Calculus of variations

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Annotation

These notes have been prepared during the author's stay in Harbin Institute of Technology (HIT) and based on the course of lectures given to students of HIT. The course contains basic problems of calculus of variations and methods for solving them. In particular, the materials cover variational problem with fixed and free ends, the isoperimetric problem, the Lagrange problem and problems with movable ends. Together with necessary conditions the attention is paid also to sufficient conditions of extrema. The theoretical problems are illustrated not only by classical examples such as propagation of light in geometrical optics, the brachystochrone problem, the problem of geodesics on sphere, Hamilton formalism, but also by extensive analysis of typical problems.

The materials can be of interest to undergraduate students studying mathematics and physics at universities, students of technical universities, and other readers wishing to better understand the mathematical foundations of physics.

Calculus of variations

1 The simplest variational problems

Calculus of variations studies the behaviour of functionals, i.e. mappings defined on some functional space $\mathcal{F}$ and taking values in $\mathbb{R}^n$. One of the main classes of functional is the class of integral functionals.

To specify this class we consider a set $\mathcal{F}$ of twice continuously differentiable functions defined on a segment of the real line and taking values in a domain $\Omega \subset \mathbb{R}^n$

$$\mathcal{F} = \left\{ x : I_x = [a_x, b_x] \to \Omega : \ x \in C^2(I_x, \Omega), \right\}$$

where $C^2(I_x, \Omega)$ stands for the set of twice continuously differentiable functions. We emphasize that each function $x \in \mathcal{F}$ has its own domain of definition I_x.

Then the integral functional $\mathcal{I}$ defined on a subset $\mathcal{F}_0 \subset \mathcal{F}$ is given by

$$\mathcal{I}(x(\cdot)) = \int_{a_x}^{b_x} \mathcal{L}(t, x(t), x'(t)) \mathrm{d}t, \tag{1}$$

where $\mathcal{L} \in C^2(I_0 \times \Omega \times \mathbb{R}^n)$ is a given function called the *Lagrangian* and $I_0 = \bigcup\limits_{x \in \mathcal{F}_0} I_x$. The arguments of this function are denoted by t, x and v and called *time*, *coordinate* and *velocity*, respectively.

1.1 Variational problem with fixed ends

In this section we consider the simplest variational problem. Let $a, b \in \mathbb{R}$ and $A, B \in \Omega$ be fixed. We consider a subset $\mathcal{F}_{a,b,A,B} \subset \mathcal{F}$

$$\mathcal{F}_{a,b,A,B} = \{ x \in \mathcal{F} : \ a_x = a, \, b_x = b, \, x(a) = A, \, x(b) = B \} .$$

Elements of $\mathcal{F}_{a,b,A,B}$ are called curves with fixed ends.

Let $\mathcal{I}$ be a functional defined on $\mathcal{F}_{a,b,A,B}$. For any two curves with fixed ends x and y consider

$$\Delta \mathcal{I}(x(\cdot), y(\cdot)) = \mathcal{I}(x(\cdot)) - \mathcal{I}(y(\cdot)).$$

Definition 1.1. *One says that a curve x_0 furnishes global minimum (resp., global maximum) to the functional $\mathcal{I}$ if*

$$\Delta \mathcal{I}(x(\cdot), x_0(\cdot)) \geq 0, \quad (\mathit{resp.}, \Delta \mathcal{I}(x(\cdot), x_0(\cdot)) \leq 0)$$

$\forall \, x \in \mathcal{F}_{a,b,A,B}$.

In addition to the notion of global minimum (maximum) we introduce the notion of local minimum (maximum). For this purpose we define two distances between any two curves with fixed ends:

$$\begin{aligned} d_s(x(\cdot), y(\cdot)) &= \max_{a \le t \le b} \|x(t) - y(t)\|_{\mathbb{R}^n}, \\ d_w(x(\cdot), y(\cdot)) &= \max_{a \le t \le b} \|x(t) - y(t)\|_{\mathbb{R}^n} + \max_{a \le t \le b} \|x'(t) - y'(t)\|_{\mathbb{R}^n}. \end{aligned} \tag{2}$$

Definition 1.2. *One says that a curve x_0 furnishes local strong (weak) minimum (resp., local strong (weak) maximum) to the functional $\mathcal{I}$ if there exists $\varepsilon > 0$ such that*

$$\Delta\mathcal{I}(x(\cdot), x_0(\cdot)) \ge 0, \quad (\textit{resp.}, \Delta\mathcal{I}(x(\cdot), x_0(\cdot)) \le 0)$$

$\forall\, x \in \mathcal{F}_{a,b,A,B}$ *satisfying* $d_s(x_0(\cdot), x(\cdot)) < \varepsilon$ *(resp.,* $d_w(x_0(\cdot), x(\cdot)) < \varepsilon$*).*

If a curve x_0 furnishes global or local minimum (resp., maximum) to the functional $\mathcal{I}$ it is called *minimal* (resp., *maximal*). If a curve x_0 is minimal or maximal then it is called *extremal* and one says in this case that x_0 furnishes extremum to the functional $\mathcal{I}$.

The variational problem with fixed ends is to find minimal (maximal) curves for an integral functional $\mathcal{I}$ defined on the subset $\mathcal{F}_{a,b,A,B}$.

1.2 The first variation of a functional defined on curves with fixed ends. Necessary condition for extremum

First we remark that for any functions $x, y \in \mathcal{F}_{a,b,A,B}$ the difference $h \equiv x - y \in C^2(I, \mathbb{R}^n)$ and $h(a) = h(b) = 0$. On the other hand, for any $x \in \mathcal{F}_{a,b,A,B}$ and $h \in C^2(I, \mathbb{R}^n)$ satisfying $h(a) = h(b) = 0$ there exists $\varepsilon_0 > 0$ such that $x + \varepsilon h \in \mathcal{F}_{a,b,A,B}$ for all ε admiting $|\varepsilon| < \varepsilon_0$.

Assume that a curve x_0 is extremal for a functional $\mathcal{I}$ defined on the subset $\mathcal{F}_{a,b,A,B}$ and consider $\Delta\mathcal{I}((x_0+\varepsilon h)(\cdot), x_0(\cdot))$ for arbitrary $h \in C^2(I, \mathbb{R}^n)$ satisfying $h(a) = h(b) = 0$ and sufficiently small ε. Using the Taylor formula for the function $\mathcal{L}$ we obtain that

$$\begin{aligned} &\Delta\mathcal{I}((x_0 + \varepsilon h)(\cdot), x_0(\cdot)) = \\ &\quad = \varepsilon \int_a^b (\mathcal{L}_x(t, x_0(t), x_0'(t))h(t) + \mathcal{L}_v(t, x_0(t), x_0'(t))h'(t))\, \mathrm{d}t + O(\varepsilon^2). \end{aligned} \tag{3}$$

One has to note that we shortened the notations in (3). Indeed, as x, v and h are vector-valued functions, the terms of the integrant in (3) have the following meaning:

$$\mathcal{L}_x(t, x_0(t), x_0'(t))h(t) = \sum_{k=1}^{n} \frac{\partial \mathcal{L}}{\partial x_k}(t, x_0(t), x_0'(t))\, h_k(t),$$
$$\mathcal{L}_v(t, x_0(t), x_0'(t))h'(t) = \sum_{k=1}^{n} \frac{\partial \mathcal{L}}{\partial v_k}(t, x_0(t), x_0'(t))\, h_k'(t).$$

We keep these shortened notations further in the text.

Integrating the second term in the integral of (3) by parts yields

$$\Delta\mathcal{I}((x_0+\varepsilon h)(\cdot), x(\cdot)) = \varepsilon\mathcal{L}_v(t, x_0(t), x_0'(t))h(t)\Big|_a^b +$$
$$+\varepsilon\int_a^b \left(\mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t))\right) h(t)\mathrm{d}t + O(\varepsilon^2). \quad (4)$$

The first term in the right hand side of (4) vanishes due to conditions $h(a) = h(b) = 0$. Since x_0 is assumed to be extremal, the expression in the left hand side of (4) is sign-definite for all sifficiently small ε. This implies

$$\int_a^b \left(\mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t))\right) h(t)\mathrm{d}t = 0. \quad (5)$$

An expression in the left hand side of (4) is called *the first variation* of the functional $\mathcal{I}$ calculated at x_0 along h and is denoted as

$$\delta\mathcal{I}(x_0(\cdot), h(\cdot)) = \int_a^b \left(\mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t))\right) h(t)\mathrm{d}t. \quad (6)$$

Thus, if x_0 is extremal, the first variation $\delta\mathcal{I}(x_0(\cdot), h(\cdot))$ vanishes along any function $h \in C^2(I, \mathbb{R}^n)$, satisfying $h(a) = h(b) = 0$.

Lemma 1 (Dubois-Raimond)**.** *Let f be a vector-valued function continuous on the segment I (i.e. $f \in C(I, \mathbb{R}^n)$). If*

$$\int_a^b f(t)h(t)\mathrm{d}t = 0$$

for all $h \in C^2(I, \mathbb{R}^n)$ satisfying $h(a) = h(b) = 0$ then $f \equiv 0$.

Proof: Assume that $f \not\equiv 0$. Then there exists $t_0 \in I$ such that $f(t_0) \neq 0$. Due to continuity of f there exists positive δ such that

$$f(t) \neq 0 \; \forall \, t \in [t_0 - \delta, t_0 + \delta].$$

Consider a function h_0 such that for all $k = 1, \ldots, n$

$$h_{0,k}(t) = \begin{cases} (t - t_0 + \delta)^3 (t_0 + \delta - t)^3, & t \in [t_0 - \delta, t_0 + \delta], \\ 0, & t \in I \setminus [t_0 - \delta, t_0 + \delta]. \end{cases}$$

Obviously, $h_0 \in C^2(I, \mathbb{R}^n)$, $h_0(a) = h_0(b) = 0$ and $h_{0,k}(t) > 0$ for all $t \in (t_0 - \delta, t_0 + \delta)$ and all $k = 1, \ldots, n$. This leads to contradiction since

$$\int\limits_a^b f(t) h_0(t) \mathrm{d}t \neq 0.$$

Thus, the lemma is proved. $\square$

Applying this lemma to the equality (5) we obtain the following necessary condition for a curve x_0 to be extremal.

Theorem 1.1. *If a curve with fixed ends x_0 is extremal then it satisfies the Euler-Lagrange equation*

$$\mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{L}_v(t, x_0(t), x_0'(t)) = 0.$$

We point out that the Lagrange equation in theorem (1.2) is written in the vector form and is equivalent to a system of equations

$$\frac{\partial \mathcal{L}}{\partial x_k}(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial v_k}(t, x_0(t), x_0'(t)) = 0, \quad k = 1, \ldots, n. \tag{7}$$

Each of equations (7) is a differential equation of order 2, hence, the total order of the system is $2n$.

1.3 One classical example

One of the classical examples illustrating application of variational methods is the propagation of light. Let A and B be two arbitrary points in Euclidian space $\mathbb{R}^3$. We assume that a point source of light is located at the point A and look for a curve γ connecting A and B along which the light emitted by the source follows to reach the point B. Denote by $T(\gamma)$ the time which light spent travelling from A to B along γ.

By Fermat principle the curve γ is a *ray of light* if and only if $T(\cdot)$ attains its minimum at γ. Let $c(x)$ stands for the speed of light at the point x. Then the time $T(\gamma)$ can be expressed as

$$T(\gamma) = \int\limits_{\gamma} \frac{\mathrm{d}l}{c(x)},$$

where $\mathrm{d}l$ is the differential of arc length and the integral is taken along γ. If one parameterises the curve γ as $x = x(t)$, $t \in I = [a, b]$ (note that t is any parameter, not necessarily the true time) then

$$T(x(\cdot)) = \int\limits_{a}^{b} \frac{\|x'(t)\|_{\mathbb{R}^3}}{c(x(t))} \mathrm{d}t,$$

where $\|x'(t)\|_{\mathbb{R}^3} = \sqrt{\sum\limits_{k=1}^{3} (x_k')^2}$ and derivatives are taken with respect to t. The functional T is called *Fermat's functional.*

Thus we arrive at the classical variational problem with fixed ends for the functional T with the Lagrangian

$$\mathcal{L}(t, x, v) = \frac{\|v\|_{\mathbb{R}^3}}{c(x)}.$$

The Lagrange equations for this functional are

$$-\frac{\|x'(t)\|_{\mathbb{R}^3}}{c^2(x(t))} \frac{\partial c}{\partial x_k}(x(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{x_k'(t)}{\|x'(t)\|_{\mathbb{R}^3} c(x(t))} \right) = 0, \quad k = 1, 2, 3.$$

If speed of light c does not depend on x then the Lagrange equations can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{x_k'(t)}{\|x'(t)\|_{\mathbb{R}^3}} \right) = 0, \quad k = 1, 2, 3.$$

Hence, the unit tangent vector $x_k'(t)/\|x'(t)\|_{\mathbb{R}^3}$ is constant along γ and we arrive at well-known fact that in a homogeneous media the light propagates along a straight line.

