Complex analysis.

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1 Complex plane.

1.1 Definitions and examples.

A complex plane $\mathbb{C}$ is a set of ordered pairs $z = (x, y)$ of real numbers. Points of complex plane are called complex numbers and are denoted by $z = (x, y)$. Real components x and y are called real and imaginary parts of a complex number $z = (x, y)$, respectively, and are denoted by

$$x = \operatorname{Re} z, \quad y = \operatorname{Im} z.$$

The complex number

$$\bar{z} = (x, -y)$$

is called a conjugate of $z = (x, y)$.

Considering a set $\mathbb{C}$ as a real plane $\mathbb{R}^2$ we can induce a structure of a vector space (over a real field $\mathbb{R}$). The essential basis in $\mathbb{C} \equiv \mathbb{R}^2$ is defined by vectors

$$1 := (1, 0), \quad i := (0, 1)$$

and every complex number $z = (x, y)$ in this basis has the form

$$z = (x, y) = x \cdot 1 + y \cdot i = x + iy.$$

A set of real numbers (real axis) $\mathbb{R}$ is usually identified with a subset $\{(x, 0) : x \in \mathbb{R}\}$ of $\mathbb{C}$, which in other way can be described as

$$\mathbb{R} = \{z \in \mathbb{C} : z = \bar{z}\}.$$

A set

$$i\mathbb{R} = \{(0,y)\} = \{z : z = -\bar{z}\}$$

is a set of pure imaginary numbers.

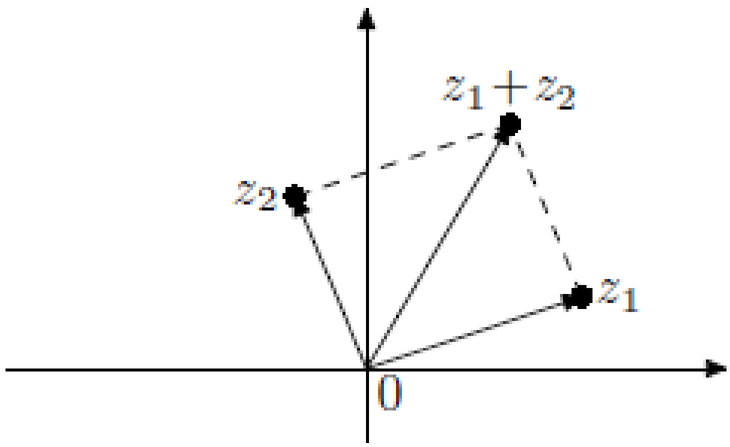

Figure 1: Sum of two complex numbers.

Theorem 1.1. *These definition of addition and multiplication turn the set of all complex numbers into a field with* $(0,0)$ *and* $(1,0)$ *in the role of* 0 *and* 1.

Remark For any real numbers $a, b \in \mathbb{R}$

$$(a,0)+(b,0)=(a+b,0), \quad (a,0)(b,0)=(ab,0).$$

This allows us to identify real number a with $(a,0)$ and to consider $\mathbb{R}$ as a subfield of $\mathbb{C}$.

Definition 1.2. *Imaginary unit is a complex number* $i=(0,1)$.

It is clear that $i^2=(0,1)(0,1)=(0-1,0)=(-1,0)=-1$.

Remark

$$\bar{\bar{z}} = z; \quad \overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2; \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2; \tag{1}$$

$$z = \bar{z} \text{ if and only if } z \in \mathbb{R}; \tag{2}$$

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}; \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}; \tag{3}$$

$$|\operatorname{Re} z| \leq |z|; \quad |\operatorname{Im} z| \leq |z|; \tag{4}$$

$$|z_1 z_2| = |z_1| \, |z_2|; \tag{5}$$

$$|z_1 \pm z_2| \leq |z_1| + |z_2|; \tag{6}$$

$$||z_1| - |z_2|| \leq |z_1 - z_2|. \tag{7}$$

If $z = a + ib \neq 0$ then

$$\frac{1}{z} = z^{-1} = \frac{a - ib}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}.$$

Definition 1.3. *Let $p \in \mathbb{C}$, $r > 0$. A set*

$$B(p, r) = \{z \in \mathbb{C} : |z - p| < r\}$$

is an open disc of radius r with center at point p or a neighborhood of p.

A closed disc is a set

$$\overline{B}(p, r) = \{z \in \mathbb{C} : |z - p| \leq r\}$$

1.2 Complex sequences and functions.

Definition 1.4. *Let $\{z_n\}$ be a sequence of complex numbers. Number z is a limit of a sequence z_n if $|z - z_n| \to 0$ as $n \to \infty$. The sequence is convergent if it has a limit.*

Theorem 1.5. *The convergence of complex sequence is equivalent to the convergence of real and imaginary parts and*

$$\lim z_n = \lim \operatorname{Re} z_n + i \lim \operatorname{Im} z_n.$$

Proof. Assume that $z_n \to z$. Then

$$|\operatorname{Re} z_n - \operatorname{Re} z| \le |z_n - z| \to 0, \quad |\operatorname{Im} z_n - \operatorname{Im} z| \le |z_n - z| \to 0$$

and $\operatorname{Re} z_n \to \operatorname{Re} z$, $\operatorname{Im} z_n \to \operatorname{Im} z$.

Assume now that $z_n = x_n + iy_n$, $x_n \to x$ and $y_n \to y$. Then

$$|z_n - (x + iy)| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0$$

and $z_n \to x + iy$. $\square$

Remark. $z_n \to z$ if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|z_n - z| < \varepsilon$ for every $n > N$. In terms of neighborhoods it can be written as following: for every neighborhood V_z of z there exists a number $N \in \mathbb{N}$ such that $z_n \in V_z$ for every $n > N$.

Remark. Let $\sum\limits_{k=1}^{\infty} c_k$ be a series with complex terms $c_k \in \mathbb{C}$. This series converges if and only if series $\sum\limits_{k=1}^{\infty} \operatorname{Re} c_k$ and $\sum\limits_{k=1}^{\infty} \operatorname{Im} c_k$ are convergent and

$$\sum_{k=1}^{\infty} c_k = \sum_{k=1}^{\infty} \operatorname{Re} c_k + i \sum_{k=1}^{\infty} \operatorname{Im} c_k.$$

Also if a series is absolutely convergent (that is $\sum\limits_{k=1}^{\infty} |c_k|$ is convergent) the it is convergent by simple comparison test since $|\operatorname{Re} c_k|,\ |\operatorname{Im} c_k| \le |c_k|$.

Remark. Let E be a set and $f_k : E \to \mathbb{C}$. Then we can define pointwise and uniform convergence of a functional sequence and of a functional series as we did it for real-valued functions. Bolzano-Cauchy theorem and Weierstrass M-test can be considered in the same way as in real case.

1.3 Limit of a function

Definition 1.6. *Let $\varepsilon > 0$. An open disk $V_p = V_p(\varepsilon) = \{z \in \mathbb{C} : |z - p| < \varepsilon\}$ is called a neighborhood and a set $\dot{V}_p = \dot{V}_p(\varepsilon) = V_p(\varepsilon) \setminus \{p\}$ is called a punctured neighborhood of a point $p \in \mathbb{C}$.*

Definition 1.7. *Let $E \subset \mathbb{C}$, $p \in \mathbb{C}$. Then*

1. *p is a limit point (cluster point, accumulation point) of E if every punctured neighbourhood $\dot{V}_p$ of p the intersection has a common point with E, i.e. $E \cap \dot{V}_p = E \cap V_p \setminus \{p\} \neq \emptyset$.*

2. *$p \in E$ is an isolated point of E if there exists a neighborhood V_p of p such that $E \cap V_p = \{p\}$.*

We note that every point $p \in E$ is either isolated or accumulation point.

Lemma 1.8. *The following assertions are equivalent.*

1. *p is a limit point of E;*

2. *Every neighborhood of p has an infinite intersection with E.*

3. *There exists a sequence $\{z_n\}$ such that $z_n \in E$, $z_n \neq p$ and $z_n \to p$.*

Definition 1.9. *Let $D, G \subset \mathbb{C}$; $f : D \to G$, and let $p \in \mathbb{C}$ to be an accumulation point of D. A is called a limit of function f at point p if one of the following equivalent assertions holds.*

1. ***Cauchy definition or ε-δ definition.***

$$\forall \varepsilon > 0 \ \exists \delta > 0 : \ \forall z \in D \setminus \{p\} \ \ |z - p| < \delta \Rightarrow \ |f(z) - A| < \varepsilon;$$

2. ***Definition by neighborhoods.***

$$\forall V_A \ \exists V_p : \ f(D \cap \dot{V}_p) = f(D \cap V_p \setminus \{p\}) \subset V_A;$$

3. ***Heine definition.***

$$\forall \{z_n\} : z_n \in D \setminus \{p\}, \ z_n \to p \Rightarrow f(z_n) \to A.$$

Notations:

$$A = \lim_{z \to p} f(z); \quad f(z) \xrightarrow[z \to p]{} A; \quad f(z) \to A \text{ as } z \to p.$$

Theorem 1.10. *All three definitions of a limit of a function are equivalent.*

Theorem 1.11. *Let $f, g : D \to G$ and $D, G \subset \mathbb{C}$, $p \in \mathbb{C}$ to be a limit point of D. Let $\lim\limits_{z \to p} f(x) = A$, $\lim\limits_{z \to p} g(x) = B$. Then*

1. $\lim\limits_{z \to p} (f + g)(x) = A + B$;

2. $\lim\limits_{z \to p} (fg)(x) = AB$;

3. $\lim\limits_{z \to p} |f(x)| = |A|$;

4. *If* $B \neq 0$ *then* $\lim\limits_{z \to p} \frac{f(z)}{g(z)} = \frac{A}{B}$.

Definition 1.12. *Let* $D, G \subset \mathbb{C}$, $f : D \to G$. *A function* f *is called continuous at* $z_0 \in D$ *if one of the following equivalent assertions holds.*

1. *Either* z_0 *is an isolated point of* D *or* z_0 *is a limit point and*

$$f(z_0) = \lim_{z \to z_0} f(z).$$

2. *Weierstrass and Jordan definitions (epsilon–delta) of continuous functions.*

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall z \in D : \; |z - z_0| < \delta \;\; |f(z) - f(z_0)| < \varepsilon.$$

3. *Definition in terms of neighborhoods*

$$\forall V_{f(z_0)} \; \exists V_{z_0} \; f(V_{z_0}) \subset V_{f(z_0)}.$$

4. *Heine definition.*

$$\forall \{z_n\} : z_n \in D, \;\; z_n \to z_0 \Rightarrow f(z_n) \to f(z_0).$$

Definition 1.13. *If* f *is not continuous in* $z_0 \in D$ *then one says that it has a discontinuity at* $z_0 \in D$.

1.4 Polar representation of complex numbers

If $z \in \mathbb{C} \setminus \{0\}$ then the angle φ measured from direction of vector 1 is called an argument of number z and is denoted by $\arg z$. Argument is not

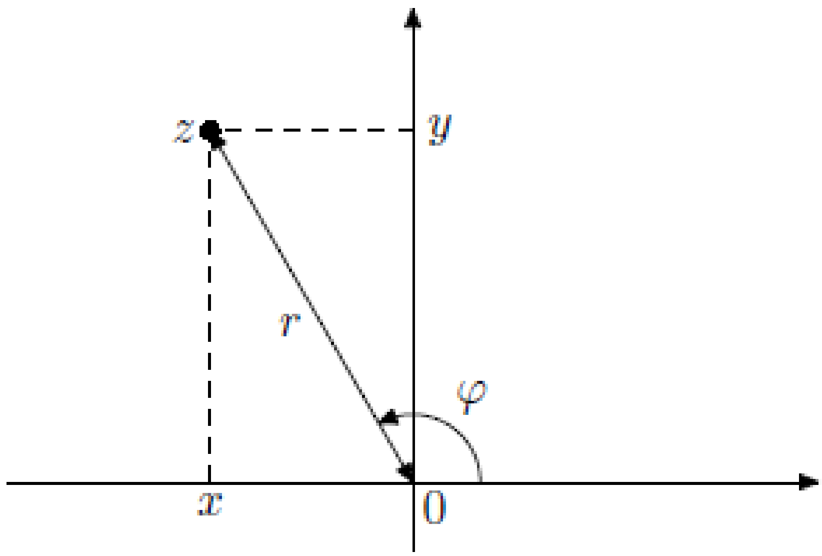

Figure 2: Polar representation of complex number.

unique and is defined up to the term multiple to 2π. A set of all values of argument is denoted by $\operatorname{Arg} z$. Then $\arg z$ is any element of this set. Sometimes we fix a semiopen interval of length 2π (usually $(-\pi, \pi]$ or $[0, 2\pi)$) and say that this value of argument is principal. Numbers $r = |z|$ and $\varphi = \arg z$ are polar coordinates of a point (x, y) and

$$x = r\cos\varphi, \quad y = r\sin\varphi;$$

$$z = r(\cos\varphi + i\sin\varphi), \ r = |z|\,, \ \varphi = \arg z;$$

$$\cos\varphi = \frac{x}{r}, \quad \sin\varphi = \frac{y}{r}.$$

$$\operatorname{Arg} z = \{\varphi + \pi k : k \in \mathbb{Z}\}.$$

Notice that if

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1),$$

$$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

then

$$z_1 z_2 = r_1 r_2(\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)).$$

In particular:

$$z^n = r^n(\cos n\varphi + i\sin n\varphi).$$

1.5 Paths in a complex plane

Definition 1.14. *A path in a complex plane is a continuous map*

$$\gamma : [\alpha, \beta] :\to \mathbb{C},$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. Two paths $\gamma_1 : [\alpha, \beta] :\to \mathbb{C}$ and $\gamma_2 : [\alpha, \beta] :\to \mathbb{C}$ are called equivalent if there exists an increasing continuous bijective function (that is called parametrization change)

$$\tau : [\alpha_1, \beta_1] \to [\alpha, \beta]$$

such that

$$\gamma_2 \circ \tau = \gamma_1.$$

Class of equivalences of paths is called curve.

