HW1. Week 11th.

Exercise 1.1. *For all the Examples 1-5 detect which condition (R, S, T, A) hold for given Relation.*

Example. 1. $X = \mathbb{N}$ and $R = \{(a, b) \in \mathbb{N}^2 \mid a \text{ divide } b\}$. Here we denote this relation by $a \mid b$.

2. $X = \mathbb{R}$ and $R = \{(a, b) \in \mathbb{R}^2 \mid a \geqslant b\}$. That means R is denoted by $\geqslant$.
3. X is the set of all humans. $R = \{(x, y) \mid x \text{ is a parent to } y\}$.
4. X is the set of all lines in a plane. $R = \{(x, y) \mid x \text{ is parallel to } y\}$. That means $xRy \Leftrightarrow x \parallel y$.
5. For the set of all triangles on Euclidean plane one can consider the similarity relation: $\triangle ABC \sim \triangle XYZ$.

1). R. T. A. holds. S not.

$\forall a, b, c \in \mathbb{N}$. $\underline{R:}$ $a|a$ holds. $\underline{T:}$ $a|b$, $b|c \Rightarrow a|c$ $\underline{A:}$ $a|b \wedge b|a \Rightarrow a = b$.

2). R. T. A holds S not.

$\forall a, b, c \in \mathbb{R}$. $\underline{R:}$ $a \le a$. $\underline{T:}$ $a \le b$, $b \le c \Rightarrow a \le c$ A: $a \le b \wedge b \le a \Rightarrow a = b$.

3) None of them hold.

4). We claim that "overlap" is not include in our cases of parallel.

R. A not holds. S. T. holds.

$\forall x, y$. 2 straight lines on plane. $\underline{S:}$ $x \parallel y \Rightarrow y \parallel x$ $\underline{T:}$ $x \parallel y$, $y \parallel z \Rightarrow x \parallel z$.

5). R. S. T. holds. A not.

$\forall \triangle A_1B_1C_1$. $\triangle A_2B_2C_2$, $\triangle A_3B_3C_3$.

$\underline{R:}$ $\triangle A_1B_1C_1 \sim \triangle A_1B_1C_1$ $\underline{S:}$ $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2 \Rightarrow \triangle A_2B_2C_2 \sim \triangle A_1B_1C_1$. $\underline{T:}$ $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$, $\triangle A_2B_2C_2 \sim \triangle A_3B_3C_3$. $\Rightarrow \triangle A_1B_1C_1 \sim \triangle A_3B_3C_3$.

Nevertheless, if there exists the greatest element g in a poset P then g is a unique maximal element in P.

Exercise 1.2. *Prove this assertion.*

Pf: if $\exists g \in P$. $\forall a \in P$. $a \prec g$.

if $\exists g_1, g_2 \in P$. g_1, g_2 are maximal element.

Since $g_1, g_2 \in P$. we have $g_1 \prec g$, $g_2 \prec g$

Since g_1 is maximal. $g \prec g_1$. (since g_1, g is comparable.).

Similarly. $g \prec g_2$. by Antisymmetry. $g_1 = g_2 = g$. thus. maximal element is unique in this case.

Exercise 1.3. *Draw the Hasse diagram for partially ordrered set that corresponds to the sequence* $5, 3, 6, 1, 4, 2, 8, 9, 7$.

Pf: we have several chains. (the partial order def by $i \prec j \Leftrightarrow (i \le j) \wedge (a_i \le a_j)$.

$5 \prec 6 \prec 8 \prec 9$.

$5 \prec 6 \prec 7$

$3 \prec 6 \prec 8 \prec 9$.

$3 \prec 4 \prec 8 \prec 9$

$3 \prec 4 \prec 7$.

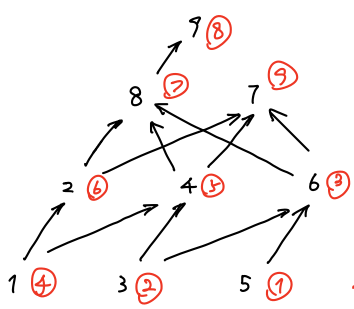
$3 \prec 6 \prec 7$

$1 \prec 4 \prec 8 \prec 9$.

$1 \prec 4 \prec 7$

$1 \prec 2 \prec 8 \prec 9$

$1 \prec 2 \prec 7$.



1	2	3	4	5	6	7	8	9
5	3	6	1	4	2	8	9	7

↘ 角标表顺序的序号.

