Chapter 2

DEFINITE INTEGRALS

2.1. The Geometric Definition of Definite Integrals

The mathematical concept of definite integrals can be understood better by considering the following problem.

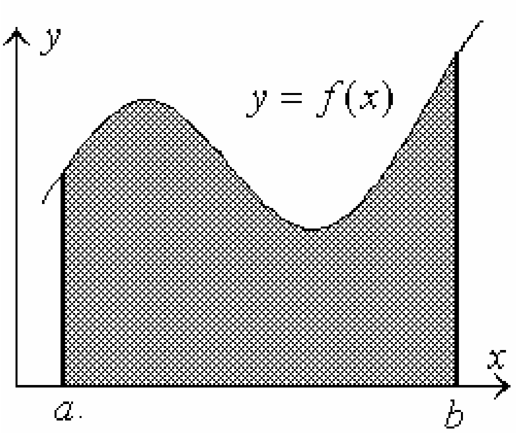


Fig. 1

Problem: Let a function $y = f(x)$ be positive defined on a closed interval $[a,b]$. Find the area of the region under the curve $y = f(x)$ bounded by the x-axis and the lines $x = a$ and $x = b$. (See Fig.1.)

Solution: The main idea is very simple: parts form a whole.

- First, we partition the interval $[a,b]$ into n subintervals $[x_0, x_1]$, $[x_1, x_2]$,…, $[x_{n-1}, x_n]$ by arbitrary points x_1, x_2, ..., x_{n-1} of the partition, as it is shown in Fig. 2.

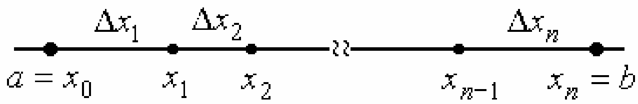


Fig. 2

- Next, we draw vertical lines at the partition points to approximate the region by n rectangles. The area of each rectangle equals the product of its base and height, and it can be easily found.

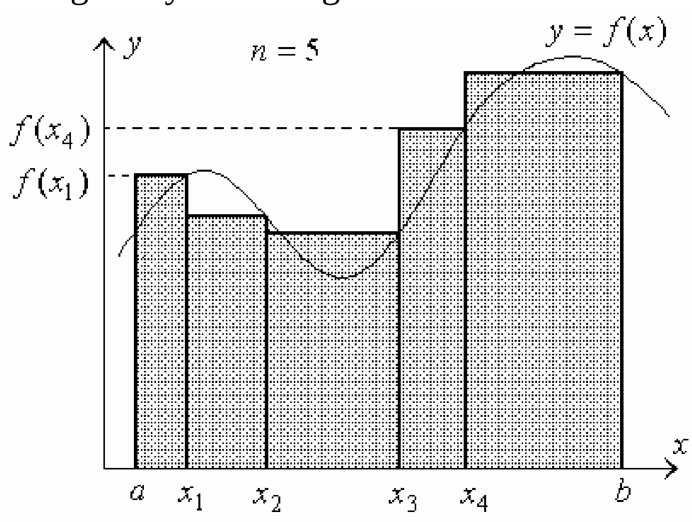


Fig. 3

The base of each rectangle is the difference between one value of x and the previous value of x:
$\Delta x_1 = x_1 - x_0$,
$\Delta x_2 = x_2 - x_1$, …,
$\Delta x_n = x_n - x_{n-1}$.
The heights of the rectangles are equal to $y_k = f(x_k)$, where index k varies from 1 to n.

- Then, we sum up the areas of all rectangles to find approximately the total area A of the region bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$.

$$A \approx \sum_{k=1}^{n} f(x_k)\Delta x_k . \qquad (1)$$

  The above sum is known as the Riemann Sum.
- By comparing Fig. 3 and Fig. 4 one can easily see that approximation (1) is getting better when the number of approximating rectangles increases.

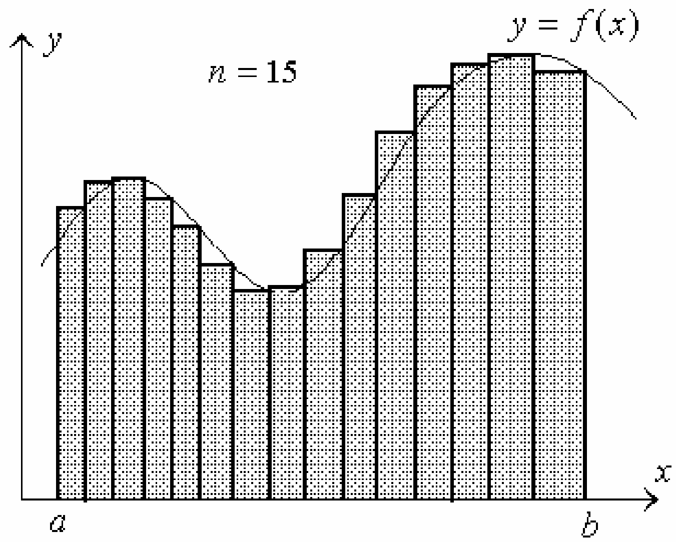


Fig. 4

If the number of the rectangles tends to infinity, so that all the bases of the rectangles tend to zero, then sum (1) gives the area under the curve **exactly**.

Note that the last condition can be written for a short as "maximum $\Delta x \to 0$" because in this case all bases $\Delta x_k \to 0$ $(k = 1, 2, ..., n)$ and the number of the rectangles $n \to \infty$.
Therefore,

$$A = \lim_{\max \Delta x \to o} \sum_{k=1}^{n} f(x_k)\Delta x_k . \qquad (2)$$

If this limit exists, no matter how the partition points x_k are chosen, then it is called a **definite integral** of $f(x)$ over the interval $[a, b]$.
A definite integral is denoted as an indefinite integral but with upper and lower limits:

$$\int_a^b f(x)dx = \lim_{\max \Delta x \to o} \sum_{k=1}^{n} f(x_k)\Delta x_k . \qquad (3)$$

The numbers a and b are said to be **lower** and **upper limits** correspondingly.

2.2. The Algebraic Definition of Definite Integrals

Let $f(x)$ be a function defined on a closed interval $[a,b]$. Consider a partition of the interval $[a,b]$ taking points $x_1, x_2, ..., x_{n-1}$ such that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b.$$

The sum of the products $f(x_k)\Delta x_k$ is called the Riemann Sum, where Δx_k denotes the difference between two successive partition points, that is, $\Delta x_k = x_k - x_{k-1}, \quad k \in N$.

Let $n \to \infty$ and all $\Delta x_k \to 0$. If the limit of the Riemann Sums exists and does not depend on a choice of the points x_k of the partition, then it is called a definite integral of the function $f(x)$ over the interval $[a,b]$:

$$\int_a^b f(x)dx = \lim_{\max \Delta x \to o} \sum_{k=1}^{n} f(x_k)\Delta x_k . \qquad (4)$$

The process of computing an integral is called **integration** and the approximate computation of an integral is called **numerical integration**.

