13 Quadratic and Bilinear forms

13.1 Old definition (remindment)

Now we discuss the notion of **quadratic form** on a vector space. Let us begin from an Old-fashioned definition and then switch to more modern one.

Definition. Quadratic form f in variables $X_1, X_2, \dots, X_n$ is just a homogeneous polynomial of second degree

$$f(X_1, X_2, \dots, X_n) = \sum_{i,j} a_{i,j} X_i X_j. \tag{1}$$

Homogeneous of second degree means each monomial in f has total degree 2 i.e. has the form $c \cdot X_i^2$ or $c \cdot X_i X_i$.

When its coefficients a_{ij} belongs to the field F then f is called real quadratic form. When $a_{ij} \in \mathbb{C}$ then f is called complex quadratic form.

Example. $2X^2 + 3XY - Y^2$ is a quadratic form in two variables.

$2X^2 + XY + 3XZ + Y^2 + 2YZ - Z^2$ is a quadratic form in three variables.

How many coefficients a quadratic form in n variables does have? One can suppose that this quantity is equal to n^2 since i and j vary from 1 to n. Nevertheless in our examples one can see that the quadratic form in two variables has 3 coefficients and quadratic form in three variables has 6 coefficients.

The point is that in the quadratic form (1) the coefficients which are symmetric to each other — a_{ij} and a_{ji}, are multiplied by the same monomial $X_i X_j$.

$$a_{11}X^2 + a_{12}XY + a_{21}YX + a_{22}Y^2 = a_{11}X^2 + (a_{12} + a_{21})XY + a_{22}Y^2.$$

Frequently, one can define quadratic form by a formula

$$\sum_{i \leqslant j} b_{ij} X_i X_j$$

in order to avoid the repetition of the same monomials in the sum. It occurs that it is more convenient to admit repetition as in the formula (1) but in this formula only the sum of the two symmetric coefficients $a_{ij} + a_{ji}$ makes sense. Usually, one suppose in the formula (1) that these coefficients are equal to each other: $a_{ij} = a_{ji}$. So the coefficients in quadratic form before a monomial $X_i X_j$ is equal to $2a_{ij}$.

13.2 Matrix expression and change of variables

Let $f(X_1, X_2, \dots, X_n) = \sum\limits_{i,j=1}^{n} a_{ij}X_iX_j$ be a quadratic form over F where $a_{ij} = a_{ji}$ for any $i, j = 1..n$. The symmetric matrix $A_f = (a_{ij})$ is called the **matrix of a quadratic form** f. Then one has

$$f(X_1, X_2, \dots, X_n) = (X_1, X_2, \dots, X_n) \cdot A_f \cdot (X_1, X_2, \dots, X_n)^T.$$

Remark. The symmetric $n \times n$ matrix A_f is uniquely defined by a homogeneous polynomial of second degree in n variables.

Suppose that there is another set of variables $X_1', X_2', \dots, X_n'$ which is connected to the former one by a linear change variables

$$\begin{array}{c} X_1 = c_{11}X_1' + c_{12}X_2' + \dots + c_{1n}X_n' \\ X_2 = c_{21}X_1' + c_{22}X_2' + \dots + c_{2n}X_n' \\ \dots\dots\dots \\ X_n = c_{n1}X_1' + c_{n2}X_2' + \dots + c_{nn}X_n' \end{array} \tag{2}$$

where transition matrix $C = (c_{ij})_{i,j=1}^{n}$ is invertible.

Then the new quadratic form $\widetilde{f}$ in variables $X_1', X_2', \dots, X_n'$ obtained from f by a substitution (2):

$$\widetilde{f}(X_1', X_2', \dots, X_n') = f(X_1, X_2, \dots, X_n)$$

is called **equivalent** to f.

In the matrix notation it means that

$$(X_1', X_2', \dots, X_n') \cdot A_{\widetilde{f}} \cdot (X_1', X_2', \dots, X_n')^T = (X_1, X_2, \dots, X_n) \cdot A_f \cdot (X_1, X_2, \dots, X_n)^T =$$

$$\begin{aligned} &= \big((X_1', X_2', \dots, X_n') \cdot C^T\big) \cdot A_f \cdot \big(C \cdot (X_1', X_2', \dots, X_n')^T\big) = \\ &= (X_1', X_2', \dots, X_n') \cdot (C^T A_f C) \cdot (X_1', X_2', \dots, X_n')^T. \end{aligned}$$

So at the beginning an at the end we see equal (not merely equivalent) quadratic forms in variables $X_1', X_2', \dots, X_n'$. The matrix of the former is equal to $A_{\widetilde{f}}$ and the matrix of the latter is equal to $C^T AC$. Since the matrix $C^T AC$ is symmetric when A is a symmetric matrix and there is one to one correspondence between symmetric matrices and quadratic forms we conclude that

$$A_{\widetilde{f}} = C^T AC \tag{3}$$

Remark. Usually when matrix of a quadratic form in variables $X_1, X_2, \dots, X_n$ is denoted simply by A then the matrix of corresponding quadratic form in variables $X'_1, X'_2, \dots, X'_n$ is denoted by A'. So the equation (3) is usually written as

$$A' = C^T AC.$$

13.3 Lagrange diagonalization theorem (remindment)

Let us begin from numerical example.

Problem 13.1. *Find a suitable linear change of variables for a quadratic form $f(X, Y, Z) = 2X^2 + 4XY + 3XZ + Y^2 + 2YZ - Z^2$ such that the new quadratic form $\widetilde{f}$ is equal to $a(X')^2 + b(Y')^2 + c(Z')^2$ for some $a, b.c \in \mathbb{R}$.*

Solution. Consider monomials containing first variable: $2X^2 + 4XY + 3XZ = 2(X^2 + 2XY + \frac{3}{2}XZ)$ and try to find a linear change of variables $X' = X + \alpha Y + \beta Z;\quad Y' = Y;\quad Z' = Z$ such that $2 \cdot (X')^2$ would have the required coefficients before XY and XZ. Of course, on have to chose $2\alpha = 2$ and $2\beta = \frac{3}{2}$:

$$2X^2+4XY+3XZ+Y^2+2YZ-Z^2 = 2\left(X + Y + \frac{3}{4}Z\right)^2 - Y^2 - YZ - \left(1 + 2 \cdot \frac{9}{16}\right) Z^2.$$

So in the new variables the coefficients before $X'Y'$ and $X'Z'$ have been eliminated. Then we proceed in the same way with the second variable:

$$-Y^2 - YZ - \frac{17}{8}Z^2 = -(Y^2 + YZ) - \frac{17}{8}Z^2 = -\left(Y + \frac{1}{2}Z\right)^2 + \left(\frac{1}{4} - \frac{17}{8}\right) Z^2.$$

So under the cnange ov variables given by
$X' = X + Y + \frac{3}{4}Z;\quad Y' = Y + \frac{1}{2}Z;\quad Z' = Z$
the quadratic form $2X^2 + 4XY + 3XZ + Y^2 + 2YZ - Z^2$
transforms into $2(X')^2 - (Y')^2 - \frac{15}{8}(Z')^2$.

□

$\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ implies that the transition matrix $C = \begin{pmatrix} 1 & 1 & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}^{-1}$.

The symmetric matrix $A = \begin{pmatrix} 2 & 2 & \frac{3}{2} \\ 2 & 1 & 1 \\ \frac{3}{2} & 1 & -1 \end{pmatrix}$ being transformed into diagonal form $A = (C^{-1})^T A' C^{-1}$:

$$\begin{pmatrix} 2 & 2 & \frac{3}{2} \\ 2 & 1 & 1 \\ \frac{3}{2} & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{3}{4} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\frac{15}{8} \end{pmatrix} \begin{pmatrix} 1 & 1 & \frac{3}{4} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Exercise 13.1. *For the same matrix* $A = \begin{pmatrix} 2 & 2 & \frac{3}{2} \\ 2 & 1 & 1 \\ \frac{3}{2} & 1 & -1 \end{pmatrix}$ *find low-triangular matrix* L *with units on the main diagonal such that* $L^T AL$ *has diagonal form.*

Theorem 13.2. Lagrange diagonalization theorem

- *For any quadratic form in variables* $X_1, X_2, \dots, X_n$ *over a field* F *(where* $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}$ *et cetera) there exist an invertible linear change of variables* $(X_1 \dots X_n)^T = C \cdot (X_1' \dots X_n')$ *such that* $f\left((X_1, \dots X_n) \cdot C^T\right) = d_1(X_1')^2 + d_2(X_2')^2 + \dots + d_n(X_n')^2$ *for some* $d_1, \dots, d_n \in F$.
- *For any symmetric matrix* A *over a field* F *there exists an invertible matrix* C *such that* $C^T AC$ *is a diagonal matrix.*

Hint to the proof. (1st Case) If the coefficient of a monomial X_1^2 is not equal to zero then we proceed as in the first step of the above example (**Problem 13.1**) eliminating all coefficients of monomials $X_1 X_i$ where $i = 2..n$. Then we obtain quadratic form that looks like $d_1 X_1^2 + g(X_2, \dots, X_n)$ (in new variables) and proceed by induction.

(2nd Case) If there exist i such that coefficients of X_i^2 is not equal to zero then we can renumber the variables (which is a special case of a linear change) in such a way that $X_1' = X_i$ to be able proceed as in the 1st Case.

(3d Case) If all the coefficients of X_i^2 are equal to zero and the coefficient of $X_1 X_2$ is not equal to zero then we make the following change of variables:

$$X_1 = (X_1' + X_2'); \quad X_2 = (X_1' - X_2'); \quad X_i = X_i' \quad \textit{where} \quad i = 3, \dots, n.$$

Thus we obtain a quadratic form in new variables satisfying condition of the 1st Case. □

13.4 Quadratic form. Actual definition

Definition. Let V be a vector space over F. A function $q : V \to F$ is called a quadratic form on V if there exists a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ and homogeneous quadratic polynomial f in variables $X_1, \dots, X_n$ such that for any $x_1, x_2, \dots, x_n \in F$ one has

$$q(x_1v_1 + x_2v_2 + \cdots + x_nv_n) = f(x_1, x_2, \dots, x_n).$$

Thus in order to compute the value of the quadratic form on an arbitrary vector v: Firstly, we take basis expansion $v = x_1v_1 + x_2v_2 + \cdots + x_nv_n$ so $(x_1, \dots, x_n)^T = [v]_{\mathcal{B}}$. Secondly, compute the value of the corresponding polynomial $f(x_1, x_2, \dots, x_n)$.

Remark. The word "there exists a basis" in the definition above could be replaced by "for any basis". Indeed, choosing another basis $\mathcal{B}'$ coordinate columns of the same vector $v \in V$ in both bases are connected by a linear change of coordinates given by a transition matrix $(X'_1, X'_2, \dots, X'_n)C^T = (X_1, X_2, \dots, X_n)$ where $C = M_{\mathcal{B}\to\mathcal{B}'}$ (see **Theorem 6.6**). For any vector $v \in V$ let $(x_1, x_2, \dots, x_n)^T = [v]_{\mathcal{B}}$ and $(x'_1, x'_2, \dots, x'_n)^T = [v]_{\mathcal{B}'}$ — corresponding coordinate columns. So if $q(v) = f(x_1, x_2, \dots, x_n)$ for some homogeneous quadratic polynomial f then $q(v) = \widetilde{f}(x'_1, x'_2, \dots, x'_n)$ where $\widetilde{f}$ is obtained from f by a linear change of variables. In other words, if a symmetric matrix $A = A_f$ corresponds to f then $\widetilde{f}$ is obtained from the matrix $A' = C^TAC$ by the formula $\widetilde{f}(x'_1, x'_2, \dots, x'_n) = (x'_1, x'_2, \dots, x'_n) \cdot A' \cdot (x'_1, x'_2, \dots, x'_n)^T$. Therefore, $\widetilde{f}$ should be also homogeneous quadratic polynomial independent of the choice of vector v.

Remark. A homogeneous quadratic polynomial f plays exactly the same role for a quadratic form $q : V \to F$ as the matrix $[L]_{\mathcal{B}}$ plays the role relative to a linear operator $L : V \to V$. It would be more rigorous to use expression like $f = [q]_{\mathcal{B}}$ or to write $f_{q,\mathcal{B}}$ stressing that the homogeneous quadratic polynomial depends on the choice of basis.

Definition. Given a quadratic form q and a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ the symmetric matrix A such that

$$q(v) = (x_1, \dots, x_n)A(x_1, \dots, x_n)^T \text{ for any vector } \quad v = x_1v_1 + \dots x_nv_n$$

is called **Gram matrix** of the quadratic form q relative to the basis $\mathcal{B}$.

Exercise 13.2. *Let q be a quadratic form, $G_{\mathcal{B}}$ and $G_{\mathcal{B}'}$ are the Gram matrices of q relative to the bases $\mathcal{B}$ and $\mathcal{B}'$ respectively. Denote by C a transition matrix $M_{\mathcal{B}\to\mathcal{B}'}$. Write and prove the equation relating matrices $G_{\mathcal{B}}, C$ and $G_{\mathcal{B}'}$.*

Remark. As the matrix and linear operator are different notions although closely related the two notions of the quadratic form (Old and new one) is also different!

13.5 Bilinear form

Our aim is to give coordinate free definition of a quadratic form, i.e. to describe axiomatically what does it mean for a function $q : V \to F$ to be a quadratic form.

Definition. Let V be a vector space over a filed F. Let $h : V \times V \to F$ be a function of two vector arguments. h is called **bilinear form** on a vector space V if it satisfies the following axioms:

- $h(au_1 + u_2, v) = ah(u_1.v) + h(u_2, v)$ for all $u_1, u_2, v \in V$ and $a \in F$.
- $h(u, av_1 + v_2) = ah(u, v_1) + h(u, v_2)$ for all $u, v_1, v_2 \in V$ and $a \in F$.

Why does it called bilinear? Because fixing the first argument, say u_0 coordinate we obtain a linear functional $v \mapsto h(u_0, v)$ and the same for the second argument.

Main example is the usual dot product on vectors in a geometric plane (or space): $h(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$.

Given basis $\mathcal{B} = \{v_1, \dots, v_n\}$ we can easily compute the value of B on an arbitrary pair $(u, v) \in V \times V$ using basis expansions $u = x_1v_1 + x_2v_2 + \cdots + x_nv_n$ and $v = y_1v_1 + y_2v_2 + \cdots + y_nv_n$ in the following way

$h(u, v) = h(x_1v_1 + x_2v_2 + \cdots + x_nv_n, y_1v_1 + y_2v_2 + \cdots + y_nv_n) = x_1h(v_1, y_1v_1 + y_2v_2 + \cdots + y_nv_n) + x_2h(v_2, y_1v_1 + y_2v_2 + \cdots + y_nv_n) + \cdots + x_nh(v_n, y_1v_1 + y_2v_2 + \cdots + y_nv_n) =$
$= x_1 \sum_{j=1}^{n} y_jh(v_1, v_j) + x_2 \sum_{j=1}^{n} y_jh(v_2, v_j) + \cdots + x_n \sum_{j=1}^{n} y_jh(v_n, v_j) = \sum_{i,j=1}^{n} x_iy_jh(v_i, v_j).$

Definition. Let $G_{h,\mathcal{B}} = (g_{ij})$ be a matrix such that $g_{ij} = h(v_i, v_j)$. Then $G_{h,\mathcal{B}}$ is called a Gram matrix of a bilinear form h relative to the basis $\mathcal{B}$.

Since $\sum_{i,j=1}^{n} x_iy_jh(v_i, v_j) = (x_1, \dots, x_n)G_{h,\mathcal{B}}(y_1, \dots, y_n)^T$ Gram matrix allows to compute $h(u, v)$ in terms of coordinate columns:

$$h(u, v) = \mathbf{x}^T \cdot G_{h,\mathcal{B}} \cdot \mathbf{y} \text{ where } \mathbf{x} = [u]_{\mathcal{B}} \text{ and } \mathbf{y} = [v]_{\mathcal{B}}. \tag{4}$$

Definition. Bilinear form $h : V \times V \to F$ is called symmetric if $h(u, v) = h(v, u)$ for any $u, v \in V$.

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We would discuss only symmetric bilinear form although *symplectic* bilinear form are also of great importance in Mathematics, Mechanics and Quantum Physics.

Exercise 13.3. *Consider a function on a vector space $\mathbb{R}^n$ given by a formula $h(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y}$ where $A \in M_n(\mathbb{R})$ is a given matrix. Prove that h is a bilinear form which is symmetric if and only if $A = A^T$.*

13.6 Quadratic and Bilinear forms

Quadratic and symmetric bilinear form are more or less the same notions: given a quadratic form one has a symmetric matrix in a given basis that define a symmetric bilinear form as in the **Exercise 13.3**. And conversely, given a symmetric bilinear form one has its Gram matrix which defines a corresponding quadratic form.

One can prove directly that this correspondence between quadratic forms and symmetric bilinear forms does not depend on the choce of basis. But it is much more instructive to present a formula connecting these notions.

Theorem 13.3.
- *Let $h : V \times V \to F$ be a (symmetric) bilinear form on a vector space V, Then $q_h(v) = h(v, v)$ is a quadratic form.*
- *Let $q : V \to F$ be a quadratic form on a vector space F. Then $h_q(u, v) = \frac{1}{2}(q(u+v) - q(u) - q(v))$ is a symmetric bilinear form which is called a* **polarization** *of the quadratic form q.*
- *Construction in the first two items are inverse to each other.*

Proof. The first assertion follows immediately from the matrix representation of symmetric bilinear form (4):

$h(x_1v_1 + \cdots + x_nv_n, x_1v_1 + \cdots + x_nv_n) = (x_1, \ldots, x_n)G(x_1, \ldots, x_n)^T$ which is a homogeneous polynomial of second degree.

The second assertion is a bit more tricky. In order to proof that h_q is linear on both its vector arguments it suffices to proof that in a certain basis $\mathcal{B}$ it is computed by a formula (4). So from the Exercise 13.3 it follows that h_q is a symmetric bilinear form. But in order to prove that h_q can be computed by this formula, let us take A to be a matrix of quadratic form q. And consider a bilinear form $h_A(u, v) = \mathbf{x}^T A\mathbf{y}$ where $\mathbf{x} = [u]_\mathcal{B}$ and $\mathbf{y} = [v]_\mathcal{B}$. In order to prove that $h_A = h_q$ it is enough to check a matrix identity:

$$\mathbf{x}^T A\mathbf{y} = \frac{1}{2}\left((\mathbf{x}+\mathbf{y})^T A(\mathbf{x}+\mathbf{y}) - \mathbf{x}^T A\mathbf{x} - \mathbf{y}^T A\mathbf{y}\right)$$

which follows directly from the distributive law (bracket expansion) and obvious identity for 1×1 matrices $\mathbf{x}^T A\mathbf{y} = (\mathbf{x}^T A\mathbf{y})^T = \mathbf{y}^T A^T\mathbf{x} = \mathbf{y}^T A\mathbf{x}$.

If we first take a quadratic form q then construct a bilinear form h_q then it is easy to check that $h_q(v, v) = q(v)$. In another direction, take first a symmetric bilinear form h, consider $q = q_h$ then the construction from the second item gives us

$h_q(u, v) = \frac{1}{2}(h(u+v, u+v) - h(u, u) - h(v, v)) = \frac{1}{2}(h(u, v) + h(v, u))$ which is equal to $h(u, v)$ since h is symmetric.

□

Definition. (Axiomatic defifnition of the quadratic form). A function $q : V \to F$ is called a quadratic form is the function $h_q : V \times V \to F$ of two vector arguments given by $h_q(u, v) = \frac{1}{2}(q(u + v) - q(u) - q(v))$ is bilinear.

13.7 Orthogonal operators

Proposition 13.4. *Let V be a quadratic space, $h : V \times V \to F$ be a symmetric bilinear form and $q : V \to F$ is a corresponding quadratic form. For an invertible linear operator $L : V \to V$ the following conditions are equivalent to each other:*

- *L preserves the bilinear form h: $h(L(u), L(v)) = h(u, v)$ for all $u, v \in V$.*
- *L preserves the quadratic form q, i.e. $q(L(v)) = q(v)$ for all $v \in V$.*

Proof. It follows immediately from the **Theorem 13.3**. Indeed, suppose that $q(L(v)) = q(v)$ for any $v \in V$. Then one has a chain of equalities

$$h_q(L(u), L(v)) = \frac{1}{2}\left(q(L(u) + L(v)) - q(L(u)) - q(L(v))\right) = \text{(since L is linear)}$$

$$= \frac{1}{2}\left(q(L(u + v)) - q(L(u) - q(L(v))\right) = 12\left(q(u + v) - q(u) - q(v)\right) = h_q(u, v)$$

demonstrating that L preseves h_q as well as q. $\square$

Definition. Invertible operator $L : V \to$ is said to be orthogonal with respect to a quadratic (symmetric bilinear) form if this operator preserves given form.

Remark. For the "main example" when h is a dot product and $q(v)$ is a squared length of a vector v the polarization formula from **Theorem.13.3** is just a cosine theorem from plane geometry. You have studied orthogonal linear transformation on a euclidean plane and three-dimensional euclidean space in Analytic geometry course.

Let us express an orthogonality condition in the matrix form. Let $G = G_{h,\mathcal{B}}$ be a Gram matrix relative to a basis $\mathcal{B}$, $\mathbf{x} = [u]_{\mathcal{B}}$ and $\mathbf{y} = [v]_{\mathcal{B}}$ for arbitrary $u, v \in V$.

Then orthogonality condition $h(u, v) = h(L(u), L(v))$ can be rewritten as

$$\mathbf{x}^T G\mathbf{y} = \left([L]_{\mathcal{B}}\mathbf{x}\right)^T G[L]_{\mathcal{B}}\mathbf{y} = \mathbf{x}^T \left(([L]_{\mathcal{B}})^T G[L]_{\mathcal{B}}\right)\mathbf{y}.$$

So the matrix $([L]_{\mathcal{B}})^T G[L]_{\mathcal{B}}$ is also the Gram matrix of symmetric bilinear form h relative to the basis $\mathcal{B}$. By uniqueness of the Gram matrix we proved

Proposition 13.5. *Let V be n-dimensional vector space over F with a basis $\mathcal{B}$ and h is symmetric bilinear form on V. Let G be a Gram matrix $A \in M_n(F)$ is the matrix of an orthogonal operator on V with respect to h relative to the basis $\mathcal{B}$ if and only if*

$$G = A^T G A \tag{5}$$

Remark. When Gram matrix is equal to E (e.g. for orthonormal basis on euclidean plane) the equation (5) transforms into $A^T A = E$. In that case matrix A is called **orthogonal**.

Exercise 13.4. *(each item costs 2 points). Let $V = \mathbb{R}^2$, $q(xe_1 + ye_2) = x^2 - y^2$ be a quadratic form and $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator given in standard basis by a multiplication on a matrix A.*

- *Prove that L is orthogonal operator with respect to q if and only if it preserves the set $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 - y^2 = 1\}$.*
- *Suppose that $\det A = 1$. Prove that L is orthogonal with respect to q if and only if for a certain parameter $t \in \mathbb{R}$ one has $A = \pm \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.* $\rightarrow$ 怎么证

  $\begin{pmatrix} \cosh t & -\sinh t \\ -\sinh t & \cosh t \end{pmatrix}$ 不行

Exercise 13.5. *Let $V = M_2(F)$ and for a given invertible matrix $C \in M_2(F)$ consider a linear operator $\gamma_C : V \to V$ such that $\gamma_C(X) = CXC^{-1}$ for any $X \in M_2(F)$.*

- *Consider a function $q : V \to F$ whose value on an arbitrary matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$*

  we can exclaim $\Rightarrow q(X) = \det X$

  is given by $q(X) = ad - bc$. Prove that q is a quadratic form on a space V and γ_C is orthogonal with respect to q.
- *Consider a function $\varphi : V \to F$ whose value on an arbitrary matrix $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $\varphi(X) = a^2 + 2bc + d^2$. Prove that φ is a quadratic form on a space V and γ_C is orthogonal with respect to φ.*

13.8 Orthogonal complement

As before we consider a vector space V of dimension n with a symmetric bilinear form h and corresponding quadratic form q. Vectors $u, v \in V$ are said to be **orthogonal** with respect to h if $h(u, v) = 0$, and in this ase one write $u \perp_h v$.

