Introduction: from Dido's problem to soft spacecraft landing

Dido's problem

This problem comes from the epic story of queen Dido, the daughter of the Phoenician king. When she had been on the Mediterranean coast and asked the local leader for as much land as could be enclosed by the piece of leather.Dido cut the leather into narrow strips, tied them together and encircled a large tract of land which became the city of Carthage. Dido faced the following mathematical problem: find among all curves of given length the one which encloses maximal area.

Spacecraft landing

The problem is how to bring a spacecraft to a soft landing, using the least amount of fuel.

Examples of optimization problems

Finite-dimensional space

Let's look at some examples of finite-dimensional optimization problems. All of them can be reduced even to the minimization of a function of one real variable.

1. triangle of smallest area
2. encircled rectangle of maximum area
3. stick of longest length carring along the coridor (the sofa problem)
4. the largest viewing angle of the painting

Functional space

A *functional* is a real-valued function whose domain is a set of functions, that is, $J\colon S \to \mathbb{R}$ for some set of functions S.

The main task of the calculus of variations is to find a function y^* on which the functional J reaches its maximum (or minimum):

$$y^* = \underset{y \in S}{\operatorname{argmax}}\, J[y],$$

that is,

$$J[y^*] \geqslant J[y]$$

for all $y \in S$. Such a function y^* is said to be an *extremizing function* or simply an *extremal*.

Let's look at some examples of functionals and corresponding problems of the calculus of variations. Next we use the standard Cartesian coordinate system Oxy.

The simplest variational problem

• **Shortest distance.** Find the shortest path joining given points (a, A) and (b, B).

The length of a curve $y(x)$ is a functional

$$J[y] = \int_a^b \sqrt{1 + y'(x)^2} dx$$

subject to minimization with restrictions $y(a) = A, y(b) = B$.

• **Soap films.** Find the surface of minimal area that is generated by revolving a curve $y(x)$ about the x-axis, where $y(x)$ passes through two given points (a, A) and (b, B).

The area of such a surface is a functional

$$J[y] = 2\pi \int_a^b y\sqrt{1 + (y')^2} dx$$

subject to minimization with restrictions $y(a) = A, y(b) = B$.

• **Brachistocrone.** Find the shape of a tube joining two given points (a, A) and (b, B), so that a ball will slide down the tube under gravity from one point to the other (without friction) in the shortest time.

Without loss of generality, let's assume that $(a, A) = (0, 1)$ and $(b, B) = (b, 0)$. If the function $y(x)$ describes the height of the ball's position, then it can be shown that time can be expressed by the functional

$$J[y] = \int_0^b \sqrt{\frac{1 + (y')^2}{2gy}} dx.$$

All the examples above have a similar structure. Here we have *the simplest variational problem*:

> find the extremum of a functional
>
> $$J[y] = \int_a^b F(x, y, y') dx. \qquad \text{(functional)}$$
>
> among smooth curves y satisfying the boundary conditions
>
> $$y(a) = A,\ y(b) = B \qquad \text{(fixed ends)}$$

Here $F(x, y, y')$ is an expression containing at least one of the functions y and y' (which depend on variable x), and possibly also the variable x.

The isoperimetric problem

Now let's go back to the problem from the introduction and rewrite it in mathematical terms.

• **Dido's problem.** Find the plane curve of fixed perimeter L and fixed ends (a, A) and (b, B) which has maximum area above x-axis. The area under the curve $y(x)$ is a functional

$$J[y] = \int_a^b y(x)dx$$

subject to minimization with additional constraint

$$\int_a^b \sqrt{1 + y'(x)^2}dx = L$$

Note that now the domain of the functional is limited only by curves of length L. Moreover, the problem may not have a solution. For example, if L is less than $b - a$ then there will be no function joining two given points.

Here we have *the isoperimetric problem*:

> find the extremum of a functional
>
> $$J[y] = \int_a^b F(x, y, y')dx. \qquad \text{(functional)}$$
>
> among smooth curves y satisfying the boundary conditions
>
> $$y(a) = A, \; y(b) = B \qquad \text{(fixed ends)}$$
>
> and the isoperimetric constraint
>
> $$\int_a^b G(x, y, y') = L \qquad \text{(iC)}$$

The last constraint is also called *the integral dependence.* A differential dependence may also naturally appear in problems of the calculus of variations.

Generalizations

If we don't fix the end(s), we get the next problem:

> find the extremum of a functional
>
> $$J[y] = \int_a^b F(x, y, y')dx + g(y(b)). \tag{functional}$$
>
> among smooth curves y satisfying the initial boundary condition
>
> $$y(a) = A \tag{fixed end}$$

Here $g\colon \mathbb{R} \to \mathbb{R}$ is a *terminal payoff.*

The general problem of the calculus of variations could be also written as follows:

> find the extremum of a functional
>
> $$J[y] = \int_a^b F(x, y, y')dx. \tag{functional}$$
>
> among smooth curves y satisfying the boundary conditions
>
> $$y(a) = A,\ y(b) = B, \tag{fixed ends}$$
>
> differential equation
>
> $$y' = f(y), \tag{dC}$$
>
> and the isoperimetric constraint
>
> $$\int_a^b G(x, y, y') = L \tag{iC}$$

The simplest variational problem

Notation

We will use a regular font for real numbers and real-valued functions (t, x, f) and bold font for vectors and vector-valued functions $(\mathbf{x}, \mathbf{f})$. We will denote the components of a vector-valued function by a superscript and write them as columns:

$$\mathbf{x}(t) = \begin{pmatrix} x^1(t) \\ x^2(t) \\ \vdots \\ x^n(t) \end{pmatrix}$$

Directional derivative

Let's recall the definition of the directional derivative from the classical multivariable analysis. Let's consider a real-valued function $f\colon \mathbb{R}^n \to \mathbb{R}$ and fix a point $\mathbf{x} \in \mathbb{R}^n$ and a vector $\mathbf{v} \in \mathbb{R}^n$. Define $\varphi(\varepsilon) = f(\mathbf{x} + \varepsilon\mathbf{v})$. Then *the directional derivative* at the point $\mathbf{x} \in \mathbb{R}^n$ along vector $\mathbf{v} \in \mathbb{R}^n$ is the limit:

$$\lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon\mathbf{v}) - f(\mathbf{x})}{\varepsilon} = \left.\frac{d\varphi}{d\varepsilon}\right|_{\varepsilon=0}$$

It is clear, that derivative in any direction must be zero at the maximum point. Therefore, this is a necessary condition for the maximum (but not sufficient). We want to define something analogous to the directional derivatives and use this to formulate a necessary condition for the maximum of the functional.

Variation of functional

Let's consider some perturbation of the real-valued function $y(x)$. Define $y_\varepsilon(x) = y(x) + \varepsilon\eta(x)$ where $\eta(x)$ is a continuously differentiable function that satisfies $\eta(a) = \eta(b) = 0$ and ε is a real number.

Fix y and η and define $J(\varepsilon) = J[y_\varepsilon(\cdot)]$. The *variation* of a functional J on y under η is the limit:

$$\lim_{\varepsilon \to 0} \frac{J[y(\cdot) + \varepsilon\eta(\cdot)] - J[y]}{\varepsilon} = \left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0}$$

A function f is said to be a *stationary path* of J, if the variation of the functional is zero on all functions η that satysfy $\eta(a) = \eta(b) = 0$.

Therefore, a necessary condition for the function to be an extremal is that the variation of functional vanishes:

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0, \text{ for any } \eta\colon \eta(a) = \eta(b) = 0$$

Euler-Lagrange equation (EL)

For the main result of this section we need the following theorem

Theorem 1 (Fundamental lemma of calculus of variations)**.** *If $f(x)$ is a continuous function on $[a, b]$ and if*

$$\int_a^b f(x)g(x)dx = 0$$

for every function $g(x) \in C(a, b)$ such that $g(a) = g(b) = 0$ then

$$f(x) \equiv 0 \text{ on } [a, b].$$

Proof. Let $f(x) \neq 0$ for some $c \in (a, b)$. Without loss of generality let, us assume that $f(c) > 0$. Because of continuity of f there exists a subinterval $[x_1, x_2] \subset [a, b]$ where $f(x) > 0$ that contains the point c. Let's define a function

$$g(x) = \begin{cases} (x - x_1)(x_2 - x) & \text{for } x \in [x_1, x_2] \\ 0 & \text{outside } [x_1, x_2] \end{cases}$$

Note that $(x - x_1)(x_2 - x)$ is positive for $x \in (x_1, x_2)$.
Now we have

$$\begin{aligned} \int_a^b f(x)g(x)dx &= \underbrace{\cancel{\int_a^{x_1} f(x)g(x)}}_{g(x)=0 \text{ on } [a,x_1]} + \int_{x_1}^{x_2} f(x)g(x)dx + \underbrace{\cancel{\int_{x_2}^{b} f(x)g(x)}}_{g(x)=0 \text{ on } [x_2,b]} \\ &= \int_{x_1}^{x_2} f(x)g(x)dx \\ &= \int_{x_1}^{x_2} f(x)(x - x_1)(x_2 - x)\, dx > 0, \end{aligned}$$

a contradiction.

