小结论

$\ln|f| = \mathrm{Re}(\ln f)$.

$c_n = \dfrac{a_n - ib_n}{2}$ $\quad c_{-n} = \dfrac{a_n + ib_n}{2}$

对 $\ln\left(\frac{1}{1-e^{ix}}\right)$ 求实/虚部. 组装 $\dfrac{1}{1-e^{ix}} = \dfrac{1}{2\sin\frac{x}{2}}\left(\cos\left(\frac{\pi}{2}-\frac{x}{2}\right) + i\sin\left(\frac{\pi}{2}-\frac{x}{2}\right)\right)$

$$\ln\left(\frac{1}{1-e^{ix}}\right) = \ln\left(2\sin\frac{x}{2}\right) + \frac{i(\pi - x)}{2}.$$

Leibniz Rule. $J(y) = \int_X f(x,y)\, d\mu(x)$.

if. $f_y' \in L_{loc}(\{y_0\})$, $J'(y_0) = \int_X f_y'(x, y_0)\, d\mu(x)$.

$*$ $F(x,y) = \int_a^b f(x,y,t)\, dt$. 可先. $\frac{\partial F}{\partial y}(x,y) = \int_a^b f_y'(x,y,t)\, dt = g(x,\beta)$. $F(x,\beta) + C = \int g(x,\beta)\, d\beta$. 根据连续性求c.

Tips

考前看泰勒.

所有傅里叶前提是 $f \in L^1/L$

题型总结

一、Fourier integral.

1. "Calculate Fourier series"

(1) trigonometric Fourier integral 前提 $L'[a, a+2l]$

$$\begin{cases} a_0 = \frac{1}{2l}\int_a^{a+2l} f(x)\,dx \\ a_n = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{n\pi x}{l}\,dx & (\text{奇函数 } a_n = 0) \\ b_n = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{n\pi x}{l}\,dx. & (\text{偶函数 } b_n = 0) \end{cases}$$

$$c_n = \frac{1}{2l}\int_a^{a+2l} f(x)\cdot e^{-i\frac{n\pi x}{l}}\,dx.$$

"$\sim$" f收敛到其 Fourier series.

(2) "decomposed by sin / cos system"

前提 $f \in L'[0, l]$ 可作偶 or 奇延拓至 $[-l, l]$.

$$f \sim a_0 + \sum_{n=1}^{+\infty} a_n \cos\frac{n\pi x}{l} \qquad a_0 = \frac{1}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$f \sim \sum_{n=1}^{+\infty} b_n \cos\frac{n\pi x}{l} \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

2. 需要根据级数反推函数. / 对不好直接计算 Fourier series 的函数作变量代换然后泰勒展开.

△ Parseval's identity

① $f: [-\pi, \pi] \to \mathbb{R}$, $f \in L^2[-\pi, \pi]$ $\quad f \sim a_0 + \sum_{n=1}^{+\infty}(a_n \cos nx + b_n \sin nx)$.

$$\|f\|_{L^2}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = a_0^2 + \sum_{n=1}^{+\infty}\frac{a_n^2 + b_n^2}{2}.$$

② $f: [-\pi, \pi] \to \mathbb{C}$, $f \in L^2[-\pi, \pi]$ $\quad f \sim \sum_{n=-\infty}^{+\infty} c_n e^{int}$

$$\|f\|_{L^2}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2(x)\,dx = \sum_{-\infty}^{+\infty}|c_n|^2$$

△ change variable. $z = e^{ix}$. $\Rightarrow \cos x = \dfrac{z + \frac{1}{z}}{2} = \dfrac{z^2+1}{2z}$ $\qquad \sin x = \dfrac{z - z^{-1}}{2i} = \dfrac{z^2-1}{2zi}$

△ Bessel inequality. $\sum_{n=1}^{\infty} |c_n(f)|^2 \|e_n\|^2 \le \|f\|$

3. "Calculate Fourier transform"

前提. $f \in L'(\mathbb{R})$.