Definition. For a subset (not necessary a subspace) $U \subset V$ we define its orthogonal complement with respect to h by

$$U^{\perp_h} = \{v \in V \mid h(v,u) = 0 \text{ for all } u \in U\}.$$

背景

When quadratic or bilinear form is given by context we omit the index in notation for orthogonal complement.

Exercise 13.6. *(each item costs 1 point).*

- *Let $A \subset V$ is an arbitrary subset in a vector space with a given symmetric bilinear form. Prove that $A^\perp = (\mathrm{Span}(A))^\perp$;*
- *Prove directly that $A^\perp$ is a subspace in V.*

Proposition 13.6. *(Basic properties of orthogonal complement). For subspaces $U, U_i \subset V$.*

1. *$U^\perp$ is a subspace in V;*
2. *$U \subset (U^\perp)^\perp$;*
3. *$U_1 \subset U_2 \Rightarrow U_1^\perp \supset U_2^\perp$;*
4. *$(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$;*
5. *$(U_1 + U_2 + \cdots + U_k)^\perp = U_1^\perp \cap U_2^\perp \cap \cdots \cap U_k^\perp$;*
6. *If $U = \mathrm{Span}(u_1, u_2, \ldots, u_k)$ then $U^\perp = u_1^\perp \cap u_2^\perp \cap \cdots \cap u_k^\perp$;*
7. *$\dim U^\perp \geqslant n - \dim U$.*

Proof. 2) follows from symmetry: given $u \in U$ for all $v \in U^\perp$ one has $h(u,v) = h(v,u) = 0$ since $u \perp v$. It means that $u \in \{x \in V \mid h(x,v) = 0 \text{ for all } v \in U^\perp\}$.

3) Let $x \in U_2^\perp$ and $U_1 \subset U_2$ then for all $y \in U_1$ one has $h(x,y) = 0$ since $y \in U_2$ and $x \in U_2^\perp$.

4) $(U_1+U_2)^\perp \subset U_i^\perp$ for $i = 1, 2$ from 3). Taking $x \in U_1^\perp \cap U_2^\perp$ and $u \in U_1 + U_2$ one has $u = u_1 + u_2$ where $u_1 \in U_1$ and $u_2 \in U_2$. Therefore, $h(x,u) = h(x, u_1 + u_2) = h(x,u_1) + h(x,u_2) = 0 + 0$.

5) Follows immediately from 4) by induction.

6) is a special case of 5) taking for U_i the subspace $\mathrm{Span}(u_i)$ and using $u_i^\perp = (\mathrm{Span}(u_i))^\perp$.

7) Let $U = \mathrm{Span}(u_1, \ldots, u_k)$ where $k = \dim U$. Consider a map $L : V \mapsto F^k$ which assign to any vector v the column $(h(v, u_1), h(v, u_2), \ldots, h(v, u_k))^T$. It is easy to check that L is a linear map since for any component of the value column $h(v + v', u_i) = h(v, u_i) + h(v', u_i)$. By definition $\mathrm{Ker}\,(L) = \bigcap\limits_{i=1}^{k} u_i^\perp$ which is equal to $U^\perp$ by 6). By **Theorem 5.6** $\dim U^\perp = \dim \mathrm{Ker}\,(L) = \dim V - \dim \mathrm{Im}\,(L)$. Since $\mathrm{Im}\,(L) \subset F^k$ then $\dim \mathrm{Im}\,(L) \leqslant k$ and consequently $\dim U^\perp \geqslant n - k$;

1) was proved in the course of the proof of 7). But it is instructive to prove it directly that can be asked on exam.

$\square$

Definition. Symmetric bilinear form $h : V \times V \to F$ is said to be **non-degenerate** if $V^{\perp_h} = 0$, i.e. for any non-zero vector u there exists a vector $v \in V$ such that $h(u, v) \neq 0$.

If there exists a non-zero vector u such that $\langle u \rangle^{\perp_h} = V$ then the bilinear form h and corresponding quadratic form q are called degenerate.

Proposition 13.7. *The following conditions on symmetric bilinear form $h : V \times V \to F$ are equivalent:*

$$h(x, v_i) = (x_1, \ldots x_n)\, G\, [0, \ldots \overset{\text{ith}}{1} \ldots 0]^T$$

$$\begin{bmatrix} h(x, v_1) \\ \vdots \\ h(x, v_n) \end{bmatrix} = \begin{bmatrix} & & \\ & L & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

1. *h is non-degenerate;*
2. *For any basis $\mathcal{B}$ the Gram matrix $G_{h,\mathcal{B}}$ has non-zero determinant;* $[L(X)]_2 = [L]_{B,2}[X]_B$
3. *there exists a basis $\mathcal{B}$ such that Gram matrix $G_{h,\mathcal{B}}$ has non-zero determinant.*

Proof. 1) $\Rightarrow$ 2). Take a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$. Consider a linear map $L : V \to F^n$, $\quad x \mapsto (h(x, v_1), h(x, v_2), \ldots, h(x, v_n))^T$. The matrix of this linear map $[L]_{\mathcal{B},\mathcal{E}}$ with respect to the $\mathcal{B}$ and standard basis $\mathcal{E}$ in F^n is exactly the Gram matrix $G_{h,\mathcal{B}}$. Non-degeneracy of h implies that for any non-zero $x \in V$ the column $L(x)$ has at least one non-zero component, otherwise by **Proposition 13.6**(6) one has $x \in v_1^\perp \cap v_2^\perp \cap \cdots \cap v_n^\perp = V^\perp = 0$. Therefore $\mathrm{Ker}\,(L) = 0$ and homogeneous linear system $[L]_{\mathcal{B},\mathcal{E}} \cdot \mathbf{x} = 0$ has only trivial solution. As the linear system $G_{h,\mathcal{B}} \cdot \mathbf{x} = 0$ with a square matrix $G_{h,\mathcal{B}}$ has only non-zero solution then this matrix is invertible and has non-zero determinant.

2) $\Rightarrow$ 3) is obvious.

3) $\Rightarrow$ 1) is proved in almost the same way as the first implication. We use the assertion in the opposite direction: if the square matrix has non-zero determinant

then the corresponding linear homogeneous system has only trivial solution. Therefore $\operatorname{Ker}(L) = 0$ and bilinear form h is non-degenerate.

$\square$

Problem 13.8. *Let $V = \mathbb{R}^4$ and h is a bilinear form having identity matrix as its Gram matrix in standard basis, i,e,* $h\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}\right) = x_1y_1+x_2y_2+x_3y_3+x_4y_4.$

Find a basis of the orthogonal complements to the subspace Span $\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix}\right)$

Solution. Denote for the sake of brevity $u_1 = (1,0,1,-1)^T$ and $u_2 = (-1,2,1,0)^T$. It is easy to see that

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}^{\perp} \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

and the same for the second vector. By **Proposition 13.6**(6)

$\mathbf{x} \in \operatorname{Span}(u_1, u_2)^{\perp} \Leftrightarrow \begin{cases} \mathbf{x} \in u_1^{\perp} \\ \mathbf{x} \in u_2^{\perp} \end{cases}$. Therefore in order to find the orthogonal complement we need just to solve a homogeneous linear system with the matrix $\begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 2 & 1 & 0 \end{pmatrix}$.
Adding first row to the second, we obtain the equivalent linear system $\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 2 & 2 & -1 \end{pmatrix}$ already in reduced row echelon form. Find the unique solution with $x_3 = 1$ and $x_4 = 0$ — it implies that $x_2 = -1$ and $x_1 = -1$. So the column $\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ is the first fundamental solution (corresponding to this echelon form). And for another choice of the free variable, taking $x_4 = 1$ and $x_3 = 0$ we get the column

$\begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$. Therefore, orthogonal complement is equal to the Span $\left(\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right)$. $\square$

Definition. For any subspace $U \subset V$ and a given bilinear form h and corresponding quadratic form q on a vector space V one can consider a **restriction** $h|_U$ or $q|_U$. It is usual restriction of a domain for arbitrary map. Trivially, the map $h|_U : U \times U \to F$ remains to be bilinear. Moreover, the quadratic form on U corresponding to $h|_U$ would be equal the restriction $q|_U$.

Remark. How the Gram matrices of quadratic form and its restriction are related to each other?

Of course Gram matrix depends of the basis choice. So the chosen bases in subspace U and 背景的，环境的 ambient space V should be somehow related to each other. in order to be able to say something about Gram matrices.

Suppose that we choose a basis in V in a way described in the **Theorem 5.6**: first, choose a basis in U and then extend it to a basis in V. So we obtain a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ in V such that $\mathcal{B}_U = \{v_1, v_2, \ldots, v_k\}$ is a basis in U. The question about relation between Gram matrices for h and $h|_U$ in these bases has a simple answer. The matrix $G_{h|_U, \mathcal{B}_U}$ is an upper-left corner of size $k \times k$ in the matrix $G_{h,\mathcal{B}}$

Theorem 13.9. *(orthogonal completion for non-degenerate quadratic form). Let V be an n-dimensional vector space and h be a non-degenerate symmetric bilinear form on V then the following holds:*

1. *For any subspace $U \subset V$ one has $\dim U^\perp = n - \dim U$;*
2. *$\left(U^\perp\right)^\perp = U$; If $U_1^\perp = U_2^\perp$ then $U_1 = U_2$ for any subspaces $U_1, U_2 \subset V$.*
3. *For any subspace $U_1, U_2 \subset V$ one has $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$.*

Proof. Proceeding as in the proof of the analogous assertion in general case **Proposition 13.6**(7) one has to prove that the linear map $L : V \to F^k$ where $k = \dim U$ is surjective. Assume for a contradiction that $\operatorname{Im}(L)$ is a proper subspace in F^k, i.e. $\operatorname{Im}(L) \neq F^k$.

Lemma 13.10. *Let $W \subset U$ is a proper subspace. Then there exists non-zero linear funtional $\varphi : U \to F$ such that $\varphi|_W \equiv 0$.*

Proof. Consider an arbitrary non-degenerate symmetric bilinear form $\widetilde{h}$ on U. Then $W^{\perp_{\tilde{h}}} \neq 0$ by **Proposition 13.6**(7). Taking a non-zero vector $u \in W^{\perp_{\tilde{h}}}$ one obtains a non-zero linear functional $x \mapsto \widetilde{h}(x, u)$ which is zero being restricted on a subspace W by construction. But it is non-zero itself due to non-degeneracy of $\widetilde{h}$. $\square$

Resuming the proof of 1) consider a non-zero linear functional $\varphi : F^k \to F$ such that $\varphi|_{\mathrm{Im}(L)} = 0$. Then $\varphi \circ L = 0$ as a linear map from V to F. We know that linear functional $\varphi : F^k \to F$ is always given by a string $(a_1, a_2, \ldots, a_k)$: $\varphi\left((x_1, x_2, \ldots, x_k)^T\right) = a_1x_1 + a_2x_2 + \cdots + a_kx_k$. Moreovere, there exists i such that $a_i \neq 0$ because φ is non-zero. Taking into account that $L(v) = (h(v, u_1), h(v, u_2), \ldots, h(v, u_k))^T$ we obtain the formula $(\varphi \circ L)(v) = a_1h(v, u_1) + a_2h(v, u_2) + \cdots + a_kh(v, u_k) = h\left(v, \sum\limits_{i=1}^{k} a_iu_i\right)$. Since $\varphi \circ L = 0$ and h is non-degenerate we conclude that $\sum\limits_{i=1}^{k} a_iu_i = 0$ which contradicts linear independence of $u_1, u_2, \ldots, u_k$.

2) By **Proposition 13.6**(2) $U \subset (U^\perp)^\perp$ and in order to prove the equality it is sufficient to check that $\dim U = \dim (U^\perp)^\perp$. Applying 1) to the subspace $U^\perp$ we obtain $\dim U^\perp + \dim (U^\perp)^\perp = n$. Together with s $\dim U + \dim U^\perp = n$ it implies $\dim U = \dim (U^\perp)^\perp$. The second assertion follows from the first immediately applying orthogonal complements to both sides of $U_1^\perp = U_2^\perp$.

3) By the previous statement we can "srtip off" the operation of an orthogonal complement from both side of an equality. So it is sufficient to check that

$$\left((U_1 \cap U_2)^\perp\right)^\perp = (U_1^\perp + U_2^\perp)^\perp.$$

By **Proposition 13.6**(4) The righthandside is equal to $(U_1^\perp)^\perp \cap (U_2^\perp)^\perp = U_1 \cap U_2$ which is equal to the lefthandside.

$\square$

The assertion **Theorem 13.9**(3) and **Proposition 13.6**(4) is crucial for us. It allows to compute in concrete terms an intersection of the given subspaces $U_1, U_2 \subset \mathbb{R}^n$ using formula $U_1 \cap U_2 = \left(U_1^\perp + U_2^\perp\right)^\perp$. This formula follows immediately from **Theorem 13.9**(2,3).

Problem 13.11. *Given subspaces* $U_1 = \operatorname{Span}\left(\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\3\end{pmatrix}\right)$ *and* $U_2 = \operatorname{Span}\left(\begin{pmatrix}1\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}-1\\2\\1\\0\end{pmatrix}\right)$ *in* $\mathbb{R}^4$ *find a basis for their intersection* $U_1 \cap U_2$.

Solution. We are going to find the intersection using formula $U_1 \cap U_2 = \left(U_1^{\perp} + U_2 \perp\right)^{\perp}$ where orthogonal complement supposed to be relative to the usual dot-product on $\mathbb{R}^n$. In order to find $U_1^{\perp}$ we have to solve the homogeneous linear system

$$\begin{cases} x_1 + \ \ x_2 + \ \ x_3 - \ \ x_4 = 0 \\ \qquad\ \ x_2 - \ \ x_3 \qquad\quad = 0 \\ x_1 - \ \ x_2 + \ \ x_3 + 3x_4 = 0 \end{cases}$$

Subtracting the first equation from the third and adding with factor 2 the second equation to the third we get homogeneous linear system with the matrix

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 4 \end{pmatrix}.$$

The variable x_4 is the only free variable with respect to this row echeloned form, so taking $x_4 = 1$ one obtain a fundamental solution $x_4 = 1; x_3 = 2; \quad x_2 = 2; x_1 = -3$. So we have just computed $U_1^{\perp} = \operatorname{Span}\left((-3, 2, 2, 1)^T\right)$. In the course of the **Problem 13.8** we computed $U_2^{\perp} = \operatorname{Span}\left(\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\2\end{pmatrix}\right)$. Since for any vectors v_1, v_2, v_3 one has $\operatorname{Span}(v_1) + \operatorname{Span}(v_2, v_3) = \operatorname{Span}(v_1, v_2, v_3)$ then

$$U_1^{\perp} + U_2^{\perp} = \operatorname{Span}\left(\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\\2\end{pmatrix}, \begin{pmatrix}-3\\2\\2\\1\end{pmatrix}\right).$$

In order to compute the orthogonal completion $(U_1^{\perp} + U_2^{\perp})^{\perp}$ we need to solve the homogeneous linear system with the matrix $\begin{pmatrix} -1 & -1 & 1 & 0 \\ 2 & 1 & 0 & 2 \\ -3 & 2 & 2 & 1 \end{pmatrix}$. Adding the doubled first row to the second and subtracting the tripled first row from the third we obtain $\begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 5 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 9 & 11 \end{pmatrix}$. So $U_1 \cap U_2 = \operatorname{Span}\left(\begin{pmatrix} 7 \\ 4 \\ 11 \\ -9 \end{pmatrix}\right)$. Solving another two linear systems you can find expansions $\begin{pmatrix} 7 \\ 4 \\ 11 \\ -9 \end{pmatrix} = 2 \cdot \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + 9 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 4 \\ 11 \\ -9 \end{pmatrix} = \frac{15}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{2} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ 3 \end{pmatrix} - 4 \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$, confirming that $(7, 4, 11, -9)^T \in U_2$ and $(7, 4, 11, -9)^T \in U_1$.

$\square$

Exercise 13.7. *Finf the basis if the intersection $U_1 \cap U_2$ of two subspaces in $\mathbb{R}^4$:*

$$U_1 = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right); \quad U_2 = \operatorname{Span}\left(\begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 7 \end{pmatrix}\right).$$

Remark. (Tiny philosophical). We can define the subspace in vector space V by choosing several vectors and forming their span, e.g. $U = \operatorname{Span}(v_1, v_2, \dots, v_k)$. Another way to define a subspace is to to define it by linear equations. You should be familiar that the plane in 3-dimensional space can be defined by one linear equation. In a bit other words we can define a subspace as orthogonal completion to the given set of vectors: $U = v_1^\perp \cap v_2^\perp \cap \dots \cap v_k^\perp = \{v_1, v_2, \dots, v_k\}^\perp$. Every vector v_k in this case puts exactly one linear equation on a vector u to belong to U.

Theorem 13.12. *(Riesz representation theorem (baby version)). Let V be an n dimensional vector space and h is a non-degenerate bilinear form. Then for any linear functional $\varphi \in \mathcal{L}(V, F)$ there exists a unique vector v_φ such that $\varphi(x) = h(x, v_\varphi)$ for any vector $x \in V$.*

Remark. In the special case of $\mathbb{R}^n$ with the bilinear form h which is equal to the usual dot-product $h(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ it is rather easy to check. Any linear functional $\varphi : \mathbb{R}^n \to \mathbb{R}$ has the matrix, actually, a row $(a_1, a_2, \ldots, a_n)$ so it is given in a concrete form as $\varphi(\mathbf{x}) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \mathbf{x} \cdot \mathbf{a} = h(\mathbf{x}, \mathbf{a})$ where we denote by $\mathbf{a}$ the column $(a_1, a_2, \ldots, a_n)$.

Proof. (will be omittes on the lecture). Recall that the set $\mathcal{L}(V, F)$ is a linear space because we can add linear maps and multiply them by scalar (see **Proposition 5.2** and subsection nearby).

For any vector $v_0 \in V$ one can consider a linear map which can be obtained from h fixing the second argument: $x \mapsto h(x, v_0)$. Denote this map by $\varphi_{v_0} : V \to F$, so $\varphi_v(x) = h(x, v)$. Of course this map is linear due to linearity of h in the first argument. Consider the auxiliary map

$$\Phi : V \to \mathcal{L}(V, F), \quad v \mapsto \varphi_v.$$

The statement of the theorem is essentially the existence of the inverse to th map Φ. Correspondences $v \mapsto \varphi_v$ and $\varphi \mapsto v_\varphi$ would be inverse to each other. First, we check that Φ is linear and then deduce by the **Theorem 5.6** that Φ is a bijection.

Let us check that Φ is linear, e.g. $\Phi(u + v) = \Phi(u) + \Phi(v)$ or equivalently, $\varphi_{u+v} = \varphi_u + \varphi_v$. On both hands of the desired equality there are linear maps from $\mathcal{L}(V, F)$. In order to prove that two linear map coincide it is sufficient to prove that for any argument $x \in V$ their values coincide:

$$\varphi_{u+v}(x) = (\varphi_u + \varphi_v)(x).$$

Linear maps are added pointwise, therefore $(\varphi_u + \varphi_v)(x) = \varphi_u(x) + \varphi_v(x) = h(x, u) + h(x, v) = h(x, u + v) = \varphi_{u+v}(x)$. And the same for multiplying by scalar.

We need to prove that for any $\varphi \in \mathcal{L}(V, F)$ there exists a vector $v \in V$ such that $\varphi = \varphi_v$. This is exactly the surjectivity of the linear map Φ that can be derived from **Theorem 5.6** in the following way.

$\dim \mathcal{L}(V, F) = n$ since in matrix form linear functionals are given by the strings of the length n. So the linear map Φ is between equidimensional vector spaces.Since $\dim \operatorname{Ker}(\Phi) = n - \dim \operatorname{Im}(\Phi)$ surjectivity follows from injectivity. What does injectivity of the map Φ mean? Since

$$v \in \operatorname{Ker}(\Phi) \Leftrightarrow \varphi_v = 0 \Leftrightarrow \text{ for all } x \in V \ \varphi_v(x) = 0 \Leftrightarrow \forall x \in V \ h(x, v) = 0 \Leftrightarrow v \in V^\perp,$$

one has $\operatorname{Ker}(\Phi) = 0 \Leftrightarrow V^\perp = 0$. So the injectivity (and surjectivity) of Φ_h is equivalent to non-degeneracy of h. $\square$

14 Orthogonal decomposition

14.1 Gram-Schmidt orthogonalization

Let V be a vector space with a given bilinear form h and q is a corresponding quadratic form.. Orthogonalization is a process of finding **orthogonal** basis, i.e. the basis $\mathcal{B}_O\{v_1, \ldots, v_n\}$ such that the Gram matrix $G_{h,\mathcal{B}}$ is diagonal. That is to say all $h(v_i, v_i) = 0$ for all $i \neq j$.

Proposition 14.1. *In any vector space V with a symmetric bilinear form h there exists an orthogonal basis.*

Proof. We procced by induction on $\dim V$.

Induction base. Suppose that $h(u, v) = 0$ for all $u, v \in V$ that is to say the Gram matrix is zero. The any basis would be orthogonal.

In the case when the Gram matrix is not equal to zero we can fulfil the induction step. Take a pair of vectors such that $h(u, v) \neq 0$. Then at least one out of three quantities $q(u+v)$, q(u), $q(v)$ is not equal to zero due to polarization identity $h(u, v) = \frac{1}{2}(q(u+v) - q(u) - q(v))$. Therefore we can choose a vector v_1 such that $q(v_1) \neq 0$. Consider its orthogonal complement $v_1^\perp$ and denote this subspace as V'. Then $\dim V' \geqslant n - 1$ by **Proposition 13.6**(1) and we conclude that $V = \mathrm{Span}(v_1) \oplus V'$. Indeed $\mathrm{Span}(v_1) \cap V' = 0$ since for $h(av_1, v_1) \neq 0$ for any $a \in F \setminus \{0\}$; so $av_1 \in V' \Leftrightarrow a = 0$. By **Theorem 4.5** $\dim\,(\mathrm{Span}(v_1) + V') = \dim\,(\mathrm{Span}(v_1)) + \dim V' \geqslant 1 + (n-1) = n$. Therefore, $\mathrm{Span}(v_1) + V'$ is equal to V being the subspace of V whose dimension is greater or equal than $\dim V$. Particularly, its dimension is actually equal to n and $\dim V' = n - 1$.

Now we are ready to make an induction step. By induction hypothesis the assertion of the proposition holds for the space V' with the bilinear form $h_{V'}$. So we can chose an orthogonal basis $\{v_2, v_3, \ldots, v_n\}$ in V'. Since the union of the bases of the direct summands is the basis of their sum on obtain that $\{v_1, v_2, \ldots, v_n\}$ is the vasis of V. The values $h(v_1, v_i)$ are equal to 0 whence $i \geqslant 2$ as $v_i \in V' = v_1^\perp$. And the values $h(v_i, v_j) = 0$ whence $i, j \geqslant 2$ and $i \neq j$ just by induction hypothesis. $\square$

Remark. This proposition is just a geometric restatement Lagrange diagonalization theorem. Essentially, these are the same assetion siad in different languages.

How to deduce Lagrange diagonalization from **Proposition 14.1**? For a given symmetric matrix A consider a bilinear form h on F^n which has matrix A as its Gram matrix relative to the standard basis $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$, that is to say, define h by $h(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T A\mathbf{y}$. Then by **Proposition 14.1** there exists an orthogonal basis

$\mathcal{E}' = \{e_1', \dots, e_n'\}$ with transition matrix $C = M_{\mathcal{E}\to\mathcal{E}'}$ such that Gram matrix $A' = G_{h,\mathcal{E}'}$ has diagonal form. Since $A' = C^T A C$ the Lagrange diagonalization theorem follows

Let us start a description how to obtain a certain orthogonal basis starting from a given basis by a simple modification. This is called **Gram-Schmidt orthogonalization process**.

