Measure and integral.

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1 Measure.

1.1 Definitions and examples.

Definition 1.1. *A family $\mathcal{P}$ of subsets of a set X is called* ***a semiring*** *if it satisfies the following three properties*

1. $\varnothing \in \mathcal{P}$.
2. *If* $A, B \in \mathcal{P}$ *then* $A \cap B \in \mathcal{P}$.
3. *if* $A, B \in \mathcal{P}$, $B \subset A$ *then* $A \setminus B = \bigcup\limits_{k=1}^{N} C_k$ *for some* $C_k \in \mathcal{P}$, *where sets* C_k *are mutually disjoint.*

Examples 1.2. 1. $\{\varnothing\}$ *is a semiring,*

2. *A set* 2^X *of all subsets of* X *is a semiring.*
3. *Let* (X, d) *be a metric space. A set of all bounded sets of a metric space is a semiring.*
4. *Segments in* $\mathbb{R}$ *(including one-point sets and* $\varnothing$*) is a semiring.*

Theorem 1.3 (Properties of a semiring.)**.** *Let X be a set, $\mathcal{P}$ be a semiring of subsets of X. The the following assertions are satisfied.*

1. *If $A_1, \ldots, A_n \in \mathcal{P}$ then*

$$A \setminus \bigcup_{k=1}^{n} A_k = \bigcup_{i=1}^{N} C_k,$$

where $C_i \in \mathcal{P}$ and sets C_i are mutually disjoint.

2. *Let $A_k \in \mathcal{P}$ for every $k \in \mathbb{N}$. Then*

$$\bigcup_{k} A_k = \bigcup_{k} \bigcup_{i=1}^{N_k} C_{ki},$$

where $C_{ki} \in \mathcal{P}$, C_{ki} are mutually disjoint and $C_{ki} \subset A_k$.

Proof. **1.** We will prove the first property by induction by n.
Base $(n = 1)$. By the second axiom $A \cap A_1 \in \mathcal{P}$. Since $A \cap A_1 \subset A$ by the third axiom

$$A \setminus A_1 = A \setminus (A \cap A_1) = \bigcup_{i=1}^{N} C_k,$$

where $C_i \in \mathcal{P}$ and sets C_i are mutually disjoint.
Step $(n \to n + 1)$. By inductional assumption and the base

$$A \setminus \bigcup_{i=1}^{n+1} A_k = \left(A \setminus \bigcup_{i=1}^{n} A_k \right) \setminus A_{n+1} = \left(\bigcup_{i=1}^{N} C_k \right) \setminus A_{n+1} = \bigcup_{i=1}^{N} \bigcup_{j=1}^{M_i} D_{ij},$$

where $C_i, D_{ij} \in \mathcal{P}$, sets C_i are mutually disjoint, $D_{ij_1} \cap D_{ij_2} = \varnothing$ when $j_1 \neq j_2$. Moreover, $D_{ij} \subset C_i$. Consequently, D_{ij} are mutually disjoint.

2. Sets $B_k = A_k \setminus \bigcup\limits_{j=1}^{k-1} A_j$ are mutually disjoint. Then, by the first assertion,

$$\bigcup_k A_k = \bigcup_k B_k = \bigcup_k \bigcup_{i=1}^{N_k} C_{ki},$$

where $C_{ki} \in \mathcal{P}$, $C_{ki} \subset B_k \subset A_k$ and sets C_{ki} are mutually disjoint. $\square$

Definition 1.4. *Nonempty family $\mathcal{A}$ of subsets of a set X is a **σ-algebra** if it satisfies the following two conditions.*

1. *If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.*
2. *if $A_k \in \mathcal{A}$ for every $k \in \mathbb{N}$ then $\bigcup\limits_{k=1}^{\infty} A_k \in \mathcal{A}$.*

These properties are called axioms of $\sigma-$algebra.

Remark 1.5. *The second axiom implies that if $A_k \in \mathcal{A}$ for every $1 \le k \le n$ then $\bigcup\limits_{k=1}^{n} A_k \in \mathcal{A}$.*

Examples 1.6. 1. *$\{\varnothing, X\}$ is a $\sigma-$algebra.*

2. *2^X is a $\sigma-$algebra.*
3. *The family that contains all at most countable subsets of X and their complements is a $\sigma-$algebra. It is called the $\sigma-$algebra of **countable or co-countable sets.***

Theorem 1.7 (Properties of $\sigma-$algebra.)**.** *Let X be a set, $\mathcal{A}$ be a $\sigma-$algebra of subsets of X. Then*

1. *If* $A_k \in \mathcal{A}$ *for every* $1 \le k \le n$ *(for every* $k \in \mathbb{N}$*) then* $\bigcap\limits_k A_k \in \mathcal{A}$.

2. *If* $A, B \in \mathcal{A}$ *then* $A \setminus B \in \mathcal{A}$.

3. $\mathcal{A}$ *is a semiring.*

Proof. **1.** By De'Morgan's law

$$\bigcap_k A_k = \left(\bigcup_k A_k^c\right)^c \in \mathcal{A}.$$

2. Using the first property we see that $A \setminus B = A \cap B^c \in \mathcal{A}$.
3. $\mathcal{A}$ is not empty and there exists $A \in \mathcal{A}$. Then $\varnothing = A \setminus A \in \mathcal{A}$. Second and the sird axiom of semiring follow from the previous two properties. □

Remark 1.8. *Assume that* $\{\mathcal{A}_t\}_{t \in T}$ *is a family of* $\sigma-$*algebras of subset of a set* X*. Then* $\mathcal{A} = \bigcap\limits_{t \in T} \mathcal{A}_t$ *is a* σ*-algebra.*

Definition 1.9. *Let* X *be a set,* M *be a family of subsets of* X*. Then there exists a minimal (by inclusion)* $\sigma-$*algebra that contains* M*. This* $\sigma-$*algebra is called a* $\sigma-$***algebra generated by*** M *and is denoted by*

$$\sigma(M) = \bigcap_{\substack{\mathcal{A} \text{ is a } \sigma\text{-algebra} \\ M \subset \mathcal{A}}} \mathcal{A} =$$
$$\Big\{A \subset X : A \in \mathcal{A} \text{ for every } \sigma - \text{algebra such that } M \subset \mathcal{A}\Big\}.$$

Definition 1.10. *Let* (X, d) *be a metric space.* ***The Borel*** $\sigma-$***algebra*** $\mathcal{B}_X$ *is the* $\sigma-$*algebra generated by the collection of open subsets of* X*. Elements of this* $\sigma-$*algebra are Borel sets. The Borel* $\sigma-$*algebra of subsets of* $\mathbb{R}^n$ *is denoted by* $\mathcal{B}_n$*.*

Definition 1.11. *Let X be a set and $\mathcal{P}$ be a semiring of subsets of X. A* ***measure*** *is a function $\mu : \mathcal{P} \to [0, +\infty]$ that satisfies the following conditions.*

1. $\mu\varnothing = 0$.
2. ***Countable additivity.*** *Let $A_k \in \mathcal{P}$ for every $k \in \mathbb{N}$ be a sequence of mutually disjoint sets such that $A = \bigcup\limits_{k=1}^{\infty} A_k \in \mathcal{P}$. Then $\mu A = \sum\limits_{k=1}^{\infty} \mu A_k$.*

Definition 1.12. *Let X be a set and $\mathcal{P}$ be a semiring of subsets of X. A* ***volume*** *is a function $\mu : \mathcal{P} \to [0, +\infty]$ that satisfies the following conditions.*

1. $\mu\varnothing = 0$.
2. ***Finite additivity.*** *Let $A_k \in \mathcal{P}$ for every $1 \le k \le n$ be a finite family of mutually disjoint sets such that $A = \bigcup\limits_{k=1}^{n} A_k \in \mathcal{P}$. Then $\mu A = \sum\limits_{k=1}^{n} \mu A_k$.*

Remark 1.13. *Every measure is a volume. Not every volume is a measure.*

Theorem 1.14 (First properties of a volume and a semiring)**.** *Let X be a set, $\mathcal{P}$ be a semiring of subsets of X and μ be a volume on $\mathcal{P}$. Then*

1. *If $A_k \in \mathcal{P}$ for every $k \in \mathbb{N}$ be mutually disjoint sets, $A \in \mathcal{P}$ and $\bigcup\limits_{k} A_k \subset A$ then $\sum\limits_{k} \mu A_k \le \mu A$. In particular, if $A, B \in \mathcal{P}$ and $B \subset A$ then $\mu B \le \mu A$.*

2. *If* $A_k \in \mathcal{P}$, $1 \le k \le n$, $A \in \mathcal{P}$, $A \subset \bigcup\limits_{k=1}^{n} A_k$*. Then* $\mu A \le \sum\limits_{k=1}^{n} \mu A_k$*.*

   If μ *is a measure then this property is true for countable faminly of sets.*

Remark 1.15. *First property is called* ***strong monotonicity****, and its special case* ***monotonicity*** *of a volume* μ*. Second property for finite family of sets is call* ***semiadditivity*** *and for countable family -* ***countable semiadditivity****.*

Remark 1.16. *Notice that in the semiadditivity we don't assume that sets* A_k *are mutually disjoint.*

Proof. **1.** First we will prove this property for a finite family of sets $\{A_k\}_{k=1}^{n}$. By Theorem 1.3

$$A \setminus \bigcup_{k=1}^{n} A_k = \bigcup_{i=1}^{N} C_i,$$

where sets $C_i \in \mathbb{P}, C_i$ are mutually disjoint. Consequently,

$$A = \left(\bigcup_{k=1}^{n} A_k\right) \cup \left(\bigcup_{i=1}^{N} C_i\right),$$

where $A_k, C_i \in \mathbb{P}, \{A_k, C_i\}_{k,i} -$ is a family of mutually disjoint sets. By additivity of μ we see that

$$\mu A = \sum_{k=1}^{n} \mu A_k + \sum_{i=1}^{N} \mu C_i \geqslant \sum_{k=1}^{n} \mu A_k.$$

Now, letting $n \to \infty$, we obtain the assertion for countable family of sets $\{A_k\}_{k=1}^{\infty}$.

2. Let $B_k = A \cap A_k$. Then $B_k \in \mathbb{P}$ and by Theorem 1.3

$$A = A \cap \bigcup_k A_k = \bigcup_k B_k = \bigcup_k \bigcup_{i=1}^{N_k} C_{ki},$$

where $C_{ki} \in \mathbb{P}, C_{ki}$ are mutually disjoint $C_{ki} \subset B_k$. By strong monotonicity of a volume we have

$$\sum_{i=1}^{N_k} \mu C_{ki} \leqslant \mu B_k \leqslant \mu A_k.$$

If k is finite then by additivity

$$\mu A = \sum_k \sum_{i=1}^{N_k} \mu C_{ki} \leqslant \sum_k \mu B_k \leqslant \sum_k \mu A_k.$$

If μ is a measure then this equality holds for countable family of subsets $\{A_k\}_{k=1}^{\infty}$. $\square$

Definition 1.17. *Assume that $X \in \mathcal{P}$ and $\mu X < +\infty$ then measure μ is called* ***finite***. *If $\mu X = 1$ then measure μ is called* ***probability*** *or* ***probability measure***. *If $X = \bigcup\limits_{k=1}^{\infty} X_k$ where $X_k \in \mathcal{P}$ have finite measure $\mu X_k < +\infty$ then measure μ is* ***σ-finite***.

Examples 1.18. ***1.*** *Let $\mathcal{P}$ be a semiring of intervals in $\mathbb{R}$. The length of the interval is a measure. This statement will be proved later during the construction of Lebesgue measure in the next part.*

2. *Let X be a set. For a set $A \subset X$ we let νA be equal to the number of elements of A if A is finite or empty and $\nu A = +\infty$ if A is infinite. Then ν is a measure on 2^X. This measure is called countable measure on X.*

3. *Let X be a set, $a \in X$,*

$$\delta_a A = \begin{cases} 1, & a \in A, \\ 0, & a \notin A, \end{cases} \quad A \subset X.$$

Then function δ_a is a measure 2^X. This measure is called δ-measure.

Definition 1.19. *Let X be a set, $A \subset X$. The function $\chi_A : X \to \mathbb{R}$, defined by*

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$$

is a characteristic function of a set A. Notice that $\delta_a A = \chi_A(a)$.

4. *let X be a set, $\{a_k\}_k$ is at most countable collection of points of X and $h_k \in (0, +\infty]$ for every k. Then the function*

$$\mu A = \sum_{k : a_k \in A} h_k, \quad A \subset X.$$

is a measure on 2^X. This measure is called ***discrete measure*** *generated by weights h_k at points a_k. Measure μ can be expressed as the linear combination of $\delta-$measures*

$$\mu = \sum_k h_k \delta_{a_k}.$$

Finiteness of this measure is equivalent to the condition $\sum\limits_k h_k < +\infty$.

Definition 1.20. *If $A_1, A_2, \ldots$ is a countable collection sets that increases to A in the sense that $A_k \subset A_{k+1}$ for all k, and $A = \bigcup_{k=1}^{\infty} A_k$, then we write $A_k \nearrow A$.*

Similarly, if $A_1, A_2, \ldots$ decreases to A in the sense that $A_k \supset A_{k+1}$ for all k, and $A = \bigcap_{k=1}^{\infty} A_k$, we write $A_k \searrow A$.

Theorem 1.21 (Continuity of a measure)**.** *Let X be a set, $\mathcal{P}$ be a semiring of subsets of X, μ be a measure on $\mathcal{P}$, $A_k \in \mathcal{P}$ for every $k \in \mathbb{N}$ and $A \in \mathcal{P}$ Then the following assertions are satisfied*

1. *If $A_k \nearrow A$ then $\mu A_k \underset{l\to\infty}{\longrightarrow} \mu A$. This property is called **lower continuity** of a measure.*

2. *If $A_k \searrow A$ and $\mu A_1 < +\infty$ then $\mu A_k \underset{l\to\infty}{\longrightarrow} \mu A$. This property is called **upper continuity** of a measure.*

Proof. **1.** If $\mu A_k = +\infty$ for some k then by monotonicity of a measure $\mu A_n = +\infty$ for every $n > k$ and $\mu A = +\infty$. Consequently, in this case the assertion is satisfied.

Assume that $\mu A_k < +\infty$ for every k. We can express A as the union of mutually disjoint sets

$$A = A_1 \cup \bigcup_{k=1}^{\infty} \left(A_{k+1} \backslash A_k\right).$$

The third axiom of semiring implies that

$$A_{k+1} \backslash A_k = \bigcup_{i=1}^{N_k} C_{ki}, \quad A_{k+1} = A_k \cup \bigcup_{i=1}^{N_k} C_{ki},$$

where $C_{ki} \in \mathcal{P}$ are mutually disjoint. By additivity we see that

$$\mu A_{k+1} = \mu A_k + \sum_{i=1}^{N_k} \mu C_{ki}.$$

Hence, by finiteness of μA_k

$$\mu A_{k+1} - \mu A_k = \sum_{i=1}^{N_k} \mu C_{ki}.$$

The set A can also be express as union of mutually disjoint sets from $\mathcal{P}$

$$A = A_1 \cup \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{N_k} C_{ki}.$$

Using countable additivity of a measure μ we see that

$$\mu A = \mu A_1 + \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} \mu C_{ki} = \mu A_1 + \sum_{k=1}^{\infty} (\mu A_{k+1} - \mu A_k) =$$
$$= \mu A_1 + \lim_{n\to\infty} \sum_{k=1}^{n-1} (\mu A_{k+1} - \mu A_k) =$$
$$\mu A_1 + \lim_{n\to\infty} (\mu A_n - \mu A_1) = \lim_{n\to\infty} \mu A_n. \quad (1)$$

2. Notice that, by monotonicity of a measure, $\mu A_k < +\infty$ for every k. Express A_1 as the union

$$A_1 = A \cup \bigcup_{k=1}^{\infty} (A_k \backslash A_{k+1}),$$

which is mutually disjoint by monotonicity of A_k. The third axiom of a semiring we see that

$$A_k \backslash A_{k+1} = \bigcup_{i=1}^{N_k} C_{ki}, \quad A_k = A_{k+1} \cup \bigcup_{i=1}^{N_k} C_{ki},$$

where $C_{ki} \in \mathcal{P}$ and the union is disjoint. Since $\mu A_{k+1} < +\infty$ we see that

$$\mu A_k - \mu A_{k+1} = \sum_{i=1}^{N_k} \mu C_{ki}$$

A set A can also be expressed as union of mutually disjoint sets from $\mathcal{P}$:

$$A_1 = A \cup \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{N_k} C_{ki}.$$

Using countable additivity of μ we obtain

$$\begin{aligned}\mu A_1 = \mu A + \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} \mu C_{ki} = \mu A + \sum_{k=1}^{\infty} (\mu A_k - \mu A_{k+1}) = \\ = \mu A + \lim_{n\to\infty} \sum_{k=1}^{n-1} (\mu A_k - \mu A_{k+1}) = \mu A + \lim_{n\to\infty} (\mu A_1 - \mu A_n) = \\ = \mu A + \mu A_1 - \lim_{n\to\infty} \mu A_n. \quad (2)\end{aligned}$$

Hence, $\lim\limits_{n\to\infty} \mu A_n = \mu A$ since $\mu A_1 < \infty$. □

Remark 1.22. *The condition $\mu A_1 < +\infty$ in the second property can be modified as $\mu A_N < +\infty$ for some N. Let's see that this condition is important. Consider a semiring of segments $\mathcal{P}$ in $X = \mathbb{R}$ with* *measure μ defined as the length of the segment. Let $A_k = [k, +\infty)$. Then $\mu A_k = +\infty$ for every $k \in \mathbb{N}$ while intersection $\bigcap_{k=1}^{\infty} A_k$ is empty and has measure 0.*

Definition 1.23. *Let X be a set, $\mathcal{P}$ be a semiring of subsets X, μ a measure on $\mathcal{P}$, $E \subset X$. The* ***exterior (outer) measure of a set*** *E* ***generated by measure*** *μ is defined by*

$$\mu^* E = \inf_{\substack{A_k \in \mathcal{P} \\ \bigcup_k A_k \supset E}} \sum_k \mu A_k.$$

As usual we assume that $\inf \varnothing = +\infty$ and that if a set E can not be covered by any countable family of sets $A_k \in \mathcal{P}$ then $\mu^* E = +\infty$.

