Title of the course:

Numerical Analysis

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Introduction

We will study methods, algorithms, for solving complex equations numerically.

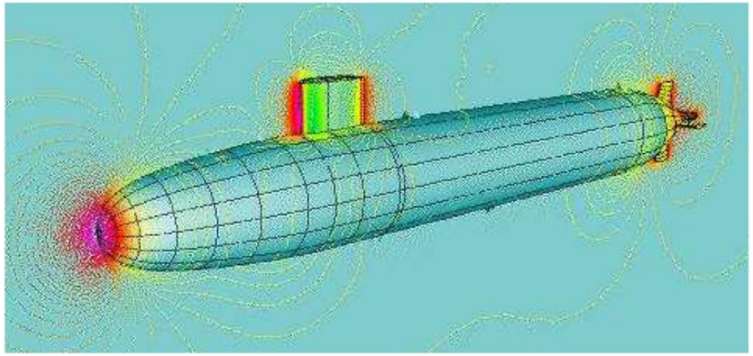
Though numerical solutions are approximate, their accuracy is very high, errors can be made extremely small.

Numerical analysis finds applications in all fields of engineering and physical sciences; and in the 21st century also in social sciences like economics, medicine, business and even the arts.

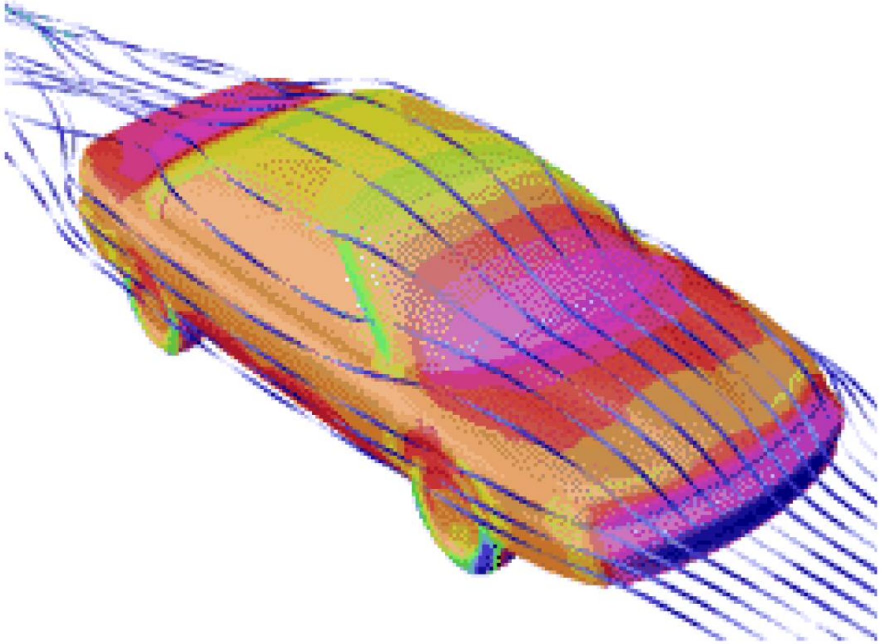
https://en.wikipedia.org/wiki/Numerical_analysis

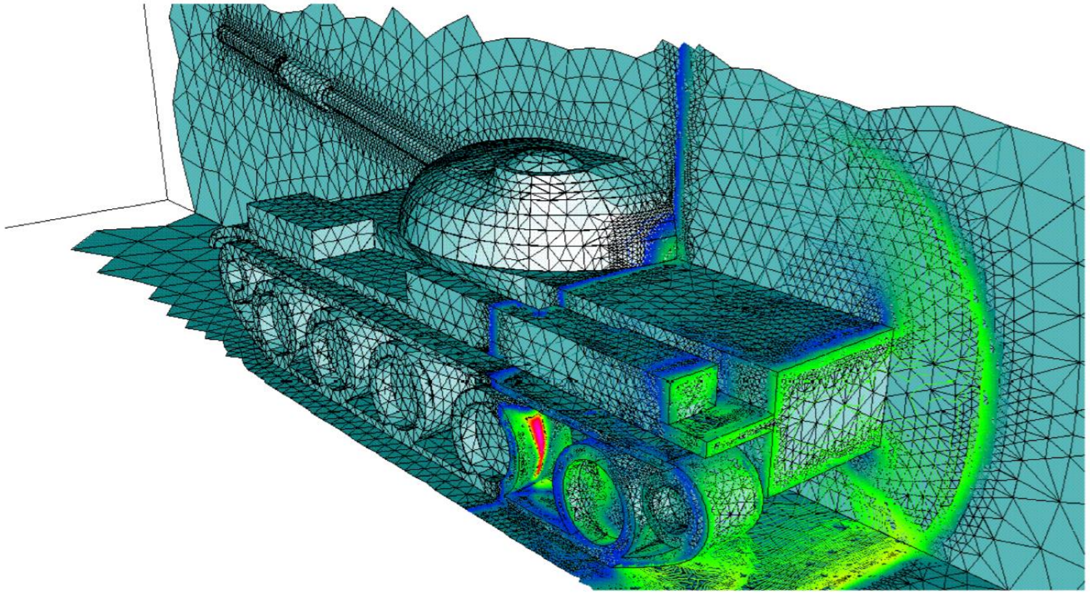
Examples of engineering problems:

A design of advanced flying or floating vehicles. It needs solutions of Fluid Dynamics and Aerodynamics equations



Also, advanced design of ground vehicles would not be possible without solving equations of Solid Mechanics which govern stresses and deformations in solid bodies





Also, Computational Mechanics of solids

is urgent for the design and construction of buildings, bridges, roads, and so on.

