Homework: Stability of Dynamical System

July 2025

Problem 1(Full Version)

Let's investigate function $f_\lambda(x) = x^3 + \lambda$

1 Fixed points

Solution

We solve

$$f_\lambda(x) = x \quad \iff \quad x^3 - x + \lambda = 0.$$

Let Δ be the cubic discriminant:

$$\Delta = -4(-1)^3 - 27\lambda^2 = 4 - 27\lambda^2.$$

$$\begin{cases} |\lambda| < \dfrac{2}{3\sqrt{3}} \approx 0.3849 & \Longrightarrow \text{three real fixed points,} \\ |\lambda| = \dfrac{2}{3\sqrt{3}} & \Longrightarrow \text{two coincident real fixed points (saddle–node),} \\ |\lambda| > \dfrac{2}{3\sqrt{3}} & \Longrightarrow \text{one real fixed point.} \end{cases}$$

For some fixed λ

λ	fixed point
0	$x = -1,\ 0,\ 1$
$\frac{2}{3\sqrt{3}}$ (≈ 0.3849)	$x = \frac{1}{\sqrt{3}}, -\frac{2}{\sqrt{3}}$
0.5 (> 0.3849)	$x \approx -1.1915$

2 Points of period two

Solution

No point of period two. In fact, we can prove a stronger proposition:

Any increasing function from $\mathbb{R}$ to $\mathbb{R}$ do not have periodic point of period more than or equal to 2.

Proof: Assume the converse. If exist $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $f^n(x_0) = x_0$, then we can denote $f(x_0) = x_1$, $f(x_1) = x_2$,...$f(x_{n-1}) = x_0$
If $x_0 = x_1$, then x_0 is a fixed point.
If $x_0 \neq x_1$, w.l.g. let $x_0 > x_1$ then we have a chain: $x_{n-1} > x_0 > x_1 > x_2 > ... > x_{n-1}$, which causes a contradiction.

3 Points of period more than two

No point of period more than two. As we have shown above.

4 Attracting/Repelling of points

Solution

For a fixed point x_*,

$$f'_\lambda(x_*) = 3x_*^2.$$

Hence the point is *attracting* if $|x_*| < \dfrac{1}{\sqrt{3}}$ and *repelling* if $|x_*| > \dfrac{1}{\sqrt{3}}$. For some fixed λ

λ	fixed λ	nature
0	$x = -1,\ 0,\ 1$	$-1, 1$ repelling; 0 attracting
$\frac{2}{3\sqrt{3}}$ (≈ 0.3849)	$x = \frac{1}{\sqrt{3}}$(double), $-\frac{2}{\sqrt{3}}$(simple)	double root bifurcation; simple root repelling
0.5 (> 0.3849)	$x \approx -1.1915$	repelling

Thus the general classification is, inside the window $|\lambda| < 2/(3\sqrt{3})$ there are three real fixed points with exactly one attractor; outside that window only one (always repelling) fixed point remains.If $\lambda = 2/(3\sqrt{3})$ the bifurcation appears,more detailed consideration is needed.

5 λ for structurally unstable

Solution

- A map $f : J \to J$ is C^1–*structurally stable* on J if there exists $\varepsilon > 0$ such that every $g : J \to J$ with $d_1(f, g) < \varepsilon$ is topologically conjugate to f.
- A necessary condition for structural stability in one dimension is that *all* periodic points are hyperbolic; the failure of hyperbolicity at any parameter value therefore signals structural *instability*.

For $f_\lambda(x) = x^3 + \lambda$

$$f_\lambda(x) = x \iff x^3 - x + \lambda = 0, \qquad f'_\lambda(x) = 3x^2.$$

1. Hyperbolicity fails when $|f'_\lambda(x_*)| = 1$, i.e. when $3x_*^2 = 1$ so $x_* = \pm\frac{1}{\sqrt{3}}$.
2. Substituting these x_* into the fixed-point equation gives

