Exercise 5.

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2022111221

1. Show that for the feasible region defined by

$$(x_1-1)^2+(x_2-1)^2 \le 2, \qquad (x_1-1)^2+(x_2+1)^2 \le 2, \qquad x_1 \ge 0,$$

the MFCQ is satisfied at $x^* = (0,0)^T$ but the LICQ is not satisfied.

Pf: denote: $g_1 = (x_1-1)^2 + (x_2-1)^2 - 2.$ $\quad \nabla g_1 = \begin{pmatrix} 2x_1-2 \\ 2x_2-2 \end{pmatrix} \quad \nabla g_1(x^*) = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$

$g_2 = (x_1-1)^2 + (x_2+1)^2 - 2 \quad \nabla g_2 = \begin{pmatrix} 2x_1-2 \\ 2x_2+2 \end{pmatrix} \quad \nabla g_2(x^*) = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$

$g_3 = -x_1 \quad \nabla g_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \nabla g_3(x^*) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

MFCQ: $\exists\, d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ s.t. $\nabla g_i(x^*)^T d < 0$ for $i = 1,2,3$. MFCQ satisfy.

LICQ: $g_1(x^*) = g_2(x^*) = g_3(x^*) = 0$. 3 vectors can't linear independent in 2-dim space. LICQ not satisfy

2. Consider the following modification of Example 6 (see page 4, LectureNotes 7.pdf), where t is a parameter to be fixed prior to solving the problem:

$$\min_x \; (x_1 - \tfrac{3}{2})^2 + (x_2 - t)^4$$

subject to

$$\begin{bmatrix} 1-x_1-x_2 \\ 1-x_1+x_2 \\ 1+x_1-x_2 \\ 1+x_1+x_2 \end{bmatrix} \ge 0.$$

(a) For what values of t does the point $x^* = (1,0)^T$ satisfy the KKT conditions?
(b) Show that when $t = 1$, only the first constraint is active at the solution, and find the solution.

(a) $\nabla f = \begin{pmatrix} 2x_1-3 \\ 4(x_2-t)^3 \end{pmatrix}$

$\nabla f(1,0) = \begin{pmatrix} -1 \\ -4t^3 \end{pmatrix}$

$c_1 = x_1+x_2-1 \le 0 \quad \nabla c_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$c_2 = x_1-x_2-1 \le 0 \quad \nabla c_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$c_3 = -x_1+x_2-1 \le 0 \quad \nabla c_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$c_4 = -x_1-x_2-1 \le 0. \quad \nabla c_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$

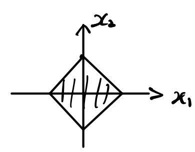
KKT for $(1,0)$:
$$\begin{cases} -1 + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \\ -4t^3 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0. \\ c_i \le 0. \quad \lambda_i c_i = 0. \end{cases}$$

$c_1 = 0 \quad c_2 = 0 \quad c_3 = -2 \quad c_4 = -2$

$\Rightarrow \lambda_3 = \lambda_4 = 0.$

$$\Rightarrow \begin{cases} \lambda_1 = \dfrac{4t^3+1}{2} \ge 0 \\ \lambda_2 = \dfrac{1-4t^3}{2} \ge 0 \end{cases} \Rightarrow t \in [-2^{-\frac{2}{3}}, 2^{-\frac{2}{3}}]$$

(b) KKT:
$$\begin{cases} 2x_1 - 3 + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \\ 4(x_2-1)^3 + \lambda_1 - \lambda + \lambda_3 - \lambda_4 = 0. \\ c_i \le 0. \quad \lambda_i c_i = 0 \end{cases}$$



i/ unconstrain minimum.
$\nabla f = 0 \Rightarrow (x_1^*, x_2^*) = (\frac{3}{2}, 1)$ outside the feasible region. thus the constrained sol must on the boundary.

ii/ if ≥ 2 constrain are active. i.e. sol. on the vertex

① $1 \notin [-2^{-\frac{2}{3}}, 2^{-\frac{2}{3}}]$, $(1,0)$ not s.t. KKT.

② $x^* = (0,1)$. $\begin{pmatrix} -3 + \lambda_1 - \lambda_3 \\ 0 + \lambda_1 + \lambda_3 \end{pmatrix} = 0 \Rightarrow$ no sol. not s.t KKT.

③ $x^* = (-1, 0)$ $\begin{pmatrix} -5 - \lambda_3 - \lambda_4 \\ -4 + \lambda_3 - \lambda_4 \end{pmatrix} = 0 \Rightarrow \lambda_4 = -\frac{9}{2}$ not s.t KKT.

④ $x^* = (0,-1)$ $\begin{pmatrix} -3 + \lambda_2 - \lambda_4 \\ -32 - \lambda_2 - \lambda_4 \end{pmatrix} = 0 \Rightarrow \lambda_4 = -\frac{35}{2}$ not s.t KKT.

iii/ only 1 constrains active $\Rightarrow \nabla f(x) = -\lambda_i \nabla C_i(x)$ for some i.

① if $c_2 = 0 \Rightarrow x_2 = x_1 - 1$.

$\nabla f = \begin{pmatrix} 2x_1 - 3 \\ 4(x_1-2)^3 \end{pmatrix} \begin{matrix} <0 \\ <0. \end{matrix}$ but $-\lambda_4 \nabla C_4(x) = \begin{pmatrix} -\lambda \\ \lambda \end{pmatrix} \begin{matrix} \leq 0 \\ \geq 0 \end{matrix}$ Contradicts.

② if $c_3 = 0 \Rightarrow x_2 = x_1 + 1$ $x_1 \in (-1, 0)$

$\nabla f = \begin{pmatrix} 2x_1 - 3 \\ 4x_1^3 \end{pmatrix} \begin{matrix} <0 \\ <0 \end{matrix}$ but $-\lambda_3 \nabla C_3(x) = \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} \begin{matrix} \geq 0 \\ \leq 0 \end{matrix}$ Contradicts.