2.3. Properties of Definite Integrals

The following properties are based on the definition of definite integrals.

1. The variable of integration is a dummy variable, that is, an integral is independent of the choice of a symbol denoting the variable of integration:
$$\int_a^b f(x)dx = \int_a^b f(t)dt .$$
2. For any constant *c* and any function $f(x)$ we have:
$$\int_a^b cf(x)dx = c\int_a^b f(x)dx .$$
3. The integral of a sum of integrable functions over the interval $[a,b]$ is equal to the sum of the integrals of the addends over $[a,b]$:
$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx .$$
4. By definition $\int_a^a f(x)dx = 0$.
5. $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

6. $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx, \qquad (a<b).$

7. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

This formula is quite evident if $c \in [a,b]$ (see Fig. 5), but it holds true when $c \notin [a,b]$ provided that all the above integrals exist.

8. $\int_a^b f(x)dx = f(\bar{x})\,(b-a), \quad (a<\bar{x}<b).$ (See Fig. 6.)

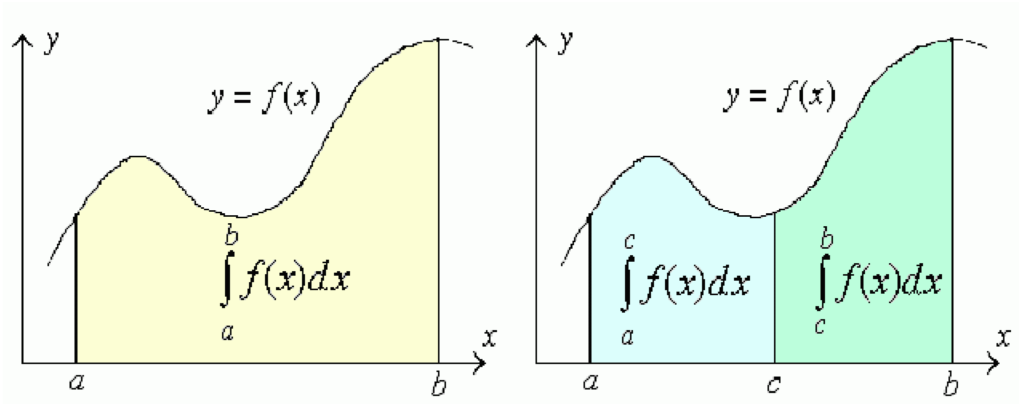


Fig. 5

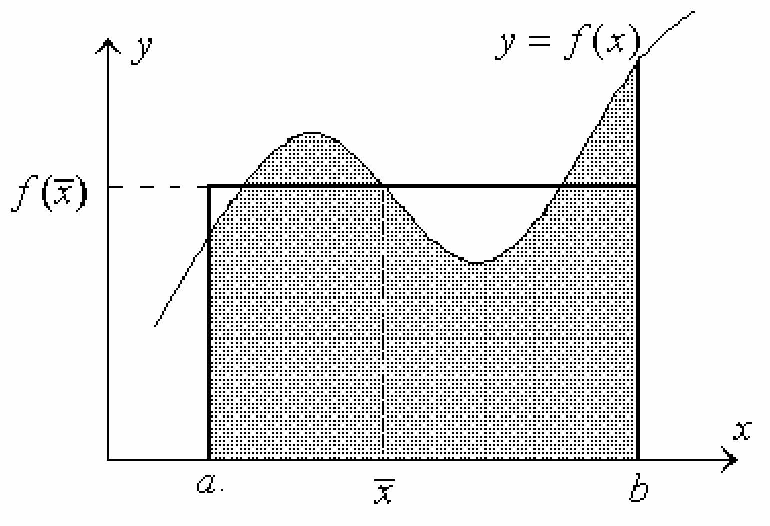


Fig. 6

2.4. The Fundamental Theorems of Calculus

1. If the function $f(x)$ is continuous on (a,b), then the function $\int_a^x f(t)dt$ is a primitive of $f(x)$ for any $x \in (a,b)$:
$$\frac{d}{dx}\int_a^x f(t)dt = f(x). \tag{5}$$
2. If the function $f(x)$ is continuous on a closed interval $[a,b]$ and $F(x)$ is a primitive of $f(x)$ on the interval $[a,b]$, then
$$\int_a^b f(t)dt = F(x)\Big|_a^b = F(b) - F(a). \tag{6}$$

Proof: Let us recall the definition of the derivative:
$$\frac{d\varphi(x)}{dx} = \lim_{\Delta x \to 0} \frac{\varphi(x+\Delta x) - \varphi(x)}{\Delta x}.$$
Therefore, by Property 7,
$$\frac{d}{dx}\int_a^x f(t)dt = \lim_{\Delta x \to 0} \frac{\int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt - \int_a^x f(t)dt}{\Delta x} = \lim_{\Delta x \to 0} \frac{\int_x^{x+\Delta x} f(t)dt}{\Delta x}.$$

Applying Property 8 to the interval $[x, x+\Delta x]$ we find that
$$\int_x^{x+\Delta x} f(t)dt = f(\bar{x})\Delta x,$$
where $\bar{x} \in (x, x+\Delta x)$ and $\bar{x} \to x$ as $\Delta x \to 0$.
By combining these results, we get
$$\frac{d}{dx}\int_a^x f(t)dt = \lim_{\Delta x \to 0} \frac{f(\bar{x})\Delta x}{\Delta x} = f(x).$$

Therefore, the function

$$F(x)=\int_a^x f(t)dt+C \tag{7}$$

is a primitive of $f(x)$.

This is the first fundamental theorem of calculus.

Setting $x=a$, we find the constant C:

$$F(a)=\int_a^a f(t)dt+C \qquad \Rightarrow \qquad C=F(a).$$

Hence,

$$\int_a^x f(t)dt=F(x)-F(a).$$

Setting $x=b$, we get the second fundamental theorem of calculus:

$$\int_a^b f(t)dt=F(b)-F(a).$$

Therefore, both fundamental theorems of calculus are proved.

The Fundamental Theorems of Calculus bind a definite integral of $f(x)$ over the interval $[a,b]$ with an indefinite integral of $f(x)$. All we need only is to evaluate $F(x)$ at b and to subtract $F(x)$ evaluated at a from it.

Examples: Evaluate each of the following integrals:

1) $\displaystyle\int_0^{\pi/12}\cos 2x dx$, 2) $\displaystyle\int_2^5 (3x^2-\frac{7}{x})dx$.

Solution:

1) $$\int_0^{\pi/12}\cos 2x dx=\frac{1}{2}\sin 2x\Big|_0^{\pi/12}=\frac{1}{2}(\sin\frac{\pi}{6}-\sin 0)=\frac{1}{4}.$$

2) $$\int_2^5 (3x^2-\frac{7}{x})dx=3\int_2^5 x^2dx-7\int_2^5\frac{dx}{x}=(x^3-7\ln x)\Big|_2^5$$

$$=(5^3-7\ln 5)-(2^3-7\ln 2)=117-7\ln\frac{5}{2}.$$

2.5. Techniques of Integration

This section contains a review of the major techniques of integration including substitution method and integration by parts.

