Metric spaces. Differential calculus of functions of several real variables.

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1 Examples of problems with solutions

1.1 Limit and iterated limit of a function.

Let $D_1, D_2 \subset \mathbb{R}$, a and b to be limit points of D_1 and D_2 respectively, $(D_1 \setminus \{a\}) \times (D_2 \setminus \{b\}) \subset D$, $f : D \to \mathbb{R}$ or $\mathbb{C}$.

1. Assume that for every $x \in D_1 \setminus \{a\}$ the limit $\varphi(x) = \lim\limits_{y\to b} f(x,y)$ exists. Then the limit of function φ at a is called an **iterated limit** of function f at (a,b) and

$$\lim_{x\to a} \varphi(x) = \lim_{x\to a} \lim_{y\to b} f(x,y).$$

2. Assume that for every $y \in D_1 \setminus \{b\}$ the limit $\psi(x) = \lim\limits_{x\to a} f(x,y)$ exists. Then the limit of function ψ at a is called an **iterated limit** of function f at (a,b) and

$$\lim_{y\to b} \psi(y) = \lim_{y\to b} \lim_{x\to a} f(x,y).$$

3. Recall that A is a limit of a function f at (a,b), if

$$\forall V_A\ \exists V_a, V_b : (x,y) \in (V_a \times V_b) \setminus \{(a,b)\} \Rightarrow f(x,y) \in V_A,$$

$$A = \lim_{\substack{x\to a \\ y\to b}} f(x,y) = \lim_{(x,y)\to(a,b)} f(x,y).$$

Theorem 1.1. *Let $D_1, D_2 \subset \mathbb{R}$, a and b are limit points D_1 and D_2 respectively, $(D_1 \setminus \{a\}) \times (D_2 \setminus \{b\}) \subset D$, $f : D \to \mathbb{R}$ or $\mathbb{C}$. Assume that*

- *There exists finite or infinite limit $A = \lim\limits_{\substack{x\to a\\ y\to b}} f(x,y)$.*
- $\forall x \in D_1 \setminus \{a\}$ $\exists \varphi(x) = \lim\limits_{y\to b} f(x,y) < \infty$.

Then the iterated limit $\lim\limits_{x\to a} \varphi(x)$ exists and is equal to A.

Proof. We will prove the theorem in cases when A is finite. Let $\varepsilon > 0$ then there exist neighborhoods V_a and V_b such that

$$|f(x,y) - A| < \varepsilon$$

for every $x \in \dot{V}_a \cap D_1$ and $y \in \dot{V}_b \cap D_2$. Considering limits at b we see that

$$|\varphi(x) - A| = \lim_{y\to b} |f(x,y) - A| \leq \varepsilon$$

for every $x \in \dot{V}_a \cap D_1$. Consequently, $\lim\limits_{x\to a} \varphi(x) = A$. □

Problem 1. Let $f(x,y) = \frac{x-y}{x+y}$. Prove that

$$\lim_{x\to 0}\left(\lim_{y\to 0} f(x,y)\right) = 1, \quad \lim_{y\to 0}\left(\lim_{x\to 0} f(x,y)\right) = -1,$$

While

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$$

doesnt exist.

Solution. First,

$$\lim_{x\to 0}\left(\lim_{y\to 0}\frac{x-y}{x+y}\right)=\lim_{x\to 0}\frac{x}{x}=1,\quad \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x-y}{x+y}\right)=\lim_{y\to 0}\frac{-y}{y}=-1.$$

Since sequences $(x_n,y_n)=\left(\frac{1}{n},\frac{1}{n}\right),(x_n',y_n')=\left(\frac{2}{n},\frac{1}{n}\right)$ converge to $(0,0)$ as $n\to\infty$, and corresponding sequences of values of the function are different,

$$f(x_n,y_n)=0\to 0,\quad f(x_n',y_n')=\frac{\frac{1}{n}}{\frac{3}{n}}\to\frac{1}{3}$$

as $n\to\infty$, then the limit

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$$

doesn't exist. □

Problem 2. Prove that for a function $f(x,y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$

$$\lim_{x\to 0}\left(\lim_{y\to 0}f(x,y)\right)=\lim_{y\to 0}\left(\lim_{x\to 0}f(x,y)\right)=0,$$

while $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ doesn't exist.

Solution. The equality of iterated limits follows from identities

$$\lim_{y\to 0}f(x,y)=0,\quad \lim_{x\to 0}f(x,y)=0.$$

The limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ doesn't exist since

$$f\left(\frac{1}{n},\frac{1}{n}\right)=\frac{\frac{1}{n^4}}{\frac{1}{n^4}}\to 1,\quad f\left(\left(\frac{1}{n},-\frac{1}{n}\right)\right)=\frac{\frac{1}{n^4}}{\frac{1}{n^4}+\frac{4}{n^2}}\to 0$$

Problem 3. Find the limit
$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^2}{x^2+y^2}$$
or prove that it doesn't exist.

Proof. Notice that
$$2xy \le x^2+y^2$$
and
$$f(x,y) = \frac{x^2y^2}{x^2+y^2} \le \frac{x^2+y^2}{4}.$$
Consequently,
$$f(x,y)\to 0,\ (x,y)\to(0,0).$$
$\square$

Definition 1.2. *We say that*
$$A = \lim_{\substack{x\to+\infty\\ y\to+\infty}} f(x,y)$$
if for every $\varepsilon>0$ there exists $L>0$ such that $|f(x,y)-A|<\varepsilon$ when $x,y>L$.

We say that
$$A = \lim_{\substack{x\to+\infty\\ y\to b}} f(x,y)$$
if for every $\varepsilon>0$ there exists $L>0$ and $\delta>0$ such that $|f(x,y)-A|<\varepsilon$ when $x>L$ and $|y-b|<\delta$.

Problem 4. Find the limit

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x-y}{x^2+xy+y^2}$$

or prove that it doesn't exist

Solution. Notice that

$$0 \leqslant \left|\frac{x-y}{x^2+xy+y^2}\right| \leqslant \frac{|x|+|y|}{|xy|} \leqslant \frac{1}{|y|}+\frac{1}{|x|}.$$

Consequently,

$$0 \leqslant \lim_{\substack{x\to\infty\\ y\to\infty}} \left|\frac{x+y}{x^2-xy+y^2}\right| \leqslant \lim_{x\to\infty y\to\infty} \left(\frac{1}{|x|}+\frac{1}{|y|}\right) = 0$$

and $\lim_{\substack{x\to+\infty\\ y\to+\infty}} \frac{x-y}{x^2+xy+y^2} = 0.$

Problem 5. Find the limit

$$\lim_{\substack{x\to0\\ y\to0}} (x^2+y^2)^{x^2y^2}$$

or prove that it doesn't exist

Solution. Notice that $x^2y^2 \le \frac{1}{4}(x^2+y^2)^2$ and, consequently,

$$(x^2+y^2)^{(x^2+y^2)^2/4} \le (x^2+y^2)^{x^2y^2} \le 1$$

when $x^2+y^2 \le 1$. Also letting $t = x^2+y^2$ we see that

$$\lim_{\substack{x\to0\\ y\to0}} (x^2+y^2)^{(x^2+y^2)^2/4} = \lim_{t\to0+} (t^2)^{\frac{t^2}{4}} = \lim_{t\to0+} e^{\frac{t^2}{2}\ln t} = 1.$$