Therefore, a researcher or engineer must be familiar with Numerical Analysis and methods for solving various equations.

We will study:

(*) **Algebraic equations**

(*) **Calculation of integrals**

(*) **Finding maxima and minima of functions**

(*) **Ordinary differential equations**

(*) **Partial differential equations**

Our aim is <u>not</u> a search of <u>true</u> (exact) solutions analytically, often this is not possible.

We will search numerical solutions, which are approximate;
however, they are very close to true solutions, because errors (tolerance) can be made extremely small.

Textbooks:

1) S. Baskar Introduction to numerical analysis. 2010, 128 pages.

2) S.S. Sastry Introductory methods of numerical analysis. 2012, 463 pages.

Chapter 1.

Nonlinear and transcendental algebraic equations

Let us consider the single equation

$$f(x)=0\,, \qquad a \le x \le b \qquad [a,b]$$

where $f(x)$ is a given function, x is unknown.

"nonlinear" means: $x^2, x^3, \ldots$

"transcendental" means: $\sin x, \cos x, e^x, \ldots$

For example: $4x^2 + \sin(4\pi x) - 10 = 0\,,\quad 1 \le x \le 2$

Problem:

if there exists a true/exact solution x^*, such that $f(x^*)=0$, then which way can we calculate an approximate value of x^* , admitting a small error, say ± 0.0001 ?

x^* is called a root of the equation

Recall that in the case of quadratic equation there is an explicit formula for calculation of x^* :

If $ax^2 + bx + c = 0 \qquad a \neq 0$

then $$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

In the above-mentioned example

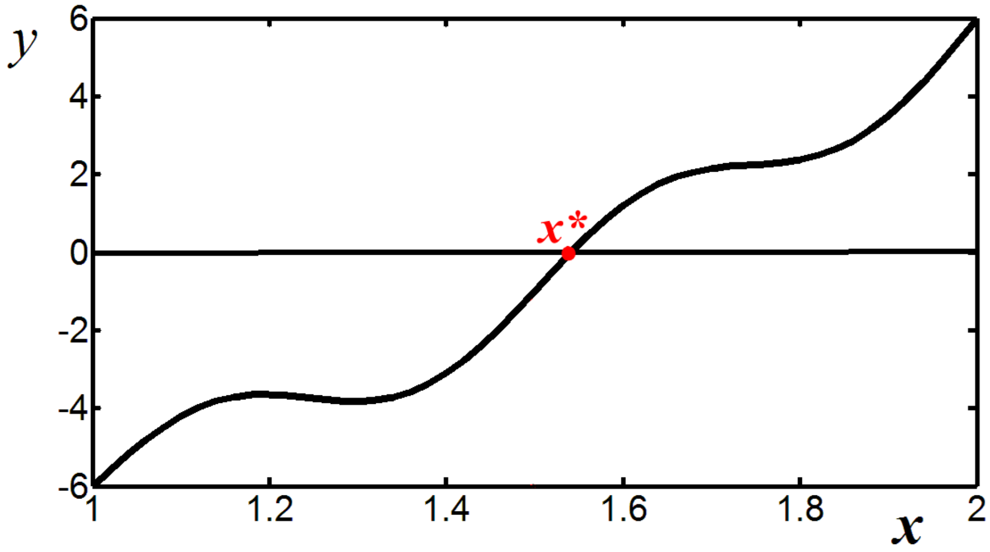
$$4\,x^2 + \sin(4\,\pi\,x)\ -10 = 0\ ,\ \ 1 \leq x \leq 2$$

there is no explicit formula for calculation of root x^* .

At first, we must understand if the root x^* exists.

Theorem. If the function $y=f(x)$ is continuous on $a \le x \le b$, and the signs of $f(a)$ and $f(b)$ are opposite, then there exists x^* such that $f(x^*)=0$.

(for a proof, see course of Math. Analysis)



$$4\,x^2 + \sin(4\,\pi\,x) - 10 = 0\,,\quad 1 \le x \le 2$$

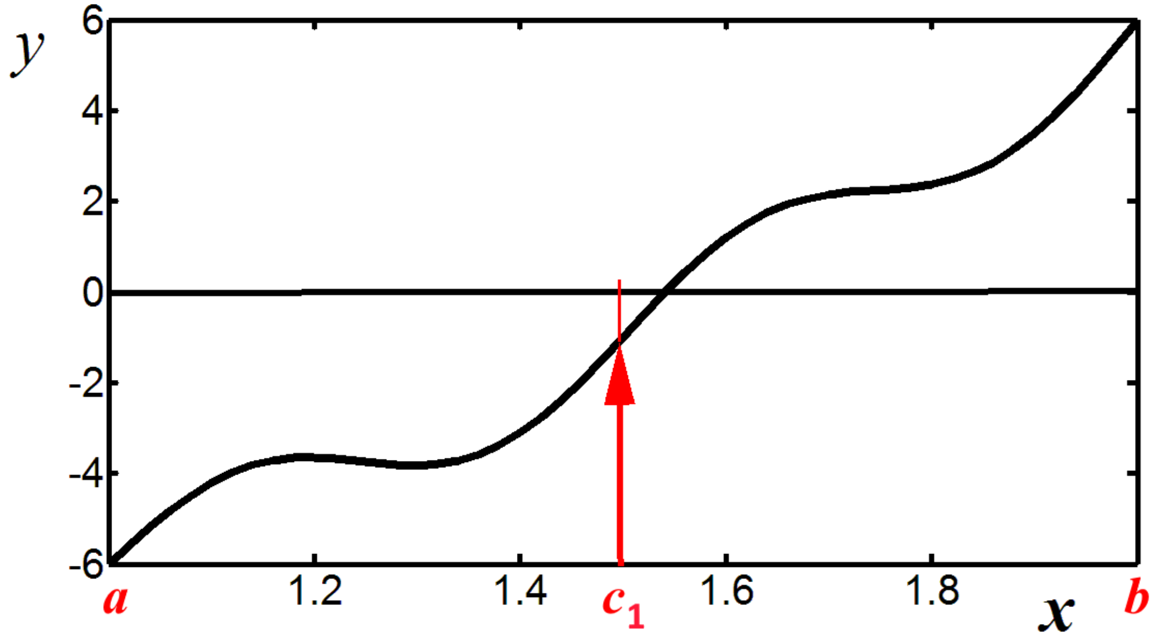
The solution exists!