2.5.1. Substitution Method

Theorem: Let $f(x)$ be a continuous function on the interval $[a,b]$. Assume that a function $x=\varphi(t)$ has a continuous derivative on the interval $[\alpha,\beta]$.

If $\varphi(\alpha)=a$ and $\varphi(\beta)=b$, then

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt \tag{8}$$

Proof: Let $F(x)$ be a primitive of $f(x)$ on the interval $[a,b]$.

Applying the fundamental theorem of calculus and the properties of primitives we have

$$\int_a^b f(x)dx=F(b)-F(a)=F(\varphi(\beta))-F(\varphi(\alpha))=\int_\alpha^\beta dF(\varphi(t))$$

$$=\int_\alpha^\beta F'(\varphi(t))d\varphi(t)=\int_\alpha^\beta f(\varphi(t))d\varphi(t)=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt.$$

Formula (8) allows us to change the variables of integration in definite integrals just as in the case of indefinite integrals, but in addition we have to replace the limits of integration.

Note that it is not necessary to return to the initial variable x.

Example 1: Evaluate $\int_1^e \frac{\ln x}{x}dx$.

Solution: Let $t=\ln x$. Then the equalities $\begin{cases} x=1 \\ x=e \end{cases}$ imply $\begin{cases} t=\ln 1=0 \\ t=\ln e=1. \end{cases}$

Therefore, the interval of integration from 1 to *e* is replaced by the interval $[0,1]$:

$$\int_1^e \frac{\ln x}{x}dx=\int_0^1 tdt=\frac{1}{2}t^2\Big|_0^1=\frac{1}{2}.$$

Example 2: Evaluate $\int_2^3 x^2 e^{x^3} dx$.

Solution: By applying the substitution $t = x^3$, we have $dt = 3x^2 dx$.
Then we find the lower and upper limits of integration:

$$\begin{cases} x = 2 \\ x = 3 \end{cases} \Rightarrow \begin{cases} t = 2^3 = 8 \\ t = 3^3 = 27. \end{cases}$$

Therefore,

$$\int_2^3 x^2 e^{x^3} dx = \frac{1}{3}\int_8^{27} e^t dt = \frac{1}{3} e^t \Big|_8^{27} = \frac{1}{3}(e^{27} - e^8) = \frac{1}{3} e^8 (e^{19} - 1).$$

2.5.2. Integration by Parts

The formula for integration by parts for definite integrals states that

$$\int_a^b u dv = uv \Big|_a^b - \int_a^b v du \tag{9}$$

for any differentiable functions $u(x)$ and $v(x)$.

The following example refers to the case when we need to use the method of integration by parts and the substitution technique.

Example: Evaluate $\int_0^1 \arcsin x dx$.

Solution: Let $u = \arcsin x$ and $dv = dx$. Then $du = \dfrac{dx}{\sqrt{1 - x^2}}$ and $v = x$.

Therefore,

$$\int_{1/2}^1 \arcsin x dx = x \arcsin x \Big|_{1/2}^1 - \int_{1/2}^1 \frac{x dx}{\sqrt{1 - x^2}}.$$

In view of the fact that $\arcsin 1 = \pi/2$ and $\arcsin(1/2) = \pi/6$, we obtain

$$x \arcsin x \Big|_{1/2}^1 = \arcsin 1 - \frac{1}{2}\arcsin\frac{1}{2} = \frac{\pi}{2} - \frac{\pi}{12} = \frac{5\pi}{6}.$$

The integral on the right-hand side can be evaluated by substitution of the variable. One natural substitution is the following. We introduce a new variable t in order to eliminate the radical sign of the integrand.

Let $t^2 = 1 - x^2$.

Then $\sqrt{1-x^2} = t$ and $tdt = -xdx$.

The new limits of integration are as follows:

The lower limit equals $\sqrt{1-(1/2)^2} = \sqrt{3/4} = \sqrt{3}/2$.

The upper limit equals $\sqrt{1-1^2} = 0$.

Thus, we get $\int\limits_{1/2}^{1} \frac{x}{\sqrt{1-x^2}}dx = -\int\limits_{\sqrt{3}/2}^{0} \frac{tdt}{t} = \int\limits_{0}^{\sqrt{3}/2} dt = t\Big|_{0}^{\sqrt{3}/2} = \frac{\sqrt{3}}{2}$.

By combining these results, we finally obtain

$$\int\limits_{1/2}^{1} \arcsin x dx = \frac{5\pi}{6} - \frac{\sqrt{3}}{2}.$$

2.6. Geometric Applications of Definite Integrals

2.6.1. The Area of a Region

One of the problems of such a kind has been considered in section 2.1.

Let us recall the main idea: The given region is represented by an infinite number of rectangles, whose altitudes depend on x-coordinate, and the definite integral of the altitude gives the area of this region.

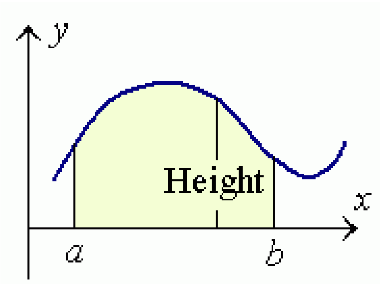


Fig. 7

Problem 1: Given a functions $y = f(x)$ defined over a closed interval $[a,b]$, find the area A of the region bounded by the graph of this function, the x-axis and the vertical lines $x = a$ and $x = b$.

Solution: The altitude of the rectangle with base in the vicinity of the point x is equal to the absolute value of $f(x)$ - it does not matter whether the curve $y = f(x)$ lies above or below the x-axis.

Therefore, we have the following formula for the area of the given region:

$$A = \int\limits_{a}^{b} |f(x)| dx.$$

Note: If the graph lies below the x-axis, then $f(x)<0$ and $\int_a^b f(x)dx=-A$.

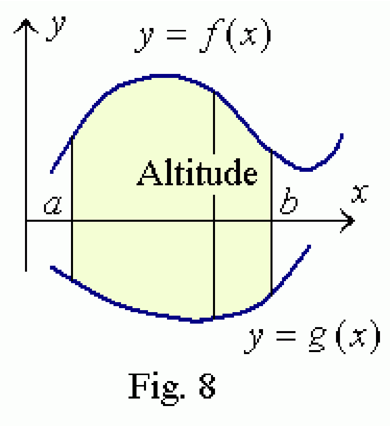


Fig. 8

Problem 2: Given two functions, $y=f(x)$ and $y=g(x)$, defined over a closed interval $[a,b]$, find the area A of the region bounded by their graphs and the vertical lines $x=a$ and $x=b$.

