient that its coordinates satisfy the relations

$$\begin{aligned} a_{11}\alpha_0 + a_{12}\beta_0 + a_{13}\gamma_0 &= 0, \\ a_{21}\alpha_0 + a_{22}\beta_0 + a_{23}\gamma_0 &= 0, \\ a_{31}\alpha_0 + a_{32}\beta_0 + a_{33}\gamma_0 &= 0. \end{aligned} \quad (4.4)$$

Proof. It follows from the equation (4.2) that the vector $\{\alpha_0, \beta_0, \gamma_0\}$ has a special direction if and only if the equality

$$\alpha_0(a_{11}\alpha + a_{12}\beta + a_{13}\gamma) + \beta_0(a_{21}\alpha + a_{22}\beta + a_{23}\gamma) + \\ + \gamma_0(a_{31}\alpha + a_{32}\beta + a_{33}\gamma) = 0,$$

or

$$\alpha(a_{11}\alpha_0 + a_{12}\beta_0 + a_{13}\gamma_0) + \beta(a_{21}\alpha_0 + a_{22}\beta_0 + a_{23}\gamma_0) + \\ + \gamma(a_{31}\alpha_0 + a_{32}\beta_0 + a_{33}\gamma_0) = 0, \quad (4.5)$$

where $\{\alpha, \beta, \gamma\}$ is any vector that does not have an asymptotic direction. Since it is always possible to choose three non-collinear vectors $\{\alpha_i, \beta_i, \gamma_i\}$, which have a non-asymptotic direction with respect to the surface (4.3), then homogeneous with respect to α, β, γ equation (4.5) has three linearly independent solutions $\{\alpha_i, \beta_i, \gamma_i\}$, and this is possible if and only if all coefficients at α, β and γ in the equation (4.5) vanish; thus, the relations (4.4) are satisfied. If the surface has a single center, then

$$\delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

and, therefore, the system (4.4) has no non-zero solutions α_0, β_0, γ_0, and no special directions.

For all other surfaces $\delta = 0$ and the system (4.4) has non-zero solutions.

If the rank of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is equal to 2 (surfaces of type II and III), then the system (4.4) has non-zero solutions α_0, β_0, γ_0, but does not have two linearly independent solutions. This means that surfaces of type II and III have only one special direction. Composing the system (4.4) for the reduced equations of these surfaces, we can be sure that the special direction is the direction of the $O'Z$ axis in the reduced equations of these surfaces.

If $rank(A) = 1$, then the system (4.4) has two linearly independent solutions $\{\alpha_0', \beta_0', \gamma_0'\}$ and $\{\alpha_0'', \beta_0'', \gamma_0''\}$. All solutions of the system (4.4) are all linear combinations of these two. This means that for surfaces of types IV and V there exists a plane such that any line parallel to this plane has a special direction, and these directions unite all special directions of these surfaces (this is the $YO'Z$ plane in the reduced equations of types IV and V).

5 Tangent plane

Let point $M_0(x_0, y_0, z_0)$ lies on a second-order surface defined with respect to the general Cartesian coordinate system by the equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0. \quad (5.1)$$

This point called *non-singular* if among three numbers:

$$a_{11}x_0 + a_{12}y_0 + a_{13}z_0 + a_1,$$
$$a_{21}x_0 + a_{22}y_0 + a_{23}z_0 + a_2,$$
$$a_{31}x_0 + a_{32}y_0 + a_{33}z_0 + a_3,$$

there is at least one that is not zero[1].

Thus, a point $M_0(x_0, y_0, z_0)$ lying on a second-order surface is singular if and only if it is the center of surface (see section 1), otherwise, when the surface is conical and the point M_0 is the vertex of this surface.

Definition. *A tangent line to a second-order surface at a given non-singular point on it is a line passing through this point, intersecting the second-order surface at a double point, or being a rectilinear generatrix of the surface.*

Theorem. *The tangent lines to a second-order surface at a given non-singular point (x_0, y_0, z_0) on it lie in the same plane, called the* ***tangent plane*** *to the surface at the point under consideration. The tangent plane equation has the form*

$$F'_x(x_0, y_0, z_0)(x - x_0) + F'_y(x_0, y_0, z_0)(y - y_0) + \\ + F'_z(x_0, y_0, z_0)(z - z_0) = 0. \quad (5.2)$$

