1 Partially ordered sets

Binary relations

This section is aimed to remind some definitions which are already known to you.

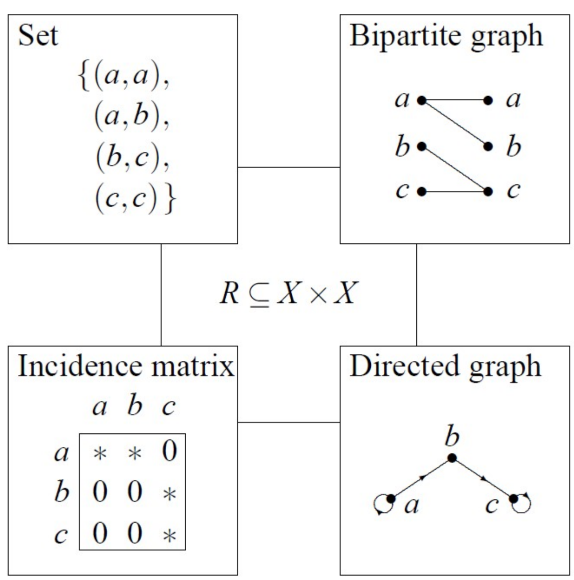
Definition. A **binary relation** R on a set X is the subset of its Cartesian square: $R \subset X \times X$.

Usually instead of writing $(x, y) \in R$ for some $x, y \in X$ one usually writes xRy. It is justified by the fact the for familiar binary relations one uses special signs in between of arguments:

Example.

1. $X = \mathbb{N}$ and $R = \{(a, b) \in \mathbb{N}^2 \mid a \text{ divide } b\}$. Here we denote this relation by $a \mid b$.
2. $X = \mathbb{R}$ and $R = \{(a, b) \in \mathbb{R}^2 \mid a \geqslant b\}$. That means R is denoted by $\geqslant$.
3. X is the set of all humans. $R = \{(x, y) \mid x \text{ is a parent to } y\}$.
4. X is the set of all lines in a plane. $R = \{(x, y) \mid x \text{ is parallel to } y\}$. That means $xRy \Leftrightarrow x \parallel y$.
5. For the set of all triangles on Euclidean plane one can consider the similarity relation: $\triangle ABC \sim \triangle XYZ$.

There are several practical ways to define or, better to say, to visualize relation.



By the very definition **directed graph** (with loops and without multiple edges) is the set X of its vertices and some subset $E \subset X \times X$ whose elements are called edges.

Incidence matrix of a relation R is the map $X \times X \to \{0,1\}$ which sends $(a,b) \in X \times X$ to the 1 if an only if aRb. Let me recall some crucial properties of abstract relations.

Definition. A relation R on a set X is called:

- **reflexive** if for any $x \in X$ one has xRx;
- **symmetric** if for any $x, y \in X$ one has $xRy \Rightarrow yRx$;
- **transitive** if for any $x, y, z \in X$ one has $xRy \wedge yRz \Rightarrow xRz$;
- **antisymmetric** if for any $x, y \in X$ one has $xRy \wedge yRx \Rightarrow x = y$.

The last one (antisymmetry condition) is a new notion for you. Considering given relation R as the directed graph it means that the graph does not contain any both directed edges. As for incidence matrix antisymmetry condition means that it does not contain two units which are on symmetric positions relative to the main diagonal apart from diagonal itself.

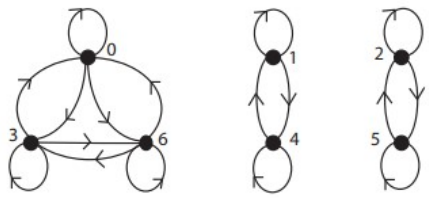
Example. Does divisibility relation $a \mid b$ on the set of all integer numbers satisfy antisymmetry condition?

Exercise 1.1. *For all the Examples 1-5 detect which condition (R, S, T, A) hold for given Relation.*

Recall that a reflexive symmetric transitive relation is called an **equivalence** relation. One of the examples is the Congruence modulo n on the set $\mathbb{Z}$: $a \equiv b \pmod{m}$ iff $m \mid (a-b)$.

Given an equivalence relation $\sim$ on a set X there is a notion of equivalence class $[a] = \{x \in X \,|\, x \sim a\}$. Different equivalence classes have empty intersection, hence equivalence relation defines a partition of the set X into disjoint union of its pieces.

Example. Which relation on the set $\{0,1,2,3,4,5,6\}$ is represented by digraph



Consider two relation R, S on the same set X. Then $R \cap S$ and $R \cup S$ are also relations on X.

Example. Prove that if R and S are both transitive then $R \cap S$ is also transitive. What can be said about $R \cup S$?

For illustration. Let X be some set of cities in China. The relation aRb means that there exist flight route from the city a to the city b with possible several changing. The relation aSb means that there exists train route from the city a to the city b. Let us suppose that there exist a pair of cities such that one cannot reach second city from the first using only planes or only trains but can reach using both types of vehicles. Then the relation $R \cup S$ is not transitive.

There is a general procedure how to obtain transitive relation starting from arbitrary one. It is called **reflexive transitive closure**.

Definition. Let R be a relation on a set X. Consider a new relation $\widetilde{R}$ which is defined as follows: $x\widetilde{R}y$ iff there exist finite sequence $x_0 = x, x_1, \ldots, x_{n-1}, x_n = y$ of non-specified length $n \geqslant 0$. such that $x_k R x_{k+1}$ for any $k < n$. In the case $n = 0$ one has automatically $x\widetilde{R}y$ for $x = x_0 = x_n = y$.

$\widetilde{R}$ is called reflexive transitive closure of R. In terms of digraph associated to R one has $x\widetilde{y}$ iff there exists a path from x to y. E.g. suppose that a given relation R is the relation on some set of cities that there exists a direct flight from one city to another. Then its closure $x\widetilde{R}y$ means that there exists some route with several changing connecting city x to city y.

Posets

Definition. A relation is said to be partial order relation if it is reflexive, antisymmetric and transitive. A set P equipped with a reflexive antisymmetric transitive relation $\prec$ is called **partially ordered set** or just poset.

- Let P be a set of humans and $a \prec b$ iff a is descendant of b.

- Boolean poset B_U. Given some set U consider the set of all its subsets 2^U putting $A \prec B$ iff $A \subset B$ for any $A, B \subset U$.

  Remark. I use the symbol $\subset$ for non-strict inclusion. Some authors in this situation use the symbol $\subseteq$.

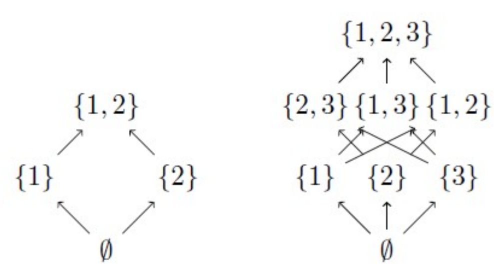
- Let $\mathcal{D}_n$ be a set of all the natural divisors of a natural number n. Then $\mathcal{D}_n$ equipped with the divisibility relation is a poset.

If one want to visualize finite partially ordered sets one of the main tools is the **Hasse diagram**.

Definition. For partially ordered set $(P, \prec)$ element $a \in P$ is said to be **covered** by an element $b \in P$ iff $a \neq b$, $a \prec b$ and for any $y \in P$ such that $a \prec y \prec b$ one has $y = a$ or $y = b$. That means that there no intermediate elements y in between a and b.

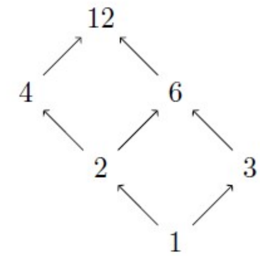
Hasse diagram is a drawing representing covering relation for a given finite poset. We draw elements of a given posets as ponts and place point b above a if $a \prec b$. Then we connect a and b by a line segment if b covers a.

Example. See on the Hasse diagram of boolean posets B_2 (i.e. $U = \{1, 2\}$ and B_3. For $X, Y \in B_U$ Y covers X means that $X \subset Y$ and $|Y \setminus X| = 1$:



B_2 B_3

There is the Hasse diagram for the poset $\mathcal{D}_{12}$:



Proposition 1.1. *Let $(P, \prec)$ is a finite poset and R is the corresponding covering relation as defined in 1. Then reflexive transitive closure $\widetilde{R}$ coincides with $\prec$.*

Proof. Informally, Hasse diagram is obtained from the digraf of a given poset by deleting obsolet arrows that can be restored by transitive property.

More formally, Let $a \prec b$ in the given poset P. Consider a path in the associated digraf of the maximum length: $a = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{k-1} \prec x_k = b$ where $x_i \neq x_{i-1}$ for $i = 1..k$. If there exists an element y such that $x_{i-1} \prec y \prec x_i$ and $y \neq x_{i-1}, x_i$ then inserting y in between x_{i-1} and x_i we would obtain a path of length $k+1$. By maximality of length of the chosen path such an element y does not exist. Therefore, for any $i = 1 \ldots k$ element x_{i-1} is covered by x_i. Hence, $a\widetilde{R}b$ where R is a covering relation on P (xRy when x is covered by y).

$\square$

Definition. Let $(P, \prec)$ be a poset and $A \subset P$. An element $g \in P$ is said to be a **greatest** element of A if $g \in A$ and for any $a \in A$ one has $a \prec g$. By switching the side of the relation one obtains the definition of a least element of A. Explicitly, an element $\ell \in P$ is said to be a **least** element of A if $\ell \in A$ and $\ell \prec a$ for any $a \in A$.

Any subset $S \subset P$ in a poset P can have at most one greatest element since given two greatest elements $g_1, g_2 \in S$ one has $g_1 \prec g_2$ and $g_2 \prec g_1$ which implies $g_1 = g_2$ by antisymmetry.

When $A = P$ in the definition then the element g is called greatest element of the poset.

Example. Let $m, n \in \mathbb{N}$. The greatest element in the subset of all common divisors of m and n with respect to divisibility relation is called the greatest common divisor $gcd(m, n)$.

It is important not to mix up the notion of the greatest element and notion of a maximal element.

Definition. For a subset $A \subset P$ in a poset P an element $m \in A$ is called **maximal** in A if there are no elements $a \in A$ such that $m \prec a$ and $m \neq a$.

One can give the definition of a **minimal** element in the same vein.

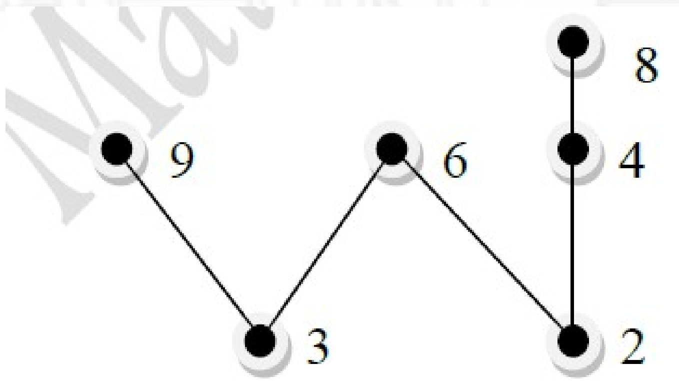
The difference between maximal and greatest elements is explained by the fact that in a partially ordered set not all two element a and b should be comparable.

Definition. Elements $a, b \in P$ are called **comparable** to each other if $a \prec b$ or $b \prec a$.

For example $6, 10 \in \mathbb{N}$ are incomparable with respect to divisibility relation.

If $m \in A$ is a maximal element then for any $a \in A$ one has $a \prec m$ or a and m are incomparable to each other.

It can be several maximal elements in a poset. For example the poset $\{2, 3, 4, 6, 8, 9\}$ with divisibility relation has three maximal and two minimal elements. Here is its Hasse diagram:



Nevertheless, if there exists the greatest element g in a poset P then g is a unique maximal element in P.

Exercise 1.2. *Prove this assertion.*

Definition. Partially ordered set P such that any two elements $x, y \in P$ are comparable to each other is called **totally ordered** set or linear order.

Totally ordered subsets in a partially ordered set are usually called **chains**.

Definition. The subset $A \subset P$ in a partially ordered set such that any two different elements incomparable to each is called antichain.

Chain-Antichain theorems

Definition. The maximal size of an antichain in a partially ordered set P is called the width of P. The maximum size of chain in a poset P is called **height** of P.

Proposition 1.2.
- *Let $A_1, A_2, \ldots, A_k$ be antichains in a partially ordered set P such that $\bigcup_{i=1}^{k} A_i = P$. Then $k \geqslant height(P)$.*
- *Let $C_1, C_2, \ldots, C_k$ be chains in a partially ordered set P such that $\bigcup_{i=1}^{k} C_i = P$. Then $k \geqslant width(P)$.*

Proof. Let C be a chain of maximum possible length $|C| = h = height(P)$. Then for $|A_i \cap C| \leqslant 1$ for any i since if there exist two distinct elements $a, b \in A_i \cap C$ then they would be simultaneously comparable to each other as $a, b \in C$ and incomparable to each other belonging to antichain A_i. Since $C = \bigcup_{i=1}^{k} C \cap A_i$ then $|C| \leqslant k$.

The second assertion is proved through the same lines. Take the antichain A of the maximal size $|A| = w = width(P)$. Then $A = \bigcup_{i=1}^{k} A \cap C_i$ where each $A \cap C_i$ consists of at most one element. Hence $w = |A| \geqslant k$. □

Actually, inverse statements are also true but the proof is a bit harder.

Theorem 1.3. *(Dilworth theorem). Let P be a poset of width k. Then there exist a family of chains $C_1, C_2, \ldots, C_k \subset P$ such that these k chains covers all the elements of P, i.e. $P = \bigcup C_i$*

We will give a proof of Dilworth theorem later at the end of section but now we will give a proof of the dual theorem.

Theorem 1.4. *Let P be a poset of height k. Then there exist a family of antichains $A_1, A_2, \ldots, A_k \subset P$ such that these k antichains covers all the elements of P, i.e. $P = \bigcup A_i$.*

Proof. The proof procced by induction on k. Let C such that $|C| = k$ be any chain of maximal length. Then the maximal element in C would be maximal in the whole poset P. Denote by A_1 the set of all maximal elements in P. Then A_1 is antichain because for any two distinct maximal elements $m_1, m_2 \in P$ cannot be comparable with each other (if $m_1 \prec m_2$ then m_1 is not maximal).

Consider the poset $P' = P \setminus A_1$. Since every chain of maximum length k in P contains an element from A_1 then every chain in P' should have smaller length, hence $height(P') \leqslant k-1$. Therefore, by induction hypothesys P' can be covered by $k-1$ antichains $A_1, A_3, \ldots, A_k$. Evidently, A_i could be regarded as antichains in P itself. The theorem follows.

□

Corollary 1.5. *(Erdős–Szekeres theorem)*

Let $a_1, a_2, \ldots, a_k$ be a sequence of distinct real numbers such that $k > n^2$. Then either there exists an increasing subsequebce of length $n+1$ or there exists a decreasing subseqence of length $n+1$.

Proof. Consider the set $P = \{1, 2, 3, \ldots, k\}$ with the following partial order:

$$i \prec j \overset{def}{\Longleftrightarrow} (i \leqslant j) \wedge (a_i \leqslant a_j).$$

In other words there is an intersection of two linear orders. First linear order is the ordinary one on the set $1, 2, \ldots, k$. The second one is induced from linear order on the set of real numbers via the inclusion $i \mapsto a_i \in \mathbb{R}$.

It is easy to see that chains in P corresponds to increasing subsequences in the given sequence.

In the same vein antichains correspond to decreasing subsequence. Indeed, consider a subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_m}$ where $i_1 < i_2 < \cdots < i_m$, Then i_p and i_q for any $1 \leqslant p < q \leqslant m$ are incomparable in P to each other if and only if $a_{i_p} > a_{i_q}$. Hence the subsequence corresponds to an antichain in P if and only if any member of this subsequence with greater number is always smaller than members with smaller numbers, that means that this subsequence is decreasing.

It remains to apply **Theorem**1.4. Suppose that the longest increasing sequence have length not greater than n. Hence the height of P is not greater than n and by the Theorem above there exist n antichains which cover all the P. If all of these n antichains would have size not greater than n then they could cover the set of maximum n^2 elements. Therefore, at least one of theses antichains should have size at least $n+1$. It gives us required decreasing sequence of size $n+1$.

□

Exercise 1.3. *Draw the Hasse diagram for partially ordrered set that corresponds to the sequence* $5, 3, 6, 1, 4, 2, 8, 9, 7$.

