Equations of Plane in Space

Assume that some plane α in the space $\mathbb{E}$ is chosen and fixed. In order to study various equations determining this plane we choose some coordinate system O, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in the space $\mathbb{E}$. Locus of plane α may be described by the radius vectors of points contained in it.

We derived and started investigate some forms of the equation of this plane.

Vectorial parametric equation:

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a} + \tau\boldsymbol{b}, \tag{1}$$

where $\boldsymbol{r}_0$ is radius vector of initial point, $\boldsymbol{a}$ and $\boldsymbol{b}$ are direction vectors of the plane.

This equation yields coordinate parametric form:

$$\begin{cases} x = x_0 + a_x t + b_x \tau \\ y = y_0 + a_y t + b_y \tau \\ x = z_0 + a_z t + b_z \tau, \end{cases} \tag{2}$$

where x_0, y_0, z_0, a_i and b_i are components of $\boldsymbol{r}_0$, $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Vectorial normal equation:

$$(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = 0, \text{ or} \tag{3}$$

$$\boldsymbol{r} \cdot \boldsymbol{n} = D, \tag{4}$$

where $\boldsymbol{n}$ is normal vector, and constant D is result of dot product of this normal vector and radius vector of initial point.

Expression normal vector as cross product of direction vectors yield vectorial form of canonical equation of plane:

$$(\boldsymbol{r} - \boldsymbol{r_0}, \boldsymbol{a}, \boldsymbol{b}) = 0 \tag{5}$$

or

$$(\boldsymbol{r}, \boldsymbol{a}, \boldsymbol{b}) = D \tag{6}$$

Writing of the dot product in coordinate form yields general equation of the plane:

$$Ax + By + C + D = 0, \tag{7}$$

where A, $B$and C are covariant coordinates of normal vector.

Solving of the mixed product in canonical equation produces coordinate canonical equation:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = 0 \tag{8}$$

or

$$\begin{vmatrix} x & y & z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = D' \tag{9}$$

Now we continue investigating various forms of plane equations and their properties.

1 Angle between planes

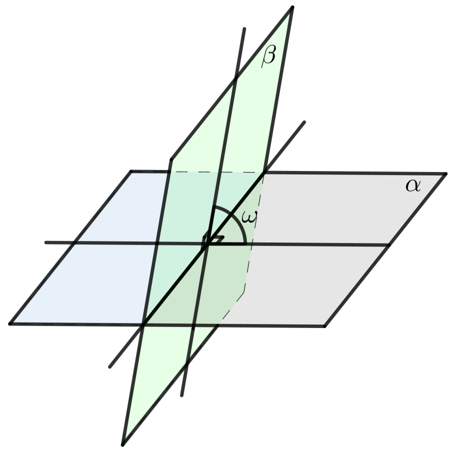


Figure 1: Angle between planes

Definition. Angle between two planes α and β is angle between two lines $a \in \alpha$, $b \in \beta$ which are perpendicular with intersection line of these planes.

Normal vectors to both lines established in the same point of intersection line shape equal angle.

Hence, cosines of this angle may be calculated with formula:

$$\cos\theta = \frac{\boldsymbol{n_1} \cdot \boldsymbol{n}_2}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}$$

If planes are expressed with general equations

$$A_1x + B_1y + C_1z + D = 0$$
$$A_2x + B_2y + C_2z + D = 0$$

with respect to right orthonormal basis, this formula has expansion:

$$\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + A_3B_3}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$$

Problem 1

Find the smallest angle between the planes:

$$3x + 2y - 5z - 4 = 0, \text{ and}$$
$$2x - 3y + 55 - 8 = 0,$$

Basis is right orthonormal.

Solution

$$\cos\theta = \left|\frac{3\cdot 2 - 3\cdot 3 - 5\cdot 5}{\sqrt{9+4+25}\sqrt{4+9+25}}\right| = \frac{25}{38}$$

Parallel and perpendicular planes

Two planes $A_1x + B_1y + C_1z + D = 0$ and $A_1x + B_1y + C_1z + D = 0$ are parallel if normal vector of these planes are collinear, hence there is $p > 0$: $p\boldsymbol{n}_2 = \boldsymbol{n}_1$.

Expressed in coordinates, this yields:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = p \tag{10}$$

For all non-zero A_i, B_i, C_i.

If any of these parameters is zero, its counterpart must be zero too, and ratio preserves for the rest of pairs.

Two planes are perpendicular, if their normal vectors are perpendicular.

