Complex Analysis

Chapter 1. Complex Plane Number and Function.

Def. complex plane $\mathbb{C}$: a set of ordered pairs $z=(x,y)$ of real numbers.

complex number : points of complex plane.

real / imaginary part : $x = \mathrm{Re}\, z$. $y = \mathrm{Im}\, z$

conjugate : $\bar{z} = (x,-y)$. (of z).

real number $\mathbb{R} = \{(x,0) \mid x \in \mathbb{R}\} = \{z \mid z = \bar{z}\}$. ($\mathbb{R}$ is a subset of $\mathbb{C}$).

$\mathbb{C} \equiv \mathbb{R}^2$. vector $1 := (1,0)$. $i := (0,1)$ → imaginary unit.

$z = (x,y) = x \cdot 1 + y \cdot i = x + iy$.

Thm 1.1. $(\mathbb{C}, +, \cdot)$ is a field with $(0,0)$ as 0 and $(1,0)$ as 1.

Review: field. $(F,+,\cdot)$ $+ : F \times F \to F$ $\times : F \times F \to F$.

axioms: 1. addition commutation. 2. addition association 3. $\exists 0 \in F$. $\forall a \in F$. $0 + a = a$.

4. $\forall a \in F$. $\exists -a \in F$. $a + (-a) = 0$ 5./6. multiplication commutation / association.

7. $\exists 1 \in F$. $\forall a \in F$ $1 \cdot a = a$. 8. $\forall a \neq 0$. $\exists a^{-1} \in F$. $a \cdot a^{-1} = 1$. 9. distribution.

(proof needs to check these axioms).

Properties of operation.

1). $\bar{\bar{z}} = z$. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$

2) $z = \bar{z}$ iff $z \in \mathbb{R}$. 3) $\mathrm{Re}\, z = \dfrac{z + \bar{z}}{2}$ $\mathrm{Im}\, z = \dfrac{z - \bar{z}}{2i}$

4) $|z_1 \pm z_2| \le |z_1| + |z_2|$ $\big||z_1| - |z_2|\big| \le |z_1 - z_2|$

5) $|z| = 0 \Leftrightarrow z = 0$

Def. limit: $\{z_n\}$ sequence. z is a limit of $|z - z_n| \xrightarrow[n\to+\infty]{} 0$ (equi. $\forall V_z$. $\exists N$, $z_n \in V_z$ for $n > N$)

convergent : sequence has a limit.

Thm 1.6. equivalence of convergence: $\lim z_n = \lim \mathrm{Re}\, z_n + \lim \mathrm{Im}\, z_n$.

series similarly: $\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \mathrm{Re}\, c_k + i \sum_{k=1}^{\infty} \mathrm{Im}\, c_k$

Def. neiborhood. an open disk $V_p(\varepsilon) = \{z \in \mathbb{C} \mid |z-p| < \varepsilon\}$.

punctured ~ : $\dot{V}_p(\varepsilon) = V_p(\varepsilon) \setminus \{p\}$.

limit point : $p \in \mathbb{C}$. $E \subset \mathbb{C}$. p is ~ of E if $\forall \dot{V}_p$, $E \cap \dot{V}_p \neq \phi$.

isolated point : $p \in \mathbb{C}$. $E \subset \mathbb{C}$, p is ~ of E if $\exists V_p$. $V_p \cap E = \{p\}$.

complentary def.

Thm. $\mathbb{C}$ is complete. coro. $\mathbb{C}$ is a Banach space. ($|\,|$ is norm)

Pf: Let $\{z_n\}_{n\geq 1}^{\infty}$ be Cauchy's.

$|Re(z_n - z_m)| \leq |z_n - z_m|$ $|Im(z_n - z_m)| \leq |z_n - z_m|$

$\Rightarrow$ $Re\,z_n$, $Im\,z_n$ Cauchy's in $\mathbb{R}$ $\Rightarrow$ conv. $\Rightarrow \{z_n\}$ conv.

Def. a limit of function (A) pre. $D, G \subset \mathbb{C}$, $f: D \to G$. $p \in \mathbb{C}$ is a limit point of D.

1) Cauchy def. (ε-δ. def). $\forall \varepsilon > 0\, \exists \delta > 0: \forall z \in D \setminus \{p\}.\ |z-p| < \delta \Rightarrow |f(z) - A| < \varepsilon$.

2). neighborhood def. $\forall V_A \exists V_p: f(D \cap \dot{V}_p) = f(D \cap \dot{V}_p) \subset V_A$.

3). Heine def. $\forall \{z_n\}: z_n \in D \setminus \{p\}$, $z_n \to p \Rightarrow f(z_n) \to A$.

(equivalent def.).

Def. countinuous (at a point) pre: $D, G \subset \mathbb{C}$. $f: D \to G$. f is cont. at $z_0 \in D$ if:

1). z_0 is an isolated point or limit point and $f(z_0) = \lim_{z \to z_0} f(z)$

2) Weierstrass - Jordan def. $\forall \varepsilon > 0\, \exists \delta > 0.\ \forall z \in D: |z - z_0| < \delta\ |f(z) - f(z_0)| < \varepsilon$.

3) neighborhood def. $\forall V_{f(z_0)}\, \exists V_{z_0}\ f(V_{z_0}) \subset V_{f(z_0)}$

4) Heine def. $\forall \{z_n\}: z_n \in D.\ z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

§. Polar Representation.

$z \in \mathbb{C} \setminus \{0\}$. argument of z: the angle φ measured from direction of vector 1.

$\arg z$ not unique.; term. 2π; $\operatorname{Arg} z = \{\varphi + 2\pi k \mid k \in \mathbb{Z} \text{ and } \varphi = \arg z\}$.

principal value of argument z: z in $(-\pi, \pi]$, $[0, 2\pi)$.

$z = x + iy$. $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $\cos\varphi = \frac{x}{r}$ $\sin\varphi = \frac{y}{r}$

$z = r(\cos\varphi + i\sin\varphi)$ $r = |z|$. $\varphi = \arg z$.

$z^n = r^n(\cos n\varphi + i\sin n\varphi)$. ($\cos n\varphi + i\sin n\varphi = (\cos\varphi + i\sin\varphi)^n$).

§. Path in Complex plane.

Def. path. a continuous map $\gamma: [\alpha, \beta] \to \mathbb{C}$. where $\alpha, \beta \in \mathbb{R}$. $\alpha < \beta$.

equivalent path. $\gamma_1: [\alpha_1, \beta_1] \to \mathbb{C}$. $\gamma_2: [\alpha_2, \beta_2] \to \mathbb{C}$. if $\exists$ increasing, continuous, bijective function $\mathcal{T}: [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$.

s.t. $\gamma_2 \circ \mathcal{T} = \gamma_1$.

curve class of equivalence of paths.

closed path γ: if $\gamma(\alpha) = \gamma(\beta)$.

Jordan path γ: if $\gamma(t_1) \neq \gamma(t_2)$ $(t_1 \neq t_2)$. closed Jordan path. if $\gamma(t_1) \neq \gamma(t_2)$. $(t_1 < t_2 < \beta)$

Def. smooth path. for every $t \in [\alpha, \beta]$. $\exists\, \dot{\gamma}(t)$ (left/right derivative for endpoints) if $\dot{\gamma}(t)$ is cont. and $\dot{\gamma}(t) \neq 0$. for $t \in [\alpha, \beta]$.

piecewise smooth path. a segment $[\alpha, \beta]$ can be decomposed by t_j: $\alpha = t_0 < t_1 < \dots < t_{n-1} < t_n = \beta$. to $[t_{j-1}, t_j]$ s.t. $\gamma|_{[t_{j-1}, t_j]}$ are all smooth.

equivalence of smooth / piecewise smooth path: $\exists$ J. J^{-1} are smooth (p.w. smooth)

§. Domain

Def. Linearly connected set. D: any two points in this set can be connected by a path. that is contained in D.

domain (a set $D \subset \mathbb{C}$): it's open and Linearly connected.

Lemma 1.19. An open set is connected $\Leftrightarrow$ linearly connected

(review: connected. $\nexists$ open G_1, G_2 s.t. $D = G_1 \cup G_2$ and $\phi = G_1 \cap G_2$).

Pf: "$\Leftarrow$" assume the converse. let $z_1 \in G_1$, $z_2 \in G_2$. $\exists\, \gamma [\alpha_1, \beta_1] \to \mathbb{C}$, $\gamma(\alpha_1) = z_1$, $\gamma(\beta_2) = z_2$.

Let $t_0 = \sup \{t.\ \gamma(t) \in G_1\}$

if. $\gamma(t_0) \in G_1 \Rightarrow \exists\, \delta > 0$. $\gamma(t_0 + \delta) \in G_1$. ($G_1$ is open). $t_0 \neq \sup$.

if $\gamma(t_0) \in G_2 \Rightarrow \exists\, \delta > 0$. $\gamma(t_0 - \theta) \in G_2$. $\forall \theta \in (0, \delta) \Rightarrow \sup \le t_0 - \delta$. (can't find sup. contradic)

"$\Rightarrow$" Let D be connected.

Let. $z \in D$. $G_1 = \{\xi \in D: \xi$ can be connected with z by a path$\}$.

Let $G_2 = D \setminus G_1$ $G_1 \neq \phi$ since $z \in D$.

if $\xi \in G_1$, $\exists\, V_\xi$, $V_\xi \subset D$. $\forall \xi_1 \in V_\xi$. ξ_1, ξ can be connected by a path.

$\Rightarrow$ z and ξ_1 can be connected. $\Rightarrow V_\xi \subset G_1$. $\Rightarrow G_1$ is open.

if $\xi' \in G_2$ $\exists\, V_{\xi'}$. s.t. $V_{\xi'} \subset D$. (D is open): 1) if $\exists\, \xi_1 \in V_{\xi'}$ s.t. z & ξ_1 can be connected. $\Rightarrow z$ & ξ' can be connected $\Rightarrow V_{\xi'} \subset G_2 \Rightarrow G_2$ - open. ; 2) if $\nexists\, \xi_1$ s.t. z & ξ_1 can be connected. then $V_{\xi'} \subset G_2 \Rightarrow G_2$ is open. Which means $G_2 = \phi$. $\Rightarrow G_1 = D$. D is Linearly connected.

Thm. 1.20 Assume $G \subset \mathbb{C}$ is a domain. $F \subset G$. and F is nonempty, open and closed in G. (F is open and $F = \bar{F} \cap G$). then $F = G$.

Pf: let $F_1 = G \setminus F$. $F_1 = G \setminus F = G \setminus (\bar{F} \cap G) = G \setminus \bar{F}$ F_1 is open.

F_1 is close and open, $F \neq \phi$. $\Rightarrow F_1 = \phi$.

Def. A set G is compactly supported in domain D. if $\bar{G} \subset D$.

Def. A domain with simple boundary ∂D: consists of finite number of closed p.w.s. paths.

orientation: D is to the left to the direction to the curve.

§. Compactification.

Def. $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

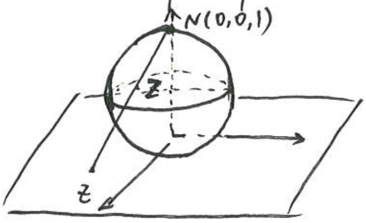
$\hat{\mathbb{C}} = \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. — an extended complex plane. $\overline{\mathbb{C}}$

base of neighborhood of ∞ in $\overline{\mathbb{C}}$. $\{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$.

§. Stereographical Projection

pre: Euclidean Space $\mathbb{R}^3$. sphere (Riemann sphere) $S = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 \mid \xi^2 + \eta^2 + (\zeta - \frac{1}{2})^2 = \frac{1}{4}\}$.

complex plane $\mathbb{C} = \{(\xi, \eta, \zeta) \in \mathbb{R}^3 \mid \zeta = 0\}$.



Nz intersects the sphere at $Z(\xi, \eta, \zeta)$.

$$\xi = \frac{x}{1+|z|^2}, \quad \eta = \frac{y}{1+|z|^2}, \quad \zeta = \frac{|z|^2}{1+|z|^2}$$

homeomorphism $S \setminus \{N\} \leftrightarrow \mathbb{C}$. $S \leftrightarrow \overline{\mathbb{C}}$

§. Spherical metric.

A metric in $\mathbb{C}$: $\rho(z_1, z_2) = \dfrac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}$ (Euc. distance between the corr. point on the sphere.)

in bounded part of $\mathbb{C}$. (disk $\{|z| < R\}$) $\rho \sim$ Euc. distance. $c|z_1 - z_2| \le \rho \le C|z_1 - z_2|$

unbounded part. $\rho(z, \infty) = \dfrac{1}{\sqrt{1+|z|^2}} \le 1$

punctured neighborhoods of ∞ in $\overline{\mathbb{C}}$ w.r.t to ρ: $\{z \in \mathbb{C} : \rho(z, \infty) < \varepsilon\} = \{z \in \mathbb{C} ; |z| > \sqrt{\varepsilon^{-2} - 1}\}$.

§ Complex differentiability.

$f: \mathbb{C} \to \mathbb{C}$. $f(z) = f(x,y) = u(x,y) + i v(x,y) = \operatorname{Re} f(x,y) + \operatorname{Im} f(x,y)$.

<u>Def.</u> $\mathbb{R}$-differentiable (at some point) pre: $f(x,y) = u(x,y) + i v(x,y)$. define in V_{z_0}.

f is $\mathbb{R}$-diff at z_0. if $u(x,y)$, $v(x,y)$ are diff. at (x_0, y_0) as function of x, y.

(review. $u(x,y) - u(x_0,y_0) = A_1 \Delta x + B_1 \Delta y + \bar{o}(|\Delta z|)$

$v(x,y) - v(x_0,y_0) = A_2 \Delta x + B_2 \Delta y + \bar{o}(|\Delta z|)$ $\quad \Delta z = d((x,y),(x_0,y_0)) \to 0$.

$\Leftrightarrow$ $f(x,y) - f(x_0,y_0) = a \Delta x + b \Delta y + \bar{o}(|\Delta z|)$ for $\Delta z \to 0$.)

$\Downarrow$ $a = A_1 + i A_2$ $\quad$ $\Downarrow$ $b = B_1 + i B_2$

$\Leftrightarrow$ $\forall \varepsilon > 0\ \exists \delta > 0$. s.t. $|\Delta f - a \cdot \Delta x - b \cdot \Delta y| < \varepsilon |\Delta z|$

$$a = \frac{\partial f}{\partial x}(z_0) = \lim_{\Delta x \to 0} \frac{f(x,y_0) - f(x_0,y_0)}{\Delta x} \qquad b = \frac{\partial f}{\partial y}(z_0) = \lim_{\Delta y \to 0} \frac{f(x_0,y) - f(x-y_0)}{\Delta y}.$$

Remark: $\Delta x = \dfrac{\Delta z + \Delta \bar{z}}{2}$ $\quad \Delta y = \dfrac{\Delta z - \Delta \bar{z}}{2}$

$$\Delta f = \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i \frac{\partial f}{\partial y}(z_0)\right)}_{\Downarrow\ \frac{\partial f}{\partial z}} \Delta z + \underbrace{\frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i \frac{\partial f}{\partial y}(z_0)\right)}_{\Downarrow\ \frac{\partial f}{\partial \bar{z}}} \Delta \bar{z} + o(\Delta z).$$

$$df(z_0) = \frac{\partial f}{\partial z}(z_0)\, dz + \frac{\partial f}{\partial \bar{z}}(z_0)\, d\bar{z}.$$

Def. $\mathbb{R}$-linear function. ($df(\alpha z) = \alpha\, df(z)$. $\alpha \in \mathbb{R}$).

$\mathbb{C}$-linear function ($df(\alpha z) = \alpha \frac{\partial f}{\partial z} z + \bar{\alpha} \frac{\partial f}{\partial \bar{z}} \bar{z}$, $\alpha \in \mathbb{C}$)

df is $\mathbb{R}$-linear, df is $\mathbb{C}$-linear iff f is R-diff and $\frac{\partial f}{\partial \bar{z}} = 0$.

Def. $\mathbb{C}$-differentiable. (at some point). pre: f define in V_{z_0}.

f is $\mathbb{C}$-diff. if $\exists a \in \mathbb{C}$. s.t. $\Delta f = f(z) - f(z_0) = a \cdot \Delta z + o(\Delta z)$ in V_{z_0}

$\Leftrightarrow$ $\dfrac{f(z) - f(z_0)}{z - z_0} = a + o(1)$ $\quad (z - z_0)$.

$\Leftrightarrow$ $a = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) =: \dfrac{df}{dz}(z_0)$: complex derivative of f at z_0.

Rem: $\mathbb{C}$-diff / $\mathbb{R}$-diff both implies the continuity.

Thm 2.3. Pre: f define in V_{z_0}.

f is $\mathbb{C}$-diff (at z_0) iff f is $\mathbb{R}$-diff (at z_0) and $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

Remark: $\frac{\partial f}{\partial \bar{z}}$ is Cauchy-Riemann condition. and $\frac{\partial f}{\partial z}(z_0) = f'(z_0)$ in this case.

Pf: "$\Rightarrow$" $\Delta f = a \cdot \Delta z + \bar{o}(\Delta z) = a\Delta x + i \cdot a\Delta y + \bar{o}(\Delta z)$

$\Rightarrow$ f is $\mathbb{R}$-diff and $a = \frac{\partial f}{\partial x}$, $ia = \frac{\partial f}{\partial y}$ $\Rightarrow$ $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}(a + i \cdot ia) = 0$

"$\Leftarrow$" R-diff: $\Delta f = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\Delta \bar{z} + o(\Delta z)$ in V_{z_0}.

then $\Delta f = \frac{\partial f}{\partial z}(z_0)\Delta z + o(\Delta z)$ denote $a = \frac{\partial f}{\partial z}(z_0)\Delta z$

the C-R condition: $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$

thus $\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \Leftrightarrow \begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{cases}$ (C-R equations)

$\Leftrightarrow \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}$.

§. Operation and Property. Elementary function.

$(f \pm g)' = f' \pm g'$ $(fg)' = f'g + fg'$ $f(g(z))' = f'(g(z))\, g'(z)$

Def. analytic: f is analytic on U if f has complex derivative at every point in U (synonym: holomorphic)

Elementary function

1. Any complex polynomial $p(z) = \sum_{k=0}^{n} c_k z^k$ is analytic in $\mathbb{C}$.

2. Any rational function $r(z) = \frac{p(z)}{q(z)}$ (p, q are complex polynomial. $q \not\equiv 0$).
is analytic on open set $\{z \in \mathbb{C} ; q(z) \neq 0\}$.