1.4 Variational problem with free ends

As in the case of variational problem with fixed ends we consider the intergal functional $\mathcal{I}$ (1), but define it on a different subset $\mathcal{F}_{a,b} \subset \mathcal{F}$:

$$\mathcal{F}_{a,b} = \{x \in \mathcal{F} : a_x = a,\ b_x = b\},$$

where a, b are given real numbers. Elements of $\mathcal{F}_{a,b}$ are called *curves with free ends* since there are no any conditions on their ends $x(a)$, $x(b)$ (they are "free").

In this case the notions of global (strong (soft) local) minimum (maximum) together with notions of minimal, maximal, extremal curves can be introduced in a similar way.

Thus, the variational problem with free ends is to find minimal (maximal) curves for the integral functional $\mathcal{I}$ defined on $\mathcal{F}_{a,b}$.

To obtain necessary conditions for a curve with free ends to be extremal we first note that for any functions $x, y \in \mathcal{F}_{a,b}$ the difference $h \equiv x - y \in C^2(I, \mathbb{R}^n)$ and for any $x \in \mathcal{F}_{a,b}$ and $h \in C^2(I, \mathbb{R}^n)$ there exists $\varepsilon_0 > 0$ such that $x + \varepsilon h \in \mathcal{F}_{a,b}$ for all ε admiting $|\varepsilon| < \varepsilon_0$.

Assume that a curve with free ends x_0 is extremal and consider $\Delta\mathcal{J}((x_0 + \varepsilon h)(\cdot), x(\cdot))$ for arbitrary $h \in C^2(I, \mathbb{R}^n)$ and sufficiently small ε. Arguing similar to the case of variational problem with fixed ends we arrive at

$$\mathcal{L}_v(t, x_0(t), x_0'(t))h(t)\Big|_a^b + \\ + \int\limits_a^b \left(\mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t))\right) h(t)\mathrm{d}t = 0. \quad (8)$$

As in the case of variational problem with fixed ends the left hand side of (8) is called *the first variation* of the functional $\mathcal{I}$ calculated at x_0 along h (compare with (6)):

$$\delta\mathcal{I}(x_0(\cdot), h(\cdot)) = \mathcal{L}_v(t, x_0(t), x_0'(t))h(t)\Big|_a^b + \\ + \int\limits_a^b \left(\mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t))\right) h(t)\mathrm{d}t. \quad (9)$$

Obviousely, the first term in the left hand side of (8) vanishes for any function $h \in C^2(I, \mathbb{R}^n)$, satisfying $h(a) = h(b) = 0$. Hence, by Dubois-Raymond lemma we conclude that x_0 satisfies the Lagrange equations (7).

If we choose $h \in C^2(I, \mathbb{R}^n)$ such that $h(b) = 0$ then

$$\mathcal{L}_v(a, x_0(a), x_0'(a))h(a) = 0.$$

Since value $h(a)$ can be arbitrary one gets

$$\mathcal{L}_v(a, x_0(a), x_0'(a)) = 0.$$

In a similar way one may show that

$$\mathcal{L}_v(b, x_0(b), x_0'(b)) = 0.$$

Thus, we arrive at the following necessary conditions for a curve with free ends to be extremal.

Theorem 1.2. *If a curve with free ends x_0 is extremal then it satisfies the Euler-Lagrange equation*

$$\mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t)) = 0$$

and natural boundary conditions

$$\mathcal{L}_v(t, x_0(t), x_0'(t))\Big|_{t=a,b} = 0.$$

1.5 Lagrange equations in special cases

In this subsection we consider integral functionals whose Lagrangians have specific dependence on their variables. The results will be valid for both variational problems with fixed and free ends.

First, we consider the case when the Lagrangian does not depend on time $\mathcal{L} = \mathcal{L}(x, v)$.

Lemma 1. *If the Lagrangian is independent of time, then the Lagrange equation possesses an integral of motion*

$$\mathcal{H}(x(t), x'(t)) = x'(t)\mathcal{L}_v(x(t), x'(t)) - \mathcal{L}(x(t), x'(t)), \tag{10}$$

i.e. $\mathcal{H}(x(t), x'(t))$ is constant along any solution of Lagrange equation.

Proof: The rpoof is very straightforward. Assume that x_0 is a solution of the Lagrange equation. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = x_0''(t)\mathcal{L}_v(x_0(t), x_0'(t)) + x_0'(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(x_0(t), x_0'(t)) - \\ - \mathcal{L}_x(x_0(t), x_0'(t))x_0'(t) - \mathcal{L}_v(x_0(t), x_0'(t))x_0''(t) = \\ - x_0'(t)\left(\mathcal{L}_x(x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(x_0(t), x_0'(t))\right) = 0.\end{aligned}$$

□

The expression $\mathcal{H}(x(t), x'(t))$ has meaning of the energy and Lemma 1 states the energy preservation law for conservative systems.

In the second case we consider the Lagrangians which do not depend on coordinate x, i.e. $\mathcal{L} = \mathcal{L}(t, v)$. Then $\mathcal{L}_x \equiv 0$ and the Lagrange equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x'(t)) = 0.$$

Taking this into account we arrive at the following lemma

Lemma 2. *If the Lagrangian is independent of coordinate x, then the Lagrange equation possesses an integral of motion*

$$\mathcal{L}_v(t, x'(t)).$$

Note that the expression $\mathcal{L}_v(t, x'(t))$ has meaning of the momentum. Hence, Lemma 2 states preservation of the momentum for homogeneous case.

Finally, we consider the case when the Lagrangian does not depend on velocity v. In this case $\mathcal{L}_v \equiv 0$ and the Lagrange equation becomes not differential, but algebraic equation:

$$\mathcal{L}_x(t, x(t)) = 0.$$

1.6 Another classical example

This subsection is devoted to another classical example, which illustrates the case of free ends variational problem in combination with existence of integral of motion. Namely, we consider the brachistochrone problem. This problem has long history started from J. Bernoulli and can be formulated as follows: find a plane curve connecting two given points A and B lying in the vertical plane (B is below A) such that a material point of mass m placed at A and subjected to the action of the garvity force only will passages from A to B in the shortest time. Such a curve is called brachistochrone originated from greek words $\beta\rho\alpha\chi\iota\sigma\tau\o\varsigma$ - the shortest and $\chi\rho\o\nu\o\varsigma$ - the time.

We introduce a rectangular coordinate system with the origin at the point A and vertical axis directed downwards (along gravity acceleration g). Let B has coordinates (X, Y), $X \geq 0, Y > 0$. Parameterizing the curve we search as $y = y(x)$ one expresses the time of motion as

$$T(y(\cdot)) = \int_0^X \frac{\sqrt{1 + (y'(x))^2}}{V(x, y(x))}\mathrm{d}x,$$

where $V(x, y)$ is the velocity of the material point at the point (x, y). Since the gravity force is conservative the full energy of the material point is preserved, i.e.

$$E = \frac{mV^2(x, y)}{2} - mgy = \mathrm{const}.$$

As the initial velocity is zero $v(0,0)=0$ then $V(x,y)=\sqrt{2gy}$ and

$$T(y(\cdot)) = \int\limits_0^X \frac{\sqrt{1+(y'(x))^2}}{\sqrt{2gy(x)}}\mathrm{d}x.$$

The problem of minimizing the time of motion T is equivalent to minimization of an integral functional $\mathcal{J}$ with the Lagrangian

$$\mathcal{L}(x,y,v) = \sqrt{\frac{1+v^2}{y}}, \tag{11}$$

where x plays the role of the time and y is the coordinate.

This functional is defined on a subset

$$\mathcal{F}_{0,X,0} = \{y \in \mathcal{F} : a_y = 0,\ b_y = X,\ y(a_y) = 0\},$$

i.e. the left end is fixed and the right one is free.

Since the Lagrangian does not depend on x there exists an integral of motion of the form (10):

$$y'\mathcal{L}_v(y,y') - \mathcal{L}(y,y') = C, \tag{12}$$

where C is a constant.

Substituting (11) into (12), one obtains

$$\frac{(y')^2}{\sqrt{y(1+(y')^2)}} - \sqrt{\frac{1+(y')^2}{y}} = C \Longleftrightarrow -\frac{1}{\sqrt{y(1+(y')^2)}} = C.$$

This implies

$$y(1+(y')^2) = C \tag{13}$$

with some new constant C.

We solve this equation in a parametric form. Introduce a parameter θ by the formula

$$y' = \cot\theta. \tag{14}$$

Substituting (14) into (13) we obtain a formula describing the dependence of coordinate y on the parameter:

$$y = \frac{C}{1+\cot^2\theta} = C\sin^2\theta.$$

On the other hand, due to

$$\mathrm{d}y = y'\mathrm{d}x = \cot\theta\mathrm{d}x = 2C\cos\theta\sin\theta\mathrm{d}\theta$$

we obtain a differential equation on x as a function of θ:

$$\mathrm{d}x = 2C\sin^2\theta\mathrm{d}\theta = C(1 - 2\cos(2\theta))\mathrm{d}\theta.$$

Integrating this equation yields

$$x = x_0 + \frac{C}{2}(2\theta - \sin(2\theta)),$$

where x_0 is a constant of integration. Since the initial position corresponds to $(x, y) = (0, 0)$ we conclude that $x_0 = 0$. Thus, the parametric description of the brachistochrone is

$$\begin{cases} x = R(2\theta - \sin(2\theta)), \\ y = R(1 - \cos(2\theta)) \end{cases} \tag{15}$$

with $R = C/2$. Finally, we introduce a new parameter $\varphi = 2\theta$ to rewrite (15):

$$\begin{cases} x = R(\varphi - \sin\varphi), \\ y = R(1 - \cos\varphi). \end{cases} \tag{16}$$

Note that formulas (16) describe a cycloid that is a curve traced by a point on a circle as it rolls along horizontal line.

Remind that we search a solution described by (16), but which belongs to $\mathcal{F}_{0,X,0}$. To do this we take into account the natural boundary condition at the right end $x = X$:

$$\mathcal{L}_v(y, y')\Big|_{x=X} = 0 \Longleftrightarrow \frac{y'}{\sqrt{y(1 + (y')^2)}}\Big|_{x=X} = 0 \Rightarrow y'(X) = 0.$$

Since $y' = \frac{\sin\varphi}{1-\cos\varphi}$ the natural boundary condition at the right end is satisfied if $\varphi = \pi m, m \in \mathbb{Z} \setminus \{0\}$ and $R = X/(\pi m)$. In addition, the brachistochrone must twice continuously differentiable on the segment $[0, X]$, but the cycloid (16) is not differentiable at $\varphi = 2\pi k, k \in \mathbb{Z}\setminus\{0\}$. Taking this into account we conclude that condition $x = X$ corresponds to $\varphi = \pi$ and $R = X/\pi$. Thus, the brachistochrone has the following form:

$$\begin{cases} x = \frac{X}{\pi}(\varphi - \sin\varphi), \\ y = \frac{X}{\pi}(1 - \cos\varphi). \end{cases}$$

1.7 Practice

In this subsection we consider some practicle problems and present their solutions.

Problem 1. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_1^2 ((x')^2 - 2xt)dt, \quad \text{if} \quad x(1) = 0, \quad x(2) = -1.$$

Solution: First, we note that this is a one-dimensional variational problem with fixed ends. The segment I is $I = [1, 2]$ and the constants A, B are $A = 0, B = -1$. Thus, the integral functional $\mathcal{I}$ is defined on a set of functions

$$\mathcal{F}_{a,b,A,B} = \left\{ x \in C^2([1, 2], \mathbb{R}) : \ x(a) = 0, \ x(b) = -1 \right\}.$$

To find all stationary curves one needs to solve the Lagrange equation in the class $\mathcal{F}_{a,b,A,B}$. The Lagrangian of the functional $\mathcal{I}$ has the form

$$\mathcal{L}(t, x, v) = v^2 - 2xt.$$

Substitute this expression into (7) to obtain the Lagrange equation

$$x'' + t = 0.$$

A solution of this differential equation is

$$x(t) = -t^3/6 + C_1 t + C_2, \tag{17}$$

where C_1, C_2 are arbitrary constants.