In orthonormal basis this has expression:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0$$

Problem 2

Show that plane $Ax + By + D = 0$ is perpendicular with coordinate plane xOy. Basis is right orthonormal basis.

Solution

Coordinates of normal vector of $Ax + By + D = 0$ are $(A, B, 0)$.

Plane xOy, in its turn, has equation $z = 0$, and normal vector with coordinates $(0, 0, 1)$ (actually, third basis vector $\boldsymbol{e}_3$)

Dot product of these vectors is $A \cdot 0 + B \cdot +0 \cdot 1 = 0$, hence the vector are perpendicular.

Definition. Planes $Ax+By+D = 0$, $By+Cz+D = 0$ and $Ax+Cz+D = 0$ are perpendicular with planes xOy, yOz and xOz respectively, while these planes are expressed in coordinate system with right orthonormal basis.

2 Equation of the plane passing through three given points

Suppose three different not laying on a single line points are specified, and we are looking for equation of the plane containing them all.

General form of this equation is $Ax + By + Cz + D = 0$.

Writing this equation with condition that paint passes through three points (x_1, y_1, z_1), (x_2, y_2, z_3), (x_3, y_3, z_3) and any point of the plane (x, y, z) different form that three yields homogenous system of equations with unknown A, B, C and D:

$$\begin{cases} Ax + By + Cz + D = 0 \\ Ax_1 + By_1 + Cz_1 + D = 0 \\ Ax_2 + By_2 + Cz_2 + D = 0 \\ Ax_3 + By_3 + Cz_3 + D = 0. \end{cases}$$

Remark. It is important to underline that we suppose that (x, y, z) is "known" a point of plane, and we actually looking for A, B, C and D.

System in matrix form:

$$\begin{pmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

For any point of the plane (x, y, z) this homogenous system must have non-zero solution, hence for each point of the plane following condition must be fulfilled:

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{11}$$

Definition. Equation (11) is the **equation of a plane passing through three given points**.

Problem 1

Find the general equation of the plane through $(1, 1, -1)$, $(-2, -2, 2)$, $(1, -1, 2)$

Solution

Substitution given points into (11) yields:

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & -1 & 1 \\ -2 & -2 & 2 & 1 \\ 1 & -1 & 2 & 1 \end{vmatrix} = 0$$

$$(-1)\begin{vmatrix}1 & 1 & -1\\ -2 & -2 & 2\\ 1 & -1 & 2\end{vmatrix}+\begin{vmatrix}x & y & z\\ -2 & -2 & 2\\ 1 & -1 & 2\end{vmatrix}-\begin{vmatrix}x & y & z\\ 1 & 1 & -1\\ 1 & -1 & 2\end{vmatrix}+\begin{vmatrix}x & y & z\\ 1 & 1 & -1\\ -2 & -2 & 2\end{vmatrix}=0 \qquad (12)$$

Calculation of these determinants and simplification yields:

$$x - 3y - 2z = 0.$$

Equation of a plane passing through three given points may be also expressed in different manner.

Suppose these points be $A(x_0, y_0, z_0)$, $B(x_1, y_1, z_1)$ and $C(x_3, y_3, z_3)$.

Let us select A as initial point, and $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as direction vector with coordinates $(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ and $(x_2 - x_0, y_2 - y_0, z_2 - z_0)$.

Rewriting the canonical equation (8) now yields:

$$\begin{vmatrix}x - x_0 & y - y_0 & z - z_0\\ x_1 - x0 & y_1 - y_0 & z_1 - z_0\\ x_2 - x0 & y_2 - y_0 & z_2 - z_0\end{vmatrix} = 0 \qquad (13)$$

Purposed here approach appears more accurate because direct solution of the system of tree equations with four unknown variables

$$\begin{cases} Ax_1 + By_1 + Cz_1 + D = 0\\ Ax_2 + By_2 + Cz_2 + D = 0\\ Ax_3 + By_3 + Cz_3 + D = 0,\end{cases}$$

requires elimination of one of the variables with change of the variables to some ratios, e.g. $\dfrac{A}{C}$, $\dfrac{B}{C}$, $\dfrac{D}{C}$, and we must demand non-zero denominator.

Writing the honest equation (11) or (13) is strongly recommended instead attempts to solve such system directly.

3 Triple intercept equation of a plane in the space

Let us consider a plane in the space that does not pass through the origin and intersects with each of the three coordinate axes.

This means all-non-zero A, B, C and D in the general equation of the line.

