Chapter 3
Systems of <u>linear</u> algebraic equations

$$a\,x + b\,y + c\,z = e$$
$$f\,x + g\,y + h\,z = l$$
$$p\,x + q\,y + s\,z = t$$

$x,\ y,\ z$ *are unknowns*

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$$
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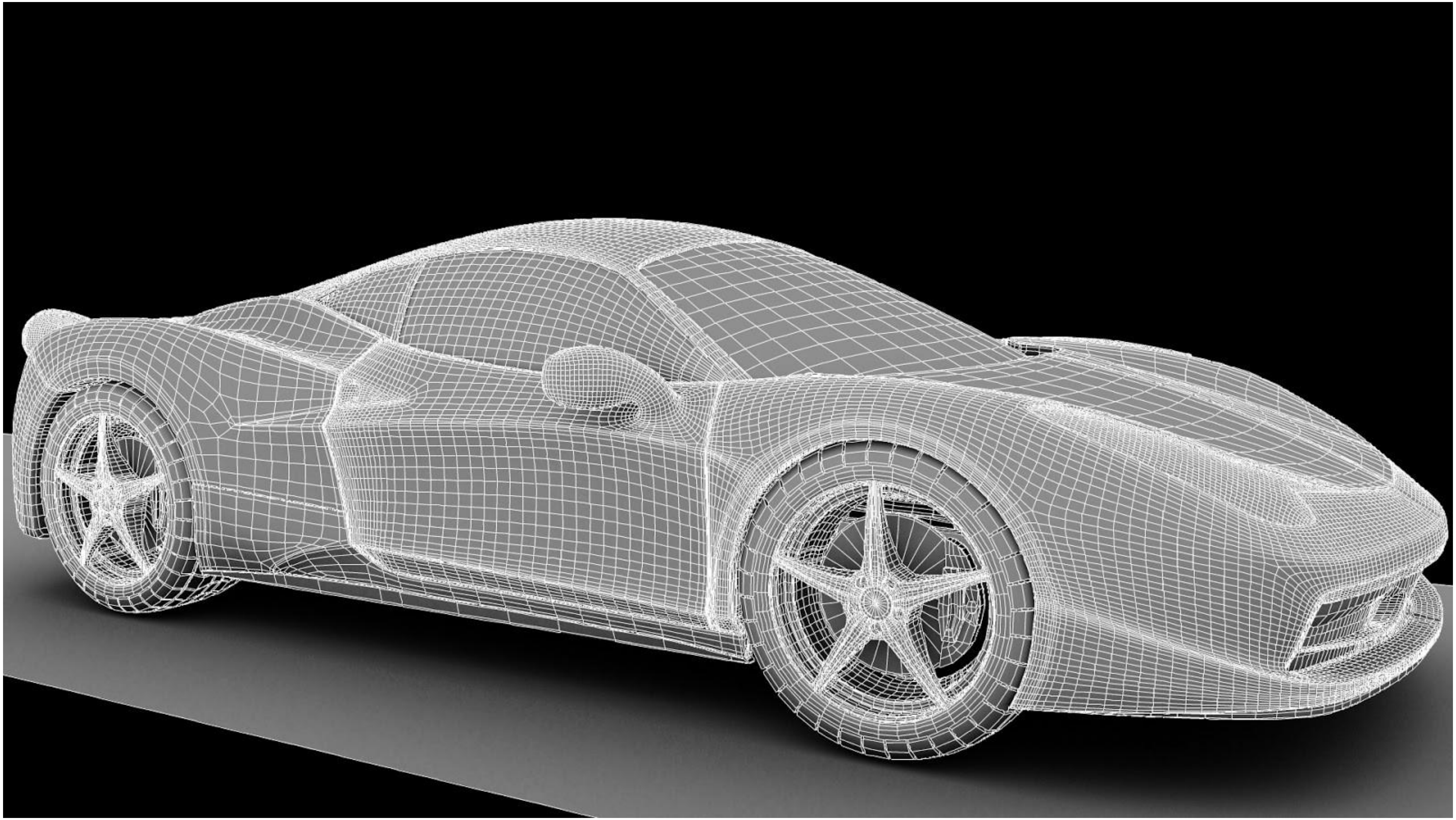
a\,x + b\,y + c\,z &= e \\
f\,x + g\,y + h\,z &= l \\
p\,x + q\,y + s\,z &= t
\end{aligned}
\qquad x,\ y,\ z \ \text{ are unknowns}
$$

Change the notations:

$$
\begin{aligned}
a\,x_1 + b\,x_2 + c\,x_3 &= b_1 \\
f\,x_1 + g\,x_2 + h\,x_3 &= b_2 \\
p\,x_1 + q\,x_2 + s\,x_3 &= b_3
\end{aligned}
\qquad x_1,\ x_2,\ x_3 \ \text{ are unknowns}
$$



$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 + \ldots + a_{1n} x_n = b_1,$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 + \ldots + a_{2n} x_n = b_2,$$

- -

$$a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3 + \ldots + a_{in} x_n = b_i,$$

- -

$$a_{n1} x_1 + a_{n2} x_2 + a_{n3} x_3 + \ldots + a_{nn} x_n = b_n.$$

where a_{ij} - given real numbers

x_i - unknowns to be found

The system can be written in matrix form:

$$\boldsymbol{Ax=b}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ - & - & - & - & - \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ -- \\ x_i \\ -- \\ x_n \end{pmatrix} \quad \text{column vector} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ -- \\ b_i \\ -- \\ b_n \end{pmatrix} \quad \text{column vector}$$

$$Ax=b$$

If $\det A \neq 0$,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

then there exists a unique solution x of the system.

