Real Analysis

Chapter 1. Integrability and Continuous Function.

Def1. <u>σ-algebra</u>. a family of sets on X. - $\mathcal{A}$ s.t.:
symmetric system of set. equivalent: $\bigcap_{n\geq 1} Y_n \in \mathcal{A}$.
1.) $\phi \in \mathcal{A}$ 2.) if $Y \in \mathcal{A}$, $X \setminus Y \in \mathcal{A}$ 3) $Y_n \in \mathcal{A}$. (countable family). $\bigcup_{n\geq 1} Y_n \in \mathcal{A}$.

Def2. <u>measurable space</u>: pair $(X, \mathcal{A})$.

Def3. <u>measure</u> $\mathcal{A}$: a set of subsets of X. $\phi \in \mathcal{A}$. ($\mathcal{A}$ can be semiring, but we only consider σ-algebra here).
a function $\mu: \mathcal{A} \to [0, +\infty)$ s.t.
1) $\mu(\phi) = 0$ 2) $\mu(\bigcup_{k\geq 1}^{\infty} A_k) = \sum_{k\geq 1}^{\infty} \mu A_k$. A_k mutually. disjoint in $\mathcal{A}$.

Def4. <u>measurable set</u> if $E \in \mathcal{A}$. then E is measurable.
⇓

§. Measurable Functions and σ-algebras.

Def5 <u>measurable function</u>: pre: $(X, \mathcal{A})$. $(X', \mathcal{A}')$ measurable spaces
$f: X \to X'$ is measurable if $\forall E \in \mathcal{A}'$. $f^{-1}(E) \in \mathcal{A}$

<u>Lemma 1.2.</u> 1). pre: $(X, \mathcal{A})$. - <u>id. map</u>: $X \to X$ is measurable.

2) pre: $(X, \mathcal{A})$. $(X', \mathcal{A}')$. - <u>constant map</u>. $f(x) \equiv c \in X'$ is measurable.

3). pre: $X \xrightarrow{f} X'$. $X' \xrightarrow{g} X''$ measurable maps.

- <u>composition map</u> of measurable maps also measurable.

Fact: we can let any map be measurable by constructing "inverse" σ-algebra.
pre: any map $f: X \to X'$. $\mathcal{A}'$ is σ-algebra on X'.
let: $\mathcal{A} = f^{-1}(\mathcal{A}') = \{f^{-1}(Y) \mid Y \in \mathcal{A}'\}$. proof $\mathcal{A}$ is σ-algebra (在任意映射中均成立).
special case: when $X \subseteq X'$. $\mathcal{A} = \{Y \cap X' \mid Y \in \mathcal{A}'\}$. f be id. map.
Fact: we can find "smaller" σ-algebra. (use intersection)
Pf: let. $(\mathcal{A}_i)_{i\in I}$. be σ-algebras on X. I is set of index. (arbitrary).

$$\bigcap_I \mathcal{A}_i = \{Y \subseteq X \mid Y \in \mathcal{A}_i \text{ for all } i \in I\} \text{ is a } \sigma\text{-algebra.}$$

Fact: $\mathcal{B}_X \otimes \mathcal{B}_{X'} = \mathcal{B}_{X\times X'}$

Def.6. σ-algebra generated by $\mathcal{M}$. / smallest σ-algebra contain $\mathcal{M}$.

pre: X be a set and $\mathcal{M}$ any family of subsets of X.

$$\sigma(\mathcal{M}) = \{Y \subset X \mid Y \in \mathcal{A} \text{ for every } \sigma\text{-algebra } \mathcal{A}.\ s.t.\ \mathcal{M} \subset \mathcal{A}\} = \bigcap_{\substack{\mathcal{A} \text{ is } \sigma\text{-algebra} \\ \mathcal{M} \subset \mathcal{A}}} \mathcal{A}$$

Example1. $(X, \mathcal{T})$ t.s. Borel σ-algebra on X : $\mathcal{B}_X$ is the σ-algebra generated by the collection $\mathcal{T}$ of open sets in X. (all open sets). 因此 Borel σ-algebra 是含 X 所有开集的最小 σ-algebra.

Example2. Let $(X, \mathcal{A})$. $(X', \mathcal{A}')$ be measurable spaces. product σ-algebra $\mathcal{A} \otimes \mathcal{A}'$ generated by all the sets $Y \times Y'$. $Y \in \mathcal{A}$ and $Y' \in \mathcal{A}'$.

sub Def. pre: $f: X \to \mathbb{R}$. $(X, \mathcal{A})$. measur.s.; f is measurable if f is measurable w.r.t. Borel σ-algebra

Borel σ-algebra 是含 X 中所有开集的最小 σ-algebra. (但在 $\mathbb{R}$ 中. $\mathcal{B}_{\mathbb{R}} \neq 2^{\mathbb{R}}$, 势不同).

Lemma 1.5. 1) pre: $(X, \mathcal{A})$. $(X', \mathcal{A}')$ $\mathcal{A}' = \sigma(\mathcal{M})$. [$\mathcal{M}$ is arbitrary collection of subsets]

A map $f: X \to X'$ is measurable $\Leftrightarrow \forall E \in \mathcal{M}$ $f^{-1}(E) \in \mathcal{A}$ (equiv. $f^{-1}(\mathcal{M}) \subset \mathcal{A}$).

2) pre: $(X, \mathcal{A})$.

$f: X \to \mathbb{R}$ is measurable $\Leftrightarrow \forall a \in \mathbb{R}$. $\underbrace{f^{-1}(-\infty, a) = X(f < a)}_{\text{(Lebesgue set)}} \in \mathcal{A}$.

3) pre: $(X, \mathcal{A})$ $(X', \mathcal{A}')$. $(X'', \mathcal{A}'')$.

$f: X'' \to X \times X'$ is measurable (w.r.t. product σ-algebra $\xrightarrow{\text{on } X \times X'}$ $\mathcal{A} \otimes \mathcal{A}' = \{E \times E' \mid E \in \mathcal{A}, E' \in \mathcal{A}'\}$)

$\Leftrightarrow$ $p_1 \circ f: X'' \to X$ and $p_2 \circ f$ $X'' \to X'$ are measurable.

"$\Leftarrow$"

Pf: 1) step 1. show Lemma $f^{-1}(\sigma(\mathcal{M})) = \sigma(f^{-1}(\mathcal{M}))$

"$\supset$"

Pf: $\sigma(\mathcal{M})$ is σ-algebra. $\Rightarrow f^{-1}(\sigma(\mathcal{M}))$ is σ-algebra $\Rightarrow f^{-1}(\sigma(\mathcal{M})) \supset \overset{\text{min } \sigma\text{-algebra}}{\sigma}(f^{-1}(\mathcal{M}))$

"$\subset$" denote $\mathcal{A}'' = \{Y \mid f^{-1}(Y) \in \sigma(f^{-1}(\mathcal{M}))\}$. is σ-algebra of X'

$\Rightarrow \mathcal{M} \subset \mathcal{A}''$ ($\forall Y \in \mathcal{M} \Rightarrow f^{-1}(Y) \in f^{-1}(\mathcal{M}) \subset \sigma(f^{-1}(\mathcal{M}))$ ($\mathcal{M}$ 满足 $\mathcal{A}''$ 的条件)

$\Rightarrow \underset{\text{最小的含}\mathcal{M}}{\sigma(\mathcal{M})} \subset \underset{\text{也含}\mathcal{M}}{\mathcal{A}''} \Rightarrow f^{-1}(\sigma(\mathcal{M})) \subset f^{-1}(\mathcal{A}'') \subset \sigma(f^{-1}(\mathcal{M}))$

step 2. Use Lemma. let $Y \in \mathcal{A}' = \sigma(\mathcal{M})$

$$f^{-1}(Y) \in f^{-1}(\sigma(\mathcal{M})) = \sigma(f^{-1}(\mathcal{M})) \subset \mathcal{A}.$$

3). "$\Rightarrow$" composition.

Generalization: $q_1: X \times X' \to X$ $\quad q_2: X \times X' \to X'$ (check q_1, q_2 are meas.)

1) $g: X'' \to X' \times X$ is meas. $\Leftrightarrow$ $q_1 \circ g$, $q_2 \circ g$ are meas.

"$\Leftarrow$" 2) $f = \langle f_1, f_2 \rangle$ is meas. $\Leftrightarrow$ f_1, f_2 are meas. (vector form)

we need: $\forall E \in \mathcal{A}$. $E' \in \mathcal{A}'$. $f^{-1}(E \times E') \in \mathcal{A}''$

$$f^{-1}(E \times E') = \{x'' \in X'' \mid f(x'') \in E \times E'\} = \{x'' \in X'' \mid p_1 \circ f \in E \text{ and } p_2 \circ f \in E'\} = \underset{\mathcal{A}''}{\underset{\Cap}{(p_1 \circ f)^{-1}(E)}} \cap \underset{\mathcal{A}''}{\underset{\Cap}{(p_2 \circ f)^{-1}(E')}} \in \mathcal{A}''$$

coro 1.5.1.1) Let $f: X \to X'$ be cont. map (w.r.t. tps.). Then f is meas. (w.r.t. $\mathcal{B}_X$ and $\mathcal{B}_{X'}$).

2). pre. $(X, \mathcal{A})$. $f, g: X \to \mathbb{C}$ meas. maps. → 默认σ-algebra是$\mathcal{B}_\mathbb{C}$.

then $f \pm g$, fg are meas. if $g(x) \neq 0$, $\frac{f}{g}$ are meas.

3) $f: X \to \mathbb{C}$ is meas. (for $\mathcal{B}_\mathbb{C}$) iff $Re(f)$ and $Im(f)$ are meas. (as $X \to \mathbb{R}$).

Pf: 1). $\mathcal{B}_{X'} = \{G' \subset X', G' \text{ is open}\}$. $\forall G' \subset X'$ open, $f^{-1}(G')$ is open (def. of conti.)

thus. $f^{-1}(G') \in \mathcal{B}_X$

2). use Lemma 1.5. by construction. (check $\mathbb{R}$ first. use +, scalar * to extend to $\mathbb{C}$).

$f+g = p \circ (f \times g)$. $p: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. $p(c_1, c_2) = c_1 + c_2$ $\quad f \times g: x \mapsto (f(x), g(x))$

p is meas. (since it's cont.). and $f \times g$ is meas.

3). $f: X \to \mathbb{C}$. $f = Re(f) + i\, Im(f)$ $\quad Re(f), Im(f): X \to \mathbb{R}$. (special case of 2)).

Thm 1.6. pre: $(X, \mathcal{A})$. $\{f_n\}$. meas. real-valued functions.

Then: 1. function: $\limsup f_n(x)$, $\liminf f_n(x)$, $\sup_n f_n(x)$. $\inf_n f_n(x)$. are meas.

2. if $f_n(x) \to f(x)$ for any $x \in X$ (pointwise conv.) then the function is meas.

(Consider the Lebesgue set to prove

$h(x) = \sup_n f_n(x)$. $E(h(x) < a) = \bigcup_{k=1}^{\infty} E(h(x) \leq a - \frac{1}{k}) = \bigcup_{k=1}^{\infty} (\bigcap_{n=1}^{\infty} E(f_n \leq a - \frac{1}{k})) \in \mathcal{A}$.) ($f_n$ are meas.)

△ $\limsup f_n(x) = \lim_{n\to\infty} \overset{(\text{递减})}{\sup_{k \geq n}} f_k(x) = \inf_n (\sup_{k \geq n} f_k(x)) = \overline{\lim}\, f_n$

$f = \lim f_n = \limsup f_n = \liminf f_n$

§. Measure on a σ-algebra. (表述: a measure μ on X.. for $\mathcal{A}$).

Def. finite measure: $\mu(X) < +\infty$.

σ-finite measure: $\exists X_n \in \mathcal{A}$ s.t. $X = \bigcup_{n \geq 1} X_n$, $\mu(X_n) < +\infty$ for all n.

probability measure: $\mu(X) = 1$ (e.g. characteristic measure. $\delta_p(E) = \begin{cases} 1. & p \in E \\ 0, & p \notin E \end{cases}$).

Borel measure: measure on $\mathcal{B}_X$.

measured space: the triple $(X, \mathcal{A}, \mu)$

Property: Monotonicity. 1) For $E, F \in \mathcal{A}$, with $E \subset F$, we have $\mu(E) \leq \mu(F)$ and $\mu(F) = \mu(E) + \mu(F \setminus E)$

2) For $E, F \in \mathcal{A}$. we have $\mu(E \cup F) + \mu(E \cap F) = \mu(E) + \mu(F)$.

lower continuity: an increasing sequence $E_1 \subset \cdots \subset E_n \subset \cdots$ of measurable sets. then:
$\mu(\bigcup_n E_n) = \lim_{n \to +\infty} \mu(E_n) = \sup_{n \geq 1} \mu(E_n)$

upper continuity: an decreasing sequence $\{E_n\}$ of measurable sets. and $\mu(E_1) < +\infty$, then:
$\mu(\bigcap_n E_n) = \lim_{n \to +\infty} \mu(E_n) = \inf_{n \geq 1} \mu(E_n)$

Property: For any countable family (E_n) of measurable sets, we have.
$$\mu\left(\bigcup_{n\geq 1} E_n\right) \leq \sum_{n\geq 1}^{\infty} \mu(E_n)$$

定义时不要求 Y 可测

Def. μ-negligible (subset). pre: $(X, \mathcal{A}, \mu)$. subset $Y \subset X$; Y is a set of measure 0.

(i.e. there exists $E \in \mathcal{A}$ s.t. $Y \subset E$ and $\mu(E) = 0$).

Rem: if $Y \in \mathcal{A}$ and negligible, then $\mu Y = 0$.

If $P(x)$ is a mathematical property (predicate) parametrized by $x \in X$, P is true μ-a.e. if: $\{x \in X \mid P(x) \text{ is not true}\}$ is μ-negligible.

complete measured space: all μ-negligible sets are measurable.

Procedure: construct a natural complete "extension" of $(X, \mathcal{A}, \mu)$

denote $\mathcal{A}_0 = \{\mu\text{-negligible sets in } (X, \mathcal{A}, \mu)\}$

$\mathcal{A}' = \{E \subset X \mid E = E_1 \cup E_0 \text{ with } E_0 \in \mathcal{A}_0 \text{ and } E_1 \in \mathcal{A}_1\}$.

define $\mu'(E) = \mu(E_1)$ if $E = E_0 \cup E_1 \in \mathcal{A}'$

then $(X, \mathcal{A}', \mu')$ is completed measured space. $(\mathcal{A} \subset \mathcal{A}', \mu' = \mu \text{ on } \mathcal{A})$

§. Operation and Construction on measures.

Lemma 1.12. Let $(X, \mathcal{A}, \mu)$ a measured space.

1. For any finite collection of measures $\{\mu_1 \dots \mu_k\}$ on $(X, \mathcal{A})$ and $\forall \{\alpha_1 \dots \alpha_k\} \subset [0, +\infty)$ the measure $\mu = \sum \alpha_i \mu_i$ is defined by $\mu(Y) = \sum_{1\leq i\leq k} \alpha_i \mu_i(Y)$. for $Y \in \mathcal{A}$.
μ is a measure on $(X, \mathcal{A})$.

2. Let $f: (X, \mathcal{A}) \to (X', \mathcal{A}')$ be a measurable map. Let $f_*(\mu)(Y) = \mu(f^{-1}(Y))$ for $Y \in \mathcal{A}'$. Then $f_*(\mu)$ is a measure on X': image measure of μ. under f (synonym. $f(\mu)$).
Moreover, measurable $g: (X', \mathcal{A}') \to (X'', \mathcal{A}'')$ then $(g \circ f)_*(\mu) = g_*(f_*(\mu))$

3. For any measurable subset $Y \subset X$, $\mathcal{A}_Y = \{E \subset Y \mid E \in \mathcal{A}\}$.
$\mu|_{\mathcal{A}_Y}$ (the restriction of μ to the σ-algebra of measurable subsets of Y) is a measure on Y for $\mathcal{A}_Y$.

Pf2: $f_*(\mu)$ is a measure: 1) $f_*(\mu)(\phi) = \mu f^{-1}(\phi) = \mu\phi = 0$.

2). Let $\{E_n'\}_{n\geq 1}^{\infty}$ $E' \in A$. $E_n' \cap E_m' \neq \phi \,(\Rightarrow f^{-1}(E_n') \cap f^{-1}(E_m') = \phi$. inverse not true).

$$f_*(\mu)\left(\bigcup_{n\geq 1}^{\infty} E_n'\right) = \mu f^{-1}\left(\bigcup_{n\geq 1}^{\infty} E_n'\right) = \mu \bigcup_{n\geq 1}^{\infty} f^{-1}(E_n) = \sum_{n\geq 1}^{\infty} \mu f^{-1}(E_n)$$

§. the Lebesgue Measure.

Def. $\exists!$ measure μ on the $\mathcal{B}_n$ in $\mathbb{R}^n$. s.t. $\mu([a_1,b_1)\times\cdots\times[a_n,b_n)) = \prod_{1\le k\le n}(b_k-a_k)$

for any cell in $\mathbb{R}^n$. this measure is. Lebesgue measure.

Remark: Lebesgue measure μ on $\mathcal{B}_n$ is not complete.

use completion procedure. $\mathcal{B}_n \xrightarrow{\text{exten.}}$ Lebesgue σ-algebra. ($\mathcal{A}$ or $\mathcal{A}_n$. ← dim.)

Property: 1) Lebesgue measure is σ-finite.

2). Lebesgue σ-algebra contains $\mathcal{B}_n$.

In particular, open, closed set. F_σ ($\bigcup_{n=1}^{\infty}$ closed set). G_δ ($\bigcap_{n=1}^{\infty}$ open set) are meas.

3) $\mu E = 0$. $E = \{$at most countable set of $\mathbb{R}^n\}$.

4). let $E \in \mathcal{A}_n$. Then $\exists H$ of type F_σ and $\exists K$ of type G_δ.

s.t. $H \subset E \subset K$. $\mu(K \setminus H) = 0$.

5) E is Lebesgue meas. if $E = \bigcup_{k=1}^{\infty} F_k \cup e$. F_k are compact. $F_k \subset F_{k+1}$. $\mu e = 0$.

6) Let $A: \mathbb{R}^n \to \mathbb{R}^n$ be. l.t. and $E \subset \mathbb{R}^n$ be measurable. then $A(E)$ is measurable

and $\mu(A(E)) = |\det A|\,\mu(A)$ (the Lebesgue measure is invariant w.r.t. to a shift.)

§. Borel measure.

Def. regular Borel. measure.: pre: Borel measure on $(X, \mathcal{T})$.

for any Borel set $Y \subset X$. we have:

$\mu(Y) = \inf\{\mu(U) \mid U$ is open. $Y \subset U\}$.

$\mu(Y) = \sup\{\mu(K) \mid K$ is compact $K \subset Y\}$.

Thm. 1.16. The Lebesgue measure on $\mathcal{B}_{\mathbb{R}^n}$ is regular.

Thm 1.17. Let X be a locally compact t.s. in which any open set is a countable union of compact subsets.

Then any Borel measure μ s.t. $\mu(K) < +\infty$ for all compact sets $K \subset X$ is regular. Particularly, any finite measure is regularly.

Chapter 2. Integration w.r.t. a measure.

Pre: $(X, \mathcal{A}, \mu)$. $E \in \mathcal{A}$.

real-valued function $f: E \to \mathbb{R}$. $f = f^+ - f^-$ $\quad |f| = f^+ + f^-$

$$f^+(x) = \max(0, f(x)) \qquad f^-(x) = \max(0, -f(x)).$$

approximation of measurable function : simple function, step function.

notation: $\int_E f(x)\, d\mu(x) = \int_E f\, d\mu = \int_E f$.

(indicates variable: x in $d\mu(x)$; measure is fixed: $\int_E f$)

Def. integral $\int_E f\, d\mu$. (of function $f \in S(E)$ w.r.t. measure μ on E):

1. f is simple: $f = \sum_{k=1}^{N} c_k \chi_{A_k}$, $A_k \in \mathcal{A}$. A_k disjoint. $c_k \in [0, +\infty)$

   Then $\int_E f\, d\mu = \sum_{k=1}^{N} c_k \mu(A_k \cap E)$ $\quad$ (c_k are values of f in A_k)

2. $f \geq 0$. Then we let $\int_E f\, d\mu = \sup_{\substack{\varphi \text{ simple} \\ \varphi \leq f \text{ on } E}} \int_E \varphi\, d\mu$.

3. f be arbitrary meas. Then we let $\int_E f\, d\mu = \int_E f_+\, d\mu - \int_E f_-\, d\mu$. if at least one of them is finite. If both infinite. symbol $\int_E f\, d\mu$ no value. (not exist).

4. $f: E \to \mathbb{C}$. meas. $\int_E f\, d\mu : \int_E \mathrm{Re}(f)\, d\mu + i \int_E \mathrm{Im} f\, d\mu \in \mathbb{C}$.

   if both integrals exist as integral of real-valued function.

Def. Lebesgue integral : integral w.r.t. Lebesgue measure

Thm 2.1. Pre: $(X, \mathcal{A})$. $E \in \mathcal{A}$. $f: E \to \mathbb{R}$. $f \in S(E)$.