□

Theorem 2 (Necessary condition for extremum). *If $y(x)$ is an extremal for the simple variational problem*

$$J[y] = \int_a^b F(x, y, y')dx. \tag{functional}$$

$$y(a) = A,\ y(b) = B, \tag{fixed ends}$$

then $y(x)$ is a solution for Euler-Lagrange equation:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 \tag{EL}$$

$$y(a) = A,\ y(b) = B, \tag{fixed ends}$$

Proof. Here we have

$$J(\varepsilon) = \int_a^b F\left(x, y_\varepsilon(x), y'_\varepsilon(x)\right) dx$$

where $y_\varepsilon(x) = y(x) + \varepsilon\eta(x)$. Since $y(x)$ is an extremal, then

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0.$$

Using the chain rule, we get

$$\frac{dJ}{d\varepsilon} = \int_a^b \left[\frac{\partial F}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial y_\varepsilon}\frac{\partial y_\varepsilon}{\partial \varepsilon} + \frac{\partial F}{\partial y'_\varepsilon}\frac{\partial y'_\varepsilon}{\partial \varepsilon}\right] dx$$

Now, such that

$$\frac{\partial x}{\partial \varepsilon} = 0$$
$$\frac{\partial y_\varepsilon}{\partial \varepsilon} = \eta(x)$$
$$\frac{\partial y'_\varepsilon}{\partial \varepsilon} = \eta'(x)$$

we have

$$\frac{dJ}{d\varepsilon} = \int_a^b \left[\frac{\partial F}{\partial y_\varepsilon}\eta(x) + \frac{\partial F}{\partial y'_\varepsilon}\eta'(x)\right] dx$$

Integrating by parts, we get

$$\int_a^b \frac{\partial F}{\partial y'_\varepsilon}\eta'(x)dx = \underbrace{\cancel{\left.\frac{\partial F}{\partial y'_\varepsilon}\eta(x)\right|_a^b}}_{\eta(a)=\eta(b)=0} - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'_\varepsilon}\right)\eta(x)dx = -\int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial y'_\varepsilon}\right)\eta(x)dx$$

So, such that $y_\varepsilon = y,\ y'_\varepsilon = y'$ for $\varepsilon = 0$ we have

$$\left.\frac{dJ}{d\varepsilon}\right|_{\epsilon=0} = \int_a^b \left.\left(\frac{\partial F}{\partial y_\varepsilon} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'_\varepsilon}\right)\right)\right|_{\varepsilon=0} \eta(x)dx = \int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)\eta(x)dx.$$

$$\left.\frac{dJ}{d\varepsilon}\right|_{\epsilon=0} = 0 \Rightarrow \int_a^b \left(\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right)\eta(x)dx = 0.$$

Since $\eta(x)$ is an arbitrary function, by applying the fundamental lemma of calculus of variations, we get

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

□

Note, that this form of Euler-Lagrange equation includes only partial derivatives with respect to y and y'.

Proposition. *There is another form of Euler-Lagrange equation*

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$$

Proof. Using the chain rule, we have

$$\frac{d}{dx}F(x,y,y') = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}y''$$

Now,

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = y'\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y'}y''$$

Let's subtract the former from the last equation:

$$\frac{dF}{dx} - \frac{d}{dx}\left(y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)y'$$

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = y'\underbrace{\cancel{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right)}}_{\text{Euler-Lagrange equation}}$$

Finally, we have another form of Euler-Lagrange equation

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial x} = 0$$

□

Note, that this form of Euler-Lagrange equation includes only partial derivatives with respect to x and y'.

Special Cases

Function F does not depend on x explicitly $\left(\frac{\partial F}{\partial x} = 0\right)$, then

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) - \cancel{\frac{\partial F}{\partial x}} = 0$$

Hence,

$$F - y'\frac{\partial F}{\partial y'} = \text{const.}$$

This equation is known as *Beltrami identity*.

Function F does not depend on y explicitly $\left(\frac{\partial F}{\partial y} = 0\right)$, then

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \cancel{\frac{\partial F}{\partial y}} = 0$$

Hence,

$$\frac{\partial F}{\partial y'} = \text{const.}$$

Function F does not depend on y' explicitly $\left(\frac{\partial F}{\partial y'} = 0\right)$, then

$$\cancel{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)} - \frac{\partial F}{\partial y} = 0$$

Hence,

$$\frac{\partial F}{\partial y} = 0.$$

Examples

• **Shortest distance.** Find the shortest smooth plane curve joining given points (a, A) and (b, B).
Here, $F(x, y, y') = \sqrt{1 + y'^2}$, so

$$\frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

and since the function F does not depend on y explicitly, then

$$\frac{\partial F}{\partial y'} = \text{const} = c.$$

It gives us

$$\frac{y'}{\sqrt{1+y'^2}} = c.$$

Solving this equation for the variable y' we get

$$y' = \sqrt{\frac{c^2}{1-c^2}} = m,$$

or

$$y = mx + b.$$

Using fixed ends conditions $y(a) = A, y(b) = B$ we find the unknown coefficients

$$m = \frac{B-A}{b-a}$$

and

$$n = A - \frac{B-A}{b-a}a.$$

Thus the curve having minimum arc length passing through the given two fixed point is a straight line.

- **Brachistocrone.**

Euler-Lagrange equation for isoperimetric problem

Let's recall isoperimetric problem:

> find the extremum of a functional
>
> $$J[y] = \int_a^b F(x, y, y')dx. \tag{functional}$$
>
> among smooth curves y satisfying the boundary conditions
>
> $$y(a) = A,\ y(b) = B \tag{fixed ends}$$
>
> and the isoperimetric constraint
>
> $$\int_a^b G(x, y, y') = L \tag{iC}$$

We will reduce the problem to the previous one using the next technique.

Lagrange Multiplier Technique:

Convert the constrained optimization problem into an unconstrained optimization problem by the *Lagrange Multiplier Technique.*

Define a new functional H by $H(x, y, y') = F(x, y, y') + \lambda G(x, y, y')$ and optimize $\int_a^b H(x, y, y')\, dx$ without constraints. That is,

> find the extremum of a functional
>
> $$J[y] = \int_a^b F(x, y, y') + \lambda G(x, y, y')\, dx \qquad \text{(functional)}$$
>
> among smooth curves y satisfying the boundary conditions
>
> $$y(a) = A,\ y(b) = B \qquad \text{(fixed ends)}$$

The problem is solved by solving the Euler-Lagrange equation:

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$
$$y(a) = A, \quad y(b) = B$$

- **Dido's problem**

$$\text{Maximize } J[y] = \int_a^b y(x)dx \text{ subject to}$$
$$\int_a^b \sqrt{1 + y'^2}dx = L \text{ and with end conditions}$$
$$y(a) = A, \quad y(b) = B$$
$$\text{Here, } F(x, y, y') = y(x), \quad G(x, y, y') = \sqrt{1 + y'^2}$$
$$H(x, y, y') = y + \lambda\sqrt{1 + y'^2}$$
$$\frac{\partial H}{\partial y} = 1, \quad \frac{\partial H}{\partial y'} = \frac{\lambda y'}{\sqrt{1 + y'^2}}$$

Euler's Equation:

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}}\right) &= 1 \\
\Longrightarrow \frac{\lambda y'}{\sqrt{1+y'^2}} &= x + c_1 \\
\Longrightarrow \frac{\lambda y'}{x+c_1} &= \sqrt{1+y'^2} \\
\Longrightarrow \lambda^2 y'^2 &= \left(1+y'^2\right)(x+c_1)^2 \\
\Longrightarrow \quad y'^2\left(\lambda^2-(x+c_1)^2\right) &= (x+c_1)^2 \\
y' &= \frac{x+c_1}{\sqrt{\lambda^2-(x+c_1)^2}} \\
\Longrightarrow \quad y &= -\sqrt{\lambda^2-(x+c_1)^2}+c_2 \\
(y-c_2)^2 &= \lambda^2-(x+c_1)^2 \\
\Longrightarrow \quad (x+c_1)^2+(y-c_2)^2 &= \lambda^2
\end{aligned}
$$

which is a circle, where the constants c_1, c_2, λ can be obtained from 3 conditions namely, one constraint condition and two end conditions. That is,

$$
\begin{aligned}
\int_a^b \sqrt{1+y'^2}dx &= L \\
y(a) = A, \quad y(b) &= B
\end{aligned}
$$

Dynamical systems and control

Let's consider an ordinary differential equation (ODE) having the form

$$\begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{f}(\mathbf{x}(t)) \quad (t > 0) \\ \mathbf{x}(0) & = x^0 \end{cases}$$

We are here given the initial point $x^0 \in \mathbb{R}^n$ and the function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. The unknown is the curve $\mathbf{x} : [0, \infty) \to \mathbb{R}^n$, which we interpret as the dynamical evolution of the state of some "system". The domain of the functional will be restricted to such curves.

Control comes into play

We generalize a bit and suppose now that $\mathbf{f}$ depends also upon some "control" parameters belonging to a set $A \subset \mathbb{R}^m$; so that $\mathbf{f} \colon \mathbb{R}^n \times A \to \mathbb{R}^n$.

1. Let's select some value $a \in A$ and consider the corresponding dynamics:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), a) \quad (t > 0) \\ \mathbf{x}(0) = x^0 \end{cases}$$

We obtain the evolution of our system when the parameter is constantly set to the value a.

