Week 1.

P23.

3. List all periodic points for each of the following maps. Then use the graph of $f(x)$ to sketch the phase portrait of $f(x)$ on the indicated interval.

a. $f(x) = -\frac{1}{2}x, \quad -\infty < x < \infty$

b. $f(x) = -3x, \quad -\infty < x < \infty$

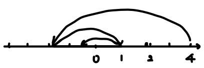
c. $f(x) = x - x^2, \quad 0 \le x \le 1$

d. $f(x) = \frac{\pi}{2}\sin x, \quad 0 \le x \le \pi$

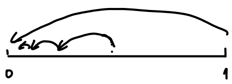
e. $f(x) = -x^3, \quad -\infty < x < \infty$

f. $f(x) = \frac{1}{2}(x^3 + x), \quad -1 \le x \le 1$

a). $(-\frac{1}{2})^n \cdot x = x \quad \Rightarrow \quad x = 0.$

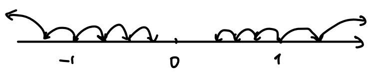


c) $f(x) \in [0, \frac{1}{4}]$. in $[0, \frac{1}{2}]$. $f(x)$ decrease. and $|f'(x)| < 1$. $\Rightarrow$ $f(x) < x$. for all $x \in [0, 1]$.



$\Rightarrow$ no periodic point except $x > 0$.

e). $f(x) = -x^3$.



$x = \pm 1$ with period 2n

$x = 0$ with any period

4. Identify the stable sets of each of the fixed points for the maps in the previous Exercise.

a). $(-\infty, +\infty)$. for $x = 0$

$\forall x \in (-\infty, +\infty) \quad \lim_{n\to\infty} (-\frac{1}{2})^n x = 0 = f(0).$

c). $[0, 1]$. for $x = 0$

$f(x) \in [0, \frac{1}{4}] \qquad f(x) = x - x^2 = x(1-x) \le (1-\varepsilon)x.$ for some $\varepsilon \to 0$.

$\lim_{n\to\infty} f^n(x) \le \lim_{n\to\infty} x \cdot (1-\varepsilon)^n = 0.$

e). $(-1, 1)$ for $x = 0$.

$$\lim_{n\to\infty} f^{in}(x) = \lim_{n\to\infty} (-1)^{in} x^{3^{in}} \begin{cases} = 0 & \text{when } x \in (-1,1). \\ = 1 & \text{when } x = 1 \quad (i = 2k) \\ = -1 & \text{when } x = -1 \quad (i = 2k). \\ \infty & \text{when } x > 1 \text{ or } x < -1. \end{cases}$$

5. For each of the following functions, list all critical points and decide whether each is degenerate or non-degenerate.

a. $f(x) = x^3 - x$

b. $S(x) = \sin(x)$

a) $f'(x) = 3x^2 - 1. \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{3}}$

$f''(x) = 6x \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{3}}$ non-degenerate.

b) $f'(x) = \cos x \quad \Rightarrow \quad x = \frac{\pi}{2} + k\pi$

$f''(x) = -\sin x. \quad \Rightarrow \quad x = \frac{\pi}{2} + k\pi$ non-degenerate.

6. Describe the phase portrait of the map of the circle given by

$$f(\theta) = \theta + \frac{2\pi}{n} + \epsilon \sin(n\theta)$$

for $0 < \epsilon < 1/n$.

Sol: find the fixed point. $\frac{2\pi}{n} + \epsilon \sin(n\theta) = 0.$

$\epsilon \in (0, \frac{1}{n})$, $\frac{1}{\epsilon} > n$. $\frac{2\pi}{n\epsilon} > 2\pi$. $-\sin(n\theta)$ can't have value $> 2\pi$. thus. no fixed point for f.

the phase portrait the a circle, no fixed point.

7. Prove that a homeomorphism of $\mathbf{R}$ can have no periodic points with prime period greater than 2. Give an example of a homeomorphism that has a periodic point of period 2.

Pf: Firstly, a homeomorphism of $\mathbb{R}$ is monotonic:

If $\exists\ a < b < c$, $f(a) < f(b)$, $f(c) < f(b)$ w.l.g.

$\forall\ y_1 \in [\max(f(c), f(a)), f(b)]$. $\exists\ x_1 \in [a, b]$. s.t. $f(x_1) = y_1$.
$\exists\ x_2 \in [b, c]$ s.t. $f(x_2) = y_1$ } by continuity. which contradicts with. f is bijective.