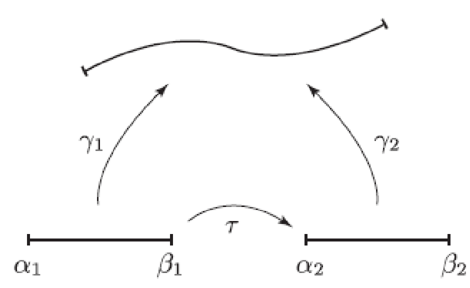

Figure 3: Two equivalent path.

Definition 1.15. *A path γ is called closed if $\gamma(\alpha) = \gamma(\beta)$.*

Definition 1.16. *A path γ is Jordan if $\gamma(t_1) \neq \gamma(t_2)$, $t_1 \neq t_2$. A closed path is called Jordan (or simple) if $\gamma(t_1) \neq \gamma(t_2)$, $t_1 < t_2 < \beta$.*

Figure 4: Examples of paths.

$$\boxed{\text{For } \gamma(t) = x(t) + iy(t) \text{ we use notation } \dot{\gamma}(t) = x'(t) + iy'(t).}$$

Definition 1.17. *Assume that a path $\gamma : [\alpha, \beta] \to \mathbb{C}$ is such that for every $t \in [\alpha, \beta]$ there exists a derivative $\dot{\gamma}(t)$ (for endpoints α, β this means that at α there exists right-hand derivative and at β left-hand side derivative). A path γ is smooth if $\dot{\gamma}(t)$ is continuous and $\dot{\gamma}(t) \neq 0$ for $t \in [\alpha, \beta]$. A path $\gamma : [\alpha, \beta] \to \mathbb{C}$ is piece-wise smooth if a segment $[\alpha, \beta]$ can be decomposed by points*

$$\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta$$

to segments $[t_{j-1}, t_j]$ such that the restriction of γ to any of these segments is a smooth path.

Equivalence of smooth (piecewise smooth) path is defined in the same way, with additional assumption that τ and τ^{-1} are smooth (piecewise smooth) functions.

1.6 Domains in a complex plane.

Definition 1.18. *A set $D \subset \mathbb{C}$ is path-conneted if any two points in this set can be connected by a path that is contained in D. A set $D \subset \mathbb{C}$ is called a domain if it is open and path-connected.*

Lemma 1.19. *An open set is connected if and only if it is path-connected.*

Proof. Assume first that D is open and path-connected. We will prove that it is connected. Assume the converse, that there exist sets $D_1, D_2 \subset D$ such that

$$D_1 \cap D_2 = \varnothing \quad \text{and} \quad D_1 \cup D_2 = D.$$

COnsider two points $a \in D_1$, $b \in D_2$, and let $\gamma : [0,1] \to D$ be a continuous path such that $\gamma(0) = a$, $\gamma(1) = b$. Consider a set

$$K := \{t \in [0,1] : \gamma(t) \in D_1\}$$

and let $t_0 := \sup\{t : t \in K\}$. By our assumption $0 < t_0 < 1$ since both D_1 and D_2 are open. The point $z_0 := \gamma(t_0)$ can not belong neither to D_1 neither to D_2. Indeed, if $z_0 \in D_1$ then there exists $\varepsilon > 0$ such that

$$[t_0, t_0 + \varepsilon) \in K,$$

and if $z_0 \in D_2$ then there exists $\varepsilon > 0$ such that

$$(t_0 - \varepsilon, t_0] \notin K.$$

Consequently, $\gamma(t_0) \notin D$ which contradicts the definition of γ as the path in D and, consequently, cotradicts our assumption that D is not connected.

Assume now that D is open and connected. Consider a point $z_0 \in D$ and define a set $D_1 \subset D$ as a set of points $z \in D$ that can be connected with z_0 by a continuous path. $\gamma : [0,1] \to D$. Let also $D_2 := D \backslash D_1$.

Since every point $z \in D$ is contained in D with some disk and every point of a disk can be connected with center by a radius we see that both D_1 and D_2 are open. Since D is connected this implies that D_2 is empty (since D_1 contains at least one point z). Hence, $D = D_1$ and D is path-connected. □

Theorem 1.20. *Assume that $G \subset \mathbb{C}$ is a domain, and $F \subset G$ be nonempty set such that it is open and closed in G (that means that F is open and $F = \overline{F} \cap G$. Then $F = G$.*

Proof. Assume that F and G satisfy the assumption of the theorem. Let $F_1 = G \backslash F = G \backslash \overline{F}$. Then F_1 is open as difference of an open and of a closed sets. Also $G = F \cup F_1$. Since G is connected this implies that one of the sets F and F_1 is empty. Since F is not empty we conclude that $F_1 = \varnothing$ and $G = F$. □

Definition 1.21. *A domain D is called a domain with simple boundary if ∂D consists of finite number of closed piecewise-smooth paths. The orientation is always chosen so that the domain D is to the left to the direction to the curve.*

Definition 1.22. *A set G is compactly supported in domain D if $\overline{G} \subset D$.*

1.7 Compactification of a complex plane

Definition 1.23. *An extended complex plane $\bar{\mathbb{C}}$ is a one-point compactification of $\mathbb{C}$ that is obtained by addition of ∞. The base of neighborhoods of a point ∞ in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is defined by complements of closed disks $\{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$.*

All basic topological notions that we know for $\mathbb{C}$ are generalized for $\overline{\mathbb{C}}$.

1.7.1 Stereographical projection of extended complex plane

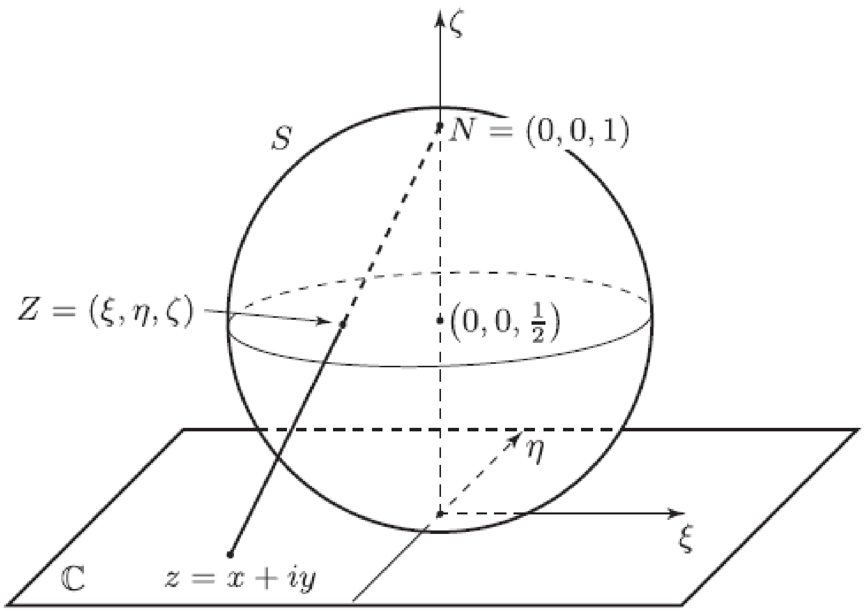

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Figure 5: Stereographical projection.

Let

$$S = \left\{ (\xi, \eta, \zeta) \in \mathbb{R}^3 : \xi^2 + \eta^2 + \left(\zeta - \frac{1}{2}\right)^2 = \frac{1}{4} \right\}$$

be a sphere in Euclidean space $\mathbb{R}^3$ with center at $\left(0, 0, \frac{1}{2}\right)$ of radius $\frac{1}{2}$. We identify a complex plane $\mathbb{C}$ with a plane $\{\zeta = 0\}$ in $\mathbb{R}^3$ and match each point $z = x + iy$ with a point $Z = (\xi, \eta, \zeta)$ of the intersection of sphere S with a ray that connects z with a north pole $N = (0, 0, 1)$ of sphere S. To express coordinates of Z we write down this ray in parametric form

$$\xi = tx, \quad \eta = ty, \quad \zeta = 1 - t.$$

Then the point of intersection is defined by parameter t that satisfies the equation

$$t^2 \left(x^2 + y^2\right) + \left(\frac{1}{2} - t\right)^2 = \frac{1}{4} \quad \Longrightarrow \quad t = \frac{1}{1 + |z|^2}.$$

Consequently, coordinates of this point $Z = (\xi, \eta, \zeta)$ are calculated by the following formulas

$$\xi = \frac{x}{1 + |z|^2}, \quad \eta = \frac{y}{1 + |z|^2}, \quad \zeta = \frac{|z|^2}{1 + |z|^2}.$$

The invese map is defined by $t = 1 - \zeta$. Hence,

$$x = \frac{\xi}{1 - \zeta}, \quad y = \frac{\eta}{1 - \zeta}.$$

This means that stereographical projection $Z \longleftrightarrow z$ defines bijection between points of a sphere $S \setminus \{N\}$ without north pole N and of the complex plane $\mathbb{C}$. Moreover, punctured neighborhoods $\{z \in \mathbb{C} : |z| > R\}$ of $\infty \in \overline{\mathbb{C}}$ are transformed to punctured neighborhoods of N on a sphere S. Thus, we can continue the projection $S \setminus \{N\} \longleftrightarrow \mathbb{C}$ to $S \longleftrightarrow \overline{\mathbb{C}}$ mapping N to $\infty \in \overline{\mathbb{C}}$. This defines a homeomorphism of S and $\overline{\mathbb{C}}$. This model of $\overline{\mathbb{C}}$ is called a Riemann sphere.

1.7.2 Spherical metric

We can define in $\mathbb{C}$ a metric defined by a Eucleden distance between the corresponding points on a sphere. This metric can be expressed in the following form

$$\rho\left(z_1, z_2\right) = \frac{|z_1 - z_2|}{\sqrt{1 + |z_1|^2}\sqrt{1 + |z_2|^2}}.$$

In a bounded part of $\overline{\mathbb{C}}$ (in a disk $\{|z| < R\}$) a spherical distance $\rho\left(z_1, z_2\right)$ is equivalent to Eucledean

$$C_1(R)\left|z_1 - z_2\right| \leqslant \rho\left(z_1, z_2\right) \leqslant C_2(R)\left|z_1 - z_2\right|$$

At the same time the distance from any point $z \in \mathbb{C}$ to ∞ in spherical metric is finite

$$\rho(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}} \leqslant 1.$$

Punctured neighbourhoods of $\infty \in \overline{\mathbb{C}}$ in this metric ρ are defined by sets

$$\{z \in \mathbb{C} : \rho(z, \infty) < \varepsilon\} = \left\{z \in \mathbb{C} : |z| > \sqrt{\varepsilon^{-2} - 1}\right\}.$$

Thus, topology of $\overline{\mathbb{C}}$ is equivalent to topology defined by ρ.

2 Complex differentiability. Geometric meaning of derivative.

2.1 $\mathbb{R}$-differentiability.

Consider a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ on a complex plane as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ that maps a point $z = x + iy$ to

$$f(z) = f(x, y) = u(x, y) + iv(x, y) = \operatorname{Re} f(x, y) + i \operatorname{Im} f(x, y).$$

Definition 2.1. *A function $f(x, y) = u(x, y) + iv(x, y)$ defined in a neighborhood of a point $z_0 = x_0 + iy_0$ is called $\mathbb{R}$-differentiable at z_0 if functions $u(x, y), v(x, y)$ are differentiable at (x_0, y_0) as functions of x, y.*

Consider a point $z = x+iy$ near z_0 and let $\Delta x := x-x_0$, $\Delta y := y-y_0$. Moreover, denote

$$\Delta z := z - z_0 = \Delta x + i\Delta y, \quad \Delta f := f(z) - f(z_0) = f(x, y) - f(x_0, y_0).$$

Then $\mathbb{R}$-differentiability of f at z_0 is equivalent to existence of constants $a, b \in \mathbb{C}$ such that

$$\Delta f = a \cdot \Delta x + b \cdot \Delta y + o(|\Delta z|) \quad \text{for} \quad \Delta z \to 0.$$

That means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\Delta f - a \cdot \Delta x - b \cdot \Delta y| < \varepsilon |\Delta z|$$

for every z such that $|z - z_0| < \delta$. In particular, this implies that function f has partial derivatives by x and y at z_0 and that

$$\frac{\partial f}{\partial x}(z_0) = \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{\Delta f}{\Delta x} = a, \quad \frac{\partial f}{\partial y}(z_0) = \lim_{\substack{\Delta y \to 0 \\ \Delta x = 0}} \frac{\Delta f}{\Delta y} = b.$$

Remark. Notice that the existence of partial derivatives doesn't imply $\mathbb{R}$-differentiability of f at z_0.

