1.4. Reduction of operators of the 2nd order with constant coefficients to the canonical form

Consider a 2nd-order operator with constant coefficients of the main part:

$$A = \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2}{\partial x_i \partial x_k} + \sum_{l=1}^{n} b_l \frac{\partial}{\partial x_l} + c, \qquad (1.11)$$

where a_{ik} are real constants, $a_{ik} = a_{ki}$ (because $\frac{\partial^2}{\partial x_i \partial x_k} = \frac{\partial^2}{\partial x_k \partial x_i}$).

We will reduce the upper part of the operator A

$$A_0 = \sum_{i,k=1}^{n} a_{ik} \frac{\partial^2}{\partial x_i \partial x_k} \qquad (1.12)$$

to a simpler form using linear variable replacement:

$$y_k = \sum_{l=1}^{n} c_{kl} x_l, \qquad (1.13)$$

where c_{kl} are real constants.

Let's consider a quadratic form:

$$Q(\xi) = \sum_{i,k=1}^{n} a_{ik} \xi_i \xi_k, \qquad (1.14)$$

that differs only by a sign from the main symbol of the operator A.

By linear substitution of variables $\eta = F\xi$, where F is a invertible constant matrix, the shape $Q(\xi)$ can be reduced to the sum of squares:

$$Q(\xi) = (\pm\eta_1^2 \pm \eta_2^2 \pm \ldots \pm \eta_r^2)\big|_{\eta=F\xi}. \tag{1.15}$$

Denote by C the replacement matrix $(c_{kl})_{k,l=1}^{n}$ (1.13).

According to Theorem 1.3. in coordinates y, the operator A will have the form of a 2nd-order operator A_1 with such a quadratic form $Q_1(\eta)$ that

$$Q(\xi) = Q_1(\eta)\big|_{\eta=({}^tC)^{-1}\xi},$$

where ${}^tC$ is a matrix transposed to C.

From here and from (1.15) it is clear that we must choose the matrix C so that it is $({}^tC)^{-1} = F$ or

$$C = ({}^tF)^{-1}. \tag{1.16}$$

Then the main part of the operator A, when replacing variables $y = Cx$ of the form (1.13), is reduced to the form

$$\pm\frac{\partial^2}{\partial y_1^2} \pm \frac{\partial^2}{\partial y_2^2} \pm \ldots \pm \frac{\partial^2}{\partial y_r^2}, \tag{1.17}$$

called **canonical**.

1.5. Characteristics. Ellipticity and hyperbolicity

Let A be a differential operator of order m, and $a_m(x, \xi)$ is its main symbol.

A nonzero covector (x, ξ) is called a characteristic, if $a_m(x, \xi) = 0$.

A surface in Ω is called *characteristic* at point x_0 if its normal at this point is a characteristic vector. It is called a characteristic if it is characteristic at each point.

If the surface S is given by the equation $\varphi(x) = 0$, where $\varphi \in C^1(\Omega)$, $\varphi_x|_S \neq 0$, then its characteristic at point x_0 means that $a_m(x_0, \varphi_x(x_0)) = 0$.

It is a characteristic if $a_m(x, \varphi_x(x))|_S \equiv 0$.

All surfaces of level $\varphi = \text{const}$ are characteristics if and only if $a_m(x, \varphi_x(x)) \equiv 0$.

It follows from Theorem 1.3 that the concept of characterization and characteristics does not depend on the choice of coordinates in Ω.

Examples.

1. The Laplace operator Δ has no real characteristic vectors.
2. For the thermal conductivity operator $\frac{\partial}{\partial t} - \Delta$, the characteristic vector is $(\tau, \xi) = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.

Surfaces of type $t = \text{const}$ are characteristics.

The surface $t = |x|^2$ (paraboloid) is characteristic at one point (origin).

3. Consider the wave operator $\frac{\partial^2}{\partial t^2} - \Delta.$ Its characteristic vectors at each point (t, x) form a cone $\tau^2 = |\xi|^2.$

Any cone $(t - t_0)^2 = |x - x_0|^2$ is a characteristic.

In particular, for $n = 1$ (that is, $x \in \mathbb{R}^1$), the characteristics are straight lines of the form $x + t = \text{const}$ and $x - t = \text{const}.$

Definition.

