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1 Fourier Series and the Fourier Transform

1.1 Orthogonal Systems in the Space $\mathscr{L}^2(X, \mu)$

In the present section, we consider only the norm in the space $\mathscr{L}^2(X, \mu)$. For brevity, we denote it by $\|\cdot\|$ without index.

1.1.1

The norm in the space $\mathscr{L}^2(X, \mu)$ has an important specific feature: just like a norm in a finite dimensional Euclidean space, it is generated by a scalar product. The scalar product of functions f and g belonging to the (in general, complex) space $\mathscr{L}^2(X, \mu)$ is defined by the formula

$$\langle f, g\rangle=\int_X f \bar{g} d \mu$$

(the product $f \bar{g}$ is summable since $2|f \bar{g}| \leqslant|f|^2+|g|^2$).

Obviously, $\langle g, f\rangle=\overline{\langle f, g\rangle}$ and $\langle f, f\rangle=\|f\|^2$. Moreover, by the CauchyBunyakovsky inequality, we have $|\langle f, g\rangle| \leqslant\|f\|\|g\|$, which implies the continuity of the scalar product with respect to convergence in norm. Indeed, if $f_n \underset{n \rightarrow \infty}{\longrightarrow} f$ and $g_n \underset{n \rightarrow \infty}{\longrightarrow} g$, then

$$\begin{aligned}
|\langle f_n, g_n\rangle-\langle f, g\rangle| & \leqslant|\langle f_n-f, g_n\rangle|+|\langle f, g_n-g\rangle| \\
& \leqslant\|f_n-f\|\|g_n\|+\|f\|\|g_n-g\| \underset{n \rightarrow \infty}{\longrightarrow} 0
\end{aligned}$$

From the continuity of the scalar product, it follows that the scalar multiplication of a series convergent in norm by a function can be carried out termwise, $\langle\sum_{n=1}^{\infty} f_n, g\rangle=\sum_{n=1}^{\infty}\langle f_n, g\rangle$. To verify this, it is sufficient to pass to the limit in the equation $\left\langle\sum_{n=1}^k f_n, g\right\rangle=\sum_{n=1}^k\langle f_n, g\rangle$ (the limit on the left-hand side of the equation exists since the series converges and the scalar product is continuous).

We point out one more property of the norm in $\mathscr{L}^2(X, \mu)$, the so-called parallelogram identity

$$\|f+g\|^2+\|f-g\|^2=2\left(\|f\|^2+\|g\|^2\right) \quad\left(f, g \in \mathscr{L}^2(X, \mu)\right)$$

which is connected with the fact that the norm is generated by a scalar product.

The reader can easily verify that if a measure is non-degenerate (more precisely, if there exist two disjoint sets of positive finite measure), then in each space $\mathscr{L}^p(X,\mu)$ with $p \neq 2$ the parallelogram identity is violated.

1.1.2

In the presence of a scalar product, as in a finite-dimensional Euclidean space, we can introduce the notion of the angle between vectors. We are not going to do this in the general setting, instead restricting ourselves to the most important case where the angle is $\pi/2$. We introduce the following definition.

Definition Functions $f, g \in \mathscr{L}^2(X,\mu)$ are called orthogonal if $\langle f, g\rangle = 0$.

We remark that if $\langle f, g\rangle = 0$, then also $\langle g, f\rangle = \overline{\langle g, f\rangle} = 0$, and so the orthogonality relation is symmetric. We denote it by $f \perp g$. A function that is zero almost everywhere is orthogonal to every function in $\mathscr{L}^2(X,\mu)$ and, obviously, the converse is also true. For orthogonal functions the Pythagorean theorem is valid: if $f \perp g$, then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$. This result remains valid for an arbitrary number of pairwise orthogonal summands: if $f_j \perp f_k$ for $j \neq k (j, k = 1, \ldots, n)$, then

$$\|f_1 + \cdots + f_n\|^2 = \|f_1\|^2 + \cdots + \|f_n\|^2 \tag{1}$$

Indeed, since $\langle f_j, f_k\rangle = 0$ for $j \neq k$, we have

$$\|f_1 + \cdots + f_n\|^2 = \langle f_1 + \cdots + f_n, f_1 + \cdots + f_n\rangle = \sum_{j,k=1}^{n} \langle f_j, f_k\rangle = \sum_{k=1}^{n} \|f_k\|^2$$

The Pythagorean theorem is also valid for an "infinite number of summands". If functions $f_1, f_2, \ldots$ are pairwise orthogonal and the series $\sum_{k=1}^{\infty} f_k$ converges, then

$$\left\| \sum_{k=1}^{\infty} f_k \right\|^2 = \sum_{k=1}^{\infty} \|f_k\|^2 \tag{1'}$$

For the proof, it remains only to pass to the limit in Eq. (1).

Due to the scalar product, every n-dimensional space L contained in $\mathscr{L}^2(X,\mu)$ is isomorphic (as a Euclidean space) to $\mathbb{R}^n$ or $\mathbb{C}^n$ (depending on the field of scalars under consideration). Therefore, we can speak of the orthogonal projection of a function f onto a subspace L. In particular, the projection of f onto the one-dimensional subspace generated by the unit vector e, is $\langle f, e\rangle e$.

In the space $\mathscr{L}^2(X,\mu)$, the families of pairwise orthogonal functions play a role similar to that of the orthogonal bases in finite dimensional Euclidean spaces.

Definition A family of functions $\{e_\alpha\}_{\alpha\in A}$ is called an orthogonal system (briefly, OS) if $e_\alpha \perp e_{\alpha'}$ for $\alpha \neq \alpha'$ and $\|e_\alpha\| \neq 0$ for every $\alpha \in A$. An orthogonal system is called orthonormal if $\|e_\alpha\| = 1$ for every $\alpha \in A$.

It follows immediately from the Pythagorean theorem (1) that the functions from an OS are linearly independent. Obviously, dividing each element of an orthogonal system by its norm, we obtain an orthonormal system.

Let the functions $e_1, \ldots, e_n$ form an OS, and let L be the subspace generated by $e_1, \ldots, e_n$ (i.e., the set of all linear combinations of these functions). It is important to know how to find the best approximation to a given function f by elements of L. The following theorem gives a solution of this extremal problem.

Theorem The minimum value of the norm $\|f - \sum_{k=1}^n a_k e_k\|$ is attained if and only if $a_k = c_k(f)$, where

$$c_k(f) = \frac{\langle f, e_k\rangle}{\|e_k\|^2} \quad (k = 1, \ldots, n) \tag{2}$$

The function $f - \sum_{k=1}^n c_k(f) e_k$ is orthogonal to every element of L.

Thus, the function $\sum_{k=1}^n c_k(f) e_k$ is the best approximation for f in the set L. The above-stated theorem can be regarded as a generalization of the following wellknown fact of school geometry: "the perpendicular dropped from a point f to L ", i.e., the difference $f - \sum_{k=1}^n c_k(f) e_k$, is shorter than any "slant" $f - \sum_{k=1}^n a_k e_k$.

Proof We begin with the second assertion of the theorem. We put $S_n = \sum_{k=1}^n c_k(f) e_k$ and verify that $f - S_n \perp \sum_{k=1}^n a_k e_k$. It is sufficient to prove that $f - S_n \perp e_m$ for all $m = 1, \ldots, n$. Indeed,

$$\begin{aligned}\langle f - S_n, e_m\rangle &= \langle f, e_m\rangle - \langle S_n, e_m\rangle = \langle f, e_m\rangle - \sum_{k=1}^n c_k(f)\,\langle e_k, e_m\rangle \\ &= \langle f, e_m\rangle - c_m(f)\,\|e_m\|^2 = 0.\end{aligned}$$

The last equality holds by the definition of $c_m(f)$.

Now, the extremal property of the sum S_n follows from the Pythagorean theorem. Indeed, if $g = \sum_{k=1}^n a_k e_k$ is an arbitrary function L, then $S_n - g \in L$, and, consequently, $f - S_n \perp S_n - g$. Therefore, by the Pythagorean theorem, we obtain

$$\begin{aligned}\|f - g\|^2 &= \|(f - S_n) + (S_n - g)\|^2 = \|f - S_n\|^2 + \|S_n - g\|^2 \\ &= \|f - S_n\|^2 + \sum_{k=1}^n |a_k - c_k(f)|^2\,\|e_k\|^2 .\end{aligned} \tag{3}$$

From this it follows that

$$\left\| f - \sum_{k=1}^n a_k e_k \right\|^2 \geqslant \left\| f - \sum_{k=1}^n c_k(f) e_k \right\|^2$$

and the equality holds only in the case where $a_k = c_k(f)$ for all k.□

For $g = 0$ Eq. (3) takes the form

$$\|f\|^2 = \left\| f - \sum_{k=1}^{n} c_k(f) e_k \right\|^2 + \sum_{k=1}^{n} |c_k(f)|^2 \, \|e_k\|^2$$

and, therefore, the Bessel inequality

$$\sum_{k=1}^{n} |c_k(f)|^2 \, \|e_k\|^2 \leqslant \|f\|^2 \tag{4}$$

holds.

1.1.3

Let $\{e_n\}_{n\in\mathbb{N}}$ be an OS in the space $\mathscr{L}^2(X,\mu)$. Obviously, there are functions in $\mathscr{L}^2(X,\mu)$ that cannot be represented as linear combinations of functions e_n. Therefore, the question naturally arises, what are the conditions under which a function $f \in \mathscr{L}^2(X,\mu)$ is the sum of a series of the form $\sum_{n=1}^{\infty} a_n e_n$. From the theorem proved above it follows that such a series can converge to f only if it coincides with the series $\sum_{n=1}^{\infty} c_n(f) e_n$ whose coefficients are calculated by formula (2). Indeed, Eq. (3) shows that if $a_m \neq c_m(f)$ and $n \geqslant m$, then

$$\left\| f - \sum_{k=1}^{n} a_k e_k \right\| \geqslant |a_m - c_m(f)| \, \|e_m\|^2 > 0$$

and, therefore, the series $\sum_{n=1}^{\infty} a_k e_k$ cannot converge to f.

The series with coefficients calculated by formula (2) play an important role, which justifies the following definition.

Definition Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal system, and let $f \in \mathscr{L}^2(X,\mu)$. The numbers $c_n(f)$ obtained by formula (2) are called the Fourier coefficients, and the series $\sum_{n=1}^{\infty} c_n(f) e_n$ is called the Fourier series of f with respect to the given OS.

As we will see, the Fourier series of an arbitrary function $f \in \mathscr{L}^2(X,\mu)$ converges in the norm $\|\cdot\|$ (but not necessarily to f).

In the case of an orthonormal system, formula (2) becomes simpler and takes the form $c_n(f) = \langle f, e_n \rangle$. If an orthogonal system $\{e_n\}_{n\in\mathbb{N}}$ is not orthonormal, then we can pass to the system $\widetilde{e}_n = e_n / \|e_n\|$ (to "normalize" the given system). The Fourier coefficients, obviously, can change, but the terms of the Fourier series do not change as the following relation shows:

$$c_n(f) e_n = \left\langle f, \frac{e_n}{\|e_n\|} \right\rangle \frac{e_n}{\|e_n\|} = \langle f, \widetilde{e}_n \rangle \, \widetilde{e}_n$$

Thus, the terms of the Fourier series of a function f are simply the projections of f onto the lines generated by the elements of the orthogonal system.

Passing to the limit in Bessel inequality (4) as $n \to \infty$, we obtain the estimate

$$\sum_{k=1}^{\infty} |c_k(f)|^2 \|e_k\|^2 \leqslant \|f\|^2 \qquad (4')$$

also called Bessel's inequality. As follows from $(1')$, inequality $(4')$ becomes an equality if $f = \sum_{n=1}^{\infty} c_n(f)e_n$.

1.1.4

We do not yet know whether a Fourier series converges or, in the case of convergence, what its sum is. The following important theorem establishes that the sum of a Fourier series always exists. As a preliminary, we prove the following lemma.

Lemma Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal system. A series

$$\sum_{n=1}^{\infty} a_n e_n \qquad (5)$$

converges in norm if and only if

$$\sum_{n=1}^{\infty} |a_n|^2 \|e_n\|^2 < +\infty \qquad (5')$$

In the case of convergence, series (5) is the Fourier series of its sum.

Proof Let S_n and T_n be the partial sums of series (5) and (5'), respectively. Then, for all $n, p \in \mathbb{N}$, we have

$$\|S_{n+p} - S_n\|^2 = \left\| \sum_{k=n+1}^{n+p} a_k e_k \right\|^2 = \sum_{k=n+1}^{n+p} |a_k|^2 \|e_k\|^2 = T_{n+p} - T_n$$

It follows that the partial sums of series (5) and (5') are fundamental simultaneously. Since the space $\mathscr{L}^2(X, \mu)$ is complete, we obtain the first assertion of the lemma. The concluding assertion follows from the fact that scalar multiplication of a convergent series by a function can be performed termwise, i.e., if S is the sum of series (5), then the relation

$$\langle S, e_m \rangle = \sum_{n=1}^{\infty} a_n \langle e_n, e_m \rangle = a_m \|e_m\|^2$$

is valid for every $m \in \mathbb{N}$. Thus, $a_m = c_m(S)$ for all m, i.e., series (5) is the Fourier series of its sum.□

Theorem (Riesz-Fischer) For every orthogonal system $\{e_n\}_{n\in\mathbb{N}}$, the Fourier series of a function $f \in \mathscr{L}^2(X, \mu)$ converges in norm and

$$f = \sum_{n=1}^{\infty} c_n(f)e_n + h, \qquad \text{where } h \perp e_n \text{ for all } n \in \mathbb{N}. \qquad (6)$$

Proof By Bessel's inequality, we obtain $\sum_{n=1}^{\infty} |c_n(f)|^2 \|e_n\|^2 \leqslant \|f\|^2 < +\infty$, and so the series $\sum_{n=1}^{\infty} c_n(f)e_n$ converges by the lemma. Let S be its sum. By the second assertion of the lemma, we have $c_n(f) \equiv c_n(S)$. Therefore, the Fourier coefficients of the difference $h = f - S$ are zero, i.e., $h \perp e_n$ for all n.□

1.1.5

Obviously, the sum of the Fourier series may not coincide with the function generating this series. For example, if we replace an OS$e_1, e_2, \ldots$ by the system $e_2, e_3, \ldots$ obtained by deleting the first vector, then the Fourier coefficients of the function e_1 with respect to the new system are zeros, and e_1 is not equal to the sum of its Fourier series (with respect to the new system).

Definition An orthogonal system $\{e_n\}_{n\in\mathbb{N}}$ is called a basis if every function in $\mathscr{L}^2(X, \mu)$ coincides with the sum of its Fourier series almost everywhere.

If $\{e_n\}_{n\in\mathbb{N}}$ is a basis, then, by $(1')$, the relation $f = \sum_{n=1}^{\infty} c_n(f)e_n$ implies that $\|f\|^2 = \sum_{n=1}^{\infty} |c_n(f)|^2 \|e_n\|^2$. Thus, for a basis, the Bessel inequality becomes an equality. We will prove that this property characterizes a basis.

We remark that if $\{e_n\}_{n\in\mathbb{N}}$ is a basis, then the scalar product of two functions can be calculated by their Fourier coefficients since

$$\langle f, g\rangle = \left\langle \sum_{n=1}^{\infty} c_n(f)e_n, g \right\rangle = \sum_{n=1}^{\infty} c_n(f) \langle e_n, g\rangle = \sum_{n=1}^{\infty} c_n(f)\overline{c_n(g)} \|e_n\|^2$$

This relation (as well as the special case where $g = f$) is called Parseval's identity.

We introduce one more important property which, like Parseval's identity, is characteristic for a basis.

Definition A family of functions $\{f_\alpha\}_{\alpha\in A}$ in $\mathscr{L}^2(X, \mu)$ is called complete if the condition

$$f \in \mathscr{L}^2(X, \mu) \quad \text{and} \quad f \perp f_\alpha \quad \text{for every } \alpha \in A$$

implies that $f = 0$ almost everywhere, i.e., $\|f\| = 0$.

Lemma A family $\{f_\alpha\}_{\alpha\in A}$ is complete if the set of all linear combinations of functions contained in this family is everywhere dense, i.e., if, for every function $f \in \mathscr{L}^2(X, \mu)$ and every $\varepsilon > 0$, there exists a linear combination $g = \sum_{k=1}^{n} c_k(f) f_{\alpha_k}$ such that $\|f - g\| < \varepsilon$.

Proof Let $f \perp f_\alpha$ for each α. If $\|f\| \neq 0$, then there is a function $g = \sum_{k=1}^{n} c_k(f) f_{\alpha_k}$ such that $\|f - g\| < \|f\|$. Since $f \perp g$, we obtain a contradiction:

$$\|f\|^2 > \|f - g\|^2 = \|f\|^2 + \|g\|^2 \geqslant \|f\|^2.$$

□

Theorem (On the characterization of bases) Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal system. The following conditions are equivalent:

(1) the system $\{e_n\}_{n\in\mathbb{N}}$ is a basis;

(2) for every function $f \in \mathscr{L}^2(X,\mu)$, Parseval's identity $\sum_{n=1}^{\infty} |c_n(f)|^2 \|e_n\|^2 = \|f\|^2$ holds;

(3) the system $\{e_n\}_{n\in\mathbb{N}}$ is complete.

Proof We prove the chain of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2) This implication was proved just after the definition of a basis.

(2) $\Rightarrow$ (3) Assume that $f \perp e_n$, i.e., $c_n(f) = 0$ for all $n = 1, 2, \ldots$ By hypothesis, $\|f\|^2 = \sum_{n=1}^{\infty} |c_n(f)|^2 \|e_n\|^2 = 0$, which means that the system $\{e_n\}_{n\in\mathbb{N}}$ is complete.

(3) $\Rightarrow$ (1) Let $f \in \mathscr{L}^2(X,\mu)$. By the Riesz-Fischer theorem, $f = g + h$, where $g = \sum_{n=1}^{\infty} c_n(f)e_n$ and $h \perp e_n$ for all n. Since the system is complete, we obtain that $h = 0$ almost everywhere. Taking account of the arbitrariness of f, we obtain that the OS in question is a basis.$\Box$

Comparing the theorem with the preceding lemma, we see that the following statement is valid.

Corollary An orthogonal system $\{e_n\}_{n\in\mathbb{N}}$ is complete if and only if the set of all linear combinations of the functions contained in this system is everywhere dense.

1.1.6

It is useful to generalize the definition of the Fourier series and coefficients. Let $\{e_\alpha\}_{\alpha\in A}$ be an arbitrary OS in the space $\mathscr{L}^2(X,\mu)$, and let $f \in \mathscr{L}^2(X,\mu)$. As above, the numbers $c_\alpha(f) = \frac{\langle f, e_\alpha\rangle}{\|e_\alpha\|^2}$ will be called the Fourier coefficients of the function f with respect to the given OS. Since Bessel's inequality $\sum_{k=1}^{n} |c_{\alpha_k}(f)|^2 \|e_{\alpha_k}\|^2 \leqslant \|f\|^2$ is valid for every finite set of indices $\alpha_1, \ldots, \alpha_n$, the family $\left\{|c_\alpha(f)|^2 \|e_\alpha\|^2\right\}_{\alpha\in A}$ is summable. Therefore, the set A_f of indices of the non-zero coefficients $c_\alpha(f)$ is at most countable, which, after enumeration, can be written in the form $\{\alpha_1, \alpha_2, \ldots\}$. By the Riesz-Fischer theorem, the series $\sum_{k=1}^{\infty} c_{\alpha_k}(f)e_{\alpha_k}$ converges, and its sum will also be called the sum of the Fourier series of f with respect to $\{e_\alpha\}_{\alpha\in A}$. To verify that the sum is well-defined, we must prove that different enumerations of the set A_f give the same sum. A change of enumeration of the set A_f results in a series obtained by rearranging the terms of the series $\sum_{k=1}^{\infty} c_{\alpha_k}(f)e_{\alpha_k}$. Therefore, it is sufficient to prove the following auxiliary statement.

Lemma Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthogonal system and $\omega : \mathbb{N} \to \mathbb{N}$ be a bijection. Then the series (a) $\sum_{n=1}^{\infty} a_n e_n$ and (b) $\sum_{k=1}^{\infty} a_{\omega(k)} e_{\omega(k)}$ converge simultaneously and, in the case of convergence, their sums are equal.

Proof As established in Lemma 1.1.4, series (a) and (b) converge simultaneously with the series $\sum_{n=1}^{\infty} |a_n|^2 \|e_n\|^2$ and $\sum_{k=1}^{\infty} \left|a_{\omega(k)}\right|^2 \left\|e_{\omega(k)}\right\|^2$, respectively. The last two series converge simultaneously because the sum of a positive series is independent of any rearrangement of the terms. This proves that series (a) and (b) converge simultaneously. Now, let series (a) and (b) converge and S_n be a partial sum of (a). By the Pythagorean theorem (see Eq. ($1'$)), we obtain

$$\left\| \sum_{k=1}^{\infty} a_{\omega(k)} e_{\omega(k)} - S_n \right\|^2 = \sum_{\omega(k)>n}^{\infty} \left| a_{\omega(k)} \right|^2 \left\| e_{\omega(k)} \right\|^2 = \sum_{j=n+1}^{\infty} \left| a_j \right|^2 \left\| e_j \right\|^2 \underset{n\to\infty}{\longrightarrow} 0$$

which implies that the sums of series (a) and (b) coincide.□

As in the case of sequences, a family $\{e_\alpha\}_{\alpha\in A}$ is called a basis if every function is the sum of its Fourier series. It can easily be seen that the theorem on the characterization of bases and its corollary remain valid in the more general setting in question.

1.1.7

Let $\{e_k\}_{k\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be orthogonal systems in the spaces $\mathscr{L}^2(X,\mu)$ and $\mathscr{L}^2(Y,v)$, respectively. We use these systems to construct an OS $\{h_{k,n}\}_{k,n\in\mathbb{N}}$ in the space $\mathscr{L}^2(X\times Y,\mu\times\nu)$ by putting

$$h_{k,n}(x,y) = e_k(x)g_n(y) \quad (x\in X, y\in Y)$$

Using Fubini's theorem, we can easily verify that the functions $h_{k,n}$ are squaresummable and pairwise orthogonal. We will prove that the above construction preserves completeness.

Theorem If orthogonal systems $\{e_k\}_{k\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ are complete, then the system $\{h_{k,n}\}_{k,n\in\mathbb{N}}$ is also complete.

Proof Let $f \perp h_{k,n}$ for all $k,n\in\mathbb{N}$. This means that

$$\begin{aligned}&\int_{X\times Y} f(x,y)\overline{e_k(x)g_n(y)}d(\mu\times v)(x,y)\\ &= \int_X \left(\int_Y f(x,y)\overline{g_n(y)}d\nu(y) \right) \overline{e_k(x)}d\mu(x) = 0\end{aligned} \tag{7}$$

for all $k,n\in\mathbb{N}$. We fix an arbitrary n and consider the function

$$x \mapsto \varphi_n(x) = \int_Y f(x,y)\overline{g_n(y)}dv(y)$$

This function is measurable by Corollary 2 to Tonelli's theorem. Moreover, $\varphi_n \in \mathscr{L}^2(X,\mu)$ since

$$|\varphi_n(x)| \leqslant \left(\int_Y |f(x,y)|^2 dv(y) \right)^{1/2} \|g_n\|$$

and, therefore,

$$\int_X |\varphi_n(x)|^2 \, d\mu(x) \leqslant \int_X \left(\int_Y |f(x,y)|^2 d\nu(y) \right) d\mu(x) \, \|g_n\|^2 < +\infty.$$

Equation (7) means that the Fourier coefficients of φ_n with respect to the system $\{e_k\}_{k\in\mathbb{N}}$ are zero. Since the system is complete, we have $\varphi_n(x) = 0$ almost everywhere. Since this is true for all indices n, we have

$$\sum_{n=1}^{\infty} |\varphi_n(x)|^2 = 0 \quad \text{almost everywhere on } X \tag{8}$$

Since $\int_X \int_Y |f(x,y)|^2 d\nu(y) d\mu(x) < +\infty$, Fubini's theorem implies that $\int_Y |f(x,y)|^2 d\nu(y) < +\infty$ almost everywhere. In other words, the function $y \mapsto f_x(y) = f(x,y)$ is square-summable for almost all x. The numbers $\varphi_n(x)$ are simply the Fourier coefficients of this function with respect to the system $\{g_n\}_{n\in\mathbb{N}}$. Since the system $\{g_n\}_{n\in\mathbb{N}}$ is complete, Eq. (8) means that

$$\int_Y |f(x,y)|^2 d\nu(y) = \|f_x\|^2 = \sum_{n=1}^{\infty} |\varphi_n(x)|^2 = 0 \quad \text{almost everywhere on } X.$$

Integrating the above equation over X, we obtain

$$0 = \int_X \int_Y |f(x,y)|^2 d\nu(y) d\mu(x) = \|f\|^2$$

Consequently, $f = 0$ almost everywhere, which proves that the system $\{h_{k,n}\}_{k,n\in\mathbb{N}}$ is complete.□

By induction, the statement just proved can obviously be carried over to the case of more than two orthogonal systems.