2. The next possibility is that we change the value of the parameter as the system evolves. For instance, suppose we define the function $\boldsymbol{\alpha} : [0, \infty) \to A$ this way:

$$\boldsymbol{\alpha}(t) = \begin{cases} a_1 & 0 \le t \le t_1 \\ a_2 & t_1 < t \le t_2 \\ a_3 & t_2 < t \le t_3 \end{cases} \qquad \text{etc.}$$

for times $0 < t_1 < t_2 < t_3 \ldots$ and parameter values $a_1, a_2, a_3, \dots \in A$; and we then solve the dynamical equation

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\alpha}(t)) \quad (t > 0) \\ \mathbf{x}(0) = x^0 \end{cases}$$

The point is that the system may behave quite differently as we change the control parameters.

3. More generally, we call a function $\boldsymbol{\alpha} : [0,\infty) \to A$ a control. Corresponding to each control, we consider the ODE
(ODE)

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\alpha}(t)) \quad (t > 0) \\ \mathbf{x}(0) = x^0, \end{cases}$$

and regard the trajectory $\mathbf{x}(\cdot)$ as the corresponding response of the system.

The basic problem

Again, we will write

$$\mathbf{f}(x,a) = \begin{pmatrix} f_1(x,a) \\ \vdots \\ f_n(x,a) \end{pmatrix}$$

to display the components of $\mathbf{f}$, and similarly put

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

Also, introduce

$$\mathcal{A} = \{\boldsymbol{\alpha} : [0,\infty) \to A \mid \boldsymbol{\alpha}(\cdot) \text{ measurable }\}$$

to denote the collection of all admissible controls, where

$$\boldsymbol{\alpha}(t) = \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_m(t) \end{pmatrix}$$

Note very carefully that our solution $\mathbf{x}(\cdot)$ of (ODE) depends upon $\boldsymbol{\alpha}(\cdot)$ and the initial condition. Consequently our notation would be more precise, but more complicated, if we were to write

$$\mathbf{x}(\cdot) = \mathbf{x}\left(\cdot, \boldsymbol{\alpha}(\cdot), x^0\right)$$

displaying the dependence of the response $\mathbf{x}(\cdot)$ upon the control and the initial value.

PAYOFFS. Our overall task will be to determine what is the "best" control for our system. For this we need to specify a specific payoff (or reward) criterion. Let us define the payoff functional

$$P[\boldsymbol{\alpha}(\cdot)] := \int_0^T r(\mathbf{x}(t), \boldsymbol{\alpha}(t))dt + g(\mathbf{x}(T)) \tag{P}$$

where $\mathbf{x}(\cdot)$ solves (ODE) for the control $\boldsymbol{\alpha}(\cdot)$. Here $r : \mathbb{R}^n \times A \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are given, and we call r *the running payoff* and g *the terminal payoff*. The terminal time $T > 0$ is given as well.

THE BASIC PROBLEM. Our aim is to find a control $\boldsymbol{\alpha}^*(\cdot)$, which maximizes the payoff. In other words, we want

$$P\left[\boldsymbol{\alpha}^*(\cdot)\right] \geq P[\boldsymbol{\alpha}(\cdot)]$$

for all controls $\boldsymbol{\alpha}(\cdot) \in \mathcal{A}$. Such a control $\boldsymbol{\alpha}^*(\cdot)$ is called *optimal control.* This task presents us with these mathematical issues:
(i) Does an optimal control exist?
(ii) How can we characterize an optimal control mathematically?
(iii) How can we construct an optimal control?

These turn out to be sometimes subtle problems, as the following collection of examples illustrates.

Brachistocrone revisited

Consider the Brachistochrone problem and let's represent the tube curve by a parametric form, i.e. $(u(t), x(t))$ where t is time. Then, the speed at time t is $s(u(t)) = s(t) = \sqrt{\dot{u}(t)^2 + \dot{x}(t)^2}$. Then, conservation of energy leads to

$$2gx(t) = \dot{x}(t)^2 + \dot{u}(t)^2$$

Now, we imagine the reverse scenario treating x, u as controls, by setting

$$\theta_1(t) = \dot{u}(t)/\sqrt{2gx(t)} \quad \theta_2(t) = \dot{x}(t)/\sqrt{2gx(t)}$$

Then, we end up with a control system that defines the equation of the ramp

$$\begin{aligned}
&\dot{u}(t) = \theta_1(t)\sqrt{2gx(t)} \\
&\dot{x}(t) = \theta_2(t)\sqrt{2gx(t)} \\
&\theta_1(t)^2 + \theta_2(t)^2 = 1 \\
&(u(t_0), x(t_0)) = (a, 0), \quad u(t_1) = b
\end{aligned}$$

The cost function in this case is the time taken, so $P = \int_{t_0}^{t_1} 1 dt = t_1 - t_0$.

It is worth noting that by formulating the original calculus of variations problem as a control problem, we actually gained some generality:

- It is no longer assumed that x can be written as a function of u
- It is not necessary for x to be differentiable with respect to u

More examples

• **Control of production and consumption.** Suppose we own, say, a factory whose output we can control. Let us begin to construct a mathematical model by setting

$$x(t) = \text{ amount of output produced at time } t \geq 0.$$

We suppose that we consume some fraction of our output at each time, and likewise can reinvest the remaining fraction. Let us denote

$$\alpha(t) = \text{ fraction of output reinvested at time } t \geq 0.$$

This will be our control, and is subject to the obvious constraint that

$$0 \leq \alpha(t) \leq 1 \quad \text{ for each time } t \geq 0$$

Given such a control, the corresponding dynamics are provided by the ODE

$$\begin{cases} \dot{x}(t) = k\alpha(t)x(t) \\ x(0) = x^0 \end{cases}$$

the constant $k > 0$ modelling the growth rate of our reinvestment. Let us take as a payoff functional

$$P[\alpha(\cdot)] = \int_0^T (1 - \alpha(t))x(t)dt$$

The meaning is that we want maximize our total consumption of the output, our consumption at a given time t being $(1 - \alpha(t))x(t)$. This model fits into our general framework for $n = m = 1$, once we put

$$A = [0, 1], f(x, a) = kax, r(x, a) = (1 - a)x, g \equiv 0.$$

$$\alpha^\star = 1$$

As we will see later, an optimal control $\alpha^*(\cdot)$ is given by

$$\alpha^*(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq t^* \\ 0 & \text{if } t^* < t \leq T \end{cases}$$

for an appropriate switching time $0 \leq t^* \leq T$. In other words, we should reinvest all the output (and therefore consume nothing) up until time t^*, and afterwards, we should consume everything (and therefore reinvest nothing). The switchover time t^* will have to be determined. We call $\alpha^*(\cdot)$ *a bang-bang control.*

• **A moon lander.** This model asks us to bring a spacecraft to a soft landing on the lunar surface, using the least amount of fuel.

We introduce the notation

$$\begin{aligned} h(t) &= \text{ height at time } t \\ v(t) &= \text{ velocity } = \dot{h}(t) \\ m(t) &= \text{ mass of spacecraft (changing as fuel is burned)} \\ \alpha(t) &= \text{ thrust at time } t \end{aligned}$$

We assume that

$$0 \leq \alpha(t) \leq 1$$

and Newton's law tells us that

$$m\ddot{h} = -gm + \alpha,$$

the right hand side being the difference of the gravitational force and the thrust of the rocket. This system is modeled by the ODE

$$\begin{cases} \dot{v}(t) = -g + \dfrac{\alpha(t)}{m(t)} \\ \dot{h}(t) = v(t) \\ \dot{m}(t) = -k\alpha(t) \end{cases}$$

We summarize these equations in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \alpha(t))$$

for $\mathbf{x}(t) = (v(t), h(t), m(t))$.

We want to minimize the amount of fuel used up, that is, to maximize the amount remaining once we have landed. Thus

$$P[\alpha(\cdot)] = m(\tau)$$

where

$$\tau \text{ denotes the first time that } h(\tau) = v(\tau) = 0.$$

This is *a variable endpoint problem*, since the final time is not given in advance. We have also the extra constraints

$$h(t) \geq 0, \quad m(t) \geq 0$$

• **Rocket railroad car.** Imagine a railroad car powered by rocket engines on each side. We introduce the variables

$$\begin{aligned} q(t) &= \text{ position at time } t \\ v(t) = \dot{q}(t) &= \text{ velocity at time } t \\ \alpha(t) &= \text{ thrust from rockets} \end{aligned}$$

where

$$-1 \leq \alpha(t) \leq 1$$

The sign depending upon which engine is firing.
We want to figure out how to fire the rockets, so as to arrive at the origin 0 with zero velocity in a minimum amount of time. Assuming the car has mass $m = 1$, the law of motion is

$$\ddot{q}(t) = \alpha(t).$$

We rewrite by setting $\mathbf{x}(t) = (q(t), v(t))^T$. Then

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x}(t) + \binom{0}{1}\alpha(t) \\ \mathbf{x}(0) = x^0 = (q_0, v_0)^T. \end{cases}$$

Since our goal is to steer to the origin (0,0) in minimum time, we take

$$P[\alpha(\cdot)] = -\int_0^\tau 1 dt = -\tau$$

for

$$\tau = \text{ first time that } q(\tau) = v(\tau) = 0.$$

Controllability

Definitions

Again, let's recall:

$$\begin{cases} \dot{\mathbf{x}}(t) & = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\alpha}(t)) \\ \mathbf{x}(0) & = x^0. \end{cases} \tag{ODE}$$

Here $x^0 \in \mathbb{R}^n, \mathbf{f} : \mathbb{R}^n \times A \to \mathbb{R}^n, \boldsymbol{\alpha} : [0, \infty) \to A$ is the control, and $\mathbf{x} : [0, \infty) \to \mathbb{R}^n$ is the response of the system.