Solution: This region can be represented by an infinite number of rectangles whose altitudes are equal to the absolute value of the difference between $f(x)$ and $g(x)$. Therefore,

$$A=\int_a^b |f(x)-g(x)|dx. \tag{10}$$

Problem 3: Let a function be specified in the polar system of coordinates as $r=r(\varphi)$; find the area A of the region bounded by the graph $r=r(\varphi)$ and the rays $\varphi=\alpha$ and $\varphi=\beta$.

Solution: This region can be represented by an infinite number of sectors. (See Fig. 9).

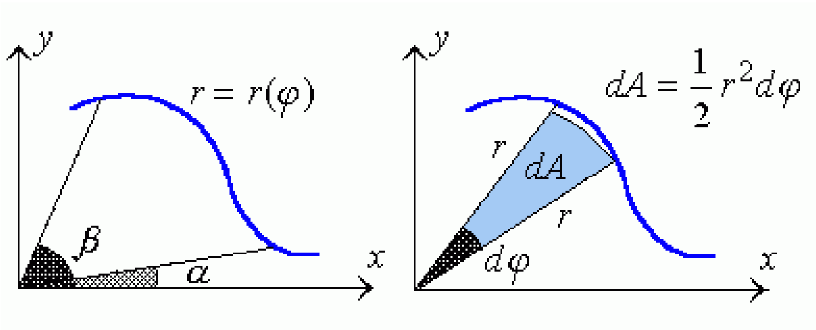


Fig. 9

The area of an arbitrary sector is $dA=\frac{1}{2}r^2d\varphi$.

Therefore, the area of the whole region is

$$A=\frac{1}{2}\int_\alpha^\beta r^2d\varphi. \tag{11}$$

Example: Find the area of the region bounded by the graphs of the functions $y = 3x$ and $y = x^2$.

Solution: First, let us make a sketch of this region.

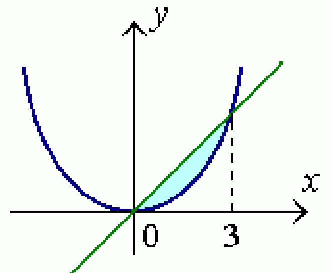


Fig. 10

Then we find the points of intersection, solving the equation $3x = x^2$. This equation has two roots: $x_1 = 0$ and $x_2 = 3$, which give the limits of integration.

Finally, we obtain

$$A = \int_0^3 (3x - x^2)dx = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^3 = \frac{27}{2} - 9 = \frac{9}{2}.$$

2.6.2. The Arc Length of a Curve

Problem 1: Given a curve $y = f(x)$ in the $xy-$plane, find the arc length of the curve between the given values of x.

Solution: The given arc can be subdivided by partition points into an infinite number of portions of the curve, and each of the portions can be represented by a line segment.

Look at the Fig. 6, where some partition of the arc is shown. There is also an arbitrary portion with the approximating line segment in expanded scale.

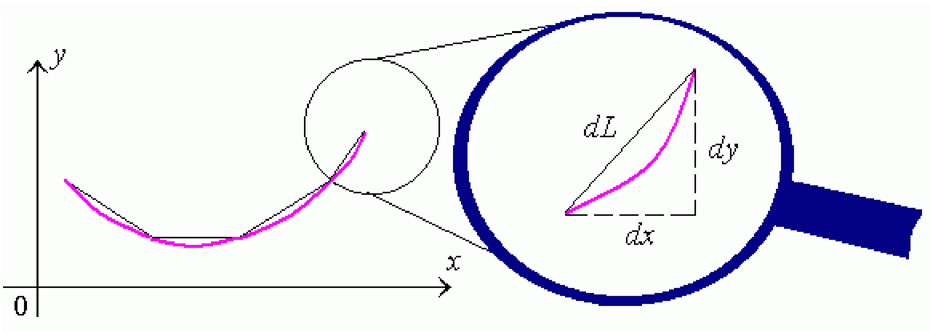


Fig. 11

Let dx and dy be Cartesian coordinates of an arbitrary segment. Then its length dL can be found by the Pythagorean Theorem: $dL = \sqrt{(dx)^2 + (dy)^2}$. Therefore,

$$dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + (y')^2}\, dx,$$

where y' is the derivative of the function $y = f(x)$ with respect to x.

The total length of the arc equals the sum of all lengths of the portions, and hence, the definite integral of $\sqrt{1+(y')^2}$ with respect to x.

Therefore, the arc length of the curve between points a and b of the x-axis is given by the following formula:

$$L = \int_a^b \sqrt{1+(y')^2}\,dx \tag{12}$$

Another solution: The length of an arbitrary portion of the arc can be written by the Pythagorean Theorem as $dL = \sqrt{(dx)^2+(dy)^2}$ which implies

$$dL = \sqrt{1+(\frac{dy}{dx})^2}\,dx = \sqrt{1+(y')^2}\,dx,$$

where y' is the derivative of the function $y = f(x)$ with respect to x.

Therefore, $L(x)$ is a primitive of $\sqrt{1+(y')^2}$:

$$L(x) = \int \sqrt{1+(y')^2}\,dx + C.$$

Since $L(a)=0$ and $L(b)=L$, so $L = \int_a^b \sqrt{1+(y')^2}\,dx$.

Problem 2: Let a given curve be defined parametrically in three-dimensional space: $\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t). \end{cases}$

Find the arc length of the curve between the given values of t.

Solution: As above by the Pythagorean Theorem, we have

$$dL = \sqrt{(dx)^2+(dy)^2+(dz)^2} \quad \Rightarrow \quad dL = \sqrt{\frac{(dx)^2+(dy)^2+(dz)^2}{(dt)^2}}dt$$

$$\Rightarrow$$

$$dL = \sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2+(\frac{dz}{dt})^2}\,dt \quad \Rightarrow$$

$$dL = \sqrt{(x')^2+(y')^2+(z')^2}\,dt,$$

where x', y' and z' are derivatives of the functions $x(t)$, $y(t)$ and $z(t)$ correspondingly with respect to t.

If the end-points of arc are determined by the values t_1 and t_2 of the parameter t, then the arc length of the curve is given as

$$L = \int_{t_1}^{t_2} \sqrt{(x')^2 + (y')^2 + (z')^2}\, dt\,. \tag{13}$$

This formula gives the general solution of finding the arc length of a curve. In a particular case when the curve lies in the $xy-$plane and the x-coordinate is considered as the parameter t, we have $x = x$, $y = y(x)$ and $z = 0$.
Hence, we return to formula (12).

2.6.3. Volumes of Solids

Problem 1: Given a solid, find the volume of the solid. (See Fig. 12.)
Solution: Assume that the solid is of such a nature that whenever we intersect the solid with a plane perpendicular to the x-axis, the cross-sectional area A is known.
This area A is a function of a point x, which we make the cross-section through. Let $A(x)$ be the cross-sectional area at the point x, and let $A(x) = 0$ for any $x \notin [a,b]$.
By intersecting the solid with planes perpendicular to the x-axis, it can be subdivided into an infinite number of layers. Each of the layers can be represented by a cylinder. The volume of an arbitrary cylinder is $dV = A(x)dx$.