Note that solution (17) does not belong to the set $\mathcal{F}_{a,b,A,B}$ for arbitrary C_1, C_2. To find the constants we substitute (17) into the boundary conditions and obtain

$$\begin{aligned} x(1) &= -1/6 + C_1 + C_2 = 0, \\ x(2) &= -4/3 + 2C_1 + C_2 = -1. \end{aligned}$$

Then $C_1 = 1/6, C_2 = 0$. This yields that the only solution of the Lagrange equation which belongs to $\mathcal{F}_{a,b,A,B}$ (i.e. the stationary curve) is

$$x(t) = -t^3/6 + t/6.$$

Problem 2. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_0^{2\pi} ((x')^2 - x^2)dt, \quad \text{if} \quad x(0) = 1, \quad x(2\pi) = 1.$$

Solution: This is also a one-dimensional variational problem with fixed ends. The Lagrangian of $\mathcal{I}$ is

$$\mathcal{L}(t, x, v) = v^2 - x^2.$$

Hence, the Lagrange equation takes the form

$$x'' + x = 0.$$

This is a linear differential equation of the second order. To solve it, we consider the corresponding characteristic equation

$$\lambda^2 + 1 = 0.$$

Solving this algebraic equation, one obtains $\lambda = \pm i$. This implies that the general solution of the Lagrange equation is

$$x(t) = C_1 \cos t + C_2 \sin t,$$

where C_1, C_2 are arbitrary constants. Taking into account the boundary conditions we deduce that $C_1 = 1$, while C_2 can be arbitrary. Thus, the set of all stationary curves is a one-parametric family of functions

$$x(t) = \cos t + C_2 \sin t, \quad C_2 \in \mathbb{R}.$$

Problem 3. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_0^1 x(2t - x)dt, \quad \text{if} \quad x(0) = 1, \quad x(1) = 2.$$

Solution. Noting that the Lagrangian of this variational problem is $\mathcal{L} = x(2t - x)$ we get the Lagrange equation

$$x - t = 0.$$

One sees that the unique solution $x(t) = t$ of this equation does not satisfy the boundary condition. Thus, the functional $\mathcal{I}$ does not have stationary curves at all.

Problem 4. *Find geodesics on the conus*

$$x^2 + y^2 = z^2.$$

Solution: We remind that for given two points A, B on a surface Γ a geodesic is those curve on the surface connecting A and B which has the minimal length. To find such a curve we fix two points $A(x_0, y_0, z_0)$ and $B(x, y, z)$ and consider the sets of curves connecting those two points. Then the length of such a curve γ can be calculated by curvalinear integral

$$\mathcal{I}(\gamma) = \int\limits_{\gamma} \mathrm{d}l,$$

where $\mathrm{d}l^2 = dx^2 + dy^2 + dz^2$. Using the cylindrical coordinates (r, φ, z) introduced by

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z,$$

one obtains

$$r^2 = z^2, \quad r\mathrm{d}r = z\mathrm{d}z, \quad (\mathrm{d}r)^2 = (\mathrm{d}z)^2.$$

We take φ as a parameter on γ, then

$$\mathcal{I}(\gamma) = \int\limits_{\varphi_0}^{\varphi} \sqrt{2z'^2 + z^2}\mathrm{d}\varphi.$$

Note that one needs to minimize $\mathcal{I}$. Thus, we reformulate the problem of geodesics finding as a variational problem with free ends. The angle φ plays the role of time, while z is the coordinate.

The Lagrangian $\mathcal{L} = \sqrt{2v^2 + z^2}$ does not depend on time, hence there exists an integral of motion in the form (10), namely

$$\frac{2z'^2}{\sqrt{2z'^2 + z^2}} - \sqrt{2z'^2 + z^2} = -\frac{z^2}{\sqrt{2z'^2 + z^2}} = C_1,$$

where C_1 is a constant. This equation can be rewritten as

$$\frac{\mathrm{d}z}{z\sqrt{z^2 - C_1^2}} = \pm\frac{1}{\sqrt{2}C_1}\mathrm{d}\varphi. \tag{18}$$

We introduce new unknown function, p, by

$$z = C_1\cosh(p).$$

In terms of this variable (18) takes the form

$$\frac{\mathrm{d}p}{\cosh(p)} = \pm\frac{\mathrm{d}\varphi}{\sqrt{2}} = \sqrt{2}\mathrm{d}\left(\arctan(\mathrm{e}^p)\right).$$

This yields

$$z = C_1\cosh(p) = \pm\frac{C_1}{\sin\left(\frac{\varphi}{\sqrt{2}} + C_2\right)}, \tag{19}$$

where C_1, C_2 are constants. Substituting this into boundary conditions one may find the constants C_1, C_2.
Remark: We proved that the obtained curve (19 is stationary, but we did not check that it is a geodesic.

Problem 5. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_0^1 (x')^2 \mathrm{d}t, \quad x(0) = 1.$$

Solution. First we note that this is a variational problem of the mixed type, since the left end is fixed and the right end is free. It means that we consider the functional $\mathcal{I}$ on a set of functions $\mathcal{F}_{a,b,A} = \{x \in C^2([0,1], \mathbb{R}) : x(0) = 1\}$.

Taking into account that the Lagrangian $\mathcal{L} = v^2$, we obtain the Lagrange equation

$$x'' = 0$$

and its general solution

$$x(t) = C_1 t + C_2,$$

where C_1, C_2 are arbitrary constants. From the boundary condition at the left end we get

$$x(0) = C_2 = 1.$$

At the right end the solution should satisfy the natural boundary condition

$$\frac{\partial \mathcal{L}}{\partial v}(1, x(1), x'(1)) = 2x'(1) = 2C_1 = 0.$$

Thus, the only stationary curve in this variational problem is

$$x(t) = 1.$$

Remark If we consider the same functional, but with both free ends, the Lagrange equation and, hence, its general solution will be the same. However, to be a stationary curve this solution should also satisfy the natural boundary condition at the left end too. For this particular functional the natural boundary condition at the left end coincide with the natural boundary condition at the right end. It means that for this varioational problem there are plenty of stationary curves:

$$x(t) = C_2,$$

where C_2 is arbitrary.

Problem 6. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_0^1 ((x')^2 + x^2 + 2x \sinh t)\mathrm{d}t, \quad x(0) = 0.$$

Solution. The Lagrangian of this functional and the Lagrange equation have the form

$$\mathcal{L} = v^2 + x^2 + 2x \sinh t, \quad x'' - x = \sinh t.$$

The characteristic equation $\lambda^2 - 1 = 0$ has two simple roots $\lambda = \pm 1$. Hence, the general solution of homogeneous equation $x'' - x = 0$ is

$$x(t) = C_1 \cosh t + C_2 \sinh t,$$

where C_1, C_2 are arbitrary constants. We will find a partial solution for the inhomogeneous equation in the form

$$x_0(t) = At \cosh t + Bt \sinh t.$$

Substituting this into the Lagrange equation and collecting like terms one obtains

$$2A \sinh t + 2B \cosh t = \sinh t.$$

This leads to $A = 1/2, B = 0$. Hence, the general solution of the Lagrange equation is

$$x(t) = \frac{1}{2} t \cosh t + C_1 \cosh t + C_2 \sinh t.$$

From the boundary conditions at the left end we get

$$x(0) = C_1 = 0.$$

The natural boundary condition at the right end gives

$$x'(1) = \frac{1}{2} \cosh(1) + \frac{1}{2} \sinh(1) + C_1 \sinh(1) + C_2 \cosh(1) = 0.$$

Since $C_1 = 0$

$$C_2 = -\frac{1}{2}\big(1 + \tanh(1)\big).$$

We conclude that the only stationary curve for the functional $\mathcal{I}$ is

$$x(t) = \frac{1}{2} t \cosh t - \frac{1}{2}\big(1 + \tanh(1)\big) \sinh t.$$

Problem 7. *Find all stationary curves of the functional*

$$\mathcal{I}(x,y) = \int_0^{\pi/2} ((x')^2 + (y')^2 + 2xy)\mathrm{d}t,$$
$$x(0) = 0, \quad y(0) = 0, \quad x(\pi/2) = 1, \quad y(\pi/2) = -1.$$

Solution. First we note that this is already a multidimensional ($n = 2$) variational problem with fixed ends. The functional $\mathcal{I}$ is defined on a set of functions

$$\mathcal{F}_{a,b,A,B} = \{(x,y) \in C^2([0,\pi/2], \mathbb{R}^2) : \\ x(0) = 0, \quad y(0) = 0, \quad x(\pi/2) = 1, \quad y(\pi/2) = -1\}.$$

Denote the components of velocity by $v = (u,w)$, then the Lagrangian of the functional is

$$\mathcal{L} = u^2 + w^2 + 2xy.$$

The system of Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial u} = 0, \qquad \frac{\partial \mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial w} = 0$$

takes the form

$$2y - 2x'' = 0, \quad 2x - 2y'' = 0.$$

Differentiating the first equation twice with respect to t yields

$$x^{(4)} - x = 0. \tag{20}$$

The characteristic equation of this fourth order linear differential equation is

$$\lambda^4 - 1 = 0.$$

It has four simple roots $\lambda = \pm 1, \pm i$. So, the general solution of (20) is

$$x(t) = C_1 \cos t + C_2 \sin t + C_3 e^t + C_4 e^{-t},$$

where C_1, C_2, C_3, C_4 are arbitrary constants. Since $y = x''$ one has

$$y(t) = -C_1 \cos t - C_2 \sin t + C_3 e^t + C_4 e^{-t}.$$

We substitute this solution to the boundary conditions and obtain an algebraic system for the constants $C_1, \ldots, C_4$:

$$C_1 + C_3 + C_4 = 0, \quad -C_1 + C_3 + C_4 = 0,$$
$$C_2 + C_3 e^{\pi/2} + C_4 e^{-\pi/2} = 1, \quad -C_2 + C_3 e^{\pi/2} + C_4 e^{-\pi/2} = -1.$$

The solution of this system is

$$C_1 = C_3 = C_4 = 0, \quad C_2 = 1.$$

Thus, we obtain that there exists only one stationary curve

$$x(t) = \sin t, \quad y(t) = -\sin t.$$

Problem 8. *Find all stationary curves of the functional*

$$\mathcal{I}(x,y) = \int_0^1 ((x')^2 + (y')^2 + 2xe^t)\mathrm{d}t, \quad x(1) = e, \quad y(1) = 1.$$

Solution. This is a two-dimensional variational problem with free left end and fixed right end. The Lagrangian $\mathcal{L} = u^2 + w^2 + 2xe^t$ and the Lagrange equations are of the form

$$e^t - x'' = 0, \quad y'' = 0.$$

The general solution can be written as

$$x = C_1 t + C_2 + e^t, \quad y = C_3 t + C_4$$

with arbitrary constants $C_1, \ldots, C_4$.

At the left end the natural boundary conditions are

$$\frac{\partial \mathcal{L}}{\partial u}(0, x(0), y(0), x'(0), y'(0)) = 2x'(0) = 0,$$
$$\frac{\partial \mathcal{L}}{\partial w}(0, x(0), y(0), x'(0), y'(0)) = 2y'(0) = 0.$$

This implies

$$C_1 + 1 = 0, \quad C_3 = 0.$$

The boundary conditions at the right end are

$$C_1 + C_2 + e = e, \quad C_3 + C_4 = 1.$$

Hence, the only stationary curve of the functional $\mathcal{I}$ is

$$x(t) = 1 - t + e^t, \quad y(t) = 1.$$

Problem 9. *Consider a set of curves $x = x(t)$, which connect two given points on the plane: $(t,x) = (a,A)$ and $(t,x) = (b,B)$. Every such a curve* *defines a surface of revolution over the t-axis. Find the curve that furnishes a minimum to the area of the correspondent surface of revolution.*

Solution. The area of a surface of revolution defined by the curve $x(t)$ can be found as an integral

$$\mathcal{I}(x) = 2\pi \int\limits_a^b \sqrt{1 + (x')^2} x \mathrm{d}t.$$

Hence, our problem becomes the variational problem with fixed ends for the functional $\mathcal{I}$ defined on the set $\mathcal{F}_{a,b,A,B}$.

