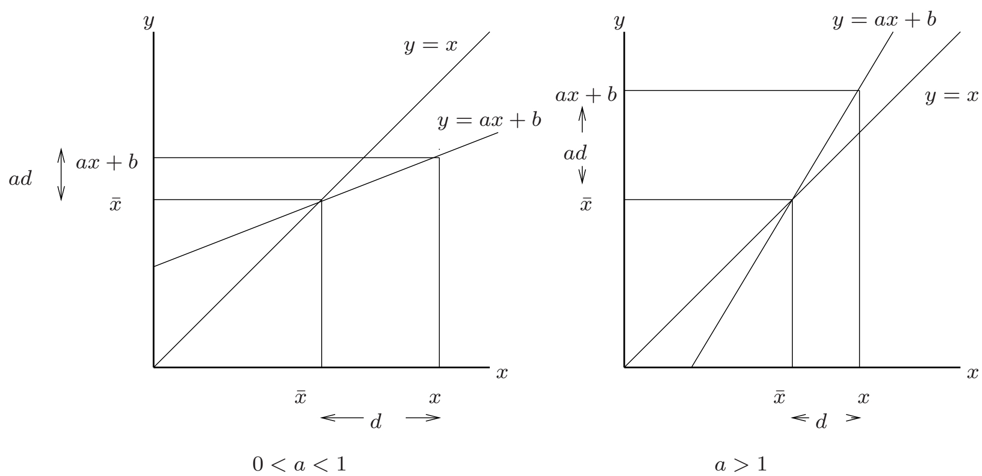
Lecture 4

Determining the nature of fixed points of nonlinear mappings

From the previous lecture:

We have examined the nature of fixed points for linear (degree one polynomial) mappings $f(x) = ax+b$ on several occasions. For $a \neq 1$, $\bar{x} = \dfrac{b}{1-a}$ is the unique fixed point of f, i.e. $f(\bar{x}) = \bar{x}$. If $|a| < 1$, $\bar{x}$ is attractive; if $|a| > 1$, $\bar{x}$ is repulsive. When $a = -1$, $\bar{x}$ is neither attractive nor repulsive and is referred to as *neutral* or *indifferent*. The attractivity of repulsivity of $\bar{x}$ as determined by the multiplier "a" can be viewed geometrically, as we show below for the case $a > 0$:



In each of the above cases, pick an $x \neq \bar{x}$ and let d denote the distance between $\bar{x}$ and x, i.e. $d = |x - \bar{x}|$. Then the distance between $f(x) = ax + b$ and $f(\bar{x}) = \bar{x} = a\bar{x} + b$ is

$$\begin{aligned} |f(x) - \bar{x}| &= |ax + b - a\bar{x} - b| \\ &= |a|\,|x - \bar{x}| \\ &= |a| d. \end{aligned} \tag{1}$$

If $|a| < 1$, then $f(x)$ is closer to $\bar{x}$ than x is. If $|a| > 1$, then $f(x)$ is farther from $\bar{x}$ than x is. This is a simple consequence of the slope a of the line $y = ax + b$. If we now repeat the procedure by replacing x in (1) with $f(x)$, we have

$$\begin{aligned} |f^2(x) - \bar{x}| &= |a|\,|f(x) - \bar{x}| \\ &= |a|^2\,|x - \bar{x}|\,. \end{aligned} \tag{2}$$

Now repeat over an over again to obtain, in general,

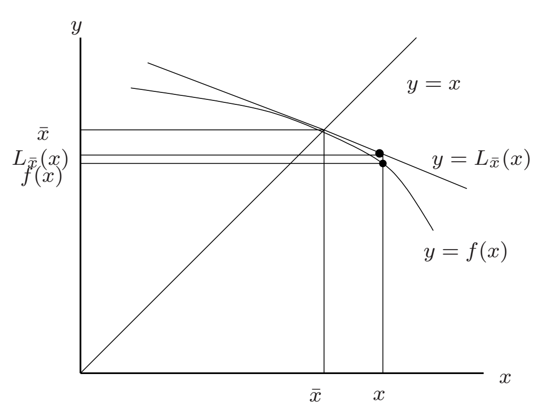
$$|f^n(x) - \bar{x}| = |a|^n \, |x - \bar{x}| \, , \quad n \geq 1 \, . \tag{3}$$

If $|a| < 1$, then $|f^n(x) - \bar{x}| \to 0$ as $n \to \infty$, implying that the points $x_n = f^n(x)$ converge to $\bar{x}$ as $n \to \infty$. If $|a| > 1$, then $|f^n(x) - \bar{x}| \to \infty$ as $n \to \infty$. The equalities in (1)-(3) are a consequence of the fact that the graph of $f(x) = ax + b$ is a **straight line** with constant slope a for all $x \in \mathbb{R}$.

Today's lecture:

Of course, the graphs of nonlinear functions $f(x)$, e.g. $f(x) = x^2 - 1$, are **not** straight lines. Nevertheless, the examples studied in the previous section suggest that we may be able to consider the graph of f as behaving roughly like a straight line in a neighbourhood of a fixed point. This, of course, has the aroma of considering the linear approximation to $f(x)$ at a fixed point $\bar{x}$.

For example, in the case $f(x) = x^2$ (Example 5 of the previous section) $\bar{x}_1 = 0$ is attractive and $\bar{x}_2 = 1$ is repulsive. Note that $f'(\bar{x}_1) = 0$ and $f'(x_2) = 2$. In Example 7, $f(x) = \sqrt{x}$, $\bar{x}_1 = 0$ is repulsive, with $f'(0) = +\infty$, and $\bar{x}_2 = 1$ is attractive, with $f'(\bar{x}_2) = \frac{1}{2}$. We may then conjecture that a linear approximation to $f(x)$ at a fixed point $\bar{x}$ may provide geometric pictures analogous to those sketched above for linear maps.



The linear approximation to $f(x)$ at $x = \bar{x}$ is given by

$$\begin{aligned} f(x) \cong L_{\bar{x}}(x) &= f(\bar{x}) + f'(\bar{x})(x - \bar{x}) \\ &= \bar{x} + f'(\bar{x})(x - \bar{x}) \end{aligned} \tag{4}$$

for x sufficiently near to $\bar{x}$. Then,

$$\begin{aligned} |f(x) - \bar{x}| &\cong |L_{\bar{x}}(x) - \bar{x}| \\ &= |f'(\bar{x})|\,|x - \bar{x}|. \end{aligned} \tag{5}$$

We see, as expected, that $|f'(\bar{x})|$ plays the role of $|a|$ for linear maps. The only problem is the occurrence of "$\cong$" in the equation above, due to the use of the linear approximation: We cannot conclude that $|f(x) - \bar{x}| < |x - \bar{x}|$ if $|f'(\bar{x})| < 1$ since we have no idea of "how good" the approximation "$\cong$" is. Fortunately, we can bypass this difficulty with the use of the famous Mean Value Theorem from first-year Calculus, which we review below.

Theorem 3.1 (Mean Value Theorem): Let f be differentiable on $[a, b]$. Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{6}$$

The Mean Value Theorem does not, in general, tell us where the point c is located: It only guarantees the existence of at least one such c.

We are now in a position to prove an important result relating the nature of a fixed point $\bar{x}$ of a function f to its derivative $f'(\bar{x})$. In what follows, we assume that f is C^1 on its domain of definition, generally an interval $[a, b] \subset \mathbb{R}$. The statement "$f \in C^1[a, b]$" means that $f'(x)$ is continuous on the interval $[a, b]$.

Theorem 3.2: Let $f \in C^1[a, b]$. Suppose that $\bar{x} \in (a, b)$ is a fixed point of f such that $|f'(\bar{x})| < 1$. Then there exists an open interval $I \subset [a, b]$ containing $\bar{x}$ such that for any $x \in I$,

$$\lim_{n \to \infty} f^n(x) = \bar{x}. \tag{7}$$

In other words, $\bar{x}$ is an **attractive fixed point** and the interval I belongs to the **basin of attraction** of $\bar{x}$.

Note: The result involving (7) can be written in another way (as done in class): For any $x_0 \in I$, the iteration sequence $x_{n+1} = f(x_n)$, $n \geq 0$, behaves as follows:

$$\lim_{n \to \infty} x_n = \bar{x}. \tag{8}$$

Proof: Since $f'(x)$ is assumed to be continuous, the fact that $|f'(\bar{x})| < 1$ implies that there exists an open interval J containing $\bar{x}$ and a constant $0 \leq K < 1$ such that

$$|f'(x)| \leq K < 1 \quad \text{for all} \quad x \in J \tag{9}$$

> **Aside for clarification:** $|f'(\bar{x})| < 1$ means that $-1 < f'(\bar{x}) < 1$. Suppose that $f(\bar{x}) = 1 - \epsilon$ where $\epsilon > 0$ is very small. The fact that $f'(x)$ is continuous implies that as we move away from $x = \bar{x}$, the value of $f'(x)$ cannot instantaneously jump from the value $1 - \epsilon < 1$ to a value greater than 1. Even if it were to increase to an eventual value greater than one at, say, $x_1 \neq \bar{x}$, it would have to assume all intermediate values between $1 - \epsilon$ and 1 before getting to $x = x_1$. This is a consequence of the Intermediate Value Theorem applied to the continuous function $f'(x)$. The same argument holds for $f(\bar{x}) = -1 + \epsilon$, $\epsilon > 0$.
>
> In fact, we could have shortened the above discussion by noting that the function $g(x) = |f'(x)|$ is continuous since the absolute value function $|x|$ is continuous. Since $g(\bar{x}) < 1$, it follows that $g(x)$ cannot instantaneously jump from the value $g(\bar{x}) < 1$ to a value greater than 1.