Theorem 1.24 (Properties of exterior measure)**.** *Let X be a set, $\mathcal{P}$ be a semiring of subsets X, μ a measure on $\mathcal{P}$, μ^* be the exterior measure generated by μ. Then the following assertions are satisfied*

1. *If $A \in \mathcal{P}$ then $\mu^* A = \mu A$.*
2. *The exterior measure is semiadditive. Assume that $E_k \subset X$ and $E \subset \bigcup_k E_k$. Then*
$$\mu^* E \leq \sum_k \mu^* E_k.$$

Proof. **1.** Since $A \in \mathcal{P}$ covers itself then $\mu^* A \leq \mu A$. To prove the inverse inequality let $A \subset \bigcup_k A_k$. Then, by semiadditivity, $\mu A \leq \sum_k \mu A_k$. Consequently,

$$\mu A \leq \inf_{\substack{A_k \in \mathcal{P} \\ \bigcup_k A_k \supset A}} \sum_k \mu A_k = \mu^* A.$$

2. If $\mu^* E_k = +\infty$ for some k then the inequality is trivial. Assume that $\mu^* E_k < +\infty$ for every k. Let $\varepsilon > 0$. For every $k \in \mathbb{N}$ by definition of infimum there exists a cover of a set E_k by at most countable family of sets $A_{ki} \in \mathcal{P}$ such that

$$\sum_i \mu A_{ki} < \mu^* E_k + \frac{\varepsilon}{2^k}.$$

Then the family $\{A_{ki}\}_{k,i}$ is at most countable cover E. Consequently,

$$\mu^* E \leqslant \sum_{k,i \in \mathbb{N}} \mu A_{ki} = \lim_{n\to\infty} \sum_{k=1}^{n} \sum_{i=1}^{n} \mu A_{ki} = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \mu A_{ki} \leq \sum_{k=1}^{\infty} \left(\mu^* E_k + \frac{\varepsilon}{2^k} \right) = \sum_{k=1}^{\infty} \mu^* E_k + \varepsilon. \quad (3)$$

Since ε is arbitrary this proves the assertion. □

Corollary 1.24.1. *Exterior measure is monotone. If $D \subset E \subset X$ then $\mu^* D \leq \mu^* E$.*

The exterior measure is defined for every subset of X. Unfortunately, it is rarely $\sigma-$additive and even additive. This motivates us to consider some collection of sets on which this measure is $\sigma-$additive.

Let $E, A \subset X$. Then

$$E = (E \cap A) \cup (E \cap A^c).$$

Consequently,

$$\mu^* E \leq \mu^*(E \cap A) + \mu^* (E \cap A^c).$$

Definition 1.25. *Assume that*

$$\mu^* E = \mu^*(E \cap A) + \mu^*\left(E \cap A^c\right).$$

*Then a set A **additively decomposes** a set E. If a set A additively decomposes every set $E \subset X$ then A is called μ^*-**measurable** or, simply, **measurable**.*

Theorem 1.26 (Carathéodory's extension theorem.)**.** *Assume that X is a set, $\mathcal{P}$ is a semiring of subsets of X, μ is a measure on $\mathcal{P}$, μ^* is exterior measure generated by μ and*

$$\mathcal{A} = \{A \subset X : A \textit{ is } \mu^*\textit{-measurable}\}.$$

Then the following assertions are satisfied.

1. *$\mathcal{A}$ is a $\sigma-$algebra.*

2. *The restriction $m = \mu^*|_{\mathrm{A}}$ is a measure.*

3. *$\mathcal{P} \subset \mathcal{A}$ and $mA = \mu A$ for every $A \in \mathcal{P}$ (that is $m|_{\mathcal{P}} = \mu$).*

Definition 1.27. *The measure m is called the **Caratheodory extension** of measure μ.*

Properties of Caratheodory extension.

C1. Let $A \subset X, \mu^* A = 0$ then $A \in \mathcal{A}$ and $mA = 0$.

Proof. Let $E \subset X$. By monotonicity of exterior measure

$$0 \leqslant \mu^*(E \cap A) \leqslant \mu^* A = 0, \quad \mu^*\left(E \cap A^c\right) \leqslant \mu^* E.$$

Consequently,

$$\mu^* E \geqslant \mu^* (E \cap A^c) + \mu^*(E \cap A),$$

that means that A additively decomposes E. $\square$

Definition 1.28. *A measure on a semiring $\mathcal{P}$ is* ***complete*** *if every subset of a set of measure* 0 *belongs to $\mathcal{P}$ (and, consequently, has measure* 0*).*

C2. Caratheodory extension is a complete measure.

Proof. Let $A \in \mathcal{A}$, $mA = 0$, $E \subset A$. Then $\mu^* E \leqslant \mu^* A = 0$ and $E \in \mathcal{A}$ by property **C1**. $\square$

C3. Measurability criterion. Assume that $E \subset X$ and that for every $\varepsilon > 0$ there exist sets $A_\varepsilon, B_\varepsilon \in \mathcal{A}$ such that $A_\varepsilon \subset E \subset B_\varepsilon$ and $m(B_\varepsilon \backslash A_\varepsilon) < \varepsilon$. Then $E \in \mathcal{A}$.

Proof. By assumption for every $n \in \mathbb{N}$ there exist sets $A_n, B_n \in \mathcal{A}$ such that $A_n \subset E \subset B_n$ and $m(B_n \backslash A_n) < \varepsilon$. Let

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcap_{n=1}^{\infty} B_n.$$

Then $A_n \subset E \subset B_n$ and, since, $\mathcal{A}$ is a σ-algebra, $A, B \in \mathcal{A}$. By monotonicity of measure m for every $n \in \mathbb{N}$

$$0 \leqslant m(B \backslash A) \leqslant m(B_n \backslash A_n) < \frac{1}{n}$$

since $B \backslash \mathcal{A} \subset B_n \backslash A \subset B_n \backslash A_n$. Hence $m(B \backslash A) = 0$. Since $E \backslash A \subset B \backslash A$ and measure m is complete we conclude that $E \backslash A \in \mathcal{A}$. Consequently, $E = A \cup (E \backslash A) \in \mathcal{A}$. $\square$

C4. assume that $E \subset X$ and for every $\varepsilon > 0$ there exists a set $B_\varepsilon \in \mathcal{A}$ be such that $E \subset B_\varepsilon$ and $mB_\varepsilon < \varepsilon$. Then $E \in \mathcal{A}$ and $mE = 0$.

Proof. this property follows from property **C3** by letting $A_\varepsilon = \varnothing$. □

C5. Properties of $\sigma-$finiteness of measure μ and its Caratheodory extension are equivalent.

Proof. Let m be σ-finite that is

$$X = \bigcup_{k=1}^{\infty} A_k, \quad A_k \in \mathcal{A}, \quad mA_k < +\infty.$$

By the definition of exterior measure for every k we see that

$$A_k \subset \bigcup_i C_{ki}, \quad C_{ki} \in \mathcal{P}, \quad \mu C_{ki} < +\infty.$$

Consequently, $X = \bigcup\limits_{k,i} C_{ki}$ that implies σ-finiteness of μ. The inverse implication is obvious. □

Remark 1.29. *Exterior measures μ^* and m^* coincide. Consequently, Caratheodory extension of measure m is the same measure m and same σ-algebra $\mathcal{A}$.*

Remark 1.30. *$\mathcal{A}$ is not always a minimal σ-algebra that contains $\mathcal{P}$. For example, this is so for Lebegue measure that will be discussed in the next paragraph.*

Theorem 1.31 (On the uniqueness of Caratheodory extension)**.** *Assume that X is a set, $\mathcal{P}$ is a semiring of subsets of X, μ is a $\sigma-$finite (!) measure on $\mathcal{P}$. Assume that m is the Caratheodory extension of* *measure μ to $\sigma-$algebra $\mathcal{A}$ and measure ν is extension of measure μ to some $\sigma-$algebra $\mathcal{B}$. Then $mA = \nu A$ for every $A \in \mathcal{A} \cap \mathcal{B}$. If, moreover, the measure ν is complete then $\mathcal{A} \subset \mathcal{B}$.*

Remark 1.32. *Theorem 1.31 implies that $\mathcal{A}$ is a minimal σ-algebra on which μ has complete extension. The condition of σ-finiteness in Theorem 1.31 can not be omitted. To see this consider a set that consists of two points $X = \{a, b\}$ and measure μ defined on a semiring $\mathcal{P} = \{\varnothing, \{a\}\}$ as $\mu\varnothing = 0$, $\mu\{a\} = 1$. Then $\mu^*\{b\} = \mu^* X = +\infty$ since these sets can not be covered by elements of $\mathcal{P}$. $\mathcal{A} = 2^X$ is the unique σ-algebra that contains $\mathcal{P}$. Consequently, Caratheodory extension m is defined by $m\{b\} = mX = +\infty$. However, this extension is not unique since*

$$\nu\varnothing = 0, \quad \nu\{a\} = 1, \quad \nu\{b\} = 2, \quad \nu X = 3$$

is also the extension of measure μ to 2^X.

2 Lebesgue measure

By $\langle \alpha, \beta \rangle$ we denote one of the segments $[a, b]$, $[a, b)$, $(a, b]$, (a, b) in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ for $\alpha, \beta \in \overline{\mathbb{R}}$. If $\alpha > \beta$ we assume that $\langle \alpha, \beta \rangle$ is empty. Denote by

$$\mathbb{I} = \mathbb{I}_n = (1, \dots, 1), \quad \mathbb{O} = \mathbb{O}_n = (0, \dots, 0).$$

Let $a, b \in \mathbb{R}^n$. We say that $a \leq b$ if $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$ and $a_k \leq b_k$ for every $1 \leq k \leq n$. The $n-$dimensional parallelepiped $\Pi = \langle a, b \rangle$ is any set such that $(a, b) \subset \Pi \subset [a, b]$. The empty set $\varnothing$ is also parallelepiped.

Definition 2.1. *Let $a, b \in \mathbb{R}^n, a \le b$. Parallelepiped $[a, b)^n$ is called a cell. The empty set is also a cell. The family of all n-dimensional cells is denoted by $\mathcal{P}_n$.*

Notice that the n-dimensional cell is a product of n semiopen intervals $[a, b) = \prod\limits_{k=1}^{n} [a_k, b_k)$.

Lemma 2.2. *$\mathcal{P}_n$ is a semiring.*

Proof. Let's check properties of a semiring.
1. By definition $\varnothing \in \mathcal{P}_n$.
2. Since for every $1 \le k \le n$

$$[a_k, b_k) \cap [c_k, d_k) = [\max\{a_k, c_k\}, \min\{b_k, d_k\}),$$

intersection $[a, b) \cap [c, d)$ is a cell (that can be empty):

$$[a, b) \cap [c, d) = \prod_{k=1}^{n} [\max\{a_k, c_k\}, \min\{b_k, d_k\}).$$

3. Let $[c, d) \subset [a, b)$ and $[c, d) \neq \varnothing$ otherwise the condition is automatically satisfied. Then $a_k \leqslant c_k < d_k \leqslant b_k$ for every $1 \le k \le n$. Since

$$\bigcup_{k=1}^{n} \bigcup_{i_k \in I_k} A_{ki_k} = \bigcup_{i \in I_1 \times \dots \times I_n} \prod_{k=1}^{n} A_{ki_k},$$

where I_k, A_{ki_k} are some sets. Moreover, if for every fixed k the union in the left side is disjoint then the union in the right side is also disjoint. Consequently, the cell

$$[a, b) = \prod_{k=1}^{n} \big([a_k, c_k) \cup \big[c_k, d_k\big) \cup [d_k, b_k)\big)$$

is a disjoint union of no more that 3^n one of which is cell $[c,d) = \prod_{k=1}^{n} [c_k, d_k)$. Substracting $[c,d)$ we conclude that $[a,b)\backslash[c,d)$ is a disjoint union of no more than $3^n - 1$ nonempty cells. $\square$

Definition 2.3. *A function $v : \mathcal{P}_n \to \mathbb{R}_+$ defined as*

$$v[a,b) = \prod_{k=1}^{n} (b_k - a_k)\,, \quad v\varnothing = 0,$$

*is a **classical volume**. To indicate the dimesion of a space we use notation v_n.*

Lemma 2.4. *Classical volume is a volume. If $\Delta, \Delta_i \in \mathcal{P}_n$ and cells Δ_i are mutually disjoint $\Delta = \bigcup_{i=1}^{N} \Delta_i$ then*

$$v\Delta = \sum_{i=1}^{N} v\Delta_i.$$

Proof. **Case 1.** Consider first the special case when

$$\Delta = [a,b) = \prod_{k=1}^{n} [a_k, b_k)$$

and the partition

$$a_k = x_k^{(0)} < x_k^{(1)} < \ldots < x_k^{(m_k)} = b_k. \tag{4}$$

of a segment $[a_k, b_k]$ is fixed.

Let $m = (m_1, \ldots, m_n)$,

$$D_l = \prod_{k=1}^{n} \left[x_k^{(l_k)}, x_k^{(l_k+1)}\right), \quad l = (l_1, \ldots, l_n) \in [\mathbb{O} : m - \mathbb{I}].$$

It is clear that cells D_l are mutually disjoint and

$$\Delta = \bigcup_{l \in [0:m-1]} D_l$$

In this case we say that cells D_l define the mesh partition of a cell Δ. By the definition of classical volume

$$\sum_{l \in [\mathbb{O}:m-\mathbb{I}]} vD_l = \sum_{l \in [\mathrm{O}:m-\mathbb{I}]} \prod_{k=1}^{n} \left(x_k^{(l_k+1)} - x_k^{(l_k)}\right) =$$
$$\prod_{k=1}^{n} \sum_{l_k=0}^{m_k-1} \left(x_k^{(l_k+1)} - x_k^{(l_k)}\right) = \prod_{k=1}^{n} (b_k - a_k) = v\Delta.$$

Case 2. Assume now that cells

$$\Delta_i = \prod_{k=1}^{n} \left[a_k^{(i)}, b_k^{(i)}\right)$$

define an arbitrary partition of a cell Δ. Then we can complete this partition to the mesh partition.

For any $1 \leq k \leq n$ arrange numbers $\left\{a_k^{(i)}, b_k^{(i)}\right\}_{i=1}^{N}$ increasingly and denote that as in (4). Obtained cells D_l form the mesh partition of Δ and those that are contained in Δ_i form mesh partition of Δ_i. Moreover, since Δ_i cells are mutually disjoint, every cell D_l is contained only in one cell

Δ_i from initial partition. By the previous case

$$v\Delta = \sum_{l\in[\mathbb{O}:m-1]} vD_l = \sum_{i=1}^{N} \sum_{\substack{l\in[\mathbb{O};m-\mathbb{I}] \\ D_l\subset\Delta_i}} vD_l = \sum_{i=1}^{N} v\Delta_i.$$

□

Lemma 2.5. *let $a, b \in \mathbb{R}^n, a < b, \Delta = [a, b)$. Then for every $\varepsilon > 0$ there exist cells $\Delta^{(\pm\varepsilon)} \in \mathcal{P}_n$ such that*

$$\operatorname{cl} \Delta^{(-\varepsilon)} \subset \operatorname{int} \Delta \subset \operatorname{cl} \Delta \subset \operatorname{int} \Delta^{(+\varepsilon)}, \tag{5}$$

$$\Delta^{(+\varepsilon)} - v\Delta < \varepsilon, \quad v\Delta - \Delta^{(-\varepsilon)} < \varepsilon. \tag{6}$$

Proof. Let $\varepsilon > 0$. Since volume of a cell is a continuous function of its vertexes then there exists such $t > 0$ that

$$v[a - t\mathbb{I}, b + t\mathbb{I}) - v[a, b) < \varepsilon, \quad v[a, b) - v[a + t\mathbb{I}, b - t\mathbb{I}) < \varepsilon.$$

Consequently, we can let

$$\Delta^{(+\varepsilon)} = [a - t\mathbb{I}, b + t\mathbb{I}), \quad \Delta^{(-\varepsilon)} = [a + t\mathbb{I}, b - t\mathbb{I}).$$

□

Theorem 2.6. *Classical volume is a measure.*

Proof. We need to prove countable additivity of classical volume. Assume that

$$\Delta, \Delta_i \in \mathcal{P}_n, \quad \Delta_i \text{ are disjoint}, \quad \Delta = \bigcup_{i=1}^{\infty} \Delta_i.$$

By the strong monotonicity of the classical volume we have

$$\sum_{i=1}^{\infty} v\Delta_i \leqslant v\Delta.$$

We need to prove the inverse inequality $v\Delta \leqslant \sum\limits_{i=1}^{\infty} v\Delta_i$. Let $\varepsilon > 0$. Consider a cell $\Delta^{(-\varepsilon)}$ and for every $i \in \mathbb{N}$ a cell $D_i = \Delta_i^{\left(+\varepsilon/2^i\right)}$ obtained by Lemma 2.5. Then

$$\operatorname{cl}\Delta^{(-\varepsilon)} \subset \Delta = \bigcup_{i=1}^{\infty} \Delta_i \subset \bigcup_{i=1}^{\infty} \operatorname{int} D_i.$$

The family $\{\operatorname{int} D_i\}_{i=1}^{\infty}$ is the open cover of the compact $\operatorname{cl}\Delta^{(-\varepsilon)}$. It has a finite subcover $\{\operatorname{int} D_i\}_{i=1}^{N}$. Consequently,

$$\Delta^{(-\varepsilon)} \subset \operatorname{cl}\Delta^{(-\varepsilon)} \subset \bigcup_{i=1}^{N} \operatorname{int} D_i \subset \bigcup_{i=1}^{N} D_i.$$

By finite semiadditivity of the volume

$$v\Delta^{(-\varepsilon)} \leqslant \sum_{i=1}^{N} vD_i \leqslant \sum_{i=1}^{\infty} vD_i.$$

Hence,

$$v\Delta < v\Delta^{(-\varepsilon)} + \varepsilon \leqslant \sum_{i=1}^{\infty} vD_i + \varepsilon \leq \sum_{i=1}^{\infty} \left(v\Delta_i + \frac{\varepsilon}{2^i} \right) + \varepsilon = \sum_{i=1}^{\infty} v\Delta_i + 2\varepsilon.$$

Letting $\varepsilon \to 0$ we finalize the proof. □

Definition 2.7. *Caratheodory extension of classical volume frome a semiring of cells $\mathcal{P}_n$ to some σ-algebra $\mathcal{A}_n$ is called the* ***Lebesgue measure*** $\mathbb{R}^n$ *and elements of $\mathcal{A}_n$ are* ***Lebesgue measurable sets****.*

Lebesgue measure is denoted by μ_n, and exterior measure generated by the classical volume (that is called exterior Lebesgue measure) by μ_n^*. Index n, that indicates the dimension of a space is often omitted. In this section, if other is not specified, μ is the Lebesgue measure.