If we express Δx and Δy in terms of $\Delta z := \Delta x + i\Delta y$ and $\Delta z := \Delta x - i\Delta y$, then the condition of $\mathbb{R}$-differentiability of function f at z_0 will have the form

$$\Delta f = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right)\Delta z + \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0)\right)\Delta \bar{z} + o(\Delta z).$$

Consider differential operators (formal partial definitions by z and $\bar{z}$)

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

Hence, we have the following representation of differential $df(z_0) : \mathbb{C} \to \mathbb{C}$ of function $\mathbb{R}$-differentiable at z_0 function f

$$df(z_0) = \frac{\partial f}{\partial z}(z_0)\, dz + \frac{\partial f}{\partial \bar{z}}(z_0)\, d\bar{z}.$$

The differential $df(z_0)$ defines a linear mapping $\mathbb{C} \to \mathbb{C}$ acting by the formula

$$df(z_0) : \zeta \in \mathbb{C} \mapsto df(z_0)\,\zeta = \frac{\partial f}{\partial z}(z_0) \cdot \zeta + \frac{\partial f}{\partial \bar{z}}(z_0) \cdot \bar{\zeta}$$

for every $\zeta \in \mathbb{C}$.

2.2 $\mathbb{C}$-differentiabilty. Cauchy-Riemann identities.

Definition 2.2. *A function f defined in a neighborhood of a point z_0 is $\mathbb{C}$-differentiable (differentiable) at z_0 if there exists a number $a \in \mathbb{C}$ such that in a neighborhood of z_0 one has*

$$\Delta f = f(z) - f(z_0) = a \cdot \Delta z + o(\Delta z).$$

This definition is equivalent to the condition

$$\frac{\Delta f}{\Delta z} = a + o(1) \quad \text{for} \quad \Delta z \to 0,$$

that is

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0).$$

A number $f'(z_0)$ is called a complex derivative of f at z_0.

Theorem 2.3. *Assume that a function f defined in the neighbourhood z_0 is $\mathbb{C}$-differentiable if and only if f is $\mathbb{R}$-differentiable at z_0 and Cauchy-Riemann condition is satisfied*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0.$$

In this case $\frac{\partial f}{\partial z}(z_0) = f'(z_0)$.

Proof. $\Longrightarrow$. By the definition $\mathbb{C}$-differentiabilty of function f at z_0 means that the function f is $\mathbb{R}$-differentiable at z_0 and its differential has the following form

$$df(z_0)(\zeta) = a\zeta \quad \text{for every } \zeta \in \mathbb{C} \approx T_{z_0}\mathbb{C}.$$

This implies that $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

$\Longleftarrow$. $\mathbb{R}$-differentiability of function f at z_0 means that

$$\Delta f = \frac{\partial f}{\partial z}(z_0)\,\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\,\Delta \bar{z} + o(\Delta z)$$

in some neighborhood of z_0. Then, by Cauchy-Riemann condition, this implies that

$$\Delta f = \frac{\partial f}{\partial z}(z_0)\,\Delta z + o(\Delta z),$$

That is function f is $\mathbb{C}$-differentiable at z_0. $\square$

Considering $f = u + iv$ in the formula

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

and real and imaginary parts independently we express Cauchy-Riemann condition in real form (that is in terms of $u = \operatorname{Re} f$, $v = \operatorname{Im} f$ and real variables $x = \operatorname{Re} z$, $y = \operatorname{Im} z$))

$$\frac{\partial f}{\partial \bar{z}} = 0 \Longleftrightarrow \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \Longleftrightarrow \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0.$$

Thus the Cauchy-Riemann identity $\frac{\partial f}{\partial \bar{z}} = 0$ is equivalent to the system

$$\boxed{\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \end{cases}}$$

Remark 2.4. *In assumption that $f = u + iv$ is $\mathbb{C}-$differentiable (and thus satisfies Cauchy-Riemann identities) we may deduce that*

$$f'(z_0) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

2.3 Cauchy-Riemann condition in terms of polar coordinates

We can rewrite the Cauchy-Riemann condition in terms of the polar variables r, φ connected with variables $z, \bar{z}$ as $z = re^{i\varphi}, \bar{z} = re^{-i\varphi}$. Differentiating these formulas by z and solving the system of linear equations we see that

$$\frac{\partial r}{\partial \bar{z}} = \frac{e^{i\varphi}}{2}, \quad \frac{\partial \varphi}{\partial \bar{z}} = \frac{ie^{i\varphi}}{2r}.$$

Consequently, by the formula of derivative of the composition

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial r}{\partial \bar{z}}\frac{\partial}{\partial r} + \frac{\partial \varphi}{\partial \bar{z}}\frac{\partial}{\partial \varphi} = \frac{e^{i\varphi}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \varphi}\right).$$

Applying this operator to $f = u + iv$ we see that

$$\frac{\partial f}{\partial \bar{z}} = 0 \iff \frac{\partial f}{\partial r} + \frac{i}{r}\frac{\partial f}{\partial \varphi} = 0 \iff \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}.$$

2.4 Examples. Elementary functions of complex variable.

Remark. A function f is differentiable iff there exists a limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Formulas for derivative of arithmetical operations and composition word-by-word proofs can be generalized for complex case.

$$(f \pm g)' = f' \pm g'; \quad (fg)' = f'g + fg'; \quad f(g(z))' = f'(g(z))g'(z)$$

1. The function $f(z) = z$ is differentiable in $\mathbb{C}$ and

$$f'(w) = \lim_{z \to w} \frac{z - w}{z - w} = 1 \quad \forall w \in \mathbb{C}.$$

2. The function $f(z) = z^n$ is differentiable in $\mathbb{C}$ for any $n \in \mathbb{N}$ and

$$f'(w)=\lim_{z\to w}\frac{z^n-w^n}{z-w}=\lim_{z\to w}\frac{(z-w)\left(z^{n-1}+z^{n-2}w+\cdots+zw^{n-2}+w^{n-1}\right)}{z-w}=$$
$$=\lim_{z\to w}\left(z^{n-1}+z^{n-2}w+\cdots+zw^{n-2}+w^{n-1}\right)=nw^{n-1}.$$

Thus, as in the case of a real variable $\left(z^2\right)'=2z$, $\left(z^3\right)'=3z^2$, etc.

3. Any complex polynomial $p(z)=\sum\limits_{k=0}^{n}c_kz^k$ is differentiable in $\mathbb{C}$. Also, any rational function $r(z)=\frac{p(z)}{q(z)}$, where p,q are complex polynomials and q is not identically zero, is differentiable on the open set $\{z\in\mathbb{C}:q(z)\neq 0\}$. Recall that by the Principal Theorem of Algebra, any polynomial of degree n has exactly n complex roots if we take into account multiplicities.

4. The complex exponential

$$e^z=e^{x+iy}=e^x(\cos y+i\sin y)$$

is differentiable in $\mathbb{C}$ and $(e^z)'=e^z$.

We have $e^z=u(x,y)+iv(x,y)$ where $u(x,y)=e^x\cos y$, $v(x,y)=e^x\sin y$. Let us verify the Cauchy–Riemann equations. Indeed,

$$u'_x=v'_y=e^x\cos y,\qquad u'_y=-v'_x=-e^x\sin y.$$

Also,

$$(e^z)'=u'_x+iv'_x=e^x\cos y+ie^x\sin y=e^z.$$

5. We can now define complex functions $\sin z$ and $\cos z$ by the formulas

$$\cos z=\frac{e^{iz}+e^{-iz}}{2},\qquad \sin z=\frac{e^{iz}-e^{-iz}}{2i}.$$

Note that in view of our definition of the complex exponential this perfectly agrees with the definition of sine and cosine on real numbers. These function are also differentiable in $\mathbb{C}$ and

$$(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z.$$

Properties of the exponent and trigonometric functions

1. Functions sin and cos are not bounded in $\mathbb{C}$.

   Let $y \in \mathbb{R}$ then

   $$\cos iy = \frac{e^{-y} + e^{y}}{2} \to +\infty, \quad y \to \pm\infty;$$

   $$|\sin iy| = \frac{|e^{-y} - e^{y}|}{2} \to +\infty, \quad y \to \pm\infty.$$

2. $\sin z$ is odd function; $\cos z$ is even function.

   $$e^{iz} = \cos z + i \sin z.$$

   For example,

   $$e^{i\pi} = -1, \quad e^{\frac{i\pi}{2}} = i, \quad e^{-\frac{i\pi}{2}} = -i.$$

3. **Exponential form of a complex number** Lets consider

   $$z = r(\cos\varphi + i \sin\varphi), \quad r = |z|\,, \ \varphi \in \operatorname{Arg} z$$

   then

   $$z = re^{i\varphi}.$$

4. **The fundamental property of an exponent**

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

This can be checked directly from the definition but we will prove it later using Taylor's decomposition of the exponent. Also we can generalize it as

$$e^{z_1+z_2+\ldots+z_n} = e^{z_1}e^{z_2}\cdot\ldots\cdot e^{z_n}.$$

5. **De Moivre's formula**

$$(\cos t + i\sin t)^n = (e^{it})^n = e^{int} = \cos nt + i\sin nt, \quad n \in \mathbb{N},\ t \in \mathbb{R}.$$

6. Exponent has no zeroes.

Proof. Assume that $z = x + iy$, $x, y \in \mathbb{R}$ and $e^z = 0$. Then

$$e^{x+iy} = e^x(\cos y + i\sin y) = 0$$

and $\cos y = \sin y = 0$ since $e^x > 0$. We obtained a contradiction because $\sin^2 y + \cos^2 y = 1$. $\square$

7. Exponent has periods equal to $2k\pi i$, $k \in \mathbb{Z} \setminus \{0\}$, and no other periods. Functions sin and cos have periods equal to $2k\pi$, $k \in \mathbb{Z} \setminus \{0\}$, and no other periods.

Proof. Notice that

$$e^{2k\pi i} = \cos 2k\pi + i\sin 2k\pi = 1.$$

Consequently,

$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z$$

Conversely, assume that $e^{z+T} = e^z$ for every z and $T = x + iy$. Then $\left|e^T\right| = e^x = 1$ and $x = 0$. Finally,

$$e^{iy} = \cos y + i \sin y = 1$$

if and only if $y = 2k\pi i$. $\square$

6. To define the complex logarithm as a single-valued function we need to choose a branch of its argument. Let

$$\log z = \log |z| + i \arg z + 2\pi k i$$

for $z \in \mathbb{C} \backslash (-\infty, 0]$, where $\arg z$ is the unique number in $(-\pi, \pi)$ such that $z = re^{i\varphi}, r > 0$ and $k \in \mathbb{Z}$ is fixed.

Then $\log z$ is differentiable in $\mathbb{C} \backslash (-\infty, 0]$,

$$e^{\log z} = z$$

and

$$(\log z)' = \frac{1}{z}$$

as in the real case.

Prove this as an exercise. Note that in this case $u(x, y) = \ln \sqrt{x^2 + y^2}$, while $v = \varphi$ can be found as an arcsin, arccos or arctan, depending on (x, y). E.g., for $x > 0$ one can define $\varphi = \arctan \frac{y}{x}$.