1.1.8

Lemma 1.1.4 shows that, for a given orthonormal system, an arbitrary sequence $\{a_n\}_n \geqslant 1$ satisfying the condition $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$ can serve as the sequence of Fourier coefficients of a square-summable function. It is natural to assume that the smaller the class of functions in question, the greater, in general, the rate of decrease of the Fourier coefficients. In Sect. 1.3, we will find more evidence for this conjecture. However, if, instead of square-summable functions, we consider arbitrary bounded functions (assuming, naturally, that they belong to $\mathscr{L}^2(X,\mu)$, i.e., that the measure μ is finite) then our conjecture is false: the Fourier coefficients of bounded functions tend to zero "no faster" than the Fourier coefficients of arbitrary functions from $\mathscr{L}^2$. A more precise formulation of this result of F.L. Nazarov is as follows.

Theorem Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal system in $\mathscr{L}^2(X,\mu), \mu(X) < +\infty$, such that $\int_X |e_n| \, d\mu \geqslant \beta > 0$, where β does not depend on n. Then, for every series $\sum_{n=1}^{\infty} a_n^2 = 1 \, (a_n > 0)$, there exists a measurable function F_a such that $|F_a| \leqslant 1$ and $|c_n(F_a)| \geqslant \theta a_n$ for all n (the coefficient $\theta > 0$ depends only on $\mu(X)$ and β).

We note that the condition $\int_X |e_n| \, d\mu \geqslant \beta > 0$ is certainly fulfilled if the orthogonal system consists of uniformly bounded functions since $1 = \int_X |e_n|^2 \, d\mu \leqslant \|e_n\|_\infty \int_X |e_n| \, d\mu$.

Proof Lets consider only the real case.

For an arbitrary sequence of signs $\varepsilon = \{\varepsilon_n\}$, where $\varepsilon_n = \pm 1$, we construct the sum

$$f_\varepsilon = \sum_{n=1}^{\infty} \varepsilon_n a_n e_n$$

(the series on the right-hand side converges by Lemma 1.1.4). Let A be the set formed by all functions f_ε. This set is compact as a continuous image of the Cantor set (the reader can verify independently the continuity of the mapping that takes a number $\sum_{n=1}^{\infty} t_n 3^{-n}$ ($t_n = 0$ or 2) from the Cantor set to the point $\sum_{n=1}^{\infty} (t_n - 1) a_n e_n$ of the set A).

Now, we consider the function Φ of class $C^2(\mathbb{R})$ such that $|\Phi'|, |\Phi''| \leqslant 1$ (the choice of Φ will be specified later). Since $|\Phi(u)| \leqslant |\Phi(0)| + |u|$ and the measure is finite, the integral $I(f) = \int_X \Phi(f) d\mu$ is finite for every function $f \in \mathscr{L}^2(X, \mu)$. Obviously, the integral continuously depends on f and so, by the Weierstrass extreme value theorem, it assumes its maximum value on A : there exists a sequence of signs $\varepsilon = \{\varepsilon_n\}$ such that $I(f_\varepsilon) \geqslant I(f)$ for every function f in A. We show that the required function has the form $F_a = \Phi'(f_\varepsilon)$ for an appropriate choice of Φ. Since $|F_a| \leqslant \sup |\Phi'| \leqslant 1$, it only remains for us to estimate the Fourier coefficients $c_n(F_a)$. To this end, we use the fact that the replacement of ε_n by $-\varepsilon_n$ leaves a function in the class A and, therefore, does not increase the integral I,

$$\int_X (\Phi(f_\varepsilon) - \Phi(f_\varepsilon - 2\varepsilon_n a_n e_n))\, d\mu \geqslant 0$$

The application of the Taylor formula to the integrand leads to the inequality

$$\int_X \left(2\varepsilon_n a_n e_n \Phi'(f_\varepsilon) - \frac{1}{2} (2\varepsilon_n a_n e_n)^2 \Phi''(g_n) \right) d\mu \geqslant 0 \tag{9}$$

where g_n is a function whose values lie between f_ε and $f_\varepsilon - 2\varepsilon_n a_n e_n$.

Dividing both sides of inequality (9) by $2a_n$, we obtain the required estimate for the Fourier coefficients of the function $F_a = \Phi'(f_\varepsilon)$,

$$|c_n(F_a)| \geqslant \varepsilon_n \int_X e_n \Phi'(f_\varepsilon)\, d\mu \geqslant a_n \int_X e_n^2 \Phi''(g_n)\, d\mu$$

Now, it is necessary to choose Φ so that the integrals $J_n = \int_X e_n^2 \Phi''(g_n)\, d\mu$ be separated from zero. If we take an antiderivative of $\frac{2}{\pi} \arctan u$ as Φ, then

$$J_n = \frac{2}{\pi} \int_X \frac{e_n^2}{1 + g_n^2} d\mu$$

To estimate this integral, we use the Cauchy-Bunyakovsky inequality,

$$\beta \leqslant \int_X |e_n|\, d\mu = \int_X \frac{|e_n|}{\sqrt{1 + g_n^2}} \cdot \sqrt{1 + g_n^2} d\mu \leqslant \sqrt{\frac{\pi}{2} J_n} \cdot \sqrt{\int_X (1 + g_n^2)\, d\mu}$$

Since $|g_n| \leqslant |f_\varepsilon| + |f_\varepsilon - 2\varepsilon_n a_n e_n|$, we obtain
$\int_X g_n^2 d\mu \leqslant 2\left(\|f_\varepsilon\|^2 + \|f_\varepsilon - \ 2\varepsilon_n a_n e_n\|^2\right) = 4.$
Therefore, $J_n \geqslant \theta = \frac{2\beta^2}{\pi(\mu(X)+4)}$ and $|c_n(F_a)| \geqslant \theta a_n$ for all n.□

1.2 Examples of Orthogonal Systems

Throughout this section, we consider the convergence of Fourier series only with respect to the $\mathscr{L}^2$-norm, which is denoted by $\|\cdot\|$. Instead of $\mathscr{L}^2(X, \lambda_m)$, where $X \subset \mathbb{R}^m$, we will write briefly $\mathscr{L}^2(X)$, omitting the indication of a measure.

1.2.1

Trigonometric Systems. The most important orthogonal systems are the following real and complex trigonometric systems in the space $\mathscr{L}^2((a, a+2\ell))$:

$$1, \quad \cos\frac{\pi x}{\ell}, \quad \sin\frac{\pi x}{\ell}, \ldots, \quad \cos\frac{\pi n x}{\ell}, \quad \sin\frac{\pi n x}{\ell}, \ldots \quad \text{and } \left\{e^{\frac{\pi i n x}{\ell}}\right\}_{n\in\mathbb{Z}}$$

The Fourier series with respect to these systems have, respectively, the form

$$A(f) + \sum_{n=1}^{\infty}\left(a_n(f)\cos\frac{\pi n x}{\ell} + b_n(f)\sin\frac{\pi n x}{\ell}\right) \text{ and } \sum_{n=-\infty}^{\infty} c_n(f) e^{\frac{\pi i n x}{\ell}},$$

where the Fourier coefficients are calculated by the formulas

$$A(f) = \frac{1}{2\ell}\int_a^{a+2\ell} f(x)dx,$$
$$a_n(f) = \frac{1}{\ell}\int_a^{a+2\ell} f(x)\cos\frac{\pi n x}{\ell}dx,$$
$$b_n(f) = \tfrac{1}{\ell}\textstyle\int_a^{a+2\ell} f(x)\sin\frac{\pi n x}{\ell}dx \quad (n\in\mathbb{N});$$
$$c_n(f) = \tfrac{1}{2\ell}\textstyle\int_a^{a+2\ell} f(x)e^{-\frac{\pi i n x}{\ell}}dx \quad (n\in\mathbb{Z}).$$

In the study of Fourier series, we may assume that the functions are defined on the intervals of the form $(0, 2\ell)$, because the general case can be reduced to the case $a = 0$ by a translation. It is often convenient to use a symmetric interval $(-\ell, \ell)$.

The study of Fourier series with respect to a trigonometric system with some period can be reduced to the study of Fourier series with a different period. Following tradition, we will consider (with rare exceptions) only the Fourier series

$$A(f) + \sum_{n=1}^{\infty}(a_n(f)\cos nx + b_n(f)\sin nx) \quad \text{ and } \sum_{n=-\infty}^{\infty} c_n(f)e^{inx}$$

with respect to more natural and convenient 2π-periodic systems

$$1, \quad \cos x, \quad \sin x, \quad \ldots, \quad \cos nx, \quad \sin nx, \quad \ldots \quad \text{and } \left\{e^{inx}\right\}_{n\in\mathbb{Z}} \tag{T}$$

In this case, the Fourier coefficient $c_n(f)$ will also be denoted by the symbol $\widehat{f}(n)$. Thus,

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \quad (n \in \mathbb{Z})$$

The transition from the expansion in one system to the expansion in a different system proceeds as follows. For a function $f \in \mathscr{L}^2((0,2\ell))$, we define a function g by putting $g(y) = f\left(\frac{\ell}{\pi}y\right)$, where $y \in (0,2\pi)$. It is clear that $g \in \mathscr{L}^2((0,2\pi))$. There is an obvious relation connecting the Fourier coefficients of these functions (with respect to the corresponding systems):

$$c_k(f) = \frac{1}{2\ell}\int_0^{2\ell} f(x)e^{-\frac{\pi ikx}{\ell}}dx = \frac{1}{2\pi}\int_0^{2\pi} f\left(\frac{\ell}{\pi}y\right)e^{-iky}dy = \widehat{g}(k)$$

for each $k \in \mathbb{Z}$. Consequently,

$$\sum_{|k|\leqslant n} c_k(f)e^{\frac{\pi ikx}{\ell}} = \sum_{|k|\leqslant n} \widehat{g}(k)e^{\frac{\pi ikx}{\ell}} = \sum_{|k|\leqslant n} \widehat{g}(k)e^{iky}$$

i.e., the partial sums of the Fourier series of the functions f and g at the corresponding points coincide. From this, it follows, in particular, that both series converge simultaneously and their sums coincide (or do not coincide) simultaneously with the values of the functions f and g. Thus, the transition from f to g makes it possible to reduce the study of a Fourier series in a system with an arbitrary period to the study of a Fourier series in the 2π-periodic system.

By Euler's formula, the systems (T) are tightly connected with each other: their linear spans coincide (the functions from these spans are called trigonometric polynomials), and the Fourier coefficients in one system are expressed in terms of the Fourier coefficients in the other one by the following formula:

$$\widehat{f}(\pm n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)(\cos nx \mp i\sin nx)dx = \frac{a_n(f) \mp ib_n(f)}{2} \quad (n \in \mathbb{N})$$

and

$$a_n(f) = \widehat{f}(n) + \widehat{f}(-n) \quad \text{and} \quad b_n(f) = i(\widehat{f}(n) - \widehat{f}(-n)) \quad (n \in \mathbb{N})$$

It follows that the Fourier series in systems (T) essentially coincide. More precisely, the relation

$$A(f) + \sum_{k=1}^{n} (a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=-n}^{n} \widehat{f}(k)e^{ikx}$$

showing that the partial sums of the Fourier series in the real system (T) coincides with symmetric partial sums of the Fourier series in the complex system, is valid for each n.

In the following theorem, we establish one of the most important properties of the systems (T).

Theorem The real and complex trigonometric systems form bases in $\mathscr{L}^2((0, 2\pi))$.

Proof The assertion of the theorem follows immediately from Corollary 1.1.5.

Since each of the systems (T) is a basis, it satisfies Parseval's identity: if $f, g \in \mathscr{L}^2((0, 2\pi))$, then

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}dx &= A(f)\overline{A(g)} + \frac{1}{2}\sum_{n=1}^{\infty}\Big(a_n(f)\overline{a_n(g)} + b_n(f)\overline{b_n(g)}\Big)\\ &= \sum_{n=-\infty}^{\infty} \widehat{f}(n)\overline{\widehat{g}(n)}\end{aligned}$$

in particular, every function f in $\mathscr{L}^2((0, 2\pi))$ satisfies the equation

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 dx = |A(f)|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\Big(|a_n(f)|^2 + |b_n(f)|^2\Big) = \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2$$

which is often called the closeness relation. As we have already noted, in these formulas and in the theorem, the interval $(0, 2\pi)$ can be replaced by an arbitrary interval of length 2π, in particular, by $(-\pi, \pi)$.

We will now give several examples that illustrate the importance of this formula.

Example 1 Let $f(x) = x$ for $x \in (-\pi, \pi)$. The Fourier series of this function has the form $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{n}\sin nx$. By Parseval's identity, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} |b_n(f)|^2 = 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Thereby we have arrived at the following result first obtained by Euler: $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. The same reasoning applied to the function $f(x) = x^2 (|x| \leqslant \pi)$ gives another result of Euler's: $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

Example 2 As we must know, 2π-periodic functions in $\tilde{\mathscr{L}}^2$, i.e., square integrable functions on $(-\pi, \pi)$ are continuous in mean. By the closeness equation, we can obtain an exact value for the deviation of a function from its translation.

We will assume that a function $f \in \mathscr{L}^2((-\pi, \pi))$ is extended by periodicity from $[-\pi, \pi]$ to $\mathbb{R}$. Let $h \in \mathbb{R}$, and let f_h be the corresponding translation of f, i.e., $f_h(x) = f(x-h)$ for $x \in \mathbb{R}$. It can easily be verified that $\widehat{f_h}(k) = e^{-ikh}\widehat{f}(k)$. Therefore, by Parseval's identity, we obtain

$$\|f_h - f\|^2 = 2\pi \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 \left|e^{-ikh} - 1\right|^2 = 8\pi \sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^2 \sin^2 \frac{kh}{2}.$$

From this formula, the continuity in the mean, $f_h \underset{h\to 0}{\longrightarrow} f$, follows directly.

Example 3 We apply Parseval's identity to prove an elegant inequality, which, in some cases, makes it possible to estimate from above the mean value of a function on an interval by its mean value on a smaller interval.

Let the Fourier coefficients of a function φ in $\mathscr{L}^2((-\pi,\pi))$ be non-negative. Then the inequality

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi(t)|^2 dt \leqslant \frac{3}{2\alpha}\int_{-\alpha}^{\alpha} |\varphi(t)|^2 dt$$

is valid for every $\alpha \in (0,\pi)$.

Since the function $h(t) = \left(1 - \frac{|t|}{\alpha}\right)_+$ does not exceed 1 , it is sufficient for us to estimate the integral $I = \int_{-\pi}^{\pi} |\varphi(t)h(t)|^2 dt$ from below. The product $f = \varphi h$, obviously, belongs to $\mathscr{L}^2((-\pi,\pi))$. We calculate its Fourier coefficients (in what follows, $e_n(t) = e^{int}$),

$$\widehat{f}(n) = \frac{1}{2\pi}\langle \varphi h, e_n\rangle = \frac{1}{2\pi}\langle \varphi, he_n\rangle = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \widehat{\varphi}(k)\langle e_k, he_n\rangle =$$
$$= \sum_{k=-\infty}^{\infty} \widehat{\varphi}(k)\widehat{h}(n-k) = \sum_{k+j=n} \widehat{\varphi}(k)\widehat{h}(j)$$

Now, by Parseval's identity we obtain

$$I = 2\pi \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = 2\pi \sum_{n=-\infty}^{\infty} \left|\sum_{k+j=n} \widehat{\varphi}(k)\widehat{h}(j)\right|^2$$

A direct calculation shows that $\widehat{h}(j) \geqslant 0$ for all $j \in \mathbb{Z}$. Therefore, replacing the square of the sum by the sum of squares (here we use the inequalities $\widehat{\varphi}(k) \geqslant 0$), we obtain

$$I \geqslant 2\pi \sum_{n=-\infty}^{\infty} \sum_{k+j=n} \widehat{\varphi}^2(k)\widehat{h}^2(j) = 2\pi \sum_{k=-\infty}^{\infty} \widehat{\varphi}^2(k) \sum_{j=-\infty}^{\infty} \widehat{h}^2(j) =$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi(t)|^2 dt \int_{-\pi}^{\pi} h^2(t) dt = \frac{\alpha}{3\pi}\int_{-\pi}^{\pi} |\varphi(t)|^2 dt.$$

Thus,

$$\int_{-\alpha}^{\alpha} |\varphi(t)|^2 dt \geqslant I \geqslant \frac{\alpha}{3\pi} \int_{-\pi}^{\pi} |\varphi(t)|^2 dt$$

Example 4 Hurwitz found an unexpected application of trigonometric series. It turns out that they can be used to obtain a very simple proof of the classical isoperimetric inequality connected with the problem of determining a closed plane curve that has a given circumference L and bounds a figure of the largest area. This inequality has the form

$$4\pi S \leqslant L^2$$

where S is the area of the figure. The equality is attained only in the case where the curve is a circle.

The proof given by Hurwitz is analytic. It uses only the closeness equation and the formula for the area in terms of a curvilinear integral.

Let $K \subset \mathbb{R}^2$ be a compact set whose boundary is a closed smooth curve. Without loss of generality, we may assume that the length of the curve is 2π. Let $z(t) = (x(t), y(t)), 0 \leqslant t \leqslant 2\pi$ be the natural parametrization of the curve ∂K. Then $z(0) = z(2\pi)$ because the curve ∂K is closed and $|z'(t)| \equiv 1$ because the parametrization is natural.

Using the closeness equation and the identity $|z'(t)| \equiv 1$, we can represent the relation $L = 2\pi$ in the form

$$L^2 = 2\pi \int_0^{2\pi} |z'(t)|^2 \, dt = 4\pi^2 \sum_{n\in\mathbb{Z}} |\hat{z}'(n)|^2 \tag{10}$$

To calculate the area $S = \lambda_2(K)$, we apply the relation

$$S = \frac{1}{2} \int_{\partial^+ K} (-y dx + x dy) = \frac{1}{2} \int_0^{2\pi} (x(t)y'(t) - y(t)x'(t)) \, dt$$

which follows from Green's formula with $P(x,y) = -y$ and $Q(x,y) = x$. Since $x(t)y'(t) - y(t)x'(t) = \mathcal{I}m \left(z'(t)\overline{z(t)} \right)$ and $\int_0^{2\pi} \mathcal{R}e\,(z'(t)z(t))\, dt = \int_0^{2\pi} \left(x^2(t) + y^2(t) \right)' dt = 0$, we have

$$S = \frac{1}{2i} \int_0^{2\pi} z'(t)\overline{z(t)} dt$$

Transforming the integral by Parseval's identity, we obtain

$$S = -\pi i \sum_{n\in\mathbb{Z}} \widehat{z}'(n)\overline{\widehat{z}(n)} \tag{11}$$

Now, we eliminate the Fourier coefficients of the derivative from Eqs. (10) and (11), expressing them in terms of the Fourier coefficients of the function z. Integrating by parts and taking into account that $z(0) = z(2\pi)$, we have

$$\widehat{z}'(n) = \tfrac{1}{2\pi} \int_0^{2\pi} z'(t) e^{-int} dt = \tfrac{1}{2\pi} z(t) e^{-int} \Big|_{t=0}^{2\pi} + \tfrac{in}{2\pi} \int_0^{2\pi} z(t) e^{-int} dt = in\widehat{z}(n).$$

Substituting the resulting expressions for $\widehat{z}'(n)$ in (10) and (11), we obtain

$$L^2 = 4\pi^2 \sum_{n\in\mathbb{Z}} n^2 |\widehat{z}(n)|^2 \quad \text{and} \quad S = \pi \sum_{n\in\mathbb{Z}} n|\widehat{z}(n)|^2$$

Consequently,

$$L^2 - 4\pi S = 4\pi^2 \sum_{n\in\mathbb{Z}} \left(n^2 - n\right) |\widehat{z}(n)|^2 \geqslant 0$$

which proves the isoperimetric inequality. Moreover, the last formula implies that the equality holds only if $\widehat{z}(n) = 0$ for $n \neq 0$, 1, i.e., only if $z(t) = \widehat{z}(0) + \widehat{z}(1)e^{it}$. We have $|\widehat{z}(1)| = 1$, since $|z'(t)| \equiv 1$. Thus, the curve of length 2π for which the isoperimetric inequality becomes an equality is the unit circle $|z - \widehat{z}(0)| = 1$.

1.2.2

Considering the product of m copies of the complex trigonometric system (see Sect. 1.1.7), we obtain its multi-dimensional analog in the space $\mathscr{L}^2(Q)$, where $Q = (-\pi, \pi)^m$ (a multi-dimensional version of the real trigonometric system is quite cumbersome and we do not consider it). The new system consists of the complex exponential functions e_n numbered by multi-indices $n = (n_1, \ldots, n_m)$:

$$e_n(x) = e^{i\langle n,x\rangle}, \quad \text{where } x \in Q, n \in \mathbb{Z}^m$$

The Fourier coefficients of a function $f \in \mathscr{L}^2(Q)$ in this system are calculated by the formulas

$$\widehat{f}(n) = \frac{\langle f, e_n\rangle}{\|e_n\|^2} = \frac{1}{(2\pi)^m} \int_Q f(x) e^{-i\langle n,x\rangle} dx \quad (n \in \mathbb{Z}^m)$$

From Theorem 1.1.7, it follows that the system $\left\{e^{i\langle n,x\rangle}\right\}_{n\in\mathbb{Z}^m}$ is complete, which implies Parseval's identity

$$\int_Q f(x)\overline{g(x)}dx = (2\pi)^m \sum_{n\in\mathbb{Z}^m} \widehat{f}(n) \cdot \overline{\widehat{g}(n)}, \quad f, g \in \mathscr{L}^2(Q).$$

Of course, the cube $Q = (-\pi, \pi)^m$ in the two last formulas can be replaced by a shifted cube.

Example Let $0 < \rho \leqslant \pi$. We consider the function $f \in \mathscr{L}^2\left((-\pi, \pi)^3\right)$ that is equal to $1/\|x\|$ for $\|x\| < \rho$ and vanishes on $(-\pi, \pi)^3 \backslash B(0, \rho)$. Its norm is easily calculated in spherical coordinates,

$$\|f\|^2 = \int_{B(0,\rho)} \frac{1}{\|x\|^2} dx = 4\pi \int_0^\rho \frac{1}{r^2} r^2 dr = 4\pi\rho.$$

To calculate the Fourier coefficients, we use the formula

$$\widehat{f}(y) = \frac{2}{\|y\|} \int_0^\infty f_0(r) r \sin(2\pi r\|y\|) dr$$

with $f_0(r)=1/r$ on $(0,\rho)$, $f_0(r)=0$ for $r\geqslant\rho$ and $y=n/2\pi$:

$$\begin{aligned}
\widehat{f}(n) &= \frac{1}{(2\pi)^3}\int_{B(0,\rho)}\frac{1}{\|x\|}e^{-i\langle n,x\rangle}dx=\frac{1}{2\pi^2\|n\|}\int_0^\rho\frac{1}{r}r\sin(\|n\|r)dr\\
&=\left(\frac{\sin\frac{\rho}{2}\|n\|}{\pi\|n\|}\right)^2
\end{aligned}$$

if $n\neq 0$ and $\widehat{f}(0)=\rho^2/4\pi^2$. By Parseval's identity for the function f, we obtain

$$4\pi\rho=(2\pi)^3\sum_{n\in\mathbb{Z}^3}\left(\frac{\sin\frac{\rho}{2}\|n\|}{\pi\|n\|}\right)^4$$

Thus, the identity

$$\frac{\pi^2}{t^3}=\sum_{n\in\mathbb{Z}^3}\left(\frac{\sin\|n\|t}{\|n\|t}\right)^4$$

is valid for $t=\frac{\rho}{2}\in\left(0,\frac{\pi}{2}\right]$ (the summand for $n=0$ is equal to 1).

1.2.3

The trigonometric system is closely connected with the orthogonal system $\{z^n\}_{n\in\mathbb{Z}}$ in the space $\mathscr{L}^2\left(S^1,\sigma\right)$, where $S^1=\{z\in\mathbb{C}||z|=1\}$ is the unit circle and σ is the arc length. Knowing that the trigonometric system is complete in $\mathscr{L}^2((-\pi,\pi))$, we use the change of variable $z=e^{ix}(-\pi<x<\pi)$ and easily verify that the system $\{z^n\}_{n\in\mathbb{Z}}$ is complete in $\mathscr{L}^2\left(S^1,\sigma\right)$. Therefore, every function f in this space is the sum of the series $\sum_{n\in\mathbb{Z}}c_nz^n$, where $c_n=\frac{1}{2\pi}\int_{S^1}f(z)\bar{z}^n d\sigma(z)$. Easily to see that this formula coincides with the formula for the nth coefficient of the Laurent expansion of f in the annulus $r<|z|<R$, where $r<1<R$. Therefore, the Fourier series in the system $\{z^n\}_{n\in\mathbb{Z}}$ can be regarded as the limit form of the Laurent series, when the annulus degenerates to a circle.