1. An operator A is called *elliptic* if $a_m(x, \xi) \neq 0$ at $x \in \Omega, \xi \neq 0$, that is, if A has no real characteristic vectors.

2. The operator A in the space $t, x, t \in \mathbb{R}^1$ is called *hyperbolic* with respect to $t,$ if the equation $a_m(t, x, \tau, \xi) = 0,$ considered as an equation with respect to $\tau,$ for any fixed t, x, ξ at $\xi \neq 0$ has exactly m real and distinct roots.

Examples.

1. The Laplace operator Δ is elliptical.

2. The thermal conductivity operator is neither elliptic nor hyperbolic with respect to $t.$

3. The wave operator is hyperbolic with respect to $t,$ since the equation $\tau^2 = |\xi|^2$ at $\xi \neq 0$ has two real and distinct roots $\tau = \pm|\xi|.$

4. The Sturm-Liouville operator $Lu \equiv \frac{d}{dx}\left(p(x)\frac{d}{dx}u\right) + q(x)u$ is elliptical on $(a, b),$ if $p(x) \neq 0$ at $x \in (a, b).$

1.6. Characteristics and reduction to the canonical form of operators and equations of the 2nd order at n=2

At $n = 2$, the characteristics are lines and are simple. Consider the 2nd order operator:

$$A = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + \dots, \qquad (1.18)$$

where a, b, c — are smooth functions of $x, y,$ defined in some domain $\Omega \subset \mathbb{R}^2$. (Three dots mean terms containing only first-order derivatives.)

Let $(x(t), y(t))$ be a line in $\Omega,$ (dx, dy) be its tangent vector, and $(-dy, dx)$ be the normal vector. A line is a characteristic if and only if when along it

$$a(x, y)\,dy^2 - 2b(x, y)\,dx\,dy + c(x, y)\,dx^2 = 0. \qquad (1.19)$$

If $a(x, y) \neq 0$, then in the vicinity of point (x, y) we assume that $dx \neq 0$ and that x is a parameter along the characteristic $y = y(x)$.

Then the equation of the characteristic takes the form

$$ay'^2 - 2by' + c = 0.$$

If $b^2 - ac > 0$, then the operator **(1.18)** is called *hyperbolic* and has 2 families of real characteristics found from ordinary differential equations

$$y' = \frac{b + \sqrt{b^2 - ac}}{a}, \tag{1.20}$$

$$y' = \frac{b - \sqrt{b^2 - ac}}{a}. \tag{1.20'}$$

Note that two nonintersecting characteristics pass through each point $(x, y) \in \Omega$ in this case.

Let's write these families of characteristics in the form $\varphi_1(x, y) = C_1$ and $\varphi_2(x, y) = C_2$, where $\varphi_1, \varphi_2 \in C^\infty(\Omega)$.

Thus, φ_1, φ_2 are the first integrals of equations (1.20) and (1.20'), respectively.

Let's assume that $\operatorname{grad}\varphi_1 \neq 0$ and $\operatorname{grad}\varphi_2 \neq 0$ in Ω.

Then $\operatorname{grad}\varphi_1$ and $\operatorname{grad}\varphi_2$ are linearly independent, since the characteristics from different families are not tangent.

Let's enter the new coordinates $\xi = \varphi_1(x, y),\ \eta = \varphi_2(x, y)$.

In them, the characteristics will be the lines $\xi = \text{const}$ and $\eta = \text{const}$,

but then the coefficients at $\frac{\partial^2}{\partial \xi^2}$ and $\frac{\partial^2}{\partial \eta^2}$ will identically turn to 0, so that the operator A will take the form

$$A = p(\xi, \eta)\frac{\partial^2}{\partial \xi \partial \eta} + \dots, \tag{1.21}$$

called canonical.

Here $p(\xi, \eta) \neq 0$. Similarly, the reduction to the canonical form (1.21) is done in the case when $c(x, y) \neq 0$.

Differential equations of the form

$$Au = f, \tag{1.22}$$

are often considered, where f — is a known function, A — is a linear differential operator, and u — is an unknown function.

If A — is a hyperbolic operator of the 2nd order with two independent variables (that is, an operator of the form (1.18), where $b^2 - ac > 0$), then after introducing the coordinates ξ, η described above and dividing by $p(\xi, \eta)$, equation (1.22) (in this case, the equation is called *hyperbolic*) is reduced to the canonical form

$$\frac{\partial^2 u}{\partial \xi \, \partial \eta} + \ldots = 0, \tag{1.23}$$

(Three dots mean the terms of the equation that do not contain the second derivatives of the function u.)