1. If $f \geq 0$. $\exists \{\varphi_n\}_{n=1}^{\infty}$ simple functions s.t. $\forall x \in E$. $\varphi_n(x) \leq \varphi_{n+1}(x)$.

   and $f(x) = \lim \varphi_n(x) = \sup_n \varphi_n(x)$

2. $\exists \langle \psi_n \rangle_{n=1}^{\infty}$ step functions s.t. $\forall x \in E$, $|\psi_n(x)| \leq |f(x)|$. and $f(x) = \lim_{n\to\infty} \psi_n(x)$

Pf: 1. fix $n \in \mathbb{N}$. denote $E_{in} = \begin{cases} E(\frac{i}{2^n} \leq f < \frac{i+1}{2^n}) & i \in \{0, 1, \dots n2^n - 1\}. \\ E(f \geq n) & i = n \cdot 2^n \end{cases}$

E_{in} are measurable and mutually disjoint. $\bigcup_{i=0}^{n2^n} E_{in} = E$ Let $\varphi_n = \sum_{i=0}^{n2^n} \frac{i}{2^n} \chi_{E_{in}}$ ($\varphi_n \leq f$ by def)

Thus we have $\varphi_n = \frac{i}{2^n}$ on E_{in} $\quad \varphi_n$ are simple.

① $f(x) = +\infty$. $\quad \varphi_n = \frac{n \cdot 2^n}{2^n} = n \xrightarrow[n\to+\infty]{} +\infty$

② $f(x) < +\infty$. $\exists N \in \mathbb{N}$: $f(x) < N \Rightarrow \forall n \geq N$. $\exists i \in \{0, \dots n \cdot 2^{n-1}\}$. s.t. $\frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}$

$\Rightarrow |f(x) - \varphi_n(x)| < \frac{1}{2^n}$. $\Rightarrow \varphi_n(x) \xrightarrow[n\to\infty]{} f(x)$

Check $\varphi_n \leq \varphi_{n+1}$. $\quad i \leq 2^n f(x) < i+1$. $\Rightarrow \frac{[2^n f(x)]}{2^n} \leq \frac{[2^{n+1} f(x)]}{2^{n+1}} \Rightarrow 2[A] \leq [2A]$.

2. by. $\exists \varphi_{n_1} \to f_+$ $\quad \varphi_{n_2} \to f_-$. $\quad \psi_n = \varphi_{n_1} - \varphi_{n_2} \to f_+ - f_- = f$.

3. If f is bounded then convergence in previous assertion is uniform.

Pf: $0\le f\le M$: $\varphi_n(x)=\frac{[2^n f(x)]}{2^n}$ for $n>M$. and $|f(x)-\varphi_n(x)|<\frac{1}{2^n}$. for every $x\in E$ ($E(f\ge n)=\phi$)

$f=f_+-f_-$ generalize it.

Thm 2.4. (Monotonicity of the integral). $f,g: E\to\bar{\mathbb{R}}$. $\int_E f\,d\mu$. $\int_E g\,d\mu$ exists.

$f\le g$ on E, then $\int_E f\,d\mu\le\int_E g\,d\mu$.

Coro 2.4.1. (Monotonicity of the integral by set.) Let $f\in S(E)$, $f\ge 0$.

$E_1\subset E$ and $E_1\in\mathcal{A}$. then $\int_{E_1} f\,d\mu\le\int_E f\,d\mu$.

(Check by step. 1) simple 2) nonnegative 3) $f\le g$ for arbitrary meas. function.)

Thm 2.5. (B. Levy. monotone convergence thm.) Let $f_n\in S(E)$. (nonnegative. increase. meas. has pointwise limit).

$0\le f_n\le f_{n+1}$, $f=\lim f_n$. then $\int_E f_n\,d\mu\xrightarrow[n\to\infty]{}\int_E f\,d\mu$.

若无$f_n\ge 0$条件,可推出 $f_n\to f$ in $L(E)$.

Pf: 1). $f_n\le f$. (by def. of f and f_n).

by thm 2.4. $\int_E f_n\,d\mu\le\int_E f\,d\mu$ $\xrightarrow[\text{for all } n \text{ holds}]{\text{pass the limit}}$ $\lim\int_E f_n\,d\mu\le\int_E f\,d\mu$.

2). "$\ge$" denote $\lim\int_E f_n\,d\mu=L$.

denote $\varphi=\sum_{k=1}^N c_k\chi_{A_k}$, $\varphi\le f$ on E. Let $q\in(0,1)$ and denote $E_n=E(f_n\ge q\varphi)$.

Then: $E_n\subset E_{n+1}$. (since $f_n\le f_{n+1}$).

∘ we have $\bigcup_{n=1}^\infty E_n=E$. "$\subset$" obvious "$\supset$" $\forall x\in E$. 1) if $\varphi(x)=0$. $x\in E_n$ for every n.

2) if $\varphi(x)>0$. then $f(x)\ge\varphi(x)>q\varphi(x)$. and $f_n(x)\to f(x)$. $x\in E_n$.

$\forall A\subset E$. $(A\cap E_n)\subset(A\cap E_{n+1})$, $A=\bigcup_{n=1}^\infty(A\cap E_n)$.

By continuity of measure μ, $\mu(A\cap E_n)\xrightarrow[n\to\infty]{}\mu(A\cap E)=\mu(A)$.

$\int_E f_n\,d\mu\ge\int_{E_n} f_n\,d\mu\ge\int_{E_n} q\varphi\,d\mu=q\sum_{k=1}^N c_k\mu(A_k\cap E_n)$.

pass the limit. $L\ge q\sum_{k=1}^N c_k\mu(A_k\cap E)=q\int_E\varphi\,d\mu$.

pass $q\to 1$. RHS to the sup. $L\ge\int_E f\,d\mu$.

Thm 2.6. (Linearity of the integral).

(1). $\alpha\in\mathbb{R}$. $f: E\to\bar{\mathbb{R}}$. $\int_E f\,d\mu$ exists. Then $\int_E \alpha f\,d\mu$ exists and $\int_E\alpha f\,d\mu=\alpha\int_E f\,d\mu$.

(2) $\int_E f\,d\mu$. $\int_E g\,d\mu$. $\int_E f\,d\mu+\int g\,d\mu$ exists. Then $\int_E(f+g)$ exists and $\int_E(f+g)\,d\mu=\int_E f+\int_E g$

Coro 2.6.1. (Levy thm. for series). (Series with non-negative term can be integral term-by-term)

If $f_k\ge 0$, $f_k\in S(E)$, then $\int_E\sum_{k=1}^\infty f_k\,d\mu=\sum_{k=1}^\infty\int_E f_k\,d\mu$.

Lemma 2.7. (Chebyshev inequality) Let $f: X \to \mathbb{R}$ be measurable function. $t \in (0, +\infty)$.

$$\mu E(|f| \geq t) \leq \frac{1}{t} \int_E |f| \, d\mu.$$

Pf: denote. $E_t = E(|f| \geq t)$.

$$\int_E |f| \, d\mu \leq \int_{E_t} |f| \, d\mu \geq \int_{E_t} t \, d\mu = t \cdot \mu E_t.$$

Coro 2.7.1. If $\int_E |f| d\mu$ is finite then f is finite a.e. on E.

2. If $\int_E |f| \, d\mu = 0$. then $f = 0$ a.e. on E ($f \sim 0$).

Pf: 1. $\mu E(|f| = +\infty) \leq \mu E(|f| \geq t) \leq \frac{1}{t} \int_E |f| d\mu$ for every $t > 0$.

$t \to +\infty$, $\frac{1}{t} \to 0$. $\mu E(|f| = +\infty) = 0$.

2. $E(f \neq 0) = \bigcup_{n=1}^{\infty} E(|f| \geq \frac{1}{n})$.

$\mu E(|f| \geq \frac{1}{n}) \leq n \int_E |f| \, d\mu = 0$.

§. Integral as a measure.

Thm 2.9. Let $(X, \mathcal{A}, \mu)$ be measured space. f is meas. $f \geq 0$. $\underline{\mu_f(E)} \overset{def.}{=} \int_E f \, d\mu$. $E \in \mathcal{A}$.

μ_f is a measure on $(X, \mathcal{A})$. s.t. any μ-neg. set is μ_f-neg. and.

$\int_E g \, d\mu_f = \int_E g f \, d\mu$. for any $E \in \mathcal{A}$ and any meas. $g \geq 0$.

Pf: 1) μ_f is a measure on $(X, \mathcal{A})$.

$f \geq 0 \Rightarrow \mu_f(E) \geq \int_E 0 \cdot d\mu \Rightarrow \mu_f(E) = 0$. and $\mu_f(\emptyset) = 0$.

Consider $\{E_n\}_{n=1}^{\infty}$ pairwise disjoint and $E = \bigcup_{n=1}^{\infty} E_n$. $\chi_E(x) = \sum_{n=1}^{\infty} \chi_{E_n}(x)$

$\mu_f(E) = \int_X f \chi_E \, d\mu = \sum_{n=1}^{\infty} \int_X f \chi_{E_n}(x) \, d\mu = \sum_{n=1}^{\infty} \mu_f(E_n)$. (show the countable additivity)

2). any μ-neg. set is μ_f-neg.

Assume $E_1 \subset E$ s.t. $\mu(E) = 0 \Rightarrow \mu_f(E) = 0$. by monotonicity of set: $\mu_f(E_1) = 0$.

3). $\int_E g \, d\mu_f = \int_E g f \, d\mu$.

denote simple $\langle s_n \rangle_{n=1}^{\infty}$ $s_n \leq s_{n+1}$. $\lim s_n = g$

$$\int_E s_n \, d\mu_f = \sum_{k=1}^{N} c_{nk} \, \mu_f(A_k \cap E) = \sum_{k=1}^{N} c_{nk} \int_E f \chi_{A_k} \, d\mu = \int_E f \cdot \sum_{k=1}^{N} c_{nk} \chi_{A_k} \, d\mu = \int_E f s_n \, d\mu.$$

$$\int_E g \, d\mu_f = \lim_{n \to \infty} \int_E s_n \, d\mu_f = \lim_{n \to +\infty} \int_E f \cdot s_n \, d\mu = \int_E f g \, d\mu \quad (\text{By B. Levy thm.})$$

Thm 2.10. (General formula for the change of the variable) Assume that $\varphi: X \to X'$ is meas. Then for any $g \geq 0$ meas on X' and any $E \in \mathcal{A}'$, we have (change variable formula).

$$\int_{\varphi^{-1}(E)} (g \circ \varphi)\, d\mu = \int_{\varphi^{-1}(E)} g(\varphi(x))\, d\mu(x) = \int_E g\, d\varphi_*(\mu).$$

$\varphi^*(\mu)$ is the image of measure μ w.r.t. the map μ.

Pf: 1. $g = \chi_F$. $\int_E \chi_F \, d\varphi_*(\mu) = \varphi_*(\mu)(E \cap F) = \mu(\varphi^{-1}(E \cap F)) = \mu(\varphi^{-1}(E) \cap \varphi^{-1}(F))$

$$= \int_{\varphi^{-1}(E)} \chi_F \circ \varphi \, d\mu \underset{\Downarrow}{=} \int_{\varphi^{-1}(E)} \chi_{\varphi^{-1}(F)} \, d\mu.$$

$$(\chi_{\varphi^{-1}(F)}(x) = 1 \Leftrightarrow x \in \varphi^{-1}(F) \Leftrightarrow \varphi(x) \in F \Leftrightarrow (\chi_F \circ \varphi)(x))$$

By linearity, the formula holds for any simple function.

2. $g \geq 0$. $\exists \langle s_n \rangle_{n \geq 1}^{\infty}$ s_n is simple $s_n \leq s_{n+1}$, $s_n \to g$ pointwisely.

By Levy thm. $\int_E g\, d\varphi_*(\mu) = \lim_{n\to\infty} \int_E s_n \, d\varphi_*(\mu) = \lim \int_{\varphi^{-1}(E)} s_n \circ \varphi \, d\mu = \int_{\varphi^{-1}(E)} g \circ \varphi \, d\mu.$

$s_n \circ \varphi$ is simple, $s_n \circ \varphi \to g \circ \varphi$ pointwisely. □.

e.g. $(X, \mathcal{A}, \mu)$. $\varphi: \mathbb{R} \to \mathbb{R}$ is meas. denote $\nu = \varphi_*(\mu)$. $g: \mathbb{R} \to \mathbb{R}$. $g(y) = |y|$.

if $\varphi \in C^1$ and strict-increasing. $\varphi(x) = y$ $x = \varphi^{-1}(y)$

$$\int_{\mathbb{R}} |\varphi(x)|\, dx = \int_{\mathbb{R}} |y| \cdot (\varphi^{-1}(y))'\, dy = \int_{\mathbb{R}} \frac{|y|}{\varphi'(\varphi^{-1}(y))}\, dy.$$

§. Integrable functions.

Def. pre: $(X, \mathcal{A}, \mu)$. $E \in \mathcal{A}$. $f: E \to \mathbb{R}$ is measurable.

f is integrable on E w.r.t. the measure μ. if the non-negative function $|f| = f_+ + f_-$ has finite integral $\int_E |f|\, d\mu < +\infty$.

If $f: E \to \mathbb{C}$. is integrable on E w.r.t. measure μ. if $|f| = \sqrt{Re(f)^2 + Im(f)^2} \geq 0$. has finite integral $\int_E |f|\, d\mu < +\infty$.

the set of all μ-integrable complex-valued functions. $L(E, \mu)$. / $L(E)$ / $L(\mu)$.

Fact: $Y \subset X$. $\Rightarrow |\chi_F f|_Y \leq |f|_X$. $\Rightarrow$ $f \in L(X)$ then $f \in L(Y)$.

Lemma 2.13. The set $L(X, \mu)$ is $\mathbb{C}$-vector space. Moreover, the map $\|f\|_1 = \int_X |f|\, d\mu$. is a semi-norm on $L(X, \mu)$. i.e. ① $\|af\|_1 = |a| \|f\|_1$. for $a \in \mathbb{C}$ and $f \in L(X, \mu)$;
② $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$ for $f, g \in L(X, \mu)$. (与norm相比，少了 positive-definiteness. $p(E) = 0$ iff $E = \phi$. 而变为$p(E)=0$ iff $\mu(E)=0$)

Moreover, $\|f\|_1 = 0$ iff f is zero μ-a.e.

Lemma 2.13.2. The map $f \mapsto \int_X f d\mu$ is a linear map, it is non-negative.

and $\left| \int_X f d\mu \right| \le \int_X |f| d\mu = \|f\|_1$.

Pf: use. $|\alpha f + \beta g| \le |\alpha||f| + |\beta||g|$

e.g.1. X be arbitrary set. μ be counting measure thus $\int_X f(x) d\mu = \sum_{x \in X} f(x)$:

1. a family $(f(x))_{x \in X}$ is absolutely summable with $S \in \mathbb{C}$ iff $\forall \varepsilon > 0$. $\exists X_0 \subset X$. X_0 is finite. s.t. $\left| \sum_{x \in X_1} f(x) - S \right| < \varepsilon$. for any finite X_1 with. $X_0 \subset X_1 \subset X$.
2. In particular. $X = \mathbb{N}$. consider $\sum_{n \ge 1} a_n$, $a_n \in \mathbb{C}$. sequence (a_n) is integrable iff series abs. conv.

e.g.2. $\mu = \delta_{x_0}$ be Dirac measure. at $x_0 \in X$. i.e. $\delta(A) = \begin{cases} 1, & x_0 \in A \\ 0, & x_0 \notin A \end{cases}$

then any function $f: X \to \mathbb{C}$ is μ-integrable and $\int_X f(x) d\delta_{x_0}(x) = f(x_0)$.

let $\{x_1, \dots x_n\}$. finite points. $x_n \in X$. $\delta = \frac{1}{n} \sum_{1 \le i \le n} \delta_{x_n}$. the probability measure.

s.t. $\int_X f(x) d\delta(x) = \frac{1}{n} \sum_{1 \le i \le n} f(x_i)$ "sample sum"

Lemma 2.14. (Fatou's lemma). pre. (f_n). sequence of non-negative. meas. function. $f_n : X \to [0, +\infty]$.
We then have $\int_X (\liminf_{n \to +\infty} f_n) d\mu \le \liminf_{n \to +\infty} \int_X f_n d\mu$. and in particular, if $f_n(x) \to f(x)$ for all x, we have $\int_X f(x) d\mu(x) \le \liminf_{n \to +\infty} \int_X f_n(x) d\mu(x)$.

Pf: 1. denote $g = \underline{\lim} f_n$. $g_n = \inf_{k \ge n} f_k$.

Then $g_n \le g_{n+1}$. $g = \lim g_n$. $g_n \le f_n$. $g_n \in S(E)$.

by monotonicity of integral. $\int_E g_n \le \int_E f_n$.

by Levy. thm. $\int_E g = \lim \int_E g_n = \underline{\lim} \int_E g_n \le \underline{\lim} \int_E f_n$.

2. $f = \underline{\lim} f_n$.

Lemma 2.15. (Lebesgue dominated conv. thm.). pre: $(X, \mathcal{A}, \mu)$. $E \subset X$. $f_n(x) \in \mathbb{C}$. $\int_E |f_n| d\mu < +\infty$ (μ-integrable).
Assume $f_n(x) \xrightarrow[n \to \infty]{} f(x) \in \mathbb{C}$. for all $x \in E$. Then f is measurable on E ① (thm 1.6).
Moreover, if $\exists \Phi \in L^1(E, \mu)$ s.t. $|f_n(x)| \le \Phi(x)$ for all $n \ge 1$ and $x \in E$. then f is μ-integrable ②,

$\int_E f(x) d\mu(x) = \lim_{n \to \infty} \int_E f_n(x) d\mu(x)$ ③ and $\int_E |f_n - f| d\mu \xrightarrow[n \to \infty]{} 0$. ④

also called f_n conv. to f in $L^1(E, \mu)$

Pf: since $f_n(x) \to f(x)$ f is measurable on E. (by thm 1.6). ①✓

since $|f_n(x)| \le \Phi(x) \xrightarrow{n \to +\infty} |f(x)| \le \Phi(x)$

(continue the pf.)

$f_n, f \in L^1(E,\mu)$ (by monotonicity of integral). //②

since $\Phi + f_n \geq 0$. $\Phi + f \geq 0$. use Fatou's thm. ($f = \underline{\lim} f$).

$$\int_E \Phi + \int_E f = \int_E (\Phi + f) \leq \underline{\lim} \int (\Phi + f_n) = \underline{\lim}\left(\int_E \Phi + \int_E f_n\right) = \int_E \Phi + \underline{\lim} \int_E f_n.$$

thus $\int_E f \leq \underline{\lim} \int_E f_n$

Analogously,

$$\int_E \Phi - \int_E f \leq \underline{\lim} \int_E (\Phi - f_n) = \int_E \Phi + \underline{\lim}\left(-\int_E f_n\right) = \int_E \Phi - \overline{\lim} \int_E f_n.$$

$\int_E f \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E f$. $\lim \int_E f_n$ exists and $\int_E f = \lim_{n\to\infty} \int_E f_n$ ③//

Finally, define $F_n(x) = |f_n(x) - f(x)|$. we have $F_n(x) \xrightarrow[n\to\infty]{} 0$ and $F_n(x) \leq 2g(x)$.

$2g(x) \in L^1(E,\mu)$. $F_n(x) \in L^1(E,\mu)$. then apply the previous result:

$\lim \int_E F_n(x) d\mu(x) = \int_E 0 \, d\mu = 0$. i.e. $\int_E |f_n(x) - f(x)| d\mu \to 0$. ④//

thm. (General Lebesgue Dominated Convergence thm.)

pre: $(X, \mathcal{A}, \mu)$. $E \subset X$. $f_n : E \to \mathbb{C}$. $f_n \in S(E)$. $f_n \to f$ on E a.e.

Suppose there exists $\{g_n\}$ s.t. $g_n \in S(E)$. $g_n \geq 0$. $g_n \to g$ on E a.e. and

$|f_n| \leq g_n$ a.e. on E. for all n. If $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$ then. $\lim_{n\to\infty} \int_E f_n = \int_E f$

coro. $f_n \in S(E)$. $f_n \to f$ a.e. on E and $f \in L^1(E,\mu)$. $\int_E |f - f_n| \to 0 \Leftrightarrow \lim_{n\to\infty} \int_E |f_n| = \int_E |f|$

"$\Rightarrow$" $\left|\int_E |f| - \int_E |f_n|\right| \leq \int_E \big||f| - |f_n|\big| \leq \int_E |f - f_n|$

"$\Leftarrow$" let $g_n = |f| + |f_n|$. $|f - f_n| \leq |f| + |f_n| = g_n$; $|f - f_n| \to 0$ a.e. on E.

$\lim \int_E g_n = \lim \int_E |f| + |f_n| = 2\int_E |f|. < \infty$ conv.

then $\lim \int_E |f - f_n| = \lim \int_E 0 = 0$.

Chapter 3. L^p-spaces.