Theorem 14.2. *Let V be a vector space over F with a given symmetric bilinear form h. Given a basis $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ suppose that the additional condition holds:*

$$\text{for any } k = 1 \dots n \text{ the restriction } h_{\mathrm{Span}(v_1,v_2,\dots,v_k)} \text{ is non-degenerate.} \tag{6}$$

Then there exists an orthogonal basis $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ such that $e_k \in \mathrm{Span}(v_1, \dots, v_k)$ for any k.

Remark. The condition means also that the upper-left $k \times k$ minor in the Gram matrix $G_{h,\mathcal{B}}$ is non-zero. The conclusion means also that the spans are actually coincide: $\mathrm{Span}(e_1, e_2, \dots, e_k) = Span(v_1, v_2, \dots, v_k)$.

Proof. Let us proceed by induction on n, The case $n = 1$ is trivial. Then we should make an induction step from $n-1$ to n. Applying induction hypothesis to the space $V' = \mathrm{Span}(v_1, \dots, v_{n-1})$ we assume that we have already found the orthogonal basis $\{e_1, e_2, \dots, e_{n-1}\}$ in V'.

Since $v_n \notin V' = \mathrm{Span}(e_1, \dots, e_{n-1})$ then adding the vector v_n to the linear independent system $' = \{e_1, \dots, e_{n-1}\}$ one obtains a linearly independent system $\{e_1, e_2, \dots, e_{n-1}, v_n\}$ in V which has to be a basis of V. Let us modify the vector v_n in this basis subtracting a linear combination of remaining basis vectors with yet unknown coefficients

$$v_n' = v_n - a_1 e_1 - a_2 e_2 - \dots - a_{n_1} e_{n_1} \tag{7}$$

in order to obtain a new vector v_n' which is orthogonal to the subspace V'. Since $h(v_n', e_k) = h(v_n, e_k) - \sum_{i=1}^{n-1} a_i h(e_i, e_k) = h(v_n, e_k) - a_k h(e_k, e_k)$ provided by $h(e_i, e_k) = 0$ for $i \neq k$, one have to chose a_k to be equal $\dfrac{h(v_n, e_k)}{h(e_k, e_k)}$ in order to get $h(v_n', e_k) = 0$. We need to check that $h(e_k, e_k) \neq 0$. Indeed, Gram matrix $G_{h|_{V'},\mathcal{E}'}$ is diagonal and invertible since the restriction $h_{V'}$ s non-degenerate. As its determinant equal to $h(e_1, e_1)\dot{h}(e_2, e_2) \cdots h(e_{n-1}, e_{n-1})$ we conclude that $h(e_k, e_k) \neq 0$ for all $k = 1 \dots n-1$. So we can take $e_n = v_n - \sum_{k=1}^{n-1} \dfrac{h(v_n, e_k)}{h(e_k, e_k)}$ and obtain the desired orthogonal basis. $\square$

Exercise 14.1. *Let $V = \mathbb{R}[X]_3$ be a space of polynomials having degree less or equal than 3. Let $v_0 = 1; v_1 = X; v_2 = X^2; v_3 = X^3$ be a standard basis in this space. Consider a symmetric bilinear form h on the space V which values are given by*

$$h\left(f(X), g(X)\right) = \int\limits_{-1}^{1} f(x)g(x)dx \quad \forall f, g \in \mathbb{R}[X]_3$$

- *(1 point) Prove that h is indeed a symmetric bilinear form.*
- *(2 points) Apply the orthogonalization process from the* **Theorem 14.2** *and obtain the orthogonal basis $\{e_0, e_1.e_2.e_3\}$ relative to h such that e_k is a polynomial of degree k.*

Definition. Let V be a vector space with a given symmetric bilinear form h. The basis $\mathcal{B} = \{v_1, \dots, v_n\}$ is said to be **orthonormal** if the Gram matrix $G_{h,\mathcal{B}}$ is equal to Identity matrix, i.e. $\mathcal{B}$ is orthogonal and $h(e_k, e_k) = 1$ for $k = 1 \dots n$.

Remark. Not always the orthonormal basis does exists. In the case $F = \mathbb{R}$ and additional condition $h(v, v) > 0$ for any $v \neq 0$ we can make a further step in **Theorem 14.2** and denoting $e'_k = \dfrac{e_k}{\sqrt{h(e_k, e_k)}}$ can obtain the orthonormal basis $\{e'_1, e'_2, \dots, e'_n\}$.

Remark. Let $V = \mathbb{R}^n$ and h is a usual dot product, i.e. bilinear form having an identity Gram matrix relative to the standard basis. Then $h(\mathbf{x}, \mathbf{x}) = x_1^2 + x_2^2 + \cdots + x_n^2 > 0$ for $\mathbf{x} \neq 0$.

Proposition 14.3. *Consider $V = \mathbb{R}^n$ with a bilinear form given by a standard dot product. For a matrix $C \in M(\mathbb{R}^n)$ the following conditions are equivalent:*

1. *A linear operator $\mathbf{x} \mapsto A \cdot \mathbf{x}$ is an orthogonal operator on $\mathbb{R}^n$;*
2. *A is orthogonal matrix, i.e. $A^T A = E$;*
3. *Columns of the matrix A constitute an orthonormal basis in $\mathbb{R}^n$;*
4. *$AA^T = E$;*
5. *If $\mathbf{x}_1^T, \mathbf{x}_2^T, \dots, \mathbf{x}_n^T$ are the rows of the matrix A then $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ is an orthonormal basis in $\mathbb{R}^n$.*

Proof. 1) $\Leftrightarrow$ 2) is just a special case of Equation (5).
2) $\Leftrightarrow$ 3) Our bilinear form h is given by $h(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T\mathbf{y}$. Let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ are the

columns of the matrix A. Then $h(\mathbf{y_i}, \mathbf{y_i}) = \mathbf{y}_i^T \mathbf{y}_j$ is equal to the usual matrix product of the i-th row of the matrix A^T and the j-th column of the matrix A. Therefore AA^T has $\mathbf{y}_i^T \mathbf{y}_j$ as its component at the intersection of i-th row and j-th column.
4) $\Leftrightarrow$ 5) is the same as the 2) $\Leftrightarrow$ 3) applied to the transposed matrix.
2) $\Leftrightarrow$ 4) Assume that $A^T A = E$. Since $rank(BC) \leqslant rank(B), rank(C)$ then $rank(A) = n$ and A is invertible. Multipying by A^{-1} from the right one obtain $A^T = A^{-1}$. Then multiplying by A from the left we obtain $AA^T = AA^{-1} = E$. $\square$

Problem 14.4. *Given a square matrix* $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$ *find an orthogonal matrix* Q *and low tringular matrix* L *such that* $A = QL$.

Solution. We can regard the multiplication $A \mapsto AL$ as the orthogonalization process being applied to the suitably ordered columns of the matrix A. First of all we find using this interpretation the low triangular matrix $\widetilde{L}$ such that the columns of $A\widetilde{L}$ form an orthogonal (not yet orthonormal) basis. Denote the columns of A by $v_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$; $v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Then we start to apply the orthogonalization process. Put $e_1 = v_1$, then

$$e_2 = v_2 - \frac{h(v_2, e_1)}{h(e_1, e_1)} e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \tfrac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$ Putting this formulae in the matrix form: $(v_3|v_2|v_1) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = (v_3|e_2|e_1)$ or, that is the same, $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, obtaining the matrix where the second and the third column are orthogonal to each other. Then we proceed by the formula

$$e_3 = v_3 - \frac{h(v_3, e_2)}{h(e_2, e_2)} \cdot e_2 - \frac{h(v_3, e_1)}{h(e_1, e_1)} \cdot e_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} - \tfrac{-1}{1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{0}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

So we obtain $\begin{pmatrix}1&0&1\\1&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}$ the matrix with orthogonal columns multiplying A by low triangular matrices:

$A\begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}$. Then multiplying on their inverse matrices:

$A=\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&1&1\end{pmatrix}=\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&1&1\end{pmatrix}=\widetilde{Q}\widetilde{L}$. In order to make the first factor to be orthogonal matrix we need to divide the columns by their lengths: $\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0\\0&1&0\\0&0&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}$. So we get

$$A=\begin{pmatrix}1&0&1\\0&1&0\\-1&0&1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}&0&0\\0&1&0\\0&0&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2}&0&0\\0&1&0\\0&0&\sqrt{2}\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\0&1&1\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\sqrt{2}&0&0\\-1&1&0\\0&\sqrt{2}&\sqrt{2}\end{pmatrix}.$$

□

Exercise 14.2. *For the same matrix A find orthogonal matrix Q' and low trangular L' such that $A = L'Q'$.*

Remark. Orthogonalization process **Theorem 14.2** can be considered as a direct geometric analogue of "completion the square" in the proof of the Lagrange diagonalization theorem.

Corrolary 14.5. *Let A be a symmetric matrix such that all its upper left minors are non-zero. Then there exists a low triangular matrix L such that LAL^T is a diagonal.*

Proof. Let A be a Gram matrix of a certain symmetric bilinear form h relative to a basis $\mathcal{B} = \{v_1, \dots, v_n\}$. Then the condition (6) holds since the Gram matrix of the restriction $h_{\mathrm{Span}(v_1,\dots,v_k)}$ is exactly an upper-left $k \times k$ submatrix in $G_{h,B} = A$. Performing orthogonalization process one can see that the transition matrix from $\mathcal{B}$ to the orthogonal basis $\mathcal{E} = \{e_1, \dots, e_n\}$ is upper triangular. Indeed, $e_k \in \mathrm{Span}(v_1, \dots, v_k)$ implies that if $(v_1, \dots, v_n)C = (e_1, \dots, e_n)$ then $c_{jj} = 0$ for $i > j$. Since the the Gram matrix $G_{h,\mathcal{E}}$ is diagonal the equality $G_{h,\mathcal{E}} = C^T G_{h,\mathcal{B}} C$ yields the desired putting $L = C^T$. □

Exercise 14.3. *(3 point for two questions). Prove the inverse statement: if $A = LDL^T$ where L is low triangular with units on diagonal and D is an invertible diagonal* *matrix then all the upper-left minors in the matrix A are non-zero. If $F = \mathbb{R}$ then for all $k = 1 \dots n$ then upper left minor $k \times k$ in the matrix A is equal to the upper left minor $k \times k$ in the matrix D, the latter being just the product of k diagonal elements.*

14.2 Orthogonal projectors

Let V be a vector space with a symmetric bilinear form h one call the $V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$ is the **decomposition into orthogonal direct sum** if it is a direct sum decomposition such that for any $x \in U_i$ and $y \in U_j$ one has $h(x, y) = 0$ provided $i \neq j$. We just call it **orthogonal decomposition**.

We shall use the notation $V = U_1 \perp U_2 \perp \cdots \perp U_k$ for the orthogonal decomposition.

Proposition 14.6. *Let $U \subset V$ be a subspace such that the restriction $h|_U$ is non-degenerate. Then there is an orthogonal direct sum decomposition $V = U \perp U^\perp$. Moreover, if there is another orthogonal decomposition $U \perp W$ then $W = U^\perp$.*

Proof. First we check that $U \cap U^\perp = 0$. Take a vector $u \in U \cap U^\perp$. Therefore, $h(u, v) = 0$ for any $v \in U$ as $u \in U^\perp$. It implies that $u = 0$ because the restriction $h|_U$ is non-degenerate. Second, $\dim U^\perp \geqslant \dim V - \dim U$ and **Theorem 4.5** yields $U + U^\perp = V$. As for the second assertion, $W \subset U^\perp$ by a very definition of orthogonal decomposition and $\dim W = \dim V - \dim U = \dim U^\perp$.

Remark. In general it is another case when an inequality $\dim U^\perp \geqslant \dim V - \dim U$ becomes an equality, slightly different from **Theorem 13.9**(1). In the theorem the bilinear form h on the whole space V is assumed to be non-degenerate but in the proposition above it need not be non-degenerate itself but non-degeneracy of its restriction is required.

□

Definition. Suppose that $h|_U$ is non-degenerate. Let $V = U \perp W$ is the decomposition into orthogonal direct sum. Define a linear operator $P : V \to V$ in the following way. Take $v \in V$ and consider its unique representation $v = u + w$ where $u \in U$ and $w \in W$. Put $P(v) = u$. Then the operator P is called an **orthogonal projector** onto subspace U. We denote it by $P = \mathrm{Proj}_U$.

Exercise 14.4. *Prove that $P = \mathrm{Proj}_U$ is a linear map such that $P^2 = P$, $\mathrm{Im}\, P = U$; $\mathrm{Ker}\, P = U^\perp$.*

Problem 14.7. *Let $V = \mathrm{M}_n(\mathbb{R})$ is the n^2-dimensional space of square matrices and $U = \mathrm{Sym}_n(\mathbb{R})$ is the subspace of symmetric matrices. Consider a bilinear form on V whose values are given by $h(A,B) = \mathrm{Tr}(A^T B)$. Find the formula giving the values of orthogonal projector Proj_U in this case.*

Solution. Recall that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$ for any pair of rectangular matrices such that AB is a square matrix. Therefore $h(B,A) = \mathrm{Tr}(B^T A) = \mathrm{Tr}((B^T A)^T) = \mathrm{Tr}((A^T B) = h(A,B)$ so h is symmetric.

Let $A = A^T$ be a symmetric matrix and $B = -B^T$ be a **skew-symmetric** matrix. In this case $h(A,B) = \mathrm{Tr}(A^T B) = \mathrm{Tr}(B^T A) = \mathrm{Tr}(-BA^T) = -h(B,A) = -h(A,B)$. Therefore $A \perp_h B$. Denote by $\mathrm{Skew}_n(\mathbb{R})$ the space of skew symmetric matrix. Show that $\mathrm{M}_n(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{Skew}_n(\mathbb{R})$. First of all, $\mathrm{Sym}_n(\mathbb{R}) \cap \mathrm{Skew}_n(\mathbb{R}) = 0$ since if $A = A^T$ and $A = -A^T$ simultaneously then $A = 0$. Secondly, any matrix A can be represented in the form $A = \dfrac{A+A^T}{2} + \dfrac{A-A^T}{2}$ where the first belongs to $\mathrm{Sim}_n(\mathbb{R})$ and the second summand belongs to $\mathrm{Skew_n}(\mathbb{R})$ as $\left(A - A^T\right)^T = A^T - A = -\left(A - A^T\right)$. Finally, we obtain the orthogonal decomposition $\mathrm{M}_n(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R}) \perp_h \mathrm{Skew}_n(\mathbb{R})$ where projection on the first summand given by $\mathrm{Proj}_{\mathrm{Sym}_n(\mathbb{R})}(A) = \dfrac{A+A^T}{2}$.

□

Proposition 14.8. *Suppose that $\{u_1, \ldots, u_k\}$ is an orthogonal basis of the subspace $U \subset U$ and $h(u_i, u_i) \neq 0$ for $i = 1 \ldots k$. Then for any vector v the value of the orthogonal projector is given by*

$$\mathrm{Proj}_U(v) = \sum_{i=1}^{k} \frac{h(v,u_i)}{h(u_i,u_i)} u_i \tag{8}$$

Proof. We have a representation $v = \sum_{i=1}^{k} \frac{h(v,u_i)}{h(u_i,u_i)} u_i + \left(v - \sum_{i=1}^{k} \frac{h(v,u_i)}{h(u_i,u_i)} u_i\right)$ where the first summand belongs to U. But the second summand belongs to $U^\perp$ as it have been checked in the proof of the **Theorem 14.2**:

$$h\left((v - \sum_{i=1}^{k} \frac{h(v,u_i)}{h(u_i,u_i)} u_i, u_k\right) = h(v,u_k) - \sum_{i=1}^{k} \frac{h(v,u_i)}{h(u_i,u_i)} h(u_i,u_k) = h(v,u_k) - \frac{h(v,u_k)}{h(u_k,u_k)} h(u_k,u_k) = 0.$$

□

Remark. The common mistake is to use this formula (8) in the case of an arbitrary basis. It does not work when the basis $\{u_1, u_2, \ldots, u_k\}$ is not orthogonal.

Exercise 14.5. *Let $V = \mathbb{R}[X]_4$ and bilinear form h is given as in the* **Exercise 14.1**. *For the subspace $U = \mathbb{R}[X]_2$ find a matrix of an orthogonal projector* Proj_U *relative to the basis $\{1, X, X^2, X^3, X^4\}$.*

Remark. In the special case $U = V$ and the basis $\{u_1 \ldots, u_n\}$ is an orthonormal basis in V the equality (8) becomes

$$v = \sum_{i=1}^{n} h(v, u_i)u_i. \tag{9}$$

This equality is called **Fourier decomposition** of the vector v with respect to the orthonormal basis $\{u_1, \ldots, u_n\}$.

Consider he case when h is the dot product on $\mathbb{R}^n$. This bilinear form and its restriction on any subspace are non-degenerate due to positivity.

Problem 14.9. *Consider the subspace* $U = \mathrm{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}\right) \subset \mathbb{R}^4$. *Find an orthogonal projection of the vector* $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ *onto the subspace $U^{\perp}$.*

Solution. Denote the spanning vectors of U by $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix}$. Apply the orthogonalization process: $u_1 = v_1$ and $\widetilde{u_2} = v_2 - \frac{h(v_2, u_1)}{h(u_1, u_1)}u_1 = \begin{pmatrix} -1 \\ 2 \\ 1 \\ 1 \end{pmatrix} - \frac{-1}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ 2 \\ \frac{4}{3} \\ \frac{2}{3} \end{pmatrix} = \frac{2}{3}\begin{pmatrix} -1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$. For the computation purposes I replace $\widetilde{u_2}$ by the proportional vector $u_2 = \begin{pmatrix} -1 \\ 3 \\ 2 \\ 1 \end{pmatrix}$. So $U =$

$\mathrm{Span}(u_1, u_2)$ where $\{u_1, u_2\}$ is an orthogonal basis in U and we can apply **Proposition 14.8**:

$$\mathrm{Proj}_{U^\perp}(e_1) = e_1 - \frac{h(e_1, u_1)}{h(u_1, u_1)}u_1 - \frac{(h(e_1, u_2)}{h(u_2, u_2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{15}\begin{pmatrix} -1 \\ 3 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore the answer is $\frac{1}{5}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix}$.

□

Remark. You see that you need not to compute the space $U^\perp$ itself as a span in order to find the projection on that space. Actually, the projection onto the subspace U itself have also almost been computed:

$$\mathrm{Proj}_U(e_1) = e_1 - \mathrm{Proj}_{U^\perp}(e_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{5}\begin{pmatrix} 3 \\ 1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 \\ -1 \\ 1 \\ -2 \end{pmatrix}.$$

Remark. From geometric point of view the point $\frac{1}{5}(2, -1, 1, -2)$ is the nearest point in the plane U to the point $(1, 0, 0, 0)$. So the length of the difference $\left\|\frac{1}{5}(3, 1, -1, 2)\right\| = \sqrt{\frac{3}{5}}$ is just a distance from the end of radius-vector e_1 and the plane U.

Proposition 14.10. *Let $U \subset \mathbb{R}^n$, h is the dot product on $\mathbb{R}^n$ and $\mathcal{Y} = \{\mathbf{y}_1, \dots, \mathbf{y}_k\}$ is the basis in U. Denote by Y the matrix with the columns $\mathbf{y}_1, \dots, \mathbf{y}_k$. Then*

1. *$k \times k$ matrix Y^TY is the Gram matrix $G_{h|_U,\mathcal{Y}}$.*
2. *If $\mathcal{Y}$ is an orthonormal basis in U then $n \times n$ matrix YY^T is the matrix of orthogonal projector Proj_U relative to the standard basis in $\mathbb{R}^n$;*
3. *In general case $Y(Y^TY)^{-1}Y^T$ is the matrix of the orthogonal projector Proj_U relative to the standard basis in $\mathbb{R}^n$.*

Proof. 1) is obvious since $h(\mathbf{y}_i, \mathbf{y}_j) = \mathbf{y}_i^T \mathbf{y}_j$.

2) Let $\{\mathbf{y}_{k+1}, \dots, \mathbf{y}_n\}$ be a basis of the subspace $U^\perp$. Then $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ is the basis in $\mathbb{R}^n$. Due to **Theorem 5.1** it suffices to check that $(YY^T) \cdot \mathbf{y}_i = \mathrm{Proj}_U(\mathbf{y}_i)$ for all $i = 1 \dots n$. Identity $(YY^T)Y = Y(Y^TY) = Y$ yields the desired for $i = 1 \dots k$. Indeed for any column $\mathbf{y}_i$ of the right factor one has $(YY^T)\mathbf{y}_i = \mathbf{y}_i = \mathrm{Proj}_U(\mathbf{y}_i)$ since $\mathbf{y}_i \in U$. For $j > k$ one has $(YY^T)\mathbf{y}_j = Y(Y^T\mathbf{y}_j) = 0$ as the i-th component in the second factor is equal to $\mathbf{y}_i^T\mathbf{y}_j$ where $\mathbf{y}_i \in U$ and $\mathbf{y}_j \in U^\perp$.

3) The following exercise. $\square$

Exercise 14.6. *Prove the assertion* **Theorem 14.10***(3).*

Proposition 14.11. *Let $\mathbb{R}^n$ be equipped with a standard dot product. The matrix $P \in$ $\mathrm{M}_n(A)$ defines an orthogonal projector on sum subspace (in the usual way $\mathbf{x} \mapsto P \cdot \mathbf{x}$) if and only if both of the following conditions hold:*

1. *$P^2 = P$;*
2. *$P^T = P$.*

Proof. The "only if" part is simple. Assume that P is an orthogonal projector onto a subspace $U \subset \mathbb{R}^n$. Then $P^2 = P$ by an **Exercise 14.4**. The second assertion follows immediately from **Proposition 14.10** since $(Y^TY)^T = Y^TY$.

So let us prove the "if" part. Assume that the conditions 1) and 2) are fulilled. Denote subspace $\mathrm{Im}\, P$ by U. Then for any $\mathbf{x} \in U$ one has $P\mathbf{x} = \mathbf{x}$. Indeed, if $\mathbf{x} \in \mathrm{Im}\, P$ then there exist $\mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{x} = P\mathbf{z}$ and $P\mathbf{x} = PP\mathbf{z} = P\mathbf{z} = \mathbf{x}$.

Further, given $\mathbf{y} \in U^\perp$ one has $P(\mathbf{y}) \in U^\perp$ since for any $x \in U$ $h(\mathbf{x}, P(\mathbf{y})) = \mathbf{x}^TP\mathbf{y} = \mathbf{x}^TP^T\mathbf{y} = h(P(\mathbf{x}), \mathbf{y}) = 0$ where the last equlity provided $P(\mathbf{x}) \in U$ and $\mathbf{y} \in U^\perp$. From other hand $P(\mathbf{y}) \in \mathrm{Im}\, P = U$. It yields $\mathbf{y} \in U^\perp \cap U = 0$. Taking arbitary $\mathbf{v} \in \mathbb{R}^n$ and its decomposition $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} \in U$ and $\mathbf{y} \in U^\perp$ we obtain $P(\mathbf{z}) = P(\mathbf{x}) + P(\mathbf{y}) = \mathbf{x} + 0 = \mathbf{x}$. It follows that P is an orthogonal projector onto U by the very definition.

$\square$

14.3 Euclidean space

There is a general concept the **metric space** which will be defined in Analysis course, Roughly speaking, in the metric space one can measure a distance between any two points. The related concept of the **normed space** which is a vector space equipped with a length function is also wil be define there. The euclidean space is the special case of the normed space which we can define now.

Definition. Let V be a real vector space equipped with a synnetric bilinear form h such that

$$h(v,v) \geqslant 0 \text{ for any } v \in V \text{ and } h(v,v) = 0 \text{ if and only if } v = 0. \tag{10}$$

Then (V,h) is called a **real inner space** or, equivalently, a **Euclidean space**.

The quadratic form corresponding to the bilinear form h satisfying condition (14.3) is called positive definite quadratic form. Its gram matrix is also called a positive definite matrix. You are familiar with this concept from the Fall semester where you have proved the Sylvester criterion.