Hence,

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)^{x^2y^2} = 1.$$

Problem 6. Find the limit

$$\lim_{\substack{x\to\infty\\ y\to 2}} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$$

or prove that it doesn't exist

Solution. The continuity of exponent and logarithm implies that

$$\lim_{\substack{x\to\infty\\ y\to 2}} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}} = \lim_{\substack{x\to\infty\\ y\to 2}} \exp\left\{\frac{1}{1+\frac{y}{x}} \ln\left(1+\frac{1}{x}\right)^x\right\} = e.$$

Problem 7. Let

$$f(x,y) = x\sin\frac{1}{y} + y\sin\frac{1}{x}.$$

Check the existence of iterated limits $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)$, $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)$, and the double limit $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$.

Solution. First, notice that

$$|f(x,y)| \le |x| + |y|$$

and, consequently,

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0.$$

Iterated limits do not exist since limits $\lim\limits_{y\to 0} x\sin\frac{1}{y}$ and $\lim\limits_{x\to 0} y\sin\frac{1}{x}$ do not exist.

1.2 Calculation of partial derivatives

Problem 1. Calculate derivatives of the first and the second order of function u

1. $u = xy + \frac{x}{y}$;
2. $u = x^y$;
3. $u = x^{y/z}$;

Solution. 1. $u'_x = y + \frac{1}{y}$; $u'_y = x - \frac{x}{y^2}$;

$$u''_{xx} = 0; \quad u''_{xy} = 1 - \frac{1}{y^2}; \quad u'_{yy} = \frac{2x}{y^3}.$$

2. $u = e^{y \ln x}$ and

$$u'_x = yx^{y-1}; \quad u'_y = x^y \ln x;$$

$$u''_{xx} = y(y-1)x^{y-2}; \quad u''_{xy} = x^{y-1} + x^y \ln^2 x; \quad u''_{yy} = x^y (\ln x)^2.$$

3. $u = e^{\frac{y}{z} \ln x}$ and

$$u'_x = \frac{y}{z} x^{y/z-1} = \frac{y}{xz} u; \quad u'_y = \frac{\ln x}{z} x^{y/z} = \frac{\ln x}{z} u; \quad u'_z = -\frac{y \ln x}{z^2} x^{y/z} = -\frac{y \ln x}{z^2} u$$

$$u''_{xx} = \frac{y}{z}\left(\frac{y}{z} - 1\right) x^{y/z-2}; \quad u''_{yy} = \left(\frac{\ln x}{z}\right)^2 x^{y/z};$$

$$u''_{zz} = \frac{2y \ln x}{z^3} x^{y/z} + \left(\frac{y}{z^2}\right)^2 x^{y/z}; \quad u''_{xy} = \frac{1}{z} x^{y/z-1} + \frac{y \ln x}{z^2} x^{y/z-1};$$

$$u''_{xz} = -\frac{y}{z^2} x^{y/z-1} - \frac{y^2 \ln x}{z^3} x^{y/z-1}; \quad u''_{yz} = -\frac{\ln x}{z^2} x^{y/z} - \frac{y \ln^2 x}{z^2} x^{y/z-1}.$$

Definition 1.3. *Assume that $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is differential at point $x = (x_1, \dots, x_n)$. The differential of function f at the point x is the linear operator $df(x) : \mathbb{R}^n \to \mathbb{R}$ defined by the formula*

$$df(x; dx) = df(x_1, \dots, x_n; dx_1, \dots, dx_n) = \sum_{k=1}^{n} f'_{x_k}(x) dx_k.$$

Notice that symbols dx_k have no relation with coordinates of the point x and have to be understood as unified symbol. If you have difficulties with this notation then you can use notation $h_k = dx_k$. However, the introduced notation is classic and widely used.

Definition 1.4. *Assume that $f : D \subset \mathbb{R}^n \to \mathbb{R}$ is $m-$times differentiable at point $x = (x_1, \dots, x_n)$. The differential of order m of function f at the point x is the operator $d^m f(x) : \mathbb{R}^n \to \mathbb{R}$ defined by the formula*

$$d^m f(x; dx) = \sum_{|k|=m} \frac{m!}{k!} \frac{\partial^m f}{\partial x^k} (dx)^k.$$

Here the summation is considered over all multiindexes $k = (k_1, \dots, k_n) \in (\mathbb{N} \cup \{0\})^n$ with length $|k| = k_1 + \dots + k_n = m$, $k! = k_1! \dots k_n!$, $dx = (dx_1, \dots, dx_n)$ and $(dx)^k = dx_1^k \dots dx_n^{k_n}$. Notice that

$$d^m f(x; dx) = d(d^{m-1} f(\cdot; dx))(x; dx).$$

Remark 1.5. *The second order differential can be expressed in the following form*

$$d^2 f = \sum_{k,j=1}^{n} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k dx_j = \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2} dx_k^2 + 2 \sum_{\substack{i,k=1 \\ i<k}}^{n} \frac{\partial^2 u}{\partial x_i \partial x_k} dx_i dx_k.$$

Remark 1.6. *The differential of order m of the function of two variables can be expressed in the following form*

$$d^m f = \sum_{k=1}^{m} C_m^k \frac{\partial^m f}{\partial x^{m-k} \partial y^k} dx^{m-k} dy^k,$$

where

$$C_m^k = \binom{m}{k} = \frac{m!}{k!(m-k)!}.$$

Problem 2. Find differential of order m of function u

1. $u = x^3 + y^3 - 3xy(x-y), \; m = 3;$
2. $u = \ln(x+y), \; m = 10;$
3. $u = \cos x \cosh y, \; m = 6.$

Solution.1. $u = x^3 + y^3 - 3x^2y + 3xy^2$ is polynomial and

$$u'''_{xxx} = 6; \; u'''_{x^2y} = -6; \; u'''_{xy^2} = 6; \; u'''_{y^3} = 6.$$

Consequently,

$$d^3u = 6dx^3 - \frac{3!}{1!2!} 6dx^2dy + \frac{3!}{1!2!} 6dxdy^2 + 6dy^3 = 6dx^3 - 18dx^2dy + 18dxdy^2 + 6dy^3.$$

2. First, notice that $u'_x = u'_y = \frac{1}{x+y}$ and

$$\frac{\partial^{10} u}{\partial x^k \partial y^{10-k}} = -\frac{9!}{(x+y)^9}, \; k = 0, \ldots, 10.$$

Consequently,

$$d^{10}u = -\sum_{k=0}^{10} \frac{10!}{k!(10-k)!} \frac{9!}{(x+y)^9} dx^k dy^{10-k} = -\frac{9!}{(x+y)^9}(dx+dy)^{10}.$$

