23.10.24

4.3. PROBLEMS ON AN INFINITE LINE FOR THE EQUATION OF THERMAL CONDUCTIVITY

4.3.1. CAUCHY PROBLEM

Let's consider a problem with initial data on an infinite line (Cauchy problem): find the function $u(x,t)$ $(t>0, -\infty<x<\infty)$ satisfying the equation of thermal conductivity (heat equation):

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \qquad (4.14)$$

and the initial condition

$$u(x,0)=\varphi(x), -\infty<x<\infty \qquad (4.15)$$

where $\varphi(x)-$ is a continuous and bounded function.

Let's first find a partial solution of equation (4.14) in the form of a product:

$$u(x,t)=X(x)T(t),$$

substituting which into equation (4.14), we have

$$X(x)T'(t)=a^2X''(x)T(t).$$

Dividing both parts of this equation by $a^2X(x)T(t)$, we obtain

$$\frac{T'(t)}{a^2T(t)}=\frac{X''(x)}{X(x)}. \qquad (4.16)$$

The right side of equality (4.16) is a function of only variable x, and the left side is only t, so the right and left sides of equality (4.16) retain a constant value when changing their arguments. It is convenient to denote this value by $-\lambda^2$, that is , we have

$$\frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 ,$$

$$X''(x) + \lambda^2 X(x) = 0 ,\ T'(t) + \lambda^2 a^2 T(t) = 0 ,$$

$$T(t) = e^{-a^2\lambda^2 t}$$

$$X(x) = A(\lambda)e^{i\lambda x}$$

We obtain a partial solution of equation (4.14):

$$u_\lambda(x,t) = A(\lambda)e^{-a^2\lambda^2 t + i\lambda x} . \qquad (4.17)$$

Here λ is any real number. Integrating (4.17) with respect to the parameter λ, we also obtain the solution of equation (4.14):

$$u(x,t) = \int_{-\infty}^{\infty} A(\lambda)e^{-a^2\lambda^2 t + i\lambda x} d\lambda . \qquad (4.18)$$

Requiring the fulfillment of the initial condition (4.15) at $t = 0$, we will

Have

$$\varphi(x) = \int_{-\infty}^{\infty} A(\lambda)e^{i\lambda x} d\lambda .$$

Let's now use the formula for the inverse transformation of the Fourier integral:

$$A(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(\xi)e^{-i\lambda\xi} d\xi .$$

Substituting this function in (4.18) and changing the order of integration, we obtain

$$u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\varphi(\xi)e^{-i\lambda\xi}d\xi\right]e^{-a^2\lambda^2t+i\lambda x}d\lambda=$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}e^{-a^2\lambda^2t+i\lambda(x-\xi)}d\lambda\right]\varphi(\xi)d\xi\,. \qquad (4.19)$$

The internal integral in (4.19):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-a^2\lambda^2t+i\lambda(x-\xi)}d\lambda=\frac{1}{2\sqrt{\pi a^2t}}e^{-\frac{(x-\xi)^2}{4a^2t}}\,. \qquad (4.20)$$

Substituting (4.20) into (4.19), we arrive at an integral representation of the desired solution:

$$u(x,t)=\int_{-\infty}^{\infty}G(x,\xi,t)\varphi(\xi)d\xi\,,$$

where

$$G(x,\xi,t)=\frac{1}{2\sqrt{\pi a^2t}}e^{-\frac{(x-\xi)^2}{4a^2t}}\,. \qquad (4.21)$$

The function (4.21) is called the fundamental solution of the thermal conductivity equation.

The fundamental solution $G(x,\xi,t)$ (4.21) gives a temperature distribution in an infinite rod if, at the initial moment of time $t=0$, an amount of heat $Q=c\rho$ is instantly released at the point $x=\xi$.