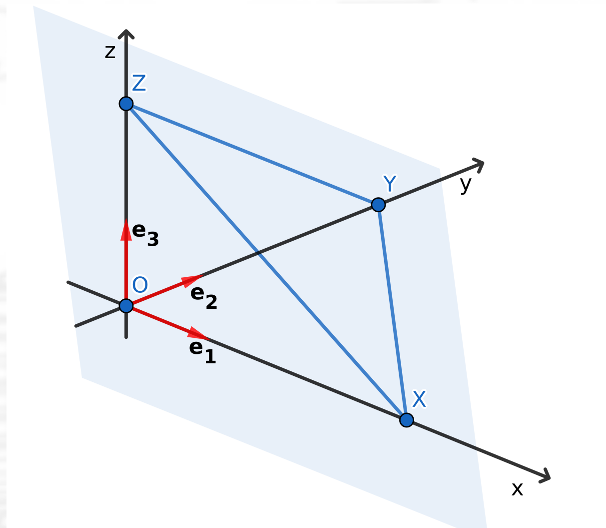


Figure 2: Triple intercept of the plane

Through $X(a, 0, 0)$, $Y(0, b, 0)$ and $Z(0, 0, c)$ denoted three intercept points of the plane.

The quantities a, b, and c have expression through the constant parameters A, B, C, and D of the general equation of the line $Ax_By + Cz + D = 0$ by means of the following formulas.

Letting $y = 0$ and $z = 0$, we yield

$$\begin{aligned} Aa + D &= 0 \\ a &= -\frac{D}{A}. \end{aligned}$$

Letting $x = 0$ and $z = 0$, we yield

$$\begin{aligned} Bb + D &= 0 \\ b &= -\frac{D}{B}. \end{aligned}$$

Letting $x = 0$ and $y = 0$, we yield

$$\begin{aligned} Cc + D &= 0 \\ c &= -\frac{D}{C}. \end{aligned}$$

With divide of the $Ax + By + Cz + D = 0$ with $-D$ we derive:

$$-\frac{A}{D}x - \frac{B}{D}y - \frac{C}{D}z - 1 = 0$$

Substitution of a, b, and c yields

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \tag{14}$$

Definition. We call equation (14) the **triple intercept equation of a plane in the space**

Definition. Lines XY, XZ and YZ are called **traces** of the plane.

To express these traces we're letting corresponding coordinate to zero and eliminating corresponding degree of freedom.

Problem 1

Express equation of the plane $2x + 3y + 6z = 12$ in triple intercept form and write equations of its traces.

Solution

General equation of this line is:

$$2x + 3y + 6z - 12 = 0$$

Intercepts of this equation are:

$$a = -\frac{D}{A} = 6$$
$$b = -\frac{D}{B} = 4$$
$$c = -\frac{D}{C} = 2$$

Equation in triple intercept form is

$$\frac{x}{6} + \frac{y}{4} + \frac{z}{2} = 1$$

Equation of the xy-trace is

$$2x + 3y - 12 = 0$$

Equation of yz-trace is

$$3y + 6z - 12 = 0$$
$$y + 2z - 4 = 0$$

Equation of xz-trace is

$$2x + 6z - 12 = 0$$
$$x + 3z - 6 = 0$$

4 Normal equation of the plane. Distance to plane from a point

Suppose plane expressed with general equation

$$Ax + By + Cz + D = 0.$$

Suppose point with coordinates $X(x, y, z)$ laying *not* on this plane.

What now means expression $Ax + By + Cz + D$, where x, y and z are coordinates of X?

Let point of the plane $X_0(x_0, y_0, z_0)$ be the base of the segment originating in X and perpendicular with plane.

Obviously, $Ax_0 + By_0 + Cz_0 + D = 0$

We can write:

$$\begin{aligned} d = Ax + By + Cz + D = Ax + By + Cz + D - (Ax_0 + By_0 + Cz_0 + D) = \\ = A(x - x_0) + B(y - y_0) + C(z - z_0) = \\ = \boldsymbol{n} \cdot \overrightarrow{X_0X} = \pm|\boldsymbol{n}||\overrightarrow{X_0X}|. \end{aligned}$$

While vectors $\boldsymbol{n}$ and $\overrightarrow{X_0X}$ are both perpendicular with the same plane, hence they are collinear.

$|\overrightarrow{X_0X}|$ constructed to have length equal with distance from the point X to the plane.

Signed length of the normal vector

$$\pm|\boldsymbol{n}|$$

is the *ratio* between this length and expression $Ax + By + Cz + D$.