1). f is increasing. , then f^n is increasing. ($[f^2(x)]' = f'(f(x)) \cdot f'(x) > 0$. by induction $(f^n)' > 0$.)

if $f^n(x_0) = x_0$. denote $f(x_0) = x_1$, $f(x_1) = x_2$ ... $f(x_{n-1}) = x_0$, $n > 2$.

if $x_0 = x_1$. it's fixed point if $x_0 \neq x_1$:

w.l.g. $x_0 > x_1$, $f(x_0) > f(x_1)$. i.e. $x_1 > x_2$, ... $x_{n-1} > x_0 > x_1 > \cdots > x_{n-2} > x_{n-1}$. contradicts.

2). f is decreasing. then f^{2n-1} decrease. f^{2n} increasing.

若周期为3:

for orbit. $x_1 < x_2 > x_3 < x_1 > x_2$

所有没有奇周期.

Assume $f^n(x_0) = x_0$. $n > 2$ denote $f(x_0) = x_1$, $f(x_1) = x_2$ ... $f(x_{n-1}) = x_0$.

w.l.g. $n = 3$. $x_0 > x_1$ $\begin{cases} f^2(x_0) = x_2 \\ f^2(x_1) = x_0 \end{cases} \Rightarrow x_2 > x_0 > x_1$ $\begin{cases} f(x_0) = x_1 \\ f(x_2) = x_0 \end{cases} \Rightarrow x_0 < x_1$. contradicts.

Thus. a homoemorphism can't have prime period greater than 2.

<u>example</u> $f(x) = -x$. $\mathbb{R} \setminus \{0\}$ is point of period 2.

p30.

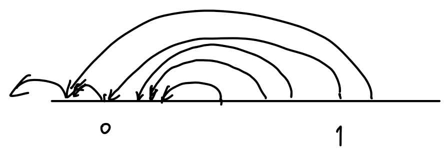
1. Find all periodic points for each of the following maps and classify them as attracting, repelling, or neither. Sketch the phase portraits.

a. $f(x) = x - x^2$

b. $f(x) = 2(x - x^2)$

a) $x = 0$ with any period.

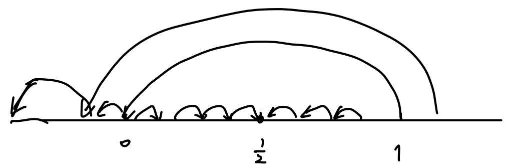
$f'(x) = 1 - 2x$ $\quad f'(0) = 1$. neither.



b). $f'(x) = 2 - 4x$.

$x = 0$ with any period $\quad f'(0) = 2 > 1$. repelling

$x = \frac{1}{2}$ fixed point. $\quad f'(\frac{1}{2}) = 0 < 1$. attracting.



2. Discuss the bifurcations which occur in the following families of maps for the indicated parameter value

a. $S_\lambda(x) = \lambda \sin x, \quad \lambda = 1$

b. $E_\lambda(x) = \lambda e^x, \quad \lambda = 1/e$

a) $\lambda = 1$. fixed point : $x = 0$. $\quad S_\lambda'(x) = \lambda \cos x = 1$ $\quad$ neither

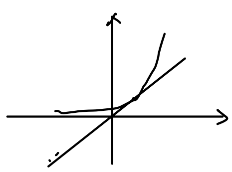
$\lambda < 1$. fixed point : $x = 0$ $\quad S_\lambda'(x) = \lambda \cos x < 1$ $\quad$ attracting.

$\lambda > 1$ fixed point : $x = 0$ and two mutually opposite. s.t. $\lambda = \frac{x}{\sin x}$ (which is > 1).

$S_\lambda' = \lambda \cos x = \lambda > 1$ $\quad$ ($x = 0$ repelling)

$= \frac{x \cos x}{\sin x} = \frac{x}{\tan x}$. $\quad$ (2 others attracting when x closed to 0 (λ close to 1). then be repelling. when x is larger).

b). the increasing continuous func. can't have prime period ≥ 2.



$\lambda = \frac{1}{e}$. fixed point $x = 1$. $\quad E_\lambda'(1) = e^{x-1} = 1$ $\quad$ neither.

$\lambda < \frac{1}{e}$ two fixed point s.t. $\lambda e^x = x$. one $x > 1$ and one $x < 1$. $\quad E_\lambda'(x) = \lambda e^x = x$.

thus two fixed point, one attracting, one repelling.