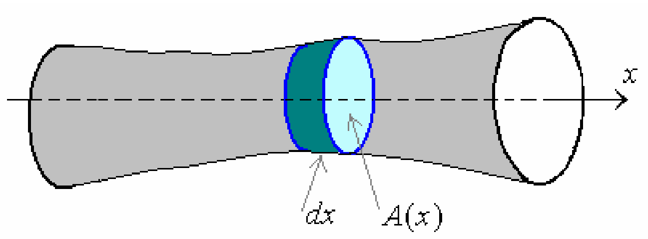


Fig. 12

In a similar way as above we can conclude that the volume of the solid between points a and b is given by the following formula:

$$V = \int_a^b A(x)dx \tag{14}$$

Note: In order to determine the values of the limits of integration, a and b, one can use the following rules:

The lower limit a of integration is the smallest number such that $A(x)=0$ for all $x<a$.
The upper limit b of integration is the largest number such that $A(x)=0$ for all $x>b$.

Problem 2: Let a curve $y=f(x)$ be defined over a closed interval $[a,b]$; find the volume of the resultant solid of revolution by rotating of the curve about the x-axis.

Solution: If we intersect the solid with a plane perpendicular to the x-axis, **t**hen the cross-section of the solid is a circular disk. The radius of this circular disk is $|f(x)|$. By the formula for the area of a circle, the cross-sectional area of the solid at x equals $A(x)=\pi f^2(x)$, provided that $a<x<b$.
Thus, in view of formula (14), the volume of the solid of revolution is given by

$$V=\int_a^b A(x)dx=\pi\int_a^b f^2(x)dx. \tag{15}$$

IMPROPER INTEGRALS

2.1. Basic Definitions

Improper integrals are either integrals with at least one infinite limit of integration or integrals of functions that are unbounded on the interval of integration. For instance, the following integrals are improper:

$$\int_a^{+\infty} f(x)dx, \quad \int_{-\infty}^{b} f(x)dx, \quad \int_{-\infty}^{+\infty} f(x)dx, \quad \int_0^1 \frac{dx}{\sqrt{x}}, \qquad \int_1^2 \frac{dx}{2-x}, \qquad \int_2^5 \frac{dx}{(x-3)^2}.$$

All improper integrals are defined as limits of the definite integrals. In particular,

$$\int_a^{+\infty} f(x)dx = \lim_{c\to+\infty} \int_a^c f(x)dx, \tag{1}$$

$$\int_{-\infty}^{b} f(x)dx = \lim_{c\to-\infty} \int_c^b f(x)dx. \tag{2}$$

Note that integrals with both infinite limits do not require a special consideration because of

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{+\infty} f(x)dx. \tag{3}$$

Note also that the integral with the lower infinite limit can be transformed into the integral with the upper infinite limit by substitution $x=-t$:

$$\int_{-\infty}^{b} f(x)dx = \int_{-b}^{+\infty} f(-t)dt. \tag{4}$$

Integrals of unbounded functions are defined in a similar way:
Let $f(x)\to\infty$ as $x\to a$. Then

$$\int_a^b f(x)dx = \lim_{c\to a} \int_c^b f(x)dx. \tag{5}$$

An improper integral is said to be **convergent**, if there exists the limit of the corresponding definite integral. Otherwise, if the limit does not exist or it is infinite, then the improper integral is called **divergent**.

Examples of convergent integrals:

- $$\int_1^{\infty}\frac{dx}{x^2} = \lim_{c\to\infty}\int_1^{c}\frac{dx}{x^2} = \lim_{c\to\infty}(-\frac{1}{x})\Big|_1^c = \lim_{c\to\infty}(1-\frac{1}{c}) = 1.$$
- $$\int_0^{+\infty} e^{-5x}dx = \lim_{c\to+\infty}\int_0^{c} e^{-5x}dx = \lim_{c\to+\infty}(-\frac{1}{5}e^{-5x})\Big|_0^c = \frac{1}{5}\lim_{c\to+\infty}(1-e^{-5c}) = \frac{1}{5}.$$
- $$\int_3^{\infty}\frac{dx}{x^2-1} = \lim_{c\to\infty}\int_3^{c}\frac{dx}{x^2-1} = \frac{1}{2}\lim_{c\to\infty}\ln\left|\frac{x-1}{x+1}\right|\Bigg|_3^c$$
$$= \frac{1}{2}\lim_{c\to\infty}(\ln\frac{c-1}{c+1} - \ln\frac{2}{4}) = \frac{1}{2}(\ln 1 + \ln 2) = \frac{\ln 2}{2}.$$

Examples of divergent integrals:

- $$\int_1^{\infty}\frac{dx}{x} = \lim_{c\to\infty}\int_1^{c}\frac{dx}{x} = \lim_{c\to\infty}\ln|c|\Big|_1^c = \lim_{c\to\infty}\ln|c| = \infty.$$
- $$\int_0^{\infty}\cos x dx = \lim_{c\to\infty}\int_0^{c}\cos x dx = \lim_{c\to\infty}\sin x\Big|_0^c = \lim_{c\to\infty}\sin c,$$
  which does not exist.

2.2. Convergence and Divergence of Improper Integrals

If $f(x)$ is a positive defined function on some interval (a,b), then $\int_a^b f(x)dx$ represents the area of the region bounded by the graph of the function $y = f(x)$, the x-axis and the vertical lines $x = a$ and $x = b$.

In order to determine whether or not some integral is convergent we can either evaluate it by the definition or use the **comparison tests**.

The idea of the simplest comparison test is based on the property of integrals: If a curve goes down as a whole, then the area under the curve decreases, and vice versa. This means that:

- If some integral converges, then the integral of a smaller positive function also has to converge.

- If some integral of a positive function diverges, then the integral of a greater function also has to be divergent.

Direct Comparison Test

Let $f(x)$ and $g(x)$ be two functions defined on (a,b) such that

$$0 \le f(x) \le g(x)$$

for any $a < x < b$.

- If $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ also converges.
- If $\int_a^b f(x)dx$ diverges, then $\int_a^b g(x)dx$ also diverges.

Here a and b are finite or infinite numbers, and the function $f(x)$ has the improper behavior at a or b.

According to this test we have to find an integral that is similar to but always less than the original one. If it diverges by any tests, then the original integral also diverges. On the contrary, we can try to find an integral similar to but always greater than the original one. If it converges, then the given integral also converges.

2.3. Convergence and Divergence of Integrals with Infinite limits

The integral $\int_a^{+\infty} |f(x)|\,dx$ equals the area of the region under the curve $y = |f(x)|$ bounded by the x-axis and the vertical line $x = a$. It is evident that this area is infinite, if the function $f(x)$ is not decreasing one.

Divergence Test

If $\lim\limits_{x \to +\infty} f(x) \ne 0$,

then the integral $\int_a^{+\infty} f(x)dx$ diverges.

Note that the implication goes only one way: if $\lim\limits_{x\to+\infty} f(x)=0$, it does not mean that the integral of $f(x)$ is convergent.