③ if $c_4 = 0 \Rightarrow x_2 = -1 - x_1$

$\nabla f = \begin{pmatrix} 2x_1 - 3 \\ 4(-2-x_1)^3 \end{pmatrix} \begin{matrix} <0 \\ <0 \end{matrix}$ but $-\lambda_4 \nabla C_4(x) = \begin{pmatrix} \lambda \\ \lambda \end{pmatrix} \begin{matrix} \geq 0 \\ > 0 \end{matrix}$ Contradicts.

thus. only $c_1 = 0$ is possible $\Rightarrow \begin{cases} \text{sol: } \lambda_2 = \lambda_3 = \lambda_4 = 0. \\ \begin{cases} 2x_1 - 3 + \lambda_1 = 0 \\ 4(x_2-1)^3 + \lambda_1 = 0 \\ x_1 + x_2 = 1. \end{cases} \end{cases}$ $\Rightarrow 4x_1^3 + 2x_1 - 3 = 0$

LHS is monotonic increase w.r.t. x_1.

$x_1 \approx 0.728$. $x_2 = 0.272$.

3. Consider the half space defined by

$$H = \{x \in \mathbb{R}^n \mid a^T x + \alpha \geq 0\},$$

where $a \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ are given. Formulate and solve the optimization problem for finding the point in H that has the smallest Euclidean norm.

Formulate: $\min_x f(x) = \|x\|_{\mathbb{R}^n} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

s.t. $-\alpha - a^T x \leq 0$.

Sol: use the func. $\tilde{f}(x) = \frac{1}{2}\|x\|^2$. the optimization problem will be equivalent w.r.t. f. $\tilde{f}$

apply the KKT condition.

$\begin{cases} x - \lambda a = 0 \\ \lambda(\alpha + a^T x) = 0 \\ \lambda \geq 0. \end{cases}$

$\lambda(\lambda \|a\|^2 + \alpha) = 0$

① $\lambda = 0$. $x = 0$

② $\lambda > 0$. $\lambda = -\frac{\alpha}{\|a\|^2}$ $x = -\frac{\alpha}{\|a\|^2} a$. $(\alpha < 0)$.

thus. $\begin{cases} \alpha \geq 0. \text{ we have } x^* = 0 \\ \alpha < 0 \text{ we have } x^* = -\frac{\alpha}{\|a\|^2} a. \end{cases}$

4. Solve the problem

$$\min_x \; x_1 + x_2 \quad \text{s.t. } x_1^2 + x_2^2 = 1$$

by eliminating the variable x_2. Show that the choice of sign for a square root operation during the elimination process is critical; the "wrong" choice leads to an incorrect answer.

Sol: $x_2 = \pm\sqrt{1-x_1^2}$ $\quad x_1 \in [-1,1]$.

① $\min_{x_1} f_1(x_1) = x_1 + \sqrt{1-x_1^2}$ $\quad$ s.t. $x_1 \in [-1,1]$

$\nabla f_1(x_1) = 0 \Rightarrow x_1 = \pm\frac{1}{\sqrt{2}} \in [-1,1]$. $\Rightarrow f_1(\frac{1}{\sqrt{2}}) = \sqrt{2}$ $\quad f_1(-\frac{1}{\sqrt{2}}) = 0$.

$\nabla f_1(x_1) = 1 - \frac{x_1}{\sqrt{1-x_1^2}}$ $\quad$ check the end point. $f_1(1) = 1$ $\quad f_1(-1) = -1$.

$\Rightarrow \min f_1 = -1$.

② $\min_{x_1} f_2(x_1) = x_1 - \sqrt{1-x_1^2}$ $\quad$ s.t. $x_1 \in [-1,1]$

$\nabla f_2(x_1) = 1 + \frac{x_1}{\sqrt{1-x_1^2}}$ $\quad$ $\left.\begin{array}{l}\nabla f_2(x_1) = 0 \Rightarrow x_1 = -\frac{1}{\sqrt{2}} \Rightarrow f_2(-\frac{1}{\sqrt{2}}) = -\sqrt{2} \\ f_2(1) = 1 \quad f_2(-1) = -1\end{array}\right\} \Rightarrow \min f_2(x_1) = -\sqrt{2}$.

only if when we choose f_2, we get the correct minimimum $-\sqrt{2}$.

5. Consider the problem of finding the point on the parabola

$$y = \tfrac{1}{5}(x-1)^2$$

that is closest to $(1,2)$ in the Euclidean norm sense. This can be written as

$$\min f(x,y) = (x-1)^2 + (y-2)^2 \quad \text{s.t. } (x-1)^2 = 5y.$$

(a) Find all the KKT points for this problem. Is the LICQ satisfied?
(b) Which of these points are solutions?
(c) By directly substituting the constraint into the objective function and eliminating x, show that the solutions of this problem cannot be solutions of the original problem.

Sol. (a). $\nabla f(x,y) = \begin{pmatrix} 2(x-1) \\ 2(y-2) \end{pmatrix}$

$c(x,y) = (x-1)^2 - 5y$ $\quad \nabla c(x,y) = \begin{pmatrix} 2(x-1) \\ -5 \end{pmatrix}$

KKT: $\begin{cases} 2(x-1) + 2\mu(x-1) = 0 \\ 2(y-2) - 5\mu = 0 \\ (x-1)^2 - 5y = 0 \end{cases}$ $\quad$ i. $\begin{cases} x = 1. \\ y = 0 \\ \mu = -\frac{4}{5} \end{cases}$ $\quad$ ii. $\mu = -1$. $\Rightarrow y = -\frac{1}{2}$ $\quad$ no real x.

the only KKT point is $(1,0)$. $\quad \nabla c = \begin{pmatrix} 2(x-1) \\ -5 \end{pmatrix} \neq 0$. thus LICQ satisfy.

(b). $f(1,0) = 4$.

(c). $(x-1)^2 = 5y$ $\quad$ original problem $\Rightarrow \min g(y) = y^2 + y + 4$. s.t. $y \geq 0$.

$\nabla g(y) = 0 \Rightarrow 2y + 1 = 0 \Rightarrow y = -\frac{1}{2}$. does not satisfy $y \geq 0$. thus. not sol. of original problem

6. Consider the problem

$$\min_{x \in \mathbb{R}^2} f(x) = -2x_1 + x_2$$

subject to

$$(1 - x_1)^3 - x_2 \geq 0, \qquad x_2 + 0.25x_1^2 - 1 \geq 0.$$

The optimal solution is $x^* = (0, 1)^T$, where both constraints are active.
(a) Does the LICQ hold at this point?
(b) Are the KKT conditions satisfied?
(c) Write down the sets $F(x^*)$ and $C(x^*, \lambda^*)$.
(d) Are the second-order necessary conditions satisfied? Are the second-order sufficient conditions satisfied?