The main example of the Euclidean space is the column space $\mathbb{R}^n$ equipped with the dot product. In some sense there are no other Euclidean spaces. Given an arbitrary Euclidean space (V,h) on can chose an orthonormal basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ and define a linear map $\theta_{\mathcal{B}} : V \to \mathbb{R}^n$ whih assign to any vector v its coordinate column $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$, i.e. $v = x_1v_1 + x_2v_2 + \cdots + x_nv_n$, see **Theorem 5.1**. The crucial point that θ is not only an isomorphism of vector spaces but also of Euclidean spaces that means that given bilinear form is preserved by θ: if $\mathbf{x} = \theta(v)$ and $\mathbf{y} = \theta(w)$ then $h(v,w) = \mathbf{x}^T\mathbf{y}$ where the righthandside is just a dot product of the columns $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Let us state more formally which Eucledean spaces are called isomorphic,

Definition. Let (V,h) and (V,h') be Euclidean space. The linear map $\theta : V \to V'$ is called an isomorphism of these Euclidean spaces if

$$\text{for any } u, v \in V \text{ one has } h(u,v) = h'\left(\theta(u), \theta(v)\right). \tag{11}$$

Remark. We do not require that h is a bijection because it is fulfilled automatically: $\operatorname{Ker}\theta = 0$ as $\theta(v) = 0 \Leftrightarrow h'\left(\theta(v), \theta(v)\right) = 0 \Leftrightarrow h(v,v) = 0 \Leftrightarrow v = 0$.

Remark. The condition (11) often cited as θ *preserve* bilinear form. As in the **Proposition 13.4** it is equivalent to the condition $q(v) = q'\left(\theta(v)\right)$ where q is a quadratic form corresponding to h and q' is a quadratic form correspondng to h'. The latter equaity means that θ preserves the length of any vector. In the **Proposition 13.4** we discussed the linear operator on the space V itself, here we switch to the case of the linear map between two different vector spaces, but the terminology is essentially the same.

The only difference between arbitrary n-dimensional Euclidean space (V,h) and the column space $(\mathbb{R}^n, \bullet)$ is the following. In the space $\mathbb{R}^n$ the orthonornmal basis is supposed to be already chosen (i.e. standard basis) but in V there are no preferable choice of the basis: all orthonormal bases are equally natural.

Definition. Let (V, h) be a Euclidean space. The quantity $\sqrt{h(v,v)}$ is called the **length** of the vector $v \in V$ and is denoted by $\|v\|$.

Proposition 14.12. *(Cauchy-Буняковский-Schwarz inequality).*
For any two vectors $u, v \in V$ in Euclidean space one has:

1. $|h(u,v)| \leqslant \|u\| \cdot \|u\|$;
2. $\|u+v\| \leqslant \|u\| + \|v\|$.

Proof. Consider an auxiliary quadratic trinomial

$$h(tu+v) = h(u,u) \cdot t^2 + 2h(u,v) \cdot t + h(v,v) \in \mathbb{R}[t].$$

It has non-negative values for any $t_0 \in \mathbb{R}$. Therefore it has non-positive discriminant $4h(u,v)^2 - 4h(u,u)h(v,v) \leqslant 0$ and the assertion 1) follows. When this inequality becomes an equality? Only if the auxiliary quadratic trinomial has a root. It is equivalent that there exists $a \in \mathbb{R}$ such that $au + v = 0$ that is v is proportional to u.

The second part is actually called a triangle inequality. It follows from the 1): $\|u+v\|^2 = h(u+v, u+v) = h(u,u) + 2h(u,v) + h(v,v) \leqslant \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2$.

□

Exercise 14.7. *Prof that for any $a, b, c, x, y, z \in \mathbb{R}$ the following inequality holds:*

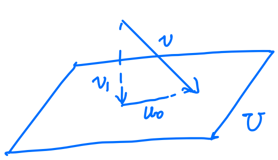
1. *(1 point)* $(ax + by + cz)^2 \leqslant (a^2 + b^2 + c^2)(x^2 + y^2 + z^2)$;
2. *(2 points)* $(2ax + 2by + 2cz + ay + bx + bz + cy)^2 \leqslant (2a^2 + 2b^2 + 2c^2 + 2ab + 2bc)(2x^2 + 2y^2 + 2z^2 + 2xy + 2yz)$.

14.4 Least squares

Definition. In Euclidean space V one can measure the **distance** between to points: for $u, v \in V$ one define $\operatorname{dist}(u,v) = \|u - v\|$.

Problem 14.13. *Consider a problem, for given $v \in V$ and subspace $U \subset V$ find in U the closest point to v, i.e. the vector $u_0 \in U$ such that* $\operatorname{dist}(v, u_0) = \min\limits_{u \in U} \operatorname{dist}(v, u)$.

Solution. The answer to this problem is that u_0 should be a base of the altitude drawn from v to U, that is $u_0 = \operatorname{Proj}_U(v)$. Let us check that for this choice of u_0 one has $\|v - u_0\| \leqslant \|v - u\|$ for any $u \in U$ and the equality holds if only $u = u_0$. Indeed, $v - u = (v - u_0) + (u_0 - u)$ is the decomposition of the vector $v - u$ into the



components corresponding to the orthogonal decomposition $V = U^{\perp} \perp U$. Therefore $\|v-u\|^2 = h(v-u, v-u) = h(v-u_0, v-u_0) + 2h(v-u_0, u_0-u) + h(u-u_0, u-u_0) = = \|v-u_0\|^2 + \|u_0-u\|^2 \geqslant \|v-u_0\|^2$. □

Remark. The closest point is unique since the above inequality $\|v-u\|^2 \geqslant \|v-u_0\|^2$ becomes an equality if and only if $\|u-u_0\|^2 = 0 \Leftrightarrow u = u_0$.

Consider the following **interpolation problem**:

x	1	2	3	4
f(x)	0	2	3	5

The *exact* solution to this problem is the function f such that

$$f(1) = 0; \quad f(2) = 2; \quad f(3) = 4; \quad f(4) = 5.$$

Usually, someone is restricted to find a solution in a given class of functions. The most classical choice is to seek solutions among polynomial functions. Since there are 4 points to interpolate we can find a third degree polynomial giving the exact solution to this interpolation problem.

Nevertheless, taking into account a measurement errors it is more natural to seek an *approximate* solution not an exact one. Moreover, frequently there is assumed linear trend in the measured quantities. Then there is a problem to find linear polynomial whose values in the given points approximate the measured values in the best possible way in certain sense. There are many different ways to measure a quality or goodness of approximation.

Problem 14.14. *Find a linear polynomial $f(x) = ax + b$ such that sum of squares of the errors*

$$(f(1) - 0)^2 + (f(2) - 2)^2 + (f(3) - 3)^2 + (f(4) - 5)^2$$

would be minimal as possible.

Solution. Consider the vector $\mathbf{b} = (0, 2, 3, 5)^T$ and the vector $(f(1), f(2), f(3), f(4))^T$ in the standard Euclidean space $\mathbb{R}^4$. The problem is to minimise the euclidean distance between these two vector when f varies through all linear polynomials. For $f(x) = ax + b$ one has $\begin{pmatrix} f(1) \\ f(2) \\ f(3) \\ f(4) \end{pmatrix} = \begin{pmatrix} a \cdot 1 + b \\ a \cdot 2 + b \\ a \cdot 3 + b \\ a \cdot 4 + b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$.

Denote the 4×2 matrix above by A. So the column $(f(1), f(2), f(3), f(4))^T$ varies through the subspace $U = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)$ i.e. the image of the linear map

$A : \mathbb{R}^2 \to \mathbb{R}^4$ which is given by A. Therefore we can restate the problem: find a vector $\mathbf{u} \in U$ such that $\|\mathbf{b} - \mathbf{u}\|$ is minimal. By the **Problem 14.13** the answer is given by $\mathbf{u} = \mathrm{Proj}_U(\mathbf{b})$ which can be computed using formula from **Proposition 14.10**(3):

$$\mathbf{u} = A(A^T A)^{-1} A^T \mathbf{b} = A \cdot \frac{1}{10} \begin{pmatrix} 2 & -5 \\ -5 & 15 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \\ 5 \end{pmatrix} = A \cdot \begin{pmatrix} 1.6 \\ 1.5 \end{pmatrix}.$$

Therefore the best approximation in the class of linear functions in the sense of least squares errors is $f(x) = 1.6x + 1.5$. $\square$

Definition. Let $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map given by the matrix A. Consider possibly inconsistent linear system

$$A \cdot \mathbf{x} = \mathbf{b} \text{ where } \mathbf{b} \in \mathbb{R}^m. \tag{12}$$

A column $\mathbf{x_0}$ is called a least square solution to the system (12) if the norm of the difference $\||A\mathbf{x}_0 - \mathbf{b}\|$ is minimal as possible:

$$\||A\mathbf{x}_0 - \mathbf{b}\| \leqslant \||A\mathbf{x} - \mathbf{b}\| \quad \text{for any } \mathbf{x} \in \mathbb{R}^n.$$

Remark. Here we assume that $\mathbb{R}^m$ and $\mathbb{R}^n$ are equipped with a standard dot product, so the norm of the vector $\mathbf{y} = (y_1, \ldots, y_m)^T$is equal to $\|\mathbf{y}\| = \sqrt{y_1^2 + y_2^2 + \cdots + y_m^2}$.

Proposition 14.15. *1. The set of all least square solutions of a given linear system (12) coincide with the set of all exact solutions of the system* $A \cdot \mathbf{x} = \mathbf{b}_0$ *where* $\mathbf{b}_0 = \mathrm{Proj}_{\mathrm{Im}(A)}(\mathbf{b}$.

2. The set of all least square solutions of a given linear system (12) coincide with the set of all exact solutions of the so called **normal** *linear system* $A^T A\mathbf{x} = A^T\mathbf{b}$.

3. When $\operatorname{rank}(A) = n$ *then the least square solution of (12) is unique and given by*

$$\mathbf{x}_0 = (A^T A)^{-1} A^T \mathbf{b}. \tag{13}$$

Proof. If $\mathbf{x}_0$ is a least square solution then the point $A\mathbf{x}_0 \in \mathbb{R}^m$ belongs to the subspace $\mathrm{Im}(A) \subset \mathbb{R}^m$ and occurs to be the closest point in this subspace to the point $\mathbf{b}$. By **Problem 14.13** the closest point is unique and coincide with the projection of the point $\mathbf{b}$ onto the subspace. Denote the latter by $\mathbf{b}_0 = \mathrm{Proj}_{\mathrm{Im}(A)}(\mathbf{b})$. Due to uniqueness of the closest point we conclude that $A\mathbf{x}_0 = \mathbf{b}_0$ and the assertion 1) follows.

The second assertion follows from the

Lemma 14.16. *Under above notations let* $U = \mathrm{Im}(A) \subset \mathbb{R}^m$. *Then for any* $\mathbf{y}, \mathbf{y}' \in \mathbb{R}^m$ *one has*

$$\mathrm{Proj}_U(\mathbf{y}) = \mathrm{Proj}_U(\mathbf{y}') \Leftrightarrow A^T\mathbf{y} = A^T\mathbf{y}' \tag{14}$$

Proof. Obviously, $U^\perp = \mathrm{Ker}(A^T)$. Let $\mathbf{y} = \mathbf{u} + \mathbf{z}$ and $\mathbf{y}' = \mathbf{u}' + \mathbf{z}'$ where $\mathbf{u}, \mathbf{u}' \in U$ and $\mathbf{z}, \mathbf{z}' \in U^\perp$ be an orthogonal decomposition according to the orthogonal direct sum $U \perp U^\perp$. Then $A^T\mathbf{u} = A^T\mathbf{u} + A^T\mathbf{z} = A^T\mathbf{y}$ and the same $A^T\mathbf{u}' = A^T\mathbf{y}'$. Therefore, $A^T\mathbf{y} = A^T\mathbf{y} \Leftrightarrow A^T(\mathbf{u} - \mathbf{u}') = 0 \Leftrightarrow (\mathbf{u} - \mathbf{u}') \in \mathrm{Ker}(A^T) = U^\perp \Leftrightarrow (\mathbf{u} - \mathbf{u}') \in U^\perp \cap U = 0 \Leftrightarrow \mathbf{u} = \mathbf{u}'$. Since $\mathbf{u} = \mathrm{Proj}_U(\mathbf{y})$ and $\mathbf{u}' = \mathrm{Proj}_U \mathbf{y}'$ the lemma follows. □

To finish the proof of the asserton 2), $\mathbf{x}$ is a least square solution of (12) if and only if $A\mathbf{x} = \mathrm{Proj}_{\mathrm{Im}(A)}(\mathbf{b})$ where the lefthandside coincides with its projection: $\mathrm{Proj}_{\mathrm{Im}(A)}(A\mathbf{x}) = A\mathbf{x}$. By the lemma above it is equivalent to say that $\mathbf{x}$ is a least square solution if and only if $\mathrm{Proj}_{\mathrm{Im}(A)}(A\mathbf{x}) = \mathrm{Proj}_{\mathrm{Im}(A)}(\mathbf{b}) \Leftrightarrow A^TA\mathbf{x} = A^T\mathbf{b}$.

The third assertion follows directly from the second. Since the columns of the matrix A are linear independent they constitute a basis of their span which is equal to $U = \mathrm{Im}(A)$. A^TA is a Gram matrix or restriction of the dot product on U with respect to this basis. Therefore it is invertible as all restrictions of positive definite bilinear form are positive definite hence non-degenerate. So on can multiply the equality $A^TA\mathbf{x} = A^T\mathbf{b}$ by $(A^TA)^{-1}$ on the left. □

Remark. The **Proposition 14.10**(3) follows directly from the formula (14) since $\mathrm{Proj}_{\mathrm{Im}(A)}(\mathbf{b}) = A\mathbf{x}_0$ where $\mathbf{x}_0$ is a least square solution of (12).

Exercise 14.8. *For a given table*

x	-1	0	1	2
$f(x)$	1	-1	0	2

find

1. *(2 points) linear function* $f(x) = ax + b$ *which minimize the value*

$$(f(-1) - 1)^2 + (f(0) + 1)^2 + (f(1) - 0)^2 + (f(2) - 2)^2;$$

2. *(3 points) quadratic function* $f(x) = a + bx + cx^2$ *which minimizes the same expression.*

It is instructive to see an interactive plots of best fit lines, parabola and other curves. `https://textbooks.math.gatech.edu/ila/least-squares.html`

14.5 Lagrange Interpolation

Consider the following interpolation problem

$$\begin{array}{c|c|c|c|c} x & c_1 & c_2 & \cdots & c_n \\ \hline f(x) & b_1 & b_2 & \cdots & b_n \end{array} \tag{16}$$

where $c_i \in F$ for $i = 1 \dots n$ are different scalars and $b_i \in F$ are arbitrary. The points c_i are called **interpolation nodes** and b_i is called the **value** at the node c_i.

Proposition 14.17. *There exists a unique $p \in F[X]_{n-1}$ such that $p(c_i) = b_i$. That is the interpolation problem has an exact solutions in the class of polynomials of degree less or equal to $n-1$.*

Proof. Consider a map $\theta : F[X]_{n-1} \longrightarrow F^n$ which assigns to any polynomial $p(X) \in F[X]_{n-1}$ the column $(p(c_1), p(c_2), \dots, p(c_n))^T$. Obviously, θ is linear map as the value (at a given point) of the sum of two polynomials is equal to the sum values. One can reformulate the proposition in the following words: the map θ is bijective. By **Theorem 5.6** it is sufficient to check that $\mathrm{Ker}(\theta) = 0$. That is the case because $p(X) \in \mathrm{Ker}(\theta) \Leftrightarrow p(c_i) = 0$ for $i = 1 \dots n$ but the polynomial of degree not greater than $n-1$ can not have n different roots in all but one case when $p = 0$.

$\square$

Remark. What about the matrix of the linear map θ with respect to the bases: standard one in F^n and $\{1, X, \dots, X^{n-1}\}$ in $F[X]_{n-1}$?

Since $\theta(X^k) = (c_1^k, c_2^k, \dots, c_n^k)^T$ then the matrix is a transposed *Vandermonde* matrix for a given values $c_1, c_2, \dots, c_n$. Its determinant is equal to $\prod\limits_{i>j}(X_i - X_j)$ as you know from the fall semester. This gives another way to prove that θ is bijective relying upon the fact that the Vandermonde determinant is not equal to zero.

Let us consider a polynomials which correspond to the vectors of standard basis in F^n under the map θ. For any $i = 1 \dots n$ there is a unique polynomial $L_i \in F[X]_{n-1}$ which is the solution to the following interpolation problem:

$$\begin{array}{c|c|c|c|c|c|c} x & c_1 & c_2 & \cdots & c_i & \cdots & c_n \\ \hline f(x) & 0 & 0 & \cdots & 1 & \cdots & 0. \end{array}$$

Since $L_i(c_k) = 0$ for $k \neq 0$ then $L_i(X)$ is divided by any $(X - c_k)$. Therefore, $L_i(X) = \dfrac{(X-c_1)\dots(X-c_{i-1})(X-c_{i+1})\dots(X-c_n)}{(c_i-c_1)\dots(c_i-c_{i-1})(c_i-c_{i+1})\dots(c_i-c_n)}$, where denominator is chosen in order to satisfy the condition $L_i(c_i) = 1$.

Suppose that $f(X)$ is a solution to the interpolation problem (16). Then $\theta(f(X)) = b_1e_1 + \dots + b_ne_n = \theta\left(b_1L_1(X) + \dots + b_nL_n(X)\right)$ where $\{e_1, \dots, e_n\}$ is a standard basis in F^n as $\theta(L_i(X)) = e_i$. We conclude that $f(X) = \sum\limits_{i=1}^{n} b_i L_i(X)$ provided by injectivity of θ. The formula we have just obtained is called **Lagrange interpolation formula** and usually is written in the following form.

Proposition 14.18. *The unique solution of the interpolation problem (16) is given by a formula*

$$f(X) = \sum_{i=1}^{n} f(c_i) \prod_{k \neq i} \frac{X - c_k}{c_i - c_k}. \tag{17}$$

Exercise 14.9. *Solve the interpolation problem*

x	0	1	2	3	4
$f(x)$	1	1	1	1	2.

15 Adjoint operator

15.1 Hermitian forms

Complex inner space is NOT a complex vector space with (positive definite) symmetric bilinear form. Since it is impossible to define positivity for complex bilinear form.

Definition. An **Hermitian** form on a complex vector space V is a function $f : V \times V \to \mathbb{C}$ such that the following axioms hold:

1. $f(au + v, w) = af(u, w) + f(v, w)$ for any $u, v, w \in V$ and $a \in \mathbb{C}$;
2. $f(v, u) = \overline{f(u, v)}$ for any $u, v \in V$.

The first is an ordinary linearity in the first argument.But analogous property for the second argument now is slightly differs:

$$f(w, au+v) = \overline{f(au + v, w)} = \overline{af(u, w) = f(v, w)} = \overline{a}\,\overline{f(u, w)} + \overline{f(v, w)} = \overline{a} f(w, u) + f(w, v).$$

Other speaking, fixing first argument w we obtain a function $\varpi(x) = f(w, x)$ which is not linear but *antilinear*. Latter means that although $\varphi(x + y) = \varphi(x) + \varphi(y)$ but $\varphi(\lambda x) = \overline{\lambda}\varphi(x)$ for $x, y \in V$ and $\lambda \in \mathbb{C}$.

As in the case of the bilinear form the Hermitian form f is completely determined by its Gram matrix. For a given basis $\mathcal{B} = \{v_1, \dots, v_n\}$ define Gram matrix as before: $G_{f,\mathcal{B}} = (f(v_i, v_j))_{i,j=1}^{n}$. Then the value of Hermitian form f on the arbitrary vectors

$u, v \in V$ which coordinates columns with respect to the basis $\mathcal{B}$ are equal to $\mathbf{x}$ and $\mathbf{y}$ is given by

$$f(u, v) = \mathbf{x}^T G_{f,\mathcal{B}} \overline{\mathbf{y}} \tag{18}$$

due to the same chain of equalities as in (4).

Due to axiom $f(v, u) = \overline{f(u, v)}$ one has

$$\mathbf{y}^T G_{f,\mathcal{B}} = \overline{\mathbf{x}^T G_{f,\mathcal{B}} \overline{\mathbf{y}}} = \overline{\mathbf{x}}^T \overline{G}_{f,\mathcal{B}} \mathbf{y} = \left(\overline{\mathbf{x}}^T \overline{G}_{f,\mathcal{B}} \mathbf{y}\right)^T = \mathbf{y}^T \overline{G^T}_{f,\mathcal{B}} \overline{x}.$$

Definition. Complex matrix $A \in \mathrm{M}_n(\mathbb{C})$ is called **hermitian symmetric** if $\overline{A^T} = A$.

Hence the Gram matrix of an hermitian form is hermitian symmetric.

If $C = M_{\mathcal{B} \to \mathcal{B}'}$ then $G_{h,\mathcal{B}'} = C^T G_{h,\mathcal{B}} \overline{C}$.

Proposition 15.1. *(Lagrange diagonalization for hermitian form) For any hermitian symmetric matrix $A \in \mathrm{M}_n(\mathbb{C})$ there exist an invertible complex matrix C such that $C^T A \overline{C}$ is a real diagonal.*

By the hermitian symmetry property $f(v, v) = \overline{f(v, v)}$ so it is real number. The same is true for the determinant of the Hermitian matrix: if $A = \overline{A^T}$ then $A^T = \overline{A}$ and $\det A = \det(A^T) = \det(\overline{A}) = \overline{\det A}$ is also a real number.

Definition. An hermitian form $f : V \times V \to \mathbb{C}$ is called positive definite if $f(v, v) > 0$ for any non-zero vector $v \in V$.

The same word can be applied to an hermitian matrix: A is called positive definite hermitian matrix if $\mathbf{x}^T A \overline{\mathbf{x}} > 0$ for any non-zero column $\mathbf{x} \in \mathbb{C}^n$. The Sylvester criterion for positive definite matrices holds also for hermitian form.

Definition. A complex inner space or unitary space is a complex vector space equipped with a positive definite hermitian form.

Remark. Positive definite hermitian form is frequently called **unitary scalar product** or even simply scalar product using the same term as in the real case.

Example. Let $V = \mathbb{C}^n$. An hermitian dot product given by $f\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}\right) = x_1\overline{y}_1 + x_2\overline{y}_2 + \cdots + x_n\overline{y}_n$ makes $\mathbb{C}^n$ into a unitary space.

Definition. The operator on a unitary space which preserves intary scalar product is called **unitary operator**. The matrix definining unitary operator on a standard column space $\mathbb{C}^n$ is called a **unitary matrix**

Remark. The term *unitary* here is a complex counterpart of the term *orthogonal* in the real case. Matrix of a unitary operator L relative to a basis $\mathcal{B}$ satisfies the equation

$$C^T G \overline{C} = G \tag{19}$$

where $C = [L]_{\mathcal{B}}$ and $G = G_{f,\mathcal{B}}$ is a Gram matrix. The equation (19) is complex counterpart of equation (6) and can be easily derived in the same way. Hence the unitary matrix U satisfies the equation $U^T \overline{U} = E$ as the Gram matrix of standard dot product is the Identity matrix. Usually, it is stated in the form $\overline{U^T} U = E$ (after applying complex conjugation) or, equivalently. $U^{-1} = \overline{U^T}$.

Definition. The matrix $\overline{A^T}$ is usually called an **hermitian conjugate** or **hermitian transpose** to a matrix A.

Remark. The complex case analogue of the **Proposition 14.3** also holds. In a unitary space we continue to use the words "orthogonal complement", "orthonormal basis" and so on. Al the properties of orthogonal complement remain the same in the complex including the Riesz representation theorem.

Remark. If (V, f) is a complex inner space (or real inner space) then $f(u_1, u_2)$ will be denoted be $(u_1, u_2)_V$.

Example. Consider a space $V = \mathbb{C}[X]_n$ equipped with a hermitian form $(p, q)_V = \int\limits_{-\infty}^{\infty} e^{-x^2} p(x) \overline{q(x)} dx$. Then V is a unitary space.

