Differential calculus of a function of one real variable. Lecture 2. Derivative of an inverse function. Mean value theorems in differential calculus.

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Examples

$$(ln\,|f(x)|)' = \frac{f'(x)}{f(x)},\ \ f(x) \neq 0;$$

$$\left(f(x)^{g(x)}\right)' = \left(e^{g(x)\ln f(x)}\right)' = \left(g'\ln f(x) + \frac{f'(x)g(x)}{f(x)}\right)e^{g(x)\ln f(x)} = f^g\left(g'\ln f + g\frac{f'}{f}\right).$$

$$(x^x)' = (e^{x\ln x})' = x^x(\ln x + 1).$$

Differentiation of the inverse function.

Theorem 1 (Theorem on derivative of the inverse function)

Let $f : \langle a, b\rangle \to \langle c, d\rangle$ be continuous, strictly monotonic and differentiable at $x \in \langle a, b\rangle$ function, $f'(x) \neq 0$. Then the inverse to f function is differentiable at $f(x)$ and

$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)}.$$

Proof.

Notice that strict monotonicity implies existence of an inverse function that is defined on an interval $I = f(\langle a, b\rangle)$, is monotonic and continuous.
Let $y = f(x)$ and $k \neq 0$ be such that $y + k \in I$. Define

$$h = h(k) = f^{-1}(y + k) - f^{-1}(y).$$

Then $h(k) \neq 0$ and $x + h = f^{-1}(y + k)$. Consequently, by the theorem on a limit of a composition,

$$\lim_{k\to 0} \frac{f^{-1}(y + k) - f^{-1}(y)}{k} = \lim_{k\to 0} \frac{h(k)}{f(x + h(k)) - f(x)} =$$

$$\lim_{h\to 0} \frac{h}{f(x + h) - f(x)} = \frac{1}{f'(x)}.$$

- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$.

  Proof. Let $x \in x \in (-1,1)$. Then $y = \arcsin x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $(\sin(y))' = \cos y > 0$. By the theorem on differentiation of inverse function we obtain

$$(\arcsin x)' = \frac{1}{(\sin(y))'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-x^2.}}.$$

- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$.
- $(\mathrm{arctg}\, x)' = \frac{1}{1+x^2}$, $x \in \mathbb{R}$.
- $(\mathrm{arcctg}\, x)' = -\frac{1}{1+x^2}$, $x \in \mathbb{R}$.

Mean value theorems.

Fermat's theorem.

Definition 2

Let $f : \langle a, b\rangle \to \mathbb{R}$ be differentiable at $x_0 \in \langle a, b\rangle$ and $f'(x_0) = 0$. Then x_0 is called **stationary** point of f.

Definition 3

Let $f : D \to \mathbb{R}$. We say that f has a **local maximum** at a point $x_0 \in D$ if there exists a neighbourhood V_{x_0} such that $f(x) \le f(x_0)$ for all $x \in V_{x_0}$. **Local minima** are define likewise. We say that $x_0 \in D$ is **(local) extremum** if function f has local maximum or minimum at x_0.

Theorem 4 (Fermat's theorem.)

Let $f : (a, b) \to \mathbb{R}$. If $x_0 \in (a, b)$ is extremum of function f and f is differentiable at x_0 then x_0 is a stationary point.

Proof. Let $f(x_0) = \max\limits_{x\in\langle a,b\rangle} f(x)$. Then $f(x) - f(x_0) \le 0$ for all x in some neighbourhood of x_0

$$f'(x_0) = \lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \le 0,$$

$$f'(x_0) = \lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \ge 0.$$

Consequently, $f'(x_0) = 0$.

Theorem 5 (Rolle's theorem.)

Let f be a continuous on $[a, b]$ and differentiable on (a, b) function. If $f(a) = f(b)$ then there exists a stationary point $c \in (a, b)$, that is $f'(c) = 0$.

Proof.

A function f is a continuous on $[a, b]$ and, consequently, reaches its maximal and minimal value, there exist $x_1, x_2 \in [a, b]$ such that $f(x_1) = \max\limits_{x \in [a,b]} f(x), \; f(x_2) = \min\limits_{x \in [a,b]} f(x)$.
If x_1, x_2 are endpoints of $[a, b]$ then $f(x_1) = f(x_2)$ and function f is constant which proves theorem with arbitrary $c \in (a, b)$.
If f is not constant then one of the points x_1, x_2 lies in the interval (a, b), and, applying Fermat's theorem, we get $f'(c) = 0$ for $c = x_1$ if $x_1 \in (a, b)$ or for $c = x_2$ if $x_2 \in (a, b)$. □

Mean value theorems.

