Dynamical Systems and Optimal Control

Derivatives and Euler-Lagrange equation (20+20)

1. Let $F = x\sqrt{y^2 + (y')^2}$, where y and y' are functions of x.

(a) Find $\dfrac{\partial F}{\partial x}$, $\dfrac{\partial F}{\partial y}$ and $\dfrac{\partial F}{\partial y'}$, and hence find $\dfrac{dF}{dx}$.

(b) If $y = \sin x$, then find $\dfrac{dF}{dx}$ by substituting $y = \sin x$ into the formula for F and differentiating using the result from part (a).

2. For the function $F = \sqrt{x^2 + y\,(y')^2}$, where y and y' are functions of x, find

$$\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial y'} \quad \text{and} \quad \frac{dF}{dx}.$$

3. For the function $F = x^2 y^3$, where y is a function of x, find

$$\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y} \quad \text{and} \quad \frac{dF}{dx}$$

Also, show that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{dF}{dx}\right)$$

4. Consider the functional

$$S[y] = \int_a^b \left((y')^2 + y\right) dx, \quad y(0) = 0, y(1) = 2$$

(a) Show that the stationary paths of this functional must satisfy

$$2y'' - 1 = 0, \quad y(0) = 0, y(1) = 2$$

(b) Solve this differential equation to show that the stationary path is

$$y = \frac{1}{4}x^2 + \frac{7}{4}x$$

5. (a) Show that the Euler-Lagrange equation for the functional

$$S[y] = \int_0^X \left((y')^2 - y^2\right) dx, \quad y(0) = 0, y(X) = 1$$

where $0 < X < \pi$, is

$$y'' + y = 0.$$

(b) Hence show that the stationary function is $y = \sin x / \sin X$.

6. Show that the Euler-Lagrange equation for the functional

$$S[y] = \int_0^1 \left((y')^2 + y^2 + 2xy\right) dx, \quad y(0) = 0, y(1) = \alpha$$

where α is a constant, is

$$y'' - y = x$$

The First Travel Problem (Euler-Lagrange equation, 40)

Fill in the gaps by answering the questions.

Consider a general (possibly non-uniform) medium in the Cartesian plane (x, y). The speed of light will vary in space, so it will depend on coordinates $v(x, y)$. The time taken for light to travel along an infinitesimal line segment of length δs is $\delta t = \frac{\delta s}{v(x,y)}$. Along a path $y = y(x)$, we have $\delta s = \sqrt{1 + y'(x)^2}\delta x$. So, the total time travel will be equal to

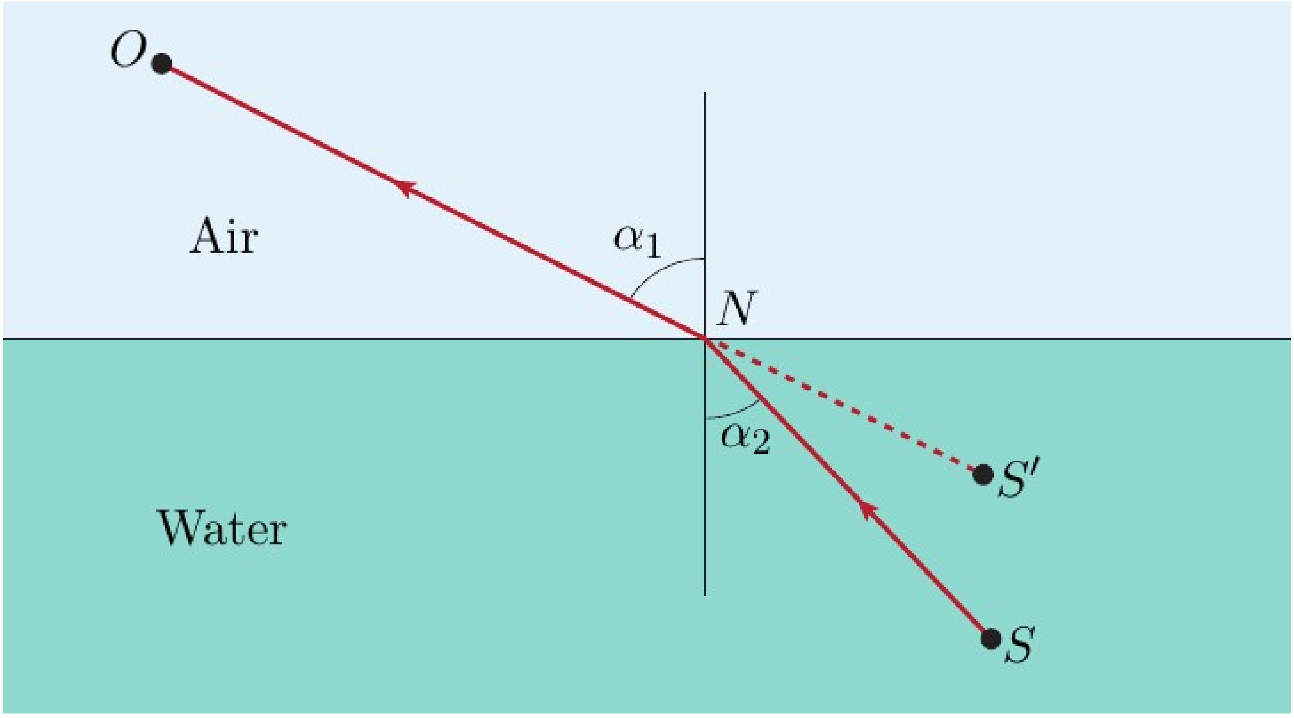
$$J[y] = \int_a^b \frac{\sqrt{1 + y'(x)^2}}{v(x, y)} dx$$

Fermat's principle says that the path taken between two points by a ray of light is a stationary path of the time functional (meaning $\frac{dJ}{d\varepsilon}\Big|_{\varepsilon=0} = 0,$ for any η: $\eta(a) = \eta(b) = 0$). Therefore, to find a solution we can apply Euler-Lagrange equation to this functional.