正变换 $F[f] = \hat{f}(y) = \int_{-\infty}^{+\infty} f(t)\,e^{-2\pi i y t}\,dt.$

逆变换 $F^{-1}[f] = \check{f}(y) = \int_{-\infty}^{+\infty} f(t)\,e^{2\pi i y t}\,dt.$

cos transform $F_c[f] = \int_{-\infty}^{+\infty} f(t)\cos(2\pi y t)\,dt$

sin transform $F_s[f] = \int_{-\infty}^{+\infty} f(t)\sin(2\pi y t)\,dt$

$F[f] = F_c[f] - iF_s[f]$

$\xRightarrow{\text{延拓}}$ $f \in L'(0, +\infty)$ $\quad F_c[f] = 2\int_0^{+\infty} f(t)\cos(2\pi y t)\,dt \qquad F_s[f] = 2\int_0^{+\infty} f(t)\sin(2\pi y t)\,dt.$

Thm 2.8 $f \in L'(0, +\infty) \cap C[0, +\infty)$. $F(f) \in L'(0, +\infty)$. $\quad F_c[F_c(f)](x) = f(x)$. $\quad x \ge 0$

$F_s[F_s(f)](x) = f(x)$. $\quad x > 0$.

常用 Fourier transformation

$f_\alpha(x) = e^{-\alpha|x|}$. $\alpha > 0$. $\quad \mathcal{F}[f_\alpha](y) = 2\int_0^{+\infty} e^{-\alpha t}\cos(2\pi y t)\,dt = \frac{2\alpha}{\alpha^2+(2\pi y)^2}$

4. " Express the Fourier integral "

① 先计算 $\hat{f}(y)$

② $J(f)(x) = \text{v.p.}\int_{-\infty}^{+\infty} \hat{f}(y)e^{2\pi i x y}\,dy = \lim_{A\to+\infty}\int_{-A}^{A}\hat{f}(y)e^{2\pi i x y}\,dy$ （若积分本就收敛，可省略 v.p.）

△ 一般对 "$2\pi y$" 作变量代换，答案中出现形如 "$\cos xy\,dy$" "$\sin xy\,dy$" 即可。

5. 用 Fourier transformation 性质

(1) 证明光滑性

$f \in L(\mathbb{R})$, for some $r \in \mathbb{N}$ a function $t \mapsto t^r f(t) \in L(\mathbb{R})$.

then $\hat{f} \in C^{(r)}(\mathbb{R})$. for every $k \in [1:r]$. $\quad \hat{f}^{(k)}(y) = (-2\pi i)^k \int_{\mathbb{R}} t^k f(t)e^{-2\pi i y t}\,dt$. $\quad \hat{f}^{(k)}(y) \xrightarrow[y\to\infty]{} 0$.

$g(x) = x^k f(x) \Rightarrow \hat{g}(y) = \frac{1}{(-2\pi i)^k}\hat{f}^{(k)}(y)$

(2) 值的讨论.

① $\hat{f} \in C(\mathbb{R})$. and $|\hat{f}(y)| \le \|f\|_1$ for every $y \in \mathbb{R}$. $\quad$ ③ $\hat{f}(y) \xrightarrow[y\to\infty]{} 0$

(3) 导数/仿射/缩放 与 变换的次序.

① 前提. $f \in C^{(r)}(\mathbb{R})$. $r \in \mathbb{N}$. $f^{(k)} \in L^1(\mathbb{R})$. for every $k \in [0, r]$.

for every $k \in [1:r]$. $\quad \widehat{f^{(k)}}(y) = (2\pi i y)^k \hat{f}(y)$. 先导再变.

② $f_h(x) = f(x+h)$. $\quad \hat{f_h}(y) = e^{2\pi i h y}\hat{f}(y)$.

$\hat{f_h}(y) = \int_{-\infty}^{+\infty} f(t+h)e^{-2\pi i y t}\,dt$.

③ $f(a\cdot)(x) = f(ax)$. $\quad \widehat{f(a\cdot)}(y) = \frac{1}{|a|}\cdot\hat{f}\left(\frac{y}{a}\right)$

$\widehat{(f(a\cdot))}(y) = \int_{-\infty}^{+\infty} f(at)\cdot e^{-2\pi i y t}\,dt$

$= \frac{1}{|a|}\int_{-\infty}^{+\infty} f(x)\cdot e^{-2\pi i y\cdot\frac{x}{a}}\,dx$

$\to x = at$. 作用于 $\int_{-\infty}^{+\infty}$ 涉及符号交替

(4) 两次变换.