Exercise 1.4. *Let $I_k = [a_k; b_k] \subset [0;1]$ where $k = 1 \dots 10$ be a family of ten segments. Prove that at least one of the following is true:*

- *there exist four segments having a point in common;*
- *there exist four segments with pairwise empty intersections.*

Pf: Firstly, we rearrange I_k by the order of starting point. s.t. $a_{k1} \le a_{k2} \le \cdots \le a_{k9} \le a_{k10}$, denote the ordered segment by $\{I_{ki}\}$.

define a new partial order: $I_{ki} \prec I_{kj}$: if $b_{ki} < a_{kj}$ $(i=j)$ or $(i \le j \wedge b_{ki} < a_{kj})$. ; other segment is incomparable.

check: 1) R: $a_{ki} < b_{ki}$.

2). S: $b_{ki} < a_{kj}$ $\quad b_{kj} < a_{kl}$. $\Rightarrow$ $b_{ki} < a_{kj} < b_{kj} < a_{kl}$.

3) A: $a_{ki} < b_{kj}$ $\Rightarrow$ $i = j$.

$a_{kj} < b_{ki}$

对逻辑运算: $A \Rightarrow B$.

if $A = 0$ $(A \Rightarrow B) = 1$.

then: the number of segment has common point $\Leftrightarrow$ width. of $P = \{I_{ki}\}$.

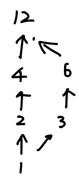
the number of segment has pairwise empty intersection. $\Leftrightarrow$ height of $P = \{I_{ki}\}$.

If both of them ≤ 3. by Dilworth thm. $\exists\, C_1, C_2, C_3 \subset P$. chains of $(P, \prec)$. s.t. $\bigcup_{i=1,2,3} C_i = P$.

but $(P, \prec)$ has height 3, $|C_i| = 3$. $|\bigcup C_i| \le 9$. $P = 10$. Contradicts.

Exercise 1.5. *How many different linear extensions does have the partial ordered set $\mathcal{D}_{12}$?*

Sol: D_{12} has element: $\{1, 2, 3, 4, 6, 12\}$. its Hasse diagram:



we have several chains: $1 \prec 2 \prec 4 \prec 12$ (1, 12 should be end points. the middle

$1 \prec 3 \prec 6 \prec 12$ 4 should perserve $2 \prec 4$, $3 \prec 6$.).

$1 \prec 2 \prec 6 \prec 12$

we have sequence 1, 2, 4, 3, 6, 12 $\Rightarrow$ thus, exists 5 Linear extensions.

1, 2, 3, 4, 6, 12

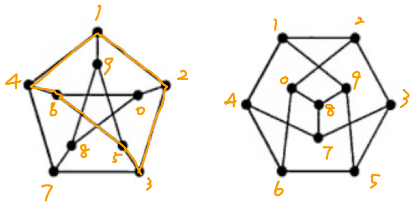
1, 2, 3, 6, 4, 12

1, 3, 2, 4, 6, 12

1, 3, 2, 6, 4, 12.

HW2. Week 12.

Exercise 2.1. *Whether these two graphs are isomorphic to each other. Either proof that they are not isomorphic or proff that they are isomorphic by labelling the vertices with the same numbers* $\{0, 1, \ldots, 9\}$*:*



first find the maximal cycle length = 6. labelled it.

Second deal with the other vertices.

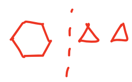
← by the label on the figure, this 2 graph are isomorphic.

Exercise 2.2. *How many non-isomorphic graphs with degree vector:*
a) $(4, 3, 3, 3, 3)$*?* *b)* $(3, 3, 3, 3, 3, 3)$*?*

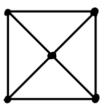
证不存在其他. isomorphic ⟺ 补图 isomorphic.

考虑补图. (1) (0, 1, 1, 1, 1).

(2) 2-regular.



a)

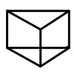


4-vertrix. adjencent to every other vertrices.

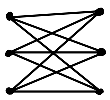
3-1 = 2. (2, 2, 2, 2) can only be a 4-length cycle.

Only 1 non-Isomorphic graph.

b).



minimal cycle length = 3.



minimal cycle length = 4

$|E(G)| = \frac{\sum deg(v)}{2} = 9.$

use minimal cycle length as " invariant".

Exercise 2.3. *Prove the converse statement. If all the cycles in the given graph have even length then the graph is bipartite.*

Pf. denote that $V(G) = V_0(G) \cup V_1(G)$.

$\forall v \in V(G)$. consider all vertices connected with v.

if $v - u$. the path have odd length. $u \in V_0(G)$

$v - u$. the path have even length. $u \in V_1(G)$. we denote $v \in V_1(G)$. as well (0 is even).

this pocedure can be repeated until all the " Connected component " are considered.

this partition is correctly defined.: 1) adjency vertices is in different set (length 1. distinct the parity).