Note that the Lagrangian $\mathcal{L} = x\sqrt{1 + v^2}$ does not depend on t. This leads to the existence of an integral of motion. Namely,

$$\frac{x(x')^2}{\sqrt{1 + (x')^2}} - x\sqrt{1 + (x')^2} = C,$$

where C is a constant. This implies

$$\frac{x}{\sqrt{1 + (x')^2}} = C_1$$

with $C_1 = -C$. Let us introduce the parameter s such that

$$x' = \sinh s.$$

Then

$$x = C_1 \cosh s,$$

and

$$\mathrm{d}t = \frac{\mathrm{d}x}{x'} = \frac{C_1 \sinh s}{\sinh s}.$$

Thus, we get the soultion of Lagrange equation in the parametric form

$$t = C_1 s + C_2, \quad x = C_1 \cosh s,$$

where C_1, C_2 are arbitraty constants or equivalently

$$x = \frac{1}{D_1} \cosh(D_1 t + D_2)$$

with some new constants D_1, D_2. Substituting this expression to the boundary conditions we obtain a system of algebraic equations

$$\cosh(D_1 a + D_2) = D_1 A, \quad \cosh(D_1 b + D_2) = D_1 B,$$

which detemines the constants D_1, D_2. This system cannot be solved explicitly, but numerically.

Problem 10. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_1^2 t^3(x')^4 \mathrm{d}t, \quad x(1) = 0, \quad x(2) = 2\ln 2.$$

Solution. The Lagrangian for our functional $\mathcal{L} = t^3 v^4$ does not depend on x. Hence, there exists an integral of motion in the form

$$\frac{\partial \mathcal{L}}{\partial v}(t, x(t), x'(t)) = 4t^3(x'(t))^3 = C$$

where C is a constant. This implies

$$tx' = C_1$$

with some other arbityrary constant C_1. Solving this equation yields

$$x = C_1 \ln t + C_2,$$

where C_1, C_2 are arbitrary constants.

Substituting this solution to the boundary condition, we obtain

$$x(1) = C_2 = 0, \quad x(2) = C_1 \ln 2 + C_2 = 2\ln 2.$$

Finally, we conclude that there exists the unique stationary curve for the functional $\mathcal{I}$

$$x(t) = 2\ln t.$$

Problems for self-control:

In the following problems (if the other is not specified) find all stationary curves of the given functionals:

Problem 11.

$$\mathcal{I}(x) = \int_{-1}^{0} (12xt - (x')^2)\mathrm{d}t, \quad x(-1) = 1, \quad x(0) = 0.$$

Problem 12. *Find a curve connecting the two given points on the plane and having the minimal length.*

Problem 13. *Find geodesics on the circular cylinder of radius* r

$$x^2 + y^2 = r^2.$$

Problem 14. *Find geodesics on the sphere of radius* r

$$x^2 + y^2 + z^2 = r^2.$$

Problem 15.

$$\mathcal{I}(x) = \int_0^{\pi/2} (2x + x^2 - (x')^2)\mathrm{d}t, \quad x(0) = 0, \ \ x(\pi/2) = 0.$$

Problem 16.

$$\mathcal{I}(x) = \int_0^1 ((x')^2 + x^2 + 4x \sinh t)\mathrm{d}t, \quad x(0) = -1, \ \ x(1) = 0.$$

Problem 17.

$$\mathcal{I}(x) = \int_0^{\pi/4} (4x^2 - (x')^2)\mathrm{d}t, \quad x(0) = 1, \ \ x(\pi/4) = 0.$$

Problem 18.

$$\mathcal{I}(x) = \int_0^1 ((x')^2 + x^2 + tx)\mathrm{d}t, \quad x(0) = 0, \ \ x(1) = 0.$$

Problem 19.

$$\mathcal{I}(x) = \int_0^1 ((x')^2 + 3x^2)e^{2t}\mathrm{d}t, \quad x(0) = 1, \ \ x(1) = e.$$

Problem 20.

$$\mathcal{I}(x) = \int_0^T ((x')^2 + x)\mathrm{d}t, \quad x(0) = 1.$$

Problem 21.

$$\mathcal{I}(x) = \int\limits_0^T ((x')^2 + x)\mathrm{d}t, \quad x(T) = T.$$

Problem 22.

$$\mathcal{I}(x) = \int\limits_{-1}^0 (12xt - (x')^2)\mathrm{d}t, \quad x(-1) = 1.$$

Problem 23.

$$\mathcal{I}(x) = \int\limits_0^1 ((x')^2 - x + 1)\mathrm{d}t, \quad x(0) = 0.$$

Problem 24.

$$\mathcal{I}(x) = \int_0^T ((x')^2 + x + 2)\mathrm{d}t, \quad x(0) = 0.$$

Problem 25.

$$\mathcal{I}(x) = \int\limits_0^1 (x')^3\mathrm{d}t, \quad x(0) = 0.$$

Problem 26.

$$\mathcal{I}(x) = \int\limits_0^{\pi/2} ((x')^2 - x^2)dt.$$

Problem 27.

$$\mathcal{I}(x,y) = \int\limits_1^2 ((x')^2 + (y')^2 + y^2)\mathrm{d}t,$$
$$x(1) = 1, \quad y(1) = 0, \quad x(2) = 2, \quad y(2) = 1.$$

Problem 28.

$$\mathcal{I}(x,y) = \int\limits_0^\pi (2xy + (x')^2 - (y')^2 + y^2 - 2x^2)\mathrm{d}t,$$

$$x(0) = 1, \quad y(0) = 0, \quad x(\pi) = 1, \quad y(\pi) = -1.$$

Problem 29.

$$\mathcal{I}(x) = \int\limits_0^1 (x + (x')^2)\mathrm{d}t, \quad x(0) = 1, \quad x(1) = 0.$$

Problem 30.

$$\mathcal{I}(x) = \int\limits_0^2 (t + 1 - (x')^2)\mathrm{d}t, \quad x(0) = 0, \quad x(2) = 2.$$

Problem 31.

$$\mathcal{I}(x) = \int\limits_0^{\pi/2} (2x + x^2 - (x')^2)\mathrm{d}t, \quad x(0) = 0, \quad x(\pi/2) = 0.$$

Problem 32.

$$\mathcal{I}(x) = \int\limits_0^1 (t^2 - (x')^2)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = 0.$$

2 Sufficient conditions for existence of extrema

Up to this section we considered only the necessary conditions on existence of extremum for integral functionals. In this section we obtain sufficient conditions for existence of extremum. In particular, we answer the question when a solution of Lagrange equation is an extremal curve.

Without loss of generality we discuss only the case of minimal curves, since any maximal curve of a functional $\mathcal{I}$ becomes a minimal one for the functional $-\mathcal{I}$.

2.1 The second variation of a functional

Let $\mathcal{I}$ be an integral functional defined on a subset $\mathcal{F}_0$ with the Lagrangian $\mathcal{L}$. To simplify exposition we assume in this section $\mathcal{L} \in C^3(I_0 \times \Omega \times \mathbb{R}^n, \mathbb{R})$ and $I_x = I = [a, b]$ for all $x \in \mathcal{F}_0$.

For a given function $x \in \mathcal{F}_0$ we define a set of *admissible* functions $\mathcal{G}_x$

$$\mathcal{G}_x = \left\{ h \in C^2(I, \mathbb{R}^n) : x + h \in \mathcal{F}_0 \right\}. \tag{21}$$

Particularly, for the variational problem with fixed ends and $\Omega = \mathbb{R}^n$ for any $x \in \mathcal{F}_{a,b,A,B}$ the set of admissible functions is

$$\mathcal{G}_x = \left\{ h \in C^2(I, \mathbb{R}^n) : h(a) = h(b) = 0 \right\}. \tag{22}$$

Introduce also

$$\|h\|_s = d_s(h, 0), \quad \|h\|_w = d_w(h, 0), \tag{23}$$

where d_s, d_w are described by formulas (2).

Let $x_0 \in \mathcal{F}_0$ and $h \in \mathcal{G}_{x_0}$ such that $\|h\|_w$ is sufficiently small. We consider the difference $\Delta\mathcal{I}(x_0 + h, x_0)$. Using the Taylor's formula one expands the Lagrangian to obtain

$$\begin{aligned}\Delta\mathcal{I}(x_0 + h, x_0) = &\int_a^b \left(\mathcal{L}_x(t, x_0(t), x_0'(t))h(t) + \mathcal{L}_v(t, x_0(t), x_0'(t))h'(t)\right) \mathrm{d}t + \\ &+ \frac{1}{2}\int_a^b \Big(\mathcal{L}_{xx}(t, x_0(t), x_0'(t))h^2(t) + 2\mathcal{L}_{xv}(t, x_0(t), x_0'(t))h(t)h'(t) + \\ &+ \mathcal{L}_{vv}(t, x_0(t), x_0'(t))(h'(t))^2 + O\left((|h(t)| + |h'(t)|)^3\right) \Big) \mathrm{d}t,\end{aligned} \tag{24}$$

where the constant in the error term is independent of h.

The terms linear in h define the first variation $\delta\mathcal{I}(x_0, h)$

$$\delta\mathcal{I}(x_0, h) = \int_a^b \left(\mathcal{L}_x(t, x_0(t), x_0'(t))h(t) + \mathcal{L}_v(t, x_0(t), x_0'(t))h'(t)\right) \mathrm{d}t. \tag{25}$$

If x_0 is extremal for the functional $\mathcal{I}$ the first variation vanishes. The opposite is not true in general. This leads to the following definition.

Definition 2.1. *Any curve $x_0 \in \mathcal{F}_0$ such that*

$$\delta\mathcal{I}(x_0, h) = 0, \quad \forall\, h \in \mathcal{G}_{x_0}$$

is called stationary (or critical).

Consider the quadratic terms with respect to h in (44). They are called *the second variation* of the functional $\mathcal{I}$ evaluated at x_0 along h and denoted as

$$\delta^2\mathcal{I}(x_0,h) = \frac{1}{2}\int\limits_a^b \Big(\mathcal{L}_{xx}(t,x_0(t),x_0'(t))h^2(t) + \\ + 2\mathcal{L}_{xv}(t,x_0(t),x_0'(t))h(t)h'(t) + \mathcal{L}_{vv}(t,x_0(t),x_0'(t))(h'(t))^2\Big)\mathrm{d}t. \quad (26)$$

As in the previous section we use the shortened notation. In particular, the expression $\mathcal{L}_{xx}(t,x_0(t),x_0'(t))h^2(t)$ means

$$\mathcal{L}_{xx}(t,x_0(t),x_0'(t))h^2(t) = \sum_{k=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\mathcal{L}}{\partial x_k \partial x_j}(t,x_0(t),x_0'(t))h_k(t)h_j(t).$$

It is convenient to represent the second variation in a different form. Taking into account that

$$\int\limits_a^b 2\mathcal{L}_{xv}(t,x_0(t),x_0'(t))h(t)h'(t)\mathrm{d}t = \mathcal{L}_{xv}(t,x_0(t),x_0'(t))h^2(t)\Big|_a^b - \\ \int\limits_a^b \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{xv}(t,x_0(t),x_0'(t))\right)h^2(t)\mathrm{d}t,$$

we rewrite (26) as

$$\delta^2\mathcal{I}(x_0,h) = \mathcal{L}_{xv}(t,x_0(t),x_0'(t))h^2(t)\Big|_a^b + \\ \frac{1}{2}\int\limits_a^b \left[\left(\mathcal{L}_{xx}(t,x_0(t),x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{xv}(t,x_0(t),x_0'(t))\right)h^2(t) + \\ \mathcal{L}_{vv}(t,x_0(t),x_0'(t))(h'(t))^2\right]\mathrm{d}t. \quad (27)$$

Thus, if x_0 is extremal (or, at least, stationary)

$$\Delta\mathcal{I}(x_0+h,x_0) = \delta^2\mathcal{I}(x_0,h) + \int\limits_a^b O\left((|h(t)|+|h'(t)|)^3\right)\mathrm{d}t. \quad (28)$$

Theorem 2.1. *If $x_0 \in \mathcal{F}_0$ is minimal for an integral functional $\mathcal{I}$ then for any admissible function $h \in \mathcal{G}_{x_0}$*

$$\delta^2\mathcal{I}(x_0, h) \geq 0.$$

Proof: Assume the opposite, i.e. there exists some $h_0 \in \mathcal{G}_{x_0}$ such that $\delta^2\mathcal{I}(x_0, h_0) < 0$. Consider $\Delta\mathcal{I}(x_0 + \varepsilon h, x_0)$ for small ε. Then

$$\Delta\mathcal{I}(x_0 + \varepsilon h, x_0) = \varepsilon^2\delta^2\mathcal{I}(x_0, h_0) + O\left(\varepsilon^3\right) < 0$$

provided ε to be sufficiently small. This contradicts the assumption that x_0 is minimal. $\square$

The following theorem provides a sufficient condition for a weak minimum of a functional.