Sol.: (a). $C_1(x_1, x_2) = (1 - x_1)^3 - x_2 \qquad C_2(x_1, x_2) = x_2 + \frac{1}{4}x_1^2 - 1$

$\nabla C_1(x_1, x_2) = (-3(1-x_1)^2, -1)^T$

$\nabla C_2(x_1, x_2) = (\frac{1}{2}x_1, 1)^T$

$\nabla C_1(0,1) = (-3, -1)^T \quad \nabla C_2(0,1) = (0,1)^T \qquad \nabla C_1(0,1) \; \nabla C_2(0,1)$ are linear independent.

LICQ hold.

(b). it suffices to check with $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0. \quad \lambda_i \geq 0$

$\nabla f(x_1, x_2) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

i.e. solve. $\begin{bmatrix} -2 \\ 1 \end{bmatrix} - \lambda_1 \begin{bmatrix} -3 \\ -1 \end{bmatrix} - \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$

$\Rightarrow \begin{cases} 3\lambda_1 = 2 \\ \lambda_1 - \lambda_2 = -1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = \frac{2}{3} > 0 \\ \lambda_2 = \frac{5}{3} > 0. \end{cases}$ KKT satisfy.

(c). denote that $d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad \begin{cases} d^T \nabla C_1 \geq 0 \\ d^T \nabla C_2 \geq 0 \end{cases} \Rightarrow \begin{cases} -3d_1 - d_2 \geq 0 \\ d_2 \geq 0 \end{cases}$

$\begin{cases} d^T \nabla C_1 \geq 0 \\ d^T \nabla C_2 \geq 0 \end{cases} \Rightarrow \begin{cases} d_1 = 0 \\ d_2 = 0 \end{cases}$

$F(x^*) = \{ d : 3d_1 + d_2 \leq 0, \; d_2 \geq 0 \}. \qquad C(x^*, \lambda^*) = \{ d^* = (0,0)^T \}.$

(d). $w \in C(x^*, \lambda^*)$ implies $w = (0,0)^T$ thus $w^T \mathcal{L}_{xx}(x^*, \lambda^*) w = 0.$

the nessary condition satisfy.

no $w \neq 0$. thus the sufficient condition also satisfy.

7. Find the minima of

$$f(x) = x_1 x_2$$

on the unit circle $x_1^2 + x_2^2 = 1$. Illustrate this problem geometrically.

Sol: $\mathcal{L} = x_1 x_2 - \lambda(x_1^2 + x_2^2 - 1) = 0$

$\nabla \mathcal{L} = \begin{bmatrix} x_2 - 2\lambda x_1 \\ x_1 - 2\lambda x_2 \end{bmatrix}$ the KKT condition gives $\begin{cases} \mathcal{L}_x = 0 \\ x_1^2 + x_2^2 - 1 = 0 \end{cases}$ $\frac{x_2}{2x_1} = \frac{x_1}{2x_2} \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1^2 = x_2^2 = \frac{1}{2}$.

$\begin{cases} x_1 = \frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$ $\begin{cases} x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$ $\begin{cases} x_1 = \frac{1}{\sqrt{2}} \\ x_2 = -\frac{1}{\sqrt{2}} \end{cases}$ $\begin{cases} x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = -\frac{1}{\sqrt{2}} \end{cases}$

when $\begin{cases} x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$ or $\begin{cases} x_1 = \frac{1}{\sqrt{2}} \\ x_2 = -\frac{1}{\sqrt{2}} \end{cases}$ f attains minima $\min f = -\frac{1}{2}$.

geometrically, we find the tangent curve in the set of hyperbola $\{x_1 x_2 = c,\ c \in \mathbb{R}\}$ and the circle $x_1^2 + x_2^2 = 1$.

8. Find the maxima of

$$f(x) = x_1 x_2$$

over the unit disk defined by

$$1 - x_1^2 - x_2^2 \geq 0.$$

Sol: denote $g(x) = -x_1 x_2$.

$\mathcal{L} = -x_1 x_2 - \lambda(1 - x_1^2 - x_2^2)$ $\nabla \mathcal{L} = \begin{bmatrix} -x_2 + 2\lambda x_1 \\ -x_1 + 2\lambda x_2 \end{bmatrix}$

the KKT condition $\begin{cases} -x_2 + 2\lambda x_1 = 0 \\ -x_1 + 2\lambda x_2 = 0 \\ \lambda(1 - x_1^2 - x_2^2) = 0 \\ \lambda \geq 0 \end{cases}$ i) $\lambda = 0$. $x_1 = x_2 = 0$

ii) $\lambda > 0$ $x_1^2 = x_2^2 = \frac{1}{2}$.

$\begin{cases} x_1 = \frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$ $\begin{cases} x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = -\frac{1}{\sqrt{2}} \end{cases}$

when $\begin{cases} x_1 = \frac{1}{\sqrt{2}} \\ x_2 = \frac{1}{\sqrt{2}} \end{cases}$ or $\begin{cases} x_1 = -\frac{1}{\sqrt{2}} \\ x_2 = -\frac{1}{\sqrt{2}} \end{cases}$, g attains minima, i.e. f attains maxima. $\max f = \frac{1}{2}$.

9. Convert the following linear program to standard form:

$$\max_{x,y}\ c^T x + d^T y \quad \text{s.t. } A_1 x = b_1,\ A_2 x + B_2 y \leq b_2,\ l \leq y \leq u.$$

$-c^T x^+ + c^T x^- - d^T y^+$

denote $x = x^+ - x^-$, $x^+, x^- \geq 0$.

$y^+ = y - l$, $y^+ + l \leq u \Rightarrow y^+ \leq u - l$.