2) if ∃ two path between some vertices. then it has same parity of length. otherwise, they construct an odd-length cycle.

Thus. $V_0(G) \cap V_1(G) = \phi$. $\Rightarrow$ $V_0(G) \cup V_1(G) = V(G)$.

Exercise 2.4. *Is there exist an oriented graph with five vertices such that 2 vertices have indegree zero, three vertices have indegree 3, two vertices have outdegree zero and three vertices have outdegree 3.*

Sol: by Hand making lemma.

$\sum dev^{+}(v) = \sum dev^{-}(v) = |E(G)| = 9.$

denote $V_{I_0}(G)$. $V_{O_0}(G)$ be the set of vertrices. with 0 in/out degree.

if $V_{I_0}(G) \cap V_{O_0}(G) = \phi$. only 1 vertrix left with. in degree 3 and out degree 3.

if $V_{I_0}(G) \cap V_{O_0}(G) \neq \phi$. $\exists v$ - isolated.

only 2 cases.

i)

	in	out
V_1	0	0
V_2	0	3
V_3	3	0
V_4	3	3
V_5	3	3.

ii)

	in	out
V_1	0	0
V_2	0	0
V_3	3	3
V_4	3	3
V_5	3	3.

both impossible. for V_4. only V_2. V_5. come in. i)

only V_3. V_5 come in ii)

can't have indegree 3.

Thus no graph satisfy the condition without loop and multi edge.

Exercise 2.5. *Consider a bipartite simple graph G such that all the vertices from $V_0(G)$ have degree 3 and all the vertices from $V_1(G)$ have degree five. Prove that the number of vertices $|V(G)|$ is divided by 8.*

Pf: by the def. of bipartite graph. each edge connect 1 vertex in $V_0(G)$. 1 vertex in $V_1(G)$.

thus. $\sum_{v \in V_1(G)} dev(v) = \sum_{v \in V_0(G)} dev(v) = |E(G)|$. by Hand making lemma. $\sum dev(v) = 2|E(G)|$

that is. $5n = 3k$. all the numbers are integral. thus we can denote $m \in \mathbb{Z}$.

$\gcd(3,5) = 1 \Rightarrow 3 | n.\ 5 | k.$ $n = 3m,\ k = 5m.$ ($5n = 3k$)

$|V(G)| = |V_1(G)| + |V_2(G)| = n + k = 8m.$ $8 | 8m.$ $m \in \mathbb{Z}$.

Exercise 2.6. *Let G be a graph such that all the vertices have an even degree. Let us fix a natural number n. Considering a suitable auxiliary graph whose vertices are paths of length at most n prove that there is an even number of simple paths of length n starting with r.*

Pf: in the auxiliary graph. P_G

denote the set of simple path. γ_k: $\{ (v_0, v_1, \dots, v_k) \mid 0 \le k \le n.\ v_i \neq v_j\ \forall i \neq j \}$.

define the "adjency" relation of path same as the proof of thm 2.7.

γ and γ' are adjancent if 1). $k \ge 1$. $\gamma'_{k-1} = (v_0, v_1, \dots, v_{k-1})$

2) $k < n$. $\gamma'_{k+1} = (v_0, v_1, \dots, v_k, v_{k+1})$.

3) lollipop. (only for the end.). v' adjencent to v_{k-1}. $w \neq v_{k-2}. v_k$.

$\gamma'_k = (v_0, v_1, \dots v_{k-1}, v')$.

thus. the pair (γ, γ'). defines the edge in P_G.

Now consider $\deg_{P_G}(\gamma_k)$. we have proved in thm 2.7. that. this number is odd iff. $k = n$.

by coro 2.4. the number of odd degree vertices is even. $\Rightarrow |$the simple path $\gamma_n| =$ even number.

Exercise 3.1. *Let G be a simple graph. Denote the minimal degree of its vertices by δ and the maximum degree of its vertices by Δ. Prove that if $\delta+\Delta \geqslant |V(G)|-1$ then G must be connected.*

Pf: Assume the converse.

in one connected component. with vertex of degree Δ. there is at least $\Delta+1$ vertice.

in another connected component. $\forall$ vertex v. $\deg(v) \geq \delta$. there is at least $\delta+1$ vertice

$|V(G)| \geq \Delta+1+\delta.+1 \Rightarrow \Delta+\delta \leq |V(G)|-2.$ contradicts.