Theorem 2.2. *Assume $x_0 \in \mathcal{F}_0$ is stationary for $\mathcal{I}$ (i.e. $\delta\mathcal{I}(x_0, h) = 0$ for all admissible functions $h \in \mathcal{G}_{x_0}$). Then, if there exists a positive constant $\varkappa$ such that*

$$\delta^2\mathcal{I}(x_0, h) \geq \varkappa\|h\|_w^2, \quad \forall\, h \in \mathcal{G}_{x_0},$$

the curve x_0 furnishes a weak minimum to the functional $\mathcal{I}$.

Proof: Fix $\varepsilon > 0$ such that

$$\left|\Delta\mathcal{I}(x_0 + h, x_0) - \delta^2\mathcal{I}(x_0, h)\right| < \frac{\varkappa}{2}\|h\|_w^2, \quad \forall\, h \in \mathcal{G}_{x_0} : \ \|h\|_w < \varepsilon. \tag{29}$$

This is possible due to (28). Then

$$\Delta\mathcal{I}(x_0 + h, x_0) > \frac{\varkappa}{2}\|h\|_w^2 > 0$$

provided $\|h\|_w < \varepsilon$. This finishes the proof. $\square$

Note that conditions of Theorem 2.2 is not sufficient for a strong minimum, since the error term in (28) does not satisfy in general to an analog of estimate (29), where the weak norm is replaced by the strong one.

2.2 Quadratic functionals. Legendre's necessary condition

We have seen in the previous subsection that the second variation plays an important role in analysis of existence problem for extrema of an integral functional. We focus in this subsection on the variational problem with fixed ends. As it was already mentioned for any $x \in \mathcal{F}_{a,b,A,B}$ the set of admissible functions $\mathcal{G}_x$ is described by formula (22). Then, by (27), the second variation of a functional defined on $\mathcal{F}_{a,b,A,B}$ is

$$\delta^2\mathcal{I}(x_0,h) = \frac{1}{2}\int\limits_a^b \Big[\Big(\mathcal{L}_{xx}(t,x_0(t),x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{xv}(t,x_0(t),x_0'(t))\Big)h^2(t)+ \mathcal{L}_{vv}(t,x_0(t),x_0'(t))(h'(t))^2\Big]\mathrm{d}t. \quad (30)$$

It can be considered as a quadratic functional on the class of admissible functions. In this subsection we study the variational problem of a quadratic functional

$$\mathcal{I}(h) = \int\limits_a^b \big(Q(t)h^2(t) + P(t)h'^2(t)\big)\,\mathrm{d}t \quad (31)$$

on a set of functions $h \in C^2(I,\mathbb{R}^n)$ satisfying $h(a) = h(b) = 0$. Here functions $Q, P \in C^2(I, M(n,\mathbb{R}))$, where $M(n,\mathbb{R})$ stands for the set of quadratic $n\times n$ matrices with real entries.

Theorem 2.3. *If a quadratic functional* (31) *is non-negative (i. e.* $I(h) \geq 0, \forall\, h$*) then*

$$P(t) \geq 0, \quad \forall\, t \in I. \quad (32)$$

Proof: We begin the proof with the case $n = 1$. Assume that there exists an internal point $t_0 \in I$ such that $P(t_0) < 0$ and consider a C^2-function $h = \sqrt{\varepsilon}h_0((t-t_0)/\varepsilon)$, where

$$h_0(t) = \begin{cases} \frac{(1-t^2)^3}{3}, & |t| \leq 1, \\ 0, & |t| > 1 \end{cases}$$

and ε is positive and sufficiently small such that $(t_0-\varepsilon, t_0+\varepsilon) \subset I$. Then

$$\mathcal{I}(h) = \int\limits_{t_0-\varepsilon}^{t_0+\varepsilon} Q(t)h^2(t)\mathrm{d}t + \int\limits_{t_0-\varepsilon}^{t_0+\varepsilon} P(t)h'^2(t)\mathrm{d}t.$$

The first integral tends to zero as $\varepsilon \to 0$, while the second one admits an estimate

$$\int\limits_{t_0-\varepsilon}^{t_0+\varepsilon} P(t)h'^2(t)\mathrm{d}t = \alpha P(t_0) + O(\varepsilon), \quad \alpha = 4\int\limits_{-1}^{1} s^2(1-s^2)^4\mathrm{d}s.$$

Note that $\alpha \approx 7.15\ldots > 0$. Hence, $\mathcal{I}(h) < 0$ for sufficiently small ε. This contradicts the fact that the functional is non-negative.

In multidimensional case ($n > 1$) the condition (32) means that a quadratic form (Ph', h') satisfies

$$\sum_{i,j=1}^{n} P_{ij}(t)h_i'(t)h_j'(t) \geq 0, \quad \forall\, t \in I.$$

It follows from Linear Algebra that there exists a continuous mapping $\mathcal{A} \in C(I, M(n, \mathbb{R}))$ such that $\mathcal{A}(t)$ is invertible for any $t \in I$ and the quadratic form (Ph', h') is diagonazable by means of $\mathcal{A}(t)$, i.e.

$$(P(t)h'(t), h'(t)) = \sum_{i=1}^{n} \lambda_i(t)\xi_i^2(t), \quad \lambda_i(t) \in \mathbb{R}, \quad \xi(t) = \mathcal{A}(t)h'(t).$$

Assume that there exists an internal point $t_0 \in I$ and $h_1 \in \mathcal{G}_{x_0}$ such that $(P(t_0)h_1'(t_0), h_1'(t_0)) < 0$. It means that there exists a natural $k \leq n$ such that $\lambda_k(t) < 0$ in a small neighborhood of t_0.

Consider a C^2-function

$$h = \sqrt{\varepsilon} h_0((t - t_0)/\varepsilon)\mathcal{A}^{-1}(t_0)\mathbf{e_k},$$

where the vector $\mathbf{e_k} \in \mathbb{R}^n$ is such that

$$(\mathbf{e_k})_i = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

Then, similarly to the case $n = 1$, one gets for sufficiently small ε

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} (Q(t)h(t), h(t))\mathrm{d}t = O(\varepsilon), \quad \int_{t_0-\varepsilon}^{t_0+\varepsilon} (P(t)h'(t), h'(t))\mathrm{d}t = \alpha\lambda_k(t_0) + O(\varepsilon)$$

and, hence, $\Delta\mathcal{I}(x_0, h) < 0$ what contradicts the Theorem assumptions. $\square$

Thus, one may note that the term $P(t)h'^2(t)$ plays a crucial role for sign-definitness of the quadratic functional (31).

We apply this theorem to the second variation of an integral functional (26). Notice that if x_0 is minimal for a functional $\mathcal{I}$ then due to Theorem 2.1 the second variation is non-negative and one obtains the following

Theorem 2.4 (Legendre's necessary condition). *Let x_0 be minimal for a functional $\mathcal{I}$ defined on the set $\mathcal{F}_{a,b,A,B}$. Then*

$$\mathcal{L}_{vv}(t, x_0(t), x_0'(t)) \geq 0, \quad \forall\, t \in I.$$

2.3 Quadratic functionals. Sufficient condition of positivity in one-dimensional case

In the previous subsection we obtained that condition $\mathcal{L}_{vv}(t, x_0(t), x_0'(t)) \geq 0$ is necessary for extremum. On the other hand such condition and even stronger condition $\mathcal{L}_{vv}(t, x_0(t), x_0'(t)) > 0$ cannot be sufficient due to its local character. We focus in this subsection on the case $n = 1$. To get a sufficient condition we consider the Lagrange equation for a quadratic functional (31):

$$Qh - \frac{\mathrm{d}}{\mathrm{d}t}(Ph') = 0. \tag{33}$$

This is a linear differential equation supplied by boundary conditions $h(a) = h(b) = 0$. Obviuosly, equation (33) has the trivial solution $h(t) \equiv 0$. However, it may also possesses some non-trivial solutions. This observation leads to an important notion of a *conjugate* point.

Definition 2.2. *A point $t_0 \in (a, b]$ is called conjugate to the point $t = a$ if equation* (33) *possesses a non-trivial solution such that $h(a) = h(t_0) = 0$.*

We remark that if $h(t)$ is such a non-trivial solution then $Ch(x)$ with any constant $C \neq 0$ is also a non-trivial solution of that kind. To fix constant C we add a condition $h'(a) = 1$.

Theorem 2.5. *Assume that $P(t) > 0$ for all $t \in I$ and there are no conjugate points to a. Then the quadratic functional* (31) *is positively defined, i.e. for all $h \in C^2(I, \mathbb{R})$ such that $h(a) = h(b) = 0$ and $h \not\equiv 0$*

$$\mathcal{I}(h) > 0.$$

Proof: Note that due to conditions $h(a) = h(b) = 0$ one has for any function $w \in C^1(I, \mathbb{R})$

$$\int_a^b \mathrm{d}(w(t)h^2(t)) = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(w(t)h^2(t))\mathrm{d}t = 0.$$

We look for a function w which satisfies

$$Ph'^2 + Qh^2 + \frac{\mathrm{d}}{\mathrm{d}t}(wh^2) = Ph'^2 + 2whh' + (Q + w')h^2 = P\left(h' + \frac{w}{P}h\right)^2.$$

The latter equality takes place if w solves the Riccati's equation

$$P(Q + w') = w^2. \tag{34}$$

To solve (34) we make a substitution

$$w = -\frac{u'}{u}P,$$

where u is a new unknown function. Then u satisfies

$$-\frac{\mathrm{d}}{\mathrm{d}t}(Pu') + Qu = 0,$$

i.e. u satisfies the Lagrange equation (33). By assumption there are no conjugate points to a and, hence, all solutions of (33), which satisfy the zeroth boundary condition at $t = a$, do not vanish on $(a, b]$. We show that there exists a solution of (33), which does not vanish on the whole segment $I = [a, b]$.

Indeed, take any solution u_1 of the Lagrange equation (33) such that $u_1(a) = 0$ and consider the Cauchy problem

$$-\frac{\mathrm{d}}{\mathrm{d}t}(Pu') + Qu = 0, \quad u(a) = \gamma, \quad u'(a) = u_1'(a), \tag{35}$$

where $\gamma \in \mathbb{R}$ and $|\gamma|$ is small. Such Cauchy problem possesses a unique solution which smoothly depends on initial data. In particular, this solution $u = u(t, \gamma)$ is a C^2-function of γ. Note that $u(\cdot, 0) = u_1(\cdot)$ and $u_1(t) \neq 0$ for all $t \in (a, b]$. Then, by the Implicit Function Theorem, there exists $\rho > 0$ and $\gamma_\rho \in \mathbb{R}$ with $|\gamma_\rho| < \rho$ such that $u(t, \gamma_\rho) \neq 0$ for all $t \in I$.

We take this solution $u(t, \gamma_\rho)$ to construct a function w satisfying the Riccati equation. In this case the functional $\mathcal{I}$ has the form

$$\mathcal{I}(h) = \int_a^b P(t)\left(h'(t) + \frac{w(t)}{P(t)}h(t)\right)^2 \mathrm{d}t \geq 0. \tag{36}$$

Finally, we remark that if a function h is such that $\mathcal{I}(h) = 0$ then h is extremal and, hence, a solution of the Lagrange equation (33). On the other hand, it has to satisfy

$$h'(t) + \frac{w(t)}{P(t)}h(t) \equiv 0.$$

Taking into account that $h(a) = 0$, one obtains $h'(a) = 0$. This implies by uniqueness of a solution of a Cauchy problem that $h(t) \equiv 0$. Thus, the functional $\mathcal{I}$ is positively defined. $\square$

2.4 Jacobi's sufficient condition for extremum

We apply the results obtained in the previous subsection to the one-dimensional variational problem with fixed ends, i.e. to a functional

$$\mathcal{I}(x) = \int\limits_a^b \mathcal{L}(t, x(t), x'(t))\mathrm{d}t$$

defined on the set of functions $\mathcal{F}_{a,b,A,B}$.

Assume that a curve x_0 is *stationary* for this functional. Then it is a solution of the Lagrange equation and the second variation of $\mathcal{I}$ is a quadratic functional

$$\delta^2(x_0, h) = \int\limits_a^b \big(P(t)h'^2(t) + Q(t)h^2(t)\big)\,\mathrm{d}t$$

with

$$\begin{aligned} P(t) &= \frac{1}{2}L_{vv}(t, x_0(t), x_0'(t)), \\ Q(t) &= \frac{1}{2}\left(L_{xx}(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}L_{xv}(t, x_0(t), x_0'(t))\right). \end{aligned} \tag{37}$$

Definition 2.3. *If $x_0 \in \mathcal{F}_{a,b,A,B}$ is a stationary curve for the integral functional $\mathcal{I}$, then an equation*

$$-\frac{\mathrm{d}}{\mathrm{d}t}(Ph') + Qh = 0,$$

where P, Q are defined by (37) *is called the Jacobi's equation for this functional.*

We formulate a sufficient condition for a curve x_0 to furnish a local weak minimum for the functional $\mathcal{I}$.