$A_2 x + B_2 y = A_2(x^+ - x^-) + B_2(y^+ + l)$

the slack variables: $A_2(x^+ - x^-) + B_2(y^+ + l) + s_1 = b_2$

$y^+ + s_2 = u - l$

$-c^T(x^+ - x^-) - d^T(y^+ + l) = -c^T x^+ + c^T x^- - d^T y^+ - d^T l$.

thus, the standard LP:

$X = (x^+\ x^-\ y^+\ s_1\ s_2)^T$

$$\begin{cases} \min f(X) = -[c\ {-c}\ d\ 0\ 0]^T X. \\ \text{s.t. } X \geq 0 \\ \begin{bmatrix} A_1 & -A_1 & 0 & 0 & 0 \\ A_2 & -A_2 & B_2 & I & 0 \\ 0 & 0 & I & 0 & I \end{bmatrix} X = \begin{bmatrix} b_1 \\ b_2 - B_2 l \\ u - l \end{bmatrix} \end{cases}$$

10. Verify that the dual

$$\max\ b^T\lambda \quad \text{s.t.} \quad A^T\lambda + s = c,\ s \geq 0$$

is the original primal problem

$$\min\ c^T x \quad \text{s.t.} \quad Ax = b,\ x \geq 0.$$

Pf: for original problem $\mathcal{L} = c^T x - \lambda^T(Ax-b) - s^T x$. $(s \geq 0)$.

the dual objective $q(\lambda) = \inf_x (c - A^T\lambda - s)^T x + b^T\lambda$.

if $c - A^T\lambda - s \neq 0$. $\inf \to -\infty$.

if $c - A^T\lambda - s = 0$ $\inf_x = b^T\lambda$.

thus. the dual problem is $\max\ b^T\lambda$ s.t. $A^T\lambda + s = c$, $s \geq 0$

11. Complete the proof of Theorem 31 (page 31, LectureNotes 8.pdf) by showing that if the dual

$$\max\ b^T\lambda \quad \text{s.t.}\ A^T\lambda \leq c$$

is unbounded above, the primal

$$\min\ c^T x \quad \text{s.t.} \quad Ax = b,\ x \geq 0$$

must be infeasible.

Pf: we need dual unbounded above $\Rightarrow$ primal infeasible.

Assume the converse, $\exists\ x^*$ s.t. $Ax^* = b$, $x^* \geq 0$.

then for any feasible sol. in dual problem λ, we have $b^T\lambda \leq c^T x^*$ by the Weak Duality thm (thm 27)

which means $b^T\lambda$ has boundary above, contradiction $\square$.

12. Theorem 31 does not exclude the possibility that both primal and dual are infeasible. Give a simple linear program for which such is the case.

Example: $\min\ -x_1$ s.t. $0 \cdot x_1 = 1$. $x_1 \geq 0$.

the primal infeasible since no feasible x_1. s.t. $0 \cdot x_1 = 1$.

Dual: $\max\ \lambda_1$. s.t. $0 \cdot \lambda_1 \leq -1$. $\lambda_1 \geq 0$

the dual problem infeasible too.

13. Show that the dual of the linear program

$$\min c^T x \quad \text{s.t. } Ax \geq b,\ x \geq 0$$

is

$$\max b^T \lambda \quad \text{s.t. } A^T \lambda \leq c,\ \lambda \geq 0.$$

Sol: for primal: $\min [c^T\ 0] \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$ s.t. $Ax - b - \tilde{x} = 0$ $\quad \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \geq 0.$

$\mathcal{L}(x, \lambda) = c^T x - \lambda^T (Ax - b) = b^T \lambda + (c - A^T \lambda)^T x$

the dual objective. $p(\lambda) = \inf_x \left(b^T \lambda + (c - A^T \lambda)^T x \right).$

if $\exists j.\ (c - A^T \lambda)^T < 0.$ $\quad x_j \to +\infty.$ $\quad p(\lambda) \to -\infty.$

thus $c - A^T \lambda \geq 0.$ $\quad (\lambda \geq 0).$

the dual problem: $\max b^T \lambda$ $\quad$ s.t. $A^T \lambda \leq c.$ $\lambda \geq 0.$

14. Consider the following linear program:

$$\min -5x_1 - x_2$$

subject to

$$x_1 + x_2 \leq 5, \quad 2x_1 + \tfrac{1}{2}x_2 \leq 8, \quad x \geq 0.$$

(a) Add slack variables x_3 and x_4 to convert this problem to standard form.
(b) Using the simplex procedure (page 16, LectureNotes 9.pdf), solve this problem showing at each step the basis and the vectors $\lambda, s_{\mathcal{N}}, x_{\mathcal{B}}$, and the value of the objective function. (The initial choice of $\mathcal{B}$ for which $x_{\mathcal{B}} \geq 0$ should be obvious once you have added the slacks in part (a).)

Sol: a). $\min\ -5x_1 - x_2.$ $\quad$ s.t. $Ax = b.$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq 0$

where $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

b)

(1). Basis $\mathcal{B} = \{3, 4\}.$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x_{\mathcal{B}} = B^{-1} b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$x_{\mathcal{N}} = (x_1, x_2) = (0, 0).$ $\quad$ solve $B^T \lambda = c_{\mathcal{B}}$ $\quad \Rightarrow \lambda = 0.$

$s_{\mathcal{N}} = c_{\mathcal{N}} - N^T \lambda = (-5, -1)^T$

since $-5, -1 < 0.$ choose $q = 1.$ $\quad Bd = A_q \Rightarrow d = (1, 2)^T > 0.$

$x_q^+ = \min_{i | d_i > 0} \frac{(x_{\mathcal{B}})_i}{d_i} = \min \{5, 4\}.$ thus $p = 2.$ $\quad x_4$ out. x_1 in.

(2) Basis $\mathcal{B} = \{1, 3\}.$ $\quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ $\quad B^{-1} = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix}$

$x_{\mathcal{B}} = B^{-1} b = \begin{bmatrix} 0 & 1/2 \\ 1 & -1/2 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\quad x_{\mathcal{N}} = (0, 0)^T$

solve $B^T \lambda = c_{\mathcal{B}}$ $\quad \Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$ $\quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{5}{2} \end{bmatrix}$

$$S_N = C_N - N^T\lambda = \begin{bmatrix} -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -\frac{5}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{5}{2} \end{bmatrix} > 0$$

thus. we found optimal point $x^* = (4, 0, 1, 0)^T$, $\min f = -20$.