Let $I \subset J$ be an open interval centered around the point $\bar{x}$, i.e. $I = (\bar{x} - \delta, \bar{x} + \delta)$ for some $\delta > 0$. Now choose any $x \in I$, $x \neq \bar{x}$. We now wish to compare the distance $|f(x) - \bar{x}|$ to the distance $|x - \bar{x}|$. From the Mean Value Theorem, with $a = \bar{x}$ and $b = x$:

$$\frac{f(x) - f(\bar{x})}{x - \bar{x}} = f'(c) \tag{10}$$

for some c between x and $\bar{x}$. Note that this implies that $c \in I$. We rearrange (10), substituting $f(\bar{x}) = \bar{x}$:

$$f(x) - \bar{x} = f'(c)(x - \bar{x}). \tag{11}$$

Now take absolute values of both sides and use (9) to obtain

$$\begin{aligned} |f(x) - \bar{x}| &= |f'(c)|\,|x - \bar{x}| \\ &\leq K\,|x - \bar{x}|\,, \quad x \in I\,. \end{aligned} \tag{12}$$

In other words, $f(x)$ is closer to $\bar{x}$ than x is. This implies that $f(x) \in I$. Note that $x \in I$ above plays the role of the "seed" x_0 in the iteration process $x_1 = f(x_0)$, $x_{n+1} = f(x_n)$. Let us now replace x in both sides of (12) with $f(x)$. (We can do this because Eq. (12) is true for all $x \in I$. But $f(x) \in I$.) The result is

$$\begin{aligned} |f(f(x)) - \bar{x}| &\leq K|f(x) - \bar{x}| && (13) \\ &\leq K^2|x - \bar{x}|, && (14) \end{aligned}$$

where the final line is a consequence of (12). We rewrite (14) as

$$|f^2(x) - \bar{x}| \leq K^2|x - \bar{x}|. \tag{15}$$

Note that $f^2(x) \in I$.

The reader should see the pattern: If we apply f to x n times, then we obtain

$$|f^n(x) - \bar{x}| \leq K^n|x - \bar{x}|. \tag{16}$$

Recall, from (9) that $0 < K < 1$. Since $K^n \to 0$ as $n \to \infty$, we have that $|f^n(x) - \bar{x}| \to 0$ as $n \to \infty$, i.e.

$$\lim_{n\to\infty} f^n(x) = \bar{x}.$$

The proof is complete.

The following theorem is an important result regarding the attractive fixed point $\bar{x}$ and interval I.

Theorem 3.3: No other fixed points of f lie in the interval I of the previous Theorem.

Proof: By contradiction. Suppose that $\bar{y} \neq \bar{x}$ is another fixed point in I. Apply the Mean Value Theorem to $a = \bar{x}$, $b = \bar{y}$:

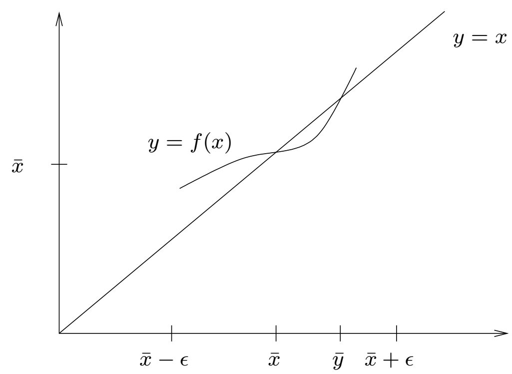
$$\frac{f(\bar{y}) - f(\bar{x})}{\bar{y} - \bar{x}} = f'(c)$$

for some $c \in I$ between $\bar{x}$ and $\bar{y}$. Since $f(\bar{y}) = \bar{y}$ and $f(\bar{x}) = \bar{x}$, we are led to the conclusion that $f'(c) = 1$. Since this contradicts (9), the assumption that $\bar{y} \neq \bar{x}$ is false and the proof is complete.

At this point, the reader may be wondering, "Why can't there be another fixed point $\bar{y}$ of $f(x)$ near the fixed point $\bar{x}$, specifically inside the interval I? One should be able to construct the graph of a function $f(x)$ with two fixed points that are close to each other." Yes, indeed, one can, but at the expense of violating the fundamental assumption in the theorem that

$$|f'(\bar{x})| \leq K < 1 \quad \text{for all} \quad x \in I\,. \tag{17}$$

The situation is sketched in the figure below. In the figure, $\bar{x}$ is attractive – the slope of the tangent to the graph of $f(x)$ at $\bar{x}$ is $0 < f'(\bar{x}) < 1$. In order that $f(x)$ have another fixed point $\bar{y}$ near $\bar{x}$, the graph of $f(x)$ must move upward sufficiently rapidly so that it can intersect the line $y = x$ at $x = \bar{y}$. This makes it necessary for $f'(x) > 1$ at some points between $\bar{x}$ and $\bar{y}$, which contradicts (17).



Repulsive fixed points: As might be expected, if the *multiplier* $|f'(\bar{x})| > 1$ then the fixed point $\bar{x}$ is *repulsive*. The reader is encouraged to prove the following result along similar lines used in the proof of the previous Theorem.

Theorem: Let $\bar{x}$ be a fixed point of f such that $|f'(\bar{x})| > 1$. Then there exists an open interval I containing $\bar{x}$ such that, if $x \in I$, $x \neq \bar{x}$, then there exists a $k > 0$ such that $f^k(x) \notin I$.

Hint: First show that there exists an interval J such that

$$|f(x) - \bar{x}| \geq L\,|x - \bar{x}|, \quad \forall x \in J,$$

where $L > 1$.

Re-examination of some examples encountered earlier:

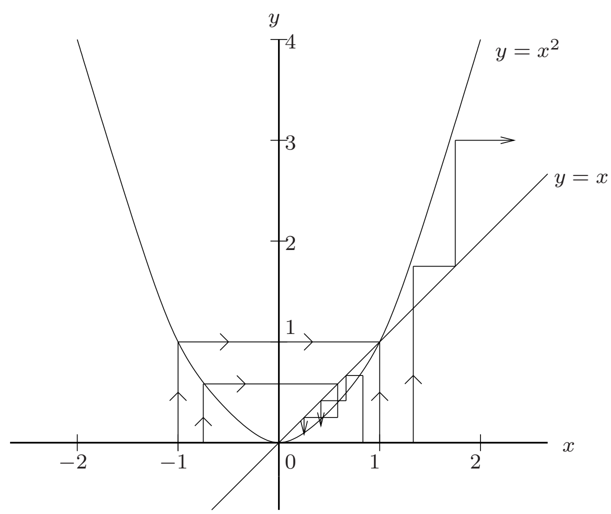
1. $f(x) = ax$. The linear map, a rather trivial case. Here the only fixed point is $\bar{x} = 0$ and $f'(\bar{x}) = f'(0) = a$. The fixed point $x = 0$ is attractive when $|a| < 1$, in agreement with our earlier analysis.

2. $f(x) = x^2$. There are two fixed points, as seen in the figure below: $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. Since $f'(x) = 2x$, we have

   (a) $f'(\bar{x}_1) = 0$, implying that $\bar{x}_1 = 0$ is **attractive**,

   (b) $f'(\bar{x}_2) = 2$, implying that $\bar{x}_2 = 1$ is **repulsive**.

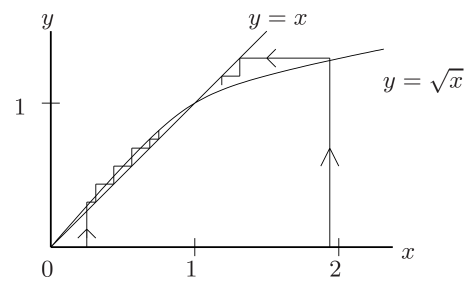
   Both of these results are consistent with our earlier graphical analysis.



3. $f(x) = \sqrt{x}$ for $x \geq 0$. There are two fixed points, as seen in the figure below: $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. Since $f'(x) = \dfrac{1}{\sqrt{x}}$, we have

   (a) $f'(0)$ is undefined, with $f'(x) \to \infty$ as $x \to 0^+$. As such, we cannot apply the above analysis directly. An argument involving the Mean Value Theorem (Exercise) shows that $x = 0$ is repulsive.

   (b) $f'(1) = \dfrac{1}{2}$, implying that $\bar{x}_2 = 1$ is **attractive**.



Once again, these results are consistent with our earlier graphical analysis.

Final remarks:

From the above Theorems, the nature of a fixed point $\bar{x}$ of a (nonlinear) function $f(x)$ can, in general, be determined only *locally*, that is, in an immediate neighbourhood of $\bar{x}$. This is often sufficient to ascertain the dynamics of the iteration process $x_{n+1} = f(x_n)$ over larger intervals. Indeed, as mentioned earlier, a function f may have several fixed points with various properties, e.g. repulsive, attractive, indifferent.

Regarding the term "indifferent": A fixed point $\bar{x}$ for which $|f'(\bar{x})| = 1$ is said to be *neutral* or indifferent. In contrast to the properties $|f'(\bar{x})| > 1$ or $|f'(\bar{x}) > 1$, which will not change dramatically as we move (infinitesimally) away from $\bar{x}$, the condition $|f'(\bar{x})| = 1$ is exceptional and subject to "violation", i.e. $|f'(x)| \neq 1$ for x even "ϵ-close" to $\bar{x}$. As such, any theories for indifferent points are much more complicated. Fortunately, indifferent points play rather minor roles in the applications we wish to consider, representing transition points from one behaviour to another, e.g. from repulsive to attractive.

An important point regarding the convergence of an iteration sequence $\{f^n(x)\}$

In last week's lectures, we examined a good number of examples in which the iterates of a discrete dynamical system,

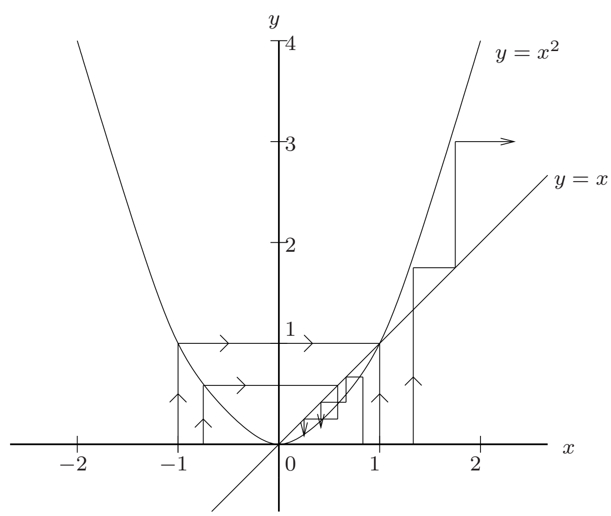
$$x_{n+1} = f(x_n)\,, \quad n = 0, 1, 2, \cdots\,, \tag{18}$$

were converging to a fixed point $\bar{x}$ of the function f. For example, in the case that $f(x) = ax$, where $|a| < 1$, we easily showed that

$$x_n = |a|^n x_0 \to 0 \quad \text{as} \quad n \to \infty\,. \tag{19}$$

And, indeed, $\bar{x} = 0$ is the (unique) fixed point of the function $f(x) = ax$.