3. Complex exponential. $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$ is analytic in $\mathbb{C}$.
and $(e^z)' = e^z$

4. Triangular. $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$; also analytic in $\mathbb{C}$. $\begin{cases} (\cos z)' = -\sin z \\ (\sin z)' = \cos z \end{cases}$

Exponential form of complex number.

$z = r(\cos\varphi + i\sin\varphi)$, $r = |z|$, $\varphi = \operatorname{Arg} z$.

then $z = r \cdot e^{i\varphi}$.

Property of exponent.

1. $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$ (use def. or Taylor's decomposition to prove).

2. De Moivre's formula. $(\cos t + i\sin t)^n = (e^{it})^n = e^{int} = \cos nt + i\sin nt$. $n \in \mathbb{N}$. $t \in \mathbb{R}$.

3. exponent has no zero. $e^z \neq 0$.

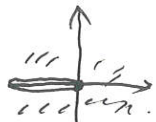
4. exponent has periods. $2k\pi i$, $k \in \mathbb{Z}\setminus\{0\}$. no other periods.

Logarithm form.

$\log z = \ln|z| + i\arg z + 2\pi k i$. $z \in \mathbb{C}\setminus(-\infty, 0]$ $\arg z$ is unique in $[-\pi, \pi)$

s.t. $z = re^{i\varphi}$. $r > 0$ and $k \in \mathbb{Z}$ is fixed.

$\log z$ is diff in $\mathbb{C}\setminus(-\infty, 0]$ and. $(\log z)' = \frac{1}{z}$.



△ $\ln z$ is not cont. on $\mathbb{C}\setminus\{0\}$ (for any $k \in \mathbb{Z}$). but on $\mathbb{C}\setminus(-\infty, 0]$ cont.

$\Downarrow$

正好在周期边界，出现了"间断点".

consider $-1 = e^{-\pi i}$

$\ln(-1) = \ln 1 - i\pi + 2k\pi i = -i\pi + 2k\pi i$

$e^{i\varphi} \xrightarrow[\varphi\to\pi]{} -1$ $\ln(e^{i\varphi}) = \ln 1 + i\varphi + 2\pi k i \xrightarrow[\varphi\to\pi]{} i\pi + 2\pi k i \neq \ln(-1)$ not cont. at (-1).

Property.

1. Let $z, w \in \mathbb{C}$. z^w may have countable numbers of values.

$z^w = e^{w\log z} = e^{w\ln|z| + iw\arg z + 2k\pi w i}$ e.g. $i^i = e^{i\log z} = e^{-\frac{\pi}{2} - 2\pi k}$. $k \in \mathbb{Z}$.

Roots of Complex number.

Let $z = re^{i\varphi} \neq 0$. $n \in \mathbb{N}$. $\omega^n = z$.

denote $\omega = \rho^n e^{in\psi}$ $\Rightarrow \rho = \sqrt[n]{r}$ $\psi = \frac{\varphi + 2\pi k}{n}$ $\Rightarrow w_k = \sqrt[n]{r}\, e^{i\frac{\varphi + 2\pi k}{n}}$ $k = 0, \dots, n-1$ (has n values)

$w_k = \omega_{n+k}$. $w_k(z)$ is continuous and $\mathbb{C}$-diff on $\mathbb{C}\setminus(-\infty, 0]$.

在复平面上，w_0 绕原点依次旋转 $\frac{2\pi}{n}, 2\cdot\frac{2\pi}{n}, \dots (n-1)\cdot\frac{2\pi}{n}$.

$(\sqrt[n]{z})' = \frac{1}{n}\frac{1}{\sqrt[n]{z^{n-1}}}$ $|\sqrt[n]{z}| = \sqrt[n]{|z|}$

In particularly, $\omega^n = 1$. for the root of 1. denote as $1, \omega, \omega^2, \dots \omega^{n-1}$. $(\omega = e^{i\frac{2\pi}{n}})$.

$(1-\omega)(1+\omega+\omega^2+\dots+\omega^{n-1}) = 0$. $\Rightarrow 1+\omega+\omega^2+\dots+\omega^{n-1} = 0$.

for $z^n = 1$, $z \neq 0$. the n number root $w_0, \omega w_0, \dots, \omega^{n-1} w_0$.

§. Directional derivative

Pre: f: $\mathbb{R}$-diff then $\Delta f = \frac{\partial f}{\partial z}(z_0)\Delta z + \frac{\partial f}{\partial \bar z}(z_0)\Delta \bar z + o(\Delta z)$.

$\Delta z = |\Delta z| e^{i\theta}$ (polar expression). $\Delta \bar z = \overline{\Delta z} = |\Delta z| e^{-i\theta} = \Delta z \cdot e^{-2i\theta}$.

thus $\Delta f = \left[\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar z}(z_0)\cdot e^{-2i\theta}\right]\Delta z + o(\Delta z)$

$\Delta z \to 0$: $\lim\limits_{\substack{\Delta z \to 0 \\ \arg \Delta z = \theta}} \frac{\Delta f}{\Delta z} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar z}(z_0) e^{-2i\theta} =: f'_\theta(z_0)$ → derivative of f by direction θ.

Lemma 2.4. f is $\mathbb{R}$-diff at z_0. Then f is $\mathbb{C}$-diff iff $f'_\theta(z_0)$ doesn't depend on θ.

In this case $f'_\theta(z_0) = \frac{\partial f}{\partial z}(z_0) = f'(z_0)$. for every $\theta \in \mathbb{R}$.

§. Holomorphic functions and conformal mappings.

<u>Def.</u> f is holomorphic at $z_0 \in \mathbb{C}$. if it's $\mathbb{C}$-diff in some neighborhood of this point.

f is holomorphic in D. if $\forall z_0 \in D$, f is holomorphic at z_0.

A set of <u>functions</u> holomorphic in D is denoted by $H(D)$

<u>Def.</u> Pre: f $\mathbb{R}$-diff at z_0.

A mapping f is <u>conformal at z_0</u> if it's $\mathbb{C}$-diff at z_0 and $f'(z_0) \neq 0$.

f is conformal in D if it's conformal at every point of D.

Domain D is <u>conformally equivalent</u> to domain G if: $\exists$ bijective conformal mapping $f: D \to G$. e.g. $\{\operatorname{Re} z > 0\} \xrightarrow{z^2} \mathbb{C} \setminus (-\infty, 0]$

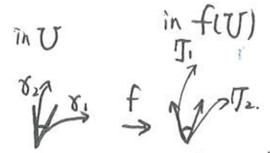
Remark: $f: D \to G$ and f^{-1} are holomorphic and $(f^{-1}(z))' = \frac{1}{f'(f^{-1}(z))}$

Thus "conformally equivalent" has symmetry. ($D \to G$ then $G \to D$).

Moreover. "conformally equivalent" is a equivalent relation.

Lemma 2.10 (old def.). f is conformal at z_0 <u>iff</u> f is $\mathbb{R}$-diff on V_{z_0} and it's differential $df(z_0)$. consider as a <u>linear mapping</u> of $\mathbb{R}^2$ to $\mathbb{R}^2$ is ① degenerate (bijective) ($df(z_0): \mathbb{R}^2 \to \mathbb{R}^2$ $df(z_0): \xi \mapsto \frac{\partial f}{\partial z}(z_0)\xi + \frac{\partial f}{\partial \bar z}(z_0)\bar\xi$) and is ② a composition of rotation and a scaling.

§. Geometric meaning of complex derivative.



perserves the angle. of tangent line.

Pre: f is conformal. in some U_{z_0}. and $f'(z)$ is cont. in U_{z_0}.

Consider a smooth path in U with a starting point at z_0 that is

$\gamma : [0,1] \to U$, $\gamma(0) = z_0$. s.t. $\gamma'(t) \neq 0$ for every $t \in [0,1]$.

$T := f \circ \gamma : [0,1] \to f(U)$. T is smooth in $f(U)$ since $T'(t) = f'(\gamma(t))\gamma'(t)$.

the length of curve γ at $\gamma(t)$ is. $ds_\gamma = |\gamma'(t)|dt$ $\quad ds_T = |T'(t)|\, dt$.

$$\frac{ds_T}{ds_\gamma} = \frac{|T'(0)|}{|\gamma'(0)|} = |f'(z_0)| \qquad \arg f'(z_0) = \arg T'(0) - \arg \gamma'(0).$$

§. Extended complex plane (both value and variable)

Def. A complex-valued function f defined in the neighborhood of $\infty \in \bar{\mathbb{C}}$ is called holomorphic (or conformal) at $z = \infty$ if $g(z) := f(\frac{1}{z})$ is holomorphic (or conformal) at 0

Def. function f defined in the neighborhood of $\infty \in \bar{\mathbb{C}}$ s.t. $f(z_0) = \infty$ is holomorphic at z_0 if the function $F(z) := \frac{1}{f(z)}$ is holomorphic at z_0.

In particular. if $f(\infty) = \infty$, f is holomorphic if $G(z) := \frac{1}{g(z)} = \frac{1}{f(\frac{1}{z})}$ holomorphic at 0 (respectively, conformal).

Chapter 3. Complex Integration.

§. Integral of complex-valued function.

$f(t) = u(t) + i\,v(t)$. $t \in [a,b] \subset \mathbb{R}$.

$$\int_a^b f(t)\,dt := \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

intuitive property:

1) $\int_a^c f\,dt + \int_c^b f\,dt = \int_a^b f\,dt$. $c \in [a,b]$

2) $\int_a^b \lambda f(t)\,dt = \lambda \int_a^b f(t)\,dt$. $\lambda \in \mathbb{C}$.

3) $\left|\int_a^b f\,dt\right| = \int_a^b |f|\,dt$.

Pf: $\theta = \arg\left(\int_a^b f(t)\,dt\right)$ $\left(\int_a^b f(t)\,dt \neq 0\right)$.

$$\left|\int_a^b f\,dt\right| = \mathrm{Re}\left(e^{-i\theta}\int_a^b f(t)\,dt\right) = \mathrm{Re}\left(\int_a^b e^{-i\theta} f(t)\,dt\right) = \int_a^b \mathrm{Re}\left(e^{-i\theta}f(t)\right)dt.$$

$$\leq \int_a^b |f(t)|\,dt.$$

§ Integration along path.

Let γ be piecewise differentiable arc in $\mathbb{C}$. with parametric equation $\gamma: z = z(t).\ t \in (a,b)$

Def. If f is cont. on γ, then $f(z(t))$ is cont. on (a,b). define integral of f on γ as the line integral $\int_\gamma f(z)\,dz := \int_a^b f(z(t))\frac{dz}{dt}\,dt$.

Remark: for $f = u + iv$. and $dz = \dot{\gamma}(t)\,dt = dx + i\,dy$.

$$\int_\gamma f\,dz = \int_\gamma (u\,dx - v\,dy) + i\int_\gamma (u\,dy + v\,dx)$$

e.g. $z = \gamma(t) = a + re^{it}$ $(0 \leq t \leq 2\pi)$. circle at a with radius r.

integral along γ of $f(z) = (z-a)^n$ for $n \in \mathbb{Z}$.

$\dot{\gamma}(t) = ire^{it}$. $f(\gamma(t)) = r^n e^{int}$.

$$\int_\gamma (z-a)^n\,dz = ir^{n+1}\int e^{i(n+1)t}\,dt = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

$n \neq -1$. $\int_0^{2\pi} e^{i(n+1)t}\,dt = \dfrac{e^{2\pi i(n+1)} - e^0}{i(n+1)} = 0$

$n = -1$ $\int_0^{2\pi} e^0\,dt = 2\pi$.

Property:

1. Linearity. f, g be cont. along the path γ and $\alpha, \beta \in \mathbb{C}$. Then: (only need p.w. diff (not smooth. maybe).)

$$\int_\gamma (\alpha f + \beta g)\,dz = \alpha\int_\gamma f\,dz + \beta\int_\gamma g\,dz.$$

2. Additivity: two piecewise smooth path: $\gamma_1: [\alpha, \beta_1] \to \mathbb{C}$, $\gamma_2: [\beta_1, \beta] \to \mathbb{C}$.

s.t. $\gamma_1(\beta_1) = \gamma_2(\beta_1)$. consider the compound: $\gamma = \gamma_1 \cup \gamma_2 : [\alpha, \beta] \to \mathbb{C}$.

Let $\gamma(t) = \begin{cases} \gamma_1(t) & t \in [\alpha, \beta_1] \\ \gamma_2(t) & t \in [\beta_1, \beta] \end{cases}$.

assume f is cont. along $\gamma_1 \cup \gamma_2$:

$$\int_{\gamma_1 \cup \gamma_2} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz.$$

generalized: the connected is not essential any p.w. smooth γ_1, γ_2 works

3. Independence of parametrization.

这一条件不是必要的 → equivalent path. 不改路径,只改速度. strictly-increase. 保证不往回走. 但回走再回来部分积分为0 (见4.)

Let $\gamma: [\alpha, \beta] \to \mathbb{C}$. p.w. smooth path. obtain from $\gamma_1: [\alpha_1, \beta_1] \to \mathbb{C}$ by change of para. $\gamma = \gamma_1 \circ J$

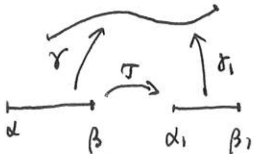
where $J: [\alpha, \beta] \to [\alpha_1, \beta_1]$ is strictly-increasing and $J \in C'$ s.t. $J(\alpha) = \alpha_1$. $J(\beta) = \beta_1$.

If $f: \gamma([\alpha, \beta]) \to \mathbb{C}$ is cont. along γ, then f is cont. along γ_1.

and $\int_{\gamma_1} f dz = \int_{\gamma} f dz$.

Remark: 1) γ_1, γ are equivalent if $\exists J$ - change of para. ($J \in C'$ and $\begin{cases} J(\alpha) = \alpha_1 \\ J(\beta) = \beta_1 \end{cases}$) s.t. $\gamma = \gamma_1 \circ J$.

2) curve is class of equivalent path.



equivalent path.

4. Dependence of orientation.

Let p.w. smooth path $\gamma^{-1}: [\alpha, \beta] \to \mathbb{C}$ is obtain from $\gamma: [\alpha, \beta] \to \mathbb{C}$ by change of orientation. i.e. $\gamma^{-1}(t) = \gamma(\alpha + \beta - t)$ for $\alpha \le t \le \beta$.

If $f: \gamma([\alpha, \beta]) \to \mathbb{C}$ is cont. along γ. then f is cont. along γ^{-1}.

$$\int_{\gamma^{-1}} f dz = -\int_{\gamma} f dz.$$

5. Estimate of the integral.

f be p.w. smooth path $\gamma: [\alpha, \beta] \to \mathbb{C}$. Then the estimation satisfied: $\left| \int_\gamma f(z) dz \right| \le \int_\gamma |f(z)| |dz|$ ①

where $\int_\gamma |f(z)| |dz| := \int_\alpha^\beta |f(\gamma(t))| |\dot\gamma(t)| dt$. (line-integral of 1-st kind of $|f|$ along γ).

In particular, if $|f(z)| \le M$ for every $z \in \gamma([\alpha, \beta])$.

then $\left| \int_\gamma f(z) dz \right| \le M \cdot |\gamma|$. $|\gamma|$ is length of a path. ②

Pf: denote $J := \int_\gamma f(z) dz$ and express J in a polar form $J = |J| e^{i\theta}$. $\theta \in \mathbb{R}$.

Then $|J| = e^{-i\theta} J = \int_\alpha^\beta e^{-i\theta} f(\gamma(t)) \dot\gamma(t) dt$

$|J| = \int_\alpha^\beta \mathrm{Re}\{ e^{-i\theta} f(\gamma(t)) \dot\gamma(t) \} dt \le \int_\alpha^\beta |e^{-i\theta} f(\gamma(t)) \dot\gamma(t)| dt$

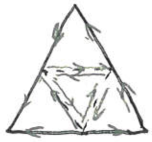
thus $\left| \int_\gamma f(z) dz \right| \le \int_\gamma |f(z)| |dz|$

Since $|\gamma| = \int_\alpha^\beta |\dot\gamma(t)| dt$, then ② holds

§. Cauchy - Goursat theorem.

Thm 3.3. (Cauchy - Goursat thm. for triangles). Let $f \in H(D)$ (holomorphic). Then for every triangle $\Delta \subset D$ with its boundary $\int_{\partial\Delta} f(z)dz = 0$.

Pf: Assume $\exists\, \Delta_0 \subset D$. s.t. $\left|\int_{\partial\Delta_0} f dz\right| = M > 0$.



partition Δ_0 into 4 equal triangle

the $\int_{\partial\Delta_0} f dz = \sum_{i=1}^{4} \int_{\partial\Delta_i} f dz$.

$\exists\, \Delta_1$. s.t. $\left|\int_{\partial\Delta_1} f dz\right| \geq \frac{M}{4}$. partition Δ_1 and repeat the procedure

$\{\Delta_n\}_{n=1}^{\infty}$, $\therefore \left|\int_{\partial\Delta_n} f dz\right| \geq \frac{M}{4^n}$. $\quad \bar{\Delta}_{n+1} \subset \bar{\Delta}_n$

and $\bigcap_{n=1}^{\infty} \Delta_n = \{z_0\}$. $z_0 \in D$ and is unique.

Then apply $\mathbb{C}$-diff of f at z_0. $\forall \varepsilon > 0$. $\exists\, \delta > 0$. s.t. in $U = U_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}$.

$f(z) = f(z_0) + f'(z_0)(z - z_0) + \alpha(z)(z - z_0)$. $\quad \alpha(z) < \varepsilon$ for every $z \in U$.

for $\bar{\Delta}_n \subset U$. ($\exists N$, $\bar{\Delta}_n \subset U$, for $n \geq N$) $\int_{\partial\Delta_n} f\, dz = \int_{\partial\Delta_n} f(z_0) dz + \int_{\partial\Delta_n} f'(z_0)(z - z_0)\, dz + \int_{\partial\Delta_n} \alpha(z)(z - z_0)\, dz$.

$\because$ integrate on closed curve. by previous e.g. $= 0$.

$\left|\int_{\partial n} f dz\right| = \left|\int_{\partial\Delta_n} \alpha(z)(z - z_n) dz\right| \leq \varepsilon \cdot \int_{\partial\Delta_n} |z - z_0|\, dz \leq \varepsilon \cdot |\partial\Delta_n|^2$. (perimeter (周长) of Δ_n)

$|\partial\Delta_n| = \frac{|\partial\Delta_0|}{2^n}$ (by geometric pro.). $\Rightarrow \frac{M}{4^n} \leq \left|\int_{\partial\Delta_n} f dz\right| \leq \varepsilon \frac{|\partial\Delta_0|^2}{4^n} \Rightarrow M \leq \varepsilon |\partial\Delta_0|^2$

$\varepsilon \to 0$. $M = 0$. which contradicts with assumption.