Theorem 2.6 (Jacobi's sufficient condition for extremum)**.** *Let $x_0 \in \mathcal{F}_{a,b,A,B}$ satisfies the following conditions:*
1. *x_0 is a stationary point of the functional $\mathcal{I}$;*
2. *Along x_0 the function P defined by* (37) *is positive*

$$P(t) > 0, \quad \forall\, t \in I;$$

3. *The segment I does not contain points conjugate to the point $t = a$.*
Then x_0 furnishes a local weak minimum to the functional $\mathcal{I}$.

Proof: Since $P(t) > 0$ and continuous its minimum, P_{min}, on the segment I is positive $P_{min} > 0$.

Consider for real α such that $\alpha^2 < P_{\min}$ a quadratic functional

$$\mathcal{J}_\alpha(x) = \int_a^b \left(Ph'^2 + Qh^2\right) \mathrm{d}t - \alpha^2 \int_a^b h'^2 \mathrm{d}t$$

and the correspondent Lagrange equation

$$-\frac{\mathrm{d}}{\mathrm{d}t}\left((P-\alpha^2)h'\right) + Qh = 0. \tag{38}$$

Let $h(\cdot,\alpha)$ be a solution of this equation satisfying initial conditions $h(a,\alpha) = 0, h'(a,\alpha) = 1$. Since coefficients of (38) smoothly depend on α, the solution $h(\cdot,\alpha)$ is a continuous function of the parameter α. Due to the Theorem assumptions the solution $h(\cdot,0)$ does not vanish on $(a,b]$. Hence, by the Implicit Function theorem there exists $\alpha_0 < \sqrt{P_{\min}}$ such that for all $\alpha \in [0,\alpha_0]$ the solution $h(t,0) \neq 0$ for any $t \in (a,b]$. Thus, for any $\alpha \in [0,\alpha_0]$ the segment I does not contain points conjugate to $t = a$.

Applying Theorem 2.5 one concludes that the functional $\mathcal{J}_{\alpha_0} > 0$, i.e. for all $h \in C^2(I,\mathbb{R})$ such that $h(a) = h(b) = 0$ and $h \not\equiv 0$

$$\mathcal{J}_{\alpha_0}(h) > 0 \Longleftrightarrow \delta^2\mathcal{I}(x_0,h) > \alpha_0^2 \int_a^b h'^2(t)\mathrm{d}t. \tag{39}$$

The Cauchy inequality yields

$$h^2(t) = \left(\int_a^t h'(s)\mathrm{d}s\right)^2 \le (t-a)\int_a^t h'^2(s)\mathrm{d}s \le (t-a)\int_a^b h'^2(s)\mathrm{d}s.$$

Then

$$\int_a^b h^2(t)\mathrm{d}t \le \frac{(b-a)^2}{2}\int_a^b h'^2(t)\mathrm{d}t.$$

Since x_0 is stationary we obtain that there exist $\varepsilon > 0$ such that for any admissible function h satisfying $\|h\|_w < \varepsilon$ (see (28))

$$\Delta(x_0+h,h) - \delta^2(x_0,h) = O\big(\varepsilon\|h\|_w^2\big) = O\left(\varepsilon\left(1+\frac{(b-a)^2}{2}\right)\int_a^b h'^2(t)\mathrm{d}t\right).$$

Since ε is arbitrary small one gets

$$\left|\Delta(x_0+h,h)-\delta^2(x_0,h)\right| < \frac{\alpha_0^2}{2}\int\limits_a^b h'^2(t)\mathrm{d}t$$

provided $\|h\|_w$ is small. Together with (39) this implies

$$\Delta(x_0+h,h) > 0$$

for any admissible function h with sufficiently small $\|h\|_w$. $\square$

2.5 Quadratic functionals. Sufficient condition of positivity in multi-dimensional case

To formulate sufficient conditions in multi-dimensional case we need to generalize the notion of a conjugate point. As in one-dimensional case we consider the Lagrange equation for a quadratic functional (31)

$$Qh - \frac{\mathrm{d}}{\mathrm{d}t}(Ph') = 0.$$

In the case $n = 1$ Definition 2.2 can be reformulated as follows. A point $t_0 \in (a, b]$ is said to be conjugate to the point $t = a$ if the unique soultion of the Lagrange equation (33) satisfying the initial conditions $x(a) = 0, x'(a) = 1$ vanishes at $t = t_0$. This formulation admits a direct generalization for multi-dimensional case.

Definition 2.4. *Let $u_k, k = 1,\dots,n$ be solutions of the Lagrange equation* (33) *satisfying initial conditions*

$$(u_k)_j(a) = 0, \quad (u_k')_j(a) = \delta_{k,j}, \quad j = 1,\dots,n,$$

where $\delta_{k,j}$ stands for the Kronecker symbol

$$\delta_{k,j} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

One says that a point $t_0 \in (a, b]$ is conjugate to the point $t = a$ if

$$\det \mathcal{U}(t_0) = 0,$$

where entries of the matrix $\mathcal{U}$ are defined as $\mathcal{U}_{j,k} = (u_k)_j$.

We remark that definition 2.4 transforms into 2.2 when $n = 1$.

Taking this general definition of a conjugate point Theorem 2.5 becomes valid also in the multi-dimensional case

Theorem 2.7. *Assume that $P(t) > 0$ for all $t \in I$ (i.e. $(P(t)\xi, \xi) > 0$ for all $\xi \in \mathbb{R}^n : \xi \neq 0$) and there are no conjugate points to a. Then the quadratic functional* (31) *is positively defined.*

Proof: We give a sketch of the proof since it repeats the proof of Theorem 2.5. Let $W \in C^1(I, M(n, \mathbb{R}))$ such that $W(t)$ is hermitian ($W^*(t) = W(t)$ for all $t \in I$). Then as in one-dimensional case one has

$$\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}(W(t)h^2(t))\mathrm{d}t = \int_a^b \big(W(t)h^2(t) + 2W(t)h(t)h'(t)\big)\,\mathrm{d}t = 0$$

for all $h \in C^2(I, \mathbb{R}^n)$ satisfying $h(a) = h(b) = 0$. Hence,

$$Ph'^2 + Qh^2 + \frac{\mathrm{d}}{\mathrm{d}t}(Wh^2) = \\ Ph'^2 + 2Whh' + (Q + W')h^2 = \big(P^{1/2}h' + P^{-1/2}Wh\big)^2,$$

if W is a solution of the matrix Riccati's equation

$$Q + W' = WP^{-1}W. \tag{40}$$

Making a substitution $W = -PU'U^{-1}$ and noting that

$$\big(U^{-1}\big)' = -U^{-1}U'U^{-1},$$

we arrive at

$$QU - \frac{\mathrm{d}}{\mathrm{d}t}(PU') = 0, \tag{41}$$

which is the matrix analog of the Lagrangian equation (33). Due to the absence of conjugate points there exists a unique soultion $\mathcal{U}$ of (41), which satisfies the initial conditions from Definition 2.4 and such that

$$\det\mathcal{U}(t) \neq 0, \quad \forall\, t \in (a, b].$$

On the other hand $\det\mathcal{U}(a) = 0$. Consider a matrix-solution $U(t, \gamma)$ of the equation (41), which satisfies the following initial conditions:

$$U_{j,k}(a) = \gamma\delta_{k,j}, \quad U'_{j,k}(a) = \delta_{k,j}, \quad j, k = 1, \ldots, n,$$

where γ is a small real parameter. This solution is a C^2-function of both variables (t, γ) and $U(t, 0) = \mathcal{U}(t)$. Hence, there exists a positive ρ such that

$$\det U(t, \gamma) = (t - a + \gamma)^n \Big(1 + o(1)\Big), \quad \forall\, t \in [a, a + \rho],\ \gamma \in [-\rho, \rho].$$

Then arguing similar to the one-dimensional case, one may show that there exists $\gamma_\rho \in [-\rho, \rho]$ and the solution $U(t, \gamma_\rho)$ such that $\det U(t) \neq 0$ for all $t \in I$. This leads to positivity of the quadratic functional. $\square$

2.6 Jacobi's sufficient condition for extremum. Multi-dimensional version

As in the case $n = 1$ we apply the results obtained in the previous subsection to the multi-dimensional variational problem with fixed ends. Definition of the Jacobi equation 2.3 is valid also for $n > 1$. Then, taking into account the modified notion of a conjugate point (2.4), one may show that the formulated condition (Theorem 2.6) being sufficient for a curve x_0 to furnish a local weak minimum in the case $n = 1$ stays true also for multi-dimensional case. Namely, one has

Theorem 2.8 (Jacobi's sufficient condition for extremum)**.** *Let $x_0 \in \mathcal{F}_{a,b,A,B}$ satisfies the following conditions:*
1. *x_0 is a stationary point of the functional $\mathcal{I}$;*
2. *Along x_0 the matrix-valued function P defined by* (37) *is positive*

$$P(t) > 0, \quad \forall\, t \in I;$$

3. *The segment I does not contain points conjugate to the point $t = a$.*
Then x_0 furnishes a local weak minimum to the functional $\mathcal{I}$.

We skip the proof of the Theorem and only mention that it follows arguments given in the proof for $n = 1$.

2.7 Practice

Problem 1. *Find all stationary curves of the functional and check by definition whether they give an extremum*

$$\mathcal{I}(x) = \int_0^1 ((x')^2 + x^2)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = 1.$$

Solution: For this functional, the Lagrange equation is of the form

$$x'' - x = 0.$$

The general solution of this differential equation is

$$x(t) = C_1 e^t + C_2 e^{-t},$$

where $C_1, C_2 \in \mathbb{R}$ are arbitrary constants. From the boundary conditions we deduce that $C_1 = 1/(2\sinh 1), C_2 = -1/(2\sinh 1)$, hence, the functional has unique stationary curve

$$x_0(t) = \sinh t / \sinh 1.$$

Now, let us prove that this function gives the minimum to the functional $\mathcal{I}$. We prove this by definition. Consider $\Delta\mathcal{I}(x_0+h, x_0)$ for any smooth function $h(t) \in C^2[0,1]$ with the following conditions $h(0) = 0,\ h(1) = 0$ and show that for any such nonzero function $h(t)$

$$\Delta\mathcal{I}(x_0 + h, x_0) > 0.$$

Indeed,

$$\begin{aligned}\Delta\mathcal{I}(x_0 + h, x_0) &= \mathcal{I}(x_0 + h) - \mathcal{I}(x_0) = \\ &= \int_0^1 ((x_0' + h')^2 + (x_0 + h)^2)\mathrm{d}t - \int_0^1 ((x_0')^2 + x_0^2)\mathrm{d}t = \\ &= \int_0^1 ((h')^2 + h^2)\mathrm{d}t + \int_0^1 (2x_0'h' + 2x_)h)\mathrm{d}t.\end{aligned}$$

The linear term is zero (due to the Lagrange equation)

$$\int_0^1 (2x_0'h' + 2x_0 h)\mathrm{d}t = 0.$$

Thus,

$$\Delta\mathcal{I}(x_0 + h, x_0) = \int_0^1 ((h')^2 + h^2)\mathrm{d}t > 0.$$

Problem 2. *Find all stationary curves of the functional and check by definition whether they give an extremum*

$$\mathcal{I}(x) = \int\limits_0^1 ((x')^2 - 2xt)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = 1.$$

Solution: The Lagrange equation for this functional is

$$x'' + t = 0$$

and its general solution has the form

$$x(t) = -\frac{t^3}{6} + C_1 t + C_2,$$

where C_1, C_2 are arbitrary constants. Substituting this expression to the boundary conditions we obtain that $C_1 = 0, C_2 = 7/6$ and

$$x_0(t) = -\frac{t^3}{6} + \frac{7}{6}$$

is the only stationary curve.