Making this ratio be 1 yields direct calculation of any length.

Suppose basis of given coordinate system is right orthonormal, hence

$$\pm|\boldsymbol{n}| = \pm\sqrt{A^2 + B^2 + C^2}$$

This unity ratio may be achieved with division of the equation with $\pm\sqrt{A^2 + B^2 + C^2}$.

Coefficients with x, y and z in this **normalized** form became *direction cosines* of the normal vector, and equation of the plane takes form:

$$x \cos\alpha + y \cos\beta + x \cos\gamma - p = 0. \tag{15}$$

To reveal meaning of p we let $x = 0$, $y = 0$ and $z = 0$. Hence, p is numerically equal with perpendicular distance to the plane from the origin.

While we're normalizing equation of the plane, we demand the sign of the $\pm\sqrt{A^2 + B^2 + C^2}$ be negative with sign of D. If D is zero, we take sign of C.

Distance from the point to the plane calculated as

$$d = \left|\frac{Ax + By + Cz + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Problem 1

Find the perpendicular distance from the point $(-2, 2, 3)$ to the plane $8x - 4y - z - 8 = 0$. Basis is right orthonormal.

Solution

Normalization ratio of this line is $\sqrt{64+4+1} = \sqrt{81} = 9$, taken positive because $D < 0$.

Distance from the point to plane is:

$$\frac{8(-2) - 4 \cdot 2 - 3 - 12}{9} = -\frac{35}{9}.$$

The negative sign shows that the point and the origin are on the same side of the plane.

Problem 2

Discuss the locus of the equation $2x + 3y + 6z = 12$ in right orthonormal basis.

Solution

Since the equation is of the first degree it represents a plane.

Normal vector of this plane is vector with coordinates $(2, 3, 6)$, or any vector with proportional coordinates.

Normalization ratio for this equation is $\sqrt{4+9+36} = \sqrt{49} = 7$, taken with $+$, as $D < 0$.

Normal form of the equation:

$$x\frac{2}{7} + y\frac{3}{7} + z\frac{6}{7} - \frac{12}{7} = 0$$

Distance from the origin to the plane is $\frac{12}{7}$.

Intercepts of this equation are:

$$a = -\frac{D}{A} = 6$$
$$b = -\frac{D}{B} = 4$$
$$c = -\frac{D}{C} = 2$$

Equation in triple intercept form is

$$\frac{x}{6} + \frac{y}{4} + \frac{z}{2} = 1$$

Equation of the xy-trace is

$$2x + 3y - 12 = 0$$

Equation of yz-trace is

$$3y + 6z - 12 = 0$$
$$y + 2z - 4 = 0$$

Equation of xz-trace is

$$2x + 6z - 12 = 0$$
$$x + 3z - 6 = 0$$

The intercepts and traces are shown on the figure.

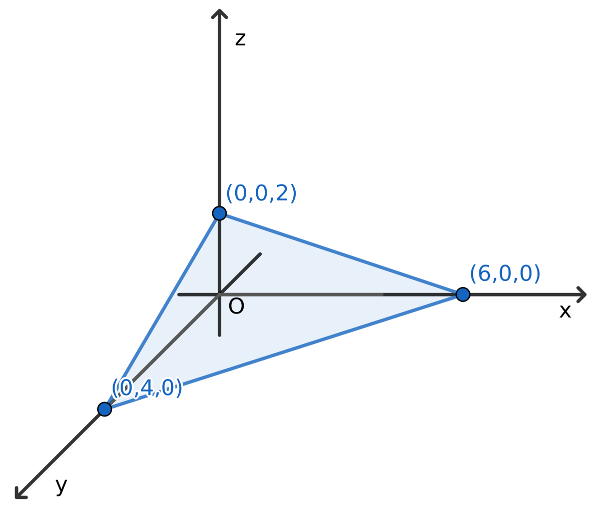


Figure 3: Sketch of the traces and intercepts for $8x - 4y - z - 8 = 0$

5 Relative position of three planes

Suppose three different non-parallel planes

$$\begin{aligned} A_1x + B_1y + C_1z + D = 0 \\ A_2x + B_2y + C_3z + D = 0 \\ A_3x + B_3y + C_3z + D = 0 \end{aligned}$$

are intersecting pairwise shaping arbitrary lines. If that lines intersect in the same point, we say that the planes are intersecting in this point.

Suppose (x, y, z) is such intersecting point.