3.

$$\begin{aligned} d^6u = &-\cos(x)\cosh(y)dx^6 - \frac{6!}{5!1!}\sin(x)\sinh(y)dx^5dy + \\ &\frac{6!}{4!2!}\cos(x)\cosh(y)dx^4dy^2 + \frac{6!}{3!3!}\sin(x)\sinh(y)dx^3dy^3 - \\ &\frac{6!}{2!4!}\cos(x)\cosh(y)dx^2dy^4 - \frac{6!}{1!5!}\sin(x)\sinh(y)dxdy^5 + \\ &\cos(x)\cosh(y)dy^6 = -\cos(x)\cosh(y)dx^6 - 6\sin(x)\sinh(y)dx^5dy + \\ &15\cos(x)\cosh(y)dx^4dy^2 + 10\sin(x)\sinh(y)dx^3dy^3 - \\ &16\cos(x)\cosh(y)dx^2dy^4 - 6dxdy^5 + \cos(x)\cosh(y)dy^6 \end{aligned}$$

1.3 Derivative of a composition. Chain rule.

Remark 1.7. *Assume that $w = f(x, y, z)$ is differentiable $x = \varphi(u, v), y = \psi(u, v)$, $z = \chi(u, v)$, functions φ, ψ, χ are differentiable. Then*

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}.$$

In general, the rule of differentiation is the following. Assume that $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \operatorname{int} D$, $f(x) \in \operatorname{int} E$ and $g = (g_1, \ldots, g_l) : E \subset \mathbb{R}^m \to \mathbb{R}$ is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and

$$D_k(g \circ f)(x) = \sum_{i=1}^{m} D_i g(f(x)) D_k f(x). \tag{1}$$

or

$$\frac{\partial(g \circ f)}{\partial x_k}(x) = \sum_{i=1}^{m} \frac{\partial g}{\partial y_k}(f(x)) \frac{\partial f}{\partial x_k}(x). \tag{2}$$

Examples 1.8. ***Example 1.*** *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$, $x, h \in \mathbb{R}^n$. *Then*

$$F'(t_0) = f'(x + t_0 h)h = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(x + t_0 h)h_k.$$

Since $x + th = (x_1 + th_1, \ldots, x)n + th_n)$.
Example 2.

$$\begin{aligned}
(f(r\cos t, r\sin t))'_r = &\quad D_1 f(r\cos t, r\sin t)\cos t + D_2 f(r\cos t, r\sin t)\sin t \\
(f(r\cos t, r\sin t))'_t = &\ D_1 f(r\cos t, r\sin t)(-r\sin t) + D_2 f(r\cos t, r\sin t)(r\cos t)
\end{aligned}$$

The derivatives of the higher order can be obtained by differentiation of these identities. For example,

$$\frac{\partial^2 w}{\partial u^2} = \left(P_1 \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y} + R_1 \frac{\partial}{\partial z}\right)^2 w + \frac{\partial P_1}{\partial u}\frac{\partial w}{\partial x} + \frac{\partial Q_1}{\partial u}\frac{\partial w}{\partial y} + \frac{\partial R_1}{\partial u}\frac{\partial w}{\partial z};$$

$$\begin{aligned}
\frac{\partial^2 w}{\partial u \partial v} = \left(P_1 \frac{\partial}{\partial x} + Q_1 \frac{\partial}{\partial y} + R_1 \frac{\partial}{\partial z}\right)\left(P_2 \frac{\partial}{\partial x} + Q_2 \frac{\partial}{\partial y} + R_2 \frac{\partial}{\partial z}\right) w + \\
\frac{\partial P_1}{\partial v}\frac{\partial w}{\partial x} + \frac{\partial Q_1}{\partial v}\frac{\partial w}{\partial y} + \frac{\partial R_1}{\partial v}\frac{\partial w}{\partial z},
\end{aligned}$$

where

$$\begin{aligned}
P_1 = \frac{\partial x}{\partial u}, \quad Q_1 = \frac{\partial y}{\partial u}, \quad R_1 = \frac{\partial z}{\partial u}, \\
P_2 = \frac{\partial x}{\partial v}, \quad Q_2 = \frac{\partial y}{\partial v}, \quad R_2 = \frac{\partial z}{\partial v}.
\end{aligned}$$

In other words,

$$\frac{\partial^2 w}{\partial u^2} = \left(\frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial x \partial y}\frac{\partial y}{\partial u} + \frac{\partial^2 w}{\partial x \partial z}\frac{\partial z}{\partial u}\right)\frac{\partial x}{\partial u} + \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial u^2} +$$
$$\left(\frac{\partial^2 w}{\partial y \partial x}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial u} + \frac{\partial^2 w}{\partial y \partial z}\frac{\partial z}{\partial u}\right)\frac{\partial y}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial u^2} +$$
$$\left(\frac{\partial^2 w}{\partial z \partial x}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial z \partial y}\frac{\partial y}{\partial u} + \frac{\partial^2 w}{\partial z^2}\frac{\partial z}{\partial u}\right)\frac{\partial z}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial^2 z}{\partial u^2} =$$
$$\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial x}{\partial u}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial y}{\partial u}\right)^2 + \frac{\partial^2 w}{\partial z^2}\left(\frac{\partial z}{\partial u}\right)^2 +$$
$$2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial x}{\partial u}\frac{\partial y}{\partial u} + 2\frac{\partial^2 w}{\partial x \partial z}\frac{\partial y}{\partial u}\frac{\partial z}{\partial u} + 2\frac{\partial^2 w}{\partial y \partial z}\frac{\partial y}{\partial u}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial w}{\partial y}\frac{\partial^2 y}{\partial u^2} + \frac{\partial w}{\partial z}\frac{\partial^2 z}{\partial u^2}$$

Problem 2. Assume that $f \in C^2$. Find first and the second order derivatives of function u

1. $u = f(x^2 + y^2 + z^2)$;
2. $u = f(x, xy, xyz)$;

Solution. 1.

$$u'_x = 2xf'(x^2+y^2+z^2);\ u'_y = 2yf'(x^2+y^2+z^2);\ u'_z = 2zf'(x^2+y^2+z^2);$$
$$u'_{xx} = 2f'(x^2+y^2+z^2) + 4x^2 f''(x^2+y^2+z^2);$$
$$u'_{yy} = 2f'(x^2+y^2+z^2) + 4y^2 f''(x^2+y^2+z^2);$$
$$u'_{zz} = 2f'(x^2+y^2+z^2) + 4z^2 f''(x^2+y^2+z^2);$$
$$u''_{xy} = 4xyf''(x^2+y^2+z^2);$$
$$u''_{xz} = 4xzf''(x^2+y^2+z^2);$$
$$u''_{yz} = 4yzf''(x^2+y^2+z^2).$$

