

# Equations of Plane in Space

## 1 Definition and features of plane

While discussing axiomatic system we defined the plane as a *primitive notion* with axioms:

- **S<sub>1</sub>**. For each triplet of points there is plane containing them
- **S<sub>2</sub>**. For each triplet of points and plane shaped by them may be demonstrated point not laying in that plane. For any plane there are point not laying on it.
- **S<sub>3</sub>**. Locus of intersection of two planes possessing a common point is a line

Combined with features of segments and lines these axioms yield list of the features which we proved:

1. If two points, say  $A$  and  $B$  are contained in the plane, containing them line also lies in this plane, and there is exactly one such line
2. Each triplet of points, say  $A$ ,  $B$  and  $C$  which do not lay on the same line shapes single and only single plane. Thus,
  - (a) Line and point not laying on the line shape single and only single plane.
  - (b) Two crossing line shape single and only single plane.
3. Each plane separates space into two non-intersection parts

Assume that some plane  $\alpha$  in the space  $\mathbb{E}$  is chosen and fixed. In order to study various equations determining this plane we choose some coordinate system  $O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  in the space  $\mathbb{E}$ . Then we can describe the points of this plane  $\alpha$  by their radius vectors.

## 2 Vectorial parametric equation of a plane in the space

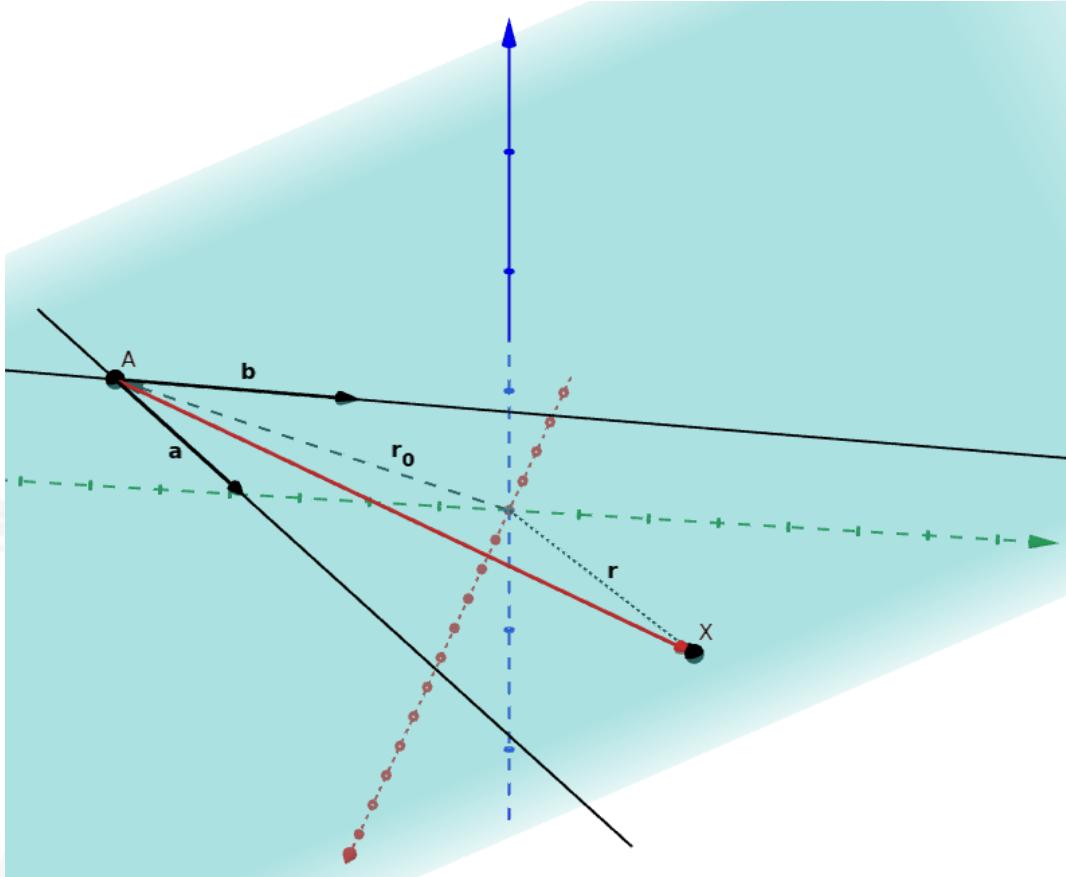


Figure 1: Parametrization of the plane

Let's denote as  $A$  some fixed point *initial* point on plane  $\alpha$ , and denote as  $X$  arbitrary point of this plane.