Proof . Let

$$\begin{aligned} x &= x_0 + \alpha t, \\ y &= y_0 + \beta t, \\ z &= z_0 + \gamma t \end{aligned} \qquad (5.3)$$

are parametric equations of a line passing through a non-singular point $M_0(x_0, y_0, z_0)$ of the second-order surface given by the equation (5.1). Substituting into the equation (5.1) instead of x, y and z relations (5.3), we get

$$(a_{11}\alpha^2 + a_{22}\beta^2 + a_{33}\gamma^2 + 2a_{12}\alpha\beta + 2a_{13}\alpha\gamma + 2a_{23}\beta\gamma)t^2 + \\ 2\left[F'_x(x_0, y_0, z_0)\alpha + F'_y(x_0, y_0, z_0)\beta + F'_z(x_0, y_0, z_0)\gamma\right] t + \\ + 2F(x_0, y_0, z_0) = 0. \quad (5.4)$$

Since the point $M_0(x_0, y_0, z_0)$ lies on the surface (5.1), then $2F(x_0, y_0, z_0) = 0$, and from the equation (5.4) we find $t = 0$ (this value of t corresponds to the point M_0). In order for the point of intersection of the line (5.3) with the surface (5.1) to be double, or for the line (5.3) to lie entirely on surface, it is necessary and sufficient that the following equality fulfills

$$F'_x(x_0, y_0, z_0)\alpha + F'_y(x_0, y_0, z_0)\beta + F'_z(x_0, y_0, z_0)\gamma = 0, \qquad (5.5)$$

[1] These numbers are the values at point $M_0(x_0, y_0, z_0)$ of the partial derivatives with respect to x, y and z of the left side of the equation (5.1); henceforth we will denote them by $F'_x(x_0, y_0, z_0)$, $F'_y(x_0, y_0, z_0)$, $F'_z(x_0, y_0, z_0)$, respectively, and in accordance with this the left side of the equation (5.1) will be denoted by 2F.

if in addition $a_{11}\alpha^2 + a_{22}\beta^2 + a_{33}\gamma^2 + 2a_{12}\alpha\beta + 2a_{13}\alpha\gamma + 2a_{23}\beta\gamma \neq 0$, then the point of intersection of the line (5.3) with the surface (5.1) is double, and if $a_{11}\alpha^2 + a_{22}\beta^2 + a_{33}\gamma^2 + 2a_{12}\alpha\beta + 2a_{13}\alpha\gamma + 2a_{23}\beta\gamma = 0$, then the line (5.3) lies entirely on the surface (5.1).

It follows from the relation (5.5) and relations (5.3) that the coordinates x, y, z of any point $M(x, y, z)$ lying on any tangent to the surface (5.1) satisfy the equation

$$F'_x(x_0, y_0, z_0)(x - x_0) + F'_y(x_0, y_0, z_0)(y - y_0) + \\ + F'_z(x_0, y_0, z_0)(z - z_0) = 0. \quad (5.6)$$

Conversely, if the coordinates of some point $M(x, y, z)$ different from M_0 satisfy this equation, then the coordinates $\alpha = x - x_0$, $\beta = y - y_0$, $\gamma = z - z_0$ of vector $\overrightarrow{M_0M}$ satisfy the relation (5.5), and hence, the line M_0M is tangent to the surface under consideration.

Since the point $M_0(x_0, y_0, z_0)$ is a non-singular point of the surface (5.1), among the numbers

$$F'_x(x_0, y_0, z_0), \quad F'_y(x_0, y_0, z_0), \quad F'_z(x_0, y_0, z_0)$$

there is at least one that is not equal to zero; hence (5.6) is an equation of the first order with respect to x, y, z, namely it is an equation of the plane tangent to the surface (5.1) at a non-singular point $M_0(x_0, y_0, z_0)$ given on it.

Based on the canonical equations of second-order surfaces and the equation (5.6), it is easy to compose equations of tangent planes to an ellipsoid, hyperboloids, paraboloids, etc. at a given point on them (x_0, y_0, z_0). Equations of the tangent plane to different surfaces have following forms:

1. ellipsoid
$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} + \frac{z_0z}{c^2} = 1,$$
2. one-sheeted and two-sheeted hyperboloids
$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} - \frac{z_0z}{c^2} = \pm 1,$$
3. the cone at the point (x_0, y_0, z_0) that is not its vertex
$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} - \frac{z_0z}{c^2} = 0,$$
4. the elliptic and hyperbolic paraboloids
$$\frac{x_0x}{p} \pm \frac{y_0y}{q} = z + z_0,$$

etc.

6 Intersection of a tangent plane with a second-order surface

Let us take a non-singular point O of the second-order surface as the origin of the general Cartesian coordinate system, place the axes Ox and Oy in the plane tangent to the surface at the point O. Then in the general surface equation

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0, \quad (6.1)$$

the free term is zero: $a_0 = 0$, and the equation of the tangent plane to the surface at the origin must be

$$z = 0.$$

According to the equation (5.6), the equation of the plane tangent to the surface (6.1) at the origin is

$$a_1x + a_2y + a_3z = 0,$$

and since this equation must be equivalent to the equation $z = 0$, then $a_1 = a_2 = 0$, $a_3 \neq 0$.