Exercise 1.4. *Let $I_k = [a_k; b_k] \subset [0; 1]$ where $k = 1 \ldots 10$ be a family of ten segments. Prove that at least one of the following is true:*

- *there exist four segments having a point in common;*
- *there exist four segments with pairwise empty intersections.*

Proof. Here we give Tverberg's beautiful proof of Dilworth theorem. Let P be a poset whose maximal antichain length is equal to k. We want to cover it by k chains. We proceed by induction on $k = width(P)$ and the size of P. Supposed that for all posets of width $k-1$ and less the assertion os true. And that assertion is true for all posets of width k having less than $|P|$ elements.

Let A be a maximal antichain of length k. The ingenious part of the Tverberg's proof is decomposition $P = P^- \cup P^+$ associated to A where P^{-1} consists of elements which are greater than some elements of A and $P^- = \{x \in P \,|\, \exists a \in A : x \prec c\}$. Of course, $P^+ \cap P^- = A$ since all the elements in A are incomparable to each other.

Moreover, P^+ and P^- do cover P. As if $x \notin P^+ \cup P^-$ then x incomparable with all elements in A hence $A \cup \{x\}$ would be greater antichain than A itself.

If both P^+ and P^- are proper subsets in P then we can apply induction hypothesis and cover each of them by k chains. Say $P^+ = \bigcup\limits_{i=1}^{k} C_i^+$ and $P^- = \bigcup\limits_{i=1}^{k} C_i^-$. It is easy to see that A is the set of all maximal elements in P^- and simultaneously, A is the set of all minimal elements in P^-.

Since every chain C_i^+ can contain at most one element that is minimal in P^+ then every element $a \in A$ belongs exactly to the one chain C_i^+ for some $i = 1 \ldots k$. In the same vein, every elements $a \in A$ belongs exactly to the one chain C_j^- for some $j = 1 \ldots k$. After suitable renumbering we can suppose that $i = j$ for a given a. Then we cam merge upper chain with minimal element a and bottom chain with maximal element a denoting $C_i = C_i^+ \cup C_i^-$. This gives us the required covering by k chains $P = \bigcup\limits_{i=1}^{k} C_i$.

But what we should do if for every maximal size antichain A one of the part P^+ or P^- is always empty? It could be only if there are at most two maximal size antichains: the set of all maximal elements in P or the set of all minimal elements.

Let us consider arbitrary maximal chain $C \subset P$. It wuld contain one minimal and one maximal element of P. If any maximal size antichain in P has nonempty intersection with C then $width(P \setminus C) = k - 1$ and we can apply induction hypothesis to $P \setminus C$. Else one can find a maximal antichain A such that $A \cap C = \varnothing$. Then A does not contain at least one maximal and at least one minimal element. Therefore P^+ and P^- associated with the antichain A would be nonempty.

□

Poset mappings and constructions

Definition. A map $f : P \to Q$ between two poset is said to be order-preserving if for any $x, y \in P$ one has $x \prec_P y \Longrightarrow f(x) \prec_Q f(y)$.

Example. In analysis order-preserving maps from $\mathbb{R}$ to $\mathbb{R}$ with usual linear order are called monotonically increasing or non-decreasing.

Identity map from $\mathbb{N}$ with divisibility relation to $\mathbb{N}$ with standard linear order is oreder preserving. But the inverse map (which is of course also identity map of underlying sets) is not order-preserving.

Definition. Partially ordered sets P and Q are said to be isomorphic if there exist mutually inverse maps $f : P \to Q$ and $g : Q \to P$ both of them order preserving.

Example. $\mathcal{D}_8 \cong \{1 < 2 < 3 < 4\}$.

Example. How many non-isomorphic posets having 3 elements?

Definition. Cartesian product of two posets $P \times Q$ is their set-theoretic Cartesian product equipped with the following relation $(a_1, b_1) \prec (a_2, b_2)$ iff $(a_1 \prec_P < a_2) \wedge (b_1 \prec_Q b_2)$.

Example. Draw the Hasse diagram of Cartesian products of two chains.

Another construction under posets is their disjoint union. In the poset $P \sqcup Q$ elements from distinct parts are incomparable to each other and element from the same part (P or Q) are compared as the elements of this given partially ordered set.

Proposition 1.6. *(Linear extension of a partial order).*

For any finite poset $(P, \prec)$ there exist a linear order $\leqslant$ on the same set P such that for any $a, b \in P$ one has $a \prec b \Rightarrow a \leqslant b$. In other words for any partial order there exist a bijective order-preserving map to linear order.

Proof. Let $|P| = n$. We proceed by descending induction on the size of the given partial order. Let $R \subset P \times P$ be a given partial order. If $|R| = \dfrac{n(n+1)}{2}$ then the order R is already linear since all $\dfrac{n(n-1)}{2}$ pairs of distinct elements are comparable.

Induction step. Consider a partial order relation R on the set P. Suppose that R is not a linear order, hence there exists (a, b) — a pair of incomparable elements. Consider new relation $R' = \widetilde{R \cup (a, b)}$. That means that we add one extra directed edge to the digraph associated to (P, R) and then take reflexive transitive closure. Let us check that R' is also partial order. It is reflexive and transitive by construction. One need to check only antisymmetry condition. Suppose that $xR'y$ and $yR'x$. It means that there are two chains $x = x_0, x_1, \ldots, x_k = y$ and $y = y_0, y_1, \ldots, y_\ell = x$ such that:

- $x_{i-1} \prec x_i$ or $(x_{i-1}, x_i) = (a, b)$ for any $i = 1 \ldots k$;
- $y_{j-1} \prec y_j$ or $(y_{j-1}, y_j) = (a, b)$ for any $j = 1 \ldots \ell$.

Suppose that $x_{i-1} = a$ and $x_i = b$ for some i. Then $x \prec a$ and $b \prec y$.

If simultaneously $y_{j-1} = a$ and $y_j = b$ for some k then $y \prec a$. Taking into account that $b \prec y$ one obtains $b \prec a$ which contradicts to the choice of (a, b).

If the second chain does not contain the segment (a, b) then $y \prec x$ and by transitivity $b \prec y \prec x \prec a$ also implies that the pair (a, b) is comparable. Therefore, both chains does not contain the segment (a, b). Therefore, $(x \prec y) \wedge (y \prec x)$, hence $x = y$ by antisymmetry property of the given relation R.

Since the identity map $(P, R) \to (P, R')$ is order preserving and by induction hypothesis there exists a linear extension of partial order R' the proposition follows.

$\square$

Exercise 1.5. *How many different linear extensions does have the partial ordered set $\mathcal{D}_{12}$?*

2 Basic Notion of Graph Theory

2.1 Various kinds of graphs

As it was already stated above directed graph is a pair of two sets (V, E) where V is called a set of vertices and E is a binary relation on V. Another way to state almost the same: E is a set which is called a set of egdes equipped with an injective map $E \mapsto V \times V$. If one want to define directed graph without loops one should require that E maps into $(V \times V) \setminus \Delta_V$ where $\Delta_V = \{(v, v) \,|\, v \in V\} \subset V \times V$ is a diagonal of Cartesian square. Being the same as binary relation digraph can be given by the matrix $V \times V \to \{0, 1\}$. This matrix is called **adjacency** matrix of the digraph.

Another kind of graphs — directed multigraphs. In this case we omit the requirement that the map $E \to V \times V$ should be injective. In this case several directed edges could have the same source and target. Adjacency matrix for directed multigraph could have arbitrary non-negative integer values. Its component A_{uv} for $u, v \in V$ is equal to the number of edges with the source u and the target v. If the loops are forbidden then the adjacency matrix has zero's on the main diagonal.

The most common type of graphs are simple graphs which are indirected graphs without loops and multiple edges. It is defined as a pair os some sets (V, E) equipped with the injective map

$$E \to \{\{u, v\} \,|\, u \neq v \text{ where } u, v \in V\} \tag{1}$$

which maps the set of edges into the set of unordered pairs of distinct vertices. Every unordered pair $\{u, v\}$ of distinct vertices correspond to the two ordered pairs (u, v) and (v, u). So simple graphs can be regarded as directed graph without loops with the same set of vertices of special kind: for every directed edge in the digraph there is an edge going in the opposite direction. This is just the symmetric binary relation on the set of vertices with additional restrictions that every vertex $v \in V$ is unrelated to itself.

Adjacency matrix of a simple graph is symmetric $V \times V$ matrix with $0-1$ components having zero's on its diagonal. Very often it is convenient to regard simple graph as reflexive symmetric relation. In this case we treate diagonal pairs $(v, v) \in V \times V$ as so called virtual edges not loops.

If we admit the graph structure map $E \to \{\{u, v\} \mid u \neq v \text{ where } u, v \in V\}$ to be arbitrary we obtain the definition of the (indirected) multigraph without loops. In order to obtain the definition of multigraph with loops we should consider the structure map $E \to \{\{u, v\} \mid \text{ where } u, v \in V\}$ into the set of all unordered pairs. Strictly speaking, $\{v, v\}$ is not a pair but just 1-element set. Nevertheless, we can consider unordered pairs of two coinciding vertices v and v which is naturally correspond to the 1-element set $\{v\}$.

Most of our course is devoted to simple graphs without loops.

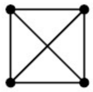
Another matrix corresponding to the graph is an **incidence** matrix. For simple graph that is in general rectangular $V \times E$ matrix with $0-1$ components whose component at the crossing of the row number $v \in V$ and the column number $e \in E$ is equal to 1 if and only if there exist $u \in V$ such that $e \in E$ maps to the unordered pair $\{u, v\}$ under the graph structure map (1). In the latter case one said that v is **incident** to e.

For directed graph $G = (V, E)$ incidence matrix $I(G)$ is a $V \times E$ matrix with component equal to $0, 1$ and -1. $I(G)_{v,e} = -1$ iff v is the source for the edge e, i.e. $e = (v, u)$ for some $u \in V$. $I(G)_{u,v} = 1$ iff v is the target for the edge e, that means $e = (u, v)$ for some $u \in V$. And $I(G)_{v,e} = 0$ in all the other cases.

Definition. For two vertices $u, v \in V(G)$ in a simple (may be directed) graph G a **path** of length k is a sequence of vertices $x_0 = u, x_1, x_2, \ldots, x_k = v$ such that x_{i-1} is adjacent to x_i for all $i = 1 \ldots k$ (in directed case (x_{i-1}, x_i) belongs to $E(G)$). For the case of multigraph one should specify the actual choice of edges connecting x_{i-1} and x_i. By the very definition 1-element sequence u defines the path of length zero from u to itself.

The path is called **simple** if all the vertices in the sequence are distinct. The path is called **closed** when it is a path which starts and ends at the same vertex. When the only repetition of vertices in the given closed path is the first and the last vertex then this closed path is called **cycle**. Strictly speaking, cycle is a graph such that all its vertices and edges constitute one simple closed path.

Example. How many cycles does the following graph have:



Proposition 2.1. *Let $A(G)$ be an adjacency matrix of a simple graph G. Then the components of its k-th power $A(G)^k_{u,v}$ are equal to number of paths of length k.*

Proof. Let us proceed by induction. Cases $k = 0$ and $k = 1$ are trivial. Let us prove the induction step $k \to k + 1$. Since $A(G)^{k+1} = A(G)^k \cdot A(G)$ then by the definition of matrix multiplication $A(G)^{k+1}_{uv} = \sum\limits_{w \in V} A(G)^k_{uw} \cdot A(G)_{wv}$. To every path $x_0 = u, x_1, \dots, x_{k+1} = v$ of length $k + 1$ from u to v one can assign the pair consisting of the path of length k from u to x_k and the edge $\{x_k, v\}$. Moreover, this is a bijective correspondence. Since for every path from u to w of length k and an edge $\{w, v\}$ there is exactly one path of length $k + 1$ from u to v such that $x_k = w$. Therefore, one summand $A(G)^k_{uw} \cdot A(G)_{wv}$ is equal to the number of path of length $k + 1$ from u to v such that $x_k = w$. And the some over all the w is equal to the number of all the paths of length $k + 1$. $\square$

Remark. The result is also valid for directed graph (where incidence matrix is not (in general) symmetric $0 - 1$ matrix and even for directed graph with loops and multiple edges.

One of the most usable notion in Graph theory is the notion ov vertex degree.

Definition. Let G be a simple graph. Then

$$\deg(v) = |\{u \in V(G) \mid \{u, v\} \in E(G) \wedge u \neq v\}| \tag{2}$$

is the number of other vertices which are adjacent to v. It is the same as the number of edges incident to a given vertex.

For digraph one have to distinguish the outdegree (number of outgoing edges), denoted by $\deg^+(v)$, from the indegree (number of incoming edges), denoted $\deg^-(v)$.

Proposition 2.2. *Let $A(G)$ be the adjacency matrix of a simple graph G with vertices $v_1, v_2, \dots, v_n$. Then $A(G) \cdot (1, 1, \dots, 1)^T = (d_1, d_2, \dots, d_n)^T$ where $d_i = \deg(\mathrm{v})$.*

Proposition 2.3. *Let $I(\overline{G})$ be an incidence matrix of directed graph $\overline{G}$ and let G be the corresponding simple graph obtaining by forgetting directions of arcs. Then $I(\overline{G})I(\overline{G})^T = D(G) - A(G)$ where and $D(G)$ is a diagonal matrix with $\deg_{\overline{G}}(v_1), \dots, \deg_{\overline{G}}(v_n)$ on its diagonal.*

Definition. Matrix $D(G) - A(G)$ for a simple or even multigraph without loops G is called **Laplacian** matrix of G and will be denoted by $L(G)$.

2.2 Graph mappings

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Morphism between map is a map that respect graph structure. In the case of simple graphs it is a map $f : V \to V'$ such that for any two adjacent vertices $u, v \in V$ of the graph G their images $f(u), f(v)$ are also adjacent.

In the case of multigraphs morphism is given by pair of maps $f_0 : V \to V'$ and $f_1 : E \to E'$ such that for any $v \in V$ and $e \in E$ such that v is incident to e the same is true for their images: $f_0(v)$ is incident to $f_1(v)$.

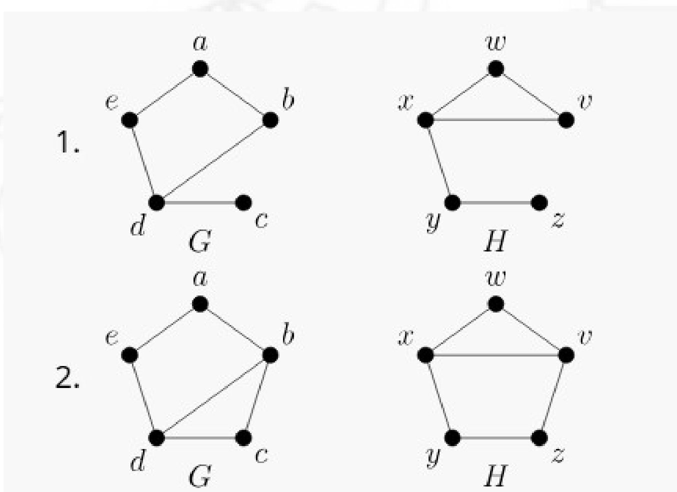
Remark. In the case of simple graphs we do not need to specify the map f_1 since it is completely defined by the map f. When $e = \{u, v\} \in E$ is an edge then $f(u), f(v) \in V'$ are adjacent to each other. Hence $\{f(u), f(v)\}$ is an edge in G', so we can define $f_1(\{u, v\}) = \{f(u), f(v)\}$.

Remark. A delicate point here is that adjacency relation is supposed to be reflexive. Distinct vertices $u, v \in V(G)$ G could have the same image in the graph G'. In this case the image of the edge $\{u, v\}$ is virtual edge corresponding to the vertex $f(u) = f(v)$. See below an example 2.2.

As usual composition of graph morphism is also graph morphism. Morphism $f : G \to G'$ is called **isomorphism** of graphs if their exists a morphism $g : G' \to G$ such that $f_0 : V \to V'$ and $g_o : V' \to V$ are inverse mappings and, in the case of multigraphs $f_1 : E \to E'$ and $g_1 : E' \to E$ are also mutually inverse.

Definition. Two graphs G, H are called isomorphic if there exist an isomorphism $f : G \to H$.

Example. Are these graphs G, H isomorphic or not?