引入等价类. 商空间. 从 $L(E,M)$ 的 seminorm 到 $L^1(X,\mu)$ 的 norm

§. Lebesgue space L^1

"quotient space" in t.p.

Def. $(X, \mathcal{M}, \mu)$. The space $L^1(X,\mu) = L^1(\mu) = \{f : X \to \mathbb{C} \mid f \text{ is integrable}\}/_N$

where. $N = \{f \mid f \text{ is measurable and } f \text{ is zero } \mu\text{-a.e.}\} = \{f \mid \int_X |f| d\mu = 0\}$.

the space $L^1(X,\mu)$ is a <u>normed vector space</u>. $\|f\|_1 = \int_X |f| d\mu$.

(in particular $\|f\|_1 = 0$ iff $f = 0$ in $L^1(X,\mu)$).

Remark:

$L(X,\mu) = \{f : \|f\|_1 = \int_X |f| d\mu < +\infty\}$.

denote equiva. relation. $f \sim g$ if $f = g$ a.e.

$N = \{f : f \sim 0\}$. $f \sim g \Leftrightarrow f - g \sim N$. the equivalent class: $f \in L(X,\mu)$. $\{f\} = \{g \in L(X,\mu), g \sim f\}$

$L(X,\mu) = \bigcup_{f \in L(X,\mu)} \{f\}$. $\quad L^1(X,\mu) = L(X,\mu)/_N = L(X,\mu)/_\sim := \{\{f\} : f \in L(X,\mu)\}$.

$L^1(X,\mu)$ is $\mathbb{C}$-vector space since $\begin{cases} \{f_1\} + \{f_2\} = \{f_1 + f_2\} \\ \alpha \in \mathbb{C}.\ \alpha\{f_1\} = \{\alpha f_1\}. \end{cases}$

norm: $\|\{f\}\|_1 = \|f\|_1 = \int_X |f| d\mu$.

Thm 3.2. Pre: $(X, \mathcal{A}, \mu)$. $f_n \in L^1(\mu)$.

Assume series $\sum f_n$ is normally conv. in $L^1(\mu)$. i.e. $\sum_{n=1}^{\infty} \|f_n\|_1 < +\infty$.

Then the series $\sum_{n=1}^{\infty} f_n(x)$ conv. a.e. in X.

And if $f(x) = \lim_{K\to\infty} \sum_{n=1}^{K} f_n(x)$. (well-define a.e.) then $f \in L^1(\mu)$ and $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$.

Moreover we have $\| \sum_{n\geq 1}^{\infty} f_n - f \| \xrightarrow[n\to\infty]{} 0$

Pf: define $h(x) = \sum_{n\geq 1}^{\infty} |f_n(x)| \in [0, +\infty]$ for $x \in X$.

h is changed at worst on $Y = \bigcup_{n\geq 1}^{\infty} Y_n$. $\mu Y_n = 0$ and f_n is modified on Y_n.

h is well-defined a.e. thus it's measurable.

$\int_X h d\mu = \sum_{n=1}^{\infty} \int_X |f_n| d\mu = \sum_{n\geq 1} \|f_n\|_1 < +\infty$. (by assumption). h is finite a.e.

$\forall x$ s.t. $h(x) < \infty$. $\sum f_n(x)$ conv. abs. $\exists f(x) = \lim_{K\to\infty} \sum_{n=1}^{K} f_n(x)$ a.e. on X (extend. f. $f(x) = 0$ when $x \in Z$, $Z = \{x \mid h(x) = \infty\}$)

in any case, we have $|f| \leq h$. thus $f \in L^1(\mu)$.

denote the partial sum $u_N = \sum_{n\geq 1}^{N} f_n(x)$, $x \notin Z$, $u_N(x) = 0$, $x \in Z$ thus $u_N \to f(x)$ $|u_N(x)| \leq h \cdot \chi_{X\setminus Z}$ for all x and N.

apply Lemma 2.15 (dominated ...)

$$\int_X f d\mu = \lim_{N\to+\infty} \int_X u_N d\mu = \lim \sum_{n=1}^{N} \int_X f_n d\mu = \sum_{n\geq 1}^{\infty} \int_X f_n d\mu.$$

and the last past: apply Lemma 2.15.

Thm 3.3. (Completeness of the space L^1). Pre: $(X, \mathcal{A}, \mu)$.

1. (f_n) is Cauchy sequence in $L^1(\mu)$. then (f_n) is conv. i.e. $\exists!\ f \in L^1(\mu)$ s.t. $f_n \to f$ in $L^1(\mu)$ ($\lim_{n\to+\infty} \|f_n - f\|_1 = 0$) ($L^1(\mu)$ is complete).

2. Moreover, $\exists (f_{n_k}) \subset (f_n)$. s.t. $\lim_{k\to+\infty} f_{n_k}(x) = f(x)$ for μ-a.e. on X.

Pf: apply the Cauchy Condition: $\forall \varepsilon > 0$. $\exists N(\varepsilon) \in \mathbb{N}$. s.t. $\|f_n - f_m\|_1 < \varepsilon$ for all $n, m > N(\varepsilon)$

consider $\varepsilon = \varepsilon_k = 2^{-k}$ for $k \geq 1$.

Then using induction on k, we see that $\exists (n_k)_k$ $n_k < n_{k+1}$. s.t. $\|f_{n_{k+1}} - f_{n_k}\|_1 < 2^{-k}$

(f_{n_k}) conv. a.e. on $L^1(\mu)$. denote $g_k = f_{n_{k+1}} - f_{n_k}$. $\sum \|g_k\|_1 \leq \sum_{k\geq 1}^{\infty} 2^{-k} < +\infty$.

by thm 3.2. $\exists g \in L^1(\mu)$ s.t. $\lim_{k\to\infty} g_k = g \Rightarrow \lim_{k\to\infty}(f_{n_{k+1}} - f_{n_1}) = g \Rightarrow \lim_{k\to\infty} f_{n_k} = f_{n_1} + g$. // ②

denote $f = f_{n_1} + g$, $f \in L^1(\mu)$; $f_{n_k} \xrightarrow[k\to\infty]{} f$ on X (a.e.)

and $|f_{n_k}| \leq f$. by lemma 2.15, $\|f - f_{n_k}\|_1 \to 0$

by the uniqueness of limit. (in metric space actually). $\|f - f_n\|_1 \to 0$ // ①

§. Lebesgue space L^p with $1 \leq p < \infty$

Def. pre: $(X, \mathcal{A}, \mu)$. $p \in [1, +\infty)$. $p \in \mathbb{R}$.

$L^p(X, \mu) = L^p(\mu)$. is the quotient vector space $\{f: X \to \mathbb{C} : |f|^p \text{ is integrable}\}/N$.

this space equipped with norm $\|f\|_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$.

Check the definition (correctness).

1. $\|\cdot\|_p$ is the norm.

1) $\|f\|_p = 0 \iff |f|^p = 0$ a.e. (by Chebyshev inequality, $\mu E_t = 0$ for any $t > 0$).

2) $\|\alpha f\|_p = |\alpha| \|f\|_p$. $\alpha \in \mathbb{C}$.

3) $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ $\quad \left(\int_X |f+g|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p d\mu\right)^{\frac{1}{p}}$

Recall: $\varphi: I \to [0, +\infty)$. I is an interval. 1) φ is convex if: $\varphi(t_1 x_1 + t_2 x_2) \leq t_1 \varphi(x_1) + t_2 \varphi(x_2)$ for any $x_1, x_2 \in I$ and $t_1, t_2 \geq 0$, $t_1 + t_2 = 1$.

2) if $\varphi \in C^2$, then φ is convex $\iff \varphi'' \geq 0$ on I.

Generalize: $\varphi(t_1 x_1 + \cdots + t_n x_n) \leq t_1 \varphi(x_1) + \cdots + t_n \varphi(x_n)$

$\forall x_1, \dots x_n \in I$. and $t_1 \dots t_n \geq 0$. $t_1 + t_2 + \cdots + t_n = 1$

Lemma 3.5. Pre: $(X, \mathcal{A}, \mu)$.

1. Assume $\mu X = 1$. Then for any $\varphi: [0,+\infty) \to [0,+\infty)$ which is non-decreasing, continuous, and convex, and any measurable function $f: X \to [0,+\infty)$.

we have Jensen's inequality $\varphi\left(\int_X f(x)\, d\mu(x)\right) \leq \int_X \varphi(f(x))\, d\mu(x)$, $\varphi(+\infty) = +\infty$.

Pf: 1) $f \equiv c$. obvious.

2) $f = \sum_{k=1}^{N} c_k \chi_{A_k}$. A_k partition X. $\sum \mu(A_k) = 1$

$$\varphi\left(\int_X f(x)\, d\mu(x)\right) = \varphi\left(\sum_{k=1}^{N} c_k \mu(A_k)\right) \overset{\text{by } \sum \mu A_k = 1 \text{ and convex of } \varphi}{\leq} \sum_{k=1}^{N} \varphi(c_k)\mu(A_k) \leq \int_X \varphi(f(x))\, d\mu(x)$$

3) $f \geq 0$. $\exists S_n$ - simple. $S_n \leq S_{n+1}$ $S_n \to f$ $\forall x \in X$. (φ is non-de, $\varphi(S_n) \leq \varphi(S_{n+1})$ $\varphi(S_n) \to \varphi \circ f$

$$\varphi\left(\int_X f\, d\mu\right) = \varphi\left(\lim_{n\to\infty} \int_X S_n\, d\mu\right) \underset{\text{cont.}}{=} \lim_{n\to\infty} \varphi\left(\int_X S_n\, d\mu\right) \overset{\text{case 2)}}{\leq} \lim_{n\to\infty} \int_X \varphi \circ S_n\, d\mu \overset{\text{cont.}}{=} \int_X \varphi \circ f\, d\mu$$

2. $\underset{\text{"dual"}}{p, q > 1}$. $p, q \in \mathbb{R}$. $\frac{1}{p} + \frac{1}{q} = 1$. Then for any meas. function $f, g: X \to [0,+\infty]$

we have Hölder's inequality: $\int_X fg\, d\mu \leq \left(\int_X f^p\, d\mu\right)^{\frac{1}{p}} \left(\int_X g^q\, d\mu\right)^{\frac{1}{q}} = \|f\|_p \|g\|_p$.

Pf: $\forall x, y \geq 0$. $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$ (Young inequality).

$\Rightarrow f(x) g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}$

Assume $\|f\|_p = \|g\|_q = 1$

$$\int_X fg\, d\mu \leq \int_X \frac{f(x)^p}{p} + \frac{g(x)^q}{q}\, d\mu = \frac{\|f\|_p^p}{p} + \frac{\|g\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1 = \|f\|_p \|g\|_q.$$

for general case. $f_1 = \frac{f}{\|f\|_p}$ $g_1 = \frac{g}{\|g\|_q}$.

we have $\int_X \frac{fg}{\|f\|_p \|g\|_q}\, d\mu \leq 1$.

3. $p > 1$. $p \in \mathbb{R}$. Then for any measurable function $f, g: X \to [0,+\infty]$

we have Minkowski's inequality: $\|f+g\|_p = \left(\int_X (f+g)^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X f^p\, d\mu\right)^{\frac{1}{p}} + \left(\int_X g^p\, d\mu\right)^{\frac{1}{p}}$

(moreover $f, g \in L^p \Rightarrow L + g \in L^p$) $= \|f\|_p + \|g\|_p$

Pf: 1). either $\|f\|_p$ or $\|g\|_p = +\infty$ obvious.

2). $(f+g)^p \leq (2\max(f,g))^p \leq 2^p(f^p + g^p) < +\infty$.

consider $h = (f+g)^{p-1}$. $(f+g)^p = fh + gh$.

$$\int_X (f+g)^p\, d\mu = \int_X (fh + gh)\, d\mu \leq \|f\|_p \|h\|_q + \|g\|_p \|h\|_q. \quad \text{(by Hölder's)}$$

$$\|h\|_q = \left(\int_X (f+g)^{q(p-1)}\, d\mu\right)^{\frac{1}{q}} = \|f+g\|_p^{\frac{p}{q}} = \|f+g\|_p^{p-1}$$

$$\|f+g\|_p^p \leq \|h\|_q (\|f\|_p + \|g\|_p) = \|f+g\|_p^{p-1} (\|f\|_p + \|g\|_p).$$

$\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$. ($\|f+g\|_p^{p-1} > 0$. diminish) e.g. $f(x) = \begin{cases} \frac{\sin x}{x} & x \in \mathbb{R}\setminus\{0\} \\ 1 & x = 0. \end{cases}$ on $(\mathbb{R}, \mathcal{B}, \mu)$.

$p>1$. $\left|\frac{\sin x}{x}\right|^p \le \frac{1}{|x|^p}$, $\left|\frac{\sin x}{x}\right|^p \le 1$. $\Rightarrow |f(x)| \le h(x) = \begin{cases} 1, & |x| \le 1 \\ \frac{1}{|x|^p}, & |x|>1. \end{cases}$

$\|f\|_p \le \int_{\mathbb{R}} h(x)\,dx \le \int_{-1}^{1} dx + 2\int_1^{+\infty} \frac{dx}{|x|^p} < +\infty$ (since $p>1$).

thus $f \in L^p$ (any $p>1$).

but $\int_{\mathbb{R}} f\,dx = \int_{-\infty}^{+\infty} \left|\frac{\sin x}{x}\right| dx = +\infty$. i.e. $f \notin L^1(\mathbb{R})$.

$\int_{-\infty}^{+\infty} \frac{\sin x}{x}\,dx$ conv. as Riemann integral. but Lebesgue integral not exist. $\int_{\mathbb{R}} \left(\frac{\sin x}{x}\right)_{\pm} dx = +\infty$

$\int_{-\infty}^{+\infty} \left|\frac{\sin x}{x}\right| dx$. not Lebesgue integrable. (Lebesgue integral exist.)

Thm 3.7. (Completeness of L^p-space). pre: $(X, \mathcal{A}, \mu)$.

1. Let $p \in (1, +\infty) \subset \mathbb{R}$. $(f_n)_{n=1}^{\infty}$ s.t. $f_n \in L^p(\mu)$.
   if $\sum_{n\ge1} \|f_n\|_p < +\infty$ then series $\sum_{n\ge1} f_n$ conv. a.e. in X and $\exists\, g \in L^p(\mu)$, s.t. $f_n \to g$ in $\|\ \|_p$.

2. For any $p \ge 1$, $L^p(\mu)$ is Banach space for norm $\|\cdot\|_p$.
   (More precisely, $\forall$ Cauchy sequence (f_n) in $L^p(\mu)$. $\exists f \in L^p(\mu)$ s.t. $f_n \to f$ in $L^p(\mu)$.)
   in addition, $\exists (f_{n_k}) \subset (f_n)$. s.t. $\lim_{k\to+\infty} f_{n_k}(x) = f(x)$ for almost all $x \in X$.

Pf: 1. Follow the proof of Thm 3.2.

Let $h(x) = \sum |f_n(x)| \in [0, +\infty]$, $x \in X$.

$h(x)^p = \lim_{N\to+\infty} \left(\sum_{n\ge1}^{N} |f_n(x)|\right)^p$ (by continuity of $y \mapsto y^p$).

h^p has non-decreasing limit of non-negative function. by thm 2.5 (B.Levy. mono. conv. thm).

$\left(\int_X h^p\,d\mu\right)^{\frac{1}{p}} = \lim_{N\to+\infty} \left(\int_X \left(\sum_{n\ge1}^{N} |f_n|\right)^p d\mu\right)^{\frac{1}{p}} = \lim_{N\to+\infty} \left\|\sum_{n\ge1}^{N} |f_n|\right\|_p \underset{\text{tri. inequal.}}{\le} \sum_{n=1}^{\infty} \|f_n\|_p \underset{\text{assumption.}}{<} +\infty$

h^p, also h is finite a.e. on X.

$f(x) = \sum_{n\ge1} f_n(x)$ conv. abs. almost everywhere.

since $\sum f_n \to f$ on X (a.e.) and $|f(x)| \le h(x)$. by Lemma 2.15. $f \in L^p(\mu)$.

and $\|f - \sum_{n\ge1}^{N} f_n\|_p = \|\sum_{n=N}^{\infty} f_n\|_p \le \sum_{n=N}^{\infty} \|f_n\|_p \to 0$ (数项级数收敛的必要条件).

2. repeat the proof of Thm 3.3. (the proof of 1. ensures we can use thm 3.2 in L^p).

thm (L^p dominated conv. thm). Let $\{f_n\} \in S(E)$. $f_n \to f$ p.w. a.e. on E.
For $p \in [1, +\infty)$. $\exists g \in L^p(E)$. s.t. $|f_n| \le g$ for all $n \in \mathbb{N}$. a.e. we have $f_n \to f$ in $L^p(E)$

Def. 3.8 A measurable function $f: X \to \mathbb{C}$ is essentially bounded by $M \geq 0$ if $\mu(\{x \mid |f(x)| > M\}) = 0$.

Lemma 3.9. 1) Let f be measurable on X. $\|f\|_\infty = \inf\{M : f \text{ is essentially bounded by } M\} \in [0, +\infty]$

Then f is essentially bounded by $\|f\|_\infty$. (infimum is attained)

2) Moreover $L^\infty(\mu) = \{f \mid \|f\|_\infty < +\infty\}/_N$ normed vector space with $\|\cdot\|_\infty$.

3) if $f \in L^1(\mu)$, $g \in L^\infty(\mu)$, we have $fg \in L^1(\mu)$ and $\int_X |fg| \, d\mu \leq \|f\|_1 \|g\|_\infty$

Pf. 1) $\{x : f(x) > \|f\|_\infty\} = \bigcup_{n=1}^{\infty} \{x : f(x) > \|f\|_\infty + \frac{1}{n}\}$.

$\|f\|_\infty + \frac{1}{n}$ is essential bound $\Rightarrow \mu\{x : f(x) > \|f\|_\infty\} = \mu e_n = 0$.

3) $$\int_X |fg| \, d\mu = \int_{|g| \leq \|g\|_\infty} |fg| \, d\mu + \underbrace{\int_{|g| > \|g\|_\infty} |fg| \, d\mu}_{0} \leq \|g\|_\infty \int |f| \, d\mu = \|g\|_\infty \|f\|_1$$

2) $f, g \in L^\infty(X, \mu)$ denote $E_f = \{x : |f(x)| > \|f\|_\infty\}$, E_g similarly.

$\mu\{x : |(f+g)(x)| > \|f\|_\infty + \|g\|_\infty\} \leq \mu\{x : |f(x)| + |g(x)| > \|f\|_\infty + \|g\|_\infty\}$.

$\leq \mu(E_f \cup E_g) = 0$. ($x \notin E_f \cup E_g \Rightarrow |f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$.
$|f+g| > \|g\|_\infty + \|f\|_\infty \Rightarrow x \in E_f \cup E_g$).

Lemma 3.10. (Completeness of L^∞). pre: $(X, \mathcal{A}, \mu)$

1. Assume $(f_n)_{n=1}^\infty \subset S(X)$. $f_n \in L^\infty(\mu)$ if $\sum \|f_n\|_\infty < +\infty$, the series $\sum_{n=1}^\infty f_n$ conv. μ-a.e. and in L^∞ to function $g \in L^\infty(\mu)$. (几乎处处收敛和依范数收敛)

2. The space $L^\infty(\mu)$ is Banach space. (i.e. $\forall$ Cauchy's $(f_n)_{n=1}^\infty \subset L^\infty(\mu)$. $\exists f \in L^\infty(\mu)$ s.t. $f_n \to f$ in $L^\infty(\mu)$ a.e.).

Pf. 1) denote $h(x) = \sum_{n=1}^\infty |f_n(x)|$ $E_n = \{x : |f_n(x)| > \|f_n\|_\infty\}$. $\mu E_n = 0$. $g(x) = \sum_{n=1}^\infty f_n(x)$

$h(x) \leq \sum_{n=1}^\infty \|f_n\|_\infty < +\infty$. $h \notin \bigcup_{n=1}^\infty E_n \Rightarrow \underset{\swarrow \text{Thm 3.2.}}{h < +\infty}$ a.e.

$|g(x)| \leq h(x) < +\infty$ and $g(x) = \sum_{n=1}^\infty f_n(x)$ is conv. a.e. $\Rightarrow g \in L^\infty(\mu)$ (Lemma 2.15).

$$\|g - \sum_{n=1}^N f_n\|_\infty = \|\sum_{n=N+1}^\infty f_n\|_\infty \leq \sum_{n=N+1}^\infty \|f_n\|_\infty \xrightarrow[N \to +\infty]{} 0$$

2). $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty$, $x \notin e_{m,n}$. $\mu e_{m,n} = 0$. (by essential bounded of $f_n - f_m$)

for all $x \notin A$. $(f_n(x))$ is Cauchy's in $\mathbb{C} \Rightarrow \exists f(x) = \lim_{n \to \infty} f_n(x)$,

check $f \in L^\infty$: Fix $\varepsilon > 0$. $\exists N = N(\varepsilon)$, $\forall n, m > N$. $\|f_n - f_m\|_\infty < \varepsilon$.