For instance, $\lim\limits_{x\to+\infty} 1/x=0$, but the integral $\int\limits_1^{\infty}\frac{dx}{x}$ is divergent. (See the example above.)

One can also compare the rates of decreasing of functions to determine whether some integral converges.

Let $\lim\limits_{x\to+\infty}\left|\frac{f(x)}{g(x)}\right|=\lambda$. There are three possible cases: $0<\lambda<\infty$, $\lambda=0$ or $\lambda=\infty$.

- If λ is a finite non-zero number, then the integral

  $\int\limits_a^{+\infty} f(x)dx$ converges if and only if $\int\limits_a^{+\infty} g(x)dx$ converges;

  $\int\limits_a^{+\infty} f(x)dx$ diverges if and only if $\int\limits_a^{+\infty} g(x)dx$ diverges.

  It follows from the fact that the area under the asymptotic part of a curve being multiplied by a finite non-zero number λ holds its finiteness or infinity.

- If $\lambda=0$, then the convergence of the integral $\int\limits_a^{+\infty} g(x)dx$ implies the convergence of the integral $\int\limits_a^{+\infty} f(x)dx$.

  However, we can say nothing if the integral of $g(x)$ is divergent.

- If $\lambda=\infty$, then the divergence of the integral $\int\limits_a^{+\infty} g(x)dx$ implies the divergence of the integral $\int\limits_a^{+\infty} f(x)dx$.

  However, we can say nothing if the integral of $g(x)$ is convergent.

Limit Comparison Test

Let $f(x)$ and $g(x)$ be two functions defined on $[a,+\infty)$ such that

$$0<\lim_{x\to+\infty}\left|\frac{f(x)}{g(x)}\right|<\infty$$

Then both integrals, $\int_a^{+\infty} f(x)dx$ and $\int_a^{+\infty} g(x)dx$, converge or diverge simultaneously.

To use this test for a given integral we have to find a second integral such that the limit of the ratio of integrands is evaluable.
If the second integral converges (diverges) by any tests, then the original integral also converges (diverges).

Example: Determine whether the integral $\int_1^{+\infty}\frac{\ln x dx}{x^3}$ converges.

Let us compare the given integral with the convergent integral $\int_1^{+\infty}\frac{dx}{x^2}$. (See the example above.)
To apply the limit comparison test we calculate:
$$\lim_{x\to+\infty}\frac{\ln x/x^3}{1/x^2}=\lim_{x\to+\infty}\frac{\ln x}{x}=0.$$
Therefore, the given integral is also convergent.

Note that there are useful integrals of the general form $\int_a^{+\infty}\frac{dx}{x^p}$, which are called p-integrals. They are helpful in comparison tests because of the following theorem:

Theorem

The p-integral $\int_a^{+\infty}\frac{dx}{x^p}$ $\begin{cases}\text{converges, if} & p>1\\ \text{diverges, if} & p\le 1\end{cases}$

Proof: If $p\neq 1$, then

$$\int_a^{+\infty} \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{-p+1}\right|_a^{+\infty} \Rightarrow \begin{cases} \text{convergence, if} \quad (-p+1) < 0, \\ \text{divergence,} \quad \text{if} \quad (-p+1) > 0. \end{cases}$$

If $p=1$, then the integral $\int_a^{+\infty} \frac{dx}{x^p}$ is divergent. (See the example above.)

Example: Determinate whether or not the integral $\int_1^{+\infty} \frac{\sqrt{x}}{2x^4+5} dx$ converges.

Solution: We use the comparison test, comparing with the p-integral:

$$\int_1^{+\infty} \frac{\sqrt{x}}{2x^4+5} dx < \int_1^{+\infty} \frac{\sqrt{x}}{2x^4} dx = \frac{1}{2} \int_1^{+\infty} \frac{dx}{x^{7/2}}.$$

The p-integral with $p>1$ converges. Hence, the given integral also converges.

2.4. Convergence and Divergence of Integrals of Unbounded Functions

The limit comparison test can be easily adapted for unbounded function. For instance, assume $f(x)$ is unbounded at the point a.

Limit Comparison Test

Let $f(x)$ and $g(x)$ be two functions defined on (a,b) such that

$$0 < \lim_{x\to a} \left|\frac{f(x)}{g(x)}\right| < \infty$$

Then both integrals, $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$, converge or diverge simultaneously.

If the function $f(x)$ is unbounded at $x=b$, then we have to operate with $\lim\limits_{x\to b}$ instead of $\lim\limits_{x\to a}$, changing nothing more. For instance, let $f(x)$ and $g(x)$ be two functions defined on (a,b) such that

$$0 < \lim_{x\to b} \left|\frac{f(x)}{g(x)}\right| < \infty.$$

Then both integrals, $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$, converge or diverge simultaneously.

We need also to modify the p-integrals.

Theorem

The p-integrals $\int_a^b \frac{dx}{(x-a)^p}$ and $\int_a^b \frac{dx}{(b-x)^p}$

$$\begin{cases} \text{converge, if} & p < 1 \\ \text{diverge, if} & p \geq 1 \end{cases}$$

Proof: If $p \neq 1$, then

$$\int_a^b \frac{dx}{(x-a)^p} = \left.\frac{(x-a)^{-p+1}}{-p+1}\right|_a^b \Rightarrow \begin{cases} \text{convergence, if} & (-p+1) > 0, \\ \text{divergence, if} & (-p+1) < 0. \end{cases}$$

If $p = 1$, then $\int_a^b \frac{dx}{x-a} = (\ln|b-a| - \lim_{x \to a} |x-a|)$. Since this limit does not exist so the given integral diverges.

Similar arguments can be used to prove the second part of the theorem.

Example: Consider the integral $\int_2^5 \frac{dx}{x(x^2-4)}$.

The function $\frac{1}{x(x^2-4)}$ is unbounded at $x = 2$. Compare the original integral with the divergent p-integral: $\int_2^5 \frac{dx}{x-2}$.

The limit of the ratio of the integrands is a finite number:

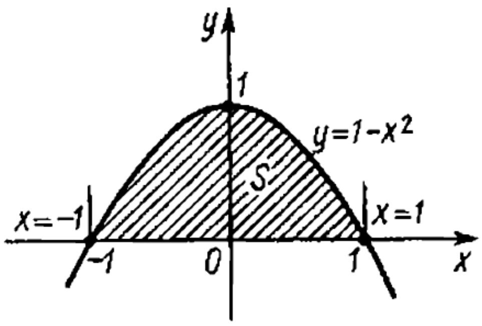
$$\lim_{x \to 2} \frac{x-2}{x(x^2-4)} = \lim_{x \to 2} \frac{1}{x(x+2)} = \frac{1}{8}.$$

Thus, we conclude that the given integral diverges by the limit comparison test.