So, in the chosen coordinate system, the surface equation (6.1) has the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + 2a_3z = 0. \quad (6.2)$$

Conversely, if $a_3 \neq 0$, then this equation is the equation of the surface passing through the origin O, and the plane $z = 0$ is tangent to this surface at the point O. The equations of the curves along which the tangent plane at the point O intersects the surface (6.2) have the form

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = 0, \quad z = 0.$$

If

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0,$$

then these are pair of imaginary intersecting lines; if $\delta < 0$, then pair of real lines; if $\delta = 0$, but at least one of the coefficients a_{11}, a_{12}, a_{22} is not equal to zero, then the curve of intersection is pair of coinciding lines. Finally, if $a_{11} = a_{12} = a_{22} = 0$, then the plane $z = 0$ is part of the given surface, and the surface itself decomposes, therefore, into a pair of planes.

7 Elliptic, hyperbolic or parabolic points of a second-order surface

In this section, we consider only real and non-decomposable surfaces of the second order.

Let O be a non-singular point of such a surface. Since we assume that the surface is non-decomposable, the tangent plane to the surface at the point O cannot be a part of the surface itself. There may be three cases.

1°. The tangent plane to the surface at the point O intersects it along pair of imaginary intersecting lines. In this case the point O is called the ***elliptic*** *point of the surface.*

2°. The tangent plane to the surface at the point O intersects it along pair of real lines intersecting at the point of contact. In this case, the point O will be called ***hyperbolic****.*

3°. The tangent plane to the surface at the point O intersects it along pair of coinciding lines. In this case, the point O will be called ***parabolic****.*

Theorem. *Let the second-order surface be given by the general equation*

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2 + \\ + 2a_1x + 2a_2y + 2a_3z + a_0 = 0, \quad (7.1)$$

with respect to the general Cartesian coordinate system $Oxyz$ and let the given equation (7.1) be the equation of a real second-order non-decomposing surface. Then if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_1 \\ a_{21} & a_{22} & a_{23} & a_2 \\ a_{31} & a_{32} & a_{33} & a_3 \\ a_1 & a_2 & a_3 & a_0 \end{vmatrix} < 0,$$

then all points of the surface are elliptic, if $\Delta > 0$, then all points of the surface are hyperbolic, and if $\Delta = 0$, then they are parabolic.

Proof. Let us introduce a new coordinate system $Mx'y'z'$, choosing any non-singular point M of the given surface as the origin of coordinates and placing axes Mx' and My' in the plane tangent to the surface at point M. The equation (7.1) is converted to the $Mx'y'z'$ coordinate system into an equation of the form (see section 6)

$$a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_{13}x'z' + 2a'_{23}y'z' + a'_{33}z'^2 + 2a'_3z' = 0, \qquad (7.2)$$

where $a'_3 \neq 0$. Let's calculate the determinant Δ' for this equation:

$$\Delta' = \begin{vmatrix} a'_{11} & a'_{12} & a'_{13} & 0 \\ a'_{21} & a'_{22} & a'_{23} & 0 \\ a'_{31} & a'_{32} & a'_{33} & a'_3 \\ 0 & 0 & a'_3 & 0 \end{vmatrix} = -a'^2_3 \begin{vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{vmatrix}.$$

Since the sign of Δ does not change when passing from one general Cartesian coordinate system to another, the signs of Δ and Δ' are the same, and, therefore, the signs

$$\Delta \text{ and } \begin{vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{vmatrix}$$

are opposite. Therefore, if $\Delta < 0$, then

$$\delta = \begin{vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{vmatrix} > 0$$

and, as follows from section 6, the tangent plane to the surface at the point M intersects the surface along pair of imaginary intersecting lines, i.e. M is an elliptical point.

If $\Delta > 0$, then $\delta < 0$, the tangent plane to the surface at the point M intersects it along pair of real lines intersecting at the point M; the point M is hyperbolic.

If, finally, $\Delta = 0$, then $\delta = 0$, the tangent plane to the surface at the point M intersects it along a pair of coinciding lines; the point M is parabolic.

Restricting ourselves, as already indicated, to real non-decomposing second-order surfaces and calculating Δ, for example, from the canonical equations of these surfaces, we can make sure that:

1. ellipsoid, two-sheet hyperbeloid and elliptical paraboloid consist of elliptical points;

2. two-sheet hyperboloid and a hyperbolic paraboloid consist of hyperbolic points;

3. real second-order cone (vertex excluded), elliptic (real), hyperbolic and parabolic cylinders consist of parabolic points.