CONTROLLABILITY QUESTION: Given the initial point x^0 and a "target" set $S \subset \mathbb{R}^n$, does there exist a control steering the system to S in finite time?

For the time being we will therefore not introduce any payoff criterion that would characterize an "optimal" control, but instead will focus on the question as to whether or not there exist controls that steer the system to a given goal. In this chapter we will mostly consider the problem of driving the system to the origin $S = \{0\}$.

We define *the reachable set for time t* to be

$\mathcal{C}(t) =$ set of initial points x^0 for which there exists a control such that $\mathbf{x}(t) = 0$

and the overall *reachable set*

$$\begin{aligned} \mathcal{C} = {} & \text{set of initial points } x^0 \text{ for which there exists a} \\ & \text{control such that } \mathbf{x}(t) = 0 \text{ for some finite time } t. \end{aligned}$$

Note that

$$\mathcal{C} = \bigcup_{t \geq 0} \mathcal{C}(t).$$

We say the system (ODE) *is controllable* if $C = \mathbb{R}^n$

Hereafter, let $\mathbb{M}^{n \times m}$ denote the set of all $n \times m$ matrices. We assume that our ODE is linear in both the state $\mathbf{x}(\cdot)$ and the control $\boldsymbol{\alpha}(\cdot)$, and consequently has the form

$$\begin{cases} \dot{\mathbf{x}}(t) = M\mathbf{x}(t) + N\boldsymbol{\alpha}(t) \quad (t > 0) \\ \mathbf{x}(0) = x^0, \end{cases} \tag{ODE}$$

where $M \in \mathbb{M}^{n \times n}$ and $N \in \mathbb{M}^{n \times m}$. We assume the set A of control parameters is a cube in $\mathbb{R}^m$:

$$A = [-1,1]^m = \{a \in \mathbb{R}^m || a_i |\leq 1, i = 1, \ldots, m\}$$

Quick review of linear ODE

DEFINITION. Let $\mathbf{X}(\cdot) : \mathbb{R} \to \mathbb{M}^{n\times n}$ be the unique solution of the matrix ODE

$$\begin{cases} \dot{\mathbf{X}}(t) = M\mathbf{X}(t) \quad (t \in \mathbb{R}) \\ \mathbf{X}(0) = I \end{cases}$$

We call $\mathbf{X}(\cdot)$ *a fundamental solution*, and sometimes write

$$\mathbf{X}(t) = e^{tM} := \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}$$

the last formula being the definition of the *exponential* e^{tM}. Observe that

$$\mathbf{X}^{-1}(t) = \mathbf{X}(-t)$$

(i) The unique solution of the homogeneous system of ODE

$$\begin{cases} \dot{\mathbf{x}}(t) = M\mathbf{x}(t) \\ \mathbf{x}(0) = x^0 \end{cases}$$

is

$$\mathbf{x}(t) = \mathbf{X}(t)x^0 = e^{tM}x^0$$

(ii) The unique solution of the nonhomogeneous system

$$\begin{cases} \dot{\mathbf{x}}(t) = M\mathbf{x}(t) + \mathbf{f}(t) \\ \mathbf{x}(0) = x^0 \end{cases}$$

is

$$\mathbf{x}(t) = \mathbf{X}(t)x^0 + \mathbf{X}(t)\int_0^t \mathbf{X}^{-1}(s)\mathbf{f}(s)ds$$

This expression is the *variation of parameters formula.*

Controllability of linear equations

According to the variation of parameters formula, the solution of (ODE) for a given control $\boldsymbol{\alpha}(\cdot)$ is

$$\mathbf{x}(t) = \mathbf{X}(t)x^0 + \mathbf{X}(t)\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}(s)ds$$

where $\mathbf{X}(t) = e^{tM}$. Furthermore, observe that

$$x^0 \in \mathcal{C}(t)$$

if and only if there exists a control $\boldsymbol{\alpha}(\cdot) \in \mathcal{A}$ such that $\mathbf{x}(t) = 0$
if and only if

$$0 = \mathbf{X}(t)x^0 + \mathbf{X}(t)\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}(s)ds \quad \text{for some control } \boldsymbol{\alpha}(\cdot) \in \mathcal{A}$$

if and only if

$$x^0 = -\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}(s)ds \quad \text{for some control } \boldsymbol{\alpha}(\cdot) \in \mathcal{A} \qquad \text{(reachability)}$$

Theorem 3 (Structure of reachable set)**.** *The next properties hold:*

- *The reachable set $\mathcal{C}$ is symmetric and convex.*
- *Also, if $x^0 \in \mathcal{C}(t)$, then $x^0 \in \mathcal{C}(\hat{t})$ for all times $\hat{t} \geq t$.*

Proof. 1. (Symmetry) Let $t \geq 0$ and $x^0 \in \mathcal{C}(t)$. Then $x^0 = -\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}(s)ds$ for some admissible control $\boldsymbol{\alpha} \in \mathcal{A}$. Therefore $-x^0 = -\int_0^t \mathbf{X}^{-1}(s)N(-\boldsymbol{\alpha}(s))ds$, and $-\boldsymbol{\alpha} \in \mathcal{A}$ since the set A is symmetric. Therefore $-x^0 \in \mathcal{C}(t)$, and so each set $\mathcal{C}(t)$ symmetric. It follows that $\mathcal{C}$ is symmetric.
2. (Convexity) Take $x^0, \hat{x}^0 \in \mathcal{C}$; so that $x^0 \in \mathcal{C}(t), \hat{x}^0 \in \mathcal{C}(\hat{t})$ for appropriate times $t, \hat{t} \geq 0$. Assume $t \leq \hat{t}$. Then

$$\begin{aligned} x^0 &= -\textstyle\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}(s)ds && \text{for some control } \boldsymbol{\alpha} \in \mathcal{A} \\ \hat{x}^0 &= -\textstyle\int_0^t \mathbf{X}^{-1}(s)N\hat{\boldsymbol{\alpha}}(s)ds && \text{for some control } \hat{\boldsymbol{\alpha}} \in \mathcal{A} \end{aligned}$$

Define a new control

$$\tilde{\boldsymbol{\alpha}}(s) := \begin{cases} \boldsymbol{\alpha}(s) & \text{if } 0 \leq s \leq t \\ 0 & \text{if } s > t \end{cases}$$

Then

$$x^0 = -\int_0^{\hat{t}} \mathbf{X}^{-1}(s)N\tilde{\boldsymbol{\alpha}}(s)ds$$

and hence $x^0 \in \mathcal{C}(\hat{t})$. Now let $0 \leq \lambda \leq 1$, and observe

$$\lambda x^0 + (1-\lambda)\hat{x}^0 = -\int_0^{\hat{t}} \mathbf{X}^{-1}(s)N(\lambda\tilde{\boldsymbol{\alpha}}(s) + (1-\lambda)\hat{\boldsymbol{\alpha}}(s))ds$$

Therefore $\lambda x^0 + (1-\lambda)\hat{x}^0 \in \mathcal{C}(\hat{t}) \subseteq \mathcal{C}$.

3. The second assertion follows from the foregoing if we take $\bar{t} = \hat{t}$. □

• **Simple example.** Let $n = 2$ and $m = 1, A = [-1,1]$, and write $\mathbf{x}(t) = \left(x^1(t), x^2(t)\right)^T$. Suppose

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = \alpha(t) \end{cases}$$

This is a system of the form $\dot{\mathbf{x}} = M\mathbf{x} + N\alpha$, for

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Clearly $\mathcal{C} = \{(x_1, x_2) \mid x_1 = 0\}$, the x_2-axis.

We next wish to establish some general algebraic conditions ensuring that $\mathcal{C}$ contains a neighborhood of the origin.

The controllability matrix is

$$G = G(M,N) := \underbrace{\left[N, MN, M^2N, \ldots, M^{n-1}N\right]}_{n\times(mn) \text{ matrix}}.$$

Theorem 4 (Controllability matrix and reachability)**.** *We have*

$$\operatorname{rank} G = n$$

if and only if

$$0 \in \mathcal{C}^\circ$$

Remark. We write $\mathcal{C}^\circ$ for the interior of the set $\mathcal{C}$. Remember that

$$\text{rank of G} \begin{array}{l} = \text{ number of linearly independent rows of } G \\ = \text{ number of linearly independent columns of } G. \end{array}$$

Clearly $\operatorname{rank} G \leq n$.

Theorem 5 (Criterion for Controllability)**.** *Let A be the cube $[-1,1]^n$ in $\mathbb{R}^n$. Suppose as well that* $\operatorname{rank} G = n$*, and* $\operatorname{Re}\lambda < 0$ *for each eigenvalue λ of the matrix M.*

Then the system (ODE) is controllable.