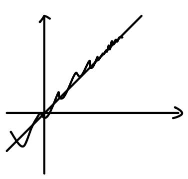
$\lambda > \frac{1}{e}$ $\quad \lambda e^x > x$ always holds $\quad$ no fixed.

3. Suppose f is a diffeomorphism. Prove that all hyperbolic periodic points are isolated.

Pf: consider diffeomorphism in $\mathbb{R}$. f is increasing or decreasing.

increasing func. can only have fixed point (no period ≥ 2), while decreasing func. have no periodic point whose period ≥ 3.

Consider increasing func. at first. $\quad$ Assume the converse. $\exists x$. limit point of hyperbolic fixed point.



i.e. $\exists \{x_n\} \to x$. s.t. $f(x_n) = x_n$. by the graph. $f(x)$ will continuously passing through. the line $y = x$.

thus. the derivative $f'(x)$. will be > 1 and < 1. alternatingly.

i.e. $f'(x_n) \xrightarrow[n\to\infty]{} 1$. by the diffeomorphicity. of f. $f'(x_n) \xrightarrow[n\to\infty]{} 1$ and $\{x_n\} \to x$ implies. $f'(x) = 1$.

which means x is non-hyperbolic, causes contradiction.

4. Show via an example that hyperbolic periodic points need not be isolated.

$$f(x) = \begin{cases} 2x & x \in [0, \frac{1}{2}) \\ 2x-1 & x \in [\frac{1}{2}, 1]. \end{cases}$$

$x = \frac{k}{2^n - 1}$ $k \in [0 : 2^n - 2]$.

for fixed n. x is periodic point. for period n (may be not prime).

thus, at least. $\{x_n = \frac{1}{2^n-1}\} \xrightarrow[n\to\infty]{} 0$. $x=0$ is. non-isolated periodic point.

5. Find an example of a C^1 diffeomorphism with a non-hyperbolic fixed point which is an accumulation point of other hyperbolic fixed points.

$f(x) = x + x^3 \sin(\frac{1}{x})$.

$f'(x) = 1 - x\cos(\frac{1}{x}) + 3x^2 \sin(\frac{1}{x})$.

$x=0$ is a fixed point. $f'(0) = 1$. is non-hyperbolic.

when $x_k = \frac{1}{k\pi}$, $f(x_k) = x_k$. a sequence of fixed point and $\{x_k\} \xrightarrow[k\to\infty]{} x=0$

Week 3-4

1. Prove that $F_2(x) = 2x(1-x)$ satisfies: if $0 < x < 1$, then $F_2^n(x) \to 1/2$ as $n \to \infty$.

Pf: $F_2(\frac{1}{2}) = \frac{1}{2}$. $x = \frac{1}{2}$ is a fixed point of F_2.

also, $x = \frac{1}{2}$ is the maximum.

i) for those $x \in (0, \frac{1}{2})$. $F_2'(x) = 2 - 4x > 0$. and $F_2((0,\frac{1}{2})) = (0, \frac{1}{2})$.

consider $g(x) = F_2(x) - x = x - 2x^2 > 0$

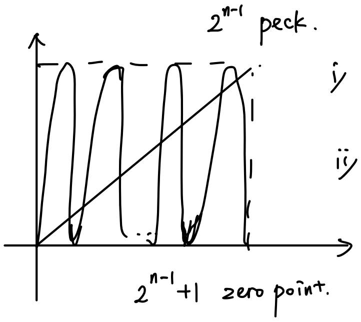
so the sequence $\{F_2(x), F_2^2(x) \dots F_2^n(x) \dots\}$ monotonic increasing and has boundary $\frac{1}{2}$. thus. $F_2^n(x) \xrightarrow[n\to\infty]{} \frac{1}{2}$ $x \in (0, \frac{1}{2})$.

ii) for those $x \in (\frac{1}{2}, 1)$. consider $y = 1-x$. by the symmetry of function. $F_2(y) = 2y(1-y)$.

similar as i). we have $F_2^n(x) \xrightarrow[n\to\infty]{} \frac{1}{2}$ when $x \in (\frac{1}{2}, 1)$.

2. Sketch the graph of $F_4^n(x)$ on the unit interval, where $F_4(x) = 4x(1-x)$. Conclude that F_4 has at least 2^n periodic points of period n.

2^{n-1} peck.