Remark 2.8. *Lebesgue measure is σ-finite.*

Proof.

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{N}} [-k,k)^n, \quad \mu[-k,k)^n = (2k)^n.$$

□

Further we will study properties of Lebesgue-measurable sets. Previously, in the definition of the parallelepiped $\langle a, b\rangle$ we assumed that $a, b \in \mathbb{R}^n$. Now we extend this definition allowing some a_k be $-\infty$ and some b_k be $+\infty$.

Theorem 2.9. *Any bounded parallelepiped is Lebesgue measurable and its measure is equal to the product of its edges.*

As usual we assume that $0 \cdot \infty = 0$. Consequently, if length of one of the edges has length 0 then its measure is 0.

Proof. Let $a, b \in \mathbb{R}^n$ and $\Pi = \langle a, b\rangle$ is a parallelepiped with bounded edges. Then

$$[a,b] = \bigcup_{p=1}^{\infty} \left[a, b + \frac{1}{p}\mathbb{I}\right), \quad a \leqslant b, \quad (a,b) = \bigcup_{p=1}^{\infty} \left[a + \frac{1}{p}\mathbb{I}, b\right), \quad a < b.$$

By Theorem 1.21 on continuity of measure

$$\mu[a,b] = \lim_{p\to\infty}\left[a, b+\frac{1}{p}\mathbb{I}\right) = \lim_{p\to\infty}\prod_{k=1}^{n}\left(b_k+\frac{1}{p}-a_k\right) = \prod_{k=1}^{n}(b_k-a_k)\,,$$
$$\mu(a,b) = \lim_{p\to\infty}\left[a+\frac{1}{p}\mathbb{I}, b\right) = \lim_{p\to\infty}\prod_{k=1}^{n}\left(b_k-a_k-\frac{1}{p}\right) = \prod_{k=1}^{n}(b_k-a_k)\,.$$

Then by criterion of measurability **C3** parallelepiped Π is measurable and by monotonicity of a measure

$$\mu\Pi = \mu[a,b] = \prod_{k=1}^{n}(b_k-a_k)\,.$$

□

Corollary 2.9.1. *Any at most countable subset of* $\mathbb{R}^n$ *is Lebesgue measurablle and has measure* 0*.*

Proof. Any one point-set has measure 0 as a parallelepiped with edges of length 0. Since measurable sets form σ-algebra any at most countable set is measurable as at most countable union of measurable sets. Its measure is 0 by countable additivity. □

Corollary 2.9.2. $\mu_1\mathbb{Q} = 0$*.*

Theorem 2.10 (Cell-decomposition of an open set)**.** *Every open nonempty set* $G \subset \mathbb{R}^n$ *can be expressed as*

$$G = \bigcup_{k=1}^{\infty}\Delta_k$$

where Δ_k *are mutually disjoint cubic cells such that* $\operatorname{cl}\Delta_k \subset G$*.*

Proof. Let $m \in \mathbb{Z}_+$. Cells $\left[\frac{k}{2^m}, \frac{k+1}{2^m}\right)$, where $k \in \mathbb{Z}^n$ are n-dimensional binary cells of rank m. For a fixed m cells of rank m form a partition of $\mathbb{R}^n$. Consequently for every $x \in \mathbb{R}^n$ and $m \in \mathbb{Z}_+$ there exists a unique cell of rank m that contains x. We denote this cell by $\Delta_{m,x}$. Moreover, two binary cells are either disjoint or one contains another. The set of all binary cells is countable.

Denote by H_0 the set of all cells of rank 0 that are contained in G with the closure. Let $m \in \mathbb{N}$ and assume that sets of cells $H_0, \dots, H_{m-1}$ are already defined. Denote by H_m the set of cells of rank m that are contained in G with the closure and that are not contained in any cell from $H_0, \dots, H_{m-1}$. Let $H = \bigcup\limits_{m=0}^{\infty} H_m$. Then H is at most countable family of mutually disjoint cells.

We will prove that $\bigcup\limits_{\Delta \in H} \Delta = G$. The inclusion $\bigcup\limits_{\Delta \in H} \Delta \subset G$ is obvious. Let's check the inverse inclusion. Let $x \in G$. Since G is open there exists $r > 0$ such that $\bar{B}(x, r) \subset G$. If m is such that $\frac{\sqrt{n}}{2^m} < r$ then $\bar{\Delta}_{m,x} \subset \bar{B}(x, r) \subset G$ and the set $\left\{m \in \mathbb{Z}_+ : \bar{\Delta}_{m,x} \subset G\right\}$ is not empty. Let m_0 be the minimum of this set. Then $\operatorname{cl} \Delta_{m,x} \not\subset G$ for $m < m_0$ and $\operatorname{cl} \Delta_{m_0,x} \subset G$. Consequently, $\Delta_{m_0,x} \in H_{m_0}$, and $x \in \bigcup\limits_{\Delta \in H} \Delta$. $\square$

Remark 2.11. *If $G \neq \varnothing$ then the union in this theorem is countable (not finite). Indeed, if $G = \bigcup\limits_{k=1}^{N} \Delta_k$, $\operatorname{cl} \Delta_k \subset G$ then*

$$G = \bigcup_{k=1}^{N} \operatorname{cl} \Delta_k.$$

Consequently, G is bounded and closed. But by assumption it is open in $\mathbb{R}^n$ and we obtained a contradiction.

Corollary 2.11.1. *1. Every open subset of $\mathbb{R}^n$ is Lebesgue-masurable.*

2. Lebesgue measure of any open set is positive.

3. Every Borel subset of $\mathbb{R}^n$ is Lebesgue-masurable.

Remark 2.12. *There exists a measurable sets that is not Borel. Moreover, σ-algebras $\mathcal{A}_n$ and $\mathcal{B}_n$ have different cardinality. Algebra $\mathcal{B}_n$ is continuum, (is equivalent to $\mathbb{R}^n$), while $\mathcal{A}_n$ has cardinality of a set of all subsets of $\mathbb{R}^n$.*

Theorem 2.13. *If $E \subset \mathbb{R}^n$ then*

$$\mu^* E = \inf_{\substack{G \supset E \\ G \text{ is open}}} \mu G.$$

In particular, if $E \in \mathcal{A}_n$ then

$$\mu E = \inf_{\substack{G \supset E \\ G \text{ is open}}} \mu G$$

Proof. Let

$$\alpha = \inf_{\substack{G \supset E \\ G \text{ is open}}} \mu G.$$

For every open set $G \supset E$ by monotonicity of exterior measure we see that

$$\mu^* E \leqslant \mu^* G = \mu G.$$

Considering infimum of the left-hand side we get the estimate $\mu^* E \leqslant \alpha$.

Now we will prove the inverse inequality. If $\mu^* E = +\infty$ then it is trivial. Assume that $\mu^* E < +\infty$ and $\varepsilon > 0$. By the definition of the exterior measure there exists such cells $\Delta_k \in \mathcal{P}_n$ that

$$E \subset \bigcup_k \Delta_k, \quad \sum_k v\Delta_k < \mu^* E + \varepsilon.$$

By Lemma 2.5 for every k there exists an open parallelepiped $\Delta'_k \supset \Delta_k$ such that $\mu\Delta'_k < v\Delta_k + \frac{\varepsilon}{2^k}$. Then the set $G = \bigcup_k \Delta'_k$ is open and $E \subset G$. Hence,

$$\mu G \leqslant \sum_k \mu\Delta'_k < \sum_k \left(v\Delta_k + \frac{\varepsilon}{2^k}\right) \leqslant \sum_k v\Delta_k + \varepsilon < \mu^* E + 2\varepsilon.$$

Consequently, $A \leqslant \mu^* E$ since ε is arbitrary. □

Corollary 2.13.1. *Let $E \in \mathcal{A}_n$, $\varepsilon > 0$. Then there exists such open set G that*

$$E \subset G \text{ and } \mu(G \backslash E) < \varepsilon.$$

Proof. If $\mu E < +\infty$ then the assertion directly follows from Theorem 2.13. Let $\mu E = +\infty$. By σ-finiteness of μ we ca express E as the union of sets of finite measure

$$E = \bigcup_{k=1}^{\infty} E_k, \quad \mu E_k < +\infty.$$

By Theorem 2.13 for every k there exists an open set $G_k \supset E_k$ such that $\mu\left(G_k \backslash E_k\right) < \frac{\varepsilon}{2^k}$. Then the set $G = \bigcup_k G_k$ is open,

$$E \subset G, \quad G \backslash E \subset \bigcup_{k=1}^{\infty} \left(G_k \backslash E_k\right)$$

and

$$\mu(G \backslash E) \leqslant \sum_{k=1}^{\infty} \mu\left(G_k \backslash E_k\right) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

□

Corollary 2.13.2. *Let $E \in \mathcal{A}_n, \varepsilon > 0$. Then there exists a closed set F such that*

$$F \subset E \text{ and } \mu(E \backslash F) < \varepsilon.$$

Proof. By Corollary 2.13.1 applied to the complement E^c there exists an open set $G \supset E^c$ such that $\mu(G \backslash E^c) < \varepsilon$. Then $F = G^c$ is closed, $F \subset E$, $E \backslash F = G \backslash E^c$ and, consequently, $\mu(E \backslash F) < \varepsilon$. □

Corollary 2.13.3. *Let $E \in \mathcal{A}_n$. Then*

$$\mu E = \sup_{\substack{F \subset E \\ F \text{ closed}}} \mu F = \sup_{F \text{ compact}} \mu F.$$

Proof. Let

$$B = \sup_{\substack{F \subset E \\ F \text{ closed}}} \mu F, \quad C = \sup_{\substack{F \subset E \\ F \text{ compact}}} \mu F.$$

It is clear that $\mu E \geqslant B \geqslant C$. The inverse to the first inequality holds by Corollary 2.13.2. it remains to prove that $B \leqslant C$. Let $F \subset E$ be closed. Let $F_p = F \cap [-p\mathbb{I}, p\mathbb{I}]$. Then every set F_p is compact and $F_p \nearrow F$. Be theorem on continuity of a measure

$$\mu F = \lim_{p \to \infty} \mu F_p \leqslant C$$

and it is enough to take supremum over all F in the right-hand side. □

Definition 2.14. *Let X be a metric space and a measure μ be defined on σ-algebra $\mathcal{A}$ that contains all open subsets of X. Measure μ is regular if for every $E \in \mathcal{A}$*

$$\mu E = \inf_{\substack{G \supset E \\ G \text{ open}}} \mu G = \sup_{\substack{F \subset E \\ \text{closed}}} \mu F.$$

Remark 2.15. *Theorem 2.13 and Corollary 2.13.3 imply regularity of Lebesgue measure.*

Definition 2.16. *A set $E \subset X$ is a set* ***of type*** G_δ *if it can be expressed as the countable intersection of open sets. Let (X, d) be a metric space. A set $E \subset X$ is a set* ***of type*** F_σ *if it can be expressed as the countable union of closed sets.*

Theorem 2.17 (Approximation of measurable sets by borel sets.)**.** *Let $E \in \mathcal{A}_n$. Then there exists a set H of type F_σ and a set K of type G_δ such that*

$$H \subset E \subset K, \quad \mu(K \backslash H) = 0.$$

Proof. By Corollaries 2.13.1 and 2.13.2 of Theorem 2.13 for every $m \in \mathbb{N}$ there exists a closed set F_m and open set G_m such that

$$F_m \subset E \subset G_m, \quad \mu\left(E \backslash F_m\right) < \frac{1}{m}, \quad \mu\left(G_m \backslash E\right) < \frac{1}{m}.$$

LEt

$$H = \bigcup_{m=1}^{\infty} F_m, \quad K = \bigcup_{m=1}^{\infty} G_m.$$

Then set H is of type F_σ and set K is of type G_δ. Moreover, $H \subset E \subset K$ and for every $m \in \mathbb{N}$

$$\mu(K \backslash H) \leqslant \mu\left(G_m \backslash F_m\right) = \mu\left(G_m \backslash E\right) + \mu\left(E \backslash F_m\right) < \frac{2}{m}.$$

Letting $m \to \infty$ we see that $\mu(K \backslash H) = 0$. □

Corollary 2.17.1. *Let $E \in \mathcal{A}_n$. Then E can be expressed as the union of increasing sequence of compact sets and a set of measure* 0, *that is*

$$E = \bigcup_{k=1}^{\infty} F_k \cup e,$$

where F_k are compact sets, $F_k \subset F_{k+1}, \mu e = 0$.

Proof. Let $e = E \backslash H$. Then $e \subset K \backslash H$ and, consequently, $\mu e = 0$. Hence, $E = H \cup e$ and $H = \bigcup\limits_{k=1}^{\infty} F_k$ where F_k is closed. Every closed set F_k is a union of compact sets

$$F_k = \bigcup_{p=1}^{\infty} \left(F_k \cap [-p, p]^n\right).$$

Consequently H can be expressed as union of compact sets as well $H = \bigcup\limits_{j=1}^{\infty} K_j$, where K_j are compact. Finally, letting $\widetilde{K}_j = \bigcup\limits_{i=1}^{j} K_i$ we obtain the increasing sequence of compact sets which union is H. □

Remark 2.18. . *Recall that image of the union is equal to union of images, that is $f\left(\bigcup_\alpha E_\alpha\right) = \bigcup_\alpha f\left(E_\alpha\right)$.*

2.1 Example of a nonmeasurable set.

Consider a relation of equivalence on $\mathbb{R}$

$$x \sim y \text{ if } x - y \in \mathbb{Q},$$

and a factor-set

$$\mathbb{R}/_\sim = \{[x] : x \in \mathbb{R}\},$$

where $[x] = \{y \in \mathbb{R} : x \sim y\}$ is a class of equivalence to which x belongs. Then, $\mathbb{R}/_\sim$ consists of a collection disjoint subsets and by the axiom of choice, there exists a map

$$f : \mathbb{R}/_\sim \to \mathbb{R}$$

which associates each equivalence class to the element in it, that is $f([x]) \in [x]$ for every $[x] \in \mathbb{R}/_\sim$ (the image of the class belongs to the class). In fact, we can assume that $f(\mathbb{R}/_\sim) \subset [0,1]$ (replace f(x) by its fractional part if needed). Let

$$E = f(\mathbb{R}/_\sim).$$

Theorem 2.19. *The set E defined above is not Lebesgue-measurable.*

Proof. Assume the converse, that E is measurable. Let

$$E_q = \{x + q : x \in E\}$$

be a rational shift of E. Then E_q is measurable and $\mu E_q = \mu E$ by invariance of Lebesgue measure with respect to shifts. The definition of E implies that $E_{q_1} \cap E_{q_2} = \varnothing$ if $q_1 \neq q_2$ and

$$\mathbb{R} = \bigcup_{q \in \mathbb{Q}} E_q.$$

Consider two cases: $\mu E = 0$ and $\mu E > 0$. If $\mu E = 0$ then $\mu E_q = 0$ and this would imply that $\mu \mathbb{R} = \sum_{q \in \mathbb{Q}} E_q = 0$ and this leads to the contradiction.