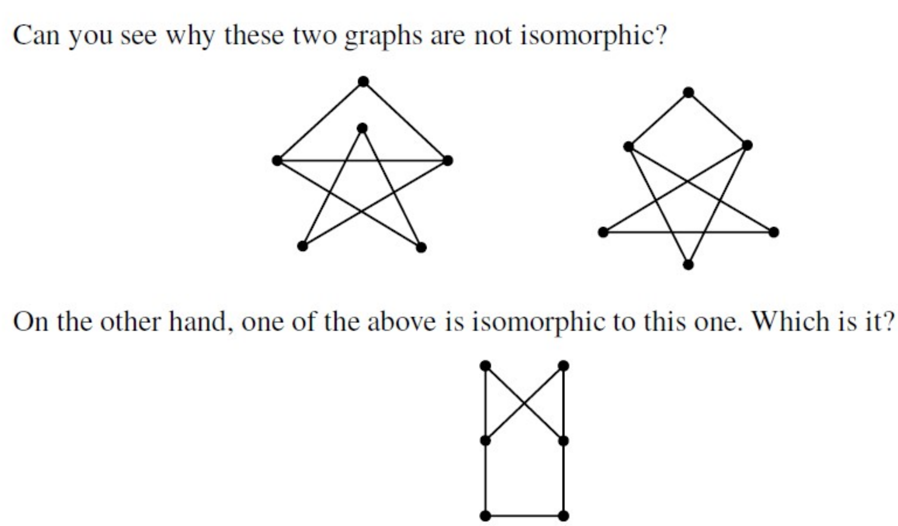


In order to prove that two graphs are isomorphic, we must find a bijection that acts as an isomorphism between them. If we want to prove that two graphs are not isomorphic, we must show that no bijection can act as an isomorphism between them.

Sometimes it can be very difficult to determine whether or not two graphs are isomorphic. It is possible to create very large graphs that are very similar in many respects, yet are not isomorphic. A common approach to this problem is to find an "invariant" that will distinguish between non-isomorphic graphs. An "invariant" is any function that can be defined on graphs, that must produce the same output for all graphs in any isomorphism class. Thus, if you can find an invariant that is different for two graphs, you know that these graphs must not be isomorphic. We say in this case that this invariant distinguishes between these two graphs.

Isomorphic graphs have all the same characteristics: number of vertices and edges, number of vertices of given degree, minimal cycle length, maximal length of simple path and so on.

Example.



Another general example of graph morphism is inclusion as a subgraph. When H, G are two graphs such that $V(H) \subset V(G)$ and there is a graph homomorphism $f : H \to G$ such that $f_0 : V(H) \to V(G)$ is just an inclusion of the subset then H is said to be a subgraph of G. In other words graph H can be obtained from G by recursively deleting some edges and some isolated vertices.

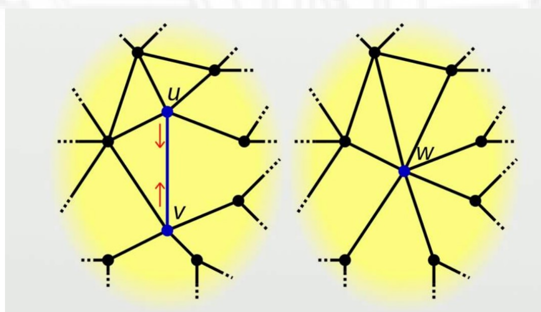
Definition. When H is a subgraph of G and for any two vertices $u, v \in V(H)$ they are adjacent in G if and only if the are adjacent in G then H is said to be **induced subgraph** of G.

Definition. Let G be a simple graph and $\{u,v\}$ is a non-degenerate edge in G. Consider a graph G' such that $V(G') = (V(G) \setminus \{u,v\}) \cup \{w\}$ where w is a new vertex with adjacency relation defined by:

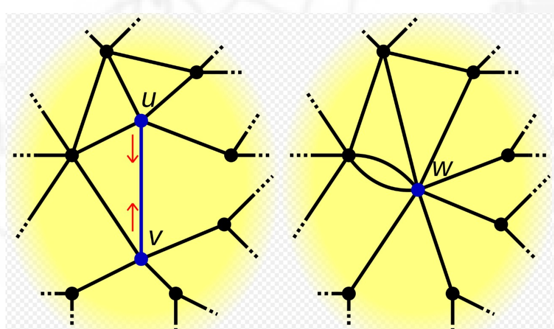
- For $x, y \neq u, v, w$ vertices x and y adjacent to each other in G' if and only they are adjacent in G;
- For $x \neq w$ vertices x and w are adjacent in G' if and only if either x and u are adjacent or x and v are adjacent in G.

Then the graph homomorphism $G \to G'$ which maps vertices u, v into w and all other vertices of the graph G into the same vertices of the graph G' is called **edge contraction**.

I should stress that in this case the edge $\{u,v\}$ maps to the virtual or degenerate edge $\{w,w\}$. See the illustration:

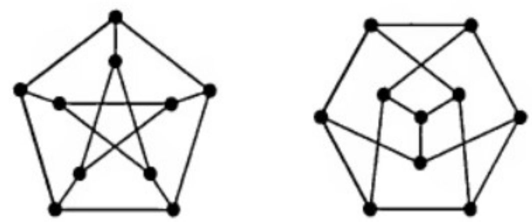


There are another versions of an edge contraction in the context of multigraphs and graphs with loops which can produced multiple edge when contracting edge in a simple graph:



We we denote by G/e the simple graph obtained by contracting of an edge e in a simple graph G. Multigraph obtained by contraction an edge e in a graph G (the latter also can be multigraphs) will be denoted by $G * e$.

Exercise 2.1. *Whether these two graphs are isomorphic to each other. Either proof that they are not isomorphic or proff that they are isomorphic by labaelling the vertices with the same numbers* $\{0, 1, \ldots, 9\}$*:*



2.3 Handshaking Lemma

For directed graph

$$\sum_{v \in V(G)} \deg^{+}(v) = \sum_{v \in V(G)} \deg^{-}(v) = |E(G)| \tag{3}$$

since every edge contributes 1 to the both sums.

For simple graph G we can orient its edges in arbitrary manner (i.e. choose direction of every edge). Thus we obtain directed graph $\widetilde{G}$ such that $\deg^{-}_{\widetilde{G}}(v) + \deg^{+}_{\widetilde{G}}(v) = \deg_G(v)$ for every vertex $v \in V(G)$. That is the poof of **Handshaking formula**:

$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|. \tag{4}$$

Definition. For a simple graph G with vertex set $V = \{v_1, \ldots, v_n\}$ its **degree sequence** or degree vector is the vector $(\deg(v_1), \deg(v_2), \ldots, \deg(v_n))$ where the vertices are renumbered in such a way that the sequence is decreasing $\deg(v_1) \geqslant \deg(v_2) \geqslant \ldots, \geqslant \deg(v_n)$.

Handshaking formula gives the fundamental constraints on the degree vector.

Example. Does their exist graph with degree vector:

- $(4, 4, 3, 3, 1)$?
- $(4, 3, 2, 1, 0)$?
- $(4, 4, 4, 3, 3)$?

It is easy to see that there is a unique up to isomorphism graph with n vertices whose degree vector is $(n-1, n-1, \dots, n-1)$. In this graph every vertex is adjacent to all others. This graph is called a **complete graph** on n vertices and denoted by K_n.

In general non-isomorphic graphs can have the same degree sequence.

Exercise 2.2. *How many non-isomorphic graphs with degree vector:*
a) $(4,3,3,3,3)$*?* *b)* $(3,3,3,3,3,3)$*?*

Corollary 2.4. *In a simple graph G there is an even number of odd degree vertices.*

Definition. A graph G is called **regular** if all the vertices have the same degree. When k is the common degree of all its vertices then G is called k-regular.

Corollary 2.5. *Let G be a k-regular graph with n vertices. The the number of edges is equal to* $\dfrac{kn}{2}$.

Remark. **Proposition 2.2** says that the column $(1,1,\dots,1)^T$ is an eigencolumn for adjacency matrix of a k-regular graph corresponding to eigenvalue k.

Proposition 2.6. *Any 2-regular graph is a disjoint union of one or more cycles.*

Proof. We proceed by induction on the number of vertices. Consider a maximal simple path $x_0, x_1, \dots, x_n$. Since $\deg(x_n) = 2$ then x_n is adjacent to some of the vertices $x_0, \dots, x_{n-2}$ apart from being adjacent to x_{n-1}. It can not be adjacent to any other verticers by maximality of the path. The end x_n ca not be adjacent to any inner vertices $x_1, \dots, x_{n-2}$ as in this case the degree of this inner vertex would be at lest 3. Therefore x_n is adjacent to x_0 and we obtain a cycle. All the vertices in this cycle are adjacent only to two vertices in the same cycle. Then we delete all the vertices of this cycle considering an induced subgraph on the complimentary set of vertices. We get the subgraph with a less number of vertices such that we can apply induction hypothesis. $\square$

Definition. Any 3-regular graph is usually called **cubic** graph.

Definition. The graph G is called **bipartite** if $V(G)$ can be decomposed into disjoint union $V(G) = V_)(G) \sqcup V_1(G)$ such that any two vertices belonging to the same part are not adjacent to each other, In other words, any edge $e \in E(G)$ has one ends in $V_0(G)$ and other end in $V_1(G)$.

Remark. Let G be a graph such that G is k-regular and G is simultaneously bipartite. Then $V_0(G)$ and $V_1(G)$ has the same size.

Proof. We can orient this graph such that all arcs goes from V_0 to V_1. Then the number of edges by the formula (3) is equal $k \cdot |V_0(G)|$ being the sum of outdegrees and equal to $k \cdot |V_1(G)|$ as the sum of indegrees. □

Since every subgraph of a bipartite graph is also bipartite then every cycle in bipartite graph should have even length since the cycle satisfy the condition of the above remark.

Exercise 2.3. *Prove the converse statement. If all the cycles in the given graph have even length then the graph is bipartite.*

Exercise 2.4. *Is there exist an oriented graph with five vertices such that* 2 *vertices have indegree zero, three vertices have indegree* 3*, two vertices have outdegree zero and three vertices have outdegree* 3.

Exercise 2.5. *Consider a bipartite simple graph G such that all the vertices from $V_0(G)$ have degree* 3 *and all the vertices from $V_1(G)$ have degree five. Prove that the number of vertices $|V(G)|$ is divided by* 8.

Despite that the Handshaking Lemma looks almost trivial it has a number of relatively deep and non-trivial corollaries where the creative part of the proof is to invent a suitable graph for applying Handshaking Lemma.

Theorem 2.7. *Consider a simple graph G such that all the vertices have even degrees. Then every edge e belongs to an odd number of cycles.*

Proof. Denote by r one of two ends of the edge e. We call it *root* vertex. Let s be other end of e. Consider an auxiliary graph P_G whose vertices are simple paths starting with vertices r, s and so on. Two paths in P_G would be adjacent by definition if one can obtain one from another by deleting the ending edge or adding an edge. There is additional situation when we consider two path as adjacent vertices in the graph P_G.

Let us call by **lollipop** with stick e the graph consisting of the cycle and a simple path joining r to some vertex of the cycle which starts with e and does not meet the cycle up to very end. Two maximal length simple paths in a lollipop with a stick e also would be adjacent.

Since all vertices in G have even degree then the path of length ! consisting only of e has an odd degree $\deg_G(s) - 1$. A path of length at lest two starting with r, s and ending at some vertex v which is adjacent only to v and not to other vertices in the path would have also an odd degree $1 + \deg_G(v) - 2$.

The path starting with r and ending at v which is not adjacent to any other vertex in this part excluding the previous vertex would have an even degree $1 + \deg(v) - 1$.

And the most interesting case. Consider the path $r = x_0, x_1, \dots, x_k = v$ where v is adjacent to some vertices x_i where $i \in \{1, 2, \dots, n-1\}$. Then for every such i we can consider another path $x_0, \dots, x_i, x_n, x_{n-1}, \dots, x_{i+1}$ which is the the second maximal path in the same lollipop containing the chosen path. So we can:

- delete the last edge;
- add the edge $\{v, u\}$ for a vertex u not belonging to the path;
- add the edge $\{v, x_i\}$ obtaining the lollipop and consequently delete the edge $\{x_i, x_{i+1}\}$ in this lollipop obtaining new path.

The chosen path would have degree $\deg_G(v)$ in the graph P_G.

The only case which i have not yet considered is the case of the path whose ending vertex v is adjacent to r and some other vertices in the path, In this case degree would be odd. Therefore odd degree paths of length at least two in P_G correspond to cycles in G containing the edge e.

By Handshaking Lemma apllied to the graph P_G the number of odd degree vertices in P_G is even.But all odd degree vertices in P_G excluding exactly one which is a path of length 1 bijectively correspond to cycles containing e. Therefore the number of these cycles is even.

$\square$

Exercise 2.6. *Let G be a graph such that all the vertices have an even degree. Let us fix a natural number n. Considering a suitable auxiliary graph whose vertices are paths of length at most n prove that there is an even number of simple paths of length n starting with r.*

Now we are going to discuss the general procedure that can detect if there exists a simple graph with a given degree sequence.

Problem 2.8. *Is there exists a graph with degree vector* $(3, 3, 3, 2, 2, 1)$.

Suppose that required graph does exist. Consider the vertex of degree 3. Deleting it we obtain a graph with five vertex whose degree sequence can be obtained by decreasing by 1 three members in the sequence $(3, 3, 2, 2, 1)$. The crucial point is that one can decrease four biggest member. Since if there exists a graph with non yet ordered degree sequence $3-1, 3, 2-1, 2, 1-1$ then by switching an edge once one can obtain a graph with a degree sequence $\{3-1, 3-1, 2, 2, 1-1\}$ and then switching second time we obtain degree sequence $\{3-1, 3-1, 2, 2-1, 1\} = \{3-1, 3-1, 2-1, 2, 1\}$.

Lemma 2.9. *(Switching lemma) Let G be a graph and $umv \in V(G)$ such that* $\deg(u) > \deg(v)$ *then there exists a vertex w such that w is adjacent to u and non-adjacent to v. Deleting the edge $\{uw\}$ and adding the edge $\{vw\}$ we obtain a graph G' such that* $\deg_{G'}(u) = \deg_G(u) - 1$ *and* $\deg_{G'}(v) = \deg_G(v) + 1$.

Therefore the sequence $(3,3,3,2,2,1)$ is realizable as degree sequence of some six vertex graph if and only if the degree sequence $\{3-1, 3-1, 2-1, 2, 1\} = (2,2,2,1,1)$ is realizable as degree sequence of five vertex graph.

The second step is reducing the question to realizibility of degree sequence $\{2-1, 2-1, 1, 1\} = (1,1,1,1)$ of four vertex graph. One can do another step or just realize that four-vertex graph all whose vertex have degree 1 is just a disjoint union of two edges.

3 Connectivity

Connected components

Definition. In a simple graph two vertices $x, y \in V(G)$ are called to be connected to each other if there is path starting with x and ending with y. This is obviously an equivalence condition. The equivalent classes are called in this cases **connected components**. That means that connected component of the vertex v is an induced subgraph on the vertex set $\{x \mid v \text{ is connected to } x\}$.

Obviously, every graph is a disjoint union of its connected components. If there is only one connected components in a graph then it is called **connected**. That means that for every two vertices in this graph there exists a path joining them.

Exercise 3.1. *Let G be a simple graph. Denote the minimal degree of its vertices by δ and the maximum degree of its vertices by Δ. Prove that if $\delta + \Delta \geqslant |V(G)| - 1$ then G must be connected.*

For directed graph one can define **reachability** relation on the vertex. It is reflexive transitive relation. For vertices $x, y \in V(G)$ to be mutually reachable one from another is the equivalence relation. Equivalence classes in this case are called **strongly connected component**.

The graph is called is strongly connected iff for any two ordered pair $(x, y) \in V \times V$ there exists a path from x to y. It is easy to see that strongly connected components of an arbitrary directed graph are just maximal (by inclusion) strongly connected subgraphs.

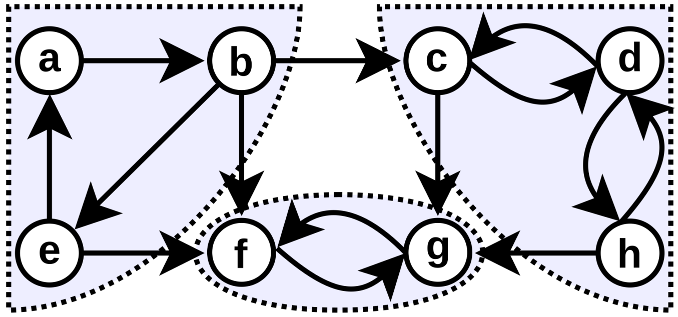


Figure 1: Enter Caption

Let C, C' be two strongly connected components of some directed graph. Then for any $x, y \in C$ and $x', y' \in C'$ the vertex y is reachable from x if and only if the vertex y' is reachable from x'. Therefore one can define the relation of reachability on the set of all connected components. This relation in addition to reflexive and transitive property satisfy also antisymmetry condition.Hence reachability relation on the set of all connected components is a partial order. Compare this with a familiar to you example of divisibility relation on the set of all classes of associated elements in a commutative ring.