Fermat's principle can be used to show that for light reflected in a mirror, the angle of incidence equals the angle of reflection. For light crossing the boundary between two media (air and water), it gives *Snell's law*, which says that

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{c_1}{c_2}$$

where α_1 and α_2 are the angles between the ray and the normal to the boundary, and c_1 and c_2 are the corresponding speeds of light in the media.



Let's prove the Snell's law. Assume, that the boundary between air and water is the x-axis. Also, note, that speed depends only on y: $v(x, y) = v(y)$. We have

$$J[y] = \int_a^b \frac{\sqrt{1 + y'(x)^2}}{v(y)} dx$$

where $v(y) = c_1, y > 0$ (speed in the air) and $v(y) = c_2, y < 0$ (speed in the water).

By applying Euler-Lagrange equation we get [**why?**]:

$$\frac{1}{v(y)\sqrt{1 + y'(x)^2}} = C$$

Such that $v(y)$ is a piecewise-linear function, it follows that $y'(x)$ is a piecewise-linear function too [**why?**].

Let's recall, that $y' = \tan\varphi$, where φ is an angle between graph of function f and x-axis (between the ray and the boundary). So, we have

$$\frac{1}{v(y)\sqrt{1 + \tan^2\varphi}} = C$$

i.e.

$$\frac{1}{c_1\sqrt{1 + \tan^2\varphi_1}} = C \text{ and } \frac{1}{c_2\sqrt{1 + \tan^2\varphi_2}} = C.$$

Therefore, for $\alpha_1 = \pi/2 - \varphi_1,\ \alpha_2 = \pi/2 - \varphi_2$ we finally get [**why?**]

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{c_1}{c_2}$$

Remark. We can always calculate the exact value of α_1 (and therefore α_2) finding the unique ray passing between two fixed points (O and S at the picture).

Remark. The situation is similar to that with my suitcase on wheels. In this case, I want to minimize the amortization (the shock absorption) of my wheels instead of time. This amortization is also propositional to the path length and has values c_1 and c_2 corresponding to the grass at the garden, or concrete at the square I cross.

The Second Travel Problem (Pontryagin Maximum Principle, up to 50)

Fill in the gaps by answering the questions.

Suppose we are driving a car on a straight road for $t \in [0, T]$. Let $s(t)$ denote the position of the car at time t. We suppose that we are initially at rest at the origin, and we want to drive forwards on the road. We have control over an accelerator, which we can use to accelerate or brake, but acceleration costs fuel. The problem statement is, suppose we want to drive far yet save fuel, how should we drive?

This problem can be formulated as a *Bolza problem* or a problem with fixed end time and free end point. We need to maximaze:

$$P[\alpha] = -\int_0^T \frac{1}{2} k \max(0, \alpha(t))^2 dt + s(T)$$

subject to

$$\begin{aligned} &\dot{s}(t) = v(t), \quad s(0) = 0 \\ &\dot{v}(t) = l\alpha(t), \quad v(0) = 0, \\ &\alpha(t) \in [-1, 1] \text{ for all } t. \end{aligned}$$

Here, the velocity is the derivative of the coordinate, the acceleration is propositional to our "normed acceleration" α with some factor l, $\alpha = 1$ means that we use acceleration pedal in full, up to 100%, $\alpha = -1$ means that we use brakes in full, up to 100%, the fuel cost is related to the "normed acceleration"s by $\frac{1}{2} k \max(0, \alpha^2)$ (braking spends no fuel) with some factor k.

Now we will apply PMP. **See lecture notes, page 33 and theorem 13**. This is a problem in $\mathbb{R}^2, \mathbf{x} = (s, v) \in \mathbb{R}^2$. Let's denote $\mathbf{p} = (p_1, p_2)$. Our Hamiltonian [**why?**]:

$$H(\mathbf{x}, \mathbf{p}, a) = p_1 v + p_2 l a - 1/2 k \max(0, a^2(t))$$

Firstly, find $\mathbf{p}^*$ using adjoint equations and terminal conditions [**why?**]:
$p_1^*(T) = 1$
$p_2^*(T) = 0$
$\dot{p^*}_1 = -\partial H/\partial s = 0$
$\dot{p^*}_2 = -\partial H/\partial v = -p_1$

Therefore, we get [**why?**]: $p^*(t)_1 = 1,\ p_2^*(t) = T - t.$

$$H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) = v^*(t) + (T-t)l\alpha^*(t) - 1/2k\max(0, \alpha^{*2}(t))$$

Now we can use the maximization principle, which tells:

$$H(\mathbf{x}^*(t), \mathbf{p}^*(t), \alpha^*(t)) = \max_{a\in[-1,1]} \underbrace{\left(v^*(t) + (T-t)la - 1/2k\max(0, a^2(t))\right)}_{\text{function of } a}$$

In other words, for each time t, our optimal solution $\alpha^*(t)$ should be equal to such a, that the expression at the right obtains its maximum value. For $a > 0$ it obtains its maximum at $a = l(T-t)/k$ [**why?**]. So, finally we get [**why?**]:

$$\alpha^*(t) = \min(l(T-t)/k, 1)$$

Thus, we should drive at maximum acceleration, and then ease off on the accelerator linearly [**draw the graph of** $\alpha^*(t)$].