15.2 Adjoint operator

Proposition 15.2. *For any linear map $L : V \to V'$ there exists a unique linear map $L^* : V' \to V$ such that for any pair ov vectors $v \in V$ and $v' \in V'$ one has*

$$(L(v), v')_{V'} = (v, L^*(v'))_V \tag{20}$$

Definition. The linear map L^* in the proposition above is called an adoint of a linear map L.

Proof. Let us choose orthonormal bases in V and V' respectively. For $\mathbf{x}$ being a coordinate column of v and $\mathbf{y}$ taken a coordinate column of v' the equality (20) becomes $\mathbf{x}^T A^T \overline{\mathbf{y}} = \mathbf{x}^T \overline{A^* \mathbf{y}}$ where A is the matrix of L relative to the chosen bases and A^* is yet unknown matrix of the desired linear map L^*. Since $\mathbf{x}^T A^T \overline{\mathbf{y}} = \mathbf{x}^T \cdot \overline{\left(\overline{A^T} \mathbf{y}\right)}$ one can take $A^* = \overline{A^T}$ and the existence of the desired adjoint linear map follows. Indeed, there exists a linear map $V' \to V$ whose matrix relative to the chosen bases is exactly $\overline{A^T}$.

The uniqueness follows from non-degeneracy of $(-,-)_V$. Indeed, for given $v' \in V'$ consider a linear map $x \overset{\theta}{\mapsto} (L(x), v')_{V'}$ from V to $\mathbb{C}$ or $\mathbb{R}$. By the Riesz representation theorem linear functional θ can be given by a scalar product by a unique vector $u \in V$: $(L(x),v') = \theta(x) = (x,u)_V$. Comparing with (20) we see that necessary $L^*(v') = u$. $\square$

Corollary 15.3. *The matrix of adoint linear map with respect to orthonormal bases is the hermitian conjugate to the matrix of a given linear map relative to the same bases.*

Exercise 15.1. *Suppose that we chose arbitrary bases in V and V'. Let G be a Gram matrix of scalar product on V and G' be a Gram matrix of a scalar product on V'. Write the equation relating matrix A of a given linear map $L : V \to V'$ and the matrix A^* of its adjoint $L^* : V' \to V$.*

Definition. A linear operator $L : V \to V$ on an inner space V is called **self-adjoint** if $L^* = L$.

Proposition 15.4. *Characteristic polynomial of the self adoint operator is always a polynomial with real coefficients, all its complex roots are real, so the characteristic polynomial is completely factorizable over the $\mathbb{R}$.*

Proof. Choose an orthonormal basis. The the matrix A of the given operator satisfies the equation $A^T = \overline{A}$. Characteristic polynomials of the matrices A and A^T coincide. From the other hand, characteristic polynomial of $\overline{A}$ can be obtained from the characteristic polynomial of A by applying complex conjugation to all the coefficients. Therefore, all the coefficients are stable under complex conjugation, hence are real.

Taking $\lambda \in \mathbb{C}$ a root of characteristic polynomial we prove that λ is real. Indeed, Let $\mathbf{x} \in \mathbb{C}^n$ be an eugen-column of A belonging to the eugenvalue λ. Then $A\mathbf{x} = \lambda\mathbf{x}$ and applying complex conjugation $\overline{A}\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$. From the other hand, applying transposition we obtain $\lambda\mathbf{x}^T = \mathbf{x}^T A^T$. Since $A^T = \overline{A}$ we can produce a chain of equalities:

$$\lambda\mathbf{x}^T\overline{\mathbf{x}} = \mathbf{x}^T A^T\overline{\mathbf{x}} = \mathbf{x}^T\left(\overline{A}\overline{\mathbf{x}}\right) = \mathbf{x}^T \cdot \overline{\lambda}\overline{\mathbf{x}} = \lambda\mathbf{x}^T\overline{\mathbf{x}}.$$

Since $\mathbf{x}^T\overline{\mathbf{x}}$ is a positive real number equal to $x_1\overline{x_1} + \cdots + x_n\overline{x_n}$ then $\lambda = \overline{\lambda}$.

$\square$

Proposition 15.5. *For a given selfadjoint operator on an inner space its eugenvectors belonging to the different eugenvalues are orthogonal to each other.*

Proof. Again, chosing an orthonormal basis we can assume that we are given standard (complex) inner space — i.e. $\mathbb{C}^n$ equipped with a hermitian dot product, and the operator is given by multiplication by hermitian symmetric matrix A. Taking $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{y}$ such that $A\mathbf{y} = \mu\mathbf{y}$ we can produce the following chain of equalities:

$$\mu\mathbf{x}^T\overline{\mathbf{y}} = \overline{\mu}\mathbf{x}^T\overline{\mathbf{y}} = \mathbf{x}^T\overline{\mu\mathbf{y}} = \mathbf{x}^T\overline{A\mathbf{y}} = \left(\mathbf{x}^T A^T\right)\overline{\mathbf{y}} = \lambda\mathbf{x}^T\overline{\mathbf{y}}.$$

Since $\lambda \neq \mu$ then $\mathbf{x}^T\overline{\mathbf{y}} = 0$ i.e. the eugencolumns of the matrix A $\mathbf{x}$ and $\mathbf{y}$ are orthogonal to each other. In the real case the chain is the same but we simply strip off the complex conjugation since all quantities are real.

□

Proposition 15.6. *(Properties of adjoint linear map). Let $A_1, A_2 : V \to U$ and $B : W \to V$ be a linear maps between (complex) inner spaces. Then:*

- $(A^*)^* = A$ *i.e. adjoint of A^* is equal to A itself;*
- $(A_1 + A_2)^* = A_1^* + A_2^*$ *and for any scalar $a \in \mathbb{C}$ one has $(aA)^* = \overline{a}A^*$;*
- $(AB)^* = B^*A^*$.

Proof. All these properties can be easily derived from the corresponding matrix equalities as we choose the orthonormal bases in U, V, W. E.g. the last statement reduces to the equality $\overline{(AB)^T} = \overline{B^T A^T}$ by the **Corollary 15.3**. Neveκtheless, this can be proved by the direct chain of equalities
$((AB)(w), u)_U = (A(B(w), u)_U = (B(w), A^*(u))_V = (w, B^*(A^*(u)))_W.$ □

Proposition 15.7. *(Kernel and Image of an adjoint operator). Let $L : V \to V'$ be a linear map between inner spaces and $L^* : V' \to V$ be an adjoint of L. Then* $\mathrm{Ker}\,(L) = (\mathrm{Im}(L^*))^\perp$ *and* $\mathrm{Im}(L) = (\mathrm{Ker}\,(L^*))^\perp$.

Proof. Th second assertion follows from the first and the properties $(U^\perp)^\perp = U$ and $(L^*)^* = L$. In order to prove the first we need to verify two inclusions.

First, $\mathrm{Ker}\,(L) \subset (\mathrm{Im}(L^*))^\perp$ since for any $v \in \mathrm{Ker}\,(L)$ and arbitrary $v' \in V'$ one has $0 = (L(v), v')_{V'} = (v, L^*(v'))_V$. Therefore v is orthogonal to any vector chosen from $\mathrm{Im}(L^*)$.

Second, if $v \in (\mathrm{Im}(L^*))^\perp$ then for any $v' \in V'$ one has $0 = (v, L^*(v'))_V = (L(v), v')_V$. That means $L(v) \in (V')^\perp$. Since the scalar product is positive definite hence non-degenerate then $L(v) = 0$. □

Proposition 15.8. *Let $L : V \to V$ be a linear operator on a unitary or euclidean space V. Then for any subspace $U \subset V$ if U is invariant under L then $U^\perp$ is invariant under L^*. See the definition before* **Proposition 8.5**.

Proof. In order to prove that $U^\perp$ is L^*-invariant let us take an arbitrary $w \in U^\perp$ and check that $L^*(v)$ is also belongs to $U^\perp$. I.e. for any $u \in U$ one has check that $(u, L^*(w))_V = 0$. That is true because $(u, L^*(w))_V = (L(u), w) = 0$ where the last equality follows from two facts: $w \in U^\perp$ and $L(u) \in U$ due to U is L-invariant. $\square$

Problem 15.9. *Let $V = \mathbb{C}[X]_n$ and inner product be given by $(f,g)_V = \int\limits_{-1}^{1} f(x)\overline{g(x)}dx$ for any $f, g \in \mathbb{C}[X]$. Prove that the linear operator $L : V \to V$ given by $L(p(X)) = (1-X^2)p''(X) - 2Xp'(X)$ is self-adjoint.*

Proof. We need to proof that $\int\limits_{-1}^{1} L(f(x))\overline{g(x)}dx = \int\limits_{-1}^{1} f(x)\overline{L(g(x))}dx$.

for any $f, g \in \mathbb{C}[X]$. We observe that $L(f(x)) = \dfrac{d}{dx}\big((1-x^2)f'(x)\big)$. Then

$$\int\limits_{-1}^{1} \overline{g(x)}L(f(x))dx = \int\limits_{-1}^{1} \overline{g}(X)\frac{d}{dx}\big((1-x^2)f'(x)\big)\,dx = \int\limits_{-1}^{1} \overline{g}(X)d\big((1-x^2)f'(x)\big) =$$

$$= \overline{g(x)}\big((1-x^2)f'(x)\big)\Big|_{-1}^{1} - \int\limits_{-1}^{1}\big((1-x^2)f'(x)\big)\,d\left(\overline{g(x)}\right) = -\int\limits_{-1}^{1}(1-x^2)f'(x)\overline{g'(x)}dx. \tag{21}$$

We can transform the righthandside of the desired equality in almost the same way:

$$\int\limits_{-1}^{1} f(x)\overline{L(g(x))}dx = \int\limits_{-1}^{1} f(x)d\left((1-x^2)\overline{g'(x)}\right) = f(x)(1-x^2)\overline{g'(x)}\Big|_{-1}^{1} -$$
$$-\int\limits_{-1}^{1}((1-x^2)\overline{g'(x)})f'(x)dx = -\int\limits_{-1}^{1} f'(x)((1-x^2)\overline{g'(x)})dx. \tag{22}$$

Since we obtained the same expression the operator L is selfadjoint. $\square$

Exercise 15.2. *Consider a space $V = \mathbb{R}[X]_n$ equipped with a real inner product $(f,g)_V = \int\limits_{-\infty}^{\infty} e^{-x^2}f(x)g(x)dx$.*

1. *(2 points) Prove that the linear operator $L : \mathbb{R}[X]_n \to \mathbb{R}[X]_n$ given by $L(f(x)) = f''(x) - 2xf'(x)$ is selfadjoint.*
2. *(2 points) Let $n = 4$. Find the eugenvalues and eugenvectors of the operator L.*
3. *(1 point) Can you find the relation between the previous item and orthogonalization process being applied to the standard basis $\{1, X, X^2, X^3, X^4\}$?*

15.3 Canonical from of selfadjoint operator

In this section we prove that for any real symmetric matrix A there exists an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal and for any hermitian symmetric matrix $B \in \mathrm{M}_n(\mathbb{C})$ there exists a unitary matrix U such that $U^{-1}BU$ is diagonal.

Let us begin from example.

Problem 15.10. *For a matrix $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$ find an orthogonal matrix Q such that $Q^{-1}AQ$ is diagonal.*

Proof. Consider a linear operator $A : \mathbb{R}^3 \to \mathbb{R}^3, \quad \mathbf{x} \mapsto A\mathbf{x}$ which is given by a matrix A and is denoted by the same letter. So A is the matrix of this linear operator relative to a standard basis in $\mathbb{R}^3$. If Q is a transition matrix from standard to another basis in $\mathbb{R}^3$ then the new basis consist of the columns of Q. And the matrix of the same operator relative to a new basis is equal to $Q^{-1}AQ$. Let us denote the columns of Q by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. And suppose that $Q^{-1}AQ$ has $\lambda_1, \lambda_2, \lambda_3$ on its diagonal. Then one has $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1; \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2; \quad A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$. Thus the problem reformulating how to find an orthonormal basis consisting of eugenvectors of A.

So let us find eugenvalues and eugenctors of the matrix A. Skipping the computation of characteristic polynomial $\chi_A(t) = -(t-1)^2(t-10)$ we obtain the eugenvalues. For $\lambda_1 = 10$ we need to solve homogeneous linear system with matrix $\begin{pmatrix} -8 & 2 & -2 \\ 2 & -5 & -4 \\ -2 & -4 & -5 \end{pmatrix} \sim \begin{pmatrix} 2 & -5 & -4 \\ -8 & 2 & -2 \\ -2 & -4 & -5 \end{pmatrix} \sim \begin{pmatrix} 2 & -5 & -4 \\ 0 & -18 & -18 \\ 0 & -9 & -9 \end{pmatrix} \sim \begin{pmatrix} 2 & -5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. So the subspace of solutions is of dimension 1 and generated by the column $u_1 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$.

For $\lambda_2 = 1$ we obtain only one independent equation $x_1 + 2x_2 - 2x_3 = 0$. The subspace of solutions is generated by $u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $u_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. The column u_1 is orthogonal to u_2 and u_3. It is just a consequence of **Proposition 15.5**. But u_2 and u_3 are not orthogonal to each other. Nevertheless, it is easy to find an orthogonal basis in the two-dimensional eugenspace applying Gram-Schmidt orthogonalization process to $\{u_2, u_3\}$: $\widetilde{u_3} = 2(u_3 - \frac{1}{2}u_2) = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$. Here we multiply by 2 in order to get rid off the denominators. Hence we obtain matrix $\widetilde{Q} = \begin{pmatrix} -1 & 0 & 4 \\ -2 & 1 & -1 \\ 2 & 1 & 1 \end{pmatrix}$ having orthogonal columns which are the eugenvectors of A. In order to gen an orthogonal matrix Q we should notmalize these columns dividing them by their lengths. Then we obtain the desired matrix $Q = \begin{pmatrix} -\frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{6} \\ \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{6} \end{pmatrix}$. Then $Q^{-1} = Q^T = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{2\sqrt{2}}{3} & -\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} \end{pmatrix}$. $\square$

Corollary 15.11. *The quadratic form* $2X^2 + 5Y^2 + 5Z^2 + 4XY - 4XZ - 8YZ$ *has the following diagonalization after the orthogonal change of coordinate:*

$$10\left(\frac{-X - 2Y + 2Z}{3}\right)^2 + \left(\frac{Y + Z}{\sqrt{2}}\right)^2 + \left(\frac{(4X - Y - Z)\sqrt{2}}{6}\right)^2.$$

Theorem 15.12. *Consider a selfadjoint operator $L : V \to V$ on an inner space V. Then there exists an orthonormal basis such that the matrix of the given operator relative to this basis is diagonal.*

Proof. We procced by induction on the dimension. The case $\dim V = 1$ is trivial.

Making induction step, we assume that for any spaces of dimension equal to $\dim V - 1$ is already proved. By **Proposition 15.4** we can chose $\lambda_1 \in \mathbb{R}$ and corresponding eugenvector $u_1 \in V$, i.e. $L(u_1) = \lambda u_1$. Dividing by its length we can assume that $\|u_1\| = 1$. The 1-dimensional subspace $\mathrm{Span}(u_1)$ is L-invariant. Therefore, by **Proposition 15.8** its orthogonal complement $W = \mathrm{Span}(u_1)^\perp$ is invariant under L^*. As $L^* = L$ then W is also L-invariant subspace of dimension $\dim V - 1$.

Therefore we can consider the restriction of the operator $L|_W$ and the restriction of the given scalar product on W. Since for any two vectors $x, y \in W$ one has $(x,y)_W = (x,y)_V$, $L|_W(x) = L(x)$ and $L|_W(y) = L(y)$ then $L|_W$ would be also self-adjoint. Applying induction hypothesis to $L_W : W \to W$ we choose an orthonormal basis in W consisting of eugenvectors of $L|_W$. I.e. there exist $u_2, \ldots, u_n \in W$ such that $L(u_2) = \lambda_2 u_2, \ldots, L(u_n) = \lambda_n u_n$. Since $V = \mathrm{Span}(u_1) \perp W$ then union of the bases of the direct summands gives us the basis of V satisfying the requirements.

$\square$

Exercise 15.3. *For a given quadratic form* $X^2 - 2Y^2 - 2Z^2 - 4XY + 4XZ + 8YZ$ *find an orthogonal coordinate change* $\begin{pmatrix} X' \\ Y' \\ Z' \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ *such that*

$$X^2 - 2Y^2 - 2Z^2 - 4XY + 4XZ + 8YZ = \lambda_1 (X')^2 + \lambda_2 (Y')^2 + \lambda_3 (Z')^2 .$$

16 Canonical form of normal operator

16.1 Normal operator on unitary space

Definition. An operator $L : V \to V$ on an inner space (complex or real) is called **normal** if $LL^* = L^*L$, i.e. the product with an adjoint does not depend on the ordering the factors.

Main examples of the normal operators are:

- selfadjoint operator $L^* = L$;
- **skew-symmetric** operator, by definitiion $L^* = -L$;
- orthogonal or unitary operator $L^*L = \mathrm{Id}$.

Lemma 16.1. *Operator $L : V \to V$ on an inner space preserves scalar product if and only if $L^{-1} = L^*$.*

Proof. L preserves scalar product means that $(u, v) = (L(u), L(v)$ for any $u, v \in V$. But $(L(u), L(v)) = (u, L^*L(v))$ implies that $(u, v) = (u, L^*L(v)) \Leftrightarrow (u, L^*L(v) - v) = 0$. By non-degeneracy of the scalar product we conclude that $L^*L(v) - v = 0$ for any $v \in V$. That means $L^*L = \mathrm{Id}_V$. Therefore, $\det(L) \neq 0$ and $L^{-1} = L^*$. The proof in other direction is easy. $\square$

Remark. There is a common word for both orthogonal and unitary operators. They are called **isometries**.

Theorem 16.2. *Let L be a normal operator on an unitary space. Then there exists an orthonormal basis consisting of eigenvectors of L.*

Corollary 16.3. *Let $A \in \mathrm{M}_n(\mathbb{C})$ be a square matrix such that $A\overline{A^T} = \overline{A^T}A$. Then there exists a unitary matrix U such that $U^T A \overline{U}$ is diagonal.*

Proof. Let λ be an eigenvalue of L. Denote by U the eigenspace $U = V_\lambda(L)$.

Lemma 16.4. *Let $T : V \to V$ be a linear operator such that $L \circ T = T \circ L$. Then the eigenspace $U = V_\lambda(L)$ is invariant under T.*

Proof. We need to prove that for any eigenvector u of L the vector $T(u)$ would be also an eigenvector of L corresponding to the same eigenvalue λ. Since $L(T(u)) = (L \circ T)(u) = (T \circ L)(u) = T(L(u)) = T(\lambda u) = \lambda T(u)$ the lemma follows. □

By this lemma U is also invariant under L^*. Applying **Proposition 15.8** we conclude that $U^\perp$ is also invariant under $(L^*)^* = L$.

Therefore we can proceed by induction on $\dim V$ as in the proof of **Theorem 15.12**. Since $V = U \perp U^\perp$ is an orthogonal direct sum decomposition into L-invariant subspaces. Moreover. both summands are also L^*-invariant. We can consider $W = U^\perp$ and $L|_W$. Then $(L|_W)^* = L^*|_W$ and the restriction $L|_W$ is also normal. By induction hypothesis one can find an orthonormal basis in W consisting of eigenvectors of L. It remains only to add to this basis arbitrarily chosen orthonormal basis of U which is automatically consists of eigenvectors. □

Remark. In the case of the complex inner space **Theorem 15.12** is the special case of the **Theorem 16.2**. But in the case of real inner space canonical form of a normal operator is not diagonal as in the case of selfadjoint operator.

Exercise 16.1. *Let u be an eigenvector for a normal operator $L : V \to V$ corresponding to the eigenvalue λ. Prove that u is also an eigenvector for L^* corresponding to eigenvalue $\overline{\lambda}$.*

16.2 Canonical form of normal operator on Euclidean vector space

Selfdadjoint operator has only real eigenvalues. But complex eigenvalues of a normal operator are not necessary all real. So given a normal operator on a real inner space with a complex eigenvalue we just can not find a corresponding eigenvector in our real vector space for this complex eigenvalue.

Lemma 16.5. *Let $L : V \to V$ be a normal operator on Euclidean space and $\lambda = a+bi$ where $b \neq 0$ its complex eigenvalue, i.e. the complex root of characteristic polynomial. Consider a polynomial $p(t) = (t-\lambda)(t-\overline{\lambda}) = t^2 - 2at + (a^2+b^2)$ and let us substitute an operator L into: $T = p(L) = L^2 - 2aL + (a^2+b^2)\mathrm{Id}$. Then the L-invariant subspace $U = \mathrm{Ker}(T)$ has an orthonormal basis such that the matrix of $L|_U$ has block-diagonal form with 2×2 blocks of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.*

Proof. First of all $\mathrm{Ker}(T) \neq 0$ because the matrix of T with respect to arbitrary basis has zero determinant: it is equal to $A^2 - 2aA + (a^2+b^2)E = (A-\lambda E)(A-\lambda E)$ where $A = [L]$. Then, $\mathrm{Ker}\, T$ is an eigenspace of T corresponding to the zero eigenvalue $\mathrm{Ker}\,(T) = V_0(T)$ and is invariant under L and L^* due to **Proposition 15.8** since $LT = TL$ and $L^*T = TL^*$. Denote by $S : U \to U$ an operator $L|_U - a \cdot \mathrm{Id}_U$. It is easy to see that $S^* = L^*|_U - \mathrm{Id}_U$ and $S^*S = SS^*$. Since $(L - a\mathrm{Id})^2|_U = -b^2 \cdot \mathrm{Id}|_W$ then $S^2 = -b^2\mathrm{Id}_U$. Consequently, $(S^*)^2 = (S^2)^* = -b^2\mathrm{Id}_U = S^2$. Consider $(S - S^*)(S + S^*) = S^2 - (S^*)^2 - (S^*S + SS^*) = 0$. Let us prove that, in fact, $S + S^* = 0$.

Proceed by contradiction and suppose S^* were not equal to $-S$. Take $x \in U$ such that $y = (S+S^*)(x) \neq 0$. Therefore, $(S - S^*)(S+S^*)(x) = 0$ implies that $(S - S^*)(y) = 0$. Recall that $S(y) \neq y$ due to the chain $-b^2 y = S^2(y) = S(S(y))$. Since $S(y) = S^*(y) \neq 0$ then $0 < (S(y), S^*(y)) = (S^2(y), y) = (-b^2 y, y) = -b^2(y,y) < 0$ and we obtain the contradiction.

Finally, take any non-zero vector $u \in U$. We can assume that $\|u\| = 1$. $S(u)$ is not proportional to u. Indeed, if $S(u) = ku$ then $S^2(u) = k^2 u$ but $S^2(u) = -b^2 u$ so the contradiction $k^2 = -b^2$. Denote $b^{-1}S(u)$ by v. One has $S(u) = bv; S(v) = -bu$ and $S^*(u) = -bv; S^*(v) = bu$. Therefore 2-dimensional subspace $U_0 = \mathrm{Span}(u,v)$ is invariant under S and S^* hence it is invariant under L and L^*. Let us check that in fact $\mathcal{B}_0 = \{u, v\}$ is an orthonormal basis in this 2-dimensional subspace. Since $b(v,u) = (S(u).u) = (u, S^*(u)) = (u, -S(u)) = -(S(u), u)$ would be equal to zero then $u \perp v$. Since $b^2(v,v) = (S(u), S(u)) = (u, S^*S(u)) = (u, -S^2(u)) = b^2(u,u) = b^2$ then $\|v\| = 1$. Matrix $[S]_{\mathcal{B}_0}$ is equal to $\begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$ and matrix $[L|_{U_0}]_{\mathcal{B}_0} = [S]_{\mathcal{B}_0} + aE = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

Finally, $U_0^\perp$ is also invariant under L^* and L by **Proposition 15.8**.

Then we can prove that for any L-invariant subspace U such that $p(L)|_U = 0$ there exists an orthonormal basis $\mathcal{B}_U$ such that the matrix $[L|_U]_{\mathcal{B}_U}$ has a desired block-diagonal form with 2×2 blocks. We proceed by induction on $\dim U$. If $U = U_0$ then we are done else we proceed as usually by induction on $\dim U$. Since $p\left(L|_{U_0}\right) = 0$ then we can apply induction hypothesis and find and orthonormal basis in U_0 such that $L|_{U_0^\perp}$ has a desired block-diagonal form. Taking union of the basis in $U_0^\perp$ and $\mathcal{B}_0$ we obtain the desired orthonormal basis in U.

