Problems

1. A mapping $f : A \to B$ is a surjection iff $\exists g : B \to A$, such that $f \circ g = 1_B$ (Hint: Axiom of Choice).

2. The converse relation of $R \subseteq X \times Y$ is the binary relation $R^{-1} \subseteq Y \times X$ defined by $(y, x) \in R^{-1}$ iff $(x, y) \in R$.

   Let $R \subseteq X \times X$ be a reflexive and transitive relation. Show that $R \cap R^{-1}$ is an equivalence relation.

3. Find the number of equivalence relations on $\{1, 2, 3, 4\}$ that contains $\{(1, 2), (3, 4)\}$.

4. Let T be a relation defined on $\mathbb{R}$ such that: $(x, y) \in T$ iff $x^2 + 2x = y^2 + 2y$. Show that T is an equivalence relation, and find the equivalence classes $[1]$ and $[0]$.

5. Find the equivalence relation (as a set of ordered pairs) on $\{a, b, c, d, e, f\}$, whose equivalence classes are $\{b, c\}, \{d\}, \{a, e, f\}$.

6. Prove that there exists an order-preserving bijection between $(\mathbb{Z}, <)$ and $(\mathbb{Z}, >)$.

7. Prove that there is NO order-preserving bijection between

   a.) $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$.

   b.) $(\mathbb{R}, <)$ and $(\mathbb{R} \setminus \{0\}, <)$.

8. If a longest anti-chain contains a subset of size m then it will contain all subsets of size m.

9. Consider the $3-$ary predicates S and P on the set of natural numbers: $S(x, y, z) \Leftrightarrow x + y = z$, and $P(x, y, z) \Leftrightarrow x \cdot y = z$

   (I) In a first-order language with predicate symbols S, P write formulas with one free variable a, true if and only if:

   (a) $a = 1$

   (b) $a = 2$

   (c) a is an even number

   (d) a is an odd number

   (Here is an example for $a = 0 : a = 0 \Leftrightarrow \forall x S(a, x, x)$)

(II) In a first-order language with predicate symbols S, P write formulas with two free variable a, b, true if and only if:

(a) $a = b$

(b) $a \leq b$

(c) a divides b

10. (i) Prove that the following formula is satisfiable :

$$\forall x \exists y P(x,y) \wedge \forall x \forall y (P(x,y) \rightarrow \neg P(y,x)) \wedge \forall x \forall y \forall z (P(x,y) \rightarrow (P(y,z) \rightarrow P(x,z)))$$

(Hint: consider $\mathfrak{M}$ with $D = \mathbb{N}$).

(ii) Prove that this formula is nonsatisfiable in a finite model.

11. Show that $\approx$ is an equivalence relation on $\mathcal{R}$ (hyperreal field). Moreover, prove that if $x \approx y$ and $u \approx v$ implies $x \pm u \approx y \pm v$.

12. Show that $\mathcal{R}$ is not a complete linear order, i.e. it has a bounded subset, namely $\mathbb{R}$, for which sup does not exist.

13. Let $\{S_n \mid n \in \mathbb{N}\}$ be a collection of finite subsets of $\mathbb{N}$ such that for each finite subset $F \subset \mathbb{N}$, there exists a subset $A_F \subseteq \mathbb{N}$ with $|A_F \cap S_n| = 1$ for all $n \in F$. Prove that there exists a subset $A \subseteq \mathbb{N}$ such that $|A \cap S_n| = 1$ for all $n \in \mathbb{N}$.

14. Suppose that $R \subset \mathbb{N} \times \mathbb{N}$ is a relation satisfying the following conditions:

(a) for all $a \in \mathbb{N}$ the set $\{b \in \mathbb{N} \mid (a,b) \in R\}$ is non-empty and finite.

(b) For every finite subset $A_0 \subseteq \mathbb{N}$, there exists an injective function $f_0 : A_0 \rightarrow \mathbb{N}$, such that $(a, f_0(a)) \in R$ for all $a \in A_0$.

Use the Compactness Theorem to prove that there exists an injective function $f : \mathbb{N} \rightarrow \mathbb{N}$, such that $(a, f(a)) \in R$ for all $a \in \mathbb{N}$.

15. Prove that the set of prime numbers is computable.

16. Prove that the following functions are primitive recursive:

(a) The factorial function $fac(n) = n!$.