15. (a) Solve the quadratic program

$$\min f(x) = 2x_1 + 3x_2 + 4x_1^2 + 2x_1x_2 + x_2^2$$

subject to $x_1 - x_2 \geq 0$, $x_1 + x_2 \leq 4$, $x_1 \leq 3$. Illustrate it geometrically.
(b) If the objective function is redefined as $q(x) = -f(x)$, does the problem have a finite minimum? Are there local minimizers?

(a) Sol: $G = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad c = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\nabla f(x) = \begin{bmatrix} 2 + 8x_1 + 2x_2 \\ 3 + 2x_1 + 2x_2 \end{bmatrix}$ $\quad \nabla f(x_1, x_2) = 0 \Rightarrow \begin{cases} x_1^* = \frac{1}{6} \\ x_2^* = -\frac{5}{3} \end{cases}$

$\nabla^2 f = \begin{bmatrix} 8 & 2 \\ 2 & 2 \end{bmatrix}$ $\quad \nabla^2 f > 0$. f - strictly convex

and $\begin{cases} x_1^* - x_2^* < 0 \\ x_1^* + x_2^* < 4 \\ x_1^* < 3. \end{cases}$ holds.

we have the minima without constraint s.t. all the constraints.
and f is strictly convex, thus. $x^* = (\frac{1}{6}, -\frac{5}{3})^T$ is the global sol.

Geometrically. the constraint is a unbounded region.
the level set of f is ellipse. the minimal ellipse tangent to the feasible region inside the region. will not attain an boundaries.

(b). $q(x) = -f(x)$

$\begin{cases} x_1 - x_2 \geq 0 \\ x_1 + x_2 \leq 4 \\ x_1 \leq 3 \end{cases}$ $\quad$ Consider a ray $\{(3, t) \mid t \leq 1\}$.
which is always inside the feasible regi

$q(t) = -[6 + 3t + 36 + 6t + t^2] = -[t^2 + 9t + 42] = -(t + \frac{9}{2})^2 - \frac{87}{4}$

$t \to -\infty$, $q \to -\infty$. thus, no finite minimum.

by a). $\nabla^2 q = \begin{bmatrix} -8 & -2 \\ -2 & -2 \end{bmatrix} \precsim 0$. concave. globally

if $\exists$ local minimum, then it will be global minimum $\Rightarrow$ no local minimum.

16. The problem of finding the shortest distance from a point x_0 to the hyperplane $\{x \mid Ax = b\}$, where A has full row rank, can be formulated as the quadratic program

$$\min \tfrac{1}{2}(x - x_0)^T(x - x_0) \quad \text{s.t. } Ax = b.$$

Show that the optimal multiplier is

$$\lambda^* = (AA^T)^{-1}(b - Ax_0),$$

and that the solution is

$$x^* = x_0 + A^T(AA^T)^{-1}(b - Ax_0).$$

Show that in the special case in which A is a row vector, the shortest distance from x_0 to the solution set of $Ax = b$ is $|b - Ax_0|/\|A\|^2$.

Sol: the Lagrangian $\mathcal{L}(x,\lambda) = \frac{1}{2}(x-x_0)^T(x-x_0) - \lambda^T(Ax-b)$

the KKT : $\begin{cases} \nabla_x \mathcal{L}_x = 0 \ (1) & \text{by}(1) \Rightarrow x - x_0 - A^T\lambda = 0 \Rightarrow x = x_0 + A^T\lambda. \\ Ax = b \ (2) & \text{then by } (2). \ A(x_0 + A^T\lambda) = b \end{cases}$

since A has full row rank. AA^T full rank. AA^T invertible.

$\lambda^* = (AA^T)^{-1}(b - Ax_0)$

$x^* = x_0 + A^T\lambda^* = x_0 + A^T(AA^T)^{-1}(b - Ax_0)$

if $A = a^T$ $\quad AA^T = a^Ta = \|a\|_2^2$

$x^* = x_0 + a\dfrac{b - a^Tx_0}{\|a\|_2^2} \Rightarrow \|x^* - x_0\|_2 = |a|\dfrac{|b - a^Tx_0|}{\|a\|_2^2} = \dfrac{|b - Ax_0|}{\|A\|_2} \quad A \in \mathbb{R}^{1\times n}$

17. Use Theorem 21 (page 2, LectureNotes 7.pdf) to verify that the first-order necessary conditions for equality-constrained QPs (page 18, LectureNotes 10.pdf) are given by the KKT system (page 18, LectureNotes 10.pdf).

Pf: Problem: $\min\limits_x q(x) = \frac{1}{2}x^TGx + c^Tx, \quad Ax = b.$

$\mathcal{L}(x,\lambda) = \frac{1}{2}x^TGx + c^Tx - \lambda^T(Ax - b)$

$\nabla\mathcal{L}_x = Gx + c - A^T\lambda$

by thm 21. x^* is minima of the problem. $\exists \lambda^*$. s.t. $\begin{cases} \nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \\ c(x^*) = 0 \end{cases}$

i.e. $\begin{cases} Gx^* + c - A^T\lambda^* = 0 \\ Ax^* - b = 0 \end{cases} \Rightarrow \begin{bmatrix} G & -A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^* \\ \lambda^* \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix}$

18. Verify that the inverse of the KKT matrix (page 25, LectureNotes 10.pdf) is given by

$$\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}^{-1} = \begin{bmatrix} C & E \\ E^T & F \end{bmatrix},$$

with

$$C = G^{-1} - G^{-1}A^T(AG^{-1}A^T)^{-1}AG^{-1}, \quad E = G^{-1}A^T(AG^{-1}A^T)^{-1}, \quad F = -(AG^{-1}A^T)^{-1}.$$

Pf: it sufficies to check $\begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} C & E \\ E^T & F \end{bmatrix} = I$. i.e. $\begin{cases} GC + A^TE^T = I \\ GE + A^TF = 0 \\ AC = 0 \\ AE = I \end{cases}$

$GC + A^TE^T = I - A^T(AG^{-1}A^T)^{-1}AG^{-1} + A^TG^{-1}A^T(AG^{-1}A^T)^{-1}$

since G symmetric, $AG^{-1}A^T$ symmetric.

thus $GC + A^TE^T = I + A^T[(AG^{-1}A^T)^{-1}AG^{-1} - (AG^{-1}A^T)^{-1}AG^{-1}] = I$

$GE + A^TF = A^T(AG^{-1}A^T)^{-1} - A^T(AG^{-1}A^T)^{-1} = 0$

$AC = AG^{-1} - AG^{-1}A^T(AG^{-1}A^T)^{-1}AG^{-1}$

$= AG^{-1} - (AG^{-1}A^T)(AG^{-1}A^T)^{-1}AG^{-1} = AG^{-1} - AG^{-1} = 0$

$AE = AG^{-1}A^T(AG^{-1}A^T)^{-1} = I.$

19. For each of the alternative choices of initial working set W_0 in the example (page 21, LectureNotes 11.pdf), $W_0 = \{3\}$, $W_0 = \{5\}$, and $W_0 = \emptyset$, work through the first two iterations of the Active-Set Method for Convex QP (page 18, LectureNotes 11.pdf).