Proof. Since $\operatorname{rank} G = n$, Theorem 4 tells us that $\mathcal{C}$ contains some ball B centered at 0. Now take any $x^0 \in \mathbb{R}^n$ and consider the evolution

$$\begin{cases} \dot{\mathbf{x}}(t) = M\mathbf{x}(t) \\ \mathbf{x}(0) = x^0 \end{cases}$$

in other words, take the control $\boldsymbol{\alpha}(\cdot) \equiv 0$. Since $\operatorname{Re}\lambda < 0$ for each eigenvalue λ of M, then the origin is asymptotically stable. So there exists a time T such that $\mathbf{x}(T) \in B$. Thus $\mathbf{x}(T) \in B \subset \mathcal{C}$; and hence there exists a control $\boldsymbol{\alpha}(\cdot) \in \mathcal{A}$ steering $\mathbf{x}(T)$ into 0 in finite time. $\square$

• **Simple example.** Consider the rocket railroad car, for which $n = 2, m = 1, A = [-1,1]$, and

$$\dot{\mathbf{x}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \alpha$$

Then

$$G = [N, MN] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Therefore

$$\operatorname{rank} G = 2 = n$$

Also, the characteristic polynomial of the matrix M is

$$p(\lambda) = \det(\lambda I - M) = \det \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix} = \lambda^2.$$

Since the eigenvalues are both 0, we fail to satisfy the hypotheses of Theorem 5. This example motivates the following extension of the previous theorem:

Theorem 6 (Improved Criterion for Controllability)**.** *Assume* $\operatorname{rank} G = n$ *and* $\operatorname{Re} \lambda \leq 0$ *for each eigenvalue* λ *of* M.

Then the system (ODE) is controllable.

Bang-bang principle

We will again take A to be the cube $[-1,1]^m$ in $\mathbb{R}^m$.
A control $\boldsymbol{\alpha}(\cdot) \in \mathcal{A}$ is called *bang-bang* if for each time $t \geq 0$ and each index $i = 1, \ldots, m$, we have $\left|\alpha^i(t)\right| = 1$, where

$$\boldsymbol{\alpha}(t) = \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_m(t) \end{pmatrix}$$

Theorem 7 (Bang-bang principle)**.** *Let $t > 0$ and suppose $x^0 \in \mathcal{C}(t)$, for the system*

$$\dot{\mathbf{x}}(t) = M\mathbf{x}(t) + N\boldsymbol{\alpha}(t)$$

Then there exists a bang-bang control $\boldsymbol{\alpha}(\cdot)$ which steers x^0 to 0 at time t.

To prove the theorem we need some tools from functional analysis.

Functional analysis

$$L^\infty = L^\infty(0, t; \mathbb{R}^m) = \{\boldsymbol{\alpha}(\cdot)\colon [0,t] \to \mathbb{R}^m \mid \sup_{0 \leq s \leq t} |\boldsymbol{\alpha}(s)| < \infty\}.$$
$$\|\boldsymbol{\alpha}\|_{L^\infty} = \sup_{0 \leq s \leq t} |\boldsymbol{\alpha}(s)|.$$

Let $\boldsymbol{\alpha}_n \in L^\infty$ for $n = 1, \ldots$ and $\boldsymbol{\alpha} \in L^\infty$. We say $\boldsymbol{\alpha}_n$ converges to $\boldsymbol{\alpha}$ in the *weak* sense*, written

$$\boldsymbol{\alpha}_n \stackrel{*}{\rightharpoonup} \boldsymbol{\alpha},$$

provided

$$\int_0^t \boldsymbol{\alpha}_n(s) \cdot \mathbf{v}(s) ds \to \int_0^t \boldsymbol{\alpha}(s) \cdot \mathbf{v}(s) ds$$

as $n \to \infty$, for all $\mathbf{v}(\cdot) : [0,t] \to \mathbb{R}^m$ satisfying $\int_0^t |\mathbf{v}(s)| ds < \infty$.
We will need the following useful weak* compactness theorem for L^∞:

Theorem 8 (Alaoglu)**.** *Let $\boldsymbol{\alpha}_n \in \mathcal{A}, n = 1, \ldots.$ Then there exists a subsequence $\boldsymbol{\alpha}_{n_k}$ and $\boldsymbol{\alpha} \in \mathcal{A}$, such that*

$$\boldsymbol{\alpha}_{n_k} \stackrel{*}{\rightharpoonup} \boldsymbol{\alpha}.$$

Remind, that:

the set $\mathbb{K}$ is convex if for all $x, \hat{x} \in \mathbb{K}$ and all real numbers $0 \leq \lambda \leq 1$,

$$\lambda x + (1-\lambda)\hat{x} \in \mathbb{K};$$

a point $z \in \mathbb{K}$ is called extreme provided there do not exist points $x, \hat{x} \in \mathbb{K}$ and $0 < \lambda < 1$ such that

$$z = \lambda x + (1-\lambda)\hat{x}.$$

Theorem 9 (Krein-Milman)**.** *Let $\mathbb{K} \subset L^\infty$ be*

- *nonempty,*
- *a convex,*
- *compact in the weak * topology.*

Then $\mathbb{K}$ has at least one extreme point.

Application to bang-bang controls

The foregoing abstract theory will be useful for us in the following setting. We will take $\mathbb{K}$ to be the set of controls which steer x^0 to 0 at time t, prove it satisfies the hypotheses of Krein-Milman Theorem and finally show that an extreme point is a bang-bang control.

So consider again the linear dynamics

$$\begin{cases} \dot{\mathbf{x}}(t) = M\mathbf{x}(t) + N\boldsymbol{\alpha}(t) \\ \mathbf{x}(0) = x^0 \end{cases}$$

Take $x^0 \in \mathcal{C}(t)$ and write

$$\mathbb{K} = \left\{ \boldsymbol{\alpha}(\cdot) \in \mathcal{A} \mid \boldsymbol{\alpha}(\cdot) \text{ steers } x^0 \text{ to } 0 \text{ at time } t \right\}$$

Theorem 10 (Geometry of set of controls)**.** *The collection $\mathbb{K}$ of admissible controls satisfies the hypotheses of the Krein-Milman Theorem.*

Proof. Since $x^0 \in \mathcal{C}(t)$, we see that $\mathbb{K} \neq \varnothing$.
Next we show that $\mathbb{K}$ is convex. For this, recall that $\boldsymbol{\alpha}(\cdot) \in \mathbb{K}$ if and only if

$$x^0 = -\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}(s)ds$$

Now take also $\hat{\boldsymbol{\alpha}} \in \mathbb{K}$ and $0 \le \lambda \le 1$. Then

$$x^0 = -\int_0^t \mathbf{X}^{-1}(s)N\hat{\boldsymbol{\alpha}}(s)ds$$

and so

$$x^0 = -\int_0^t \mathbf{X}^{-1}(s)N(\lambda\boldsymbol{\alpha}(s) + (1-\lambda)\hat{\boldsymbol{\alpha}}(s))ds$$

Hence $\lambda\boldsymbol{\alpha} + (1-\lambda)\hat{\boldsymbol{\alpha}} \in \mathbb{K}$.

Lastly, we confirm the compactness. Let $\boldsymbol{\alpha}_n \in \mathbb{K}$ for $n = 1, \ldots$. According to Alaoglu's Theorem there exist $n_k \to \infty$ and $\boldsymbol{\alpha} \in \boldsymbol{\mathcal{A}}$ such that $\boldsymbol{\alpha}_{n_k} \overset{*}{\to} \boldsymbol{\alpha}$. We need to show that $\boldsymbol{\alpha} \in \mathbb{K}$.

Now $\boldsymbol{\alpha}_{n_k} \in \mathbb{K}$ implies

$$x^0 = -\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}_{n_k}(s)ds \to -\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}(s)ds$$

by definition of weak-* convergence. Hence $\boldsymbol{\alpha} \in \mathbb{K}$. □

We can now apply the Krein-Milman Theorem to deduce that there exists an extreme point $\boldsymbol{\alpha}^* \in \mathbb{K}$. What is interesting is that such an extreme point corresponds to a bang-bang control.

Theorem 11 (Extremality and bang-bang pronciple)**.** *The control* $\boldsymbol{\alpha}^*(\cdot)$ *is bang-bang.*

Proof. 1. For the extreme point $\boldsymbol{\alpha}^*$ we must show that for almost all times $0 \le s \le t$ and for each $i = 1, \ldots, m$, we have

$$|\alpha_i^*(s)| = 1$$

Suppose not. Then there exists an index $i \in \{1, \ldots, m\}$ and a subset $E \subset [0, t]$ of positive measure such that $|\alpha_i^*(s)| < 1$ for $s \in E$. In fact, there exist a number $\varepsilon > 0$ and a subset $F \subseteq E$ such that

$$|F| > 0 \text{ and } |\alpha_i^*(s)| \le 1 - \varepsilon \text{ for } s \in F.$$

Define

$$I_F(\beta(\cdot)) := \int_F \mathbf{X}^{-1}(s)N\boldsymbol{\beta}(s)ds$$

for

$$\boldsymbol{\beta}(\cdot) := (0,\ldots,\beta(\cdot),\ldots,0)^T$$

the function β in the i^{th} slot. Choose any real-valued function $\beta(\cdot) \not\equiv 0$, such that

$$I_F(\beta(\cdot)) = 0$$

and $|\beta(\cdot)| \leq 1$. Define

$$\begin{aligned}\boldsymbol{\alpha}_1(\cdot) &:= \boldsymbol{\alpha}^*(\cdot) + \varepsilon\boldsymbol{\beta}(\cdot)\\ \boldsymbol{\alpha}_2(\cdot) &:= \boldsymbol{\alpha}^*(\cdot) - \varepsilon\boldsymbol{\beta}(\cdot)\end{aligned}$$

where we redefine β to be zero off the set F

2. We claim that

$$\boldsymbol{\alpha}_1(\cdot), \boldsymbol{\alpha}_2(\cdot) \in \mathbb{K}$$

To see this, observe that

$$\begin{aligned}-\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}_1(s)ds &= -\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\alpha}^*(s)ds - \varepsilon\int_0^t \mathbf{X}^{-1}(s)N\boldsymbol{\beta}(s)ds\\ &= x^0 - \varepsilon\underbrace{\int_F \mathbf{X}^{-1}(s)N\boldsymbol{\beta}(s)ds}_{I_F(\beta(\cdot))=0} = x^0\end{aligned}$$