$\forall x \notin A$, $|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon \Rightarrow f_n \rightrightarrows f$ on $X \setminus A$.

$$\underset{\substack{\Downarrow \\ \text{inf essential bound}}}{\|f_n - f\|_\infty} \leq \underset{\substack{\Downarrow \\ \text{essential bound.}}}{\sup_{x \notin A} |f_n - f|} < \varepsilon \Rightarrow f_n \to f \text{ in } L^\infty(\mu)$$

§. Comparison of Lebesgue spaces.

Thm 3.12. (Comparsion of L^p space for finite measure). pre: $(X, \mathcal{A}, \mu)$. $\mu X < +\infty$.

For any p_1, p_2 with $1 \le p_2 \le p_1 \le +\infty$, there exists a cont. inclusion map $L^{p_1}(\mu) \hookrightarrow L^{p_2}(\mu)$

and $\|f\|_{p_2} \le \mu(X)^{\frac{1}{p_2} - \frac{1}{p_1}} \|f\|_{p_1}$ (for any $f \in L^{p_1}$, $\Rightarrow f \in L^{p_2}$)

Pf: Assume $f \ge 0$ (can be simply generalize). for any $p \in [1, +\infty]$. $\frac{1}{p} + \frac{1}{q} = 1$.

$$\int_X |f|^{p_2} d\mu \le \int_X f^{p_2} \cdot 1 \, d\mu \le \left(\int f^{pp_2} d\mu\right)^{\frac{1}{p}} \left(\int 1^q d\mu\right)^{\frac{1}{q}}$$

$p = \frac{p_1}{p_2} \ge 1$. with $q^{-1} = \frac{p_2}{p_1} - 1$. $\Rightarrow \int_X f^{p_2} d\mu \le \|f\|_{p_1}^{p_2} \cdot \mu(X)^{\frac{p_2}{p_1} - 1}$

$\Rightarrow \|f\|_{p_2} \le \mu(X)^{\frac{1}{p_2} - \frac{1}{p_1}} \|f\|_{p_1}$

if $p_1 = +\infty$. $\|f\|_{p_2} = \left(\int |f|^{p_2} d\mu\right)^{\frac{1}{p_2}} \underset{|f| \le \|f\|_\infty}{\le} \|f\|_\infty (\mu X)^{\frac{1}{p_2}}$

Remark: $\mu X < +\infty$. $L^1(X) \supsetneq L^2(X) \supsetneq \cdots \supsetneq L^\infty(X)$.

e.g. $\frac{1}{x^{\frac{2}{3}}} \in L^1[0,1] \notin L^2(0,1)$

Lemma: $(X, \mathcal{A}, \mu)$. $L^2(X, \mu)$ is a Hilbert space (complete space with scalar product).

$\langle f, g \rangle = \int_X f(x) \overline{g(x)} d\mu$. Moreover, $\underset{\text{(Cauchy. inequality)}}{|\langle f, g \rangle| \le \|f\|_2 \|g\|_2}$. $\langle f, g \rangle = \overline{\langle g, f \rangle}$

Review: in $\mathbb{C}^n$: $\langle x, y \rangle = \sum_{k=1}^{n} x_k \overline{y_k}$ in ℓ^2: $\langle \{a_k\}, \{b_k\} \rangle = \sum_{n=1}^{\infty} a_k \overline{b_k}$

Chapter 4. Integration w.r.t. Lebesgue measure.

consider measure space $(\mathbb{R}^n, \mathcal{A}_n, \mu)$.

e.g. $(\mathbb{R}, \mathcal{A}, \mu)$. $\mathcal{A}, \mu$ is Lebesgue's. a method to check integrability of f on X.

find X_1, X_2 s.t. $X_1 \cup X_2 = X$, $X_1 \cap X_2 = \phi$. find g_1, g_2 s.t.

$$f(x) \leq \begin{cases} g_1(x) & x_1 \in X_1 \\ g_2(x) & x \in X_2 . \end{cases} \quad \text{if } g_1(x) \in L^1(X_1), g_2(x) \in L^1(X_2), \text{ thus } f \in L^1(X).$$

Thm4.1. $f \in R[a,b] \Rightarrow f \in L^1[a,b]$ and $(L)\int_a^b f \, d\mu = (R)\int_a^b f \, dx$..

Pf: consider a partition $a = x_0 < x_1 < \cdots < x_n = b$.

$$S_+(f) = \sum_{i\geq 1}^{n-1} (y_i - y_{i-1}) \sup_{y_{i-1} \leq x \leq y_i} f(x) = \int \sum_{k=1}^{n} M_k \chi_{[y_k, y_{k+1}]}$$

$$S_-(f) = \sum_{i\geq 1}^{n-1} (y_i - y_{i-1}) \min_{y_{i-1} \leq x \leq y_i} (f(x)) = \int \sum_{k=1}^{n} m_k \chi_{[y_k, y_{k+1}]}$$

we have $S_-(f) \leq (L)\int f \leq S_+(f)$ $\quad S_-(f) \leq (R)\int f \leq S_+(f)$.

Thm4.2. (Restatement of integrability of Riemann Integral).

Let $f: [a,b] \to \mathbb{C}$ be any function. Then f is Riemann-integrable iff f is bound and $\mu\{\text{discontinunity point}\} = 0$ w.r.t Lebesgue measure.

Thm4.3. Let $I = [a, +\infty)$ and $f: I \to \mathbb{C}$. s.t. $f \in R_{loc}[a, +\infty)$ and $\int_a^{+\infty} f \, dx$ conv. abs.

Then $f \in L^1(I, \mu)$ and $\int_a^{+\infty} f(x)\, dx = \int_I f \, d\mu$.

Pf: $X_n = [a, a+n)$ $\quad f_n = f \cdot \chi_{[a, a+n)}$, $(L)\int_I f_n \, d\mu = (R)\int_a^{a+n} f$.

Then $|f_n| \leq |f_{n+1}| \to f$. by Levy. thm. $\int_I |f| d\mu = \lim_{n\to\infty} \int_I |f_n| \, d\mu = \lim_{n\to\infty} \int_a^{a+n} |f| dx$

thus $f \in L^1(I, \mu)$. $\qquad = \int_a^{+\infty} |f| \, dx < +\infty$.

Remark: for $f \in R_{loc}[a,b)$. and $f \underset{x\to b}{\to} \infty$. if $\int_a^b f(x)\, dx$ conv. abs. we have same result.

§. Multiple Lebesgue Integral.

*提供了一个判定基准.

Thm4.5. Let B be a ball in $\mathbb{R}^m$ of radius r, center a. Give $q>0$.
set $f(x) = \frac{1}{\|x-a\|^q}$ for $x \in \mathbb{R}^m \setminus \{a\}$. Then:

1. $f \in L^1(B)$ iff $q<m$
2. $f \in L^1(\mathbb{R}^m \setminus B)$ iff $q>m$.

Pf: denote: $E(R) = \{x \in \mathbb{R}^m \mid \frac{R}{2} \le \|x-a\| \le R\}$. $x \in E(R)$, $\frac{1}{R^q} \le \frac{1}{\|x-a\|^q} \le \frac{2^q}{R^q}$

$\mu E(R) = \alpha_m R^m - \alpha_m \left(\frac{R}{2}\right)^m = \alpha_m (2^m-1)\left(\frac{R}{2}\right)^m$ α_m is the volume of unit radius ball.

partition the Ball $B(a,R) = \{a\} \cup \bigcup_{k=0}^{\infty} E(2^{-k}R)$ $\mu E(2^{-k}R) = \alpha_m (2^m-1)\cdot\left(\frac{R}{2^k}\right)^m$

$$\int_{B(a,R)} \frac{dx}{\|x-a\|^q} = \sum_{k=0}^{\infty} \int_{E(2^{-k}R)} \frac{dx}{\|x-a\|^q} \le \sum_{k=0}^{\infty} \frac{\mu(E(2^{-k}R))\, 2^q}{(2^{-k}R)^{mq}} = 2^q \cdot \alpha_m (2^{m-1}) \cdot \sum_{k=0}^{\infty} \frac{(2^{-k}R)^m}{(2^{-k}R)^{mq}}$$

$$= 2^q \alpha_m (2^m-1)\cdot R^{m(1-q)} \sum_{k=0}^{\infty} 2^{-k(m-q)} < +\infty \text{ (when } q<m)$$

$$\int_{B(a,R)} \frac{dx}{\|x-a\|^q} \ge \alpha_m R^{m(1-q)} \sum_{k=0}^{\infty} 2^{-k(m-q)}$$

coro4.5.1. In the same condition as thm4.5. 1) $f \in L^p(B) \Leftrightarrow q < \frac{m}{p}$ 2) $f \in L^p(\mathbb{R}^m \setminus B) \Leftrightarrow q > \frac{m}{p}$

Pf: notice that $f \in L^p(X)$ iff $|f|^p \in L^1(X)$.

Remark: method to test integrability. (w.r.t. Lebesgue measurable)

1) Tonelli theorem. (multiple integral → iterate integral). *前提是 $f \in S(E \to [0,+\infty])$.

e.g: Let $E(x) = \{y \in \mathbb{R} : (x,y) \in E\}$.

$$\iint_E |f(x,y)|\,dx\,dy = \int_{\mathbb{R}} dx \int_{E(x)} |f(x,y)|\,dy.$$ LHS is finite $\Leftrightarrow$ RHS is finite.

2). change variable. Suppose $\Phi: G \to V$ is diffeomorphism of G and $V \subset \mathbb{R}^n$. $E \subset G$ is meas.

Then $$\iint_{\Phi(E)} |f|\,d\mu_n = \iint_E |f\circ\Phi|\,|\det \Phi'|\,d\mu_n$$ LHS is finite $\Leftrightarrow$ RHS is finite.

Remark: $\int_E f\,d\mu$ is finite $\Leftrightarrow \int_E |f|\,d\mu$ is finite.

($\int_E f\,d\mu = \int_E f_+ - \int_E f_-$ and $\int_E |f|\,d\mu = \int_E f_+ + \int_E f_-$ if both infinite then $\nexists \int_E f\,d\mu$).

Problems.

— Find condition. ( ) that integrals are finite.

e.g. 4.1. $I = \iint_{|x|+|y|\geq 1} \frac{dx\,dy}{|x|^p + |y|^q}$

f is conv. and even. (w.r.t. every variable) and bounded on compacts. that do not contain 0. (有限性等价时. 积分域可自由放缩)

$\{(x,y) \mid |x|+|y| \geq 1\} \Rightarrow \{(x,y) \mid x^p + y^p \geq 1,\ x,y > 0\}$ (same integrability).

then consider polar change $\begin{cases} x = u^{\frac{2}{p}} \\ y = v^{\frac{2}{q}} \end{cases}$ $\quad |J| = \frac{4}{pq} u^{\frac{2}{p}-1} v^{\frac{2}{q}-1}$.

$$I = \frac{4}{pq} \int_1^{+\infty} r^{2(\frac{1}{p}+\frac{1}{q})-3}\,dr \underbrace{\int_0^{\frac{\pi}{2}} \cos^{\frac{2}{p}-1} t \sin^{\frac{2}{q}-1} t\,dt}$$

Beta. func. $B(\frac{m+1}{2}, \frac{n+1}{2}) = 2\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\,dx$ $\qquad$ finite iff $p,q>0$, $\begin{cases} \frac{2}{p}-1 > -1 \\ \frac{2}{q}-1 > -1 \end{cases}$.

$\int_1^{+\infty} r^{2(\frac{1}{p}+\frac{1}{q})-3}\,dr$. finite iff. $\frac{1}{p}+\frac{1}{q} < 1$.

e.g. 4.2. $I = \iint_{x^2+y^2\leq 1} \frac{\varphi(x,y)}{(1-x^2-y^2)^p}\,dx\,dy$. $\quad 0 < m < \varphi(x,y) < M$. $\varphi(x,y)$ is cont.

(放缩部分函数. 成 $m g(x,y) \leq f(x,y) \leq M g(x,y)$. g为更易积分的函数. 做倍数夹逼)

denote. $\tilde{I} = \iint_{x^2+y^2\leq 1} \frac{dx\,dy}{(1-x^2-y^2)^p}\,dx\,dy$. $\qquad m\tilde{I} \leq I \leq M\tilde{I}$.

$$\tilde{I} = \int_0^{2\pi} d\theta \int_0^1 \frac{r\,dr}{(1-r^2)^p} = 2\pi \int_0^1 \frac{r\,dr}{(1-r^2)^p} \xlongequal{s=1-r^2} 2\pi \int_0^1 \frac{ds}{s^p}.$$ conv. iff $p<1$.

Base:

$$\begin{cases} \int_0^1 \frac{dr}{r^p}. & p\geq 1 \text{ div.} \quad p<1 \text{ conv.} \\ \int_1^{+\infty} \frac{dr}{r^p} & p\leq 1 \text{ div.} \quad p>1 \text{ conv.} \end{cases}$$

§. Integration and continuous functions. (Chapter 5*)

Def 5.1. Pre. $(X, \mathcal{T})$. $f \in C(X \to \mathbb{C})$.

the support of f. ($\mathrm{supp}(f)$) is the closed subset $\mathrm{supp}(f) = \overline{\{x \mid f(x) \neq 0\}} \subset X$.

if $\mathrm{supp}(X) \subset X$ is compact. f is compactly supported. (denote $f \in C_c(X)$)

$C_c(X)$: space of compactly supported continuous functions on X.

1) if X is compact. $C_c(X) = C(X)$. otherwise, $C_c(X) \neq C(X)$. ($f \equiv c \neq 0 \in C(X)$. but $f \in C_c(X)$ only if X is compact).

2) $C_c(X)$ is vector space

Check $\mathrm{supp}(\alpha f + \beta g) \subset \mathrm{supp}(f) \cup \mathrm{supp}(g)$

$\alpha f + \beta g \neq 0 \Rightarrow f \neq 0$ or $g \neq 0 \Rightarrow x \in \mathrm{supp}(f) \cup \mathrm{supp}(g)$

$\Rightarrow \{\alpha f + \beta g \neq 0\} \subset \underbrace{\mathrm{supp}(f) \cup \mathrm{supp}(g)}$ $\to$ closed $\Rightarrow \mathrm{supp}(\alpha f + \beta g) \subset \mathrm{supp}(f) + \mathrm{supp}(g)$

3) if $f \in C_c(X)$. $\Rightarrow \sup f < +\infty$ (bounded on X).

since $|f(x)| \leq \sup\limits_{x \in \mathrm{supp}(f)} |f(x)| = \max\limits_{x \in \mathrm{supp}(f)} |f(x)|$ (in compact set $\exists \max f$).

denote $\|f\|_\infty = \sup\{f(x) \mid x \in X\}$. for $f \in C_c(X)$. a norm on $C_c(X)$.

Def. 5.2. A t.s. is locally compact if every point has a neighborhood which closure is compact.

Thm 5.3. Pre $(X, \mathcal{T})$ locally compact.

1). $\forall$ compact set $K \subset X$. and $\forall$ open neighborhood V. s.t. $K \subset V \subset X$.

$\exists f \in C_c(X)$. s.t. $\chi_K \leq f \precsim \chi_V$. ($f \precsim V \Rightarrow \begin{cases} f \leq \chi_V \\ \mathrm{supp} f \subset V \end{cases}$).

2) Urysohn's Lemma. Let K_1, K_2 be disjoint compact subsets of X. $\exists f \in C_c(X)$ s.t. $0 \leq f \leq 1$ and $f(x) = \begin{cases} 0 & \text{if } x \in K_1 \\ 1 & \text{if } x \in K_2. \end{cases}$

Pf: 1). prove the case when X is metric space.

$\exists W$ open, $\overline{W}$ compact $K \subset W \subset \overline{W} \subset V$. $F = X \setminus W$.

$$f(x) = \frac{d(x, F)}{d(x, F) + d(x, K)} \quad (d(x, F) = \inf\{d(x, y),\ y \in F\}).$$

① $\mathrm{supp} f \subset \overline{W} \subset V$. (if $f(x) = 0 \Rightarrow d(x, F) = 0. \Rightarrow d(x, K) > 0. \Rightarrow x \notin K \Rightarrow x \notin W$).

$f(x) \leq 1$. $x \in V$, $f(x) \leq \chi_V$.

$x \notin V$. $x \in X \setminus W$. $d(x, F) = 0$. $f(x) = 0 = \chi_V$.

② $x \in K$. $x \notin F$ ~~and~~, $d(x, K) = 0$ $d(x, F) > 0$. $\Rightarrow f(x) = 1$.

2) Let $K_2 = K$. and V be open neighbor of K_2 s.t. $V \cap K_1 \neq \phi$. and $K_1 \subset X \setminus V$

$$f(x) = \frac{d(x, K_1)}{d(x, K_1) + d(x, K_2)}$$

Thm 5.3. 3) Let $K \subset X$. K-compact. $V_1, \dots V_n$ - open s.t. $K \subset V_1 \cup V_2 \cup \dots \cup V_n$.

For any $g \in C_c(X)$. there exists $g_i \in C_c(X)$. s.t. $\mathrm{supp}(g_i) \subset V_i$ for all i.

and $\sum_{i=1}^{n} g_i(x) = g(x)$ for all $x \in K$.

Pf: $\exists W_i$-open. $\overline{W_i} \subset V_i$ s.t. $K \subset W_1 \cup \dots \cup W_n$.

1) construct $0 \le f_i \prec \chi_{V_i}$. and. $\sum_{i=1}^{n} f_i(x) = 1$:

$\tilde{f_i}$: $\tilde{f_i}(x) = 1 \quad x \in \overline{W_i} \cap K$.

$\tilde{f_i}(x) = 0. \quad x \notin V_i$

$f_i = \dfrac{\tilde{f_i}}{\sum_{j=1}^{i} \tilde{f_j}}$ $\quad \mathrm{supp}(f_i) \subset V_i$.

2) $g_i = g f_i$. and $g_i \le \chi_{V_i}$

Def.5.4. $(X, \mathcal{T})$ is Locally compact if every point has neighborhood V s.t. $\overline{V}$ is compact

Def (regular measure). μ is regular if $\forall E$-meas. $\forall \varepsilon > 0$. $\exists K$-compact, G-open s.t. $K \subset E \subset G$. $\mu(E \setminus K)$, $\mu(G \setminus E) < \varepsilon$

Def 5.5. (radon measure) X is l.c.t.s. radon measure μ is regular Borel measure and finite on compact set. (i.e. $\forall K$-compact $\subset X$, $\mu K < +\infty$).

Def 5.6. (dense) a set of function $V \subset L^p(X, \mu)$ is dense is $L^p(X, \mu)$ if: $\forall f \in L^p(X, \mu) \forall \varepsilon > 0$. $\exists g \in V$. s.t. $\|f - g\|_p \le \varepsilon$

Lemma 5.7 Let μ be a Radon measure on l.c.t.s. X. and $1 \le p < \infty$. Then any $E \subset X$ with $\mu E < +\infty$, and any $\varepsilon > 0$. $\exists f \in C_c(X)$. s.t. $\|f - \chi_E\|_p < \varepsilon$

Pf: by regularity: $\exists K \subset E \subset G$. $\mu(G \setminus K) < \varepsilon^p$.

by lemma 5.3.1. $\exists f \in C(X)$. s.t. $\chi_K \le f \le \chi_G$ and $\mathrm{supp} f \subset G$.

$\|f - \chi_E\|_p^p = \int_X |f - \chi_E|^p d\mu \le \int_{G \setminus K} |f - \chi_E|^p d\mu \le \mu\, G \setminus K < \varepsilon^p$

Thm 5.8. (Density of continuous function in L^p). μ - Radon measure. X - l.c.t.s. $p \in [1, +\infty)$.

TFAS: (1) The space of step function is dense in $L^p(x, \mu)$.

(2) The space $C_c(X)$ is dense in $L^p(X, \mu)$.

Pf: (1) $\forall f \in L^p(X, \mu)$. $\exists \psi_n$ - step. s.t. $|\psi_n| \le |f|$ $\psi_n \to f$ p.w.

By Lebesgue thm. (thm 2.5. in L^p) $\int |\psi_n - f|^p d\mu \to +\infty$.

(2) $\forall f \in L^p(X, \mu)$. $\forall \varepsilon > 0$. let $\psi = \sum_{k=1}^{n} c_k \chi_{E_k}$ s.t. $\|f - \psi\| < \frac{\varepsilon}{2}$.

For every $k \in \{1, 2, \dots n\}$. $\exists g_k \in C_c(X)$ s.t. $\|g_k - \chi_{E_k}\| < \frac{\varepsilon}{n 2 c_k}$ (by lemma 5.7).

let $g = \sum_{k=1}^{n} c_k g_k$ $\quad \|g - \psi\|_p \le \sum_{k=1}^{n} c_k \|\chi_{E_k} - g_k\|_p < \frac{\varepsilon}{2}$. $\Rightarrow \|g - f\|_p < \varepsilon$.

($g \in C_c(X)$. since $C_c(X)$ is a vector space).

Chapter 6. Convergence in measure.

premise. $(X, \mathcal{A}, \mu)$. $L^0(E) = L^0(E,\mu) = \{f: E \to \mathbb{R},\ f$ is meas. f is finite a.e.$\}$.