Hence, this point is the only solution for system

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_3z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0. \end{cases}$$

This condition means that determinant of this system is non-zero:

$$\begin{cases} B_1C_2 = B_2C_1 \\ B_2C_3 = B_3C_2 \\ B_1C_3 = \end{cases}$$

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0$$

Therefore, vectorial meaning of this determinant is the mixed product of the normal vectors of these planes.

This non-zero product means that all three normal vectors are *non-coplanar*.

Coplanarity of these normals yields a variety of cases.

First, suppose that one of the minors of Δ is non-zero. rank.

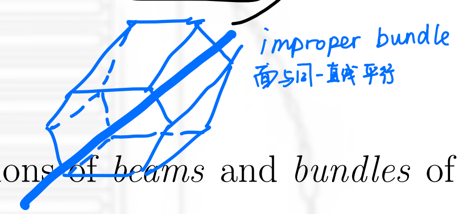
If at least on of the Δ_i determinants of the system (see Cramer's rule) is non-zero, then one plane is parallel to the intersection line of two others.

If all Δ_i are zeros, planes intersect with a single line.

Second, suppose all minors of Δ are zeros.

~~If there is non-zero Δ_i~~, all three planes are parallel, ~~and oversize~~ or they coincide.

Correct: non-zero minor of Δi



6 Bundles and beams of planes

To describe relative position of family of planes, we introduce notions of *beams* and *bundles* of planes.

bundle → normal vector non-coplanar

Definition. We call family of planes sharing common single point the **proper bundle** of planes.

Definition. We call family of planes coplanar with a single line the **improper bundle** of planes.

Definition. We call family of planes intersecting with common line the **proper beam** of planes.

Definition. We call family of parallel planes the **improper beam** of planes.

7 Equation of proper beam of planes

Theorem 7.1. *Any plane in proper beam may be expressed as*

$$p(A_1x + B_1y + C_1z + D_1) + s(A_2x + B_2y + C_2z + D_2) = 0,$$

*where $A_ix + B_iy + C_iz + D_i$ are equations of two planes intersecting by the line (**main planes**). Parameter p and q are a real numbers.*

To derive this formula, we need to proof some lemmas first.

Lemma 7.2. *Suppose planes $A_ix + B_iy + C_iz + D_i$, $i = 1..3$ are parts of the beam, proper or improper. Then rank of the matrix*

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{vmatrix} = 0$$

is 2 or 1.

Proof. While planes are part of the beam the system

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_3z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0. \end{cases}$$

Has infinite variety of solutions (proper beam) or has no one solution (improper beam), hence rank of the matrix is 2 or 1. □

Lemma 7.3. *If for family of planes $A_ix + B_iy + C_iz + D_i$, $i = 1..3$ rank of the matrix*

$$M = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{vmatrix}$$

is 2 or 1, then these planes are part of proper or improper beam.

Proof. Let rank of M be 2. Hence, rank of

have solution(s)

augment / coefficient matrix have same ranl

$$m = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

is also 2. This yields infinite variety of solutions of valid system

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_3z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0. \end{cases}$$

It is granted only with crossing planes shaping the system. If rank is 1 this system has no solutions and planes (at least two of them) are parallel. □

Now we return to the theorem 7.1.

Suppose that $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parts of the beam. For any other plane of this beam $A_3x + B_3y + C_3z + D_2 = 0$

System

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_3z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases}$$

will have the infinite variety of solutions, and rank of the matrix

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{vmatrix}$$

is 2. Hence, third row is linear combination of first and second:

$$A_3x + B_3y + C_3z + D = p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_3z + D_2).$$

p and q are not zeros in the same time. Equation of the third plane is

$$p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_3z + D_2) = 0.\Box$$

If we are given with expression

$$A_3x + B_3y + C_3z + D = p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_3z + D_2),$$

then each term on the left side is a linear combination of the terms in the right:

$$\begin{aligned} A_3 &= pA_1 + qA_2 \\ A_3 &= pB_1 + qB_2 \\ A_3 &= pC_1 + qC_2, \end{aligned}$$

thus third equation in the system

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_3z + D_2 = 0 \\ A_3x + B_3y + C_3z + D_3 = 0 \end{cases}$$

is linear combination of first and second. And, while first and second row express non-collinear planes, rank of the matrix of this system is 2, and plane

$$p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_3z + D_2)$$

is part of the proper beam along with $A_1x+B_1y+C_1z+D_1 = 0$ and $A_2x+B_2y+C_3z+D_2 = 0$.