Note also $\boldsymbol{\alpha}_1(\cdot) \in \mathcal{A}$. Indeed,

$$\begin{cases}\boldsymbol{\alpha}_1(s) = \boldsymbol{\alpha}^*(s) & (s \notin F)\\ \boldsymbol{\alpha}_1(s) = \boldsymbol{\alpha}^*(s) + \varepsilon\boldsymbol{\beta}(s) & (s \in F)\end{cases}$$

But on the set F, we have $|\boldsymbol{\alpha}_i^*(s)| \leq 1-\varepsilon$, and therefore

$$|\boldsymbol{\alpha}_1(s)| \leq |\boldsymbol{\alpha}^*(s)| + \varepsilon|\boldsymbol{\beta}(s)| \leq 1-\varepsilon+\varepsilon = 1.$$

Similar considerations apply for $\boldsymbol{\alpha}_2$. Hence $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2 \in \mathbb{K}$, as claimed above.

3. Finally, observe that

$$\begin{cases} \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}^* + \varepsilon\boldsymbol{\beta}, & \boldsymbol{\alpha}_1 \neq \boldsymbol{\alpha}^* \\ \boldsymbol{\alpha}_2 = \boldsymbol{\alpha}^* - \varepsilon\boldsymbol{\beta}, & \boldsymbol{\alpha}_2 \neq \boldsymbol{\alpha}^* \end{cases}$$

But

$$\frac{1}{2}\boldsymbol{\alpha}_1 + \frac{1}{2}\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}^*$$

and this is a contradiction, since $\boldsymbol{\alpha}^*$ is an extreme point of $\mathbb{K}$.

□

Time-optimal control

Consider the linear system of ODE:

$$\begin{cases} \dot{\mathbf{x}}(t) = M\mathbf{x}(t) + N\boldsymbol{\alpha}(t) \\ \mathbf{x}(0) = x^0, \end{cases} \tag{ODE}$$

for given matrices $M \in \mathbb{M}^{n\times n}$ and $N \in \mathbb{M}^{n\times m}$. We will again take A to be the cube $[-1,1]^m \subset \mathbb{R}^m$.

Define next

$$P[\boldsymbol{\alpha}(\cdot)] := -\int_0^\tau 1ds = -\tau \tag{P}$$

where $\tau = \tau(\boldsymbol{\alpha}(\cdot))$ denotes the first time the solution of our ODE hits the origin 0. (If the trajectory never hits 0, we set $\tau = \infty$.)

OPTIMAL TIME PROBLEM: We are given the starting point $x^0 \in \mathbb{R}^n$, and want to find an optimal control $\boldsymbol{\alpha}^*(\cdot)$ such that

$$P\left[\boldsymbol{\alpha}^*(\cdot)\right] = \max_{\boldsymbol{\alpha}(\cdot)\in\mathcal{A}} P[\boldsymbol{\alpha}(\cdot)]$$

Then

$$\tau^* = -\mathcal{P}\left[\boldsymbol{\alpha}^*(\cdot)\right] \text{ is the minimum time to steer to the origin.}$$

Theorem 12 (Existence of time-optimal control)**.** *Let $x^0 \in \mathbb{R}^n$. Then there exists an optimal bang-bang control $\boldsymbol{\alpha}^*(\cdot)$.*

Proof. Let $\tau^* := \inf\left\{t \mid x^0 \in \mathcal{C}(t)\right\}$. We want to show that $x^0 \in \mathcal{C}(\tau^*)$; that is, there exists an optimal control $\boldsymbol{\alpha}^*(\cdot)$ steering x^0 to 0 at time τ^*.

Choose $t_1 \geq t_2 \geq t_3 \geq \ldots$ so that $x^0 \in \mathcal{C}(t_n)$ and $t_n \to \tau^*$. Since $x^0 \in \mathcal{C}(t_n)$, there exists a control $\boldsymbol{\alpha}_n(\cdot) \in \mathcal{A}$ such that

$$x^0 = -\int_0^{t_n} \mathbf{X}^{-1}(s) N \boldsymbol{\alpha}_n(s) ds$$

If necessary, redefine $\boldsymbol{\alpha}_n(s)$ to be 0 for $S \geq t_n$. By Alaoglu's Theorem, there exists a subsequence $n_k \to \infty$ and a control $\boldsymbol{\alpha}^*(\cdot)$ so that

$$\boldsymbol{\alpha}_{n_k} \overset{*}{\rightharpoonup} \boldsymbol{\alpha}^*.$$

We assert that $\boldsymbol{\alpha}^*(\cdot)$ is an optimal control. It is easy to check that $\boldsymbol{\alpha}^*(s) = 0$, $s \geq \tau^*$. Also

$$x^0 = -\int_0^{t_{n_k}} \mathbf{X}^{-1}(s) N \boldsymbol{\alpha}_{n_k}(s) ds = -\int_0^{t_1} \mathbf{X}^{-1}(s) N \boldsymbol{\alpha}_{n_k}(s) ds$$

since $\boldsymbol{\alpha}_{n_k} = 0$ for $s \geq t_{n_k}$. Let $n_k \to \infty$:

$$x^0 = -\int_0^{t_1} \mathbf{X}^{-1}(s) N \boldsymbol{\alpha}^*(s) ds = -\int_0^{\tau^*} \mathbf{X}^{-1}(s) N \boldsymbol{\alpha}^*(s) ds$$

because $\boldsymbol{\alpha}^*(s) = 0$ for $s \geq \tau^*$. Hence $x^0 \in \mathcal{C}(\tau^*)$, and therefore $\boldsymbol{\alpha}^*(\cdot)$ is optimal.
According to Theorem 11 there in fact exists an optimal bang-bang control. $\square$

The Pontryagin Maximum Principle

Lagrange multiplicators

Unconstrained optimization

- **Function of several real variables**

$$f(\mathbf{x}) \to \max \ \Rightarrow \ \nabla f(\mathbf{x}) = 0$$

- **Variational problem**

$$J[y] = \int_a^b F(x, y, y')dx \to \max \ \Rightarrow \ \left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0$$

$$\Rightarrow_{\text{NL equation}} \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Constrained optimization

- **Function of several real variables**
  Constrained optimization problem reduces to the unconstrained case.

$$f(\mathbf{x}) \to \max, \text{ where } g(\mathbf{x}) = c$$

$$\text{Compose } f(\mathbf{x}) + \lambda g(\mathbf{x})$$

$$\text{Find } \nabla \left(f(\mathbf{x}) + \lambda g(\mathbf{x})\right) = 0$$

- **Variational problem**

$$J[y] = \int_a^b F(x, y, y')dx \to \max, \text{ where } \int_a^b G(x, y, y') = L$$

$$\text{Compose } \int_a^b F(x, y, y')dx + \lambda \int_a^b G(x, y, y') = \int_a^b \left(F\left(x, y, y'\right) + \lambda G\left(x, y, y'\right)\right) dx$$

$$\text{Define } H\left(x, y, y'\right) = F\left(x, y, y'\right) + \lambda G\left(x, y, y'\right)$$

$$\Rightarrow_{\text{NL equation}} \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) - \frac{\partial H}{\partial y} = 0$$

Statement of Pontryagin Maximum Principle (PMP)

Free endpoint problem

Let's recall that we consider the system of ordinary differential equations

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \boldsymbol{\alpha}(t)) \quad (t \geq 0) \\ \mathbf{x}(0) = x^0 \end{cases} \tag{ODE}$$

where $A \subseteq \mathbb{R}^m$ and also $\mathbf{f} \colon \mathbb{R}^n \times A \to \mathbb{R}^n$, $x^0 \in \mathbb{R}^n$, $\boldsymbol{\alpha}(\cdot) \colon [0, \infty) \to A$ is measurable (admissible control).

We would like to maximaze the payoff functional

$$P[\boldsymbol{\alpha}(\cdot)] = \int_0^T r(\mathbf{x}(t), \boldsymbol{\alpha}(t))dt + g(\mathbf{x}(T)) \to \max_{\boldsymbol{\alpha}(\cdot) \in \mathcal{A}}, \tag{P}$$

where the terminal time $T > 0$, running payoff $r \colon \mathbb{R}^n \times A \to \mathbb{R}$ and terminal payoff $g \colon \mathbb{R}^n \to \mathbb{R}$ are given.