Theorem. For any bounded continuous function j(x), there is a unique solution to the Cauchy problem (4.14)–(4.15), which has the form

$$u(x,t)=\int_{-\infty}^{\infty} G(x,\xi,t)\varphi(\xi)d\xi\,. \tag{4.22}$$

Example 1

$$4u_t = u_{xx}$$

$$u\big|_{t=0} = e^{2x-x^2}$$

<u>Solution:</u>

$$\begin{cases} u_t = \dfrac{1}{4}u_{xx} \\ u\left(x,0\right) = e^{2x-x^2} \end{cases}$$

Let's use the Poisson formula and write down the answer:

$$u\left(x,t\right)=\frac{1}{2\sqrt{\pi\frac{1}{4}t}}\int_{-\infty}^{\infty} e^{2\xi-\xi^2}e^{-\frac{\left(x-\xi\right)^2}{4\frac{1}{4}t}}\,d\xi =$$

$$=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{2\xi-\xi^2-\frac{x^2}{t}+\frac{2\xi x}{t}-\frac{\xi^2}{t}}\,d\xi$$

$$2\xi-\xi^2-\frac{x^2}{t}+\frac{2\xi x}{t}-\frac{\xi^2}{t}=-\left(\xi^2\left(1+\frac{1}{t}\right)-2\xi\left(1+\frac{x}{t}\right)+\frac{x^2}{t}\right)=$$

$$=-\frac{t+1}{t}\left(\xi^2-2\left(\frac{t+x}{t+1}\right)\xi+\frac{x^2}{t+1}\right)=$$

$$=-\frac{t+1}{t}\left[\xi^2-2\left(\frac{t+x}{t+1}\right)\xi+\left(\frac{t+x}{t+1}\right)^2-\left(\frac{t+x}{t+1}\right)^2+\frac{x^2}{t+1}\right]=$$

$$=-\frac{t+1}{t}\left[\left(\xi-\frac{t+x}{t+1}\right)^2-\left(\frac{t+x}{t+1}\right)^2+\frac{x^2}{t+1}\right]$$

$$u(x,t)=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{2\xi-\xi^2-\frac{x^2}{t}+\frac{2\xi x}{t}-\frac{\xi^2}{t}}d\xi=$$

$$=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty}e^{-\frac{t+1}{t}\left[\left(\xi-\frac{t+x}{t+1}\right)^2-\left(\frac{t+x}{t+1}\right)^2+\frac{x^2}{t+1}\right]}d\xi=$$

$$=\frac{1}{\sqrt{\pi t}}e^{-\frac{t+1}{t}\left[\frac{x^2}{t+1}-\left(\frac{t+x}{t+1}\right)^2\right]}\int_{-\infty}^{\infty}e^{-\frac{t+1}{t}\left(\xi-\frac{t+x}{t+1}\right)^2}d\left(\xi-\frac{x+t}{t+1}\right)=$$

$$=\frac{1}{\sqrt{\pi t}}e^{-\frac{t+1}{t}\left[\frac{x^2}{t+1}-\left(\frac{t+x}{t+1}\right)^2\right]}\cdot\sqrt{\frac{t}{t+1}}\int_{-\infty}^{\infty}e^{-\left[\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)\right]^2}d\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)$$

$$\int_{-\infty}^{\infty}e^{-\left[\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)\right]^2}d\sqrt{\frac{t+1}{t}}\left(\xi-\frac{t+x}{t+1}\right)=\sqrt{\pi}$$

$$u(x,t)=\frac{1}{\sqrt{t+1}}e^{-\frac{t+1}{t}\left(\frac{x^2}{t+1}-\left(\frac{x+t}{t+1}\right)^2\right)}$$

4.3.2. BOUNDARY VALUE PROBLEM FOR A SEMI-BOUNDED LINE

In cases where the temperature distribution near one of the ends of the rod is interesting, and the influence of the other is insignificant, it is assumed that this end is at infinity, this leads to the problem of determining the solution of the thermal conductivity equation on a semi-bounded straight line.

So, the following first boundary value problem is considered.

