Example 4

Find a solution of the differential equation

$$x'''(t)+2x''(t)+5x'(t)=0,$$

satisfying the conditions: $x(0)=-1,\ x'(0)=2,\ x''(0)=0$.

<u>Solution:</u>

Let $x(t)\leftrightarrow X(p)$.

Since, take into account the given conditions, we have

$$x'(t)\leftrightarrow pX(p)-x(0)=pX(p)-(-1)=pX(p)+1,$$

$$x''(t)\leftrightarrow p^2X(p)-px(0)-x'(0)=p^2X(p)-p(-1)-2=p^2X(p)+p-2,$$

$$x'''(t)\leftrightarrow p^3X(p)-p^2x(0)-px'(0)-x''(0)=$$

$$=p^3X(p)-p^2(-1)-p2-0=p^3X(p)+p^2-2p,$$

then, after applying the Laplace transform for a given equation, we obtain the following operator equation:

$$p^3X(p)+p^2-2p+2p^2X(p)+2p-4+5pX(p)+5=0,$$

or after the transformations:

$$X(p)(p^3+2p^2+5p)=-p^2-1.$$

Solving this equation for $X(p)$, we obtain

$$X(p)=\frac{-p^2-1}{p(p^2+2p+5)}.$$

The resulting expression is decomposed into the simplest fractions:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{A}{p}+\frac{Bp+C}{p^2+2p+5}.$$

Using the method of undefined coefficients, we find A, B, C.

To do this, we bring the fractions to a common denominator and equate the coefficients with the same degrees of p:

$$\frac{-p^2-1}{p(p^2+2p+5)}=\frac{Ap^2+2Ap+5A+Bp^2+Cp}{p(p^2+2p+5)}.$$

We obtain a system of algebraic equations for A,B,C:

$$A+B=-1,\ 2A+C=0,\ 5A=-1,$$

the solution of which will be: $A=-\frac{1}{5}, B=-\frac{4}{5}, C=\frac{2}{5}$.

Then

$$X(p)=-\frac{1}{5p}+\frac{1}{5}\frac{-4p+2}{p^2+2p+5}.$$

To find the original of the second fraction, select the full square in its denominator: $p^2+2p+5=(p+1)^2+4$, then select the summand $p+1$ in the numerator:

$$-4p+2=-4(p+1)+6,$$

and decompose the fraction into the sum of two fractions:

$$\frac{1}{5}\frac{-4p+2}{p^2+2p+5}=-\frac{4}{5}\frac{p+1}{(p+1)^2+4}+\frac{3}{5}\frac{2}{(p+1)^2+4}.$$

Next, using the displacement property and the table of correspondence between images and originals, we obtain a solution to the original equation:

$$x(t) = -\frac{1}{5} - \frac{4}{5}e^{-t}\cos 2t + \frac{3}{5}e^{-t}\sin 2t.$$

Example 5

Solve the Cauchy problem:

$$x''' - x'' - 6x' = 0,$$
$$x(0) = 15, x'(0) = 2, x''(0) = 56.$$

Solution:

Let's move on from the originals to the images:

$x(t) \leftrightarrow X(p),$

$x'(t) \leftrightarrow pX(p) - 15,$

$x'''(t) \leftrightarrow p^3X(p) - 15p^2 - 2p - 56.$

Let's solve the equation for images

$$\left(p^3 - p^2 - 6p\right)X(p) = 15p^2 - 13p - 36,$$

$$X(p) = \frac{15p^2 - 13p - 36}{p(p-3)(p+2)}.$$

The function $X(p)$ is a proper rational irreducible fraction for which the points $p_1 = 0,\ p_2 = 3,\ p_3 = -2$ are simple poles. Since

$$P(p) = 15p^2 - 13p - 36,$$

$$Q(p) = p^3 - p^2 - 6p,$$

$$Q'(p) = 3p^2 - 2p - 6,$$

that's for

$p_1 = 0$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_1=0} = \frac{-36}{-6} = 6,$$

for

$p_2 = 3$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_2=3} = \frac{60}{15} = 4,$$