Chapter 4 Antiderivative.

Lemma 4.1. Let $D \subset \mathbb{C}$ be a domain, $f \in H(D)$. Then TFAS:

1) if $\mathrm{Re} f$ is constant f is constant.

2) if $\mathrm{Im} f$ is constant. f is constant

3) if $|f|$ is constant then f is constant.

Pf: $f = u + iv$.

1) $u \equiv c$ $\quad u'_x = u'_y = 0 \Rightarrow$ by C-R condition $v'_x, v'_y = 0. \Rightarrow v \equiv c \Rightarrow f$ is constant.

2) similarly

3) $|f|$ is constant $\Rightarrow |f|^2 = u^2 + v^2$ $\quad$ if $u^2 + v^2 = 0$ obvious, if $u^2 + v^2 \neq 0$:

take the derivative $\Rightarrow \begin{cases} 0 = uu'_x + vv'_x \\ 0 = uu'_y + vv'_y \end{cases} \xRightarrow{\text{C-R condition}} \begin{cases} uu'_x - vu'_y = 0 \\ vu'_x + uu'_y = 0 \end{cases} \Rightarrow \begin{matrix} v'_y, u'_y = 0 \\ v'_x, u'_x = 0 \end{matrix}$

Remark: 4). $\arg f$ is const. then f is constant.

[证存在原函数:
设$F = \int_\gamma f(z)\,dz$.
1)证路径无关 (correctly defined).
2)证F可微. 即有$F' = f$ in D].

Def 4.2 Let $D \subset \mathbb{C}$. $f \in C(D)$. $F \in H(D)$ s.t. $F'(z) = f(z)$. $z \in D$.
F is the anti-derivative of f in D.

Lemma 4.3 (uniqueness problem). Assume F is an antiderivative of f in D. Then all antiderivative of f in D differ from F by a constant. i.e. has form $F(z) + c$, $c \in \mathbb{C}$.

Pf: $\forall F, F_1 \in H(D)$ s.t. $F'(z) = F_1'(z) = f(z)$ $\quad$ denote $F - F_1 = \varphi$. in D.

we have $\varphi'(z) = (F - F_1)'(z) = 0 \Rightarrow \varphi'_z = \frac{1}{2}(\varphi'_x - i\varphi'_y) = 0$

and $\varphi \in H(D)$. $\varphi'_{\bar{z}} = \frac{1}{2}(\varphi'_x + i\varphi'_y) = 0$.

$\Rightarrow \varphi'_x = \varphi'_y = 0 \Rightarrow \varphi$ is const.

Lemma 4.4. (existence on a disk) Let $U = \{z \in \mathbb{C} : |z - a| < r\}$ $f: U \to \mathbb{C}$. be cont. in U and for every triangle Δ s.t. $\int_{\partial\Delta} f\,dz = 0$. Then the function $F(z) = \int_a^z f(\xi)\,d\xi$. $z \in U$. is an antiderivative of f in U. ("global" F exists for disk case)

Pf: Let $z \in U$. $\exists \delta > 0$. s.t. $D = \{z + h : h \in \mathbb{C}, |h| \le \delta\} \subset U$.

apply thm 3.3 for the triangle: $\underbrace{\int_a^z f(\xi)\,d\xi}_{F(z)} + \int_z^{z+h} f(\xi)\,d\xi - \underbrace{\int_a^{z+h} f(\xi)\,d\xi}_{F(z+h)} = 0.$

$\Rightarrow F(z+h) - F(z) = \int_z^{z+h} f(\xi)\,d\xi.$

$\frac{F(z+h) - F(z)}{h} = f(z) + \frac{1}{h}\int_z^{z+h}(f(\xi) - f(z))\,d\xi$ $\quad$ apply uniform cont. of f (in closure of disk D)

$\left|\frac{1}{h}\int_z^{z+h}(f(\xi) - f(z))\,d\xi\right| \le \frac{1}{|h|} \cdot |h| \cdot \max_{\xi \in [z, z+h]} |f(\xi) - f(z)| \xrightarrow[h \to 0]{} 0 \quad \Rightarrow F'(z) = f(z)$

§. Antiderivate along the path.

Def. Let $\gamma: I \to D$ be arbitrary path in D and $f: D \to \mathbb{C}$.

Then function $\Phi: I \to \mathbb{C}$. is an antiderivative f along the path γ if

1. Φ is cont. on I.

2. for every $t_0 \in I$. $\exists \overset{\text{disk.}}{U} \subset D$, with a center at $z_0 = \gamma(t_0)$ and F_U of a function f in this disk s.t. $\Phi(t) = F_U(\gamma(t))$ for every t in some neighborhood $(t_0 - \delta, t_0 + \delta) \cap I$ 。

△ Φ是t的函数.不是z的函数($z = \gamma(t)$). 若 $U_{\gamma(t_1)} \cap U_{\gamma(t_2)} \neq \phi$. $\Rightarrow F_1 = F_2 + C$ on $U_{\gamma(t_1)} \cap U_{\gamma(t_2)}$

$\not\Rightarrow F_1 \equiv F_2$ on $U_{\gamma(t_1)} \cap U_{\gamma(t_2)}$

△ If $f: D \to \mathbb{C}$. has global antiderivative $F: D \to \mathbb{C}$ in D then the function $\Phi = F(\gamma(t))$ is an a.d. along f.

Thm 4.8. (Existence and uniqueness of antiderivative along path) for any path $\gamma: I \to D$. (此时Φ也是z的函数).
Let f be holomorphic in D and $\gamma: I \to D$ be a path in D. Then the antiderivative of f along γ exists and is unique (up to a constant)

Pf: Exist: $I = [\alpha, \beta]$. partition $\alpha = t_0 < t_1 < \dots < t_{n-1} < t_n = \beta$. $I_j := [t_{j-1}, t_j]$

$\gamma(I_j) \subset U_j \subset D$. U_j is some disk.

Let $F_1' = f$ in U_1. $F_2' = f$ in U_2. then in $U_1 \cap U_2 \neq \phi$. $F_1 - F_2 = \text{const.}$

w.l.g. $F_2 \equiv F_1$ (by uniqueness on domain) on $U_1 \cap U_2 \neq \phi$. similarly. $F_j \equiv F_{j-1}$ on $U_{j-1} \cap U_j \neq \phi$.

define $\Phi: I \to \mathbb{C}$. as $\Phi(t) = F_j(\gamma(t))$. $t \in [t_{j-1}, t_j]$.

Φ is cont. on I. and is antiderivative of f along γ.

Unique: if $\exists \Phi_1, \Phi_2$. in some disk $U \subset D$. $F_1 - F_2 = \text{const}$. then $\Phi_1 - \Phi_2 = \text{const}$ on $u(t_0) \subset I$

$\Phi_1 - \Phi_2$ is loc. constant. on I. I is connected $\Rightarrow \Phi_1, \Phi_2$ is constant on I.

Thm 4.9 (Newton–Leibniz formula) Let $\gamma: [\alpha, \beta] \to D$ be a piecewise smooth path in D. and $f \in H(D)$. Let Φ be antiderivative of f along γ.

Then: $\int_\gamma f\,dz = \Phi(\beta) - \Phi(\alpha)$

Pf. $I = [\alpha, \beta]$. $\alpha = t_0 < \dots < t_n = \beta$. $I_j = [t_{j-1}, t_j]$. $\gamma(I_j) \subset$ disk $U_j \subset D$.

in U_j. $\exists F_j$ s.t. $\Phi(t) = F_j(\gamma(t))$ $t \in I_j$

$$\int_\gamma f\,dz = \sum_{j=1}^{n} \int_{\gamma_j} f\,dz = \sum_{j=1}^{n} \left(F_j(\gamma(t_j)) - F_j(\gamma(t_{j-1})) \right) = \sum_{j=1}^{n} \left(\Phi(t_j) - \Phi(t_{j-1}) \right)$$

$$= \Phi(b) - \Phi(a)$$

thm: $f \in H(D)$.
f has anti-de $\Leftrightarrow \int_\gamma f\,dz = 0$ for every closed path in D.
(i.e. independence of path).

Remark: function may not have global antiderivative in some domain.
(global ~~de~~ antiderivative × global antiderivative along path ✓).

e.g $f(z) = \frac{1}{z}$ on $D = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$ $f \in H(D)$.

(Since $(\ln z)' = \frac{1}{z}$ holds only for $\mathbb{C} \setminus (-\infty, 0]$. if $F(z) = \ln z$. $F(z) \notin H(D)$)

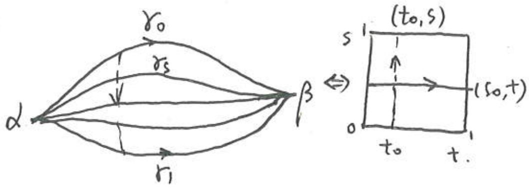
Chapter 5. Homotopy(同伦) Cauchy thm.

For simplicity. $D \subset \mathbb{R}^2(\mathbb{C})$. $I = [0,1]$

Def 5.1. Two path $\gamma_0, \gamma_1 : I \to D$ with common endpoints. $\gamma_0(0) = \gamma_1(0) = \alpha$. $\gamma_0(1) = \gamma_1(1) = \beta$ are homotopic in domain D (as paths with common endpoints) if: $\exists\, \Gamma \in C(I \times I \to D)$ s.t:

1. $\Gamma(0,t) = \gamma_0(t)$ and $\Gamma(1,t) = \gamma_1(t)$ for every $t \in I$.
2. $\Gamma(s,0) = \alpha$, $\Gamma(s,1) = \beta$. for every $s \in I$.

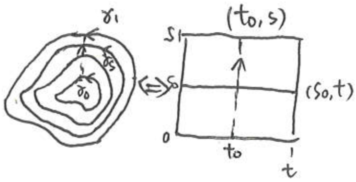
γ_0, γ_1 are Γ-homotopic.



Def 5.2. Two closed path $\gamma_0, \gamma_1 : I \to D$. are homotopic in domain D as closed paths.

if $\exists\, \Gamma \in C(I \times I \to D)$ s.t.

1. $\Gamma(0,t) = \gamma_0(t)$ and $\Gamma(1,t) = \gamma_1(t)$ for every $t \in I$.
2. $\Gamma(s,0) = \Gamma(s,1)$ for every $s \in I$.



Remark: in both case. map Γ is called a homotopy of γ_0 and γ_1

An intermediate map is denoted by $\gamma_s(\cdot) = \Gamma(s, \cdot)$

Homotopy is equivalence relation.

Def. 5.5. A path $\gamma : I \to \mathbb{C}$. is a constant path if $\gamma(t)$ is constant: $\gamma(t) = \gamma(0)$ for every $t \in I$.

A closed path contractible if it's homotopic to a constant path

i.e. $\exists\, \Gamma \in C(I \times I \to D)$ and $\exists\, z_0 \in D$ s.t:

1.) $\Gamma(0,t) = \gamma_0(t)$ and $\Gamma(1,t) = z_0$ for every $t \in I$

2.) $\Gamma(s,0) = \Gamma(s,1)$ for every $s \in I$

Def 5.6. A domain D is simply connected if every closed path in D is contractible.
(don't have holes).

Lemma(1) D is simply connected $\Leftrightarrow$ ∂D is connected $\Leftrightarrow$ $\mathbb{C} \setminus D$ is connected.

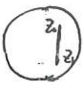
Lemma(2). D is convex $\Rightarrow$ D is simply connected.

Pf: denote $\gamma_0(t) : I \to D$. $\forall z_0 \in D$.

$\Gamma(s,t) = (1-s)\gamma_0(t) + s \cdot z_0 \in D$ (by convexness).

(对每个确定 $\gamma_0(t)$, $(1-s)\gamma_0(t) + s\, z_0$ 是 s 到 $\gamma_0(t)$ 的线段上所有点)

Remark: Def of convex set: $D \subset \mathbb{C}$ is convex. if $\forall z_0, z \in D$. $\forall s \in [0,1]$. $(1-s)z_0 + s z_1 \in D$
(simply. $[z_0, z_1] \subset D$).



Def 5.7 A domain D is star-shaped if there exists a point $z \in D$ s.t. for every $w \in D$, a segment connects z and D in contained in D. i.e.

$\exists z \in D$: $tw + (1-t)z \in D$ for every $z \in D$. and $t \in [0,1]$.

Remark: convex $\Rightarrow$ star-shaped $\Rightarrow$ simply-connected

e.g. $0 \le r < R \le \infty$. $z_0 \in \mathbb{C}$. annulus: $K_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R\}$.

not simply-connected — inner and outer radius.

Thm 5.9. (Cauchy's thm. on homotopy) Let f be holomorphic in domain D and γ_0, γ_1 be two paths homotopic in D. Then $\int_{\gamma_0} f dz = \int_{\gamma_1} f dz$.

Pf: Let $\gamma_s(t) = T(s,t) : I \to D$. (homotopy of γ_0, γ_1)

Let $J(s) := \int_{\gamma_s} f dz$ for $s \in I$ we need: $J(s)$ is locally constant. (then $J(0) = J(1)$)

i.e. we need: $\forall s \in I_0$. $\exists V(s_0) \subset I$ s.t. $J(s) = J(s_0)$ for every $s \in V(s_0)$

Let $\Phi: I \to \mathbb{C}$. be antiderivative of f along γ_{s_0}. consider partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$.

in $I_j := [t_{j-1}, t_j]$. 1) $\exists$ disk $U_j \subset D$ s.t. $\gamma_{s_0}(I_j) \subset U_j \subset D$.

2) $\exists F_j \in H(D)$. s.t. $F_j' = f$ in U_j and $F_j \circ \gamma_{s_0} = \Phi$. on I_j.

$F_{j-1} = F_j$ in $U_j \cap U_{j-1}$.

T is cont. in $I \times I$ $\Rightarrow$ uni. cont. $\Rightarrow \exists V(s_0)$ s.t. $T(I_j \times V(s_0)) \subset U_j$ ($\Rightarrow \gamma_s(t) \subset U_j$.

$\forall s \in V(s_0)$. $\gamma_s(t) \subset U_j$. thus $\exists F_j' \in H(D)$. s.t. $F_j'' = f$. by uniqueness. $F_j' \equiv F_j$. in U_j.

thus $F_j \circ \gamma_s = \Phi_s = \Phi$ on I_j. (i.e. Φ_s is the antiderivative of f along γ_s.)

by NL-formula: $J(s) := \int_{\gamma_s} f dz = \Phi_s(1) - \Phi_s(0) = F_n(\gamma_s(1)) - F_1(\gamma_s(0))$

1) $= F_n(\beta) - F_1(\alpha)$. (path)

2) $= F_n(z) - F_1(z)$ (closed path)

$\Rightarrow$ not depend on s. in $V(s_0)$. Loc. constant.

coro. 5.9.1. Let $f \in H(D)$ and $\gamma: I \to D$ be contractible. Then $\int_\gamma f dz = 0$.

In particular. if D is simply connected. $\gamma: I \to D$ is closed. Then $\int_\gamma f dz = 0$.

coro 5.9.2. Let $D \subset \mathbb{C}$. be simply connected, $\forall f \in H(D)$ has antiderivative.

Pf: Fix $a \in D$. $\forall z \in D$ consider p.w.s. path $\gamma: I \to D$ connects a with z. $F(z) := \int_\gamma f(\xi) d\xi$.

$F(z)$ doesn't depend on γ. ($\forall \gamma_1, \gamma_2$. homotopic. construct $\gamma_1 \cup \gamma_2^{-1} \Leftrightarrow \int_{\gamma_1 \cup \gamma_2^{-1}} f dz = 0$).

In particular, if $z_0 \in D$ and U is a disk; $U \subset D$; center z_0.

γ_0 is any path connects $a \sim z_0$.

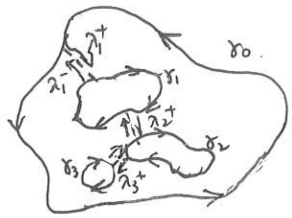
for $z \in U$. $F(z) = \int_{\gamma_0} f(\xi) d\xi + \int_{z_0}^{z} f(\xi) d\xi = F(z_0) + \int_{z_0}^{z} f(\xi) d\xi$.

$\Rightarrow F$ is diff in U and $F'(z) = f(z)$ for every $z \in U$.

Since z_0 is arbitrary, F is antiderivative of f in D.

Thm 5.10.

Cauchy-~~Goursat~~ (multiple connected domain).



(Assume anti-clockwise +)

Suppose $D \subset \mathbb{C}$ is a bounded domain with simple boundary.

$f \in H(G)$. $\bar{D} \subset G$. Then $\int_{\partial D} f\,dz = 0$.

Pf: (as the figure shows). $\Gamma = \gamma_0 \cup \gamma_1^{-1} \cup \cdots \cup \gamma_n^{-1} \cup \lambda_1^+ \cup \lambda_1^- \cup \cdots \lambda_n^+ \cup \lambda_n^-$

$$0 = \int_\Gamma f\,dz = \int_{\partial D} f\,dz + \sum \int_{\lambda_j^+} f + \sum \int_{\lambda_j^-} f = \int_{\partial D} f\,dz.$$

$\triangle$ "∂D" $= \gamma_0 \cup \gamma_1^{-1} \cup \cdots \cup \gamma_n^{-1}$

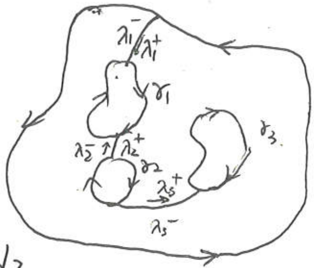
Remark: for multiple connected domain, we have:

$$\int_{\partial D} f\,dz = \int_{\gamma_0} f\,dz - \sum_{j=1}^{n} \int_{\gamma_j} f\,dz$$

§. Cauchy thm.

Thm. $D \subset \mathbb{C}$ is a bounded domain with a simple border $f \in H(G)$. $\bar{D} \subset G$.

Then $\int_{\partial D} f dz = 0$



Thm. $D \subset \mathbb{C}$. is a bounded multiple connected domain

$$\int_{\partial D} f dz = \int_{\gamma_0} f dz + \sum_{j\geq 1}^{n} \int_{\gamma_j} f dz = 0.$$

Pf: $0 = \int_J f dz = \int_{\gamma_0} f dz + \sum_{j\geq 1}^{n} \int_{\gamma_j} f dz + \sum_{i\geq 1}^{n} \int_{\lambda_i^+} f dz + \sum_{i\geq 1}^{n} \int_{\lambda_i^-} f dz$ $\quad J = \gamma_0 \cup \gamma_1 \cup \cdots \cup \gamma_n$.