8 Equation of proper bundle of planes

Theorem 8.1. *Any plane in proper bundle may be expressed as*

$$p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_2z + D_2) + s(A_3x + B_3y + C_3z + D_3) = 0,$$

where $A_ix+B_iy+C_iz+D_i$ are equations of three planes having the ***center*** *of bundle as common point (**main planes**). Parameters p, q and s are real numbers.*

To derive this formula first we proof some lemmas.

Lemma 8.2. *Suppose planes $A_ix + B_iy + C_iz + D_i$, $i = 1..4$ are parts of the bundle, proper or improper. Then*

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0$$

Proof. Suppose four planes $A_ix + B_iy + C_iz + D_i$, $i = 1..4$ are parts of the proper bundle with center (x_0, y_0, z_0). Hence, system of linear equations is valid

$$\begin{cases} A_1x_0 + B_1y_0 + C_1z_0 + D_1 = 0 \\ A_2x_0 + B_2y_0 + C_2z_0 + D_2 = 0 \\ A_3x_0 + B_3y_0 + C_3z_0 + D_3 = 0 \\ A_4x_0 + B_4y_0 + C_4z_0 + D_4 = 0 \end{cases}$$

This yields linearly dependent columns in the determinant of system's matrix

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

Hence, $\Delta = 0$.

Suppose four planes $A_ix + B_iy + C_iz + D_i$, $i = 1..4$ are parts of the proper bundle and (n_x, n_y, n_z) is the vector coplanar with all these planes.

Hence,

$$\begin{aligned} A_1n_x + B_1n_y + C_1n_z = 0 \\ A_2n_x + B_2n_y + C_2n_z = 0 \\ A_3n_x + B_3n_y + C_3n_z = 0 \\ A_4n_x + B_4n_y + C_4n_z = 0, \end{aligned}$$

Thus, Δ contains tree linearly dependent columns and $\Delta = 0$ too. $\square$

Lemma 8.3. *If for family of planes $A_ix + B_iy + C_iz + D_i$, $i = 1..4$*

$$\Delta = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} = 0,$$

then these planes are part of the bundle of planes, proper or improper.

Proof. $\Delta = 0$ yields, that rank of the matrix

$$M = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \\ A_4 & B_4 & C_4 \end{pmatrix}$$

is 3 or less than 3.

If it is 3, for any three planes, say for 1, 2 and 3

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \neq 0$$

We do not leak any generalization, as we are free to rearrange numbers.

There is their intersection point (x_0, y_0, z_0).

Rank 3 yields that three first lines in the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{pmatrix}$$

are linearly independent and express fourth as linear combination:

$$\begin{aligned} A_4 &= pA_1 + qA_2 + sA_3 \\ B_4 &= pB_1 + qB_2 + sB_3 \\ C_4 &= pC_1 + qC_2 + sC_3, \end{aligned}$$

and

$$A_4x + B_4y + C_4z + D_4 = \tag{16}$$
$$p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_2z + D_2) + s(A_3x + B_3y + C_3z + D_3). \tag{17}$$

This yields that (x_0, y_0, z_0) is also point of the fourth plane and is center of bundle.

If rank of M is 2, then only two normal vectors of these planes are linearly independent.

Suppose that vectors are $\boldsymbol{n}_1$, and $\boldsymbol{n}_2$. Then we have expression for two other normal vectors.

$$\boldsymbol{n}_3 = a\boldsymbol{n}_1 + b\boldsymbol{n}_2,$$
$$\boldsymbol{n}_4 = a'\boldsymbol{n}_1 + b'\boldsymbol{n}_2.$$

The system

$$\begin{cases} A_1 n_x + B_1 n_y + C_1 n_z = 0 \\ A_2 n_x + B_2 n_y + C_2 n_z = 0 \end{cases} \tag{18}$$

has non-trivial solution (l, m, n) (rank of M is 2).

The system

$$\begin{cases} A_3 n_x + B_3 n_y + C_3 n_z = 0 \\ A_4 n_x + B_4 n_y + C_4 n_z = 0 \end{cases}$$

is derived form (18) with combining of equations:

$$\begin{cases} (a+b)A_1 n_x + (a+b)B_1 n_y + (a+b)C_1 n_z = 0 \\ (a'+b')A_1 n_x + (a'+b')B_1 n_y + (a'+b)'C_1 n_z = 0, \end{cases}$$

and also has this solution (l, m, n).

Hence, vector (l, m, n) is normal for all four planes, and they are coplanar.