$E(f>a) = \{x \in E : f(x) > a\}$.

Def. (types of convergence on E).

1). p.w. conv. ($f_n \to f$). if $f_n(x) \underset{n\to\infty}{\to} f(x)$ for every $x \in E$.

2). uni. conv. ($f_n \rightrightarrows f$). if $\sup_{x\in E} |f_n(x) - f(x)| \underset{n\to\infty}{\longrightarrow} 0$

3) conv. a.e. ($f_n \to f$ a.e.) if $\exists e \subset E$. s.t. $\mu e = 0$. $f_n(x) \underset{n\to\infty}{\to} f(x)$ for every $x \in E \setminus e$.

4). conv. in measure. ($f_n \xrightarrow{\mu} f$). if $\forall \varepsilon > 0$. $\mu(E(|f_n - f| > \varepsilon)) \underset{n\to\infty}{\longrightarrow} 0$

4) subse $\Rightarrow$ 3).

Remark. 2) $\Rightarrow$ 1). 2) $\Rightarrow$ 4). 1) $\not\Rightarrow$ 4). 4) $\not\Rightarrow$ 3). 3) $\overset{\mu X < +\infty}{\Longrightarrow}$ 4).

2) $\Rightarrow$ 4). $f_n \rightrightarrows f$. $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$. s.t. $\forall n > N$. $\forall x \in E$. $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$. i.e. $E(|f_n - f| > \varepsilon) = \phi$.

$\mu E(|f_n - f| > \varepsilon) = 0$. for $n > N$.

1) $\not\Rightarrow$ 4). consider $E = \mathbb{R}$. and func. $\chi_{(n,n+1)}$.

$f_n = \chi_{(n,n+1)} \to 0$ p.w. $\mu E(|f_n - f| > \frac{1}{2}) = 1$.

4) $\not\Rightarrow$ 3) (even in finite measure) $E = \mathbb{R}$. μ. 1-dim Lebesgue measure.

for every $k \in \mathbb{N}$. $\Delta(k,p) = [\frac{p}{2^k}, \frac{p+1}{2^k})$. $p = 0, 1, \dots, 2^k - 1$. $f_n = \chi_{\Delta(k,p)}$. $k = [\log_2 n]$

$X(f_n \neq 0) = \Delta(k,p)$. $\mu(\Delta(k,p)) = \frac{1}{2^k} \leq \frac{2}{n} \to 0$.

$\{f_n\}$. no limits. (infinitely many 1 and 0 for any $x \in [0,1)$).

Thm 6.6. (Lebesgue). If $\mu X < +\infty$. conv. a.e. implies conv. in measure.

Pf: Let $f_n \underset{n\to\infty}{\xrightarrow{a.e.}} f$ on X. $f_n \underset{n\to\infty}{\to} f$ on $X \setminus e$. $\mu e = 0$.

redefine on e. by 0. $f_n \to f$ on X. p.w.

case 1. $\{f_n\}$ is monotonic. and conv. to $f = 0$.

$\forall \varepsilon > 0$. set $E(|f_n| \geq \varepsilon)$ decrease. i.e. $E(|f_1| > \varepsilon) \supset E(|f_2| > \varepsilon) \supset \cdots$

and $\bigcap_{n=1}^{\infty} E(|f_n| > \varepsilon) = \phi$.

$\mu(E(|f_n| \geq \varepsilon)) \underset{n\to\infty}{\longrightarrow} \mu\phi = 0$. (upper continuity of measure. premise: $\mu X < +\infty$).

case 2. $f_n \underset{n\to\infty}{\to} f$ for all $x \in X$. let $\varphi_n(x) = \sup_{k \geq n} |f_k(x) - f(x)|$

$\varphi_n(x) \underset{n\to\infty}{\to} 0$ monotonically. by case 1. $\mu E(\varphi_n > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$

since $E(|f_k(x) - f(x)| > \varepsilon) \subset E(\varphi_n > \varepsilon) \to 0$.

$\mu E(|f_k(x) - f(x)| > \varepsilon) \leq \mu E(\varphi_n > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$

Lemma 6.7. (Borel–Cantelli) Let $\{E_n\}_{n\geq 1}$ be a sequence of measurable sets and $E = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k = \{x \in X \mid x \in E_n \text{ for infinitely many } n\}$. If $\sum_{n=1}^{\infty} \mu(E_n) < +\infty$ then $\mu E = 0$.

Pf: since $E \subset \bigcup_{n=k}^{\infty} E_k$, we have $\mu E \leq \sum_{n\geq k}^{\infty} \mu E_n \xrightarrow[k\to\infty]{} 0$.

coro 6.7.1. Let $\varepsilon_n > 0$. $\varepsilon_n \xrightarrow[n\to\infty]{} 0$, $g_n \in L^0(X,\mu)$. and $X_n = X(|g_n| > \varepsilon_n)$. $\sum \mu X_n < +\infty$,

then: 1. $g_n \xrightarrow[n\to\infty]{a.e.} 0$.

2. $\forall \varepsilon > 0$. $\exists e \subset X$. s.t. $\mu e < \varepsilon$. and $g_n \rightrightarrows 0$. on $X \setminus e$.

Pf: 1. $E = \{x \mid \exists \varepsilon > 0, \exists \{n_k\}; |g_{n_k}(x)| > \varepsilon\} = \bigcup_{\varepsilon > 0} E_\varepsilon$.

$\exists N: \forall k \geq N.\ \varepsilon_k < \varepsilon$
$E_k \subset X_k$.

fix $\varepsilon > 0$. $E_\varepsilon = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} X(|g_k| > \varepsilon)$ and $\sum_{k\geq N}^{\infty} \mu X(|g_k| > \varepsilon) < +\infty$

by Lemma 6.7, $\mu E_\varepsilon = 0$. $E = \bigcup_{m\geq 1}^{\infty} E_{1/m}$.

2. fix $\varepsilon > 0$. $\exists N \in \mathbb{N}$. $\sum_{k\geq N}^{\infty} \mu X_k < \varepsilon$. put $e = \bigcup_{n\geq N}^{\infty} X_n$. $\mu e < \varepsilon$.

then $|g_n(x)| < \varepsilon_n$. for $x \in X \setminus e$ and $n > N$.

Thm 6.8 (F. Riesz). Every sequence that conv. in measure contains a subsequence conv. a.e. to the same limit.

Pf: Let $f_n \xrightarrow[n\to\infty]{\mu} f$. then $\mu(X(|f_n - f| > \frac{1}{k})) \xrightarrow[n\to\infty]{} 0$. for every $k \in \mathbb{N}$.

$\exists$ increasing n_k. s.t. $\mu(X(|f_n - f| > \frac{1}{k})) < \frac{1}{2^k}$. for all $n \geq n_k$.

denote $g_k = |f_{n_k} - f|$. by coro 6.7.1.1. $f_{n_k} \xrightarrow[n_k\to\infty]{a.e.} f$

Remark: Moreover. $\forall \varepsilon > 0$. $\exists e \subset E$. s.t. $\mu e < \varepsilon$. and $f_{n_k} \underset{k\to\infty}{\rightrightarrows} f$ on $X \setminus e$.

Def 6.10. A sequence $\{f_n\}_{n\geq 1}$. conv. to f. almost uniformly. if for every $\varepsilon > 0$. there exists a set A_ε s.t. $\mu(A_\varepsilon) < \varepsilon$ and $f_n \underset{n\to\infty}{\rightrightarrows} f$ on $X \setminus A_\varepsilon$.

coro 6.9.1. If a sequence $\{f_n\}_{n\geq 1}$ conv. in measure to f and g. then $f = g$. a.e.

Pf: by thm 6.8. $\exists \{f_{n_k}\}$, $f_{n_k} \xrightarrow{a.e.} f$.

by thm 6.8 on $\{f_{n_k}\}$. $\exists \{f_{n_{k_j}}\}$, $f_{n_{k_j}} \xrightarrow{a.e.} g$.

$f = g$. a.e. as the limit of. a.e. convergent sequence $\{f_{n_{k_j}}\}$.

Coro 6.9.2. If $f_n \xrightarrow[n\to\infty]{\mu} f$ and $f_n \le g$ a.e. for every n. then $f \le g$ a.e on E.

Pf: $\exists \{f_{n_k}\}$, $f_{n_k} \xrightarrow[n\to\infty]{a.e.} f$

$f_{n_k} \le g$. on $E \setminus e_k$. $\mu e_k = 0$.

put $e = \bigcup_{k\ge1}^{\infty} e_k$. $f_{n_k} \le g(x)$ on $E\setminus e$. for every $k \in \mathbb{N}$. then pass the limit. $k\to\infty$.

Rem: Almost uni. conv. $\Rightarrow$ conv. a.e.

Thm. 6.11. (Egorov). Assume $f_n . f \in L^0(X, \mu)$ and that $f_n \xrightarrow[n\to\infty]{a.e.} f$.

if $\mu X < +\infty$. then $f_n \xrightarrow[n\to\infty]{} f$ almost uniformly on X.

Pf: $g_n(x) = \sup_{k\ge n} |f_k(x) - f(x)|$. then $g_n \xrightarrow[n\to\infty]{a.e.} 0$.

since $\mu X < +\infty$. by Lebesgue thm. (6.6) $g_n \xrightarrow[n\to\infty]{\mu} 0$.

thus $\exists \{g_{n_k}\}$ s.t. $\mu\left(X\left(g_{n_k} > \frac{1}{k}\right)\right) < \frac{1}{2^k}$.

$\Rightarrow \forall \varepsilon > 0$. $\exists e \subset X$. $\mu e < \varepsilon$. $g_{n_k} \underset{k\to\infty}{\rightrightarrows} 0$. on $X \setminus e$. (coro 6.7.1.2)

$\forall n \ge n_k$. $|f_n(x) - f(x)| \le |g_{n_k}(x) - g(x)| \rightrightarrows 0$. on $X\setminus e$.

thus. $|f_n - f| \rightrightarrows 0$. on $X \setminus e$.

$$\begin{array}{lr} f_1^{(1)} \; f_2^{(1)} \cdots & \to g_1 \\ f_1^{(2)} \; f_2^{(2)} \cdots f_i^{(2)} \cdots & \to g_2 \\ f_1^{(3)} \; f_2^{(3)} \cdots f_i^{(3)} \cdots f_j^{(3)} & \to g_3 \\ \cdots & \cdots \\ f_1^{(n)} \; f_2^{(n)} \cdots f_i^{(n)} \cdots f_j^{(n)} \cdots f_{(k)}^{(n)} & \to g_n \\ & \downarrow \\ & h \end{array}$$

Thm 6.12. (Diagonal sequence). Let μ be σ-finite. $f_k^{(n)} \in L^0(X, \mu)$ $g_n \in L^0(X, \mu)$.

suppose that. $f_k^{(n)} \xrightarrow[k\to\infty]{a.e.} g_n$ for every $n \in \mathbb{N}$. and $g_n \xrightarrow[n\to\infty]{a.e.} h$.

Then there exists a strictly increasing sequence of indices k_n s.t. $f_{k_n}^{(n)} \xrightarrow[n\to\infty]{a.e.} h$.

Pf: 1) assume. $\mu X < +\infty$.

then $f_k^{(n)} \xrightarrow[k\to\infty]{\mu} g_n$. (Thm 6.6). i.e. $\mu\left(X(|f_k^{(n)} - g_n| > \varepsilon)\right) \xrightarrow[k\to\infty]{} 0$ for every $n \in \mathbb{N}$. $\varepsilon > 0$.

thus for every n, $\exists \{k_n\}$. s.t. $k_n < k_{n+1}$ and $\mu\left(X\left(|f_{k_n}^{(n)} - g_n| > \frac{1}{n}\right)\right) < \frac{1}{2^n}$.

by coro 6.7.1.1. $f_{k_n}^{(n)} - g_n \xrightarrow[n\to\infty]{a.e.} 0$. thus $f_{k_n}^{(n)} = (f_{k_n}^{(n)} - g_n) + g_n \xrightarrow[n\to\infty]{a.e.} h$.

2). $\mu X = +\infty$. define a new measure $\nu(E) = \sum_{n\ge1} \frac{1}{2^n} \frac{\mu(E\cap X_n)}{\mu X_n}$. $\nu X = 1$.

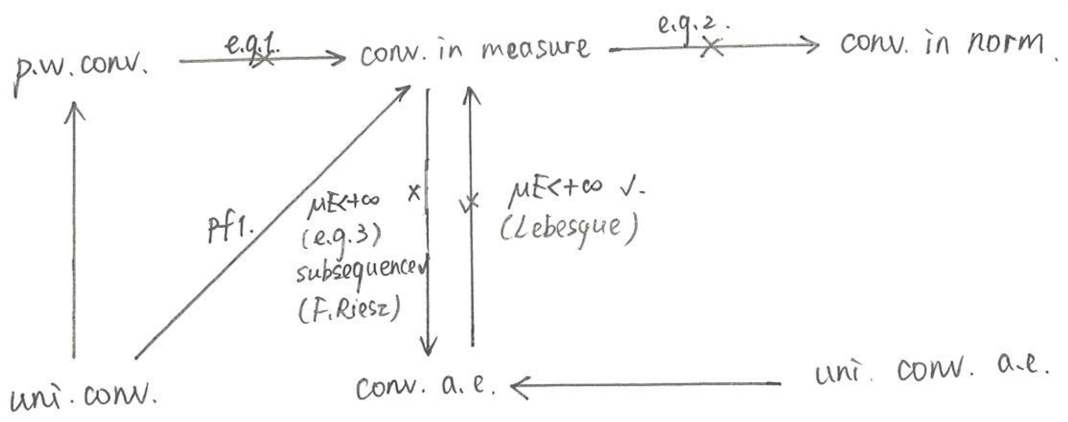
$X = \bigcup_{n=1}^{\infty} X_n$. $0 < \mu X_n < +\infty$. $\nu E = 0 \Leftrightarrow \mu E = 0$. i.e. $f_{k_n}^{(n)} \xrightarrow[n\to\infty]{a.e. on\ \nu} h \Leftrightarrow f_{k_n}^{(n)} \xrightarrow[n\to\infty]{a.e. on\ \mu} h$.

Lemma. 6.13. If μ is σ-finite. $\exists$ ν-finite. s.t. $\nu E = 0$ iff $\mu E = 0$.

Remark1: h can take infty values on positive measure. (unlike $f_k^{(n)}$ and g_n).

Remark2: conv. in measure $\not\Rightarrow$ conv. in $L^1(0,1)$ (e.g. $f_n = \begin{cases} n, & x \in [0, \frac{1}{n}] \\ 0 & , \text{other}. \end{cases}$ $f_n \xrightarrow{\mu} 0$. $\int_0^1 f_n = 1$).

Type of convergences.



p.w. conv. $f_n \xrightarrow[n\to\infty]{} f$ $\forall \varepsilon>0 \,\forall x\in E, \exists N\in\mathbb{N}$. for any $n>N$. $|f_n(x)-f(x)|<\varepsilon$.

uni. conv. $f_n \rightrightarrows f$ $\forall \varepsilon>0. \exists N\in\mathbb{N}$. for any $n>N$, $x\in E$. $|f_n(x)-f(x)|<\varepsilon$.

conv. a.e. $f_n \xrightarrow[n\to\infty]{a.e.} f$ $\exists e\subset E. \ \mu e=0.$ $f_n \xrightarrow[n\to\infty]{} f$ on $E \setminus e$.

conv. a. uni. $f_n \overset{a.e.}{\rightrightarrows} f$. $\forall \varepsilon>0. \exists A_\varepsilon \in \mathcal{A}. \ \mu A_\varepsilon<\varepsilon.$ $f_n \rightrightarrows f$ on $E\setminus A_\varepsilon$

conv. in meas. $f_n \xrightarrow{\mu} f$. $\forall\varepsilon>0.$ $\mu(E|f_n-f|>\varepsilon) \xrightarrow[n\to\infty]{} 0$

conv. in norm. f_n conv. to f in L^p. $\forall\varepsilon>0. \exists N\in\mathbb{N}$. $\|f_n-f\|_p<\varepsilon$. $p\in[1,+\infty]$

e.g. 1. $E=\mathbb{R}$. $f_n=\chi_{[n,n+1)}$

$f_n \to 0$. $\mu E(|f_n - f|>\frac{1}{2})=1$.

e.g. 2. $E=(0,1)$. $f_n=\begin{cases} n, & (0,\frac{1}{n}) \\ 0 & [\frac{1}{n},1) \end{cases}$. $L^1(E,\mu)$.

$f_n \xrightarrow{\mu} 0$. $\int_E |f_n|d\mu = 1$.

e.g. 3. $E=(0,1)$. $\mu=\lambda$ (Lebesgue). $f_n=\chi_{\Delta(k,p)}$, $k=[\log_2 n]$

$\forall k\in\mathbb{N}$. $\Delta(k,p)=[\frac{p}{2^k}, \frac{p+1}{2^k})$. $p=0,1,\dots,2^k-1$.

$E_n' = E(f_n\neq 0) = \Delta(k,p)$. $\mu(\Delta(k,p))=\frac{1}{2^k}\le\frac{2}{n}\to 0$.

$f_n \xrightarrow{\mu} 0$. f_n no limit.

Pf1. $f_n\rightrightarrows f$: $\forall\varepsilon>0. \exists N\in\mathbb{N}$. s.t. $\forall n>N$. $\forall x\in E$. $|f_n-f|<\frac{\varepsilon}{2}$.

i.e. $E(|f_n-f|>\varepsilon)=\emptyset$. (for $n>N$).

Chapter 7. Approximation of Measurable Function by Continuous Functions.

Premise : measurable function on $\mathbb{R}^m$. Lebesgue measure λ.

Def 7.1. $A \subset \mathbb{R}^m$. $x \in \mathbb{R}^m$. the value. $\mathrm{dist}(x, A) = \inf_{y \in A} \{\|x-y\|\}$ is distance from x to A.

Lemma 7.2. The function $x \mapsto \mathrm{dist}(x, A)$. is cont. on $\mathbb{R}^m$.

Pf: Let $y \in A$. $x, x_1 \in \mathbb{R}^m$. $\|x-y\| \le \|x-x_1\| + \|x_1 - y\|$.

$\Rightarrow \mathrm{dist}(x, A) \le \mathrm{dist}(x_1, A) + \|x - x_1\|$.

$\Rightarrow |\mathrm{dist}(x, A) - \mathrm{dist}(x_1, A)| \le \|x - x_1\|$

Lemma 7.3. A closed set $F \subset \mathbb{R}^m$. $\exists f_n \in C(\mathbb{R}^m)$ s.t. $f_n(x) \to \chi_F(x)$

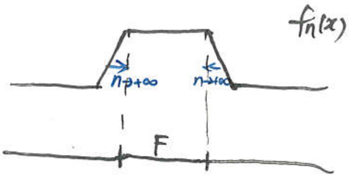
Pf: $\mathbb{R}^m \setminus F = \bigcup_{n \ge 1}^{\infty} H_n$, $H_n = \{x \in \mathbb{R}^m \mid \mathrm{dist}(x, F) \ge \frac{1}{n}\}$.

define $f_n(x) = \dfrac{\mathrm{dist}(x, H_n)}{\mathrm{dist}(x, F) + \mathrm{dist}(x, H_n)}$, $x \in \mathbb{R}^m$. $f_n \in C(\mathbb{R}^m)$

by Lemma 7.2. and $\mathrm{dist}(x, F) + \mathrm{dist}(x, H_n) \ne 0$. ($x$ can't belongs to F, H_n simultaneously)

$x \in F$. $f_n(x) = \dfrac{\mathrm{dist}(x, H_n)}{\mathrm{dist}(x, H_n)} \le 1$.

$x \notin F. \Rightarrow \exists N$, $x \in H_n$ for $n > N$. $f_n(x) = 0$.



Coro. Let $E \subset \mathbb{R}^m$. E is measurable. $\Rightarrow \exists f_n \in C(\mathbb{R}^m)$. $f_n \to \chi_E$ a.e.

(Pf is the first part of Pf. thm 7.4).

Thm 7.4. (Fréchet). Every (Lebesgue) measurable function. f on $\mathbb{R}^m$ is the limit of a sequence of continuous functions. conv. a.e.

Pf: 1). $f = \chi_E$. E is a measurable set.

$E = e \cup \bigcup_{n \ge 1}^{\infty} K_n$ $\lambda e = 0$. $\{K_n\}$. s.t. K_n - compact and $K_n \subset K_{n+1}$ (regularity of λ).

$\chi_{K_n} \to \chi_E$. by lemma 7.3 χ_{K_n} is the limit of a sequence of cont. func.

by diagonal thm (6.12). χ_E is also a limit of sequence of cont. func. conv. a.e.

2). f is step. $f = \sum_{k \ge 1}^{N} c_k \chi_{E_k}$ E_k - measurable. ($\forall k\ \exists f_{(n)}^{(k)} \xrightarrow[n\to\infty]{a.e.} \chi_{E_k} \Rightarrow \sum_{k=1}^{N} c_k f_n^{(k)} \xrightarrow[n\to\infty]{a.e.} f$).

approximate χ_{E_k} by cont. func.