Exercise 3.2. *(3 points) Prove that $z(G) = 1$ if and only if graph G has exactly one cycle.*

Pf: "$\Leftarrow$" Let $e \in G$. e is contained in the only cycle. ;

$c(G \setminus \{e\}) = c(G)$. $v(G \setminus \{e\}) = v(G)$. $e(G \setminus \{e\}) = e(G)-1 \Rightarrow z(G \setminus \{e\}) = z(G)-1$.

$G \setminus \{e\}$ has no cycle. $\Rightarrow z(G \setminus \{e\}) = 0 \Rightarrow z(G) = 1$.

"$\Rightarrow$" let $G = \bigcup_{i=1}^{n} G_i$ G_i be it's connected component.

by def of z. $z(G) = n - v(G) + e(G) = n - \sum v(G_i) + \sum e(G_i) = \sum z(G_i)$.

by the non-negativeness of z. $\exists! k \in [1:n]$ s.t. $z(G_k) = 1$. $\wedge$ $z(G_i) = 0$ $i \neq k$.

thus there is no cycle in G_i $(i \neq k)$. and at least 1 cycle in G_k (otherwise $z(G_k)$ also equals to 0).

if $\exists$ more the 1 cycle in G_k. let $e_k \in G_k$ s.t. e_k belongs to some cycle while not belongs to another cycle in G_k.

$z(G_k \setminus \{e_k\}) = 1 - v(G_k) + e(G_k) - 1 = 0 \Rightarrow G_k \setminus \{e_k\}$ has no cycle. contradicts. $\Rightarrow$ only 1 cycle in G_k, as well as in G.

Exercise 3.3. *Suppose that G_1 and G_2 are complimentary graphs. Prove that at least one of them must be connected.*

Pf: w.l.g. Assume G_1 - not connected.

$\exists V(G_1') \subseteq V(G_1)$. G_1' is a connected component and $V(G_1'') := V(G_1 \setminus G_1') \neq \phi$.

i) $\forall v_1' \in V(G_1')$ $v_1'' \in V(G_1'')$. the edge $e_1 = \{v_1', v_1''\} \notin E(G_1)$. thus. $e_1 \in E(G_2)$.

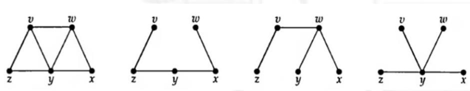
ii) $\forall v_1', v_2' \in V(G_1')$. we can let $v_2'' \in V(G_1'')$. the edge $e_2 = \{v_1', v_2''\}$ $e_2' = \{v_1'', v_2''\}$. $\in E(G_2)$.

(since it not belongs to $E(G_1)$ and the complementarity.). thus. $v_1' - v_2'' - v_2'$ a path is constructed.

iii) $\forall v_1''. v_2'' \in V(G_1'')$ similar as ii)

since $V(G_2) = V(G) = V(G_1) = V(G_1') \sqcup V(G_1'')$. and. for any two vertices we can find a path in $E(G_2)$. G_2 is connected.

Example. Here is an example of graph and several spanning trees in it:



the order : v-w-x-y-z. 1-5.

Exercise 3.4. *How many spanning trees does this graph have?*

write Laplacion matrix by graph.

$$L = \begin{pmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

$$t(G) = \det(L_{1;1}) = \begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & -1 & 0 \\ 0 & \frac{5}{3} & -\frac{4}{3} & 0 \\ 0 & -\frac{4}{3} & \frac{11}{3} & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 3 \cdot [\frac{5}{3} \cdot (\frac{22}{3} - 1) + \frac{4}{3}(-\frac{8}{3})] = 21.$$

HW3. Week 13th.

Exercise 3.5. *Prove the converse statement: if for any three distinct vertices u, v, w in a graph G with $v(G) \geqslant 3$ there exists a simple path joining u and w and containing v then G is biconnected.*

Pf: 1) connectness is already given.

2). no articulation point:

Assume the converse. $\exists a \in V(G)$. a is articulation point.

then $G \setminus \{a\}$ contain at least 2 connected component. w.l.g. let $c(G\setminus\{a\})$ is exactly 2 and $G \setminus \{a\} = G_1 \sqcup G_2$.

Let $u, w \in V(G_1)$. $v \in V(G_2)$. the simple path $u - v - w$ not exist since it can't cross a twice. Contradicts

Exercise 3.6. *(Each items cost 2 points) Let G be any simple graph.*

- *Prove that any biconnected component in the graph $B(G)$ is a complete graph;*
- *Describe vertices and adjacency relation for graph $B(B(G))$ in terms of elements of graph G.*

1). denote the biconnected component by $V(B(G)) = \{B_1, B_2, \dots, B_n\}$. which are blocks. in G.

firstly show the articulation point for block is unique. in biconnected component.

if. $\exists$ $B_i \cap B_j = \{v_{ij}\}$. $B_i \cap B_k = \{v_{ik}\}$. and $v_{ij} \neq v_{ik}$.

then 1). B_j, B_k not adjacent. otherwise $B_j \cap B_k \supseteq \{v_{ij}, v_{ik}\}$, contradicts to Prop 3.1b.