2. let $f = f(u, v, w)$ then

$$\begin{aligned}
u'_x &= f'_u(x, xy, xyz) + yf'_v(x, xy, xyz) + yzf'_w(x, xy, xyz);\\
u'_y &= xf'_v(x, xy, xyz) + xzf'_w(x, xy, xyz);\\
u'_z &= xyf'_w(x, xy, xyz);
\end{aligned}$$

$$\begin{aligned}
u''_{xx} &= f''_{uu} + y^2 f''_{vv} + y^2 z^2 f''_{ww} + 2yf''_{uv} + 2yzf''_{uw} + 2y^2 zf''_{vw};\\
u''_{yy} &= x^2 f''_{vv} + 2x^2 zf''_{vw} + x^2 z^2 f''_{ww};\\
u''_{zz} &= x^2 y^2 f''_{ww};
\end{aligned}$$

$$\begin{aligned}
u''_{xy} &= f'_v + zf'_w + xf''_{uv} + xyf''_{vv} + 2xyzf''_{vw} + xzf''_{uw}xyz^2 f''_{ww};\\
u''_{xz} &= yf'_w + xyf'_{uw} + xy^2 f''_{vw} + xy^2 zf''_{ww};\\
u''_{yz} &= xf'_w + x^2 yf''_{vw} + x^2 yzf''_{ww};
\end{aligned}$$

Problem 3. Prove that if C^2-function $u = u(x, y)$ satisfies Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

then the function $v = u\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ also satisfies this identity.

Solution. Let $\varphi = \frac{x}{x^2+y^2}, \psi = \frac{y}{x^3+y^2}$.

$$\frac{\partial v}{\partial x} = u'_1 \frac{\partial \varphi}{\partial x} + u'_2 \frac{\partial \psi}{\partial x}, \quad \frac{\partial v}{\partial y} = u'_1 \frac{\partial \varphi}{\partial y} + u'_2 \frac{\partial \psi}{\partial y},$$

$$\frac{\partial^2 v}{\partial x^2} = u''_{11}\left(\frac{\partial\varphi}{\partial x}\right)^2 + 2u''_{12}\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + u''_{22}\left(\frac{\partial\psi}{\partial x}\right)^2 + u'_1\frac{\partial^2\varphi}{\partial x^2} + u'_2\frac{\partial^2\psi}{\partial x^2},$$
$$\frac{\partial^2 v}{\partial y^2} = u''_{11}\left(\frac{\partial\varphi}{\partial y}\right)^2 + 2u''_{12}\frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + u''_{22}\left(\frac{\partial\psi}{\partial y}\right)^2 + u'_1\frac{\partial^2\varphi}{\partial y^2} + u'_2\frac{\partial^2\psi}{\partial y^2},$$

where

$$u'_1 = \frac{\partial u}{\partial\varphi},\ u'_2 = \frac{\partial u}{\partial\psi},\ u''_{11} = \frac{\partial^2 u}{\partial\varphi^2},\ u''_{12} = \frac{\partial^2 u}{\partial\varphi\partial\psi},\ u''_{22} = \frac{\partial^2 u}{\partial v^2}.$$

Calculating derivatives of φ, ψ we see that

$$\Delta v = u''_{11}\left(\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right) + u''_{22}\left(\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right) +$$
$$+ 2u''_{12}\left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y}\right) + v'_1\Delta\varphi + u'_2\Delta\psi.$$

Hence,

$$\frac{\partial\varphi}{\partial x} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2},\quad \frac{\partial\varphi}{\partial y} = -\frac{2xy}{\left(x^2+y^2\right)^2},\quad \frac{\partial^2\varphi}{\partial x^2} = \frac{2x\left(x^2 - 3y^2\right)}{\left(x^2+y^2\right)^3},$$

$$\frac{\partial^2\varphi}{\partial y^2} = \frac{2x\left(3y^2 - x^2\right)}{\left(x^2+y^2\right)^3},\quad \frac{\partial\psi}{\partial x} = -\frac{2xy}{\left(x^2+y^2\right)^2},\quad \frac{\partial\psi}{\partial y} = \frac{x^2 - y^2}{\left(x^2+y^2\right)^2},$$
$$\frac{\partial^2\psi}{\partial x^2} = \frac{2y\left(3x^2 - y^2\right)}{\left(x^2+y^2\right)^3},\quad \frac{\partial^2\psi}{\partial y^2} = \frac{2y\left(y^2 - 3x^2\right)}{\left(x^2+y^2\right)^3},$$

and

$$\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} = 0,\quad \Delta\varphi = 0,\quad \Delta\psi = 0.$$

Consequently, since $\Delta u = 0$ then

$$\Delta v = \frac{1}{\left(x^2 + y^2\right)^2} \Delta u = 0.$$

Problem 3. Let $u, v \in C^2(\mathbb{R}^2)$. Prove that

$$\Delta(uv) = u\Delta v + v\Delta u + 2(u'_x v'_x + u'_y v'_y).$$

Solution. Consider application of Leibniz rule for higher-order derivatives of the product of two functions

$$(uv)''_{xx} = uv''_{xx} + 2u'_x v'_x + uv'_{xx};$$

$$(uv)''_{yy} = uv''_{yy} + 2u'_y v'_y + uv'_{yy};$$

Summarizing these identities we obtain the assertion of the problem.

1.4 Taylor's series of multivariate function

$$f(x, y) = \sum_{k=0}^{m} \sum_{\alpha_1 + \alpha_2 = k} \frac{1}{\alpha_1! \alpha_2!} \frac{\partial^k f\left(x_0; y_0\right)}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \left(x - x_0\right)^{\alpha_1} \left(y - y_0\right)^{\alpha_2} + o\left(\rho^m\right) =$$

$$\sum_{k=0}^{m} \frac{1}{k!} \sum_{i=0}^{k} C_k^i \frac{\partial^k f\left(x_0; y_0\right)}{\partial x^{k-i} \partial y^i} \left(x - x_0\right)^{k-i} \left(y - y_0\right)^i + o\left(\rho^m\right),$$

where

$$\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}, \quad (x, y) \to (x_0, y_0).$$

Problem 1. Find Taylor's decomposition of function

$$f(x, y) = \operatorname{arctg} \frac{1 + x}{1 + y}$$

at $(x_0, y_0) = (0, 0)$ with residue $o(\rho^2)$, where $\rho = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.
Solution 1. Function f has continuous partial derivatives of arbitrary order if $y \neq -1$.

$$f'_x = \frac{1}{1 + \left(\frac{1+x}{1+y}\right)^2} \frac{1}{1+y} = \frac{1+y}{(1+x)^2 + (1+y)^2}\Bigg|_{x=y=0} = \frac{1}{2};$$

$$f'_y = \frac{1}{1 + \left(\frac{1+x}{1+y}\right)^2} \frac{-(1+x)}{(1+y)^2} = -\frac{1+x}{(1+x)^2 + (1+y)^2}\Bigg|_{x=y=0} = -\frac{1}{2};$$

$$f'_{xx} = \frac{-2(1+y)(1+x)}{((1+x)^2 + (1+y)^2)^2}\Bigg|_{x=y=0} = -\frac{1}{2};$$

$$f'_{xy} = \frac{1}{(1+x)^2 + (1+y)^2} - \frac{2(1+y)^2}{((1+x)^2 + (1+y)^2)^2}\Bigg|_{x=y=0} = 0;$$

$$f'_{xx} = \frac{2(1+y)(1+x)}{((1+x)^2 + (1+y)^2)^2}\Bigg|_{x=y=0} = \frac{1}{2}.$$