TASKS SOLUTION

№1. Find the area of a shape bounded by lines $f(x)=1-x^2$ and $y=0.$

Solution:

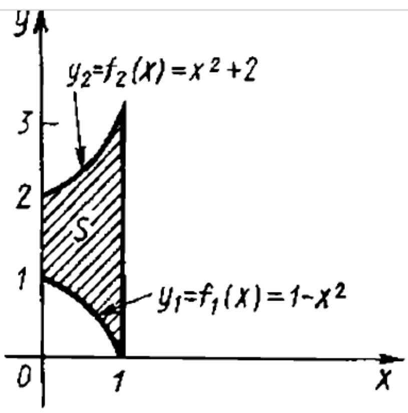


$$S=\int_{-1}^{1}(1-x^2)\,dx=\left(x-\frac{x^3}{3}\right)\Bigg|_{-1}^{1}=\frac{4}{3}.$$

№2. Find the area of a shape bounded by lines

$$y_1=f_1(x)=1-x^2,\ y_2=f_2(x)=x^2+2,\ x=0,\ x=1.$$

Solution:

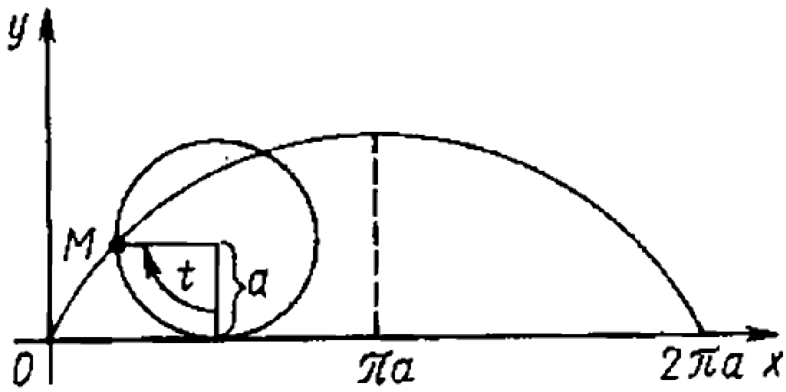


$$S=\int_{0}^{1}[(x^2+2)-(1-x^2)]\,dx=\left(\frac{2}{3}x^3+x\right)\Bigg|_{0}^{1}=\frac{2}{3}+1=\frac{5}{3}.$$

№3. Find the area of the figure bounded by one arch of the cycloid

$x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \leqslant t \leqslant 2\pi$ and the Ox-axis.

Solution:



$$S = \int_0^{2\pi} a(1 - \cos t)\, a(1 - \cos t)\, dt = a^2 \int_0^{2\pi} (1 - \cos t)^2\, dt =$$

$$= a^2 \int_0^{2\pi} \left(1 - 2\cos t + \frac{1 + \cos 2t}{2}\right) dt = a^2 \left[\frac{3}{2} t - 2\sin t + \frac{1}{4}\sin 2t\right]_0^{2\pi} = 3\pi a^2.$$

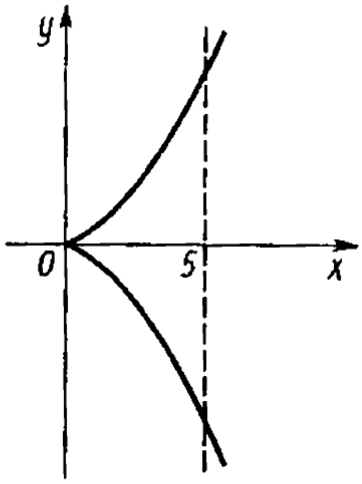
(What we used:

$$S = \int_\alpha^\beta \psi(t)\, \varphi'(t)\, dt$$

)

№4. Find the arc length of a semicubical parabola $y = x^{\frac{3}{2}}$ from $x = 0$ to $x = 5$.

Solution:



The curve is symmetrical with respect to the x-axis.

Let's find the length of the upper branch of the curve.

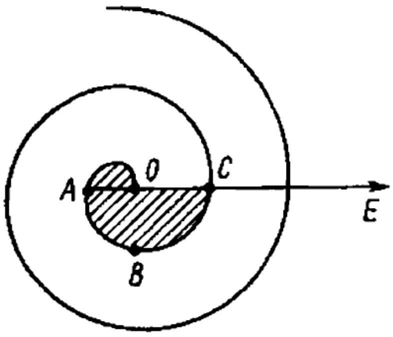
We find $y' = \frac{3}{2}x^{1/2}$.

$$L = \int_0^5 \sqrt{1 + y'^2(x)}\,dx = \int_0^5 \sqrt{1 + \frac{9x}{4}}\,dx = \frac{8}{27}\left(1 + \frac{9x}{4}\right)^{3/2}\Bigg|_0^5 = \frac{335}{27}.$$

№5. Find the length of the first turn of the Archimedean spiral $\rho = a\varphi$

Solution:

The first turn of the Archimedean spiral is formed when the polar angle changes from 0 to 2π.



We get

$$L=\int_0^{2\pi}\sqrt{a^2\varphi^2+a^2}\,d\varphi=a\int_0^{2\pi}\sqrt{\varphi^2+1}\,d\varphi.$$

Let

$$u=\sqrt{\varphi^2+1},\quad du=\frac{\varphi}{\sqrt{\varphi^2+1}}d\varphi;\quad dv=d\varphi,\ v=\varphi.$$

Then

$$L=a\int_0^{2\pi}\sqrt{\varphi^2+1}\,d\varphi=a\left[\varphi\sqrt{\varphi^2+1}\Big|_0^{2\pi}-\int_0^{2\pi}\frac{\varphi^2}{\sqrt{\varphi^2+1}}d\varphi\right]=$$

$$=a\left[\varphi\sqrt{\varphi^2+1}\Big|_0^{2\pi}-\int_0^{2\pi}\frac{\varphi^2+1-1}{\sqrt{\varphi^2+1}}d\varphi\right]=$$

$$=a\left[\varphi\sqrt{\varphi^2+1}\Big|_0^{2\pi}-\int_0^{2\pi}\sqrt{\varphi^2+1}\,d\varphi+\int_0^{2\pi}\frac{d\varphi}{\sqrt{\varphi^2+1}}\right]=$$

$$=a\left[\frac{1}{2}\varphi\sqrt{\varphi^2+1}+\frac{1}{2}\ln(\varphi+\sqrt{\varphi^2+1})\right]_0^{2\pi}=$$

$$=a\left[\pi\sqrt{4\pi^2+1}+\frac{1}{2}(2\pi+\sqrt{4\pi^2+1})\right].$$

(What we used:

$$L=\int_\alpha^\beta\sqrt{\rho(\varphi)+\rho'^2(\varphi)}\,d\varphi,$$

$\rho=\rho(\varphi),\ \alpha\leqslant\varphi\leqslant\beta.$)

№6. Find the volume of the body formed by the rotation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the Ox-axis

Solution:

Since the ellipse is symmetrical with respect to the coordinate axes, it is enough to find half of the desired volume.