Assume that $\mu E > 0$. Notice that $E_q \subset [0,2]$ if $q \in [0,1]$. Then

$$\bigcup_{q \in \mathbb{Q} \cap [0,1]} E_q \subset [0,2]$$

and by the monotonicity of a measure

$$\sum_{q \in \mathbb{Q} \cap [0,1]} \mu E_q \leq 2$$

which is impossible since by our assumption the left-hand-side is $+\infty$. $\square$

2.2 Cantor's set

The Cantor set $\mathcal{C}$ is created by iteratively deleting the open middle third from a set of line segments. We start by deleting interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ from $[0, 1]$ leaving intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$, that is

$$C_1 := [0,1] \setminus \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, 1\right].$$

Next, the open middle intervals are deleted from each of the remaining parts,

$$C_2 = \left(\left[0, \tfrac{1}{3}\right] \setminus \left(\tfrac{1}{9}, \tfrac{2}{9}\right)\right) \bigcup \left(\left[\tfrac{2}{3}, 1\right] \setminus \left(\tfrac{7}{9}, \tfrac{8}{9}\right)\right) = \\ \left[0, \frac{1}{9}\right] \bigcup \left[\frac{2}{9}, \frac{1}{3}\right] \bigcup \left[\frac{2}{3}, \frac{7}{9}\right] \bigcup \left[\frac{8}{9}, 1\right].$$

Continuing this process on every step we delete 2^{n-1} intervals of length $\frac{1}{3^n}$. That is

$$C_n = [0,1] \setminus \bigcup_{m=1}^{n} \bigcup_{k=0}^{3^m-1} \left(\frac{3k+1}{3^{m+1}}, \frac{3k+2}{3^{m+1}}\right).$$

The Cantor's set is defined as the result of this infinite process

$$C = \bigcup_{n=1}^{\infty} C_n = [0,1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{k=0}^{3^m-1} \left(\frac{3k+1}{3^{m+1}}, \frac{3k+2}{3^{m+1}}\right).$$

Theorem 2.20 (Measure of the Cantor's set)**.**

$$\mu C = 0.$$

Proof. Lebesgue measure is continuous, consequently,

$$\mu C = \lim \mu C_n,$$

while

$$\mu C_n = 1 - \sum_{m=0}^{n} \frac{2^m}{3^{m+1}} = 1 - \frac{1}{3}\frac{1-(2/3)^{n+1}}{1-(2/3)} = \frac{2^n}{3^n} \to 0,\ n \to \infty.$$

□

Remark 2.21. *Cantor's set is closed, nowhere dense and uncountable. Numbers that belong to C have no 1 in their ternary expansions*

$$C = \left\{ \sum_{k=1}^{\infty} \frac{\varepsilon_k}{3^k} : \varepsilon_k \in \{0, 2\} \right\}.$$

3 Measurable functions.

Let E be a set, $\mathcal{P}$ be a proposition that depends on $x \in E$. We will use the following short notation

$$E(\mathcal{P}) = \{x \in E : \mathcal{P}(x) \text{ is true}\}.$$

For example, for function $f : E \to \mathbb{R}$, $a \in \mathbb{R}$.

$$E(f < a) = \{x \in E : f(x) < a\}.$$

Definition 3.1. *A measured space is the triple* $(X, \mathcal{A}, \mu)$, *where* X *is a set,* $\mathcal{A}$ *is a* $\sigma-$*algebra of subsets of* X *and* μ *is the measure on* $\mathcal{A}$.

Definition 3.2. *Let* $f : E \to \mathbb{R}$, $a \in \mathbb{R}$. *Sets* $E(f > a)$, $E(f \geq a)$, $E(f < a)$, $E(f \leq a)$ *are* ***Lebesgue sets*** *of function* f. *In other words, Lebesgue sets are preimages of intervals* $(a, +\infty)$, $[a, +\infty)$, $(-\infty, a)$, $(-\infty, a]$.

Definition 3.3. *A function* $f : E \to \overline{\mathbb{R}}$ *is measurable or, measurable with respect to* $\mathcal{A}$, *on a set* E *if for every number* $a \in \mathbb{R}$ *all four types of Lebesgue spaces are measurable.*

The set of all measurable on E *functions is denoted by* $S(E)$ *or* $S_{\mathrm{A}}(E)$ *if we need to specify* σ*-algebra.*

A function measurable with respect to the σ*-algebra* $\mathcal{A} = \mathcal{A}_n$ *of Lebesgue measurable sets is called Lebesgue measurable. A function measurable with respect to a Borel* σ*-algebra is called Borel measurable function or Borel function.*

Remark 3.4. *If* f *is measurable on* E *then* E *is measurable. It follows from the following identities*

$$E = \bigcup_{n=1}^{\infty} E(f < n) \cup E(f = +\infty), \quad E(f = +\infty) = \bigcup_{n=1}^{\infty} E(f > n).$$

Lemma 3.5. *Let* $E \in \mathcal{A}$ $f : E \to \overline{\mathbb{R}}$. *For measurability of* f *it is enough to check measurability of one of the types of th Lebesgue sets.*

Proof. The proof follows from the following identities

$$E(f \leqslant a) = \bigcup_{n=1}^{\infty} E\left(f < a + \frac{1}{n}\right), \quad E(f > a) = E \backslash E(f \leqslant a),$$

$$E(f \geqslant a)=\bigcup_{n=1}^{\infty} E\left(f>a-\frac{1}{n}\right), \quad E(f<a)=E \backslash E(f \geqslant a).$$

□

Remark 3.6. *The condition of measurability of a set E in Lemma 3.5 can not be omitted. A function f that is equal to $+\infty$ on nonmeasurable set E is not measurable, while all sets $E(f<a)$ are empty.*

3.1 Properties of measurable functions

S1. A constant function on a measurable set is measurable.

Proof. Let $E \in \mathcal{A}$, $c \in \overline{\mathbb{R}}$, $f \equiv c$ on E, $a \in \mathbb{R}$. Then the Lebesgue set

$$E(f<a)= \begin{cases}E, & a>c, \\ \varnothing, & a \leqslant c\end{cases}$$

is measurable. □

S2. The restriction of a measurable function is measurable.

Proof. Let $f \in S(E), E_1 \subset E, E_1 \in \mathcal{A}, a \in \mathbb{R}$. Then

$$E_1(f<a)=E_1 \cap E(f<a) \in \mathcal{A}.$$

□

If f is defined at least on E, then by measurability of f on E we mean measurability of restriction .

S3. If $f: E \rightarrow \overline{\mathbb{R}}$, $E=\bigcup_k E_k$, $f \in S\left(E_k\right)$ for every k then $f \in S(E)$.

Proof. Indeed, $E(f<a)=\bigcup_k E_k(f<a)$. □

S4. Let $f : E \to \overline{\mathbb{R}}$, $E = \bigcup_k E_k$, $E_k \in \mathcal{A}$, and f is constant on E_k. Then $f \in S(E)$.

Proof. $f \in S(E_k)$ for every k by property **S1** then $f \in S(E)$ by **S3**. □

S5. Measurability of a set E is equivalent to the measurability of a function χ_E.

Proof. If E is measurable then

$$\chi_E(x) = \begin{cases} 1, & x \in E, \\ 0, & x \in E^c \end{cases}$$

is measurable by property. **S4**. Assume now that χ_E is measurable then the set $E = X(\chi_E \geqslant 1)$ is measurable. □

S6. If f is measurable then the preimage of any interval of $\overline{\mathbb{R}}$ is measurable.

Proof. Measurability of images of points follows from formulas

$$E(f = a) = E(f \leqslant a) \cap E(f \geqslant a), \quad a \in \mathbb{R},$$
$$E(f = +\infty) = \bigcup_{n=1}^{\infty} E(f > n), \quad E(f = -\infty) = \bigcup_{n=1}^{\infty} E(f < -n),$$

and measurability of open intervals from

$$E(a < f < b) = E(f > a) \cap E(f < b), \quad a, b \in \overline{\mathbb{R}},$$
$$E(f < +\infty) = E \backslash E(f = +\infty), \quad E(f > -\infty) = E \backslash E(f = -\infty).$$

Other intervals are obtained from open intervals adding boundary points.

□

S7. Function that is contionuous on a measurable subset of $\mathbb{R}^n$ is Lebesgue measurable.

Proof. Let $E \in \mathcal{A}_n$, $f \in C(E)$. Then the set $E(f < a)$ is open in E as a preimage of an open set by continuous mapping. Consequently, there exists a set G that is open in $\mathbb{R}^n$, then $E(f < a) = E \cap G$ that implies continuity $E(f < a)$. $\square$

S8. Let μ be a complete measure on $\mathcal{A}$ such that $\mu e = 0$ then every function $f : e \to \overline{\mathbb{R}}$ is measurable.

Proof. Since $e(f < a) \subset e$, then by completeness of measure the set $e(f < a)$ is mesurable and has measure 0. $\square$

Remark 3.7. *Applying property **S8** and completeness of Lebesgue measure we can generalize property **S7**. Let $E \in \mathcal{A}_n$, $f : E \to \overline{\mathbb{R}}$, $E_1 \subset E$, $\mu_n (E \backslash E_1) = 0$, $f|_{E_1} \in C(E_1)$. Then f is Lebesgue measurable.*

*Indeed, $f \in S(E_1)$ by property **S7**, $f \in S(E \backslash E_1)$ by property **S8** and then $f \in S(E)$ by property **S3**.*

Theorem 3.8 (Measurability of upper and lower boundaries and limit of a sequence of measurable functions)**.** .
1. *Let $\{f_n\}_n$ be the finite or countable family of functions, $f_n \in S(E)$. Then $\sup\limits_n f_n$, $\inf\limits_n f_n \in S(E)$.*
2. *Let $f_n \in S(E)$, $n \in \mathbb{N}$. Then $\varlimsup\limits_{n\to\infty} f_n$, $\varliminf\limits_{n\to\infty} f_n \in S(E)$. In particular, if sequence $\{f_n\}$ converges pointwise to f on E then $f \in S(E)$.*

Proof. **1.** Let $g = \sup\limits_n f_n$, $h = \inf\limits_n f_n$. Measurability of Lebesgue sets of functions g and h follows from identities

$$E(g > a) = \bigcup_n E\left(f_n > a\right), \quad E(h < a) = \bigcup_n E\left(f_n < a\right).$$

We will prove the first identity. The condition $x \in E(g > a)$ means that $\sup f_n(x) > a$, or, equivalently, $f_n(x) > a$ for some n, that is $x \in \bigcup\limits_n E\left(f_n > a\right)$.

2. We will prove the measurability of the upper limit. Since $g_n = \sup_{k \geqslant n} f_k$ is decreasing, then by the definition of upper limit

$$\varlimsup_{n\to\infty} f_n = \lim_{n\to\infty} g_n = \inf_{n\in\mathbb{N}} g_n.$$

Applying the first assertion of the theorem twice we see that functions g_n and $\varlimsup\limits_{n\to\infty} f_n$ are measurable. □

Definition 3.9. *Let $f : E \to \overline{\mathbb{R}}$. The functions $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$ are **positive and negative parts** of f.*

Remark 3.10. *By definition we see that*

$$f_+ - f_- = f, \quad f_+ + f_- = |f|, \quad 0 \leqslant f_\pm \leqslant |f|.$$

Moreover, if $f(x) \neq -\infty$ then $f_+(x) = \frac{1}{2}\left(|f(x)| + f(x)\right)$, and if $f(x) \neq +\infty$ then $f_-(x) = \frac{1}{2}\left(|f(x)| - f(x)\right)$.

Remark 3.11. *Measurability of f is equivalent to measurability of both functions f_+ and f_-.*

Proof. Since $E(-f < a) = E(f > -a)$ then measurability of f and $-f$ is equivalent. Consequently, measurability of f implies measurability of $f_{\pm}$. The inverse assertion follows from identities

$$E(f > a) = \begin{cases} E\left(f_{+} > a\right), & a \geqslant 0, \\ E\left(-f_{-} > a\right), & a < 0. \end{cases}$$

$\square$

Definition 3.12. *A function $\varphi : X \to \mathbb{R}$ is a **simple function** if it is measurable, nonnegative, and has finite number of values. A function $\varphi : X \to \mathbb{R}$ is **step function** if it is measurable and has finite number of values.*

Let $c_1, \ldots, c_N$ be all values of a step function φ and $A_k = X\left(\varphi = c_k\right)$. Then

$$\varphi = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad A_k \in \mathcal{A}, A_k \text{ are disjoint, } c_k \in \mathbb{R}. \tag{7}$$

Moreover,

$$\bigcup_{k=1}^{N} A_k = X; \quad A_k \neq \varnothing; \quad c_k \neq c_j,\ k \neq j. \tag{8}$$

Inversely, any function of the form (7) is a step function. It is measurable by property **S4.** Without condition (8) the expression of the step function in the form (7) is not unique.

This is also true for simple functions with additional condition $c_k \geqslant 0$.

Remark 3.13. *If $\alpha \in \mathbb{R}$ and φ and ψ are step functions then functions $\varphi + \psi$, $\alpha\varphi$, $\varphi\psi$, $|\varphi|$ are also step functions.*

Proof. Indeed, if $A_k, B_i \in \mathcal{A}$, A_k are mutually disjoint, B_i are mutually disjoint,

$$\varphi = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad \psi = \sum_{i=1}^{M} d_i \chi_{B_i}$$

then $A_k \cap B_i \in \mathcal{A}$, $A_k \cap B_i$ are mutually disjoint and

$$\varphi + \psi = \sum_{k=1}^{N} \sum_{i=1}^{M} (c_k + d_i) \, \chi_{A_k \cap B_i}, \quad \alpha\varphi = \sum_{k=1}^{N} (\alpha c_k) \, \chi_{A_k},$$

$$\varphi\psi = \sum_{k=1}^{N} \sum_{i=1}^{M} (c_k d_i) \, \chi_{A_k \cap B_i}, \quad |\varphi| = \sum_{k=1}^{N} |c_k| \, \chi_{A_k}. \quad \square$$

$\square$

Theorem 3.14 (Approximation of measurable functions by simple functions)**.** *Let $f : E \to [0, +\infty]$, $f \in S(E)$. Then there exists an increasing sequence of simple functions $\{\varphi_n\}_{n=1}^{\infty}$ that converges to f pointwise on E.*

Proof. Let $n \in \mathbb{N}$ and denote

$$E_{in} = \begin{cases} E\left(\frac{i}{2^n} \leqslant f < \frac{i+1}{2^n}\right), & i \in [0 : n2^n - 1], \\ E(f \geqslant n), & i = n2^n. \end{cases}$$

Sets E_{in} are measurable, disjoint and $\bigcup\limits_{i=0}^{n2^n} E_{in} = E$. Let

$$\varphi_n = \sum_{i=0}^{n2^n} \frac{i}{2^n} \chi_{E_{in}}.$$

In other words, $\varphi_n = \frac{i}{2^n}$ on E_{in} and $\varphi_n = 0$ on E. Functions φ_n are simple.

Let $x \in E$. Assume that $[+\infty] = +\infty$ and check that

$$\varphi_n(x) = \min\left\{\frac{[2^n f(x)]}{2^n}, n\right\}.$$

If $x \in E_{in}$ for $i = n2^n$ then $f(x) \geqslant n$. Hence, $\frac{[2^n f(x)]}{2^n} \geqslant n = \varphi_n(x)$. If $x \in E_{in}$ for $i \in [0 : n2^n - 1]$ then $i \leqslant 2^n f(x) < i + 1$. Hence, $n > \frac{[2^n f(x)]}{2^n} = \frac{i}{2^n} = \varphi(x)$.

Now, we will prove that $\varphi_n(x) \leqslant \varphi_{n+1}(x)$. It is enough to check that

$$\frac{[2^n f(x)]}{2^n} \leqslant \frac{\left[2^{n+1} f(x)\right]}{2^{n+1}}.$$

Let $A = 2^n f(x)$. Then, this estimate follows from general estimate for the integer part of real number:

$$[2A] \geq [2[A]] = 2[A].$$

It remains to prove that $\varphi_n(x) \underset{n\to\infty}{\longrightarrow} f(x)$. If $f(x) = +\infty$ then $\varphi_n(x) = n$ for every $n \in \mathbb{N}$ and $\varphi_n(x) \to +\infty$. Assume that $f(x) \in [0, +\infty)$. Then for every $n > f(x)$ there exists such $i \in [0 : n2^n - 1]$ that $\frac{i}{2^n} \leqslant f(x) < \frac{i+1}{2^n}$, that is $x \in E_{\text{in}}$. Consequently, $\varphi_n(x) = \frac{i}{2^n}$ and

$$0 \leqslant f(x) - \varphi_n(x) < \frac{1}{2^n}.$$

This proves (pointwise) convergence and finalizes the proof. $\square$

Corollary 3.14.1 (Approximation of a measurable function by step functions)**.** *Let $f \in S(E)$. Then there exists an increasing sequence of functions $\{\psi_n\}_{n=1}^{\infty}$ that is pointwise convergent to f on E and such that $|\psi_n| \leqslant |f|$ for every n.*

Proof. By the previous theorem there exist the increasing sequences of functions φ_n^* and φ_n^{**} pointwise convergent to f_+ and f_-. Functions $\psi_n = \varphi_n^* - \varphi_n^{**}$ are step functions. Moreover, $\psi_n \to f_+ - f_- = f$.

It remains to check the estimate. If $f(x) \geqslant 0$ then $f_-(x) = 0$ and $\varphi_n^{**}(x) = 0$. Consequently,

$$|\psi_n(x)| = \varphi_n^*(x) \leqslant f_+(x) = f(x).$$

If $f(x) < 0$ then $f_+(x) = 0$ and $\varphi_n^*(x) = 0$, Consequently,

$$|\psi_n(x)| = \varphi_n^{**}(x) \leqslant f_-(x) = -f(x).$$

□

Corollary 3.14.2. *If in Theorem 3.14 and Corollary 3.14.1 function F is bounded then the convergence is uniform.*

Proof. Let $M = \sup|f| < \infty$. Then estimate (11.7) is true for every $n > M$ and $x \in E$ that implies uniform convergence of φ_n to f. To prove uniform convergence in corollary it is enough to notice that difference of uniformly convergent sequences is uniformly convergent. □

Let $f, g : E \to \overline{\mathbb{R}}$. Then function $f + g$ is defined on the set $E_1 = E \setminus (e_1 \cup e_2)$, where

$$e_1 = E(f = +\infty) \cap E(g = -\infty), \quad e_2 = E(f = -\infty) \cap E(g = +\infty),$$

and function $f - g$ on the set $E_1' = E \setminus (e_1' \cup e_2')$, where

$$e_1' = E(f = +\infty) \cap E(g = +\infty), \quad e_2' = E(f = -\infty) \cap E(g = -\infty).$$

Function fg is defined everywhere on E by assumption $0 \cdot (\pm\infty) = 0$. We assume that the function $\frac{f}{g}$ is defined on the set $E(g \neq 0)$ and is equal to 0, when g is infinite even if f is infinite too. Then $\frac{f}{g} = f \cdot \frac{1}{g}$ on $E(g \neq 0)$.

