Homework 2.

1. $I = \int \frac{(x^3+1)\,\mathrm{d}x}{x^3-5x^2+6x}$

$$\frac{x^3+1}{x^3-5x^2+6x} = 1 + \frac{5x^2-6x+1}{x^3-5x^2+6x}$$

$$\frac{5x^2-6x+1}{x^3-5x^2+6x} = \frac{5x^2-6x+1}{x(x-2)(x-3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-3}$$

$$A = \frac{5x^2-6x+1}{(x-2)(x-3)}\Big|_{x=0} = \frac{1}{6}\,, B = \frac{5x^2-6x+1}{x(x-3)}\Big|_{x=2} = -\frac{9}{2}\,, \frac{5x^2-6x+1}{x(x-2)}\Big|_{x=3} = \frac{28}{3}\,,$$

$$I = x + \frac{1}{6}\log|x| - \frac{2}{9}\log|x-2| + \frac{28}{3}\log|x-3| + c.$$

2. $\int \frac{x\,\mathrm{d}x}{x^3-1}$

$$\frac{x}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}, \quad x = A(x^2+x+1) + (Bx+C)(x-1)$$

$$\begin{cases} A+B=0, \\ A-B+C=1, \\ A-C=0. \end{cases} \Leftrightarrow \begin{cases} A=1/3, \\ B=-1/3, \\ C=1/3. \end{cases}$$

$$I = \frac{1}{3}\log|x-1| - \frac{1}{3}\int\frac{(x-1)\,\mathrm{d}x}{x^2+x+1} = \frac{1}{3}\log|x-1| - \frac{1}{6}\log|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + c.$$

3. $I = \int \frac{\mathrm{d}x}{x^6+1}$

$$\frac{1}{x^6+1} = \frac{(x^4+1)+(1-x^4)}{2(x^6+1)} = \frac{x^4+1}{2(x^6+1)} + \frac{1-x^4}{2(x^6+1)} = \frac{(x^4-x^2+1+x^2)}{2(x^2+1)(x^4-x^2+1)} + \frac{(1-x^2)(1+x^2)}{2(x^2+1)(x^4-x^2+1)}$$

$$= \frac{1}{2(x^2+1)} + \frac{x^2}{2(x^6+1)} - \frac{x^2-1}{2(x^4-x^2+1)}$$

$$I_1 = \int\frac{\mathrm{d}x}{2(x^2+1)} = \frac{1}{2}\arctan x + c, \quad I_2 = \int\frac{x^2\,\mathrm{d}x}{2(x^6+1)} = \frac{1}{6}\arctan x^3 + c, \quad I_3 = \int\frac{-x^2+1}{2(x^4-x^2+1)}\,\mathrm{d}x$$

$$\frac{-x^2+1}{2(x^4-x^2+1)} = \frac{Ax+B}{x^2+\sqrt{3}x+1} + \frac{Cx+D}{x^2-\sqrt{3}x+1}$$

$$-\frac{x^2}{2} + \frac{1}{2} = (Ax+B)(x^2-\sqrt{3}x+1) + (Cx+D)(x^2+\sqrt{3}x+1)$$

$$\begin{cases} A+C=0, \\ -\sqrt{3}A+B+\sqrt{3}C+D=-\frac{1}{2}, \\ A-\sqrt{3}B+C+\sqrt{3}D=0 \\ B+D=\frac{1}{2}. \end{cases} \Leftrightarrow \begin{cases} A=-C=\frac{1}{2\sqrt{3}}, \\ B=D=\frac{1}{4}. \end{cases}$$

$$I_2 = \frac{1}{2\sqrt{3}}\int\frac{x+\frac{\sqrt{3}}{2}}{x^2+\sqrt{3}x+1}\,\mathrm{d}x - \frac{1}{2\sqrt{3}}\int\frac{x-\frac{\sqrt{3}}{2}}{x^2-\sqrt{3}x+1}\,\mathrm{d}x = \frac{1}{4\sqrt{3}}\log\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1} + c, \quad I = I_1 + I_2 + I_3.$$

4. For what condition $I = \int \frac{ax^2+bx+c}{x^3(x+1)^2}\,\mathrm{d}x$ is rational?

$$\frac{ax^2+bx+c}{x^3(x+1)^2} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{(x+1)^2} + \frac{E}{x+1}$$

$$I \text{ is rational} \Leftrightarrow C = E = 0.$$

$$\begin{cases} B + D = 0, \\ A + 2B = a, \\ 2A + B = b, \\ A = c. \end{cases} \Rightarrow a - 2b + 3c = 0.$$

5. For what condition $I = \int \frac{\alpha x^2 + 2\beta x + \gamma}{(x-x_1)^4}\,\mathrm{d}x$ is rational?

$$I = \int \frac{\alpha(t+x_1)^2 + 2\beta(t+x_1)\gamma}{t^4}\,\mathrm{d}t = \alpha \int \frac{\mathrm{d}t}{t^2} + (2\alpha x_1 + 2\beta)\int \frac{\mathrm{d}t}{t^3} + (\alpha x_1^2 + \gamma)\int \frac{\mathrm{d}t}{t^4}.$$

It is always rational.

6. $I = \int \frac{2x^4 - 4x^3 + 24x^2 - 40x + 20}{(x-1)(x^2-2x+2)^3}\,\mathrm{d}x$

$$I = \frac{Ax^3 + Bx^2 + Cx + D}{(x^2-2x+2)^2} + \int \frac{E\,\mathrm{d}x}{x-1} + \int \frac{(Fx+G)\,\mathrm{d}x}{x^2-2x+2}$$

$$\frac{2x^4 - 4x^3 + 24x^2 - 40x + 20}{(x-1)(x^2-2x+2)^3} = \left(\frac{Ax^3 + Bx^2 + Cx + D}{(x^2-2x+2)^2}\right)' + \frac{E}{x-1} + \frac{Fx+G}{x^2-2x+2}$$

$$2x^4 - 4x^3 + 24x^2 - 40x + 20 = (3Ax^2 + 2Bx + C)(x^2-2x+2)(x-1) - (Ax^3 + Bx^2 + Cx + D)2(2x-2)(x-1)$$
$$+E(x^2-2x+2)^3 + (Fx+G)(x-1)(x^2-2x+2)^2$$

$$\begin{cases} E + F = 0, \\ -A - 6E - 5F + G = 0, \\ -A - 2B + 18E + 12f - 5G = 2, \\ 8A + 2B - 3C - 32E - 16F + 12G = -4, \\ -6A + 4B + 5C - 4D + 36E + 12F - 16G = 24, \\ -4B + 8D - 24E - 4F + 12G = -40, \\ -2C - 4D + 8E - 4G = 20. \end{cases} \Rightarrow \begin{cases} A = 2, \\ B = -6, \\ C = 8, \\ D = -9, \\ E = 2, \\ F = -2, \\ G = 4. \end{cases}$$

$$I = \frac{2x^3 - 6x^2 + 8x - 9}{(x^2-2x+2)^2} + \int \frac{9\,\mathrm{d}x}{x-1} - \int \frac{(2x-4)\,\mathrm{d}x}{x^2-2x+2} = \frac{2x^3 - 6x^2 + 8x - 9}{(x^2-2x+2)^2} + \log\frac{(x-1)^2}{x^2-2x+2} + 2\arctan(x-1) + c.$$