We also examined the dynamics associated with the iteration of the function $f(x) = x^2$, shown graphically below,



The fixed points of $f(x) = x^2$ are $\bar{x}_1 = 0$ and $\bar{x}_2 = 1$. We saw that the fixed point $\bar{x}_1 = 0$ is attractive and the fixed point $\bar{x}_2 = 1$ is repulsive.

To repeat, we have seen a number of examples in which iterates $\{x_n\}$ of the dynamical system in Eq. (18) converge to a fixed point $\bar{x}$ of the function $f(x)$. But there remains an important question: Is it possible that a sequence of iterates $\{x_n\}$ in (18) converges to a limit p which is **not** the fixed point of the function f? The answer is **no**, as we now prove.

Theorem: Let f be a continuous function on some domain D and for an $x_0 \in D$, define

$$x_{n+1} = f(x_n)\,, \quad n = 0, 1, 2, \cdots\,. \tag{20}$$

Furthermore suppose that the sequence of iterates $\{x_n\}$ has a limit, i.e.,

$$\lim_{n\to\infty} x_n = p\,. \tag{21}$$

Then p is a fixed point of f, i.e., $f(p) = p$.

This theorem can be stated in another way, as it is in the book by Gulick (Theorem 1.3, Page 8):

Theorem: Let f be a continuous function on some domain D. If, for an $x \in D$, the sequence of iterates $\{f^n(x)\}$ has a limit, i.e.,

$$\lim_{n\to\infty} f^n(x) = p\,, \tag{22}$$

then p is a fixed point of f, i.e., $f(p) = p$.

Before we prove the theorem, let us recall the mathematical definition of the limit in Eq. (21): Given an $\epsilon > 0$, there exists an $N > 0$ – which will probably depend on ϵ, so we'll write it as $N(\epsilon)$ - such that

$$|x_n - p| < \epsilon \quad \text{for all} \quad n > N(\epsilon). \tag{23}$$

In other words, for all $n > N(\epsilon)$, the sequence $\{x_n\}_{n=N(\epsilon)+1}^{\infty}$ lies inside the "ϵ-ribbon" centered around p, i.e.,

$$p - \epsilon < x_n < p + \epsilon\,, \quad n > N(\epsilon)\,. \tag{24}$$

Since this is true for all $\epsilon > 0$, we can then push ϵ toward zero to allow smaller and smaller deviations from p – which, of course, is the very idea of "limit."

Proof: We'll prove the first form of the Theorem. Noting that Eq. (20) holds for all $n \geq 0$,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n)\,, \tag{25}$$

provided that the two limits exist.

LHS: We have that

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = p\,, \tag{26}$$

since we are taking limits of the same sequence.

RHS: Since f is assumed to be continuous,

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(p)\,. \tag{27}$$

(This property was almost certainly discussed, but perhaps not proved, in your first-year Calculus course. It can be proved using an ϵ-δ argument. In fact, it's a nice exercise.)

We have established that the two limits in (25) exist. From Eq. (21), these two limits are equal, i.e.,

$$p = f(p)\,, \tag{28}$$

and the proof is complete.

Appendix: Monotone sequences and the "Monotone Sequence Theorem"

In Problem Set No. 1, you will have to make use of the "Monotone Sequence Theorem" that you most probably saw in Year 1 Calculus. Here is a brief review of the ideas behind the theorem as well as the theorem itself. No proofs will be presented here. For more details, you may consult a standard first-year Calculus text, e.g., the book by J. Stewart, Chapter 11, Section 11.4 entitled, "Sequences."

We are concerned with (infinite) sequences of the form $\{a_1, a_2, a_3, \cdots\}$, where $a_n \in \mathbb{R}$ for $n \geq 1$. Such sequences are also denoted as

$$\{a_n\}_{n=1}^{\infty} \quad \text{or simply} \quad \{a_n\}\,. \tag{29}$$

Recall the definition of the **limit** of a sequence.

Definition: A sequence $\{a_n\}$ has the limit L, denoted as

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \;\text{ as }\; n \to \infty\,, \tag{30}$$

if for every $\epsilon > 0$ there exists a corresponding integer N (often written as $N(\epsilon)$ to denote the dependence of N on ϵ) such that

$$|a_n - L| < \epsilon \quad \text{for all} \quad n > N\,, \tag{31}$$

which can also be written as follows,

$$n > N \qquad \Longrightarrow \qquad |a_n - L| < \epsilon\,. \tag{32}$$

Definition: If a sequence $\{a_n\}$ has a (finite) limit L, it is said to be **convergent**.

Definition: A sequence $\{a_n\}$ is called **increasing** if $a_n \leq a_{n+1}$ for all $n \geq 1$, i.e., $a_1 \leq a_2 \leq a_3$. It is **strictly increasing** if $a_n < a_{n+1}$ for all $n \geq 1$.

Definition: A sequence $\{a_n\}$ is called **decreasing** if $a_n \geq a_{n+1}$ for all $n \geq 1$, i.e., $a_1 \geq a_2 \geq a_3$. It is **strictly decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.

Note: In some books, including the text by Stewart, an increasing sequence is defined as $a_n < a_{n+1}$ and a decreasing sequence as $a_n > a_{n+1}$. We need not be concerned with such differences in definitions.

Unless otherwise stated, we'll employ the definitions given earlier.

Definition: A sequence $\{a_n\}$ is **monotone** or **monotonic** if it is either increasing or decreasing.

Definition: A sequence $\{a_n\}$ is **bounded above** if there is a number $M \in \mathbb{R}$ such that

$$a_n \leq M \quad \text{for all} \quad n \geq 1\,. \tag{33}$$

It is **bounded below** if there is a number $m \in \mathbb{R}$ such that

$$m \leq a_n \quad \text{for all} \quad n \geq 1\,. \tag{34}$$

A sequence $\{a_n\}$ is said to be **bounded** if it is bounded both above and below, i.e., there exist numbers m and M, with $m \leq M$, such that

$$m \leq a_n \leq M \quad \text{for all} \quad n \geq 1\,. \tag{35}$$

Theorem: An **increasing** sequence $\{a_n\}$ which is **bounded above** is **convergent**, i.e., it has a limit L.

Theorem: A **decreasing** sequence $\{a_n\}$ which is **bounded below** is **convergent**.

These two theorems can be combined into one theorem, often called the "Monotone (or Monotonic) Sequence Theorem":

Monotone Sequence Theorem: Every bounded, monotonic sequence is convergent.

Brief explanation: If the sequence $\{a_n\}$ is increasing and bounded from above, then

$$a_1 \leq a_n \leq M \quad \text{for all} \quad n \geq 1\,, \tag{36}$$

which implies that the sequence is bounded (from above and below) with $m = a_1$. If the sequence $\{a_n\}$ is decreasing and bounded from below, then

$$a_1 \geq a_n \geq m \quad \text{for all} \quad n \geq 1\,, \tag{37}$$

which implies that the sequence is bounded (from above and below) with $M = a_1$.

Lecture 5

Dynamics associated with a simple, discrete population model – the "logistic equation"

A brief review of continuous models

In a first-year calculus course you most probably saw mathematical models that describe the evolution of a population over a continuous time parameter $t \geq 0$. For example, the model of "Malthusian growth" is as follows:

[Rate of population growth] is proportional to [population at time t]

This turns out to be a reasonable model for some populations, e.g. single celled organisms and humans alike, when unlimited resources – food and space – are available. For example, the model describes well the population "explosion" of bacteria introduced into a Petri dish full of agar nutrient, at least until "crowding" begins to takes place.

So let $x(t)$ denote the population of a species of bacteria "X" (how unimaginative!) at time $t \geq 0$ and suppose that $x(0) = x_0$. However, we encounter another potential problem – $x(t)$ should be integer valued, shouldn't it? Organisms come in discrete units. In order to get around this "glitch", we can assume that $x(t)$ is large and that single increments in x are so small that they may be considered as infinitesimal changes. (We may also let $x(t)$ represent the *concentration* of organisms per unit volume of Petri dish solution.) The translation of the Malthusian model of growth into mathematics yields the following DE

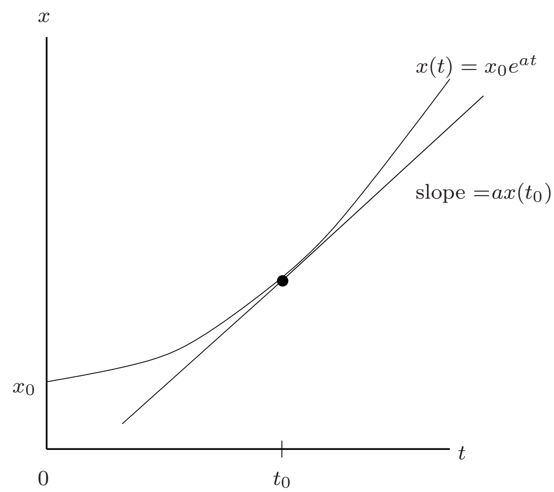
$$\frac{dx}{dt} = ax, \qquad a > 0. \tag{38}$$

The proportionality constant "a" is assumed to incorporate both birth and death processes. The solution to this DE satisfying the initial condition $x(0) = x_0$ is

$$x(t) = x_0 e^{at}. \tag{39}$$

In other words, the Malthusian model predicts exponential growth in time. (If $x_0 = 0$, then $x(t) = 0$ for all $t \geq 0$, the rather uninteresting case of zero population.)