$= \int_{\gamma_0} f dz + \sum_{j\geq 1}^{n} \int_{\gamma_j} f dz.$ $\quad$ (since $\int_{\lambda_i} f dz = - \int_{\lambda_i^-} f dz$).

Thm. (Cauchy integral thm). Let $D \subset \mathbb{C}$. D-open and let $f \in H(D)$. Let G-domain with oriented boundary γ_0. $\bar{G} \subset D$. ; $\forall z \in G \Rightarrow f(z) = \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(\xi)}{\xi - z} d\xi$.

Pf: let $z \in G$. $U_z(r) = \{\xi \in \mathbb{C} \mid |\xi - z| < r\}$. $\overline{U_z(r)} \subset G$.

apply Cauchy thm. in $G_r = G \setminus \overline{U_z(r)}$ and let $g(\xi) = \frac{f(\xi)}{\xi - z}$ $\quad g \in H(G_r)$.

$$\int_{\partial G} \frac{f(\xi)}{\xi - z} d\xi = \int_{\partial U_r} \frac{f(\xi)}{\xi - z} d\xi.$$

$$2\pi i f(z) - \int_{\partial U_z(r)} \frac{f(\xi)}{\xi - z} d\xi = \int_{\partial U_z(r)} \frac{f(z) - f(\xi)}{\xi - z} d\xi \qquad \left(\int_{\partial U_z(r)} \frac{d\xi}{\xi - z} = 2\pi i \right).$$

$$\left| \int_{\partial U_z(r)} \frac{f(z) - f(\xi)}{\xi - z} d\xi \right| \leq \max_{\xi \in \partial U_z(r)} \frac{|f(z) - f(\xi)|}{r} \cdot 2\pi r \leq 2\pi \cdot \max_{\xi \in \partial U_z(r)} |f(z) - f(\xi)| \xrightarrow[r \to 0]{} 0$$

thus, $2\pi i f(z) - \int_{\partial U_z(r)} \frac{f(\xi)}{\xi - z} d\xi = 0. \Rightarrow f(z) = \frac{1}{2\pi i} \int_{\partial U_z(r)} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\partial G} \frac{f(\xi)}{\xi - z} dz.$

Remark: if $z \in D \setminus G$ $\quad \frac{1}{2\pi i} \int_{\gamma_0} \frac{f(\xi)}{\xi - z} = 0.$ since γ_0 closed $\frac{f(\xi)}{\xi - z} \in H(G)$.

Thm. (Mean value theorem for holomorphic functions).

Let $f \in H(D)$. $\forall a \in D$. $U_r(a) = \{z \in \mathbb{C}; |z-a| < r\}$. f is compactly supported in D.

$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(z + e^{i\theta})\, d\theta$.

Pf: for $U_r(a)$. by Cauchy Integral thm. $f(a) = \frac{1}{2\pi i} \int_{\partial U_r} \frac{f(\xi)}{\xi - a} d\xi$.

Let $\xi = a + re^{i\theta}$. $d\xi = ire^{i\theta} d\theta$.

Chapter 6. Taylor Series.

Def. 1) The series $\sum_{n=1}^{\infty} a_n$ with $a_n \in \mathbb{C}$. conv. with it's sum $s \in \mathbb{C}$. if it's partial sum conv. to s. $S_n = \sum_{k=1}^{n} a_k \xrightarrow[n\to\infty]{} s$. i.e. $\lim_{n\to\infty} |s - a_n| = 0$.

2). The series $\sum_{n=1}^{\infty} f_n(z)$ defined on $K \subset \overline{\mathbb{C}}$. f_n conv. to $f: K \to \mathbb{C}$: $\lim_{n\to\infty} \|f - \sum_{j=1}^{n} f_j\|_K = 0$.

where $\|\varphi\|_K := \sup_{z\in K} |\varphi(z)|$.

Property:

1). Integration of the uniformly conv. series. $\gamma: I \to \mathbb{C}$. p.w.s. path. $f_n: \gamma(I) \to \mathbb{C}$ cont. and $\sum_{n=1}^{\infty} f_n(z)$ conv. uni. on $\gamma(I)$. Hence it's sum $f(z)$ is also cont. on $\gamma(I)$ and.

$$\int_\gamma f dz = \lim_{n\to\infty} \int_\gamma f_n dz.$$

2). Weierstrass criterion for uni. conv. Consider a series $\sum_{n=1}^{\infty} f_n(z)$ of complex-valued functions $f_n: K \to \mathbb{C}$. $K \subset \overline{\mathbb{C}}$. Assume $\|f_n\|_K \le C_n$. and $\sum_{n=1}^{\infty} C_n$ conv. Then $\sum_{n=1}^{\infty} f_n(z)$ is uniformly conv. on K.

Thm 6.2. (Taylor decomposition).

Let $f \in H(D)$. $D \subset \overline{\mathbb{C}}$, $U_R(a) = \{z \in \mathbb{C}: |z-a| \le R\} \subset D$.

Let $C_n := \frac{1}{2\pi i} \int_{|\xi - a| = r} \frac{f(\xi)\, d\xi}{(\xi - a)^{n+1}}$, $n = 0, 1, \ldots$ $0 < r < R$.

C_n not depend on r are Taylor coefficient (of f at a)

$\sum_{n=1}^{\infty} C_n (z-a)^n$ are Taylor series. of function centered at a.

it conv. for every $z \in U_R(a)$ and its sum is equal to $f(z) = \sum_{n=0}^{\infty} C_n (z-a)^n$. $|z-a| < R$.

Pf: 1). independence of C_n.

by Cauchy thm. on homotopy: $\{\zeta \mid |\zeta-a|=r_1\}$ $\{\zeta \mid |\zeta-a|=r_2\}$. $0<r_1<r_2<R$.

are homotopy on D as closed path.

2). convergence.

fix $z \in U_R(a)$. $r \in (0,R)$. s.t. $|z-a|<r<R$.

by Cauchy integral formula. $f(z) = \frac{1}{2\pi i}\int_{\partial U_r(a)} \frac{f(\zeta)}{\zeta - z} d\zeta$.

$$\frac{f(\zeta)}{\zeta-z} = \frac{f(\zeta)}{(\zeta-a)-(z-a)} = \frac{f(\zeta)}{\zeta-a}\cdot\frac{1}{1-\frac{z-a}{\zeta-a}} = \sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n\cdot\frac{f(\zeta)}{\zeta-a} = \sum_{n=0}^{\infty}\frac{(z-a)^n f(\zeta)}{(\zeta-a)^{n+1}}$$

(since $|z-a|<r=|\zeta-a|$ for every $\zeta \in \partial U_r(a)$).

$$\left|\frac{(z-a)^n f(\zeta)}{(\zeta-a)^{n+1}}\right| \le \frac{M(r)}{r}\Big(\underbrace{\frac{|z-a|}{r}}_{<1}\Big)^n, \text{ where } M(r) := \max_{|\zeta-a|=r}|f(\zeta)|.$$

by Weierstrass criterion, $\sum_{n=1}^{\infty}\frac{(z-a)^n f(\zeta)}{(\zeta-a)^{n+1}}$ uni. conv. w.r.t. $\zeta \in U_r(a)$.

$$\int_{\partial U_r(a)}\frac{f(\zeta)}{\zeta-z}d\zeta = \int_{\partial U_r(a)}\sum_{n=0}^{\infty}\frac{(z-a)^n f(\zeta)}{(\zeta-a)^{n+1}}d\zeta. = 2\pi i\sum_{n=0}^{\infty}C_n(z-a)^n$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}C_n(z-a)^n$$

Thm 6.3 In terms of previous thm. $r \in (0,R)$. $n=0,1,\dots$

$|C_n| \le \frac{M(r)}{r^n}$, $M(r) := \max_{|\zeta-a|=r}|f(\zeta)|$. (Cauchy inequality).

Pf: $|C_n| = \left|\frac{1}{2\pi i}\int_{\partial U_r(a)}\frac{f(\zeta)}{(\zeta-a)^{n+1}}d\zeta\right| \le \frac{1}{2\pi}\cdot\frac{M(r)}{r^{n+1}}\cdot 2\pi r = \frac{M(r)}{r^n}$

Thm 6.4. (Liouville thm). $f \in H(D)$ and bounded in $\mathbb{C}$, i.e. $\exists M>0$. s.t. $|f(z)| \le M$ $z \in \mathbb{C}$. Then f is constant.

Pf: Let C_n be Taylor coefficient at $a=0$. $f(z) = \sum_{n=0}^{\infty}C_n z^n$. $z \in \mathbb{C}$.

Then by Cauchy inequalities $|C_n| \le \frac{M}{r^n}$ for every $r>0$. $n=0,1,2\dots$ let $r \to \infty$.

thus $C_n = 0$, $n \in \mathbb{N}$. thus $f(z) = C_0$.

Thm 6.5. (fundamental thm of algebra). Every polynomial of degree $n \ge 1$ has exactly n roots. (n zeros).

Pf: $P(z) = \sum_{n=0}^{\infty}a_n z^n$ Assume $P(z)$ has no roots ($P(z) \ne 0$).

Then $g(z) := \frac{1}{P(z)}$ is an entire function (define on $\mathbb{C}$).

$g(z)$ is bounded. $\lim_{|z|\to\infty}\frac{|P(z)|}{|z|^n} = a_n \Rightarrow \lim_{|z|\to\infty}\frac{1}{P(z)} = 0$. (since $\lim \frac{1}{|z|^n} = 0$).

(continue the proof).

By Liouville's Thm. $g(z)$ is constant and $g(z)=0$. impossible.

Hence $P(z)$ has at least one root. write $P(z)=Q(z)(z-\alpha)$.

Repeat the procedure for $Q(z)$. P must have exactly n roots. $\square$

Coro. If $f\in H(\mathbb{C})$. $|f(z)|\xrightarrow[|z|\to\infty]{}\infty$ f must have at least one root.

Def 6.6 Let $\{b_n\}$ be complex sequence. $a\in\mathbb{C}$. consider $\sum_{n=0}^{\infty} b_n(z-a)^n$.

A value $R=[0,+\infty]$. $R=(\overline{\lim}\sqrt[n]{|b_n|})^{-1}$ is a radius of conv.

A disk. $U_R(a)=\{z\in\mathbb{C}: |z-a|<R\}$. is a disk of conv. (of this series).

Remark: $R=(\overline{\lim}\sqrt[n]{|b_n|})^{-1}$ is Cauchy-Hadamart's formula.

Thm 6.7 A power series $\sum_{n=0}^{\infty} b_n(z-a)^n$ conv. for every $z\in U_R(a)$. And this convergence is uniform on every compact $K\subset U_R(a)$. the series div. for $z\in\mathbb{C}\setminus\overline{U_R(a)}$

Pf: Assume $R\in(0,+\infty)$. $R=(\overline{\lim}_{n\to\infty}\sqrt[n]{|b_n|})^{-1}\Rightarrow \overline{\lim}_{n\to\infty}(|b_n|R^n)^{\frac{1}{n}}=1$

1) conv. on $U_R(a)$: $z\in U_R(a)$. $|z-a|<R$. $\forall\varepsilon>0$. $\exists N\in\mathbb{N}$. s.t. $|b_n|R^n<(1+\varepsilon)^n$ for $n>N$.

denote $q=\frac{|z-a|}{R}(1+\varepsilon)$. choose some ε s.t. $q<1$.

Then $|b_n(z-a)^n|<\frac{(1+\varepsilon)^n}{R^n}|z-a|^n=q^n$ for $n>N$. then $\sum_{n=0}^{\infty}b_n(z-a)^n$ conv. (Weierstrass).

2) uni. conv. on K: $K\subset U_R(a)$. $\max_{z\in K}|z-a|=:r<R$.

denote $q=\frac{r}{R}(1+\varepsilon)$. choose some ε s.t. $q<1$.

3) Div. on $\mathbb{C}\setminus\overline{U_R(a)}$: let $z\in\mathbb{C}\setminus\overline{U_R(a)}$. $|z-a|>R$. $\forall\varepsilon>0$. $\{n_k\}_{k=1}^{\infty}$ $|b_n|R^n>(1-\varepsilon)^n$ for $n=n_k$

Let $q=\frac{|z-a|}{R}(1-\varepsilon)$. , choose some ε $q>1$.

Then for every $n=n_k$. $|b_n(z-a)^n|>\frac{(1-\varepsilon)^n}{R^n}|z-a|^n=q^n\to\infty$.

Coro 6.7.1. (Uniqueness of Taylor series) Assume that $f\in H(U)$ $U=\{z:|z-a|<r\}$. $f(z)=\sum_{n=0}^{\infty}b_n(z-a)^n$. it coincide with the Taylor series of f.

Pf: $U=\{z:|z-a|<r\}$. we fix. $k=0,1,\dots$ and $\rho\in(0,r)$

$$\frac{1}{(z-a)^{k+1}}\sum_{n=0}^{\infty}b_n(z-a)^n=\frac{f(z)}{(z-a)^{k+1}} \quad \text{conv. uni. on } \{z:|z-a|=\rho\}.$$

$\Rightarrow$ integrate both side. along $\gamma:\{z\,|\,|z-a|=\rho\}$. $2\pi b_k=2\pi c_k$ (c_k is Taylor coefficient)

Thm 6.8 $f(z) = \sum_{n\geq 0}^{\infty} b_n (z-a)^n$ is holomorphic in its disk of conv.

Moreover, $f^{(m)}(z) = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} b_n (z-a)^{n-m}$

Pf: $R = \left(\overline{\lim} \sqrt[n]{|b_n|}\right)^{-1}$ $r<R$. $z, w \in B(0,r)$.

$$\frac{f(w)-f(z)}{w-z} = \sum_{k\geq 1}^{\infty} C_k \frac{w^k - z^k}{w-z} = \sum_{k\geq 1}^{\infty} C_k (z^{k-1} + z^{k-2} w + \cdots + z w^{k-2} + w^{k-1}).$$

$|C_k(z^{k-1} + z^{k-2}w + \cdots + zw^{k-2} + w^{k-1})| \leq k|C_k| \cdot r^{k-1}$, series $\sum_{k\geq 1}^{\infty} k C_k r^{k-1}$ conv.

~~thus~~. the LHS. uni.conv. for $w \in B(0,r) \setminus \{z\}$.

$$\lim_{w\to z} \frac{f(w)-f(z)}{w-z} = \sum_{k\geq 1}^{\infty} C_k \lim_{w\to z} (z^{k-1} + z^{k-2} w + \cdots + z w^{k-2} + w^{k-1}) = \sum_{k\geq 1}^{\infty} k C_k z^{k-1}$$

Thm 6.9 pre: $f \in H(D)$. $f^{(m)} \in H(D)$ for any $m \in \mathbb{N}$. (infinite diff.)

the Taylor series of $f^{(n)}(z)$ at $a \in D$. is obtained by n-fold differentiation of the Taylor series for $f(z)$ at a.

Thm 6.10 Let $f(z) = \sum_{n\geq 0}^{\infty} c_n (z-a)^n$. $U_R(a) = \{z \mid |z-a| < R\}$. $f \in U_R(a)$.

Then its Taylor coefficients can be calculated as $c_n = \frac{f^{(n)}(a)}{n!}$, $n \in \{0, 1, 2, \dots\}$

Pf: by thm 6.9. $f^{(n)}(z) = \sum_{k\geq n}^{\infty} \frac{k!}{(k-n)!} C_k (z-a)^{k-n}$ let $z=a$. $f^{(n)}(a) = n! \, C_n$.

Thm 6.11. f be holomorphic at $a \in \mathbb{C}$. $f(a)=0$. but $f \not\equiv 0$ in neighborhood of point a.

Then in some neighborhood U of a point a f can be expressed: $f(z) = (z-a)^n g(z)$. ①

g is holomorphic ② and $g(z) \neq 0$ in U. ③

Pf: $f(z) = \sum_{k\geq 1}^{\infty} c_k (z-a)^k$ (express in Taylor series's form).

in the disk of conv. $U_R(a)$ ($c_0 = f(a) = 0$).

denote. $n := \min\{m \geq 1;\ c_m \neq 0\}$. (if every $c_k = 0$ $f(z) \equiv 0$.).

$g(z) := c_n + c_{n+1}(z-a) + \cdots$ ① g has same R. as f. $g(z) \in H(U_R(a))$. ②

$f(z) = (z-a)^n g(z) \in U_R(a)$. since g cont. in $U_R(a)$. $g(a) \neq 0$.

$\exists\, U \subset U_R(a)$. s.t. $g(z) \neq 0$ for $z \in U$. ③

Thm 6.12. Let $D \Subset \mathbb{C}$ be a domain with a simple boundary and f is holomorphic in the neighborhood of $\bar{D}$. Then for all $n = 0, 1, 2, \dots$ and all $z \in D$.

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D}\frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}$$

Pf: $\forall z \in D$. choose $r > 0$. $U_r(z) \Subset D$. (compactly belongs to D).

$$c_n = \frac{1}{2\pi i}\int_{\partial U_r}\frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}} \quad \text{and} \quad c_n = \frac{f^{(n)}(z)}{n!} \quad (\text{thm. 7.2 and 7.11})$$

$$\Rightarrow \quad f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial U_r}\frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}$$

(by Cauchy thm in multiple connected set. $\partial U_r \to \partial D$)

Thm 6.13. (Morer's thm) if $f \in C(D)$ and $\int_{\partial\Delta} f\,d\mu = 0$. for any triangle s.t. $\Delta \Subset D$. then $f \in H(D)$.

Pf: According to the existence of a primitive. on a disk (Lemma 4.4).

f has anti-derivative F in $U \Rightarrow F \in H(U) \Rightarrow f$ is holomorphic in U. by thm 7.8

Thm 6.14. (Equivalent definition of holomorphic func.) at point $a \in \mathbb{C}$.

1). f is $\mathbb{C}$-diff in some neiborhood U of point a. (original def.)

2) f is analytic at point a.

(i.e. decomposes f into a power series centered at point a. converging in some neighborhood U of point a;)

3). $f \in C(U)$. U is some neighborhood of a. and $\int_{\partial\Delta} f\,d\mu = 0$ for any triangle $\Delta \Subset D$.

Pf: 1) $\Rightarrow$ 2). thm 7.2.

2) $\Rightarrow$ 1) thm 7.8.

1) $\Rightarrow$ 3). Cauchy - Goursat thm.

3) $\Rightarrow$ 1) Morer's thm

Def. The number $n = \min\{m \geq 1 : c_m \neq 0\} = \min\{m \geq 1 : f^{(m)}(a) \neq 0\}$ from thm 7.11. is called multiplicity of zero. of holomorphic f at a.