Example.

$$\log(i) = \left(\frac{\pi}{2} + 2\pi k\right) i, \; k \in \mathbb{Z}.$$

6. Let $z, w \in \mathbb{C}$. The expression z^w may have countable number of values

$$z^w = e^{w \log z} = e^{w \ln |z| + iw \arg z + 2\pi kwi}.$$

For example,

$$i^i = e^{i \log z} = e^{-\frac{\pi}{2} - 2\pi k}, \quad k \in \mathbb{Z}.$$

7. Roots of complex number. Let $z = re^{i\varphi} \neq 0$ be a complex number and $n \in \mathbb{N}$. Assume that we want to find the solution of equation

$$w^n = z.$$

If we denote $w = \rho e^{i\psi}$ then this equation has the following form:

$$\rho^n e^{in\psi} = re^{i\varphi}.$$

First, this implies that $\rho = \sqrt[n]{r}$ and that ψ can obtain the following values:

$$\psi = \frac{\varphi + 2\pi k}{n}.$$

However, among these values generate only n different solutions

$$w = \sqrt[n]{r} e^{i \frac{\varphi + 2\pi k}{n}}, \quad k = 0, 1, \dots n - 1.$$

For example, the equation

$$w^n = 1$$

has n solutions

$$w_k = e^{\frac{2\pi k}{n} i} = \cos \frac{2\pi}{k} + i \sin \frac{2\pi k}{n}, \quad k = 0, \dots, n - 1.$$

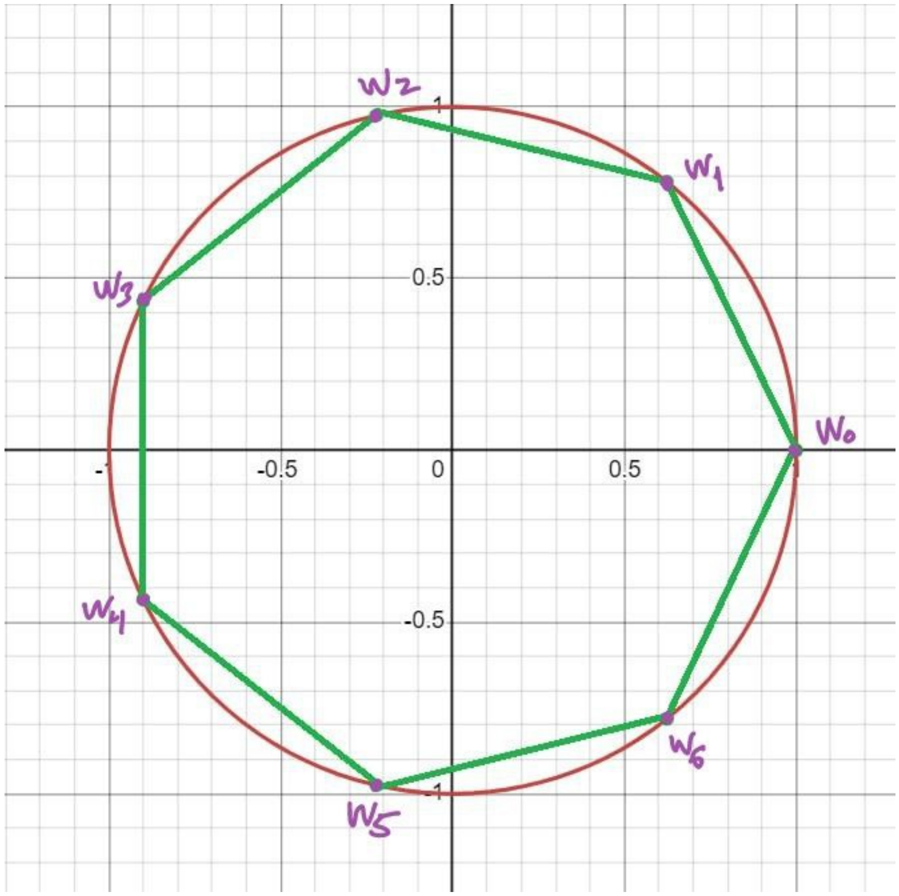


Figure 6: Roots of unity of order 7.

The seventh roots of unity are the points $w_k = e^{\frac{2\pi k}{n}i}$, $k = 0, 1, \ldots, 6$. In other words, they're just the seven points corresponding to a regular heptagon inscribed on the unit circle, with one vertex at 1.

Hence, the polynomial $z^n - 1$ has exactly n roots. We will later prove the following theorem that generalizes this conclusion to every polynomial.

Theorem 2.5 (Fundamental theorem of algebra)**.** *If P is a polymomial of degree $n \in \mathbb{N}$ with complex coefficients, then the equation $P(z) = 0$ has exactly n solutions counting multiplicities.*

In addition we prove the discontinuity of principal square root function.

Theorem 2.6. *The principal value of a square root function*

$$f(z) = \sqrt{z} = \sqrt{|z|}e^{i \arg z/2},$$

is discontinuous at the point $z_0 = -1$.

Proof. We show that $f(z) = \sqrt{z}$ is discontinuous at $z_0 = -1$ by demonstrating that the limit $\lim_{z\to -1} \sqrt{z}$ does not exist. In order to do so, we present two ways of letting z approach -1 that yield different values of this limit.

Now consider z approaching -1 along the quarter of the unit circle lying in the second quadrant. That is, consider the points $|z| = 1$, $\pi/2 < \arg(z) < \pi$. In exponential form, this approach can be described as $z = e^{i\theta}$, $\pi/2 < \theta < \pi$, with θ approaching π:

$$\lim_{z\to -1} \sqrt{z} = \lim_{z\to -1} \sqrt{|z|}e^{i \arg(z)/2} = \lim_{\theta\to\pi} e^{i\theta/2} = i.$$

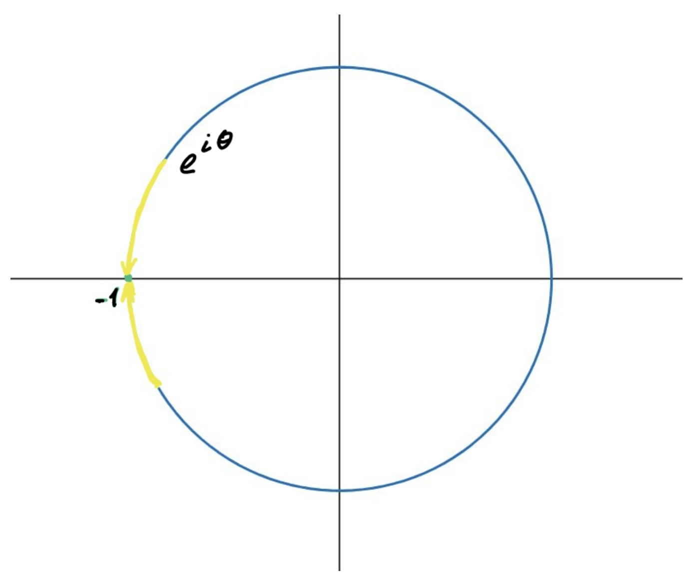


Figure 7: Discontinuity of $\sqrt{z}$.

Next, we let z approach -1 along the quarter of the unit circle lying in the third quadrant. Again refer to Figure 2.54. Along this curve we have the points $z = e^{i\theta}$, $-\pi < \theta < -\pi/2$, with θ approaching $-\pi$:

$$\lim_{z\to-1} \sqrt{z} = \lim_{z\to-1} \sqrt{|z|}e^{i\arg(z)/2} = \lim_{\theta\to-\pi} e^{i\theta/2} = -i.$$

Because these values do not agree, we conclude that $\lim_{z\to-1} z^{1/2}$ does not exist. Therefore, the principal square root function $f(z) = z^{1/2}$ is discontinuous at the point $z_0 = -1$. □

2.5 Directional derivative

Let function f be $\mathbb{R}$-differentiable at z_0. Then

$$\Delta f = \frac{\partial f}{\partial z}(z_0)\,\Delta z + \frac{\partial f}{\partial \bar{z}}(z_0)\,\Delta \bar{z} + o(\Delta z).$$

Consider polar expression of the variable $\Delta z = |\Delta z|e^{i\theta}$. Then

$$\Delta \bar{z} = \overline{\Delta z} = |\Delta z|e^{-i\theta} = \Delta z \cdot e^{-2i\theta},$$

and

$$\Delta f = \left(\frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\,e^{-2i\theta}\right)\Delta z + o(\Delta z).$$

Dividing both parts by Δz and considering the limit $\Delta z \to 0$ with a fixed argument $\arg\Delta z = \theta = \text{const}$. Consequently, the $\mathbb{R}$-differentiability of f at z_0 implies the existence of the limit

$$\lim_{\substack{\Delta z\to 0\\ \arg\Delta z=\theta}} \frac{\Delta f}{\Delta z} = \frac{\partial f}{\partial z}(z_0) + \frac{\partial f}{\partial \bar{z}}(z_0)\,e^{-2i\theta} =: f'_\theta(z_0),$$

That is called the derivative of f by the direction θ.

The last identity implies that the point $f'_\theta(z_0)$ circumscribes twice the circle with center at $\frac{\partial f}{\partial z}(z_0)$ of radius $\left|\frac{\partial f}{\partial \bar z}(z_0)\right|$ when θ changes from 0 to 2π. This proves the following assertion.

Lemma 2.7. *Assume that a function f is $\mathbb{R}$-differentiable at z_0. Then its derivative $f'_\theta(z_0)$ by direction θ doesn't depend on θ if and only if $\frac{\partial f}{\partial \bar z}(z_0) = 0$. In this case*

$$f'_\theta(z_0) = \frac{\partial f}{\partial z}(z_0) = f'(z_0) \qquad \textit{for every} \qquad \theta \in \mathbb{R}.$$

2.6 Holomorhic functions and conformal mappings

Definition 2.8. *A function f is holomorphic at $z_0 \in \mathbb{C}$ if it is $\mathbb{C}-$differentiable in some neighborhood of this point. A function f is holomorphic in domain D if it is holomorphic at every point of this domain.*

A set of functions holomorphic in D is denoted by $H(D)$.

Definition 2.9. *A mapping f is called conformal at z_0 if it is $\mathbb{C}-$differentiable at z_0 and $f'(z_0) \neq 0$.*

The mapping, defined by the function f, is conformal in domain D if it is conformal at every point of D.

We say that domains D is conformally equivalent to domain G if there exists a bijective conformal mapping $f : D \to G$.

Remark 2.10. *Later we will prove that if $f : D \to G$ is holomorphic bijective then the inverse $f^{-1} : G \to D$ is also holomorphic and*

$$(f^{-1}(z))' = \frac{1}{f'(f^{-1}(z))}.$$

This will imply that if D is conformally equivalent to G the G is conformally equivalent to D. This together with formula for derivative of composition and conformality of identical map implies that conformal equivalence is actually equivalence relation.

Lemma 2.11. *The mapping f is conformal at z_0 if it is $\mathbb{R}-$differentiable and its differential $df\,(z_0)$ that is considered as the linear mapping of the plane $\mathbb{R}^2$ to itself is nondegenerate (that is bijective) and is a composition of the rotation and a scaling.*

Proof. $\Longrightarrow$. Assume that a function f is $\mathbb{C}$-differentiable at z_0 and $f'\,(z_0) \neq 0$. Then its differential

$$df\,(z_0) : \zeta \mapsto f'\,(z_0)\,\zeta = |f'\,(z_0)|\, e^{i \arg f'(z_0)} \zeta$$

is the composition of the rotation by the angle $\arg f'\,(z_0)$ and scaling with the coefficient $|f'\,(z_0)|$. Moreover, its nondegenerate, since this composition maps $\mathbb{R}^2$ onto itself.

$\Longleftarrow$. Assume that f is $\mathbb{R}$-differentiable at z_0. Then it differential has the following form

$$df\,(z_0) : \zeta \mapsto A\zeta + B\bar{\zeta},$$

where $A := \frac{\partial f}{\partial z}\,(z_0)$ and $B := \frac{\partial f}{\partial \bar{z}}\,(z_0)$. The map $\zeta \mapsto i\zeta$ is the counterclockwise rotation by $90°$. Since all rotations and scalings commute then the differential $df(z_0)$ commutes with it since f is conformal, that means that

$$Ai\zeta + B\overline{i\zeta} = i(A\zeta + B\bar{\zeta}) \quad \text{for every } \zeta \in \mathbb{C}.$$

Consequently, $2iB\bar{\zeta} = 0$ for every $\zeta \in \mathbb{C}$. Hence $B = 0$ and every conformal map is $\mathbb{C}$-differentiable. Moreover, $f'(z_0) \neq 0$ since otherwise $df(z_0)$ would be identicaly 0 and degenerate. $\square$

2.7 Conformal maps. Examples.

Example 2.1. *The function* $w = z^2$ *maps the right half-plane* $\{\operatorname{Re} z > 0\}$ *conformally onto the slit plane* $\mathbb{C}\backslash(-\infty, 0]$*. For any fixed* $\theta_0, 0 < \theta_0 \leq \pi/2$*, it maps the sector* $\{|\arg z| < \theta_0\}$ *conformally onto the sector* $\{|\arg z| < 2\theta_0\}$ *of twice the aperture.*

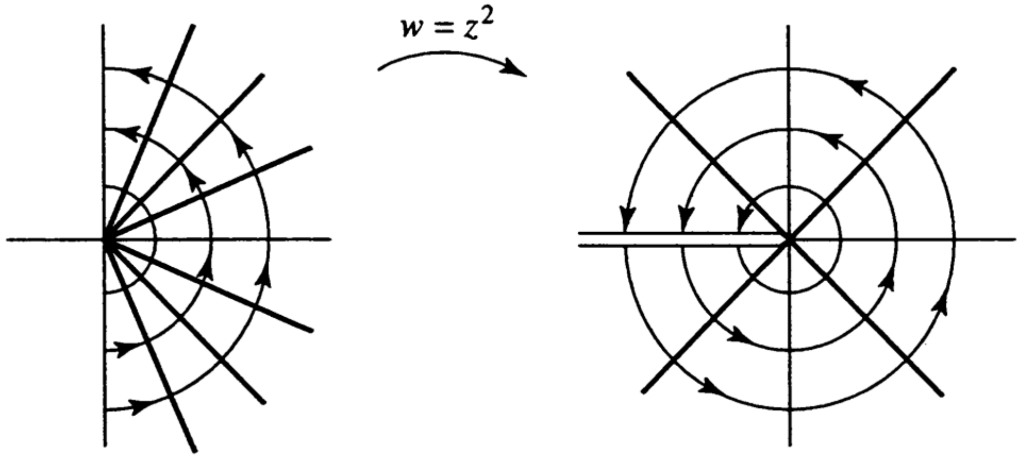


Figure 8: z^2 as a map of $\{\operatorname{Re} z > 0\} \to \mathbb{C}\backslash(-\infty, 0]$.