Theorem 3.15 (Arithmetic properties of measurable functions)**.** *Let* $f, g \in S(E)$*. Then*

1. $\alpha f \in S(E)(\alpha \in \mathbb{R})$;
2. $|f| \in S(E)$;
3. $f^p \in S(E)(f \geqslant 0, p > 0)$;
4. $f + g \in S\left(E_1\right)$;
5. $f - g \in S\left(E_1'\right)$;
6. $fg \in S(E)$;
7. $\frac{f}{g} \in S(E(g \neq 0))$.

Proof. We prove all assertions of the theorem, except 7th, by approximations by step functions ant the second assertion of Theorem about measurability of pointwise limit. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of step functions such that $\varphi_n \to f$, $\psi_n \to g$, $|\varphi_n| \leqslant |f|$, $|\psi_n| \leqslant |g|$. Moreover, if $f \geqslant 0$ then functions are φ_n simple, $\varphi_n \geqslant 0$.

Notice also that sets E_1, E_1' and $E(g \neq 0)$ are measurable.

1. Functions $\alpha\varphi_n$ are step functions and $\alpha\varphi_n \to \alpha f$.
2. Functions $|\varphi_n|$ are simple functions and $|\varphi_n| \to |f|$.
3. Functions φ_n^p are simple functions and $\varphi_n^p \to f^p$.
4. Functions $\varphi_n + \psi_n$ are step functions and $\varphi_n + \psi_n \to f + g$ on E_1.
5. Functions $\varphi_n - \psi_n$ are step functions and $\varphi_n - \psi_n \to f + g$ on E_1'.
6. Functions $\varphi_n\psi_n$ are step functions and $\varphi_n\psi_n \to fg$. It is obvious except the case when $f(x) = 0, g(x) = \pm\infty$ or $f(x) = \pm\infty$, $f(x) = 0$.

If, for example, $f(x) = 0$ then $\varphi_n(x) = 0$ for every n and $\varphi_n(x)\psi_n(x) = 0 \to 0 = f(x)g(x)$.

7. Since $\frac{f}{g} = f \cdot \frac{1}{g}$ it is enough to check measurability of $\frac{1}{g}$ and apply the 6th assertion. We can assume that $g \neq 0$ on E, otherwise we can substitute E by $E(g \neq 0)$. If $a \in \mathbb{R}$ then

$$E\left(\frac{1}{g} > a\right) = \begin{cases} E(g > 0) \cap E\left(g < \frac{1}{a}\right), & a > 0, \\ E(g > 0) \cap E(g < +\infty), & a = 0, \\ E(g > 0) \cup E\left(g < \frac{1}{a}\right), & a < 0. \end{cases}$$

Consequently, Lebesgue sets of $\frac{1}{g}$ are measurable. □

Corollary 3.15.1. *Sum of a series of measurable functions is measurable. Assume that $f_k \in S(E)$ and the series $\sum\limits_{k=1}^{\infty} f_k$ has sum F on E. Then $F \in S(E)$.*

Proof. By Theorem 3.15 partial sums $F_n = \sum_{k=1}^{n} f_k$ are measurable. Then, by measurability of pointwise limits $F = \lim\limits_{n\to\infty} F_n$ is also measurable. □

Corollary 3.15.2. *Let $f, g \in S(E)$. Then sets $E(f < g), E(f \leqslant g)$, $E(f = g), E(f \neq g)$ are measurable.*

Proof. The measurabilty follows from measurability of $f - g$ and identities

$$E(f < g) = E(f - g < 0), \quad E(f = g) = e_1' \cup e_2' \cup E(f - g = 0),$$
$$E(f \leqslant g) = E(f < g) \cup E(f = g), \quad E(f \neq g) = E \backslash E(f = g).$$

□

In many problem concerning measurable functions and measurability we can neglect sets of measure 0. To formalize this we will introduce the following definition.

Definition 3.16. *Let $(X, \mathcal{A}, \mu)$ be measured space, $E \subset X$, $\mathcal{P}(x)$ be a predicate (condition) that depends on $x \in E$. if there exists a set $e \subset X$ with zero measure, $\mu e = 0$, such that for every $x \in E \backslash e$ the predicate $\mathcal{P}(x)$ is true then $\mathcal{P}$ is true almost everywhere (a.e.) on E or for almost every (a.e.) $x \in E$.*

For example, we say that a sequence $\{f_n\}$ converges to f a.e. on E, if there exists a set $e \subset X$ such that $\mu e = 0$ and $f_n(x) \to f(x)$ for every $x \in E \backslash e$.

Remark 3.17. *If we consider two different measures the same statement can be true almost everywhere in one measure and do not hold almost everywhere for another. In this case we specify the measure and write "μ-almost everywhere".*

Remark 3.18. *For a complete measure $\mathcal{P}$ is true a.e. if*

$$\mu E(\mathcal{P} \text{ is not true}) = 0.$$

Remark 3.19. *Assume that we have a sequence of predicates $\{\mathcal{P}_k\}$. If every predicate $\mathcal{P}_k$ is true a.e. on E then they all are true simultaneously a.e. on E.*

Proof. For every k there exists a set e_k such that $\mu e_k = 0$ and such that $\mathcal{P}_k$ is true. Let $e = \bigcup_k e_k$. Then $\mu e = 0$ and all predicates $\mathcal{P}_k$ are true on $E \backslash e$ simultaneously.

$\square$

Definition 3.20. *Functions f and g that are equal a.e. on E are called equivalent on E and we write $f \sim g$.*

A set E and measure μ are usually known from the context and are omitted. The introduced equivalence is actually the relation of equivalence on a set of measurable functions.

Remark 3.21. *If μ is a complete measure, $f, g : E \to \overline{\mathbb{R}}$, $f \in S(E)$ and $f \sim g$ then $g \in S(E)$.*

Proof. By the definition sets E and $E(f \neq g)$ are measurable, moreover, $\mu E(f \neq g) = 0$. Then

$$E = E(f = g) \cup E(f \neq g).$$

Consequently, $E(f = g)$ is measurable as difference of two measurable sets. On $E(f = g)$ function g is measurable since f is measurable. On $E(f \neq g)$ function g is measurable as the function defined on the set of measure 0. By property **S3** of measurable functions $f \in S(E)$. □

Remark 3.22. *We can consider functions that are defined a.e. on E and consider measurability of such functions on E. If $E_1 \subset E$, $E \backslash E_1$ is a subset of some set of measure 0 and $f \in S(E_1)$ then we say that f is measurable on E.*

This definition can be formalized in the same way as the initial definition of measurable function on E. Indeed, if f is not defined otside of E_1 then $E(f > a) = E_1(f > a)$ and same holds for other types of Lebesgue sets. However, in case of noncomplete measure measurability of f that is defined only on E_1 implies measurability of E_1 but not measurability of E.

Remark 3.23. *If μ is a complete measure, $f_n \in S(E)$, $f_n \to f$ a.e. on E. then $f \in S(E)$.*

Proof. By assumption sets E and $E\left(f_n \not\to f\right)$ are measurable, while

$$\mu E\left(f_n \not\to f\right) = 0.$$

Consequently, the set $E_1 = E \setminus E\left(f_n \not\to f\right)$ is measurable. Function f is measurable on E_1 by Theorem 1 and is measurable on $E\left(f_n \not\to f\right)$ since this set has measure 0. Property **S3** implies that $f \in S(E)$. $\square$

4 Integration with respect to a measure

In this section we assume that $(X, \mathcal{A}, \mu)$ is a measured space, $E \in \mathcal{A}$ and that functions that we consider have values in $\overline{\mathbb{R}}$.

Definition 4.1. *Integral $\int_E f d\mu$ of function $f \in S(E)$ with respect to a measure μ on E is defined in the following way.*

1. *Let f be simple:*

$$f = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad A_k \in \mathcal{A}, A_k \text{ are disjoint, } c_k \in [0, +\infty).$$

Then we let

$$\int_E f d\mu = \sum_{k=1}^{N} c_k \mu\left(A_k \cap E\right).$$

2. *let $f \geqslant 0$. Then we let*

$$\int_E f d\mu = \sup_{\substack{\varphi \text{ is simple} \\ \varphi \leqslant f \text{ on } E}} \int_E \varphi d\mu.$$

3. *Let f be arbitrary measurable function. Then we let*

$$\int_E f d\mu = \int_E f_+ d\mu - \int_E f_- d\mu$$

if at least one of the integrals $\int_E f_\pm d\mu$ is finite. If both integrals are infinite then symbol $\int_E f d\mu$ has no value.

If $\int_E^E f d\mu$ is finite then f is integrable on E with respect to the measure μ. A set of functions integrable on E with respect to the measure μ is denoted $L(E,\mu)$. We also use notation $\int_E f(x)d\mu(x)$ to indicate the variable of integration. If the measure is fixed we write $\int_E f$.

Integral with respect to Lebesgue measure is called **Lebesgue integral** and the function that is integrable with respect to Lebesgue measure is called **Lebesgue integrable**.

Remark 4.2. *Integrability of f is equivalent to finiteness of integral $\int_E F_\pm$.*

Theorem 4.3 (Monotonicity of the integral)**.** *Assume that $f, g : E \to \overline{\mathbb{R}}$, $f \leqslant g$ on E and integrals $\int_E f$, $\int_E g$ exist. Then*

$$\int_E f \leqslant \int_E g.$$

Proof. **Case 1.** Assume that functions f and g are simple,

$$f = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad g = \sum_{i=1}^{M} d_i \chi_{B_i},$$

where $c_k, d_i \in [0, +\infty)$, $A_k, B_i \in \mathcal{A}$, A_k are mutually disjoint B_i are mutually disjoint, $\bigcup\limits_{k=1}^{N} A_k = \bigcup\limits_{i=1}^{M} B_i = X$. Let $D_{ki} = A_k \cap B_i$. Then D_{ki}

are mutually disjoint, $\bigcup_{i=1}^{M} D_{ki} = A_k$, $\bigcup_{k=1}^{N} D_{ki} = B_i$. By definition of the integral and additivity of a measure we see that

$$\int_E f = \sum_{k=1}^{N} c_k \mu\left(A_k \cap E\right) = \sum_{k=1}^{N} c_k \mu \bigcup_{i=1}^{M} \left(D_{ki} \cap E\right) =$$
$$\sum_{k=1}^{N} \sum_{i=1}^{M} c_k \mu\left(D_{ki} \cap E\right),$$

and, analogously,

$$\int_E g = \sum_{i=1}^{M} \sum_{k=1}^{N} d_i \mu\left(D_{ki} \cap E\right).$$

In this sum we can take into account only such terms for which $\mu\left(D_{ki} \cap E\right) > 0$. If $\mu\left(D_{ki} \cap E\right) > 0$ then there exists $x \in D_{ki} \cap E$. Then $f(x) = c_k, g(x) = d_i$ and, consequently, $c_k \leqslant d_i$. Thus, $\int_E f \leqslant \int_E g$.

Case 2. Let $f, g \geqslant 0$. The condition $f \leqslant g$ implies the inclusion

$$\{\varphi : \varphi \text{ is simple}, \varphi \leqslant f \text{ on } E\} \subset \{\varphi : \varphi \text{ is simple}, \ \varphi \leqslant g \text{ on } E\}.$$

Consequently the supremum of the integrals $\int_E \varphi$ over the set in the left-hand side of the inclusion is less or equal then over the set in the right-hand side.

Case 3. Let f and g be arbitrary. The inequality $f \leqslant g$ is equivalent to the system of inequalities $f_+ \leqslant g_+$, $f_- \geqslant g_-$. By the previous case

$$\int_E f_+ \leqslant \int_E g_+, \quad \int_E f_- \geqslant \int_E g_-.$$

Substracting the second inequality from the first one we obtain the assertion the theorem. $\square$

Remark 4.4. *This theorem also proves the correctness of the definition of the integral. To clarify this denote integrals introduced on the first, on the second and on the third steps by romanian numbers.*

1. *Integral of the simple function doesn't depend on its expression. We checked this in the first case of the proof of the theorem.*

2. *Let's check that the integral* $(I)\int_E f$ *defined in the first and integral* $(II)\int_E f$ *defined in the second step coincide for simple functions* f*. If* φ *is simple,* $\varphi \leqslant f$ *on* E *then, by monotonicity of the integral* $(I)\int_E \varphi \leqslant (I)\int_E f$*. Considering supremum over* φ *we see that* $(II)\int_E f \leqslant (I)\int_E f$*. The inverse inequality holds by the definition of supremum since* f *si simple and* $f \leqslant f$*.*

3. *For nonnegative functions* f *integrals* $(II)\int_E f$ *and* $(III)\int_E f$ *coincide. Indeed, in this case* $f_+ = f$ *and* $f_- = 0$ *and, consequently,*

$$\textit{(III)} \int_E f = \textit{(II)} \int_E f_+ - \textit{(II)} \int_E f_- = \textit{(II)} \int_E f.$$

Lemma 4.5. *Let* $f \in S(E)$*,* $E_1 \subset E$*,* $E_1 \in \mathcal{A}$*,* $f = 0$ *on* $E \backslash E_1$*. Then* $\int_{E_1} f d\mu = \int_E f d\mu$ *(If one of the integrals exists then another exists as well and they are equal).*

Proof. **1.** Let f be simple function (11.8). Then if $c_k \neq 0$ then $A_k \cap (E \backslash E_1) = \varnothing$. Consequently,

$$\int_E f = \sum_{k=1}^{N} c_k \mu\left(A_k \cap E\right) = \sum_{k=1}^{N} c_k \mu\left(A_k \cap E_1\right) = \int_{E_1} f.$$

2. Let $f \geqslant 0$. Then, by the previous case,

$$\int_E f = \sup_{\substack{\varphi \text{ is simple} \\ \varphi \leqslant f \text{ on } E}} \int_E \varphi = \sup_{\substack{\varphi \text{ is simple} \\ \varphi \leqslant f \text{ on } E_1, \varphi|_{E \setminus E_1} = 0}} \int_E \varphi =$$

$$= \sup_{\substack{\varphi \text{ is simple} \\ \varphi \leqslant f \text{ on } E_1, \varphi|_{E \setminus E_1} = 0}} \int_{E_1} \varphi = \sup_{\substack{\varphi \text{ is simple} \\ \varphi \leqslant f \text{ on } E_1}} \int_{E_1} \varphi = \int_{E_1} f.$$

3. Let f be of arbitrary sign. Then, by the previous case

$$\int_E f_\pm = \int_{E_1} f_\pm$$

and it remains to substract one equality from another. □

Lemma 1 allows ud to consider integral on E as the integral on the whole space X assuming that $f = 0$ on $X \setminus E$.

Corollary 4.5.1 (Monotonicity of the integral by set.)**.** *Let* $f \in S(E)$, $f \geqslant 0$, $E_1 \subset E$, $E_1 \in \mathcal{A}$. *Then* $\int_{E_1} f \leqslant \int_E f$.

Proof. Indeed, by Lemma

$$\int_{E_1} f = \int_E f \chi_{E_1} \leqslant \int_E f$$

□

Corollary 4.5.2. *Let* $f \in L(E, \mu), E_1 \subset E, E_1 \in \mathcal{A}$. *Then* $f \in L(E_1, \mu)$.

Proof. To prove this it is enough to notice that

$$\int_{E_1} f_\pm \leqslant \int_E f_\pm < +\infty$$

□

Theorem 4.6 (B.Levy)**.** *Let* $f_n \in S(E)$*,* $f_n \geqslant 0$*,* $f_n \leqslant f_{n+1}$*,* $f = \lim\limits_{n\to\infty} f_n$*. Then*

$$\int_E f_n \underset{n\to\infty}{\longrightarrow} \int_E f$$

Proof. By Theorem on the limit of measurable functions $f \in S(E)$. The monotonicity of the integral implies that the sequence $I_n = \int_E f_n$ is increasing. Consequently it has finite or infinite limit $\alpha \in [0, +\infty]$. Since $f_n \leqslant f$ then $\int_E f_n \leqslant \int_E f$ for every n and $\alpha \leqslant \int_E f$.

It remains to prove the inverse inequality. Let φ be simple function, $\varphi = \sum\limits_{k=1}^{N} c_k \chi_{A_k}$, $\varphi \leqslant f$ on E. Let $q \in (0, 1)$ and denote $E_n = E\,(f_n \geqslant q\varphi)$. Then $E_n \subset E_{n+1}$ since the sequence $\{f_n\}$ is increasing. We will prove that $\bigcup\limits_{n=1}^{\infty} E_n = E$. the inclusion of the left-hand side into the right-hand side is obvious. Let $x \in E$. If $\varphi(x) = 0$ then $x \in E_n$ for every n. If $\varphi(x) > 0$ then $f(x) \geqslant \varphi(x) > q\varphi(x)$. By the definition of the limit starting from dome number $f_n(x) > q\varphi(x)$, that is $x \in E_n$.