$$\lambda = x_* - x_*^3 = \pm\frac{2}{3\sqrt{3}}.$$

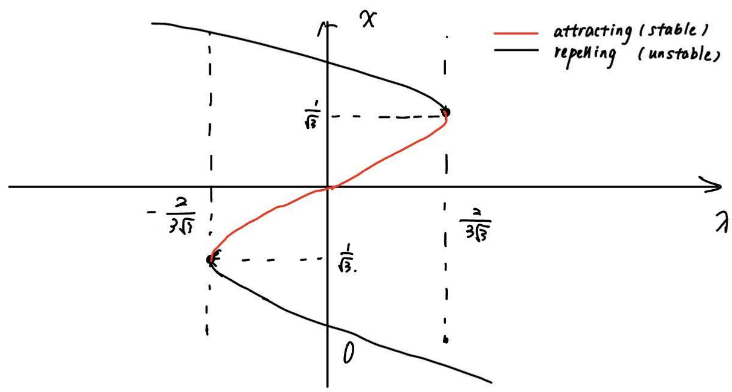
At these two values f_λ possesses a non-hyperbolic fixed point produced by a saddle–node collision; hence the family is C^1–*structurally unstable* precisely at $\lambda = \pm 2/(3\sqrt{3})$ and structurally stable for all other parameter values.

6 Bifurcation Diagram

Solution

The figure below plots all real fixed points x_* of f_λ for $\lambda \in [-1, 1]$; the vertical dashed lines mark the structurally-unstable parameters $\lambda_{\text{SN}} = \pm 2/(3\sqrt{3})$. Inside the interval $|\lambda| < 2/(3\sqrt{3})$ three branches exist (middle one attracting, outer two repelling); outside it only a single repelling branch remains.

bifurcation map: $x_{n+1} = x_n^3 + \lambda$.



Problem 2.1

$$A_1 = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$$

Solution

- Eigenvalues: $\lambda_1 = -2,\ \lambda_2 = 2$ (both satisfy $|\lambda| > 1$).
- Corresponding eigenvectors: $v_1 = (1,0)^\mathsf{T},\ v_2 = (0,1)^\mathsf{T}$.
- Unstable set: $W^u = \text{span}\{v_1, v_2\} = \mathbb{R}^2$.
- Stable set: $W^s = \{0\}$ (no eigenvalues with $|\lambda| < 1$).
- All eigenvalues lie off the unit circle $\Rightarrow$ the map is hyperbolic. The origin is a totally repelling fixed point.

Problem 2.6

$$A_6 = \begin{pmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Solution

- Eigenvalues: $\lambda_u = 2$ (unstable), $\lambda_s = \frac{1}{2}$ (stable).
- Eigenvectors:

$$v_u = (1,0)^\mathsf{T}, \qquad v_s = (2,-3)^\mathsf{T} \text{ (solves } (A - \tfrac{1}{2}I)v = 0).$$

- Unstable set: $W^u = \text{span}\{v_u\}$ (the x–axis).
- Stable set: $W^s = \text{span}\{v_s\}$ (a one-dimensional line through the origin with slope $-3/2$).
- Because no eigenvalue lies on the unit circle, A_6 is hyperbolic. Trajectories on W^u diverge exponentially; those on W^s converge exponentially to the origin.

Problem 2.11

$$A_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Solution

- Eigenvalues: $\lambda_1 = 1,\ \lambda_2 = -1$ (both satisfy $|\lambda| = 1$).
- Eigenvectors: $v_1 = (1,1)^\mathsf{T}$ for $\lambda = 1$; $v_2 = (1,-1)^\mathsf{T}$ for $\lambda = -1$.
- Moduli equal to one $\Rightarrow$ the map is non-hyperbolic.
- Dynamics: points on $\operatorname{span}\{v_1\}$ are fixed; points on $\operatorname{span}\{v_2\}$ alternate sign and have period 2. No trajectories converge to or diverge from the origin, so the classical stable and unstable sets reduce to $\{0\}$.

Problem 3.1

Let $T_{-1}(x) = x^3 + x$ on $\mathbb{R}$.Prove that T_{-1} is not structurally stable.