Methods for computation of the solution:

1. Using the inverse $\mathrm{inv}(A)$ of matrix A
2. Gaussian elimination of unknowns
3. LU factorization
4. Iteration method
5.
6.

Scilab (www.scilab.org):

```
-->A= [2 3 4 5 ; 2 1 3 4; 5 6 8 9 ; 5 6 4 3]
 A  =
    2.    3.    4.    5.
    2.    1.    3.    4.
    5.    6.    8.    9.
    5.    6.    4.    3.

-->A(4,2)=60
 A  =
    2.    3.     4.    5.
    2.    1.     3.    4.
    5.    6.     8.    9.
    5.    60.    4.    3.
```

```
--> b= [ 3; 2; 5; 7]
 b  =
   3.
   2.
   5.
   7.
```

column vector

```
--> b= [4  3  0   9 ]'
 b  =
   4.
   3.
   0.
   9.
```

column vector

Product of a matrix and column vector:

--> A*b ➔ column vector

```
--> x=0 : 0.2 : 0.6
 x  =   0.    0.2    0.4    0.6
```

row vector (1 row, 4 columns)

Product of a row vector and a column vector:

```
--> w=x*b
    w = x(1)*b(1)+x(2)*b(2)+x(3)*b(3)+x(4)*b(4)=
```

result is a number

det(A)

1) Method of the inverse of matrix A

$$Ax=b$$

we denote by inv(A) or A^{-1} the inverse of matrix A

How to compose A^{-1} : by computing its cofactors, see Algebra

If we got A^{-1}, then multiplying the system by A^{-1}

$$A^{-1} A\, x = A^{-1} b$$

$$A^{-1} A = I = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ . & . & . & . & \dots & . \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

identity matrix

$I\, x = A^{-1} b$ **and we obtain the solution:** $x = A^{-1} b$

Scilab:

```
A=
b=
det(A)
x=inv(A)*b
```

x=A\b is an equivalent way of using inv(A)

1)

A				b
- 2.	4.	2.	- 2.	14
1.	2.	3.	1.	-21
0.	2.	3.	1.	19
1.	4.	3.	1.	-11

3)

A				b
3.	4.	2.	-2.	14
1.	-2.	3.	0.	-22
0.	2.	3.	0.	18
1.	4.	3.	1.	14

2)

A				b
-6.	4.	2.	- 2.	24
1.	2.	3.	5.	-21
-4.	2.	3.	1.	19
1.	4.	3.	1.	-11

4)

A				b
4.	4.	2.	-2.	4
1.	-2.	6.	2.	-12
-2.	2.	3.	0.	18
1.	4.	3.	-2.	14

2. Gaussian elimination

Carl Friedrich Gauss 1777-1855

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2,$$

- -

$$a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + \ldots + a_{in}x_n = b_i,$$

- -

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n.$$

Suppose that $a_{11} \neq 0$. Then

$$x_1 = (b_1 - a_{12} x_2 - a_{13} x_3 - \ldots - a_{1n} x_n)/a_{11}$$

we multiply this by a_{i1}:

$$a_{i1} x_1 = a_{i1}(b_1 - a_{12} x_2 - a_{13} x_3 - \ldots - a_{1n} x_n)/a_{11} \qquad (*)$$

and replace $a_{i1} x_1$ in i^th equation by (*):

$$a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{ij} x_j + \ldots + a_{in} x_n = b_i \qquad i^{th} \text{ equation}$$

$$a_{i1}(b_1 - a_{12} x_2 - a_{13} x_3 - \ldots - a_{1n} x_n)/a_{11}$$
$$+ a_{i2} x_2 + \ldots + a_{ij} x_j + \ldots + a_{in} x_n = b_i$$

We now sum up the coefficients in front of the same x_j

$$a_{i2}^{(1)} x_2 + \ldots + a_{ij}^{(1)} x_j + \ldots + a_{in}^{(1)} x_n = b_i^{(1)}$$

where $a_{i2}^{(1)} = a_{i2} - a_{i1}a_{12}/a_{11}$

$a_{i3}^{(1)} = a_{i3} - a_{i1}a_{13}/a_{11}$

$a_{ij}^{(1)} = a_{ij} - a_{i1}a_{1j}/a_{11}$

$b_i^{(1)} = b_i - a_{i1}b_1/a_{11}$ $\quad i=2,3,\ldots,n$

After elimination of x_1, the system becomes

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1j} x_j + \ldots + a_{1n} x_n = b_1$$

$$a_{22}^{(1)} x_2 + \ldots + a_{2j}^{(1)} x_j + \ldots + a_{2n}^{(1)} x_n = b_2^{(1)}$$