Definition. A **bridge** (or isthmus) in a simple graph G is an edge such that its deleting strongly increases the number of connected components.

Slightly more general notion is the **cut-set** — it is some set of edges in a given graph such that deleting all the edge belonging to a cut-set strongly increases the number of connected components.

Proposition 3.1. *Minimal (by inclusion) cut-set (for example, any bridge) could increase the number of connected components only by* 1*.*

Proof. It is enough to check that deleting one edge could increase the number of connected component at least by 1. Of course every connected component C of the graph $G \setminus e$ is a subset of some connected component C' of graph G. If $C \subsetneq C'$ then there is an edge in G connecting C and $C' \setminus C$. The only candidate for this role is e. It is easy to see that $C' \setminus C$ is also a connected component in $G \setminus e$ since every simple path between $x, y \in C' \setminus C$ in graph G can not use the edge e. Moreover, any other connected component of the graph G remains to be connected after deleting the edge e by the same reason: any path connecting some vertices $a, b \notin C'$ in graph G can not use the edge e. $\square$

Proposition 3.2. *For any edge e in a graph G such that e is not a bridge there exists a cycle in G containing e.*

Proof. Let $e = \{u, v\}$. Consider a simple path in $G \setminus e$ starting with u and ending with v. It does exist since the connected components in G and $G \setminus e$ are the same (as the sets of vertices). Adding the edge e to the chosen simple path one obtains the required cycle. □

Proposition 3.3. *Let G be a graph without cycles. Then the number of connected components in G is equal to $c(G) = v(G) - e(G)$ where $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For arbitrary graph G one has inequality $c(G) \geqslant v(G) - e(G)$.*

Proof. Proceeding by induction on $e(G)$. If $e(G) = 0$ then every connected component is an isolated vertex, hence the number of connected components is equal to the number of vertices.

Induction step. Suppose that G has no cycles. Deleting every edge e would increase the number of connected components by 1 since e is necessarily a bridge. By induction hypothesis $c(G) + 1 = (G \setminus e) = v(G \setminus e) - e(G \setminus e) = v(G) - (e(G) - 1) = (v(G) - e(G) + 1$.

For arbitrary graph considerations are the same but $c(G) + 1 \geqslant e(G \setminus e) \geqslant v(G \setminus e) - e(G \setminus) = v(G) - e(G) + 1$. □

Remark. If at least in one induction step we deleted non-bridge then the inequality $c(G) \geqslant v(G) - e(G)$ would become strict. Hence an equality $c(G) = v(G) - e(G)$ is equivalent to the property of G to be without cycles.

Corollary 3.4. *The following conditions are equivalent for a graph G.*

- *Graph G is connected and has no cycles;*
- *Graph G is connected and $e(G) = v(G) - 1$;*
- *Graph G has no cycles and $e(G) = v(G) - 1$.*

Definition. A graph G satisfying one of the equivalent condition of **Corollary 3.4** is called a **tree**. Arbitrary graph without cycles is called a **forest**.

Obviously every connected component in the forest is necessarily is a tree.

Corollary 3.5. *In any tree and forest with exclusion of the forest consisted only of isolated points there exists a vertex of degree* 1.

Proof. It is enough to consider tree. In a connected graph that is not an isolated point every vertex have degree at least 1. Suppose that all the vertices have degree at least 2. Then the sum of degrees would be at least $2 \cdot v(G) = 2(e(G) + 1)$ which contradicts to Handshaking formula. □

Definition. For any graph G the number $c(G) - v(G) + e(G)$ is called a cyclomatic number or circuit rank of G and denoted usually by $z(G)$.

By **Proposition 3.3** cyclomatic number is always non-negative and is equal to zero if and only if G is a forest.

Exercise 3.2. *(3 points) Prove that $z(G) = 1$ if and only if graph G has exactly one cycle.*

Theorem 3.6. *Let G be a simple graph and G' be a directed graph being obtained form G by choosing somehow an orientation on every edge. Then the nullity of the incidence matrix $I(G')$ is equal to the cyclomatic number.*

Proof. Here the nullity of the matrix $A \in M_{m,n}(F)$ is just a dimKer(L_A) where $L_A : F^n \to F^m$ is a linear map sending $\mathbf{x} \in F_n$ into $A \cdot \mathbf{x} \in F^m$. By the Kernel and Image theorem equality $rank(I(G')) = v(G) - c(G)$ is equivalent to the assertion of the theorem.

Main idea is that any cycle $x_0, x_1, \ldots, x_n = x_0$ in G defines a linear dependence relation between columns of the matrix $I(G')$. Let $e_i = \{x_{i-1}, x_i\}$ for $i = 1 \ldots n$. Reversing orientation of the edge does not change the rank of the incidence matrix. Therefore we can suggest that there is a directed cycle $x_0, x_1, \ldots, x_n$ in the graph G'. Then the sum of the columns in $I(G')$ corresponding edges $e_1, \ldots, e_n$ is equal to zero since for every vertex we summing up 1 and -1.

Then deleting an edge which is not a bridge does not change the $rank(I(G')$ and decreases nullity of $I(G')$ and cyclomatic number of G by 1.

It reduces theorem to the case when G is a forest. For the forest one can find a vertex w of degree 1. Then the row of the incidence matrix corresponding to w contains exactly one non-zero elements. Deleting w and the only edge which is incident to w decreases rank by one, does not change nullity and does not change cyclomatic number. This step reduces the theorem to the case when G is a set of isolated vertex and incidence matrix is zero. In this case the assertion is trivial. □

Corollary 3.7. *The number of connected components $c(G)$ is equal to $v(G) - rank\,(L(G))$ where $L(G)$ is a Laplacian matrix for graph G.*

Proof. $L(G) = I(G') \cdot I(G')^T$ is a non-negative definite real symmetric matrix. The nullity of $L(G)$ is equal to the nullity of $I(G')^T$ since
$I(G')I(G')^T \cdot \mathbf{x} = 0 \Rightarrow \mathbf{x}^T I(G')I(G')^T \mathbf{x} = 0 \Rightarrow \|I(G')^T \mathbf{x}\|^2 = 0 \Rightarrow I(G')^T \mathbf{x} = 0$.
But the nullity of I^T is equal to $v(G) - rank\left(I(G')^T\right) = v(G) - rank(I(G')) = v(G) - (v(G) - c(G)) = c(G)$. □

Remark. Laplacian matrix $L(G)$ has zero eigenvalue. The corresponding eigenvector is $(1, 1 \ldots, 1)^T$. When G is connected the multiplicity of a zero eigenvalue is 1 and all other eigenvalues of $L(G)$ are strictly positive.

Definition. Two graphs G_1 and G_2 with the same set of vertices $V = V(G_1) = V(G_2)$ where $|V| = n$ are called **complimentary** to each other if $E(K_n) = E(G_1) \sqcup E(G_2)$.

Exercise 3.3. *Suppose that G_1 and G_2 are complimentary graphs. Prove that at least one of them must be connected.*

Another simple characterization of the the trees is path uniqueness property.

Proposition 3.8. *Let G be a simple graph. Then G is a tree if and only of for any two vertices $x, y \in V(G)$ there exists a unique simple path joining x and y.*

Proof. Let G be a tree, Let us choose two vertices $x, y \in V(G)$ an consider a simple path $x = x_0, x_1, \ldots, x_k = y$. Recall that every edge in a tree must be a bridge. Therefore for any i deleting the edge we obtain disconnected graph $G \setminus \{x_{i-1}x_i\}$ with two connected components by **Proposition 3.1**. Obviously, x_{i-1} and x_i lie in different components hence x and y also belong to the different components of the graph with deleted edge $\{x_{i-1}x_i\}$. That means that every path in G joining x and y must use the edge $\{x_{i-1}x_i$ for any i. We prove that any simple path contains all the edges from the chosen simple path. Therefore, simple path in a tree is unique.

Conversely, if in G there exists only ane simple path for any pair of vertices then G is connected and every edge in G must be a bridge being the only simple path connecting its ends.

□

Spanning trees

Definition. For any graph G the maximal by inclusion of the edge-set subgraph in G with the same vertex set which contains mo cycles is called **spanning forest** of G.

Proposition 3.9. *Spanning forests of G has exactly the same connected components as G and therefore has $v(G) - c(G)$ edges.*

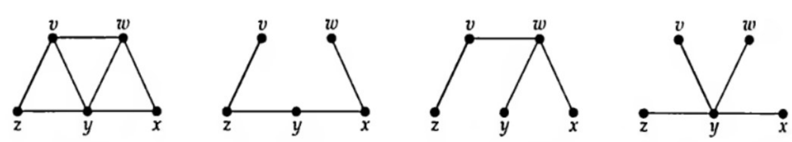
Proposition 3.10. *For any subgraph with the same vertex set number of connected components is at least $c(G)$. Therefore by* **Proposition 3.3** *the number of edge in any subgraph of G without cycles is at most $v(G) - c(G)$. Actually for spanning forest here is an equality.*

Proof. We can obtain arbitrary spanning forest $G_0 = G\backslash\{e_1, e_2, \ldots, e_m\}$ by consecutive deleting edges $e_1, e_2, \ldots, e_m$ in G in a suitable order. Suppose that we have deleted several edges and number of connected components remains the same. If there exists an edge e_i for some $i = 1 \ldots k$ in a graph G' that we obtained at this step which is not a bridge then we can delete e_i on the next step. If all the remaining edges e_i are the bridges in G' then we can prove that G' is already a forest. Since if G would contain cycle then this cycle can not use the remaining edges as all of them are bridges. Therefore this cycle is actually a cycle in a forest G_0 — a contradiction. Hence G' is a forest and $E(G') \supset E(G_0)$. Since G_0 is a maximal forest then $G' = G_0$ but the $c(G') = c(G)$ by the very construction. □

Corollary 3.11. *The cyclomatic number $z(G)$ is equal to a minimal number of edges we must delete form G in order to obtain acyclic graph.*

The number $t(G)$ of spanning trees in a connected graph G is a well studied invariant. For example for cyclic graph C_n one has $t(C_n) = n$ since we can delete exactly one of n edges in a cycle in order to obtain a spanning tree.

Example. Here is an example of graph and several spanning trees in it:



Exercise 3.4. *How many spanning trees does this graph have?*

Proposition 3.12. *(Cut and join equation for t(G)). For any connected graph or multigraph G and an edge $e \in E(G)$ the number of spanning trees satisfies equation:*

$$t(G) = t(G \setminus e) + t(G * e) \tag{5}$$

where $G * e$ is a multigraph graph obtained from G by contracting the edge e and for disconnected graph $t(G)$ is supposed to be zero.

Proof. There are two kinds of spanning trees in G. The first kind are the spanning trees which do not contain e. And the second kind are the spanning trees which contain e. Obviously, first kind spanning trees in G give us all the spanning trees in $G \setminus e$.

Let $T \subset G$ be the spanning tree of the second kind. Then $T * e$ would be the spanning tree in $G * e$. Moreover, assignment $T \mapsto T * e$ is a bijection between the set of all second kind spanning trees in G and the set of all spanning trees in $G * e$.

□

The following result is called Kirchhoff matrix tree theorem.

Theorem 3.13. *The number of spanning trees in every multigraph G is equal to every cofactor of its Laplacian matrix.*

Proof. Denote by L the Laplacian matrix of a given graph G. We will prove here only that the namber of spanning tree is equal to every diagonal cofactor $\det(L_{i;i})$ where by $L_{i;j}$ we denoting $(n-1)\times(n-1)$ submatrix obtained from L by deleting i-th row and j-th column. Rearranging vertices induces the same permutation both on columns and rows of the Laplacian matrix. Hence it does not change diagonal cofactors.

We proceed by induction on the number of edges using cut and join equation (5). Choose an edge $e \in E(G)$. Rearranging vertices we can suppose that e joins first and second vertices in our labeling. Denote by L' the Laplacian matrix of a graph $G\setminus e$ and by L^* the Laplacian matrix of $G*e$. The size of L^* is $(n-1)\times(n-1)$ where $n = v(G)$. It is easy to see that $L_{1;1}$ differs from $L'_{1;1}$ only at ane component: $\ell_{22} = \ell'_{22} + 1$.

The second crucial observation is that $(n-2)\times(n-2)$ matrices $L^*_{1;1}$ and $L_{(1,2);(1,2)}$ are the same where by $L_{(1,2);(1,2)}$ we denote the submatrix obtained from L deleting first two rows and first two columns.

Let us compute $\det(L_{1;1})$ using linearity property of determinant with respect to first row. Since first row in $L_{1;1}$ is equal to the first row of $L'_{1;1}$ plus $(1,0,0,\dots,0)$ and all other rows in these two $(n-1)\times(n-1)$ matrices are the same then

$$\det(L_{1;1}) = \det(L'_{1;1}) + \det\left(\widetilde{L}_{(1,2);(1,2)}\right).$$

Here $\widetilde{L}_{(1,2);(1,2)}$ is some block triangular $(n-1)\times(n-1)$ matrix such that its first row equal $(1,0,0,\dots,0)$ and by deleting its first row and column one obtain $L_{(1,2);(1,2)}$. Surely, $\det\left(\widetilde{L}_{(1,2);(1,2)}\right) = \det\left(L_{(1,2);(1,2)}\right)$. Now we obtained $\det(L_{1;1}) = \det(L'_{1;1}) + \det(L^*_{1;1})$. As $\det(L'_{1;1}) = t(G\setminus e)$ and $\det(L^*_{1;1}) = t(G*e)$ by induction hypothesis we conclude by formula (5) that $\det(L_{1;1}) = t(G)$.

□

Remark. Another purely algebraic formula for counting spanning trees

$$t(G) = \frac{1}{n}\lambda_2\lambda_3\dots\lambda_n$$

where $0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \dots \leqslant \lambda_n$ denote the eigenvalues of Laplacian matrix $L(G)$. It is easy to prove computing coefficient before t in the characteristic polynomial $\det(A - tE) = \prod_{i=1}^{n}(\lambda_i - t)$ in two ways.

Corollary 3.14. *(Caley formula) The number of labelled trees with n vertices is equal to n^{n-2}.*

Proof. Counting labelled trees with n vertices is equivalent to counting spanning trees in the complete graph K_n. By Kirchhoof's theorem this number is equal to $\det(L(K_n)_{1;1})$ that is the deteminant of $(n-1)\times(n-1)$ matrix which has $(n-1)$ on the diagonal and minus units at all other positions. So we need to compute the determinant of the matrix $nE_{n-1} - I_{n-1}$ where I_{n-1} is $(n-1)\times(n-1)$ matrix with all units as its components. The nullity of I is equal to $(n-2)$ and the only non-zero eigenvalue of is $n-1$ with corresponding eigenvactor $(1, 1, \dots, 1)$. Hence, I is similar to the diagonal matrix $\mathrm{diag}(n-1), 0, \dots, 0)$ and $(n-1)E - I$ is similar to a diagonal matrix $\mathrm{diag}(n-(n-1), n, \dots, n)$. Therefore, $\det(nE_{n-1} - I_{n-1}) = n^{n-2}$ and the Caley formula follows. □

There is a purely algebraic proof that all cofactors (not only diagonal) of Laplacian of connected graph are equal. We know that $rank(L) = n-1$ and $(1,1,\dots,1)^T$ is an eigencolumn of the matrix L corresponding zero eigenvalue. For an adjoint matrix $\mathrm{adj}(L)$ one has $L \cdot \mathrm{adj}(L) = 0$ as $\det(L) = 0$. Since space of solution of linear homogeneous system $L \cdot \mathbf{x} = 0$ has dimension 1 then all the columns of the adjoint matrix $\mathrm{adj}(L)$ are proportional to $(1, 1 \dots, 1)^T$. That means $L_{ji} = L_{jk}$ for any i, j, k. Where by L_{ij} we denote here a cofactor that is $L_{ij} = (-1)^{j+j}\det(L_{i;j})$. Recall that the component at the intersesction of i-th row and j-th column of the adjoint matrix $\mathrm{adj}(L)$ is equal L_{ji}. Since L is symmetric them $\mathrm{adj}(L)$ is also symmetric. That means that not only every column in $\mathrm{adj}(L)$ contains equal elements but every row as well contains equal elements. Therefore, all elements in $\mathrm{adj}(L)$ are euqal to each other, thus all cofactors in L are equal to each other.

Blocks and articulation points

Now we define a notion which is in a sense dual to the notion of bridge.

Definition. An **articulation point** (also called a cut vertex or separating vertex) is a vertex in a graph whose removal would increase the number of connected components in the graph.

For example, in a tree every vertex of degree at least 2 is an articulation point.