Equivalently, it's unique number n s.t. in some neighborhood U of point a. $f(z) = (z-a)^n g(z)$. for some $g \in \mathcal{O}(U)$ s.t. $g(a) \neq 0$.

Coro 6.12.1. If $f \in H(D)$ and has 0 at $a \in D$. then:

1) $f \equiv 0$ in some neighborhood of a.

or 2). $\exists\, U \subset D$. U is neighborhood of a s.t. $f(z) \neq 0$ for every $z \in U \setminus \{a\}$.

Thm. 6.15. (Uniqueness. thm).

Let $f, g \in H(D)$. Assume. $E = \{z \in D : f(z) = g(z)\}$. has a limit point in D.

Then $f \equiv g$ in D

Pf: Let $A = \{z \in D : f \equiv g$ in some neighborhood of $z\}$.

then A is · open. (by def of A)

· closed. if $z_0 \in D$ is a limit point of A then z_0 is nonisolated zero of function $f-g$ and $f-g$ is identically zero in some neighborhood of z_0.

· not empty. limit point of E is not isolated zero of $f-g$. thus it belongs to A.

since D is connected. $A = D$

Property of elementary function of complex variable.

$f(z) = \sum_{k=0}^{\infty} c_k z^k \qquad g(z) = \sum_{k=0}^{\infty} d_k z^k$, $|z| < R$. both are abs. conv. for $|z| < R$.

then for $\lambda \in \mathbb{R}$. $(f+g)(z) = \sum_{k=0}^{\infty} (c_k + d_k) z^k$. $\lambda f(z) = \sum_{k=0}^{\infty} \lambda c_k z^k$.

$(fg)(z) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} c_j d_{k-j} \right) z^k$.

$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad z \in \mathbb{C} \qquad (e^z)' = e^z$

$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1} \qquad z \in \mathbb{C} \qquad (\sin z)' = \cos z$

$\cos z = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}. \qquad z \in \mathbb{C}. \qquad (\cos z)' = -\sin z$.

$$e^{z_1} \cdot e^{z_2} = \left(\sum_{k=0}^{\infty} \frac{z_1^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{z_2^k}{k!} \right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{z_1^j}{j!} \frac{z_2^{k-j}}{(k-j)!} \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} z_1^j z_2^{k-j} = \sum_{k=0}^{\infty} \frac{(z_1+z_2)^k}{k!} = e^{z_1+z_2}$$

$\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$. $\Leftarrow$ since $\mathbb{R}$ has limit point in $\mathbb{C}$.

$\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$ by uniqueness thm.

$\sin^2 z + \cos^2 z = 1$.

Coro. $A \in \mathbb{C}$. if $f(p) = A$ then p is A-point of f.

if f is not constant, then for every $A \in \mathbb{C}$. A-points of function f are isolated.

Thm 6.16. (Maximum modulus principle).

Let $f \in H(D)$. If there exists a point $p \in D$. s.t. $|f(z)| \le |f(p)|$ for every $z \in D$ then f is constant.

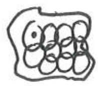
Pf: denote $|f(z_0)| = M$. Let $\rho > 0$ s.t. $B(z_0, \rho) \subset D$. by mean value thm. for every $r \in [0,\rho)$

$$M = |f(z_0)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z_0 + re^{it}) dt \right| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_0 + re^{it})| dt \le M.$$

$$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} (M - |f(z_0 + re^{it})|)\, dt = 0.$$

integrand nonnegative. and cont. $\Rightarrow M = |f(z_0 + re^{it})|$. for $t \in (-\pi, \pi)$

$\Rightarrow |f(z)| = M$ for every $z : |z - z_0| = r$. $\Rightarrow$ r is arbitrary. $|f(z)| = M$ for every in $B(z_0, \rho)$.

f is constant on the disk. $\Rightarrow$ constant in D $\Leftarrow$ 

Coro 6.16.1. An absoluti value of a nonconstant holomorphic function does not have a maximum (even a local maximum) in the domain.

Coro 6.16.2 $D \subset \mathbb{C}$. D is bounded. $f \in H(D) \cap C(\bar{D})$. Then $|f|$ obtains its maximum on ∂D.

Coro 6.13.3. Let $f \in H(D)$. f is not constant $f \ne 0$ in D.
$|f|$ does not have a minimum. (even a local maximum) in the domain.

Chapter 7. Laurent series and singular points.

Thm 7.1. (Laurent series) $f(z) \in H(V)$. $V = \{z \in \mathbb{C} : r < |z-a| < R\}$. $a \in \mathbb{C}$. $0 \le r < R \le +\infty$.

$$c_n := \frac{1}{2\pi i} \int_{|\zeta - a| = \rho} \frac{f(\zeta)\, d\zeta}{(\zeta - a)^{n+1}} \quad \text{for all } n \in \mathbb{Z}. \text{ and } r < \rho < R.$$

the numbers c_n are independent of ρ. and are called Laurent coefficient of f in ring V.

$\sum_{-\infty}^{\infty} c_n (z-a)^n$ is called Laurent series of f in ring V. conv. for all $z \in D$ and its sum is $f(z)$ (i.e. $f(z) = \sum_{-\infty}^{\infty} c_n (z-a)^n$ for $r < |z-a| < R$).

$\triangle$ regular part $\sum_{n \ge 0}^{\infty} c_n (z-a)^n$. main part $\sum_{n=-\infty}^{-1} c_n (z-a)^n$ they conv. separately.

Pf: 1) independence: Cauchy thm on homotopy.

$\{|\zeta - a| = \rho_1\}$ and $\{|\zeta - a| = \rho_2\}$. $r < \rho_1 < \rho_2 < R$. homotopy in V. as closed path.

2) convergence fix $z \in V$. and choose s.t $\in (r, R)$. $s < |z-a| < t$. V 

for ring $\{\zeta \in \mathbb{C} : s < |\zeta - a| < t\}$. bounded by circle γ_s, γ_t

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_t} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(\zeta)\, d\zeta}{\zeta - z} =: I_1 - I_2$$

For I_1:

since $|z-a|<|\zeta-a|$ for all $\zeta \in \gamma_t$.

$$\frac{f(\zeta)}{\zeta-z} = \frac{f(\zeta)}{(\zeta-a)-(z-a)} = \frac{f(\zeta)}{\zeta-a}\cdot\frac{1}{1-\frac{z-a}{\zeta-a}} = \sum_{n=0}^{\infty}\frac{(z-a)^n f(\zeta)}{(\zeta-a)^{n+1}} \qquad (1)$$

$$\left|\frac{(z-a)^n f(\zeta)}{(\zeta-a)^{n+1}}\right| \leq \frac{M(t)}{t}\left(\frac{|z-a|}{t}\right)^n, \quad M(t) := \max_{|\zeta-a|=t}|f(\zeta)|$$

by Weierstrass criterion, the series conv. uni. over $\zeta \in \gamma_t$.

integrating both side. (1). $I_1 = \sum_{n=0}^{\infty} c_n (z-a)^n \quad c_n := \frac{1}{2\pi i}\int_{\gamma_t}\frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}}$

For I_2:

$I_2 = \sum_{m=0}^{\infty} b_m (z-a)^{-(m+1)} \quad b_m = -\frac{1}{2\pi i}\int_{\gamma_s}(\zeta-a)^m f(\zeta)\,d\zeta$ (replace. $-(m+1)=n$.

△ Laurent series is unique in a fixed ring V for fixed function f.

Thm 7.2. (convergence of series by integer degree $z-a$).

For an arbitrary set $\{c_n : n \in \mathbb{Z}\}$. $c_n \in \mathbb{C}$. let $R := \{\varlimsup_{n\to\infty}|c_n|^{\frac{1}{n}}\}^{-1}$ $r := \varlimsup_{n\to\infty}|c_{-n}|^{\frac{1}{n}}$.

Then Laurent's row $f(z) := \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ 1) conv. abs. and uni. on compacts in ring $\{r<|z-a|<R\}$ and its sum $f(z)$ is holomorphic 2) in this ring and satisfy: 3)

$\frac{1}{2\pi i}\int_{|\zeta-a|=\rho}\frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} = c_n$. for all $n \in \mathbb{Z}$. $r<\rho<R$. (also implies the uniqueness of Laurent series).

if $|z-a|>R$. the regular part div. if $|z-a|<r$. the main part div.

Pf: 1) use Cauchy - Hadamart formula. (to obtain the conv.).

2) denote $f_1(z) = \sum_{n\geq 0} c_n (z-a)^n \quad f_2(z) = \sum_{n=-\infty}^{-1} c_n (z-a)^n = \underline{\sum_{m=1}^{\infty} c_{-m} Z^m}$ where $Z := \frac{1}{z-a}$ → 转化为泰勒级数.

$V_1 = \{z \mid r<|z-a|<R\}$.

$f_1(z)$ is holomorphic in V_1. (by thm 6.8).

$f_2(z)$ is holomorphic in $|Z|<r^{-1}$. $\Leftrightarrow |z-a|>r$.

3) $f(\zeta) = \sum_{m=-\infty}^{+\infty} c_m (\zeta-a)^m \quad \zeta \in V_1$.

$$\frac{f(\zeta)}{(\zeta-a)^{n+1}} = \frac{\sum_{m=-\infty}^{+\infty} c_m (\zeta-a)^m}{(\zeta-a)^{n+1}}$$ integral along $\{|\zeta-a|=\rho\}$.

$$\Rightarrow \int_{|\zeta-a|=\rho}\frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} = 2\pi i\, c_n. \quad \left(\text{the integral } \int_\gamma (z-a)^n dz = \begin{cases} 0, & n\neq -1 \\ 2\pi i, & n=-1 \end{cases}\right).$$

Thm 7.3 (Cauchy inequalities for Laurent coefficients).

Let $f(z)=\sum_{n=-\infty}^{+\infty} c_n(z-a)^n$ $f\in H(V)$. $V=\{r<|z-a|<R\}$. Then for all $n\in\mathbb{Z}$. $\rho\in(r,R)$ the inequality are valid: $|c_n|\le\frac{M(\rho)}{\rho^n}$. where $M(\rho):=\max_{|\xi-a|=\rho}|f(\xi)|$

Pf: see thm 6.3.

Remark 7.4. Each convergent Laurent series can be considered as Fourier series.

e.g. $f\in H(V)$. $V=\{1-\varepsilon<|z|<1+\varepsilon\}$ $\varepsilon>0$.

Laurent coefficient. $c_n=\frac{1}{2\pi i}\int_{|\xi|=1} f(\xi)\,\xi^{-(n+1)}\,d\xi=\frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\,e^{-int}\,dt$.

Fourier coefficient. $c_n=\frac{1}{2\pi}\int_0^{2\pi}\varphi(t)\,e^{-int}\,dt$. $t\in[0,2\pi]$

Let $z=e^{it}$, $f(z)=f(e^{it})=\varphi(t)$

△But not every Fourier series is a Laurent series of some function.

§. Isolated singular points.

Def. A point $a\in\mathbb{C}$ is isolated singular point for function $f(z)$ if $f\in H(V)$.

$V=\{0<|z-a|<\varepsilon\}$. $\varepsilon>0$. (punctured neighborhood of point a);

Three and only three type of isolated singular point.

(1) fixable. $\exists$ (finite) limit. $\lim_{z\to a} f(z)\in\mathbb{C}$. (removable/disposable/eliminated.

(2) by a pole. $\exists\lim_{z\to a} f(z)=\infty$.

(3) essentially a singular point. $\nexists$ limit of $f(z)$ for $z\to a$.

e.g. for $a=0$. (1) fixable. $f=\frac{\sin z}{z}$. $\lim_{z\to0}\frac{\sin z}{z}=1$.

(2) pole $f=\frac{1}{z}$. $\lim_{z\to0}\frac{1}{z}=\infty$

(3) essentially. $f=e^{\frac{1}{z}}$ ($x\to0$. $f\to+\infty$; $iy\to0$ $f=e^{\frac{1}{iy}}=e^{-i\frac{1}{y}}=\cos(\frac{1}{y})-i\sin(\frac{1}{y})$.

(4) unisolated. $f(z)=\mathrm{ctg}\,\frac{1}{z}$. poles. $z_n=(\pi n)^{-1}$

(不止一个pole, 找不到 $\dot V_\varepsilon(a)$. 不是 isolated singular point)

Thm. 7.6 (Description of fixable .. singular point).

△ fixable, disposable. eliminated.
synonym.

For $V = \{0<|z-a|<\varepsilon\}$. $f \in H(V)$. TFAE:

(1) $z=a$ is a fixable'' . singular point;

(2) $f(z)$ is bounded in some punctured neighborhood. $V' = \{0<|z-a|<\varepsilon'\}$. $\varepsilon'>0$.

(3). the Laurent coefficient c_n of the function f in the punctured neighborhood V satisfy: $c_n = 0$ at $n<0$. (只有非负次项)

(4). it's possible to define $f(z)$ at $z=a$. that results f be holomorphic in $\{|z-a|<\varepsilon\}$

Pf: (1) $\Rightarrow$ (2). obvious.

(2) $\Rightarrow$ (3): if $|f(z)| \le M$. for $z \in V'$

by Cauchy inequality (7.3). $|c_{-k}| \le M\rho^k$ for all $k=1,2,\dots$ all $\rho \in (0,\varepsilon')$

Let $\rho \to 0$. get $c_{-k}=0$. for $k=1,2,\dots$

(3) $\Rightarrow$ (4). $f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ for $z \in V$. put $f(a) = c_0$.

$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$ for $z \in \{|z-a|<\varepsilon\}$

(4) $\Rightarrow$ (1) obvious

Thm 7.7. (Description of the poles)

$f \in H(\dot V_\varepsilon(a))$. point a is the pole of f.. $\Longleftrightarrow$ the main part $\left(\sum_{n=-\infty}^{-1}\right)$ of the Laurent series of f in $\dot V$. contains only a finite number (nonzero). non-zero numbers.

i.e. $f(z) = \sum_{n=-N}^{\infty} c_n (z-a)^n$. $N \in \mathbb{N}$. $c_{-N} \neq 0$.

Pf: "$\Rightarrow$" By def. of the pole $\lim_{z\to a} f(z) = \infty$ so $f(z) \neq 0$. at $0<|z-a|<\varepsilon'$

the function $g(z) := \frac{1}{f(z)}$ holomorphic in $\dot V'_{\varepsilon'}(a)$ $\lim_{z\to a} g(z) = 0$.

$g \in H(U')$ $U' = \{|z-a|<\varepsilon'\}$ by thm 7.6.

assume $g(a)=0$. for $0<|z-a|<\varepsilon'$, $g(z) = (z-a)^N h(z)$ (N - the order of zero for g at $z=a$)

where $h \in H(U')$. and $h(z) \neq 0$ at $0<|z-a|<\varepsilon''$.

the function $\frac{1}{h} \in H(U'')$, $U'' = \{|z-a|<\varepsilon''\}$ $\Rightarrow$ $\frac{1}{h(z)} = b_0 + b_1(z-a) + \dots$ $b_0 = \frac{1}{h(a)}$

(有Taylor series 0 1 2 ...
对应到 Laurent $-N, -N+1, \dots$)$\times -N$.

$f(z) = (z-a)^{-N} \frac{1}{h(z)}$ on $\{0<|z-a|<\varepsilon\}$

"$\Leftarrow$" $f(z) = (z-a)^{-N} g(z)$, $g \in H(U)$. $U = \{|z-a|<\varepsilon\}$ and $g(a) \neq 0$ in U.

Thus $\lim_{z\to a} f(z) = 0$.

pole order $f(z)$ at A = order of zero $\frac{1}{f(z)}$ at A.

Remark: N - pole order. $c_{-n} = 0$ for $n>N$. $c_{-N} \neq 0$.

Fact (find in the proof) $f(z)$ has a pole at point a iff $1/f(z)$ is holomorphic and $\frac{1}{f(z)} = 0$ at a.

Rem (inverse of thm 7.7). $f(z)=\sum_{n=-\infty}^{+\infty} c_n (z-a)^n$. $f\in H(\dot{V}_\varepsilon(a))$.

f has an essential feature at a iff there are infinitely many $n\geq 1$. s.t. $c_{-n}\neq 0$.

△ main part $c_n = 0$ fixable

$c_{-N}\neq 0$. finite many N. pole.

$c_{-n}\neq 0$. infinite many n. essential.

thm 7.8 (Sokhotsky's thm).

If $a\in\mathbb{C}$. is an essentially singular point of f, then for any $A\in\bar{\mathbb{C}}$

$\exists\{z_n\}_{n\in\mathbb{N}}$. $z_n\to a$. s.t. $\lim_{n\to\infty} f(z_n)=A$.

Pf: 1° $A=\infty$. f can't be bounded in any $\dot{V}_\varepsilon(a)$. (otherwise a is fixable).

$\exists\{z_n\}\to a$. s.t. $f(z_n)\xrightarrow[n\to+\infty]{}\infty$

2° $A\in\mathbb{C}$. if $\exists\dot{V}_\varepsilon(a)$, $\exists z\in\dot{V}_\varepsilon(a)$. $f(z)=A$. obvious.

if not. $g(z):=\frac{1}{f(z)-A}$ has an isolated point at $z=a$. (since $f(z)$ has an isolated point at $z=a$. and $f(z)-A\neq 0$ on $\dot{V}$.)

point a can't be pole or fixable, since $f(z)=A+\frac{1}{g(z)}$ would have limit at a.

(in that case, contradicts with essentially singular).

therefore, a is essentially singular point for $g(z)$, let $A'=\infty$, by 1°:

$\exists\{z_n\}\to a$ s.t. $g(z_n)\xrightarrow[n\to+\infty]{}\infty$

it follows that $f(z_n)=A+\frac{1}{g(z_n)}\to A$. for $n\to\infty$

Def. the point $a=\infty$ is isolated singular point for the function f if $f\in H(\{|z|>R\})$ for some $R>0$

the type of $a=\infty$. (of f).

1) fixable. $\Leftrightarrow$ $c_n=0$ for all $n\geq 1$.

2) the pole $\Leftrightarrow$ $\exists N\geq 1$. s.t. $c_N\neq 0$ but $c_n=0$ when $n>N$. (N-pole order)

3). essential $\Leftrightarrow$ $c_n\neq 0$. for an infinite set of natural $n\geq 1$.

the main part of Laurent series of the function f in $\dot{V}_\varepsilon(\infty)$: $\sum_{n\geq 1}^{\infty} c_n z^n$

regular part $\sum_{n=-\infty}^{0} c_n z^n$

(main / regular : 0与∞正好相反. 可取倒数互推).