For

$p_3 = -2$

$$\left.\frac{P(p)}{Q'(p)}\right|_{p_3=-2} = \frac{50}{10} = 5,$$

and by the second decomposition theorem we get

$$x(t) = 6 + 5e^{-2t} + 4e^{3t}.$$

<u>Remark</u>

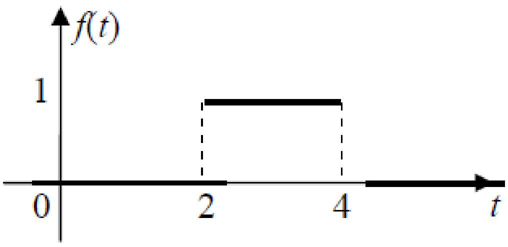
In many practical problems, the right side of the differential equation is given graphically. In this case, the solution algorithm does not change, and the delay theorem and methods from properties are used to find the image of the original given by the graph.

Example 6

Solve the Cauchy problem for a differential equation with the right-hand side given graphically:

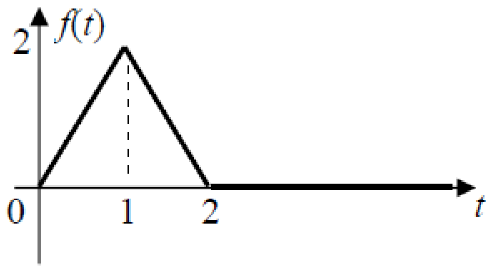
a)

$$x'' + x = f(t),$$
$$x(0) = x'(0) = 0,$$



b)

$$x'' + 4x = f(t),$$
$$x(0) = x'(0) = 0,$$



<u>Solution:</u>

a) Let's move on from the originals to the images:

$$x(t) \leftrightarrow X(p)$$

$$x'(t) \leftrightarrow pX(p)$$

$$x''(t) \leftrightarrow p^2 X(p)$$

$$f(t) = \theta(t-2) - \theta(t-4) \leftrightarrow \frac{1}{p}\left(e^{-2p} - e^{-4p}\right)$$

Let's solve the equation for images

$$\left(p^2+1\right)X(p) = \frac{1}{p}\left(e^{-2p} - e^{-4p}\right),$$

$$X(p)=\frac{1}{p\left(p^2+4\right)}\left(e^{-2p}-e^{-4p}\right).$$

Since

$$\frac{1}{p\left(p^2+1\right)}=\frac{1}{p}-\frac{p}{p^2+1}\leftrightarrow 1-cos\,t\,,$$

then

$$x(t)=\left(1-cos\left(t-2\right)\right)\theta\left(t-2\right)-\left(1-cos\left(t-4\right)\right)\theta\left(t-4\right).$$

The solution of the Cauchy problem can be presented in an analytical form:

$$x(t)=\begin{cases} 0, & t<2, \\ 1-\cos(t-2), & 2\le t<4, \\ \cos(t-4)-\cos(t-2), & t\ge 4. \end{cases}$$

b) Let's move on from the originals to the images:

$$x(t)\leftrightarrow X(p)$$

$$x'(t)\leftrightarrow pX(p)$$

$$x''(t)\leftrightarrow p^2X(p)$$

$$f(t)=2t\theta(t)-2t\theta(t-1)+(4-2t)\theta(t-1)-(4-2t)\theta(t-2)=$$

$$=2t\theta(t)-4(t-1)\theta(t-1)+2(t-2)\theta(t-2)\leftrightarrow$$

$$\leftrightarrow\frac{2}{p^2}-\frac{4}{p^2}e^{-p}+\frac{2}{p^2}e^{-2p}=\frac{2}{p^2}\left(1-2e^{-p}+e^{-2p}\right)$$

Let's solve the equation for images

$$\left(p^2+4\right)X(p)=\frac{2}{p^2}\left(1-2e^{-p}+e^{-2p}\right),$$

$$X(p)=\frac{2}{p^2\left(p^2+4\right)}\left(1-2e^{-p}+e^{-2p}\right).$$

Since

$$\frac{2}{p^2\left(p^2+4\right)}=\frac{1}{2}\left(\frac{1}{p^2}-\frac{1}{p^2+4}\right)\leftrightarrow\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right),$$

Then

$$x(t)=\frac{1}{2}\left(t-\frac{1}{2}\sin 2t\right)\theta(t)-\left(t-1-\frac{1}{2}\sin 2(t-1)\right)\theta(t-1)+$$

$$+\frac{1}{2}\left(t-2-\frac{1}{2}\sin 2(t-2)\right)\theta(t-2).$$

<u>Remark</u>

If $t=t_0\neq 0$ is taken for the initial time in the Cauchy problem, then a new variable $\tau=t-t_0$ is introduced. Then $\tau=0$ for $t=t_0$.