To prove that this function furnishes the minimum to the functional I we again consider $\Delta\mathcal{I}(x_0 + h, x_0)$ for any smooth function $h(t) \in C^2[0, 1]$ with the following conditions $h(0) = 0,\ h(1) = 0$ and show that for any such nonzero function $h(t)$ it is positive

$$\Delta\mathcal{I}(x_0 + h, x_0) > 0.$$

One has

$$\Delta\mathcal{I}(x_0 + h, x_0) = \int\limits_0^1 ((x_0' + h')^2 - 2(x_0 + h)t)\mathrm{d}t - \int\limits_0^1 ((x_0')^2 - 2x_0 t)\mathrm{d}t =$$

$$= \int\limits_0^1 ((h')^2 + h^2)\mathrm{d}t + \int\limits_0^1 (2x_0' h' - 2th)\mathrm{d}t.$$

The linear term is zero (due to the Lagrange equation)

$$\int\limits_0^1 (2x_0' h' - 2ht)\mathrm{d}t = 0.$$

Thus,

$$\Delta\mathcal{I}(x_0+h,x_0)=\int\limits_0^1((h')^2+h^2)\mathrm{d}t>0.$$

Problem 3. *Find all stationary curves of the functional and check by definition whether they give an extremum*

$$\mathcal{I}(x)=\int\limits_0^1((x')^2+(y')^2)\mathrm{d}t,$$
$$x(0)=0,\quad x(1)=1,\quad y(0)=0,\quad y(1)=2.$$

Solution: Note that it is two-dimensional variational problem. The Lagrangian reads

$$\mathcal{L}=u^2+w^2,$$

where u,w are the components of the velocity $v=(u,w)^{tr}$. Hence, the system of Lagrange equations is

$$x''=0,\quad y''=0.$$

The general solution of these differential equations has the form

$$x(t)=C_1t+C_2,\quad y(t)=C_3t+C_4,$$

where C_1,C_2,C_3,C_4 are arbitrary constants. Substituting these expressions into the boundary conditions yields $C_1=1,C_2=0,C_3=2,C_4=0$. Thus, there exists a unique stationary curve

$$x_0(t)=t,\quad y_0(t)=2t.$$

To check whether this stationary curve furnishes extremum to the functional $\mathcal{I}$ we consider $\Delta\mathcal{I}(x_0+h_1,y_0+h_2,x_0,y_0)$ for any admissible vector-function

$$\mathbf{h}(t)=\begin{pmatrix}h_1(t)\\h_2(t)\end{pmatrix},\quad h_{1,2}\in C^2[0,1]:\ h_{1,2}(0)=0,\ h_{1,2}(1)=0.$$

$$\Delta\mathcal{I}(x_0+h_1,y_0+h_2,x_0,y_0)=I(x_0+h_1,y_0+h_2)-I(x_0,y_0)=$$
$$\int\limits_0^1((x_0'+h_1')^2+(y_0'+h_2')^2)\mathrm{d}t-\int\limits_0^1((x_0')^2+(y_0')^2)\mathrm{d}t=$$
$$\int\limits_0^1(2x_0'h_1'+2y_0'h_2')\mathrm{d}t+\int\limits_0^1((h_1')^2+(h_2')^2)\mathrm{d}t,$$

where we integrated by parts the terms linear with respect to $h'_{1,2}$. Notice that

$$\int_0^1 (2x'_0 h'_1 + 2y'_0 h'_2)\mathrm{d}t = 0$$

due to the Lagrange equations.

Thus,

$$\Delta\mathcal{I}(x_0 + h_1, y_0 + h_2, x_0, y_0) = \int_0^1 ((h'_1)^2 + (h'_2)^2)\mathrm{d}t > 0$$

for any admissible function. This means that x_0 is minimal.

Problem 4. *Find all stationary curves of the functional and check by use of the Jacobi theorem whether they give an extremum to the functional for different values of the parameter* a

$$\mathcal{I}(x) = \int_0^a ((x')^2 - x^2)\mathrm{d}t, \quad a > 0, \quad x(0) = 0, \quad x(a) = 1.$$

Solution:. Let us consider the Lagrangian $\mathcal{L} = v^2 - x^2$ and the corresponding Lagrange equation

$$x'' + x = 0.$$

The general solution of this equation is as follows

$$x(t) = C_1 \cos t + C_2 \sin t.$$

Substituting this expression into the boundary conditions, we deduce that the functional $\mathcal{I}$ has a unique stationary curve

$$x_0(t) = \frac{\sin t}{\sin a}.$$

To prove that this function gives the minimum to the functional $\mathcal{I}$ we use the Jacobi sufficient condition. First we note that

$$\mathcal{L}_{vv}(t, x_0(t), x'_0(t)) = 2 > 0 \quad \forall\, t \in [0, a].$$

Hence, the first condition of the Jacobi theorem is satisfied.

Constitute the Jacobi equation. Since

$$\mathcal{L}_{xx}(t, x_0(t), x'_0(t)) = -2, \quad \mathcal{L}_{xv}(t, x_0(t), x'_0(t)) = 0$$

we obtain that the Jacobi equation is

$$-2h - 2h'' = 0$$

or equivalently

$$h'' + h = 0.$$

The solution of this equation, which satisfies the initial conditions $h(0) = 0, h'(0) = 1$ is

$$h_0(t) = \sin t.$$

Note that

$$h_0(t) = 0 \Longleftrightarrow t = \pi k,\ k \in \mathbb{Z}.$$

Thus, we conclude that if $a < \pi$, then there are no conjugate points on the interval $(0, a]$. So, we prove that function $x_0(t) = \sin t / \sin a$ gives the minimum to the functional $\mathcal{I}$.

If $a \geq \pi$ then there is at least one conjugate point to $t = a$ and the Jacobi theorem is not applicable.

Problem 5. *Find all stationary curves of the functional and check by use of the Jacobi theorem whether they give an extremum to the functional for different values of the parameter $\varepsilon \geq 0$*

$$\mathcal{I}(x) = \int_0^1 (\varepsilon(x')^2 + x^2 + t^2)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = 1.$$

Solution: Since the Lagrangian is $\mathcal{L} = \varepsilon v^2 + x^2 = t^2$, the corresponding Lagrange equation has the form

$$2x - 2\varepsilon x'' = 0.$$

The general solution of this equation (for $\varepsilon \neq 0$) is as follows

$$x(t) = C_1 \cosh \frac{t}{\sqrt{\varepsilon}} + C_2 \sinh \frac{t}{\sqrt{\varepsilon}}$$

with arbitrary constants C_1, C_2.

Taking into account the boundary conditions, we obtain that there exists a unique stationary curve

$$x_0(t) = \frac{\sinh \frac{t}{\sqrt{\varepsilon}}}{\sinh \frac{1}{\sqrt{\varepsilon}}}.$$

We apply the Jacobi sufficient condition to check whether x_0 furnishes the minimum to $\mathcal{I}$. Since

$$\mathcal{L}_{vv}(t, x_0(t), x_0'(t)) = \varepsilon > 0,$$

the first condition of the Jacobi theorem is satisfied.

The second derivatives of $\mathcal{L}$ have the form

$$\mathcal{L}_{xx}(t, x_0(t), x_0'(t)) = 2, \quad \mathcal{L}_{xv}(t, x_0(t), x_0'(t)) = 0.$$

Hence, the Jacobi equation is

$$\varepsilon h_1'' + h_1 = 0.$$

It's solution h_0 satisfying the initial conditions $h(0) = 0, h'(0) = 1$ reads as

$$h_0(t) = \frac{\sinh \frac{t}{\sqrt{\varepsilon}}}{\cosh \frac{1}{\sqrt{\varepsilon}}}.$$

It vanishes only at $t = 0$. This implies that there are no conjugate points on the interval $(0, 1]$ and x_0 is minimal.

Finally, we note that for $\varepsilon = 0$ the only solution of the Lagrange equation is $x = 0$. Since it does not satisfy the boundary conditions the variational problem does not have a solution in this case.

Problems for self-control:

In the following problems find (if the other is not specified) all stationary curves of the given functionals and check by both methods (definition and the Jacobi theorem) whether they give an extremum to the functionals.

Problem 6.

$$\mathcal{I}(x) = \int\limits_1^2 ((x')^2 + 3x^2)\mathrm{d}t, \quad x(1) = 0, \quad x(2) = 2.$$

Problem 7.

$$\mathcal{I}(x) = \int\limits_0^1 ((x')^2 + x^2 + 3tx)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = 1.$$

Problem 8. *Study existence of extrema of a functional for different ε*

$$\mathcal{I}(x) = \int_0^1 ((x')^2 + \varepsilon x^2 + t^2)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = 1.$$

Problem 9. *Study existence of extrema of a functional for different T*

$$\mathcal{I}(x) = \int_0^T ((x')^2 - 2x^2)\mathrm{d}t, \quad x(0) = 0, \quad x(T) = 0.$$

Problem 10.

$$\mathcal{I}(x) = \int_1^2 (t^2(x')^2 + 12x^2)\mathrm{d}t, \quad x(1) = 1, \quad x(2) = 8.$$

Problem 11.

$$\mathcal{I}(x) = \int_1^2 (x - t(x')^2)\mathrm{d}t, \quad x(1) = 0, \quad x(2) = 1.$$

Problem 12.

$$\mathcal{I}(x) = \int_0^{\pi/12} ((x')^2 + 2xx' - 9x^2)\mathrm{d}t, \quad x(0) = 0, \quad x(\pi/12) = \sqrt{2}.$$

Problem 13.

$$\mathcal{I}(x) = \int_0^{2\pi} ((x')^2 - x^2)\mathrm{d}t, \quad x(0) = 0, \quad x(2\pi) = 0.$$

Problem 14.

$$\mathcal{I}(x) = \int_1^2 (t^2(x')^2 + 2x^2 + 2\frac{x}{t})\mathrm{d}t, \quad x(1) = 5/2, \quad x(2) = 4.$$

Problem 15.

$$\mathcal{I}(x) = \int\limits_1^2 (t^2(x')^4 - 2x(x')^3)\mathrm{d}t, \quad x(1) = 0, \quad x(2) = 1.$$

Problem 16.

$$\mathcal{I}(x) = \int\limits_0^1 \mathrm{e}^t \left(x^2 + \frac{1}{2}(x')^2 \right) \mathrm{d}t, \quad x(0) = 1, \quad x(1) = \mathrm{e}.$$

3 Some generalizations of the simplest variational problem

In this section we describe some generalizations of the simplest variational problem when the subset of functions $\mathcal{F}_0$ (on which we considered an integral functional) was specified only as a subset of those functions from the $\mathcal{F}$

$$\mathcal{F} = \left\{ x : I_x = [a_x, b_x] \to \Omega : \; x \in C^2(I_x, \Omega), \right\}$$

which have common domain of definition $I_x = [a, b]$. We consider two kinds of generalizations. The first kind concerns some modifications of the subset $\mathcal{F}_0$, the second one deals with modifications of the Lagrangian.

3.1 General formula for the first variation

The aim of this subsection is to establish a formula for the first variation of an integral functional

$$\mathcal{I}(x) = \int\limits_{a_x}^{b_x} \mathcal{L}(t, x(t), x'(t))\mathrm{d}t$$

defined on a subset $\mathcal{F}_0 \subset \mathcal{F}$ without an assumption that all functions from $\mathcal{F}_0$ have common domain of definition.

For this purpose for any $x \in \mathcal{F}_0$ we consider its C^2-continuation, $\hat{x}$, on the whole real line

$$\hat{x}(t) = \begin{cases} x(a_x) + x'(a_x)(t - a_x) + \frac{1}{2}x''(a_x)(t - a_x)^2, & t < a_x, \\ x(t), & t \in [a_x, b_x], \\ x(b_x) + x'(b_x)(t - b_x) + \frac{1}{2}x''(b_x)(t - b_x)^2, & t > b_x. \end{cases}$$

For any given $x, y \in \mathcal{F}_0$ we define the *extended common segment of definition* $I_{x,y}$:

$$I_{x,y} = \big[\min\{a_x, a_y\}, \max\{b_x, b_y\}\big].$$

This allows us to modify definitions of distances d_s, d_w as

$$d_s^2(x(\cdot), y(\cdot)) = \max_{t \in I_{x,y}} \|\hat{x}(t) - \hat{y}(t)\|_{\mathbb{R}^n} + |a_x - a_y| + |b_x - b_y|, \tag{42}$$

$$\begin{aligned} d_w^2(x(\cdot), y(\cdot)) = \max_{t \in I_{x,y}} \|\hat{x}(t) - \hat{y}(t)\|_{\mathbb{R}^n} + \max_{t \in I_{x,y}} \|\hat{x}'(t) - \hat{y}'(t)\|_{\mathbb{R}^n} + \\ + |a_x - a_y| + |b_x - b_y|. \end{aligned} \tag{43}$$

Notice that for any given $x \in \mathcal{F}_0$ there exists $\varepsilon > 0$ such that for any $y \in B_\varepsilon(x) = \{z \in \mathcal{F}_0 : d_w(x, z) < \varepsilon\}$ one has $\hat{x}|_{I_{x,y}}, \hat{y}|_{I_{x,y}} \in \mathcal{F}_0$ and $h = (\hat{x} - \hat{y})|_{I_{x,y}} \in C^2(I_{x,y}, \mathbb{R}^n)$.