前提. $f \in C(\mathbb{R}) \cap L(\mathbb{R})$. $\hat{f} \in L(\mathbb{R})$. then $f = \tilde{(\hat{f})}$ （也可以写作 $f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)$. / $f = J(f)$

6. 算实级数. （含 $\sin nx$ / $\cos nx$ 等）.

补实部/虚数. $z = e^{ix}$, $|z|<1$. 看级数的形式. 找对应泰勒. 算 Im. Re.

$\ln(1+e^{ix}) = \ln\left|2\cos\frac{x}{2}\right| + i\frac{x}{2}$ $\quad$ $\ln(i+e^{ix}) = \ln\left|2\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + \left(\frac{x}{2}+\frac{\pi}{4}\right)i$.

$\ln(i-e^{ix}) = \ln\left|2\sin\left(\frac{x}{2}-\frac{\pi}{4}\right)\right| + \left(\frac{x}{2}-\frac{\pi}{4}\right)i$.

$\arctan z = \frac{1}{2i}\ln\frac{i-z}{z+i} =$

$\arcsin z = -i\ln\left(\sqrt{1-z^2}+iz\right) = i\ln\left(\sqrt{1-z^2}-iz\right)$.

Theoretical Part.

1. Aleksei Part.

$\langle f, g\rangle = \int_X f\bar{g}\, d\mu.$ $\qquad \langle f, g\rangle = 0 \Leftrightarrow f \perp g$

approximation of f: $\{e_\alpha\}$. orthogonal system $e_\alpha \perp e_{\alpha'}$. $\alpha \neq \alpha'$ $\qquad$ define $c_k(f) = \dfrac{\langle f, e_k\rangle}{\|e_k\|^2}$

Bessel inequality. $\sum_{k=1}^{n} |c_k(f)|\, \|e_k\|^2 \leq \|f\|^2$

推论 1 若 f 为可积函数,则

$$\left.\begin{aligned} \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0, \\ \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0. \end{aligned}\right\} \tag{5}$$

因为(1)式的左边级数收敛,所以当 $n\to\infty$ 时,通项 $a_n^2+b_n^2\to 0$,亦即有 $a_n\to 0$ 与 $b_n\to 0$,这就是(5)式. 这个推论也称为**黎曼-勒贝格定理**. (黎曼引理)

2. Measure theory.

family of subsets of X $\quad$ (always contain ϕ)

ring: $A, B \in$ it. $A\cup B$, $A\cap B$, $A\setminus B \in$ it.

semiring $\mathcal{P}$: 1) $\phi \in \mathcal{P}$ 2) $A, B \in \mathcal{P}$. $A\cap B \in \mathcal{P}$ 3) $A, B \in \mathcal{P}$, $A\setminus B = \bigcup_{n=1}^{N} C_n$, $C_n \in \mathcal{P}$ (C_n is mut. disj.)

algebra: 1) $\phi \in$ it 2) $A \in$ it. $A^c \in$ it 3) $A, B \in$ it. $A\cap B \in$ it. (equiv. $A\cup B \in$ it)

σ-algebra $\mathcal{A}$: 1) $\phi \in \mathcal{A}$ 2) $A \in \mathcal{A}$, $A^c \in \mathcal{A}$ 3) $A_k \in \mathcal{A}$, $\bigcup_{k\geq 1}^{\infty} A_k \in \mathcal{A}$ (equiv. $\bigcap_{k\geq 1}^{\infty} A_k \in \mathcal{A}$)

Borel

Borel σ-algebra: $\mathcal{B}_X$ in $(X, \mathcal{T})$. $\mathcal{B}^m$ in $\mathbb{R}^m$ $\quad$ minimal σ-algebra contain all open subsets of X. ($\mathbb{R}^m$).

Borel (sub)set: element in Borel σ-algebra. ($\triangle$ product of two Borel set is alway Borel).

