Equations of Mathematical Physics Homework 6

Problem 1

Solve the Cauchy problem for the equation

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2xy,$$

satisfying the conditions $y=x \quad u=x^2$
Solution:
Let's make up a characteristic system:

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}u}{2xy}$$

We will find the first integral by solving the equation:

$$\frac{\mathrm{d}x}{x}=\frac{\mathrm{d}y}{y} \quad\Longrightarrow\quad \frac{x}{y}=C_1$$

then we have $x=C_1y$, substitute C_1y into x ,we have the second integral:

$$\frac{\mathrm{d}x}{C_1y}=\frac{\mathrm{d}u}{2xy} \quad\Longrightarrow\quad 2x\mathrm{d}x=C_1\mathrm{d}u \quad\Longrightarrow\quad x^2=C_1u+C_2 \quad\Longrightarrow\quad C_2=\frac{xu}{y}-x^2$$

thus we find the general solution without considering the integral surface:

$$\Phi(\frac{x}{y},\frac{xu}{y}-x^2)=0$$

Solving the Cauchy problem, take x as the parameter,we obtain:

$$C_1=1 \qquad C_2=0$$

This gives $\begin{cases}\frac{x}{y}=1\\ \frac{xu}{y}-x^2=0\end{cases}$, i.e. $\begin{cases}x=y\\ u=xy\end{cases}$.

Thus the solution can be given as a parametric form: $\begin{cases}x=t\\ y=t\\ u=t^2\end{cases}$, where t is a parameter.