Theorem 3.15. *Let G be a connected graph. Then for any $v \in V(G)$ the following conditions are equivalent :*

1. *v is an articulation point in G;*
2. *There exist $u, w \in V(G)$ such that $u, w \neq v$ and any path joining u and w contains v.*
3. *There exist two nonempty disjoint subsets $U, W \subset V(G)$ such that $U \sqcup W = V(G) \setminus \{v\}$ and any path joining a vertex in U to a vertex in W contains v.*

Proof. $1 \Rightarrow 3$. Since v is an articulation point the $G \setminus v$ has at least two connected components. Take U to be one connected component of $G \setminus v$ and W to be a union of all other connected components of $G \setminus v$.

As there is no path joining arbitrary vertices $u \in U$ and $w \in W$ in the graph $G \setminus v$ then any path connected them in the graph G must contain vertex v.

$3 \Rightarrow 2$ is trivial.

$2 \Rightarrow 1$. Obviously, u and w would belong to different connected components of $G \setminus v$. □

Graph is called **biconnected** or 2-connected if it is connected and does not have any articulation vertices.

Definition. A **block** in a graph G is a maximal (by inclusion) biconnected subgraph in G. Sometimes blocks are called biconnected components.

Obviously, block must be an induced subgraph in an ambient graph.

Proposition 3.16. *Any two blocks in the graph can have at most one vertex in common. If the vertex is a common point of two blocks then it is an articulation point.*

Proof. If B_1, B_2 are two blocks such that $x, y \in V(B_1) \cap V(B_2)$ are different vertices then we prove that an induced subgraph G' with the vertex set $V(B_1) \cup V(B_2)$ is biconnected that contradicts to maximality of B_1 and B_2. Check that deleting one vertex $p \in V(G')$ preserves connectivity. We can suppose that $p \neq x$ (if $p = x$ then $p \neq y$ and we can switch x and y). Take arbitrary $u, w \in V(G' \setminus p)$. Since u and v both belongs to one of the blocks G_1 or G_2 one can choos a path joining u and x and avoiding $p \neq x$, Similarly, one can choose a path connecting w and x which avoids p since w and x are also belong to the block common for them. Union of the chosen paths gives us a path joining u and w.

As for the second assertion. Let v be a common point of two different blocks. Take vertex u from one block which is adjacent to v and vertex w from another block which is also adjacent to v. Suppose that v is not articulation point. Then there exists a path starting with u which ends at w and avoids v. Joining this path with the path wvu of length 2 we obtain a cycle containing v. Cycle is a biconnected subraph. Consider a block (which is a maximal biconnected subgraph) containing this cycle. This new block have at least two common point with the first chosen block and have at least two common points with the second. Therefore all three blocks should be the same.

□

Corollary 3.17. *Let G_0 be a biconnected subgraph of a given graph G and $v(G_0) \geqslant 2$. Then there exists a unique block of G containing G_0.*

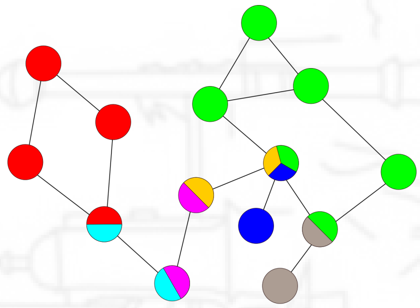
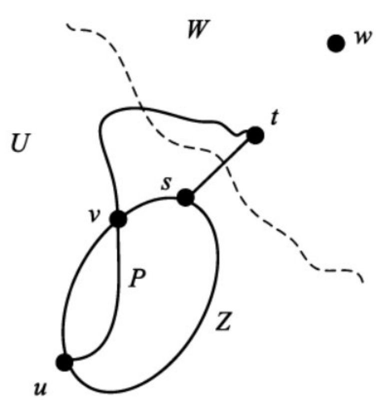


Figure 2: Each color corresponds to a block. Multi-colored vertices are articulation points

Theorem 3.18. *If a biconnected graph G contains at least 3 vertices then for any two vertices $u, w \in V(G)$ there exist at least two simple paths joining u and w which do not intersect each other at inner points.*

Proof. We will prove that there exists a cycle containing both u and w. Consider a set $U \subset V(G)$ of all the vertices belonging to all the cycles which contain u. Denote $V(G) \setminus U$ by W. One can find an edge $\{s, t\}$ such that $s \in U$ and $t \in W$ else U would be a union of several connected components in G. Consider a cycle Z containing u and s.



Since s is not an articulated point one can choose a path P joining t and u which avoiding s, Let v be a first vertex on this path belonging to the cycle Z. Then we construct a new cycle starting from $st \dots v$ and using one of two paths $v \dots s$ in a cycle Z that contains u. We obtain a cycle containing t and u that contradicts to the assumption $t \notin U$.

□

The inverse statement is also true and simple. If for any two vertices in the graph with more than one vertex lie on a cycle then graph is biconnected. Basically, because every cycle is biconnected.

Remark. **Theorem 3.18** can be strengthened with essentially the same proof. Not only two vertices in a biconnected graph lie on the same cycle but every edges and every lies on the same cycle. In the proof one should consider cycles containing given edge.

Proposition 3.19. *In every biconnected graph G for any edge e and two distinct vertices u and w which are not incident to e there exist a simple path joining u and w and using the edge e.*

Proof. Consider a cycle containing w and e. If v belongs to this cycles the proposition follows. If not — consider the shortest path avoiding w joining u with some vertex p in the cycle (it could be an end of the edge e). We start with this path $u \dots p$ and prolong it using those segment $p \dots w$ of the cycle which contains the edge e.

□

Corollary 3.20. *In a biconnected graph every two edges lie on the same cycle.*

Proof. The case of two edges having a common vertex follows immediately from the definition of a biconnected graph. The case of two edges e and $\{u, w\}$ that do not share a vertex follows immediately from the proposition being applied to the vertices u, w and the edge e. □

Corollary 3.21. *For any three distinct vertices u, v, w in a biconnected graph there exists a simple path joining u and w and containing v.*

Exercise 3.5. *Prove the converse statement: if for any three distinct vertices u, v, w in a graph G with $v(G) \geqslant 3$ there exists a simple path joining u and w and containing v then G is biconnected.*

Definition. If G is any simple graph, the block graph of G, denoted $B(G)$, is the intersection graph of the blocks of G: vertices in $B(G)$ are blocks in the given graph G, and two vertices of $B(G)$ are adjacent if the corresponding two blocks meet at an articulation vertex.

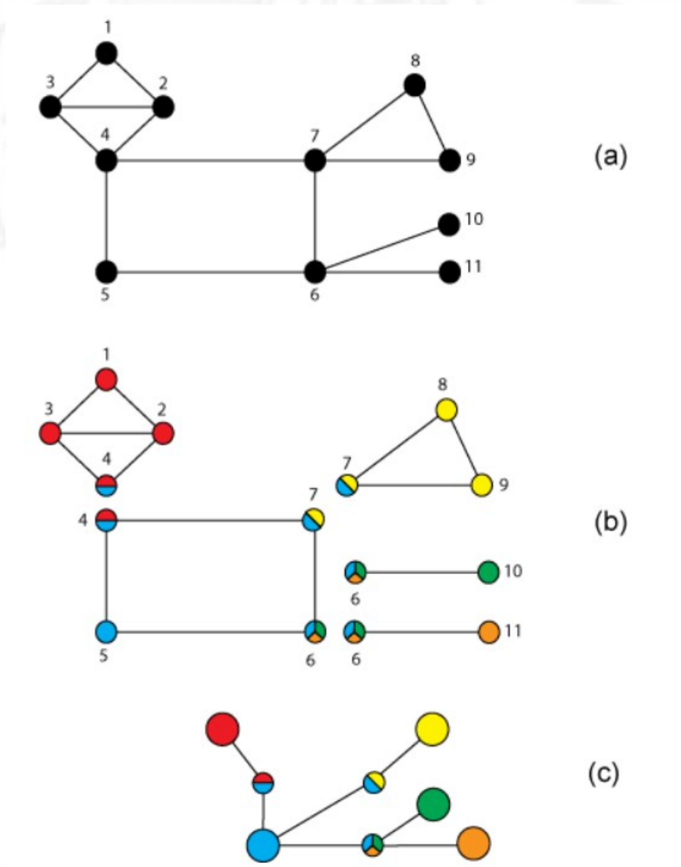
Exercise 3.6. *(Each items cost 2 points) Let G be any simple graph.*

- *Prove that any biconnected component in the graph $B(G)$ is a complete graph;*

- *Describe vertices and adjacency relation for graph $B(B(G))$ in terms of elements of graph G.*

Please, do not mix up the graph $B(G)$ with the following very important notion.

Definition. For a connected graph its **block-cut tree** is a bipartite graph that contains a node for each block of a given graph and for each articulation point. An edge connects node representing an articulation point to the node representing block if and only if articulation point belongs to this block.



Proposition 3.22. *The block-cut tree for every connected graph is indeed a tree.*

Proof. Assume that there is a cycle in block-cut tree. It has a form $v_1B_1v_2B_2\dots v_nB_n$ where $v_i \in B_i \cap B_{i-1}$ for every $i = 1 \dots n$ and $B_0 = B_n$. Since v_i and v_{i+1} (where $v_{n+1} = v_1$) for $i = 1 \dots n$ belongs to the common block B_n then there exists a path joining them. Taking union of these n paths we obtain a cycle in a given graph that have two common points v_i and v_{i+1} with a block B_i for every i. Therefore, this cycles belongs to each of the block and $n = 1$. $\square$

Definition. A **leaf** block in a graph is a block having exactly one articulation point.

Exercise 3.7. *Prove that any connected graph which is not biconnected has at least two leaf-blocks.*

One of the application of block-cut tree is the following lemma which we will need much later.

Lemma 3.23. *Let G be non-complete biconnected graph such that each vertex has degree at least 3. Then there exist a length two path abc such that a is not adjacent to c and $G \setminus \{a, b\}$ is connected.*

Proof. If for any vertex v graph $G \setminus v$ is biconnected then one can find two vertices a, c such that the shortest path connecting them is of length 2. Then $G \setminus a$ is biconnected, hence $(G \setminus a) \setminus c$ is connected. Deleting a and c does not break connectivity as $\deg(b) \geqslant 3$.

Otherwise we can choose a vertex b such that $G \setminus b$ is not biconnected. For every leaf block B it should be an inner vertex $p \in B$ adjacent to b otherwise deleting the only articulation point in block B cuts out p from b. Since there are at least two leaf blocks take vertex a in one block an vertex c in another.

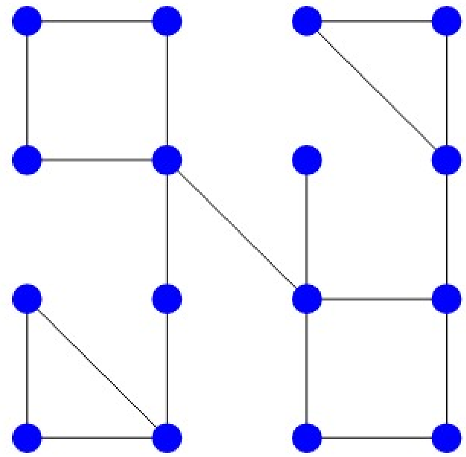
□

Yet another application of block structure of a connected but non-biconnected graph is to counting spanning trees.

Proposition 3.24. *Let G be a connected graph with blocks $B_1, B_2, \dots, B_k$. Then $t(G) = t(B_1) \cdot t(B_2) \cdot \dots \cdot t(B_k)$.*

Proof. This is almost trivial because for any spanning tree $T \subset G$ its intersection with any block B would be a connected subgraph $T \cap B \subset B$. Indeed any simple path joining two vertices from B and using only edges from T can not go through vertices not belonging to the B (if so then a path would use some articulation point twice). Therefore, there is a bijection assigning to a spanning tree $T \subset G$ a k-tuple $(T \cap B_1, T \cap B_2, \dots, T \cap B_k)$ of spanning trees each one in each block. □

Example. How many spanning trees in the following graph:



4 Hamiltonian and Eulerian graphs

Definition. A path in a non-necessary simple connected graph is called an **Euler** path if it uses every edge of the given graph exactly once. A **Hamiltonian** path is that visits every vertex exactly once.

Obviously, every Hamilton path is a simple path. If first and last vertex in a Hamilton path are adjacent to each other then adding the edge joining them one obtains a **Hamilton** cycle.

Usually, Euler path is not simple. Closed Euler path is called **Euler circuit**.

A Graph is said to be **Eulerian** if it admits an Euler circuit. A graph is called Hamiltonian if there is a Hamiltonian cycle in this graph.

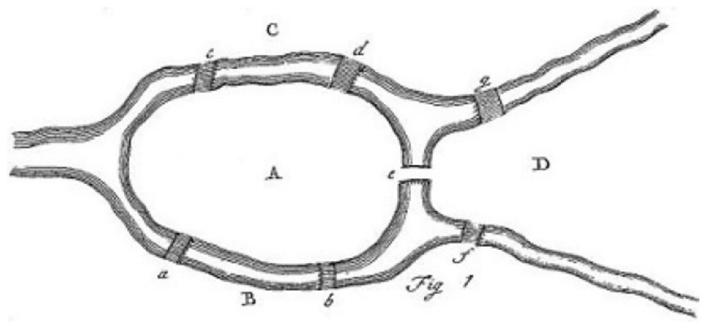
Hamilton cycle is a subgraph of a given graph. Euler circuit defines a homomorphism of a cycle graph C_n to a given graph G that is bijective on the edge sets.

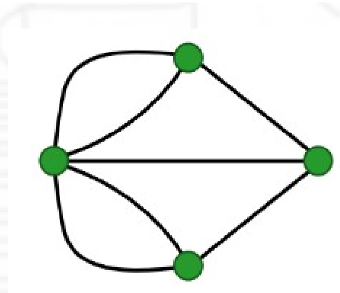
Euler circuits and paths

Euler solution to Königsberg bridges problem

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands—Kneiphof and Lomse—which were connected to each other, and to the two mainland portions of the city, by seven bridges. The problem was to devise a walk through the city that would cross each of those bridges once and only once.

Historically it was one of the first publication on a Graph theory dated back to 1736.





In modern terms it is a question whether exists an Euler path in the following graph: We start with a criteria of existing an Euler circuit.

Proposition 4.1. *Suppose that graph G is Eulerian. Then it is connected and every vertex has even degree.*

Proof. Consider an Euler circuit. We can regard that it is oriented. It defines orientation on the given graph G. Consider indegree and outdegree ov every vertex $v \in V(\widetilde{G})$. We have a homomorphism $f : \widetilde{C} \to \widetilde{G}$ where in the cycle $\widetilde{C}$ outdegree and degree of every vertex is equal to 1. Since $f_1 : E(\widetilde{C}) \to E(\widetilde{G})$ is a bijection then for every $v \in V(G)$ one has $\deg^+(v) = \sum\limits_{u \in f_0^{-1}(v)} \deg^+(u) = \left|f_0^{-1}(v)\right|$ that means how many times our Euler circuit visit the vertex v. Taking into account the same equality for indegrees $\deg^-(v) = \left|f_0^{-1}(v)\right|$ we conclude that $\deg^+(v) = \deg^-(v)$ for every v. Therefore, $\deg_G(v)$ is even. $\square$

Corollary 4.2. *If an Euler path exists in a graph or multigraph then the number of vertices of odd degree is* 0 *or* 2.

Proof. If a graph is Eulerian then number of odd degree vertices is zero. If a graph is non-Eulerian but admits an Euler path then choose one. Consider an auxiliary graph adding new edge joining the end and the start of the chosen Euler path. The new graph is Eulerian hence all degrees in the new graph are even. The only two vertices whose degree differs in the old and new graphs are just the start and the end of the chosen path. And only these two vertices have odd degree. $\square$

Theorem 4.3. *The necessary and sufficient condition for connected graph G to be Eulerian is that all vertex degree are even.*

Proof. We need to prove only sufficiency. Let us proceed by induction on a number of edges. Let G be a given graph. Choose a closed circuit of maximal possible length. Suppose that it does not use all the edges and lead to a contradiction. Deleting all the edges belonging to the chosen circuit we obtain a graph G_0 which could be disconnected but all the vertices in G_0 have even degree. That is true since the subgraph in G that is the image of the chosen cycle is Eulerian hence all its vertices are of even degree.

In graph G_0 at least one of the vertices belonging to the first chosen circuit should have non-zero degree. Consider the connected component of the graph G_0 containing this vertex of positive degree. We can apply induction hypothesis to it and find an Eulerian circuit in the component. Eulerian circuit in the connected component intersects firstly chosen circuit in at least one point. But given two circuits intersecting at some vertex we can easily construct a new circuit that contains all the edges of the given two. hence we obtain a circuit containing more edges than the first chosen which contradicts to maximality of the first chosen circuit. □

Remark. The same proof works for directed graph with loops. There exists a directed Eulerian circuit in directed graph of and only if this graph is connected and for every vertex $\deg^+(v) = \deg^-(v)$.