Def. integer: function that is holomorphic in entire complex plane $\mathbb{C}$.
(根据定义. 整函数有限点不为奇点, 只讨论 $z=\infty$ 奇点类别 (一定是孤立奇点)

Thm 7.10. If the integer function f has a fixable singular point or pole for $z=\infty$, then f is a polynomial

Pf: Denote: $P(z)=\sum_{n=1}^{N} c_n z^n$ finite main part of the Laurent series in $\dot{V}_\varepsilon(\infty)$

denote: $g(z):=f(z)-P(z)=\sum_{n=0}^{\infty} c_{-n} z^{-n}$ $\quad g\in H(\mathbb{C})$

and $g(z)$ has fixable singularity (by thm 7.6). denote $g(\infty)=\lim_{z\to\infty} g(z)$

then $g(z)$ is bounded on $\overline{\mathbb{C}}$. by Liouville thm. $g(z)\equiv C$ (同理可证明整函数的在 $z=\infty$ L-S. 的 regular part 一定是常数).

(对整函数 f. $z=\infty$. fixable $\Leftrightarrow f\equiv c_0$.
$z=\infty$ pole $\Leftrightarrow f=\sum_{n=1}^{N} c_n z^n$. N 是 order of pole.
$z=\infty$ essential $\Leftrightarrow$ infinite many $c_n\neq 0$. $n\geq 0$. 超越整函数 $\Leftrightarrow R=\infty$ 的无穷幂级数).

Def. (meromorphic). A function f is called memomorphic in the domain $D\subset\overline{\mathbb{C}}$. if it is holomorphic except at its poles. 否则对 compact set $\overline{\mathbb{C}}$ 找不到有限子覆盖.
denote $M=\{a \mid f \text{ has a pole at each point of } a\in M\}$ M is no more than countable
对 fixable singular point

Thm 7.11. If f is meromorphic in $\mathbb{C}$ and has a fixable singular point or pole at $z=\infty$ (i.e. f is meromorphic in $\overline{\mathbb{C}}$) then f is rational.

Pf: poles of f in $\overline{\mathbb{C}}$ is isolated $\Rightarrow$ the set of poles is finite.

Let poles be $\{a_1,\dots a_n\}$. denote $R_j(z)=\sum_{k=1}^{n_j} C_{jk}(z-a_j)^{-k}$. $j=1,2,\dots n$.

the main part of Laurent series is f in the punctured neighborhood of ∞.

denote by $P(z)=\sum_{k=1}^{n} C_k z^k$

then denote $g:=f-(P+R_1+\dots+R_N)$. $g\in H(\mathbb{C})$ and has fixable singularity at $z=\infty$.

by Liouville thm. $g\equiv$ const.

$\Rightarrow f\equiv C+P(z)+R_1(z)+\dots+R_n(z)$

△有理函数 $Q(x)=\dfrac{P_1(x)}{P_2(x)}$.
分子分母都是多项式, 多项式系数可不有理.

Laurent Series Practical Parts（求洛朗级数）.

if $z=z_0$ is a singularity of f. f cannot be expanded as Taylor series at z_0.

if $z=z_0$ is an isolated singularity of f. f can be expanded as Laurent series at z_0.

$$f(z) = \underbrace{\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}}_{\text{principal part}} + \underbrace{\sum_{k=0}^{\infty} a_k (z-z_0)^k}_{\text{analytic part.}}$$

principal part will converge for $|z-z_0|>r$ (or $\left|\frac{1}{z-z_0}\right| < r^*$, $r^* := \frac{1}{r}$).

analytic part will converge for $|z-z_0|<R$.

Thus, the Laurent series converges. when $z \in \{r<|z-z_0|<R\}$.

e.g. Expand $f(z)=\frac{1}{z(z-1)}$ in a Laurent series. conv. in following annular domain

(a) $0<|z|<1$. ← 展开的元.

$$f(z) = -\frac{1}{z}\cdot\frac{1}{1-z} = -\frac{1}{z}(1+z+z^2+\cdots) = -\frac{1}{z}-1-z-z^2-z^3-\cdots$$

(b) $|z|>1 \Rightarrow \left|\frac{1}{z}\right|<1$.

$$f(z) = \frac{1}{z^2}\cdot\left(\frac{1}{1-\frac{1}{z}}\right) = \frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right) = \frac{1}{z^2}+\frac{1}{z^3}+\cdots$$

(c). $0<|z-1|<1$

$$f(z) = \frac{1}{z-1}\,\frac{1}{1+(z-1)} = \frac{1}{z-1}\left[1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right]$$

$$= \frac{1}{z-1}-1+(z-1)-(z-1)^3+\cdots$$

(d) $|z-1|>1$

$$f(z) = \frac{1}{(z-1)^2}\,\frac{1}{1+\frac{1}{z-1}} = \frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}-\frac{1}{(z-1)^5}+\cdots$$

e.g. Expand $f(z)=\frac{8z+1}{z(1-z)}$ in Laurent series conv. in $0<|z|<1$.

$$f(z) = \frac{1}{z}+\frac{9}{1-z}. = \frac{1}{z}+9+9z+9z^2+\cdots$$

e.g. $f(z)=\frac{1}{z(z-1)}$. conv. for $1<|z-2|<2$.

$$f(z) = -\frac{1}{z}+\frac{1}{z-1} = -\frac{1}{2}\,\frac{1}{1+\frac{z-2}{2}} + \frac{1}{z-2}\cdot\frac{1}{1+\frac{1}{z-2}}$$

$$= \cdots - \frac{1}{(z-2)^4}+\frac{1}{(z-2)^3}-\frac{1}{(z-2)^2}+\frac{1}{z-2}-\frac{1}{2}+\frac{z-2}{2^2}-\frac{(z-2)^2}{2^3}+\frac{(z-2)^3}{2^4}-\cdots$$

Deduction. Practical Parts.

case 1. deduction in a simple pole. (1st order pole).

Let a be a simple pole of f.

$$f(z) = \frac{C_{-1}}{z-a} + \sum_{n=0}^{\infty} C_n (z-a)^n \qquad C_{-1} = \mathrm{res}_a f = \lim_{z\to a} (z-a) f(z)$$

suppose $f(z) = \frac{\varphi(z)}{\psi(z)}$ in some $\dot{V}_\varepsilon(a)$, where $\varphi(z), \psi(z) \in H(V_\varepsilon(a))$.

and s.t. $\varphi(a) \neq 0$. $\psi(a) = 0$ but $\psi'(a) \neq 0$.

thus $\mathrm{res}_a f = \lim_{z\to a} (z-a)\frac{\varphi(z)}{\psi(z)} = \lim_{z\to a} \varphi(z) \frac{z-a}{\psi(z)-\psi(a)} = \frac{\varphi(a)}{\psi'(a)}$

case 2. deduction in a n-th power pole.

$$f(z) = \frac{C_{-n}}{(z-a)^n} + \dots + \frac{C_{-1}}{z-a} + \sum_{m=0}^{\infty} C_n (z-a)^m$$

multiple $(z-a)^n$, take derivative $n-1$ times.

$$\mathrm{res}_a f = C_{-1} = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}[(z-a)^n f(z)]}{dz^{n-1}}$$

case 3. deduction at $a=\infty$ ($f \in H(\{|z| \le R_0\})$. have isolated singular point ∞.)

Def. the deduction of f at infinity is a number $\mathrm{res}_\infty f = \frac{1}{2\pi i}\int_{\gamma_R^{-1}} f\,dz$.

($\gamma_R = \{|z| = R\}$, R is sufficiently large. $R > R_0$. clockwise).

f has Laurent expansion in $\{|z| > R_0\}$. $f(z) = \sum_{n=-\infty}^{\infty} C_n z^n$

$\Rightarrow \mathrm{res}_\infty f = -C_{-1}$

Example 1 use case 1 to compute residue.

$f(z) = \frac{1}{z^4+1}$ 4 simple poles $z_i = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}}$.

$$\mathrm{Res}(f(z), z_k) = \frac{\varphi(z_k)}{\psi'(z_k)} = \frac{1}{3z_k^3}$$

Chapter 8 Deduction. (Residue).

Def. (deduction) the deduction of the function f at an isolated singular point $a \in \mathbb{C}$ is a number $\mathrm{res}_a f = \frac{1}{2\pi i} \int_{|\zeta - a| = r} f(\zeta)\, d\zeta$, where $r \in (0, \varepsilon)$.

(by Cauchy thm. the integral does not depend on the choice of r).

Thm 8.1.

Cauchy thm on deduction: $D \subset \mathbb{C}$ be a domain, with simple boundary. $\bar{D} \subset G \subset \mathbb{C}$. G be some domain. Suppose $f \in H(G)$. except a finite set (singular point) $\{a_1, \dots a_n\} \subset D$.

Then: $\int_{\partial D} f(\zeta)\, d\zeta = 2\pi i \sum_{j=1}^{n} \mathrm{res}_{a_j} f$ (用留数计算积分，注意仅考虑在域内的奇点)

Pf: choose $\varepsilon > 0$. $B_j = \{ z \in \mathbb{C} : |z - a_j| < \varepsilon \}$ did not intersect in pairs. and $\bar{B_j} \subset D$.

then we have a multiply connected $D_\varepsilon := D \setminus \bigcup_{j=1}^{n} \bar{B_j}$, use Cauchy thm.

$$0 = \int_{\partial D_\varepsilon} f(\zeta)\, d\zeta = \int_{\partial D} f(\zeta)\, d\zeta - \sum_{j=1}^{n} \int_{\partial B_j} f(\zeta)\, d\zeta = \int_{\partial D} f(\zeta)\, d\zeta - \sum_{j=1}^{n} 2\pi i\, \mathrm{res}_{a_j} f$$

Thm 8.2. (Deduction in terms of the Laurent series).

If the function $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n$ is holomorphic in the punctured neighborhood of $\dot{V}_\varepsilon(a)$.

then $\mathrm{res}_a f = c_{-1}$

Pf: direct proof:

$$\mathrm{res}_a f = \frac{1}{2\pi i} \int_{|z-a|=r} f(z)\, dz = \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} c_n \int_{|z-a|=r} (z-a)^n\, dz = \frac{1}{2\pi i} \cdot 2\pi i\, c_{-1} = c_{-1}$$

(since $\int_{|z-a|=r} (z-a)^n\, dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$)

coro. If $a \in \mathbb{C}$ is a fixable isolated singular point of f. then $\mathrm{res}_a f = 0$.

Practical Part. compute deduction in isolated singular points.

1) simple pole a. $\mathrm{res}_a f = c_{-1} = \frac{\varphi(a)}{\psi'(a)}$, where $\frac{\varphi(z)}{\psi(z)} = f(z)$. $\varphi, \psi \in H(V_\varepsilon(a))$.

2) n-th pole a. $\mathrm{res}_a f = c_{-1} = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1} [(z-a)^n f(z)]}{dz^{n-1}}$

3) $a = \infty$ $\quad \mathrm{res}_\infty f = -c_{-1}$.

Thm 8.5. (on the total amount of deductions). Let the function $f \in H(\mathbb{C})$. except for a finite number of points $\{a_v\}_{v=1}^{n}$. Then the sum of the deductions at the points $\{a_v\}$ and at infinity is zero: $\operatorname{res}_{\infty} f + \sum_{v} \operatorname{res}_{a_v} f = 0$.

Pf: Let $U_R = \{|z| < R\}$ be a circle of sufficiently large radius contains all singular points $\{a_v\}$.

Applying Cauchy's deduction thm. $\frac{1}{2\pi i}\int_{\partial U_R} f dz = \sum_{v} \operatorname{res}_{a_v} f$.

$LHS = -\operatorname{res}_{\infty} f$.

Thm 8.6. (Jordan lemma).

Let f be defined and continuous on $\{z \in \mathbb{C} : \operatorname{Im} z \geq 0, |z| \geq R_0\}$.

Let $R \geq R_0$. $M(R) := \max_{z \in \gamma_R} |f(z)|$. γ_R is a semicircle: $\gamma_R = \{z = Re^{i\theta} : 0 \leq \theta \leq \pi\}$

Suppose f tends to zero at infinity, so that $\lim_{R \to \infty} M(R) = 0$. Then for every $t > 0$ the ratio is valid. $\lim_{R \to \infty} \int_{\gamma_R} f(z) e^{itz} dz = 0$.

Pf: $$\left|\int_{\gamma_R} f(z) e^{itz} dz\right| = \left|\int_0^{\pi} f(Re^{i\theta}) e^{-tR\sin\theta + itR\cos\theta} iRe^{i\theta} d\theta\right|$$
$$\leq \int_0^{\pi} M(R) R e^{-tR\sin\theta} d\theta.$$

since $\sin\theta \geq \frac{2}{\pi}\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. by changing variable $\tau := \frac{2R\theta}{\pi}$.

$$\int_0^{\pi} Re^{-tR\sin\theta} d\theta = 2\int_0^{\frac{\pi}{2}} Re^{-tR\sin\theta} d\theta \leq 2\int_0^{\frac{\pi}{2}} Re^{-\frac{2tR\theta}{\pi}} d\theta = \pi \int_0^{R} e^{-t\tau} d\tau$$
$$= \frac{\pi}{t}(1 - e^{-tR}).$$

E.g.1. Fourier transformation of rational function

Compute $I(t) := \int_{-\infty}^{\infty} \frac{x \sin tx}{x^2+a^2} dx$. $t>0$, $a>0$.

it can be considered as $\lim_{R\to+\infty} \int_{-R}^{R} \frac{x \sin tx}{x^2+a^2} dx$.

denote $J(t) := \lim_{R\to+\infty} \int_{-R}^{R} \frac{x e^{itx}}{x^2+a^2} dx$ we have $I(t) = \mathrm{Im}\, J(t)$ (Fourier trans.).

denote the integrand $f(z) := \frac{z e^{itz}}{z^2+a^2}$; and the region $D_R := \{z\in\mathbb{C} : \mathrm{Im}\, z>0, |z|<R\}$.

the border: $\{[-R,R]\} \cup \gamma_R := \{z\in\mathbb{C} : \mathrm{Im}\, z \geq 0, |z|=R\}$

$$\int_{-R}^{R} f(x)\,dx = \int_{\partial D_R} f(z)\,dz - \int_{\gamma_R} f(z)\,dz \xrightarrow[R\to\infty]{} \pi i e^{-at} + o(1)$$

$$\int_{\partial D_R} f(z)\,dz = 2\pi i \,\mathrm{res}_{z=ai} f = \pi i e^{-at}$$

$\int_{\gamma_R} f(z)\,dz = o(1)$ by Jordan lemma (since $g(z) = \frac{z}{z^2+a^2} \xrightarrow[z\to\infty]{} 0$).

$J(t) = \pi i e^{-at}$, $I(t) = \mathrm{Im}\, J(t)$.

and since the integrand of $I(t)$ is odd. $I(t) = \begin{cases} \pi e^{-at} & t>0 \\ 0 & t=0 \\ -\pi e^{at} & t<0. \end{cases}$

Remark1:

E.g.2. Residue on essential singular point.

$\oint_C e^{3/z} dz$, contour $C: |z|=1$

$z=0$ is essential singularity. calculate Laurent expansion (by unique thm)

$e^{\frac{3}{z}} = 1 + \frac{3}{z} + \frac{3^2}{2!\,z^2} + \frac{3^3}{3!\,z^3} + \cdots$ $\qquad a_{-1} = \mathrm{Res}(f(z), 0) = 3$.

§. Practical Part: Calculate real integral.

Forms: $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$; $\int_{-\infty}^{\infty} f(x) dx$; $\int_{-\infty}^{\infty} f(x) \cos\alpha x\, dx$; $\int_{-\infty}^{\infty} f(x) \sin\alpha x\, dx$.

F and f are ration functions ; and for $f(x) = \frac{p(x)}{q(x)}$, assume p, q no common factors. :.

(1) Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$. (1). contour : $C: |z| = 1$.

convert form (1) (real, trigonometric) into complex integral.

$dz = ie^{i\theta} d\theta$. $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ $\Longrightarrow$ let $z = e^{i\theta}$. $\Rightarrow$

$d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}(z + z^{-1})$, $\sin\theta = \frac{1}{2i}(z - z^{-1})$ (replacing formula)

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_C F\left(\tfrac{1}{2}(z+z^{-1}), \tfrac{1}{2i}(z-z^{-1})\right) \frac{dz}{iz}$$

(2) Form $\int_{-\infty}^{+\infty} f(x) dx$ (2)

if f is cont. on $[0, +\infty)$ or $(-\infty, 0]$.

define $I_1 = \int_0^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_0^R f(x) dx$ (limit may not exist, that is f div.)

$I_2 = \int_{-\infty}^0 f(x)\, dx = \lim_{R\to\infty} \int_{-R}^0 f(x) dx$

if f is cont. on $(-\infty, +\infty)$. $\int_{-\infty}^{+\infty} f(x)\, dx = I_1 + I_2$.

If already knows $\int_{-\infty}^{+\infty} f(x)$ conv. $\int_{-\infty}^{+\infty} f = \lim_{R\to\infty} \int_{-R}^{R} f$ (Cauchy P.V.).

Thm1. $f(z) = \frac{p(z)}{q(z)}$ is a rational function, where $\deg p = n$. $\deg q = m \geq n+2$. If C_R is a semi-circular contour $z = Re^{i\theta}$. $\theta \in [0, \pi]$. then $\int_{C_R} f(z) dz \to 0$ as $R \to \infty$.

(3). From $\int_{-\infty}^{\infty} f(x) \cos\alpha x\, dx$ $\int_{-\infty}^{\infty} f(x) \sin\alpha x\, dx$.

Fourier integral: $\int_{-\infty}^{\infty} f(x) e^{i\alpha x} = \int_{-\infty}^{\infty} f(x) \cos\alpha x\, dx + i \int_{-\infty}^{\infty} f(x) \sin\alpha x$.

Method: using symmetry to compute. P.V. or conv. integral.

Thm2. $f(z) = \frac{p(z)}{q(z)}$ is a rational fun. $\deg p = n$. $\deg q = m \geq n+2$. If C_R is a semi-circular contour $z = Re^{i\theta}$. $\theta \in [0, \pi]$. $\alpha > 0$. then $\int_{C_R} f(z) e^{i\alpha z} dz \to 0$ as $R \to \infty$.

E.g. Evaluate the P.V. $\int_0^{\infty} \frac{x \sin x}{x^2+9} dx$.