Solution

1. Fixed point and derivative
The fixed point equation $T(x) = x$ gives

$$x^3 + x = x \implies x^3 = 0 \implies x_0 = 0.$$

The derivative is $T'(x) = 3x^2 + 1$; at the fixed point

$$T'(x_0) = 1.$$

2. Non-hyperbolicity
A periodic point is hyperbolic when the derivative of the first-return map has modulus different from 1. Here $|T'(x_0)| = 1$, so x_0 is a non-hyperbolic fixed point. Structural stability in one dimension requires every periodic point to be hyperbolic, therefore T fails this necessary condition and is not structurally stable.
3. Explicit C^1 small change f that change the dynamics
Choose $\varepsilon \neq 0$ with $|\varepsilon|$ small and define

$$T_\varepsilon(x) = x^3 + (1+\varepsilon)x.$$

Then $x = 0$ is still a fixed point and $T'_\varepsilon(0) = 1 + \varepsilon$. For $\varepsilon < 0$ we have $|T'_\varepsilon(0)| < 1$, so the fixed point is attracting; for $\varepsilon > 0$ we have $|T'_\varepsilon(0)| > 1$, so the fixed point is repelling. Both T_ε lie arbitrarily close to T in the C^1 norm, yet the qualitative behaviour (attracting versus repelling) differs. Attracting fixed points are preserved by topological conjugacy, hence T_ε with $\varepsilon < 0$ is not conjugate to T_δ with $\delta \geq 0$, nor to T itself.
4. Conclusion
Since arbitrarily small C^1 change of T_{-1} can produce maps that are not topologically conjugate to T_{-1}, the map $T_{-1}(x) = x^3 + x$ is not structurally stable.

Problem 3.4

Prove that $F_4(x) = 4x(1-x)$ is not structurally stable.

Solution

1. All periodic points of F_4 are repelling.
For the logistic family $F_\mu(x) = \mu x(1-x)$ we have

$$F'_\mu(x) = \mu\,(1-2x), \qquad 0 < x < 1.$$

If p is a periodic point of period n for F_4,

$$\left|(F_4^n)'(p)\right| = \prod_{k=0}^{n-1} \left|F'_4(F_4^k(p))\right| = \prod_{k=0}^{n-1} 4\left|1-2F_4^k(p)\right|.$$

Each factor is at least 2, so the product is at least 2^n, hence larger than 1.Every periodic point is therefore repelling and F_4 has no attracting cycles.
2. A C^1-small change with an attracting cycle exists.
For parameters $\mu < 4$ but arbitrarily close to 4 the map F_μ possesses stable periodic windows. Choose $\mu = 4 - \delta$ with $\delta > 0$ as small as desired so that F_μ has an attracting periodic point q of prime period m and $\left|(F_\mu^m)'(q)\right| < 1$. Define $G(x) = F_\mu(x)$. Then

$$\|F_4 - G\|_{C^1} \le \sup_{x\in[0,1]} |(4-\mu)x(1-x)| + \sup_{x\in[0,1]} |(4-\mu)(1-2x)| \le \tfrac{5}{4}\,\delta,$$

which can be made arbitrarily small by taking δ very small.
3. Structural instability.
F_4 has no attracting periodic points, while G has one. Attracting cycles are preserved by topological conjugacy, so F_4 and G are not conjugate. Since such non-conjugate maps can be found in every C^1 neighbourhood of F_4, the map $F_4(x) = 4x(1-x)$ is not structurally stable.

Problem 3.11

Let $f : [0,1] \to [0,1]$ be a diffeomorphism. Prove that, if $f'(x) > 0$, then f has only fixed points and no periodic points. Prove that, if $f'(x) < 0$, then f has a unique fixed point and all other periodic points have period two.

Solution

The 1st part has been proved in Problem 1. It remains to check the 2nd part.
Similarly, assume the converse. If exist $x_0 \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $f^n(x_0) = x_0$, then we can denote $f(x_0) = x_1$, $f(x_1) = x_2$,...$f(x_{n-1}) = x_0$
If n is odd. Firstly let $n = 3$. By the decreasing of the function, we have $x_0 < x_1 > x_2 < x_0 > x_1$ ... which we find both $x_0 < x_1$ and $x_0 > x_1$ appears. If $n = 5$ add x_3 and x_4 into the chain, where we obtain the same result.All of the odd number is impossible after this consideration.
If n is even, we have f^{2n} is increasing.We can consider f^2 then only fixed point can appear, which is the period 2 point of the original function.