By monotonicity of the integral we see that

$$\int_E f_n \geqslant \int_{E_n} f_n \geqslant \int_{E_n} q\varphi = \sum_{k=1}^{N} q \cdot c_k \mu\left(A_k \cap E_n\right).$$

Properties of E_n imply the following relations

$$\left(A_k \cap E_n\right) \subset \left(A_k \cap E_{n+1}\right), \quad \bigcup_{n=1}^{\infty} \left(A_k \cap E_n\right) = A_k \cap E$$

and, by continuity of the measure

$$\mu\left(A_k \cap E_n\right) \underset{n\to\infty}{\longrightarrow} \mu\left(A_k \cap E\right).$$

Consequently letting n to ∞ and q to 1 we see that

$$\alpha \geqslant \sum_{k=1}^{N} q \cdot c_k \mu \left(A_k \cap E\right) = q \int_E \varphi, \quad \alpha \geqslant \int_E \varphi.$$

Taking supremum over φ we finalize the proof of the theorem, $\alpha \geqslant \int_E f$.
□

Lemma 4.7. *Let $E, E_1 \in \mathcal{A}, E_1 \subset E, \mu\left(E \backslash E_1\right) = 0, f \in S(E)$. Then $\int_{E_1} f d\mu = \int_E f d\mu$ (If one of the integrals exists then another exists as well and they are equal).*

Proof. **1.** Let f be simple. Then

$$\int_{E_1} f = \sum_{k=1}^{N} c_k \mu\left(A_k \cap E_1\right) = \sum_{k=1}^{N} c_k \mu\left(A_k \cap E\right) = \int_E f.$$

2. Let $f \geqslant 0$. Consider the increasing sequence of simple functions $\{\varphi_n\}$ that converges to f on E. Consequently,

$$\int_{E_1} \varphi_n = \int_E \varphi_n.$$

Then, considering the limit, by Levy theorem we obtain the assertion of the theorem
3. let f be of arbitrary sign. Then, by the previous case

$$\int_{E_1} f_{\pm} = \int_E f_{\pm}$$

and this proves the lemma. □

Remark 4.8. *If in the condition of the lemma the measure μ is complete then $f : E \to \overline{\mathbb{R}}$ then measurability of f on E_1 and E is equivalent. This follows from the property **S8** of measurable functions.*

Remark 4.9. *If $\mu e = 0$ and $f \in S(e)$ then $\int_e f = 0$. If μ is the complete measure, $\mu e = 0, f : e \to \overline{\mathbb{R}}$, mo $\int_e f = 0$.*

Lemma 4.7 motivates the definition of the integral on E of the function f that is defined a.e. on E.

Definition 4.10. *Let $E, E_1 \in \mathcal{A}$, $E_1 \subset E$, $\mu\left(E \backslash E_1\right) = 0$, $f \in S\left(E_1\right)$. Then we let $\int_E f d\mu = \int_{E_1} f d\mu$ if the integral in the right hand side exists.*

Remark 4.11. *The possibility of neglecting the sets of measure zero allows us to extend the properties of the integral to integral of the measurable function that is defined almost everywhere on E. It suffices to apply conditions of function (for example, such as non-negativity or increasing succession in Levy's theorem) not everywhere, but only almost everywhere.*

Corollary 4.11.1. *Let $f, g \in S(E)$, $f \sim g$. Then $\int_E f d\mu = \int_E g d\mu$ (If one of the integrals exists then another exists as well and they are equal).*

Proof. The set $E(f = g)$ is measurable. Then by Lemma 4.7

$$\int_E f = \int_{E(f=g)} f = \int_{E(f=g)} g = \int_E g.$$

□

Theorem 4.12 (Uniformity of the integral)**.** *Assume that* $\alpha \in \mathbb{R}$, $f : E \to \overline{\mathbb{R}}$ *and the integral* $\int_E f$ *exists. Then the integral* $\int_E \alpha f$ *exists and*

$$\int_E \alpha f = \alpha \int_E f.$$

Proof. If $\alpha = 0$ the assertion is trivial. Consider other cases.

1. Let $\alpha > 0$.

1.1. Assume that f is simple. Then the function αf is also simple and

$$\int_E \alpha f = \sum_k (\alpha c_k)\, \mu\,(A_k \cap E) = \alpha \sum_k c_k \mu\,(A_k \cap E) = \alpha \int_E f.$$

1.2. Let $f \geqslant 0$. Consider the increasing sequence of simple functions $\{\varphi_n\}$ that converges to f. Then $\{\alpha\varphi_n\}$ is the increasing sequence of simple functions that converges to αf. By the previous case

$$\int_E \alpha\varphi_n = \alpha \int_E \varphi_n$$

Then, considering the limit, by Levy theorem we obtain the assertion of the theorem.

1.3. Let f then

$$\int_E (\alpha f)_\pm = \int_E \alpha f_\pm = \alpha \int_E f_\pm$$

and this finalizes the proof of the case $\alpha > 0$.

2. Let $\alpha < 0$. Since $\alpha = (-1) \cdot |\alpha|$ it is enough to consider the case $\alpha = -1$. Notice that

$$(-f)_+ = \max\{-f, 0\} = f_-, \quad (-f)_- = \max\{-(-f), 0\} = f_+,$$

and by the definition of the integral

$$\int_E (-f) = \int_E (-f)_+ - \int_E (-f)_- = \int_E f_- - \int_E f_+ = -\int_E f.$$

□

Theorem 4.13 (Additivity of the integral by the function)**.** *Assume that integrals $\int_E f$, $\int_E g$ exist and are finite. Then the integral $\int_E (f+g)$ exists and*

$$\int_E (f+g) = \int_E f + \int_E g.$$

Proof. **1.** Let f and g be simple. Without loss of generality we may assume that they are equal to 0 outside of E :

$$f = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad g = \sum_{i=1}^{M} d_i \chi_{B_i},$$

where $c_k, d_i \in [0, +\infty)$, $A_k, B_i \in \mathcal{A}$, A_k are mutually disjoint, B_i are mutually disjoint, $\bigcup\limits_{k=1}^{N} A_k = \bigcup\limits_{i=1}^{M} B_i = E$. Denote $D_{ki} = A_k \cap B_i$. Then D_{ki} are mutually disjoint, $\bigcup\limits_{i=1}^{M} D_{ki} = A_k$, $\bigcup\limits_{k=1}^{N} D_{ki} = B_i$. Then by the definition of the integral and by the additivity of the measure

$$\int_E (f+g) = \sum_{k,i} (c_k + d_i)\,\mu D_{ki} = \sum_k c_k \sum_i \mu D_{ki} + \sum_i d_i \sum_k \mu D_{ki} =$$

$$= \sum_k c_k \mu A_k + \sum_i d_i \mu B_i = \int_E f + \int_E g.$$

2. Let $f, g \geqslant 0$. Consider increasing sequences of simple functions $\{\varphi_n\}$ and $\{\psi_n\}$ that converge to f and g respectively. Then $\{\varphi_n + \psi_n\}$ is the

increasing sequence of simple functions that converges to $f + g$. By the previous case
$$\int_E (\varphi_n + \psi_n) = \int_E \varphi_n + \int_E \psi_n$$
and the result follows from Levy theorem.

3. Let f and g be of arbitrary sign and $h = f + g$. Since f, g have finite integrals then they are a.e. finite and h is defined and finite a.e. By Lemma 4.7 integral on E can be substituted by the integral on $E_1 = E(|h| < +\infty\}$. In the following we consider functions f, g and h on E_1. It is clear that
$$h_+ - h_- = h = f + g = f_+ - f_- + g_+ - g_- = (f_+ + g_+) - (f_- + g_-)$$
and
$$h_+ + f_- + g_- = h_- + f_+ + g_+.$$
Hence, by the previous reasoning
$$\int_{E_1} h_+ + \int_{E_1} f_- + \int_{E_1} g_- = \int_{E_1} h_- + \int_{E_1} f_+ + \int_{E_1} g_+$$
and since integrals of f and g are finite then
$$\int_{E_1} h = \int_{E_1} h_+ - \int_{E_1} h_- =$$
$$\int_{E_1} f_+ + \int_{E_1} g_+ - \left(\int_{E_1} f_- + \int_{E_1} g_-\right) = \int_{E_1} f + \int_{E_1} g.$$
$\square$

Remark 4.14. *One can generalize the previous result as following. Assume that integrals $\int_E f$, $\int_E g$ exist and their sum is defined (at* *least one of the integrals is finite or both are infinite and have the same sign). Then the integral $\int_E (f+g)$ exists and*

$$\int_E (f+g) = \int_E f + \int_E g.$$

Further we may apply this theorem in this form.

Corollary 4.14.1 (Levy theorem for series)**.** *Series with nonnegative terms can be integrated term-by-term. If $f_k \geqslant 0, f_k \in S(E)$ then*

$$\int_E \sum_{k=1}^{\infty} f_k = \sum_{k=1}^{\infty} \int_E f_k.$$

Proof. By Theorem 4.13

$$\int_E \sum_{k=1}^{n} f_k = \sum_{k=1}^{n} \int_E f_k, \ n \in \mathbb{N}.$$

The sequence $F_n = \sum\limits_{k=1}^{n} f_k$ is increasingly convergent to f. Then, by the definition of the sum and Levy theorem the left hand-side converges to $\int_E \sum_{k=1}^{\infty} f_k$ and the right-hand side to $\sum\limits_{k=1}^{\infty} \int_E f_k$. □

Corollary 4.14.2 (Integrability of a function and its absolute value)**.** .

1. *If $f \in S(E)$ then integrability of f and $|f|$ is equivalent.*

2. *If the integral $\int_E f$ exists then*

$$\left| \int_E f d\mu \right| \leqslant \int_E |f| d\mu.$$

Proof. **1.** Recall that $|f|$ is measurable by Theorem 3.15. Since $|f| = f_+ + f_-$ then by Theorem 4.13

$$\int_E |f| = \int_E f_+ + \int_E f_-.$$

The integrability of f is equivalent to the finiteness of the integrals $\int_E f_\pm$ and this is equivalent to fineteness of $\int_E |f|$.

2. By the identity

$$\left|\int_E f\right| = \left|\int_E f_+ - \int_E f_-\right| \leqslant \int_E f_+ + \int_E f_- = \int_E |f|.$$

□

Corollary 4.14.3. *Let $f \in S(E)$ and there exists such function $g \in L(E, \mu)$ such that $|f| \leqslant g$ a.e E. Then $f \in L(E, \mu)$.*

Proof. Indeed, by the monotonicity of the integral

$$\int_E |f| = \int_{E(|f|\leqslant g)} |f| \leqslant \int_{E(|f|\leqslant g)} g = \int_E g < +\infty$$

□

Corollary 4.14.4. *Bounded measurable function is integrable on a set of finite measure.*

Proof. Indeed, if $|f| \leqslant M$ on E then $\int_E |f| d\mu \leqslant M \cdot \mu E < +\infty$. □

Lemma 4.15 (Chebyshev inequality)**.** *Let $f \in S(E), t \in (0, +\infty)$. Then*

$$\mu E(|f| \geqslant t) \leqslant \frac{1}{t} \int_E |f| d\mu.$$

Proof. By monotonicity of the integral with respect to a function and a set
$$\int_E |f| \geqslant \int_{E(|f|\geqslant t)} |f| \geqslant \int_{E(|f|\geqslant t)} t d\mu = t\mu E(|f| \geqslant t).$$
□

Corollary 4.15.1. *If $f \in L(E, \mu)$ then f is finite a.e. on E.*

Proof. By the Chebyshev inequality for every $t \in (0, +\infty)$ we have the following estimate
$$\mu E(|f| = +\infty) \leqslant \mu E(|f| \geqslant t) \leqslant \frac{1}{t} \int_E |f|.$$

It tends to 0 as $t \to +\infty$ since the integral in the right-hand side is finite. Considering the limit we obtain $\mu E(|f| = +\infty) = 0$. □

Corollary 4.15.2. *Let $f \in S(E)$, $f \geqslant 0$, $\int_E f d\mu = 0$. Then $f \sim 0$.*

Proof. By the Chebyshev inequality for every $n \in \mathbb{N}$ we see that
$$\mu E\left(f > \frac{1}{n}\right) \leqslant n \int_E f d\mu = 0,$$
hence $\mu E\left(f > \frac{1}{n}\right) = 0$. Consequently, the measure of the set
$$E(f > 0) = \bigcup_{n=1}^{\infty} E\left(f > \frac{1}{n}\right)$$
is equal to 0. □

Theorem 4.16 (Countable additivity of the integral with respect to a set)**.** *Let $f : E \to \overline{\mathbb{R}}$, $E = \bigcup_k E_k$, $E_k \in \mathcal{A}$, E_k be mutually disjoint. Assume that the integral $\int_E f d\mu$ exists. Then*

$$\int_E f d\mu = \sum_k \int_{E_k} f d\mu.$$

Proof. **1 .** Let $f \geqslant 0$. Then

$$\chi_E = \sum_k \chi_{E_k}, \quad f = f\chi_E = \sum_k f\chi_{E_k}.$$

since sets E_k are mutually disjoint. By the Levy theorem for series

$$\int_E f = \sum_k \int_E f\chi_{E_k} d\mu = \sum_k \int_{E_k} f d\mu = \sum_k \int_{E_k} f d\mu.$$

2. Let f be of arbitrary sign. By the previous case

$$\int_E f_\pm d\mu = \sum_k \int_{E_k} f_\pm d\mu.$$

This finalizes the proof since at least one of the sums in the right-hand side is finite. □

Corollary 4.16.1. *Let $f \in S(X), f \geqslant 0$,*

$$\nu E = \int_E f d\mu, \; E \in \mathcal{A}.$$

Then ν is the measure on $\mathcal{A}$.

Now we will prove two more theorems on the limit and the integral.

Theorem 4.17 (N. Fatu)**.** *1. Let $f_n \in S(E)$, $f_n \geqslant 0$. Then*

$$\int_E \varliminf_{n\to\infty} f_n d\mu \leqslant \varliminf_{n\to\infty} \int_E f_n d\mu.$$

2. Let $f_n, f \in S(E), f_n \geqslant 0, f_n \to f$ a.e. on E. Then

$$\int_E f \leqslant \varliminf_{n\to\infty} \int_E f_n.$$

Proof. **1.** Denote $g = \varliminf f_n$, $g_n = \inf\limits_{k\geqslant n} f_k$. Then $g_n \leqslant g_{n+1}$, $g = \lim g_n$ and $g_n \leqslant f_n$. Moreover, $g_n \in S(E)$ by Theorem 3.8 on the limit of measurable functions. By monotonicity of the integral

$$\int_E g_n \leqslant \int_E f_n.$$

Consequently, by Levy theorem

$$\int_E g = \lim \int_E g_n = \varliminf \int_E g_n \leqslant \varliminf \int_E f_n.$$

2. This assertion follows from the first one since $f = \varliminf f_n$ a.e. on E. □

Theorem 4.18 (H. Lebesgue theorem on dominated convergence)**.** *Assume that $f_n, f \in S(E)$, $f_n \to f$ a.e E and there exists a function $\Phi \in L(E,\mu)$ such that $|f_n| \leqslant \Phi$ a.e. E. Then*

$$\int_E f_n \xrightarrow[n\to\infty]{} \int_E f.$$

Proof. The following assertions hold simultaneously a.e.

1. $|f_n(x)| \leqslant \Phi(x)$ for every $n \in \mathbb{N}$;

2. $f_n(x) \to f(x)$.

Considering the limit in the first one we see that $|f(x)| \leqslant \Phi(x)$ for a.e. $x \in E$. Consequently, $f_n, f \in L(E, \mu)$ and by the Chebyshev inequality functions f_n, f, Φ are finite a.e.. Noticing that $\Phi + f_n \geqslant 0$ a.e. and applying Fatu's theorem we obtain

$$\int_E \Phi + \int_E f = \int_E (\Phi + f) \leqslant \underline{\lim} \int_E (\Phi + f_n) =$$
$$= \underline{\lim} \left(\int_E \Phi + \int_E f_n \right) = \int_E \Phi + \underline{\lim} \int_E f_n.$$

Hence
$$\int_E f \leqslant \underline{\lim} \int_E f_n. \tag{9}$$
Analogously,
$$\int_E \Phi - \int_E f = \int_E (\Phi - f) \leqslant \underline{\lim} \int_E (\Phi - f_n) =$$
$$= \int_E \Phi + \underline{\lim} \left(- \int_E f_n \right) = \int_E \Phi - \overline{\lim} \int_E f_n.$$
Consequently,
$$\overline{\lim} \int_E f_n \leqslant \int_E f. \tag{10}$$
Considering estimates (9) and (10) we see that

$$\int_E f \leqslant \underline{\lim} \int_E f_n \leqslant \overline{\lim} \int_E f_n \leqslant \int_E f.$$

This implies the existence of the limit $\lim \int_E f_n$ and that this limit is equal to the integral $\int_E f$. $\square$

Corollary 4.18.1. *Let $\mu E < +\infty$, $f_n, f \in S(E)$, and assume that as sequence $\{f_n\}$ is uniformly bounded on E, $f_n \to f$ a.e. on E. Then*

$$\int_E f_n d\mu \underset{n\to\infty}{\longrightarrow} \int_E f d\mu.$$

Proof. By assumptions of the theorem there exists $K \in (0, +\infty)$ such that $|f_n| \leqslant K$ a.e. on E for every n. Then with can apply Lebesgue theorem with $\Phi = K$ that is integrable on a set of finite measure. □

4.1 Comparison of Lebesgue and Riemann integrals.

Theorem 4.19 (Lebesgue's criterion for Riemann integrability)**.** *Let $f : [a, b] \to \mathbb{R}$. Then $f \in R[a, b]$ if and only if f is bounded and the set of discontinuities of f has measure* 0.