Consequently,

$$f(x, y) = \frac{\pi}{4} + \frac{x}{2} - \frac{y}{2} - \frac{x^2}{4} + \frac{y^2}{4} + o(\rho^2).$$

Solution 2. We can use Taylor's decompositions of functions of one variables

$$\arctan(1 + t) = \frac{\pi}{4} + \frac{1}{2}t - \frac{1}{4}t^2 + o(t^2), \quad t \to 0;$$

$$f(x,y) = \operatorname{arctg}((1+x)(1-y+y^2+o(\rho^2))) =$$
$$\operatorname{arctg}(1+x-y-xy+y^2+o(\rho^2)) =$$
$$\frac{\pi}{4} + \frac{1}{2}(x-y-xy+y^2) - \frac{1}{4}(x-y-xy+y^2)^2 + o(\rho^2) =$$
$$\frac{\pi}{4} + \frac{x}{2} - \frac{y}{2} - \frac{x^2}{4} + \frac{y^2}{4} + o(\rho^2).$$

Problem 2. Find Taylor's decomposition of function

$$f(x,y) = \arcsin\left(2x - \frac{3}{2}xy\right)$$

at $(x_0, y_0) = (-1, 1)$ with residue $o(\rho^2)$, where $\rho = \sqrt{(x-x_0)^2 + (y-y_0)^2}$.
Solution. Function f has continuous partial derivatives of arbitrary order $|2x - 3xy/2| < 1$. Consider the change $u = x+1$ and $v = y-1$. Then

$$f(u,v) = \arcsin(2(u-1) - 3(u-1)(v+1)/2) = \arcsin(-1/2 + u/2 + 3v/2 - 3uv/2).$$

Let

$$g(t) = \arcsin(-1/2 + t).$$

Then

$$g'(t) = \left.\frac{1}{\sqrt{1-(-1/2+t)^2}}\right|_{t=0} = \frac{2}{\sqrt{3}};$$
$$g''(t) = \left.\frac{(-1/2+t)}{(1-(-1/2+t)^2)^{3/2}}\right|_{t=0} = -\frac{4}{3\sqrt{3}}$$
$$\arcsin(-1/2+t) = -\frac{\pi}{6} + \frac{2}{\sqrt{3}}t - \frac{2}{3\sqrt{3}}t^2 + o(t^2), \quad t \to 0.$$

Consequently,

$$f(u,v)=-\frac{\pi}{6}+\frac{2}{\sqrt{3}}(u/2+3v/2-3uv/2)-\frac{2}{3\sqrt{3}}(u/2+3v/2-3uv/2)^2+o(\rho^2)=$$
$$-\frac{\pi}{6}+\frac{u}{\sqrt{3}}+\sqrt{3}v-\frac{u^2}{6\sqrt{3}}-\frac{4}{\sqrt{3}}uv-\frac{\sqrt{3}}{2}v^2+o(\rho^2)=$$
$$-\frac{\pi}{6}+\frac{x+1}{\sqrt{3}}+\sqrt{3}(y-1)-\frac{(x+1)^2}{6\sqrt{3}}-\frac{4}{\sqrt{3}}(x+1)(y-1)-\frac{\sqrt{3}}{2}(y-1)^2+o(\rho^2)$$

□

Problem 3. Find Taylor's decomposition of function

$$f(x,y,z)=\cos x\cos y\cos z-\cos(x+y+z)$$

at $(x_0,y_0,z_0)=(0,0,0)$ with residue $o(\rho^2)$, where

$$\rho=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}.$$

Solution. Applying Taylor's formula for cos we see that

$$f(x,y,z)=\left(1-\frac{x^2}{2}+o(x^2)\right)\left(1-\frac{y^2}{2}+o(y^2)\right)\left(1-\frac{z^2}{2}+o(z^2)\right)-$$
$$\left(1-\frac{(x+y+z)^2}{2}+o((x+y+z)^2)\right)=$$
$$-\frac{x^2+y^2+z^2}{2}+\frac{x^2+y^2+z^2+2xy+2xz+2yz}{2}+o(\rho^2)=$$
$$xy+xz+yz+o(\rho^2).$$

1.5 Derivative of the implicit function

Recall the theorem on the implicit function considering example with two variables. Assume that

1. $F(x_0, y_0, z_0) = 0$;

2. function F is C^1 smooth in some neighbourhood of (x_0, y_0, z_0)

3. $F'_z(x, y, z) \neq 0$.

Then there exists a function $z = f(x, y)$ that is defined and C^1-smooth in some neighbourhood of (x_0, y_0) and that satisfies in this neighborhood the equation

$$F(x, y, z) = 0$$

and initial condition $z_0 = f(x_0, y_0)$. Moreover,

$$f'_x(x, y) = -\frac{F'_x}{F'_z} = -\frac{F'_x(x, y, f(x, y))}{F'_z(x, y, f(x, y))}; \quad f'_y(x, y) = -\frac{F'_y}{F'_z} = -\frac{F'_y(x, y, f(x, y))}{F'_z(x, y, f(x, y))}$$

To obtain higher-order derivatives we need to differentiate these identities taking in account that $z = f(x, y)$ is a function. For example,

$$\begin{aligned} f'_x(x, y) = -\left(\frac{F'_x}{F'_z}\right)'_x &= -\frac{F''_{xx} + F''_{xz} f'_x}{F'_z} + \frac{F'_x(F''_{xz} + F''_{zz} f'_x)}{(F'_z)^2} = \\ &- \frac{F''_{xx}F'_z - F''_{xz}F'_x}{(F'_z)^2} + \frac{F'_x(F''_{xz}F'_z - F''_{zz}F'_x)}{(F'_z)^3} \end{aligned}$$

However, actual problems involve usually more compact calculations.

Problem 1. Find first and second-order partial derivatives of implicit function $z(x, y)$ defined by the equation

$$x + y + z = e^z.$$

Solution.

$$F(x, y, z) = e^z - x - y - z.$$

$$F'_z = e^z - 1 \neq 0,\ \ z \neq 0.$$

$$z'_x = -\frac{F'_x}{F'_z} = \frac{1}{e^z - 1},\ \ z'_y = -\frac{F'_y}{F'_z} = \frac{1}{e^z - 1};$$

$$z''_{xx} = -\frac{e^z}{(e^z - 1)^2} z'_x = -\frac{e^z}{(e^z - 1)^3} = z''_{xy} = z''_{yy}.$$

Problem 2. Find derivatives y', y'', y''' of the implicit function $y(x)$ defined by

$$x^2 + xy + y^2 = 3$$

and find extremal points of this function.