We consider an example connected with the system $\{z^n\}_{n\in\mathbb{Z}}$.
Let $T:S^1\rightarrow S^1$ be a rotation of the circle, i.e., the map $z\mapsto T(z)=\zeta z$, where $\zeta\in S^1$ is a fixed number. We now address the question of how much the points of the circle "mix" under the iterations of T. Does there exist an invariant subset of the circle, that is, a set which retains all of its points after rotation? More precisely, a set $E\subset S^1$ is called invariant if it differs from its image only on a set of measure zero, i.e., if $\chi_E=\chi_{T(E)}$ almost everywhere. Of course, such sets exist: the circle S^1 and the set $\{\zeta^n\}_{n\in\mathbb{Z}}$ are examples. It is easy to construct more examples of invariant sets of measure 2π or zero. Therefore, we are interested in the question of whether there are non-trivial invariant sets, i.e., sets satisfying the condition $0<\sigma(E)<2\pi$. If $\zeta^m=1$ for some m, then the map T is repeated after m iterations $\left(T^{m+1}=T\right)$, and a non-trivial invariant subspace can easily be constructed. We leave this construction to the reader.

However, if ζ is not a root of unity, then the map T has no non-trivial invariant sets (such maps are called ergodic). Let us prove this.

Let $E \subset S^1$ be an invariant set. Then $\chi_E = \chi_{T(E)}$ almost everywhere, and therefore, $c_n\left(\chi_{T(E)}\right) = c_n\left(\chi_E\right)$. At the same time, by a change of variable, we obtain

$$c_n\left(\chi_{T(E)}\right) = \frac{1}{2\pi}\int_{T(E)} \bar{z}^n d\sigma(z) = \frac{1}{2\pi}\int_E \overline{(\zeta z)}^n d\sigma(z) = \zeta^{-n} c_n\left(\chi_E\right).$$

Thus, $c_n\left(\chi_E\right)\left(1-\zeta^{-n}\right) = 0$ for all $n \in \mathbb{Z}$. Since $1-\zeta^{-n} \neq 0$ for $n \neq 0$, it follows that all Fourier coefficients of χ_E, except, possibly, $c_0\left(\chi_E\right)$, are zero. Since the system $\{z^n\}_{n\in\mathbb{Z}}$ is complete, the function χ_E coincides with the sum of its Fourier series almost everywhere. Therefore, χ_E is a constant almost everywhere. Consequently, either $\chi_E(x) = 0$ almost everywhere (the invariant set has measure zero) or $\chi_E(x) = 1$ almost everywhere (the invariant set is a set of full measure).

1.2.4

We will now give other examples of orthogonal systems. Let $P_n(x) = \left(\left(x^2-1\right)^n\right)^{(n)}$, $n = 0, 1, \ldots$ The polynomials P_n are called the Legendre polynomials. Obviously, $\deg P_n = n$, and so every polynomial is a linear combination of Legendre polynomials, which form an orthogonal system in the space $\mathscr{L}^2((-1,1))$. Indeed, for $m < n$, we have

$$\begin{aligned}
\langle P_m, P_n\rangle &= \int_{-1}^{1} P_m(x)\left(\left(x^2-1\right)^n\right)^{(n)} dx \\
&= P_m(x)\left(\left(x^2-1\right)^n\right)^{(n-1)}\Big|_{-1}^{1} - \int_{-1}^{1} P_m'(x)\left(\left(x^2-1\right)^n\right)^{(n-1)} dx \\
&= -\int_{-1}^{1} P_m'(x)\left(\left(x^2-1\right)^n\right)^{(n-1)} dx
\end{aligned}$$

Integrating by parts n times, we arrive at the equation

$$\langle P_m, P_n\rangle = (-1)^n \int_{-1}^{1} P_m^{(n)}(x)\left(x^2-1\right)^n dx$$

where $P_m^{(n)}(x) \equiv 0$, since $\deg P_m < n$. Thus, $\langle P_m, P_n\rangle = 0$ for $m \neq n$.

Theorem The Legendre polynomials form a basis in the space $\mathscr{L}^2((-1,1))$.

Proof As in the proof of Theorem 1.2.1, we use Corollary 1.1.5. We must verify that every function in $\mathscr{L}^2((-1,1))$ can be approximated arbitrarily closely (in the $\mathscr{L}^2$-norm) by linear combinations of polynomials P_n, i.e., by arbitrary algebraic polynomials.

We mention one more useful orthogonal system. In the space $\mathscr{L}^2(\mathbb{R})$, we consider the Hermite functions

$$h_n(x) = e^{x^2/2}\left(e^{-x^2}\right)^{(n)}, \quad n = 0, 1, \ldots$$

It is easy to verify that $h_n(x) = H_n(x)e^{-x^2/2}$, where H_n is an nth degree polynomial called a Hermite polynomial. The orthogonality of the Hermite functions can be established by integrating by parts the equation

$$\langle h_m, h_n\rangle = \int_{-\infty}^{\infty} H_m(x)\left(e^{-x^2}\right)^{(n)} dx$$

in the same way as in the proof of the orthogonality of the Legendre polynomials. It is obvious that the orthogonality of the Hermite functions in $\mathscr{L}^2(\mathbb{R})$ implies the orthogonality of the Hermite polynomials in $\mathscr{L}^2(\mathbb{R}, \mu)$ with measure $d\mu(x) = e^{-x^2}dx$.

Later on (see the corollary in Sect. 1.4.6) we prove that the system of functions h_n is complete in $\mathscr{L}^2(\mathbb{R})$ or, equivalently, the system of polynomials H_n is complete in $\mathscr{L}^2(\mathbb{R}, \mu)$.

1.2.5

In the applications of probability theory in analysis, the sequence of Rademacher functions r_n plays an important role. The Rademacher functions $r_n (n \in \mathbb{N})$ are defined on $\mathbb{R}$ by the formula $r_n(x) = \operatorname{sign} \sin 2^n \pi x$ As has already been proved, these functions are independent. Since, in addition, $\int_0^1 r_n(x)dx = 0$, we see that the relation

$$\int_0^1 r_{n_1}(x)r_{n_2}(x)\cdots r_{n_m}(x)dx = \prod_{k=1}^{m}\int_0^1 r_{n_k}(x)dx = 0 \tag{12}$$

holds for $1 \leqslant n_1 < n_2 < \cdots < n_m$.

In particular, the Rademacher functions form an orthonormal system in the space $\mathscr{L}^2((0,1))$. Of course, this system is not complete: for example, the pairwise products $r_j r_k$ are orthogonal to all Rademacher functions. To obtain a complete system containing the Rademacher functions, we proceed as follows. For every non-empty finite set $A \subset \mathbb{N}$, we consider the function $w_A = \prod_{n\in A} r_n$. Furthermore, we will assume, by definition, that $w_\varnothing \equiv 1$. The functions w_A are called the Walsh functions. The Rademacher functions are the Walsh functions corresponding to the oneelement sets. By Eq. (12), the functions w_A are pairwise orthogonal. The system of Walsh functions is complete in $\mathscr{L}^2((0,1))$. To prove this, we need the following lemma.

Lemma Let $n \in \mathbb{N}$. The set of linear combinations of the functions w_A such that $A \subset \{1, 2, 3, \ldots, 2^n\}$ coincides with the set of linear combinations of the characteristic functions of the intervals $\Delta_{n,k} = (k2^{-n}, (k+1)2^{-n})$ for $k = 0, 1, \ldots, 2^n - 1$.

Proof Let L_1 and L_2 be the linear spans of the first and second systems, respectively. Since the functions $r_1, \ldots, r_n$ are constant on the intervals $\Delta_{n,k}$, the Walsh functions in question are also constant on these intervals. Therefore,

$L_1 \subset L_2$. At the same time, the dimensions of L_1 and L_2 are, obviously, equal (to 2^n). Hence it follows that $L_1 = L_2$.

Theorem The system of Walsh functions is complete in the space $\mathscr{L}^2((0,1))$.

Proof We use Corollary to Theorem 1.1.5 on the characterization of bases. We will prove that every function f in $\mathscr{L}^2((0,1))$ can be approximated arbitrarily closely in norm by linear combinations of Walsh functions. If f is the characteristic function of an interval $(p,q) \subset (0,1)$, then, for a given ε, we can find a large n such that p and q can be approximated by the points $j/2^n$ and $k/2^n$ within ε. Then $\|f - \chi_\Delta\|^2 < 2\varepsilon$, where χ_Δ is the characteristic function of the interval $(j/2^n, k/2^n)$, which almost everywhere coincides with the sum $\sum_{s=j}^{k-1} \chi_{\Delta_{n,s}}$ equal, by the lemma, to a certain linear combination of Walsh functions. Being able to approximate the characteristic functions of the intervals, we can also approximate their linear combinations, i.e., the step functions. Now, we consider the general case. For each ε, we can find a step function g such that $\|f-g\| < \varepsilon$. Approximating g within ε by a linear combination h of Walsh functions, we obtain $\|f-h\| \leqslant \|f-g\| + \|g-h\| < 2\varepsilon$. Since ε was arbitrary, this completes the proof.

1.2.6

From the viewpoint of probability theory, the Rademacher functions give an example of a sequence of independent trials with two equiprobable outcomes (the simplest "Bernoulli scheme"). Here, a "simple" random event is a roll of a number $x \in (0,1)$, and the probability that a point will fall in the interval (p,q) is the length of the interval. A "trial" consists of the calculation of the values of the Rademacher functions: the first trial is the calculation of $r_1(x)$, the second trial is the calculation of $r_2(x)$, etc. Taking into account the connection between the values of a Rademacher function at a given point and the binary digits of the point, we can replace $r_n(x)$ by $\varepsilon_n(x)$ (the binary digits of x).

One of the first results of probability theory is Bernoulli's law of large numbers, which says that in the scheme described above the frequency of occurrence of 0 or 1 becomes close to 1/2 with probability arbitrarily close to 1 . In the language of measure theory, this result means that, on the interval $(0,1)$, the arithmetic mean $\frac{1}{n}\left(\varepsilon_1(x) + \cdots + \varepsilon_n(x)\right)$ (the frequency of occurrence of the digit 1 in the binary expansion of a point x) tends to 1/2 in measure. Returning to the Rademacher functions, we can say that

$$\frac{r_1(x) + \cdots + r_n(x)}{n} \underset{n\to\infty}{\longrightarrow} 0 \text{ in measure.}$$

This assertion follows from the fact that

$$\frac{1}{n}\|r_1 + \cdots + r_n\| = \frac{1}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} 0$$

and the convergence in norm implies the convergence in measure.

Two centuries after Bernoulli, Borel proved a stronger statement.

Theorem (Strong law of large numbers)

$$\frac{r_1(x) + \cdots + r_n(x)}{n} \underset{n\to\infty}{\longrightarrow} 0 \text{ almost everywhere on } (0,1).$$

Proof We put $S_n(x) = r_1(x)+\cdots+r_n(x)$ and estimate the integral $\int_0^1 S_n^4(x)dx$. Obviously,

$$S_n^2(x) = \sum_{k=1}^{n} r_k^2(x) + 2 \sum_{1\leqslant j<k\leqslant n} r_j(x)r_k(x) = n + 2 \sum_{1\leqslant j<k\leqslant n} w_{\{j,k\}}(x).$$

Since the Walsh functions $w_{\{j,k\}}$ form an orthonormal system, the Pythagorean theorem implies

$$\int_0^1 S_n^4(x)dx = \left\| n w_\varnothing + 2 \sum_{1\leqslant j<k\leqslant n} w_{\{j,k\}} \right\|^2 = n^2 + 4 \sum_{1\leqslant j<k\leqslant n} 1 < 3n^2.$$

Consequently, $\sum_{n=1}^{\infty} \int_0^1 \left(\frac{1}{n} S_n(x)\right)^4 < \sum_{n=1}^{\infty} \frac{3}{n^2} < +\infty$, and, therefore, the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} S_n(x)\right)^4$ converges almost everywhere. This implies the assertion of the theorem since the terms of a convergent series tend to zero.

1.2.7

The theorem just proved admits various generalizations also called the strong laws of large numbers. The statements concerning sequences of independent functions with zero mean values (obviously, these functions form an orthogonal system) are of most interest. Before passing to this question, we consider an inequality playing a decisive role in the study of series of such functions.

Throughout this section, we consider real functions in the space $\mathscr{L}^2(X,\mu)$, assuming that the measure μ is normalized ($\mu(X) = 1$).

Theorem (Kolmogorov's inequality) Let $f_1, \ldots, f_n$ in $\mathscr{L}^2(X,\mu)$ be independent and have zero means, $\int_X f_1 d\mu = \cdots = \int_X f_n d\mu = 0$. Then the inequality

$$\mu\left(\left\{x \in X \,\middle|\, \max_{1\leqslant k\leqslant n} \left| f_1(x) + \cdots + f_k(x) \right| \geqslant t \right\}\right) \leqslant \frac{1}{t^2} \sum_{k=1}^{n} \int_X f_k^2 d\mu$$

holds for every $t > 0$.

Proof We put $S_k = f_1 + \cdots + f_k, S_k^* = \max_{1\leqslant j\leqslant k} |S_j|$ and $R_k = S_n - S_k$. We need to estimate the measure of the set $E = \{x \in X \mid S_n^*(x) \geqslant t\}$. To this end, we divide the set into disjoint parts $E_k = \left\{x \in X \mid S_{k-1}^*(x) < t \leqslant S_k^*(x)\right\}$ (we assume that $S_0^* \equiv 0$). Then

$$\sum_{k=1}^{n}\int_X f_k^2 d\mu = \int_X S_n^2 d\mu \geqslant \int_E S_n^2 d\mu = \sum_{k=1}^{n}\int_{E_k} (S_k+R_k)^2\, d\mu =$$
$$= \sum_{k=1}^{n}\left(\int_{E_k} S_k^2 d\mu + 2\int_{E_k} S_k R_k d\mu + \int_{E_k} R_k^2 d\mu\right) \geqslant$$
$$\geqslant \sum_{k=1}^{n}\int_{E_k} S_k^2 d\mu + 2\sum_{k=1}^{n}\int_{E_k} S_k R_k d\mu.$$

The functions $S_k\chi_{E_k}$ and R_k are independent. Therefore,

$$\int_{E_k} S_k R_k d\mu = \int_X S_k\chi_{E_k} R_k d\mu = \int_X S_k\chi_{E_k}\, d\mu \cdot \int_X R_k d\mu = 0.$$

Since $|S_k| = S_k^* \geqslant t$ on the set E_k, we obtain the required inequality,

$$\sum_{k=1}^{n}\int_X f_k^2 d\mu \geqslant \sum_{k=1}^{n}\int_{E_k} S_k^2 d\mu \geqslant \sum_{k=1}^{n} t^2\mu\,(E_k) = t^2\mu(E).$$

We supplement the theorem (preserving the notation) and verify that, for a sequence of independent functions f_n satisfying the assumptions of the theorem, the following statement is true.

Corollary If $A^2 = \sum_{n=1}^{\infty}\int_X f_n^2 d\mu < +\infty$, then the function $S^* = \sup_{k\geqslant 1}|S_k| = \sup_{k\geqslant 1} S_k^*$ is summable and $\int_X S^* d\mu \leqslant 2A$.

Proof For every $t>0$, the set $X\,(S^*\geqslant t)$ is exhausted by the expanding sequence of sets $X\,(S_k^*\geqslant t)$. By the theorem, the measure of each of these sets does not exceed A^2/t^2. Consequently, $\mu\,(X\,(S^*\geqslant t)) \leqslant A^2/t^2$. Thus, $F(t)\leqslant A^2/t^2$, where F is the decreasing distribution function for S^*. Using the formula $\int_X h^p d\mu = p\int_0^\infty t^{p-1}\tilde{H}(t)dt$ with $p=1$, we see that

$$\int_X S^* d\mu = \int_0^\infty F(t)dt = \int_0^A \cdots + \int_A^\infty \cdots \leqslant$$
$$\leqslant AF(0) + \int_A^\infty \frac{A^2}{t^2}dt \leqslant A + A = 2A.$$

1.2.8

The estimate for the integral of the function S^* established in the previous corollary leads to an important result concerning the behavior of the series $\sum_{n=1}^{\infty} f_n$, which, in turn, implies a generalization of Borel's theorem.

Theorem 1 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of independent functions with zero means. If $\sum_{n=1}^{\infty}\int_X f_n^2 d\mu < +\infty$, then the series $\sum_{n=1}^{\infty} f_n$ converges almost everywhere.

Proof We put

$$S_n = f_1 + \cdots + f_n \quad\text{ and }\quad R_n = \sup_{p\geqslant 1}|S_{n+p} - S_n|\,.$$

Since $|S_{n+p} - S_n| \leqslant 2R_m$ for $n \geqslant m$ and all m and p, we must verify only that $\inf_n R_n = 0$ almost everywhere. For this, it is sufficient to verify the relation $\int_X R_n d\mu \underset{n\to\infty}{\longrightarrow} 0$, which follows immediately from Corollary 1.2.7,

$$\int_X R_n d\mu \leqslant 2 \left(\sum_{k=n+1}^{\infty} \int_X f_k^2 d\mu \right)^{1/2} \underset{n\to\infty}{\longrightarrow} 0$$

Corollary Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of independent functions with zero means. If $\sum_{n=1}^{\infty} \frac{1}{n^2} \int_X f_n^2 d\mu < +\infty$, then $\sigma_n = \frac{1}{n} \sum_{k=1}^{n} f_k \underset{n\to\infty}{\longrightarrow} 0$ almost everywhere.

Proof By the theorem, the sums $T_n = \sum_{k=1}^{n} \frac{1}{k} f_k$ have a finite limit almost everywhere. The quantities $\theta_n = \frac{1}{n+1}(T_1 + \cdots + T_n)$ have the same limit. Therefore, the difference $T_n - \theta_n$ tends to zero almost everywhere. At the same time, it is easy to verify that $T_n - \theta_n = \frac{1}{n+1}(f_1 + \cdots + f_n)$, which completes the proof.

A similar statement can be obtained for an arbitrary orthogonal system if we drop the independence requirement and strengthen the restriction on the quantities $\|f_n\|$.

If we impose quite natural additional restrictions on the independent functions f_n, then the condition $\sum_{n=1}^{\infty} \int_X f_n^2 d\mu < +\infty$ will turn out to be not only sufficient but also necessary for the convergence of the series $\sum_{n=1}^{\infty} f_n$ almost everywhere (or, equivalently by the zero-one law, on a set of positive measure).

Theorem 2 Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of independent bounded functions with zero means. If the series $\sum_{k=1}^{\infty} f_k$ converges almost everywhere, then $\sum_{k=1}^{\infty} \int_X f_k^2 d\mu < +\infty$.

Proof We put $S = \sum_{k=1}^{\infty} f_k$ and $S_n = \sum_{k=1}^{n} f_k (n = 1, 2, \ldots)$. Since the sum S is finite almost everywhere, the sequences $\{S_n(x)\}_n$ are bounded for almost all x. They are uniformly bounded on some set of positive measure. Therefore, for a sufficiently large t, the intersection $E = \bigcap_{n=1}^{\infty} E_n$ of the sets $E_n = \{x \in X \,|\, |S_k(x)| \leqslant t \text{ for } k = 1, \ldots, n\}$ has a positive measure. We find a recurrence estimate for the integrals

$$I_n = \int_{E_n} S_n^2 d\mu$$

For this, we use the independence of the functions f_{n+1} and $S_n \chi_{E_n}$. This gives us the relations

$$\int_{E_n} S_n f_{n+1} d\mu = \int_X \chi_{E_n} S_n d\mu \cdot \int_X f_{n+1} d\mu = 0$$

and

$$\int_{E_n} f_{n+1}^2 d\mu = \int_X \chi_{E_n} f_{n+1}^2 d\mu = \mu(E_n) \int_X f_{n+1}^2 d\mu \geqslant \mu(E) \int_X f_{n+1}^2 d\mu.$$

Therefore, putting $F_n = E_n \setminus E_{n+1}$, we arrive at the inequality

$$I_{n+1} = \int_{E_n} (S_n + f_{n+1})^2 \, d\mu - \int_{F_n} S_{n+1}^2 d\mu \geqslant I_n + \mu(E) \int_X f_{n+1}^2 d\mu - \int_{F_n} S_{n+1}^2 d\mu.$$

By assumption, there is a number c such that, for all n, the inequality $|f_n| \leqslant c$ holds almost everywhere. Then

$$|S_{n+1}(x)| \leqslant |S_n(x)| + |f_{n+1}(x)| \leqslant t + c \quad \text{for almost all } x \text{ in } E_n.$$

Thus,

$$I_{n+1} - I_n + (t+c)^2 \mu\,(F_n) \geqslant \mu(E) \int_X f_{n+1}^2 d\mu$$

Since $\sum_{k=1}^{n} (I_{k+1} - I_k) \leqslant I_{n+1} \leqslant t^2$ and $\sum_{k=1}^{n} \mu\,(F_k) \leqslant 1$, it follows that the series $\sum_k \mu(E) \int_X f_{k+1}^2 d\mu$ converges, which is equivalent to the assertion of the theorem since $\mu(E) > 0$.

1.3 Trigonometric Fourier Series

The present and following sections are devoted to harmonic analysis. Without striving to expose this important and vast subject in its entirety, we restrict ourselves to the exposition of selected topics the choice of which is motivated only by the desire to demonstrate the methods developed above.

In Sect. 1.1, we established important properties of Fourier series in arbitrary orthogonal systems. Now, we consider the properties of Fourier series in trigonometric systems in more detail. This is historically the first example of an orthogonal system, and the problem of the representability of a function as the sum of a trigonometric series was one of the central problems in mathematics for nearly two hundred years.

Suffice to say that the lively discussion in the 18th century devoted to this problem provided an important impetus for the formulation of the modern concept of function. Riemann introduced his definition of an integral in connection with the study of trigonometric series, and Cantor, studying the uniqueness of the expansion of a function as a trigonometric series, came up with his foundation of set theory.

1.3.1

We recall that the Fourier series of a function $f \in \mathscr{L}^2((0, 2\pi))$ in the systems

$$1, \quad \cos x, \quad \sin x, \quad \ldots, \quad \cos nx, \quad \sin nx, \quad \ldots, \quad \text{and} \quad \left\{e^{inx}\right\}_{n\in\mathbb{Z}}$$

have, respectively, the forms

$$A(f) + \sum_{n=1}^{\infty} (a_n(f)\cos nx + b_n(f)\sin nx) \tag{1}$$

and

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inx} \tag{1'}$$

where the Fourier coefficients are calculated by the formulas

$$A(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$$

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nxdx \tag{2}$$

$$b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nxdx \quad (n \in \mathbb{N})$$

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx \quad (n \in \mathbb{Z}). \tag{2'}$$

Unlike the previous section, where we considered only functions of class $\mathscr{L}^2$, here we will deal with arbitrary functions summable on $(0, 2\pi)$. It is obvious that, in this case, the integrands in formulas (2) and (2') will also be summable. Therefore, we keep the terminology introduced above (a Fourier coefficient, a Fourier series) for the functions in $\mathscr{L}^1((0, 2\pi))$. We are now interested not in convergence in the $\mathscr{L}^2$-norm, but in other types of convergence, and first of all, pointwise convergence. Here, by the sum of the series $(1')$, we always mean the limit of the symmetric partial sums

$$S_n(f, x) = \sum_{|k|\leqslant n} \widehat{f}(k)e^{ikx} \tag{3}$$

which are also called the Fourier sums of the function f. The partial sums of series (1) and $(1')$ are equal. Thus, all results obtained for one of the series are valid for the other one. In the sequel, we will mainly consider series $(1')$ because this leads to some technical simplifications.

In conclusion, we touch on a question that may arise when solving the problem of the expansion of a function as a trigonometric series. Up to now, the choice of its coefficients have been dictated by geometric considerations presented in Sect. 1.1 and has led to formulas (2) and $(2')$. Can it happen that, for a different mode of convergence (e.g., pointwise or in measure) the coefficients of the trigonometric series must be chosen in a different way? It is easy to verify, however, that, under mild additional assumptions, there is essentially no freedom in the choice of the coefficients. Indeed, if, for example, a trigonometric series $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converges to a function f almost everywhere or in measure and its partial sums $S_n(x) = \sum_{|k|\leqslant n} c_k e^{ikx}$ have a summable majorant, i.e., a function $g \in \mathscr{L}^1((0,2\pi))$ such that $|S_n(x)| \leqslant g(x)$ for all $x \in (0,2\pi)$ and $n \in \mathbb{N}$, then the coefficients of the series coincide with the Fourier coefficients of the function $f, c_k \equiv \widehat{f}(k)$. Indeed, by Lebesgue's theorem, the integral $\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx$ is the limit (as $n \to \infty$) of the integrals $\frac{1}{2\pi}\int_0^{2\pi} S_n(x)e^{-ikx}dx$, each of which is equal to c_k for $n \geqslant |k|$.