1) Bysection method for calculation of x^*

It consists of repeatedly bisecting the given interval, and then selecting subinterval in which the function changes sign, and therefore must contain a root.

The method is also called the **interval halving** method, and the **dichotomy method**.

At each step the method divides the interval in two parts by computing the midpoint $c_1 = (a+b)/2$ and then selecting the part in which function $f(x)$ changes sign.



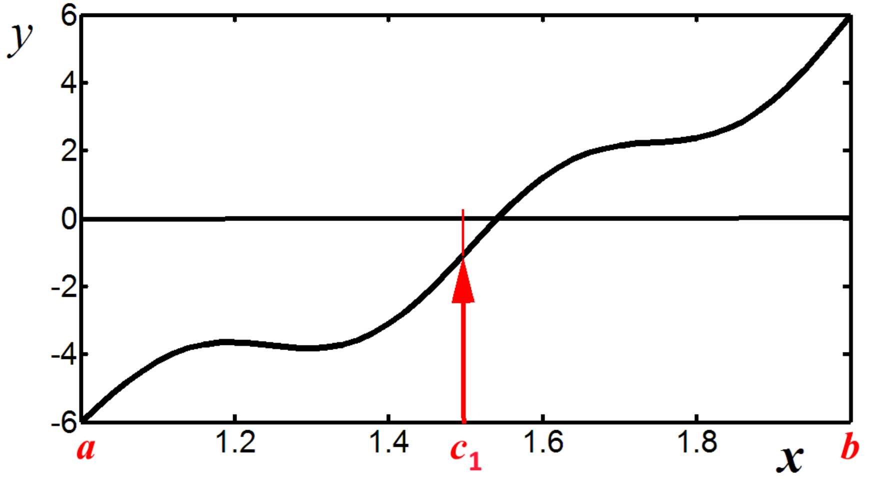
Algorithm of calculations: **Suppose** $f(a)<0, \ f(b)>0.$

Calculate $c_1 = (a+b)/2$ *and* $f(c_1)$

If $f(c_1)<0$ *, then root is in the right subsegment, therefore we denote* $a_2=c_1$, $b_2=b$

If $f(c_1)>0$ *, then we denote*

$a_2 = a$, $b_2 = c_1$



Then procedure repeats:

$c_k = (a_k+b_k)/2 \quad , \qquad k = 2, \ldots$

If $f(c_k)<0$ *, then* $a_{k+1} = c_k$, $b_{k+1} = b_k$

If $f(c_k)>0$ *, then* $a_{k+1} = a_k$, $b_{k+1} = c_k$

(If $f(c_k)=0$ *, then* c_k ***is a root*****).**

At each step, $f(a_k)<0$, $f(b_k)>0$. The procedure continues until the subinterval is sufficiently small.

Length of the subinterval $b_k - a_k = (b - a) / 2^{k-1}$

shows a maximum error in the calculated approximate solution c_k **:**

$$|c_k - x^*| \leq (b - a) / 2^k$$

where x^* **is the "true" solution.**

$k=10 \Rightarrow (b-a)/1024$

$k=20 \Rightarrow (b-a)/1048576$

In engineering problems, typically, the error (tolerance) of 0.0001 or 0.00001 is O.K.

That is, calculations can be stopped if in successive approximations, c_k **and** c_{k+1} **, 4 or 5 digits to the right of decimal point remain the same after rounding.**

Now we consider another method, which often needs smaller number of steps for obtaining a solution of the same accuracy