Solution.

$$F(x, y) = x^2 + xy + y^2.$$

$$F'_y = 2y + x \neq 0,\ \ x \neq -2y,\ \ (x, y) \neq (-2, 1),\ \ (x, y) \neq (2, -1);$$

$$F'_x = 2x + y;$$

$$y' = -\frac{2x + y}{2y + x};$$

$$y'' = -\frac{(2 + y')(2y + x) - (2y' + 1)(2x + y)}{(2y + x)^2} = 3\frac{xy' - y}{(2y + x)^2} =$$

$$3\frac{-x(2x + y) - y(2y + x)}{(2y + x)^2} = -6\frac{x^2 + xy + y^2}{(2y + x)^3} = -\frac{18}{(2y + x)^3};$$

$$y''' = -54\frac{(2y' + 1)}{(2y + x)^4} = \frac{-162x}{(x + 2y)^5}.$$

Now, we can find extremal points. Indeed, $y' = 0$ iff $y = -2x$, hence, applying the equation $F(x, y) = 3$ we see that either $x = 1$ and $y = -2$ or $x = -1$ and $y = 2$.

Moreover, $y''(1) = 2/3 > 0$ and $x = 1$ is a point of local minimum with $y(1) = -2$; $y''(-1) = -2/3 > 0$ and $x = -1$ is a point of local maximum with $y(-1) = 2$;

Remark 1.9. *The general statement of the implicit function theorem can be understood in terms of the solution of the system of equations*

Assume that functions $F_i(x_1, \dots, x_m, y_1, \dots, y_n)$, $i = 1, 2 \dots, n$, are such that

1. $F_i(x^0, y^0) = F_i(x_1^0, \dots, x_m^0, y_1^0, \dots, y_n^0) = 0$, $i = 1, 2 \dots, n$;
2. F_i *is* C^1-*smooth near* (x_0, y_0);
3. *determinant of the Jacobi matrix* $\left(\frac{\partial F_i}{\partial y_j}\right)_{i,j=1}^{n}$ *is not zero at* (x^0, y^0),

$$\det \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \dots & \frac{\partial F_1}{\partial y_n} \\ \dots & \dots & \dots \\ \frac{\partial F_n}{\partial y_1} & \dots & \frac{\partial F_n}{\partial y_n} \end{pmatrix} (x^0, y^0) \neq 0,$$

Then in some neighborhood of $x^0 = (x_1^0, \dots, x_m^0)$ there exists a unique family of C^1-smooth functions

$$y_i = f_i(x_1, \dots, x_m), \ i = 1, \dots, n,$$

such that

1. $y_i^0 = f_i(x_1^0, \dots, x_m^0)$, $i = 1, \dots, n$;
2. $F_i((x_1, \dots, x_m, f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)) = 0.$

Moreover, derivatives of f_i satisfy the following system of equations

$$\frac{\partial F_i}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial F_i}{\partial y_k}\frac{\partial f_k}{\partial x_j} = 0, \;\; k = 1, \dots, m, \;\; i = 1, \dots, n.$$

Problem 3. Find partial derivatives of the first order of implicit functions $u(x, y)$ and $v(x, y)$ defined by the system

$$\begin{cases} xu - yv = 0 \\ yu + xv = 1 \end{cases}. \tag{3}$$

Solution. First, calculate the Jacobian

$$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \neq 0.$$

Differentiating the system (3) by x we obtain

$$\begin{cases} u + xu'_x - yv'_x = 0 \\ yu'_x + v + xv'_x = 0 \end{cases}.$$

Hence,

$$u'_x = -\frac{ux + vy}{x^2 + y^2}, \quad v'_x = \frac{uy - vx}{x^2 + y^2}.$$

Differentiating the system (3) by y we obtain

$$\begin{cases} xu'_y - yv'_y = v \\ yu'_y + v + xv'_y = -u \end{cases}.$$

Consequently,

$$u'_y = \frac{vx - uy}{x^2 + y^2}, \quad v'_y = -\frac{ux + vy}{x^2 + y^2}.$$

1.6 Extremal points of functions of several variables (Unconstrained optimization).

Consider 2-dimensional case. Assume that $D \subset \mathbb{R}^2$, $f \in C^2(D)$ and (x_0, y_0) is a stationary point of f, that is

$$\operatorname{grad} f(x_0, y_0) = (f'_x(x_0, y_0), f'_y(x_0, y_0)) = (0, 0).$$

Then

$$d^2 f(x_0, y_0) = f''_{x^2}(x_0, y_0)dx^2 + 2f''_{y^2}(x_0, y_0)dxdy + f''_{y^2}(x_0, y_0)dy^2 = Adx^2 + 2Bdxdy + Cdy^2.$$

and the matrix of the second differential (as of the quadratic form) has the folllowing form

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

Then $\Delta_1 = A,\ \Delta_2 = AC - B^2$ and at (x_0, y_0) we have

1. strict minimum if $A > 0,\ AC - B^2 > 0$;
2. strict maximum if $A < 0,\ AC - B^2 > 0$;
3. has no extremum if $AC - B^2 < 0$;

Problem 1. Find extremal points of the function

$$u = e^{x^2 - y}(5 - 2x + y).$$

Solution. First, we find stationary points of u

$$u'_x = e^{x^2-y}(2x(5 - 2x + y) - 2) = 0;$$
$$u'_y = e^{x^2-y}(-(5 - 2x + y) + 1) = 0.$$

Hence, $5 - 2x + y = 1,\ x = 1,\ \ y = -2.$

To calculate partial derivatives of the second order at stationary point $(1, -2)$. we will use the following simple rule

$$g(a) = 0 \Rightarrow (fg)'(a) = f'(a)g(a) + f(a)g'(a) = f(a)g'(a).$$

$$u'_{xx}(1,-2) = e^{x^2-y}(10 - 8x + 2y)\big|_{(x,y)=(1,-2)} = -2e^3;$$
$$u'_{xy}(1,-2) = 2e^{x^2-y}\big|_{(x,y)=(1,-2)} = 2e^3;$$
$$u'_{yy}(1,-2) = -e^{x^2-y}\big|_{(x,y)=(1,-2)} = -e^3;$$

Consequently, $\Delta_1 = -2e^3 < 0,\ \Delta_2 = -2e^6 < 0$ and $(2, 0)$ is not extremal.

□

Problem 2. Find extremal points of the function

$$u = x^2 + y^2 + z^2 + 2x + 4y - 6z.$$

Solution. First, we find stationary points of u

$$u'_x = 2x + 2 = 0;$$
$$u'_y = 2x + 4 = 0;$$
$$u'_z = 2z - 6 = 0;$$

Consequently, $(-1, -2, 3)$ is a unique stationary point. The second differential is equal to

$$d^2u = 2(dx^2 + dy^2 + dz^2),$$

and is a positive definite form. Hence, $(-1, -2, 3)$ is a point of strict minimum.