Borel measure: a measure μ: $\mathcal{B}_X \to [0, +\infty)$

measurable.

measurable set E : $E \in \mathcal{A}$ (measurable w.r.t $\mathcal{A}$)

measurable space : $(X, \mathcal{A})$

measurable function: (w.r.t. $\mathcal{A}, \mathcal{A}'$): $(X, \mathcal{A})$. $(X', \mathcal{A}')$. $f: X \to X'$, $\forall E \in \mathcal{A}'$. $f^{-1}(E) \in \mathcal{A}$.

measure.

volume $\uparrow\downarrow$ measure μ $\Big\rangle$ set X. semiring $\mathcal{P}$ on X. function $\mu: \mathcal{P} \to [0, +\infty)$

s.t. 1) $\mu\phi = 0$ 2) $\begin{cases} \text{finite add.} & \mu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mu A_n \\ \text{countable add.} & \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu A_n \end{cases}$

regular measure. if $\forall E$-measurable. $\forall \varepsilon > 0$ $\exists K$-compact, G-open s.t. $K \subset E \subset G$.

and $\mu(G\setminus E)$, $\mu(E\setminus K) < \varepsilon$.

image measure of μ under $f: (X, \mathcal{A}) \to (X', \mathcal{A}')$. $f_*(\mu)(Y) = \mu(f^{-1}(Y))$.

set.

定义中不要求 $Y \in \mathcal{A}$.

μ-negligible $(X, \mathcal{A}, \mu)$. $Y \subset X$. Y is set of measure 0. (i.e. $\exists E \in \mathcal{A}$. s.t. $Y \subset E$, $\mu E = 0$).

(particularly, $Y \in \mathcal{A}$ and negligible. $\mu Y = 0$)

Space

measurable space : $(X, \mathcal{A})$

measured space : $(X, \mathcal{A}, \mu)$

complete measured space : all negligible set are measurable.

3. Convergence

Convergence.

1) series of function $\{f_n\}$. on set E.

① p.w. conv. $\quad f_n \xrightarrow[n\to\infty]{} f$. $\quad \forall \varepsilon>0.\ \forall x\in E.\ \exists N\in\mathbb{N}.$ for any $n>N$. $|f_n(x)-f(x)|<\varepsilon$.

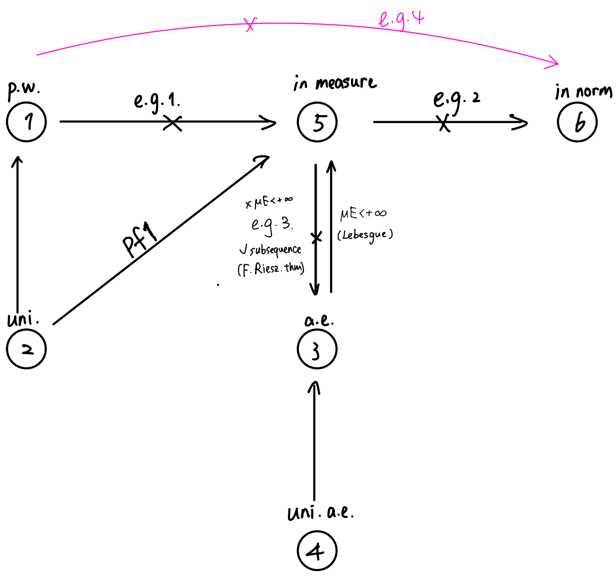
② uni. conv. $\quad f_n \rightrightarrows f$ $\quad \forall\varepsilon>0.\ \exists N\in\mathbb{N}.$ for any $n>N$ and $x\in E$. $|f_n(x)-f(x)|<\varepsilon$ $\left(\text{i.e. } \sup_{x\in E}|f_n(x)-f(x)| \xrightarrow[n\to\infty]{} 0\right)$.

③ conv. a.e. $\quad f_n \xrightarrow[n\to\infty]{a.e.} f$ $\quad \exists e\subset E.\ \mu e=0.\ f_n \xrightarrow[n\to\infty]{} f$ on $E\setminus e$

④ conv. almost uni. $\quad f_n \overset{a.e.}{\underset{n\to\infty}{\rightrightarrows}} f$ $\quad \forall\varepsilon>0.\ \exists A_\varepsilon.\ \mu A_\varepsilon<\varepsilon.$ $f_n\rightrightarrows f$ on $E\setminus A_\varepsilon$

⑤ conv. in measure. $\quad f_n \xrightarrow[n\to\infty]{\mu} f$ $\quad \forall\varepsilon>0.\ \mu(E(|f_n-f|>\varepsilon)) \xrightarrow[n\to\infty]{} 0$

⑥ conv. in norm (e.g. in $L^1(\ )$). $\quad$ in $L^1(E,\mu)$. $\int_E |f_n-f|\,d\mu \xrightarrow[n\to\infty]{} 0$ (注意依范数收敛



e.g.1. $E=\mathbb{R}$. $f_n=\chi_{[n,n+1)}$.