Example 7

Solve the Cauchy problem

a)

$$x''+x'=t,$$
$$x(1)=1,\quad x'(1)=0,$$

b)

$$x'' + x = -2\sin t,$$
$$x\left(\frac{\pi}{2}\right) = 0, \quad x'\left(\frac{\pi}{2}\right) = 1.$$

<u>Solution:</u>

a) Let $t = \tau + 1,\ x(t) = x(\tau + 1) = z(\tau)$.

Then the equation and the initial conditions will take the form

$$z'' + z = \tau + 1, \quad z(0) = 1, \quad z'(0) = 0.$$

Let's move on from the originals to the images

$$z(\tau) \leftrightarrow Z(p)$$

$$z'(\tau) \leftrightarrow pZ(p) - 1$$

$$z''(\tau) \leftrightarrow p^2 Z(p) - p$$

$$\tau + 1 \leftrightarrow \frac{1}{p^2} + \frac{1}{p}.$$

Let's write down the equation for the images

$$p^2 Z(p) - p + pZ(p) - 1 = \frac{1}{p^2} + \frac{1}{p}.$$

Solving the operator equation and moving on to the originals, we get

$$Z(p) = \frac{1}{p^3} + \frac{1}{p} \leftrightarrow 1 + \frac{\tau^2}{2} = z(\tau).$$

Returning to the original variable t, we obtain a solution to the Cauchy problem

$$x(t) = 1 + \frac{(t-1)^2}{2}.$$

b) Let

$$t=\tau+\frac{\pi}{2},\quad x(t)=x\left(\tau+\frac{\pi}{2}\right)=z(\tau).$$

Then the equation and the initial conditions will take the form

$$z''+z=-2\sin\left(\tau+\frac{\pi}{2}\right),$$

$$z(0)=0,\quad z'(0)=1.$$

Let's move on from the originals to the images

$$z(\tau)\leftrightarrow Z(p)$$

$$z'(\tau)\leftrightarrow pZ(p)$$

$$z''(\tau)\leftrightarrow p^2Z(p)-1$$

$$-2\sin\left(\tau+\frac{\pi}{2}\right)=-2\cos\tau\leftrightarrow\frac{-2p}{p^2+1}.$$

Let's write down the equation for the images

$$p^2Z(p)-1+Z(p)=\frac{-2p}{p^2+1}.$$

Let's solve the equation for images

$$Z(p)=\frac{1}{p^2+1}-\frac{2p}{\left(p^2+1\right)^2}.$$

Turning to the originals, we get

$$\frac{1}{p^2+1}\leftrightarrow\sin\tau$$

$$\frac{p}{\left(p^2+1\right)^2} \leftrightarrow \sin\tau * \cos\tau = \frac{1}{2}\tau\sin\tau$$

$$z(\tau) = (1-\tau)\sin\tau\ .$$

Returning to the original variable t, we obtain a solution to the original Cauchy problem

$$x(t) = \left(1 - t + \frac{\pi}{2}\right)\sin\left(t - \frac{\pi}{2}\right) = \left(t - 1 - \frac{\pi}{2}\right)\cos t\ .$$

Example 8

Solve systems of differential equations with given initial conditions:

a)

$$\begin{cases} x' + y = 2e^t, \\ y' + x = 2e^t, \end{cases}$$
$$x(0) = y(0) = 1,$$

b)

$$\begin{cases} x'' + x' + y'' - y = e^t, \\ x' + 2x - y' + y = e^{-t}, \end{cases}$$

$$x(0) = y(0) = y'(0) = 0,\ \ x'(0) = 1.$$

<u>Solutions:</u>

a) Let

$$x(t) \leftrightarrow X(p)$$

$$y(t) \leftrightarrow Y(p)$$

Considering that

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p) - 1$$

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p) - 1$$

$$e^t \leftrightarrow \frac{1}{p-1}$$

we obtain an operator system of linear equations

$$\begin{cases} pX(p) - 1 + Y(p) = \dfrac{2}{p-1}, \\ pY(p) - 1 + X(p) = \dfrac{2}{p-1}. \end{cases} \Rightarrow \begin{cases} pX(p) + Y(p) = \dfrac{p+1}{p-1}, \\ pY(p) + X(p) = \dfrac{p+1}{p-1}. \end{cases}$$

Solving the system, we get $X(p) = Y(p) = \dfrac{1}{p-1}$.