Problem 2. Find extremal points of the function

$$u = x^2y^3(6 - x - y).$$

Solution. First, we find stationary points of u

$$u'_x = 2xy^3(6 - x - y) - x^2y^3 = xy^3(12 - 3x - 2y) = 0;$$
$$u'_y = 3x^2y^2(6 - x - y) - x^2y^3 = x^2y^2(18 - 3x - 4y) = 0.$$

Points $(0, y)$, $(x, 0)$, $(2, 3)$ are stationary.
Case 1. Let $x = 2$, $y = 3$. Then

$$u''_{xx} = -3^4 2, \; u''_{xy} = -3^3 2^2, \; u''_{yy} = -3^2 2^4,$$

and $\Delta_1 = -3^4 2 < 0$, $\Delta_2 = 3^6 2^4 > 0$. Consequently, $(2, 3)$ is a point of strict maximum, $u(2, 3) = 108$.
Case 2. Let $xy = 0$. Then $u''_{xy} = u''_{yy} = 0$ and d^2u is insefinite form and $u(x, y) = 0$.
Case 2.1. Consider a function u in the neighborhood of the point $(0, y)$. If (h, k) is small enough

- $u(h, y + k) = h^2(y + k)^3(6 - y - k - h) \geq 0 = u(0, y)$ and $(0, y)$ is a point of nonstrict minimum if $0 < y < 6$.
- $u(h, y + k) = h^2(y + k)^3(6 - y - k - h) \leq 0 = u(0, y)$ and $(0, y)$ is a point of nonstrict maximum if $y < 0$ and $y > 6$.
- If $y = 6$ then $u(h, 6 + k) - u(0, 6) = h^2(6 + k)^3(-k - h)$ changes sign and $(0, 6)$ is not an extremal point.
- If $y = 0$ then $u(h, k) - u(0, 0) = h^2k^3(6 - h - k)$ changes sign and $(0, 0)$ is not extremal.

Case 2.2. Consider a function u in the neighborhood of the point $(x,0)$ If (h,k) is small enough

$$u(x+h,0+k)-u(x,0)=(x+h)^2k^3(6-x-h-k)$$

changes sign and the point $(x,0)$ is not extremal.
Answer: $(2,3)$ is a point of strict maximum; points $(0,y)$ are points of nonstrict if $0<y<6$, and are points of nonstrict maximum if $y<0$ or $y>6$.

Problem 3. Find points of local extrema of the implicit function $z(x,y)$ defined by the equation

$$F(x,y,z)=x^2+y^2+z^2-2x+2y-4z-10=0. \tag{4}$$

Solution. First, $F'_z=2z-4\neq 0$ if $z\neq 2$. Then

$$z'_x=-\frac{x-1}{z-2}=0,$$
$$z'_y=\frac{y+1}{z-2}=0,$$

and $x=1, y=-1$ is a stationary point. The value $z(1,-1)$ is obtained from the equation

$$z^2-4z-12=0.$$

Hence, $z=-2$ or $z=6$. This means that there exist two implicit functions z_1 and z_2 satisfying equation (4) and conditions $z_1(1,-1)=-2$ and $z_2(1,-1)=6$.

To apply the sufficient condition for local extremum we calculate second order derivatives of these functions

$$z''_{xx}=z''_{yy}=1/4,\ \ z''_{xy}=0,\ \ z=-2.$$

$$z''_{xx} = z''_{yy} = -1/4,\ z''_{xy} = 0,\ z = 6.$$

Consequently, at $(1, -1)$ the value $z = -2$ is the local minimum (of the mplicit function that obtains this value at $(-1, 1)$), and $z = 6$ is the local maximum (of the mplicit function that obtains this value at $(-1, 1)$).

1.7 Conditional extremum (Constrained optimization).

We set the problem of optimization (investigation for extremum) of a function $f(x) = f(x_1, \ldots, x_n)$ with respect to constraints $\varphi_i(x) = 0,\ 1 \leq i \leq m,\ m < n$.

Definition 1.10. *Let $m < n,\ f, \varphi_1, \ldots, \varphi_m : D \subset \mathbb{R}^n \to \mathbb{R}$,*

$$M = \{x \in D : \varphi_i(x) = 0,\ 1 \leq i \leq m\}.$$

*A point $x_0 \in M$ is a **(strict) conditional extremum (maximum, minimum)** of function f with respect to constraints $\varphi_i(x) = 0,\ 1 \leq i \leq m$, if $x_0 \in M$ is a (strict) extremum (maximum, minimum) of function $f\big|_M$.*

1.7.1 Algorithm of constrained optimization by Lagrange's function.

We assume that $f, \varphi_i \in C^2$. The problem of investigation of f for conditional extremum can be reduced to investigation of Lagranges function

$$L(x) = f(x) + \sum_{i=1}^{m} \lambda_i \varphi_i(x),$$

where numbers λ_i are constant multipliers.
Step 1. The method of Lagrange's function can be used only for such points that the rank of the matrix $\left(\frac{\partial\varphi_i}{\partial x_k}\right)_{i=1,k=1}^{m,n}$ is maximal:

$$\operatorname{rank}\left(\frac{\partial\varphi_i}{\partial x_k}\right)_{i=1,k=1}^{m,n} = m. \tag{5}$$

Step 2. Then we find the stationary points of Lagrange's function

$$\begin{cases}\operatorname{grad} L(x) = 0;\\ \varphi_i(x) = 0,\ 1 \le i \le m\end{cases}. \tag{6}$$

Solution of this system is a family of stationary points $x \in \mathbb{R}^n$ and corresponding Lagrange's multipliers. We exclude points that do not satisfy condition (5).
Step 3. Now we investigate second differential d^2L of Lagrange's function at stationary points. Let $x^0 = (x_1^0, \dots, x_n^0)$ and $\lambda^0 = (\lambda_1^0, \dots, \lambda_m^0)$ be stationary point and corresponding Lagrange's multipliers (6).

If in assumption that variables $dx_1, \dots, dx_n$ satisfy the following relations

$$d\varphi = \sum_{k=1}^{n} \frac{\partial\varphi_i(x^0)}{\partial x_k} dx_k = 0,\ 1 \le i \le m,$$

1. d^2L is positive definite then x^0 is a point of strict conditional minimum;
2. d^2L is negative definite then x^0 is a point of strict conditional maximum.

Problem 1. Find points of local extrema of the function

$$f = x^2 + 12xy + 2y^2, \text{ if } \varphi(x, y) = 4x^2 + y^2 = 25.$$

Solution. Consider Lagrange's function

$$L(x, y) = x^2 + 12xy + 2y^2 + \lambda(4x^2 + y^2)$$

and find its stationary points

$$\begin{aligned} L'_x &= 2x + 12y + 8\lambda x = (2 + 8\lambda)x + 12y = 0; \\ L'_x &= 4y + 12x + 2\lambda y = 12x + (4 + 2\lambda)y = 0; \\ &4x^2 + y^2 = 25 \end{aligned}$$

The system of two first equations has nonzero solution if and only if it determinant is zero, that is

$$(1 + 4\lambda)(2 + \lambda) - 36 = 4\lambda^2 + 9\lambda - 34 = 0.$$

That with the third equation means that $\lambda_1 = 2$ or $\lambda_2 = -17/4$.