$2^{n-1}+1$ zero point.

i) $F_4(x) = 4x(1-x)$ has fixed point. $x = \frac{1}{4}$ and $\frac{3}{4}$. (1 peck, 2 intersects point)

and for point $x = \frac{1}{2}$. $F_4(\frac{1}{2}) = 1$. ... then $F_4^n(\frac{1}{2}) = 0$. $(n \geq 2)$.

ii) $F_4^2(x)$, $\exists\ x_1^{(1)} \in (0, \frac{1}{2}), x_2^{(1)} \in (\frac{1}{2}, 1)$ s.t. $F_4^2(x_1) = F_4^2(x_2) = 1$. then $F_4^n(x_1) = F_4^n(x_2) = 0$ for $n \geq 3$.

...

iii) consider $F_4^n(x)$. $\exists . \{x_1^{(n-1)}, \dots x_{2^{n-1}}^{(n-1)}\}$. s.t. $F_4^n(x_i^{(n-1)}) = 1$. $F_4^{n+1}(x_i^{(n-1)}) = 0$.

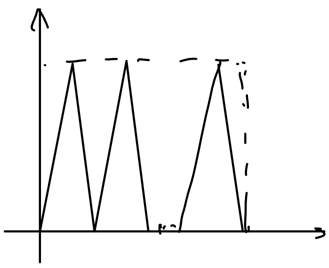
so we obtain 2^{n-1} pecks and $2^{n-1}+1$. zero point for $F_4^n(x)$ in $[0,1]$.

each peck has 2. intersection point with $y = x$. i.e. F_4^n has 2^n fixed point.

3. Sketch the graph of the tent map

$$T_2(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

on the unit interval. Use the graph of T_2^n to conclude that T_2 has exactly 2^n periodic points of period n.



Similar as 2.

For $T_2^n(x)$. $\exists \{x_1^{(n-1)} \dots x_{2^{n-1}}^{(n-1)}\}$. s.t. $T_4^n(x_i^{(n-1)}) = 1$. $F_4^{n+1}(x_i^{(n-1)}) = 0$

we also have 2^{n-1} pecks. each peck intersects $y = x$ twice.

by the procedure (we partition each segment. in each step), the intersecting point can't be equal.

Thus. $T_2^n(x)$ has 2^n periodic point.

4. Prove that the set of all periodic points of $T(x)$ are dense in $[0,1]$.

Pf: by 3. we have the set of all periodic point is. $\{\frac{k}{2^n - 1}\}_{k \in [1 : 2^n - 2]}^{n \geq 1}$

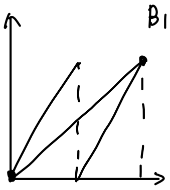
This set is dense in $[0,1]$. when we take sufficiently large n.

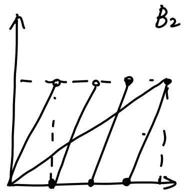
5. Sketch the graph of the baker map

$$B(x)=\begin{cases}2x & 0\le x<1/2\\ 2x-1 & 1/2\le x<1\end{cases}.$$

How many periodic points of period n does B have?

Sol:





we can write B_n explicitly.

$$B_n = \begin{cases} 2^n x & x\in[0,\frac{1}{2^n}). \\ 2^n(x-\frac{1}{2^n}) & x\in[\frac{1}{2^n},\frac{2}{2^n}). \\ \cdots \\ 2^n(x-\frac{k}{2^n}) & x\in[\frac{k}{2^n},\frac{k+1}{2^n}) \\ 2^n(x-\frac{2^n-1}{2^n}). & x\in(\frac{2^n-1}{2^n},1).\end{cases}$$

B_n has 2^n intersection point with $y=x$.

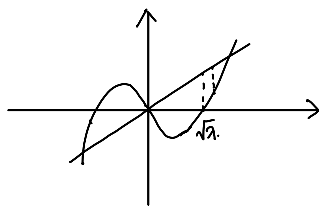
i.e. Baker mapping has. 2^n periodic point of order n.

6. The following exercises deal with the family of functions $F(x)=x^3-\lambda x$ for $\lambda>0$.

a. Find all periodic points and classify them when $0<\lambda<1$.
b. Prove that, if $|x|$ is sufficiently large, then $|f^n(x)|\to\infty$.
c. Prove that if λ is sufficiently large, then the set of points which do not tend to infinity is a Cantor set.

a) $x^3-\lambda x = x$.