1.3.2

Instead of functions defined only on the interval $(0,2\pi)$, it will be more convenient for us to deal with 2π-periodic functions. Since every function defined on $(0,2\pi)$ can be extended to a periodic function, we will assume in what follows that all functions in question are periodic (in the sequel, periodicity means 2π-periodicity). Being summable on an interval of length 2π, such functions are summable on each finite interval. We will repeatedly use the fact that the integral $\int_a^{a+2\pi} f(x)dx$ does not depend on the parameter a (the reader is invited to prove this independently). Often, especially when dealing with odd and even functions, it is more convenient to integrate over the interval $(-\pi,\pi)$ in formulas (2) and (2′). By $\widetilde{C}$ and $\widetilde{C}^r (1 \leqslant r \leqslant +\infty)$, we denote the classes of periodic functions that are continuous and, respectively, r times continuously differentiable on $\mathbb{R}$; by $\widetilde{\mathscr{L}^p}$, we denote the class of periodic functions summable on $(-\pi,\pi)$ with power $p \geqslant 1$. For a function $f \in \widetilde{\mathscr{L}^p}$, by $\|f\|_p$ we mean the $\mathscr{L}^p$-norm of its restriction to $(-\pi,\pi)$.

We note the following elementary properties of the Fourier coefficients.

(a) $|\widehat{f}(n)| \leqslant \frac{1}{2\pi}\|f\|_1$ (see formula (2′)).

(b) $\widehat{f}(n) \underset{|n|\to+\infty}{\longrightarrow} 0$ (see the Riemann-Lebesgue theorem).

This qualitative result can be supplemented by an estimate connected with the continuity in the mean.

The properties connecting Fourier coefficients with translation, differentiation, and convolution play an important role. We recall that the translation f_h of a function $f \in \widetilde{\mathscr{L}^1}$ corresponding to a number h is defined by the formula $f_h(x) = f(x-h)$. Making the change of variable $x - h \mapsto x$ in the integral $\int_0^{2\pi} f(x-h)\times\, e^{-inx}dx$, we arrive at the formula

(c) $\widehat{f_h}(n) = e^{-inh}\widehat{f}(n)$.

(d) If a periodic function f is absolutely continuous on $\mathbb{R}$ (in particular, if it is piecewise differentiable), then

$$\widehat{f'}(n) = in\widehat{f}(n) \quad (n \in \mathbb{Z})$$

(for the proof, it is sufficient to integrate by parts). In particular, $\widehat{f}(n) = o(1/n)$. We note a weak version of this estimate for a function of bounded variation.

(d′) If f is a function of bounded variation on the interval $[0,2\pi]$, then $\widehat{f}(n) = O(1/n)$. Indeed, integrating by part, we obtain

$$2\pi\widehat{f}(n) = \int_0^{2\pi} f(x)e^{-inx}dx = f(x)\frac{e^{-inx}}{-in}\Big|_0^{2\pi} + \frac{1}{in}\int_0^{2\pi} e^{-inx}df(x)$$
$$= O\left(\frac{1}{n}\right)$$

(e) Let $f, g \in \widetilde{\mathscr{L}^1}$. Then

$$\widehat{f * g}(n) = 2\pi\widehat{f}(n) \cdot \widehat{g}(n) \quad \text{for all } n \in \mathbb{Z}$$

(for the definition of the convolution of periodic functions, see Sect. 1.2.5). The proof is obtained by direct calculation using the change of the order of integration,

$$\begin{aligned}
\widehat{f * g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f * g)(x)e^{-inx}dx = \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(x-t)g(t)dt\right)e^{-inx}dx = \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}\left(\int_{-\pi}^{\pi} f(x-t)e^{-in(x-t)}dx\right)dt = \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-int}\left(\int_{-\pi}^{\pi} f(u)e^{-inu}du\right)dt = 2\pi\widehat{g}(n) \cdot \widehat{f}(n).
\end{aligned}$$

1.3.3

The problem of the Fourier series expansion of a function is rather complicated and has a long history. The famous work "The analytical theory of heat" by Fourier, in which the series that were later named after him were first studied and used systematically, did not contain an explicit formulation of a condition providing the expandability of a function as a Fourier series. Such criteria arose later. Still later it became clear that the Fourier series of a continuous function can diverge at some points, and, as Kolmogorov proved, the Fourier series of a summable function can diverge everywhere.

So far, even knowing that a Fourier series of a differentiable function converges at a point, we cannot be sure that its sum coincides with the value of the function.

At the moment, if f is a square-summable function, then series $(1')$ converges in the $\mathscr{L}^2$-norm and its sum is equal to f. If a function f is only assumed to be summable, the question of the convergence of a Fourier series (pointwise, in an $\mathscr{L}^p$-norm, or in some other sense) remains open for the time being.

We begin the investigation of a Fourier series' convergence with the derivation of an important formula for its partial sums discovered by Dirichlet. Relying on formula $(2')$, we transform Eq. (3) as follows:

$$S_n(f,x) = \sum_{|k|\leqslant n}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt\right)e^{ikx} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{|k|\leqslant n} e^{ik(x-t)}dt.$$

The function

$$D_n(u) = \frac{1}{2\pi}\sum_{|k|\leqslant n} e^{iku} = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{n}\cos ku \qquad (4)$$

is called the nth Dirichlet kernel. Obviously, the Dirichlet kernel is even and periodic. Summing the geometric sequence $\sum_{|k|\leqslant n} e^{iku}$, we obtain

$$D_n(u) = \frac{\sin\left(n+\frac{1}{2}\right)u}{2\pi\sin\frac{u}{2}} \quad \text{for } u \notin 2\pi\mathbb{Z} \qquad (4')$$

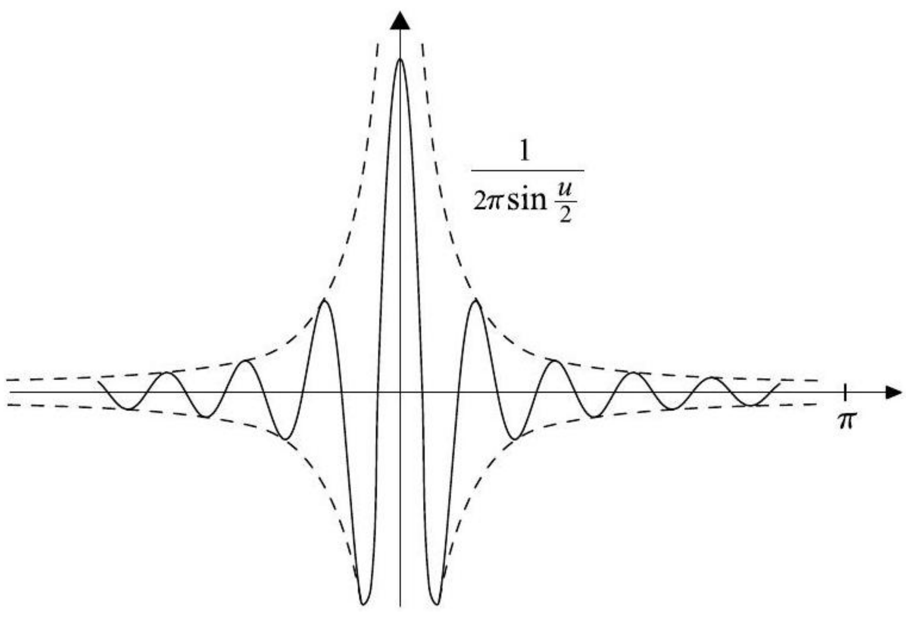

Fig. 1.1 Graph of the Dirichlet kernel

From this, we see that the function D_n is strongly oscillating for large n, and, in a neighborhood of zero, it takes extreme values with alternating signs and absolute values comparable with $\max D_n = D_n(0) = \frac{1}{\pi}\left(n+\frac{1}{2}\right)$ (see Fig. 1.1).

It follows directly from the definition that the sum of the Fourier series is the convolution of the function and the Dirichlet kernel,

$$S_n(f,x) = \int_{-\pi}^{\pi} f(t)D_n(x-t)dt = (f * D_n)(x)$$

Since the integrands are periodic, we can also represent the above equation in the form

$$S_n(f,x) = \int_{-\pi}^{\pi} f(x-u)D_n(u)du \qquad (5)$$

Considering periodic approximate identities, we have encountered similar formulas (see Sect. 1.3.5 MA(3)). The Dirichlet kernels satisfy conditions (b) and (c) of the definition of a periodic approximate identity; it immediately follows from Eq. (4) that

$$\int_{-\pi}^{\pi} D_n(u)du = 1$$

Moreover, we have

$$\int_{\delta<|u|<\pi} D_n(u)du = \int_{\delta<|u|<\pi} \frac{\sin\left(n+\frac{1}{2}\right)u}{2\pi\sin\frac{u}{2}}du \underset{n\to\infty}{\longrightarrow} 0$$

for each $\delta \in (0,\pi)$ (the passage to the limit can be justified by integration by parts or by referring to the Riemann-Lebesgue theorem).

However, D_n does not satisfy the most important property of an approximate identity, namely, the positivity. Moreover, the Dirichlet kernels do not satisfy the pe- riodic analog of condition (a′) of Sect. 1.3.1 MA3, i.e., they have unbounded $\mathscr{L}^1$-norms. Indeed,

$$\begin{aligned}\int_{-\pi}^{\pi}|D_n(u)|\,du &= \int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)u\right|}{\pi\sin\frac{u}{2}}du \geqslant \frac{2}{\pi}\int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)u\right|}{u}du\\ &= \frac{2}{\pi}\int_0^{\pi\left(n+\frac{1}{2}\right)}\frac{|\sin v|}{v}dv \geqslant \frac{2}{\pi}\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin v|}{k\pi}dv = \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}.\end{aligned}$$

Since $\sum_{k=1}^{n}\frac{1}{k} \geqslant \int_1^n \frac{1}{x}dx = \ln n$, we have $\|D_n\|_1 \geqslant \frac{4}{\pi^2}\ln n$.

Thus, the general theorems connected with the use of approximate identities cannot be applied here. This is the cause of considerable difficulties in the study of the convergence of Fourier series. Here, we meet not just technical questions, but those of a fundamental nature.

At the same time, in many problems, it is essential that the norms $\|D_n\|_1$ increase quite slowly. Indeed, the estimate from above for $\|D_n\|_1$ just obtained is exact in order,

$$\|D_n\|_1 = \int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)u\right|}{\pi\sin\frac{u}{2}}du \leqslant \int_0^{\pi}\frac{\left|\sin\left(n+\frac{1}{2}\right)u\right|}{u}du = \int_0^{\pi\left(n+\frac{1}{2}\right)}\frac{|\sin v|}{v}dv.$$

Consequently, $\|D_n\|_1 \leqslant 1 + \int_1^{\pi\left(n+\frac{1}{2}\right)}\frac{dv}{v}$, and, therefore, $\|D_n\|_1 \leqslant 2\ln n$ for $n \geqslant 10$. Since $S_n(f) = f * D_n$, we obtain the following estimate for the Fourier sums of a bounded function $(n \geqslant 10)$:

$$\|S_n(f)\|_\infty \leqslant \|f\|_\infty \, \|D_n\|_1 \leqslant 2\|f\|_\infty \ln n. \qquad (6)$$

The partial sums of the Fourier series are calculated by formula (5), and so depend on the values of the function on an interval of length 2π. It is all the more surprising that, as we will now verify, the convergence of the Fourier series at a point x and the value of its sum are local properties of the function, i.e., they are preserved under an arbitrary change of the function outside an arbitrarily small neighborhood of the point. More formally, we have the following.

Theorem (Riemann's localization principle) If functions $f_1, f_2 \in \widetilde{\mathscr{L}}^1$ coincide in a neighborhood of a point x, then their Fourier series have the same behavior at x, $S_n(f_1, x) - S_n(f_2, x) \to 0$ as $n \to \infty$.

Proof From the assumptions it follows that the function $\varphi_x(u) = \frac{f_1(x+u) - f_2(x-u)}{\sin(u/2)}$ (equal to zero in a neighborhood of the point $u = 0$) is summable on $(-\pi, \pi)$. Since

$$S_n(f_1, x) - S_n(f_2, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_x(u) \sin\left(n + \frac{1}{2}\right) u du$$

by Eq. (5), it remains to refer to the Riemann-Lebesgue theorem according to which the integral on the right-hand side of this equation tends to zero.$\square$

1.3.4

Among a great variety of convergence tests for Fourier series, we mention only two of the most applicable ones, the Dini test and the Dirichlet-Jordan test. They supplement each other and can be applied to a wide range of cases.

First, we establish a useful property of the Dirichlet kernel.

Lemma Let $n \in \mathbb{N}$. Then:

(a) $D_n(u) = \frac{\sin nu}{\pi u} + \frac{1}{2\pi}(\cos nu + \Delta(u) \sin nu)$,

where Δ is a function independent of n and $|\Delta(u)| < 1$ for $|u| \leqslant \pi$;

(b) $\left|\int_0^x D_n(u) du\right| \leqslant 2$ for $|x| \leqslant 2\pi$.

Proof (a) It is clear that

$$D_n(u) = \frac{\sin nu}{2\pi \tan \frac{u}{2}} + \frac{1}{2\pi} \cos nu = \frac{\sin nu}{\pi u} + \frac{1}{2\pi}\left(\cos nu + \left(\frac{1}{\tan \frac{u}{2}} - \frac{2}{u}\right) \sin nu\right).$$

It remains to observe that the difference $\Delta(u) = \frac{1}{\tan \frac{u}{2}} - \frac{2}{u} (\Delta(0) = 0)$ decreases on $[-\pi, \pi]$, and, therefore, $|\Delta(u)| \leqslant |\Delta(\pi)| = \frac{2}{\pi} < 1$.

(b) It is sufficient to consider the case where $x \in (0, 2\pi)$. First let $x \in (0, \pi]$. Then assertion (a) proved above implies the inequality

$$\left| \int_0^x D_n(u) du - \int_0^x \frac{\sin nu}{\pi u} du \right| \leqslant \frac{1}{2\pi} \int_0^x 2du \leqslant 1.$$

Now, we prove that the integral

$$J_n(x)=\int_0^x \frac{\sin nu}{\pi u}du=\int_0^{nx}\frac{\sin v}{\pi v}dv$$

lies between 0 and 1 . To verify this, we divide the interval of integration $[0,nx]$ into parts on which $\sin v$ preserves its sign. Then the integral $J_n(x)$ splits into the alternating sum of terms whose absolute values decrease since $\frac{1}{v}$ decreases. Therefore,

$$0\leqslant J_n(x)\leqslant \int_0^{\pi}\frac{\sin v}{\pi v}dv\leqslant \int_0^{\pi}\frac{dv}{\pi}=1$$

Thus, the integral $\int_0^x D_n(u)du$ lies between -1 and 2 provided $0<x\leqslant\pi$.

For $x\in(\pi,2\pi)$, we use the easily verifiable relation

$$\int_0^x D_n(u)du=1-\int_0^{2\pi-x}D_n(u)du$$

from which it follows that the inequality $-1\leqslant\int_0^x D_n(u)du\leqslant 2$ also holds in this case.□

Using the first assertion of the lemma, we can represent Eq. (5) in the following form:

$$S_n(f,x)=\int_{-\pi}^{\pi}f(x-u)\frac{\sin nu}{\pi u}du+\varepsilon_n \qquad (5')$$

where the quantity $\varepsilon_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x-u)(\cos nu+\Delta(u)\sin nu)du$ tends to zero by the Riemann-Lebesgue theorem.

In particular, if $f\equiv 1$, then

$$1=\int_{-\pi}^{\pi}\frac{\sin nu}{\pi u}du+o(1) \qquad (5'')$$

Making the change of variable $nu=t$ and passing to the limit as $n\to\infty$, we once again obtain the equality $\int_0^{\infty}\frac{\sin t}{t}dt=\frac{\pi}{2}$ established in Sect. 1.1.6 by a different method.

Theorem (Dini test) If a function $f\in\tilde{\mathscr{L}}^1$ satisfies the Dini condition

$$\int_0^{\pi}\left|\frac{f(x+u)+f(x-u)}{2}-C\right|\frac{du}{u}<+\infty$$

at a point $x\in\mathbb{R}$ for some $C\in\mathbb{C}$, then its Fourier series converges to C at the point x.

In particular, if f is differentiable at x, then the Dini condition is fulfilled with $C=f(x)$, and so the sum of the Fourier series is equal to $f(x)$. However, if only the one-sided limits $f(x\pm 0)$ exist and

$$|f(x\pm u)-f(x\pm 0)|=O\left(u^{\alpha}\right)\qquad \text{as } u\to+0$$

for some $\alpha>0$, then the Fourier series of f at x converges to the average $\frac{f(x-0)+f(x+0)}{2}$.

Proof From (5'), it follows that

$$S_n(f,x) = \int_{-\pi}^{\pi} f(x-u)\frac{\sin nu}{\pi u}du + o(1) = \int_{-\pi}^{\pi} f(x+u)\frac{\sin nu}{\pi u}du + o(1)$$

as $n \to \infty$. Thus,

$$S_n(f,x) = \int_{-\pi}^{\pi} \frac{f(x-u)+f(x+u)}{2}\frac{\sin nu}{\pi u}du + o(1)$$

Subtracting Eq. ($5''$) multiplied by C from the above equation, we see that

$$\begin{aligned} S_n(f,x) - C &= \int_{-\pi}^{\pi} \left(\frac{f(x-u)+f(x+u)}{2} - C\right)\frac{\sin nu}{\pi u}du + o(1) \\ &= \frac{2}{\pi}\int_0^{\pi} g_x(u)\sin nu du + o(1), \end{aligned}$$

where $g_x(u) = \frac{f(x-u)+f(x+u)-2C}{2u}$. Since the function g_x is summable on $(0,\pi)$ by the assumptions of the theorem, the integral on the right-hand side of this equation tends to zero by the Riemann-Lebesgue theorem.□

Theorem (Dirichlet-Jordan test) If a periodic function f has bounded variation on the interval $[-\pi,\pi]$, then, for each $x \in \mathbb{R}$, the Fourier series of f converges to the average $(f(x+0)+f(x-0))/2$. Moreover, $|S_n(f,x)| \leqslant \sup_{\mathbb{R}}|f| + 2\mathbf{V}_{-\pi}^{\pi}(f)$.

We remark that the convergence of a Fourier series at a point x is preserved by the localization principle if we assume that f has bounded variation only locally, in a neighborhood of this point.

Proof By Eq. (5'), we must find the limit of the integrals

$$I_n = \int_{-\pi}^{\pi} f(x-u)\frac{\sin nu}{\pi u}du = \int_0^{\pi} \varphi(u)\frac{\sin nu}{\pi u}du$$

where $\varphi(u) = f(x-u) + f(x+u)$. This function has bounded variation on $[0,\pi]$, and so can be represented as the difference of decreasing functions. Therefore, it is sufficient for us to find the limit of the integrals I_n under the assumption that the function φ is non-negative and decreases on the interval $[0,\pi]$. To this end, we represent I_n in the form

$$I_n = \int_0^{\infty} \Phi(u)\frac{\sin nu}{\pi u}du = \int_0^{\infty} \Phi\left(\frac{t}{n}\right)\frac{\sin t}{\pi t}dt$$

where $\Phi(u) = \varphi(u)\chi_{(0,\pi)}(u)$. The integral on the righthand side of the above equation tends to $\varphi(+0)/2 = (f(x-0)+f(x+0))/2$.

To obtain a uniform estimate for the sums $S_n(f)$, we put $H_n(u) = \int_0^u D_n(t)dt$. Then

$$S_n(f,x) = \int_{-\pi}^{\pi} f(x-u)D_n(u)du =$$
$$= H_n(u)f(x-u)|_{u=-\pi}^{\pi} - \int_{-\pi}^{\pi} H_n(u)df(x-u).$$

Since $H_n(\pm\pi) = \pm\frac{1}{2}$, the first summand is equal to $(f(x-\pi)+f(x+\pi))/2$. Furthermore, $|H_n(u)| \leqslant 2$ by the lemma, and so

$$\left|\int_{-\pi}^{\pi} H_n(u)df(x-u)\right| \leqslant 2\mathbf{V}_{x-\pi}^{x+\pi}(f) = 2\mathbf{V}_{-\pi}^{\pi}(f)$$

hence the required estimate for the sums $S_n(f,x)$ follows.□

In conclusion, we prove the Dini test in a different way, without using Dirichlet kernels. The Dini condition means that the function

$$g(u) = \left(\frac{f(x+u)+f(x-u)}{2} - C\right)\frac{1}{e^{iu}-1} \quad (u \notin 2\pi\mathbb{Z})$$

belongs to the class $\tilde{\mathscr{L}}^1$. Multiplying both sides of the equation

$$\frac{f(x+u)+f(x-u)}{2} - C = \left(e^{iu}-1\right)g(u)$$

by $\frac{1}{2\pi}e^{-iku}$ and then integrating with respect to $u \in (-\pi,\pi)$, we obtain

$$\begin{aligned} \tfrac{1}{2}\left(\widehat{f}(k)e^{ikx} + \widehat{f}(-k)e^{-ikx}\right) &= \widehat{g}(k-1) - \widehat{g}(k), && \text{if } k \neq 0,\\ \widehat{f}(0) - C &= \widehat{g}(-1) - \widehat{g}(0), && \text{if } k = 0. \end{aligned}$$

It remains to sum all these equations for $|k| \leqslant n$,

$$S_n(f,x) - C = \sum_{k=-n}^{n} \widehat{f}(k)e^{ikx} - C = \widehat{g}(-n-1) - \widehat{g}(n) \underset{n\to+\infty}{\longrightarrow} 0$$

If we sum them for $k = 0,1,\dots,n$ and for $k = -1,\dots,-n$ separately, it becomes clear that the Dini condition implies the convergence of not only the symmetric sums $\sum_{k=-n}^{n}\widehat{f}(k)e^{ikx}$, but also the "one-sided" sums $\sum_{k=0}^{n}\widehat{f}(k)e^{ikx}$ and $\sum_{k=-n}^{-1}\widehat{f}(k)e^{ikx}$. In other words, the Dini condition ensures the convergence of each of the series $\sum_{k=0}^{\infty}\widehat{f}(k)e^{ikx}$ and $\sum_{k=-\infty}^{-1}\widehat{f}(k)e^{ikx}$. In particular, it ensures the convergence of the series $\sum_{n\in\mathbb{Z}}\operatorname{sign}(n)\widehat{f}(n)e^{inx}$ called the conjugate to series $(1')$.

1.3.5

We give some examples of Fourier series expansions.

Example 1 Since the periodic function equal x on $(-\pi,\pi)$ is differentiable at all points distinct from $(2k+1)\pi$ $(k \in \mathbb{Z})$, its Fourier series converges not only in the $\mathscr{L}^2$-norm, but also pointwise,

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nx \quad \text{ for } x \in (-\pi, \pi)$$

At the points $(2k+1)\pi$, the sum of the series is equal to the average of the one-sided limits of the function. At $x = \frac{\pi}{2}$, the Fourier series expansion yields the relations

$$\frac{\pi}{4} = \sum_{m=0}^{\infty} (-1)^m \frac{1}{2m+1}$$

Considering the Fourier series expansion of the function equal to x^2 on $[-\pi, \pi]$ at the point π, we again obtain the Euler identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Example 2 Let $w \in \mathbb{C}\backslash\mathbb{Z}$. We consider the periodic function equal to $\cos wx$ on the interval $[-\pi, \pi]$. This function has finite one-sided derivatives everywhere on $\mathbb{R}$ and, therefore, can be expanded in a Fourier series. After elementary transformations, we obtain that the equation

$$\cos wx = \frac{\sin \pi w}{\pi w} + \frac{2}{\pi} w \sin \pi w \sum_{n=1}^{\infty} \frac{(-1)^n}{w^2 - n^2} \cos nx$$

holds for all $|x| \leqslant \pi$

For $x = \pi$ and $x = 0$, the equation implies the following expansions of cotangent and cosecant as sums of partial fractions:

$$\cot \pi w = \frac{1}{\pi w} + \frac{2w}{\pi} \sum_{n=1}^{\infty} \frac{1}{w^2 - n^2} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{w-n}$$

$$\frac{1}{\sin \pi w} = \frac{1}{\pi w} + \frac{2w}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{w^2 - n^2} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{w-n}$$

Example 3 Here, we verify the existence of a convergent non-zero numerical series $\sum_{n=1}^{\infty} a_n$ with the unusual property $\sum_{m=1}^{\infty} a_{km} = 0$ for every k. In the construction, we follow F.L. Nazarov who suggested the use of Fourier series for this purpose. We consider periodic functions equal to zero in a neighborhood of each point of the form $\pi t, t \in \mathbb{Q}$. Among them, we can, obviously, find an even function f satisfying the conditions $\widehat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = 0$ and $0 < \int_{-\pi}^{\pi} |f(x)|^2 dx < +\infty$. We take the required series equal to $\sum_{n=1}^{\infty} \widehat{f}(n)$. This is a non-zero series since

$$0 < \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 = \frac{1}{\pi} \sum_{n=1}^{\infty} |\widehat{f}(n)|^2$$

(here we have used Parseval's identity).