2) no path connect B_j and B_k without B_i. If so $\exists$ sequence of articulation point. $v_{ij} = v_0, v_1, \dots v_n = v_{ik}$.

and must exists some block contain v_i, v_{i+1}. again contradicts to Prop. 3.1b.

Thus delete B_i increases the number of connected components. contradicts to the biconnectness.

Secondly, by inclusion. and. the connectness $\exists$ articulation point $\{v\} = \bigcap_{i=1}^{n} B_i$. ("share the same articulation").

$\forall B_i, B_j \in V(B'(G))$. $\{v\} = B_i \cap B_j$. thus they are adjacent. thus $B'(G)$ is complete.

2).

by 1). each block shares unique articulation point in G.

vertex: $V(B(B(G))) = \{v \mid v \in G.\ v \text{ is articulation point}\}$

adjacency: $v, w \in V(B(B(G)))$ are adjancent if. $\{v, w\} \subseteq B$ B is a block of G.

Exercise 3.7. *Prove that any connected graph which is not biconnected has at least two leaf-blocks.*

Pf: Construct the block-cut tree for the initial graph.

not biconnected means this block-cut tree has least 3 points. (2 blocks and 1 articulation).

Leaf-blocks on the block-cut tree means the block point has degree 1.

for any tree with at least 2 vertices, there must exist at least 2 vertices with degree 1. (as the end point)

That is, 2 leaf-blocks in the initial graph

Exercise 4.1. *Let G be a multigraph such that all vertices have even degree. Prove that there is a family of edge-disjoint cycles (not circuits!) $C_1, \dots, C_k$ such that $E(G) = E(C_1) \sqcup E(C_2) \sqcup \cdots \sqcup E(C_k)$.*

Pf: the graph with all vertices have even degree has at least 1 cycle.

consider the graph. $G' := G - C_1$. s.t. $E(G') = E(G)$. $V(G') = V(G) \setminus V(E_1)$

i) for vertex $v \notin V(C_1)$. $\deg_{G'}(v) = \deg_G(v)$.

ii) for vertex $v \in V(C_1)$ $\deg_{G'}(v) = \deg_{G'}(v) - 2$.

denote $G_1' = G' \setminus I'$ I' is isolated point of G'.

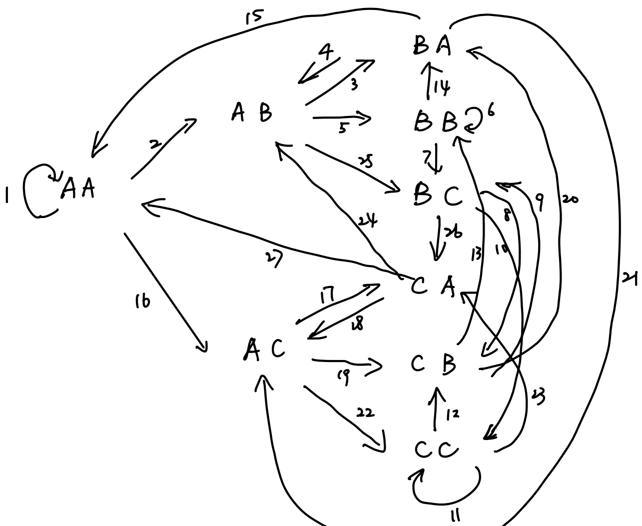
thus G' perserves all vertices. have even degree. denote $G'' := G_1' - C_2$

this procedure can be repeated until every point be isolated point.

Thus, we obtain a family of edge-disjoint cycle s.t. $E(G) = E(C_1) \sqcup E(C_2) \sqcup \dots \sqcup E(C_k)$.

Exercise 4.2. *Prove that there exist a cyclic word of length 27 in three letters A, B, C such that all 27 ordered triples of consecutive letters in this word are different.*

Pf: Similarly, we construct 3-dim De Bruijn directed graph. $DB_3 = (\{A,B,C\}^2, \{A,B,C\}^3)$.