If the rank is 1, all normal vectors are collinear, thus planes are coplanar.

□

Now return to the theorem 8.1.

Let

$$A_1 x + B_1 y + C_1 z + D_1 = 0,$$
$$A_2 x + B_2 y + C_2 z + D_2 = 0 \text{ and}$$
$$A_3 x + B_3 y + C_3 z + D_3 = 0$$

be three planes having common point and not being a part of some proper beam.

Rank of the matrix

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{pmatrix}$$

is 3, and any row added to it is linear combination of that three rows.

Hence,

$$A_4 = pA_1 + qA_2 + sA_3$$
$$B_4 = pB_1 + qB_2 + sB_3$$
$$C_4 = pC_1 + qC_2 + sC_3,$$

and

$$A_4x + B_4y + C_4z + D_4 =$$
$$p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_2z + D_2) + s(A_3x + B_3y + C_3z + D_3).$$

Numbers p, q and s can not be zeros in the same time. If we let p be non-zero and write equation of this new plane, we yield:

$$p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_2z + D_2) + s(A_3x + B_3y + C_3z + D_3) = 0$$
$$A_1x + B_1y + C_1z + D_1 + k(A_2x + B_2y + C_2z + D_2) + l(A_3x + B_3y + C_3z + D_3) = 0.$$

Here $k = \dfrac{q}{p}$, $l = \dfrac{s}{p}$. $\square$

If we are given with expression

$$A_4x + B_4y + C_4z + D_4 =$$
$$p(A_1x + B_1y + C_1z + D_1) + q(A_2x + B_2y + C_2z + D_2) + s(A_3x + B_3y + C_3z + D_3)$$

with at least one non-zero p, q, or s, then each coefficient $*_4$ ($*$ is A B, C or D) is linear combination of $*_1$, $*_2$ and $*_3$, thus, fourth row in the

$$\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{pmatrix}$$

is linear combination of first three, rank of this matrix is 3, and these planes are parts of the proper bundle. $\square$

9 Problems corner

In problems above we deal with right orthonormal bases.

Problem 1

Find the point of intersection of the planes:

$$\begin{aligned} x + 2y - z - 6 &= 0 \\ 2x - y + 3z + 13 &= 0 \\ 3x - 2y + 3z + 16 &= 0 \end{aligned}$$

Solution

Here we have three linear equations. The solution of these simultaneous equations determines the coordinates of the point of intersection of the three planes.

Determinant of the matrix of this system:

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & -2 & 3 \end{vmatrix} = 10$$

System has single honest solution.

$$\Delta_1 = \begin{vmatrix} 6 & 2 & -1 \\ -13 & -1 & 3 \\ -16 & -2 & 3 \end{vmatrix} = -10$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & -1 \\ 2 & -13 & 3 \\ 3 & -16 & 3 \end{vmatrix} = 20$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & -1 & -13 \\ 3 & -2 & -16 \end{vmatrix} = -30$$

$$\begin{aligned} x &= \frac{\Delta_1}{\Delta} = \frac{-10}{10} = -1 \\ y &= \frac{\Delta_2}{\Delta} = \frac{20}{10} = 2 \\ z &= \frac{\Delta_3}{\Delta} = \frac{-30}{10} = -3 \end{aligned}$$

Hence, intersection point is $(-1, 2, -3)$

Problem 2

Find the equation of the plane passing through the line of intersection $3x + y - 5z + 7 = 0$ and $x - 2y + 42 - 3 = 0$ and through the point $(-3, 2, -4)$.

Solution

We are looking for a plane form the proper beam shaped with $3x + y - 5z + 7 = 0$ and $x - 2y + 4z - 3 = 0$.

Substituting coordinates of point into the equations of given lines, we yield:

$$3 \cdot (-3) + 2 + -5 \cdot (-4) + 7 = 20$$
$$-3 - 2 \cdot 2 + 4 \cdot (-4) - 3 = -26.$$

Hence, this point is not part of both that planes.