Position of the point  $A$  defined with respect to origin  $O$  with radius vector  $\overrightarrow{OA} = \mathbf{r}_0$ .  
And position of  $X$  within the plane defined with vector  $\overrightarrow{AX}$ .

Radius vector of the  $X$  with respect the origin  $O$  is

$$\overrightarrow{OX} = \mathbf{r} = \overrightarrow{OA} + \overrightarrow{AX} = \mathbf{r}_0 + \overrightarrow{AX}$$

To parametrize position of the  $X$  let us introduce two non-collinear vectors  $\mathbf{a} \nparallel \mathbf{b}$  coplanar with  $\alpha$ . Expansion of the vector  $AX$  by these two is explicit and has representation with formula (1).

$$\overrightarrow{AX} = \overrightarrow{AA'} + \overrightarrow{AA''} = t\mathbf{a} + \tau\mathbf{b} \quad (1)$$

In other words, we expressed point  $X$  in planar skew-angular coordinate system with origin  $A$  and basis vectors  $\mathbf{a}$  and  $\mathbf{b}$ .

Each point of the plane  $\alpha$  has sych expression with unique values of parameters  $t$  and  $\tau$ .

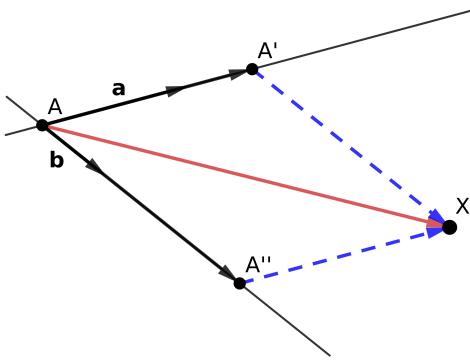


Figure 2: Parametrization of point  $X$

Thus, we have expression for arbitrary point  $X$  of plane  $\alpha$  with respect to coordinate system with origin  $O$ :

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a} + \tau\mathbf{b} \quad (2)$$

This writing is pretty similar with expressions to change coordinate system, but with overlook of third coordinate. This overlook demonstrates us that *we lost one degree of freedom while we assigned point  $X$  to the plane  $\alpha$ .*

**Definition.** The equality (2) is called the **vectorial parametric equation of a plane in the space**.

The non-collinear vectors  $\mathbf{a}$  and  $\mathbf{b}$  in it are called **direction vectors of a plane**, while  $t$  and  $\tau$  are called **parameters of the plane**.

The fixed vector  $\mathbf{r}_0$  is the **radius vector of an initial point**.

### Problem 1

Express plane passing through the points  $A(1, 1, 1)$ ,  $B(1, 2, 3)$  and  $C(3, 3, 3)$  with vectorial parametric equation.

### Solution

Let us take point  $A$  as initial.

As direction vectors we take  $\overrightarrow{AB}$  with coordinates  $(0, 1, 2)$  and  $\overrightarrow{AC}$  with coordinates  $(2, 2, 2)$

$$\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \tau \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

To demonstrate alternative equivalent form of the equation of this plane let us take  $C$  as initial point and vectors  $\overrightarrow{CA}$  and  $\overrightarrow{CB}$  as direction vectors.

Coordinates of  $\overrightarrow{CA}$  and  $\overrightarrow{CB}$  are  $(-2, -2, -2)$ , and  $(-2, -1, 0)$ . Equation is

$$\mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} + t \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} + \tau \begin{pmatrix} -2 \\ -1 \\ 0 \end{pmatrix}$$

### Problem 2

Write vectorial parametric equation of the plane passing through the origin, containing second coordinate axis and intersection of this plane with coordinate plane build on  $\mathbf{e}_1$  and  $\mathbf{e}_3$  shapes angle  $\pi/3$  with positive direction of first coordinate axis, and angle  $\pi/6$  with positive direction of the third coordinate axis. Basis is right orthonormal.

### Solution

We take the origin as initial point, thus  $\mathbf{r}_0 = \mathbf{0}$ , and we overlook it.

It is natural to take second basis vector as one of the direction vector of the line, as our plane contains this axis. Coordinates of it are  $(0, 1, 0)$ .