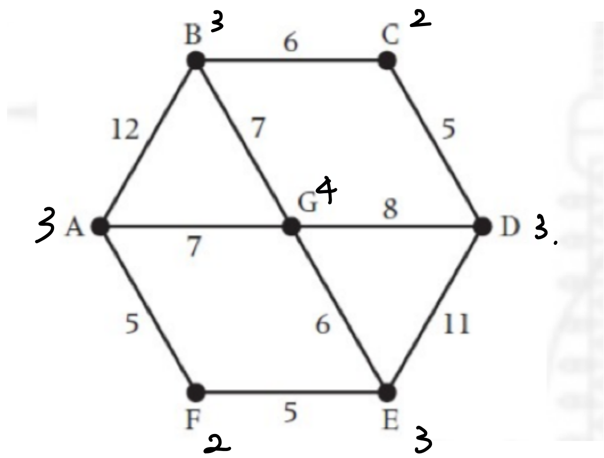


(each vertex has in and out degree 3)

by the graph. the sequence meets the requirement.

AAABABBBCBCCCBBAACACBACCABC

Exercise 4.3. *Find the length of the optimal Chinese postman route for the networks below.*



4 vertices with odd degree A. B. D. E.

$\sum w(e) = 12+6+7+5+7+8+6+11+5+5 = 72$

$\min(A-B) = 12$ $\min(B-E) = 13$

$\min(A-E) = 10.$ $\min(B-D) = 11$

$\min(A-D) = 15.$ $\min(E-D) = 11.$

to minimize the distance, we doubled path A-E and B-D.

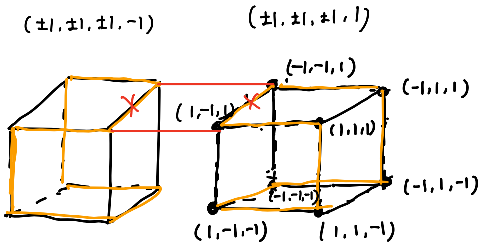
so the length of the post route is $72 + 21 = 93$.

HW4. Week 14th.

Exercise 4.4. *Each item cost 2 points.*

- *Prove that graph Q_4 is Hamiltonian.*
- *Prove that for every n graph Q_n is Hamiltonian.*

Pf: (1) we have proved Q_3 is Hamiltonian. for Q_4. we just extend Q_3 by last digit.



that is. $(\pm1, \pm1, \pm1, 1)$ and $(\pm1, \pm1, \pm1, -1)$ both construct Q_3.

we put them mirror-symmetrical, only those symmetric vertex. are adjacent in Q_4 (differs only in last digits).

Then we can detele one of the edge in Q_3 - Hamiltonian cycle and connected the corr. adjacent points from the last digit

For example. detele: $(-1,-1,1,1) - (1,-1,1,1)$ $(-1,-1,1,-1) - (1,-1,1,-1)$

add: $(-1,-1,1,1) - (-1,-1,1,-1)$ $(1,-1,1,1) - (1,-1,1,-1)$

the new path is simple, close, pass each vertex once and only once. thus is Hamiltonian.

(2). The procedure in (1) can be done inductively. that is

i) base: Q_2 has Hamiltonian cycle.

$$\begin{matrix} (1,1) & - & (1,-1) \\ | & & | \\ (-1,1) & - & (-1,-1) \end{matrix}$$

ii) hypothesis: Q_{n-1} has Hamiltonian cycle.

iii) for Q_n. consider two Q_{n-1} Hamiltonian cycle. $\{(\pm1, \dots, \pm1, 1)\}$ and $\{(\pm1, \dots, \pm1, -1)\}$.

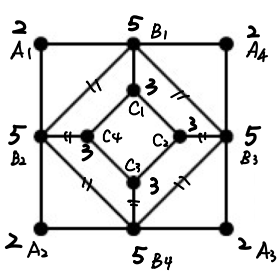
delete some edge in Q_{n-1} Hamiltonian cycle. e.g. ith place $(x, \dots, 1, \dots, 1) - (x, \dots, -1, \dots, 1)$

$(x, \dots, 1, \dots -1) - (x, \dots, -1, \dots, -1)$.

add: $(x, \dots, 1, \dots, 1) - (x, \dots, 1, \dots -1)$ $(x, \dots, -1, \dots, 1) - (x, \dots, -1, \dots, -1)$.

Then we construct the Q_n Hamiltonian cycle.

Exercise 4.5. *Prove that both of the graphs below are not Hamiltonian (3 points):*



labeled and classify the vertices.

that is $\deg(A_i) = 2$ $\deg(B_i) = 5$ $\deg(C_i) = 3$.

if it contains Hamiltonian cycle. then we can "delete" some edges until each vertex has degree 2.