Theorem 4.20 (Comparison of Lebesgue and Riemann integrals)**.** *Let $f \in R[a, b]$. Then $f \in L[a, b]$ and $(L) \int_a^b f = (R) \int_a^b f$.*

5 Multiple integral and iterated integrals

Recall that $\mathcal{A}_n$ denotes the σ-algebra of Lebesgue-measurable subsets of $\mathbb{R}^n$, and μ_n the Lebesgue measure in $\mathbb{R}^n$. In this section $S(E)$ denotes the set of Lebesgue-integrable functions. Lebesgue integral in $\mathbb{R}^n$ is called n-multliple, in $\mathbb{R}^2$ and $\mathbb{R}^3$ double and triple, respectively. In addition to notations $\int_E f d\mu_n$, $\int_E f(x) d\mu_n(x)$ we will use the following notation

$$\int_E f(x)dx, \quad \int \cdots \int_E f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n,$$

and in two- and three-dimensional case

$$\iint_E f(x,y)dxdy, \quad \iiint_E f(x,y,z)dxdydz.$$

As previously, if

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad y = (y_1, \ldots, y_m) \in \mathbb{R}^m,$$

then (x, y) is the $(n+m)$-dimensional point

$$(x,y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}.$$

Definition 5.1. *Let $E \subset \mathbb{R}^{n+m}, x \in \mathbb{R}^n$. The set*

$$E(x) = \{y \in \mathbb{R}^m : (x,y) \in E\}$$

is a ***section*** *of the set E by the coordinate x.*

Analogously, we defined the section $E(y)$ of the set E by the coordinate $y \in \mathbb{R}^m$. Sometimes, to avoid misinterpretation, if $n = m$ we write $E_1(x)$ and $E_2(y)$, indicating by the first or the second coordinate the section is considered.

Theorem 5.2 (Calculation of the measure of a set via measures of sections)**.** *Let $E \in \mathcal{A}_{n+m}$. Then the following assertions are satisfied.*

1. *$E(x) \in \mathcal{A}_m$ for a.e. $x \in \mathbb{R}^n$.*

2. *Function $x \mapsto \mu_m E(x)$ is measurable on $\mathbb{R}^n$.*

3. *$\mu_{n+m} E = \int_{\mathbb{R}^n} \mu_m E(x) dx$.*

Remark 5.3. *From the formula for $\mu_{n+m}E$ we see that if $\mu_{n+m}E < +\infty$ then $\mu_m E(x) < +\infty$ for a.e. $x \in \mathbb{R}^n$.*

Examples 5.4. ***Example 1. The measure of a ball.*** *We will calculat the measure of n-dimensional ball*

$$\bar{B}_n(a,R)=\{x\in\mathbb{R}^n: |x-a|\leqslant R\}\,,\quad a\in\mathbb{R}^n, R>0.$$

By the formula on the transformation of the Lebesgue by affine transform

$$\mu_n\bar{B}_n(a,R)=\mu_n\bar{B}_n(\mathbb{O},R)=R^n\mu_n\bar{B}_n(\mathbb{O},1).$$

Let $\bar{B}_n=\bar{B}_n(\mathbb{O},1)$ and $V_n=\mu_n\bar{B}_n$. To find the section $\bar{B}_n\left(x_n\right)$ we solve the inequality $\sum_{k=1}^n x_k^2\leqslant 1$ that defines a ball as $\sum_{k=1}^{n-1}x_k^2\leqslant 1-x_n^2$. Hence,

$$\bar{B}_n\left(x_n\right)=\begin{cases}\bar{B}_{n-1}\left(\mathbb{O},\sqrt{1-x_n^2}\right), & |x_n|\leqslant 1,\\ \varnothing, & |x_n|>1.\end{cases}$$

Applying Theorem 5.2 we see that

$$V_n=\int_{\mathbb{R}}\mu_{n-1}\left(\bar{B}_n\left(x_n\right)\right)dx_n=\int_{-1}^{1}\mu_{n-1}\left(\bar{B}_{n-1}\left(\mathbb{O},\sqrt{1-x^2}\right)\right)dx=$$
$$=V_{n-1}\int_{-1}^{1}\left(1-x^2\right)^{\frac{n-1}{2}}dx=2V_{n-1}\int_0^{\frac{\pi}{2}}\cos^n t dt.$$

This integral was calculated in the previous semester:

$$J_n=\int_0^{\frac{\pi}{2}}\cos^n t dt=\frac{(n-1)!!}{n!!}\cdot\begin{cases}\frac{\pi}{2}, & n \text{ is even},\\ 1, & n \text{ is odd}.\end{cases}$$

Applying the recurrent relation $V_n=2J_nV_{n-1}$ and noticing that $V_1=2$ we see that

$$V_n=2^{n-1}J_n\cdot\ldots\cdot J_2V_1=\frac{2^n}{n!!}\left(\frac{\pi}{2}\right)^{\left[\frac{n}{2}\right]}.$$

Hence

$$\mu_n \bar{B}_n(a,R) = \frac{2^n}{n!!}\left(\frac{\pi}{2}\right)^{\left\lfloor \frac{n}{2} \right\rceil} R^n.$$

In particular,

$$\mu_2 \bar{B}_2(a,R) = \pi R^2, \quad \mu_3 \bar{B}_3(a,R) = \frac{4}{3}\pi R^3, \quad \mu_4 \bar{B}_4(a,R) = \frac{\pi^2}{2} R^4$$

Notice that the measure of the open ball coincides with the measure of the closed since the change of weak inequality to the strict one in the formula for section leads to the same recurrent relation. This, in particular means that a sphere hase measure 0*. Notice that a sphere* $\mathbb{S}^{n-1}$ *is the union of graphs of two measurable functions*

$$(x_1,\dots,x_{n-1}) \mapsto \pm\sqrt{1 - \sum_{k=1}^{n-1} x_k^2}, \quad (x_1,\dots,x_{n-1}) \in \bar{B}_{n-1}(\mathbb{O},1).$$

Consequently, the identity $\mu_n\, \mathrm{S}^{n-1} = 0$ *follow also from theorem 5.7 on the measure of the graph.*

Example 2. Measure of a cone. *Let* $E \subset \mathbb{R}^n, h \in [0,+\infty), A = (\mathbb{O}_n, h) \in \mathbb{R}^{n+1}$*. The set*

$$K = \left\{(x,y) \in \mathbb{R}^{n+1} : x = (1-\lambda)\xi, y = \lambda h, \xi \in E, 0 \leqslant \lambda \leqslant 1\right\}$$

is a $(n+1)$*-dimensional* ***cone****, a set* E *is the base of cone* K*,* A *is the vertex and* h *is the height of the cone.*

If $E \in \mathcal{A}_n$ *then* $K \in \mathcal{A}_{n+1}$ *as the image of measurable set* $E \times [0,1]$ *by the smooth map*

$$\Phi(\xi,\lambda) = \lambda A + (1-\lambda)(\xi, \mathbb{O}_n).$$

We will calculate the measure of a cone. If $y \in [0,h]$ then

$$K(y) = \left\{x \in \mathbb{R}^n : x = \left(1 - \frac{y}{h}\right)\xi, \xi \in E\right\} = \left(1 - \frac{y}{h}\right)E,$$

and if $y \notin [0,h]$ then $K(y) = \varnothing$. By Theorem 5.2 and by the formula for transformation of the measure by homothety

$$\mu_{n+1}K = \int_{\mathbb{R}} \mu_n K(y)dy = \int_0^h \left(1 - \frac{y}{h}\right)^n \mu_n E dy = \frac{h}{n+1}\mu_n E. \quad (11)$$

Consider the particular case of this formula. Let $a > 0$. The set

$$S_n(a) = \left\{x \in \mathbb{R}^n : x_k \geqslant 0,\ \sum_{k=1}^n x_k \leqslant a\right\}$$

is the n-dimensional simplex with edge a. It is clear from the definition that $S_n(a)$ is a cone with the base $S_{n-1}(a)$ and height a. in particular, for $n = 1$ simplex is a segment, for $n = 2$ is triangle and for $n = 3$ tetrahedron. Then formula (11) is a recurrent relation $\mu_n S_n(a) = \frac{a}{n}\mu_{n-1}S_{n-1}(a)$. Applying it $n-1$ times and noticing that $\mu_1 S_1(a) = a$ we see that

$$\mu_n S_n(a) = \frac{a^n}{n!}.$$

Theorem 5.5 (Measure of the product of two sets)**.** *Let $A \in \mathcal{A}_n, B \in \mathcal{A}_m$. Then $A \times B \in \mathcal{A}_{n+m}$ and*

$$\mu_{n+m}(A \times B) = \mu_n A \cdot \mu_m B.$$

Proof. It is enough to estabilish measurability of $A \times B$ and the formula will follow from Theorem 5.2 with remark that

$$(A \times B)(x) = \begin{cases} B, & x \in A, \\ \varnothing, & x \notin A. \end{cases}$$

In particular, the assertion is true, if both sets A and B are open or closed, since the product of two open sets is open and the closed set is the difference of two open sets.

Assume that $\mu_n A, \mu_m B < +\infty$. Let $\varepsilon > 0$ then by the regularity of the Lebesgue measure there exist open sets G_1, G_2 and closed sets F_1, F_2 such that

$$F_1 \subset A \subset G_1, F_2 \subset B \subset G_2, \mu_n\left(G_1 \backslash F_1\right) < \varepsilon, \mu_m\left(G_2 \backslash F_2\right) < \varepsilon.$$

Denote $F = F_1 \times F_2, G = G_1 \times G_2$. Then F is closed and G is open, $F \subset A \times B \subset G$ and

$$G \backslash F = \left(\left(G_1 \backslash F_1\right) \times F_2\right) \cup \left(F_1 \times \left(G_2 \backslash F_2\right)\right) \cup \left(\left(G_1 \backslash F_1\right) \times \left(G_2 \backslash F_2\right)\right).$$

Consequently,

$$\mu_{n+m}(G \backslash F) \leqslant \varepsilon\left(\mu_n F_1 + \mu_m F_2\right) + \varepsilon^2 \leqslant \varepsilon\left(\mu_n A + \mu_m B\right) + \varepsilon^2.$$

By criterion of measurability we see that $A \times B \in \mathcal{A}_{n+m}$.

Assume that $\mu_n A = +\infty$ or $\mu_m B = +\infty$. Then by σ-finiteness of Lebesgue measure we can express A and B as $A = \bigcup\limits_{k=1}^{\infty} A_k$, $B = \bigcup\limits_{k=1}^{\infty} B_j$, where $\mu_n A_k, \mu_m B_j < +\infty$. Then $A \times B = \bigcup\limits_{k,j=1}^{\infty} \left(A_k \times B_j\right)$. And by the previous case $A \times B \in \mathcal{A}_{n+m}$. $\square$

Definition 5.6. .

1. *Let $f : E \subset \mathbb{R}^n \to [0, +\infty]$. The set*

$$Q_f = \left\{(x, y) \in \mathbb{R}^{n+1} : x \in E,\ 0 \leqslant y \leqslant f(x)\right\}$$

is a ***subgraph*** *of function f.*

2. Let $f : E \subset \mathbb{R}^n \to \overline{\mathbb{R}}$. The set

$$\Gamma_f = \left\{(x, y) \in \mathbb{R}^{n+1} : x \in E,\ y = f(x)\right\}$$

is a ***graph*** *of function f.*

The graph of the function that obtains infinite values is defined as a subset $\mathbb{R}^n \times \overline{\mathbb{R}}$.

Theorem 5.7 (Measure of a graph)**.** *Let $E \subset \mathbb{R}^n, f \in S(E)$. Then $\Gamma_f \in \mathcal{A}_{n+1} u\ \mu_{n+1}\Gamma_f = 0$.*

Proof. Assume that $\mu_n E < +\infty$. Let $\varepsilon > 0$ and

$$e_k = E(k\varepsilon \leqslant f < (k+1)\varepsilon)$$

for $k \in \mathbb{Z}$. Then

$$\Gamma_f = \Gamma_{f|_{E(|f|<+\infty)}} = \bigcup_{k\in\mathbb{Z}} \Gamma_{f|_{e_k}} \subset \bigcup_{k\in\mathbb{Z}} \left(e_k \times [k\varepsilon, (k+1)\varepsilon)\right).$$

Denote the right-hand side by H_ε. Then, by Theorem 5.5 $H_\varepsilon \in \mathcal{A}_{n+1}$ and, since e_k are mutually disjoint sets,

$$\mu_{n+1} H_\varepsilon = \sum_{k\in\mathbb{Z}} \mu_n e_k \cdot \varepsilon \leqslant \mu_n E \cdot \varepsilon.$$

Consequently, by criterion of measurability the graph of f is measurable ond its measure is 0.

Let $\mu_n E = +\infty$. The Lebesgue measure is σ-finite and, consequently, $E = \bigcup\limits_{k=1}^{\infty} E_k$, where $\mu_n E_k < +\infty$. Then $\Gamma_f = \bigcup\limits_{k=1}^{\infty} \Gamma_{f|_{E_k}}$. By the previous case $\Gamma_{f|_{E_k}}$ is measurable and has measure 0 for every k. Hence Γ_f has same properties. $\square$

Theorem 5.8 (Measure of a subgraph)**.** *Let $E \in \mathcal{A}_n$, $f : E \to [0, +\infty]$. Then measurability of f is equivalent to measurability of Q_f and*

$$\mu_{n+1} Q_f = \int_E f d\mu_n.$$

Proof. By the definition of a subgraph

$$Q_f(x) = \begin{cases} [0, f(x)], & x \in E, \\ \varnothing, & x \notin E. \end{cases}$$

If $Q_f \in \mathcal{A}_{n+1}$ the formula for $\mu_{n+1} Q_f$ follows from Theorem 5.2 since

$$\mu_1 Q_f(x) = \begin{cases} f(x), & x \in E, \\ 0, & x \notin E. \end{cases}$$

It remains to prove equivalence of measurability of function f and and of set Q_f. If the set Q_f is measurable then, by the second assertion of Theorem 5.2 the function $\mu_1 Q_f(\cdot)$ is measurable on $\mathbb{R}^n$. Consequently, the function $f = \mu_1 Q_f(\cdot)\big|_E$ is measurable as the restriction of a measurable function to a measurable set.

Let $f \in S(E)$. Assume first that function $f = \varphi$ is simple

$$\varphi = \sum_{k=1}^{N} c_k \chi_{A_k}, \quad c_k \in [0, +\infty), A_k \in \mathcal{A}_n.$$

We assume that the function φ is defined on E and $A_k \subset E$. then the set

$$Q_\varphi = \bigcup_{k=1}^{N} (A_k \times [0, c_k])$$

is measurable by Theorem 5.5

In the general case we consider the increasing sequence $\{\varphi_k\}$ that converges to f on E. We will prove that

$$Q_f = \bigcup_{k=1}^{\infty} Q_{\varphi_k} \cup \Gamma_f. \tag{12}$$

This and Theorem 5.7 will imply measurability of Q_f.

Since the sequence $\{\varphi_k\}$ is increasing the inequality $y \leqslant \varphi_k(x)$ implies that $y \leqslant f(x)$. Hence, $Q_{\varphi_k} \subset Q_f$ for every k. Moreover, $\Gamma_f \subset Q_f$. Consequently, the right-hand side of (12) is contained in the left-hand side. To check the inverse inclusion let $(x, y) \in Q_f$, that is $x \in E, y \in [0, f(x)]$. If $y = f(x)$ then $(x, y) \in \Gamma_f$. If $y \in [0, f(x))$ then there exists $K \in \mathbb{N}$ such that $y < \varphi_K(x)$ and $(x, y) \in Q_{\varphi_K}$. □

Theorem 5.9 (Tonelli's theorem)**.** *Let $E \subset \mathbb{R}^{n+m}$, $f \in S(E \to [0, +\infty])$. Then the following assertions are satisfied.*

1. *$f(x, \cdot) \in S(E(x))$ for a.e. $x \in \mathbb{R}^n$.*

2. *Function I defined by $I(x) = \int_{E(x)} f(x, y)dy$ is measurable on $\mathbb{R}^n$.*

3. *$\int_E f d\mu_{n+m} = \int_{\mathbb{R}^n} I(x)dx$.*

Proof. By Theorem 5.8 the subgraph of f is measurable, $Q_f \in \mathcal{A}_{n+m+1}$, and

$$\mu_{n+m+1} Q_f = \int_E f d\mu_{n+m}. \tag{13}$$

From the other side, by Theorem 5.2 $Q_f(x) \in \mathcal{A}_{m+1}$ for a.e. $x \in \mathbb{R}^n$, $\mu_{m+1}Q_f(\cdot) \in S\left(\mathbb{R}^n\right)$ and

$$\mu_{n+m+1}Q_f = \int_{\mathbb{R}^n} \mu_{m+1}Q_f(x)dx. \tag{14}$$

By the definition of a section and a subgraph we see that

$$\begin{aligned} Q_f(x) &= \left\{(y,z) \in \mathbb{R}^{m+1} : (x,y,z) \in Q_f\right\} = \\ &= \left\{(y,z) \in \mathbb{R}^{m+1} : (x,y) \in E, 0 \leqslant z \leqslant f(x,y)\right\} = \\ &= \left\{(y,z) \in \mathbb{R}^{m+1} : y \in E(x), 0 \leqslant z \leqslant f(x,y)\right\} = Q_{f(x,\cdot)}. \end{aligned}$$

Hence, for a.e. $x \in \mathbb{R}^n$ the subgraph of $f(x,\cdot)$ is measurable. This function is defined on the set $E(x)$ and $E(x)$ is measurable for a.e. x by Theorem 5.2. Applying Theorem 5.8 we see that $f(x,\cdot)$ is measurable for a.e. x. This porves the first assertion of the theorem. Next, by Theorem 5.8

$$\mu_{m+1}Q_f(x) = \mu_{m+1}Q_{f(x,\cdot)} = \int_{E(x)} f(x,y)dy = I(x), \tag{15}$$

and the function I is measurable. Concluding identities (13)-(15) we obtain the last assertion of the theorem. $\square$

Theorem 5.10 (Fubini's theorem)**.** *Let $E \subset \mathbb{R}^{n+m}$, $f \in L(E)$. Then the following assertions are satisfied.*

1. *$f(x,\cdot) \in L(E(x))$ for a.e. $x \in \mathbb{R}^n$.*

2. *Function I defined by $I(x) = \int_{E(x)} f(x,y)dy$ is integrable on $\mathbb{R}^n$.*

3. *$\int_E f d\mu_{n+m} = \int_{\mathbb{R}^n} I(x)dx$.*

Proof. Assume that $f \geqslant 0$. Then the assumptions of Tonelli's theorem are satisfied and the third assertion is satisfied. Since $f \in L(E)$,

$$\int_{\mathbb{R}^n} I(x)dx = \int_E f d\mu_{n+m} < +\infty$$

and $I \in L(\mathbb{R}^n)$. Consequently, function I is a.e. finite and the first assertion of Fubini's theorem is satisfied.