The DE in Eq. (refe1.7) represents a *continuous* dynamical model of population evolution. Once again, "continuous" refers here to the time variable - the fact that t assumes continuous real values, as opposed to discrete time values t_k, $k = 0, 1, 2, \ldots$. Without even knowing the solution of this DE (Eq. (39)) we can conclude that populations will increase: If $x_0 > 0$ then $x'(t)$ must be positive for $t > 0$, implying that $x(t)$ is an increasing function. As $x(t)$ increases, $x'(t)$ must also increase. (In fact, if we differentiate both sides of Eq. (38) with respect to t, we find that $x''(t) > 0$, implying that the graph of $x(t)$ is concave upward.) All of this proceeds in a continuous fashion in time. This reflects the basic underlying assumption of this population model: that birth/death processes are taking place continually in time.



Of course, the Malthusian model is a very simple model. It assumes that populations can, indeed, grow without limit which, of course, is not the case. There are "limits to growth". The Malthusian model may work well for low populations/concentrations, in which the resources available to the population are seemingly infinite. But as the population increases, the reality of limited resources will play more and more of a role. There will be increased competition among members of the species, thereby limiting growth.

Let us now recall a rather simple continuous model from Calculus – a slight modification of the Malthusian equation known as the *logistic* equation, given by the differential equation (DE)

$$\frac{dx}{dt} = ax - bx^2, \tag{40}$$

where $a, b > 0$. The *nonlinear* term $-bx^2$ is a simple model for competition: for small x, $x \ll 1$, it is negligible but as x increases, it becomes more significant. If x is "too large" then it will dominate the growth term ax, causing the population to decrease. We can understand the qualitative behaviour of solutions to Eq. (40) without actually solving the DE. (This is the essence of qualitative analysis.) First, let us rewrite this DE as

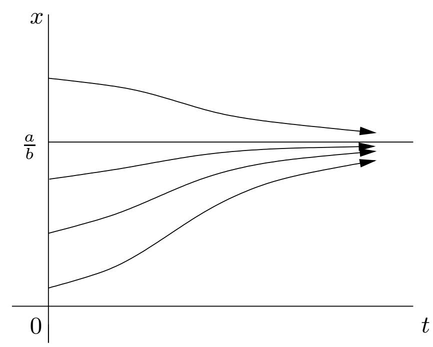
$$\frac{dx}{dt} = ax\left(1 - \frac{b}{a}x\right) \tag{41}$$

and note that there are two constant or "equilibrium" solutions: (1) the trivial solution $\bar{x}_1(t) = 0$ and (2) the solution $\bar{x}_2(t) = \dfrac{a}{b}$. The trivial solution, of course, is not a desirable solution as far as the species is concerned. As for the second solution $\bar{x}_2(t) = \dfrac{a}{b}$, note that as $b \to 0$, i.e. with lesser effects of competition/limited resources, $\frac{a}{b} \to \infty$. As for other solutions $x(t)$ to this DE:

1) If $x(t) > \dfrac{a}{b}$, then $\dfrac{dx}{dt} < 0$, implying that $x(t)$ is decreasing, approaching $\bar{x}_2(t)$ asymptotically.

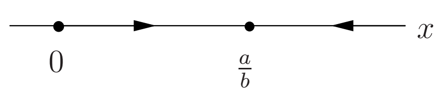
2) If $0 < x(t) < \dfrac{a}{b}$, then $\dfrac{dx}{dt} > 0$, implying that $x(t)$ is increasing, approaching $\bar{x}_2(t)$ asymptotically.

The qualitative behaviour of solution curves is sketched below.



Solution curves to logistic differential equation

(Negative solutions were not sketched since they are not physically relevant here.) The "phase portrait" summary of this solution behaviour may be sketched as follows:



Phase portrait for solution curves of logistic DE

In other words, $x = 0$ is an *unstable* fixed-point or equilibrium solution: If we begin with an initial condition $x(0) = x_0$ near zero, then the corresponding solution $x(t)$ will increase away from 0. On the other hand, the equilibrium solution $x_0 = \dfrac{a}{b}$ is *stable*. In fact, with the exception of the solution $\bar{x}_1(t) = 0$, all solutions $x(t)$ approach it as $t \to \infty$.

In summary, we can see that all solutions $x(t)$ to the logistic DE in Eq. (41) demonstrate quite **regular** behaviour. Assuming that we start with a nonzero population $x(0) > 0$, all solutions converge asymptotically to the stable equilibrium value $\bar{x}_2 = \frac{a}{b}$.

Just one final comment about this equilibrium solution – it depends on the **ratio** of the growth/death parameter a to the competition parameter b. As b decreases, i.e., the negative competitive effects are smaller, the equilibrium population $\frac{a}{b}$ increases, as expected.

Discrete population model analogues

As was discussed in a lecture last week, there are biological systems whose birth/death process are better described by discrete models - for example, animal populations. Another example is plant populations, in particular, annual plants. Each year, at a certain time, typically the fall, the number of annual plants in generation n, say x_n, will produce seeds. Let us assume for simplicity that each plant produces M seeds. A fraction $0 \leq \alpha \leq 1$ of these seeds will survive the winter and a fraction β of these survivors will germinate to produce the plants of generation $n+1$. A fraction γ of these plants will survive until the fall and produce a new set of seeds, etc.. In this very simple picture, we can postulate the relationship

$$x_{n+1} = cx_n, \qquad \text{where} \quad c = \alpha\beta\gamma M \,. \tag{42}$$

We know this dynamical system very well: Given an initial population x_0, the population at year n is

$$x_n = c^n \, x_0 \,. \tag{43}$$

If $c > 1$, then the population x_n is guaranteed to increase with n.

Of course, this **discrete Malthusian model** is an oversimplified one. As the plant population increases, there will be more competition – for nutrients, space, and even sunlight (except, perhaps in the case of fungi). As in the continuous case, we should add some kind of term that models competition. Along the lines of what was done in the continuous case, we expect that such a term (i) will be negative for all positive populations and (ii) will increase in magnitude with population. With the continuous logistic DE in mind, we therefore consider the following nonlinear discrete model,

$$x_{n+1} = cx_n - dx_n^2 \,, \tag{44}$$

where $c, d > 0$. It may be tempting to call this dynamical system a "discrete logistic dynamical system," but that is actually not the case. In keeping with terminology that is standard in the mathematical literature, the term "discrete logistic dynamical system" will be reserved for a special case of the above equation.

Note that Eq. (44) has the form

$$x_{n+1} = f(x_n)\,, \tag{45}$$

where

$$f(x) = cx - dx^2\,. \tag{46}$$

It is helpful to first analyze the qualitative behaviour of this function. First of all, very trivially,

$$f(0) = 0\,. \tag{47}$$

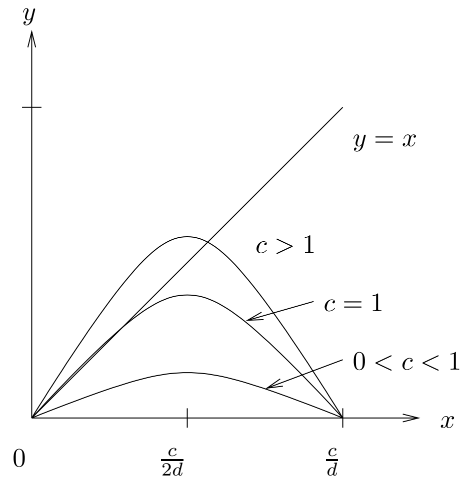
In other words $x = 0$ is a fixed point of f – if we start with zero population, then we stay at zero population. Nothing too surprising! Secondly, from the fact that

$$f'(x) = c - 2dx\,, \tag{48}$$

we see that $x = \dfrac{c}{2d}$ is a critical point. From the fact that $f''(x) = -2d < 0$ as well as the fact that the graph of $f(x)$ is a downward-pointing parabola, it follows that the critical point $x = \dfrac{c}{2d}$ is the global maximum of $f(x)$. In fact, writing $f(x)$ as follows,

$$f(x) = cx - dx^2 = cx\left(1 - \frac{d}{c}x\right)\,, \tag{49}$$

we see that $f(x)$ is zero at $x = 0$ and $x = \dfrac{c}{d}$ and $f(x) > 0$ between these two zeros. In the figure below are sketched the graphs of $y = f(x)$ for $0 \leq x \leq \dfrac{c}{d}$ for three cases of $c > 0$: (i) $0 < c < 1$, (ii) $c = 1$ and $c > 1$ but not too much larger than 1. Note that we are interested only in non-negative values of x since it represents the population. Furthermore, we are interested only in nonnegative values of $f(x)$.



Graphically, we see that for $0 < c \leq 1$, $f(x)$ has only one fixed point, $\bar{x}_1 = 0$. For $c > 1$, there is another fixed point $\bar{x}_2$ that is situated in the interval $[0, c/d]$. It is easily computed:

$$f(x) = x \quad \Longrightarrow \quad x = cx - dx^2 \quad \Longrightarrow (c-1)x = dx^2 \,. \tag{50}$$

Assuming that $x \neq 0$, we have

$$c - 1 = dx \,, \tag{51}$$

which leads to

$$\bar{x}_2 = \frac{c-1}{d} < \frac{c}{d} \,. \tag{52}$$

Once again, we see that $\bar{x}_2 > 0$ for $c > 1$.

One final fact: The maximum value of $f(x)$ on the interval $[0, c/d]$ occurs at $x = \dfrac{c}{2d}$:

$$f_{\max} = f\left(\frac{c}{2d}\right) = \frac{c^2}{4d} \,. \tag{53}$$

Determining the nature of the fixed points of $f(x)$:

We could use the graphs in the above figure to deduce the stability properties of the fixed points of f for values of c and d. Instead, however, we'll use the results from the previous section: A fixed point $\bar{x}$ of a function $f(x)$ is

- **attractive** if $|f'(\bar{x})| < 1$,
- **repulsive** if $|f'(\bar{x})| > 1$,

- **indifferent** if $|f'(\bar{x})| = 1$. Further analysis required.

Fixed point No. 1, $\bar{x}_1 = 0$: From Eq. (48),

$$f'(0) = c\,, \tag{54}$$

which implies that $\bar{x}_1 = 0$ is

- **attractive** for $0 < c < 1$. This is expected – even without competition, the iterates behave as

  $$x_n = c^n x_0 \to 0 \quad \text{as} \quad n \to \infty\,. \tag{55}$$

  If there is no possibility for growth without competition, then one wouldn't expect anything better with competition.