3). f in general case. consider step. function $\{f_n\}_{n \ge 1}^{\infty}$ $f_n \to f$. p.w.

Def. (Luzin property). "almost continuity"

$\forall \delta > 0$ there exists a set $e \subset \mathbb{R}^m$ s.t. $\mu e < \delta$ and the restriction of f to $\mathbb{R}^m \setminus e$ is continuous.

Thm 7.5. (Luzin). Every Lebesgue measurable function f on $\mathbb{R}^m$ and $f \in L^0(\mathbb{R}^m)$. satisfied Luzin property.

Pf. by Frechet thm. (thm 7.4). $\exists \{f_k\} \subset C(\mathbb{R}^m)$ and $f_k \xrightarrow{a.e.} f$

denote. $E_n = \{x \in \mathbb{R}^m \mid n-1 \leq \|x\| < n\}$. (spherical layers)

by Egorov thm (thm 6.11). $\exists e_n$. s.t. $\mu(e_n) < \frac{\delta}{2^n}$ and $f_k \rightrightarrows f$ on $E_n \setminus e_n$.

$\Rightarrow f|_{E_n \setminus e_n}$ is cont. (as uniform limit of cont. func.).

Put $e = \bigcup_{n=1}^{\infty} (e_n \cup S_n)$. $S_n = \{x \in \mathbb{R}^n \mid \|x\| = n\}$. , $\mu S_n = 0$.

$\mu e < \delta$. and $f_k \rightrightarrows f$ on $\mathbb{R}^m \setminus e$. i.e. $f|_{\mathbb{R}^m \setminus e}$ is cont. on $\mathbb{R}^m \setminus e$.

Thm 7.6. A closed subset $F \subset \mathbb{R}^m$. Every function $f \in C(F)$. is the restriction of F of a function continuous on $\mathbb{R}^m$.

(i.e. $F = \bar{F} \subset \mathbb{R}^m$. $\forall f \in C(F)$. $\exists \tilde{f} \in C(\mathbb{R}^m)$ s.t. $\tilde{f}|_F = f$).

Thm 7.7. $\forall f \in L^0(\mathbb{R}^m) \cup S(\mathbb{R}^m)$ $\forall \delta > 0$. $\exists \varphi_\delta \in C(\mathbb{R}^m)$. s.t. $\mu(\{x \in \mathbb{R}^m \mid f(x) \neq \varphi_\delta(x)\}) < \delta$.

Pf: fix $\delta > 0$. and consider a set e. s.t. $\mu e < \delta$. $f|_{\mathbb{R}^m \setminus e} \in C(\mathbb{R}^m \setminus e)$ (Luzin pro.).

$\exists G$-open. $e \subset G$ and $\mu G < \delta$. (regularity of Lebesgue measure)

Let $F = \mathbb{R}^m \setminus G$. thus. we have $f|_F \in C(F)$

denote $\varphi_\delta = \begin{cases} f|_F. & x \in F. \\ \text{extension}. & x \in G \end{cases}$

Chapter V Fourier Series and Transform.

§.1. Orthogonal Systems in Space $L^2(X,\mu)$.

$\|\cdot\|$ - norm in $L^2(X,\mu)$.

scalar product.

$f,g\in L^2(X,\mu)$ $\quad \langle f,g\rangle = \int_X f\bar{g}\,d\mu$. ($f\cdot\bar{g}$ is integrable since $2|f\cdot\bar{g}|\le |f|^2+|g|^2$)

property (1) $\langle f,g\rangle = \overline{\langle g,f\rangle}$. $\quad \langle f,f\rangle = \|f\|^2$

(2) $|\langle f,g\rangle| \le \|f\|\|g\|$ (by Cauchy inequality).

(3) if $f_n\to f$. $g_n\to g$

then $|\langle f_n,g_n\rangle - \langle f,g\rangle| \le |\langle f_n-f,g_n\rangle| + |\langle f,g_n-g\rangle| \le \underbrace{\|f_n-f\|}_{\to 0}\cdot\|g_n\| + \|f\|\cdot\underbrace{\|g_n-g\|}_{\to 0}$

thus $|\langle f_n,g_n\rangle - \langle f,g\rangle| \underset{n\to\infty}{\to} 0$.

(4) $\langle \sum_{n=1}^{\infty} f_n, g\rangle = \sum_{n\ge1}^{\infty}\langle f_n,g\rangle$

property of norm.

(1) parallelogram identity. (actually holds for L^p-space. $p\in[1,+\infty)$).

$f,g\in L^2(X,\mu)$. $\|f+g\|^2+\|f-g\|^2 = 2(\|f\|^2+\|g\|^2)$

Def. $f,g\in L^2(X,\mu)$. is orthogonal if $\langle f,g\rangle = 0$. denote $f\perp g$.

Thm (Pythagoras) if $f\perp g$, then $\|f+g\|^2 = \|f\|^2+\|g\|^2$

Moreover, if $f_i\perp f_j$. $(i\ne j)$ $\quad \|\sum_{i\ge1}^{n} f_i\|^2 = \sum_{i\ge1}^{n}\|f_i\|^2$ (also $\|\sum_{i\ge1}^{\infty} f_i\|^2 = \sum_{i\ge1}^{\infty}\|f_i\|^2$).

Pf: $\langle f_i,f_j\rangle = 0$. $(i\ne j)$

$LHS = \langle \sum f_i, \sum f_i\rangle = \sum_{i,j\ge1}^{n}\langle f_i,f_j\rangle = \sum_{i\ge1}^{n}\|f_i\| = RHS$.

projection (of f onto 1-dim space)

Def. A family of functions $\{e_\alpha\}_{\alpha\in A}$. is orthogonal system. (OS).

if $e_i\perp e_j$ $\forall i\ne j$ and $\|e_\alpha\|\ne 0$ $\forall\alpha\in A$.

Moreover OS is orthonormal system if $\|e_\alpha\| = 1$. $\forall\alpha\in A$.

Let $\{e_1,\dots e_n\}$ be OS. L - be space generated by OS.

Thm the minimum value of a norm $\|f-\sum_{\alpha=1}^{n} a_\alpha e_\alpha\|$ is attained iff.

$a_\alpha = c_\alpha(f)$ where $c_\alpha(f) = \dfrac{\langle f,e_\alpha\rangle}{\|e_\alpha\|^2}$ $(\alpha=1,2,\dots,n)$ $\quad \approx$ moreover. equal to 0.

(the function $f-\sum_{\alpha\ge1}^{n} c_\alpha e_\alpha$ is orthogonal for $\forall$ element from L)

Def. Let $\{e_n\}_{n\in\mathbb{N}}$ - OS. $f\in L^2(X,\mu)$. $C_n(f)=\dfrac{\langle f,e_n\rangle}{\|e_n\|^2}$.

numbers $\{C_n(f)\}_{n\in\mathbb{N}}$ are Fourier coefficience.

the series $\sum_{n\geq1}^{\infty} C_n(f)e_n$ - Fourier series of f w.r.t. OS - $\{e_n\}_{n\in\mathbb{N}}$.

Remark. for orthonormal system, $C_n(f)=\langle f,e_n\rangle$.

in general: $C_n(f)\cdot e_n=\langle f,\frac{e_n}{\|e_n\|}\rangle\cdot\frac{e_n}{\|e_n\|}=\langle f,\tilde{e_n}\rangle\cdot\tilde{e_n}$.

$$\sum_{n=1}^{\infty}|C_n(f)|^2\|e_n\|^2\leq\|f\|^2 \quad \text{(Bessel inequality)}.$$

(the equality holds iff $f=\sum_{n\geq1}^{\infty}C_n(f)e_n$).

Lemma. Let $\{e_n\}_{n\in\mathbb{N}}$ - OS. series $\sum_{n\geq1}^{\infty}a_ne_n$ is conv. iff. $\sum_{n\geq1}^{\infty}|a_n|^2\cdot\|e_n\|^2<+\infty$ ②

Remark: if $\sum_{n\geq1}^{\infty}a_ne_n$ conv. $\sum_{n\geq1}^{\infty}a_ne_n$ is the Fourier series of its sum. ①

Pf: denote the partial sum $S_n=\sum_{n\geq1}^{\infty}a_ne_n$ $T_n=\sum_{n=1}^{\infty}|a_n|^2\cdot\|e_n\|^2$

$L^2(X,\mu)$ is complete. $S_n\to S$, $T_n\to T$.

$\forall m\in\mathbb{N}$, $\langle S,e_m\rangle=\sum_{n=1}^{\infty}a_n\langle e_n,e_m\rangle=a_m\|e_m\|^2\Rightarrow a_m=C_m(S)$, series $\sum_{n=1}^{\infty}a_ne_n$ is Fourier series (of it's sum) ①

"$\Leftrightarrow$" $\|S_{n+p}-S_n\|=\|\sum_{k=n+1}^{n+p}a_ne_n\|^2=\sum_{k=n+1}^{n+p}|a_k|^2\|e_k\|^2=T_{n+p}-T_n$.

$\Rightarrow$ 函数可积，其Fourier series 就收敛.

Thm (Riesz-Fischer) $\forall$ OS $\{e_n\}_{n\in\mathbb{N}}$, the Fourier series of function f from $L^2(X,\mu)$. conv. in norm and $f=\sum_{n\geq1}^{\infty}C_n(f)e_n+h$. where $h\perp e_n$ for $\forall n\in\mathbb{N}$.

Pf: by Bessel inequality $\sum_{n\geq1}^{\infty}|C_n(f)|^2\|e_n\|^2\leq\|f\|^2<+\infty$.

series. $\sum_{n\geq1}^{\infty}C_n(f)e_n$ conv. by the previous lemma. Let S - partial sum series.

$C_n(f)\equiv C_n(S)$. (conv. to same limit, same base, no other choice.)

$\Rightarrow$ Fourier coefficient of $h=f-S$. is $0\Rightarrow h\perp e_n$, $\forall n\in\mathbb{N}\Rightarrow f=S+h$.

(h 对正交基的系数 (Fourier's) 是 0，本身不一定是 0).

Def. OS $\{e_n\}_{n\in\mathbb{N}}$ is basis. if $\forall$ function from $L^2(X,\mu)$ coincide with sum of its Fourier series almost everywhere.

In the case $\{e_n\}_{n\in\mathbb{N}}$ - basis. $f=\sum_{n\geq1}^{\infty}C_n(f)e_n$, $\|f\|^2=\sum_{n=1}^{\infty}|C_n(f)|^2\|e_n\|^2$

$\langle f,g\rangle=\langle\sum_{n=1}^{\infty}C_n(f)e_n,g\rangle=\sum_{n\geq1}^{\infty}C_n(f)\langle e_n,g\rangle=\sum_{n\geq1}^{\infty}C_n(f)\overline{C_n(g)}\|e_n\|^2$

Parseval Identity

Def. (completeness) A family of function $\{f_\alpha\}_{\alpha\in A}$ in $L^2(X,\mu)$ is complete if: $f\in L^2(X,\mu)$ and $f\perp f_\alpha$ for every $\alpha\in A$ implies $f=0$ a.e. (i.e. $\|f\|=0$).
(给出了构造基的一种途径).

Lemma. A family $\{f_\alpha\}_{\alpha\in A}$ is complete if the set of all linear combinations of function contained in this family is everywhere dense. (i.e. for every function $f\in L^2(X,\mu)$ and every $\varepsilon>0$, there exists a linear combination $g=\sum_{k=1}^{n} c_k f_{\alpha_k}$ s.t. $\|f-g\|<\varepsilon$).

$\{\sum \alpha_i f_i \mid \alpha_i\in\mathbb{R},\ f_i\in\{f_\alpha\}\}$

Pf: Let $f\perp f_\alpha$ for every $\alpha\in A$. assume $\|f\|\neq 0$.
$\exists g=\sum c_k f_{\alpha_k}$ s.t. $\|f-g\|<\|f\|$.
but since $f\perp g$, $\|f\|^2>\|f-g\|^2=\|f\|^2+\|g\|^2\geq\|f\|^2$ contradicts.

Thm. (On the characterization of bases). Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal system. TFAE:
(1). the OS $\{e_n\}_{n\in\mathbb{N}}$ is basis.
(2) $\forall f\in L^2(X,\mu)$. Parseval's identity holds: $\sum_{n\geq1}^{\infty}|c_n(f)|^2\|e_n\|^2=\|f\|^2$ holds.
(3) system $\{e_n\}_{n\in\mathbb{N}}$ is complete.

Pf: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)
(1) $\Rightarrow$ (2). proved after the def. of basis.
(2) $\Rightarrow$ (3). let $f\perp e_n$, $n\in\mathbb{N}$. i.e. $c_n(f)=0$. for all $n\in\mathbb{N}$.
$\|f\|^2=\sum_{n\geq1}^{\infty}|c_n(f)|^2\|e_n\|^2=0$.
(3) $\Rightarrow$ (1). let $f\in L^2(X,\mu)$. by Riesz-Fischer thm. $f=\sum_{n=1}^{\infty}c_n(f)e_n+h$. $h\perp e_n$, $n\in\mathbb{N}$.
by completeness. $h=0$ a.e. i.e. $f=\sum_{n=1}^{\infty}c_n(f)e_n$ a.e.

Coro. OS-$\{e_n\}_{n\in\mathbb{N}}$ completely $\Leftrightarrow$ the set of all l.c. of function contained in this system in everywhere dense. (previous lemma "$\Leftarrow$": +condition $\{e_n\}$ - OS. we receive "$\Rightarrow$").

Lemma. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal system and $\omega:\mathbb{N}\to\mathbb{N}$ be bijection.
Then series (a) $\sum_{n=1}^{\infty}a_n e_n$ and (b) $\sum_{k=1}^{\infty}a_{\omega(k)}e_{\omega(k)}$ converges simultaneously.
if they converge, their sum are equal.

Pf: (a) conv. iff $\sum_{n=1}^{\infty}|a_n|\,\|e_n\|<+\infty$ (b) conv. iff $\sum_{k=1}^{\infty}|a_{\omega(k)}|\,\|e_{\omega(k)}\|<+\infty$
For positive numerical series, independent under rearrangement.
$\left\|\sum_{n=1}^{\infty}a_{\omega(k)}e_{\omega(k)}-S_n(a)\right\|^2=\sum_{\omega(k)>n}\|a_{\omega(k)}e_{\omega(k)}\|^2=\sum_{j>n+1}|a_j|\,\|e_j\|^2\xrightarrow[n\to+\infty]{}0$ sum (a), (b) coincide.

<u>Thm</u> If orthogonal system $\{e_k\}_{k\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$. are complete then the system

$\{h_{k,n}\}_{k,n\in\mathbb{N}}$. is also complete

$\{e_k\} \subset L^2(X,\mu)$. $\{g_n\} \subset L^2(Y,\nu)$, $\{h_{k,n}\} \subset L^2(X\times Y, \mu\times\nu)$

$h_{k,n}(x,y) = e_k(x)\, g_n(y) \quad (x\in X,\ y\in Y)$.

→ 一种"复合关系"，而不是积

Pf: Let $f \perp h_{k,n}$ for all $k,n\in\mathbb{N}$. this means: $f\in L^2(X\times Y, \mu\times\nu)$.

$$\int_{X\times Y} f(x,y)\,\overline{e_k(x)}\,\overline{g_n(y)}\, d(\mu\times\nu)(x,y) = \int_X \left(\int_Y f(x,y)\,\overline{g_n(y)}\, d\nu(y)\right)\overline{e_k(x)}\, d\mu(x) = 0.$$

fix $n\in\mathbb{N}$. consider the function $x \mapsto \varphi_n(x) = \int_Y f(x,y)\,\overline{g_n(y)}\, d\nu(y)$.

$$|\varphi_n(x)| \le \left(\int_Y |f(x,y)|^2 d\nu(y)\right)^{\frac{1}{2}} \|g_n\| .$$

$$\int_X |\varphi_n(x)|\, d\mu(x) \le \int_X \left(\int_Y |f(x,y)|^2 d\nu(y)\right) d\mu(x)\, \|g_n\|^2 < +\infty. \Rightarrow \varphi_n \in L^2(X,\mu).$$

for all $n\in\mathbb{N}$.

Fourier coefficients of φ_n w.r.t. $\{e_k\}_{k\in\mathbb{N}}$ are zero. $\{e_k\}$ complete. $\varphi_n(x) = 0$ a.e.

$$\sum_{n=1}^{\infty} |\varphi_n(x)|^2 = 0 \quad \text{a.e. on } X.$$

Since $\int_X\int_Y |f(x,y)|^2 d\nu(y)\, d\mu(x) < +\infty$. by Fubini's thm. $\int_Y |f(x,y)|^2 d\nu(y) < +\infty$ a.e.

i.e. function $\psi(y) = f_x(y) = f(x,y). \in L^2(Y,\nu)$. for almost all x.

$\varphi_n(x)$ are Fourier coefficients of $\psi(y)$ w.r.t. to $\{g_n\}_{n\in\mathbb{N}}$.

$\{g_n\}_{n\in\mathbb{N}}$ is complete $\Rightarrow \int_Y |f(x,y)|^2 d\nu(y) = \|f_x\|^2 = \sum_{n=1}^{\infty} |\varphi_n(x)|^2 = 0$ a.e. on X.

integrating the above equation over X. we obtain $0 = \int_X\int_Y |f(x,y)|^2 d\nu(y)\, d\mu(x) = \|f\|^2$.

$f=0$ a.e. $\{h_{n,k}\}_{n,k\in\mathbb{N}}$ is complete. □.

§. Example of orthogonal systems.

1. Triangular systems. $L^2((a, a+2l))$.

$\{1, \cos\frac{\pi x}{l}, \sin\frac{\pi x}{l}, \dots, \cos\frac{n\pi x}{l}, \sin\frac{n\pi x}{l}, \dots\}$ real. and $\{e^{\frac{\pi i n x}{l}}\}_{n\in\mathbb{Z}}$ complex.

Fourier series.

$$A(f) + \sum_{n=1}^{\infty}\left(a_n(f)\cos\frac{\pi n x}{l} + b_n(f)\sin\frac{\pi n x}{l}\right) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} c_n(f)\, e^{\frac{\pi i n x}{l}}.$$

Fourier coefficients:

$$A(f) = \frac{1}{2l}\int_a^{a+2l} f(x)\,dx$$

$$a_n(f) = \frac{1}{l}\int_a^{a+2l} f(x)\cos\frac{\pi n x}{l}\,dx \quad (n\in\mathbb{N}).$$

$$b_n(f) = \frac{1}{l}\int_a^{a+2l} f(x)\sin\frac{\pi n x}{l}\,dx \quad (n\in\mathbb{N})$$

$$c_n(f) = \frac{1}{2l}\int_a^{a+2l} f(x)\cdot e^{-\frac{\pi i n x}{l}}\,dx \quad (n\in\mathbb{Z})$$

usually the function defined on $(0, 2l)$ (or symmetric $(-l, l)$).

and the period is 2π (for simplicity). instead of $2l$.

$$\hat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} f(x)\cdot e^{-inx}\,dx \quad (n\in\mathbb{Z}). \qquad (c_n(f) \text{ in period } 2\pi).$$

transition $\displaystyle c_k(f) = \frac{1}{2l}\int_0^{2l} f(x)\cdot e^{\frac{-ikx}{l}}\,dx = \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{l}{\pi}y\right) e^{-iky}\,dy = \hat{g}(k)$

$$(f\in L^2((0,2l)) \Leftrightarrow g\in L^2((0,2l))$$

partial sum: $\displaystyle \sum_{|k|\le n} c_k(f)e^{\frac{\pi i k x}{l}} = \sum_{|k|\le n}\hat{g}(k)\,e^{\frac{\pi i k x}{l}} = \sum_{|k|\le n}\hat{g}(k)^{iky}$

$$\hat{f}(\pm n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,(\cos nx \mp i\sin nx)\,dx = \frac{a_n(f)\mp i b_n(f)}{2} \quad (n\in\mathbb{N}).$$

$$a_n(f) = \hat{f}(n) + \hat{f}(-n) \qquad b_n(f) = i(\hat{f}(n) - \hat{f}(-n))$$

$$A(f) + \sum_{k=1}^{n}(a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=-n}^{n}\hat{f}(k)\,e^{ikx}.$$

<u>Thm.</u> The real and complex trigonometric systems form bases. in $L^2((0,2\pi))$.

Pf. if $f, g\in L^2((0,2\pi)) \Rightarrow \displaystyle \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx = A(f)\overline{A(g)} + \frac{1}{2}\sum_{n\ge1}\left(a_n(f)\overline{a_n(g)} + b_n(f)\overline{b_n(g)}\right)$

$\forall f\in L^2((0,2\pi))$ $\displaystyle = \sum_{n=-\infty}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}$ (each $\{\ \}$ real. $\{\ \}$ complex are basis of $L^2(X, M)$. they satisfy Parseval).

$$\frac{1}{2\pi}\underbrace{\int_0^{2\pi}|f(x)|^2\,dx}_{\|f\|^2} = |A(f)|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left((a_n(f))^2 + (b_n(f))^2\right) = \sum_{n\ge1}^{\infty}|\hat{f}(n)|^2.$$

(every function satisfy the Parseval's identity.)

e.g.1 (Hurwitz). For closed plane curve. $4\pi S \le L^2$.