□

Exercise 16.2. *Let $A \in \mathrm{M}_n(\mathbb{R})$ be a normal matrix. i.e. $AA^T = A^T A$. Suppose that $\mathbf{x} \in \mathbb{C}^n$ is an eigencolumn of A corresponding to a non-real eigenvalue $a + bi$. Consider a representation $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.*

- *(2 points) Prove that $\mathrm{Span}(\mathbf{u}, \mathbf{v}) \subset \mathbb{R}^n$ is invariant under multiplication by A.*
- *(2 points) Prove that $\|\mathbf{u}\| = \|\mathbf{v}\|$ and $\mathbf{u} \perp \mathbf{v}$.*

Theorem 16.6. *Let L be a normal operator on the Euclidean space V. Then there exists an orthonormal basis $\mathcal{B}$ such that the matrix $[L]_{\mathcal{B}}$ has a block-diagonal form with the bloks of size 1×1 and 2×2 where 1-dimensional blocks corresponds to real eigenvalues and 2×2 blocks of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ correspond to pairs of complex eigenvalues $a \pm bi$.*

Proof. We proceed by induction on $\dim V$.

Let λ is a root of characteristic polynomial. If λ is real than we can make an induction step as in **Theorem 16.2**. If $\lambda = a + bi$ is not real them consider U as in the **Lemma 16.5**. Since U is invariant under L and L^* then $U^\perp$ is invariant under L^* and L. So we can apply induction hypothesis to $U^\perp$ and apply **Lemma 16.5** to U. □

Problem 16.7. *Find an orthonormal basis in $\mathbb{R}^3$ such that a skew-symmetric matrix $A = \begin{pmatrix} 0 & 7 & 4 \\ -7 & 0 & -4 \\ -4 & 4 & 0 \end{pmatrix}$ has a canonical form relative to this basis.*

Proof. Compute a characteristic polynomial $\chi_A(t) = -t(t^2 + 81)$. Therefore A has only zero real eigenvalue and two non-real eigenvalue. It is easy to find an eigenvector corresponding to zero eigenvalue: $u_1 = \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$. Eigenspace $U = V_0(A)$ is one-dimensional and invariant under A and A^T. Therefore two-dimensional subspace $U^\perp$

is also invariant under A^T and under A. Taking arbitrary orthonormal basis $\{u_2, u_3\}$ in $U^\perp$ we obtain an orthonormal basis $\mathcal{B} = \{u_1, u_2, u_3\}$ such that the matrix of the operator $\mathbf{x} \mapsto A\mathbf{x}$ with respect to $\mathcal{B}$ has the form $A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. Since $A' = Q^T AQ$ where Q is orthogonal matrix having columns u_1, u_2, u_3 then A' is also skew-symmetric: $(A')^T = \left(Q^T AQ\right)^T = Q^T A^T Q = -Q^T AQ = -A'$. Therefore $\pm A' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & -9 & 0 \end{pmatrix}$.

It remains only to find u_2 and u_3. Take arbitrary basis of $U^\perp$: $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix}$ and apply Gram-Schmidt orthogonalization process: $\begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix} - \frac{7}{2}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 7 \\ 7 \\ 8 \end{pmatrix}$. We obtain $Q = \begin{pmatrix} \frac{4}{9} & \frac{1}{\sqrt{2}} & \frac{7}{9\sqrt{2}} \\ \frac{4}{9} & -\frac{1}{\sqrt{2}} & \frac{7}{9\sqrt{2}} \\ -\frac{7}{9} & 0 & \frac{8}{9\sqrt{2}} \end{pmatrix}$.

Since $Au_2 = \dfrac{1}{\sqrt{2}}\begin{pmatrix} -7 \\ -7 \\ -8 \end{pmatrix} = -9u_3$ then $Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 9 \\ 0 & -9 & 0 \end{pmatrix}$.

$\square$

Exercise 16.3. *Prove that the operator* $\mathbf{x} \mapsto A\mathbf{x}$ *for a given matrix* $A = \begin{pmatrix} 12 & 3 & -4 \\ -3 & 12 & 0 \\ 4 & 0 & 12 \end{pmatrix}$ *is normal. Find the orthogonal matrix* Q *such that* $Q^T AQ$ *is a matrix of canonical form for normal operator.*

Exercise 16.4. *Let* L *be a skew-symmetric operator on an inner space. Prove that all the complex eigenvalues of* L *are pure imaginary, i.e. have zero real part.*

16.3 Canonical form of isometries

Lemma 16.8. *Let* λ *be an eigenvalue of unitary (or orthogonal) matrix* A. *Then* $\lambda \cdot \overline{\lambda} = 1$. *i.e.* λ *belongs to the unit circle in the complex plane.*

Proof. We know that $A^T \cdot \overline{A} = E$. Then $A\mathbf{x} = \lambda\mathbf{x}$ implies that $\mathbf{x}^T A^T = \lambda\mathbf{x}^T$ and $\overline{A}\overline{\mathbf{x}}^T = \overline{\lambda}\overline{\mathbf{x}}$. Therefore,

$$\mathbf{x}^T\overline{\mathbf{x}} = \mathbf{x}^T A^T \overline{A}\overline{\mathbf{x}} = \lambda \mathbf{x}^T \overline{\lambda}\overline{\mathbf{x}} = \lambda\overline{\lambda}\mathbf{x}^T\overline{\mathbf{x}}.$$

Since $\mathbf{x}^T\overline{\mathbf{x}} = |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 > 0$ then $\lambda\overline{\lambda} = 1$.

□

Corollary 16.9. *Let $L : V \to V$ be a unitary operator on a complex inner space V. Then there exists an orthonormal basis $\mathcal{B}$ in V such that the matrix $[L]_{\mathcal{B}}$ is diagonal where all the components belong to a unit circle.*

Theorem 16.10. *Let $L : V \to V$ be an orthogonal operator on a Euclidean space. Then there exists an orthonormal basis such that the matrix of L relative to this basis is block diagonal with blocks of size 1×1 and 2×2 where 2×2 blocks have the form $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ and 1×1 blocks are just ± 1. Moreover, assuming that φ can be equal to π one can regard that there is at most one 1×1 block. In the case* $\det(L) = 1$ *this 1×1 block necessarily contains $+1$.*

Proof. Recall that for orthogonal operator $L^* = L^{-1}$ hence L is normal. If $\lambda = a + bi$ is a non-real eigenvalue then by the **Lemma 16.8** λ is situated in the unit circle, hence $a = \cos\varphi$ and $b = \sin\varphi$ for a certain angle φ. If λ is real eigenvalue then as $|\lambda| = 1$ one has $\lambda = 1$ or $\lambda = -1$. Therefore 1×1 blocks contain only ± 1. And 2×2 blocks of the canonical form in **Theorem 16.6** are just rotation 2×2 matrices. When $\det(L) = 1$ then there are even number of -1 since the determinant of each rotation matrix is equal to 1. We can assume that all the -1's are situated one by one and we can organize a pair of two -1 into the block $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ which can be regard as the 2×2 block corresponding to the angle $\varphi = \pi$.

□

Proof. There is another proof that is not rely upon more general case of normal operator but just repeat the proof of the **Theorem 16.6** for the special case of orthogonal operator with significant simplifications.

For real eigenvalue $\lambda \in \mathbb{R}$ we consider an eigenvector $u \in V$. Since $L(u) = \lambda u$ then $L^{-1}(L(u)) = L^{-1}(\lambda u)$ and $\lambda^{-1}u = L^*(u)$ as $L^{-1} = L^*$. So u is also eigenvector of L^* and by the **Lemma 15.8** the subspace $W = \mathrm{Span}(u)^\perp$ is also invariant under L^* and L and we can apply induction hypothesis to W and $L|_W$.

So we can assume that the matrix of the given operator L does not have real eigenvalues. For a non-real eigenvalue $\lambda = a + bi$ consider the subspace

$$U = \mathrm{Ker}\,\left(L^2 - 2aL + (a^2 + b^2)\mathrm{Id}\right).$$

Take an arbitrary non-zero vector $u \in U$ denote by v the vector $b^{-1}(L(u) - au)$. Subspace $U_0 = \mathrm{Span}(u, v) = \mathrm{Span}(u, L(u))$ is invariant under L as $L(L(u)) = 2aL(u) - (a^2 + b^2)u$. It is easy to see that U_0 is also invariant under L^{-1}. Indeed for any $v \in U_0$ there exists $w \in U_0$ such that $L(w) = v$ because $L|_{U_0} : U_0 \to U_0$ is bijection. So $L^{-1}(u) = L^{-1}(L(w)) = w \in U_O$ and invariance follows. As before, $U_0^{\perp}$ would be invariant under L^* and L and we can apply induction hypothesis. As for 2-dimensional subspace U_0, the operator $L|_{U_0}$ is just an isometry of Euclidean plane which does not have real eigenvalues. Take arbitrary orthonormal basis in U_0 and write a matrix A of an operator $L|_{U_0}$ relative to this basis. It is an orthogonal 2×2 matrix which does not have real eigenvalues. Since two eigenvalues of this 2×2 matrix are conjugate to each other complex numbers situated in the unit circle then there are equal $\cos\varphi \pm \sin\varphi$ for some angle φ. The $\det(A) = (\cos\varphi + \sin\varphi)(\cos\varphi - \sin\varphi) = 1$ and classification of the orthogonal 2×2 matrices with unit determinant yields the theorem.

Here I recall it. Let $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$ an orthogonal matrix of $\det(A) = 1$. The equality $A^{-1} = A^T$ implies $\begin{pmatrix} t & -y \\ -z & x \end{pmatrix} = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$ means that $x = t$ and $y = -z$. Since $1 = \det(A) = xt - yz = x^2 + z^2$ then $x = \cos\varphi$ and $z - \sin\varphi$ for a certain angle φ. $\square$

Corollary 16.11. *(Euler rotation theorem). Every isometry in 3-dimensional Euclidean space preserving origin and orientation (i.e. having positive determinant) is a rotation around a certain axis.*

Problem 16.12. *Find the canonical form of an orthogonal operator on $\mathbb{R}^3$ given by a matrix* $A = \dfrac{1}{3}\begin{pmatrix} 2 & 1 & 2 \\ 2 & -2 & -1 \\ 1 & 2 & -2 \end{pmatrix}$.

Proof. We can check that $\det(A) = 1$ and $AA^T = E$. By the **Theorem 16.10** it has eigenvalue which is equal to $+1$. Let us find a corresponding eigenvector: $\begin{pmatrix} -1 & 1 & 2 \\ 2 & -5 & -1 \\ 1 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix}$. Since $rank(A - E) = 2$ we have the only eigenvector $u_1 = \dfrac{1}{\sqrt{11}}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ up to sign. We can proceed as in the **Problem 16.7**, i.e. take a basis in the orthogonal complement $\mathrm{Span}(u_1)^\perp$ but we make use here of an another idea.

We know that A has canonical form $A' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$, so its eigenvalues equal to $1, \cos\varphi + i\sin\varphi, \cos\varphi - i\sin\varphi$ for a certain angle φ which we need to find out.

trace does not depend on the choice of basis

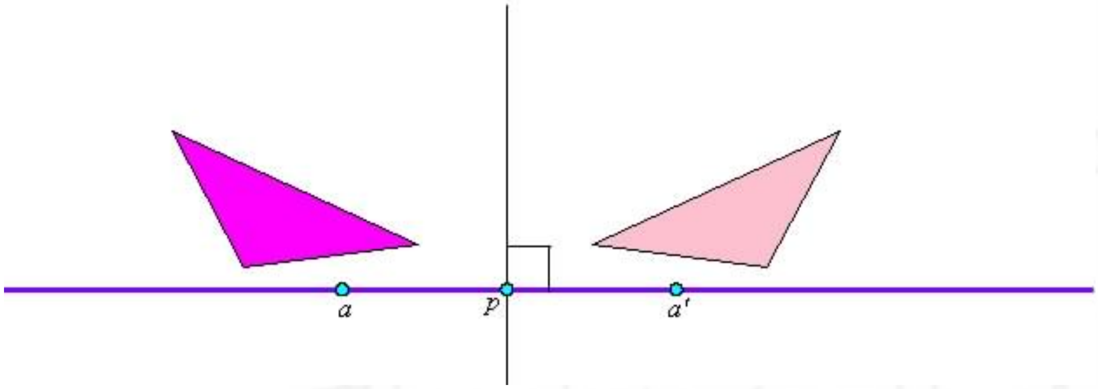
Consider $\mathrm{Tr}(A) = \mathrm{Tr}(A')$. Lefthandside is equal to $\frac{1}{3}(2-2-2) = -\frac{2}{3}$. The righthandside is equal to $1 + 2\cos\varphi$. Hence, $1 + 2\cos\varphi = -\frac{2}{3}$ and $\cos\varphi = -\frac{5}{6}$. $\sin\varphi = \pm\frac{\sqrt{11}}{6}$ an the sign of the sine depends on the orientation of the chosen basis in the plane $\mathrm{Span}(u_1)^\perp$.

E.g. taking an arbitrary basis in $v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ and performing Gram-Schmidt orthogonalization we obtain an orthonormal basis $\{u_1, u_2, u_3\}$ such that $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $u_3 = \frac{1}{\sqrt{22}}\begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix}$.

Since $A \cdot u_2 = \frac{1}{3\sqrt{2}}\begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix} = -\frac{5}{6\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{1}{6\sqrt{2}}\begin{pmatrix} 2 \\ -3 \\ -3 \end{pmatrix} = -\frac{5}{6}u_2 - \frac{\sqrt{11}}{6}u_3$ then $Q^TAQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{5}{6} & \frac{\sqrt{11}}{6} \\ 0 & -\frac{\sqrt{11}}{6} & -\frac{5}{6} \end{pmatrix}$ where $Q = \begin{pmatrix} \frac{3}{\sqrt{11}} & 0 & \frac{2}{\sqrt{22}} \\ \frac{1}{\sqrt{11}} & \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{22}} \\ \frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{22}} \end{pmatrix}$. If we replace vector u_3 by $-u_3$ then the sign of the sine would change. $\square$

16.4 Reflections

The important example of the isometry operators on an inner space is given by reflections. Reflections are closely related with orthogonal projectors.



Definition. Let $V = U \perp W$ be orthogonal decomposition for any $v \in V$ consider a unique representation $v = u + w$ where $u \in U$ and $w \in W$. The linear map $S_W : V \to V$ which assigns $u - w$ to the vector $v = u + w$ is called a **reflection** in the subspace U. The subspace U is called a *mirror* of the reflection S_U.

Proposition 16.13. *Reflection is an isometry, i.e. it preserves scalar product.*

Proof. We have to check that for any $v, v' \in V$ one has $(v, v') = (S_W(v), S_W(v'))$. Take representations $v = u + w$ and $v' = u' + w'$ where $u, u' \in U$ and $w.w' \in W$. Then $(S_W(v), S_W(v')) = (u - w, u' - w') = (u, u') + (w, w') - (w, u') - ((u, w') = (u, u') + (w, w')$ since u, u' are orthogonal to w, w'. Similarly $(v, v') = (u + w.u' + w') = (u, u') + (w, w')$ and the proposition follows. We omit to verify that S_W is linear map as it is true for the same reason as for the projector. □

There is a particular case of reflection when the subspace W is one-dimensional. If $W = \mathrm{Span}(v_0)$ then the reflection S_W is usually denoted by s_{v_0} and called a reflection along v_0.

Proposition 16.14. *For any $x \in V$ one has $s_{v_0} = x - 2\dfrac{(x, v_0)}{(v_0, v_0)} \cdot v_0$.*

Proof. Let $x = kv_0 + u$ where $u \perp v_0$. Then $(x, v_0) = k(v_0, v_0) + (u, v_0) = k(v_0, v_0)$. Therefore,

$$x - 2\frac{(x, v_0)}{(v_0, v_0)} \cdot v_0 = kv_0 + u - 2\frac{k(v_0, v_0)}{(v_0, v_0)} \cdot v_0 = kv_0 + u - 2kv_0 = u - kv_0 = s_{v_0}(x).$$

□

Proposition 16.15. *Let $P = \mathrm{Proj}_U$ where $V = U \perp W$. Then $\mathrm{Id}_V - 2P$ is the reflection S_U in the subspace W.*

Proof. For $v = u+w$ where $u \in U$ and $w \in W$ one has $P(v) = u$. Then $(\mathrm{Id} - 2P)(v) = v - 2u = u + w - 2u = w - u = S_U(v)$. $\square$

Proposition 16.16. *Reflection L is a selfadjoint operator such that $L^2 = L$.*

Proof. Let $L = S_U$ where $V = U \perp W$. Since $(\mathrm{Id} - 2P)^* = \mathrm{Id} - 2P^*$ it is enough to check that orthogonal projector $P = \mathrm{Proj}|_U$ is selfadjoint. For any $v, v' \in V$ take representations $v = u + w$ and $v' = u' + w'$. Then $(P(v), v') = (u, v') = (u, u' + w') = (u, u')$. Similarly, $(v, P(v')) = (u + w, u') = (u, u')$, hence $(P(v), v') = (v, P(v'))$. Since $L^2(v) = L(L(v)) = L(w - u) = w - (-u) = w + u = v$ the proposition follows. $\square$

Remark. Since $LL^* = \mathrm{Id}$ then the property $L^2 = \mathrm{Id}$ is equivalent to the equality $L = L^*$. I.e. the selfadjoint orthogonal operator L satisfies the equation $L^2 = \mathrm{Id}$.

Exercise 16.5. *Suppose that selfadjoint operator $L : V \to V$ on euclidean space V satisfies $L^2 = \mathrm{Id}$. Prove that $L = S_U$ for a certain subspace U.*

Exercise 16.6. *Find a matrix in standard basis for a reflection in $\mathbb{R}^3$ which mirror is defined by equation $x + 2y - 3z = 0$.*

17 Polar decomposition

17.1 Positive operators

Consider a coordinate space $\mathbb{R}^n$. Then a symmetric matrix A defines two different geometric objects. One is selfadoint operator on $(\mathbb{R}^n, \cdot)$ fiven by $\mathbf{x} \mapsto a\mathbf{x}$. The second is a quadratic form given by $q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ or the corresponding bilinear form. It occurs that correspondence between selfadjoint operators and quadratic forms can be reformulated abstractly avoiding matrix notations.

Proposition 17.1. *Let V be a real inner space. Then given a selfadjoint operator $L : V \to V$ the formula $h_L(u, v) = (L(u), v)$ define a symmetric bilinear form. From the other hand, given a symmetric bilinear form h one can uniquely define an operator L_h such that $h(u, v) = (u, L_h(v))$.*

Proof. The map $V \times V \to \mathbb{R}, \quad (u, v) \mapsto (L(u), v)$ is linear in the first argument as L is linear map and scalar product is linear in the first argument. h_L is symmetric by symmetry of scalar product and $L^* = L$: $h_L(u, v) = (L(u), v) = (u, L(v)) = (L(v), u) = h_L(v, u)$.

In other direction, given a symmetric bilinear form h we can consider for a fixed vector v a linear functional $x \mapsto h(x,v)$. By the Riesz representation theorem any liner functional on inner space can be given by a sclar product with a suitable vector, i.e. there exists a unique $w \in V$ such that $h(x,v) = (x,w)$ for all $x \in V$. Denote w by $L(v)$ and prove that $L(v+v') = L(v) + L(v')$. Indeed, $(x, L(v) + L(v')) = (x, L(v)) + (x, L(v')) = h(x,v) + h(x,v') = (x, h(x, v+v') = (x, L(v+v'))$ implies that $(x, L(v) + L(v') - L(v+v')) = 0$ for all x. Therefore, $L(v) + L(v') - L(v+v') = 0$ □

Proposition 17.2. *Let V be a complex inner space. For a given hermitian form $f : V \times V \to \mathbb{C}$ there exists a unique linear operator $L_f : V \to V$ such that $(x, L_f(v)) = f(x,v)$ for any $x, v \in V$. Moreover, L_f is selfadjoint.*

And in the opposite direction, given a selfadjoint operator L on an unitary space ona can define another hermitian form $f_L(u,v) = (u, L(v))$.

Proof. The proof is omitted. □

Exercise 17.1. *Check that the operator L satisfying the condition $(x, L_f(v)) = f(x,v)$ for any $x, v \in V$ is indeed selfadjoint.* Q: how to prove $f(x,v) = f(v,x)$.

Remark. If $\mathcal{B} = \{e_1, e_2 \ldots, e_n\}$ is an orthonormal basis in an inner space V and $L : V \to V$ is a selfadjoint operator then the matrix $[L]_{\mathcal{B}}$ is equal to transpose of the Gram matrix $G_{h_L, \mathcal{B}}$. Indeed, $L(e_j) = \sum_{i=1}^{n} (L(e_j), e_i) e_i$ by equation (10). Therefore, $[L]_{ij} = (L(e_j), e_i) = h_L(e_j, e_i)$ which is equal to (i,j)-component of transposed Gram matrix of h_L.

Definition. A selfadjoint operator $L : V \to V$ on an inner space V is called strictly **positive** if the corresponding bilinear (hermitian) form is positive definite, i.e. $(L(u), u) > 0$ for any vector $u \neq 0$.

Proposition 17.3. *Let $L : V \to V$ be an invertible operator on an inner space. Then the operator LL^* is strictly positive (selfadjoint) operator.*

Proof. That is easy becuase $(LL^*(u), u) = (L^*(u), L^*(u)) > 0$ since $L^*(u) \neq 0$ for $u \neq 0$. □

Proposition 17.4. *The following conditions on a given selfadjoint operator L are equivalent:*

- *L is positive;*
- *All the roots of characteristic polynomial $\chi_L(t)$ are positive real numbers;*

- *All the upper-left minors of the matrix of L relative to an orthonormal basis are positive.*

Proof. 1) $\Rightarrow$ 2). We know that all the roots are real. Consider an eigenvalue $\lambda \in \mathbb{R}$ and corresponding eigenvector $v \in V$. Then $0 < (L(v), v) = \lambda(v, v) \Rightarrow \lambda > 0$.

2) $\Rightarrow$ 1). Consider an orthonormal basis $\{v_1, \dots, v_n\}$ consisting of eigenvectors of L. Then $L(v_i) = \lambda_i v_i$ where $\lambda_i > 0$ and $(v_i, v_i) = 0$ for $i \neq j$. Then for any $v = \sum_{i=1}^{n} x_i v_i$ one has $(L(v), v) = \left(\sum_{i=1}^{n} \lambda_i x_i v_i, \sum_{i=1}^{n} x_i v_i\right) = \sum_{i=1}^{n} \lambda_i |x_i|^2 > 0$.

1) $\Rightarrow$ 3. The question is about minors of the Gram matrix of corresponding bilinear or hermitian form which is positive definite. Since the restriction of the positive definite form is positive definite then it sufficient to prove that determinant of the Gram matrix of positive definite form is positive. But determinant is just the product of all eigenvalues of the corresponding selfadjoint operator which are already proven to be positive.

4) $\Rightarrow$ 1). Consider a corresponding hermitian or bilinear form and let G be its Gram matrix. Given orthonormal basis in the inner space need not be orthogonal with respect to this form f_L. But we can apply the Gram-Shmidt orthogonalization process and obtain a diagonal matrix $G' = C^T G \overline{C}$ where C is upper triangular with units on the diagonal. Then G' has the same upper left minors as G since $G'_k = C_k G_k \overline{C_k}$ where index k denotes the upper-left submatrix. Therefore G' is a diagonal matrix with positive components hence it defines a positive definite hermitian (or bilinear) form. $\square$

Proposition 17.5. *For s positive selfadjoint operator L on an inner space V there is exists a unique positive operator $T : V \to V$ such that $T^2 = L$.*

Proof. It is very easy to derive existence from the canonical form of the selfadjoint operator. There exists an orthonormal basis such that the matrix $[L]$ with respect to this basis is diagonal with positive elements since they are equal to eigenvalues. So we can find positive square roots, i.e. consider the matrix with $\sqrt{\lambda_1}, \sqrt{\lambda_2}, \dots, \sqrt{\lambda_n}$ on ots diagonal which wuold be the matrix of the certain selfadjoint positive operator relative to the same orthonormal basis.