- **repulsive** for $c > 1$. This is somewhat expected – without competition, the iterates

  $$x_n = c^n x_0 \tag{56}$$

  increase with n.

- **indifferent** for $c = 1$. Nothing can be concluded from this result, so we have to resort to some further analysis. From the property that $f(x) < x$, it follows that $f^n(x) \to 0$ as $n \to \infty$. (Exercise.)

The above three results are exactly as expected from a look at the graphs of $f(x)$ for the different cases involving c. Note that the stability properties depend only on the parameter c and not on d.

Fixed point No. 2, $\bar{x}_2 = \dfrac{c-1}{d}$: From Eq. (48),

$$\begin{aligned} f'(\bar{x}_2) &= c - 2d \cdot \frac{c-1}{d} && (57)\\ &= c - 2c + 2 && (58)\\ &= 2 - c\,. && (59) \end{aligned}$$

Recalling that this fixed point exists only when $c > 1$, we obtain the following results:

- When $1 < c < 3$, $|f'(\bar{x}_2)| = |2 - c| < 1$ implying that $\bar{x}_2$ is **attractive**,

- When $c = 3$, $|f'(\bar{x}_2)| = 1$, implying that $\bar{x}_2$ is **indifferent**.

- When $c > 3$, $|f'(\bar{x}_2)| > 1$, implying that $\bar{x}_2$ is **repulsive**.

Numerical experiments easily confirm the first result above regarding stability of the fixed point $\bar{x}_2$ when $1 < c < 3$.

Numerical Experiment No. 1:

We set $c = 2$ and vary d. For each value of d reported in the table below, the iteration procedure was begun with $x_0 = 0.25$. The iterates x_1 to x_{1000} were examined. Note that for $d = 0$, $x_n \to +\infty$, as expected since $x_n = c^n x_0 = 2^n x_0$ in this case. For $0 < d \leq 5.0$, iterates x_n are observed to converge to the fixed point $\frac{c-1}{d}$. (Note that as d is increased, the equilibrium value $\frac{c-1}{d}$ decreased, in accordance with the idea that d is a measure of competition, hence the limitedness of the resources.)

c	d	Behaviour of x_n as $n \to \infty$
2.0	0.0	$x_n \to \infty$
2.0	0.1	$x_n \to 10.0$
2.0	0.2	$x_n \to 5.0$
2.0	0.5	$x_n \to 2.0$
2.0	1.0	$x_n \to 1.0$
2.0	2.0	$x_n \to 0.5$
2.0	5.0	$x_n \to 0.2$

We now investigate the effect of varing c. The big question is, "What happens to the iterates x_n when $c > 3$, so that the fixed point $\bar{x}_2$ is no longer attractive?"

Numerical Experiment No. 2:

We fix the parameter $d = 2.5$ and now vary c from 2.0 upwards, once again with the initial condition $x_0 = 0.25$. For $2.0 \leq c \leq 2.8$, we observe that the iterates x_n approach the fixed point $\dfrac{c-1}{d}$. However, at $c \cong 3.0$, an interesting phenomenon occurs. The iterates x_n no longer approach a *single* fixed point. Instead, they approach an oscillation between two distinct values – a two-cycle $\{p_1, p_2\}$. The fixed point value observed at $c = 2.8$ appears to have split into two values, for example, $\{0.773, 0.836\}$ at $c = 3.0$. As c is increased to 3.2, the two values comprising the two-cycle $\{p_1, p_2\}$ are moving away from each other. This continues up to $c = 3.4$. At $c = 3.5$, the two-cycle disappears and the iterates x_n approach a four-cycle of values $\{p_1, p_2, p_3, p_4\}$. At $c \cong 3.55$, the four-cycle is replaced by an eight-cycle. At $c \cong 3.56$, the eight-cycle becomes a sixteen-cycle. At $c \cong 3.59$, the iterates x_n are observed to travel over the interval $[0.334, 1.289]$ in a seemingly random, unperiodic manner. The results of this experiment are summarized in the next table.

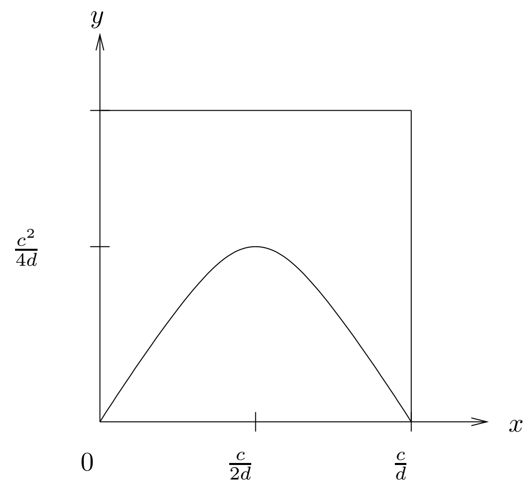
c	d	$x_n \to$	
2.0	2.5	0.4	fixed point
2.5	2.5	0.6	"
2.75	2.5	0.7	"
2.8	2.5	0.72	"
3.0	2.5	{0.773, 0.836}	2-cycle
3.1	2.5	{0.692, 0.941}	"
3.2	2.5	{0.657, 1.023}	"
3.4	2.5	{0.615, 1.145}	"
3.5	2.5	{0.536, 1.158, 0.701, 1.225}	4-cycle
3.55	2.5	8-cycle	
3.56	2.5	16-cycle	
3.59	2.5	"chaotic motion" over [0.334, 1.289]	

The reader is invited to explore this behaviour in more detail. For example, with a little more care (and precision in the floating point computations) one may be able to locate more accurately the value of c at which asymptotic motion towards a fixed point is replaced by motion towards a two-cycle.

The evolution of behaviour observed in the above table is an example of the classic "pitchfork bifurcation and cascade to chaos," a subject of intense research over the past twenty or so years. We shall attempt to analyze a little piece of this scheme in a later section.

Numerical experiments such as those presented above indicate that there is some rich dynamical behaviour associated with the iteration of nonlinear functions. Of course, numerical experiments are fascinating but they do not provide a full understanding of what is responsible for such interesting behaviour. A better understanding will be provided with the help of (a) some Calculus as well as (b) some graphical methods of analyzing iteration schemes. We begin with a discussion of (b) in the next section.

First of all, however, we shall perform one step in order to simplify our analysis. The discrete dynamical system in Eq. (44) that we have studied has **two** parameters c and d. We can essentially reduce the number of independent parameters to one and still capture all of the dynamics associated with this system. Let us see how by returning to the graph of $f(x)$, shown below.



Note that the ratio $\dfrac{c}{d}$ appears everywhere in the figure. The size of the square box in which we have enclosed the graph of $f(x)$ is $\dfrac{c}{d} \times \dfrac{c}{d}$. The height of the graph is $\dfrac{c}{4} \times \dfrac{c}{d}$. The height of the graph is still determined by c. This suggests that we may be able to scale the variable x so that the two parameters c and d are replaced by one parameter. To do this, let us first write our original discrete dynamical system involving the populations, x_n:

$$x_{n+1} = cx_n - dx_n^2 \,. \tag{60}$$

Now define a new set of population variables, y_n, as follows,

$$x_n = \alpha y_n \,, \tag{61}$$

where $\alpha > 0$ is a parameter to be determined. Substitution into Eq. (60) yields

$$\alpha y_{n+1} = c\alpha y_n - d\alpha^2 y_n^2 \,, \tag{62}$$

Dividing by α (assumed to nonzero – otherwise the exercise is meaningless) yields

$$\begin{aligned} y_{n+1} &= cy_n - d\alpha y_n^2 \\ &= cy_n \left(1 - \frac{\alpha d}{c} y_n\right) \,. \end{aligned} \tag{63}$$

If we let

$$\alpha = \frac{c}{d} \tag{64}$$

then Eq. (63) becomes the so-called **logistic discrete dynamical system**,

$$y_{n+1} = cy_n(1 - y_n)\,. \tag{65}$$

We have achieved our desired result: Eq. (65) is a **one-parameter** dynamical system in the modified population variables y_n. As such, we need to examine the behaviour of $\{y_n\}$ sequences only as the single parameter c is varied. And nothing is lost: For a given value of c, the behaviour of the $\{y_n\}$ – whether they approach a fixed point, or an N-cycle, or demonstrate chaotic behaviour – is exactly the same as the behaviour of the $\{x_n\}$ sequence for any d since, from Eq. (61),

$$x_n = \alpha y_n = \frac{c}{d}\, y_n\,. \tag{66}$$

The function $f_c(y)$ associated with the logistic discrete dynamical system in Eq. (63),

$$f_c(y) = cy(1 - y) \tag{67}$$

is known as the **logistic quadratic map**. Note that the domain of interest of this function will be $y \in [0, 1]$. This is consistent with the fact that our domain of interest in the x population variable was $\left[0, \frac{c}{d}\right]$.

Dynamics of the Logistic Quadratic Map on the interval $[0, 1]$

In this section we shall study some dynamics associated with iteration of the logistic quadratic map,

$$f_a(x) = ax(1 - x) \tag{68}$$

i.e.,

$$x_{n+1} = f_a(x_n)\,. \tag{69}$$

Note that we have changed "y" from the previous section to "x". Furthermore, we have changed the parameter to a to conform with standard notation. Note that the parameter a is kept fixed for a given iteration process.