Assume that f has arbitrary sign. Apply the previous case to functions $f_\pm$. Then $f_\pm(x,\cdot) = (f(x,\cdot))_\pm$ is integrable on $E(x)$ for a.e. x. Consequently, the function $f(x,\cdot) = (f(x,\cdot))_+ - (f(x,\cdot))_-$ is also integrable. Denote $I^\pm(x) = \int_{E(x)} f_\pm(x,y)dy$. Then $I^\pm \in L(\mathbb{R}^n)$ and $I = I^+ - I^-$ a.e. and $I \in L(\mathbb{R}^n)$. Substracting on of the inequalities

$$\int_E f_\pm d\mu_{n+m} = \int_{\mathbb{R}^n} I^\pm(x)dx$$

from another we obtain the third assertion of the theorem. □

Third assertions of Tonneli's and Fubini's theorem can be written as

$$\int_E f d\mu_{n+m} = \int_{\mathbb{R}^n} \int_{E(x)} f(x,y)dydx.$$

The integral in the right-hand side of this identity is the **iterated integral**. Sometimes it is convinient to write it as

$$\int_{\mathbf{R}^n} dx \int_{E(x)} f(x,y)dy.$$

Remark 5.11. *Statements Tonelli's and Fubini's theorems are symmetric with respect to the coordinates x and y. Consequently,*

$$\int_E f d\mu_{n+m} = \int_{\mathbb{R}^m} J(y)dy,$$

where

$$J(y) = \int_{E(y)} f(x,y)dx.$$

Hence, both iteradet integrals are equal to $(n+m)$*-dimensional and*

$$\int_E f d\mu_{n+m} = \int_{\mathbb{R}^n} \int_{E(x)} f(x,y)dydx = \int_{\mathbb{R}^m} \int_{E(y)} f(x,y)dxdy.$$

Remark 5.12. *In some cases the outer integral in the iterated integral can be considered not on the whole space* $\mathbb{R}^n$ *but on some subset. If* $E(x) = \varnothing$ *then* $I(x) = 0$*. The set*

$$\operatorname{Pr}_1 E = \{x \in \mathbb{R}^n : E(x) \neq \varnothing\}$$

is a ***projection*** *of* E *to the first subspace* $\mathbb{R}^n$ *of* $\mathbb{R}^{n+m}$*. If the set* $\operatorname{Pr}_1 E$ *is measurable, then we can calculate the outer integral on this set*

$$\int_E f d\mu_{n+m} = \int_{\operatorname{Pr}_1 E} \int_{E(x)} f(x,y)dydx.$$

However, this projection is not always measurable. Let e *be a non-measurable set in* $\mathbb{R}^n, y \in \mathbb{R}^m, E = e \times \{y\}$*. Then* $E \in \mathcal{A}_{n+m}$ *and* $\mu_{n+m}E = 0$ *while* $\operatorname{Pr}_1 E = e \notin \mathcal{A}_n$*. Thus we need to modify this notion. The set*

$$\operatorname{Pr}_1^* E = \{x \in \mathbb{R}^n : \mu_m E(x) > 0\}$$

is the essential projection of the set E *to the first subspace.*

Essential projection is measurable as the Lebesgue function of the measurable function $\mu_m E(\cdot)$*. Moreover, if* $x \in \mathbb{R}^n \setminus \operatorname{Pr}_1^* E$ *and* $I(x)$ *exists then* $I(x) = 0$*. Consequently,*

$$\int_E f d\mu_{n+m} = \int_{\operatorname{Pr}_1^* E} \int_{E(x)} f(x,y)dydx. \tag{16}$$

Examples 5.13. ***Example 1.*** *Let* $E = [0,1]^2, f(x,y) = \frac{x^2-y^2}{\left(x^2+y^2\right)^2}$. *Then*

$$\int_0^1 \int_0^1 f(x,y)dydx = \int_0^1 \left[\frac{y}{x^2+y^2}\right]_{y=0}^1 dx = \int_0^1 \frac{dx}{x^2+1} = \frac{\pi}{4},$$
$$\int_0^1 \int_0^1 f(x,y)dxdy = \int_0^1 \left[\frac{-x}{x^2+y^2}\right]_{x=0}^1 dy = -\int_0^1 \frac{dy}{1+y^2} = -\frac{\pi}{4}.$$

Iterated integrals are different. Consequently, $f \notin L\left([0,1]^2\right)$ *and, moreover,* f *has no (finite or infinite) integral on* $[0,1]^2$.

Example 2. *Let* $E = [-1,1]^2, g(x,y) = \frac{2xy}{\left(x^2+y^2\right)^2}$. *Since* g *is odd with respect to every coordinate iterated integrals are equal to* 0 *while*

$$\int_{-1}^1 \int_{-1}^1 |g(x,y)|dydx = 4\int_0^1 \int_0^1 \frac{2xy}{\left(x^2+y^2\right)^2}dydx =$$
$$= 4\int_0^1 \left[\frac{-x}{x^2+y^2}\right]_{y=0}^1 dx = 4\int_0^1 \left(\frac{1}{x} - \frac{1}{1+x^2}\right) dx = +\infty.$$

Consequently, $g \notin L\left([-1,1]^2\right)$. *Moreove,* g *has no integral on* $[-1,1]^2$ *since its existence would imply that it is* 0 *and integrability of* g.

Remark 5.14. *Let* $E \subset \mathbb{R}^n$, $f : E \to \overline{\mathbb{R}}$ *and the integral* $\int_E f d\mu_n$ *exists. Applying the formula (16)* $(n-1)$ *times the* n*-dimensional repeated multiple integral can be written as iterated integral composed* *of n integrals in $\mathbb{R}$:*

$$\int_E f d\mu_n = \int_{\mathrm{Pr}_1^* E} dx_1 \int\limits_{E(x_1)} f(x) dx_2 \dots dx_n =$$

$$= \int_{\mathrm{Pr}_1^* E} dx_1 \int_{\mathrm{Pr}_2^* E(x_1)} dx_2 \int_{E(x_1,x_2)} f(x) dx_3 \dots dx_n = \dots =$$

$$= \int_{\mathrm{Pr}_1^* E} dx_1 \int_{\mathrm{Pr}_2^* E(x_1)} dx_2 \dots \int_{\mathrm{Pr}_{n-1}^* E(x_1,\dots,x_{n-2})} dx_{n-1} \int_E f(x) dx_n.$$

Here Pr_i^* denotes the essential projection to i-th coordinate. We used the obvious identity $E(x_1, \dots, x_k)(x_{k+1}) = E(x_1, \dots, x_k, x_{k+1})$.

For functions f defined on a set X and g defined on a set Y let

$$(f \otimes g)(x, y) = f(x) g(y), \quad x \in X, y \in Y.$$

Lemma 5.15. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $f \in S(X)$, $g \in S(Y)$. Then $f \otimes g \in S(X \times Y)$.*

Proof. Let $\widetilde{f}(x, y) = f(x)$, $\widetilde{g}(x, y) = g(y)$. Then $\widetilde{f}, \widetilde{g} \in S(X \times Y)$ since

$$(X \times Y)(\tilde{f} < a) = X(f < a) \times Y, \quad (X \times Y)(\tilde{g} < a) = X \times Y(g < a).$$

Consequently, the function $f \otimes g$ is measurable as the product of measurable functions. □

Corollary 5.15.1. *Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$, $f \in S(X \to [0, +\infty])$, $g \in S(Y \to [0, +\infty])$, $h = f \otimes g$. Then*

$$\int_{X \times Y} h d\mu_{n+m} = \int_X f d\mu_n \cdot \int_Y g d\mu_m.$$

Proof. Measurability h was estabilished in Lemma 5.15. By Tonelli's theorem

$$\int_{X\times Y} h d\mu_{n+m} = \int_X \left(\int_Y f(x)g(y)dy\right) dx =$$

$$= \int_X f(x) \left(\int_Y g(y)dy\right) dx = \int_X f d\mu_n \cdot \int_Y g d\mu_m.$$

□

6 Change of the variable in the integral with respect to a measure

In this section we consider functions on various σ-algebras. The class of functions measurable on a set E with respect to a σ-algebra $\mathcal{A}$ is denoted by $S_{\mathrm{A}}(E)$. In this paragraph μ further denotes the Lebesgue measure and the dimension is usually omitted. In this chapter we neglect the proofs.

Definition 6.1. *Let $u \in \mathbb{R}^n$. The transform $T_u : \mathbb{R}^n \to \mathbb{R}^n$ defined as $T_u x = x + u$ is called a* ***shift by the vector*** *u.*

Theorem 6.2 (Lebesgue measure is invariant with respect to a shift.)**.** *If $u \in \mathbb{R}^n$ and $E \in \mathcal{A}_n$ then $T_u(E) \in \mathcal{A}_n$ and $\mu T_u(E) = \mu E$.*

Proof. Since the shift doesn't change the edges of the cell the identity holds for cells and, consequently, it is enough to prove that $T_u(E)$ is measurable for every measurable set E.

The shift is smooth and invertible $T_u^{-1} = T_{-u}$, consequently, images of open, closed and compact sets are open, closed and compact, respectively.

In particular, the assertion of theorem holds for open, closed and compact sets.

Let E be measurable let $\varepsilon > 0$ then there exist an open set G and closed set F such that

$$F \subset E \subset G, \ \mu(E \setminus F) < \varepsilon/2, \ \mu(G \setminus E) < \varepsilon/2.$$

Hence,

$$T_u(F) \subset T_u(E) \subset T_u(G)$$

and, since $G \setminus F$ is open,

$$\mu T_u(G) \setminus T_u(F) = \mu T_u(F \setminus G) = \mu(F \setminus G) < \varepsilon.$$

Consequently, by criterion of measurability (property **C3** of Caratheodory extension) $T_u(E)$ is measurable. $\square$

Definition 6.3. *Let G and V be open subsets of $\mathbb{R}^n$. The mapping $\Phi : G \to V$ is a **diffemorphism** of sets G and V if Φ is bijective, $\Phi \in C^1(G \to V)$ and $\Phi^{-1} \in C^1(V \to G)$. We say that $\Phi : G \to \mathbb{R}^n$ is diffeomorphism if Φ is diffeomorphism of G and $\Phi(G)$.*

The Jacobian of diffomorfism is not 0 at every point. If G is open in $\mathbb{R}^n$ the mapping $\Phi \in C^{(1)}(G \to \mathbb{R}^n)$ is invertible and $\det \Phi'(x) \neq 0$ for every $x \in G$ then the set $V = \Phi(G)$ is open and Φ is diffeomorphism of G and V.

Theorem 6.4 (Image of Lebesgue measure with respect to diffeomotphism)**.** *let G be open in $\mathbb{R}^n$, $\Phi : G \to \mathbb{R}^n$ be diffeomorphism. Then for every measurable subset $E \subset G$ its image $\Phi(E)$ is measurable and*

$$\mu\Phi(E) = \int_E |\det \Phi'| \, d\mu.$$

Theorem 6.5 (Change of the variable in Lebesgue integral)**.** *Let G be open $\mathbb{R}^n$, $\Phi : G \to \mathbb{R}^n$ be diffeomorphism, $E \in \mathcal{A}_n(G), f \in S(\Phi(E))$. Then*

$$\int_{\Phi(E)} f d\mu = \int_E (f \circ \Phi) \left|\det \Phi'\right| d\mu$$

(If one of the integrals exists then another exists as well and they are equal).

Remark 6.6. *The integral in Theorem 6.5 automatically exists if $f \geqslant 0$ or $f \in L\left(\Phi(E)\right)$. The latter is equivalent to condition $(f \circ \Phi)\left|\Phi'\right| \in L(E)$ (recall that the function f is assumed to be measurable).*

Remark 6.7. *We can let the assumpion in theorems 6.4 and 6.5 on Φ loose and assume that it is diffeomorphism and bijective on the complement to a set of measure 0. Consider the following generalization of Theorem 6.5*

Assume that $G \subset H \subset \mathbb{R}^n$, G is open, $\Phi : H \to \mathbb{R}^n$, $\Phi|_G$ is diffeomorphism, $\mu(H \backslash G) = \mu(\Phi(H) \backslash \Phi(G)) = 0$, $E \in \mathcal{A}_n(H)$, $f \in S(\Phi(E))$. Then

$$\int_{\Phi(E)} f d\mu = \int_E (f \circ \Phi) \left|\det \Phi'\right| d\mu.$$

Examples 6.8. ***Example 1. Shift and reflection*** *If $f \in L\left(\mathbb{R}^n\right)$ then $f(a \pm \cdot) \in \mathbb{R}^n$ for every $a \in \mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} f(a \pm x) dx = \int_{\mathbb{R}^n} f(x) dx.$$

Indeed, if $\Phi(x) = a \pm x$ then Φ is diffeomorphism of $\mathbb{R}^n$ to itself and $\left|\det \Phi'\right| = 1$.

Example 2. Polar coordinates in $\mathbb{R}^2$***.*** *Cartesian coordinates* x, y *are expressed via polar coordinates* r, φ *by the following formulas*

$$x = r\cos\varphi, \quad y = r\sin\varphi.$$

Let

$$\Phi(r,\varphi) = (r\cos\varphi, r\sin\varphi),$$
$$G = \{(r,\varphi) : r \in (0,+\infty), \varphi \in (-\pi,\pi)\}.$$

The mapping Φ *is invertible on* G *and*

$$\Phi(G) = \mathbb{R}^2 \backslash \{(x,0) : x \leqslant 0\}.$$

The Jacobian of the polar change is

$$\det\Phi'(r,\varphi) = \det\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix} = r.$$

It is clear that it is not 0 *in* G*. Consequently,* $\Phi|_G$ *is diffeomorphism. Moreove,* $\mu_2 \partial G = \mu_2 \Phi(\partial G) = 0$ *and we can apply remark 6.7.*

Now, we will show the application of spherical change to Euler-Poisson integral

$$I = \int_0^{+\infty} e^{-x^2} dx$$

Applying the first corollary of Tonelli theorem and the polar change we see that

$$I^2 = \int_0^{+\infty} e^{-x^2} dx \cdot \int_0^{+\infty} e^{-y^2} dy = \iint_{(0,+\infty)^2} e^{-\left(x^2+y^2\right)} dxdy =$$
$$= \int_0^{+\infty} \int_0^{\frac{\pi}{2}} re^{-r^2} d\varphi dr = \frac{\pi}{2} \int_0^{+\infty} re^{-r^2} dr = \frac{\pi}{2} \frac{e^{-r^2}}{2} \Bigg|_0^{+\infty} dr = \frac{\pi}{4}.$$

Hence,
$$I = \frac{\sqrt{\pi}}{2}.$$

Example 3. Cylindrical change in $\mathbb{R}^3$***.*** *Cartesian coordinates* x, y, z *are expressed via Cylindrical coordinate* r, φ, h *as*

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = h.$$

In other words, we apply spherical change to coordinates x *and* y*, and the third coordinate* z *is not changing. Denote*

$$\Phi(r, \varphi, h) = (r\cos\varphi, r\sin\varphi, h),$$
$$G = \{(r, \varphi) : r \in (0, +\infty), \varphi \in (-\pi, \pi), h \in \mathbb{R}\}.$$

The mapping Φ *is invertible on* G*,*

$$\Phi(G) = \mathbb{R}^3 \backslash \{(x, 0, z) : x \leqslant 0, z \in \mathbb{R}\}$$

and $\det \Phi'(r, \varphi, h) = r \neq 0$ *in* G*. Consequently,* $\Phi|_G$ *is diffeomorphism. Moreover,* $\mu_3 \partial G = \mu_3 \Phi(\partial G) = 0$ *and we can apply Remark 6.7.*

Example 4. Spherical coordinates in $\mathbb{R}^3$***.*** *Cartesian coordinates* x, y, z *are expressed via spherical* r, φ, ψ *as*

$$x = r\cos\varphi\cos\psi, \quad y = r\sin\varphi\cos\psi, \quad z = r\sin\psi.$$

Let

$$\Phi(r, \varphi) = (r\cos\varphi\cos\psi, r\sin\varphi\cos\psi, r\sin\psi),$$
$$G = \left\{(r, \varphi, \psi) : r \in (0, +\infty), \varphi \in (-\pi, \pi), \psi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right\}.$$

The mapping Φ *is invertible on* G,

$$\Phi(G) = \mathbb{R}^3 \backslash \{(x, 0, z) : x \leqslant 0, z \in \mathbb{R}\}.$$

Let's find the Jacobian of spherical change

$$\det \Phi'(r, \varphi, \psi) = \det \begin{pmatrix} \cos\varphi \cos\psi & -r\sin\varphi\cos\psi & -r\cos\varphi\sin\psi \\ \sin\varphi\cos\psi & r\cos\varphi\cos\psi & -r\sin\varphi\sin\psi \\ \sin\psi & 0 & r\cos\psi \end{pmatrix} = r^2\cos\psi$$

It is clear that it is not 0 *in* G. *Consequently,* $\Phi|_G$ *is diffeomorphism. Moreover,* $\mu_3 \partial G = \mu_3 \Phi(\partial G) = 0$ *and we can apply Remark 6.7.*