At each point $x = 2\pi \frac{j}{k} (j \in \mathbb{Z}, k \in \mathbb{N})$, the function f satisfies the Dini condition and, therefore,

$$\sum_{n=1}^{\infty} \widehat{f}(n) \cos\left(2\pi \frac{j}{k} n\right) = 0$$

Summing these equations for $j = 0, 1, \ldots, k-1$, we obtain

$$\sum_{n=1}^{\infty} \widehat{f}(n) \sum_{j=0}^{k-1} \cos\left(2\pi \frac{j}{k} n\right) = 0$$

If the index n is divisible by k, then the inner sum is equal to k, since otherwise this sum is obviously zero. Consequently,

$$k \sum_{m=1}^{\infty} \widehat{f}(km) = 0 \quad \text{for all } k \in \mathbb{N}$$

1.3.6

As we have already noted, the Fourier series of a summable, or even of a continuous, function may diverge. However, such a series has the remarkable property that it can be integrated termwise over an arbitrary finite interval without worrying about convergence.

Theorem 1 Let $f \in \tilde{\mathscr{L}}^1$. Then the equation

$$\int_a^b f(x)dx = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \int_a^b e^{inx} dx$$

(where the sum is regarded as the limit of the symmetric partial sums) is valid for all $a, b \in \mathbb{R}$.

Proof Taking into account the periodicity, we restrict ourselves, without loss of generality, to the case where $-\pi \leqslant a < b \leqslant \pi$. Let χ be the characteristic function of the interval (a, b). Then a partial sum of the series on the right-hand side of the required equation can be represented in the form

$$\begin{aligned}\sum_{k=-n}^{n} \widehat{f}(k) \int_a^b e^{ikx} dx &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt\right) 2\pi \widehat{\chi}(-k) \\ &= \int_{-\pi}^{\pi} f(t) S_n(\chi, t) dt\end{aligned}$$

By Dini's test, we have $S_n(\chi, t) \underset{n\to\infty}{\longrightarrow} \chi(t)$ for $t \in (-\pi, \pi)$ and $t \neq a, b$. Moreover,

$$\begin{aligned}S_n(\chi, t) &= \int_a^b D_n(x-t)dx = \int_{a-t}^{b-t} D_n(u)du = \\ &= \int_0^{b-t} D_n(u)du - \int_0^{a-t} D_n(u)du\end{aligned}$$

Therefore, Lemma 1.3.4 gives use the uniform estimate $|S_n(\chi,t)| \leqslant 4$. By Lebesgue's theorem, we can pass to the limit on the right-hand side of Eq. (7) and obtain

$$\sum_{k=-n}^{n} \widehat{f}(k) \int_a^b e^{ikx}dx = \int_{-\pi}^{\pi} f(t)S_n(\chi,t)dt \underset{n\to\infty}{\longrightarrow} \int_{-\pi}^{\pi} f(t)\chi(t)dt = \int_a^b f(t)dt$$

$\square$

Theorem 1 allows us to considerably strengthen the assertion on the completeness of the trigonometric system, according to which two functions of the class $\widetilde{\mathscr{L}}^2$ that have the same Fourier coefficients coincide almost everywhere. Now, we can extend this result to the class $\widetilde{\mathscr{L}}^1$.

Corollary 1 Functions $f, g \in \tilde{\mathscr{L}}^1$ having the same Fourier coefficients coincide almost everywhere on $\mathbb{R}$.

Proof By the theorem, the integrals of f and g are equal on every finite interval. Therefore, f and g coincide almost everywhere.$\square$

Corollary 2 For every function $f \in \tilde{\mathscr{L}}^1$, the series $\sum_{n=1}^{\infty} b_n(f)/n$ converges.

We recall that $b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = i(\widehat{f}(n) - \widehat{f}(-n))$ is the Fourier sine coefficient of f.

Proof As established in the theorem, the equation

$$\int_0^u f(x)dx = \sum_{n=-\infty}^{\infty} \widehat{f}(n) \int_0^u e^{inx}dx$$

holds for all u. From (7) and the estimate $|S_n(\chi,t)| \leqslant 4$, it follows that the symmetric partial sums of this series are uniformly bounded for $u \in [-\pi,\pi]$. Therefore, we can integrate the series termwise,

$$\int_{-\pi}^{\pi}\left(\int_0^u f(x)dx\right)du = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\int_{-\pi}^{\pi}\left(\int_0^u e^{inx}dx\right)du = -2\pi\sum_{n\neq 0}\frac{\widehat{f}(n)}{in}.$$

The convergence of the symmetric partial sums of the series $\sum_{n\neq 0}\frac{\widehat{f}(n)}{n}$ is equivalent to the required statement.$\square$

Corollary 2 gives a necessary condition for a trigonometric series $\sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$ to be a Fourier series. The everywhere convergent series $\sum_{n=2}^{\infty}(\sin nx)/\ln n$ does not satisfy this condition and, therefore, cannot be the Fourier series of a summable function. It is interesting to note that, in contrast to the sine coefficients, the cosine coefficients can tend to zero arbitrarily slowly. For example, the series $\sum_{n=2}^{\infty}(\cos nx)/\ln n$ is the Fourier series of a summable function.

The relation obtained in Theorem 1 can be regarded as a new version of Parseval's identity in which the assumption about one function is weakened (it belongs to $\widetilde{\mathscr{L}}^1$ but not to $\widetilde{\mathscr{L}}^2$) and the assumption concerning the other is strengthened considerably (it is the characteristic function of an interval). At the same time, the proof of the theorem uses the properties of the function χ only partially. This makes it possible to extend considerably the applicability conditions of Parseval's identity.

Theorem 2 Let $f \in \widetilde{\mathscr{L}^1}$, and let g be a bounded (measurable and periodic) function whose Fourier sums $S_n(g,x)$ are uniformly bounded (with respect to x and n). Then the following Parseval identity is valid:

$$\int_{-\pi}^{\pi} f(x)\bar{g}(x)dx = 2\pi \sum_{n=-\infty}^{\infty} \widehat{f}(n)\bar{g}(n)$$

The class of functions with uniformly bounded partial sums of Fourier series is sufficiently wide. In particular, it contains all smooth functions on $[-\pi,\pi]$. As follows from the Dirichlet-Jordan test, this class also contains all functions with finite variation on $[-\pi,\pi]$.

The assumption that the function g is bounded is superfluous.

Proof Since $g \in \widetilde{\mathscr{L}^2}$, the sums $S_n(g)$ converge to g in the $\mathscr{L}^2$-norm and, a fortiori, in measure. This implies, as one can easily verify, that

$$f(x)\overline{S_n}(g,x) \to f(x)\bar{g}(x) \quad \text{in measure.}$$

Therefore, we can use Lebesgue's theorem and pass to the limit on the right-hand side of the equation

$$\int_{-\pi}^{\pi} f(x)\overline{S_n}(g,x)dx = 2\pi \sum_{|k|\leqslant n} \widehat{f}(k)\overline{\widehat{g}(k)}$$

as required.□

1.3.7

To obtain a further generalization of the uniqueness theorem for Fourier series, (see Corollary 1 of the previous section), we introduce the notion of Fourier coefficients and Fourier series for a measure.

Definition Let μ be a finite Borel measure on the interval $[-\pi,\pi]$. The Fourier coefficients of μ are defined by the formula

$$\widehat{\mu}(n) = \frac{1}{2\pi} \int_{[-\pi,\pi]} e^{-inx} d\mu(x) \quad (n \in \mathbb{Z})$$

The series $\sum_{n=-\infty}^{\infty} \widehat{\mu}(n)e^{inx}$ is called the Fourier series of μ.

If a measure μ has density f with respect to Lebesgue measure, then $\widehat{\mu}(n) = \widehat{f}(n)$ for all $n \in \mathbb{Z}$ and, consequently, the Fourier series of the measure μ and of the function f coincide. As in the case of Fourier series, it follows directly from definition that the nth (symmetric) partial sum of the Fourier series of a measure, which will be denoted by $S_n(\mu,x)$, is the convolution of this measure and a Dirichlet kernel,

$$S_n(\mu, x) = \int_{[-\pi,\pi]} D_n(x-t)d\mu(t) = (D_n * \mu)(x).$$

Extending Corollary 1 to measures, we must take into account the relation

$$\widehat{\mu}(n) = \frac{(-1)^n}{2\pi}(\mu(\{-\pi\}) + \mu(\{\pi\})) + \frac{1}{2\pi}\int_{(-\pi,\pi)} e^{-inx}d\mu(x)$$

Thus, the Fourier coefficients do not change under redistribution of the loads (preserving their sum) at the points $\pm\pi$. This will be the case when we replace these loads by, for example, $\mu(\{-\pi\}) + \mu(\{\pi\})$ (at the point $-\pi$) and by 0 (at the point π). Therefore, it makes sense to pose the question of whether a measure is uniquely determined by its Fourier coefficients only if we fix the load at one of the points $\pm\pi$. For definiteness, we will consider only the measures that have zero load at the point π.

Theorem Let μ and v be finite Borel measures on the interval $[-\pi,\pi]$ satisfying the condition $\mu(\{\pi\}) = v(\{\pi\}) = 0$. If the Fourier coefficients of these measures coincide, then the measures also coincide.

Proof First, we verify that the Fourier series of a measure, as well as the Fourier series of a function, can be integrated termwise, i.e., if
$\mu(\{a\}) = \mu(\{b\}) = 0$, then

$$\sum_{n=-\infty}^{\infty} \widehat{\mu}(n)\int_a^b e^{inx}dx = \mu([a,b)) \qquad (8)$$

for $[a,b) \subset [-\pi,\pi)$. Indeed, let $\chi = \chi_{[a,b)}$. Then

$$\begin{aligned}\sum_{|k|\leqslant n} \widehat{\mu}(k)\int_a^b e^{ikx}dx &= \sum_{|k|\leqslant n} \widehat{\chi}(-k)\int_{[-\pi,\pi]} e^{-ikx}d\mu(x) \\ &= \int_{[-\pi,\pi]} S_n(\chi, x)d\mu(x)\end{aligned} \qquad (9)$$

In the proof of Theorem 1 of Sect. 1.3.6, we have established that $|S_n(\chi,t)| \leqslant 4$. Moreover, $S_n(\chi,t) \underset{n\to\infty}{\longrightarrow} \chi(t)$ for $t \neq a,b$ and, consequently, μ-almost everywhere. Therefore, we can use Lebesgue's theorem and pass to the limit on the right-hand side of Eq. (9), which leads to Eq. (8). Thus, if measures μ and v have the same Fourier coefficients, then $\mu([a,b)) = v([a,b))$ for every interval $[a,b) \subset [-\pi,\pi)$ satisfying the condition $\mu(\{a\}) = \mu(\{b\}) = v(\{a\}) = v(\{b\}) = 0$. Since the set of points of non-zero measure is at most countable this condition is fulfilled on a dense subset of $(-\pi,\pi)$. Hence it follows that the measures μ and v coincide on all Borel subsets of the interval $(-\pi,\pi)$. At the same time, $\mu(\{\pi\}) = \nu(\{\pi\}) = 0$ and $\mu([-\pi,\pi]) = \widehat{\mu}(0) = \widehat{v}(0) = \nu([-\pi,\pi])$ by assumption. Consequently, the measures μ and v have the same loads at the point $-\pi$, which completes the proof of the theorem.□

1.3.8

Considering Fourier series with coefficients that tend to zero sufficiently fast, we must take into account that if the Fourier series of a function f converges uniformly, then its sum coincides with f almost everywhere by the uniqueness theorem. Therefore, if the Fourier series of a continuous function converges uniformly, then its sum coincides with the function. Taking this into account, we consider only continuous functions in the theorems of this section. Lifting the assumption of continuity, we must replace the equality of a function and its Fourier series by their equivalence.

The Fourier coefficients of smooth functions tend to zero sufficiently fast. For example, if a function satisfies the Lipschitz condition of order α, then $\widehat{f}(n) = O\left(|n|^{-\alpha}\right)$. Indeed, if $h = \frac{\pi}{n}$, then property (c) of Sect. 1.3.2 implies that $\widehat{f_\pi}(n) = -\widehat{f}(n)$. Consequently, $2\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi}\left(f(x) - f\left(x - \frac{\pi}{n}\right)\right)e^{-inx}dx$, and, therefore,

$$2|\widehat{f}(n)| \leqslant \frac{1}{2\pi}\int_0^{2\pi}\left|f(x) - f\left(x - \frac{\pi}{n}\right)\right| dx \leqslant L\left|\frac{\pi}{n}\right|^{\alpha},$$

where L is a Lipschitz constant for f.

The repeated application of the relation $\widehat{f'}(n) = \text{in}\widehat{f}(n)$ (see property (d) of Sect. 1.3.2) shows that the Fourier coefficients of a function f of class $\widetilde{C}^r$ satisfy the relation $\widehat{f}(n) = o\left(|n|^{-r}\right)$ as $|n| \to +\infty$. The converse is "almost true": if $\widehat{f}(n) = O\left(|n|^{-r-2}\right)$ for some $r \in \mathbb{N}$, then the continuous function f coincides with a function of class $\widetilde{C}^r$. Indeed, the series $\sum_n \widehat{f}(n)e^{inx}$ converges uniformly, and, by the above remark, its sum coincides with f. Moreover, since the coefficients decrease fast, the Fourier series admits r-fold differentiation, which implies that $f \in \widetilde{C}^r$. For infinitely smooth functions, this gives a complete description.

Theorem 1 In order that a function $f \in \widetilde{C}$ be infinitely differentiable it is necessary and sufficient that the limit relation $n^r\widehat{f}(n) \to 0$ as $|n| \to +\infty$ be fulfilled for every $r \in \mathbb{N}$.

The smaller class of holomorphic periodic functions can also be well described in terms of the Fourier coefficients: these coefficients must tend to zero not slower that a geometric sequence. We note that a periodic function f is analytic at all points of the line $\mathbb{R}$ if and only if, on $\mathbb{R}$, the function f coincides with a function holomorphic in some horizontal strip $\{z \in \mathbb{C} || \mathcal{I}mz \,|< L\}$. In the proof of the following theorem, we use some elementary properties of holomorphic functions.

Theorem 2 Let $f \in \widetilde{C}$. The following two statements are equivalent:

(a) there is a function F holomorphic in a strip $|\mathcal{I}mz| < L$ and coinciding with f on the real axis;

(b) the relation $\widehat{f}(n) = O\left(e^{-a|n|}\right)$ as $|n| \to +\infty$ holds for every $a \in (0, L)$.

Proof (a) $\Rightarrow$ (b). Assuming that $n > 0$ and $0 < a < L$, we consider the integral $\int_C F(z)e^{-inz}dz$, where C is the boundary of the rectangle P with vertices at the points $\pm\pi, \pm\pi - ai$ lying in the strip $|\mathcal{I}mz| < L$. Since the function F is holomorphic in a neighborhood of P, this integral is equal to zero. Moreover,

F has period 2π, and, therefore, the sum of the integrals over the vertical sides of P are equal to zero. Consequently,

$$\widehat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi-ai}^{\pi-ai} F(z)e^{-inz}dz$$

Therefore,

$$|\widehat{f}(n)| \leqslant \max_{x\in\mathbb{R}} |F(x-ai)| \left|e^{-in(x-ai)}\right| = e^{-an}\max_{x\in\mathbb{R}} |F(x-ai)| = C_a e^{-an}.$$

The coefficients with negative indices can be estimated in the same way, only in this case the rectangle is replaced by a rectangle symmetric with respect to the real axis.

(b) $\Rightarrow$ (a). The series $\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{inz}$ converges uniformly in the strip $|\mathcal{I}mz| \leqslant a$ if $0 < a < L$. By Weierstrass's theorem the sum of the series is holomorphic in the strip $|\operatorname{Im} z| < L$ and coincides with the function f on the real axis.□

1.3.9

As we have already mentioned, the Fourier series of a periodic continuous function may diverge. There are several such examples. We give here a slight modification of an example suggested by Schwartz. We define an even function $f \in \widetilde{C}$ whose oscillation frequency increases rapidly when approaching zero. More precisely, we will assume that

$$f(0) = 0 \quad \text{and} \quad f(t) = \frac{1}{\sqrt{k}}\sin n_k t \quad \text{for } t \in [t_k, t_{k-1}]\,, k = 2, 3, \ldots$$

where $n_k = 2^{k!}, t_k = 2\pi/n_k$ for $k \in \mathbb{N}$ (see Fig. 1.2).

We prove that the sums $S_n(f,0)$ tend to infinity along the indices n_k. Since

$$S_n(f,0) = \frac{2}{\pi}\int_0^{\pi} \frac{\sin nt}{t} f(t)dt + o(1)$$

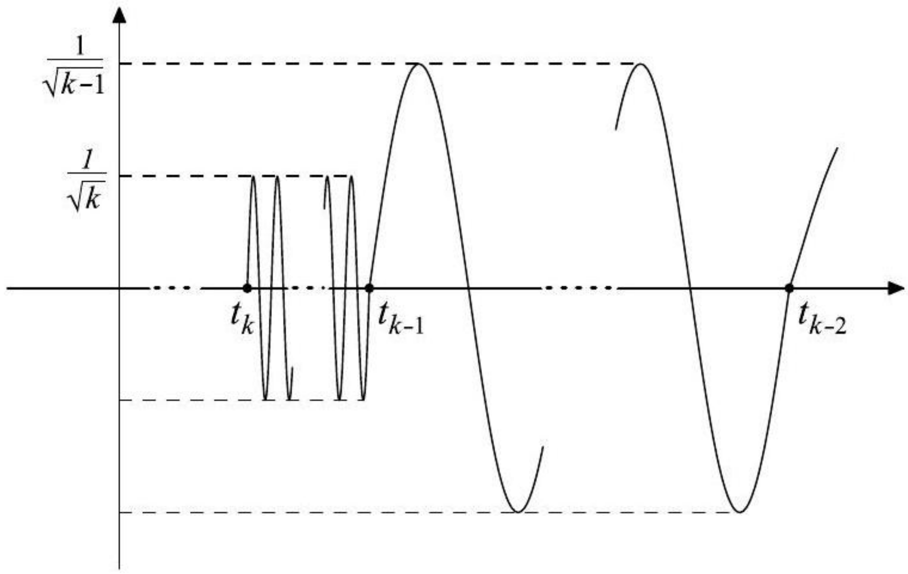

Fig. 1.2 Sketch of the graph of f

by $(5')$, it is sufficient to prove that the integrals

$$I_k = \int_0^{\pi} \frac{\sin n_k t}{t} f(t)dt = \int_0^{t_k} \cdots + \int_{t_k}^{t_{k-1}} \cdots + \int_{t_{k-1}}^{\pi} \cdots = F_k + J_k + H_k$$

tend to infinity. We verify that the main contribution comes from the integral J_k. Indeed, since $|\sin n_k t| \leqslant n_k t$ and $|f(t)| < \frac{1}{\sqrt{k}}$ on $(0, t_k)$, we have

$$|F_k| = \left| \int_0^{t_k} \cdots \right| \leqslant \frac{n_k}{\sqrt{k}} t_k = \frac{2\pi}{\sqrt{k}} \to 0$$

Since the absolute value of the integrand does not exceed $1/t$, we have

$$|H_k| \leqslant \int_{t_{k-1}}^{\pi} \frac{1}{t} dt = \ln \pi / t_{k-1} = \ln \frac{n_{k-1}}{2} < (k-1)! \ln 2.$$

Now, we calculate the integral over the middle interval,

$$J_k = \int_{t_k}^{t_{k-1}} \frac{\sin n_k t}{t} f(t)dt = \frac{1}{\sqrt{k}} \int_{t_k}^{t_{k-1}} \frac{\sin^2 n_k t}{t} dt = \frac{1}{\sqrt{k}} \int_{2\pi}^{A_k} \frac{\sin^2 u}{u} du$$

where $A_k = n_k t_{k-1} = 2\pi n_k / n_{k-1}$. Consequently, for sufficiently large k, we have

$$J_k = \frac{1}{2\sqrt{k}} \int_{2\pi}^{A_k} \frac{1 - \cos 2u}{u} du = \frac{\ln A_k + O(1)}{2\sqrt{k}} > \frac{k! \ln 2}{3\sqrt{k}}.$$

Thus,

$$I_k = F_k + J_k + H_k \geqslant \frac{k! \ln 2}{3\sqrt{k}} - (k-1)! \ln 2 + o(1) \to +\infty$$

and, therefore, $S_{n_k}(f, 0) = \frac{2}{\pi} I_k + o(1) \to +\infty$.

In the above example, we could select a subsequence $\{S_{n_k}(f, 0)\}$ that tends to $+\infty$. We must know that it is impossible to construct a continuous function for which $S_n(f, 0) \underset{n \to \infty}{\longrightarrow} +\infty$. We also remark that, in the above example, estimate (6) for the Fourier sums is almost attained (in order) on the sequence $\{n_k\}$.

By a slightly more complicated construction it is possible to give an example of a continuous function whose Fourier series diverges on a countable set. Must this series converge almost everywhere? This famous problem was open for more than half a century. It was answered in the affirmative only in 1966 by L. Carleson. It turned out that the Fourier series of an arbitrary function of class $\widetilde{\mathscr{L}}^2$ (for example a continuous function) converges to the function almost everywhere (see [C]). Since that time, several modifications and strengthenings of the original proof have been obtained, but all of them are quite difficult and lie far beyond the scope of this book.

1.3.10

Using the Riemann-Lebesgue theorem, we can obtain an important result concerning arbitrary trigonometric series, i.e., series of the form

$$A + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (A, a_n, b_n \in \mathbb{C}) \qquad (10)$$

As we know, (see Sect. 1.3.6) even an everywhere convergent trigonometric series may not be a Fourier series. At the same time, the following statement holds:

Theorem (Denjoy-Luzin) If series (10) converges absolutely on a set of positive measure, then

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < +\infty \qquad (11)$$

In particular, if a trigonometric series converges absolutely on a set of positive measure, then it converges uniformly on $\mathbb{R}$, and, therefore, is the Fourier series of its sum.

Proof Without loss of generality, we may assume that the coefficients a_n and b_n are real. We put $\varphi_n(x) = |a_n \cos nx + b_n \sin nx|$. Since the series $\sum_{n=1}^{\infty} \varphi_n$ converges on a set of positive measure, its sum is bounded on a smaller set X of positive measure,

$$\sum_{n=1}^{\infty} \varphi_n(x) \leqslant C \quad \text{for all } x \in X$$

Consequently (in what follows, λ is the one-dimensional Lebesgue measure),

$$\sum_{n=1}^{\infty} \int_X \varphi_n(x) dx \leqslant C\lambda(X)$$

We represent the functions φ_n in the form $\varphi_n(x) = c_n \left|\sin\left(nx + \theta_n\right)\right|$, where $c_n = \sqrt{a_n^2 + b_n^2}$ and $\theta_n \in \mathbb{R}$. Using the obvious inequality $|\sin t| \geqslant \sin^2 t$ and the Lebesgue-Riemann theorem, we see that

$$\int_X \frac{1}{c_n} \varphi_n(x) dx \geqslant \int_X \sin^2\left(nx + \theta_n\right) dx = \int_X \frac{1 - \cos 2\left(nx + \theta_n\right)}{2} dx \underset{n \to \infty}{\longrightarrow} \frac{\lambda(X)}{2}$$

Therefore,

$$0 < \frac{\lambda(X)}{3} \leqslant \int_X \frac{1}{c_n} \varphi_n(x) dx \text{ for } n \geqslant N$$

for some N, and, consequently,

$$\sum_{n=N}^{\infty} \frac{\lambda(X)}{3} c_n \leqslant \sum_{n=N}^{\infty} \int_X \varphi_n(x) dx \leqslant C\lambda(X)$$

Thus, the following estimate is valid for the remainder of series (11):

$$\sum_{n=N}^{\infty} \left(\left|a_n\right| + \left|b_n\right|\right) \leqslant 2 \sum_{n=N}^{\infty} \sqrt{a_n^2 + b_n^2} = 2 \sum_{n=N}^{\infty} c_n \leqslant 6$$

□

1.4 The Fourier Transform

1.4.1

We introduce one of the main concepts of this chapter.

Definition The Fourier transform $\widehat{f}$ of a function f in $\mathscr{L}^1\left(\mathbb{R}^m\right)$ is defined by the formula

$$\widehat{f}(y) = \int_{\mathbb{R}^m} f(x) e^{-2\pi i\langle y, x\rangle} dx \quad \left(y \in \mathbb{R}^m\right)$$

(here, as before, $\langle y, x\rangle$ is the scalar product of vectors y and x).