$\Rightarrow$ $x=0$ or $x=\pm\sqrt{\lambda+1}$ $\to$ fixed point.



$F'(x)=3x^2-\lambda$. $x=0$. $|F'(0)|=\lambda<1$. attracting.

$x=\pm\sqrt{\lambda+1}$. $|F'(\pm\sqrt{\lambda+1})|=3\cdot 2\lambda>1$ repelling

for points $x\in[-\sqrt{\lambda},\sqrt{\lambda}]$, it will be attracted into $x=0$ by iteration.

for point $x\in[\sqrt{\lambda},\sqrt{\lambda+1})$, it will tend to $[0,\sqrt{\lambda}]$ then attracted to 0. $x\in(-\sqrt{\lambda+1},-\sqrt{\lambda}]$. similarly

for point $x\in(-\infty,-\sqrt{\lambda+1})$, it goes to $-\infty$. after sufficient iteration; $x\in(\sqrt{\lambda+1},+\infty)$ similarly (goes to $+\infty$).

thus. we have $f^n(x)\to 0$ or $+\infty$ or $-\infty$ ($x\ne 0,\pm\sqrt{\lambda+1}$). thus no periodic point other than fixed points.

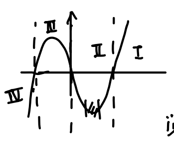
b). consider $\left|\frac{F(x)}{x}\right|=|x^2-\lambda|$

We $|x|$ sufficiently large. for example, let $|x|>\sqrt{2+\lambda}$, we have $\left|\frac{F(x)}{x}\right|>2$.

thus $|F^n(x)|>2^n|x|$ when $n\to\infty$, $|F^n(x)|\to\infty$.

c). when λ is sufficiently large. Then func. is odd, we only consider $x\in[0,+\infty)$.

at least we need $F(\sqrt{\frac{\lambda}{3}})<-\sqrt{\lambda}$. $\Rightarrow$ $\frac{\lambda}{3}\cdot\sqrt{\frac{\lambda}{3}}-\lambda\cdot\sqrt{\frac{\lambda}{3}}=\frac{-2\lambda^{\frac{3}{2}}}{3\sqrt{3}}<-\sqrt{\lambda}$ $\Rightarrow$ $\lambda>\frac{3\sqrt{3}}{2}$.



i) for $F(x)$. point $(\sqrt{\lambda},+\infty)$ goes to $-\infty$.

ii) when $\lambda>\frac{3\sqrt{3}}{2}$. $\exists$ A_1 - a neighborhood of $\sqrt{\frac{\lambda}{3}}$. s.t. $\forall$ $a_1\in A_1$. $F(a_1)<-\sqrt{\lambda}$.

thus. point $F^2(a_1)$ is in IV. it goes to $-\infty$ by iteration.

iii) the procedure can be repeated. $\exists A_2,\dots A_n,\dots$ s.t. $F^3(a_2),\dots,F^{n+1}(a_n)$ in IV.

each step. the interval from "middle" is delete, so the remaining set is. totally disconnected by inifite many iterations.

Week 3.

1. Let

$$\mathbf{s} = (001\,001\,001\ldots)$$
$$\mathbf{t} = (01\,01\,01\ldots)$$
$$\mathbf{r} = (10\,10\,10\ldots).$$

Compute:
a. $d[\mathbf{s},\mathbf{t}]$
b. $d[\mathbf{t},\mathbf{r}]$
c. $d[\mathbf{s},\mathbf{r}]$.

a). $d[s,t] = \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left[\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} \right] = \frac{7}{8} \cdot \frac{1}{1-(\frac{1}{2})^6} = \frac{7}{8} \cdot \frac{64}{63} = \frac{8}{9}.$

b) $d[t,r] = \sum_{k=0}^{\infty} \frac{1}{2^{2k}} \left[1 + \frac{1}{2} \right] = \frac{3}{2} \cdot \frac{1}{1-\frac{1}{4}} = 2$

c) $d[s,r] = \sum_{k=0}^{\infty} \frac{1}{2^{6k}} \left[\frac{1}{2^0} + \frac{1}{2^4} + \frac{1}{2^5} \right] = \frac{35}{32} \cdot \frac{64}{63} = \frac{10}{9}$

2. Identify all sequences in Σ_2 which are periodic points of period 3 for σ. Which sequences lie on the same orbit under σ?

