

Equations of Mathematical Physics Homework 8

Task 1

Find solutions of the equation of free vibration of a string with initial condition:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = x^2, \quad \frac{\partial u(x, 0)}{\partial t} = 1; \end{cases}$$

Solution:

Apply the D'Alembert's formula, we know that the solution has the form:

$$u(x, t) = \frac{\varphi(x + at) + \varphi(x - at)}{2} + \frac{1}{2a} \int_{x-at}^{x+at} \psi(y) dy$$

in this problem, we have $\varphi(x) = x^2$ and $\psi(x) = 1$, thus,

$$\begin{aligned} \frac{\varphi(x + at) + \varphi(x - at)}{2} &= \frac{(x + at)^2 + (x - at)^2}{2} = x^2 + a^2 t^2 \\ \int_{x-at}^{x+at} \psi(y) dy &= \frac{1}{2a} \cdot y \Big|_{x-at}^{x+at} = t \end{aligned}$$

the solution is $u(x, t) = x^2 + a^2 t^2 + t$

Task 2

Find solutions of the equation with initial and boundary condition:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = 1, \quad \frac{\partial u(x, 0)}{\partial t} = 0, \\ u(0, t) = 0, \quad u\left(\frac{\pi}{2}, t\right) = 0; \end{cases}$$

Solution:

By the Fourier method, the solution of the equation has the form:

$$u(x, t) = \sum_{k=1}^{\infty} u_k(x, t) = \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{ak\pi}{l}t\right) + b_k \sin\left(\frac{ak\pi}{l}t\right) \right) \sin\left(\frac{k\pi}{l}x\right).$$

the only things we need to do is compute the Fourier coefficients a_k and b_k ,

$$\begin{aligned} a_k &= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin \frac{k\pi}{2} x \, dx = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin(2kx) \, dx \\ &= \frac{2}{k\pi} (1 - (-1)^k) = \begin{cases} \frac{4}{k\pi} & k = 2n + 1 \\ 0 & k = 2n \end{cases} \\ b_k &= 0 \end{aligned}$$

the solution is $u(x, t) = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \cos(4k+2)t \cdot \sin(4k+2)x$, where $0 < x < \frac{\pi}{2}, t > 0$