Theorem on the continuity of an integral depending on a parameter

If a function f satisfies condition $\left(L_{\mathrm{loc}}\right)$ at a point $y_0 \in Y$ and is continuous with respect to the second variable at almost all $x \in X$, i.e.,

$$f(x, y) \underset{y \rightarrow y_0}{\longrightarrow} f\left(x, y_0\right) \quad \text{for almost all } x \in X$$

then the function J is continuous at y_0 :

$$J(y)=\int_X f(x, y) d \mu(x) \underset{y \rightarrow y_0}{\longrightarrow} J\left(y_0\right)=\int_X f\left(x, y_0\right) d \mu(x)$$

implies that the function $\widehat{f}$ is continuous. This function is bounded since

$$|\widehat{f}(y)| \leqslant\|f\|_1 \quad \text{for all } y \in \mathbb{R}^m$$

Moreover, by the Riemann-Lebesgue theorem, we have $\widehat{f}(y) \rightarrow 0$ as $\|y\| \rightarrow+\infty$.

We recall that the translation f_h of a function f by a fixed vector $h \in \mathbb{R}^m$ is defined by the equation $f_h(x)=f(x-h)$. An easy calculation shows that $\widehat{f}$ and $\widehat{f}_h$ are related as follows:

$$\widehat{f_h}(y)=\int_{\mathbb{R}^m} f(x-h) e^{-2 \pi i\langle y, x\rangle} d x=\int_{\mathbb{R}^m} f(t) e^{-2 \pi i\langle y, t+h\rangle} d t=e^{-2 \pi i\langle y, h\rangle} \widehat{f}(y)$$

Another operation with the argument of a function, a contraction, is also connected with the Fourier transform: if $a \in \mathbb{R} \backslash\{0\}$ and $g(x)=f(a x)$, then

$$\widehat{g}(y)=\int_{\mathbb{R}^m} f(a x) e^{-2 \pi i\langle y, x\rangle} d x=\frac{1}{|a|^m} \int_{\mathbb{R}^m} f(t) e^{-2 \pi i \frac{1}{a}\langle y, t\rangle} d t=\frac{1}{|a|^m} \widehat{f}\left(\frac{y}{a}\right)$$

An important property of the Fourier transform relates the operations of convolution and multiplication.

Theorem If $f, g \in \mathscr{L}^1\left(\mathbb{R}^m\right)$, then $\widehat{f * g}(y)=\widehat{f}(y) \widehat{g}(y)\ \left(y \in \mathbb{R}^m\right)$. Moreover, $\int_{\mathbb{R}^m} \widehat{f}(y) g(y) d y=\int_{\mathbb{R}^m} f(y) \widehat{g}(y) d y$.

Proof The proof is an almost verbatim repetition of the corresponding reasoning for Fourier coefficients (see property (e)) of Sect. 1.1.2),

$$\begin{aligned}
\widehat{f * g}(y) & =\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^m} f(u) g(x-u) d u\right) e^{-2 \pi i\langle y, x\rangle} d x= \\
& =\int_{\mathbb{R}^m} f(u) e^{-2 \pi i\langle y, u\rangle}\left(\int_{\mathbb{R}^m} g(x-u) e^{-2 \pi i\langle y, x-u\rangle} d x\right) d u= \\
& =\int_{\mathbb{R}^m} f(u) e^{-2 \pi i\langle y, u\rangle}\left(\int_{\mathbb{R}^m} g(v) e^{-2 \pi i\langle y, v\rangle} d v\right) d u=\widehat{f}(y) \widehat{g}(y)
\end{aligned}$$

The second relation is proved similarly.□

We consider some examples.

Example 1 The Fourier transform of the characteristic function χ of the interval $(-1,1)$ is calculated very simply:

$$\widehat{\chi}(y)=\int_{-\infty}^{\infty}\chi(x)e^{-2\pi iyx}dx=\int_{-1}^{1}e^{-2\pi iyx}dx=\frac{\sin 2\pi y}{\pi y}.$$

We remark that $\widehat{\chi}\notin\mathscr{L}^1(\mathbb{R})$.

Example 2 We consider the function $f_t(x)=e^{-\pi t^2x^2}(x\in\mathbb{R}, t>0)$. Its Fourier transform is actually calculated in Example 1 of Sect. 1.1.6:

$$\widehat{f}_t(y)=\int_{-\infty}^{\infty}e^{-\pi t^2x^2}e^{-2\pi iyx}dx=2\int_0^{\infty}e^{-\pi t^2x^2}\cos 2\pi yx dx=\frac{1}{t}e^{-\frac{\pi}{t^2}y^2} \tag{1}$$

It is interesting to note that $\widehat{f_t}=\frac{1}{t}f_{\frac{1}{t}}$ and, in particular, $\widehat{f_1}=f_1$.

From Eq. (1), we immediately obtain its multi-dimensional counterpart,

$$\int_{\mathbb{R}^m}e^{-\pi t^2\|x\|^2}e^{-2\pi i\langle y,x\rangle}dx=\frac{1}{t^m}e^{-\frac{\pi}{t^2}\|y\|^2} \tag{1'}$$

Example 3 Let $f(x)=e^{-|x|}(x\in\mathbb{R})$. Then

$$\begin{aligned}\widehat{f}(y)&=\int_{-\infty}^{\infty}e^{-|x|}e^{-2\pi iyx}dx=\\&=2\mathcal{R}e\left(\int_0^{\infty}e^{-(1+2\pi iy)x}dx\right)=\mathcal{R}e\frac{2}{1+2\pi iy}=\frac{2}{1+(2\pi y)^2}\end{aligned}$$

Example 4 It is considerably harder to obtain a multi-dimensional generalization of Example 3, i.e., to calculate the Fourier transform of the function $f(x)=e^{-\|x\|}$ $(x\in\mathbb{R}^m)$ because, in this case, it is impossible to use separation of variables. The complication can be overcome by an artificial trick based on an integral representation of the function $e^{-\|x\|}$. We need the formula

$$e^{-t}=\frac{2}{\sqrt{\pi}}\int_0^{\infty}e^{-u^2-\frac{t^2}{4u^2}}du \quad \text{for every } t>0$$

To obtain it, we must represent the integral on the right-hand side in the form $e^{-t}\int_0^{\infty}e^{-\left(u-\frac{t}{2u}\right)^2}du$. After the change of variables $v=u-\frac{t}{2u}$, this integral reduces to the Euler-Poisson integral $\int_{-\infty}^{\infty}e^{-v^2}dv=\sqrt{\pi}$.

Now, we use the relation established above to calculate $\widehat{f}$,

$$\widehat{f}(y)=\int_{\mathbb{R}^m}e^{-\|x\|}e^{-2\pi i\langle y,x\rangle}dx=\frac{2}{\sqrt{\pi}}\int_{\mathbb{R}^m}\left(\int_0^{\infty}e^{-u^2-\frac{\|x\|^2}{4u^2}}du\right)e^{-2\pi i\langle y,x\rangle}dx$$

Changing the order of integration and applying Eq. (1′) with $t=\frac{1}{2\sqrt{\pi}u}$, we obtain

$$\begin{aligned}\widehat{f}(y) &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}\left(\int_{\mathbb{R}^m} e^{-\frac{\|x\|^2}{4u^2}}e^{-2\pi i\langle y,x\rangle}dx\right)du = \\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}(2\sqrt{\pi}u)^m e^{-4\pi^2u^2\|y\|^2}du = \\ &= 2^{m+1}\pi^{\frac{m-1}{2}}\int_0^\infty u^m e^{-\left(1+4\pi^2\|y\|^2\right)u^2}du\end{aligned}$$

Now, the change of variables $v = \left(1+4\pi^2\|y\|^2\right)u^2$ allows us to express the last integral in terms of the Gamma function, and we obtain the required result

$$\widehat{f}(y) = 2^m\pi^{\frac{m-1}{2}}\frac{\Gamma\left(\frac{m+1}{2}\right)}{(1+4\pi^2\|y\|^2)^{\frac{m+1}{2}}}$$

Example 5 Let $a, u > 0$, and let $f(x) = x^{a-1}e^{-ux}$ for $x > 0$ and $f(x) = 0$ for $x < 0$. Then $\widehat{f}(y) = \frac{\Gamma(a)}{(u+2\pi iy)^a}$ (we use the branch of the power function z^a equal to 1 at $z = 1$).

Before passing to a more detailed study of the properties of the Fourier transform, we show the usefulness of this notion by one more example.

Example 6 Let f be a function in $\mathscr{L}^1(\mathbb{R})$ equal to zero outside $(-\pi,\pi)$, and let f_0 be its 2π-extension from this interval to $\mathbb{R}$ $\left(f_0 \in \widetilde{\mathscr{L}^1}\right)$. The Fourier coefficients of f_0 can easily be expressed in terms of the Fourier transform of f, namely, $\widehat{f_0}(n) = \frac{1}{2\pi}\widehat{f}\left(\frac{n}{2\pi}\right)$ for all $n \in \mathbb{Z}$. We also consider the function $g(x) = e^{itx}$ for $x \in (-\pi,\pi)$, where t is a fixed number. Since the Fourier coefficients of g are equal to

$$\widehat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(t-n)x}dx = \frac{\sin\pi(t-n)}{\pi(t-n)},$$

we obtain by Parseval's generalized identity

$$\begin{aligned}\widehat{f}\left(\frac{t}{2\pi}\right) &= \int_{-\pi}^{\pi} f_0(x)\bar{g}(x)dx = 2\pi\sum_{n=-\infty}^{\infty}\widehat{f_0}(n)\overline{\widehat{g}(n)} = \\ &= \sum_{n=-\infty}^{\infty}\widehat{f}\left(\frac{n}{2\pi}\right)\frac{\sin\pi(t-n)}{\pi(t-n)}\end{aligned}$$

Thus, the following sampling formula is valid for the function $F(t) = \widehat{f}(t/2\pi)$:

$$F(t) = \sum_{n=-\infty}^{\infty} F(n)\frac{\sin\pi(t-n)}{\pi(t-n)} = \frac{\sin\pi t}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{t-n}F(n)$$

This formula allows one to recover the value of a function F at an arbitrary point t knowing the values of F on the integer lattice. This fact plays a fundamental role in optics and radio engineering because it is easier to deal with a discrete system of values than with a continuously varying signal.

1.4.2

We establish elementary relations between differentiation and the Fourier transform.

Theorem Let $f \in \mathscr{L}^1(\mathbb{R}^m)$. Then:

(1) if the partial derivative $g = \frac{\partial f}{\partial x_k}$ is summable and continuous for some $k = 1, \ldots, m$, then

$$\widehat{g}(y) = 2\pi i y_k \widehat{f}(y) \quad (y \in \mathbb{R}^m)$$

(2) if the product $\|x\| f(x)$ is summable, then $\widehat{f} \in C^1(\mathbb{R}^m)$ and the equation

$$\frac{\partial \widehat{f}(y)}{\partial y_k} = -2\pi i \widehat{f_k}(y), \qquad \text{where } f_k(x) = x_k f(x)\,(x \in \mathbb{R}^m)$$

holds for all $y \in \mathbb{R}^m$ and $k = 1, \ldots, m$

Proof

(1) Without loss of generality, we will assume that $k = m$. We identify a point $x = (x_1, \ldots, x_{m-1}, t)$ with the pair (u, t), where $u = (x_1, \ldots, x_{m-1}) \in \mathbb{R}^{m-1}$. First, we verify that $f(u,t) \to 0$ as $t \to \pm\infty$ for almost all points $u \in \mathbb{R}^{m-1}$. Indeed, since the derivative $f'_t = g$ is continuous, we have

$$f(u,t) - f(u,0) = \int_0^t g(u,s)ds$$

From Fubini's theorem, it follows that the function $t \mapsto g(u,t)$ is summable for almost all u, and, therefore,

$$f(u,t) - f(u,0) = \int_0^t g(u,s)ds \underset{t\to\pm\infty}{\longrightarrow} \int_0^{\pm\infty} g(u,s)ds$$

Thus, the limits $\lim_{t\to\pm\infty} f(u,t)$ exist and are finite for almost all $u \in \mathbb{R}^{m-1}$. However, since (again, by Fubini's theorem) the function $t \mapsto f(u,t)$ is summable for almost all u, we see that the limits are zero for such u and, therefore,

$$\int_{-\infty}^{\infty} g(u,t)e^{-2\pi i y_m t}dt = f(u,t)e^{-2\pi i y_m t}\big|_{-\infty}^{\infty} - (-2\pi i y_m)\int_{-\infty}^{\infty} f(u,t)e^{-2\pi i y_m t}dt$$
$$= 2\pi i y_m \int_{-\infty}^{\infty} f(u,t)e^{-2\pi i y_m t}dt.$$

To obtain the required result, it only remains to multiply both sides of this equation by $e^{-2\pi i(y_1x_1+\cdots+y_{m-1}x_{m-1})}$ and integrate with respect to u.

To obtain the equation of (2), we must apply the Leibnitz rule.

Let $Y \subset \mathbb{R}$ be an arbitrary interval. Assume that:

(a) *the derivative*

$$f'_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

exists for almost all $x \in X$ and every $y \in Y$;
(b) *the function f_y' satisfies condition (L_{loc}) at a point $y_0 \in Y$. Then the function J is differentiable at y_0 and*

$$J'(y_0) = \int_X f_y'(x, y_0)\, d\mu(x)$$

The functions $f_1, \ldots, f_m$ are summable by assumption. Therefore, their Fourier transforms and the first order partial derivatives of $\widehat{f}$ are continuous everywhere. Consequently, $\widehat{f} \in C^1(\mathbb{R}^m)$.

Corollary

If $f \in \mathscr{L}^1(\mathbb{R}^m)$ is a compactly supported function, then $\widehat{f} \in C^\infty(\mathbb{R}^m)$; if $f \in C_0^\infty(\mathbb{R}^m)$, then the product $\|y\|^p \widehat{f}(y)$ is summable in $\mathbb{R}^m$ for every $p > 0$.

Proof The fact that $\widehat{f}$ is infinitely differentiable follows directly from the second assertion of the theorem because the product $\|x\|^n f(x)$ is summable for every $n \in \mathbb{N}$.

If $f \in C_0^\infty(\mathbb{R}^m)$, then the derivatives of all orders of f are summable and the relation

$$\left(\frac{\partial^n f}{\partial y_k^n}\right)\widehat{(y)} = (2\pi i y_k)^n \widehat{f}(y)$$

is fulfilled for all $k = 1, \ldots, m$ and $n \in \mathbb{N}$. Since the functions $\left(\frac{\partial^n f}{\partial y_k^n}\right)^{-}(y)$ are bounded, we obtain the estimate

$$|\widehat{f}(y)| \leqslant \text{ const } \cdot \left(1 + |y_1|^n + \cdots + |y_m|^n\right)^{-1}$$

providing (if we take sufficiently large n) the summability of $\|y\|^p \widehat{f}(y)$.□

In many problems, it is important to know the rate of decrease of the Fourier transform at infinity. The theorem proved above shows that a fast decrease can be provided by the smoothness of the function in question. How accurate are these conditions? What can be expected if smoothness fails on a "small" set? The following examples are devoted to such results.

Example 1 Supplementing Examples 2 and 3 of Sect. 1.4.1, we investigate the asymptotic behavior of the Fourier transform of the function $f(x) = e^{-|x|^p}$ at infinity for $0 < p < 2$. After integrating by parts, we see that

$$\widehat{f}(y) = 2\int_0^\infty e^{-x^p} \cos(2\pi x y) dx = \frac{p}{\pi y}\int_0^\infty e^{-x^p} x^{p-1} \sin(2\pi x y) dx,$$

which implies the crude estimate $\widehat{f}(y) = o(1/y)$ as $y \to +\infty$. We study the behavior of $\widehat{f}(y)$ for large y in detail. If $0 < p < 1$, then the change $2\pi x y = u$ leads to the equation

$$\widehat{f}(y) = \frac{2p}{(2\pi y)^{p+1}}\int_0^\infty e^{-\left(\frac{u}{2\pi y}\right)^p} \frac{\sin u}{u^{1-p}} du$$

The integral $\int_0^\infty \frac{\sin u}{u^{1-p}} du$ of the limit function (as $y \to +\infty$) converges, and justifies the passage to the limit,

$$\int_0^\infty e^{-\left(\frac{u}{2\pi y}\right)^p} \frac{\sin u}{u^{1-p}} du \underset{y\to+\infty}{\longrightarrow} \int_0^\infty \frac{\sin u}{u^{1-p}} du = \Gamma(p) \sin \frac{\pi p}{2}$$

Consequently, the estimate

$$\widehat{f}(y) \sim \frac{C_p}{y^{p+1}} \quad \text{as } y \to +\infty$$

is valid for $0 < p < 1$ with constant $C_p = \frac{2\Gamma(p+1)}{(2\pi)^{p+1}} \sin \frac{\pi p}{2}$. This, in particular, implies that the function $\widehat{f}$ is summable.

Now, let $1 < p < 2$ (the case where $p = 1$ was considered in Example 3 of Sect. 1.4.1). Integrating the right-hand side of (2) one more time, we arrive at the equation

$$\begin{aligned}\widehat{f}(y) =& \frac{p}{2(\pi y)^2} \left((p-1) \int_0^\infty x^{p-2} e^{-x^p} \cos(2\pi y x) dx \right. \\ & \left. -p \int_0^\infty x^{2(p-1)} e^{-x^p} \cos(2\pi y x) dx \right).\end{aligned}$$

Here, the second integral admits the estimate $o(1/y)$ as $y \to +\infty$, but the first integral tends to zero more slowly. Indeed, by an almost verbatim repetition of the reasoning given in the case where $0 < p < 1$, we obtain

$$\int_0^\infty x^{p-2} e^{-x^p} \cos(2\pi y x) dx \underset{y\to+\infty}{\sim} \frac{1}{(2\pi y)^{p-1}} \Gamma(p-1) \sin \frac{\pi p}{2}.$$

Thus, we again come to relation (3), which is also valid for $p = 1$; for $p = 2$, the coefficient C_p vanishes, and the asymptotic of $\widehat{f}$ changes completely (see Examples 2 and 3 of Sect. 1.4.1).

It can be proved that $\widehat{f}(y) > 0$ for $0 < p \leqslant 2$ (for $0 < p \leqslant 1$.

Example 2 Let us determine how fast the Fourier transform of the characteristic function of the unit ball

$$\widehat{\chi}_B(y) = \int_B e^{-2\pi i \langle y,x \rangle} dx = \int_B e^{-2\pi i \|y\| x_1} dx = \alpha_{m-1} \int_{-1}^1 \left(1-t^2\right)^{\frac{m-1}{2}} e^{-2\pi i \|y\| t} dt$$

decreases at infinity (α_{m-1} is the volume of the unit ball in $\mathbb{R}^{m-1}$). For odd m the "integral can be calculated" and $\widehat{\chi}_B$ can be expressed explicitly in terms of $\|y\|$. In particular, in the one-dimensional case, we have $B = (-1, 1)$ and $\widehat{\chi}_B(y) = \frac{\sin 2\pi y}{\pi y}$. For $m = 3$, we have $\widehat{\chi}_B(y) = \frac{1}{\pi \|y\|^2} \left(\frac{\sin 2\pi \|y\|}{2\pi \|y\|} - \cos 2\pi \|y\| \right)$.

For even m, the situation is more complicated. In this case, χ_B can be expressed in terms of the Bessel function. However, an exact formula for $\widehat{\chi}_B(y)$ is not our main concern here. We want to study the asymptotic behavior of this function as $\|y\| \to +\infty$. To this end, we put $r = 2\pi\|y\|$ and consider the integrals

$$I_m(r) = \int_{-1}^{1} \left(1-t^2\right)^{\frac{m-1}{2}} e^{-irt} dt \quad (m = 0, 1, 2, \ldots)$$

The larger m is, the more derivatives of the function $\left(1-t^2\right)^{\frac{m-1}{2}}$ vanish at the endpoints of the interval of integration. Therefore, as m increases, the rate at which the integrals $I_m(r)$ tend to zero as $r \to +\infty$ also increases. To describe this in more detail, we use the recurrence formula

$$I_m(r) = \frac{m-1}{r^2}\left((m-2)I_{m-2}(r) - (m-3)I_{m-4}(r)\right) \quad (m \geqslant 4)$$

which can easily be obtained by twofold integration by parts. From this relation, we see that, to obtain the asymptotic of the integral $I_m(r)$, it is sufficient to know only the asymptotics of the integrals $I_0(r)$ and $I_2(r)$ or of $I_1(r)$ and $I_3(r)$, depending on the parity of m. The integrals $I_1(r)$ and $I_3(r)$ can easily be calculated,

$$I_1(r) = 2\frac{\sin r}{r}, \quad I_3(r) = \frac{4}{r^2}\left(\frac{\sin r}{r} - \cos r\right)$$

The integrals $I_0(r)$ and $I_2(r)$ coincide with the integrals $C(r)$ and $S(r)$, respectively:

$$I_0(r) = C(r) = \sqrt{\frac{\pi}{r}}(\sin r + \cos r) + O\left(\frac{1}{r}\right),$$
$$I_2(r) = S(r) = \frac{\sqrt{\pi}}{r^{3/2}}(\sin r - \cos r) + O\left(\frac{1}{r^2}\right).$$

The last four formulas can be written uniformly as follows:

$$I_m(r) = \frac{\gamma_m}{r^{\frac{m+1}{2}}}\cos\left(r - \varphi_m\right) + O\left(\frac{1}{r^{\frac{m}{2}+1}}\right) \quad (r \to +\infty)$$

where $m = 0, 1, 2, 3$, $\varphi_m = \frac{\pi}{4}(m+1)$, and γ_m is a positive coefficient depending only on m. The recurrence formula allows us to extend this relation to all positive integers m.

Returning to the Fourier transform of the function χ_B, we see that

$$\widehat{\chi}_B(y) = \alpha_{m-1} I_m(2\pi\|y\|) = \frac{C_m}{\|y\|^{\frac{m+1}{2}}}\cos\left(2\pi\|y\| - \varphi_m\right) + O\left(\frac{1}{\|y\|^{\frac{m}{2}+1}}\right).$$

It can be verified that $C_m = 1/\pi$ for all m.

It is interesting to compare $\widehat{\chi}_B$ with the function $\widehat{\chi}_Q$, where $Q = (-1,1)^m$. It is clear that

$$\widehat{\chi}_Q(y) = \prod_{j=1}^{m} \frac{\sin 2\pi y_j}{\pi y_j}$$

If the angles between the vector y and the coordinate axes are non-zero, then this function admits the estimate $O\left(\|y\|^{-m}\right)$. Thus, for most directions, the function decreases considerably faster than $\widehat{\chi}_B$. One possible sharpening of this assertion is as follows: the integrals $L_B(R) = \int_{\|y\|<R} |\widehat{\chi}_B(y)|\,dy$ grow considerably faster than the integrals $L_Q(R) = \int_{\|y\|<R} |\widehat{\chi}_Q(y)|\,dy$ as $R \to +\infty$. Indeed,

$$\begin{aligned} L_Q(R) &\leqslant \int_{[-R,R]^m} |\widehat{\chi}_Q(y)|\,dy = \prod_{j=1}^{m} \int_{-R}^{R} \left| \frac{\sin 2\pi y_j}{\pi y_j} \right| dy_j \\ &= \left(\frac{2}{\pi} \int_0^{2\pi R} \frac{|\sin t|}{t} dt \right)^m . \end{aligned}$$

It follows that $L_Q(R) = O\left(\ln^m R\right)$ as $R \to +\infty$. At the same time

$$\begin{aligned} L_B(R) &= \alpha_m \int_0^R \left| \frac{C_m}{r^{\frac{m+1}{2}}} \cos\left(2\pi r - \varphi_m\right) + O\left(\frac{1}{r^{\frac{m}{2}+1}} \right) \right| r^{m-1} dr \\ &= \alpha_m C_m \int_0^R r^{\frac{m-3}{2}} \left|\cos\left(2\pi r - \varphi_m\right)\right| dr + O\left(R^{\frac{m}{2}-1}\right) \end{aligned}$$

for $m > 2$ (for $m = 2$, the remainder term has order $O(\ln R)$). Therefore, $L_B(R) = O\left(R^{\frac{m-1}{2}}\right)$, and the estimate is exact by order,

$$\begin{aligned} L_B(R) &\geqslant \alpha_m C_m \int_{R/2}^{R} r^{\frac{m-3}{2}} \cos^2\left(2\pi r - \varphi_m\right) dr + O\left(R^{\frac{m}{2}-1}\right) \\ &\geqslant \ \text{const}\ R^{\frac{m-3}{2}} \int_{R/2}^{R} \left(1 + \cos 2\left(2\pi r - \varphi_m\right)\right) dr + O\left(R^{\frac{m}{2}-1}\right) \\ &= \frac{\text{const}}{2} R^{\frac{m-1}{2}} + O\left(R^{\frac{m}{2}-1}\right) \end{aligned}$$

Example 3 It follows from the theorem that the condition $f(x) = O\left(\|x\|^{-p}\right)$ as $\|x\| \to +\infty$ implies the smoothness of the Fourier transform for $p > m + 1$. It turns out that this restriction cannot be weakened essentially. To verify this, we show that if $f(x) \sim \|x\|^{-p}$ as $\|x\| \to +\infty$, then the differentiability of $\widehat{f}$ at zero implies the inequality $p > m + 1$.