Problem $\min_{x_1, x_2} q(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2.5)^2$ s.t. $\begin{cases} x_1 - 2x_2 + 2 \geq 0 \\ -x_1 - 2x_2 + 6 \geq 0 \\ -x_1 + 2x_2 + 2 \geq 0 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$

$\Rightarrow$ standard form $G = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $c = \begin{bmatrix} -2 \\ -5 \end{bmatrix}$

$a_i^T x \geq b_i$ where $\begin{cases} a_1 = (1, -2)^T & b_1 = -2 \\ a_2 = (-1, -2)^T & b_2 = -6 \\ a_3 = (-1, 2)^T & b_3 = -2 \\ a_4 = (1, 0)^T & b_4 = 0 \\ a_5 = (0, 1)^T & b_5 = 0. \end{cases}$

initial $x^0 = (2, 0)^T$

define the gradient $g^k := Gx^k + c$.

Sol: (1) $W_0 = \{3\}$.

1). $g^0 = 2(2, 0)^T + (-2, -5)^T = (2, -5)^T$

$a_3^T p = 0 \Rightarrow p_1 = 2p_2$.

$\phi(p_1, p_2) = \frac{1}{2} p^T G p + g^{0T} p = p_1^2 + p_2^2 + 2p_1 - 5p_2$ $\qquad \phi(p_2) = 5p_2^2 - p_2$

$\phi'(p_2) = 10p_2 - 1 = 0 \Rightarrow p_2 = \frac{1}{10} \rightarrow p^0 = (\frac{1}{5}, \frac{1}{10})^T$

$a_1^T p^0 = 0 \quad a_2^T p^0 = -0.4 \quad a_3^T p^0 = 0.2 \quad a_4^T p^0 = 0.1$

$a_2^T x_0 = -2. \quad b_2 = -6. \qquad \dfrac{b_2 - a_2^T x_0}{a_2^T p^0} = \dfrac{-4}{-0.4} = 10$

$\alpha_0 = \min\{1, 10\} = 10.$ no new block.

$x^1 = x^0 + \alpha_0 p^0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 2.2 \\ 0.1 \end{bmatrix}$

check: $\begin{cases} 2.2 - 0.2 + 2 = 4 > 0 \\ -2.2 - 0.2 + 6 = 1.6 \\ -2.2 + 0.2 + 2 = 0 \quad \text{active} \\ 2.2 > 0 \\ 0.1 > 0 \end{cases}$ thus $W_1 = W_0 = 3.$

2) $g^1 = Gx^1 + c = (2.4, -4.8)^T$

similarly $a_3^T p = 0 \Rightarrow p_1 = 2p.$

$\phi(p_2) = (2p_2)^2 + p_2^2 + 2.4 \cdot 2p_2 - 4.8 p_2 = 5p_2^2 \qquad \min \phi(p_2) = \phi(0).$

thus $p_1 = p_2 = 0$ solve $Gx^1 + c = \lambda_3 a_3. \Rightarrow \lambda_3 = -2.4.$ not satisfy KKT.

$x^2 = x^1 = (2.2, 0.1)^T \quad W_2 = W_1 \setminus \{3\} = \phi.$

(2) i) $W_0 = \{5\}.$

$a_5^T p = 0 \Rightarrow p_2 = 0. \qquad g^0 = (2, -5)^T \quad \phi(p_1, p_2) = p_1^2 + p_2^2 + 2p_1 - 5p_2$

$\phi(p_1) = p_1^2 + 2p_1. \quad \phi_{min} = \phi(-1).$ thus $p = (-1, 0)^T$

check: $a_1^T p^0 = -1 < 0, \quad a_2^T p^0 = 1, \quad a_3^T p^0 = 1 \quad a_4^T p^0 = -1 < 0$

for 1: $\dfrac{b_1 - a_1^T x^0}{a_1^T p^0} = \dfrac{-4}{-1} = 4.$

for 4: $\dfrac{b_4 - a_4^T x^0}{a_4^T p^0} = \dfrac{0-2}{-1} = 2.$

$\Rightarrow \alpha_0 = \min\{1, 4, 2\} = 1.$

$x^1 = x^0 + \alpha_0 p^0 = (1, 0)^T.$

check constraints. $\begin{cases} 1 + 2 = 3 > 0 \\ -1 + 6 = 5 > 0 \\ -1 + 2 = 1 > 0 \\ 1 > 0 \\ 0 = 0 \quad \text{active} \end{cases}$ $W_1 = \{5\}.$

ii) $W_1 = \{5\}. \qquad g^1 = Gx^1 + c = (0, -5)^T$

Similarly $p_2 = 0$. $\phi(p_1) = p_1^2$ $\quad \phi_{min} = \phi(0)$. $\quad p^1 = (0,0)^T$

solve. $\cancel{G} x^1 + c = \lambda_5 a_5$ $\quad \lambda_5 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \Rightarrow \lambda_5 < 0$.

$x^2 = x^1 = (1,0)^T$ $\quad W_2 = W_1 \setminus \{5\} = \phi$.

(3) $W_0 = \phi$.

$\min\limits_p \frac{1}{2} p^T G p + g^{0T} p$. $\quad g^0 = (2,-5)^T$ $\quad$ no constraints. convex. $Gp + g^0 = 0$

i) $\Rightarrow p^0 = -G^{-1} g^0 = \begin{bmatrix} -1 \\ 2.5 \end{bmatrix}$

check. $a_1^T p^0 = -6 < 0$ $\quad a_2^T p^0 = -4 < 0$ $\quad a_3^T p^0 = 6$ $\quad a_4^T p^0 = -1 < 0$ $\quad a_5^T p^0 = 2.5 > 0$

for 1: $\dfrac{b_1 - a_1^T x^0}{a_1^T p^0} = \dfrac{-2-2}{-6} = \dfrac{2}{3}$ $\quad \alpha_0 = \min\{1, \frac{2}{3}, 1, 2\} = \frac{2}{3}$.