Intersection of two planes is a line laying in  $xOz$  and shaping angle  $\pi/3$  with positive direction of  $Ox$ , and angle  $\pi/6$  with positive direction of the  $Oz$  axis. We take unit vector collinear to this line as second direction vector. Its coordinates are  $(\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$ .

$$\mathbf{r} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

### Problem 3

Write vectorial parametric equations of the coordinate planes

### Solution

We take origin as initial point and overlook  $\mathbf{r}_0$ .

Each pair of basis vectors satisfies conditions on directional vectors and is coplanar with one of the planes in question.

Hence, equations are:

$$\mathbf{r} = t\mathbf{e}_1 + \tau\mathbf{e}_2$$

$$\mathbf{r} = t\mathbf{e}_1 + \tau\mathbf{e}_3$$

$$\mathbf{r} = t\mathbf{e}_2 + \tau\mathbf{e}_3.$$

### 3 Vectorial parametric equation of a plane in the space

Let us recap some definitions and statements first.

**Definition.** We say that arbitrary line  $a$  is perpendicular with plane  $\alpha$  if this line is perpendicular with all lines laying in  $\alpha$ .

**Definition.** Direction vector of any line  $a$  perpendicular with plane  $\alpha$  is said **normal vector of plain**.

Perpendicularity of arbitrary line  $a$  with both of arbitrary crossing lines  $b$  and  $c$  is the evidence for fact that line  $a$  is perpendicular with plane  $\alpha$  containing these lines  $b$  and  $c$ .

Let  $\mathbf{n} \neq \mathbf{0}$  be normal vector of the plane expressed with vectorial parametric equation:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a} + \tau\mathbf{b}$$

Dot product of both sides of this equation by vector  $\mathbf{n}$  yields:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{r}_0 \cdot \mathbf{n} + t\mathbf{a} \cdot \mathbf{n} + \tau\mathbf{b} \cdot \mathbf{n}$$

Terms  $\mathbf{a} \cdot \mathbf{n}$  and  $\mathbf{b} \cdot \mathbf{n}$  are equal with zero because  $\mathbf{n}$  perpendicular with both  $\mathbf{a}$ , and  $\mathbf{b}$ . Term  $\mathbf{r}_0 \cdot \mathbf{n}$  is constant value usually denoted as  $D$ .

Hence, our equation takes one of forms:

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0, \text{ or} \tag{3}$$

$$\mathbf{r} \cdot \mathbf{n} = D \tag{4}$$

**Definition.** Any of equations (4) is called the **normal vectorial equation of a plane in the space**.

#### Problem 1

Give the vectorial normal equation of the plane with normal vector  $(10, 8, 3)$  that contains the point  $(10, 5, 5)$ . Basis is right orthonormal.

#### Solution

We are given the initial point and normal vector, hence we can write the equation:

$$\left( \mathbf{r} - \begin{pmatrix} 10 \\ 5 \\ 5 \end{pmatrix} \right) \cdot \begin{pmatrix} 10 \\ 8 \\ 3 \end{pmatrix} = 0.$$

With provided information about basis we can calculate  $D$ :

$$D = 10 \cdot 10 + 8 \cdot 5 + 3 \cdot 5 = 155$$

Hence, equation in provided basis has form:

$$\mathbf{r} \cdot \begin{pmatrix} 10 \\ 8 \\ 3 \end{pmatrix} = 155$$

### Problem 2

Find the vectorial normal equation of the plane that passes through the point  $(5, 1, -1)$  and is parallel to the two vectors  $(9, 7, -8)$  and  $(-2, 2, -1)$ . Basis right orthonormal.

### Solution

We are given with initial point and two direction vectors  $\mathbf{a}$  and  $\mathbf{b}$  of the plane.

All we need is to express normal vector  $\mathbf{n}$ .

While we recap definition of the **cross product** of two vectors, we notice that result of dot product is perpendicular with both multiply.

Hence, we take  $\mathbf{n}$  as

$$\begin{aligned} \mathbf{n} = \mathbf{a} \times \mathbf{b} &= (7 \cdot (-1) - (-8) \cdot 2)\mathbf{e}_1 - (9 \cdot (-1) - (-8) \cdot (-2))\mathbf{e}_2 + (9 \cdot 2 - 7 \cdot (-2))\mathbf{e}_3 = \\ &= 9\mathbf{e}_1 + 25\mathbf{e}_2 + 32\mathbf{e}_3 \end{aligned}$$

Value of  $D$  is  $D = 5 \cdot 9 + 25 - 32 = 38$ .