Without loss of generality, we may assume that $f \geqslant 0$. Indeed, we know that $f(x) \geqslant 0$ for large $\|x\|$, but changing the function on an arbitrary ball (for example, putting $f(x) = 0$ inside the ball), we change the Fourier transform of f by an infinity differentiable function.

Assuming that $f \geqslant 0$, we study the mean value of the difference $\widehat{f}(0) - \widehat{f}$ in the vicinity of zero (in what follows, B is the unit ball centered at zero and v is the volume of B). We put

$$I(r) = \frac{1}{v} \int_B (\widehat{f}(0) - \widehat{f}(ry))dy$$

Since $\widehat{f}$ is differentiable at zero, we obtain that $I(r) = o(r)$ as $r \to +0$. Now, we estimate the integral $I(r)$ from below. Since

$$\widehat{f}(0) - \widehat{f}(ry) = \int_{\mathbb{R}^m} f(x) \left(1 - e^{-2\pi i r\langle y,x\rangle}\right) dx$$

we obtain, by Fubini's theorem, that

$$I(r) = \int_{\mathbb{R}^m} f(x) \left(1 - \frac{1}{v} \int_B e^{-2\pi irs\langle y,x\rangle} dy\right) dx = \int_{\mathbb{R}^m} f(x) \left(1 - \frac{1}{v} \widehat{\chi}(rx)\right) dx,$$

where χ is the characteristic function of B. Obviously, $\widehat{\chi}(x) \in \mathbb{R}$ and $|\widehat{\chi}(x)| \leqslant v$, and the Riemann-Lebesgue theorem implies that $\widehat{\chi}(x) \to 0$ as $\|x\| \to +\infty$. We take a sufficiently large radius R so that $f(x) > \frac{1}{2\|x\|^p}$ and $|\widehat{\chi}(x)| < \frac{v}{2}$ for $\|x\| > R$. Then, since $f \geqslant 0$, we have

$$\begin{aligned} I(r) &\geqslant \int_{\|x\|>R/r} f(x) \left(1 - \frac{1}{v} \widehat{\chi}(rx)\right) dx \\ &\geqslant \frac{1}{4} \int_{\|x\|>R/r} \frac{dx}{\|x\|^p} = \frac{\sigma\left(S^{m-1}\right)}{4} \int_{R/r} \frac{dt}{t^{p-m+1}}. \end{aligned}$$

Thus, $I(r) \geqslant \text{const } r^{p-m}$. Since $I(r) = o(r)$ for $r \to 0$, we obtain that $p > m + 1$.

1.4.3

In the one-dimensional case, for a function f differentiable at a point x, there is an important formula allowing one to find the value of $f(x)$ from $\widehat{f}$. This formula is called the inversion formula and has the following form:

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(y) e^{2\pi ixy} dy$$

The integral on the right-hand side of this equation is called the Fourier integral of f. In general, this is an improper integral because the Fourier transform can be non-summable on $\mathbb{R}$ (see Sect. 1.4.1). We will say that the integral converges if there exists a limit of the partial integrals

$$I_A(f, x) = \int_{-A}^{A} \widehat{f}(y) e^{2\pi ixy} dy$$

as $A \to +\infty$.

There is an obvious analogy between the expansion of a periodic function in a Fourier series and the Fourier integral representation of a non-periodic function. The following theorem shows that these two problems share not only some superficial analogies but are connected in essence. To show this, we need the following easy lemma.

Lemma Let $f \in \mathscr{L}^1(\mathbb{R})$ and $x \in \mathbb{R}$. Then the following holds for every $A > 0$

$$I_A(f,x) = \int_{-A}^{A} \widehat{f}(y) e^{2\pi i x y} dy = \int_{-\infty}^{\infty} f(x-t) \frac{\sin 2\pi A t}{\pi t} dt.$$

Proof It is clear that

$$I_A(f,x) = \int_{-A}^{A} \left(\int_{-\infty}^{\infty} f(u) e^{2\pi i (x-u) y} du \right) dy$$

Since the function $(y,u) \mapsto f(u) e^{2\pi i (x-u)y}$ is summable in the strip $(-A, A) \times \mathbb{R}$, we may use Fubini's theorem,

$$I_A(f,x) = \int_{-\infty}^{\infty} \left(\int_{-A}^{A} f(u) e^{2\pi i (x-u)y} dy \right) du = \int_{-\infty}^{\infty} f(u) \frac{\sin 2\pi A (x-u)}{\pi (x-u)} du$$

It remains to change the integration variable $t = x - u$.$\square$

By the Riemann-Lebesgue theorem, the integral $\int_{|t| \geqslant \delta} f(x-t) \frac{\sin 2\pi A t}{\pi t} dt$ tends to zero as $A \to +\infty$ for every $\delta > 0$. Therefore, the lemma implies the asymptotic relation

$$I_A(f,x) = \int_{-\delta}^{\delta} f(x-t) \frac{\sin 2\pi A t}{\pi t} dt + o(1) \qquad \text{as } A \to +\infty \tag{4}$$

(we already know a similar result for the partial sums of Fourier series; see Eq. $(5')$ of Sect. 1.2.4). Thus, the behavior of the integrals $I_A(f,x)$ as $A \to +\infty$ is determined only by the values of f in the vicinity of x. In other words, we have the same localization principle for Fourier integrals as for Fourier series. Furthermore, it is easy to prove the equiconvergence of the expansions in the Fourier series and the Fourier integral. More precisely, the following statement holds.

Theorem If functions $f \in \mathscr{L}^1(\mathbb{R})$ and $f_0 \in \tilde{\mathscr{L}}^1$ coincide in a neighborhood of a point x, then the convergence of the Fourier integral of f at x is equivalent to the convergence of the Fourier series of f_0 at x, and, in the case of convergence, the following holds:

$$\int_{-\infty}^{\infty} \widehat{f}(y) e^{2\pi i x y} dy = \sum_{n=-\infty}^{\infty} \widehat{f_0}(n) e^{inx}$$

From the theorem, it obviously follows that the convergence tests for Fourier series, obtained in Sect. 1.2.4, can be carried over to the Fourier integrals. In particular, the inversion formula is valid at a point x if Dini's condition is fulfilled at x with $C = f(x)$. We leave it to the reader to state an analog of the Dirichlet-Jordan test.

Proof We show that the following holds:

$$I_A(f,x) - S_{[2\pi A]}(f_0,x) \underset{A\to+\infty}{\longrightarrow} 0$$

where $[u]$, as usual, is the integer part of u.

Let $f(x-t) = f_0(x-t)$ for $|t| < \delta$, where $0 < \delta < \pi$. By Eq.(4) and Eq.($5'$) of Sect. 1.2.4, we have

$$I_A(f,x) = \int_{-\delta}^{\delta} f(x-t)\frac{\sin 2\pi At}{\pi t}dt + o(1) = \int_{-\delta}^{\delta} f_0(x-t)\frac{\sin 2\pi At}{\pi t}dt + o(1),$$

$$S_n(f_0,x) = \int_{-\pi}^{\pi} f_0(x-t)\frac{\sin nt}{\pi t}dt + o(1) = \int_{-\delta}^{\delta} f_0(x-t)\frac{\sin nt}{\pi t}dt + o(1)$$

as $A, n \to +\infty$. If $2\pi A = n \in \mathbb{N}$, then we immediately obtain the required relation. If $2\pi A$ is not integer, then we have $n < 2\pi A < n+1$ for $n = [2\pi A]$, and, therefore,

$$\begin{aligned}\left|I_A(f,x) - I_{n/2\pi}(f,x)\right| &\leqslant \int_{A-\frac{1}{2\pi}}^{A} |\widehat{f}(y)|dy + \int_{-A}^{-A+\frac{1}{2\pi}} |\widehat{f}(y)|dy \\ &\leqslant 2\max_{|y|\geqslant A-1} |\widehat{f}(y)| \underset{A\to+\infty}{\longrightarrow} 0\end{aligned}$$

Thus,

$$I_A(f,x) - S_n(f_0,x) = \big(I_A(f,x) - I_{n/2\pi}(f,x)\big) + \big(I_{n/2\pi}(f,x) - S_n(f_0,x)\big)$$

where each of the two differences on the right-hand side tend to zero as $A \to +\infty$.□

Now, we once again turn to Examples 2 and 3 considered in Sect. 1.2.1.

Example 1 From the theorem, it follows that the inversion formula is valid for the function $f_t(x) = e^{-\pi t^2 x^2} (x \in \mathbb{R}, t > 0)$. However, this already follows from the relation $\widehat{f_t} = \frac{1}{t} f_{\frac{1}{t}}$ established in Example 2 of Sect. 1.2.1. Indeed, since the function $\widehat{f_t}$ is even and summable, we have

$$\int_{-\infty}^{\infty} \widehat{f_t}(y)e^{2\pi ixy}dy = \left(\widehat{f_t}\right)\widehat{(x)} = \frac{1}{t}\left(f_{\frac{1}{t}}\widehat{)}(x) = f_t(x)$$

Example 2 The function $f(x) = e^{-|x|} (x \in \mathbb{R})$ satisfies Dini's condition at every point (in particular, at zero). The Fourier transform of f was calculated in Example 3 of Sect. 1.2.1. By the inversion formula, we obtain

$$
\begin{aligned}
e^{-|x|} &= \int_{-\infty}^{\infty} \widehat{f}(y) e^{2\pi i y x} dy = \int_{-\infty}^{\infty} \frac{2e^{2\pi i y x}}{1+4\pi^2 y^2} dy \\
&= \int_{0}^{\infty} \frac{4\cos 2\pi y x}{1+4\pi^2 y^2} dy = \frac{2}{\pi}\int_{0}^{\infty} \frac{\cos xt}{1+t^2} dt
\end{aligned}
$$

Thus, we again obtain the value of the Laplace integral

$$
\int_{0}^{\infty} \frac{\cos xt}{1+t^2} dt = \frac{\pi}{2} e^{-|x|}
$$

.

1.4.4

Generalizing the inversion formula to functions of several variables, we confine ourselves to the most important case where the Fourier transform is summable. In this connection, we note that Dini's condition providing the validity of the inversion formula in the one-dimensional case is a local property of a summable function, whereas the summability of the Fourier transform is a global property.

In contrast to the one-dimensional setting, now, when deriving the inversion formula, we cannot use the equiconvergence of the expansions in the Fourier series or Fourier integral since Theorem 1.2.3 cannot be carried over to the multidimensional case).

The transformation that assigns the function $\check{g}$ defined by the formula

$$
\check{g}(x) = \int_{\mathbb{R}^m} g(y) e^{2\pi i \langle x,y \rangle} dy \quad (x \in \mathbb{R}^m)
$$

to $g \in \mathscr{L}^1(\mathbb{R}^m)$ is called the inverse transform. Obviously, $\check{g}(x) = \widehat{g}(-x)$, and so the properties of the Fourier transform can easily be carried over to the inverse transform. Using the inverse transform, we can represent the inversion formula proved in the one-dimensional case in the following form:

$$
f(x) = (\widehat{f})\check{}(x) \qquad (5)
$$

This justifies the choice of the term "inverse transform".

Theorem Let $f \in \mathscr{L}^1(\mathbb{R}^m)$. If $\widehat{f} \in \mathscr{L}^1(\mathbb{R}^m)$, then inversion formula (5) is valid for almost all x in $\mathbb{R}^m$.

We remark that the right-hand side of Eq. (5) continuously depends on x since $\widehat{f}$ is summable. Therefore, the condition of the theorem (the summability of $\widehat{f}$) can be fulfilled only if the function f is equivalent to a continuous function. Moreover, Eq. (5) is valid at all points where f is continuous because it is valid on a set of full measure. In particular, if f is continuous and its Fourier transform is summable, then $f(x) = (\widehat{f})\check{}(x)$ for all $x \in \mathbb{R}^m$.

Proof We use the approximate identity W_t, which played an important role in the proof of the Weierstrass theorem. We recall that $W_t(x) = \frac{1}{t^m} e^{-\frac{\pi}{t^2}\|x\|^2}$ $(x \in \mathbb{R}^m, t > 0)$. Our proof of the theorem is based on inversion formula $(1')$ for this function,

$$W_t(x) = \int_{\mathbb{R}^m} e^{-\pi t^2 \|y\|^2} e^{2\pi i \langle x, y\rangle} dy \qquad (6)$$

First, for the smoothened function $f * W_t$, we obtain an equation close to (5). Then, we obtain the statement of the theorem by passage to the limit.

Using Eq. (6) and changing the order of integration, we obtain, for every $t > 0$ that

$$\begin{aligned}
(f * W_t)(x) &= \int_{\mathbb{R}^m} f(y) W_t(x - y) dy \\
&= \int_{\mathbb{R}^m} f(y) \left(\int_{\mathbb{R}^m} e^{-\pi t^2 \|u\|^2} e^{2\pi i \langle x-y, u\rangle} du \right) dy \\
&= \int_{\mathbb{R}^m} e^{-\pi t^2 \|u\|^2} e^{2\pi i \langle x, u\rangle} \left(\int_{\mathbb{R}^m} f(y) e^{-2\pi i \langle y, u\rangle} dy \right) du.
\end{aligned}$$

Thus, we have established the required relation,

$$(f * W_t)(x) = \int_{\mathbb{R}^m} e^{-\pi t^2 \|u\|^2} e^{2\pi i \langle x, u\rangle} \widehat{f}(u) du \qquad (7)$$

Since the absolute value of the integrand in the last integral does not exceed $|\widehat{f}|$, we obtain, by Lebesgue's theorem, that, for every x, this integrals tends to $(\widehat{f})\check{}(x)$ as $t \to +0$.

Now, we can finish the proof, referring to Theorem 1.2.4, from which it follows that the limit on the left-hand side of Eq. (7) coincides with $f(x)$ almost everywhere. However, it is possible to dispense with the use of the theorem based on the notion of a Lebesgue point and on Theorem on differentiation of an integral with respect to a set. We show that the left-hand side of Eq. (7) tends to $f(x)$ almost everywhere as t tends to zero along a sequence.

Indeed, let $\{t_n\}$ be a sequence such that $t_n \underset{n\to\infty}{\longrightarrow} 0$. By Riesz's theorem, there is a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that, almost everywhere $f * W_{t_{n_k}} \to f$ as $k \to \infty$. Replacing t by t_{n_k} in Eq. (7) and passing to the limit, we obtain the required result.

Example We give the inversion formula for the function $f(x) = e^{-\|x\|}$ $(x \in \mathbb{R}^m)$ whose Fourier transform is calculated in Example 4 of Sect. 1.3.1,

$$\begin{aligned}
e^{-\|x\|} &= 2^m \pi^{\frac{m-1}{2}} \Gamma\left(\frac{m+1}{2}\right) \int_{\mathbb{R}^m} \frac{e^{2\pi i \langle x, y\rangle} dy}{(1 + 4\pi^2 \|y\|^2)^{\frac{m+1}{2}}} \\
&= \frac{\Gamma\left(\frac{m+1}{2}\right)}{\pi^{\frac{m+1}{2}}} \int_{\mathbb{R}^m} \frac{\cos\langle x, t\rangle dt}{(1 + \|t\|^2)^{\frac{m+1}{2}}}
\end{aligned}$$

In the one-dimensional case, this formula was obtained in Example 2 of Sect. 1.4.3.

The summability of the Fourier transform is important in many problems. This condition is necessarily fulfilled if $\widehat{f} \geqslant 0$ and the function f is continuous (or at least bounded in a neighborhood of zero). In this connection, we recall that $\widehat{f} \geqslant 0$ if f is an even function summable on $\mathbb{R}$ and convex on $(0, +\infty)$. Together with the inversion formula, this proves the following statement.

Corollary If an even continuous function f is summable on the real line and is convex on the positive semi-axis, then f is the Fourier transform of a non-negative summable function.

The fact just proved remains valid even if, instead of the summability of f, we assume only that $f(x) \underset{x \to +\infty}{\longrightarrow} 0$, but, in this case, the proof invokes a subtler reasoning

1.4.5

Here, we discuss one more important property of the Fourier transform, its injectivity on the entire set of summable functions. Of course, there is no injectivity in the literal sense because distinct equivalent (i.e., coinciding almost everywhere) functions have the same Fourier transform. However, Theorem 1.4.4 shows that the injectivity holds up to equivalence on the set of functions with summable Fourier transform. To strengthen this result, we generalize Definition 1.4.1 somewhat.

Definition Let μ be a finite Borel measure on $\mathbb{R}^m$. The function $y \mapsto \widehat{\mu}(y) \equiv \int_{\mathbb{R}^m} e^{-2\pi i \langle y,x \rangle} d\mu(x)$ is called the Fourier transform of μ.

If a measure μ has a density f with respect to Lebesgue measure, then $\widehat{\mu} = \widehat{f}$.

Now, we establish an important result connected with the injectivity of the Fourier transform of a measure.

Theorem If two finite Borel measures μ and v have the same Fourier transform, then the measures coincide.

Proof Let $H_j(t) = \{(x_1, \ldots, x_m) \mid x_j = t\}$ be a plane perpendicular to the j th coordinate axis, and let

$$E = \{t \in \mathbb{R} \mid \mu\left(H_j(t)\right) = v\left(H_j(t)\right) = 0 \text{ for every } j = 1, \ldots, m\}$$

The set E is everywhere dense because the set $\{t \in \mathbb{R} \mid \mu\left(H_j(t)\right) > 0\}$ is at most countable for each j. Therefore, the Borel hull of the semiring $\mathscr{P}_E^m$ consisting of the cells whose vertices have coordinates belonging to E coincides with the σ-algebra of Borel subsets of the space $\mathbb{R}^m$. We express the measure of the cell $P = \prod_{j=1}^m \Delta_j$ in terms of $\widehat{\mu}$, assuming that $P \in \mathscr{P}_E^m$.

Obviously, $\chi_P(x) = \prod_{j=1}^m \chi_{\Delta_j}(x_j)$, where $x_1, \ldots, x_m$ are the coordinates of a vector x. By Fubini's theorem, $\widehat{\chi}_P(y) = \prod_{j=1}^m \widehat{\chi}_{\Delta_j}(y_j)$ for $y = (y_1, \ldots, y_m)$, and, therefore,

$$I_A\left(\chi_\Delta, x\right) = \int_{(-A,A)^m} \widehat{\chi}_P(y) e^{2\pi i \langle x,y \rangle} dy = \prod_{j=1}^m \int_{-A}^A \widehat{\chi}_{\Delta_j}\left(y_j\right) e^{2\pi i x_j y_j} dy_j$$

The characteristic function of an interval satisfies Dini's condition everywhere except the endpoints of the interval. Therefore, we have

$$\int_{-A}^{A} \widehat{\chi}_{\Delta_j}(y_j)\, e^{2\pi i x_j y_j} dy_j \underset{A\to+\infty}{\longrightarrow} \chi_{\Delta_j}(x_j)$$

for all $j = 1, \ldots, m$, provided that x_j is distinct from the endpoints of the interval Δ_j. Since $P \in \mathscr{P}_E^m$, we see that $I_A(\chi_\Delta, x) \underset{A\to+\infty}{\longrightarrow} \chi_P(x)\mu$-almost everywhere. Moreover, putting $\Delta_j = [a_j, b_j\ )$, we obtain that

$\int_{-A}^{A} \widehat{\chi}_{\Delta_j}(y_j)\, e^{2\pi i x_j y_j} dy_j = \int_{-\infty}^{\infty} \chi_{\Delta_j}(x_j - t)\, \frac{\sin 2\pi A t}{t} dt = \int_{A(x_j-b_j)}^{A(x_j-a_j)} \frac{\sin 2\pi u}{u} du.$

All these integrals are bounded (since the integral $\int_0^{\infty} \frac{\sin 2\pi u}{u} du$ converges), and so, the integral $I_A(\chi_\Delta, x)$ is also bounded (uniformly with respect to x and A).

Therefore, we can use Lebesgue's theorem on passing to the limit under the integral sign,

$$\begin{aligned}\mu(P) &= \int_{\mathbb{R}^m} \chi_P(x) d\mu(x) = \lim_{A\to+\infty} \int_{\mathbb{R}^m} I_A(\chi_\Delta, x)\, d\mu(x)\\ &= \lim_{A\to+\infty} \int_{\mathbb{R}^m} \left(\int_{(-A,A)^m} \widehat{\chi}_P(y) e^{2\pi i\langle x,y\rangle} dy \right) d\mu(x)\end{aligned}$$

Changing the order of integration, we obtain

$$\begin{aligned}\mu(P) &= \lim_{A\to+\infty} \int_{(-A,A)^m} \widehat{\chi}P(y) \left(\int_{\mathbb{R}^m} e^{2\pi i\langle x,y\rangle} d\mu(x) \right) dy\\ &= \lim_{A\to+\infty} \int_{(-A,A)^m} \widehat{\chi}P(y) \widehat{\mu}(-y) dy\end{aligned}$$

This relation shows that the values of the measure on the cells belonging to $\mathscr{P}_E^m$ can be expressed in terms of its Fourier transform. Since the measures μ and v have the same Fourier transform, they coincide on the semiring $\mathscr{P}_E^m$, and, consequently, (by the uniqueness theorem) on all Borel sets.□

It follows from the above theorem that the Fourier transform is injective up to equivalence on the set of summable functions.

Corollary 1 If two summable functions f and g have the same Fourier transform, they coincide almost everywhere.

Proof It is clear that the Fourier transform of the functions $\bar{f}$ and $\bar{g}$ also coincide. Consequently, the functions $\mathcal{R}ef = (f + \bar{f})/2$ and $\mathcal{R}eg = (g + \bar{g})/2$, as well as the imaginary parts of the functions f and g, have the same Fourier transform. Therefore, we may assume that the functions f and g are real.

If they are non-negative, the theorem just proved implies that the measures with the densities f and g coincide. It was proved that, in this case, the densities coincide almost everywhere.

In the general case, we represent f and g in the form $f = f_+ - f_-$ and $g = g_+ - g_-$, where $f_\pm, g_\pm \geqslant 0$. Then

$$\widehat{f} = \widehat{f}_+ - \widehat{f}_- = \widehat{g} = \widehat{g}_+ - \widehat{g}_-$$

Consequently, the non-negative functions $f_+ + g_-$and $f_- + g_+$have the same Fourier transform, and, therefore, they coincide almost everywhere, which is equivalent to the assertion of the corollary.□

Corollary 2 If finite Borel measures μ and v have the same values on all halfspaces (in $\mathbb{R}^m$), then they coincide.

Proof By the theorem, it is sufficient to verify that $\widehat{\mu}(y) = \widehat{v}(y)$ for all $y \in \mathbb{R}^m$. For $y = 0$, the equality holds since $\widehat{\mu}(0) = \mu(\mathbb{R}^m)$ and $\widehat{v}(0) = \nu(\mathbb{R}^m)$, and if two measures coincide on half-spaces, they coincide on the entire space. For $y \neq 0$, we consider the half-spaces

$$H_t = \{x \in \mathbb{R}^m \mid \langle x, y\rangle < t\} \quad (t \in \mathbb{R})$$

and put $g(t) = \mu(H_t) = v(H_t)$ and $\Phi(x) = \langle x, y\rangle$. The function g increases, and the Stieltjes measure μ_g is the Φ-image of the measures μ and v since $\Phi^{-1}((-\infty, t)) = H_t$. It remains to use Theorem on integration with respect to a weighted image of a measure,

$$\begin{aligned}\widehat{\mu}(y) &= \int_{\mathbb{R}^m} e^{-2\pi i\langle x,y\rangle} d\mu(x) = \int_{\mathbb{R}} e^{-2\pi i t} dg(t) \\ &= \int_{\mathbb{R}^m} e^{-2\pi i\langle x,y\rangle} d\nu(x) = \widehat{v}(y)\end{aligned}$$

□

1.4.6

Using the results of the previous section, we will prove here that the system of Hermite polynomials is complete. The method we use enables us to consider a more general situation and prove that the family of monomials in m variables, i.e., the products $x^n = x_1^{n_1} \cdots x_m^{n_m}$, where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and $n = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$, is complete in $\mathscr{L}^2(\mathbb{R}^m, \mu)$ for a wider class of measures.

Theorem

If a Borel measure μ on $\mathbb{R}^m$ satisfies the condition $\int_{\mathbb{R}^m} e^{a\|x\|} d\mu(x) < +\infty$ for some $a > 0$, then the family of all monomials is complete in the space $\mathscr{L}^2(\mathbb{R}^m, \mu)$.