Example 2.2. *Fix* $\theta_0, 0 < \theta_0 \leq \pi$*. If* $0 < a < \pi/\theta_0$*, the function* z^a *maps the sector* $\{|\arg z| < \theta_0\}$ *conformally onto the sector* $\{|\arg z| < a\theta_0\}$*. In particular, the function* $z^{\pi/2\theta_0}$ *maps the sector* $\{|\arg z| < \theta_0\}$ *conformally onto the right half-plane.*

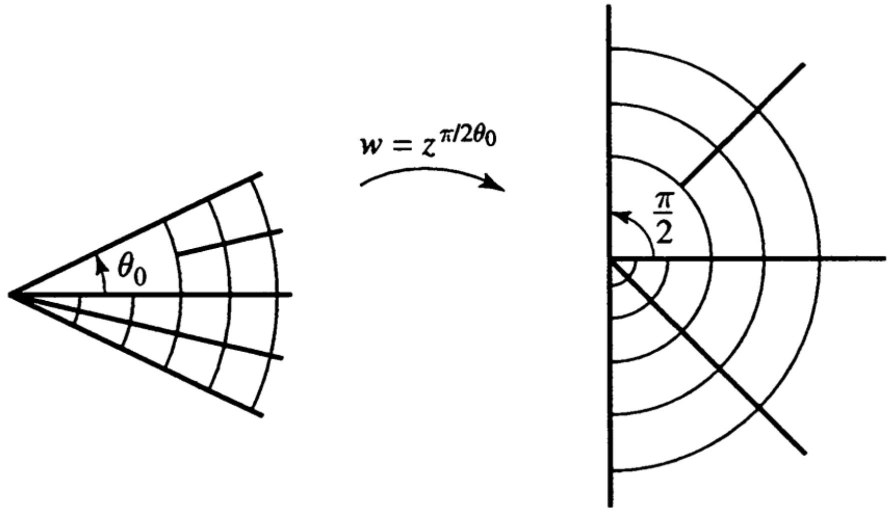


Figure 9: z^a as a map of $\{|\arg z| < \theta_0\} \to \{\operatorname{Re} z > 0\}$.

Example 2.3. *The exponential function e^z is conformal at each point $z \in \mathbb{C}$, since its derivative does not vanish at z. Its image is the punctured plane $\mathbb{C}\backslash\{0\}$. However, it is not a conformal mapping of the plane onto the punctured plane, since it is not one-to-one. Its restriction to the horizontal strip $\{|\operatorname{Im} z| < \pi\}$ is a conformal mapping of the strip onto the slit plane $\mathbb{C}\backslash(-\infty, 0]$.*

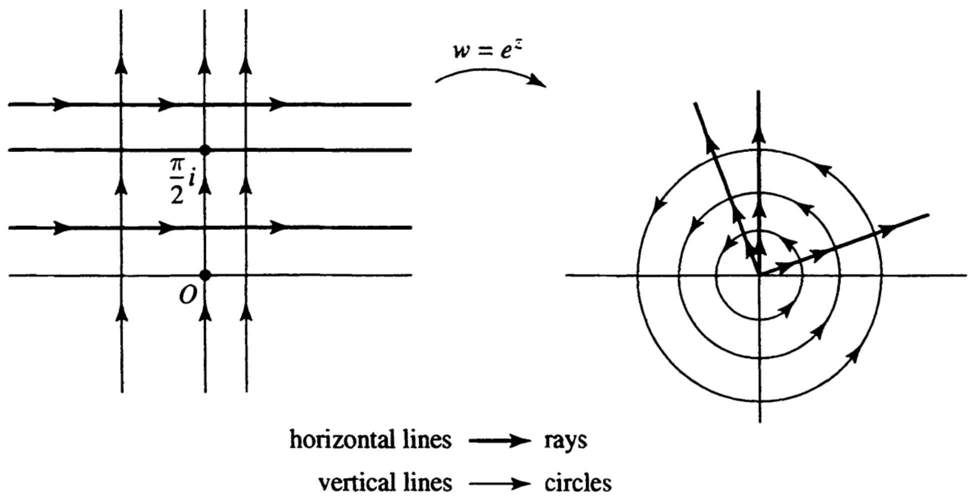


Figure 10: Exponent as a map of $\{|\operatorname{Im} w| < \pi\} \to \mathbb{C}\backslash(-\infty, 0]$.

Example 2.4. *The principal value*

$$\ln z = \ln|z| + i \arg z$$

of the logarithm is a conformal mapping of the slit plane $\mathbb{C}\backslash(-\infty, 0]$ *onto the horizontal strip* $\{|\operatorname{Im} w| < \pi\}$.

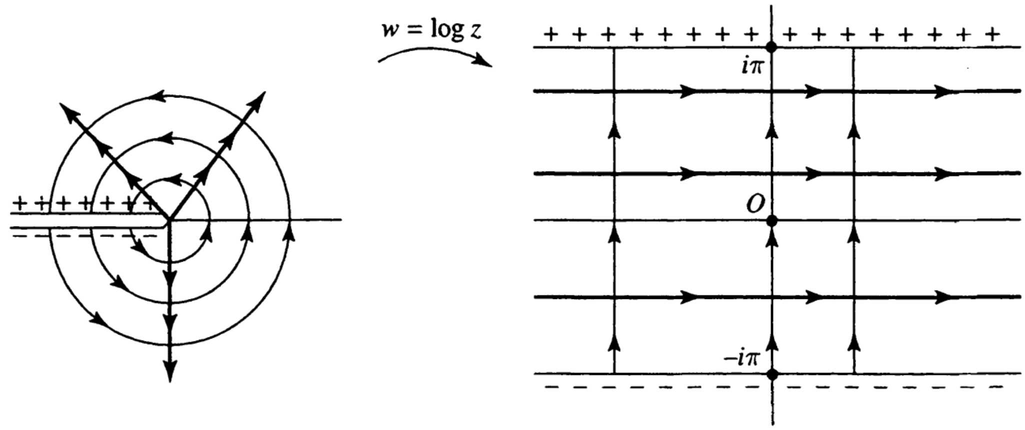


Figure 11: Logarithm as a map of $\mathbb{C}\backslash(-\infty, 0] \to \{|\operatorname{Im} w| < \pi\}$.

2.8 Harmonic functions

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Then u and v satisfy the Cauchy-Riemann equations:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Let us take for granted that u and v have continuous higher order partial derivatives (we will prove this later in this course). We can write

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0$$

In other words, $\Delta u = 0, \Delta v = 0$. The real and imaginary parts of an analytic function satisfy Laplace's equation and, consequently, are harmonic functions.

Definition 2.12. *A C^2-smooth function $u : D \subset \mathbb{R}^2 \to \mathbb{R}$ is harmonic at in domain D if it satisfies Laplace's equation*

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \textit{ in } D.$$

If two harmonic functions u and v satisfy the Cauchy-Riemann equations, then we say that v is a conjugate harmonic function of u.

Example: It is easy to see that $f(z) = z^3$ is analytic on $\mathbb{C}$. We can write

$$u(x, y) = x^3 - 3y^2 x, \quad v(x, y) = 3x^2 y - y^3,$$

and compute

$$\Delta u = 6x - 6x = 0, \quad \Delta v = 6y - 6y = 0.$$

2.9 Geometric meaning of a complex derivative

In this section we will study geometric properties of conformal mappings. Assume that f is conformal in some neighborhood U of a point z_0 and the derivative $f'(z)$ is continuous in U. Consider a smooth path in U with starting point at z_0, that is

$$\gamma : [0,1] \to U, \quad \gamma(0) = z_0$$

such that $\gamma'(t) \neq 0$ for every $t \in [0,1]$. The composition

$$\Gamma := f \circ \gamma : [0,1] \to f(U)$$

is a smooth path in $f(U)$ since

$$\Gamma'(t) = f'(\gamma(t))\gamma'(t). \tag{8}$$

In the geometric sence $\gamma'(t)$ is a tangent vector to the curve $g([0,1])$ at the point $\gamma(t)$, analogously $\Gamma'(t)$. Hence the element of the length of a curve γ at $\gamma(t)$ is equal to

$$ds_\gamma = |\gamma'(t)|\, dt \quad \text{and,} ds_\Gamma = |\Gamma'(t)|\, dt.$$

Hence

$$\frac{ds_\Gamma}{ds_\gamma} = \frac{|\Gamma'(0)|}{|\gamma'(0)|} = |f'(z_0)|\,.$$

This means that the absolute value of derivative $f'(z_0)$ is the coefficient of the scaling of the length of a path at z_0 by the mapping f.

In particular, this implies that all curves passing through z_0 are scaled at this point with the same coefficient. Hence the map f translates small circles centered at z_0 to the smooth curves that coincide in the first order with circles centered at $f(z_0)$.

Formula (8) implies also that

$$arg f'(z_0) = arg \Gamma'(0) - arg \gamma'(0),$$

that is the argument of the derivative $f'(z_0)$ is the angle of rotation of tangent vectors to the curves at z_0 by the mapping f.

In particular all curves passing through z_0 are rotated to the same angle. In other words, conformal mappings preserve angles, an angle between two curves passing through z_0 is equal to the angle between their images.

Remark. Geometric properties of conformal mapings f can not be generalized to holomorphic mapings f with $f'(z_0) = 0$. For example, the mapping $f(z) = z^2$ is holomorphic at $z_0 = 0$ but doesn't preserve angles.

2.10 Holomorphic and conformal of mapping of extended complex plane

Definition 2.13. *A complex-valued function f defined in the neighborhood of $\infty \in \overline{\mathbb{C}}$ is called holomorphic (or, respectively, conformal) at $z = \infty$ if the function*

$$g(z) := f\left(\frac{1}{z}\right)$$

is holomorphic (or, respectively, conformal) at 0.

Exercise. Prove that if f is holomorphic at ∞ then $\lim\limits_{z\to\infty} f'(z) = 0$.

Definition 2.14. *A function f defined in the neighborhood of $\infty \in \overline{\mathbb{C}}$ such that $f(z_0) = \infty$ is called holomorphic (or, respectively, confor-*

mal) at $z = z_0$ if the function

$$F(z) := \frac{1}{f(z)}$$

is holomorphic (or, respectively, conformal) at z_0. In particular, if $f(\infty) = \infty$ then f is holomorphic (or, respectively, conformal) if the function

$$G(z) := \frac{1}{g(z)} = \frac{1}{f(1/z)}$$

is holomorphic (or, respectively, conformal) at 0.

3 Complex Integration

A natural way to construct the integral of a complex function over a curve in the complex plane is to link it to line integrals in R^2 as already seen in vector calculus.

3.1 Integration of a complex-valued function

Consider a complex function $f(t) = u(t) + iv(t)$, for $t \in [a, b] \subset \mathbb{R}$, where u and v real valued functions. If f is an integrable function, we may define

$$\int_a^b f(t)dt := \int_a^b u(t)dt + i\int_a^b v(t)dt$$

This definition, combined with the elementary properties of addition and multiplication in $\mathbb{C}$ means that the integral has many intuitive properties that are reminiscent of the properties of integrals of real functions. Let us mention a few without proof, as these proofs are elementary.

- $\int_a^c f(t)dt + \int_c^b f(t)dt = \int_a^b f(t)dt$, $c \in [a, b]$.
- $\int_a^b \lambda f(t)dt = \lambda \int_a^b f(t)dt$, $\lambda \in \mathbb{C}$.
- $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|\, dt$

Proof. If $\int_a^b f(t)dt = 0$, the inequality is trivial.