Hence, vectorial normal equation of this plane for right orthonormal basis is:

$$\mathbf{r} \cdot \begin{pmatrix} 9 \\ 25 \\ 32 \end{pmatrix} = 38$$

If we are given with a triplet of points, approach is the same:

### Problem 3

Write in vectorial normal form the equation of the plane containing points  $(1, 0, 3)$ ,  $(1, 2, -1)$ , and  $(6, 1, 6)$ .

Basis is right orthonormal.

### Solution

Let's first determine a normal vector to the plane.

We can obtain two direction vectors of the plane by subtracting the position vectors of pairs of points on the plane:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \quad (5)$$

$$\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ -3 \end{pmatrix} \quad (6)$$

Normal vector is cross product of these two:

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = ((-2) \cdot (-3) - 4 \cdot (-1))\mathbf{e}_1 - (0 \cdot (-3) - 4 \cdot (-5))\mathbf{e}_2 + (0 \cdot (-1) - (-2) \cdot (-5))\mathbf{e}_3 = 10\mathbf{e}_1 - 20\mathbf{e}_2 - 10\mathbf{e}_3.$$

Value of  $D$  is  $D = 10 + 0 - 30 = -20$ .

Hence, equation is:

$$\mathbf{r} \cdot \begin{pmatrix} 10 \\ -20 \\ -10 \end{pmatrix} = -20,$$

or

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = -2,$$

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## 4 Coordinate parametric equation of a plane in the space

Let us write coordinate representation for vectors  $\mathbf{r}$ ,  $\mathbf{r}_0$ ,  $\mathbf{a}$  and  $\mathbf{b}$  in vectorial parametric equation of the line (2):

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

This lets us to rewrite vectorial parametric equation as a system of equations for each coordinate:

$$\begin{cases} x = x_0 + a_x t + b_x \tau \\ y = y_0 + a_y t + b_y \tau \\ z = z_0 + a_z t + b_z \tau \end{cases} \quad (7)$$

**Definition.** The equations (7) are called the **coordinate parametric equations of the plane in space**.

To ensure that vectors  $\mathbf{a}$  and  $\mathbf{b}$  are not collinear, triplets of parameters  $\{a_x, a_y, a_z\}$  and  $\{b_x, b_y, b_z\}$  must not be proportional

### Problem 1

Write coordinate parametric equations for the plane containing triangle  $\triangle ABC$  with vertices  $A(1, 3, 1)$ ,  $B(7, 7, 7)$ ,  $C(5, 5, 5)$

### Solution

We take point  $A$  as initial, thus  $x_0 = 1$ ,  $y_0 = 3$ ,  $z_0 = 1$ .

As direction vectors we take  $\mathbf{a} = \overrightarrow{AB}$  and  $\mathbf{b} = \overrightarrow{AC}$ . Their coordinates are  $(6, 4, 6)$  and  $(4, 2, 4)$ .

Hence,  $a_x = 6$ ,  $a_y = 4$ ,  $a_z = 6$ ,  $b_x = 4$ ,  $b_y = 2$ ,  $b_z = 4$ . We can not express triplet  $\{b_x, b_y, b_z\}$  with  $\{a_x, a_y, a_z\}$  by multiplication with a constant, thus the set of parameters is valid.

Coordinate parametric equations of this plane are:

$$\begin{cases} x = 1 + 6t + 4\tau \\ y = 3 + 4t + 2\tau \\ z = 1 + 6t + 4\tau \end{cases}$$

## Coordinate parametric equations of the coordinate planes

Each of these plane passes through the origin, thus  $x_0 = y_0 = z_0 = 0 = 0$

1. Plane directed with  $\mathbf{e}_1(1, 0, 0)$ , and  $\mathbf{e}_2(0, 1, 0)$ :

$$a_x = 1, a_y = a_z = 0, b_y = 1, b_x = b_z = 0$$

$$\begin{cases} x = t \\ y = \tau \\ z = 0 \end{cases} \quad \text{or} \quad \begin{cases} x = \tau \\ y = t \\ z = 0 \end{cases}$$

or

2. Plane directed with  $\mathbf{e}_1(1, 0, 0)$ , and  $\mathbf{e}_3(0, 0, 1)$ :

$$a_x = 1, a_y = a_z = 0, b_z = 1, b_x = b_y = 0$$

$$\begin{cases} x = t \\ y = 0 \\ z = \tau \end{cases} \quad \text{or} \quad \begin{cases} x = \tau \\ y = 0 \\ z = t \end{cases}$$

3. Plane directed with  $\mathbf{e}_2(0, 1, 0)$ , and  $\mathbf{e}_3(0, 0, 1)$ :

$$a_y = 1, a_x = a_z = 0, b_z = 1, b_x = b_y = 0$$

$$\begin{cases} x = 0 \\ y = t \\ z = \tau \end{cases} \quad \text{or} \quad \begin{cases} x = 0 \\ y = \tau \\ z = t \end{cases}$$

## 5 Canonical equation of a plane in the space

Let us express vectorial parametric equation of the plane in space (2)

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{a} + \tau\mathbf{b}$$

in vectorial normal form.