Corollary 4.4. *There exists an open Euler path in a connected Graph if and only if there are exactly two vertices of odd degree. The start and the end of any Euler path are necessary these two vertices of odd degree.*

Proof. Joining two vertices of odd degree by an extra edge we can apply **Theorem 4.3** and find an Euler circuit in a new graph. Removing the extra edge from this circuit we obtain an Euler path in an old graph. □

Exercise 4.1. *Let G be a multigraph such that all vertices have even degree. Prove that there is a family of edge-disjoint cycles (not circuits!) $C_1, \ldots, C_k$ such that $E(G) = E(C_1) \sqcup E(C_2) \sqcup \cdots \sqcup E(C_k)$.*

Problem 4.5. *Let G be a 4-regular graph. Prove that there are exist two 2-regular subgraphs $G_0, G_1 \subset G$ with the same vertex set such that $E(G) = E(G_o) \sqcup E(G_1)$.*

Proof. We can assume that G is connected. Consider an Euler circuit. Since sum of the degrees is divisible by 4 then the number of edges which is equal to the length of an Euler circuit is even. Consider an alternative 2-coloring of the edges of the Euler cycle that is all the edges of G, say blue and red colors. Then for any vertex $v \in V(G)$ its blue degree and red degree are the same. This is the answer to the question how many times we visit the vertex v while traversin the Euler cycle.

Under our assumption that means the the graph generated by blue edges is 2-regular subgraph in G and the same for the graph generated by red edges.

□

Fleury's Algorithm for finding the Eulerian circuit in a multigraph G

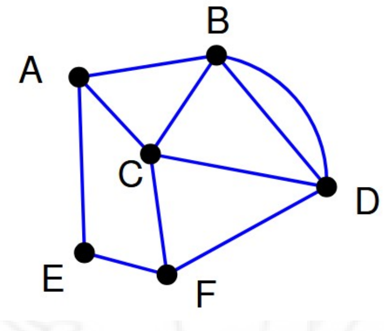
Start at any vertex v_0 and recursively construct a path $P_i = v_o e_1 v_1 e_1 \dots e_i v_i$ subject to the following rules:

- Consider remaining graph $G_i = G \setminus \{e_1, \dots, e_i\}$ removing all the already being traversed edges and then isolated vertices and choose $e_{i+1} \in E(G_i)$.
- unless there is no alternative e_{i+1} should not be a bridge in G_i.

Stop when $\deg_{G_i}(v) = 0$. Obviously, graph G_i has only two vertices of odd degree v_0 and v_i (or does not have at all when $v_i = v_0$). So the ultimate constructed path P_k would be a circuit.

Let us explain that Fleury algorithm actually produce an Eulerian circuit. Suppose that there are some adges in G not belonging to P_k that is G_k contains at least one edge. There exist i such that $\deg_{G_k}(v_i) \geqslant 1$ else the subgraph in G generated by the edges from our circuit P_k would be a connected component in G. Choosing i to be maximal satisfying this condition. Obviously, $i \neq k$ else the algorithm would not stop at v_k. Then this extra edge $\{v_i w\}$ going out from v_i is a bridge in G_i since it was not chosen instead of e_{i+1}. In this case graph $G_k\{v_i w\}$ has exactly two odd vertices — v_i and w belonging to different connected components, a contradiction.

Example. Find an Euler path using Fleury algorithm in a graph below:



Another standard application of Euler circuits is to so called *De Bruijn* sequence.

Definition. A binary De Bruijn sequence of order n is a cyclic $0-1$ sequence in which every possible length-n string A occurs exactly once as a substring.

Example. A De Bruijn sequence of order 3 is 00010111.

Definition. A De Bruijn graph is a directed graph $DB_n = \left(\{0,1\}^{n-1}, \{0,1\}^n\right)$, that is, $V(DB_n) = \{0,1\}^{n-1}$ and $E(DB_n) = \{0,1\}^n$ where there is an edge from $v = (a_1, ..., a_{n-1})$ to $v' = (a'_1, \ldots, a'_{n-1})$ if and only if $a_2 = a'_1, a_3 = a'_2, \ldots, a_{n-1} = a'_{n-2}$.

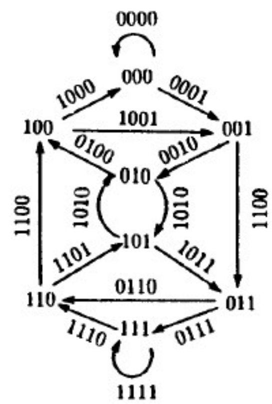


Figure 3: Graph DB_4

That means that only vertices of the form $(a_1, \ldots, a_{n_1})$ and $(a_2, a_3 \ldots, a_n)$ join by an edge. We can label this edge by a string $(a_1, a_2, \ldots, a_n)$.

Every vertex in De Bruijn graph has outdegree and indegree 2. Threfore it is an Eulerian directed graph. Euler circuits in De Bruijn graph corresponds to De Bruijn sequences and vice versa. So we proved that binary De Bruijn sequences of order n do exists.

Exercise 4.2. *Prove that there exist a cyclic word of length* 27 *in three letters* A, B, C *such that all* 27 *ordered triples of consecutive letters in this word are different.*

Chinese postman problem

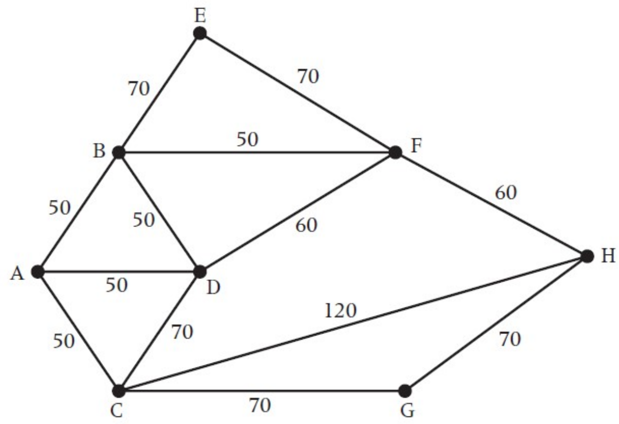
There is a classical problem in the branch of Discrete optimization where one can easy apply Euler theory.

Given some net of streets how the postman could walk all the streets such that the total distance walked by the postman was as short as possible?

More formally speaking, given a weighted graph with some weight function $w : E(G) \to \mathbb{R}_{\geqslant 0}$ we need to find a closed path $f : C \to G$ that visit every edge at least once and minimize $\sum\limits_{e \in E(C)} w(f_1(e))$.

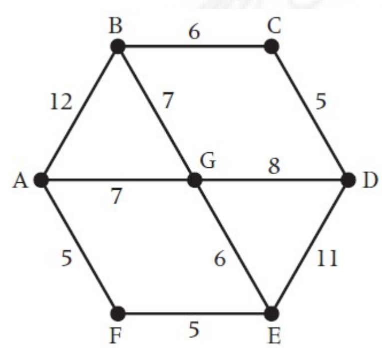
If given graph is Eulerian then all Euler circuits has the same total length that is the sum of weight of all edges in G. But if G is not Eulerian we has to walk through some edges at least twice.

Example. In the example postman has to start at E walk along all 13 street and returns to E.



The solution to Chinese postman problem is Eulerian circuit in some multigraph $\widetilde{G}$ that is obtained from G by doubling several edges. In the given graph there are only 2 odd degree vertices A and H. Hence the complimentary graph G' with the edge set $E(\widetilde{G}) \setminus E(G)$ aslo would have these two odd degree vertices. Therefore, G' consist of a simple path from A to H and possibly a few cycles. Since our aim to minimize sum of weights then we can assume that G' is just a simple path. The simple path of minimal total length here is $ABFH$. So the minimal length of the Chinese postman route is 1000.

Exercise 4.3. *Find the length of the optimal Chinese postman route for the networks below.*



Hamiltonian graphs

One of the most basic result about hamiltonian path and cycles is the following.

Proposition 4.6. *For any orientation of the complete gtaph K_n there exist a directed hamiltonian path.*

Proof. Let us proceed by induction on n. Deleting one vertex v we obtain some orientation of complete graph K_{n-1}. By induction hypothesis there exists a simple directed path $x_1x_2\dots x_{n-1}$ of length $n-2$ avoiding v. Then consideration orientation of edges $\{x_0v\}$ and

$\{x_{n-1}v\}$ in our directed graph one immediately concludes see that there is only case when it is not obvious haw to construct a path of length n: when corresponding directed edges are (x_0, v) and (v, x_{n-1}).

In this case find a maximal i such that there is outgoing edge from x_i to v. By assumption $i \geqslant 0$ since there is outgoing edge from x_0 to v and $i \leqslant n-2$ since there is no outgoing edge from x_{n-1} to v. By maximality of i there is no outgoing edge from x_{i+1} to v, hence the orientation of this undirected edge was chose to be (v, x_{i+1}). It easy to see that $x_0x_1\dots x_ivx_{i+1}x_{i+2}\dots w$ is a desired Hamiltonian path.

□

If a directed graph admits a (directed) Hamilton cycle the the graph is necessary strongly connected since any two points in a cycle are reachable from each other.

Theorem 4.7. *Let G be an orientation of complete graph K_n such that G is strongly connected then G admits a directed Hamiltonian cycle.*

Remark. There is a name for a vertex of indegree zero — **source** and fro a vertex of outdegree zero — **target**

Proof. Step 1. Prove by induction on n that G admits a directed cycle of length 3. Base $n = 3$ it is obvious. Let $n > 3$. Consider an arbitrary vertex v, If $G \setminus v$ does not have neither source nor target then we can apply induction hypothesis to it. So suppose that $G \setminus v$ has a source u (the case of target is analogous). Hence the edge $\{u, v\}$ is oriented from v to u else u would be a source in a given graph G. Take a Hamiltonian path in $G \setminus v$ which is obviously starts with $u = x_0x_1\dots x_{n-1}$. Since v is not a target there is at least one i such that there an oriented edge (x_i, v). Choose a minimal i with this property so $i \geqslant 1$. Then $vx_{i-1}x_iv$ is a cycle in G.

Step 2 Prove by induction on n that the maximal length of a directed cycle in a graph satisfying condition of the theorem is equal to n. Consider a cycle of maximal possible length $C \subset G$. We prove that for any $u \in G \setminus C$ all the edges $\{u, v\}$ when $v \in C$ are of the same kind: either all outgoing or all ingoing. If it is not the case then there are two consecutive vertices v_1, v_2 such that $\{u, v_1\}$ is ingoing to u and $\{u, v_2\}$

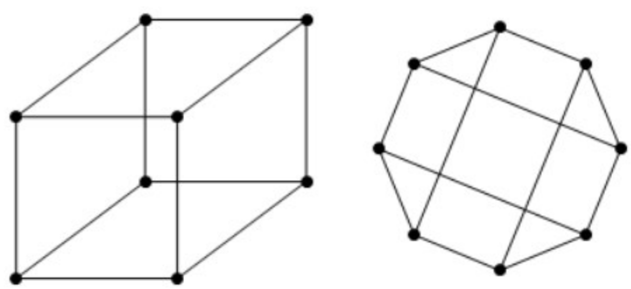
is outgoing from u. Replacing the edge $\{v_1v_2\}$ in C by the length two path v_1uv_2 one obtains a cycle of greater length than C.

Step 3. Decompose the set of vertices $V(G\backslash C)$ into disjoint union of two subsets $V_+ \sqcup V_-$ where $V_+ = \{u\,|\,(u,v) \in E(G) \quad, \forall v \in C\}$ and $V_- = \{u\,|\,(v,u) \in E(G) \quad \forall v \in C\}$. If there exists at least one edge $(y,x) \in E(G)$ such that $y \in V_-$ and $x \in V_+$ one can replace some edge (v_i, v_{i+1}) in the cycle C by the path of length 3 $v_i y x x_{i+1}$ and obtain a cycle of greater length.

Hence we can assume that all edges joining a vertex in V_+ to a vertex in V_- is going from V_+ to V_-. At least one of the set V_+ and V_- is non-empty. We can not reach a point in V_+ starting fron a point in C and can not reach a point in C starting from a pont in V_-. Hence G is not strongly connected — a contradiction. $\square$

One of the very basic non-trivial examples of Hamiltonian graphs is a graph Q_3. Its vertices are eight vertices of 3-dimensional cube and its edges are 12 edges of 3-dimensional cube.

In analyzing Hamiltonicity itis useful to describe Hamiltonian graph as a cycle with additional chords. For example Q_3 can be drawn as follows:



Definition. A graph Q_n is defined as follows. Its vertices are 2^n points in $\mathbb{R}^n$ with coordinates $(\pm 1, \pm 1, \ldots, \pm 1)$ and two vertices are adjacent if the corresponding points differs only at one position.

Exercise 4.4. *Each item cost* 2 *points.*

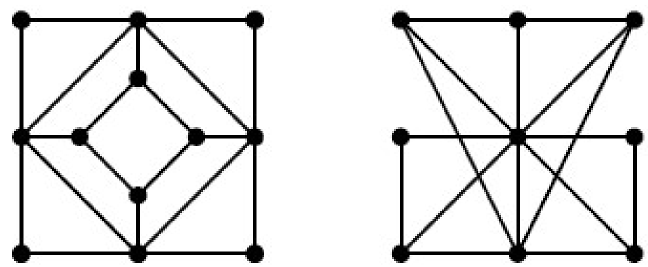
- *Prove that graph Q_4 is Hamiltonian.*
- *Prove that for every n graph Q_n is Hamiltonian.*

Here are two simple necessary condition for graph to be Hamiltonian:

- Hamiltonian graph does not have any articulation points.

- Removing k distinct points in a Hamiltonian graph one obtains a graph with at most k connected components.

Exercise 4.5. *Prove that both of the graphs below are not Hamiltonian (3 points):*



Problem 4.8. *Prove that Petersen graph is not Hamiltonian.*

Proof. Petersen graph is a complement to the intersection graph of $\binom{5}{2}$ 2-element subsets in 5-element set. That means that its vertices are just ten unordered pairs $\{i, j\} \subset \{a, b, c, d, e\}$ and two pairs are adjacent to each other if they have an empty intersection. Therefore it is easy to check that for every two non-adjacent vertices in Petersen graph there is a unique path of length 2 joining them. Indeed let $\{ab\}$ and $\{ac\}$ be two non-adjacent point then the unique point adjacent to both is a 2-element subset having empty intersection with $\{a, b, c\}$. Therefore there are no 4-cycles in Petersen graph. For the same reason there is no 3-cycles.

Let us assume that Petersen graph is Hamiltonian. Presenting it as a cycle $v_0, v_1, \ldots, v_9$ with chords we conclude that there five extra chords and any vertex is joint to only one vertex by a chord. Since there are no cycles of length 3 and 4 the vertex v_0 ca not be joined with v_2, v_3, v_8, v_7. If all chords would be the diameters of a cycle then the graph obviously has 4-cycle. Hence one can assume that v_0 is joint with v_4. If v_5 is adjacent to v_1 then there is obvious 4-cycle. If v_5 is adjacent to v_9 then there is also obvious 4-cycle. We obtain a contradiction which proves that Petersen graph is Non-Hamiltonian.

□

Another beautiful result for 3-regular graphs was proved by Smith and Tutte in 1946: the number of Hamiltonian cycles containing fixed edge in such a graph is even. Here is a generalization which is easily proved by Lollipop-method of **Theorem 2.7**

Theorem 4.9. *Let G be a graph where all the vertices have odd degree. Choose an edge e. Then the number of Hamiltonian cycles containing e is even.*

Proof. Choose an end r of the edge e. Let us construct an exchange graph whose vertices are Hamiltonian paths starting with vertex $r = x_1$ and edge e. Two Hamiltonian paths would be adjacent in the exchange graph if an only if they can be transformed into each other by Lollipop construction. If the end v of a Hamilton path P does not adjacent to r then there are even number of inner vertices x_i in the path such that x_i and v are adjacent to each other. Therefore the number of adjacent to P Hamilton paths in the exchange graph is also even since we add an edge $x_i v$ and remove the edge $x_i x_{i+1}$ in order to obtain new Hamilton path. Hamilton paths whose ends adjacent to r bijectively correspond to Hamilton cycles containing e. Points in the exchange graph corresponding to these paths have odd degrees as there is an odd number of adjacent to v inner vertices of the path P remains.

Applying Handshaking lemma to the constructed exchange graph we conclude the theorem. □

Remark. The proof remains valid if all the vertices non incident to e are of odd degree.