For $\int_a^b f(t)dt \neq 0$, let $\theta = \arg\left(\int_a^b f(t)dt\right)$. Then

$$\left|\int_a^b f(t)dt\right| = \mathrm{Re}\left(e^{-i\theta}\int_a^b f(t)dt\right) = \mathrm{Re}\left(\int_a^b e^{-i\theta} f(t)dt\right) =$$

$$\int_a^b \mathrm{Re}\left(e^{-i\theta} f(t)\right) dt \leq \int_a^b |f(t)|dt$$

□

3.2 Integration of a complex-valued function along path

Definition 3.1. *Let γ be a piecewise differentiable arc in the complex plane, with parametric equation*

$$\gamma : \quad z = z(t), a < t < b$$

If the function f is continuous on γ, then $f(z(t))$ is continuous on (a, b), and we define the integral of f on γ as the line integral

$$\int_\gamma f(z)dz := \int_a^b f(z(t))\frac{dz}{dt}dt$$

where the integral $\int_a^b$ may have to be split to match the intervals in which z is differentiable.

Remark 1. Notice that the integral $\int_\gamma f dz$ can be considered as the line-integral of a complex-valued differential form. That is for $f = u + iv$ and $dz = \dot\gamma(t)dt = dx + idy$ we have

$$\int_\gamma f dz = \int_\gamma (udx - vdy) + i \int_\gamma (udy + vdx),$$

where in the right-hand side we consider line-integrals of real differential forms.

Remark 2. The definition of the integral given above preserves its meaning for a rectified path, that is for a path $\gamma : I \to \mathbb{C}$, define by a function $\gamma(t)$ such that the derivative $\dot\gamma(t)$ exists a.e. on I and the function $|\dot\gamma(t)|$ is Lebesgue-integrable on I. Integral $\int_\gamma f dz$ is defined by the same formula, where in the right-hand side we consider Lebesgue integral $f(\gamma(t))\dot\gamma(t)$. Moreover, it is enough to assume that the composition $f \circ \gamma$ is measurable and bounded on I.

Example 1. Recall the definition of a complex exponent

$$e^{x+iy} := e^x(\cos y + i \sin y) \quad x, y \in \mathbb{R}.$$

Hence,

- $e^{z+2\pi i} = e^z$ for every $z \in \mathbb{C}$;
- For every $\alpha \in \mathbb{R}$ the derivative of a function $e^{i\alpha t}$ by parameter $t \in \mathbb{R}$ is equal to $i\alpha e^{i\alpha t}$.

Consider the circle of radius r centered at $a \in \mathbb{C}$ in a parametric form

$$z = \gamma(t) = a + re^{it}, \quad 0 \leqslant t \leqslant 2\pi,$$

and calculate the integral along γ of a function $f(z) = (z-a)^n$ foe every $n \in \mathbb{Z}$. We see that

$$\dot{\gamma}(t) = ire^{it}, \quad f(\gamma(t)) = r^n e^{int},$$

where

$$\int_\gamma (z-a)^n dz = ir^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt.$$

Applying to the obtained integral the Newton-Leeibniz formula we see that for every $n \neq -1$

$$\int_0^{2\pi} e^{i(n+1)t} dt = \frac{e^{2\pi i(n+1)} - 1}{i(n+1)} = 0.$$

For $n = -1$

$$\int_0^{2\pi} e^{i(n+1)t} dt = \int_0^{2\pi} dt = 2\pi.$$

Hence,

$$\int_\gamma (z-a)^n dz = \begin{cases} 0, & n \in \mathbb{Z} \setminus \{-1\} \\ 2\pi i, & n = -1 \end{cases}$$

Example. Let $\gamma : I \to \mathbb{C}$ be a smooth path. Consider the integralof function $f(z) = z^n$ for $n = 0, 1, 2, \ldots$ along γ. Applying Newton-Leibniz formula for complex-valued functions we see that

$$\begin{aligned}\int_\gamma z^n dz = \int_\alpha^\beta \gamma^n(t)\dot{\gamma}(t) dt = \frac{1}{n+1} \int_\alpha^\beta \frac{d}{dt} \left[\gamma^{n+1}(t)\right] dt \\ = \frac{\gamma^{n+1}(\beta) - \gamma^{n+1}(\alpha)}{n+1} = \frac{b^{n+1} - a^{n+1}}{n+1}.\end{aligned}$$

Thus the integral $\int_\gamma z^n dz$ depends only on the beginning a and the endpoint b of a path γ. In particular, the integral along closed path is 0.

3.3 Properties of the integral along the path.

1. Linearity. Let f, g be continuous along the path γ and $\alpha, \beta \in \mathbb{C}$. Then

$$\int_\gamma (\alpha f + \beta g) dz = \alpha \int_\gamma f dz + \beta \int_\gamma g dz.$$

2. Additivity. Consider two piecewise smooth path

$$\gamma_1 : [\alpha, \beta_1] \to \mathbb{C}, \quad \gamma_2 : [\beta_1, \beta] \to \mathbb{C}$$

such that $\gamma_1(\beta_1) = \gamma_2(\beta_1)$. Consider the compound of these two paths

$$\gamma = \gamma_1 \cup \gamma_2 : [\alpha, \beta] \to \mathbb{C},$$

letting

$$\gamma(t) = \begin{cases} \gamma_1(t) & \text{for } \alpha \leqslant t \leqslant \beta_1, \\ \gamma_2(t) & \text{for } \beta_1 \leqslant t \leqslant \beta. \end{cases}$$

Assume that f is continuous along $\gamma = \gamma_1 \cup \gamma_2$. Then

$$\int_{\gamma_1 \cup \gamma_2} f dz = \int_{\gamma_1} f dz + \int_{\gamma_2} f dz.$$

Remark 3.2. *Using this formula we can generalize the definition of the integral to "not connected" paths $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$ that consist of several connected piecewise smooth components $\gamma_1, \ldots, \gamma_n$ as a sum of integrals over a paths $\gamma_1, \ldots, \gamma_n$. With this definition it will be additive with respect to union $\gamma = \gamma_1 \cup \gamma_2$ of any piecewise paths γ_1, γ_2.*

3. Independence of parametrization. Let

$$\gamma : [\alpha, \beta] :\to \mathbb{C}$$

be a piecewise smooth path obtained from $\gamma_1 : [\alpha_1, \beta_1] \to \mathbb{C}$ by the change of parametrization

$$\gamma = \gamma_1 \circ \tau,$$

where $\tau : [\alpha, \beta] \to [\alpha_1, \beta_1]$ is strictly increasing C^1-function such that $\tau(\alpha) = \alpha_1$ and $\tau(\beta) = \beta_1$.

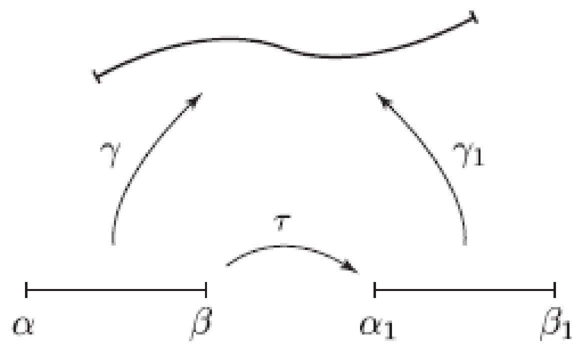


Figure 12: Equivalent paths.

If $f : \gamma([\alpha, \beta]) \to \mathbb{C}$ is continuous along γ then it is continuous along γ_1 and

$$\int_{\gamma_1} f dz = \int_{\gamma} f dz.$$

Proof. Indeed,

$$\int_{\gamma_1} f dz = \int_{\alpha_1}^{\beta_1} f(\gamma_1(\tau))\gamma_1'(\tau)d\tau = \left[{\scriptstyle \tau=\tau(t), \atop d\tau=\tau'(t)dt} \right] =$$

$$\int_{\alpha}^{\beta} f(\gamma_1(\tau(t)))\gamma_1'(\tau(t))\tau'(t)dt = \int_{\alpha}^{\beta} f(\gamma(t))\gamma(t)dt$$

since

$$\gamma_1'(\tau(t))\tau'(t) = (\gamma_1 \circ \tau)'(t) = \gamma'(t).$$

$\square$

4. Dependence of the orientation. Assume that piecewise-smooth path

$$\gamma^{-1} : [\alpha, \beta] \to \mathbb{C}$$

is obtained from a path $\gamma : [\alpha, \beta] \to \mathbb{C}$ by the change of orientation, i.e.

$$\gamma^{-1}(t) = \gamma(\alpha + \beta - t) \quad \text{for} \quad \alpha \leqslant t \leqslant \beta.$$

If $f : \gamma([\alpha, \beta]) \to \mathbb{C}$ is continuous along γ then it is continuous along γ^{-1} and

$$\int_{\gamma^{-1}} f dz = -\int_{\gamma} f dz.$$

5. Estimate of the integral. Let f be a piecewise smooth path $\gamma : \alpha, \beta] \to \mathbb{C}$. Then the following estimate is satisfied

$$\left| \int_{\gamma} f(z) dz \right| \leqslant \int_{\gamma} |f(z)| \, |dz| \, ,$$

where

$$\int_{\gamma} |f(z)| \, |dz| := \int_{\alpha}^{\beta} |f(\gamma(t))| \, |\dot{\gamma}(t)| \, dt$$

is the line-integral of the first kind of function $|f|$ along the path γ. In particular, if

$$|f(z)| \leqslant M \quad \text{for every} \quad z \in \gamma([\alpha, \beta])$$

then

$$\left| \int_{\gamma} f(z) dz \right| \leqslant M \cdot |\gamma|,$$

where $|\gamma|$ is the length of a path γ.

Proof. Let $J := \int_\gamma f(z)dz$ and express J in a polar form $J = |J|e^{i\theta}$, $\theta \in \mathbb{R}$. Then

$$|J| = e^{-i\theta} J = \int_\alpha^\beta e^{-i\theta} f(\gamma(t))\dot\gamma(t)dt.$$

Considering the real part we see that

$$\begin{aligned}|J| = \int_\alpha^\beta \operatorname{Re}\left\{e^{-i\theta} f(\gamma(t))\dot\gamma(t)\right\} dt\\ \leqslant \int_\alpha^\beta \left|e^{-i\theta} f(\gamma(t))\dot\gamma(t)\right| dt = \int_\alpha^\beta |f(\gamma(t))||\dot\gamma(t)|dt.\end{aligned}$$

Consequently,

$$\left|\int_\gamma f(z)dz\right| \leqslant \int_\gamma |f(z)|\,|dz|\,.$$

The second assertion follow from this estimate since

$$|\gamma| = \int_\alpha^\beta |\dot\gamma(t)|\,dt$$

□

3.4 The Cauchy-Goursat theorem

Theorem 3.3 (The Cauchy-Goursat theorem for triangles.)**.** *Let $f \in H(D)$. Then for every triangle Δ that is contained in D with its boundary*

$$\int_{\partial\Delta} f(z)dz = 0.$$

Proof. Assume that there exists a triangle $\Delta_0 \Subset D$ such that

$$\left|\int_{\partial\Delta_0} f dz\right| = M > 0. \tag{9}$$

We consider partition of Δ_0 into four equal triangles (see fig.). Then the integral of f along $\partial\Delta_0$ is equal to the sum of integrals of f along boundaries of these four triangles. Hence, the estimate (9) implies that absolute value of one of these integrals is greater or equal than $\frac{M}{4}$. We denote the corresponding triangle by Δ_1 so that

$$\left|\int_{\partial\Delta_1} f dz\right| \geqslant \frac{M}{4}.$$

Triangle Δ_1 will be also decomposed into the union of four equal triangles and choose triangle Δ_2 such that

$$\left|\int_{\partial\Delta_2} f dz\right| \geqslant \frac{M}{4^2}$$

Continuing this construction we obtain a sequence of triangles Δ_n such that

$$\left|\int_{\partial\Delta_n} f dz\right| \geqslant \frac{M}{4^n} \tag{10}$$

and

$$\overline{\Delta_{n+1}} \Subset \overline{\Delta_n}.$$

Consequently, their intersection contains a unique point $z_0 \in D$.