Find a solution to the thermal conductivity equation:

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \ x > 0, \ t > 0, \tag{4.23}$$

satisfying the initial condition

$$u(x,0) = \varphi(x), \ x > 0 \tag{4.24}$$

and homogeneous boundary condition

$$u(0,t) = 0, \ t > 0. \tag{4.25}$$

Let's say:

$$\varphi(x) = \begin{cases} \varphi(x), & x > 0, \\ -\varphi(-x), & x < 0 \end{cases}$$

and function

$$\upsilon(x,t) = \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}} \phi(\xi) d\xi .$$

It is easy to verify that

$$\upsilon(0,t)=0 .$$

Thus, according to formulas (4.21) and (4.22), the function $u(x,t)=\upsilon(x,t)$ for $x>0$ gives a solution to the boundary value problem (4.23)–(4.25).

We will have

$$\upsilon(x,t)=\frac{1}{2\sqrt{\pi a^2 t}}\left[\int_{-\infty}^{0} e^{-\frac{(x-\xi)^2}{4a^2 t}}\phi(\xi)d\xi+\int_{0}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}}\phi(\xi)d\xi\right]=$$

$$=\frac{1}{2\sqrt{\pi a^2 t}}\left[-\int_{0}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}}\varphi(\xi)d\xi+\int_{0}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2 t}}\varphi(\xi)d\xi\right].$$

Combining both integrals together, we get the desired function

$$u(x,t)=\frac{1}{2\sqrt{\pi a^2 t}}\int_{0}^{\infty}\left[e^{-\frac{(x-\xi)^2}{4a^2 t}}-e^{-\frac{(x+\xi)^2}{4a^2 t}}\right]\varphi(\xi)d\xi .$$

Example 2

$$\begin{cases} u_t=a^2u_{xx} \\ u(x,0)=xe^{-x^2} \qquad\qquad 0\le x<\infty \\ u(0,t)=0 \end{cases}$$

<u>Solution:</u>

We want to move on to integration along an infinite straight line. We write the same equation. The initial conditions are set in the form $u(x,0)=\varphi(x)$. Which we want to get by an odd continuation of the function xe^{-x^2} into the negative region. But if we say that $\varphi(x)=xe^{-x^2}$, then we will see that it is odd. If we put $-x$ instead of x, we just get a minus, the exponent does not change in any way, since we have a square in degree, and x has a minus. This function is odd in itself, so we solve this problem:

$$\begin{cases} u_t = a^2 u_{xx} \\ u(x,0) = xe^{-x^2} \end{cases} \qquad -\infty < x < \infty$$

We solve the problem on an infinite straight line, and then we say that due to the fact that the function is odd, we always have for positive $0 \le x < \infty$, the solution of the second problem will coincide with the solution of the original one. We use the Poisson formula.

$$u(x,t) = \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} \xi e^{-\xi^2} e^{-\frac{(x-\xi)^2}{4a^2 t}} d\xi =$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} \xi e^{-\left(\xi^2 + \frac{x^2}{4a^2 t} - \frac{2x\xi}{4a^2 t} + \frac{\xi^2}{4a^2 t}\right)} d\xi =$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} \xi e^{-\frac{1}{4a^2 t}\left[\xi^2 (4a^2 t + 1) - 2x\xi + x^2\right]} d\xi =$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} \xi e^{-\frac{1}{4a^2 t}\left[(4a^2 t + 1)\xi^2 - 2x\xi + \frac{x^2}{(1+4a^2 t)} - \frac{x^2}{1+4a^2 t} + x^2\right]} d\xi =$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} \int_{-\infty}^{\infty} \xi e^{-\frac{1}{4a^2 t}\left(\sqrt{1+4a^2 t}\,\xi - \frac{x}{\sqrt{1+4a^2 t}}\right)^2} e^{-\frac{1}{4a^2 t}\frac{4a^2 t x^2}{1+4a^2 t}} d\xi =$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} e^{-\frac{x^2}{1+4a^2t}} \int_{-\infty}^{\infty} \xi e^{-\left(\frac{\sqrt{1+4a^2t}\ \xi}{2\sqrt{a^2t}} - \frac{x}{2\sqrt{a^2t}\sqrt{1+4a^2t}}\right)^2} d\xi =$$