As for uniqueness, let $\lambda_1, \lambda_2, \dots, \lambda_k$ would be all different eigenvalues of L. Since L is diagonalizible operator by the theorem on canonical form of selfadjoint operator then by **Theorem 10.1** we have a direct sum decomposition $V = V_{\lambda_1}(L) \perp \cdots \perp V_{\lambda_k}(L)$. This decomposition is in fact the orthogonal decomposition since the eigenvectors of selfadjoint operators corresponding to different eigenvalues are orthogonal to each other.

We prove that if $T^2 = L$ and T is positive then necessarily $T|_{V_{\lambda_i}(L)}$ acts by multiplication by $\sqrt{\lambda_i}$ hence for any $v = v_1 + v_2 + \cdots + v_k$ where $v_i \in V_{\lambda_i}(L)$ one has $T(v) = \sum_{i=1}^{k} \sqrt{\lambda_i} v_k$ and T is uniquely defined by the latter equality. For this purpose we consider the same decomposition for T into the orthogonal direct sum of eigenspaces. Take any μ which is eigenvalue of T. Then $L = T^2$ being restricted on the subspace $V_\mu(T)$ acts by multiplication by μ^2. Therefore, $V_\mu(T) \subset V_{\mu^2}(L)$ hence $\mu = \sqrt{\lambda_i}$ for a certain λ_i. Moreover, different eigenspaces of T include into different eigenspaces of L. After suitable renumbering we can assume that $\mu_i = \sqrt{\lambda_i}$ and $V_{\mu_i}(T) \subset V_{\lambda_i}(L)$. Since we have two direct sum decompositions $V = \bigoplus V_{\mu_i}(T)$ and $V = \bigoplus V_{\lambda_i}(T)$ the summands in these decompositions coincide respectively.

$\square$

17.2 Polar decomposition

We start with the problem which you are familiar with form analityc geometry course.

Consider a unit circle on the euclidean plane with Cartesian coordinates. What we obtained by appling to it the some linear linear transformation?

E.g. take $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1.26 & 0 \\ 0 & 0.77 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. This is so called *scaling* transformation. Of course the image of the unite circle is given by equation

$$\frac{(x')^2}{1.26^2} + \frac{(y')^2}{0.77^2} = 1$$

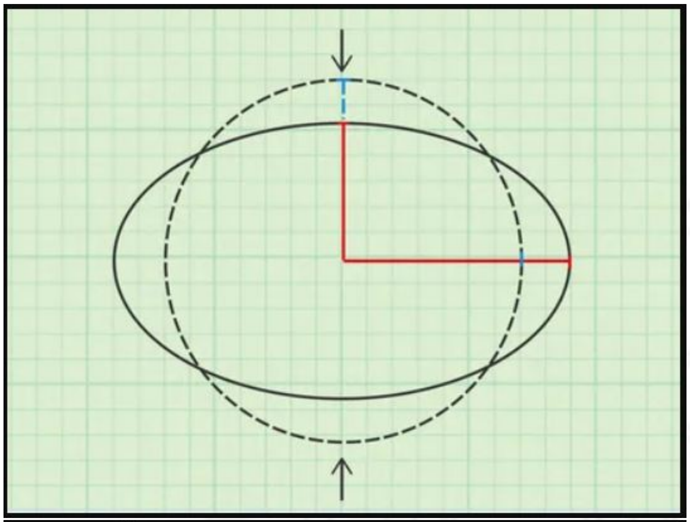
which is the ellipse with length of semi-axes 1.26 and 0.77.

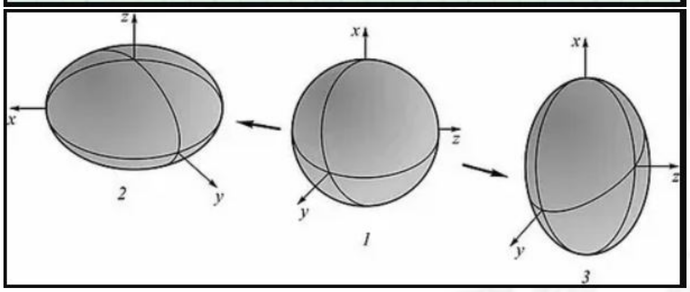
You can ask what is the figure would be the image of the unit circle or the unit sphere under arbitrary invertible linear transformation $\mathbf{x}' = A\mathbf{x}$?

Since the unit sphere is given by equation $\|\mathbf{x}\| = 1 \Leftrightarrow (\mathbf{x}, \mathbf{x}) = 1$. Then $\mathbf{x}' = A\mathbf{x}$ belongs to the image of the unit sphere if and only if $\|A^{-1}\mathbf{x}'\| = 1$. Denote $B = A^{-1}$ and rewrite it as $(B\mathbf{x}')^T \cdot (B\mathbf{x}') = 1 \Leftrightarrow (\mathbf{x}')^T B^T B\mathbf{x}' = 1$. Since $B^T B$ is positive symmetric matrix, then there exists an orthogonal transformation C such that $C^T(B^T B)C$ is diagonal with positive elements. Consider another Cartesian coordinate system such that $\mathbf{x} = C\mathbf{y}$. Then the same set $\{\mathbf{x} \,|\, \mathbf{x}^T B^T B\mathbf{x} = 1\}$ is given by $\{\mathbf{y} \,|\, \mathbf{y}^T(C^T B^T BC)\mathbf{y} = 1\} = \left\{\mathbf{y} \,\middle|\, \frac{y_1}{\lambda_1^2} + \cdots + \frac{y_n^2}{\lambda_n^2} = 1\right\}$ (where $\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2$ are all eigenvalues of the matrix AA^T (so $C^T B^T BC$ is diagonal matrix whith λ_i^{-1} on its diagonal).

$A^T B$

$A^{-1T} A^{-1}$





That means that the image of the unit sphere is ellipsoid.

Theorem 17.6. *(Polar decomposition in inner space). For any invertible operator $L : V \to V$ on an inner space there exists a positive selfadjoint operator $S : V \to V$ and isometry $Q : V \to V$ such that $L = SQ$.*

Remark. Geometric meaning for real inner space of this decomposition is the following. Given an arbitrary L we see that it maps a unit sphere to some ellipsoid that can be obtained from unit sphere by some scaling which is given in suitable orthonormal basis by diagonal matrix. Denote this scaling operator by S. Then $S^{-1}L$ is an operator which maps unit sphere to itslef hence orthogonal operator.

Proof. More formal proof uses **Proposition 17.5**. By **Proposition 17.3** and **Proposition 17.5** there exists a positive selfadjoint operator S such that $S^2 = LL^*$. Consider $Q = S^{-1}L$ and prove that Q is an isometry. $QQ^* = S^{-1}L(S^{-1}L^*) = S^{-1}LL^*(S^{-1})^* = S^{-1}(LL^*)S^{-1} = S^{-1} \cdot S^2 \cdot S^{-1} = \mathrm{Id}_V$.

Therefore, $L = SQ$ and the theorem follows. $\square$

Remark. The factors in the polar decomposition $L = SQ$ are uniquely defined by L, since given this decomposition one has $LL^* = SQQ^*S^* = SS^* = S^2$, hence S should be a square root of LL^* which is unique when we searched it in the class of positive operators.

Remark. There are two different polar decomposition (one is called right and another is called left). Since we can decompose L in the similar way $L = Q_1 S_1$. In general, $S \neq S_1$ the equality holds if and only if L is normal. The second polar decomposition $L = Q_1 S_1$ can be easily obtained bu applying **Theorem 17.6** to the operator L^*. In fact, $Q = Q_1$ in two kinds of polar decomposition. That is good, not very hard exercise. $A = SQ = C^T D_n C Q = C^T C Q (\underbrace{Q^T C^T D_n C Q}_{= S_1}) = Q(Q^T C^T D_n C Q) = Q S_1$

Corollary 17.7. *For any invertible matrix $A \in \mathrm{M}_n(\mathbb{R})$ there exists a symmetric positive definite matrix $S \in \mathrm{M}_n(\mathbb{R})$ and orthogonal matrix $Q \in M_n(\mathbb{R})$ such that $A = SQ$.*

Corollary 17.8. *For any invertible matrix $A \in \mathrm{M}_n(\mathbb{C})$ there exists an hermitian symmetric matrix $S \in \mathrm{M}_n(\mathbb{C})$ and unitary matrix $U \in \mathrm{M}_n(\mathbb{C})$ such that $A = SU$.*

Corollary 17.9. *For any invertible matrix $A \in \mathrm{M}_n(\mathbb{R})$ there exists a diagonal matrix $D\,\mathrm{M}_n(\mathbb{R})$ with positive elements and orthogonal matrices $Q_1, Q_2 \in M_n(\mathbb{R})$ such that $A = Q_1 D Q_2$.*

Corollary 17.10. *For any invertible matrix $A \in \mathrm{M}_n(\mathbb{C})$ there exists a diagonal matrix $S \in \mathrm{M}_n(\mathbb{R})$ with positive elements and unitary matrices $U_1, U_2 \in \mathrm{M}_n(\mathbb{C})$ such that $A = U_1 D U_2$.*

Proof. Let us prove **Corollary 17.9**. We ought to just combine polar decomposition and canonical form of selfadjoint operator. For invertible $A \in \mathrm{M}_n(\mathbb{R})$ take polar decomposition $A = SQ$ and apply the matrix counterpart of the main result of the subsection 15.3 to S obtaining $S = CDC^{-1}$ where $C \in \mathrm{M}_n(\mathbb{R})$ is orthogonal and D is diagonal (i.e. canonical form of selfadjoint operator).

Then $A = CDC^{-1}Q = Q_1 D Q_2$ where $Q_1 = C$ and $Q_2 = C^{-1}Q$. Exactly the same but replacing orthogonal by unitary is valid in the complex case.

Example. Consider 2×2 matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Recall that it corresponds to the linear map on 2-dimensional real vector space $\mathbb{C}$ which is given by multiplication by a complex number $a + bi$. Then the trigonometric form of the complex number

$$a + bi = \sqrt{a^2+b^2}\,(\cos\varphi + i\sin\varphi) \text{ where } \cos\varphi = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\varphi = \frac{b}{\sqrt{a^2+b^2}}$$

gives the polar decomposition $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$.

□

17.3 Arbitrary operator on an inner space

Consider a problem for a given operator $L : V \to V$ on a complex inner space to find a orthonormal basis such that the matrix of L relative to this basis would have the simplest form as possible.

In the case when we do not oblige to find an orthonormal basis but can choose an arbitrary basis the answer is Jordan form. The additional restriction that the basis must be orthonormal does not allow us to reduce the matrix to the Jordan form. Nevertheless,

Theorem 17.11. *For any operator $L : V \to V$ on a complex inner space there exists an orthonormal basis $\mathcal{B}$ such that the matrix $[L]_{\mathcal{B}}$ is upper triangular.*

Proof. We proceed by induction on $\dim V$. The case $\dim V = 1$ is trivial. So we can assume that for the case $\dim V = n-1$ the theorem holds and try to deduce it in the case $\dim V = n$.

Take an eigenvector $v_1 \in V$ which can be assumed to have a unit length: $L(v_1) = \lambda_1 v_1$. For $U = \mathrm{Span}(v)^{\perp}$ let is consider an orthogonal projector $P = \mathrm{Proj}_U$. If we take an arbitrary orthonormal basis $\{u_2, \ldots, u_n\}$ in U than the matrix of L relative to the basis $\{v_1, u_2, \ldots, v_n\}$ would be block-triangular with two blocks of sizes 1×1 and $(n-1) \times (n-1)$. Obviously, 1×1 block is just λ_1.

As for the big block, it is equal to the matrix of the restriction of the operator PLP on the subspace U. Indeed, for any basis vector $u_i \in U$ one has $PLP(u_i) = PL(u_i) = P(a_{1i}v_1 + a_{2i}u_2 + \cdots + a_{ni}u_n) = a_{2i}u_2 + \cdots + a_{ni}u_n$ so we obtain just the i-the column of the big diagonal block of size $n-1$.

We can apply the induction hypothesis to the supspace U and operator $L' = (PLP)|_U$ and find another orthonormal basis $\mathcal{B}' = \{v_2, v_2, \ldots, v_n\}$ in U such that the matrix of L' relative to this basis is upper triangular. Consider the matrix $[L]_{\mathcal{B}}$ where $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$. What was said before about arbitrary orthonormal basis $\{u_2, \ldots, u_n\}$ can be applied to $\mathcal{B}'$. So the matrix $[L]_{\mathcal{B}}$ is block triangular with two blocks and its big block is equal to $[L']_{\mathcal{B}'}$, i.e. upper triangular matrix of size $(n-1) \times (n-1)$. Hence, $[L]_{\mathcal{B}}$ is also upper triangular.

□

Corollary 17.12. *For any matrix $A \in \mathrm{M}_n(\mathbb{C})$ there exists a unitary matrix U such that $U^{-1}AU$ is an upper triangular.* $x \mapsto Ax$. U be the transition matrix of standard basis to orthonormal basis. Show it is unitary. (by computing the elements)

Remark. For the case of the real inner space there is analogous result asserting existence of not pure triangular form but block triangular form with blocks of maximum size 2. But we do not dive into details.

Exercise 17.2. *Find one (2 points) or both (3 points) polar decompositions for the real matrix* $\begin{pmatrix} -4 & -6 \\ 3 & -8 \end{pmatrix}$.

18 Singular value decomposition

Here we consider an arbitrary operator $L : V \to U$ between different inner spaces.

It is easy to see that operator $L^*L : V \to V$ and the operator $LL^* : U \to U$

both non-negative (or non-strictly positive) selfadjoint operators since $(LL^*(u), u) = (L^*(u), L^*(u)) \geqslant 0$ where the equality could hold in the case $u \neq 0$.

Proposition 18.1. *Under above circumstances strictly positive eigenvalues of L^*L and LL^* are the same.*

Proof. Take an eigenvalue λ of L^*L and the corresponding eigenvector $v \in V$. We should check that λ is also an eigenvalue of LL^*. Since $L^*L(u) = \lambda u \neq 0$ as $\lambda > 0$ and $u \neq 0$ then $LL^*(L(u)) = L(\lambda u) = \lambda L(u)$. Hence $L(u)$ is an eigenvector of LL^*. □

Remark. There is more general fact that for any two triangular matrices A and B of sizes $m \times n$ and $n \times m$ correspondingly the characteristic polynomials of square matrices AB and BA are closely related. I.e. there exists a monic polynomial $p(t)$ of certain degree k such that $\chi_{AB}(t) = (-1)^m t^{m-k} p(t)$ and $\chi_{BA}(t) = (-1)^n t^{n-k} p(t)$. So the non-zero eigenvalues of the matrices AB and BA coincide and their multiplicities are the same.

Definition. Positive square roots of non-zero eigenvalues of LL^* are called **singular values** of the linear map L.

Theorem 18.2. *(Singular value decomposition). Consider a linear operator $L : V \to V$. Then there are exist orthonormal bases $\mathcal{B}_V$ in V and $\mathcal{B}_U$ in U such that the matrix $[L]_{\mathcal{B},\mathcal{B}'}$ has the block form $\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$ where D_r is a diagonal $r \times r$ matrix with singular values of L on its diagonal.*

Proof. The block-diagonal form is very similar to the one in **Theorem 6.9** but we make use the method of another proof of the cited theorem which relies upon **Theorem 5.6**.

Denote by r the $\operatorname{rank}(L) = \dim \operatorname{Im}(L)$. Then $\dim \operatorname{Ker}(L) = n - r$ and $\dim \operatorname{Ker}(L)^{\perp} = r$. Let us choose an orthonormal basis in $\operatorname{Ker}(L)^{\perp}$ and denote it by $\{v'_1, \ldots, v'_r\}$ and choose an orthonormal basis in $\operatorname{Ker}(L)$ and denote it by $\{v_{r+1}, \ldots v_n\}$.

Choose an orthonormal basis in $\operatorname{Im}(L)$ and denote it by $\{u'_1, \ldots, u'_r\}$ and choose an orthonormal basis in $\operatorname{Im}(L)^{\perp}$ and denote it by $\{u_{r+1}, \ldots, u_m\}$. Then the matrix of L in chosen bases would have bock form $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ where A is the square matrix of rank and size r, hence invertible.

The matrix of L^*L has analogous block form with an upper left $r \times r$ block is equal to $\overline{A^T}A$. It means that $\overline{A^T}A$ is the matrix of restriction $L^*L|_{\operatorname{Ker}(L)^{\perp}}$ relative to a basis $\{v'_1, \ldots, v'_r\}$. Eigenvalue of the matrix $\overline{A^T}A$ are just all non-zero eigenvalue of an operator L^*L with the same multiplicities. We can apply **Theorem 15.12** to the restriction $L^*L|_{\operatorname{Ker}(L)^{\perp}}$ and find another orthonormal basis $\{v_1, \ldots, v_r\}$ in the subspace $\operatorname{Ker}(L)^{\perp}$ consisting of eigenvectors of L^*L corresponding to non-zero eigenvalues $\sigma_1^2, \sigma_2^2 \ldots, \sigma_r^2$. By $\sigma_1, \sigma_2 \ldots, \sigma_r$ we denote the singular values of the linear map L.

Since $L^*L(v_i) = \sigma_i^2 v_i$ then $\sigma_i^2 = (v_i, L^*L(v_i)) = (L(v_i), L(v_i))$. Moreover, $0 = \sigma_j^2(v_i, v_j) = (v_i, L^*L(v_j)) = (L(v_i), L(v_j))$ for $i \neq j$. Hence $\dfrac{1}{\sigma_i}L(v_i)$ for $i = 1 \ldots r$ form an orthonormal basis in $\operatorname{Im}(L)$ since they are pairwise orthogonal vectors in $\operatorname{Im}(L)$ of unit length. Denoting $u_i = \dfrac{1}{\sigma_i}L(v_i)$ we obtain another orthonormal basis $\{u_1, \ldots, u_r\}$ in $\operatorname{Im}(L)$ hence $\{u_1, \ldots, u_m\}$ would be an orthonormal basis in U. By the construction $L(v_i) = \sigma_i u_i$ for $i = 1 \ldots r$ and $L(v_i) = 0$ for $i \geqslant r + 1$. Therefore the matrix of L relative to these bases has the required form. $\square$

Corollary 18.3. *Let $A \in \mathrm{M}_{m.n}(\mathbb{R})$ an arbitrary matrix of rank r. Then there exist orthogonal matrices $Q_1 \in \mathrm{M}_m(\mathbb{R})$ and $Q_2 \in \mathrm{M}_n(\mathbb{R})$ such that $Q_1 A Q_2$ has the form $\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$ where D_r is a diagonal $r \times r$ matrix.*

Exercise 18.1. *Prove that if for some orthogonal matrices Q_1 and Q_2 the matrix $Q_1 A Q_2$ has the form as in the* **Corollary 18.3** *then the diagonal matrix D_r has singular values of A on its diagonal.*

For invertible operator. $Q_1 A Q_2 = D_r$. $(r = n)$.

Remark. The polar decomposition could be easily deduced from Singular value decomposition just applying it to the matrix A.

$Q_1 D_n Q_2 = (Q_1 D_n Q_1^{-1})(Q_1 Q_2)$

positive. self-adjoint. orthogonal

$Q_1 A Q_2$

18.1 Operator norm and relative errors

Informally, the operator norm measures the "size" of certain linear operators by assigning each a real number called its **operator norm**. There are different context when it is possible to give slightly different definitions.

We define the operator norm for a linear map between two inner spaces $L : V \to U$.

Definition. For $L : V \to U$ its norm $\|L\|_2$ is equal to supremum $\sup\limits_{\|x\|=1} \|L(x)\|$.

By the very definition for all $v \in V$ one has $\|L(v)\| \leqslant \|L\|_2 \cdot \|v\|$.

Proposition 18.4. *Let $L, L_1 : V \to U$ and $L_2 : W \to V$ be linear maps between inner spaces. Then*

1. $\|L + L_1\|_2 \leqslant \|L\|_2 + \|L_1\|_2$ *and* $\|\alpha L\|_2 \leqslant |\alpha| \cdot \|L\|_2$*;*
2. $\|L_1 L_2\|_2 \leqslant \|L_1\|_2 \cdot \|L_2\|_2$.

Proof. Since $\|L(x)\| \leqslant \|L\|_2 \cdot \|x\|$ and $\|L_1(x)\| \leqslant \|L_1\|_2 \cdot \|x\|$ then

$$\|(L + L_1)(x)\| = \|L(x) + L_1(x)\| \leqslant \|L(x)\| + \|L_1(x)\| \leqslant \|L\|_2 \cdot \|x\| + \|L_1\|_2 \cdot \|x\| = \\ = (\|L\|_2 + \|L_1\|_2) \cdot \|x\|, \quad (25)$$

where we use triangle inequality **Proposition 14.12(2)** in the second step.

The second assertion follows from the cnain of inequalities

$\|L_1 L_2(x)\| = \|L_1(L_2(x)\| \leqslant \|L_1\|_2 \cdot \|L_2(x)\| \leqslant \|L_1\|_2 \cdot \|L_2\|_2 \cdot \|x\|$. Therefore, for any $x \in W$ such that $\|x\| = 1$ one has $\|L_1 L_2(x)\| \leqslant \|L_1\|_2 \cdot \|L_2\|_2$, hence

$$\sup_{\|x\|=1} \|L_1 L_2(x)\| \leqslant \|L_1\|_2 \cdot \|L_2\|_2.$$

□

One of the main application of this notion is to control the error in measurement when applying linear maps.

Suppose that there are two vector $x, \widetilde{x} \in V$ where $\widetilde{x}$ is assumed to be an approximation to x. E.g. $\widetilde{x}$ is a result of measurement of unknown desired value of x. Then the difference $\|x - \widetilde{x}\|$ is called an absolute error.

The relative error is just the fraction $\dfrac{\|x - \widetilde{x}\|}{\|x\|}$. Then the absolute error in the value $\|L(x) - L(\widetilde{x})\|$ can be easily estimated as

$\|L(x) - L(\widetilde{x})\| \leqslant \|L\|_2 \cdot \|x - \widetilde{x}\|$.

So the operator norm serves as the estimate to magnifying coefficient for absolute error. There is an important question of sensitivity to a relative errors. How big the relative errors in the result can be caused by the relative errors in data. In the case $L(x) = 0$ the relative error in result is undefined so it makes sense to consider problem of estimating relative error in result only for invertible maps.

Since $\frac{\|L(x) - L(\widetilde{x})\|}{\|L(x)\|} \leqslant \|L\|_2 \frac{\|x - \widetilde{x}\|}{\|x\|} \frac{\|x\|}{\|L(x)\|}$ we need t estimate the fraction on the right. But $x = L^{-1}(L(x))$ implies that $\|x\| \leqslant \|L^{-1}\|_2 \cdot \|L(x)\|$. Therefore,

$$\frac{\|L(x) - L(\widetilde{x})\|}{\|L(x)\|} \leqslant \|L\|_2 \|L^{-1}\|_2 \cdot \frac{\|x - \widetilde{x}\|}{\|x\|},$$

so the magnification factor for relative error is estimated by $\|L\|_2 \cdot \|L^{-1}\|_2$.

Definition. The number $\kappa(L) = \|L\|_2 \cdot \|L^{-1}\|_2$ is called the **condition number** for L.

When the condition number $\kappa(L)$ is big then the behaviour of linear map L from computational viewpont is very similar to the behaviour of non-invertible map. In this case small relative errors in data could cause big relative error in the result.

Proposition 18.5. *For any operator $L : V \to U$ between inner spaces the operator norm $\|L\|_2$ is equal to the greatest singular number of L, it is denoted usually by $\sigma_1(L)$.*

If L is an invertible linear map between the inner spaces of dimension n then the condition number is equal to $\kappa(L) = \dfrac{\sigma_1(L)}{\sigma_n(L)}$ where $\sigma_1(L) \geqslant \sigma_2(L) \geqslant \cdots \geqslant \sigma_n(L)$ are all singular values of A in decreasing order.

