Real Analysis 2024. Homework 11.

1. Find Fourier series for the function $f(x) = x\sin x$ on $[-\pi, \pi]$.

$$x\sin x = 1 - \frac{1}{2}\cos x - 2\sum_{n=2}^{\infty}\frac{(-1)^n\cos nx}{n^2-1}, \quad -\pi \leqslant x \leqslant \pi$$

2. Find cos and sin series for the function $f(x) = x - x^2/2$, $0 \le x \le 1$.

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\left(1 + \frac{4}{\pi^2(2n-1)^2}\right)\frac{\sin(2n-1)\pi x}{2n-1} - \frac{\sin 2\pi n x}{2n}\right).$$

$$\frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\pi n x;$$

3. Prove that

$$\sin x \ln\left(2\cos\frac{x}{2}\right) = \frac{1}{4}\sin x + \sum_{n=2}^{\infty}\frac{(-1)^n}{n^2-1}\sin nx, \quad -\pi < x < \pi$$

Proof. Consider Fourier series for $\ln\left(2\cos\frac{x}{2}\right)$ multiply bi $\sin x$ and use formulas for product of *cos* and *sin* □

4. Calculate the sum of the trigonometric series

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n(n+1)}.$$

$$\frac{1}{2}x(1+\cos x) - \sin x\ln\left(2\cos\frac{x}{2}\right).$$

5. Calculate the sum of the trigonometric series

$$f(x) = \frac{\cos 3x}{1\cdot 3\cdot 5} - \frac{\cos 5x}{3\cdot 5\cdot 7} + \frac{\cos 7x}{5\cdot 7\cdot 9} + \dots.$$

$$f(x) = \frac{\pi}{8}\cos^2 x - \frac{1}{3}\cos x$$

6. Express function $f(x)$ by the Fourier integral

$$f(x) = \begin{cases} \sin x, & |x| \le \pi; \\ 0, & |x| > \pi. \end{cases}$$

Fourier transform is equal to:

$$F[f] = -2i \frac{\sin(2\pi^2 y)}{1 - 4\pi^2 y^2}$$

and

$$f(x) = -2i \int_{-\infty}^{+\infty} \frac{\sin(2\pi^2 y) e^{2\pi i x y}}{1 - 4\pi^2 y^2} dy = 2 \int_{-\infty}^{+\infty} \frac{\sin(2\pi^2 y)\cos(2\pi x y)}{1 - 4\pi^2 y^2} dy.$$

7. Prove that

$$\mathcal{F}[f(x)\cos(2\pi\alpha x)] = \frac{\mathcal{F}[f](y-\alpha) + \mathcal{F}[f](y+\alpha)}{2}, \quad \alpha \in \mathbb{R}.$$

Proof.

$$\mathcal{F}[f(x)\cos(2\pi\alpha x)] = \int_{\mathbb{R}} f(x)\cos(2\pi\alpha x) e^{-2\pi i x y} dx = \int_{\mathbb{R}} f(x) \frac{e^{2\pi\alpha x i} + e^{2\pi\alpha x i}}{2} e^{-2\pi i x y} dx =$$
$$\frac{1}{2}\int_{\mathbb{R}} f(x) e^{-2\pi i x(y-\alpha)} dx + \frac{1}{2}\int_{\mathbb{R}} f(x) e^{-2\pi i x(y+\alpha)} dx = \frac{\mathcal{F}[f](y-\alpha) + \mathcal{F}[f](y+\alpha)}{2}.$$

□

8. Prove that Fourier transform of function $\frac{1}{1+x^{12}}$ is $C^{10}-$smooth.

Proof. This follows from the theorem on derivative of Fourier transform:

$$F[f]^{10}(y) = F[(-2\pi i x)^{10} f(x)](y)$$

if $\int_{\mathbb{R}} x^1 0\, |f(x)|\, dx$ is finite. □

9. Calculate Fourier transform of the function

$$f(x) = \begin{cases} 2 - x^2, & |x| \le 1; \\ 1, & 1 < |x| < 2; \\ 0, & |x| \ge 2. \end{cases}$$

$$F[f] = 2\frac{(2\pi y)^2 \sin(4\pi y) + 2\sin(2\pi y) - 4\pi y \cos(2\pi y)}{(2\pi y)^3}.$$