For $a > 0$, the functions $f_a(x)$ in (68) are downward pointing parabolas that pass through the points $(0, 0)$ and $(1, 0)$. At $x = \dfrac{1}{2}$ they achieve their global maximum values of $\dfrac{a}{4}$. (Exercise.) We now determine the fixed points of $f_a(x)$, which must satisfy the equation,

$$f_a(x) = x \qquad \Longrightarrow \qquad ax(1 - x) = x\,. \tag{70}$$

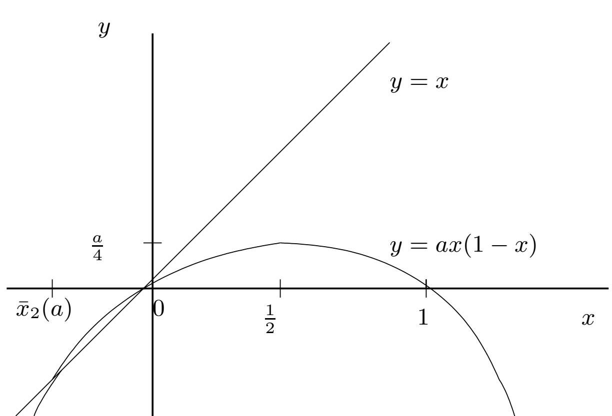
The above equation is clearly satified by $x = 0$. For $x \neq 0$, we divide by x to obtain

$$a(1 - x) = 1 \qquad \Longrightarrow \qquad 1 - x = \frac{1}{a} \qquad \Longrightarrow \qquad x = 1 - \frac{1}{a}\,. \tag{71}$$

In summary, there are two fixed points of $f_a(x)$:

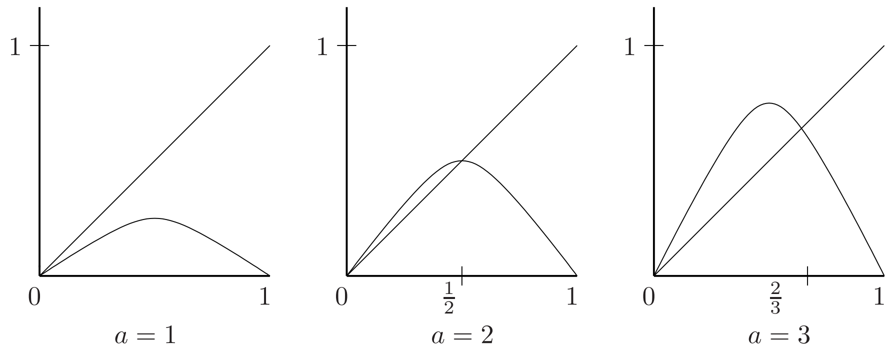
$$\bar{x}_1 = 0, \qquad \bar{x}_2(a) = 1 - \frac{1}{a} = \frac{a-1}{a}\,. \tag{72}$$

For $0 < a < 1$, $\bar{x}_2 < 0$. As $a \to 0^+$, $\bar{x}_2(a) \to -\infty$:

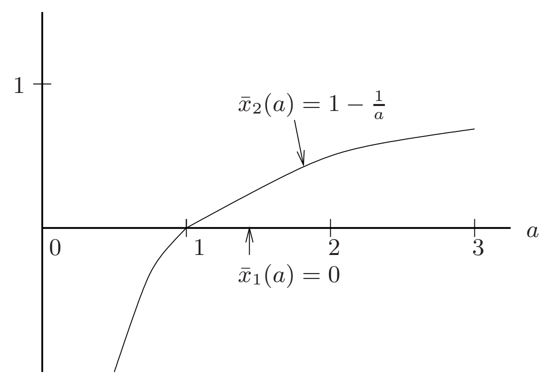


$0 < a < 1$

When $a = 1$, $\bar{x}_1 = \bar{x}_2 = 0$, i.e. the fixed point $\bar{x}_2$ has moved to coincide with $\bar{x}_1$. As a increases, $a > 1$, the fixed point $\bar{x}_2(a)$ moves away from 0 toward the right. At $a = 2$, $\bar{x}_2(2) = \frac{1}{2}$. At $a = 3$, $\bar{x}_2(3) = \frac{2}{3}$.



The positions of the two fixed points as a function of a are plotted below:



We now compute the multipliers $|f_a'(\bar{x}_i(a))|$ of these fixed points in order to determine if and when each of them is attractive or repulsive. From (68),

$$f_a'(x) = a - 2ax.$$

Therefore, from (72):

i) $\bar{x}_1 = 0$: $\qquad f_a'(\bar{x}_1) = a.$

Therefore $\bar{x}_1 = 0$ is attractive for $0 \leq a < 1$, indifferent for $a = 1$, and repulsive for $a > 1$.

ii) $\bar{x}_2 = \dfrac{a-1}{a}$: $\qquad f_a'(\bar{x}_2) = a - 2(a-1) = 2 - a.$

Therefore $\bar{x}_2$ is repulsive for $0 < a < 1$, indifferent for $a = 1$, attractive for $1 < a < 3$, indifferent for $a = 3$, and repulsive for $a > 3$.

Lecture 6

Dynamics of the logistic map $f_a(x) = ax(1-x)$ on [0,1]

We continue with the discussion from the previous lecture.

We shall be primarily interested in the iteration dynamics of iteration of f_a over the interval $[0,1]$, where $f_a(x)$ is non-negative. One could argue that this is the only region of concern as far as applications to population dynamics are concerned, since populations are non-negative. Also note that if $x \notin [0,1]$, then $f_a(x) \notin [0,1]$. In other words, if we start with a point outside [0,1], then we can never get into the interval [0,1]. As we shall see below, the dynamics of iteration outside the interval [0,1] is rather straightforward and uninteresting.

Dynamics of iteration outside the interval [0,1]

For $0 < a < 1$ (see plot on previous page), points $x \in (\bar{x}_2(a), 0)$ will travel to $\bar{x}_1 = 0$ under iteration. Likewise, points $x \in (1, 1 - \bar{x}_2(a))$ are mapped to the interval $(\bar{x}_2(a), 0)$, after which they travel to $\bar{x}_1 = 0$ under iteration. Of course, $\bar{x}_2(a)$ is a fixed point and the point $x = 1 - \bar{x}_2(a)$ is mapped to $\bar{x}_2(a)$. All other points $x \notin [\bar{x}_2(a), 1-\bar{x}_2(a)]$ travel to $-\infty$ under iteration of $f_a(x)$ (graphical exercise).

When $a \geq 1$, all points $x \notin [0,1]$ travel toward $-\infty$ under iteration of f_a, i.e. $f_a^n(x) \to -\infty$ as $n \to \infty$ for $x \notin [0,1]$.

Dynamics of iteration inside the interval [0,1]

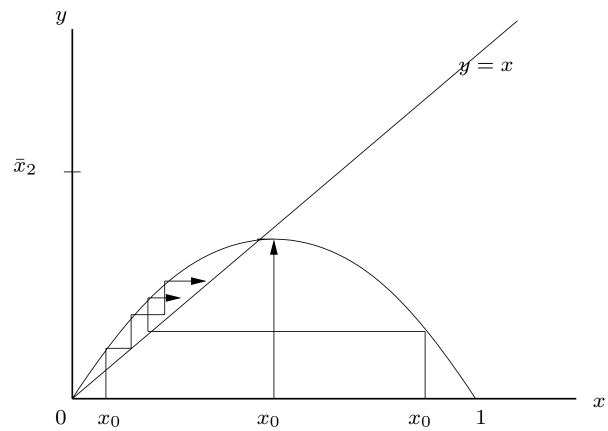
We can now focus on the dynamics of the iteration process $x_{n+1} = f_a(x_n)$ for $x_0 \in [0,1]$. This is possible since f_a maps $[0,1]$ into itself for $0 \leq a \leq 4$. When $a = 4$, f_a maps $[0,1]$ **onto** itself. (We'll return to this later.)

The case $0 \leq a \leq 1$

Recall that for $0 \leq a < 1$, the fixed point $\bar{x}_1 = 0$ is attractive. From a look at the graph of $f_a(x)$ for these a-values, we see that for *any* $x_0 \in [0,1]$, $x_n \to 0$ as $n \to \infty$, where $x_{n+1} = f_a(x_n)$. (From a population viewpoint, the species is destined for extinction, as is expected, since $0 < a < 1$.) When $a = 1$, the fixed point $\bar{x}_1 = 0$ is neutral. Nevertheless, all iterates $x_n \to 0$ for all $x_0 \in [0,1]$.

The case $1 < a < 3$

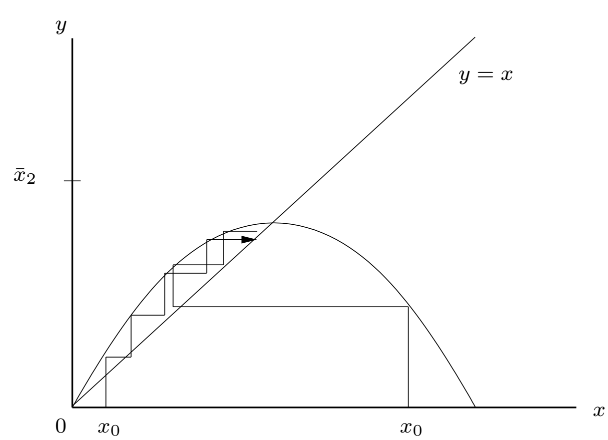
As a increases from 1, the fixed point $\bar{x}_1 = 0$ is repulsive and the fixed point $\bar{x}_2 = \dfrac{a-1}{a}$ assumes the role of the attractive fixed point. The reader is invited to determine the motion of various sets of points on the interval $[0,1]$ as they approach $\bar{x}_2$ under iteration. There are a few cases to consider but in all of these cases, if $x_0 \in (0,1)$, then $x_n \to \bar{x}_2$ as $n \to \infty$. Some are shown in the figure below. In all of these cases, the x_n approach $\bar{x}_2$ monotonically, either from the right or from the left.



$1 < a \leq 2$: Iterates x_n approach $\bar{x}_2$ monotonically.

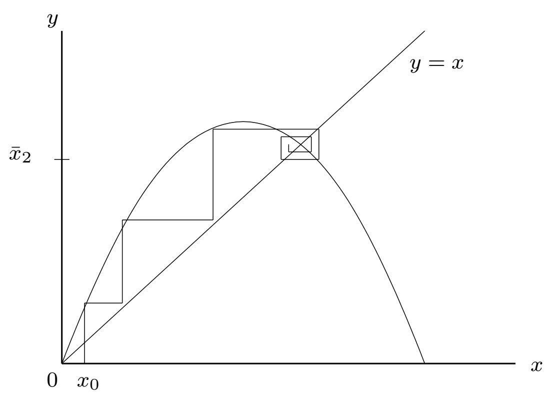
In the special case $a = 2$, the fixed point $\bar{x}_2 = \frac{1}{2}$ is the global maximum of $f_2(x)$ on $[0,1]$. Furthermore,

- (i) if $x_0 \in \left(0, \frac{1}{2}\right)$, then $x_1 < x_2 < x_3, \ldots$, i.e. $x_n \to \frac{1}{2}$ monotonically;
- (ii) if $x_0 \in \left(\frac{1}{2}, 1\right)$, then $x_1 \in \left(0, \frac{1}{2}\right)$ and $x_n \to \frac{1}{2}$ monotonically.