To obtain normal vector  $\mathbf{n}$  we cross product vectors  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\mathbf{n} = \mathbf{a} \times \mathbf{b}$$

Hence, vectorial normal equation (4) takes form:

$$\begin{aligned} (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} &= 0 \\ (\mathbf{r} - \mathbf{r}_0) \cdot (\mathbf{a} \times \mathbf{b}) &= 0 \end{aligned}$$

$$(\mathbf{r} - \mathbf{r}_0, \mathbf{a}, \mathbf{b}) = 0 \quad (8)$$

or

$$\begin{aligned} \mathbf{r} \cdot \mathbf{n} &= D \\ \mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) &= D \end{aligned}$$

$$(\mathbf{r}, \mathbf{a}, \mathbf{b}) = D \quad (9)$$

**Definition.** Equations (8) and (9) are representations for the **vectorial canonical equation of the plane in space**.

To obtain coordinate form of canonical equation, we need to calculate that mixed product. Let us write coordinate representation for vectors  $\mathbf{r}$ ,  $\mathbf{r}_0$ ,  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} & \mathbf{r}_0 &= \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \\ \mathbf{a} &= \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} & \mathbf{b} &= \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} \end{aligned}$$

Now we apply formula for the mixed product in arbitrary basis for left side of (8):

$$(\mathbf{r} - \mathbf{r}_0, \mathbf{a}, \mathbf{b}) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Mixed product  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  is the *orientated volume* of the basis, and it is never zero. Hence, canonical equation of the plane in space has first form:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = 0 \quad (10)$$

Formula for mixed product applied for the right side of (9) yields:

$$(\mathbf{r}, \mathbf{a}, \mathbf{b}) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \begin{vmatrix} x & y & z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\frac{1}{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)}$  now may be merged in to constant parameter  $D$ , and canonical equation of the plane in space has second form:

$$\begin{vmatrix} x & y & z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = D' \quad (11)$$

Prime with  $D'$  in this writing is often overlooked.

**Definition.** Equations (10) and (11) are representations for the **canonical equation of the plane in space**.

## Canonical equations of the coordinate planes

All these planes are passing through the origin

$$1. (\mathbf{r}, \mathbf{e}_1, \mathbf{e}_2) = 0$$

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0$$

$$z = 0$$

$$2. (\mathbf{r}, \mathbf{e}_1, \mathbf{e}_3) = 0$$

$$\begin{vmatrix} x & y & z \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$y = 0$$

$$3. (\mathbf{r}, \mathbf{e}_2, \mathbf{e}_3) = 0$$

$$\begin{vmatrix} x & y & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$x = 0$$

### Problem 1

Find canonical equation of the plane passing through the point  $(1, 2, 3)$  and having direction vectors  $(5, 6, 7)$  and  $(9, 9, 9)$ .

### Solution

The canonical equation is:

$$\begin{vmatrix} x - 1 & y - 2 & z - 3 \\ 5 & 6 & 7 \\ 9 & 9 & 9 \end{vmatrix} = 0$$

Let us expand the determinant:

$$\begin{vmatrix} x - 1 & y - 2 & z - 3 \\ 5 & 6 & 7 \\ 9 & 9 & 9 \end{vmatrix} = (x - 1) \cdot 6 \cdot 9 + (y - 2) \cdot 7 \cdot 9 + (z - 3) \cdot 5 \cdot 9 - (x - 1) \cdot 7 \cdot 9 - (y - 2) \cdot 5 \cdot 9 - (z - 3) \cdot 6 \cdot 9 = -9 \cdot (x - 1) + 18 \cdot (y - 2) - 9 \cdot (z - 3) = -9x + 18y - 9z$$

Equation is

$$\begin{aligned} -9x + 18y - 9z &= 0 \\ x - 2y + z &= 0 \end{aligned}$$

## 6 Resolving canonical equation. General equation of plane.

Let us calculate determinant on the left side of equation (10) in general case:

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_x b_y - a_y b_x) (z - z_0) - (a_x b_z - a_z b_x) (y - y_0) + (a_y b_z - a_z b_y) (x - x_0)$$

$$A'(x - x_0) + B'(y - y_0) + C'(z - z_0) = 0$$

expanding and combining free terms into  $D$ , we yield:

$$Ax + By + C + D = 0 \quad (12)$$

**Definition.** We call equation (12) the **general equation of the plane**. Parameters  $A$ ,  $B$  and  $C$  in this equation can not be zero in the same time.