Now we can apply $\mathbb{C}$-differentiability of function f at z_0. For every $\varepsilon > 0$ there exists such $\delta > 0$ that in neighborhood

$$U = U_\delta(z_0) := \{z \in \mathbb{C} : |z - z_0| < \delta\}$$

of a point z_0 function f can be expressed as

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \alpha(z)(z - z_0), \tag{11}$$

where $|\alpha(z)| < \varepsilon$ for every $z \in U$. Now, using this equation (11 we can calculate the integral over the boundary of any triangle Δ_n such that $\bar{\Delta}_n \Subset U$ as following

$$\int_{\partial\Delta_n} f dz = \int_{\partial\Delta_n} f(z_0)\, dz + \int_{\partial\Delta_n} f'(z_0)(z - z_0)\, dz \\ + \int_{\partial\Delta_n} \alpha(z)(z - z_0)\, dz.$$

First two integrals are equal to zero (see example). Third integral can be estimated as follows

$$\left|\int_{\partial\Delta_n} \alpha(z)(z - z_0)\, dz\right| \leqslant \varepsilon \int_{\partial\Delta_n} |z - z_0|\, |dz| \leqslant \varepsilon\, |\partial\Delta_n|^2,$$

where $|\partial\Delta_n|$ is perimeter of triangle Δ_n. (Notice that we used the property $|z - z_0| < |\partial\Delta_n|$ for $z \in \partial\Delta_n$.) Hence,

$$\left|\int_{\partial\Delta_n} f dz\right| \leqslant \varepsilon\, |\partial\Delta_n|^2. \tag{12}$$

Triangles Δ_n and Δ_0 are similar and the ratio of lengthes of edges is $1/4^n$. Consequently,

$$|\partial\Delta_n| = \frac{|\partial\Delta_0|}{2^n}$$

and estimate (12) can has the following form

$$\left|\int_{\partial\Delta_n} f dz\right| \leqslant \varepsilon \frac{|\partial\Delta_0|^2}{4^n}.$$

Comparing this with the estimate (10) we see that

$$M \leqslant \varepsilon\, |\partial\Delta_0|^2$$

for every $\varepsilon > 0$ and $M = 0$, which contradicts our assumption. $\square$

4 Antiderivative.

4.1 Criterion for constancy of a holomorphic function

Lemma 4.1. *Let $D \subset \mathbb{C}$ be a domain, $f \in H(D)$. Then the following assertions are satisfied.*

1. *If $\operatorname{Re} f$ is constant then f is constant.*
2. *If $\operatorname{Im} f$ is constant then f is constant.*
3. *If $|f|$ is constant then f is constant.*

Proof. Assume that $f = u + iv$.

1. Since u is constant then

$$u'_x = u'_y = 0.$$

Then by Cauchy-Riemann condition

$$v'_x = v'_y = 0$$

Consequently, v is constant. Hence, f is also constant.

2. This assertion can be proved analogously to the first.

3. By the assumption the function $|f|^2 = u^2 + v^2$ is constant. If $u^2 + v^2 = 0$ then $f = 0$. Assume that the constant function $u^2 + v^2$ is not zero. Then its partial derivatives are equal to 0, that is

$$\begin{cases} 2uu'_x + 2vv'_x = 0, \\ 2uu'_y + 2vv'_y = 0. \end{cases}$$

Applying Cauchy-Riemann condition we see that

$$\begin{cases} uu'_x - vu'_y = 0, \\ vu'_x + uu'_y = 0 \end{cases}$$

Consider these identities as the system of equations with respect to u'_x, u'_y. The determinant of this system is equal $u^2+v^2 \neq 0$. Consequently, this system has only zero solution $u'_x = u'_y = 0$. Hence, u is constant, and f is constant by the first assertion. □

4.2 Antiderivative of a holomorphic function.

Definition 4.2. *Let D be domain in $\mathbb{C}$, $f \in C(D)$, $F \in H(D)$. The function $F \in H(D)$ is antiderivative of function f if*

$$F'(z) = f(z), \quad z \in D.$$

Consider a question on uniqueness of antiderivative.

Lemma 4.3. *Assume that F is an antiderivative of a function f in domain D. Then all antiderivatives of f in domain D differ from F by a constant, that is they have the following form*

$$F(z) + c, \ c \in \mathbb{C}.$$

Proof. Assume that F_1 is an antiderivative of function f in D. then the function $\Phi := F - F_1$ is holomorphic in D and

$$\Phi'(z) = 0, \quad z \in D.$$

Applying Cauchy-Riemann condition to Φ we see that

$$\frac{\partial \Phi}{\partial x} = -i\frac{\partial \Phi}{\partial y} = \Phi'(z) = 0, \quad z \in D.$$

Consequently, Φ is constant. □

Solving the question on existence of antiderivative, we will first consider the case of a disk.

Lemma 4.4. *Let* $U = \{z \in \mathbb{C} : |z - a| < r\}$, $f : U \to \mathbb{C}$ *be continous in* U *and for every triangle* Δ

$$\int_{\partial\Delta} f dz = 0.$$

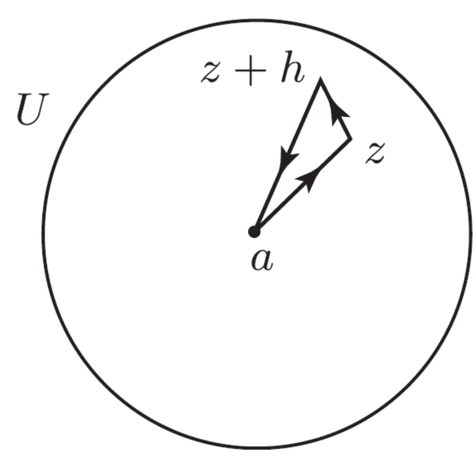
Then the function

$$F(z) = \int_a^z f(\xi)d\xi, \quad z \in U,$$

(where the integral is considered by a segment that connects center a *of a circle and point* z*) is an antiderivative of the function* f *in* U.

Proof. Let $z \in U$ and $\delta > 0$ be such that a disk $\{z+h : h \in \mathbb{C},\ |h| \le \delta\}$ is contained in U. Since the integral along the triangle with vertexes at points a, z and $z+h$, $|h| < \delta$, is zero we see that

$$F(z+h) - F(z) = \int_z^{z+h} f(\xi)d\xi.$$



At the same time

$$f(z) = f(z)\frac{1}{h}\int_z^{z+h} d\xi = \frac{1}{h}\int_z^{z+h} f(z)d\xi.$$

Consequently,

$$\frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(\xi)d\xi =$$
$$f(z) + \frac{1}{h}\int_z^{z+h} (f(\xi)-f(z))d\xi.$$

Hence, applying uniform continuity of f in a closure of a disk

$$\{z+h : h \in \mathbb{C}, |h| < \delta\}$$

we see that

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_z^{z+h} (f(\zeta)-f(z))d\zeta\right|$$
$$\leqslant \frac{1}{|h|}\cdot|h| \max_{\zeta\in[z,z+h]} |f(\zeta)-f(z)| \to 0 \quad \text{as} \quad h \to 0.$$

Hence F is $\mathbb{C}$-differentiable at z and $F'(z) = f(z)$. □

4.3 Antiderivative along the path.

From the previous consideration and Cauchy-Goursat's theorem we see that a function f holomorphic in domain D has antiderivative in every disk $U \subset D$. In other words, it has local antiderivative in D. The essential question is whether it has global antiderivative in D (defined in the whole domain D)? As we will see the answer to this question is negative and the disk U can not be substituted by any domain $D \subset \mathbb{C}$. It turns out that there are topological obstacles for this. Nevertheless the local antiderivatives can be glued in the antiderivative of f along the path $\gamma : I \to D$. Let's provide the strict definition.

Definition 4.5. *Let $\gamma : I \to D$ be arbitrary path in domain D and $f : D \to \mathbb{C}$. The function $\Phi : I \to \mathbb{C}$ is an antiderivative of function f along the path γ if*

1. *Φ is continuous on I;*
2. *for every $t_0 \in I$ there exists a disk $U \subset D$ with a center at $z_0 = \gamma(t_0)$ and antiderivative F_U of a function f in this disk such that*
$$\Phi(t) = F_U(\gamma(t)).$$
for every t in some neighborhood $u = u(t_0) \subset I$ of t_0.

Remark 4.6. *Notice that function Φ is a function of t but not of a point $z = \gamma(t)$. In particular, if disks U' and U'' for points $z' = \gamma(t')$ and $z'' = \gamma(t'')$ have nonempty intersection this doesn't imply that antiderivatives $F_{U'}$ and $F_{U''}$ coincide on $U' \cap U''$. They may differ by a constant.*

Remark 4.7. *If $f : D \to \mathbb{C}$ has a global antiderivative $F : D \to \mathbb{C}$ in D then the function*
$$\Phi = F(\gamma(t))$$
is an antiderivative of f along path γ for any path $\gamma : I \to D$.

Theorem 4.8 (On existence and uniqueness of the antiderivative along a path)**.** *Let f be holomorphic in D and $\gamma : I \to D$ be a path in D.*

Then the antiderivative of f along γ exists and is unique up to a constant.

Proof. **Existence.** Consider a partition

$$\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta$$

of a segment $I = [\alpha, \beta]$, such that the image of every segment $I_j := [t_{j-1}, t_j]$ by a map γ is contained in some disk $U_j \subset D$ (see Fig. 13).

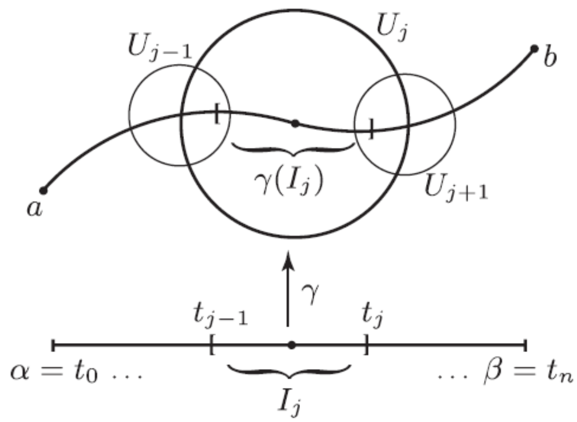


Figure 13: Partition of a path.

We will construct an antiderivative of f along path γ in inductive way starting form disk U_1. First, fix an antiderivative F_1 of function f in disk U_1. Then any antiderivative F_2 of function f in U_2 differs from F_1 by a constant on $U_1 \cap U_2$:

$$F_2 - F_1 \equiv \text{const}, \quad \text{on } U_1 \cap U_2 \neq \varnothing.$$

Substracting this constant from F_2 we may assume that

$$F_2 \equiv F_1, \quad \text{on } U_1 \cap U_2 \neq \varnothing.$$

Continuing this construction we obtain for every j antiderivative F_j of function f in a disk U_j such that

$$F_j \equiv F_{j-1}, \quad \text{on } U_{j-1} \cap U_j \neq \varnothing.$$

Now we can define $\Phi : I \to \mathbb{C}$ as

$$\Phi(t) = F_j(\gamma(t)), \quad t \in [t_{j-1}, t_j].$$

By the construction Φ is continuous on I and is antiderivative of f along γ.

Uniqueness. Assume that Φ_1 and Φ_2 are two antiderivatives of f along path γ. Let $t_0 \in I$. Then in some neighborhood $u \subset I$ of a point t_0 we have

$$\Phi_1(t) = F_1(\gamma(t)), \quad \Phi_2(t) = F_2(\gamma(t)),$$

where F_1 and F_2 are antiderivatives of f in some disk $U \subset D$ with center at point $\gamma(t_0)$. Since $F_1 - F_2 \equiv \text{const}$ in U then $\Phi_1 - \Phi_2$ is constant on u. This means that function $\Phi_1 - \Phi_2$ is locally constant on I and, since I is connected, $\Phi_1 - \Phi_2$ is constant on I. □

Theorem 4.9 (Newton-Leibniz formula.)**.** *Let $\gamma : [\alpha, \beta] \to D$ be a piecewise smooth path in D and $f \in H(D)$. Let Φ be antiderivative of f along γ. Then*

$$\int_\gamma f dz = \Phi(\beta) - \Phi(\alpha)$$

Proof. Consider a partition

$$\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta$$

of a segment $I = [\alpha, \beta]$, such that the image of every segment $I_j := [t_{j-1}, t_j]$ by a map γ is contained in some disk $U_j \subset D$ and f has in U_j an antiderivative F_j such that

$$\Phi(t) = F_j(\gamma(t)), \quad t \in I_j.$$

Consequently, with $\gamma_j = \gamma|_{I_j}$, we see that

$$\int_\gamma f dz = \sum_{j=1}^n \int_{\gamma_j} f dz = \sum_{j=1}^n (F_j(\gamma(t_j)) - F_j(\gamma(t_{j-1}))) = \\ \sum_{j=1}^n (\Phi(t_j) - \Phi(t_{j-1})) = \Phi(b) - \Phi(a).$$

□

Example 4.1. *The function $f(z) = \frac{1}{z}$ that is holomorphic in domain*

$$D = \left\{ z \in \mathbb{C} : \frac{1}{2} < |z| < 2 \right\}$$

but doesn't have in this domain a global antiderivative.

Proof. If f has an antiderivative F in domain D then for every closed path $\gamma : [\alpha, \beta] \to D$ we have

$$\int_\gamma f dz = F(\gamma(\alpha)) - F(\gamma(\beta)) = 0$$

while

$$\int_{|z|=1} \frac{dz}{z} = 2\pi i.$$

□

5 Homotopy. Cauchy theorem.

In this part we let D be a domain in $\mathbb{R}^2$ or in $\mathbb{C}$ and for simplicity assume that all paths are defined on a segment $I = [0, 1]$ (see property **L3**).