Define *the control theory Hamiltonian:*

$$H(x, p, a) := \mathbf{f}(x, a) \cdot p + r(x, a) \quad (x, p \in \mathbb{R}^n, a \in A) \tag{Hamiltonian}$$

Theorem 13 (PMP, free endpoint problem)**.** *Assume $\boldsymbol{\alpha}^*(\cdot)$ is optimal for (ODE), (P) and $\mathbf{x}^*(\cdot)$ is the corresponding trajectory.*

Then there exists a function $\mathbf{p}^* : [0, T] \to \mathbb{R}^n$ such that

$$\dot{\mathbf{x}}^*(t) = \nabla_p H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), \boldsymbol{\alpha}^*(t)\right), \tag{ODE}$$

$$\dot{\mathbf{p}}^*(t) = -\nabla_x H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), \boldsymbol{\alpha}^*(t)\right), \tag{ADJ}$$

$$\mathbf{p}^*(T) = \nabla g\left(\mathbf{x}^*(T)\right), \tag{T}$$

and

$$H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), \boldsymbol{\alpha}^*(t)\right) = \max_{a \in A} H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), a\right) \quad (0 \leq t \leq T) \tag{M}$$

(ADJ) — *adjoint equations*
(T) — *terminal conditions*
(M) — *maximization principle*

Fixed endpoint problem

Here, we would like to maximaze the payoff functional

$$P[\boldsymbol{\alpha}(\cdot)] = \int_0^{\tau} r(\mathbf{x}(t), \boldsymbol{\alpha}(t))dt \to \max_{\boldsymbol{\alpha}(\cdot)\in\mathcal{A}} \tag{P}$$

Here $r\colon \mathbb{R}^n \times A \to \mathbb{R}$ is the given running payoff, and $\tau = \tau[\boldsymbol{\alpha}(\cdot)] \leq \infty$ denotes the first time the solution of (ODE) hits the given target point $x^1 \in \mathbb{R}^n$.

Theorem 14 (PMP, fixed endpoint problem)**.** *Assume* $\boldsymbol{\alpha}^*(\cdot)$ *is optimal for (ODE), (P) and* $\mathbf{x}^*(\cdot)$ *is the corresponding trajectory.*

Then there exists a function $\mathbf{p}^*\colon [0,\tau^*] \to \mathbb{R}^n$ *such that*

$$\dot{\mathbf{x}}^*(t) = \nabla_p H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), \boldsymbol{\alpha}^*(t)\right), \tag{ODE}$$

$$\dot{\mathbf{p}}^*(t) = -\nabla_x H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), \boldsymbol{\alpha}^*(t)\right), \tag{ADJ}$$

and

$$H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), \boldsymbol{\alpha}^*(t)\right) = \max_{a\in A} H\left(\mathbf{x}^*(t), \mathbf{p}^*(t), a\right) \quad (0 \leq t \leq \tau^*) \tag{M}$$

Here τ^* denotes the first time the trajectory $x^*(\cdot)$ hits the target point x^1. We call $\mathbf{x}^*(\cdot)$ *the state* of the optimally controlled system and $\mathbf{p}^*(\cdot)$ *the costate.*

Linear time-optimal control

Let A denote the cube $[-1,1]^n$ in $\mathbb{R}^n$. We consider again the linear dynamics:

$$\begin{cases} \dot{\mathbf{x}}(t) = M\mathbf{x}(t) + N\boldsymbol{\alpha}(t) \\ \mathbf{x}(0) = x^0 \end{cases} \tag{ODE}$$

for the payoff functional

$$P[\boldsymbol{\alpha}(\cdot)] = -\int_0^{\tau} 1dt = -\tau \tag{P}$$

where τ denotes the first time the trajectory hits the target point $x^1 = 0$. We have $r \equiv -1$, and so

$$H(x,p,a) = \mathbf{f} \cdot p + r = (Mx + Na)\cdot p - 1 \tag{Hamiltonian}$$

Substitute (Hamiltonian) to maximization principle (M)

$$\mathbf{p}^*(t)^T\left(M\mathbf{x}^*(t)+N\boldsymbol{\alpha}^*(t)\right)-1=\max_{a\in A}\left\{\mathbf{p}^*(t)^T\left(M\mathbf{x}^*(t)+Na\right)-1\right\}.$$

Also, we have adjoint equation

$$\dot{\mathbf{p}}^*(t)=-M^T\mathbf{p}^*(t), \tag{ADJ}$$

The solution of (ADJ) is $\mathbf{p}^*(t)=e^{-tM^T}h$ for some nonzero vector h; and hence

$$\mathbf{p}^*(t)^T=h^T\mathbf{X}^{-1}(t),$$

since $\left(e^{-tM^T}\right)^T=e^{-tM}=\mathbf{X}^{-1}(t)$.

Substitute the solution to maximization principle, we finally get

$$h^T\mathbf{X}^{-1}(t)N\boldsymbol{\alpha}^*(t)=\max_{a\in A}\left\{h^T\mathbf{X}^{-1}(t)Na\right\}$$

for some nonzero vector h;

Examples

Rocket railroad car

We have

$$\dot{\mathbf{x}}(t)=\underbrace{\begin{pmatrix}0&1\\0&0\end{pmatrix}}_{M}\mathbf{x}(t)+\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{N}\alpha(t) \tag{ODE}$$

for

$$\mathbf{x}(t)=\begin{pmatrix}x^1(t)\\x^2(t)\end{pmatrix},\quad A=[-1,1].$$

According to the PMP for linear time-optimal control, there exists $h\neq 0$ such that

$$h^T\mathbf{X}^{-1}(t)N\alpha^*(t)=\max_{|a|\leq 1}\left\{h^T\mathbf{X}^{-1}(t)Na\right\} \tag{M}$$

We will extract the interesting fact that an optimal control α^* switches at most one time.

We must compute e^{tM}. To do so, we observe

$$M^0 = I, M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

and therefore $M^k = 0$ for all $k \geq 2$. Consequently,

$$e^{tM} = I + tM = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Then

$$\mathbf{X}^{-1}(t) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$$
$$\mathbf{X}^{-1}(t)N = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -t \\ 1 \end{pmatrix}$$
$$h^T \mathbf{X}^{-1}(t)N = (h_1, h_2) \begin{pmatrix} -t \\ 1 \end{pmatrix} = -th_1 + h_2$$

The Maximum Principle asserts

$$(-th_1 + h_2)\, \alpha^*(t) = \max_{|a| \leq 1} \{(-th_1 + h_2)\, a\}$$

and this implies that

$$\alpha^*(t) = \operatorname{sgn}(-th_1 + h_2)$$

for the sign function

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Therefore the optimal control α^* switches at most once; and if $h_1 = 0$, then α^* is constant.

Control of production and consumption

Let's recall:

$$\begin{aligned} x(t) &= \text{ output of economy at time } t \\ \alpha(t) &= \text{ fraction of output reinvested at time } t \end{aligned}$$

We have the constraint $0 \leq \alpha(t) \leq 1$; that is, $A = [0, 1] \subset \mathbb{R}$. The economy evolves according to the dynamics

$$\begin{cases} \dot{x}(t) = \alpha(t)x(t) \quad (0 \le t \le T) \\ x(0) = x^0 \end{cases} \tag{ODE}$$

where $x^0 > 0$ and we have set the growth factor $k = 1$. We want to maximize the total consumption

$$P[\alpha(\cdot)] := \int_0^T (1 - \alpha(t))x(t)dt \tag{P}$$

To characterize an optimal control $\alpha^*(\cdot)$ we apply Pontryagin Maximum Principle. To simplify notation we will not write the superscripts $*$ for the optimal control, trajectory, etc.

We have $n = m = 1$,

$$f(x, a) = xa, \ g \equiv 0, \ r(x, a) = (1 - a)x$$

and therefore

$$H(x, p, a) = f(x, a)p + r(x, a) = pxa + (1 - a)x = x + ax(p - 1)$$

The dynamical equation is

$$\dot{x}(t) = H_p = \alpha(t)x(t) \tag{ODE}$$

and the adjoint equation is

$$\dot{p}(t) = -H_x = -1 - \alpha(t)(p(t) - 1) \tag{ADJ}$$

The terminal condition reads

$$p(T) = g_x(x(T)) = 0 \tag{T}$$

Lastly, the maximality principle asserts

$$H(x(t), p(t), \alpha(t)) = \max_{0 \le a \le 1} \{x(t) + ax(t)(p(t) - 1)\} \tag{M}$$

We now deduce useful information from (ODE), (ADJ), (M) and (T).

According to (M), at each time t the control value $\alpha(t)$ must be selected to maximize $a(p(t) - 1)$ for $0 \le a \le 1$. This is so, since $x(t) > 0$. Thus

$$\alpha(t) = \begin{cases} 1 & \text{if} \quad p(t) > 1 \\ 0 & \text{if} \quad p(t) \le 1 \end{cases}$$

Hence if we know $p(\cdot)$, we can design the optimal control $\alpha(\cdot)$.
So next we must solve for the costate $p(\cdot)$. We know from (ADJ) and (T) that

$$\begin{cases} \dot p(t) = -1 - \alpha(t)[p(t) - 1] & (0 \le t \le T) \\ p(T) = 0 \end{cases}$$

Since $p(T) = 0$, we deduce by continuity that $p(t) \le 1$ for t close to $T, t < T$. Thus $\alpha(t) = 0$ for such values of t. Therefore $\dot p(t) = -1$, and consequently $p(t) = T - t$ for times t in this interval. So we have that $p(t) = T - t$ so long as $p(t) \le 1$. And this holds for $T - 1 \le t \le T$

But for times $t \le T - 1$, with t near $T - 1$, we have $\alpha(t) = 1$; and so (ADJ) becomes

$$\dot p(t) = -1 - (p(t) - 1) = -p(t)$$

Since $p(T-1) = 1$, we see that $p(t) = e^{T-1-t} > 1$ for all times $0 \le t \le T-1$. In particular there are no switches in the control over this time interval.