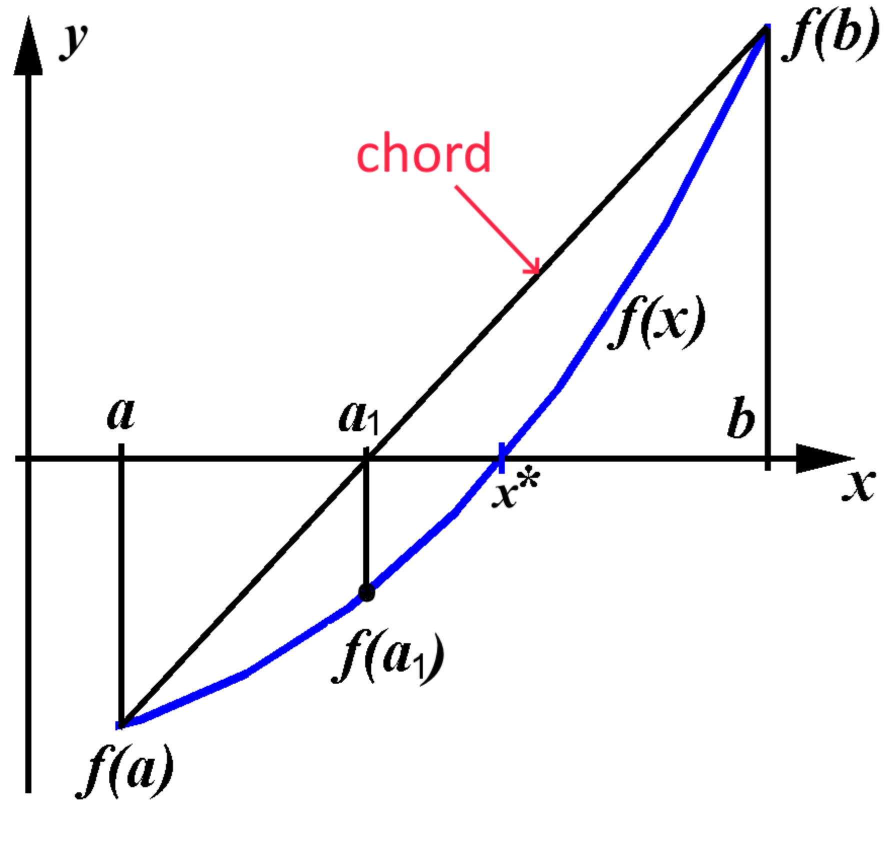
2) Method of chords

Suppose that the curve

$y=f(x)$ is convex: $f''(x)>0$

Draw a straight segment (chord) connecting its endpoints:

$$y=f(a)+(x-a)[f(b)-f(a)]/(b-a)$$



Find intersection of the segment with the axis $y=0$:

$$0=f(a)+(a_1-a)[f(b)-f(a)]/(b-a)$$

$$-(b-a)f(a)=(a_1-a)[f(b)-f(a)]$$

$$a_1-a=-(b-a)f(a)/[f(b)-f(a)]$$

$$a_1 = a - (b-a)f(a)/[f(b)-f(a)]$$

- first step towards the solution

$$a_2 = a_1 - (b-a_1)f(a_1)/[f(b)-f(a_1)]$$ **- second step**

$$a_{k+1} = a_k - (b-a_k)f(a_k)/[f(b) - f(a_k)] \qquad k=1,2,\ldots$$

We get a sequence of approximate solutions

$$a_k \longrightarrow x^*$$

the sequence is monotonously increasing as $f(a_k)<0$

Example: let's solve the equation

$$e^x + 2x^2 = 2, \qquad 0 \leq x \leq 1$$

$$e^x + 2x^2 - 2=0 \qquad f''(x)>0$$

$$a_{k+1} = a_k - f(a_k)\,(b-a_k)/\,[\,f(b) - f(a_k)\,]$$

Scilab, command window:

→ a=0

→ b=1

→ fa=exp(a)+2*a*a-2

→ fb=exp(b)+2*b*b-2

→ a=a-fa*(b-a)/(fb-fa)

→ fa=exp(a)+2*a*a-2

→ a= **iterate**

3) Method of tangent lines (Newton's method)

The same problem: solve

$$f(x)=0\ , \quad a \le x \le b$$

Suppose curve y=*f(x)* is convex: $f''(x)>0$

Line tangent to curve at x=b, y=f(b):