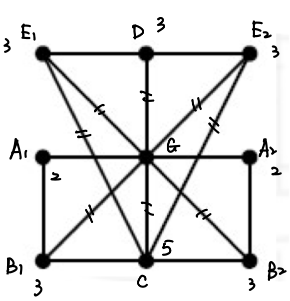
thus. each edge connects A_i can't be deleted. (edge A_iB_j)

thus. those edge like B_iB_j. B_iC_j need to be deleted. in order to satisfy that $\deg(B_i') = 2$.

after that the graph has already not connected.

thus there not exists. Hamiltonian cycle.

(or use property above. remove B_1, B_2, B_3, B_4 causes. 5 connected component: $\{A_1\}$. $\{A_2\}$. $\{A_3\}$. $\{A_4\}$. $\{C_1, C_2, C_3, C_4\}$.



similarly as the 1st one. labeled those vertices

i) $\deg(A_i) = 2$. A_iG, A_iB_i can't be delete.

thus detele E_1G, E_2G, B_1G, B_2G, DG, CG

ii) Then $\deg(B_i') = 2$. B_iC can't be delete.

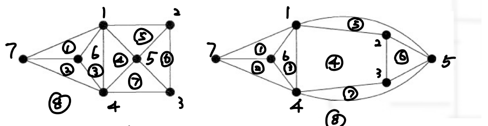
thus detele CE_1. CE_2

then $\deg(E_i') = 1$. which is already impossible to construct a Hamiltonian cycle.

(or use property. remove D, G. obtains 3 connected component $\{A_1, B_1\}$. $\{A_2, B_2\}$. $\{E_1, D, E_2\}$.

HW5. Week 15th.

Exercise 5.1. *Prove that two plane graphs shown in the picture are isomorphic as abstract graphs but their duals are not isomorpic:*

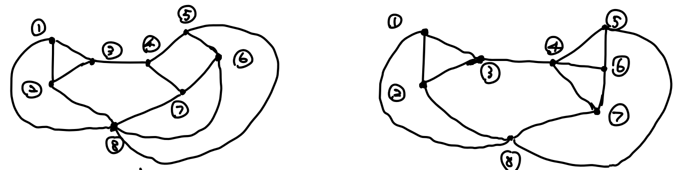


(1) (2)

pf: 1). isometry.

labelled the vertex and show the correspondence.

2). find their duals



(1)* (2)*

with degree sequence (1)* : $\{5,3,3,3,3,3,3,3\}$ distinct degree sequence $\Rightarrow$ (1)* and (2)* not isomorphic.

(2)* : $\{4,4,3,3,3,3,3,3\}$.

Exercise 5.2. *Given 20 points in the interior of a given square one draws somehow non intersection segments connecting theses points with each other ond four vertex of the square. It occurs that the square was dissected into triangles. Find the number of triangles.*

Sol: it's equivalent to a planar connected graph with 24 points.

the number of face = the number of triangles + 1

denote $T :=$ the number of triangles

each edge, excepts the edge of square, is the edge of 2 triangles.

thus we have $2e = 3T + 4$. $\Rightarrow$ $e = \frac{3(f-1)+4}{2}$.

by Euler's formula : $v - e + f = 2$.

$\Rightarrow$ $24 - \frac{3(f-1)+4}{2} + f = 2$ $\Rightarrow$ $f = 43$ $\Rightarrow$ number of triangle $= 42$.

Exercise 5.3. *Let G be a simple plane 3-regular graph. Prove that $3f_3 + 2f_4 + f_5 - f_7 - 2f_8 - 3f_9 - \cdots = 12$ where f_k is the number of ksided edges. Deduce that there exists a face with at most 5 sides.*

pf: 1). by Handshaking lemma. $3v = 2e$

by Euler's formula. $v - e + f = 2$ $\Rightarrow$ $e = 3f - 6$.

by def. of f_k. $\sum_{k\geq 3} k \cdot f_k = 2e = 6f - 12$.

$\Rightarrow$ $\sum_{k\geq 3} (6-k) f_k = 12$. $\Rightarrow$ $3f_3 + 2f_4 + f_5 - f_7 - 2f_8 - \cdots = 12$.

if there exists no face with sides at most 5. then we have $-(f_7 + 2f_8 + \cdots) = 12$.

LHS ≤ 0. RHS > 0. whichs is contradictory.

Exercise 5.4. *Let G ne a graph with 11 vertices. Prove that G and its complement $\overline{G}$ could not be both planar.*

Pf. $e(K_{11}) = \frac{11 \times (11-1)}{2} = 55$.

Assume the converse, both G and $\overline{G}$ are planar.