(L is the circumference, S is area. equality attained only in curve is circle).

Pf: let $K \subset \mathbb{R}^2$ K - compact. with closed, smooth boundary.

w.l.g. let the length is 2π.

$z(t) = (x(t), y(t))$, $0 \le t \le 2\pi$. be parametrization of the curve ∂K. (natural).

we have $z(0) = z(2\pi)$. $|z'(t)| = 1$.

$$L^2 = 2\pi \cdot \int_0^{2\pi} |z'(t)|^2 dt = 4\pi^2 \sum_{n\in\mathbb{Z}} |\hat{z}'(n)|^2$$

$$S = \frac{1}{2}\int_{\partial K} (-y\,dx + x\,dy) = \frac{1}{2}\int_0^{2\pi} (x(t)y'(t) - x'(t)y(t))\,dt.$$

$$= \frac{1}{2i}\int_0^{2\pi} z'(t)\overline{z(t)}\,dt = -\pi i \sum_{n\in\mathbb{Z}} \hat{z}'(n)\overline{\hat{z}(n)}$$ (by Parseval's identity).

$$\hat{z}'(n) = \frac{1}{2\pi}\int_0^{2\pi} z'(t) e^{-int} dt \Big|_0^{2\pi} + \frac{in}{2\pi}\int_0^{2\pi} z(t) e^{-int} dt = in\,\hat{z}(n)$$

$$L^2 = 4\pi^2 \sum_{n\in\mathbb{Z}} n^2 |\hat{z}(n)|^2 \qquad S = \pi \sum_{n\in\mathbb{Z}} n |\hat{z}(n)|^2$$

$$L^2 - 4\pi S = 4\pi^2 \sum_{n\in\mathbb{Z}} (n^2 - n)|\hat{z}(n)|^2 \ge 0.$$

equality holds for $|\hat{z}(n)| = 0$. or $n = 0, 1$.

$\Rightarrow z(t) = \hat{z}(0) + \hat{z}(1) e^{it}$ $\quad (|\hat{z}(1)| = |z'(t)| = 1)$.

$\Rightarrow |z - \hat{z}(0)| = 1$. $\Rightarrow$ unit circle. (边界上的点到中心距离 $=1$).

e.g.2 consider the space $L^2(Q)$. $Q = (-\pi, \pi)^m$ $\quad n = (n_1, \dots, n_m)$. multi-indices.

the new system $e_n(x) = e^{i\langle n, x\rangle}$. $x \in Q$, $n \in \mathbb{Z}^m$. $\Rightarrow$ complete thm (product basis)

For some $f \in L^2(Q)$.

$$\hat{f}(n) = \frac{\langle f, e_n\rangle}{\|e_n\|^2} = \frac{1}{(2\pi)^m}\int_Q f(x) e^{-i\langle n, x\rangle} dx. \ (n \in \mathbb{Z}^m).$$

By Parseval identity $\int_Q f(x)\overline{g(x)}\,dx = (2\pi)^m \sum_{n\in\mathbb{Z}^m} \hat{f}(n)\cdot\overline{\hat{g}(n)}$, $f, g \in L^2(Q)$.

e.g.3. (Lebendre Polynomial). $P_n(x) = ((x^2-1)^n)^{(n)}$. $\deg P_n = n$.

$\{P_n(x)\}$ - OS of $L^2((-1,1))$:

Orthogonality: W.l.g. $n>m$

$$\langle P_n, P_m\rangle = \int_{-1}^{1} P_m(x)((x^2-1)^n)^{(n)}dx = P_m(x)((x^2-1)^n)^{(n-1)}\Big|_{-1}^{1} - \int_{-1}^{1} P_m^{(1)}(x)((x^2-1)^n)^{(n-1)}dx$$

$$= -\int_{-1}^{1} P_m^{(1)}(x)((x^2-1)^n)^{(n-1)}dx.$$ integrate by parts n times.

$$= (-1)^n \int_{-1}^{1} P_m^{(n)}(x)(x^2-1)^n dx \quad (n>m,\ P_m^{(n)} \equiv 0)$$

$\{P_n(x)\}$ - forms basis in $L^2((-1,1))$:

idea: everywhere dense $\Leftrightarrow$ complete $\Leftrightarrow$ basis.

* coro. Let $X \subset \mathbb{R}^m$ be a bounded measurable set. $1 \le p < +\infty$. and $f \in L^p(X)$.

$\forall \varepsilon > 0$. $\exists$ Polynomial P. s.t. $\|f-P\|_p < \varepsilon$.

e.g.4. (Hermite Function/Polynomial)

function: $h_n(x) = e^{\frac{x^2}{2}}(e^{-x^2})^{(n)}$. polynomial $H_n(x)$: $h_n(x) = H_n(x)e^{-\frac{x^2}{2}}$

$\{h_n(x)\}$ - OS. $\{H_n(x)\}$ - OS in $L^2(\mathbb{R})$

$$\langle h_m, h_n\rangle = \int_{-\infty}^{\infty} H_m(x)(e^{-x^2})^{(n)}dx$$

$\{h_n\}$ - OS in $L^2(\mathbb{R}) \longrightarrow \{H_n\}$ - OS in $L^2(\mathbb{R},\mu)$, $d\mu = e^{-x^2}dx$.

$\{h_n\}$, $\{H_n\}$ is complete in $L^2(\mathbb{R})$, $L^2(\mathbb{R},\mu)$ respectively.

§. 10.3. Trigonometric Fourier Series.

recall def. Fourier series of $f \in L^2((0,2\pi))$

$1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots$ and $\{e^{inx}\}_{n\in\mathbb{Z}}$.

Fourier series has form. $A(f) + \sum_{n\geq 1}^{\infty}(a_n(f)\cos nx + b_n(f)\sin nx)$ (1) and $\sum_{n=-\infty}^{\infty}\hat{f}(n)e^{inx}$. (1')

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx. \quad (n\in\mathbb{Z}).$$

denote the symmetric partial sums $S_n(f,x) = \sum_{|k|\leq n}\hat{f}(k)e^{ikx}$ (Fourier sums of f).

recall: $A(f) + \sum_{k\geq 1}^{n}(a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=-n}^{n}\hat{f}(k)e^{ikx}$

(the partial sum of series (1) and (1') equals).

f defined on $(0,2\pi) \xrightarrow{\text{extend}} 2\pi$-periodic function. $\int_a^{a+2\pi} f(x)dx$. not depend on a.

notation: $\tilde{C}^r / \tilde{C}$ class of periodic func. cont. diff r times / cont. on $\mathbb{R}$.

$\tilde{L}^p$ class of periodic func. integrable on $(-\pi,\pi)$. with power $p\geq 1$.

($f\in\tilde{L}^p$, $\|f\|_p$ means L^p-norm of its restriction to $(-\pi,\pi)$).

Elementary properties of Fourier coefficients:

(a) $|\hat{f}(n)| \leq \frac{1}{2\pi}\|f\|_1$.

(b) $\hat{f}(n) \xrightarrow[|n|\to+\infty]{} 0$

recall: translation: f_h of $f\in\tilde{L}^1$ corresponding to number h. $f_h(x) = f(x-h)$.

(c). $\hat{f_h}(n) = e^{-inh}\hat{f}(n)$.

(d). if periodic f is abs. cont. on $\mathbb{R}$ then $\hat{f'}(n) = in\hat{f}(n)$ $n\in\mathbb{Z}$. and $\hat{f}(n) = o(\frac{1}{n})$.

(remark: piecewise diff $\Rightarrow$ abs. cont.)

(d') if f is a func. of bounded variation on $[0,2\pi]$. $\bigvee_0^{2\pi} f < +\infty$

then. $\hat{f}(n) = O(\frac{1}{n})$.

Pf: $2\pi\hat{f}(n) = \int_0^{2\pi} f(x)e^{-inx}dx = f(x)\frac{e^{-inx}}{-in}\Big|_0^{2\pi} + \frac{1}{in}\int_0^{2\pi} e^{-inx}df(x) = O(\frac{1}{n})$

(e). Let $f,g\in\tilde{L}^1$ then $\widehat{f*g}(n) = 2\pi\hat{f}(n)\cdot\hat{g}(n)$ for all $n\in\mathbb{Z}$.

Pf: $\widehat{f*g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f*g)(x)\, e^{-inx}\,dx$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(x-t)g(t)\,dt\right) e^{-inx}\,dx.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, e^{-int}\left(\int_{-\pi}^{\pi} f(x-t)\, e^{-in(x-t)}\,dx\right) dt.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\, e^{-int}\left(\int_{-\pi}^{\pi} f(u)\, e^{-inu}\,du\right) dt = 2\pi\,\hat{g}(n)\cdot\hat{f}(n).$$

def. Dirichlet kernel $D_n(u) = \frac{1}{2\pi}\sum_{|k|\le n} e^{iku} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k\ge 1}^{n} \cos ku. = \frac{\sin(n+\frac{1}{2})u}{2\pi \sin\frac{u}{2}},\ u \notin 2\pi\mathbb{Z}.$

(n-th Dirichlet kernel).

D_n is strongly oscillating for large n.

$\max D_n = D_n(0) = \frac{1}{\pi}(n+\frac{1}{2}).$

$\Rightarrow$ the sum of the Fourier series. is the convolution of the function. and the Dirichlet kernel.

$$S_n(f,x) = \int_{-\pi}^{\pi} f(t)\, D_n(x-t)\,dt = (f*D_n)(x)$$

$\Rightarrow \int_{-\pi}^{\pi} D_n(u)\,du = 1. \quad \left(\int_0^{\pi} D_n\,du = \frac{1}{2}\right)$

moreover $\int_{\delta<|u|<\pi} D_n(u)\,du = \int_{\delta<|u|<\pi} \frac{\sin(n+\frac{1}{2})u}{2\pi\sin\frac{u}{2}}\,du \xrightarrow[n\to\infty]{} 0$

$\triangle$ Dirichlet kernel have unbounded L^1-norms:

$$\int_{-\pi}^{\pi} |D_n(u)|\,du = \int_0^{\pi} \frac{|\sin(n+\frac{1}{2})u|}{\pi\sin\frac{u}{2}}\,du \ge \frac{2}{\pi}\int_0^{\pi}\frac{|\sin(n+\frac{1}{2})u|}{u}\,du.$$

$$= \frac{2}{\pi}\int_0^{\pi(n+\frac{1}{2})}\frac{|\sin v|}{v}\,dv \ge \frac{2}{\pi}\sum_{k\ge 1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin v|}{k\pi}\,dv = \frac{4}{\pi^2}\sum_{k\ge 1}^{n}\frac{1}{k}$$

i.e. $\|D_n\|_1 \ge \frac{4}{\pi}\ln n.$

$\triangle$ the norm $\|D_n\|_1$ increase "quite slowly": $\|D_n\|_1 \le 2\ln n$ for $n \ge 10$.

by approximation: $\|D_n\|_1 = \int_0^{\pi}\frac{|\sin(n+\frac{1}{2})u|}{\pi\sin\frac{u}{2}}\,du \le \int_0^{\pi}\frac{|\sin(n+\frac{1}{2})u|}{u}\,du = \int_0^{\pi(n+\frac{1}{2})}\frac{|\sin v|}{v}\,dv.$

$\|D_n\|_1 \le 1 + \int_1^{\pi(n+\frac{1}{2})}\frac{dv}{v} = 1 + \ln\pi(n+\frac{1}{2}). \le 2\ln n$ for $n \ge 10$.

A estimate for the Fourier sums: $\|S_n(f)\|_\infty \le \|f\|_\infty \|D_n\|_1 \le 2\|f\|_\infty \ln n.$

Thm. (Riemann's localization principle). If function $f_1, f_2 \in \tilde{L}^1$ coincide in a neighborhood of a point x. then their Fourier series have same behavior at x.

$$S_n(f_1, x) - S_n(f_2, x) \xrightarrow[n\to\infty]{} 0$$

Pf: denote $\varphi_x(u) = \dfrac{f_1(x+u) - f_2(x-u)}{\sin(\frac{u}{2})}$

$\varphi_x(u) = 0$ in some neighborhood of $u=0$.

$$S_n(f_1, x) - S_n(f_2, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_x(u)\sin(n+\tfrac{1}{2})u\,du, \quad \varphi_x(u) \in L^1((-\pi,\pi)).$$

by Riemann-Lebesgue thm. RHS $\to 0$.

Lemma (Property of Dirichlet kernel). Let $n \in \mathbb{N}$.

(a) $D_n(u) = \dfrac{\sin nu}{\pi u} + \dfrac{1}{2\pi}(\cos nu + \delta(u)\sin nu)$.

$|\delta(u)| < 1$. for $|u| \le \pi$; δ is a function independent of n.

(b) $\left|\int_0^x D_n(u)\,du\right| \le 2$. for $|x| \le 2\pi$.

Pf: (a). $D_n(u) = \dfrac{\sin nu \cos\frac{u}{2} + \cos nu \sin\frac{u}{2}}{2\pi \sin\frac{u}{2}} = \dfrac{\sin nu}{2\pi \tan\frac{u}{2}} + \dfrac{\cos nu}{2\pi}$

$$= \frac{\sin nu}{\pi u} + \frac{1}{2\pi}\left(\cos nu + \left(\frac{1}{\tan\frac{u}{2}} - \frac{2}{u}\right)\sin nu\right)$$

denote. $\delta(u) = \dfrac{1}{\tan\frac{u}{2}} - \dfrac{2}{u}$ $\quad(\delta(0) = 0)$.

$\delta(u)$ decrease on $[-\pi, \pi]$. $|\delta(u)| \le \max\{|\delta(\pi)|, |\delta(-\pi)|\} = \frac{2}{\pi} < 1$.

(b) consider $x \in (0, 2\pi)$.

First let $x \in (0, \pi]$.

by (a) $\left|\int_0^x D_n(u)\,du - \int_0^x \dfrac{\sin nu}{\pi u}\,du\right| \le \int_0^x \dfrac{1}{2\pi}|\cos nu + \delta(u)\sin nu|$.

$< \dfrac{1}{2\pi}\int_0^x 2\,dx$. ($\cos nu \le 1$, $\delta(u) < 1$, $\sin nu \le 1$). ≤ 1.

denote the integral $J_n(x) = \displaystyle\int_0^x \frac{\sin nu}{\pi u}\,du = \int_0^{nx}\frac{\sin v}{\pi v}\,dv \underset{\Downarrow}{\le} \int_0^{\pi}\frac{\sin v}{\pi v}\,dv \le \int_0^{\pi}\frac{dv}{\pi} = 1.$

$J_n(x) \in [0,1]$. (split $[0, nx]$ into parts. which sign($\sin v$) preserves $|\frac{\sin v}{v}| \downarrow$ since $\frac{1}{v}\downarrow$).

$\Rightarrow \int_0^x D_n(u)\,du \in [-1, 2]$. for $x \in (0, \pi]$.

Second let $x \in (\pi, 2\pi)$. $\underbrace{\int_0^x D_n(u)\,du}_{[-1,2]} = 1 - \underbrace{\int_0^{2\pi - x} D_n(u)\,du}_{[-1,2]}$.

$\left|\int_0^x D_n(u)\,du\right| \le 2.$

Remark: by lemma (a). $S_n(f,x) = \int_{-\pi}^{\pi} f(x-u)\dfrac{\sin nu}{\pi u}\,du + \varepsilon_n$.

$$\varepsilon_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\,(\cos nu + \Delta(u)\sin nu)\,du \xrightarrow[n\to+\infty]{} 0.$$

e.g. use the remark to compute $\int_0^{\infty} \frac{\sin t}{t}\,dt = \frac{\pi}{2}$.

if $f \equiv 1$. then $1 = \int_{-\pi}^{\pi} \frac{\sin nu}{\pi u}\,du \xRightarrow[n\to+\infty]{nu=t} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin t}{t}\,dt = 1$

Thm. (Dini test - conv. of Fourier series).

If a function $f \in \tilde{L}^1$ satisfy the Dini condition $\int_0^{\pi} \left|\frac{f(x+u)+f(x-u)}{2} - C\right|\frac{du}{u} < +\infty$.

at a point $x \in \mathbb{R}$. for some $C \in \mathbb{C}$ then its Fourier series converges to C at point x.

If f is diff at x. $C = f(x)$ (the Fourier series $= f(x)$ at x)

If only $f(x\pm 0)$ exists. and $|f(x\pm u) - f(x\pm 0)| = O(u^{\alpha})$ as $u \to 0^+$. for some $\alpha > 0$.

then $C = \dfrac{f(x+0)+f(x-0)}{2}$.

Pf: $S_n(f,x) = \int_{-\pi}^{\pi} f(x-u)\frac{\sin nu}{\pi u}\,du + o(1) = \int_{-\pi}^{\pi} f(x+u)\frac{\sin nu}{\pi u}\,du + o(1)$

$$\Rightarrow S_n(f,x) = \int_{-\pi}^{\pi} \frac{f(x-u)+f(x+u)}{2}\,\frac{\sin nu}{\pi u}\,du + o(1)$$

$$\xRightarrow{1=\int_{-\pi}^{\pi}\frac{\sin nu}{\pi u}du} S_n(f,x) - C\cdot 1 = \int_{-\pi}^{\pi} \left(\frac{f(x-u)+f(x+u)}{2} - 2C\right)\cdot\frac{\sin nu}{\pi u}\,du + o(1) \to g_x(u)$$

$$= \frac{2}{\pi}\int_0^{\pi} g_x(u)\sin nu\,du + o(1)$$

$g_x(u) \in L((0,\pi))$. RHS $\to 0$. (by Riemann - Lebesgue thm.)

Thm. (Dirichlet - Jordan test).

If f - periodic. and $\overset{\pi}{\underset{-\pi}{V}}(f)$ exists. Then for each $x \in \mathbb{R}$, the Fouries series of f conv. to the average $\dfrac{f(x+0)+f(x-0)}{2}$. Moreover, $|S_n(f,x)| \le \sup_{\mathbb{R}}|f| + 2\overset{\pi}{\underset{-\pi}{V}}f$

Pf: 1) convergence:

denote $I_n = \int_{-\pi}^{\pi} f(x-u)\dfrac{\sin nu}{\pi u}\,du = \int_0^{\pi} \varphi(u)\dfrac{\sin nu}{\pi u}\,du$. $\varphi(u) = f(x+u) + f(x-u)$

$\varphi \in V[0,\pi]$. φ can be represented as the difference of to decreasing func. on $[0,\pi]$.

此时u是变量. 而x是取定. 可取x使得 φ 单调减.

denote $\Phi(u) = \varphi(u)\chi_{(0,\pi)}(u)$. Φ decrease, non-negative. $I = \int_0^{\infty} \frac{\sin t}{\pi t}\,dt = \frac{1}{2}$.

$$I_n = \int_0^{\infty} \Phi(u)\frac{\sin nu}{\pi u}\,du = \int_0^{\infty} \Phi\left(\frac{t}{n}\right)\frac{\sin t}{\pi t}\,dt \xrightarrow[n\to\infty]{} \Phi(0+)\cdot I = \frac{\Phi(0+)}{2} = \frac{f(x_+)+f(x_-)}{2}$$

(by coro2. 7.4.7 p367)

2) bound: denote $H_n(u) = \int_0^u D_n(t)\,dt$

$$S_n(f,x) = \int_{-\pi}^{\pi} f(x-u)\,D_n(u)\,du = \overset{①\Rightarrow}{H_n(u) f(x-u)\Big|_{-\pi}^{\pi}} - \overset{②}{\int_{-\pi}^{\pi} H_n(u)\,d f(x-u)}$$

since $H_n(\pm\pi) = \pm\frac{1}{2}$. $① = \dfrac{f(x-\pi)+f(x+\pi)}{2}$.

since $\left|\int_0^x D_n(u)\,du\right| \le 2$. e.g. $|H_n(u)| \le 2$.

so $\left|\int_{-\pi}^{\pi} H_n(u)\,d f(x-u)\right| \le 2 \overset{x+\pi}{\underset{x-\pi}{V}}(f) = 2 \overset{\pi}{\underset{-\pi}{V}}(f)$. (since f-periodic)

<u>Thm.</u> (Riemann - Lebesgue)

Let $X \in \mathbb{R}^m$ and $f \in L^1(X)$. Then. $If(y) = \int_X f(x)\,e^{i\langle x,y\rangle}\,dx \xrightarrow[\|y\|\to+\infty]{} 0$.

if $f \in L^1[a,b]$. then $\int_a^b f(x)\,e^{ixy}\,dx \xrightarrow[|y|\to+\infty]{} 0$

Remark: more useful form: $\int_{-\pi}^{\pi} f(x)\sin nx\,dx \xrightarrow[n\to\infty]{} 0$

$$\int_{-\pi}^{\pi} f(x)\cos nx\,dx \xrightarrow[n\to\infty]{} 0$$

Thm 1. Let $f \in \widetilde{L}^1$ Then for any $a,b \in \mathbb{R}$, $\int_a^b f(x)\,dx = \sum_{n=-\infty}^{\infty} \hat{f}(n) \int_a^b e^{inx}\,dx$.