Corollary 4.10. *If a 3-regular graph is Hamiltonian then it has at least 3 different Hamiltonian cycles.*

Corollary 4.11. *In an arbitrary simple graph G and chosen vertex u there is even number of Hamilton paths starting with u and ending in a (non-fixed) vertex of even degree.*

Proof. Consider a new graph $G' = G \cup \{r\}$ with an extra vertex r which we choose to be adjacent to u and all the even vertices in G. Then there is an obvious bijection between Hamilton cycles on G' containing the edge $\{ru\}$ and Hamilton paths in G starting with u and ending in an even vertex. □

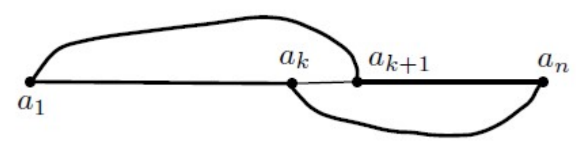
In general this is rather hard problem to test whether given graph is Hamiltonian or not. There is a bunch of sufficient conditions which says that given graph is Hamiltonian when it has relatively many edges. The most simple one belongs to Gabriel Dirac.

Theorem 4.12. *Let G be a graph with n vertices. Assume that $\deg(v) \geqslant \dfrac{n}{2}$ for every $v \in V(G)$. Then G is Hamiltonian.*

Proof. By contradiction. Take a non-Hamiltonian graph satisfying condition on degrees. If it is possible to add an extra edge such that the bigger graph is also non-Hamiltonian then do it. We stop when we obtain new non-Hamiltonian graph G such that for any two non-adjacent vertices $u, v \in V(G)$ graph $G \cup \{u, v\}$ would be Hamiltonian. Since G itself is not Hamiltonian then Hamiltonian cycle in a graph with an extra edge should use this edge $\{u, v\}$.

That is G possess a Hamiltonian path $u = a_1 a_2 \dots a_n = v$. Consider the set of u-neighbours $N(u) = \{x \in V(G) \,|\, \{u, x\} \in E(G)\}$ and the set of v-neighbours $N(v) = \{x \in V(G) \,|\, \{v, x\} \in E(G)\}$. By $N'(u)$ we denote the set of predecessors in the path of elements in $N(u)$ that is $N'(u) = \{a_i \,|\, a_{i+1} \in N(u)\}$.

By assumption $|N(u)|+|N(v)| \geqslant \frac{n}{2}+\frac{n}{2} = n$ and $N'(u) \cup N(v) \subset \{a_1, \dots, \dots, a_{n-1}\}$ as u and $v = a_n$ are not adjacent. Choose $a_k \in N'(u) \cap N(v)$ and construct a Hamilton cycle in G as it is shown in the picture:



□

5 Planarity

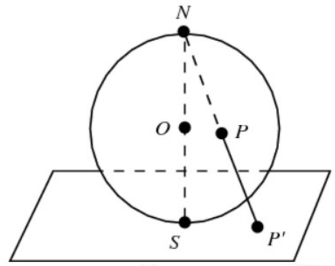
Definition. A **planar** graph is a graph that can be embedded in the plane, that is it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Strictly speaking, it means that there exists a continiuos embedding into the plane of topological realization of a given graph G which is obtained in the following way. Choose arbitrary orientation $\overline{G}$ then consider a disjoint union $\bigsqcup_{e \in E(G)} [0; 1] = [o; 1] \times E$ and glue points:

- $(0, e) \sim (0, e')$ if $s(e) = s(e')$; $(0, e) \sim (1, e')$ if $s(e) = t(e')$;
- $(1, e) \sim (1, e')$ if $t(e) = t(e')$; $(1, e) \sim (0, e')$ if $t(e) = s(e')$.

Plane graph is a planar graph with a fixed embedding of its topological realization into the plane. Strictly speaking up to isotopy but we do not go into details in topology.

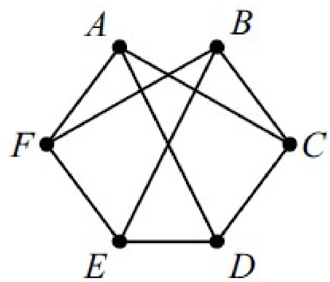
In the definition of the planar graph one could consider an embedding to the sphere S^2 since for any point $p \in S^2$ there is a homeomorphism $S^2 \setminus \{p\} \cong \mathbb{R}^2$:



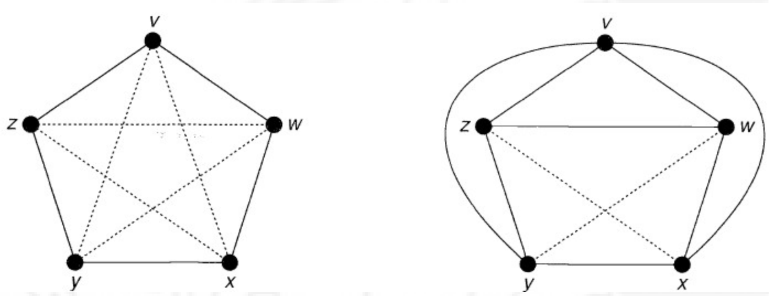
Therefore one of the main examples of plane graphs are the graph of edges and vertices of convex polyhedra.

In general it is not an easy problem to test whether is a given planar or not.

Example. Try to test the following graph.



Let us give a direct "proof" that graph K_5 is not planar. Consider a cycle in $xyztuvx$ in K_5. In the plane diagram this cycle would be a boundary of some pentagon. There exists at least one of 5 other edges which is in the inner region of the pentagon, say zw. Then the edges vx and vy should lie in the outer region. Consequently, zx and wy lie in the inner region and should intersect each other.



If G is a planar graph then any plane diagram divide whole plane into the regions namely topological connected components of the complements $\mathbb{R}^2 \setminus G$. Such regions are usually called **faces** of a plane graph. One of the faces is unbounded but actually it does not matter. In the spheric realization of a graph all faces are bounded and there is no distinguished face. We denote by $F(G)$ the set ov

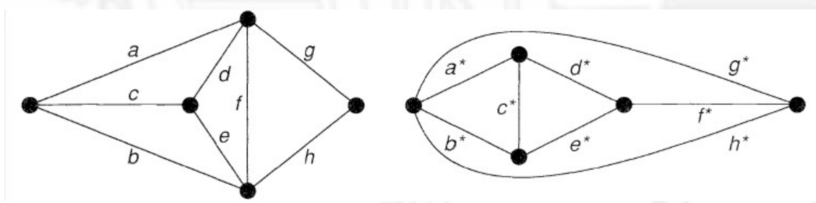
Definition. For a plane graph G consider a set of its faces and call two faces incident to each other if they share common edges. The corresponding graph G^* with $E(G^*) = E(G)$ and $V(G^*) = F(G)$ is called **dual** to G.

Actually G^* is also planar since one could choose an interior point in every face of G and join points in adjacent faces by an arcs crossing only their common edges (one arc for every common edge).

An edge which is incident to a vertex of degree one in a given graph become a loop in a dual graph. Actually, every bridge in a given graph produce a loop in a dual graph.

Remark. If G is connected multigraph then G^* is also connected plane multigraph. Moreover, $F(G^*) = E(G)$ and $(G^*)^* = G$.

Indeed for a vertex $v \in V(G)$ traversing incident vertices in a cyclic order we obtain a boundary of a face in G^* which corresponds to v.



Yet another pair of dual notions is that edge-removing is dual to edge-contraction.

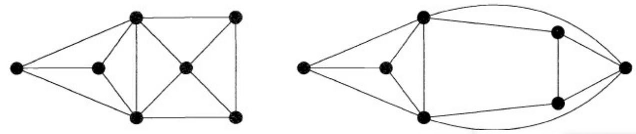
Proposition 5.1. *If e is not a bridge in a connected plane graph G then $(G \setminus e)^* \cong G^* * e^*$.*

Proof. Since e is not a bridge then it belongs to two different faces $a, b \in F(G)$. Corresponding vertices $a^*, b^* \in E(G^*)$ glue together when contracting the edge e^*. The union of faces a and b become a one big face in $G \setminus e$ whose sides are just former sides of a and b except e. $\square$

Proposition 5.2. *If C is a cycle in a planar graph G then the corresponding set of egdes in dual grap G^* forms a cutset and vice versa.*

Proof. Consider the set of faces in the interior of a given cycle. Any curve joining a point in the cycle interior to the outer point must cross at least on edge of a cycle. But for any path in G^* joining vertices corresponding to interior and outer faces of G one could easily construct such a curve. Therefore, C is a cut-set in G^*. $\square$

Exercise 5.1. *Prove that two plane graphs shown in the picture are isomorphic as abstract graphs but their duals are not isomorpic:*



Another example of a connection between properties of the graph and its dual is the following proposition.

Theorem 5.3. *Let G be an Eulerian plane graph. Then G^* is bipartite. The inverse implication is also true: if G^* bipartite then G is Eulerian.*

Proof. It is enough to prove that every cycle in G^* has even length. Consider a cycle C of length ℓ in G^* that is a sequence $f_0, f_1, \ldots, f_n = f_0$ of faces of G such that f_i and f_{i+1} share a common edge $e_i \in E(G)$. The family of arcs corresponding to $e_i^* \in E(G*)$ constitutes some closed curve. Consider the set S of all the vertices of G in the interior of this curve and find sum of their degrees that is an even number since G is Eulerian. The cycle C serves as boundary to the union of corresponding set S^* of faces in G^*. Therefore, $\sum\limits_{v \in V(G)} \deg_G(v)$ is equal to $\ell + 2r$ where r is the number of interior edges. Edges in the cycle correspond to edges in G joining vertices in S to the vertices not in S. Therefore they contribute only 1 to degree sum above.

The opposite implication is easy because every vertex in G corresponds to a face in G^* and vertex degree is equal to the number of edges in the corresponding face. But the boundary of any face in a plane graph is a cycle hence every face in G^* must have an even number of edges since G^* is bipartite. That means each vertex in G has an even degree.

□

Proposition 5.4. *Hamiltonian path in a plane graph (when it does exist) dissects a set of faces into two parts, each of them a connected subgraph in G^* without inner faces.*

Problem 5.5. *How many Hamiltonian cycles in cube graph Q_3?*

Euler formula

Theorem 5.6. *For any connected plane graph the numbers of vertices v, edges e and faces f satisfy the relation*

$$v - e + f = 2 \tag{6}$$

Proof. When G has no cycles then $f = 1$ and the formula becomes $v - e = 1$ which is a basic tree property. Then proceed by induction on f. Removing edge in the boundary of some faces decreases the number f by one and number e by one, hence does not change $v - e + f$. By induction hypothesis in a new graph $v - e + f = 2$. □

Exercise 5.2. *Given* 20 *points in the interior of a given square one draws somehow non intersection segments connecting theses points with each other ond four vertex of the square. It occurs that the square was dissected into triangles. Find the number of triangles.*

Corollary 5.7.
- *In any simple plane graph* $e \leqslant 3v - 6$ *and there exists a vertex of degree* 5 *or less.*
- *In any simple plane bipartite graph* $e \leqslant 2v - 4$.

Proof. Every face has at least three edges hence applying Handshaking lemma to the dual graph implies $2e = \sum\limits_{k \geqslant 3} k f_k \geqslant 3 \sum\limits_{k \geqslant 3} f_k = 3f$. Applying Euler formula $6 = 3v - 3e + 3f \geqslant 3v - e$. As for the second assertion denote by r_k the number of vertices of degree k. Then be Handshaking lemma $2e = \sum\limits_{x \in V(G)} \deg(v) \geqslant 6v$ if all the vertices have degree at least six. That contradicts to inequality $3v \geqslant e - 6$. If a degree for every vertex were at least 6 them sum of their degrees would have at least $6v$. hence $2e \geqslant 6v$ that contradicts inequality $e \leqslant 3v - 6$.

As for plane bipartite graph all its faces has even number of sides. Hence the minimal number is 4. Therefore, Handshaking lemma for a dual graph implies $2e \leqslant 4f$. Then $4 = 2v - 2e + 2f \geqslant 2v - 2e + e = 2v - e$. □

Corollary 5.8. *Graph* K_5 *and bipartite graph* $K_{3,3}$ *are not both planar.*

Proof. $e(K_5) = 2$ and $v(K_5) = 5$ that contradicts to inequality $e \leqslant 3v - 6$ for planar graphs.

Suppose that $K_{3,3}$ is planar. Then $e = 9$ and $v = 6$ contradicts inequality $e \leqslant 2v - 4$. □

Exercise 5.3. *Let* G *be a simple plane* 3*-regular graph. Prove that* $3f_3 + 2f_4 + f_5 - f_7 - 2f_8 - 3f_9 - \cdots = 12$ *where* f_k *is the number of* k*sided edges. Deduce that there exists a face with at most* 5 *sides.*

Exercise 5.4. *Let* G *ne a graph with* 11 *vertices. Prove that* G *and its complement* $\overline{G}$ *could not be both planar.*

Another simple illustration how one can apply Euler formula to the Hamiltonicity problem is Grinberg theorem.

Theorem 5.9. *Suppose that planar graph G admits a Hamilton cycle. Let f_k be a number of k-gonal faces in the interior of Hamilton cycle and g_k be a number of k-gonal faces in the outer region relative to the cycle. Then one has*

$$\sum_{k\geqslant 3}(k-2)(f_k-g_k)=0. \tag{7}$$

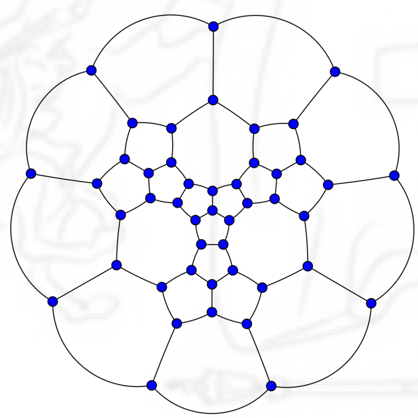
Proof. Consider interior subgraphs of the Hamilton cycle C subject to the dissection from **Proposition 5.4**. By Euler formula $v - e_1 + \sum_k f_k + 1 = 2$ where e_1 is the number of edges in this subgraph and $v = v(G)$. By counting edges in each face (that is applying Handshaking formula to the dual of interior subgraph) one has $\sum_k kf_k = 2e_1 - v$ since v is the number of edges in the unbounded face, Subtracting one equation from another in order to eliminate e_1 we obtain $\sum_k(k-2)f_k = v-2$. The same equality is true for the second component $\sum_k(k-2)g_k = v-2$. Subtracting again we obtain the required equality.

□

Corollary 5.10. *If all but one faces in a given plane graph have number of edges $k \equiv 2 \pmod 3$ then graph is not Hamiltonian.*

Proof. Under condition of the corollary lefthandside in (7) is not divisible by 3 since for the only $k_0 \not\equiv 2 \pmod 3$ one has $f_{k_0} - g_{k_0} = \pm 1$. □

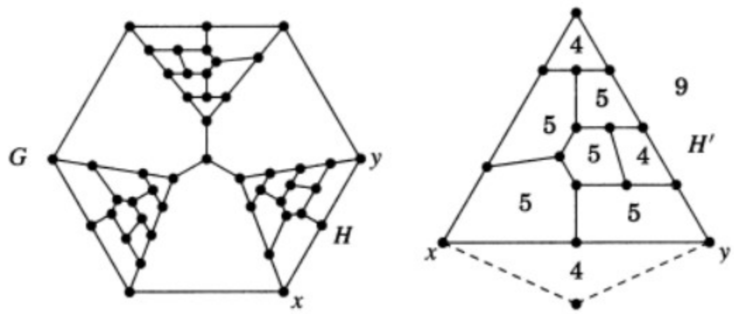
Example. Prove that the graph in the picture is non-Hamiltonian. Indeed, it has faces



with 5 and 8 edges and only an unbounded face has 9 edges.

Exercise 5.5. *Each item costs 2 points.*

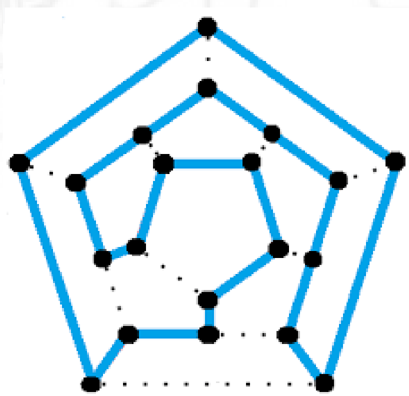
- *Consider a graph H' in the picture with 16 vertices and 9 faces. Prove that it is non-Hamiltonian,*
- *Consider graph G in the picture with 46 vertices which is constituted of three similar fragments H. Using the previous item prove that G is non-Hamiltonian.*



In 1884 Tait gave incorrect proof of 4-colour theorem taking for granted that any 3-regular polyhedral graph admits Hamiltonian cycle. In 1946 William Tutte gave an example of non-Hamiltonan 46-vertices polyhedral 3-regular graph which disproves Tait conjecture. Tutte example is given in **Exercise 5.5**. Nevertheless we can correctly state Tait result as follows.