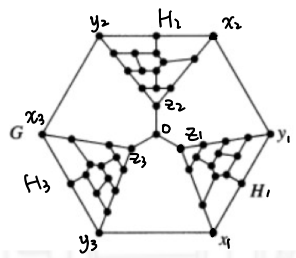
by coro 5.7. for both G and $\overline{G}$, $e \leq 3v - 6 = 27$

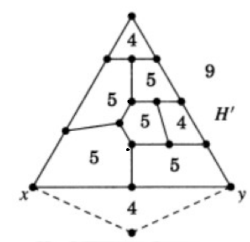
thus $e(G) + e(\overline{G}) \leq 54$

but $e(G) + e(\overline{G}) = e(K_{11}) = 55$ contradicts.

Ex 5.5. • *Consider a graph H' in the picture with 16 vertices and 9 faces. Prove that it is non-Hamiltonian,*

• *Consider graph G in the picture with 46 vertices which is constituted of three similar fragments H. Using the previous item prove that G is non-Hamiltonian.*





(1). Assume the converse. denote the Hamilton cycle by C.

the 9-side face must be outer region of C.

thus. by thm 5.9.

$$\begin{cases} 2(f_4 - g_4) + 3(f_5 - g_5) = 7 \\ f_4 + g_4 = 3. \\ f_5 + g_5 = 5. \end{cases}$$

this linear system has sol. in $[0:9]$

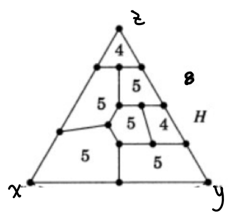
$$\begin{cases} f_4 = 1 & g_4 = 2 \\ f_5 = 4 & g_5 = 1. \end{cases}$$

by prop. 5.4. consider the face outside C. there are connected.

in the picture, if we select. 1× "9-face." 2× "4-face" 1× "5-face". there always some vertex come inner vertex of this part. which means the Hamilton cycle could not dissect them. contradicts. $\square$.

(2) by (1). H' is not Hamilton. thus, in fragment H. there is no Hamilton path connect x and y.

(if exists. the Hamilton path with 2 dashed line construct Hamilton cycle of H')



Moreover. in H. only 3 extrance, x, y, z. that is, it's impossible to entre H. pass some vertices in H. then go out then come in again to pass the remaining. thus we need to enter H. then go out before all vertices are passed.

Thus. w.l.g. if we starting from x_1, assume there exists Hamilton path. cross all vertice in H_1

the exit can't be x_1. since can't crosses x_1 twice.
can't be y_1, since H' not Hamilton.
$\Rightarrow$ could only be z_1
(if no Hamilton path $x_1 - z_1$. the proof is ended).

After similarly analysis. we construct the only possible path.

$$x_1 \xrightarrow[\text{(include } y_1)]{\text{all vertice in } H_1} z_1 \to 0 \to z_2 \xrightarrow[\text{(include } x_2)]{\text{all vertice in } H_2} y_2 \to x_3 \xrightarrow{\text{all vertice in } H_3} z_3$$ but no path $z_3 \to x_1$. remaining.

Thus. this graph G is not Hamilton.

HW 6.

Exercise 6.1. *Let G be a graph such that the number of vertices whose degree is greater than $k-1$ is less or equal than k. Then $\chi(G) \leqslant k$.*

Pf: $|\{v \mid \deg v > k-1\}| \leq k$.

For those $\deg v > k-1$. w.l.g. let then be $\{v_1, v_2, \dots, v_t\} \subseteq G$. $t \leq k$ s.t. $\deg v_1 \geq \deg v_2 \geq \cdots \deg v_t \geq k$.

colored v_1, v_2, up to v_t. use t distinct color. ($t \leq k$).

For those $\deg v \leq k-1$, one of kth color can be colored on that vertex.

thus G is k-colorable. $\chi(G) \leq k$.

Exercise 6.2. *Compute chromatic number for the Petersen graph.*

Sol: the peterson graph. non-complete, connected. $\Delta(G) = 3$. $\chi(G) \leq 3$ by Brooks thm.

and Peterson graph is not bipartite thus $\chi(G) \neq 2$.

thus $\chi(G) = 3$.

Exercise 6.3. *Prove that $\chi'(K_{2m}) = 2m-1$.*

Pf. $\forall v \in V(K_{2m})$ $\deg v = 2m-1$. thus $\chi'(G) \geq 2m-1$.

consider K_{2m-1}, it can be edge coloring properly by $2m-1$ colors. by prop 6.5.

then add 1 vertex. to the graph. as well as $2m-1$ edge.

$\forall v \in V(K_{2m-1})$. $\deg(v) = 2m-2$. from $2m-1$ color, always exists one color yet to use, coloring the new edge.

thus K_{2m} is $2m-1$ edge colorable i.e. $\chi'(G) \leq 2m-1$

In conclusion $\chi'(K_{2m}) = 2m-1$.