In any coordinate system any plane will be expressed in form (12).

## 7 Explanation of the coefficients in general equation of the plane

Suppose, plane was expressed in vectorial normal form:

$$\mathbf{r} \cdot \mathbf{n} = D, \quad (13)$$

in coordinate system with specified Gram matrix  $G$  with components  $g_{ij}$ .

Let coordinates of  $\mathbf{r}$  be  $(r^1, r^2, r^3)$ , and coordinates of  $\mathbf{n}$  be  $(n^1, n^2, n^3)$ .

$$\mathbf{r} \cdot \mathbf{n} = \sum_{i=1}^3 \sum_{j=1}^3 r^i n^j g_{ij} = \sum_{i=1}^3 \left( \sum_{j=1}^3 n^j g_{ij} \right) r^i = \sum_{i=1}^3 n_i r^i$$

**Definition.** The quantities  $n_1$ ,  $n_2$ , and  $n_3$  produced from the coordinates of the normal vector  $\mathbf{n}$  by mens of the formula

$$n_i = \sum_{j=1}^3 n^j g_{ij}$$

are called the **covariant coordinates of the vector  $\mathbf{n}$** .

*Remark.* Any vector is perpendicular with plane if its covariant coordinates are proportional with the coefficients  $A$ ,  $B$ , and  $C$ .

As particular case, any vector or line having direction numbers  $a$ ,  $b$  and  $c$  in arbitrary orthonormal basis, is perpendicular with plane  $Ax + By + Cz + D = 0$  if

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C},$$

while all numbers are non-zero. These parameters in this case may be zeros only in pair ( $A \leftrightarrow a$ ,  $B \leftrightarrow b$ ,  $C \leftrightarrow c$ ).

## 8 Dual basis in space

Similar to the plane, covariant coordinates correspond to dual basis, which in space has basis vectors:

$$\mathbf{e}^1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} \quad \mathbf{e}^2 = \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} \quad \mathbf{e}^3 = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)}$$

Let us check correctness of this basis.

First, dot products  $\mathbf{e}^i \cdot \mathbf{e}_i$  yield the same oriented volume  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  in numerator and denominator. Hence,  $\mathbf{e}^i \cdot \mathbf{e}_i = 1$

Second, dot products  $\mathbf{e}^i \cdot \mathbf{e}_j$ ,  $i \neq j$  yield coincide indexes in numerator, hence  $\mathbf{e}^i \cdot \mathbf{e}_j = 0$ ,  $i \neq j$ . Generally,

$$\mathbf{e}^i \cdot \mathbf{e}_j = \delta_j^i$$

It remains to check that this triplet of vectors is linearly independent.

Suppose they are linearly dependent, and at least one  $a_i$  is non-zero in expression:

$$\frac{\mathbf{e}_2 \times \mathbf{e}_3}{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} a_1 + \frac{\mathbf{e}_3 \times \mathbf{e}_1}{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} a_2 + \frac{\mathbf{e}_1 \times \mathbf{e}_2}{(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)} a_3 = 0$$

Overlooking denominators and expressing cross products with effectivized formula for cross product

$$\mathbf{a} \times \mathbf{b} = \pm \sqrt{\det G} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 a^i b^j C_{ij}^k \mathbf{e}_k = \pm \sqrt{\det G} \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 \sum_{q=1}^3 a^i b^j \varepsilon_{ijq} g^{qk} \mathbf{e}_k,$$

we yield:

$$\mathbf{e}_2 \times \mathbf{e}_3 = \pm \sqrt{\det G} \sum_{k=1}^3 \sum_{q=1}^3 \varepsilon_{23q} g^{qk} \mathbf{e}_k = \pm \sqrt{\det G} \sum_{k=1}^3 g^{1k} \mathbf{e}_k$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \pm \sqrt{\det G} \sum_{k=1}^3 \sum_{q=1}^3 \varepsilon_{31q} g^{qk} \mathbf{e}_k = \pm \sqrt{\det G} \sum_{k=1}^3 g^{2k} \mathbf{e}_k$$

$$\mathbf{e}_1 \times \mathbf{e}_2 = \pm \sqrt{\det G} \sum_{k=1}^3 \sum_{q=1}^3 \varepsilon_{12q} g^{qk} \mathbf{e}_k = \pm \sqrt{\det G} \sum_{k=1}^3 g^{3k} \mathbf{e}_k$$