let's replace the variable

$$\frac{\sqrt{1+4a^2t}\ \xi}{2\sqrt{a^2t}} - \frac{x}{2\sqrt{a^2t}\sqrt{1+4a^2t}} = \theta$$

$$\xi = \left(2\sqrt{a^2t}\ \theta + \frac{x}{\sqrt{1+4a^2t}}\right)\frac{1}{\sqrt{1+4a^2t}} = \frac{2\sqrt{a^2t}\ \ \theta}{\sqrt{1+4a^2t}} + \frac{x}{1+4a^2t}$$

$$d\xi = \frac{2\sqrt{a^2t}}{\sqrt{1+4a^2t}} d\theta$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} e^{-\frac{x^2}{1+4a^2t}} \int_{-\infty}^{\infty} \left(\frac{2\sqrt{a^2t}\ \ \theta}{\sqrt{1+4a^2t}} + \frac{x}{1+4a^2t}\right) e^{-\theta^2} \frac{2\sqrt{a^2t}}{\sqrt{1+4a^2t}} d\theta =$$

$$= \frac{1}{2\sqrt{\pi a^2 t}} e^{-\frac{x^2}{1+4a^2t}} \frac{2\sqrt{a^2t}}{\sqrt{1+4a^2t}} \left[\frac{2\sqrt{a^2t}}{\sqrt{1+4a^2t}} \int_{-\infty}^{\infty} \theta e^{-\theta^2} d\theta + \frac{x}{1+4a^2t} \int_{-\infty}^{\infty} e^{-\theta^2} d\theta\right]$$

This integral $\int_{-\infty}^{\infty} \theta e^{-\theta^2} d\theta$ is the integral of an odd function. The function is odd, gives zero in the integral.

We know this integral $\int_{-\infty}^{\infty} e^{-\theta^2} d\theta$, it gives us $\sqrt{\pi}$. This is the Poisson integral.

$$=\frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{1+4a^2t}}\cdot\frac{1}{\sqrt{1+4a^2t}}\cdot\frac{x}{1+4a^2t}\cdot\sqrt{\pi}=$$

$$=\frac{x}{\left(1+4a^2t\right)^{\frac{3}{2}}}e^{-\frac{x^2}{1+4a^2t}}$$

4.3.3. APPLICATION OF THE LAPLACE. TRANSFORM TO SOLVING BOUNDARY VALUE PROBLEMS

Let it be required to solve the following first boundary value problem.

Find a solution to the thermal conductivity equation (heat equation):

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2},\ x>0,\ t>0, \tag{4.26}$$

satisfying the initial condition

$$u(x,0)=0,\ x>0$$

and the boundary condition

$$u(0,t)=\mu(t),\ t>0.$$

We assume that the constraints on the parameters of the problem allow the application of the Laplace transform.

We apply to equation (4.26) the Laplace transform with respect to the time variable t, assuming $u(x,t)\leftrightarrow U(x,p)$. Since

$$\frac{\partial u}{\partial t}\leftrightarrow pU(x,p)-u(x,0)=pU(x,p),$$

$$\frac{\partial^2 u}{\partial x^2} \leftrightarrow \frac{\partial^2 U(x,p)}{\partial x^2},$$

$$u(0,t) = \mu(t) \leftrightarrow U(0,p) = M(p),$$

then the specified transformation gives the operator equation

$$pU(x,p) = a^2 \frac{\partial^2 U(x,p)}{\partial x^2},$$

to which the condition $U(0,p) = M(p)$ should be added.