Case 1. Let $\lambda_1 = 2$. Then $y = -3x/2$, $4x^2 + y^2 = 25x^2/4 = 25$, and $x = \pm 2$. Consequently, $(2, -3)$ and $(-2, 3)$ are stationary points.

Case 2. Let $\lambda_1 = -17/4$. Then $y = 8x/3$, $4x^2 + y^2 = 100x^2/9 = 25$, and $x = \pm\frac{3}{2}$ Consequently, $(3/2, 4)$ and $(-3/2, -4)$ are stationary points.

Consider the second differential at stationary points

$$\frac{1}{2}d^2L = dx^2 + 12dxdy + 2dy^2 + \lambda(4dx^2 + dy^2)$$

in assumption that

$$d\varphi(x, y) = 8xdx + 2ydy = 0 \iff dy = -\frac{4x}{y}dx = \begin{cases} \frac{8}{3}dx, & \lambda = 2; \\ -\frac{3}{2}dx, & \lambda = -17/3; \end{cases} .$$

Consequently,

$$\frac{1}{2}d^2L = dx^2+12dx\left(-\frac{4x}{y}dx\right)+2\left(-\frac{4x}{y}dx\right)^2+\lambda\left(4dx^2+\left(-\frac{4x}{y}dx\right)^2\right) = \begin{cases} 625/9dx^2, \ \lambda = 2; \\ -575/12dx^2, \ \lambda = -17/3; \end{cases}.$$

Answer.

1. $(2,-3)$ and $(-2,3)$ are points of strict conditional minimum $f(2,-3) = f(-2,3) = -50$.
2. $(3/2,4)$ and $(-3/2,-4)$ are points of strict conditional maximum $f(3/2,4) = f(-3/2,-4) = 425/4$.

1.7.2 Substitution method of constrained optimization.

If the equation

$$\varphi_j(x) = 0$$

can be solved with respect to one of the variables

$$x_k = g(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n),$$

then we can substitute x_k by g in f and ϕ_i and investigate a new problem with lower number of variables and constraints.
Problem 2. Find points of local extrema of the function

$$f = xy \text{ if } x + y = 1.$$

Solution. Express y in terms of x

$$y = 1 - x.$$

Now, the problem is reduced to investigation of the function of one variable

$$g(x) = f(x, 1 - x) = x(1 - x).$$

Here, $g'(x) = 1 - 2x = 0$ iff $x = \frac{1}{2}$ and $g''(x) = -2$. Consequently, $x = \frac{1}{2}$ is point of strict minimum of function g, and $\left(\frac{1}{2}, \frac{1}{2}\right)$ is point of strict conditional maximum of $f = xy$ with constraint $x + y = 1$.

Problem 3. Find points of local extrema of the function

$$f = xyz \text{ if } x^2 + y^2 + z^2 = 1,\ x + y + z = 0.$$

Solution. Solving the second equation with respect to variable z as $z = -x - y$ we reduce the proble to the investigation for local extrema of the function

$$f = -xy(x + y) \text{ if } \varphi(x, y) = x^2 + xy + y^2 = \frac{1}{2}.$$

Consider Lagrange's function

$$L(x, y) = -xy(x + y) + \lambda(x^2 + xy + y^2),$$

calculate the gradient

$$\begin{cases} L'_x = -2xy - y^2 + \lambda(2x + y) = (2x + y)(\lambda - y) = 0; \\ L'_y = -2xy - x^2 + \lambda(x + 2y) = (x + 2y)(\lambda - x) = 0; \\ x^2 + xy + y^2 = \frac{1}{2}; \end{cases}$$

and the second differential

$$d^2L = -2ydx^2 - 4(x + y)dxdy - 2xdy^2 + 2\lambda(dx^2 + dxdy + dy^2).$$

The condition $d\varphi = 0$ has the following form

$$d(x^2 + xy + y^2) = (2x + y)dx + (x + 2y)dy.$$

There are three cases.

Case 1. Let $x = y = \lambda = \pm\frac{1}{\sqrt{6}}$. In this case

$$d\phi = 3x(dx + dy) \Rightarrow dy = -dx$$

and

$$d^2L = 2(\lambda + 4x)dx^2 = 10x dx^2.$$

Hence, $x = y = z = \frac{1}{\sqrt{6}}$ is a point of conditional minimum and $x = y = z = -\frac{1}{\sqrt{6}}$ is a point of conditional maximum.

Case 2. Let $2x + y = 0$ and $x = \lambda$. Then $(x, y, \lambda) = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$, $(x, y, \lambda) = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ are stationary.

In this case $d\varphi = (x + 2y)dy$, consequently, $dy = 0$ and

$$d^2L = 2(\lambda - y)dx^2.$$

Hence, $(x, y, z) = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ is a point of conditional minimum, and $(x, y, z) = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ is a point of conditional maximum.

Case 3. The case $2y + x = 0$ is obtained from the previous by the symmetry of f and φ :

$$f(x, y) = f(y, x),\ \varphi(x, y) = \varphi(y, x).$$

Hence, $(x, y, z) = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ is a point of conditional minimum, and $(x, y, z) = \left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ is a point of conditional maximum.

Problem 4. Find points of local extrema of the function

$$f = xy + yz \text{ if } x^2 + y^2 = 2,\ y + z = 2,\ x, y, z > 0.$$

Solution. Solving the second equation with respect to variable z as $z = 2 - y$ we reduce the problem to the investigation for local conditional extrema of the function

$$f = xy + 2y - y^2 \text{ if } x^2 + y^2 = 2.$$

Consider Lagrange's function

$$L(x, y) = xy + 2y - y^2 + \lambda(x^2 + y^2)$$

and find its stationary points

$$\begin{cases} L'_x = y + 2\lambda x = 0; \\ L'_y = x + 2 - 2y + 2\lambda y = 0; \\ x^2 + y^2 = 2 \end{cases} .$$

From the first equation we see that $2\lambda = -y/x$. Performing this substitution in the second equation we obtain

$$y^2 - x^2 - 2x + 2xy = 0.$$

Noticing that $x^2 = 2 - y^2$ we see that

$$2y^2 - 2 - 2x(1-y) = 2(y-1)(y+1) - 2x(1-y) = 2(y+x+1)(y-1) = 0.$$

Consequently, $y = 1$ since $x, y > 0$ and $x = 1$, $\lambda = -1$.

Now, we will investigate the second differential

$$d^2L = 2dxdy - 2dy^2 + 2\lambda(dx^2 + dy^2)$$

assuming the condition

$$d(x^2 + y^2) = 2xdx + 2ydy = 2dx + 2dy = 0.$$

Substituting $dy = -dx$ in d^2L we see that

$$d^2L = 4(\lambda - 1)dy^2 = -8dy^2 < 0.$$

Consequently, the point $x = y = z = 1$ is a point of local maximum.