$$\frac{1}{2}V=\pi\int_0^a y^2(x)\,dx=\pi\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx=\pi b^2\int_0^a dx-\frac{\pi b^2}{a^2}\int_0^a x^2\,dx=$$
$$=\left(\pi b^2x-\frac{\pi b^2x^3}{3a^2}\right)\Bigg|_0^a=\pi b^2a-\frac{\pi b^2a^3}{3a^2}=\frac{2}{3}\pi ab^2.$$

Therefore

$$\frac{1}{2}V=\frac{2}{3}\pi ab^2, \qquad V=\frac{4}{3}\pi ab^2.$$

(What we used:

$$V=\pi\int_a^b y^2(x)\,dx,$$

$0\leqslant y\leqslant f(x),\ a\leqslant x\leqslant b$)

№7. Investigate convergence

$$\int_{-\infty}^{0}\frac{dx}{1+x^2}.$$

Solution:

$$\int_{-\infty}^{0}\frac{dx}{1+x^2}=\lim_{R\to-\infty}\int_R^0\frac{dx}{1+x^2}=\lim_{R\to-\infty}\operatorname{arctg}x\Big|_R^0=\lim_{R\to-\infty}(\operatorname{arctg}0-\operatorname{arctg}R)=$$
$$=0-\left(-\frac{\pi}{2}\right)=\frac{\pi}{2},$$

That is, the integral converges.

№8. Investigate convergence

$$\int_{-\infty}^{+\infty} e^{x}\,dx.$$

Solution:

Assuming $c=0$, by definition, we have

$$\int_{-\infty}^{+\infty} e^{x}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{+\infty} e^{x}\,dx;$$

$$\int_{0}^{+\infty} e^{x}\,dx = \lim_{R\to+\infty}\int_{0}^{R} e^{x}\,dx = \lim_{R\to+\infty} e^{x}\Big|_{0}^{R} = \lim_{R\to+\infty}(e^{R}-1)=\infty.$$

The integral diverges.

№9. Investigate convergence

$$\int_{1}^{+\infty} \frac{dx}{x^2(1+x)}.$$

Solution:

Let's compare the integrand function $f(x)=\dfrac{1}{x^2(1+x)}$ with the function

$f(x)=\dfrac{1}{x^2}$ on the interval $[1,+\infty)$

It is obvious that:

$$\frac{1}{x^2(1+x)}<\frac{1}{x^2}.$$

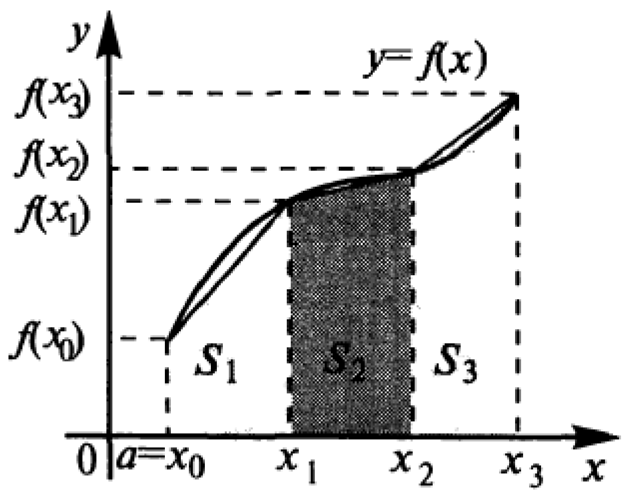
But the integral $\int_1^{+\infty} \frac{dx}{x^2}$ converges because $\alpha = 2 > 1$.

Therefore, according to the comparison feature, this integral also converges.

APPROXIMATE CALCULATION OF DEFINITE INTEGRALS

An important means of calculating definite integrals is the Newton-Leibniz formula. However, its application in practice is associated with significant difficulties that arise when finding a primitive in the case of complication of the integrand function. Therefore, applications use so-called numerical methods to find the approximate value of the desired integral with the required accuracy. This approach turns out to be even more preferable due to the increasing capabilities of modern computing technology that implements algorithms with the necessary accuracy. In this section, we will consider one of the approximate formulas for calculating a definite integral **- the trapezoidal rule formula.**

Let a continuous function $y = f(x)$ be defined on the segment $[a,b]$. Additionally, let's assume that $f(x) \geq 0$ is on the segment $[a,b]$. Then the integral $\int_a^b f(x)dx$ is numerically equal to the area under the curve $y = f(x)$ on the segment $[a,b]$. We will get an approximate value of the desired integral if, instead of the area under the curve, we take the area under the polyline located close enough to the original curve. To construct this polyline, we proceed as follows: we divide the integration segment into n equal parts of length $h = \frac{b-a}{n}$ and replace the section $[x_{i-1}, x_i]$, where $i = 1,2,..n;$ $x_i = x_0 + ih$, of the curve $y = f(x)$ with a chord on each of the segments of the partition, contracting the end points.



Then $\int_a^b f(x)dx \approx S_1 + S_2 + ... + S_n$, where $S_1, S_2, ...S_n$ the area of the trapezoids.

But

$$S_1 = \frac{f(x_0)+f(x_1)}{2}h \,;\; S_2 = \frac{f(x_1)+f(x_2)}{2}h \,;\; ...;\; S_n = \frac{f(x_{n-1})+f(x_n)}{2}h \,.$$

Then

$$\int_a^b f(x)\,dx \approx \frac{f(x_0)+f(x_1)}{2}h + \frac{f(x_1)+f(x_2)}{2}h + \ldots + \frac{f(x_{n-1})+f(x_n)}{2}h =$$

$$= h\left(\frac{f(x_0)}{2} + \frac{f(x_1)}{2} + \frac{f(x_1)}{2} + \frac{f(x_2)}{2} + \ldots + \frac{f(x_{n-1})}{2} + \frac{f(x_n)}{2}\right).$$

Taking out the multiplier h, we note that all the terms of this sum, other than $f(x_0)/2$ and $f(x_n)/2$, occur twice in it. By giving similar terms and considering that $h = \frac{b-a}{n}$, we finally get

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(x_0)+f(x_n)}{2} + f(x_1) + \ldots + f(x_{n-1})\right),$$

where $x_0 = a,\; x_i = x_0 + ih,\; i = 1, 2, \ldots, n.$

The formula is called the trapezoid formula. We obtained it under the assumption that the function $y = f(x)$ is non-negative, but we can prove that this result remains valid also in the general case.

Let's now consider the issue of estimating the error from using the trapezoid formula. Let's denote by $S(n)$ the expression on the right side of the formula. Then

$$\Delta = \left| \int_a^b f(x)\,dx - S(n) \right|$$

is the absolute error from applying **the trapezoidal rule formula**. Denote by M_2 the maximum value of the modulus of the second derivative $f''(x)$ of the integrand $y = f(x)$ on the segment $[a,b]$, that is, $M_2 = \max_{x\in[a,b]} |f''(x)|$.

It is proved that the absolute error Δ depends on the application of the trapezoid formula

$$\Delta \le \frac{(b-a)^3}{12n^2} M_2 \,.$$