(b) $max(a, b)$ and $min(a, b)$.

1. A mapping $f : A \to B$ is a surjection iff $\exists g : B \to A$, such that $f \circ g = 1_B$ (Hint: Axiom of Choice).

Pf: "$\Rightarrow$" f is surjection. $\forall b \in B$. $\exists a \in A$. s.t. $f(a) = b$

thus we can let $g : B \to A$. $g(b) = a$.

s.t. $\forall b \in B$. $(f \circ g)(b) = f(g(b)) = f(a) = b$.

"$\Leftarrow$" Assume the converse. if f is not a surjection. $\exists b_0 \in B$. s.t. not exist. $a \in A$. $f(a) = b_0$.

By condition. $f \circ g(b_0) = b_0 \Rightarrow f(g(b_0)) = b_0$. we have $g(b_0) \in A$. Contradictory.

2. The converse relation of $R \subseteq X \times Y$ is the binary relation $R^{-1} \subseteq Y \times X$ defined by $(y, x) \in R^{-1}$ iff $(x, y) \in R$.

Let $R \subseteq X \times X$ be a reflexive and transitive relation. Show that $R \cap R^{-1}$ is an equivalence relation.

Pf: since $R \cap R^{-1} \sqsubset R$. the reflexive and transitive holds trivially.

now show the symmetry.

$\forall (x_1, x_2) \in R \cap R^{-1}$.

since $(x_1, x_2) \in R$. then $(x_2, x_1) \in R^{-1} \sqsubset R \cap R^{-1}$ $\square$.

3. Find the number of equivalence relations on $\{1, 2, 3, 4\}$ that contains $\{(1, 2), (3, 4)\}$.

Pf: 1) by symmetry, $(2,1)$, $(3,4)$ must contain.

by reflexivity. $(1,1)$ $(2,2)$, $(3,3)$ $(4,4)$ contains.

t - transitivity.

s - symmetry.

2) If contains $(1,3)$. then $(1,3).(3,4) \overset{t}{\Rightarrow} (1,4) \overset{s}{\Rightarrow} (4,1)$

$(1,3) \overset{s}{\Rightarrow} (3,1) \Rightarrow (3,1)\ (1,2) \overset{t}{\Rightarrow} (3,2) \overset{s}{\Rightarrow} (2,3)$

thus all pair of $(*, *)$. $* \in \{1,2,3,4\}$ are contain.

i.e. 2 relation sets can contain $\{(1,2), (3,4)\}$

4. Let T be a relation defined on $\mathbb{R}$ such that: $(x, y) \in T$ iff $x^2 + 2x = y^2 + 2y$. Show that T is an equivalence relation, and find the equivalence classes $[1]$ and $[0]$.

Pf: reflexivity: $x^2+2x \equiv x^2+2x$ for any $x \in \mathbb{R}$.

symmetry : if $x^2+2x = y^2+2y$ then $y^2+2y = x^2+2x$.

transitivity: if $\begin{matrix} x^2+2x = y^2+2y \\ y^2+2y = z^2+2z \end{matrix}$ $\Rightarrow$ $x^2+2x = z^2+2z \Leftrightarrow (x, z) \in \mathbb{R}$.

$x^2+2x = 3 \Rightarrow x = 1$ or -3 $\qquad [1] = \{1, -3\}$

$x^2+2x = 0 \Rightarrow x = 0$ or -2. $\qquad [0] = \{0, -2\}$

5. Find the equivalence relation (as a set of ordered pairs) on $\{a, b, c, d, e, f\}$, whose equivalence classes are $\{b, c\}, \{d\}, \{a, e, f\}$.

S: For $\{b, c\}$, we have $(b,c).\ (c,b).\ (b,b),\ (c,c)$

For $\{d\}$. (d,d) contain

For $\{a,e,f\}$. reflexive $(a,a).\ (e,e).\ (f,f)$

symmetry $(a,e).\ (e,a).\ (a,f),\ (f,a).\ (e,f).\ (f,e)$

and transitivity.

