Home assignement 3

Problem 1. Find the equation of a cone passing through the lines $y = \pm x$, $z = 0$ and the point $(1, 2, 3)$, for which the axis Oz is the axis of symmetry.

Solution.

From problem description we can conclude that the coordinate system is canonical and equation of cone has form:

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} - \frac{y^2}{b^2} = 0.$$

The lines

$$y = \pm x, \quad z = 0$$

belong to the surface and can be presented in form:

$$(x - y)(x + y) = 0 \quad \text{or} \quad x^2 - y^2, \quad z = 0.$$

The cone section by plane $z = 0$ gives us following curve:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

hence $a^2 = b^2$. So, let's rewrite equation of cone in form:

$$x^2 + \frac{z^2}{c^2/a^2} - y^2 = 0.$$

Substituting the coordinates of point in last equation we find:

$$1^2 + \frac{3^2}{c^2/a^2} - 2^2 = 0 \quad \Rightarrow \quad \frac{c^2}{a^2} = 3.$$

Equation of cone is:

$$x^2 + \frac{z^2}{3} - y^2 = 0 \quad \Rightarrow \quad 3x^2 - 3y^2 + z^2 = 0.$$

Problem 2. Find the equation for a plane parallel to the yOz plane and intersecting a one-sheet hyperboloid

$$\frac{x^2}{9}+\frac{y^2}{4}-z^2=1$$

along a hyperbola whose real semiaxis is equal to 1.

Solution.

Let equation of desired plane is $x=3\alpha$. Then we have:

$$\alpha^2+\frac{y^2}{4}-z^2=1$$

or

$$\frac{y^2}{4}-z^2=1-\alpha^2.$$

Let $1-\alpha^2=\beta^2>0$:

$$\frac{y^2}{4\beta^2}-\frac{z^2}{\beta^2}=1 \quad\Rightarrow\quad \beta=\frac{1}{2} \quad\Rightarrow\quad 1-\alpha^2=\frac{1}{4} \quad\Rightarrow\quad \alpha^2=\frac{3}{4} \quad\Rightarrow\quad x=\pm\frac{3}{2}\sqrt{3}.$$

Let $1-\alpha^2=-\beta^2<0$:

$$\frac{z^2}{\beta^2}-\frac{y^2}{4\beta^2}=1 \quad\Rightarrow\quad \beta=1 \quad\Rightarrow\quad 1-\alpha^2=1 \quad\Rightarrow\quad \alpha^2=2 \quad\Rightarrow\quad x=\pm3\sqrt{2}.$$

So, we have four parallel planes: $x=\pm3\sqrt{2}$ and $x=\pm\frac{3}{2}\sqrt{3}$.

Problem 3. Determine the type of each of the following surfaces, write its canonical equation and find the canonical coordinate system:

a. $4x^2 + y^2 + 4z^2 - 4xy + 4yz - 8zx - 28x + 2y + 16z + 45 = 0$;

b. $2x^2 + 2y^2 - 5z^2 + 2xy - 2x - 4y - 4z + 2 = 0$.

Solution.

a. $4x^2 + y^2 + 4z^2 - 4xy + 4yz - 8zx - 28x + 2y + 16z + 45 = 0$

Invariants:

$$S = 9, \quad \delta = \begin{vmatrix} 4 & -2 & -4 \\ -2 & 1 & 2 \\ -4 & 2 & 4 \end{vmatrix} = 0, \quad \Delta = \begin{vmatrix} 4 & -2 & -4 & -14 \\ -2 & 1 & 2 & 1 \\ -4 & 2 & 4 & 8 \\ -14 & 1 & 8 & 45 \end{vmatrix} = 0,$$

$$K_1 = \begin{vmatrix} 4 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & -4 \\ -4 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0,$$

$$K_3 = \begin{vmatrix} 4 & -2 & -14 \\ -2 & 1 & 1 \\ -14 & 1 & 45 \end{vmatrix} + \begin{vmatrix} 4 & -4 & -14 \\ -4 & 4 & 8 \\ -14 & 8 & 45 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 2 & 4 & 8 \\ 1 & 8 & 45 \end{vmatrix} = -324.$$

Hence the surface is *parabolic cylinder*.
Reduced equation:

$$9x'^2 - 2\sqrt{-\frac{-324}{9}}y' = 0.$$

Canonical equation:

$$x'^2 = \frac{4}{3}y'.$$

Characteristic equation:

$$\begin{vmatrix} 4-\lambda & -2 & -4 \\ -2 & 1-\lambda & 2 \\ -4 & 2 & 4-\lambda \end{vmatrix} = -\lambda^3 + 9\lambda^2.$$

Roots of characteristic equation:

$$\lambda_1 = 9, \quad \lambda_{2,3} = 0.$$

Principal direction of cylinder:

$$\lambda_1 = 9: \quad \begin{pmatrix} -5 & -2 & -4 \\ -2 & -8 & 2 \\ -4 & 2 & -5 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_1, \beta_1, \gamma_1\} = \{2, -1, -2\}.$$