sequence : $S_1 = (111\,111\ldots)$ $S_2 = (011\,011\ldots)$. $S_3 = (101\,101\ldots)$ $S_4 = (110\,110\ldots)$
$S_5 = (100\,100\ldots)$ $S_6 = (010010\ldots)$ $S_7 = (001001\ldots)$ $S_8 = (000\ldots)$

$\sigma(S_1) = (111111\ldots) = S_1$
$\sigma(S_2) = (110110\ldots) = S_4$
$\sigma(S_3) = (011011\ldots) = S_2$.
$\sigma(S_4) = (101101\ldots) = S_3$.
$\sigma(S_5) = (001001\ldots) = S_7$
$\sigma(S_6) = (100100\ldots) = S_5$
$\sigma(S_7) = (010010\ldots) = S_6$.
$\sigma(S_8) = S_8$.

Orbit. $\{S_1\}$. $\{S_8\}$, $\{S_2, S_3, S_4\}$. $\{S_5, S_6, S_7\}$

3. Rework Exercise 2 for periods four and five.

For period 4: 16 sequence.

$S_1 = (0000\ldots)$
$S_2 = (1000\ldots)$
$S_3 = (0100\ldots)$
$S_4 = (0010\ldots)$
$S_5 = (0001\ldots)$
$S_6 = (1100\ldots)$
$S_7 = (1010\ldots)$
$S_8 = (1001\ldots)$
$S_9 = (0110\ldots)$
$S_{10} = (0101\ldots)$
$S_{11} = (0011\ldots)$
$S_{12} = (1110\ldots)$
$S_{13} = (1101\ldots)$
$S_{14} = (1011\ldots)$
$S_{15} = (0111\ldots)$
$S_{16} = (1111\ldots)$

Orbit $\{S_1\}$ $\{S_2, S_3, S_4, S_5\}$. $\{S_6, S_8, S_9, S_{11}\}$ $\{S_7, S_{10}\}$. $\{S_{12}, S_{13}, S_{14}, S_{15}\}$. $\{S_{16}\}$

For period 5 : 32 sequence.

Orbit. $\{S_1\}$ $\{S_{32}\}$. other 6 group of orbits with 5 distinct elements each.

2 fixed point sequence.

4. Let Σ' consist of all sequences in Σ_2 satisfying: if $s_j = 0$ then $s_{j+1} = 1$. In other words, Σ' consists of only those sequences in Σ_2 which never have two consecutive zeros.

a. Show that σ preserves Σ' and that Σ' is a closed subset of Σ.
b. Show that periodic points of σ are dense in Σ'.
c. Show that there is a dense orbit in Σ'.

a). $\forall s \in \Sigma'$. $\sigma(s)$ erase the first number.

$\sigma(s)$. satisfy that. if $s_{j-1} = 0$ then $s_j = 1$. denote $i = j-1$. we have $\sigma(s) \in \Sigma'$.

$\forall \{s_n\} \subseteq \Sigma'$. and $s_n \to s$. show that $s \in \Sigma'$

assume $s \notin \Sigma'$. then $\exists i \in \mathbb{N}$ s.t. $s_i = 0$ and $s_i = 0$

but. for s_n. if $s_{n_i} = 1$. then $d[s, s_n] > \frac{1}{2^i}$

if $s_{n_i} = 0 \Rightarrow s_{n_{i+1}} = 1$. then $d[s, s_n] > \frac{1}{2^{i+1}}$. which contradicts to $s_n \to s$.

b). denote set Σ_p as all periodic points of σ.

$\forall s \in \Sigma'$. and $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$.

then there exists $s_p \in \Sigma_p$ of period N. s.t. $s_i = s_{p_i}$ for $i \in [1:N]$.

thus we have $d(s_p, s) < \varepsilon$. by the arbitariness of s, Σ_p is dense in Σ'.

c) consider $s_0 = \{ 01 \mid 01\ 10\ 11 \mid 011\ 010\ 110\ 101\ 111 \mid \cdots$

$\underbrace{\quad}_{\text{1block}}$ $\underbrace{\quad}_{\text{2block}}$ $\underbrace{\quad}_{\text{3 block}}$.

s_0 contains all blocks of 1's and 0's with non-consecutive 0.

$\forall s \in \Sigma'$. $\forall \varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{2^N} < \varepsilon$. $\exists k$, s.t $\sigma^k(s_0)_i = s_i$ for all $i \in [1:N]$.

thus. $d[s, \sigma^k(s_0)]$. the orbit $O^+(s_0)$ is dense in Σ'.