- -

$$a_{i2}^{(1)} x_2 + \ldots + a_{ij}^{(1)} x_j + \ldots + a_{in}^{(1)} x_n = b_i^{(1)}$$

- -

$$a_{n2}^{(1)} x_2 + \ldots + a_{nj}^{(1)} x_j + \ldots + a_{nn}^{(1)} x_n = b_n^{(1)}$$

Suppose that $a_{22}^{(1)} \neq 0$

Keeping on the elimination of unknowns, in the same way we obtain

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1j}\, x_j + \ldots + a_{1n}\, x_n = b_1$$

$$a_{22}^{(1)}\, x_2 + \ldots + a_{2j}^{(1)}\, x_j + \ldots + a_{2n}^{(1)}\, x_n = b_2^{(1)}$$

- -

$$a_{n-1,\, n-1}^{(n-2)}\, x_{n-1} + a_{n-1,n}^{(n-2)}\, x_n = b_{n-1}^{(n-2)}$$

$$a_{nn}^{(n-1)}\, x_n = b_n^{(n-1)}$$

If $a_{nn}^{(n-1)} \neq 0$, then $\quad x_n = b_n^{(n-1)} / a_{nn}^{(n-1)}$

$$x_{n-1} =$$

- It could happen that $a_{ii}^{(i-1)}=0$.

<u>Example</u>. $A=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$0\cdot x_1 + x_2 = b_1$$

$$x_1 + 0\cdot x_2 = b_2$$

<u>Theorem</u> The system of n linear algebraic equations with $\det A \neq 0$ can be transformed to an equivalent system with nonzero $a_{ii}^{(i-1)}$ by transposition (interchange) of columns or rows.

Scilab:

C=rref([A b]) ; - **Gaussian elimination**

(Reduced Row Echelon Form)

```
-->C=rref([A b])
C =
1. 0. 0. 0.   2.5028409
0. 1. 0. 0.  - 0.1382955
0. 0. 1. 0.  - 0.72625
0. 0. 0. 1.  - 0.0971591
```

$\leftarrow x_1$

$\leftarrow x_2$

$\leftarrow x_3$

$\leftarrow x_4$

x=C(:,5)

The number of arithmetic operations necessary for obtaining a solution with Gaussian elimination is

$$2n(n+1)(n+2)/3 + n(n-1)$$

(a proof is available in textbooks).

3) LU factorization method

Let ***Ax=b*** denote the linear system to be solved, where ***A*** is ***n***×***n*** size matrix. In Gaussian elimination, the system was reduced to the upper triangular system ***Ux=g*** with

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ . & & \cdots & . \\ . & & \cdots & . \\ . & & \cdots & . \\ 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

Let us introduce an auxiliary lower triangular matrix *L* :

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ . & & \cdots & . \\ . & & \cdots & . \\ . & & \cdots & . \\ m_{n1} & \cdots & m_{nn-1} & 1 \end{bmatrix}$$

The relationship of the matrices *L* and *U* to the original *A* is given by the following theorem:

<u>Theorem</u>. Let *A* be a matrix with *det A* ≠ 0. Then if *U* is produced as Gaussian elimination without interchange of rows/columns, then there exists triangular matrix *L* such that *LU=A* and this is called <u>factorization</u> of *A*.

The factorization leads to a slightly different way of solving the system $Ax=b$. It can be rewritten as $LUx=b$. We denote $Ux=g$ and obtain the two simple systems

$$Lg=b \quad \text{and} \quad Ux=g \; .$$

Both L and U are triangular, therefore solutions can be easily calculated by substitution. The computational cost is here reduced drastically as compared to Gaussian one.

Instead of constructing L and U by using the elimination steps, it is possible to solve directly elements of these matrices.

We will illustrate the direct computation of L and U in the case n=3:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

1st row of A: $1=1\cdot u_{11}$ $\quad 1=1\cdot u_{12}$ $\quad -1=1\cdot u_{13}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$1= m_{21}\cdot 1$

$-2= m_{31}\cdot 1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & m_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$-2 = -1 + u_{23}$

$2 = 1 + u_{22}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & m_{32} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & u_{33} \end{bmatrix}$$

$1 = 2 - m_{32} + u_{33}$

$1 = -2 + m_{32}$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -2 \\ -2 & 1 & 1 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}}$$

Take, for example, $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$LUx = b$$

$$Lg = b$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 3 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \begin{matrix} \Rightarrow & g_1 = 1 \\ \Rightarrow & g_2 = 0 \\ \Rightarrow & g_3 = 3 \end{matrix}$$

Finally, we solve $Ux=g$:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$\Rightarrow \quad x_1 + x_2 - x_3 = 1 \quad \Rightarrow x_1 = 1$

$\Rightarrow \quad x_2 = 3/2$

$\Rightarrow \quad x_3 = 3/2$

x=linsolve(A, d) ***solves the system*** **Ax+d=0** **with LU factorization**

x=linsolve(A,-b) ***solves the system*** **Ax=b**