Δ: $\int_0^{\infty} \frac{x\sin x}{x^2+9} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+9} dx$

form contour integral : $\oint_C \frac{z}{z^2+9} e^{iz} dz$ C: 

$$\int_{C_R} \frac{z}{z^2+9} e^{iz} dz + \int_{-R}^{R} \frac{x}{x^2+9} e^{ix} dx = \oint_C = 2\pi i \,\mathrm{Res}\,(f(z) e^{iz}, 3i) = \frac{e^{-3}}{2} \cdot 2\pi i = \frac{\pi}{e^3} i$$

by thm2. ($\alpha = 1 > 0$). $\int_{C_R} f(z) e^{iz} dz \xrightarrow[R\to\infty]{} 0$. P.V. $\int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+9} dx = \frac{\pi}{e^3} i$

$$\text{P.V.} \int_0^{\infty} \frac{x \sin x}{x^2+9} dx = \frac{1}{2} \mathrm{Im}\left(\text{P.V.} \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+9} dx\right) = \frac{\pi}{2e^3}.$$

Thm 3. f has a simple pole $z = c \in \mathbb{R}$. If $C_r : \{ z \mid z = c + re^{i\theta} \}$, $\theta \in [0, \pi]$.

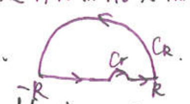
then $\lim_{r \to 0} \int_{C_r} f(z)\,dz = \pi i \, \mathrm{Res}(f(z), c)$.

(Behavior of integral as $r \to 0$).

(与前2个example的区别是有极点在实轴上)

E.g. Intended contour.

evaluate the Cauchy principle value of $\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2-2x+2)}\,dx$.



Δ: consider $\oint_C \frac{e^{iz}}{z(z^2-2z+2)}\,dz$. $f(z) = \frac{1}{z(z^2-2z+2)}$ pole on upper half plane: $z=0,\ 1+i$.

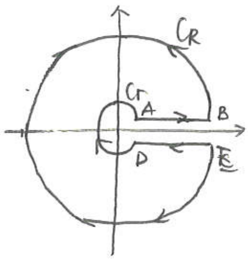
$$\oint_C = \int_{C_R} + \int_{-R}^{-r} + \int_{-C_r} + \int_r^R = 2\pi i \, \mathrm{Res}(f(z)\cdot e^{iz}, 1+i).$$

$R \to \infty$. $r \to 0$. $\mathrm{P.V.} \int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x(x^2-2x+2)} - \pi i \, \mathrm{Res}(f(z)e^{iz}, 0) = 2\pi i \, \mathrm{Res}(f(z)e^{iz}, i+1)$

§. Integration along a Branch Cut. (对多值函数).

Method: restrict the argument. guarantee the integrand is single valued.

e.g. evaluate $\int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)}$.



$x=0$. infinite discountinuity. but the integral conv.

$\oint_C \frac{1}{z^{1/2}(z+1)}\,dz$. pole $z=-1$.

$$\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i \, \mathrm{Res}(f(z), -1).$$

on AB $z = xe^{0i}$ on DB $z = xe^{2\pi i}$

$$\int_{ED} = \int_{AB} : \int_{ED} = \int_R^r \frac{(xe^{2\pi i})^{-1/2}}{xe^{2\pi i}+1} e^{2\pi i}\,dx = -\int_R^r \frac{x^{-1/2}}{x^0+1}\,dx = \int_r^R \frac{x^{-1/2}}{x+1}\,dx$$

$$\Rightarrow \int_0^{\infty} \frac{dx}{\sqrt{x}(x+1)} = \frac{1}{2} \cdot 2\pi i \, (\mathrm{Res}\, f(z), -1)$$

§. The argument Principle and Rouché's Theorem.

thm. (count number of zeros and poles within simple closed contour C)

Let C be a simple closed contour lying entirely within a domain D. Supposed $f \in H(D)$ except at a finite number of poles inside C. and $f(z) \neq 0$ on C.

Then $\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)} dz = N_0 - N_p$. 辐角原理.

number of 0s / number of poles inside C.

Pf: $\frac{f'(z)}{f(z)}$ - analytic in and on C. except poles and zeros of f in C.

1) if z_0 is a zero of order n of f inside C, we can write $f(z) = (z-z_0)^n \varphi(z)$

$\varphi(z_0) \neq 0$ and φ is analytic at z_0.

$$\frac{f'(z)}{f(z)} = \frac{(z-z_0)^n \varphi'(z) + n(z-z_0)^{n-1}\varphi(z)}{(z-z_0)^n \varphi(z)} = \frac{\varphi'(z)}{\varphi(z)} + \frac{n}{z-z_0}$$

$\cdot \operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_0\right) = \lim_{z \to z_0} (z-z_0)\left[\frac{\varphi'(z)}{\varphi(z)} + \frac{n}{z-z_0}\right] = 0 + n = n. \to$ order of zero z_0.

2) if z_p is a pole of order m of f inside C. $f(z) = \frac{g(z)}{(z-z_p)^m}$ $g(z_p) \neq 0$. g is analytic at z_0.

$$\frac{f'(z)}{f(z)} = \frac{(z-z_p)^{-m} g'(z) - m(z-z_p)^{-m-1} g(z)}{(z-z_p)^{-m} g(z)} = \frac{g'(z)}{g(z)} + \frac{-m}{z-z_p}.$$

$\operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_p\right) = -m.$

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i \left[\sum_{k=1}^{r} \operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_{0k}\right) + \sum_{k=1}^{s} \operatorname{Res}\left(\frac{f'(z)}{f(z)}, z_{pk}\right)\right]$$

$$= 2\pi i \left[\sum_{k=1}^{r} n_k + \sum_{k=1}^{s} (-m_k)\right] = 2\pi i [N_0 - N_p]$$

Remark: $N_0 - N_p = \frac{1}{2\pi}$[change in $\arg(f(z))$ as z tranverses C once in positive direction]

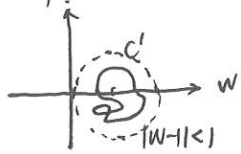
(令 $z = e^{i\theta}$ $0 \le \theta \le 2\pi$. 最终θ转的圈数).

Thm. (Rouché's Thm.).

Let C be a simple closed contour lying entirely within a domain D.
f, g are analytic in D. if $|f(z) - g(z)| < |f(z)|$ hold for all $z \in C$.
then f and g have the same number of zeros inside C (the count include the order)

Pf: denote. $F(z) = \dfrac{g(z)}{f(z)}$. the strict inequality $|F(z) - 1| < 1$.

function $F(z)$ does not enclose $w = 0$. $\frac{1}{w}$ analytic on C'.



By Cauchy-Goursat thm. $\int_{C'} \frac{dw}{w} = 0$. or $\oint_C \frac{F'(z)}{F(z)} dz = 0$.

$\Rightarrow \oint_C \frac{g'(z)}{g(z)} dz = \oint_C \frac{f'(z)}{f(z)} dz$. then by. argument principle thm.

$N_p = 0$. N_0. same.

Example. Location of zero.

e.g. $g(z) = z^9 - 8z^2 + 5$

denote $f(z) = z^9$. $z = 0$. 9-order of zero.

$|f(z) - g(z)| < |f(z)|$ holds for $z \in \{ |z| = \frac{3}{2} \}$.

thus. $g(z)$ has 9-zero inside circle $|z| = \frac{3}{2}$.

☆ (Uniqueness thm. for Holomorphic function).

$f, g \in H(D)$. $E = \{z \in D : f = g\}$ has a limit point in D. Then $f \equiv g$ in D.

常用 Taylor's series.

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad z \in \mathbb{C}.$$

$$\ln(1+z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} (-1)^{n+1} \qquad |z| < 1$$

$$\sin z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} (-1)^n \qquad z \in \mathbb{C}.$$

$$\cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} (-1)^n \qquad z \in \mathbb{C}.$$

$$\arctan z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2n+1} (-1)^n \qquad |z| < 1$$

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n. \qquad |z| < 1$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad |z| < 1.$$

$$\tan z = z + \frac{z^3}{3} + \frac{2}{15} z^5 + \cdots$$

$$\arcsin z = z + \frac{z^3}{6} + \frac{3}{40} z^5 + \cdots$$

§ Chapter 9 Analytic Continuation.

§. Analytic continuation (direct).

<u>Def. 9.1.1.</u> (direct analytic continuation, 2 function).

Let $f_1 \in H(D_1)$, $f_2 \in H(D_2)$. Δ is connected component of $D_1 \cap D_2$. If $f_1(z) = f_2(z)$ for every $z \in \Delta$ then f_2 is direct analytic continuation (DAC) of f_1 from domain D_1 to domain D_2. along the domain Δ.

Rem: If the AC exists then it's unique. (from ... to ... along ...)

If Δ^* is another connect component of $D_1 \cap D_2$.

may happen $f_1|_{\Delta^*} \neq f_2|_{\Delta^*}$, AC along Δ and Δ^* maybe not equal.

e.g. (AC to a fixable point. by continuity).

$f(z) = \frac{\sin z}{z}$. $\quad z \in \mathbb{C} \setminus \{0\}$

$f(z) = \frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \qquad f(z) = \begin{cases} \frac{\sin z}{z} & z \in \mathbb{C} \setminus \{0\} \\ 1 & z = 0 \end{cases}$

<u>Lemma 9.1.2.</u> domain $D \subset \mathbb{C}$. and $\varphi = \varphi(t,z) : [a,b] \times D \to \mathbb{C}$. is cont. func. that is holomorphic in $z \in D$ for every fixed parameter $t \in [a,b]$.

Consider a function $f(z) = \int_a^b \varphi(t,z)\, dt$. then $f \in H(D)$.

Pf: f is uni. conv. on $[a,b] \times K$, $\forall K$ - compact. $K \subset D$. $\Rightarrow f$ is cont. in D.

$\forall \Delta \subset D$. $\int_{\partial\Delta} f\, dz = \int_{\partial\Delta} \int_a^b \varphi(t,z)\, dt\, dz = \int_a^b \int_{\partial\Delta} \varphi(t,z)\, dt\, dz = \int_a^b 0\, dt = 0$

by Morer's thm. $f \in H(D)$.

§. Weierstrass theory of AC.

<u>Def 9.2.1.</u> (analytic continuation along chain. 2 function).

Let $f \in H(D)$. $g \in H(E)$ domain $D_0, D_1 .. D_n$ functions, $f_k \in H(D_k)$.

s.t. $D_0 = D$. $f_0 = f$. $D_n = E$. $f_n = g$. for every $k \in [1,n]$. f_k is DAC of f_{k-1} from D_{k-1} to D_k. Then g is analytic continuation of function f along a chain of domains $D, D_1, \ldots D_{n-1}, E$.

<u>Remark:</u> for simplicity. don't mention connected components.

(By induction AC along the fixed chain of domains and connected components of overlaps is unique).

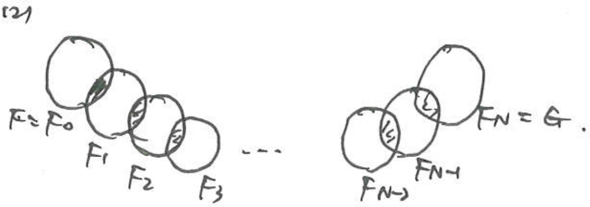
Def.9.2.3 Element is a pair $F=(U,f)$ that consist of a disk $U=\{|z-a|<R\}$ centered at a and the function $f\in H(U)$. A point a is called a center of an element and R its radius. Element F is called canonical if U coincides with disk of conv. of Taylor series of f with center at a.

Def 9.2.4 (DAC of 2 elements / DAC of elements along the chain)

(1) Elements $F=(U,f)$ and $G=(V,g)$ are DAC of each other if $U\cap V\neq\phi$ and $f(z)=g(z)$, $z\in U\cap V$.

(2) An element G is analytic continuation of an element F along the chain $F_0=F, F_1, F_2, \ldots F_{N-1}, F_N=G$ if F_{n+1} is DAC of F_n for every $n=0,1,\ldots,N-1$.

(1) 

(2) 

Property

1. Weierstrass Property If $G=(V,g)$ is DAC of an element $F=(U,f)$ and center b of a disk V belong to U then Taylor's series of function g is obtained by Taylor's decomposition of f at point b.

$$g(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(b)}{n!}(z-b)^n \quad b\in U.\ z\in V.$$

Inversely, if for every $b\in U$. if $g(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(b)}{n}(z-b)^n$ and V is the disk of conv. of this series. then (V,g) is DAC of (U,f).

2. Triangle Property Suppose that element $F_1=(U_1,f_1)$ is DAC of an element $F_0=(U_0,f_0)$ and element $F_2=(U_2,f_2)$ is DAC of an element F_1.
If $U_0\cap U_1\cap U_2\neq\phi$. then F_2 is DAC of F_0.

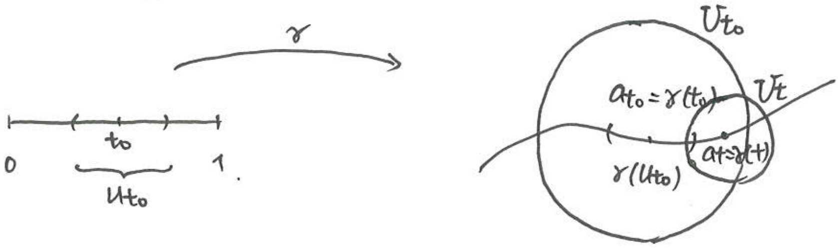
Pf: $U_0\cap U_1\cap U_2\subset U_0\cap U_2$. $f_2\equiv f_1\equiv f_0$. on $U_0\cap U_1\cap U_2$.

by uniqueness thm. $f_2\equiv f_0$ on $U_0\cap U_2$. $\Rightarrow$ F_2 is DAC of F_0.

Def. 9.2.5. (AC along path (of a canonical element))

A family of canonical elements $F_t = (U_t, f_t)$, $t \in I = [0,1]$, is analytic continuation of a canonical element F_0 along a path $\gamma : I \to \mathbb{C}$ if:

1. center at of an element F_t coincide with $\gamma(t)$ and radius $R(t)$ is strictly positive for every $t \in I$.
2. for every $t_0 \in I$, there exists a neiborhood $\mathcal{U}_{t_0} \subset I$ of a point t_0, s.t. for every $t \in \mathcal{U}_{t_0}$, $\gamma(t) \in U_{t_0}$ and F_t is DAC of F_{t_0}



Lemma 9.2.6. If $\{F_t : t \in I\}$ and $\{\tilde{F}_t : t \in I\}$ are two AC of canonical elements $F_0 = \tilde{F}_0$ along a path γ, then $F_1 = \tilde{F}_1$

Lemma 9.2.7 Let $R(t)$ be radius of an element F_t. Then either $R(t) = +\infty$ or $R : I \to \mathbb{R}$ is cont. function.

Lemm 9.2.8 Suppose $f \in H(D)$, and $a \in D$. $R = \text{dist}(a, \partial D)$. $U = \{z \in \mathbb{C} \mid |z-a| < R\}$. Then the Taylor's series of function f with center at a conv. to f in U_0.

Thm 9.2.9. (Ostrowski - Hadamard Gap thm).

Let $s > 1$. let $\{n_k\}_{k=1}^{\infty}$ be sequence in $\mathbb{N}$. s.t. $\frac{n_{k+1}}{n_k} \geq s$ for all k. and let $\{a_k\}_{k=1}^{\infty}$ be sequence in $\mathbb{C}$. If $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ and $R = 1$ (radius of conv.) then f has no AC outside of $|z| < 1$.

Def. 9.2.10. (regular point) antonym - singular point.

Let $f \in H(D)$. A point $a \in D$ is a regular point of f.

e.g. $f(z) = \sum_{k=1}^{\infty} z^{2^k}$. analytic in $|z| < 1$. $R = 1$. $\Rightarrow$ no AC outside $\{|z| = 1\}$

Thm 9.2.11. (Equivalence of AC along a chain and along a path).

(1). $\{F_t : t\in I\}$ – family of canonical elements. defines an AC of F_0 along γ.
Then there exists a finite sequence of points $0=t_0<t_1<\dots<t_n=1$
s.t. F_1 coincides with AC of F_0 along a chain $F_0=F_{t_0}, F_{t_1}, \dots F_{t_n}=F_1$.

(2) if F, G – canonical elements. G is AC of F by some chain of canonical – elements $F=F_0, F_1, F_2, \dots, F_n = G$.
Denote by $\gamma : I\to\mathbb{C}$ polygonal curve that connects centers of $F_0, F_1 \dots, F_n$.
Then there exists a family of canonical elements. F_t, $t\in I$, performs that AC of F_0 along path γ s.t. $F_{t=1}=F_n$.

Thm 9.2.12. (Construction of AC along path).

Let $D\subset\mathbb{C}$. a domain. $f\in H(D)$. G_0 and G_1. are two subdomain of D.
f has antiderivative in G_1 and G_2. Consider a path γ. connects $z_1\in G_1$. $z_2\in G_2$. and let

$$\Phi_1(z) = \int_{z_1}^{z} f(\xi)\,d\xi. \quad z\in G_1$$

$$\Phi_2(z) = \int_{\gamma} f(\xi)\,d\xi + \int_{z_2}^{z} f(\xi)\,d\xi. \quad z\in G_2.$$

Then Φ_2 is AC of Φ_1 along path γ.

Rem: $\int_{z_1}^{z} f$. $\int_{z_2}^{z} f$ can be consider along any path in G_1, G_2. since f has a.d. both in G_1 and G_2. Function Φ_1, Φ_2 are a.d. of f in G_1. G_2 respectively.

Thm 9.2.13. (Monodromy thm).

Suppose γ_0, γ_1 paths with common endpoints. $\gamma_0(0)=\gamma_0(1)=a$. $\gamma_0(1)=\gamma_1(1)=b$.
homotopic to each other in $\mathbb{C}$. Let $T: I\times I\to\mathbb{C}$ be a homotopy of these two paths
$\gamma(s,0)=a$. $\gamma(s,1)=b$. for every $s\in I$. path $\gamma_s: I\to\mathbb{C}$, $\gamma_s(t):=T(s,t)$
perform "deformation of γ_0 to γ_1".
Suppose canonical element $F=(U,f)$ centered at a has AC $\{F_{s,t} : t\in I\}$.
along every path γ_s, $s\in I$. Then the AC of F along γ_0 and γ_1 coincide. $F_{01}=F_{11}$.

Thm 9.2.14. (Another statement of Monodromy thm)

Let D – simply connected domain. $D\subset\mathbb{C}$. $F=(U,f)$ – canonical. center $a\in D$.
Suppose F has AC along any path γ in D. with startpoint at a. Then for every $b\in D$.
AC of F along any path $\gamma: a\rightsquigarrow b$. provides the same result. $G=(V,g)$.
(i.e. AC of F along all possible path define a holomorphic func. in D).
Moreover, in V (center at b) this function in define by Taylor's series of g. while $G=(V,g)$ obtained by AC of F along arbitrary path $\gamma: a\rightsquigarrow b$.