In conclusion. equivalence relation have

$\{(b,c).\ (c,b).\ (b,b),\ (c,c),\ (d,d),\ (a,a).\ (e,e).\ (f,f),\ (a,e).\ (e,a).\ (a,f),\ (f,a).\ (e,f).\ (f,e)\}$

6. Prove that there exists an order-preserving bijection between $(\mathbb{Z}, <)$ and $(\mathbb{Z}, >)$.

Pf: follow the irreflexive. asymmetric. transitive, we can prove that "<" and ">" are strictly partial order.

$\forall (x,y) \in (\mathbb{Z}, <)$. i.e. $x<y$.

$\Leftrightarrow y > x. \Leftrightarrow (y,x) \in (\mathbb{Z}, >)$

For any $(x,y) \in (\mathbb{Z}, <)$. (y,x) is unique in $(\mathbb{Z}, >)$.

i.e. $f: (\mathbb{Z}, <) \to (\mathbb{Z}, >)$. $f(x,y) = (y,x)$ is injective.

$\forall (y,x) \in (\mathbb{Z}, >). \Leftrightarrow y>x \Leftrightarrow x<y \Leftrightarrow (x,y) \in (\mathbb{Z}, <)$.

f is surjective.

In conclusion. f is bijection.

7. Prove that there is NO order-preserving bijection between

a.) $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$.

b.) $(\mathbb{R}, <)$ and $(\mathbb{R} \setminus \{0\}, <)$.

Pf: a) Assume there is a bijection. $f: (\mathbb{N}, <) \to (\mathbb{Z}, <)$

Let $k_0 \in \mathbb{N}$. $f(k_0) = 0$.

For any $n < k_0$. $f(n) < 0$. actually we only have $k_0 - 1$ number of n s.t. $f(n) < 0$.

Since for any $n > k_0$. $f(n) > 0$.

we have infinite number of element in $(\mathbb{Z}, <)$ which less than 0.

but our inverse-projection only have $k_0 - 1$ numbers. it's impossible to construct bijection between infinite and finite set.

b) Assume there is a bijection

$f(0) = k \in \mathbb{R} \setminus \{0\}$. by definition. $k \in (\mathbb{R}, <)$ also. w.l.g. let $k < 0$.

then $f(k) < f(0) \Rightarrow f(k) < k < 0 \Rightarrow f(k) < 0$ which is contradictory with $f(k) = 0$.

thus the bijection is not existent.

8. If a longest anti-chain contains a subset of size m then it will contain all subsets of size m.

Pf: Assume there are n elements in the whole set

By the theorem, the longest antichange have size $C_n^{[\frac{n}{2}]}$.

we know that $[\frac{n}{2}] = m$ holds. (if n is odd. $m = [\frac{n}{2}] + 1$ also possible)

the family $\mathcal{F}_m$, which contains all subsets of size m contains C_n^m numbers of subsets

since $C_n^m = C_n^{[\frac{n}{2}]}$. $\mathcal{F}_m$ satisfied the condition of antichain.

Sperner's theorem asserts that there is no longer antichain.

Thus. the $\mathcal{F}_m$ which contains all subsets of size m is exactly the longest anti-chain. $\square$

9. Consider the 3−ary predicates S and P on the set of natural numbers: $S(x, y, z) \Leftrightarrow x + y = z$, and $P(x, y, z) \Leftrightarrow x \cdot y = z$

(I) In a first-order language with predicate symbols S, P write formulas with one free variable a, true if and only if:

(a) $a = 1$

(b) $a = 2$

(c) a is an even number

(d) a is an odd number

(Here is an example for $a = 0$: $a = 0 \Leftrightarrow \forall x S(a, x, x)$)

(a). $a=1 \Leftrightarrow \forall x\, P(a, x, x)$

(b). $a=2 \Leftrightarrow \forall x \quad S(x, x, ax)$

(c) a is even $\Leftrightarrow \exists x \in \mathbb{Z},\ S(x, x, a)$

(b) a is odd $\Leftrightarrow \exists x \in \mathbb{Z},\ S(x, x+1, a)$

(II) In a first-order language with predicate symbols S, P write formulas with two free variable a, b, true if and only if:

(a) $a = b$

(b) $a \leq b$

(c) a divides b

(a) $a = b \Leftrightarrow \exists x=0\ S(a, x, b)$

(b) $a \leq b \Leftrightarrow \exists x \geq 0\ S(a, x, b)$

(c) $a | b \Leftrightarrow \exists x \in \mathbb{Z}\ P(a, x, b)$

10. (i) Prove that the following formula is satisfiable :

$\forall x \exists y P(x, y) \wedge \forall x \forall y (P(x, y) \to \neg P(y, x)) \wedge \forall x \forall y \forall z (P(x, y) \to (P(y, z) \to P(x, z)))$

(Hint: consider $\mathfrak{M}$ with $D = \mathbb{N}$).