Restoring the superscript * to our notation, we consequently deduce that an optimal control is

$$\alpha^*(t) = \begin{cases} 1 & \text{if} \quad 0 \le t \le t^* \\ 0 & \text{if} \quad t^* \le t \le T \end{cases}$$

for the optimal switching time $t^* = T - 1$.

Moon Lander

Let's recall:

$$\begin{aligned} h(t) &= \text{ height at time } t \\ v(t) &= \text{ velocity } = \dot h(t) \\ m(t) &= \text{ mass of spacecraft (changing as fuel is used up)} \\ \alpha(t) &= \text{ thrust at time } t \end{aligned}$$

The thrust is constrained so that $0 \le \alpha(t) \le 1$; that is, $A = [0, 1]$. There are also the constraints that the height and mass be nonnegative: $h(t) \ge 0, m(t) \ge 0$.

The dynamics are

$$\begin{cases} \dot h(t) = v(t) \\ \dot v(t) = -g + l\frac{\alpha(t)}{m(t)} \\ \dot m(t) = -k\alpha(t) \end{cases} \tag{ODE}$$

with initial conditions

$$\begin{cases} h(0) = h_0 > 0 \\ v(0) = v_0 \\ m(0) = m_0 > 0. \end{cases}$$

The goal is to land on the moon safely, maximizing the remaining fuel $m(\tau)$, where $\tau = \tau[\alpha(\cdot)]$ is the first time $h(\tau) = v(\tau) = 0$. Since $\alpha = -\frac{\dot{m}}{k}$, our intention is equivalently to minimize the total applied thrust before landing; so that

$$P[\alpha(\cdot)] = -\int_0^\tau \alpha(t)dt \tag{P}$$

This is so since

$$\int_0^\tau \alpha(t)dt = \frac{m_0 - m(\tau)}{k}$$

In terms of the general notation, we have

$$\mathbf{x}(t) = \begin{pmatrix} h(t) \\ v(t) \\ m(t) \end{pmatrix}, \ \mathbf{f} = \begin{pmatrix} v \\ -g + la/m \\ -ka \end{pmatrix}.$$

Hence the Hamiltonian is

$$\begin{aligned} H(x, p, a) &= \mathbf{f} \cdot p + r \\ &= (v, -g + la/m, -ka) \cdot (p_1, p_2, p_3) - a \\ &= -a + p_1 v + p_2 \left(-g + l\frac{a}{m}\right) + p_3(-ka) \end{aligned}$$

We next have to figure out the adjoint dynamics (ADJ). For our particular Hamiltonian,

$$H_{x_1} = H_h = 0, H_{x_2} = H_v = p_1, H_{x_3} = H_m = -\frac{p_2 la}{m^2}$$

Therefore

$$\begin{cases} \dot{p}_1(t) = 0 \\ \dot{p}_2(t) = -p_1(t) \\ \dot{p}_3(t) = \frac{p_2(t) l\alpha(t)}{m(t)^2} \end{cases} \tag{ADJ}$$

The maximization condition (M) reads

$$
\begin{aligned}
H(\mathbf{x}(t), \mathbf{p}(t), \alpha(t)) &= \max_{0 \leq a \leq 1} H(\mathbf{x}(t), \mathbf{p}(t), a) \\
&= \max_{0 \leq a \leq 1} \left\{ -a + p_1(t) v(t) + p_2(t) \left[-g + \frac{la}{m(t)} \right] + p_3(t)(-ka) \right\} \\
&= p_1(t) v(t) - p_2(t) g + \max_{0 \leq a \leq 1} \left\{ a \left(-1 + \frac{p_2(t) l}{m(t)} - k p_3(t) \right) \right\}
\end{aligned}
$$

Thus the optimal control law is given by the rule:

$$
\alpha(t) = \begin{cases} 1 & \text{if} \quad 1 - \frac{p_2(t) l}{m(t)} + k p_3(t) < 0 \\ 0 & \text{if} \quad 1 - \frac{p_2(t) l}{m(t)} + k p_3(t) > 0 \end{cases}
$$

Let us first leave rocket engine of (i.e., set $\alpha \equiv 0$) and turn the engine on only at the end. Denote by τ the first time that $h(\tau) = v(\tau) = 0$, meaning that we have landed. We guess that there exists a switching time $t^* < \tau$ when we turned engines on at full power (i.e., set $\alpha \equiv 1$). Consequently,

$$
\alpha(t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq t^* \\ 1 & \text{for} \quad t^* \leq t \leq \tau \end{cases}
$$

Therefore, for times $t^* \leq t \leq \tau$ our ODE becomes

$$
\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + \frac{l}{m(t)} \qquad (t^* \leq t \leq \tau) \\ \dot{m}(t) = -k \end{cases}
$$

with $h(\tau) = 0, v(\tau) = 0, m(t^*) = m_0$. We solve these dynamics:

$$
\begin{cases} m(t) = m_0 + k(t^* - t) \\ v(t) = g(\tau - t) + \frac{l}{k} \log \left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)} \right] \\ h(t) = -\frac{g(t-\tau)^2}{2} - \frac{l m_0}{k^2} \log \left[\frac{m_0 + k(t^* - \tau)}{m_0 + k(t^* - t)} \right] + l \frac{t - \tau}{k} \end{cases} \qquad \text{(second part)}
$$

Now put $t = t^*$:

$$
\begin{cases} m(t^*) = m_0 \\ v(t^*) = g(\tau - t^*) + \frac{1}{k} \log \left[\frac{m_0 + k(t^* - \tau)}{m_0} \right] \\ h(t^*) = -\frac{g(t^* - \tau)^2}{2} - \frac{l m_0}{k^2} \log \left[\frac{m_0 + k(t^* - \tau)}{m_0} \right] + l \frac{t^* - \tau}{k} \end{cases} \qquad \text{(switching point)}
$$

Suppose the total amount of fuel to start with was m_1; so that $m_0 - m_1$ is the weight of the empty spacecraft. When $\alpha \equiv 1$, the fuel is used up at rate k. Hence

$$k\left(\tau - t^*\right) \leq m_1$$

and so $0 \leq \tau - t^* \leq \frac{m_1}{k}$.
Before time t^*, we set $\alpha \equiv 0$. Then (ODE) reads

$$\begin{cases} \dot{h} = v \\ \dot{v} = -g \\ \dot{m} = 0 \end{cases}$$

$$\begin{cases} m(t) \equiv m_0 \\ v(t) = -gt + v_0 \\ h(t) = -\frac{1}{2}gt^2 + tv_0 + h_0 \end{cases} \tag{first part}$$

We combine the formulas for $v(t)$ and $h(t)$, to discover

$$h(t) = h_0 - \frac{1}{2g}\left(v^2(t) - v_0^2\right) \quad (0 \leq t \leq t^*)$$

We deduce that the freefall trajectory $(v(t), h(t))$ therefore lies on a parabola

$$h = h_0 - \frac{1}{2g}\left(v^2 - v_0^2\right).$$

So, we are moving along the parabola and at the time t^* turn on the rocket engine and land safely.

To justify our guess about the structure of the optimal control, let us now find the costate $\mathbf{p}(\cdot)$ so that $\alpha(\cdot)$ and $\mathbf{x}(\cdot)$ described above satisfy (ODE), (ADJ), (M). To do this, we will have have to figure out appropriate initial conditions

$$p_1(0) = \lambda_1, p_2(0) = \lambda_2, p_3(0) = \lambda_3.$$

We solve (ADJ) for $\alpha(\cdot)$ as above, and find

$$\begin{cases} p_1(t) \equiv \lambda_1 & (0 \leq t \leq \tau) \\ p_2(t) = \lambda_2 - \lambda_1 t & (0 \leq t \leq \tau) \\ p_3(t) = \begin{cases} \lambda_3 & (0 \leq t \leq t^*) \\ \lambda_3 + l\int_{t^*}^{t} \frac{\lambda_2 - \lambda_1 s}{\left(m_0 + k(t^* - s)\right)^2} ds \end{cases} & (t^* \leq t \leq \tau) \end{cases}$$

Define

$$r(t) := 1 - \frac{p_2(t)l}{m(t)} + p_3(t)k$$

then

$$\dot{r} = -\frac{\dot{p}_2 l}{m} + \frac{p_{2l}\dot{m}}{m^2} + \dot{p}_3 k = \frac{\lambda_1 l}{m} + \frac{p_2 l}{m^2}(-k\alpha) + \left(\frac{p_2 l\alpha}{m^2}\right) k = \frac{\lambda_1 l}{m(t)}.$$

Choose $\lambda_1 < 0$, so that r is decreasing. We calculate

$$r\left(t^*\right) = 1 - \frac{\left(\lambda_2 - \lambda_1 t^*\right) l}{m_0} + \lambda_3 k$$

and then adjust λ_2, λ_3 so that $r\left(t^*\right) = 0$.
Then r is nonincreasing, $r\left(t^*\right) = 0$, and consequently $r > 0$ on $\left[0, t^*\right), r < 0$ on $\left(t^*, \tau\right]$. But (M) says

$$\alpha(t) = \begin{cases} 1 & \text{if } r(t) < 0 \\ 0 & \text{if } r(t) > 0. \end{cases}$$

Thus an optimal control changes just once from 0 to 1 and so our solution of $\alpha(\cdot)$ does indeed satisfy the PMP.