Hence, we have equation:

$$\pm \sqrt{\det G} a_1 \sum_{k=1}^3 g^{1k} \mathbf{e}_k + \pm \sqrt{\det G} a_2 \sum_{k=1}^3 g^{2k} \mathbf{e}_k + \pm \sqrt{\det G} a_3 \sum_{k=1}^3 g^{3k} \mathbf{e}_k = 0$$

Multiply  $\pm \sqrt{\det G}$  is taken with the same sign in all positions, hence we overlook it and combine coefficients for  $\mathbf{e}_i$ :

$$(a_1 g^{11} + a_2 g^{21} + a_3 g^{31}) \mathbf{e}_1 + (a_1 g^{12} + a_2 g^{22} + a_3 g^{32}) \mathbf{e}_2 + (a_1 g^{13} + a_2 g^{23} + a_3 g^{33}) \mathbf{e}_3 = 0$$

Since,  $\mathbf{e}_i$  are linearly independent, this equation satisfies only if

$$\begin{cases} a_1 g^{11} + a_2 g^{21} + a_3 g^{31} = 0 \\ a_1 g^{12} + a_2 g^{22} + a_3 g^{32} = 0 \\ a_1 g^{13} + a_2 g^{23} + a_3 g^{33} = 0 \end{cases}$$

Matrix of this system is transposed inverse of Gram matrix for original basis, hence it is non-degenerate, and  $(0, 0, 0)$  is its only solution.

Therefore, purposed basis is honest dual basis for given basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ .

It must be underlined, that *orthonormal basis is dual for itself*.

## Scalar form of equation of the plane

While expanding normal equation in form

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0,$$

we obtain form of the plane equation:

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0. \quad (14)$$

**Definition.** Equation of the plane in space in form (14) is called **scalar equation of the plane**.

### Problem 1

Give the equation of the plane with normal vector  $(10, 8, 3)$  that contains the point  $(10, 5, 5)$ . Basis is right orthonormal.

## Solution

We write the scalar form of the equation for this plane:

$$\begin{aligned}10(x - 10) + 8(y - 5) + 3(z - 5) &= 0 \\10x - 100 + 8y - 40 + 3z - 15 &= 0 \\10x + 8y + 3z - 155 &= 0\end{aligned}$$

Thus, general equation of this plane is

$$10x + 8y + 3z - 155 = 0$$

## Problem 2

Give the equation of the plane with normal vector  $(6, 8, 4)$  that contains the point  $(10, 5, 5)$ . Gram matrix of the basis is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}$$

## Solution

We start with calculating the covariant coordinates of normal vector.

$$n_i = \sum_{j=1}^3 n^j g_{ij}$$

$$\begin{aligned}n_1 &= n^1 g_{11} + n^2 g_{12} + n^3 g_{13} = 6 + \frac{8}{2} + \frac{4}{2} = 12 \\n_2 &= n^1 g_{21} + n^2 g_{22} + n^3 g_{23} = \frac{6}{2} + 8 + \frac{4}{2} = 13 \\n_3 &= n^1 g_{31} + n^2 g_{32} + n^3 g_{33} = \frac{6}{2} + \frac{8}{2} + 4 = 11\end{aligned}$$

Scalar form of the equation is:

$$12(x - 10) + 13(y - 5) + 11(z - 5) = 0$$

general form of the equation is

$$12x + 13y + 11z - 240 = 0$$

### Problem 3

The equation of a plane has the general form  $5x + 6y + 9z - 28 = 0$ . What is its vector form? Basis is orthonormal.

### Solution

We want to determine the equation of the plane in vector form by using the given equation of the plane in general form.

Recall that the general form of the equation of a plane is

$$Ax + By + Cz + D = 0$$

where  $A$ ,  $B$ , and  $C$  are the covariant coordinates of the normal vector  $\mathbf{n}$ , which is perpendicular to the plane or any vector parallel to the plane. For orthonormal basis covariant and original coordinates coincide.