2: $\dfrac{b_2 - a_2^T x^0}{a_2^T p^0} = \dfrac{-6+2}{-4} = 1$ $\quad x^1 = x^0 + \alpha_0 p^0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} -1 \\ 2.5 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \end{bmatrix}$

4: $\dfrac{b_4 - a_4^T x^0}{a_4^T p^0} = \dfrac{-2}{-1} = 2$.

check: $\begin{cases} \frac{4}{3} - 2 \times \frac{5}{3} + 2 = 0 \quad \Rightarrow \text{active} \quad W_1 = \{1\}. \\ -\frac{4}{3} - 2 \times \frac{5}{3} + 6 = \frac{4}{3} > 0 \\ -\frac{4}{3} + 2 \times \frac{5}{3} + 2 > 0 \\ \frac{4}{3} > 0 \quad \frac{5}{3} > 0 \end{cases}$

ii) $g^1 = Gx^1 + c = 2 \begin{bmatrix} \frac{4}{3} \\ \frac{5}{3} \end{bmatrix} + \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ -\frac{5}{3} \end{bmatrix}^T$

$a_1^T p = 0 \Rightarrow p_1 = 2p_2$.

$\phi(p_1, p_2) = p_1^2 + p_2^2 + \frac{2}{3} p_1 - \frac{5}{3} p_2$ $\quad \phi(p_2) = 5p_2 - \frac{1}{3} p_2$.

$\phi'(p_2) = 10 p_2 - \frac{1}{3} \Rightarrow p_2 = \frac{1}{30}$ $\quad p^1 = (\frac{1}{15}, \frac{1}{30})^T$

check $a_2^T p^1 = -\frac{2}{15} < 0$ $\quad a_3^T p^1 = 0$ $\quad a_4^T p^1 = \frac{1}{15}$ $\quad a_5^T p^1 = \frac{1}{30}$

for 2: $\dfrac{b_2 - a_2^T x^1}{a_2^T p^1} = \dfrac{-6 - (-\frac{14}{3})}{-\frac{2}{15}} = 10$ $\quad \alpha_1 = \min\{1, 10\} = 1$. $\quad x^2 = x^1 + \alpha_1 p^1 = (\frac{7}{5}, \frac{17}{10})^T$

check: $\begin{cases} 1.4 - 2 \times 1.7 + 2 = 0 \quad \Rightarrow W_2 = \{1\}. \\ -1.4 - 2 \times 1.7 + 6 > 0 \\ -1.4 + 2 \times 1.7 + 2 > 0 \\ 1.4 > 0 \quad 1.7 > 0 \end{cases}$

20. Program the Active-Set Method for Convex QP (page 18, LectureNotes 11.pdf) and use it to solve the problem

$$\min x_1^2 + 2x_2^2 - 2x_1 - 6x_2 - 2x_1x_2$$

subject to

$$\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 \le 1, \quad -x_1 + 2x_2 \le 2, \quad x_1, x_2 \ge 0.$$

Choose three starting points: one in the interior, one at a vertex, and one on a non-vertex boundary point.

Pls. see the Appendix.

21. Consider equality-constrained QPs (page 18, LectureNotes 10.pdf), and assume that A has full row rank and that Z is a basis for the null space of A. Prove that there are no finite solutions if Z^TGZ has negative eigenvalues.

Pf: Assume the converse

$\exists u \neq 0.$ $\quad u^T(Z^TGZ)u < 0.$

denote x^* is the feasible sol.

$\forall t \in \mathbb{R}.$ denote $x = x^* + tZu$ $\quad Ax = Ax^* + tAZu = Ax^* = b.$

thus x is feasible for any $t \in \mathbb{R}.$

$q(x) = \frac{1}{2}(x^* + tZu)^T G (x^* + tZu) + c^T(x^* + tZu)$

$\quad = \frac{1}{2}t^2 (Zu)^T G (Zu) + t(Gx^* + c)^T Z + q(x^*)$

$(Zu)^T G (Zu) < 0.$ $\quad q(x) \xrightarrow[t \to \infty]{} -\infty$ $\quad$ thus q has no finite minima.

22. (a) Assume that $A \neq 0$. Show that the KKT matrix

$$K = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \quad \text{is indefinite.}$$

(b) Prove that if the KKT matrix is nonsingular, then A must have full rank.

(a). For the convex QP oroblem $G \succ 0.$

since $A \neq 0$ (Assume A not have full row rank).

$\exists\, \tilde{u} \in \mathbb{R}^n \setminus \{0\}$ s.t. $A\tilde{u} = 0$ denote $u = \begin{bmatrix} \tilde{u} \\ 0 \end{bmatrix}$ $\rightarrow$ same number of row as K

then $u^T K u = u^T G u > 0.$

consider $v = \begin{bmatrix} G^{-1}A^T y \\ -y \end{bmatrix}$, $y \neq 0$

$v^T K v = \begin{bmatrix} G^{-1}A^T y \\ -y \end{bmatrix}^T \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} G^{-1}A^T y \\ -y \end{bmatrix} = \begin{bmatrix} G^{-1}A^T y \\ -y \end{bmatrix}^T \begin{bmatrix} A^T y - A^T y \\ AG^{-1}A^T y \end{bmatrix} = -y^T A G^{-1} A^T y$

$G^{-1} \succ 0$ $\quad A \neq 0.$ $\quad AG^{-1}A^T \succ 0.$ $\quad \Rightarrow v^T K v < 0.$

thus K is indefinite.

(b) Assume the converse. if A has not full row rank

$\exists \tilde{y} \in \mathbb{R}^m$ s.t. $A^T\tilde{y} = 0$. consider $y = \begin{bmatrix} 0 \\ \tilde{y} \end{bmatrix}$

$$Ky = \begin{bmatrix} G & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} A^T y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

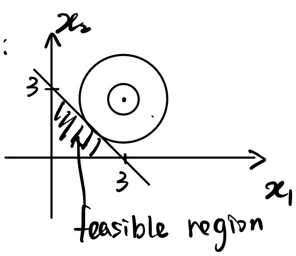
$y \in \ker(K)$ $y \neq 0$ $\Rightarrow$ K is non-singular.