The resulting equation can be considered as an ordinary second - order differential equation with constant coefficients for the function U, with an independent variable x and a parameter p. The general solution of this ordinary differential equation has the form

$$U(x,p) = c_1(p)e^{x\frac{\sqrt{p}}{a}} + c_2(p)e^{-x\frac{\sqrt{p}}{a}}.$$

To determine the coefficients $c_1(p)$ and $c_2(p)$, we use the ratio $U(0,p) = M(p)$ and the fact that $U(x,p) \to 0$ and $p \to \infty$. We get that $c_1(p) = 0,\ c_2(p) = M(p)$. Thus,

$$U(x,p) = M(p)e^{-x\frac{\sqrt{p}}{a}}.$$

By performing the inverse Laplace transform, we find

$$e^{-x\frac{\sqrt{p}}{a}} \leftrightarrow \frac{x}{2a\sqrt{\pi t^3}} e^{-\frac{x^2}{4a^2 t}}$$

$$M(p) \leftrightarrow \mu(t)$$

Next, applying the convolution image property, we get

$$U(x,p)=M(p)e^{-x\frac{\sqrt{p}}{a}}\leftrightarrow u(x,t)=\frac{x}{2a\sqrt{\pi}}\int_0^t\frac{\mu(\tau)}{\sqrt{(t-\tau)^3}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau\,.$$

Now let's assume that we need to solve the second boundary value problem.

Find a solution to the thermal conductivity equation (heat equation):

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}\,,\ x>0\,,\ t>0\,, \qquad (4.27)$$

satisfying the initial condition

$$u(x,0)=0\,,\ x>0$$

and the boundary condition

$$\frac{\partial u(0,t)}{\partial x}=\upsilon(t)\,,\ t>0\,.$$

We apply to equation (4.27) the Laplace transform with respect to the time variable t, assuming $u(x,t)\leftrightarrow U(x,p)$. Since

$$\frac{\partial u}{\partial t}\leftrightarrow pU(x,p)-u(x,0)=pU(x,p)\,,$$

$$\frac{\partial^2 u}{\partial x^2}\leftrightarrow\frac{\partial^2 U(x,p)}{\partial x^2}\,,$$

$$\frac{\partial u(0,t)}{\partial x}=\upsilon(t)\leftrightarrow\frac{\partial U(0,p)}{\partial x}=N(p)\,,$$

then the specified transformation gives the operator equation

$$pU(x,p)=a^2\frac{\partial^2 U(x,p)}{\partial x^2},$$

to which the condition should be added

$$\frac{\partial U(0,p)}{\partial x}=N(p).$$

The resulting equation can be considered as an ordinary second - order differential equation with constant coefficients for the function U, with an independent variable x and a parameter p. The general solution of this ordinary differential equation has the form

$$U(x,p)=c_1(p)e^{x\frac{\sqrt{p}}{a}}+c_2(p)e^{-x\frac{\sqrt{p}}{a}}.$$

To determine the coefficients $c_1(p)$ and $c_2(p)$, we use the condition $\frac{\partial U(0,p)}{\partial x}=N(p)$ and the fact that $U(x,p)\to 0$ at $p\to\infty$. We get that $c_1(p)$, $c_2(p)=-\frac{a}{\sqrt{p}}N(p)$. Thus,

$$U(x,p)=-\frac{a}{\sqrt{p}}N(p)e^{-x\frac{\sqrt{p}}{a}}.$$

By performing the inverse Laplace transform, we find

$$-\frac{a}{\sqrt{p}}e^{-x\frac{\sqrt{p}}{a}}\leftrightarrow-\frac{a}{\sqrt{\pi t}}e^{-\frac{x^2}{4a^2t}},$$

$$N(p)\leftrightarrow \upsilon(t).$$

Next, applying the convolution image property, we get

$$U(x,p)=-\frac{a}{\sqrt{p}}N(p)e^{-x\frac{\sqrt{p}}{a}} \leftrightarrow u(x,t)=-\frac{a}{\sqrt{\pi}}\int_0^t \frac{\upsilon(\tau)}{\sqrt{(t-\tau)}}e^{-\frac{x^2}{4a^2(t-\tau)}}d\tau .$$