Using the image table, we will find $x(t) = y(t) = e^t$.

b) We have

$$x(t) \leftrightarrow X(p)$$

$$x'(t) \leftrightarrow pX(p) - x(0) = pX(p)$$

$$x''(t) \leftrightarrow p^2 X(p) - x'(0) = pX(p) - 1$$

$$y(t) \leftrightarrow Y(p)$$

$$y'(t) \leftrightarrow pY(p) - y(0) = pY(p)$$

$$y''(t) \leftrightarrow p^2 Y(p) - y'(0) = p^2 Y(p)$$

$$e^{t} \leftrightarrow \frac{1}{p-1}$$

$$e^{-t} \leftrightarrow \frac{1}{p+1}$$

Let's write down a system of operator equations

$$\begin{cases} p^2X - 1 + pX + p^2Y - Y = \dfrac{1}{p-1}, \\ pX + 2X - pY + Y = \dfrac{1}{p+1}. \end{cases} \Rightarrow \begin{cases} \left(p^2 + p\right)X + \left(p^2 - 1\right)Y = \dfrac{p}{p-1}, \\ (p+2)X + (1-p)Y = \dfrac{1}{p+1}. \end{cases}$$

Let's solve a system of linear equations with respect to X and Y using Kramer's formulas:

$$\Delta = \begin{vmatrix} p^2 + p & p^2 - 1 \\ p+2 & 1-p \end{vmatrix} = p(p+1)(1-p) - (p+2)\left(p^2 - 1\right) =$$

$$= 2(1+p)^2(1-p),$$

$$\Delta_x = \begin{vmatrix} \dfrac{p}{p-1} & p^2 - 1 \\ \dfrac{1}{p+1} & 1-p \end{vmatrix} = 1 - 2p, \quad \Delta_y = \begin{vmatrix} p^2 + p & \dfrac{p}{p-1} \\ p+2 & \dfrac{1}{p+1} \end{vmatrix} = \frac{3p}{1-p}.$$

Thus

$$X(p) = \frac{\Delta_x}{\Delta} = \frac{1-2p}{2(p+1)^2(1-p)} = \frac{1}{8}\frac{1}{p-1} + \frac{3}{4}\frac{1}{(p+1)^2} - \frac{1}{8}\frac{1}{p+1},$$

$$Y(p) = \frac{\Delta_y}{\Delta} = \frac{3p}{2(p+1)^2(1-p)^2} = \frac{3p}{2\left(p^2 - 1\right)^2}.$$

Let's move on to the originals. Since

$e^{-t} \leftrightarrow \dfrac{1}{p+1}$ and $\operatorname{sh} t \leftrightarrow \dfrac{1}{p^2 - 1}$

then, according to the image differentiation theorem, we find

$$\left(\frac{1}{p+1}\right)' = -\frac{1}{(p+1)^2} \leftrightarrow -te^{-t}$$

$$\left(\frac{1}{p^2-1}\right)' = -\frac{2p}{(p^2-1)^2} \leftrightarrow -t\,\mathrm{sh}\,t$$

Therefore, the solution of the system will be

$$x(t) = \frac{1}{4}\mathrm{sh}\,t + \frac{3}{4}te^{-t}, \quad y(t) = \frac{3}{4}t\,\mathrm{sh}\,t\,.$$

Example 9 (HOMEWORK)

Solve system of differential equations with given initial conditions:

$$\begin{cases} x'(t) = -x(t) + y(t) + e^t, \\ y'(t) = x(t) - y(t) + e^t \end{cases}$$

$$x(0) = y(0) = 1\,.$$

* This homework and homework from 09.09.24 should be done by September 15, 2024.