23. Consider the quadratic program

$$\max\ 6x_1 + 4x_2 - 13 - x_1^2 - x_2^2 \quad s.t. \quad x_1 + x_2 \leq 3, \quad x_1 \geq 0, \quad x_2 \geq 0.$$

First solve it graphically, and then use your implementation of the Active-Set Method for Convex QP.

Sol:



$$f(x_1, x_2) = 6x_1 + 4x_2 - 13 - x_1^2 - x_2^2$$
$$= -(x_1-3)^2 - (x_2-2)^2$$

$\Rightarrow$ $\min\ g(x_1, x_2) = (x_1-3)^2 + (x_2-2)^2$.

Level set is circle center at $(3,2)$.

$\Rightarrow$ the problem is equivalent to find the tangent circle with line $x+y-3=0$ center at $(3,2)$.

$\Rightarrow$ when they tangent, the radius $d = (c, l)$. $c: (3,2)$ $l: x+y-3=0$

$d = \frac{|3+2-3|}{\sqrt{2}} = \sqrt{2}$. thus $g_{min} = R^2 = 2$

solve $\begin{cases} x_1 + x_2 - 3 = 0 \\ x_2 = x_1 - 1 \end{cases}$ $\Rightarrow$ $\begin{cases} x_1 = 2 \\ x_1 = 1 \end{cases}$ i.e. $f_{max} = f(2,1)^T = -2$.

Active-Set Method.

$\min\limits_{x}\ q(x) = \frac{1}{2}x^T G x + c^T x + p$. $G = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $c = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$

s.t. $a_i^T x \geq b_i$. $\begin{cases} a_1 = (-1,-1)^T & b_1 = -3 \\ a_2 = (1,0)^T & b_2 = 0 \\ a_3 = (0,1)^T & b_3 = 0 \end{cases}$ initial $x^0 = (0,0)^T$ $W_0 = \phi$.

i/ $g^0 = Gx^0 + c = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$

$\phi(p_1, p_2) = p_1^2 + p_2^2 - 6p_1 - 4p$. no constraint. $Gp + g^0 = 0$ $\Rightarrow$ $p^0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Check $a_1^T p^0 = -5 < 0$ $a_2^T p^0 = 3 > 0$ $a_3^T p^0 = 2 > 0$

for 1: $\frac{b_1 - a_1^T x_0}{a_1^T p^0} = \frac{3}{5}$ $\alpha_0 = \min\{1, \frac{3}{5}\} = \frac{3}{5}$. $x^1 = x^0 + \alpha_0 p^0 = (\frac{9}{5}, \frac{6}{5})^T$

check $\begin{cases} -\frac{9}{5} - \frac{6}{5} + 3 = 0 \\ \frac{9}{5} > 0 \quad \frac{6}{5} > 0. \end{cases}$ $\Rightarrow$ $W_1 = \{1\}$.

iv) $g^1 = Gx^1 + c = \begin{bmatrix} -\frac{12}{5} \\ -\frac{8}{5} \end{bmatrix}$

$a_1^T p = 0 \Rightarrow p_1 + p_2 = 0$

$\phi(p_1, p_2) = \frac{1}{2} p^T G p + g^{1T} p = p_1^2 + p_2^2 - \frac{12}{5} p_1 - \frac{8}{5} p_2 \Rightarrow \phi(p_2) = 2p_2^2 + \frac{4}{5} p_2$

$\phi'(p_2) = 4p_2 + \frac{4}{5} \Rightarrow p^1 = (\frac{1}{5}, -\frac{1}{5})$

check: $a_2^T p^1 = \frac{1}{5}$ $\quad a_3^T p^1 = -\frac{1}{5} < 0$.

For 3: $\frac{b_3 - a_3^T x^1}{a_3^T p_1} = \frac{0 - \frac{6}{5}}{-\frac{1}{5}} = 6 > 1$. $\quad \alpha_1 = 1$. no blocking

$x^2 = x^1 + \alpha_1 p_1 = (2,1)^T$

check $\begin{cases} -1-2+3 = 0 \\ 1 > 0 \quad 2 > 0 \end{cases}$ $\quad W_2 = W_1 = \{1\}$.

iii) $g^2 = Gx^2 + c = (-2,-2)^T$

$a_1^T p = 0 \Rightarrow p_1 + p_2 = 0$

$\phi(p_1, p_2) = p_1^2 + p_2^2 - 2(p_1 + p_2) \quad \phi(p_2) = 2p_2^2 \Rightarrow p^2 = (0,0)^T$

solve $Gx^2 + c = \lambda_1 a_1 \Rightarrow \lambda_1 = 2 > 0$. other constraints slack.

thus $x^* = x^2 = (2,1)^T$. is the sol

24. Let W be an $n \times n$ symmetric matrix, and suppose that Z is of dimension $n \times t$. Suppose that $Z^T W Z$ is positive definite and that $\bar{Z}$ is obtained by removing a column from Z. Show that $\bar{Z}^T W \bar{Z}$ is positive definite.

Pf: let $Z = [z_1, \dots z_t]$. $z_j \in \mathbb{R}^n$.

w.l.g. $\bar{Z} = [z_1, \dots z_{t-1}]$

$\forall u \in \mathbb{R}^{t-1} \setminus \{0\}$ $\quad \exists$ corresponding $v \in \mathbb{R}^t \setminus \{0\}$ s.t. $v = \begin{bmatrix} u \\ 0 \end{bmatrix}$

$Zv = Z\begin{pmatrix} u \\ 0 \end{pmatrix} = \sum_{j=1}^{t-1} z_j u_j = \bar{Z} u$.

thus $v^T Z W Z v = u^T \bar{Z}^T W \bar{Z} u$.

since $Z^T W Z \succ 0$. $\forall v$. $v^T Z^T W Z v > 0$.

thus $\forall u \in \mathbb{R}^{t-1} \setminus \{0\}$. $u^T \bar{Z}^T W \bar{Z} u > 0$. i.e. $\bar{Z}^T W \bar{Z} \succ 0$.