Definition 5.1. *Two paths $\gamma_0, \gamma_1 : I \to D$ with common endpoints*

$$\gamma_0(0) = \gamma_1(0) = \alpha, \quad \gamma_0(1) = \gamma_1(1) = \alpha$$

are homotopic in domain D as paths with common (or fixed) endpoints if there exists a map $\Gamma \in C(I \times I \to D)$ such that

1. *$\Gamma(0, t) = \gamma_0(t)$ and $\Gamma(1, t) = \gamma_1(t)$ for every $t \in I$;*
2. *$\Gamma(s, 0) = \alpha$ and $\Gamma(s, 1) = \alpha$ for every $s \in I$;*

Definition 5.2. *Two closed paths $\gamma_0, \gamma_1 : I \to D$ are homotopic in domain D as closed paths if there exists a map $\Gamma \in C(I \times I \to D)$ such that*

1. *$\Gamma(0, t) = \gamma_0(t)$ and $\Gamma(1, t) = \gamma_1(t)$ for every $t \in I$;*
2. *$\Gamma(s, 0) = \Gamma(s, 1)$ for every $s \in I$;*

Remark 5.3. *In both cases map Γ is called a homotopy of paths γ_0 and γ_1. An intermediate path is denoted by $\gamma_s(\cdot) = \Gamma(s, \cdot)$.*

Remark 5.4. *Homotopy is equivalence relation. Equivalent (in sense of reparametrization) paths are homotopic.*

Definition 5.5. *A path $\gamma : I \to \mathbb{C}$ is a constant path if $\gamma(t)$ is constant, $\gamma(t) = \gamma(0)$ for every $t \in I$.*

A closed path is contractible if it is homotopic to a constant path, i.e. exists a map $\Gamma \in C(I \times I \to D)$ and a point $z_0 \in D$ such that

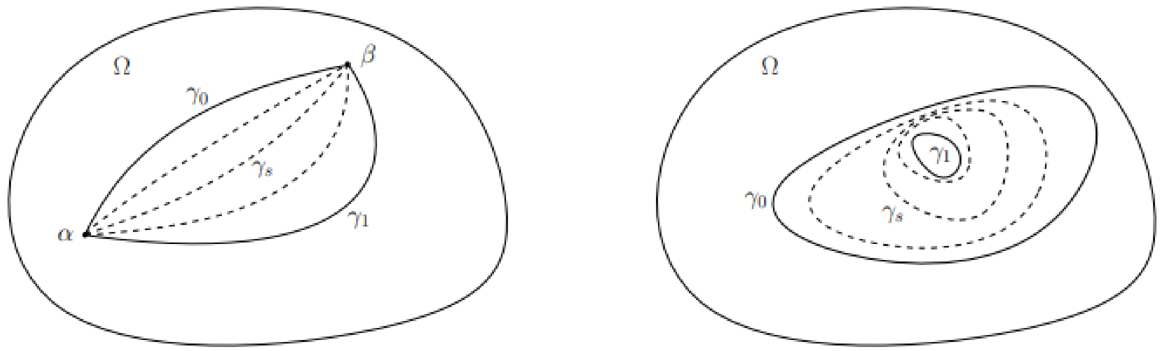


Figure 14: Left: homotopy of curves with same endpoints. Right: homotopy of closed curves.

1. $\Gamma(0, t) = \gamma_0(t)$ *and* $\Gamma(1, t) = z_0$ *for every* $t \in I$;

2. $\Gamma(s, 0) = \Gamma(s, 1)$ *for every* $s \in I$;

Definition 5.6. *A domain D is simply connected if every closed path in D is contractible.*

Definition 5.7. *A domain D is star-shaped if there exists a point $z \in D$ such that for every $w \in D$ a segment that connects z and w is contained in D, i.e.*

$$\exists z \in D : tw + (1-t)z \in D \text{ for every } z \in D \text{ and } t \in [0, 1].$$

Example 5.1. *Every star-shaped domain (in particular, a disk) is simply connected. Every convex domain is star-shaped and, consequently, simply connected.*

Example 5.2. *Let $0 \leq r < R \leq \infty$, $z_0 \in \mathbb{C}$. An annulus is a set*

$$K_{r,R}(z_0) = \{z \in \mathbb{C} : r < |z - z_0| < R.$$

Numbers r, R are inner and outer radii and z_0 is a center of the annulus $K_{r,R}(z_0)$.

Annulus is not simply connected.

Lemma 5.8. *In a simply connected domain any two paths with common endpoints are homotopic.*

Remark. Let D be a bounded domain. TFAE

1. D is simply connected;
2. ∂D is connected;
3. D^c is connected;

Theorem 5.9 (Cauchy's theorem on homotopy.)**.** *Let f be holomorphic in domain D and γ_0, γ_1 be two paths homotopic in D. Then*

$$\int_{\gamma_0} f dz = \int_{\gamma_1} f dz.$$

Proof. Let

$$\gamma_s(t) = \Gamma(s, t) : I \to D$$

is a homotopy of paths γ_0 and γ_1. Let

$$J(s) := \int_{\gamma_s} f dz \quad \text{for } s \in I.$$

To prove that $J(1) = J(0)$ is enough to show that $J(s)$ is locally constant on I, that is every point $s_0 \in I$ has a neighborhood $v = v(s_0) \subset I$ such that $J(s) = J(s_0)$ for every $s \in v$.

Let $\Phi : I \to \mathbb{C}$ be an arbitrary antiderivative of function f along path γ_{s_0}. Consider partition of a segment I by the points

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

into segments $I_j = [t_{j-1}, t_j]$ such that there exist

1. disks $U_j \subset D$ such that $\gamma_{s_0}(I_j) \subset U_j$;
2. antiderivatives $F_j \in \mathcal{O}(U_j)$ of functions f in U_j such that

$$\Phi = F_j \circ \gamma_{s_0} \quad \text{on } I_j \quad \text{for every } j = 1, \ldots, n.$$

In particular, the second condition implies that $F_j \equiv F_{j-1}$ on $U_j \cap U_{j-1}$. Moreover, the uniform continuity of $\Gamma(s,t)$ on $I \times I$ implies that there exists a neighborhood $v \subset I$ of s_0 such that $\gamma(v \times I_j) \subset U_j$ for every j.

Consider a family of functions $\Phi_s : I \to \mathbb{C}$ of a variable t letting

$$\Phi_s := F_j \circ \gamma_s \quad \text{on } I_j \quad \text{for } j = 1, \ldots, n.$$

Then for every $s \in v$ function Φ_s is continuous on I and coincides with $F(\gamma_s(t))$ in some neighborhood $t_0 \in I$ for some antiderivative F of function f in the neighborhood of $\gamma(t_0)$ (recall that $F_j \equiv F_{j-1}$ on $U_j \cap U_{j-1}$!. Thus Φ_s is an antiderivative of f along γ_s.

By the Newton-Leibniz formula (or by the definition of $\int_{\gamma_s} f dz$ for continuous paths γ_s) we see that

$$J(s) := \int_{\gamma_s} f dz = \Phi_s(1) - \Phi_s(0).$$

We will prove that this function doesn't depend on $s \in v$ which will finalize the proof of the Theorem.

Consider cases of closed paths and paths with common endpoints independently.

1. Assume that γ_0 and γ_1 are homotopic as paths with common endpoints (s.t. $\gamma_s(0) = a$ and $\gamma_s(1) = b$ for every $s \in I$). Then values

$$\Phi_s(0) = F_1(\gamma_s(0)) = F_1(a) \quad \text{and} \quad \Phi_s(1) = F_n(\gamma_s(1)) = F_n(b)$$

do not depend on $s \in v$. Consequently, their difference $J(s)$ also doesn't depend on $s \in v$.

2. Assume that γ_0 and γ_1 are homotopic as closed paths (s.t. $\gamma_s(0) = \gamma_s(1)$ for every $s \in I$), then functions (that do not depend on s) F_1 and F_n as two antiderivatives of f in the neighborhood $U_1 \cap U_n$ of a point $z_s := \gamma_s(0) = \gamma_s(1)$ differ by a constant (that doesn't depend on s)

$$F_n(z) - F_1(z) = C \quad \text{for every} \quad z \in U_1 \cap U_n.$$

Hence,

$$J(s) = F_n(\gamma_s(1)) - F_1(\gamma_s(0)) = F_n(z_s) - F_1(z_s) = C$$

doesn't depend on $s \in v$. □

Corollary 5.9.1 (Cauchy-Goursat's theorem for a contractible path)**.** *Let f be holomorphic in D and $\gamma : I \to D$ be contractible. Then*

$$\int_\gamma f dz = 0.$$

In particular, in the simply connected domain D the integral of function $f \in H(D)$ along every closed path $\gamma : I \to D$ is equal to zero.

The proof follows from the theorem on homotopy since the integral over the constant path is always zero.

Corollary 5.9.2. *Let $D \subset \mathbb{C}$ be simply connected. Then every function f holomorphic in D has antiderivative.*

Proof. Let $a \in D$. for every $z \in D$ consider a piecewise smooth path $\gamma : I \to D$ that connects a with z and let

$$F(z) := \int_{\gamma} f(\zeta) d\zeta.$$

The value $F(z)$ doesn't depend on γ. Indeed, if γ_1, γ_2 are two such paths then the integral of f along the closed path $\gamma_1 \cup \gamma_2^{-1}$ is equal to zero by the previous corollary

$$\int_{\gamma_1} f(\zeta) d\zeta - \int_{\gamma_2} f(\zeta) d\zeta = 0$$

In particular, if $z_0 \in D$ and U is a disk centered at z_0 contained in D then for $z \in U$ a function $F(z)$ can be written in the following form

$$F(z) = \int_{\gamma_0} f(\zeta) d\zeta + \int_{z_0}^{z} f(\zeta) d\zeta = F(z_0) + \int_{z_0}^{z} f(\zeta) d\zeta,$$

where the integral $\int_{z_0}^{z} f(\zeta) d\zeta$ is taken over the segment that connects z_0 and z and γ_0 is any path that connects a and z_0. Hence F is differentiable in U and

$$F'(z) = f(z) \quad \text{for every} \quad z \in U.$$

Since z_0 is arbitrary this implies that F is the antiderivative of function f in domain D. $\square$

5.1 Cauchy's theorem for multiple connected domains.

Recall that the bounded domain $D \subset \mathbb{C}$ is a domain with a simple boundary if its boundary is a union of a finite number of nonintersecting piecewise smooth simple closed curves $\gamma_0, \gamma_1, \ldots, \gamma_n$, where γ_0 denotes the outer boundary of domain D, and $\gamma_1, \ldots, \gamma_n$ are inner components of ∂D (see. Figure 15). For function f defined on ∂D the integral along the boundary is defined as follows

$$\int_{\partial D} f dz = \int_{\gamma_0} f dz + \sum_{j=1}^{n} \int_{\gamma_j} f dz$$

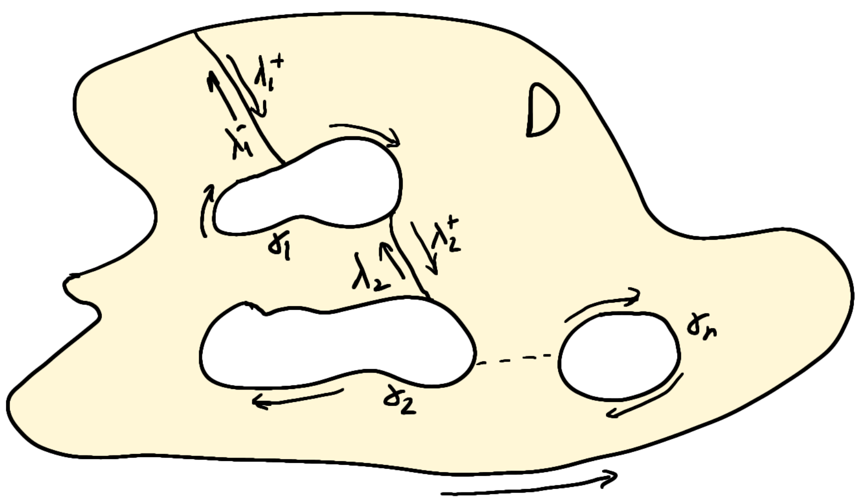


Figure 15: Multliple connected domain.

Theorem 5.10 (Cauchy-Goursat's theorem for multiple connected domain)**.** *Suppose $D \subset \mathbb{C}$ is a bounded domain with simple boundary, f*

is a holomorphic function in some domain $G \supset \overline{D}$. Then

$$\int_{\partial D} f dz = 0.$$

Proof. Consider in domain D a finite number of "slits" $\lambda_1, \dots, \lambda_n$ such that λ_k connects a point on curve γ_{k-1} with a point on γ_k and denote by λ_k^+ the path oriented from γ_{k-1} to γ_k and by λ_k^- the opposite path. We can choose slits such that the closed path Γ composed of arcs of boundary ∂D and paths $\lambda_k^{\pm}$ is contractible in G.

Then, by Cauchy-Goursat's theorem we see that

$$0 = \int_\Gamma f dz = \int_{\partial D} f dz + \sum_{j=1}^{n} \int_{\lambda_j^+} f dz + \sum_{j=1}^{n} \int_{\lambda_j^-} f dz == \int_{\partial D} f dz.$$

Since

$$\int_{\lambda_j^+} f dz = -\int_{\lambda_j^-} f dz.$$

□