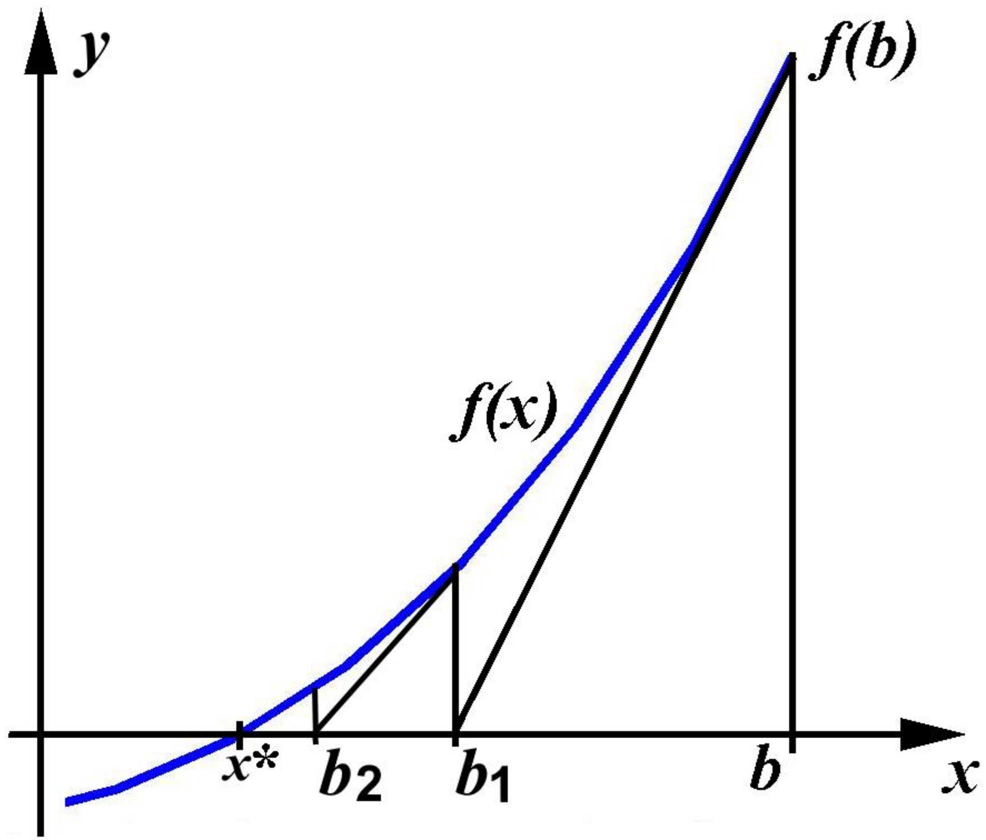
$$y=f(b)+(x-b)f'(b)$$

intersection: $0=f(b)+(b_1-b)f'(b)$

$$-f(b)=(b_1-b)f'(b)$$

$$-f(b)/f'(b)=(b_1-b)$$

$b_1=b-f(b)/f'(b)$ - ***first step towards solution,***



$b_1 = b - f(b)/f'(b)$ - *first step towards solution,*

$b_2 = b_1 - f(b_1)/f'(b_1)$

$b_{k+1} = b_k - f(b_k)/f'(b_k), \quad k=2, \ldots$

sequence is monotonously decreasing

Example: finding square root

$$x^2 = d$$

$$f(x) = x^2 - d \qquad f'(x) = 2x$$

$$b_{k+1} = b_k - (b_k^2 - d)/(2b_k)$$

$$b_{k+1} = b_k - b_k/2 + d/(2b_k) \qquad b_{k+1} = (b_k + d/b_k)/2$$

initial b – arbitrary value

Example: $e^x + 2x^2 - 2 = 0$

(same as in the method of chords)

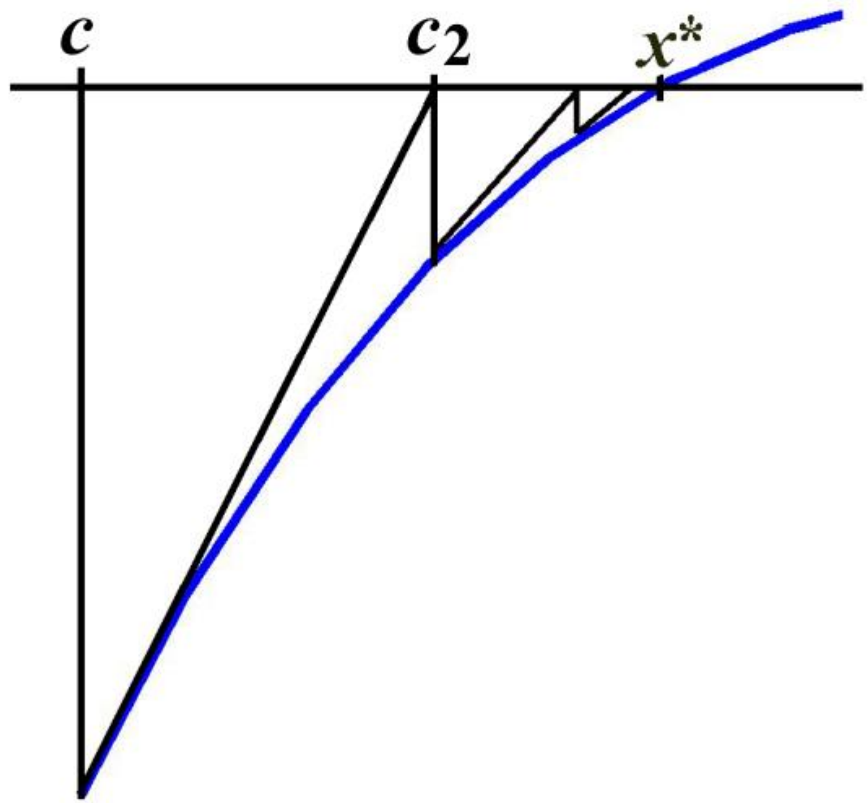
If curve y=*f(x)* is concave:
f''(x)<0

then the left endpoint *a* of the given segment is recommended as initial point for the start of iterations.