$f_n\to 0$. p.w. $\quad \mu E(|f_n-f|>\frac{1}{2})=1$.

e.g.2. $E=(0,1)$. $f_n=\begin{cases} n. & (0,\frac{1}{n}) \\ 0. & [\frac{1}{n},1) \end{cases}$

$f_n \xrightarrow{\mu} 0 \qquad \int_E |f_n| \not\to 0$.

e.g.3. $E=[0,1]$. $\mu=\lambda$.

$\forall k\in\mathbb{N}.\ \Delta(k,p)=[\frac{p}{2^k},\frac{p+1}{2^k})$. $p=0,1,2,\dots$ $f_n=\chi_{\Delta(k,p)}$. where $k=[\log_2 n]$.

$E_n=E(f_n\neq 0)=\Delta(k,p)$. $\mu(\Delta(k,p))=\frac{1}{2^k}\le\frac{2}{n}\to 0$.

$f_n \xrightarrow{\mu} 0$. ; f_n no limits.

Pf1: $f_n\rightrightarrows f$. $\forall\varepsilon>0.\ \exists N\in\mathbb{N}.$ s.t. $\forall n>N.\ \forall x\in E.\ |f_n-f|<\frac{\varepsilon}{2}$.

i.e. $E(|f_n-f|>\varepsilon)=\phi$. i.e. $\mu E(|f_n-f|>\varepsilon)=0$. for $n>N$

<u>Lebesgue thm.</u> $\mu X<+\infty$. conv. a.e $\Rightarrow$ conv in meas.

<u>Riesz thm.</u> $\forall\{f_n\}$ conv. in measure $\exists\{f_{n_k}\}$. conv. a.e. (to the same limit)

<u>Borel-Cantelli Lemma.</u> $(X,\mathcal{A},\mu)$ $\{E_n\}_{n\ge 1}\subset\mathcal{A}$.

$E=\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}E_k=\{x\in X.\mid x\in E_n \text{ for infinitely many } n\}$. if $\sum_{n=1}^{\infty}\mu(E_n)<+\infty$, then $\mu E=0$.

<u>Def.</u> conv. a. uni. $\forall\varepsilon>0.\ \exists A_\varepsilon$ s.t. $\mu(A_\varepsilon)<\varepsilon$. $f_n\rightrightarrows f$ on $X\setminus A_\varepsilon$.

<u>Egorov. thm.</u> $f.\ f_n\in L^0(X,\mu)$. $f_n\xrightarrow{a.e.} f$. $\mu X<+\infty$. then $f_n \overset{a.u.}{\underset{n\to\infty}{\rightrightarrows}} f$

<u>Diagonal. thm.</u> μ- σ-finite. $f_k^{(n)}.\ g_n\in L^0(X,\mu)$. $f_k^{(n)}\xrightarrow[k\to\infty]{a.e.} g_n$. for every n. $g_n\xrightarrow{a.e.} h$

e.g.4. **3.3.3 Example.** Let H be a Hilbert space, and let $(e_k)_{k\in\mathbb{N}}$ is an infinite orthonormal system. Consider the following sequence of linear functionals $(f_k)_{k\in\mathbb{N}}\subset H^*$:

$$\forall\, k\in\mathbb{N},\quad \forall\, x\in H \qquad f_k(x)=\langle x,e_k\rangle.$$

Than $(f_k)_{k\in\mathbb{N}}$ pointwise convergent to $\mathbf{0}$, because for all $x\in H$ Bessel's inequality gives

级数收敛的必要条件.

$$\sum_{k=1}^{\infty}|f_k(x)|^2=\sum_{k=1}^{\infty}\langle x,e_k\rangle^2\le\|x\|^2 \quad\Rightarrow\quad f_k(x)\xrightarrow[k\to\infty]{}0.$$

But $\|f_k\|=\|e_k\|=1$ for all k, so it does not tend to zero by the norm.