(where the sum is regarded as the limit of symmetric partial sums)

Pf: consider f on $[-\pi,\pi]$. $-\pi \le a < b \le \pi$ (since the periodicity).

denote χ be characteristic fun. of interval (a,b).

$$\sum_{k=-n}^{n} \hat{f}(k) \int_a^b e^{ikx}\,dx = \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, e^{-ikt}\,dt \right) 2\pi\, \hat{\chi}(-k) = \int_{-\pi}^{\pi} f(t)\, S_n(\chi,t)\,dt.$$

By Dini test, $S_n(\chi,t) \xrightarrow[n\to\infty]{} \chi(t)$. for $t \in (-\pi,\pi)$

Moreover $S_n(\chi,t) = \int_a^b D_n(x-t)\,dx = \int_{a-t}^{b-t} D_n(u)\,du = \int_0^{b-t} D_n(u)\,du - \int_0^{a-t} D_n(u)\,du$.

by previous Lemma. $\left| \int_0^x D_n(u)\,du \right| \le 2$. for $|x| \le 2\pi$. $\Rightarrow |S_n(\chi,t)| \le 4$.

By Lebesgue thm. we obtain the convergence. (

then pass the limit. $\int_{-\pi}^{\pi} f(t)\, S_n(\chi,t)\,dt \xrightarrow[n\to\infty]{} \int_{-\pi}^{\pi} f(t)\,\chi(t)\,dt = \int_a^b f(t)\,dt$.

Coro 1. Functions $f,g \in \widetilde{L}^1$ having the same Fourier coefficients. coincide a.e. on $\mathbb{R}$.

Pf: By thm 1. f,g equals on every finite interval.

then f,g. coincide a.e.

coro 2. For every $f \in \widetilde{L}^1$, $\sum_{n=1}^{\infty} \frac{b_n(f)}{n}$ conv.

Pf $b_n(f) = \frac{1}{n} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = i(\hat{f}(n) - \hat{f}(-n))$

Thm 2. Let $f \in \widetilde{L}^1$, g bounded with $S_n(g,x)$ uniformly bounded (w.r.t. x and n)

Then the Parseval Identity holds: $\int_{-\pi}^{\pi} f(x)\,\bar{g}(x)\,dx = 2\pi \sum_{n=-\infty}^{\infty} \hat{f}(n)\, \overline{\hat{g}(n)}$

Remark:

Pf: $g \in \widetilde{L}^2$. (

$S_n(g)$ conv. to g in L^2-norm. $\Rightarrow$ $S_n(g)$ conv. to g in measure.

thus $f(x)\, \overline{S_n(g,x)} \longrightarrow f(x)\,\bar{g}(x)$ in measure.

$\int_{-\pi}^{\pi} f(x)\, \overline{S_n(g,x)}\,dx = \sum_{|k|\le n} \overline{\hat{g}(k)} \int_{-\pi}^{\pi} f(x)\, e^{-ikx} = 2\pi \sum_{|k|\le n} \hat{f}(k)\, \overline{\hat{g}(k)}$, then pass the limit. (by Lebesgue thm)

§. Fourier coefficients and series for a measure.

Def. Let μ be a finite Borel measure on $[-\pi, \pi]$.

Fouries coefficient of μ. $\hat{\mu}(n) = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{-inx} d\mu(x)$. $(n \in \mathbb{Z})$.

Fouries series of μ: $\sum_{n=-\infty}^{\infty} \hat{\mu}(n) e^{inx}$

Thm. Let μ, ν be finite Borel measures on the interval $[-\pi,\pi]$. satisfy the condition $\mu(\{\pi\}) = \nu(\{\pi\}) = 0$. If the Fourier coefficients of these measures coincide, then the measures also coincides.

Pf: integrated $\hat{\mu}(n)$ termwisely. $\sum_{n=-\infty}^{\infty} \hat{\mu}(n) \int_a^b e^{inx} dx = \mu([a,b))$. $(\mu(\{a\}) = \mu(\{b\}) = 0)$

for $[a,b) \subset [-\pi, \pi)$.

denote $\chi = \chi_{[a,b)}$. then $\sum_{|k| \le n} \hat{\mu}(k) \int_a^b e^{ikx} dx = \sum_{|k| \le n} \hat{\chi}(-k) \int_{[-\pi,\pi]} e^{-ikx} d\mu(x) = \int_{[-\pi,\pi]} S_n(\chi, x) d\mu(x)$

§. Uniform convergence of Fourier series of f.

Fact: the Fourier coefficient of smooth functions tend to 0. sufficiently fast.

Thm 1. $f \in \tilde{C}$ be infinitely diff $\Leftrightarrow$ for every $r \in \mathbb{N}$, $n^r \hat{f}(n) \xrightarrow[|n|\to+\infty]{} 0$.

Thm 2. Let $f \in \tilde{C}$. TFAE:

(1) there is $F \in H(\{z \in \mathbb{C} \mid |\mathrm{Im}\, z| < L\})$ and $F = f$ on real axis.

(2) the relation $\hat{f}(n) = O(e^{-a|n|})$ as $|n| \to +\infty$ holds for every $a \in (0, L)$.

Pf: (1) $\Rightarrow$ (2).

consider $\int_C F(z) e^{-inz} dz$ C is boundary of rectangle P. vertices $\pm\pi$, $\pm\pi - ai$.

F is holomorphic in a neighborhood of P. $\int_C F(z) e^{-inz} dz = 0$.

F has period 2π. $\int_{-\pi}^{\pi} F(z) e^{-inz} dz = 0$.

therefore. $\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \cdot e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi-ai}^{\pi-ai} F(z) e^{-inz} dz$

$|\hat{f}(n)| \leq \max\limits_{x\in\mathbb{R}} |F(x-ai)| |e^{-in(x-ai)}| = e^{-an} \max\limits_{x\in\mathbb{R}} |F(x-ai)| = C_a e^{-an}$

(2) $\Rightarrow$ (1). $\sum\limits_{n=-\infty}^{\infty} \hat{f}(n) e^{inz}$ conv. uni. in strip $|\mathrm{Im}\, z| \leq a$. $a \in (0, L)$.

By Weierstrass's thm. (1) holds.

Remark:

§. Examples of diverge Fourier series of a periodic cont. function.

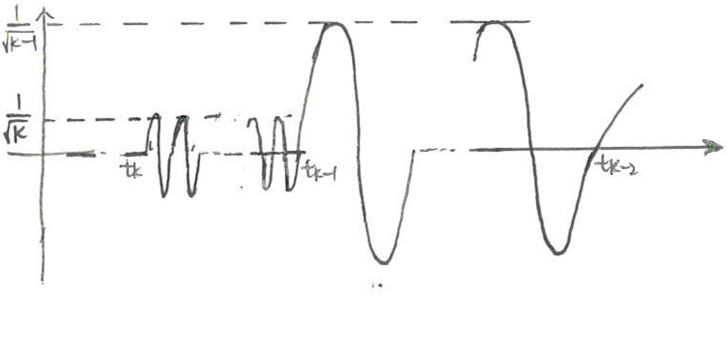
e.g.1 (Schwartz). $f \in \tilde{C}$ $\quad f(t) = \begin{cases} \frac{1}{\sqrt{k}} \sin n_k t, & t \in [t_k, t_{k-1}], \ k = 2, 3, \ldots \\ 0 & , t = 0. \end{cases}$

where $n_k = 2^{k!}$, $t_k = \frac{2\pi}{n_k}$ for $k \in \mathbb{N}$.

(f is even, oscillation frequency increases rapidly when $\to 0$).

we can see this Fourier series, periodic. cont. but diverge.

$S_n(f, 0) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin nt}{t} f(t) dt + o(1)$.



$$I_k = \int_0^{\pi} \frac{\sin n_k t}{t} f(t) dt = \int_0^{t_k} \cdots + \int_{t_k}^{t_{k-1}} + \int_{t_{k-1}}^{\pi} = F_k + J_k + H_k.$$

we have:

$$|F_k| = \left|\int_0^{t_k} \cdots\right| \leq \frac{n_k}{\sqrt{k}} t_k = \frac{2\pi}{\sqrt{k}} \to 0. \qquad (|\sin n_k t| \leq n_k t,\ |f(t)| < \frac{1}{\sqrt{k}}).$$

$$|H_k| \leq \int_{t_{k-1}}^{\pi} \frac{1}{t} dt = \ln \frac{\pi}{t_{k-1}} = \ln \frac{n_{k-1}}{2} < (k-1)!\ln 2.$$

$$|J_k| = \int_{t_k}^{t_{k-1}} \frac{\sin n_k t}{t} f(t)\, dt = \frac{1}{\sqrt{k}} \int_{t_k}^{t_{k-1}} \frac{\sin^2 n_k t}{t} dt = \frac{1}{\sqrt{k}} \int_{2\pi}^{A_k} \frac{\sin^2 u}{u} du.$$

$$A_k = n_k t_{k-1} = \frac{2\pi n_k}{n_{k-1}}.$$

for sufficiently large k,

$$J_k = \frac{1}{2\sqrt{k}} \int_{2\pi}^{A_k} \frac{1-\cos 2u}{u} du = \frac{\ln A_k + O(1)}{2\sqrt{k}} > \frac{k!\ln 2}{3\sqrt{k}}$$

Thus $I_k \geq \frac{k!\ln 2}{3\sqrt{k}} - (k-1)!\ln 2 + O(1) = (k-1)!\ln 2\left(\frac{\sqrt{k}}{3} - 1\right) + O(1) \to +\infty$

$S_{n_k}(f, 0) \to +\infty.$

Thm. (Denjoy–Luzin). If $a_0 + \sum_{n\geq 1}^{\infty}(a_n \cos nx + b_n \sin nx)$ conv. abs. on a set of positive measure. then $\sum_{n=1}^{\infty}(|a_n| + |b_n|) < +\infty$.

Remark: A everywhere conv. trigonometric series maybe not be a Fourier series.
A conv. abs. on positive measure set. trigononmetric series. is the Fourier series of it's sum (because it conv. uni. on $\mathbb{R}$).

§. Practical Part of Fourier series. ☆

Def. 1.1. Let $f \in L^1[-\pi,\pi]$. A (trigonometric) Fourier series of f is a trigonometric series $a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$ with Fouries coefficients.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \quad n \in \mathbb{N}.$$

partial sums of Fourier series of function f are denoted by $S_n(f,x)$

Notation: $f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$

Remark: Fourier series of arbitrary function $f \in L^1[-\pi,\pi]$ tend to 0.

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} b_n = 0.$$

Def. 1.2. Let $f: [-\pi,\pi] \to \mathbb{C}$, $f \in L^1[-\pi,\pi]$. transform a Fourier series of func. to the following form: $\sum_{n=-\infty}^{+\infty} c_n e^{inx}$, $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$, $n \in \mathbb{Z}$.

Then $f \sim \sum_{n=-\infty}^{+\infty} c_n e^{inx}$

if $f: [-\pi,\pi] \to \mathbb{R}$. then $c_0 = a_0$. $c_n = \frac{a_n - ib_n}{2}$, $c_{-n} = \frac{a_n + ib_n}{2}$. $(c_n = \bar{c}_{-n})$.

Def. 1.3. (Fourier Series on arbitrary segment). $f: [a; a+2l] \to \mathbb{R}$.

$f \sim a_0 + \sum_{n=1}^{+\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$. or $f \sim \sum_{n=-\infty}^{+\infty} c_n e^{int/l}$.

Coro 1.3.1. (continued extension). $f \in L^1[0,l]$ can be continued to the segment $[-l,l]$ in even and odd way and be decomposed by sin and cos system.

1. even $f(x) := f(-x)$, $x \in [0,-l]$. $f \sim a_0 + \sum_{n=1}^{+\infty} a_n \cos\frac{\pi n x}{l}$ "cos system"
2. odd $f(x) := -f(-x)$, $x \in [0,-l]$ $f \sim a_0 + \sum_{n=1}^{+\infty} b_n \sin\frac{\pi n x}{l}$. "sin system"

Thm 1.4 (derivative of Fourier series). Let $f \in C[-\pi,\pi]$ be diff. with $f' \in L^1[-\pi,\pi]$ $f(\pi) = f(-\pi)$ and $f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$. Then $f' \sim \sum_{n=1}^{+\infty} (n b_n \cos nx - n a_n \sin nx)$

Thm 1.5. (integration of Fourier Series).

Let $f \in L^1$ and $f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$.

then $\int_0^x f(t)\,dt = a_0 x + \sum_{n=1}^{+\infty} \left(a_n \frac{\sin nx}{n} + b_n \frac{1-\cos nx}{n}\right)$.

Thm (Parseval's identity)

Let $f: [-\pi,\pi] \to \mathbb{R}$, $f \in L^2[-\pi,\pi]$, $f \sim a_0 + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$

Then $\|f\|_{L^2}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)\,dx = a_0^2 + \sum_{n=1}^{+\infty} \frac{a_n^2 + b_n^2}{2}$

If $f: [-\pi,\pi] \to \mathbb{C}$. $f \in L^2[-\pi,\pi]$. $f \sim \sum_{-\infty}^{+\infty} c_n e^{int}$

Then $\|f\|_{L^2}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2(x)\,dx = \sum_{-\infty}^{+\infty} |c_n|^2$

△ orthonormal normalization:

$\tilde{L}_p$: $f(x+2\pi) = f(x)$ a.e. $\|f\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^p\,dx\right)^{\frac{1}{p}} < +\infty$

Thm 1.6. (Uniqueness of Fourier Series).

Let. $f, g \in L^1$. $x \in \mathbb{R}$. $\delta \in (0,\pi)$. f, g. coincide $(x-\delta, x+\delta)$. Then Fourier series of functions f and g behave at x in the same way, i.e. $S_n(f,x) - S_n(g,x) \xrightarrow[n\to\infty]{} 0$

Thm 1.7. (Dini test for convergence of Fourier Series).

Let $f \in L^1$, $x \in \mathbb{R}$. $S \in \mathbb{C}$. s.t. $\int_0^{\pi} \frac{|f(x+t) - 2S + f(x-t)|}{t}\,dt < \infty$.

Then Fourier series of function f conv. at point $x \in \mathbb{R}$ to the sum S.

that is $S_n(f,x) \xrightarrow[n\to\infty]{} S$.

Coro 1.7.1. Let $f \in L^1$, $x \in \mathbb{R}$. Assume these limit exists.

$$f(x\pm) = \lim_{t \to x\pm} f(t) \ ; \quad \alpha_\pm = \lim_{t \to x\pm} \frac{f(x+t) - f(x\pm)}{t}$$

then Fourier series of f conv. at point $x \in \mathbb{R}$. to the sum $S = \frac{f(x+) + f(x-)}{2}$.

In particular, if f is cont. and has one side derivatives $(\alpha_\pm)$ at x.

then the Fourier series conv. to $f(x)$

△ 求Fourier series. 还可做 $\cos x = \frac{z+z^{-1}}{2}$ $\sin x = \frac{z+z^{-1}}{2i}$. $(e^{ix} = z, |z|<1)$

对 $f(z)$ 做泰勒展开, z^n 换回 $(\cos nx + i\sin nx)$ 形式. 即为所求 Fourier series.

§. Practical Part of Fourier Transform. ☆

<u>Def.1.1.</u> Let $f \in L^1(\mathbb{R})$. $\hat{f}(y) = \int_{-\infty}^{+\infty} f(t)\, e^{-2\pi i y t}\, dt$ is Fourier transform (of function f). denote by $\mathcal{F}[f]$.

<u>Def.1.2.</u> Let $f \in L^1(\mathbb{R})$ $\check{f}(y) = \int_{-\infty}^{+\infty} f(t)\, e^{2\pi i y t}\, dt$ is inverse Fourier transform. (of function f). denote by $\mathcal{F}^{-1}[f]$.

Property:

1. $\hat{f} \in C(\mathbb{R})$ and $|\hat{f}(y)| \le \|f\|_1$ for every $y \in \mathbb{R}$.
2. $\hat{f}(y) \longrightarrow 0$ as $y \to \infty$.
3. If $f \in L^1(\mathbb{R})$. and for some $r \in \mathbb{N}$. function $t \mapsto t^r f(t)$ is integrable on $\mathbb{R}$. then $\hat{f} \in C^{(r)}(\mathbb{R})$ and for every $k \in [1:r]$. $(\mathcal{F}[f](y))^{(k)} \cdot \frac{1}{(-2\pi i)^k} = \mathcal{F}[x^k f(x)](y)$.
   $\hat{f}^{(k)}(y) = (-2\pi i)^k \int_{\mathbb{R}} t^k f(t)\, e^{-2\pi i y t}\, dt$. moreover, $\hat{f}^{(k)}(y) \xrightarrow[y\to\infty]{} 0$.
4. Suppose $r \in \mathbb{N}$. $f \in C^{(r)}(\mathbb{R})$. and $f^{(k)} \in L^1(\mathbb{R})$ for every $k \in [0:r]$.
   Then for every $k \in [1:r]$ $\widehat{f^{(k)}}(y) = (2\pi i y)^k \hat{f}(y)$
5. shift and scaling $f_h(x) = f(x+h)$. Then $\hat{f}_h(y) = e^{2\pi i h y} \hat{f}(y)$ (shift)
   $\widehat{(f(a\cdot))}(y) = \frac{1}{|a|} \hat{f}(\frac{y}{a})$, $a \neq 0$. (scale) $\qquad \check{f}(\cdot a)(y) = f(ay)$

<u>Def.1.3.</u> Let $f, g \in L^1(\mathbb{R})$. The function $(f * g)(x) = \int_{\mathbb{R}} f(x-t)\, g(t)\, dt$. convolution of function f and g.

<u>Def.1.4.</u> The integral. $J(f)(x) = \text{P.V.} \int_{-\infty}^{+\infty} \hat{f}(y)\, e^{2\pi i xy}\, dy = \lim_{A\to+\infty} \int_{-A}^{A} \hat{f}(y)\, e^{2\pi i x y}\, dy$. is called Fourier integral.

<u>Thm1.5.</u> (Dini test for convergence of Fourier integral).
Let $f \in L^1(\mathbb{R})$. $x \in \mathbb{R}$. $S \in \mathbb{R}$. and $\int_0^\delta \frac{|f(x+t) - 2S + f(x-t)|}{t}\, dt < +\infty$. for some $\delta > 0$.
Then $S = J(f)(x)$

<u>coro1.5.1.</u> Let $f \in L^1(\mathbb{R})$. $x \in \mathbb{R}$. there exists. $f(x_\pm) = \lim_{t\to x_\pm} f(t)$ $\alpha_\pm = \lim_{t\to 0_\pm} \frac{f(x+t) - f(x_\pm)}{t}$
Then $J(f) = \frac{f(x_+) + f(x_-)}{2}$ (不连续点.表示方法) In particular, if f is cont. at x. and has finite right and left side derivatives then $\underline{J(f)(x) = f(x)}$

<u>coro1.5.2.</u> Let $f \in L(\mathbb{R}) \cap C(\mathbb{R})$. If, moreover. $\hat{f} \in L^1(\mathbb{R})$. then $\underline{f = J(f)}$.
or equivalently. $\underline{f(x) = \mathcal{F}^{-1}[\mathcal{F}[f]](x)}$

§. Sine and Cosine Fourier transforms.

Def. 1.6. Let $f \in L^1(\mathbb{R})$.

Fourier cosine transform. $\mathcal{F}_c[f](y) = \int_{-\infty}^{+\infty} f(x) \cos(2\pi xy)\, dx$

Fourier sine transform $\mathcal{F}_s[f](y) = \int_{-\infty}^{+\infty} f(x) \sin(2\pi xy)\, dx$.

Notice that. $\mathcal{F}[f](y) = \mathcal{F}_c[f](y) - i\mathcal{F}_s[f](y)$.

f - even. $\mathcal{F}_s[f](y) = 0$. $\mathcal{F}[f] = \mathcal{F}_c[f](y) = 2\int_0^{+\infty} f(x) \cos(2\pi xy)\, dx$

f - odd $\mathcal{F}_c[f](y) = 0$. $\mathcal{F}[f] = \mathcal{F}_s[f](y) = -i2\int_0^{+\infty} f(x) \sin(2\pi xy)\, dx$.

Def. 1.7. Let $f \in L^1(0, +\infty)$

Fourier cosine transform $\mathcal{F}_c[f](y) = 2\int_0^{+\infty} f(x) \cos(2\pi xy)\, dx$.

Fourier sine transform. $\mathcal{F}_s[f](y) = 2\int_0^{+\infty} f(x) \sin(2\pi xy)\, dx$.

Thm 1.8 $f \in L^1(0,+\infty) \cap C[0,+\infty)$. If. $\hat{f} \in L^1(0,+\infty)$. then.

$$\mathcal{F}_c[\mathcal{F}_c(f)](x) = f(x) \qquad x \geq 0$$

$$\mathcal{F}_s[\mathcal{F}_s(f)](x) = f(x) \qquad x > 0.$$

常用 Fourier transformation:

$$\mathcal{F}[e^{-\alpha|x|}] = \frac{2\alpha}{\alpha^2 + (2\pi y)^2} \qquad \alpha > 0.$$