Now let $b^2 - ac \equiv 0$ (then operator (1.18) and equation (1.22) with this operator are called *parabolic*). Let's assume that $a \neq 0.$ Then the differential equation for the characteristics

$$y' = \frac{b}{a}. \tag{1.24}$$

is obtained.

Let's find the characteristics and write them in the form $\varphi(x, y) = \text{const}$, where φ — is the first integral (1.24), and $\operatorname{grad} \varphi \neq 0.$

Let's choose a function $\psi \in C^\infty(\Omega)$ such that $\operatorname{grad}\varphi$ and $\operatorname{grad}\psi$ are linearly independent, and introduce new coordinates $\xi = \varphi(x, y), \eta = \psi(x, y)$.

In the new coordinates, the operator A will not have a member $\frac{\partial^2}{\partial \xi^2}$, since the lines $\varphi = \mathbf{const}$ are characteristics.

But then the member $\frac{\partial^2}{\partial \xi \partial \eta}$ will also disappear, since the main character must be a quadratic form of rank **1.**

So, we get the canonical form of the parabolic operator

$$A = p(\xi, \eta)\frac{\partial^2}{\partial \eta^2} + \dots \qquad \mathbf{(1.25)}$$

For the parabolic equation (1.22), the canonical form will be

$$\frac{\partial^2}{\partial \eta^2} + \dots = 0. \qquad \mathbf{(1.26)}$$

Comment.

Note that if $b^2 - ac = 0,$ but $a^2 + b^2 + c^2 \neq 0,$ then a and c cannot turn to ***0*** at the same time, because then there will be and $b = 0.$ Therefore, it is always either $a \neq 0$ or $c \neq 0$ and the described procedure is always applicable.

Consider the case $b^2 - ac < 0,$ that is, the operator (1.18) is *elliptical*; equation (1.22) in this case is also called elliptical.

Let's assume for simplicity that the functions a, b, c are real analytic.

Then (from the theorem of the existence of holomorphic solutions to a complex equation)

$$y' = \frac{b + \sqrt{b^2 - ac}}{a}$$

it is possible to deduce the existence of a local first integral

$$\varphi(x, y) + i\psi(x, y) = C,$$

where φ, ψ — are real-valued analytical functions, $\operatorname{grad}\varphi, \operatorname{grad}\psi$ are linearly independent.

After the introduction of new coordinates $\xi = \varphi(x, y), \eta = \psi(x, y)$, the operator A is reduced to the canonical form

$$A = p(\xi, \eta)\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) + \dots, \tag{1.27}$$

where $p(\xi, \eta) \neq 0$.

1.7. General solution of a homogeneous hyperbolic equation with constant coefficients at n=2

As follows from the above, the hyperbolic equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x\,\partial y} + c\frac{\partial^2 u}{\partial y^2} = 0, \qquad (1.29)$$

where $a, b, c \in \mathbb{R}, b^2 - ac > 0,$ is reduced by replacing the variables $\xi = y - \lambda_1 x,\ \eta = y - \lambda_2 x,$ where λ_1, λ_2 — are the roots of the quadratic equation $a\lambda^2 - 2b\lambda + c = 0,$ to the form

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0. \qquad (1.30)$$

Assuming that $u \in C^2(\Omega),$ where Ω — is a convex region in $\mathbb{R}^2$, we get that

$$\frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right) = 0,$$

from where $\frac{\partial u}{\partial \eta} = F(\eta)$ and then

$$u = f(\xi) + g(\eta), \qquad (1.31)$$

where f, g — are arbitrary functions of class C^2. In variables x, y, then we have

$$u(x, y) = f(y - \lambda_1 x) + g(y - \lambda_2 x). \qquad (1.32)$$

It is useful to consider functions $u(x, y)$ of the form (1.32), where f, g are not necessarily of class C^2, but of a wider class of functions. Such functions are called *generalized solutions of equation* (1.29).

Note that the lines $y - \lambda_1 x = \text{const}, y - \lambda_2 x = \text{const}$ are characteristics. Thus, the discontinuities of solutions in this case extend along the characteristics. This is also the case for general hyperbolic equations.