The vector equation of the plane can be written as

$$\mathbf{r} \cdot \mathbf{n} = -D$$

From the given equation of the plane,  $5x + 6y + 9z - 28 = 0$ , we can identify the normal vector as  $(5, 6, 9)$  and  $D = -28$ . The vector equation of the plane can be written as

$$\mathbf{r} \cdot \begin{pmatrix} 5 \\ 6 \\ 9 \end{pmatrix} = 28$$

### Problem 4

Show that plane  $Ax + D = 0$  is perpendicular with first basis vector  $\mathbf{e}_1$  in right orthonormal basis.

### Solution

Coefficients with variables  $x$ ,  $y$ ,  $z$  in the general equation of the line represent covariant components of normal vector.

Hence, normal vector to expressed plane is  $\begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = A\mathbf{e}_1$  and collinear with  $\mathbf{e}_1$ .

Therefore,  $Ax + D = 0$  is perpendicular with first basis vector  $\mathbf{e}_1$ .

**Definition.** Planes  $Ax + D = 0$ ,  $By + D = 0$ ,  $Cx + D = 0$  are perpendicular with basis vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ ,  $\mathbf{e}_3$  respectively, while these planes are expressed in coordinate system with right orthonormal basis

## 9 Angle between planes

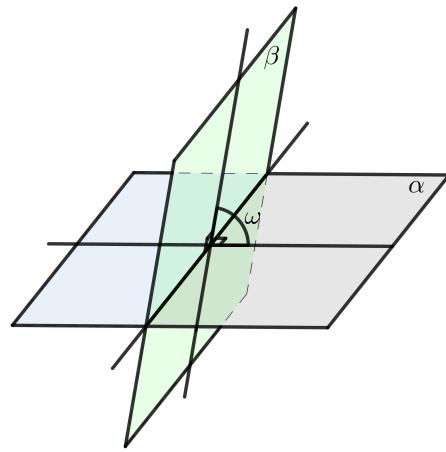


Figure 3: Angle between planes

**Definition.** Angle between two planes  $\alpha$  and  $\beta$  is angle between two lines  $a \in \alpha$ ,  $b \in \beta$  which are perpendicular with intersection line of these planes.

Normal vectors to both lines established in the same point of intersection line shape equal angle.

Hence, cosines of this angle may be calculated with formula:

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$$

If planes are expressed with general equations

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0 \\ A_2x + B_2y + C_2z + D_2 &= 0 \end{aligned}$$

with respect to right orthonormal basis, this formula has expansion:

$$\cos \theta = \left| \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

### Problem 1

Find the smallest angle between the planes:

$$\begin{aligned} 3x + 2y - 5z - 4 &= 0, \text{ and} \\ 2x - 3y + 5z - 8 &= 0, \end{aligned}$$

Basis is right orthonormal.

## Solution

$$\cos \theta = \left| \frac{3 \cdot 2 - 3 \cdot 3 - 5 \cdot 5}{\sqrt{9+4+25}\sqrt{4+9+25}} \right| = \frac{25}{38}$$

Two planes  $A_1x + B_1y + C_1z + D = 0$  and

$$A_1x + B_1y + C_1z + D = 0$$

are parallel if normal vector of these planes are collinear, hence there is  $p > 0$ :  $p\mathbf{n}_2 = \mathbf{n}_1$ .

Expressed in coordinates, this yields:

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = p \quad (15)$$

For all non-zero  $A_i, B_i, C_i$ .

If any of these parameters is zero, its counterpart must be zero too, and ratio preserves for the rest of pairs.

Two planes are perpendicular, if their normal vectors are perpendicular.

In orthonormal basis this has expression:

$$A_1A_2 + B_1B_2 + C_1C_2 = 0$$

## Problem 2

Show that plane  $Ax + By + D = 0$  is perpendicular with coordinate plane  $xOy$  if coordinate system has right orthonormal basis.

## Solution

Coordinates of normal vector of  $Ax + By + D = 0$  are  $(A, B, 0)$ .

Plane  $xOy$ , in its turn, has equation  $z = 0$ , and normal vector with coordinates  $(0, 0, 1)$  (actually, third basis vector  $\mathbf{e}_3$ )

Dot product of these vectors is  $A \cdot 0 + B \cdot 0 + 0 \cdot 1 = 0$ , hence the vector are perpendicular.

**Definition.** Planes  $Ax + By + D = 0$ ,  $By + Cz + D = 0$  and  $Ax + Cz + D = 0$  are perpendicular with planes  $xOy$ ,  $yOz$  and  $xOz$  respectively, while these planes are expressed in coordinate system with right orthonormal basis.