Theorem 6 (Cauchy's mean value theorem.)

Let f, g be a continuous on $[a, b]$ and differentiable on (a, b) functions. Then there exists a point $c \in (a, b)$ such that

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

Proof.

Let

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

Then h is a continuous on $[a, b]$, differentiable on (a, b) function such that

$$h(a) = h(b) = f(a)g(b) - f(b)g(a).$$

Consequently, by Rolle's theorem there exists a stationary point $c \in (a, b)$ of h that is

$$h'(c) = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a)) = 0.$$

This finishes the proof of the theorem. □

Consequence.

In statements of the previous theorem if $g'(t) \neq 0$ on (a, b) then $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Theorem 7 (Lagrange's Mean value theorem.)

Let f be a continuous on $[a, b]$ and differentiable on (a, b) function. Then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof.

This theorem follows from Cauchy's mean value theorem with $g(x) = x$. □

Consequence 1. Estimation of increment.

Let f be a continuous on $\langle a, b\rangle$ and differentiable on (a, b) function. If there exists $M > 0$ such that $|f'(t)| < M$ then for every points $x, x + h \in \langle a, b\rangle$

$$|f(x + h) - f(x)| \leq Mh. \tag{1}$$

Consequence 2.

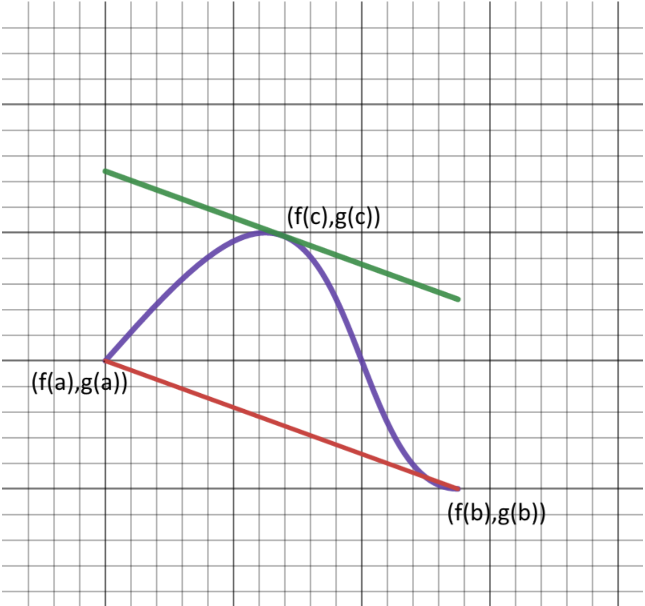
If function f is differentiable on $\langle a, b\rangle$ and has a bounded derivative then it is uniformly continuous.

Proof. Let $|f'(t)| \leq M,\ t \in \langle a, b\rangle$. Let $\varepsilon > 0$ and $\delta = \frac{\varepsilon}{M}$. Then for every $x, y \in \langle a, b\rangle$ such that $|x - y| < \delta$ by estimate 1

$$|f(x) - f(y)| \leq M\,|x - y| < M\delta = \varepsilon$$

which proves the uniform continuity.

Geometric meaning of Cauchy and Lagrange's mean theorem.



Darboux's theorem

Theorem 8 (Darboux's theorem)

Let f be a differentiable function on $[a, b]$. Then for every number C that lies between $f'(a)$ and $f'(b)$ there exists $c \in (a, b)$ such that $f'(c) = C$.

Proof.

Without loss of generality we may assume that $f'(a) < C < f'(b)$.
Case 1. Suppose that $f'(a) < C = 0 < f'(b)$. Function f is differentiable and, consequently, continuous on $[a, b]$. Notice that both endpoints a, b are local maxima of f. So $f(x) < f(a)$, $a < x < a + \delta$ for some $\delta > 0$ since

$$f'(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a} < 0.$$

Similarly, $f(x) < f(b)$, $b - \delta < x < b$ for some $\delta > 0$.
Then f has a local minimum $c \in (a, b)$ and, by Fermat's theorem, $f'(c) = 0 = C$.
Case 2. Let $f'(a) < C < f'(b)$ and introduce supplementary function $g(x) = f(x) - Cx$. Then

$$g'(a) = f'(a) - C < 0 < f'(b) - C$$

and by the previous case there exist $c \in (a, b)$ such that $g'(c) = f'(c) - C = 0$.

□

Consequence. Let f be a differentiable function on $\langle a, b \rangle$. Then $f'(\langle a, b \rangle)$ is an interval.

Example.