(ii) Prove that this formula is nonsatisfiable in a finite model.

(i)

$\forall x \forall y (P(x,y) \to \neg P(y,x))$ implies the asymmetric.

$\forall x \forall y \forall z (P(x,y) \to (P(y,z) \to P(x,z)))$ implies the transitivity.

thus we can interpret $P(x,y)$ as $x<y$.

i.e. consider $\mathfrak{M} = (\mathbb{N}, <)$

we can check the formula is true.

(ii) if the field is finite. let it be D_0.

then we can also interpret $P(x,y)$ as $x<y$. or other strictly partial order.

then extract the comparable elements under P, which is also a finite number

we can find a maximal element of this chain.

let it be x. we can't find $y \in D_0$. s.t. $x<y$. thus the formula is false.

11. Show that $\approx$ is an equivalence relation on $\mathcal{R}$ (hyperreal field). Moreover, prove that if $x \approx y$ and $u \approx v$ implies $x \pm u \approx y \pm v$.

Pf: $x \approx y \Leftrightarrow x - y = \delta_1$ $u \approx v \Leftrightarrow u - v = \delta_2$. $|\delta_1|, |\delta_2| < \frac{1}{m}$ for any $m \in \mathbb{N}$.

$(x+u) - (y+v) = (x-y) + (u-v) = \delta_1 + \delta_2$.

since δ_1, δ_2 are infinitesimal, the sum is also infinitesimal.

Indeed. if $\exists M \in \mathbb{N}$. s.t. $|\delta_1 + \delta_2| \geq \frac{1}{M} \Rightarrow |\delta_1| + |\delta_2| \geq \frac{1}{M} \Rightarrow |\delta_1| \geq \frac{1}{M} - |\delta_2|$

since $|\delta_2| < \frac{1}{m}$ of any $m \in \mathbb{N}$. $|\delta_2| < \frac{1}{2M}$. $|\delta_1| \geq \frac{1}{M} - |\delta_2| > \frac{1}{M} - \frac{1}{2M} = \frac{1}{2M}$, but $2M \in \mathbb{N}$.

thus we have $x+u \approx y+v$.

$(x-u) - (y-v) = (x-y) - (u-v) = \delta_1 - \delta_2$ $\quad |\delta_1 - \delta_2| \leq |\delta_1| + |\delta_2|$, which is infitesimal

thus. $x - u \approx y - v$ $\square$

12. Show that $\mathcal{R}$ is not a complete linear order, i.e. it has a bounded subset, namely $\mathbb{R}$, for which sup does not exist.

Pf. denote the real number set by $\mathbb{R}$.

bounded: $\forall x \in \mathbb{R}$, $-\infty < x < +\infty$. , $+\infty, -\infty \in \mathcal{R}$ (hyperreal field)

If $\mathbb{R}$ has a supremum, let it be m. we can always find $x \in \mathbb{R}$. s.t. $x > m$

Which causes a contradictory.

15. Prove that the set of prime numbers is computable.

Pf: First construct the characteristic function.

Let P be the prime numbers set.

$$\chi_P(n) = \begin{cases} 1 . & \text{if } n \in P \\ 0 . & \text{if } n \notin P \end{cases}$$

Now shows it's computable.

16. Prove that the following functions are primitive recursive:

(a) The factorial function $fac(n) = n!$.

(b) $max(a, b)$ and $min(a, b)$.

Pf. (a) define $f(0) = 1$.

$f(x) = x\, f(x-1)$, which is recursive.

(b) define a characteristic function:

$$P(x) = \begin{cases} 1, & a \leq b \\ 0, & a > b \end{cases}$$, $P(x)$ is recursive.

$max(a,b) = P(x)\, b + (1 - P(x))\, a$.

$min(a,b) = P(x)\, a + (1 - P(x))\, b$.

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