§ Analytic Function.

Def 9.3.1. (Analytic function). $\mathcal{F}$ - 函数集合/等价类.

Let $D \subset \mathbb{C}$ be a domain and $F_0 = (U_0, f_0)$. be a canonical element with center at point $a \in D$ and s.t. $U_0 \subset D$ that has analytic continuation along any path γ. in D with start point a. A set $\mathcal{F}$ of all canonical elements that can be obtained by continuation F_0 of all such paths is called an. analytic function in domain D. generated by element F_0.

Def. 9.3.2. (branch)

Suppose that $\mathcal{F}$ is analytic function in domain D. and $D_1 \subset D$ is subdomain. If there exists a canonical element $F_1 = (U_1, f_1) \in \mathcal{F}$. and AC of F_1 along all possible path $\gamma \subset D_1$ defines function g. $g \in H(D_1)$, then analytic function $\mathcal{F}$ allows single-valued (holomorphic) branch in domain D_1 and pair (D_1, g) is holomorphic branch (analytic element) of analytic function $\mathcal{F}$ in domain D_1.

Def 9.3.3. (germ)

Analytic elements (D_1, f_1). (D_2, f_2) are called equivalent at point $a \in D_1 \cap D_2$. if $f_1 \equiv f_2$ in some neighborhood of a.

Classes of equivalence are called germs at point a. ⇒ 单点的"AE"为元"的等价类

⇒ 等价类环.

Remark 9.3.4. A set of all germs at point a. generates a ring denoted by $\mathcal{O}_a$.

If the germ of function f at point a denote by $\{f\}_a$ .then:

the operation rules: $\{f\}_a + \{g\}_a := \{f+g\}_a$. $\{f\}_a \{g\}_a := \{fg\}_a$.

Def 9.3.5 Analytic continuation of a germ φ_0 along path $\gamma : I \to \mathbb{C}$ is a family of germs $\varphi_t \in \mathcal{O}_{\gamma(t)}$ s.t. $\forall t_0 \in I$. $\exists U_{t_0} \subset I$. $\exists D_{t_0} \subset \mathbb{C}$. $\exists f \in H(D_{t_0})$ s.t. $\gamma(U_{t_0}) \subset D_{t_0}$ and $\{f\}_{\gamma(t)} = \varphi_t$ for every $t \in U_{t_0}$.

Rem 9.3.6. (general views of analytic function) (3 equivalent def.).

(1) A family of canonical elements obtained by AC of some initial element.

(2) A set of branches obtain by AC of some initial element (along chain/path)

(3) A set of germs obtained by AC (along path)

Def 9.3.7 Analytic function $\mathcal{F}$ on D is called complete if no germ of $\mathcal{F}$ has AC outside of D.

(此时，$\mathcal{F}$ 包含任一解析函数分支的所有延拓．D 为存在区域．∂D 为 $\mathcal{F}$ 的自然边界).

Def 9.3.8. Values of an analytic function $\mathcal{F}$ at point z are values of all its elements (branches, germs) defined at z. A set of all values of $\mathcal{F}$ at point z is denoted by $\mathcal{F}(z)$, that is $\mathcal{F}(z) = \{f(z) : (U, f) \in \mathcal{F},\ z \in U\}$.

Remark: $\mathcal{F}$ is single-valued iff $\forall z \in D$ $\mathcal{F}(z)$ consist of one value.

Lemma 9.3.9. Canonical element (U_0, f_0) has AC along any path $\gamma \subset \mathbb{C} \setminus \{0\}$ with startpoint at $z=1$ and doesn't have AC along any path $\gamma \subset \mathbb{C}$, that passes through 0.

$f_0(z) = \sqrt{|z|}\ e^{i\frac{\arg z}{2}}, \quad -\pi < \arg z < \pi. \qquad U_0 = \{|z-1| < 1\}.$

§. Isolated Singular Points of analytic function.

Def.9.3.10. A point $a \in \mathbb{C}$ is isolated singular point of analytic function F.

is F is analytic in $\dot{V}_a$. where $\begin{cases} \dot{V}_a = \{z \in \mathbb{C} : 0 < |z-a| < \varepsilon\} & a \in \mathbb{C} \\ \dot{V}_\infty = \{z \in \mathbb{C} : |z| > \varepsilon^{-1}\}. & a = \infty \end{cases}$

Lemma 9.3.11. Let $\dot{V}_a$ be a punctured neighborhood of a. Then for every closed path $\gamma : I \to V$ with $\gamma(0) = \gamma(1) = z_0 \in \dot{V}_a$. $\exists! \, n \in \mathbb{Z}$ s.t. γ is homotopic to γ_0^n.

where $\gamma_0^n(t) = a + (z_0 - a)\, e^{2\pi n i t}$, $0 \le t \le 1$. (circle of radius $|z_0 - a|$ centered at a.)

Pf: Existence $z_0 - a = |z_0 - a| e^{i\varphi_0}$. $\varphi_0 = Arg\,(z_0 - a)$.

Then $\gamma_0^n(t) = a + |z_0 - a| e^{i(\varphi_0 + 2\pi n t)}$, $0 \le t \le 1$.

path. $\gamma(t) = a + |\gamma(t) - a| \, e^{i\varphi(t)}$. $t \in I = [0,1]$.

and $\varphi(t) = Arg\,(\varphi(t) - a)$. is cont. on I. with $\varphi(0) = \varphi_0$.

$\gamma(0) = \gamma(1) = z_0$. $\varphi(1) - \varphi(0) = 2\pi n$ for some $n \in \mathbb{Z}$.

$T : I \times I \to V$. $T(s,t) = a + |z_0 - a|^s |\gamma(t) - a|^{1-s} e^{i\{(\varphi_0 + 2\pi n t)s + \varphi(t)(1-s)\}}$.

T is cont. $\begin{aligned} T(0,t) &= a + |\gamma(t) - a| \, e^{i\varphi(t)} \\ T(1,t) &= a + |z_0 - a| \, e^{i(\varphi_0 + 2\pi n t)} \end{aligned}$ T is homotopy of γ to γ_0^n.

Uniqueness If $\gamma \sim \gamma_0^n \sim \gamma_0^m$ then $n = \frac{1}{2\pi i}\int_{\gamma_0^n} \frac{dz}{z-a} = \frac{1}{2\pi i}\int_{\gamma_0^m} \frac{dz}{z-a} = m$.

(argument principle).

Lemma 9.3.12. Let a - an isolated singular point of F defined in $\dot{V}_a$. if the AC of some canonical element $F_0 \in F$. (with center at $z_0 \in \dot{V}_a$). along circle γ_0. coincides with F_0. then F. is single-valued on $\dot{V}_a$.

Pf: By previous lemma. $\forall$ path γ_1, γ_2. $z \to z_0$. (same start and end point).

$\gamma_1 \cup \gamma_2^{-1}$ is closed $\exists n \in \mathbb{Z}$. $\gamma_1 \cup \gamma_2^{-1}$ is homotopic to γ_0^n.

$\Rightarrow \gamma_1$ is homotopic to $\gamma_2 \cup \gamma_0^n$. ⇙. $\gamma_1 = \gamma_1 \cup \gamma_2^{-1} \cup \gamma_2$ is homotopic to $\gamma_0^n \cup \gamma_2$

$\Rightarrow$ AC along γ_1 and $\gamma_2 \cup \gamma_0^n$ coincide.

$\Rightarrow$ continuation along the circle doesn't change.

AC along γ_1 and γ_2 coincide.

Classification of isolated singular point.

Suppose $\mathcal{F}$ analytic in $\dot{V}_a$. a is generated by canonical elements $F_0 \in \mathcal{F}$. with center at $z_0 \in \dot{V}_a$.

For $n \in \mathbb{Z}$. denote F_n. as the result of analytic continuation along γ_0^n defined above.

Def. 9.3.13. (1) If $F_0 = F_1$. then analytic function $\mathcal{F}$ is single-valued and holomorphic in $\dot{V}_a$. In this case a is isolated singular point of single-value character of $\mathcal{F}$.

(2). If $F_0 \neq F_1$. then a is branch point of $\mathcal{F}$. If $F_n = F_0$ for some $n \geq 2$. then is branch point of finite order of $\mathcal{F}$. and $m = \min\{n \geq 2 : F_n = F_0\}$. is the order of branch point. Else, a is logarithmic branch point.

("转圈". n 是转圈数).

Lemma. 9.3.14. The definition given above doesn't depend on the choice of the initial canonical element.

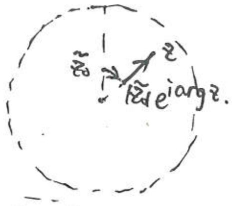
Pf: consider another canonical element $\tilde{F}_0$ (of analytic function $\mathcal{F}$. centered at $\tilde{z}_0 \in \dot{V}_a$).

$\tilde{\gamma}_0$-closed : $\gamma(t) = a + (\tilde{z}_0 - a)e^{2\pi i t}$, $t \in I = [0,1]$

Denote by $\tilde{F}_n$ the result of analytic continuation of $\tilde{F}_0$ along $\tilde{\gamma}_0^n$.

$\lambda \subset \dot{V}_a$. path, connected z and $\tilde{z}_0$. $\tilde{F}_0$ is AC of F_0 along λ.

$\lambda_0 \subset \dot{V}_a$, path. $\tilde{z}_0 \rightarrow |\tilde{z}_0| e^{i \arg z} \rightarrow z$. (along the radius of $\dot{V}_a$)



$\lambda \cup \lambda_0^{-1} \sim \gamma_0^m \Rightarrow \lambda \sim \gamma_0^m \cup \lambda_0$ and $\tilde{\gamma}_0 \sim \lambda_0^{-1} \cup \gamma_0 \cup \lambda_0$.

hence $\tilde{\gamma}_0^n \sim \lambda^{-1} \cup \gamma_0^n \cup \lambda$. $n \in \mathbb{Z}$. (continuation along λ coincides with F_n for every $n \in \mathbb{Z}$)

$\Rightarrow F_n = F_0$ and $\tilde{F}_n = \tilde{F}_0$ are equivalent.

Def. 9.3.15. $\mathcal{F}$ is analytic in D. and $V_a \subset D$. is a punctured neighborhood of $a \in \mathbb{C}$.

the restriction $\mathcal{F}|_{V_a}$ is the class. consist of several analytic functions on V with it's own singularity on a (may be different).

Singularity points of these branches are called singular points of function $\mathcal{F}$ above point a.

§. Multi-Valued Functions and Analytic Continuation.

§1. Logarithm $\operatorname{Ln} z$.

initial element. $f_0(z) = \ln z$. $U_0 = \{|z-1|<1\}$.

$$\ln z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n} \qquad z \in U_0.$$

$$(\ln z)' = \sum_{n=1}^{\infty} (1-z)^{n-1} = \frac{1}{1-(1-z)} = \frac{1}{z}. \qquad z \in U$$

by N-L. formula. $f(z) = \int_1^z \frac{d\xi}{\xi}$ $z \in U$; $f_0(z) = \int_1^z \frac{d\xi}{\xi}$. $z \in \mathbb{C} \setminus (-\infty, 0]$.

$f_0 \in H(\mathbb{C} \setminus (-\infty, 0))$. $f_0 \equiv f$ in U.

but limit on upper and lower edge of slit $(-\infty, 0]$ not coincide.
f_0 can't extended without breaking analyticity / continuity. of f_0.

AC: $f_0(z) = \ln|z| + i\arg z$. $-\pi < \arg z < \pi$. $U_0 = \{|z-1|<1\}$.

path. $\gamma \subset \mathbb{C} \setminus \{0\}$. start. $z=1$.

$\Rightarrow f_t(z) = \ln|z| + i\arg z$. $-\pi + \theta(t) < \arg z < \pi + \theta(t)$.

or $f_t(z) = \int_{\gamma[0,t]} \frac{d\xi}{\xi} + \int_{\gamma(t)}^{z} \frac{d\xi}{\xi}$. $\qquad z \in U_t = \{z \in \mathbb{C} : |z - \gamma(t)| < |\gamma(t)|\}$.

$\Rightarrow$ obtain function (has countable number of elements) at every point of $\mathbb{C} \setminus \{0\}$.

Property

1. If (G, f) is a holomorphic branch of logarithm then $f'(z) = \frac{1}{z}$. for every $z \in G$.
   In particular $0 \notin G$
2. If G is open, connected in $\mathbb{C}$. TFAE:
   (1) There exist analytic branch of logarithm in G.
   (2) There exist antiderivative of $1/z$. in G.
3. Domain of $\operatorname{Ln} z$ is $\mathbb{C} \setminus \{0\}$. branch point. $z=0$ and ∞ (logarithmic)
4. If (G, f) is a holomorphic branch of logarithm then $e^{f(z)} = z$ $\forall z \in G$.
5. If (G, f) is a holomorphic brance of logarithm. $k \in \mathbb{Z}$. then $(G, f + 2\pi k i)$ is also holomorphic branch of logarithm and no other branches in G.
6. $\operatorname{Ln} z = \{w \in \mathbb{C} : e^w = z\}$, $z \neq 0$.
7. $\operatorname{Ln} z = \ln|z| + i\operatorname{Arg} z$, $z \neq 0$. In particular for principal branches of logarithm $\ln z = \ln|z| + i\arg z$.

§2. Power function z^α.

(1). Special case $\sqrt{z}$.

initial element: $f_0(z) = \sqrt{|z|}\, e^{i\frac{\arg z}{2}}$. $-\pi < \arg z < \pi$.

f_0 is holomorphic in a slit plane $\mathbb{C} \setminus (-\infty, 0]$.

$f_0'(z)^2 = z \Rightarrow f_0'(z) = \frac{1}{2 f_0(z)}$ $\quad f'(z) \xrightarrow[z\to\infty]{} \infty$

(2) general case (def.).

$$\Phi_\alpha(z) = e^{\alpha \operatorname{Ln} z}, \quad z \neq 0. \quad \alpha \in \mathbb{C}.$$

(composition of analytic func. $\operatorname{Ln} z$ and holomorphic func. e^w).

$\varphi_\alpha(z) = e^{\alpha \ln z}$. $z \in G_0 = \mathbb{C} \setminus (-\infty, 0]$. - primary branch with degree α.

(notation: z^α. all values / particular value / primary value) 都有可能

Property:

1. $\Phi_\alpha(z)$ is complete analytic function generated by the element (U, φ_α).
   $U = \{z \in \mathbb{C} : |z-1| < 1\}$.
2. If (D, f) is a holomorphic branch of logarithm then $(G, e^{\alpha f})$ is a holomorphic branch of the power func.
3. Since $z^\alpha = e^{\alpha \operatorname{Ln} z} = e^{\alpha(\ln z + 2k\pi i)} = e^{\alpha \ln z} e^{2k\pi i \alpha}$. $k \in \mathbb{Z}$. three cases:
   1). $\alpha \in \mathbb{Z}$. $e^{2k\pi i \alpha} = 1$. function is single-valued.
   if $\alpha \geq 0$. then it is defined at $z=0$ (or fixable); $\alpha<0$. $z=0$ is $|\alpha|$-order pole.
   2) $\alpha \in \mathbb{Q}$. $\alpha = \frac{p}{q}$. $q \in \mathbb{N}$. $p \in \mathbb{Z}$. $\frac{p}{q}$ is irreducible. then z^α obtains exactly q different values. choice $k = 0, 1, \dots q-1$. (branch point: $z=0$ and $z=\infty$. q-order)
   3). If $\alpha \notin \mathbb{Q}$. z^α obtains countable number of values.
4. For every branch: $(z^\alpha)' = \frac{\alpha}{z} z^\alpha$.
   Pf: $\alpha \notin \mathbb{Z}$, $z \neq 0$. $(z^\alpha)' = (e^{\alpha \ln z})' = e^{\alpha \ln z}(\alpha \ln z)' = \frac{\alpha}{z} z^\alpha$.
5. If (D, f) is a holomorphic branch of power func. of degree α, $\alpha \in \mathbb{C}$. $k \in \mathbb{Z}$. Then $(D, e^{2k\pi i \alpha} f)$. also a holomorphic branch of power function and no other branches in D.

Rem: $z^{\alpha+\beta} \neq z^\alpha z^\beta$.

§3. Gamma Function.

initial def: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$, $t^{z-1} := e^{(z-1)\ln t}$. $t>0$. $\mathrm{Re}\, z > 0$.

$$\Gamma(z) = \int_0^1 e^{-t} t^{z-1} dt + \int_1^\infty e^{-t} t^{z-1} dt = I_1(z) + I_2(z).$$

$I_1(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n}$ for $\mathrm{Re}\, z > 1$.

RHS conv. for every $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$

$$\Rightarrow \Gamma(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty e^{-t} t^{z-1} dt \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

(AC from $\{\mathrm{Re}\, z > 0\} \to \mathbb{C} \setminus \{0, -1, -2 \dots\}$). $\Gamma(z)$ meromorphic in $\mathbb{C}$.

$z = -n$. $n \in \mathbb{N} \cup \{0\}$. are poles of 1st-order. residue: $\frac{(-1)^n}{n!}$

§4. Operation Rules and Property.

Notation: $\mathcal{F}, \mathcal{G}$. analytic function in $D \subset \mathbb{C}$

$F = (U, f)$. $G = (V, g)$. two canonical elements. centered at $a \in D$.

<u>Rules</u>: f', $f+g$, fg allows AC along any path in D. denote by $\mathcal{F}'$, $\mathcal{F}+\mathcal{G}$, $\mathcal{F}\mathcal{G}$.

But the sets may not define the unique AC. the sets may decompose into some number of different analytic function.

e.g. (1) sum. $\sqrt{z} + \sqrt{z}$. two analytic function $2\sqrt{z}$ and 0.

$\mathrm{Ln}\, z + i\,\mathrm{Ln}\, z$. countable number analytic functions.

(2) derivative $(\mathrm{Ln}\, z)' = \frac{1}{z}$. unique, single-valued function.

(3) composition. $\Phi(w) = w^2$. $\mathcal{F}(z) = \sqrt[4]{z}$

$\Phi \circ \mathcal{F} = \sqrt{z}$ unique analytic function.

(doesn't coincide with. $\sqrt[4]{z}\sqrt[4]{z} = \{\sqrt{z}, i\sqrt{z}\}$.

<u>Lemma</u>. Suppose that $\mathcal{F}$ is analytic in $D \subset \mathbb{C}$, $\mathcal{F}(D) \subset G$ and $\Phi \in H(G)$. Then the composition $\Phi \circ \mathcal{F}$ defines unique analytic function in D.

(最后一步单值函数. 即有唯一解析函数).