Theorem 5.11. *For a given 3-regular Hamiltonian plane graph there exists a regular 4-colour face colouring. That is colouring the faces in four colours such that each edge belong to two differently coloured faces.*

Proof. By **Proposition 5.4** a Hamiltonian cycle dissects graph and its faces into two part . Use two colour for regular colouring one part and another two colour for regular colouring second part:



When we consider the set of faces S belonging to the one part then in dual graph G^* the induced subgraph on the vertex set $S^* \subset V(G^*)$ would be a tree. Since any tree is a bipartite graph then it admits regular vertex covering in 2-colours. Therefore one obtains a regular 2-colouring of faces in one part.

□

6 Graph colouring

Definition. Regular (or proper) vertex colouring of graph G is a map $c : V(G) \to A$ to some set A which is treated as a "set of colours" such that for any two adjacent vertices $u, w \in V(G)$ their colours are different: $c(u) \neq c(w)$.

Graph with a loop could never be properly coloured. For a proper vertex colouring it does not matter whether there are multiple edges joining some edges or not.

For a plane graph G one can consider proper face colouring. That is the same as proper vertex colouring of G^*.

Remark. Graph admitting proper 2-colouring is a bipartite graph.

Definition. If a a graph G could admits be properly colored with k colours then G is called k-colorable. The minimal number k such that G is k-colorable is called a **chromatic number** of the graph G and denoted by $\chi(G)$.

For example, chromatic number of any tree with at least 2 vertex is 2. Chromatic number of an odd cycle is 3.

Theorem 6.1. *Let G be an Eulerian plane graph such that its every face is the triangle. Then G is 3-colorable.*

Proof. Since G is Eulerian then G^* is bipartite by **Theorem 5.3** and 2-colourable. Therefore, there exists a face colouring for G with two colours: black and white.

For every black face choose clockwise orientation on its boundary and for every white face choose counterclockwise orientation on the boundary. Then every edge obtain an orientation. Let $\{0, 1, 2\}$ are the colours considered as $\mathbb{Z}/3\mathbb{Z}$ elements. Take arbitrary vertex v and color it with 0. Using orientation of edges one could extend colouring to the whole graph using the following idea. For any path $P = v_0 v_1 v_2 \ldots v_k$ consider its "sign-length" assigning $+1$ to the path edge (v_{i-1}, v_i) this edge agree with the previously chosen orientation and assigning -1 if it does not agree with the orientation. The sign-length $\ell_\pm(P)$ of the path P is just a sum of k summand corresponding to the edges each of them $= 1$ or -1.

Them for any vertex w we chose a path $v = v_0v_1 \ldots v_k = w$ and assign to w a colour which is equal to $\ell_\pm(P)\ mod\, 3$. One should prove that it is well defined rule that is does not depend on the choice of a path. It follows from the fact that for any closed path $\ell_\pm(C) \equiv 0 \pmod 3$ exactly in the same manner as in the proof that in the graph without odd-cycles all the paths joining two vertices have the same length-pairity.

The proof of the equation $\ell_\pm(C) \equiv 0 \pmod 3$ is easy. It is easily reduced to the case of simple closed path. but for the circle C its sigh-length $\ell_\pm(C)$ is equal to the sum of sign-lengths of all the triangles inside the cycle.

$\square$

6.1 Brooks theorem

Let $\Delta(G) = \max\limits_{v \in V(G)} \deg(v)$ is the maximal degree of a vertex in a simple graph G. Then it is easy to see that $\chi(G) \leqslant \Delta(G) + 1$. We can just start assigning colours to vertices one buy one in arbitrary order. The only condition to obtain a proper colouring is to assign such a colour to the current vertex which is not yet used for its neighbours. Since the number of neighbours is strictly less than the number of colours we always can choose a colour which is not presented in the adjacent vertices.

Exercise 6.1. *Let G be a graph such that the number of vertices whose degree is greater than $k-1$ is less or equal than k. Then $\chi(G) \leqslant k$.*

From other side $\Delta(K_n) = n - 1$ and $\chi(K_n) = n$. So the estimate $\chi(G) \leqslant \Delta(G) + 1$ is sharp for the case of complete graph. It occurs that excluding the case of complete graphs the estimate can be sharpened a bit, see **Theorem 6.4**.

Exercise 6.2. *Compute chromatic number for the Petersen graph.*

Problem 6.2. *Let G and $\overline{G}$ be two complementary graphs on n vertices. Then $\chi(G) \cdot \chi(\overline{G}) \geqslant n$.*

Proof. Consider the vertex set V, which is comon for both G and $\overline{G}$. Consider a coloring $c_1 : V \to A_1$ which is optomal proper colouring for G, hence $|A_1| = \chi(G)$. And consider optimal proper colouring $c_2 : V \to A_2$ with $|A_2| = \chi(\overline{G})$. Then we can construct new colouring $V \to A_1 \times A_2, \quad v \mapsto (c_1(v), c_2(v))$. Any two vertices $v \neq w$ either adjacent in G and in this case $c_1(v) \neq c_1(w)$ or they are adjacent in $\overline{G}$ and $c_2(v) \neq c_2(w)$. Hence pairs $(c_1(v), c_2(v))$ and $(c_1(w), c_2(w))$ are always distinct and we obtain a proper colouring of the complete graph K_n with colour set $A_1 \times A_2$ of $\chi(G) \cdot \chi(\overline{G})$ colours. $\square$

Lemma 6.3. *Let G be a graph with blocks $B_1, B_2, \dots, B_r$. Then $\chi(G) = \max\limits_{i=1}^{r} \chi(B_i)$.*

Proof. Induction by a number of blocks. Let B is a leaf block with an articulation point p and $G' = G \setminus (B \setminus p)$ is a union of all other blocks. We claim that $\chi(G) = \max\{\chi(B), \chi(G')\}$. Suppose that $\chi(B), \chi(G') \leqslant k$. We want to find a proper colouring if G with k colours. Indeed, $V(G') \cap V(B)\{p\}$. switching colours if necessary we can choose proper colouring of B and of G' with k colours such that the colour of p is the same in both colourings. Combining them we obtain a required proper colouring of G. □

Theorem 6.4. *(Brooks) Let G be a connected graph which is non-complete. Then $\chi(G) \leqslant \Delta(G)$.*

Proof. By **Lemma 6.3** we can reduce theorem to the case when G is biconnected. When number of blocks is at least two then even in the case when some of the blocks with k vertices is the complete graph the degree of an articulation point belonging to this block is at least $(k-1)+1 \leqslant \Delta$ and block easily k-colorable, hence Δ-colorable.

For the case of biconnected graph We shall use **Lemma 6.4**. Choose vertices u, v, w such that shortest path from u to v is uwv and $G \setminus \{u, v\}$ is connected. Put all vertices of G in a sequence starting with $a_1 = u, a_2 = v$ and ending with $w = a_n$ in the following way. Let T be a spanning tree of $G \setminus \{u, v\}$. Denote by a_3 some vertex of degree 1 in T which is distinct from w. Then choose vertex of degree one in a tree $T \setminus a_3$ which is distinct from w and so on. For any $k = 3, \dots, n-2$ we denote by a_{k+1} the vertex of degree one in a tree $T \setminus \{a_3, \dots, a_k\}$ which is distinct from w.

Then one starts colouring process. Let us colour u and v with the same colour and then colour a_3, a_4 and so on one by one. For any k new vertex a_{k+1} is adjacent to some vertex in $T \setminus \{a_3, \dots, a_k\}$ which is not coloured yet, hence the numver of already colored neighbours of a_{k+1} in G is strictly less than Δ. So we can choose a colour for a_{k+1} such that all the same-colour vertices that are already colored are not adjacent to each other.

At the end we need to choose colour for the vertex w. It can have Δ neighbours which are all already colored. But at least to of them — u and v were coloured with the same colour. Hence the number of colours used for the neihbours of w is strictly less than Δ again. □

6.2 Edge colouring

Definition. Function $c : E(G) \to A$ is called a proper edge colouring of a simple graph G if for any $v \in V(G)$ all the edge incident to v have distinct colours.

The minimal k such that there exists a proper edge colouring with k colours is called chromatic index of G and denoted by $\chi'(G)$. Obviously, $\chi'(G) \geqslant \Delta(g)$.

If C_{2k+1} is an add cycle then $\chi'(G) = 3$.

Proposition 6.5. *Let n be an odd number. Then $\chi'(K_n) = n$.*

Proof. Let us construct a proper edge n-coloring of the graph K_n. Drawing it as a regular n-gon with all diagonals use colours $1, 2, 3 \ldots, n$ for colouring all n sides and for every diagonal we use the same colour that we used for the unique side which is parallel to this diagonal. In this colouring every two non-parallel edges would have different colours. Hence that is a proper edge colouring.

Prove by contradiction that there is no proper edge colouring with $n-1$ colours. If not then there exists at least $\dfrac{n}{2}$ having the same colour. Since $\dfrac{n}{2}$ is non-integer then actually there exists at least $\dfrac{n+1}{2}$ edges coloured with the same colour. But the number of their ends $2 \cdot \dfrac{n+1}{2} = n+1$ is greater than the number of vertices. Therefore at least to of them have to share common end and the colouring is improper. $\square$

Exercise 6.3. *Prove that $\chi'(K_{2m}) = 2m - 1$.*

Theorem 6.6. *Let G be a bipartite graph. Then $\chi'(G) = \Delta(G)$.*

Proof. We start colouring the edges one by one. Let A where $|A| = \Delta$ is the set of colours. Choosing not yet coloured edge $\{u, v\}$ consider a set of colours $A(u)$ represented in the vertex u that is the set of colours which are already used to colour edges incident to u. If $A(u) \cup A(v) \neq A$ then we can choose a colour which is not represented both in u and v and colour $\{u, v\}$ with this colour. What shall we do if $A(u) \cup A(v) = A$?. Obviously, $|A(u)| < \Delta$ since there is not yet coloured edge incident to u and we can choose a colour $\alpha \notin A(u)$ and can choose $\beta \notin A(v)$.

The idea is to reassign colours to some edges in such a way that β would not be represented in u in modified colouring.

Consider a graph $H_{\alpha\beta} \subset G$ which has only those edges which are colored with α and *beta*. Then the degree of every vertex in $H_{\alpha\beta}$ is not greater than 2. Moreover, $\deg_{H_{\alpha\beta}}(u) = 1 = \deg_{H_{\alpha\beta}}(v)$ as α is not represented at the vertex u and β is not represented at v. Connected component of $H_{\alpha\beta}$ containing u is just a path with alternatively coloured edges $\beta, \alpha, \beta, \dots$. This path can not be ended at v due to pairity. In this case it would have odd length and the last edge would be colored with β which is not represented in v. Take this path and switch colours α and β only for the edges in this path. New partial edge-colouring would be proper and has the advantage that β is not represented in u as it remains to be not represnted in v.

So we can choose β to colour edge $\{u, v\}$. □

Five colour theorem

Four colour theorem asserts that every planar Map is 4-colorable, that is for any plane connected graph without bridges there exists proper face colouring with four colours. It was stated as a conjecture in 1852 and was not correctly proved until 1976 with several false proof meanwhile.

Recall that proper face colouring is the same as proper vertex colouring of the dual graph. Hence in the following we will speak about vertex colouring.

Theorem 6.7. *Any plane simple graph admits proper vertex colouring with* 5 *colours.*

Proof. Consider a minimal counterexample that is a plane graph G after removing any vertex $v \in V(G)$ resulting graph $G \setminus v$ ais 5-colurable.

By **Corollary 5.7** there exists a vertex v of degree 5 of less. If $\deg_G(v) \leqslant 4$ then we can easily choose a colour for v_0 different from the colours of its neihgbours.

Supppose that the neighbours v_1, v_2, v_3, v_4, v_5 of v_0 are coloured in the colours $1, 2, 3, 4, 5$ in clockwise order. We want to rearrange colours in the same way as in the proof of **Theorem 6.6** such that one of five colours would be duplicated among the neighbours of v.

Consider an induced subgraph G_{13} with the vertices whose colour is 1 and 3. If v_1 and v_3 belongs to the different connected components of G_{13} then we can switch the colours $1 \leftrightarrow 3$ only in the component component containing v_1. It does not affect the properness of the colouring. Then the colours of v_1 would be 3 and we can colour v_0 with the first colour.

If v_1 and v_3 belongs to the same connected component of G_{13} then they connected by a some path $v_1 = u_1 u_2 u_3 \dots u_k = v_3$ whose all vertices coloured with 1 and 3 colours. Then v_2 and v_4 lies in the different connected components of the graph G_{24} as they belongs to two different regions of the plane dissected by the cycle $v_0 v_1 u_2 \dots v_3 v_0$. In this case we can switch colours $2 \leftrightarrow 4$ in the connected component of G_{24} containing v_2 and consequently colour v_0 with colour 2. □

Chromatic polynomial

Definition. For a given graph G define a function $\chi_G : \mathbb{N} \to \mathbb{Z}$ by the rule $\chi_G(k)$ is equal to the number of proper vertex colouring of G with k-colours.

Obviously, $\chi_G(1) = 0$ if G has at least one edge. For graph G with n vertices and no edges $\chi_G(k) = k^n$. $\chi_{K_n}(k) = n(n-1)(n-2)\cdots(n-k+1)$.

We will prove that chromatic function χ_G is actually a polynomial function, that is there exists a polynomial $\chi_G(X)$ with integer coefficients such that $\chi_G(k)$ is the number of proper k-colouring of the graph G.

Proposition 6.8. *For any edge $e \in G$ in a simple graph G*

$$\chi_G(k) = \chi_{G\setminus e}(k) - \chi_{G/e}(k). \tag{8}$$

Proof. Let $e = \{u, v\}$. Consider a proper vertex colouring of $G \setminus e$. If u and v are coloured with the same colour then we obtaing of proper vertex colouring of the graph with contracted edge G/e. If u and v are coloured with distinct colours then we obtain a proper vertex colouring of the given graph G. □

Corollary 6.9. *Chromatic function is indeed a polynomial. It is a polynomial with unit leading coefficient and degree is equal to the number of vertices.*

Proof. Let G be a graph with n vertices. We proceed by induction on a number of edges. If there is no edges then $\chi_G(X) = X^n$. Induction step. Since $G \setminus e$ and G/e has smaller number of edges one can apply induction hypothesis to them. So their chromatic functions are polynomial functions. Moreover $\chi_{G/e}(X)$ is a polynomial of degree $n-1$. Hence $\chi_{G\setminus e}(X) - \chi_{G/e}(X)$ is a polynomial of degree n with a unit leading coefficient. By equation (8) the corresponding polynomial function is the chromatic function χ_G. □

Remark. One could define chromatic polynomial by recursion $\chi_G(X) = \chi_{G\setminus e}(X) - \chi_{G/e}(X)$ but in this case it is necessary to prove somehow that the resilt does not depend of the way how we are deleting edges.

Remark. Cromatic polynomial of disconnected graph is equal to the product of the chromatic polynomials of its connected components.

Proposition 6.10. *Let G is a connected graph with blocks $B_1, B_2, \ldots, B_r$. Then* $\chi_G(X) = \dfrac{\prod_{i=1}^{r} \chi_{B_i}(X)}{X^{r-1}}$.

Proof. Induction on r. Consider a leaf block, say B_1 with an articulation point v. Then $G' = G \setminus (B_1 \setminus v)$ is a connected graph with blocks $B_2, \ldots, B_r$ and we can apply induction hypothesis to it.

Proper colourings of G bijectively correspond to pairs of proper colouring of B_1 and G' such that the colours of $v = B_1 \cap G'$ are the same in both colourings in the pair. So to properly colour graph G one can properly colour G' and choose a proper colouring for B_1 with already fixed for the vertex v.

The number of proper coloring of B_1 where v is coloured with one fixed colour is equal to the number of proper coloring of B_1 where v is coloured with another fixed colour. Therefore the number of proper k-colouring of B_1 for which the colour of v is fixed is equal $\frac{1}{k}\chi_{B_1}(k)$.

Therefore, using induction hypothesis.

$$\chi_G(k) = \frac{1}{k}\chi_{B_1}(k)\chi'_G(k) = \frac{1}{k}\chi_{B_1}(k)\frac{1}{k^{r-2}}\prod_{i=2}^{r}\chi_{B_i}(k) = \frac{1}{k^{r-1}}\prod_{i=1}^{r}\chi_{B_i}(k).$$

□

Corollary 6.11. *Cromatic polynomial of any tree with n vertices is equal to $X(X-1)^{n-1}$.*