Therefore, general expression for the equation of this plane must be written:

$$p(3x + y - 5z + 7) + q(x - 2y + 4z - 3) = 0;$$

While both p and q are not zeros, we will look for their ratio $k = \dfrac{q}{p}$ instead:

$$(3x + y - 5z + 7) + k(x - 2y + 4z - 3) = 0$$

Substitution of calculated quantities of first and second expressions yield:

$$20 - 26k = 0$$
$$k = \frac{10}{13}.$$

Equation of the plane is

$$(3x + y - 5z + 7) + \frac{10}{13}(x - 2y + 4z - 3) = 0$$

Multiply it by 13 and simplify:

$$49x - 7y - 25z + 61 = 0.$$

Problem 3

Find the equations of the planes which bisect the dihedral angles between the planes

$$6z - 6y + 7z + 21 = 0$$

and

$$2x + 3y - 6z - 12 = 0.$$

Solution

Let (x_1, y_1, z_1) be any point on the bisecting plane. Then the distances of it from the two planes must be equal in magnitude.

Normalization ratio for the first plane is $-\sqrt{36+36+49} = -\sqrt{21} = -11$. Minus because $D = 21$ and is positive.

Normalization ratio for the first plane is $\sqrt{2+9+36} = -\sqrt{21} = 7$. Plus because $D = -12$ and is negative

Equivalence of distances:

$$\frac{6z - 6y + 7z + 21}{-11} = \pm\frac{2x + 3y - 6z - 12}{7}$$

Sign $\pm$ means that we are looking for both bisectors.

Simplifying and separating signs, we obtain two planes:

$$64x - 9y - 17z + 15 = 0;$$
$$20x - 75y + 115z + 279 = 0.$$

Problem 4

Find the equation of the plane through the points $A(1, -2, 2)$, $B(-3, 1, -2)$ and perpendicular to the plane $2x + y - 2 + 6 = 0$.

Solution

While plane in question is perpendicular with $2x + y - 2 + 6 = 0$, normal vector of $2x + y - 2 + 6 = 0$ is collinear with plane in question.

Hence, one of direction vector of this plane has coordinates $(2, 1, -2)$.

As a second direction vector we take $\overrightarrow{AB}$ with coordinates $(-4, 3, -4)$.

We took A s initial point.

Coordinates of these two vectors are not proportional, hence we can start writing canonical equation:

$$\begin{vmatrix} (x-1) & (y+2) & (z-2) \\ 2 & 1 & -1 \\ -4 & 3 & -4 \end{vmatrix} =$$

Calculating this determinant yields:

$$10(z-2) + 12(y+2) - x + 1 = 0$$
$$x - 12y - 10z - 5 = 0$$

Problem 5

Find the equations of the planes parallel to $2x - 3y - 6z - 14 = 0$ and distant 5 units from the origin.

Solution
Improper beam containing given plane may be expressed as

$$2x - 3y - 6z - k = 0$$

Normalization ratio of this family of planes is $\sqrt{4+9+36} = \sqrt{49} = 7$.
Distance of 5 units from any has now expression:

$$\pm 5 = \frac{2x - 3y - 6z - k}{7}$$

Substituting coordinates of the origin $(0,0,0)$ into it we yield:

$$\pm 5 = \frac{2\cdot 0 - 3\cdot 0 - 6\cdot 0 - k}{7},$$

Thus, $k = \pm 35$
Planes are

$$2x - 3y - 6z + 35 = 0$$
$$2x - 3y - 6z - 35 = 0$$

Problem 6
Show that the planes

$$\Pi_1 : \ 7x + 4y - 4z + 30 = 0,$$
$$\Pi_2 : \ 36x - 51y + 12z + 17 = 0,$$
$$\Pi_3 : \ 14x + 8y - 8z - 12 = 0,$$
$$\Pi_4 : \ 12x - 17y + 4z - 3 = 0$$

form four faces of a rectangular parallelepiped.

Solution
First, we notice, that Π_1 and Π_3 have proportional A, B, and C:

$$\frac{7}{14} = \frac{4}{8} = \frac{-4}{-8}\left(=\frac{1}{2}\right),$$

as well Π_2 and Π_4:

$$\frac{36}{12} = \frac{-51}{-17} = \frac{12}{4}\,(=3)\,.$$

Further, the first and second planes are perpendicular since

$$7\cdot 36 + 4\cdot(-51) - 4\cdot 12 = 252 - 204 - 48 = 0.$$

Problem 7
Determine the locus of the equation $x^2 - 2xy_y^2 - 4z^2 = 0$.

Solution
We isolate full squared terms in the left side of equation:

$$x^2 - 2xy_y^2 - 4z^2 = (x-y)^2 - 4z^2 = (x-y-2z)(x-y+2z).$$

Letting this transformed term to be zero yields two equations:

$$(x-y-2z)(x-y+2z) = 0$$

$$\begin{cases} x - y - 2z = 0; \\ x - y + 2z = 0 \end{cases}$$