We have the same formula

$c_{k+1}=c_k - f(c_k)/ f'(c_k),$

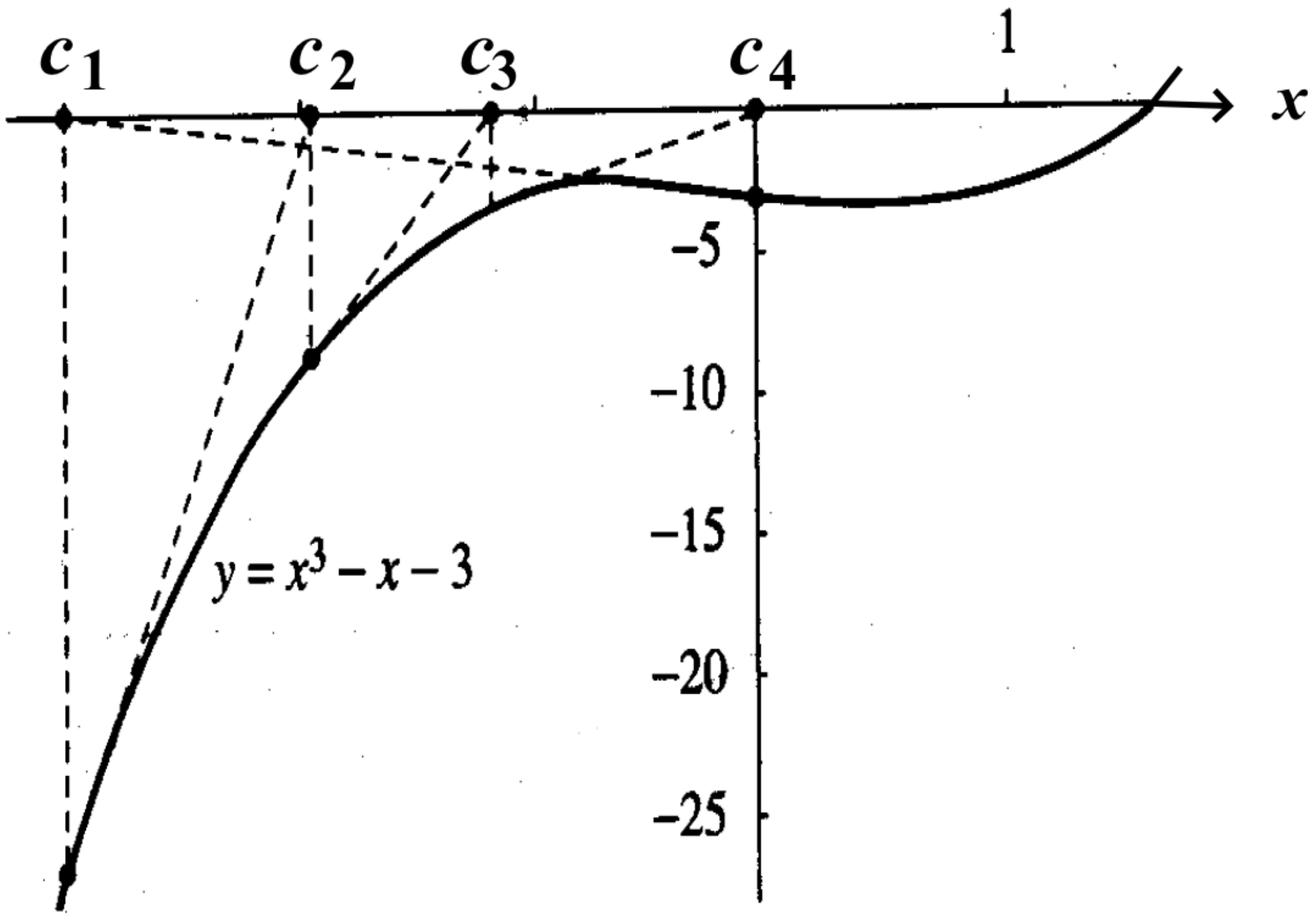
where $f(c_k)<0$; therefore, the sequence c_k is monotonously increasing.



Sometimes, the method does not work (bad cases) :

(*) If $f'(c_k)=0$ for some k, then the method can no longer be applied.

(**) Iterations can stuck in a cycle:



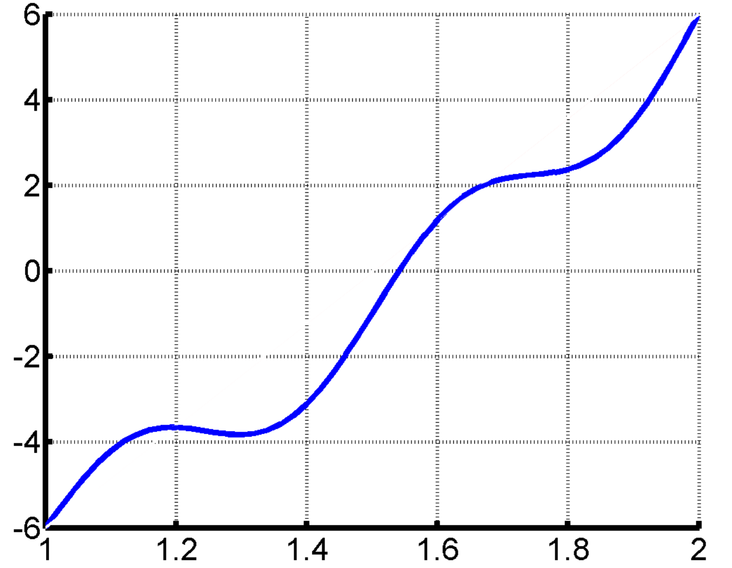
The case f(a)>0 and f(b)<0 can be reduced to the previous one by multiplying the equation by -1 :

$$-f(x)=0$$

<u>Notice</u> on the number of roots:

$$4\,x^2 + \sin(4\,\pi\,x) - 10 = 0\,, \qquad 1 \leq x \leq 2$$

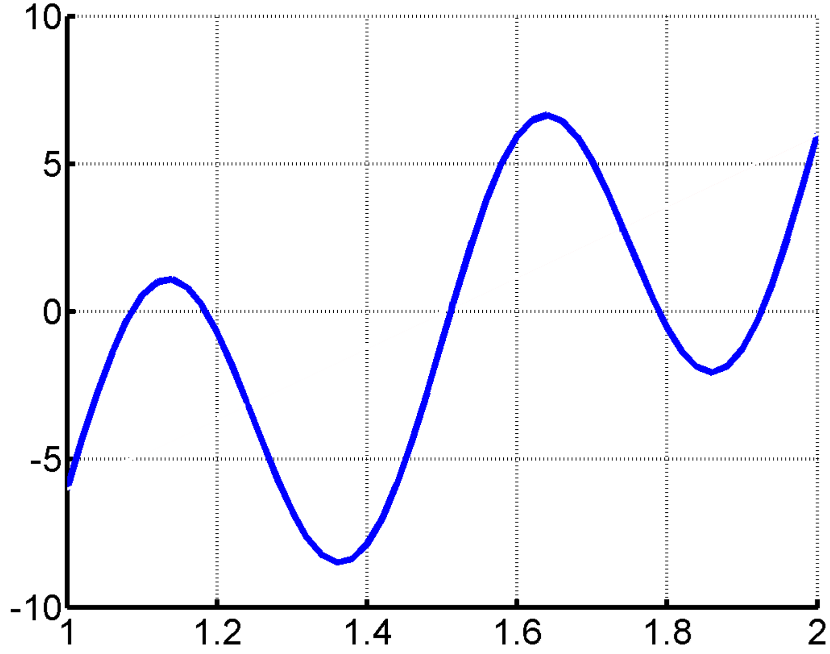
1 root:



Possibly, there exist non-unique roots

$$4x^2 + 6*\sin(4\pi x) - 10 = 0\,, \qquad 1 \le x \le 2$$

5 roots:



It is recommended to "separate" roots by plotting a graph

4) Iteration method

Example 1.

$$x - 0.1 \sin x - 2 = 0$$

$$x = 0.1 \sin x + 2$$

$c_1 = 2$ - initial approximation

$c_2 = 0.1 \sin c_1 + 2 = 2.0909297$

$c_3 = 0.1 \sin c_2 + 2 = 2.0867753$

$c_4 = 0.1 \sin c_3 + 2 = 2.0869810$

$c_5 = 0.1 \sin c_4 + 2 = 2.0869709$

$c_6 = 0.1 \sin c_5 + 2 = 2.0869714$

$c_7 = 0.1 \sin c_6 + 2 = 2.0869713$

$c_8 = 0.1 \sin c_7 + 2 = 2.0869713$

- approximate solutions

In general: The equation $f(x)=0$ can be transformed to the form $x = \varphi(x)$ by a simple addition of x to both sides:

$$x = \underline{x+f(x)}$$

$$x = \varphi(x)$$

Then choose an initial value c_1 and start iterations:

$$c_{k+1} = \varphi(c_k)\,, \qquad k=1, 2,\ldots$$

If c_{k+1} and c_k become very close to each other, then

$$c_k \approx x^* \text{ - solution}$$

Example 2. $x + 0.3(1+x^4) - 0.4 = 0$

clear $0 \leq x \leq 1.4$

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> c=0
> c= 0.4- 0.3*(1+c^4)
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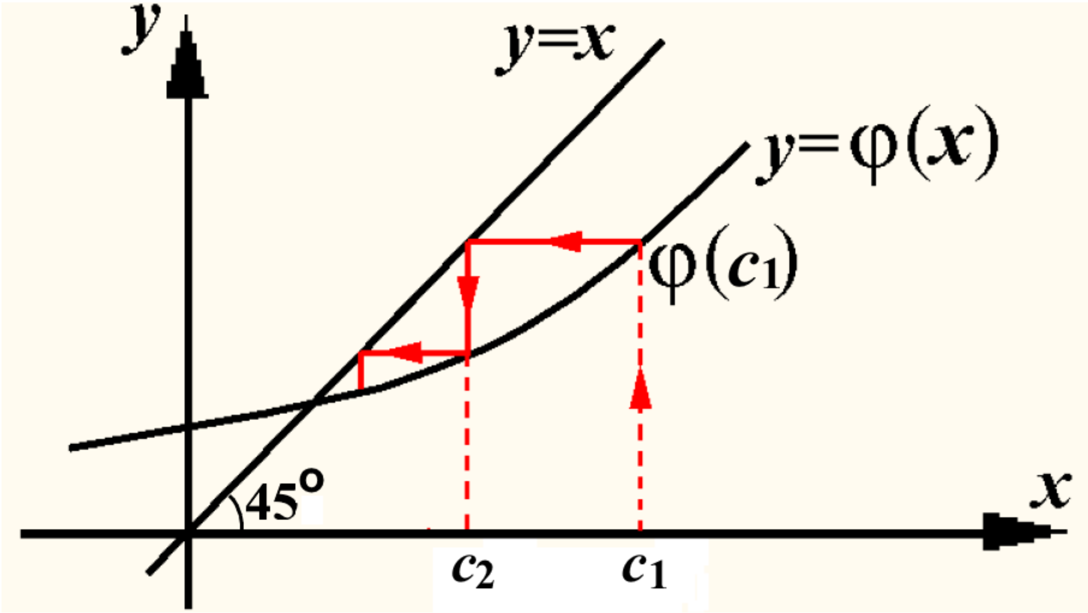
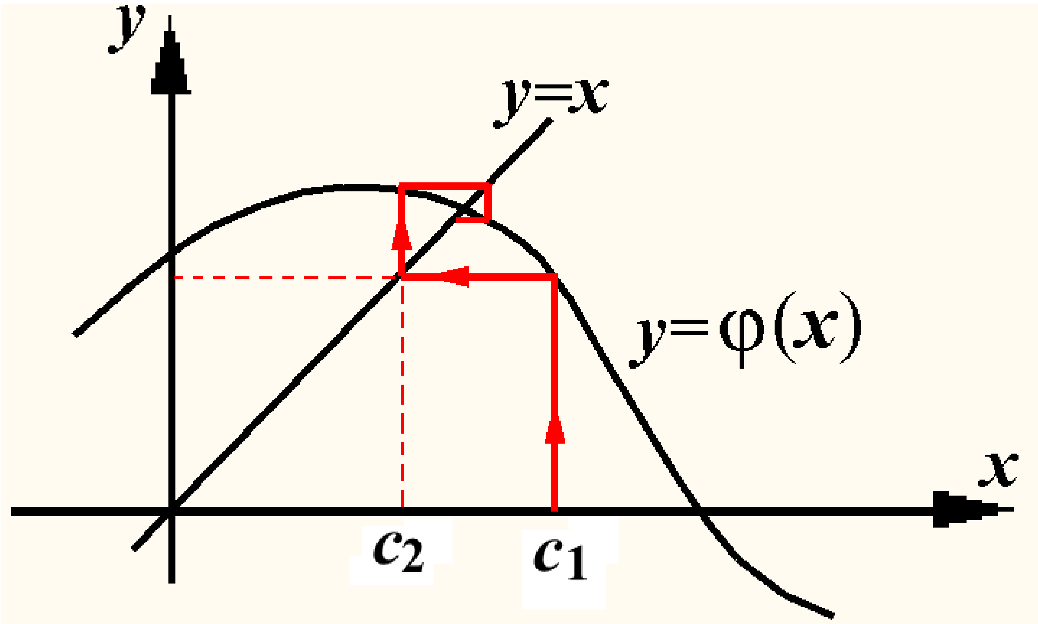
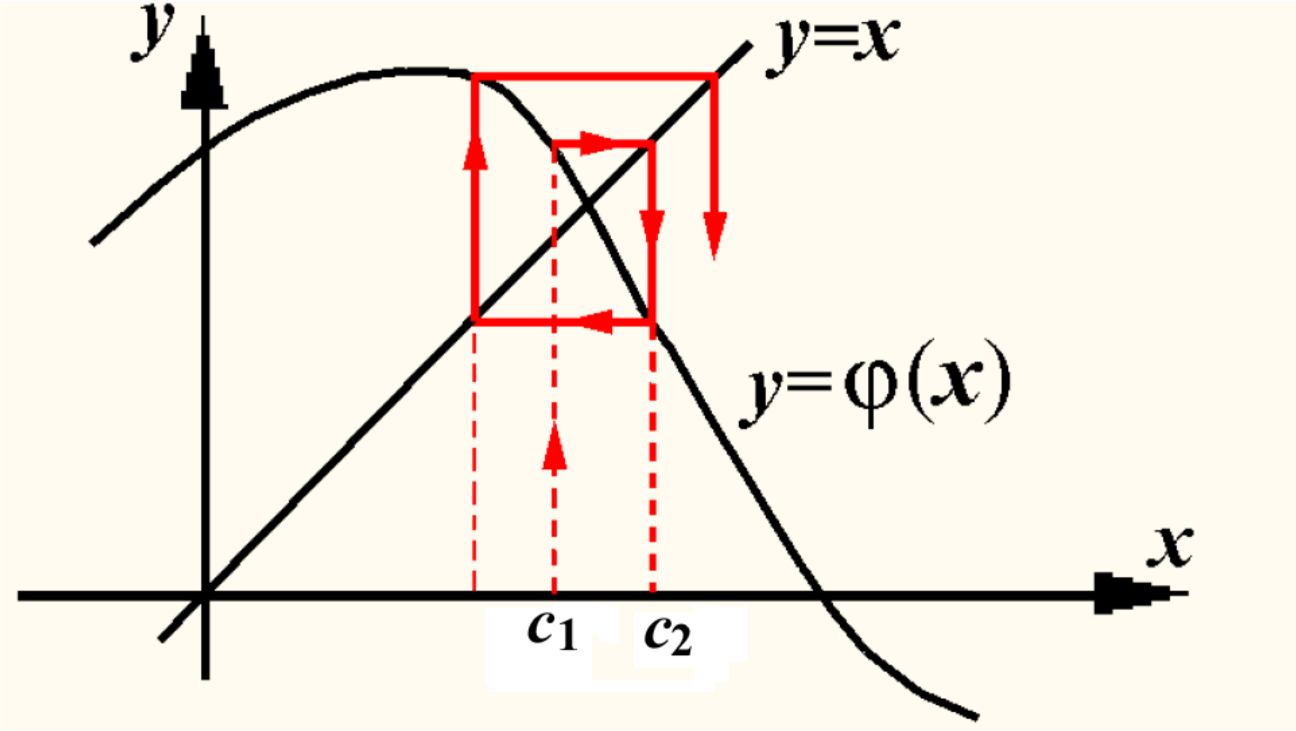
Geometric illustration of solving $x=\varphi(x)$:

Let us plot $y=x$, $y=\varphi(x)$



$$c_2 = \varphi(c_1)$$





In this figure, we see a <u>divergence</u> of successive approximations c_k, which move away from the exact solution x^*.

The convergence or divergence of the sequence c_k *depends on the slope of curve* $y=\varphi(x)$ *to the x-axis, that is on the module of the first derivative :*

$$|\varphi'(x)|$$

Theorem *(sufficient condition for the convergence of iterations):*

If $|\varphi'(x)| < 1$ **at** $a \le x \le b$, *then*

$c_k \longrightarrow x^*$ *at* $k \longrightarrow \infty$, *and*

x^* *is the unique root of the equation* $x=\varphi(x)$.