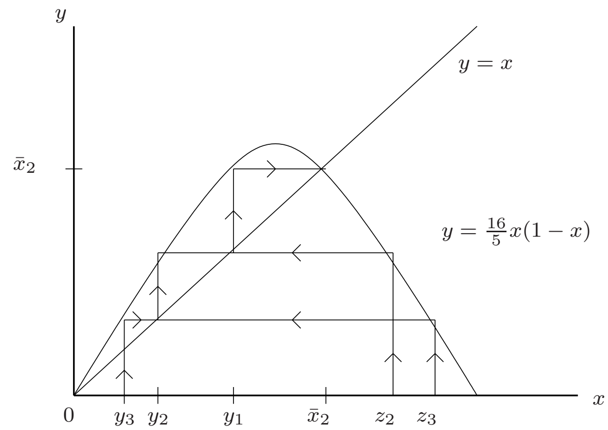
$1 < a \leq 2$: Iterates x_n approach $\bar{x}_2$ monotonically.

For the case $2 < a < 3$, the motion of iterates is not as straightforward since the fixed point value $\bar{x}_2(a)$ now lies below the maximum value $f_a\left(\frac{1}{2}\right) = \frac{a}{4}$, as shown in the figure below. As a result, the graph of $f_a(x)$ now intersects the line $y = x$ with a negative slope at the fixed point $\bar{x}_2$. For seed points x_0 close to 0, the iterates x_n increase monotonically, at least for a while, since $f(x) > x$. But at a certain point, they will be mapped to the other side of the fixed point $\bar{x}_n$, after which they approach $\bar{x}_2$ in an oscillatory manner.



$2 < a < 3$: Iterates x_n begin to oscillate above $\bar{x}_2$ as they approach it.

There is, however, a special set of points that avoid this oscillatory behaviour. By running the iteration of f "backwards" as was done in Example 8 of Section 3.3, we discover a set of points $y_1, y_2, \ldots$ such that $f^k(y_k) = \bar{x}_2$. These points y_k are called **preperiodic** points since they eventually land on a periodic orbit – in this case the fixed point $\bar{x}_2$ – after a finite number of iterations. Note that $y_k \to 0$ as $k \to \infty$, indicating that we can find points y_k arbitrarily close to the point $x = 0$. And for each point y_k, $k \geq 2$, there is a point z_k on the other side of the fixed point $\bar{x}_2$, that maps to $\bar{x}_2$ after k iterations, as shown in the diagram below. Also note that $z_k \to 1$ as $k \to \infty$. This rather complicated behaviour is made possible by the fact that the function $f(x)$ is "2-to-1": for any value $y \in \left[0, \frac{a}{4}\right)$, there are two distinct points x_1 and x_2 such that $f(x_1) = f(x_2) = y$. As in an earlier lecture, the reader is invited to construct a numerical algorithm to determine the points y_k and z_k that are sent to $\bar{x}_2$ after exactly k iterations.



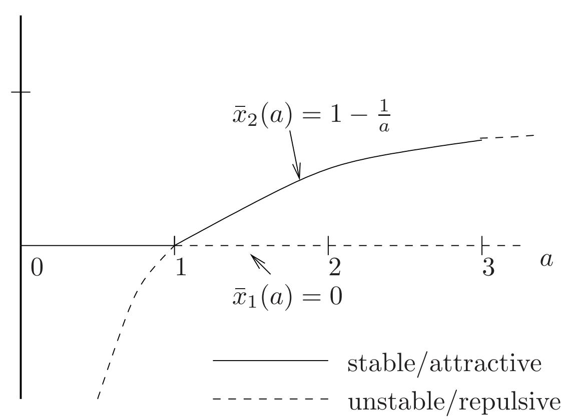
$a > 2$: Preperiodic points y_k and z_k that map to $\bar{x}_2$ after k iterations.

All other points $x \notin \{0, 1, y_k, z_k, k \geq 1\}$ approach $\bar{x}_2$ as f_a is iterated, but never reach $\bar{x}_2$. When they are mapped sufficiently close to x_2, they oscillate about $\bar{x}_2$ as they approach it. Graphically, they trace out a "cobweb" that spirals inward to the point $(\bar{x}_2, \bar{x}_2)$: The iterates x_n jump from one side of the fixed point to the other approaching it in the process.

The case $a = 3$

When $a = 3$ - see the rightmost plot at the top of Page 68 – the fixed point $\bar{x}_2 = \frac{2}{3}$ is no longer attractive but "neutral". For an $x \in (0, 1)$, the forward orbit, $x_n = f^n(x)$ does converge to $\bar{x}_2$. However, for any $a > 3$, the fixed point $\bar{x}_2$ is repulsive. The natural question is, "Where do the iterates go?" A glimpse of this behaviour was given in the numerical experiments shown in the previous lectures. We shall analyze the dynamics in a later section.

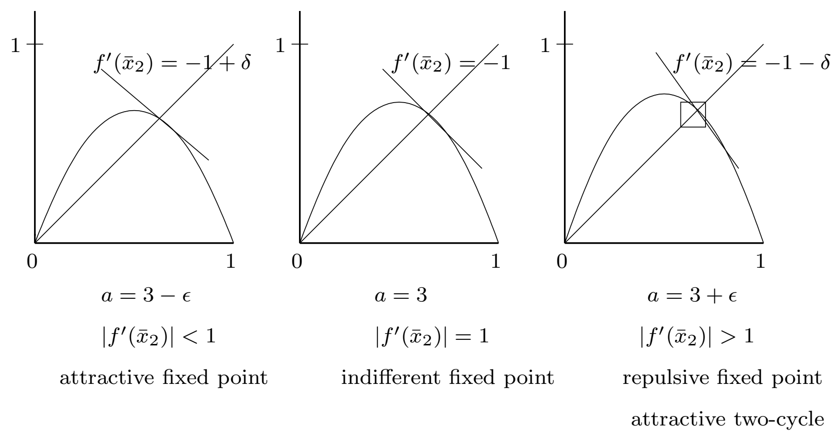
Finally, we may include the information on the attractive/repulsive properties of the fixed points $\bar{x}_1 = 0$ and $\bar{x}_2 = \frac{a-1}{a}$ of the logistic map in addition to their locations, as shown earlier. Dotted lines indicate repulsive/unstable behaviour and solid lines denote attractive/stable behaviour.



The Appearance of "Two-Cycles"

As stated earlier, for values $a > 3$, the fixed point $\bar{x}_2 = \dfrac{a-1}{a}$ is repulsive. If we begin with a seed value $x_0 \neq 0$, then, for values of a slightly greater than 3, e.g. $a = 3.1$, sequences produced by the iteration scheme $x_{n+1} = f_a(x_n)$ are observed to approach an oscillation between two numbers $\{p_1, p_2\}$. When $a = 3.2$, $p_1 \cong 0.799$, $p_2 \cong 0.513$. As "a" increases these two values are observed to move farther apart. At $a \cong 3.45$, the iterates no longer approach an oscillation between two numbers, but rather four values, as was observed in our initial numerical experiments with the nonlinear population model (Pages 63-64).

A graphical illustration of why the fixed point $\bar{x}_2$, which was attractive for $1 < a < 3$, becomes repulsive for $a > 3$ is shown below. At an a-value slightly less than 3, say, $1 - \epsilon$, where $\epsilon > 0$ is small,



the slope of the tangent line at the fixed point is negative and slighly greater than -1, say $-1 + \delta$, where $\delta > 0$ is small. This implies that the multiplier, $|f_a'(\bar{x}_2(a))| = 1 - \delta < 1$ so that the fixed point $x_2(a) = 1 - \dfrac{1}{a}$ is attractive. At $a = 3$, the slope is -1 so its magnitude is 1, implying that the fixed point $x_2(a) = \dfrac{2}{3}$ is indifferent. (It will still attract iterates.) When a is slightly less than 3, say $3 + \epsilon$, the slope of the tangent at $\bar{x}_2(a)$ is $-1 - \delta$ for some small $\delta > 0$. This implies that $|f'(\bar{x}_2(a))| = 1 + \delta > 1$ so that the fixed point $\bar{x}_2(a)$ is now repulsive. An attractive two-cycle, shown as a box in the rightmost graph, now "captures" most iterates.

As mentioned above, when $a = 3.2$, $p_1 \cong 0.799$, $p_2 \cong 0.513$. As "a" increases these two values are observed to move farther apart. At $a \cong 3.45$, the iterates no longer approach an oscillation between two numbers, but rather four values, as was observed in our initial numerical experiments with the nonlinear population model (Page 63-64).

In order to understand this behaviour, we recall that the forward orbit of a point x under iteration by a function $f(x)$ can be written as

$$x_0 = x, \quad x_1 = f(x_0), \quad x_2 = f^2(x_0), \ldots \; x_n = f^n(x_0), \tag{73}$$

where $f^n(x)$ denotes the n-fold composition of the function f with itself. The "two-cycle" $\{p_1, p_2\}$ mentioned above is an example of a **periodic orbit**: The period of this cycle is two: If $f^n(x) = p_1$, then $f^{n+1}(x) = p_2$, $f^{n+2}(x) = p_1$ so that $x_{n+2} = x_n$ for all n. (In this way, a fixed point $p = f(p)$ of f is trivially a periodic orbit with period 1.)

Note that if $f(p_1) = p_2$ and $f(p_2) = p_1$, then $f(f(p_1)) = f(p_2) = p_1$. In other words, p_1 is a fixed point of the map $g(x) = f^2(x)$. In the same way, p_2 is also a fixed point for $g(x) = f^2(x)$ since $g(p_2) = f(f(p_2)) = f(p_1) = p_2$. (The function g represents a two-fold application of f on a point.)