Darboux theorem proves extends one of the most remarkable properties of continuous functions (see Bolzano-Cauchy theorem on intermediate value)to derivatives. However it may happen that f' is not continuous. Consider

$$f(x) = x^2 \sin \frac{1}{x},\ x \neq 0, f(0) = 0.$$

Then

$$f'(0) = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x \to 0} x \sin \frac{1}{x} = 0;$$

and for $x \neq 0$

$$f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.$$

Consequently, f is differentiable on $\mathbb{R}$, while f' has no limit and is not continuous at 0.

Criteria for monotonicity of differentiable function.

Definition 9

- A function $f : E \subset \mathbb{R} \to \mathbb{R}$ is **increasing** on E if $f(x) \leq f(y)$ whenever $x \leq y$ in E. If the inequality is strict, i.e. $f(x) < f(y)$ whenever $x < y$, then f is **strictly increasing.**
- A function $f : E \subset \mathbb{R} \to \mathbb{R}$ is **decreasing** on E if $f(x) \geq f(y)$ whenever $x \leq y$. If the inequality is strict, i.e. $f(x) > f(y)$ whenever $x < y$, then f is **strictly decreasing**.
- A function $f : E \subset \mathbb{R} \to \mathbb{R}$ is **(strictly) monotonic** if it is (strictly) increasing or decreasing.

Theorem 10 (Criteria of monotonicity of function.)

Let $f \in C(\langle a, b\rangle)$ be differentiable on (a, b). Then

1. *function f is increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$;*
2. *function f is decreasing on (a, b) if and only if $f'(x) \leq 0$ for all $x \in (a, b)$.*

Proof.

1. Necessity. Assume that f increases, then

$$f'(x) = f'_+(x) = \lim_{y \to x+} \frac{f(y) - f(x)}{y - x} \geq 0.$$

2. Sufficiency. Let $f'(x) \geq 0$ on (a, b) and $x_1, x_2 \in (a, b)$, $x_1 < x_2$. The, by Lagrange's mean value theorem there exists $c > 0$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0.$$

□

Consequences

1. Let $f : \langle a, b \rangle \to \mathbb{R}$. Then f is constant if and only if $f \in C\langle a, b \rangle$ and $f'(x) = 0$ for all $x \in (a, b)$.

2. Let $f \in C(\langle a, b \rangle)$ be differentiable on (a, b). Then f is strictly increasing (strictly decreasing) on (a, b) if and only if

1. $f'(x) \geq 0$ ($f'(x) \leq 0$) for all $x \in (a, b)$.
2. $f'(x)$ is not identically equal to 0 on every interval.

Definition 11

Let $f : \langle a, b\rangle \to \mathbb{R}$ be differentiable at $p \in \langle a, b\rangle$ and $f'(p) = 0$. Then p is called **stationary** point of f.

Definition 12

Let $f : D \to \mathbb{R}$.

- We say that f has a **local maximum** at a point $p \in D$ if there exists a neighbourhood V_p such that $f(x) \leq f(p)$ for all $x \in V_p$.
- We say that f has a **strict local maximum** at a point $p \in D$ if there exists a neighbourhood V_p such that $f(x) < f(p)$ for all $x \in V_p \setminus \{p\}$.
- Local minima and strict local minima are defined likewise.
- We say that $p \in D$ is a **(strict) (local) extremum** if function f has (strict) local maximum or minimum at p.

Theorem 13 (Fermat's theorem.)

Let $f : \langle a, b\rangle \to \mathbb{R}$. If p is extremum of function f and f is differentiable at p then p is a stationary point.

Sufficient conditions for extremum.

Theorem 14

Let $f : \langle a, b\rangle \to \mathbb{R},\ p \in (a, b)$, *function* f *is continuous at* p, *differentiable on* $(a, b) \setminus \{p\}$. *If there exists such* $\delta > 0$ *that* f' *preserves a sign on* $(p-\delta, p)$ *and* $(p, p+\delta)$

1. $f' < 0$ *on* $(p-\delta, p)$ *and* $f' > 0$ *on* $(p, p+\delta)$ *then* p *is a point of strict minimum;*
2. $f' > 0$ *on* $(p-\delta, p)$ *and* $f' < 0$ *on* $(p, p+\delta)$ *then* p *is a point of strict maximum;*
3. $f' < 0$ *on* $(p-\delta, p)$ *and* $f' < 0$ *on* $(p, p+\delta)$ *then* f *is decreasing on* $(p-\delta, p+\delta)$;
4. $f' > 0$ *on* $(p-\delta, p)$ *and* $f' > 0$ *on* $(p, p+\delta)$ *then* f *is increasing on* $(p-\delta, p+\delta)$;