Rewrite the equation of cylinder in form:

$$(2x - y - 2z)^2 - 28x + 2y + 16z + 45 = 0,$$

the rectilinear generatrix of cylinder is section of two planes:

$$2x - y - 2z = 0 \quad \text{and} \quad -28x + 2y + 16z + 45 = 0.$$

Direction of generatrix is

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & -1 & -2 \\ -28 & 2 & 16 \end{vmatrix} = -12\mathbf{e}_1 + 24\mathbf{e}_2 - 24\mathbf{e}_3 \quad \Rightarrow$$

$$\{\alpha_3, \beta_3, \gamma_3\} = \{-1, 2, -2\}.$$

Direction $\{\alpha_2, \beta_2, \gamma_2\}$ can be found as

$$\{\alpha_2, \beta_2, \gamma_2\} = \{\alpha_3, \beta_3, \gamma_3\} \times \{\alpha_1, \beta_1, \gamma_1\} = \{-2, -2, -1\}.$$

Check the direction of $\{\alpha_2, \beta_2, \gamma_2\}$ along the axis of the parabola:

$$S(a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2) = -9(-14 \cdot 2 + 1 \cdot 2 + 8 \cdot 1) = 162 > 0,$$

so, the vector $\{\alpha_2, \beta_2, \gamma_2\}$ is directed in the opposite direction of the parabola axis (section of surface by plane perpendicular to generatrix of cylinder).

Unit vectors:

$$\mathbf{e}_1' = \left\{\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right\}, \quad \mathbf{e}_2' = \left\{-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right\}, \quad \mathbf{e}_3' = \left\{-\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\}.$$

b. $2x^2 + 2y^2 - 5z^2 + 2xy - 2x - 4y - 4z + 2 = 0$

Invariants:

$$S = -1, \quad \delta = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -5 \end{vmatrix} = -15, \quad \Delta = \begin{vmatrix} 2 & 1 & 0 & -1 \\ 1 & 2 & 0 & -2 \\ 0 & 0 & -5 & -2 \\ -1 & -2 & -2 & 2 \end{vmatrix} = -12.$$

Hence the surface is *two-sheet hyperboloid.*
Characteristic equation:

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & -5-\lambda \end{vmatrix} = -\lambda^3 - \lambda^2 + 17\lambda - 15.$$

Roots of characteristic equation:

$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad \lambda = -5.$$

Reduced equation:

$$x'^2 + 3y'^2 - 5z'^2 + \frac{4}{5} = 0.$$

Canonical equation:

$$\frac{x'^2}{4/5} + \frac{y'^2}{4/15} - \frac{z'^2}{4/25} = -1.$$

Center:

$$\begin{cases} 4x + 2y - 2 = 0 \\ 2x + 4y - 4 = 0 \\ -10z - 4 = 0 \end{cases} \quad \Rightarrow \quad O'\left(0, 1, -\frac{2}{5}\right).$$

Axes directions:

$$\lambda_1 = 1: \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_1, \beta_1, \gamma_1\} = \{1, -1, 0\},$$

$$\lambda_2 = 3: \quad \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -8 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_2, \beta_2, \gamma_2\} = \{1, 1, 0\},$$

$$\lambda_3 = -5: \quad \begin{pmatrix} 7 & 1 & 0 \\ 1 & 7 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = 0 \quad \Rightarrow \quad \{\alpha_3, \beta_3, \gamma_3\} = \{0, 0, 1\}.$$

Unit vectors:

$$\mathbf{e}'_1 = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right\}, \quad \mathbf{e}'_2 = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right\}, \quad \mathbf{e}'_3 = \{0, 0, 1\}.$$

Problem 4. Find the equation for the diametral plane of a hyperbolic paraboloid

$$\frac{x^2}{6} - \frac{y^2}{9} = 2z,$$

passing through the line $x = y$, $z = 1$, and find the direction of chords to which the plane is conjugate.

Solution.

Equation of diametral plane conjugate to the chords with direction vector $\{\alpha, \beta, \gamma\}$:

$$F'_x = \frac{x}{3}, \quad F'_y = -\frac{2y}{9}, \quad F'_z = -2,$$

$$\frac{\alpha x}{3} - \frac{2\beta y}{9} - 2\gamma = 0.$$

From problem description one can conclude that diametral plane contains the Oz-axis and line $x = y$, $z = 1$ and orthogonal to xOy-plane. Hence $\gamma = 0$.

Section of diametral plane and plane $z = 1$ gives us:

$$\frac{\alpha x}{3} - \frac{2\beta y}{9} = 0.$$

From here

$$\alpha = \frac{1}{3}, \quad \beta = \frac{1}{2},$$

so the equation of diametral plane is:

$$x - y = 0,$$

the direction of chords is $\{2, 3, 0\}$.