We may generalize this idea as follows: Let $\{p_1, p_2, \ldots, p_N\}$ be distinct points such that

$$f(p_1) = p_2, \quad f(p_2) = p_3, \ldots, f(p_{N-1}) = p_N, \quad f(p_N) = p_1. \tag{74}$$

Then the set $\{p_1, p_2, \ldots, p_N\}$ comprises a **periodic orbit of period** N, or simply an **"N-cycle"**. (We are assuming that this cycle is "indecomposable", i.e. that "N" is the smallest integer for which such a periodic orbit exists. For example, if $N = 6$ and $p_1 = p_4$, $p_2 = p_5$, $p_3 = p_6$, then we really don't have a 6-cycle but a 3-cycle.) Each of the points $p_1, p_2, \ldots p_N$ is a fixed point of the map $g(x) = f^N(x)$, i.e. $g(p_1) = p_1, \ldots, g(p_N) = p_N$.

We are now in a position to be able to understand how periodic orbits $\{p_1, p_2, \ldots p_N\}$ can be attractive or repulsive, as is the case for fixed points (which are periodic orbits of period 1). For a two-cycle, $\{p_1, p_2\}$, we need to consider the mapping

$$g(x) = f^2(x) = f(f(x)) \,. \tag{75}$$

Since p_1 and p_2 are fixed points of $g(x)$, i.e.,

$$g(p_1) = p_1 \quad \text{and} \quad g(p_2) = p_2 \,, \tag{76}$$

we need to examine the multipliers $g'(p_1)$ **and** $g'(p_2)$. If $|g'(p_1)| < 1$, then we expect iterates $g^n(x)$ to be attracted to p_1. Of course, as these points get closer to p_1, we expect the points "in between" (since $g = f^2$) to get closer to p_2. Therefore, if $|g'(p_1)| < 1$, we expect $|g'(p_2)| < 1$ as well.

Likewise, if p_1 is a repulsive fixed point of g, then we expect p_2 to be repulsive as well. In fact, we may compute the multipliers of g in terms of multipliers of f as follows: Since $g(x) = f(f(x))$, we have, by the Chain Rule

$$g'(x) = f'(f(x))f'(x). \tag{77}$$

For $x = p_1$, $f(x) = p_2$ so that $g'(p_1) = f'(p_2)f'(p_1)$. For $x = p_2$, $f(x) = p_1$ so that $g'(p_2) = f'(p_1)f'(p_2)$. In summary, for a two-cycle $\{p_1, p_2\}$ of f, we have

$$g'(p_1) = g'(p_2) = f'(p_1)f'(p_2). \tag{78}$$

In general, for an N-cycle $\{p_1, p_2, \ldots, p_N\}$ of f, we have, for $g = f^N$,

$$g'(p_i) = f'(p_1)f'(p_2)\ldots f'(p_N), \quad 1 \leq i \leq N.$$

Let us now return to the logistic map $f_a(x) = ax(1-x)$. We first look for values of a for which two-cycles $\{p_1, p_2\}$, $p_1 \neq p_2$, can exist:

$$\begin{aligned} ap_1(1-p_1) &= p_2 \qquad (f(p_1) = p_2), && (79)\\ ap_2(1-p_2) &= p_1 \qquad (f(p_2) = p_1). && (80) \end{aligned}$$

There are a number of ways to achieve our goal. Here is one method. First we subtract the second equation from the first to obtain,

$$a(p_1 - p_2) - a(p_1^2 - p_2^2) = p_2 - p_1\,. \tag{81}$$

We see that $p_1 = p_2$ satisfies this equation. But that would imply that our two-cycle is, in fact, a one-cycle, i.e., a fixed point. So we assume that $p_1 \neq p_2$ and divide both sides of this equation by $p_1 - p_2$ to obtain

$$a - a(p_1 + p_2) = -1\,. \tag{82}$$

A rearrangement of this equation produces the result,

$$p_1 + p_2 = \frac{1+a}{a}. \tag{83}$$

If we write p_2 in terms of p_1 (or vice versa) and substitute into either of the above equations, we obtain quadratic equations in the p_i with roots

$$\boxed{p_{1,2} = \frac{a+1}{2a} \pm \frac{1}{2a}\sqrt{(a-1)^2 - 4}.} \tag{84}$$

For real roots, we must have $(a-1)^2 - 4 \geq 0$, which implies that $a \geq 3$. Note that at $a = 3$, $p_1 = p_2 = \frac{2}{3}$. This was precisely the value of a beyond which the fixed point $\bar{x}_2(a) = \frac{a-1}{a}$ became repulsive.

Note: Another way to find the above two-cycle is to make use of the fact that each point p_1 and p_2 is a fixed point of the map $g(x) = f^2(x)$. In this case,

$$\begin{aligned} g(x) &= f(f(x)) \\ &= a(ax(1-x))(1-ax(1-x)) \\ &= -a^3x^4 + 2a^3x^3 - (a^3+a^2)x^2 + a^2x\,. \end{aligned} \tag{85}$$

One must then find solutions of the fixed point equation,

$$g(x) = x \quad \Longrightarrow \quad g(x) - x = 0\,. \tag{86}$$

After a little work, we find that

$$g(x) - x = -x(ax+1-a)(a^2x^2 - (a^2+a)x + a + 1) = 0\,, \tag{87}$$

Two solutions of this equation are

$$x_1 = 0\,, \quad x_2 = \frac{a-1}{a}\,, \tag{88}$$

which are the two fixed points of $f(x)$. (Clearly, fixed points of f are also fixed points of $f^n(x)$.) The zeros of the final quadratic term are p_1 and p_2 found earlier.

Now we know that two-cycles exist for $a > 3$. It remains to determine the a-values, if any, for which such two-cycles are attractive or repulsive. Since $f'_a(x) = a - 2ax$, we have

$$\begin{aligned} f'(p_1)f'(p_2) &= a^2[1-2p_1][1-2p_2] && (89) \\ &= a^2\left[1 - \frac{2(1+a)}{a} + \frac{4(1+a)}{a^2}\right] && (90) \\ &= -a^2 + 2a + 4. && (91) \end{aligned}$$

Thus, the two-cycle $\{p_1, p_2\}$ will be attractive for a-values such that

$$|f'(p_1)f'(p_2)| < 1 \quad \Longrightarrow \quad |-a^2 + 2a + 4| < 1\,. \tag{92}$$

We may consider this inequality as being satisfied by those positive a-values (recall that we do not consider $a < 0$) for which the graph of the function,

$$h(a) = -a^2 + 2a + 4\,, \tag{93}$$

an "upside down" parabola, lies between the lines $y = 1$ and $y = -1$. First of all, since

$$h'(a) = -2a + 2\,, \tag{94}$$

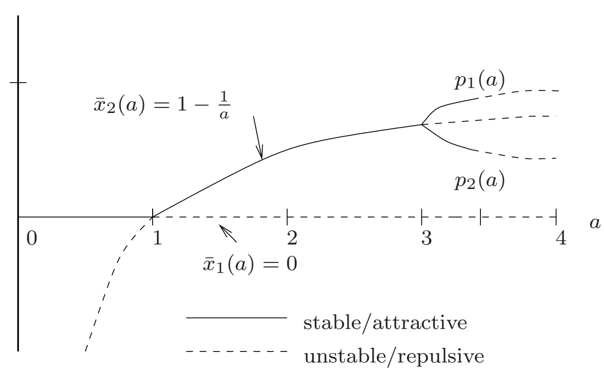
we see that $h'(1) = 0$ and $h(a)$ is increasing for $a < 1$ and decreasing for $a > 1$, i.e., $h(a)$ achieves a global maximum value of 5 at $a = 1$. As such, we're going to want to investigate values of $h(a)$ for $a > 1$. The reader may wish to sketch a graph of $h(a)$ vs. a but here we simply examine some particular values of $h(a)$ for $a > 1$ that will provide the answer to our question:

- $h(2) = 4$
- $h(3) = 1$
- $h\left(1 + \sqrt{5}\right) = 0$ (by solving quadratic equation $h(a) = 0$)
- $h\left(1 + \sqrt{6}\right) = -1$ (by solving quadratic equation $h(a) = -1$)

Since only positive values of a are relevant we conclude that the two-cycle $\{p_1, p_2\}$ for the logistic map $f_a(x) = ax(1 - x)$ is attractive, or stable, for

$$3 < a < 1 + \sqrt{6} \cong 3.44948.$$

At $a = 1 + \sqrt{6}$, the two-cycle is neutral. For $a > 1 + \sqrt{6}$, the two-cycle is repulsive. We may now update the diagram on Page 57 with this new information:



At $a = 1 + \sqrt{6}$, iterates of the logistic equation (assuming $x_0 \neq 0$) will still approach the neutral two-cycle $\{p_1, p_2\}$. The next question is, naturally, "What happens to iterates for $a > 1 + \sqrt{6}$?" We have already mentioned that numerical experiments indicate that an **attractive four-cycle** or periodic orbit of period 4 appears. One could try to perform an analysis similar to above to find a-values for which 4-cycles could occur, and then to determine their stability characteristics. However, this is quite complicated since we are working with *four* equations in the unknowns p_1, p_2, p_3, p_4. No closed-form expressions exist for the p_i. It turns out that a deeper analysis of the transition, or *bifurcation*, that occurs at $a = 3$, from a stable 1-cycles to a stable 2-cycle, will give us an idea of what goes on at $a = 1 + \sqrt{6}$. We perform such an analysis, in terms of the graphs of $f_a(x)$ and $g_a(x) = f_a^2(x)$, below.

Explaining the bifurcation from a fixed point to a two-cycle

Given a function $f(x)$, we can obtain an idea of what the graph of its iterate $g(x) = f^2(x)$ looks like from the graph of f: For every point x, we find $y = f(x)$, then travel horizontally to the line $y = x$ and then input $f(x)$ into f to obtain $f^2(x)$. This is illustrated in the figure below. Clearly, fixed points of f, $\bar{x} = f(\bar{x})$, are fixed points of $g(x)$. The point $x = a$ gets mapped, after two applications of f, to $g(a) = f^2(a)$. The point $x = b$ has been chosen so that $f(b) = \bar{x}$, i.e. b is a preperiodic point of $\bar{x}$. A point just to the right of $x = b$, say $x = b + \epsilon$, will be mapped to a value $f(b + \epsilon) > f(b) = \bar{x}$. However, from the graph, $f(f(b+\epsilon)) < \bar{x}$. The reader is encouraged to examine other x values in this example.