Proof. Using singular value decomposition take a an orthonormal basis $v_1, \ldots v_r, \ldots, v_n$ such that the values $\|L(v_i)\| = \sigma_i$ (here $\sigma_i = \sigma_i(L)$ for $i = 1 \ldots r$ and $\sigma_i = 0$ for $i > r$) and all the vectors $L(v_1), \ldots, L(v_n)$ are orthogonal to each other.

Then for any $x \in V$ let us take decomposition $x = c_1 v_1 + \cdots + c_n v_n$. Hence $\|x\|^2 = |c_1|^2 + \ldots + |c_n|^2$. Let us estimate $\|L(x)\|^2 = \|c_1 L(v_1) + \ldots c_n L(v_n)\|^2 = |c_1|^2\sigma_1^2 + |c_2|^2\sigma_2^2 + \cdots + |c_n|^2\sigma_n^2$ by $\sigma_1^2 \cdot (|c_1|^2 + \ldots + |c_n|^2)$. Therefore, $\|L(x)\|^2 \leqslant \sigma_1^2 \|x\|^2$ and the first assertion follows.

As for the second assertion it is easy to see that eigenvalues of $L^{-1}(L^{-1})^*$ are equal to the inverse of the eigenvalues of L^*L. Therefore the greatest singular value of L^{-1} is equal to σ_n^{-1}. Hence, $\|L\|_2 \cdot \|L^{-1}\|_2 = \sigma_1(L) \cdot \dfrac{1}{\sigma_n(L)}$ and the second assertion follows. $\square$

18.2 Approximation by low rank linear map

One of the prominent applications of singular values in real life/numerical methods is implemented in most search engine machines, e.g. Yandex, Google and so on. I recommend you to look what is written in Example 5.12.4 on page 419 in Carl.D.Meyer textbook "Matrix Analysis and Applied linear algebra".

Since the underlying mathematics is quite simple we discuss it now. Suppose that we are given a linear map $L : V \to U$ between inner spaces of very big dimensions. How to find an approximation of L by a certain linear map $L_k : V \to$ such that $\operatorname{rank}(L_k) = k$ and L_k is close to L. In practice handling with matrices of small rank take significally less machine time when the size of the matrices is very big.

Theorem 18.6. *Let $L : V \to U$ be a linear map between inner spaces and $\sigma_1 \geqslant \sigma_2 \geqslant \dots$ are the singular values of L. Then for any $k < \operatorname{rank}(L)$:*

- *given am arbitrary linear map $T : V \to U$ such that $\operatorname{rank}(T) = k$ one has $\|L - T\|_2 \geqslant \sigma_{k+1}$;*
- *there exists a linear map $L_k : V \to U$ such that $\operatorname{rank}(L_k) = k$ and $\|L - L_k\|_2 = \sigma_{k+1}$.*

Proof. Let $v_1, v_2, \dots, v_{k+1}$ be a part of orthonormal basis in V such that $L(v_1), \dots, L(v_i)$ are pairwise orthogonal vectors in U such that $\|L(v_i)\| = \sigma_i$. Denote by V_{k+1} the subspace $V_{k+1} = \operatorname{Span}(v_1, \dots, v_{k+1})$.

Then for any $x \in V_{k+1}$ one has $\|L(x)\| \geqslant \sigma_{k+1}\|x\|$. Indeed, let $x = c_1v_1 + c_2v_2 + \dots c_{k+1}v_{k+1}$. Then $\|x\|^2 = |c_1|^2 + |c_2|^2 + \cdots + |c_{k+1}|^2$ but
$||L(x)\|^2 = (c_1L(v_1) + \cdots + c_{k+1}L(v_{k+1}), c_1L(v_1) + \cdots + c_{k+1}L(v_{k+1}))_U =$
$= \sum_{i=1}^{k+1} |c_i|^2(L(v_i), L(v_i))_U = \sigma_1^2|c_1|^2 + \cdots + \sigma_{k+1}|c_{k+1}|^2 \geqslant \sigma_{k+1}^2(|c_1|^2 + \cdots + |c_{k+1}|^2)$.

Consider the subspaces V_{k+1} and $\operatorname{Ker}(T)$. As $\dim \operatorname{Im}(T) = rank(T) = k$ we obtain that $\dim \operatorname{Ker}(T) = \dim V - k$ by **Theorem 5.6**. Therefore, $\dim \operatorname{Ker}(T) + \dim V_{k+1} > \dim V$. Hence by **Theorem 4.5** $\dim V_{k+1} \cap \operatorname{Ker}(T) > 0$. Therefore there exists a unit vector $v \in V_{k+1} \cap \operatorname{Ker}(T)$. We already proved that $\|L(v)\| \geqslant \sigma_{k+1}$. From other hand $\|(L - T)(v)\| = \|(L(v) - T(v)\| = \|L(v)\| \geqslant \sigma_{k+1}$. It follows that $\|L - T\|_2 \geqslant \sigma_{k+1}$ as assuming that $\|L - T\|_2 < \sigma_{k+1}$ we obtain $\|(L - T)(v)\| \leqslant \|L - T\|_2 < \sigma_{k+1}$ which contradicts to what we have already proven.

For the second assertion, consider a basis corresponding to singular value decomposition of L, i.e. the basis $\{v_1, \dots, v_n\}$ such that:

- $L(v_{r+1}) = \cdots = L(v_n) = 0$ for $r = \operatorname{rank}(L)$;

- $\|L(v_i)\| = \sigma_i$ for $i = 1 \dots r$;
- $L(v_1), \dots, L(v_r)$ are orthogonal to each other.

Define a linear map L_k in term of this basis using **Theorem 5.1** by conditions:

- $L_k(v_i) = L(v_i)$ for $i = 1 \dots k$;
- $L_k(v_i) = 0$ for $i = k+1 \dots n$.

Then $\operatorname{rank}(L_k) = k$ as $\operatorname{Im}(L_k) = \operatorname{Span}(L(v_1), \dots, L(v_k))$ and vectors $L(v_1), \dots, L(v_k)$ are linear independent. We claim that $\|L - L_k\|_2 = \sigma_{k+1}$. Since $\|(L - L_k)(v_{k+1})\| = \|L(v_{k+1})\| = \sigma_{k+1}$ then $\|L - L_k\|_2 \geqslant \sigma_{k+1}$.

It remains to check that for any unit vector $x \in V$ one has $\|(L - L_k)\|_2 \leqslant \sigma_{k+1}$. Take decomposition $x = c_1 v_1 + \dots + c_n v_n$. Since $\|x\| = 1$ and the basis is orthonormal one has $|c_1|^2 + \dots + |c_n|^2 = 1$. By construction of L_k we have $(L - L_k)(v_i) = 0$ for $i = 1 \dots k$ and $(L - L_k)(v_i) = L(v_i)$ for $i > k$. Hence, $(L - L_k)(x) = c_{k+1}L(v_{k+1}) + \dots + c_n L(v_n)$. Therefore, $\|(L - L_k)(x)\|^2 = \| \sum\limits_{i=k+1}^{n} (L(v_i), L(v_i))_U \|^2 = \sigma_{k+1}^2 |c_{k+1}|^2 + \dots + \sigma_n^2 |c_n|^2$ where we denote by simplicity $\sigma_i = 0$ for $i > r$. The theorem follow from the usual estimate

$\sigma_{k+1}^2 |c_{k+1}|^2 + \dots + \sigma_n^2 |c_n|^2 \leqslant \sigma_{k+1}^2 \left(|c_{k+1}|^2 + \dots + |c_n|^2\right) \leqslant \sigma_{k+1}^2 \left(|c_1|^2 + \dots + |c_n|^2\right).$

We prove that $\|(L - L_k)(x)\|^2 \leqslant \sigma_k^2$ for any unit vector $x \in V$. $\square$

Problem 18.7. *Consider a linear operator $L = L_A : \mathbb{R}^2 \to \mathbb{R}^2$ on a coordinate space equipped by a standard dot-product which is given by a multiplication by a matrix $\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$. Find a matrix $B \in \mathrm{M}_2(\mathbb{R})$ of rank equal to 1 such that operator norm $\|A - B\|_2$ is as minimal as possible.*

Solution. We proceed as in the proof of the **Theorem 18.6** and **Theorem 18.2**. Consider an operator L^*L and find its eigenvectors. It is given by a matrix $\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ having eigenvalues 2 and 8. The eigenvector belonging to $\sigma_1^2 = 8$ is equal to $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the eigenvectors belonging to $\sigma_2^2 = 2$ is $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Of course, we can normalize them but there is no need. We see that $L_A(v_1) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ so the length being multiplied by $\dfrac{4}{\sqrt{2}} = \sqrt{8} = \sigma_1$. And $L_A(v_2) = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and the magnification factor of the length here is equal to $\dfrac{2}{\sqrt{2}} = \sigma_2$. Take an operator $L_B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L_B(v_1) = L_A(v_1)$ and $L_B(v_2) = 0$. It means that $B \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$.

Hence $B = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$. Indeed, $A - B = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$ so the difference has operator norm which is equal to $\sqrt{2}$. $\square$

Exercise 18.2. *Prove that for a normal operator on a complex inner space with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ its singular values (including those that are equal to zero and were not called as singular values earlier) are equal to $|\lambda_1|, |\lambda_2| \ldots, |\lambda_n|$.*

Exercise 18.3. *Let $A \in \mathrm{M}_{m,n}(\mathbb{R})$ and $L_A : \mathbb{R}^n \to \mathbb{R}^m, \mathbf{x} \mapsto A \cdot \mathbf{x}$ be a corresponding linear map between Euclidean spaces. Consider singular value decomposition $A = Q_1 \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} Q_2$ where D_r is a diagonal matrix with singular values of L_A on its diagonal and $Q_1 \in \mathrm{M}_m(\mathbb{R})$, $Q_2 \in \mathrm{M}_n(\mathbb{R})$ are orthogonal matrices.*

Denote by $A^\dagger = Q_2^{-1} \begin{pmatrix} D_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} Q_1^{-1} \in \mathrm{M}_{n,m}(\mathbb{R})$.

1. *(2 points). Prove that for any $\mathbf{b} \in \mathbb{R}^m$ one of the least square solution to the linear system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}_0 = A^\dagger \mathbf{b}$;*

2. *(2 points). Prove that for any other least square solution $\widetilde{x}$ one has $\|\widetilde{x}\| \geqslant \|A^\dagger \mathbf{b}\|$.*

19 Angles in Euclidean space

Recall that an angle $\alpha \in [0, \pi]$ between two nonzero vectors u, v in the real inner space V is defined by $\cos\alpha = \dfrac{(u,v)_V}{\|u\| \cdot \|v\|}$. Definition is correct due to Cauchy-Bunyakovsky inequality.

Remark. Multiplying by positive scalars $k, \ell > 0$ does not affect the value of the angle as $\dfrac{(ku, \ell v)_V}{\|ku\| \cdot \|\ell v\|} = \dfrac{(u,v)_V}{\|u\| \cdot \|v\|}$.

How to define an angle between a non-zero vector $v \in V$ in the Euclidean s[ace and some subspace $U \subset V$ such that $U \neq 0$.

There are two definitions which are equivalent to each other. One is of geometric nature (see below) and another is an algebraic one. We take an algebraic definition as the basic one and deduce that it gives the same value as the geometric one.

Definition. The angle between v and U is equal to $\dfrac{\pi}{2}$ if $v \in U^{\perp}$ and is equal to the angle between v and orthogonal projection $\mathrm{Proj}_U(v)$ when $v \notin U^{\perp}$.

Remark. If v is a unit vector then the cosine of the angle between v and U is equal to $\|\mathrm{Proj}_U(v)\|$.

Proposition 19.1. *The angle between v and U is equal to the the minimum angle between v and $u \in U$ when $u \neq 0$ ranges the subspace U.*

Proof. If $v \in U^{\perp}$ then v is orthogonal to each vector in U so the minimal angle is equal to $\dfrac{\pi}{2}$.

So let us take $u_0 = \mathrm{Proj}_U(v)$. For any $u \in U$ consider an orthogonal decomposition $u = ku_0 + w$ where $w \in \mathrm{Span}(u_0)^{\perp} \cap U$. Then w is orthogonal to v (it is High School theorem of Three Perpendiculars) since $(w, v)_V = (w, u_0 + (v - u_0))_V = (w, u_0)_V + (w, (v - u_0))_V$ and the second summand is also equal to zero as $v - \mathrm{Proj}_U(v)$ belongs to $U^{\perp}$. Moreover, $(v, u_0)_V = (u_0 + (v - u_0), u_0)_V = (u_0, u_0)_V > 0$ so the angle is acute. When $k = 0$ then u is orthogonal to v hence the angle between v and u is not minimal. If $k < 0$ then $(u, v) = (ku_0 + w, v) = k(u_0, v) < 0$ and the angle between v and u is obtuse, so can not be minimal. Assume that $k > 0$ and let us divide u by k. So the angle remain unchanged but now we can assume $u = u_0 + w$.

One has $(u, v) = (u_0 + w, v) = (u_0, v)$. Comparing two quantities $\dfrac{(u_0, v)_V}{\|u_0\| \cdot \|v\|}$ and $\dfrac{(u, v)_V}{\|u\| \cdot \|v\|}$ we see that numerators are the same but the denominators in the second fraction is greater than in the first. Indeed, $\|u\|^2 = \|u_0\|^2 + \|w\|^2 \geqslant \|u_0\|^2$. Hence the cosine of the first of this two acute angles is greater than the cosine of the second. Since the cosine function is decreasing on $\left[0, \frac{\pi}{2}\right]$ the proposition follows.

$\square$

In the case of two arbitrary non-zero subspaces $U_1, U_2 \subset V$ in the Euclidean space on we can define in deometric way so called **minimal angle** between U_1 and U_2.

Definition. The minimal angle θ_{min} between U_1 and U_2 is the minimal angle between two non-zero vectors $u_1 \in U_1$ and $u_2 \in U_2$ where u_1 ranges U_1 and u_2 ranges U_2.

In other words, $\cos\theta_{min} = \max\limits_{u_1 \in U_1; u_2 \in U_2} \dfrac{(u_1, u_2)_V}{\|u_1\| \cdot \|u_2\|}$.

Remark. We can reformulate the definition in slightly less symmetric way: the minimal angle between U_1 and U_2 is equal to the minimal value of the angle between $v \in U_1$ and the subspace U_2 when $v \neq 0$ ranges U_1. So $\cos\theta_{min} = \max\limits_{v \in U_1; \|v\|=1} \|\mathrm{Proj}_{U_2}(v)\|$.

Proposition 19.2. *Let $P_1 = \mathrm{Proj}_{U_1}$ and $P_2 = \mathrm{Proj}_{U_2}$ are orthogonal projectors onto two subspaces in the Euclidean space V. Then $\cos\theta_{min}$ is equal $\|P_1P_2\|_2$.*

Proof. Take arbitrary unit vector $u_1 \in U_1$. Then $\cos\theta_{min} \geqslant \|P_2(u_1)\|$ as the righthand-side is equal to the angle between the unit vector u_1 and the subspace U_2.

Hence by the very definition of the operator norm $\cos\theta_{min} \geqslant \|P_2|_{U_1}\|_2$ where we consider a restricted linear map (not an operator) $P_2|_{U_1} : U_1 \to V$. We claim that the operator norm of this restriction is equal to operator norm of $P_2P_1 : V \to V$. Indeed, for any $v \in V$ one has $\|P_2P_1(v)\| \leqslant \|P_2|_{U_1}\|_2 \cdot \|P_1(v)\| \leqslant \|P_2|_{U_1}\|_2 \cdot \|v\|$, hence $\|P_2P_1\|_2 \leqslant \|P_2|_{U_1}\|_2$. But taking arbitrary $u \in U_1$ one has $\|P_2|_{U_1}(u)\| = \|P_2(P_1(u))\| \leqslant \|P_2P_1\|_2 \cdot \|u\|$, hence $\|P_2|_{U_1}\|_2 \leqslant \|P_2P_1\|_2$.

It is remained to prove that $\cos\theta_{min} \leqslant \|P_2|_{U_1}\|_2 = \|P_2P_1\|_2$. By the remark after definition we can find a unit $v \in U_1$ such that $\cos\theta_{min} = \|P_2(v)\| \leqslant \|P_2|_{U_1}\|_2$.

□

Remark. $\|P_1P_2\|_2$ and $\|P_2P_1\|_2$ coincide with each other. Indeed, $(P_1P_2)^* = P_2^*P_1^* = P_2P_1$ as orthogonal projectotrs are selfadjoint. But for any linear map L one has $\|L\|_2 = \|L^*\|_2$ because the operator norm is equal to the greatest singular value and the sets of singular values of L and L^* are the same.

Remark. Geometrically the number $\|P_2|_{U_1}\|_2 = \|P_2P_1\|_2$ is equal to the greatest semi-axes of the ellipsoid obtained from the unit ball in U_1 by orthogonal projection on U_2.

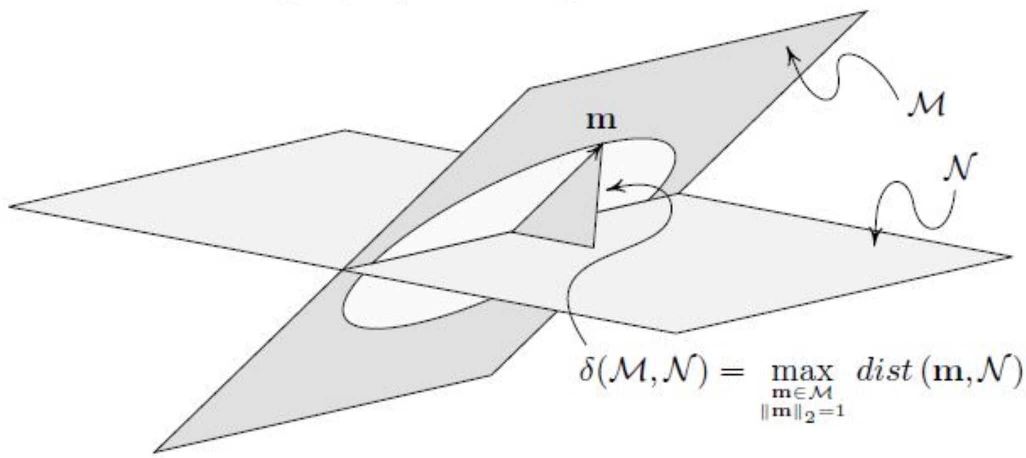
Exercise 19.1. *Suppose that $V = U_1 \oplus U_2$. Prove that minimal angle between U_1 and U_2 is equal to the minimal angle between $U_1^\perp$ and $U_2^\perp$.*

When two subspases have non-zero intersection then the minimal angle between this subspaces is equal to 0. For example, minimal angle between any two planes in three dimensional planes is always zero. What is called by the angle between two planes in solid space is, in fact, the maximum angle which have not yet defined. The mutual arrangement of two subspace in Euclidean spaces is defined not by one but by series of numbers which are usually called "principal angles".

Definition. $\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_k$ are called principal angles between U_1 and U_2 if $\cos\theta_1 \geqslant \cos\theta_2 \geqslant \cdots \geqslant \cos\theta_k$ is a decreasing sequence of singular values of P_1P_2.

By the singular value decompositions it means that there exist an orthonormal basis $v_1, v_2 \ldots, v_k$ in U_1 such that the vectors $P_2(v_1), \ldots, P_2(v_k)$ are orthogonal to each other and $\|P_2(v_i)\| = \cos\theta_i$ for $i = 1 \ldots k$.

$\int_{-b}^{b} f(x)\,dx \quad b \to +\infty$



For the case of two planes in $\mathbb{R}^3$ there are two singular values: one is equal to 1 and another is equal to $\cos\alpha$ where α is the "usual" angle between these planes.

Let us consider the case of two planes in $U_1, U_2 \subset \mathbb{R}^4$. Then for $L = P_2P_1$ one has $L^*L = P_1P_2P_1$ and $LL^* = P_2P_1P_2$ have at least two zero eigenvalues and two other common eigenvalues $\sigma_1^2 \geqslant \sigma_2^2$ (which are allowd to be zero).

Proposition 19.3. *There exist an orthonormal basis $\{v_1, v_2\}$ in U_1 and an orthonormal basis $\{u_1, u_2\}$ in U_2 such that:*

$$P(v_1) = \sigma_1 u_1; \quad P(v_2) = \sigma_2 u_2; \tag{26}$$

$$P(u_1) = \sigma_1 v_1; \quad P(u_2) = \sigma_2 v_2. \tag{27}$$

Proof. Since $U_1^\perp \subset V_0(L^*L)$ we can choose orthonormal basis v_1, v_2, v_3, v_4 consisting of eigenvectors of L^*L such that $v_3, v_4 \in U^\perp$. Then by orthogonality $v_1, v_2 \in U_1$ and $P_1P_2P_1(v_1) = \sigma_2(v_1)$ and $P_1P_2P_1(v_2) = \sigma_2^2 v_2$. In the basis $\{u_1, u_2, u_3, u_3\}$ of U_2 corresponding to singular value decomposition we can always take $u_1, u_2 \in U_2$. Indeed, when $\operatorname{Im} P_2P_1 = \mathrm{P}_2 = U_2$ than first two vectors of the basis have to belong to U_2 by the construction in the **Theorem 18.2**. If $\operatorname{Im}(P_2P_1) \subsetneq U_2$ we can take additional basis vectors in $\operatorname{Im}(L)^\perp$ choosing firstly vectors in U_2, namely given a basis in the first summand of orthogonal decomposition $U_2 = \operatorname{Im}(L) \perp \big(\operatorname{Im}(L)^\perp \cap U_2\big)$ we add the basis of the second summand and at the end add u_3, u_4 constituting the basis of $U_2^\perp$.

Anyway, we have orthonormal bases $\{v_1, v_2\}$ in U_1 and $\{u_1, u_2\}$ in U_2 such that $P_2(v_1) = L(v_1) = \sigma_1 u_1$ and $P_2(v_2) = L(v_2) = \sigma_2 u_2$ (where σ_1 and σ_2 are allowed to be equal to zero). If $\sigma_1 \neq 0$ then

$$P_1(u_1) = \frac{1}{\sigma_1}P_1(\sigma_1 u_1) = \frac{1}{\sigma_1}P_1P_2(v_1) = \frac{1}{\sigma_1}P_1P_2P_1(v_1) = \frac{1}{\sigma_1}(\sigma_1^2 v_1) = \sigma_1 v_1$$

and the same holds for u_2. If $\sigma_2 = 0$ then $u_2 \in \mathrm{Im}(L)^\perp \cap U_2$ and consequently $u_2 \in \mathrm{Ker}(L^*) = \mathrm{Im}(L)^\perp$. Hence, $0 = L^*(u_2) = P_1P_2(u_2) = P_1(u_2)$. Anyway, $P_2(u_2) = \sigma_2 v_2$ and $P_2(u_1) = \sigma_1 v_1$ in both cases when σ_i is non-zero and when σ_i is equal to zero.

$\square$

Definition. $\theta_{min}, \theta_{max} \in \left[0; \frac{\pi}{2}\right]$ such that $\cos\theta_{max} = \sigma_1$ and $\cos\theta_{max} = \sigma_2$ are called minimal and maximum angle between two planes in $\mathbb{R}^4$.

Almost the same definition holds for two planes in the space of any dimension including the case $\dim V = 3$. In the last case $\theta_{min} = 0$ and $v_1 = u_1$ in the **Proposition 19.3**. If $\theta_{min} = \theta_{max} = \dfrac{\pi}{2}$ then two planes U_1 and U_2 are orthogonal to each other.

Problem 19.4. *Let U_1, U_2 are two planes in $\mathbb{R}^4$ and θ_1, θ_2 are the minimal ans maximum angles between this planes. Prove that there exists an isometry $T \in \mathrm{M}_4(\mathbb{R})$ having a followong canonical form relative to some orthonormal basis:*

$$\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & \cos\theta_2 & -\sin\theta_2 \\ 0 & 0 & \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

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