Proof Let a function f in $\mathscr{L}^2(\mathbb{R}^m, \mu)$ be orthogonal to all monomials. Obviously, $f \perp P$ for every polynomial P in m variables. We put

$$F(y) = \int_{\mathbb{R}^m} f(x) e^{i\langle y,x\rangle} d\mu(x)$$

Since $\left|e^{i\langle y,x\rangle}\right| \equiv 1$ and all polynomials are summable with respect to μ, the function F is infinitely differentiable and, for each y, the derivatives of F can be found by the Leibnitz rule.

We prove that $F \equiv 0$. If $\|y\| < a/2$, then expanding the exponential factor in a Taylor series and integrating termwise, we obtain that $F(y) = 0$. The legitimacy of termwise integration follows from the fact that the partial sums of the series have a summable majorant, namely, $|f(x)| e^{\|x\|\|y\|}$ (this function is summable because the functions $|f|$ and $e^{\|x\|\|y\|}$ belong to $\mathscr{L}^2\left(\mathbb{R}^m, \mu\right)\big)$. To prove that $F \equiv 0$, we show that the interior G of the set where $F(y) = 0$ coincides with $\mathbb{R}^m$. Since $G \neq \varnothing$ (because it contains a neighborhood of zero), it is sufficient to verify that the set G is closed, in which case the equality $G = \mathbb{R}^m$ will follow from the fact that the space $\mathbb{R}^m$ is connected. Let $y \in \bar{G}$. The function F and all its derivatives vanish at y by continuity. Calculating the derivatives by Leibnitz's rule, we see that

$$0 = F^{(n)}(y) = \int_{\mathbb{R}^m} f(x)(ix)^n e^{i\langle x,y\rangle} d\mu(x) \quad \left(n \in \mathbb{R}^m_+\right)$$

Thus, the function $f_1(x) = f(x)e^{i\langle x,y\rangle}$ is orthogonal to all monomials. Replacing f by f_1, we may assert by what has just been proved that the function $F_1(\eta) = \int_{\mathbb{R}^m} f_1(x)e^{i\langle x,\eta\rangle} d\mu(x)$ assumes only zero values in a neighborhood of zero. However, $F_1(\eta)$ is nothing but $F(y+\eta)$. Therefore, $F \equiv 0$ in a neighborhood of y, i.e., $y \in G$. Thus, $G = \bar{G} = \mathbb{R}^m$ and, consequently, $F \equiv 0$. Now, we can easily complete the proof. Indeed, without loss of generality, we may assume that the function f is real. The identity $F \equiv 0$ means that the measures $f_+ d\mu$ and $f_- d\mu$ have the same Fourier transform. Consequently, these measures coincide by Theorem 1.4.5, which implies that the functions $f_+$and f_-coincide almost everywhere with respect to μ.□

Corollary
The Hermite polynomials are complete in $\mathscr{L}^2(\mathbb{R}, \mu)$ with $d\mu(x) = e^{-x^2}dx$

This is a special case of the theorem for $m = 1$. We also remark that the theorem implies that the Laguerre functions are complete.

The following example shows that the result obtained in the theorem is quite sharp.

Example We verify that the polynomials are not complete in the space $\mathscr{L}^2(\mathbb{R}, \mu)$ with measure μ having density $e^{-|x|^p}(0 < p < 1)$ with respect to the onedimensional Lebesgue measure (for $p \geqslant 1$ this effect is ruled out by the theorem just proved).

We will need the following formula from Example 1 of Sect. 1.1.7: if $a > 0$ and $z = e^{i\theta}$, where $\theta \in \left(0, \frac{\pi}{2}\right)$, then $z^{-a}\Gamma(a) = \int_0^\infty t^{a-1}e^{-zt}dt$. Comparing the imaginary parts and using the substitution $t = x^p/\cos\theta$, we obtain

$$\begin{aligned}\Gamma(a)\sin a\theta &= \int_0^\infty t^{a-1}e^{-t\cos\theta}\sin(t\sin\theta)dt \\ &= \frac{p}{\cos^a\theta}\int_0^\infty x^{ap-1}e^{-x^p}\sin\left(x^p\tan\theta\right)dx\end{aligned}$$

Now, we use the freedom in the choice of the parameters a and θ. Putting $\theta = \frac{\pi}{2}p$ and $a = \frac{2}{p}(n+1)$, we obtain

$$\int_0^\infty x^{2n+1}e^{-x^p}\sin\left(x^p\tan\frac{\pi}{2}p\right)dx = 0 \quad \text{for } n = 0, 1, 2\ldots$$

This means that the odd function equal to $\sin\left(x^p\tan\frac{\pi}{2}p\right)$ for $x \geqslant 0$ is orthogonal to all polynomials in the space $\mathscr{L}^2(\mathbb{R}, \mu)$ with measure

$d\mu(x) = e^{-|x|^p} dx.$

1.4.7

The present and two following sections are devoted to an important theorem, due to Plancherel, and its corollaries. The traditional formulation of the theorem would require us to invoke some concepts from functional analysis and operator theory. To avoid this, we first establish an analytic fact constituting the core of the theorem.

Theorem (Plancherel) If $f \in \mathscr{L}^1(\mathbb{R}^m) \cap \mathscr{L}^2(\mathbb{R}^m)$, then $\widehat{f} \in \mathscr{L}^2(\mathbb{R}^m)$ and $\|\widehat{f}\|_2 = \|f\|_2$.

Proof Let $\{\omega_t\}_{t>0}$ be a Sobolev approximate identity in $\mathbb{R}^m$ (see Sect. 1.4.2) and $f_t = f * \omega_t$.

First, we prove the assertion of the theorem for the smoothened function f_t. By properties of convolution, we have $f_t \in \mathscr{L}^1(\mathbb{R}^m) \cap \mathscr{L}^2(\mathbb{R}^m)$. By Theorem 1.3.1, we obtain $\widehat{f_t} = \widehat{f}\widehat{\omega}_t$. This product is summable since the function $\widehat{f}$ is bounded and $\widehat{\omega}_t \in \mathscr{L}^1(\mathbb{R}^m)$ by Corollary 1.3.2. Using Fubini's theorem and inversion formula (5), we obtain

$$\begin{aligned}\int_{\mathbb{R}^m} \widehat{f}_t(y)\widehat{\overline{f_t(y)}}dy = &= \int_{\mathbb{R}^m} \widehat{f}_t(y)\overline{\left(\int_{\mathbb{R}^m} f_t(x)e^{-2\pi i\langle y,x\rangle}dx\right)}dy = \\ &= \int_{\mathbb{R}^m} \widehat{f}_t(y)\left(\int_{\mathbb{R}^m} \overline{f_t(x)}e^{2\pi i\langle y,x\rangle}dx\right)dy = \\ &= \int_{\mathbb{R}^m} \overline{f_t(x)}\left(\int_{\mathbb{R}^m} \widehat{f}_t(y)e^{2\pi i\langle y,x\rangle}dy\right)dx = \int_{\mathbb{R}^m} \overline{f_t(x)}f_t(x)dx\end{aligned}$$

Thus,

$$\left\|\widehat{f}_t\right\|_2^2 = \|f_t\|_2^2 \qquad (8)$$

It remains to verify that we can pass to the limit in this equation as $t \to 0$.

Since $f_t \underset{t\to 0}{\longrightarrow} f$ in the $\mathscr{L}^2$-norm, the continuity of the norm implies $\|f_t\|_2 \underset{t\to 0}{\longrightarrow} \|f\|_2$. We verify that $\widehat{f} \in \mathscr{L}^2(\mathbb{R}^m)$ and $\left\|\widehat{f}_t\right\|_2 \underset{t\to 0}{\longrightarrow} \|\widehat{f}\|_2$. To this end, we write the left-hand side of Eq. (8) in more detail,

$$\left\|\widehat{f}_t\right\|_2^2 = \int_{\mathbb{R}^m} \left|\widehat{f}_t(y)\right|^2 dy = \int_{\mathbb{R}^m} |\widehat{f}(y)|^2 \left|\widehat{\omega}_t(y)\right|^2 dy \qquad (9)$$

Since $\widehat{\omega}_t(y) \underset{t\to 0}{\longrightarrow} 1$ (see Corollary 1.4.3 with $t_0 = 0$ and $g(x) = e^{-2\pi i\langle y,x\rangle}$), Fatou's theorem and Eq. (8) imply

$$\begin{aligned}\int_{\mathbb{R}^m} |\widehat{f}(y)|^2 dy \leqslant \lim_{t\to 0}\int_{\mathbb{R}^m} |\widehat{f}(y)|^2 \left|\widehat{\omega}_t(y)\right|^2 dy = \lim_{t\to 0}\left\|\widehat{f}_t\right\|_2^2 = \lim_{t\to 0}\|f_t\|_2^2 = \|f\|_2^2 \\ < +\infty\end{aligned}$$

Thus, $\widehat{f} \in \mathscr{L}^2(\mathbb{R}^m)$. Returning to Eq. (9), we see that the integrand in the integral on the right has a summable majorant, namely, $|\widehat{f}|^2$. Therefore, we can pass to the limit in this integral by Lebesgue's theorem,

$$\int_{\mathbb{R}^m} |\widehat{f}(y)|^2 \, |\widehat{\omega}_t(y)|^2 \, dy \underset{t\to 0}{\longrightarrow} \int_{\mathbb{R}^m} |\widehat{f}(y)|^2 dy$$

Now, the passage to the limit in Eq. (8) leads to the required result.□

The concluding part of the proof can be somewhat shortened. Indeed, since $|\widehat{\omega}_t| \leqslant \int_{\mathbb{R}^m} \omega(x)dx = 1$, we have $\left|\widehat{f_t}\right| \leqslant |\widehat{f}|$. Since $\widehat{f_t} \underset{t\to 0}{\longrightarrow} \widehat{f}$, we can pass to the limit on the right-hand side of Eq. (8) by the generalization of B. Levi's theorem.

1.4.8

We show how Plancherel's theorem can be used to generalize the concept of the Fourier transform to functions in $\mathscr{L}^2(\mathbb{R}^m)$.

Lemma

Let $f \in \mathscr{L}^2(\mathbb{R}^m)$. If $\{f_n\}_n \geqslant 1$ is a sequence of functions in $\mathscr{L}^1(\mathbb{R}^m) \cap \mathscr{L}^2(\mathbb{R}^m)$ convergent to f in the $\mathscr{L}^2$-norm, then the sequence $\left\{\widehat{f_n}\right\}_n \geqslant 1$ also converges in the $\mathscr{L}^2$-norm. Its limit does not depend (up to equivalence) on the choice of the sequence $\{f_n\}_n \geqslant 1$.

Proof From Plancherel's theorem, it follows that the sequence $\left\{\widehat{f_n}\right\}_{n\geqslant 1}$ is fundamental,

$$\left\|\widehat{f_n} - \widehat{f_k}\right\|_2 = \left\|\widehat{f_n - f_k}\right\|_2 = \|f_n - f_k\|_2 \underset{n,k\to\infty}{\longrightarrow} 0$$

The limit exists because the space $\mathscr{L}^2(\mathbb{R}^m)$ is complete. If $\{g_n\}_{n\geqslant 1}$ is another sequence of functions in $\mathscr{L}^1(\mathbb{R}^m) \cap \mathscr{L}^2(\mathbb{R}^m)$ convergent to f in the $\mathscr{L}^2$-norm, then the sequence $f_1, g_1, f_2, g_2, \dots$ obtained by "shuffling" the sequences $\{f_n\}_n \geqslant 1$ and $\{g_n\}_n \geqslant 1$ converges to f. By what has just been proved, the sequence $\widehat{f}_1, \widehat{g}_1, \widehat{f}_2, \widehat{g}_2, \dots$ has a limit, which is unique up to equivalence and coincides with the limits of its subsequences.□

The lemma just proved allows us to extend the definition of the Fourier transform to the functions in $\mathscr{L}^2(\mathbb{R}^m)$.

Definition By the Fourier transform of a function $f \in \mathscr{L}^2(\mathbb{R}^m)$, we mean the limit in the $\mathscr{L}^2$-norm of the functions $\widehat{f}_n$, where $\{f_n\}_n \geqslant 1$ is an arbitrary sequence of functions in $\mathscr{L}^1(\mathbb{R}^m) \cap \mathscr{L}^2(\mathbb{R}^m)$ such that $\|f_n - f\|_2 \underset{n\to\infty}{\longrightarrow} 0$.

Thus, the Fourier transform of a function in $\mathscr{L}^2(\mathbb{R}^m)$ is also square-summable. As before, we will denote the Fourier transform of f by $\widehat{f}$. However, one must keep in mind that now the Fourier transform is defined up to equivalence and the symbol $\widehat{f}$ refers to many functions. If f is summable, then, among these functions, is the

Fourier transform defined in Sect. 1.4.1. For definiteness, the latter is sometimes called the classical Fourier transform. What has just been said also applies to the inverse transform, which, as before, is denoted by $\check{f}$.

Elementary properties of the Fourier transform of square-summable functions can be obtained from the properties of the classical Fourier transform by a passage to the limit.

Theorem Let $f \in \mathscr{L}^2(\mathbb{R}^m)$. Then:

(1) $\|\widehat{f}\|_2 = \|f\|_2$;

(2) if $f_n \in \mathscr{L}^2(\mathbb{R}^m)$ and $\|f_n - f\|_2 \underset{n\to\infty}{\longrightarrow} 0$, then $\left\|\widehat{f_n} - \widehat{f}\right\|_2 \underset{n\to\infty}{\longrightarrow} 0$, and a similar statement holds for the inverse transform;

(3) we have $(\widehat{f})^{\smile} = (\check{f})^{\widehat{}} = f$;

(4) $\langle \widehat{f}, \widehat{g}\rangle = \langle f, g\rangle$ for every function $g \in \mathscr{L}^2(\mathbb{R}^m)$. In particular, the Fourier transform preserves orthogonality: if $f \perp g$, then $\widehat{f} \perp \widehat{g}$.

Proof Let $\{\varphi_n\}_{n\geqslant 1} \in C_0^\infty(\mathbb{R}^m)$ be a sequence of functions converging to f in the $\mathscr{L}^2$-norm. It is obvious that these functions and their Fourier transforms belong to $\mathscr{L}^1(\mathbb{R}^m) \cap \mathscr{L}^2(\mathbb{R}^m)$.

(1) It is clear that $\left\|\widehat{\varphi}_n - \widehat{f}\right\|_2 \underset{n\to\infty}{\longrightarrow} 0$ by the definition of $\widehat{f}$. By Plancherel's theorem, we have $\|\widehat{\varphi}_n\|_2 = \|\varphi_n\|_2$. Therefore, it is sufficient for us to use the continuity of the norm and pass to the limit in this equation.

(2) Obviously, $\left\|\widehat{f_n} - \widehat{f}\right\|_2 = \left\|\widehat{f_n - f}\right\|_2 = \|f_n - f\|_2 \underset{n\to\infty}{\longrightarrow} 0$.

(3) We will prove only the equality $(\widehat{f})^{\vee} = f$ (the other one is proved similarly). Since $\varphi_n \underset{n\to\infty}{\longrightarrow} f$, we obtain by definition that $\widehat{\varphi}_n \underset{n\to\infty}{\longrightarrow} \widehat{f}$, and, by property 2) applied to the inverse transform, we have $(\widehat{\varphi_n})^{\vee} \underset{n\to\infty}{\longrightarrow} (\widehat{f})^{\vee}$. At the same time, $(\widehat{\varphi_n})^{\vee} \underset{n\to\infty}{\longrightarrow} (\widehat{\varphi_n})^{\vee}$ by Theorem 1.4.4. Thus, it only remains to pass to the limit (in the $\mathscr{L}^2$-norm) in the last equality.

(4) For the proof, we must use the identity
$4f\bar{g} = |f+g|^2 + |f+ig|^2 - |f-g|^2 - |f-ig|^2$ and apply relation (1) to the functions $f \pm g$ and $f \pm ig$.

1.4.9

Plancherel's theorem implies an inequality known as the uncertainty principle. Without touching on its physical meaning (the impossibility of simultaneously determining the exact values of the coordinates and impulse of a quantum object), we mention only its consequence: if $f \neq 0$ only in the vicinity of the origin, then the quantity $|\widehat{f}|$ is not small at some remote points (the Fourier transform "blurs"). In the one-dimensional case, the reader can see this effect in the example of functions $\frac{1}{2t}\chi_{(-t,t)}$ forming an approximate identity.

In the precise formulation of the uncertainty principle, we confine ourselves to infinitely differentiable compactly supported functions of one variable.

Theorem If $f \in C_0^\infty(\mathbb{R})$ and $\|f\|_2 = 1$, then

$$\int_{-\infty}^{\infty} x^2|f(x)|^2 dx \cdot \int_{-\infty}^{\infty} x^2|\widehat{f}(x)|^2 dx \geqslant \frac{1}{16\pi^2}$$

Proof Since

$$\int_{-\infty}^{\infty} x\left(|f(x)|^2\right)' dx = x|f(x)|^2\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} |f(x)|^2 dx = -1$$

the Cauchy-Bunyakovsky inequality implies

$$1 = \left|\int_{-\infty}^{\infty} x\left(|f(x)|^2\right)' dx\right| \leqslant 2\int_{-\infty}^{\infty} |xf(x)| \cdot |f'(x)|\, dx \leqslant 2\|g\|_2 \, \|f'\|_2$$

where $g(x) = |xf(x)|$. By Plancherel's theorem, we have $\|f'\|_2 = \left\|\widehat{f'}\right\|_2$, and, by Theorem 1.4.2, we obtain $\widehat{f'}(y) = 2\pi i y \widehat{f}(y)$. Consequently,

$$1 \leqslant 4\|g\|_2^2 \, \|f'\|_2^2 = 4\|g\|_2^2 \left\|\widehat{f'}\right\|_2^2 = 4\int_{-\infty}^{\infty} x^2|f(x)|^2 dx \cdot 4\pi^2 \int_{-\infty}^{\infty} y^2|\widehat{f}(y)|^2 dy.$$

□

1.4.10

In the conclusion of this section, we apply the Fourier transform to estimate the Dirichlet kernels for a ball

$$D_R(x) = \frac{1}{(2\pi)^m} \sum_{\|k\|<R} e^{-i\langle k,x\rangle} \quad (x \in \mathbb{R}^m)$$

(the summation is taken over the points k of the integer lattice $\mathbb{Z}^m$). We show that, in the case where $m > 1$, their $\mathscr{L}^1$-norms (in contrast to the norms of the Dirichlet kernels for cubes $(-R, R)^m$) have not a logarithmic, but a power order of growth as $R \to +\infty$,

$$\|D_R\|_1 = \frac{1}{(2\pi)^m} \int_{[-\pi,\pi]^m} \left|\sum_{\|k\|<R} e^{-i\langle k,x\rangle}\right| dx \asymp R^{\frac{m-1}{2}}$$

Being unable to represent the kernel D_R in a compact form, we obtain for D_R an approximate integral representation, replacing the sum over the ball $B(R)$ by an integral over a set close to $B(R)$. For this, we use the fact that the mean value of the exponential function e^{-iat} on the interval $(a-1/2, a+1/2)$ differs from the function itself only by a factor independent of a,

$$e^{-iat} = \frac{t/2}{\sin t/2} \int_{a-1/2}^{a+1/2} e^{-ist} ds$$

Therefore, in the multiple integral for the shifted unit cube $Q_k = k + \left[-\frac{1}{2}, \frac{1}{2}\right]^m$ at the point $x = (x_1, \ldots, x_m)$, we have

$$e^{-i\langle k,x\rangle} = \theta(x) \int_{Q_k} e^{-i\langle y,x\rangle} dy, \qquad \text{where } \theta(x) = \prod_{j=1}^{m} \frac{x_j/2}{\sin x_j/2}$$

Putting $T(R) = \bigcup_{\|k\|<R} Q_k$, we arrive at the equation

$$D_R(x) = \frac{\theta(x)}{(2\pi)^m} \int_{T(R)} e^{-i\langle y,x\rangle} dy$$

Thus,

$$\|D_R\|_1 = \frac{1}{(2\pi)^m} \int_{[-\pi,\pi]^m} \theta(x) \left| \int_{T(R)} e^{-i\langle y,x\rangle} dy \right| dx$$

Since $1 \leqslant \frac{t}{\sin t} \leqslant \frac{\pi}{2}$ for $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we obtain $1 \leqslant \theta(x) \leqslant \left(\frac{\pi}{2}\right)^m$ in this integral, and, therefore,

$$\|D_R\|_1 \asymp \int_{[-\pi,\pi]^m} \left| \int_{T(R)} e^{-i\langle y,x\rangle} dy \right| dx = (2\pi)^m \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^m} \left| \widehat{\chi\chi\chi}_{T(R)}(u) \right| du.$$

We show that, for $m > 1$, the integral on the right-hand side of this relation grows as $R^{\frac{m-1}{2}}$. It is more convenient to deal with the integral over a ball rather than over a cube. Therefore, we consider the integral

$$I_R(\rho) = \int_{B(\rho)} \left| \widehat{\chi}_{T(R)}(u) \right| du$$

Since

$$I_R\left(\frac{1}{2}\right) \leqslant \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^m} \left| \widehat{\chi}_{T(R)}(u) \right| du \leqslant I_R\left(\frac{\sqrt{m}}{2}\right)$$

it is sufficient to verify that $I_R(\rho) \asymp R^{\frac{m-1}{2}}$ as $R \to +\infty$ for every fixed $\rho > 0$.

For large R, the set $T(R)$ is close to the ball $B(R)$. Therefore, it is natural to replace $\widehat{\chi}_{T(R)}$ with $\widehat{\chi}_{B(R)}$ and compare the integral $I_R(\rho)$ with a "similar" integral

$$J_R(\rho) = \int_{B(\rho)} \left| \widehat{\chi}_{B(R)}(u) \right| du$$

The rate of its growth was essentially found in Example 2 of Sect. 1.3.2. Indeed, since

$$\widehat{\chi}_{B(R)}(u) = \int_{\|x\|<R} e^{-2\pi i\langle u,x\rangle} dx = R^m \int_{\|x\|<1} e^{-2\pi i R\langle u,x\rangle} dx = R^m \widehat{\chi}_B(Ru)$$

the integral $J_R(\rho)$ can be reduced to the integral $L_B(R) = \int_{\|y\|<R} |\widehat{\chi}_B(y)|\, dy$ considered in this Example,

$$J_R(\rho) = \int_{B(\rho)} |R^m \widehat{\chi}_B(Ru)|\, du = \int_{B(\rho R)} |\widehat{\chi}_B(y)|\, dy = L_B(\rho R) \asymp (\rho R)^{\frac{m-1}{2}}.$$

Therefore,

$$0 < C_m (\rho R)^{\frac{m-1}{2}} \leqslant J_R(\rho) \leqslant C'_m (\rho R)^{\frac{m-1}{2}}$$

To estimate the difference $I_R(\rho) - J_R(\rho)$, we introduce the function $\eta_R = \chi_{B(R)} - \chi_{T(R)}$. It is clear that

$$|I_R(\rho) - J_R(\rho)| \leqslant \int_{B(\rho)} |\widehat{\eta}_R(u)|\, du \leqslant \sqrt{\alpha_m \rho^m \int_{B(\rho)} |\widehat{\eta}_R(u)|^2\, du} \leqslant \sqrt{\alpha_m \rho^m}\, \|\widehat{\eta}_R\|_2 \,.$$

The next step is possible due to Plancherel's theorem allowing us to pass from the norm $\widehat{\eta}_R$ to the norm η_R, which can easily be estimated (since $|\eta_R| \leqslant 1$ and the function η_R differs from zero only in the spherical layer $R - \sqrt{m} \leqslant \|x\| \leqslant R + \sqrt{m}$),

$$\|\widehat{\eta}_R\|_2 = \|\eta_R\|_2 \leqslant \sqrt{\alpha_m \left((R + \sqrt{m})^m - (R - \sqrt{m})^m\right)}$$

Therefore, we obtain for $R > 1$

$$|I_R(\rho) - J_R(\rho)| \leqslant \alpha_m \rho^{\frac{m}{2}} \sqrt{2m^{\frac{3}{2}}} (R + \sqrt{m})^{\frac{m-1}{2}} \leqslant A_m \rho^{\frac{m}{2}} R^{\frac{m-1}{2}}$$

where A_m is a coefficient depending only on the dimension m. Taking into account inequality (10), we obtain the following estimate from above: $I_R(\rho) = O\left(R^{\frac{m-1}{2}}\right)$ as $R \to +\infty$.

Because the integrals $I_R(\rho)$ grow with the growth of ρ, it is sufficient to establish an estimate from below for small ρ. For this, we again use inequalities (10) and (11),

$$\begin{aligned} I_R(\rho) &\geqslant J_R(\rho) - |I_R(\rho) - J_R(\rho)| \geqslant C_m (\rho R)^{\frac{m-1}{2}} - A_m \rho^{\frac{m}{2}} R^{\frac{m-1}{2}} \\ &= (C_m - A_m \sqrt{\rho})\, (\rho R)^{\frac{m-1}{2}}. \end{aligned}$$

We obtain the required result if we take, for example, $\rho = C_m^2 / \left(2A_m\right)^2$.