We generalize the notion of an admissible function. Namely, for a given $x \in \mathcal{F}_0$ we define a set of *admissible* functions $\mathcal{G}_x$ as

$$\mathcal{G}_x = \left\{ h \in C^2([a_h, b_h], \mathbb{R}^n) : (\hat{x} + \hat{h})|_{I_{x,h}} \in \mathcal{F}_0 \right\}.$$

Then for a given $x \in \mathcal{F}_0$ and $h \in \mathcal{G}_x$ such that $d_w((\hat{x} + \hat{h})|_{I_{x,h}}, x)$ is sufficiently small consider

$$\begin{aligned} \Delta\mathcal{I}((\hat{x} + \hat{h})|_{I_h}, x) &= \mathcal{I}((\hat{x} + \hat{h})|_{I_h}) - \mathcal{I}(x) = \\ &= \int_{a_h}^{b_h} \mathcal{L}(t, \hat{x} + h, \hat{x}' + h')\mathrm{d}t - \int_{a_x}^{b_x} \mathcal{L}(t, x, x')\mathrm{d}t = \\ &= \int_{a_x}^{b_x} \Big(\mathcal{L}(t, x + \hat{h}, x' + \hat{h}') - \mathcal{L}(t, x, x')\Big)\,\mathrm{d}t + \\ &\quad + \int_{b_x}^{b_h} \mathcal{L}(t, \hat{x} + \hat{h}, \hat{x}' + \hat{h}')\mathrm{d}t - \int_{a_x}^{a_h} \mathcal{L}(t, \hat{x} + \hat{h}, \hat{x}' + \hat{h}')\mathrm{d}t. \end{aligned}$$

Expanding the integrants by the Taylor formula and integrating the first integral by parts we obtain

$$\Delta\mathcal{I}(x+h,x) = \int\limits_{a_x}^{b_x} \Big(\mathcal{L}_x(t,x,x')\hat{h} + \mathcal{L}_v(t,x,x')\hat{h}'\Big)\,\mathrm{d}t +$$

$$+ \mathcal{L}(t,x,x')\big|_{t=b_x}\delta b_x - \mathcal{L}(t,x,x')\big|_{t=a_x}\delta a_x + O\big(d_w^2((\hat{x}+\hat{h})|_{I_{x,h}},x)\big) =$$

$$= \int\limits_{a_x}^{b_x} \left(\mathcal{L}_x(t,x,x') - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t,x,x')\right)\hat{h}\,\mathrm{d}t + \mathcal{L}_v(t,x,x')\hat{h}\Big|_{t=a_x}^{t=b_x} +$$

$$+ \mathcal{L}(t,x,x')\big|_{t=b_x}\delta b_x - \mathcal{L}(t,x,x')\big|_{t=a_x}\delta a_x + O\big(d_w^2((\hat{x}+\hat{h})|_{I_{x,h}},x)\big), \quad (44)$$

where $\delta a_x = a_h - a_x$ and $\delta b_x = b_h - b_x$.

Introduce the following notations

$$\delta x_a = \hat{x}(a_h) + h(a_h) - x(a_x), \quad \delta x_b = \hat{x}(b_h) + h(b_h) - x(b_x).$$

Then one has

$$\hat{h}(a_x) = \delta x_a - x'(a_x)\delta a_x + O\big(d_w^2(x+h,x)\big),$$
$$\hat{h}(b_x) = \delta x_b - x'(b_x)\delta b_x + O\big(d_w^2(x+h,x)\big).$$

Substituting this into (44) we obtain a formula for the first variation of a functional in general case. By definition, the *first variation* of the functional $\mathcal{I}$ evaluated at x along admissible function h is called

$$\delta\mathcal{I}(x,h) = \int\limits_{a_x}^{b_x} \left(\mathcal{L}_x(t,x,x') - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t,x,x')\right)\hat{h}\,\mathrm{d}t +$$
$$+ \mathcal{L}_v(t,x,x')\big|_{t=b_x}\delta x_b + \big(\mathcal{L}(t,x,x') - x'\mathcal{L}_v(t,x,x')\big)\big|_{t=b_x}\delta b_x -$$
$$- \mathcal{L}_v(t,x,x')\big|_{t=a_x}\delta x_a - \big(\mathcal{L}(t,x,x') - x'\mathcal{L}_v(t,x,x')\big)\big|_{t=a_x}\delta a_x - \quad (45)$$

Note that this general formula of the first variation contains formulas (6) and (9) obtained when we studied the variational problems with fixed and free ends. Indeed, in the case of variational problem with fixed ends one has $\delta a_x = \delta b_x = \delta x_a = \delta x_b = 0$. This converts (45) into (6). In the case of the free ends variational problem conditions $\delta a_x = \delta b_x = 0$ transforms (45) into (9).

Finally, we remark that the notion of a stationary curve stays without changes, i.e. $x_0 \in \mathcal{F}_0$ is called a *stationary curve* for the functional $\mathcal{I}$ if $\delta\mathcal{I}(x_0,h) = 0$ for any admissible function $h \in \mathcal{G}_x$.

3.2 Variational problem with movable ends

In this subsection we consider the one-dimensional variational problem with movable ends. Let φ_A, φ_B be C^2-functions of variables (t, x), defining smooth curves γ_A, γ_B on the plane (t, x) by equations $\varphi_A(t, x) = 0$, $\varphi_B(t, x) = 0$, respectively.

Consider an integral functional $\mathcal{I}$ defined on a subset of functions

$$\mathcal{F}_0 = \left\{ x \in C^2(I_x, \mathbb{R}) :\ \varphi_A(a_x, x(a_x)) = 0,\ \varphi_B(b_x, x(b_x)) = 0 \right\}.$$

The problem of minimization of such functional $\mathcal{I}$ is called the variational problem with movable ends.

We formulate a necessary condition for $x_0 \in \mathcal{F}_0$ to be an extremal curve for the problem with movable ends.

Theorem 3.1. *If $x_0 \in \mathcal{F}_0$ is an extremal curve for the problem with movable ends, then x_0 satisfies the Lagrange equation*

$$\frac{\partial \mathcal{L}}{\partial x}(t, x, x') - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial v}(t, x, x') = 0$$

and transversality conditions

$$\begin{aligned}
&\big(\mathcal{L}_v(t, x, x')\varphi_{A,t}(t, x) - \big(\mathcal{L}(t, x, x') - x'\mathcal{L}_v(t, x, x')\big)\varphi_{A,x}(t, x)\big)\big|_{t=a_x} = 0,\\
&\big(\mathcal{L}_v(t, x, x')\varphi_{B,t}(t, x) - \big(\mathcal{L}(t, x, x') - x'\mathcal{L}_v(t, x, x')\big)\varphi_{B,x}(t, x)\big)\big|_{t=b_x} = 0.
\end{aligned}$$

Remark: We notice that $\varphi_{A,x}$, $\varphi_{A,t}$ (respectively, $\varphi_{B,x}$, $\varphi_{B,t}$) stand for partial derivatives of φ_A (φ_B) with respect to x and t.

Proof: We consider the case when x_0 is minimal. The case of maximal curve x_0 can be proved in a similar way.

Then for any admissible function $h \in \mathcal{G}_{x_0}$ one has

$$\Delta\mathcal{I}((\hat{x}_0 + \hat{h})|_{I_h}, x_0) = \delta\mathcal{I}(x_0, h) + O\big(d_w^2((\hat{x} + \hat{h})|_{I_{x,h}}, x)\big) \geq 0.$$

In particular, this inequality holds for those $h \in \mathcal{G}_{x_0}$, which satisfy the following conditions

$$I_h = I_{x_0}, \quad h(a_h) = h(b_h) = 0.$$

For such functions

$$\Delta\mathcal{I}((\hat{x}_0 + \hat{h})|_{I_h}, x_0) = \Delta\mathcal{I}(x_0 + h, x_0)$$

and we may argue as in the case of variational problem with fixed ends to conclude that x_0 should satisfies the Lagrange equations.

Now consider such admissible functions $h \in \mathcal{G}_{x_0}$ that

$$a_h = a_{x_0}, \quad h(a_h) = 0.$$

In this case

$$\Delta\mathcal{I}((\hat{x}_0 + \hat{h})|_{I_h}, x_0) = \mathcal{L}_v(t, x_0, x_0')\big|_{t=b_{x_0}} \delta x_b + \\ + \big(\mathcal{L}(t, x_0, x_0') - x_0'\mathcal{L}_v(t, x_0, x_0')\big)\big|_{t=b_{x_0}} \delta b_x + O\big(d_w^2((\hat{x} + \hat{h})|_{I_{x,h}}, x)\big) \geq 0.$$

Let $t = t_B(s), x = x_B(s)$ be a parameterization of the curve γ_B such that $t_B(0) = b_{x_0}, x_B(0) = x_0(b_{x_0})$. Then

$$\delta b_x = \frac{\mathrm{d}t_B}{\mathrm{d}s}(0) \cdot s + O(s^2), \quad \delta x_b = \frac{\mathrm{d}x_B}{\mathrm{d}s}(0) \cdot s + O(s^2).$$

Taking this into account, one obtains

$$\Delta\mathcal{I}((\hat{x}_0 + \hat{h})|_{I_h}, x_0) = \bigg[\mathcal{L}_v(t, x_0, x_0')\big|_{t=b_{x_0}} \cdot \frac{\mathrm{d}x_B}{\mathrm{d}s}(0) + \\ + \big(\mathcal{L}(t, x_0, x_0') - x_0'\mathcal{L}_v(t, x_0, x_0')\big)\big|_{t=b_{x_0}} \cdot \frac{\mathrm{d}t_B}{\mathrm{d}s}(0)\bigg] s + O(s^2) \geq 0.$$

Since this inequality is valid for all sufficiently small s, we conclude that

$$\mathcal{L}_v(t, x_0, x_0')\big|_{t=b_{x_0}} \cdot \frac{\mathrm{d}x_B}{\mathrm{d}s}(0) + \\ + \big(\mathcal{L}(t, x_0, x_0') - x_0'\mathcal{L}_v(t, x_0, x_0')\big)\big|_{t=b_{x_0}} \cdot \frac{\mathrm{d}t_B}{\mathrm{d}s}(0) = 0. \quad (46)$$

Differentiating the equality $\varphi_B\big(t_B(s), x_B(s)\big) = 0$ with respect to s and setting $s = 0$ yields

$$\varphi_{B,t}(t, x_0(t))\big|_{t=b_{x_0}} \cdot \frac{\mathrm{d}t_B}{\mathrm{d}s}(0) + \varphi_{B,x}(t, x_0(t))\big|_{t=b_{x_0}} \cdot \frac{\mathrm{d}x_B}{\mathrm{d}s}(0) = 0.$$

We substitute this relation into (46) and obtain that x_0 satisfies the transversality condition at the right end:

$$\big(\mathcal{L}_v(t, x_0, x_0')\varphi_{B,t}(t, x_0) - \big(\mathcal{L}(t, x_0, x_0') - x_0'\mathcal{L}_v(t, x_0, x_0')\big)\varphi_{B,x}(t, x_0)\big)\,\big|_{t=b_{x_0}} = 0.$$

The transversaility condition at the left end can be varified in a similar way. □

Remark It has to be noted that there exist two special cases when one of the derivatives $\varphi_{A,t}(t, x_0)$ or $\varphi_{A,x}(t, x_0)$ vanishes at the end point. In this case the transversality conditions degenerate and have the form:

$$\begin{aligned}
&\big(\mathcal{L}(t, x_0, x_0') - x_0'\mathcal{L}_v(t, x_0, x_0')\big)\big|_{t=a_{x_0}}, \quad \text{if } \varphi_{B,t}(t, x_0)\big|_{t=a_{x_0}} = 0,\\
&\mathcal{L}_v(t, x_0, x_0')\big|_{t=a_{x_0}} = 0, \quad \text{if } \varphi_{B,x}(t, x_0)\big|_{t=a_{x_0}} = 0.
\end{aligned}$$

Such conditions are called natural boundary conditions. Note that in the case $\varphi_{A,x}(t, x_0) = 0$ the natural boundary conditions coincide with conditions obtained for the variational problem with free ends.

3.3 Isoperimetric problem

In this subsection we descibe another generalization of the simplest variational problem. We consider an integral functional

$$\mathcal{I}(x) = \int_{a_x}^{b_x} \mathcal{L}(t, x(t), x'(t))\mathrm{d}t$$

defined on the following set of functions $\mathcal{F}_0 \subset \mathcal{F}$:

$$\mathcal{F}_0 = \left\{x \in C^2(I_x, \mathbb{R}) : \, a_x = a, \, b_x = b, \, x(a) = A, \, x(b) = B, \, \mathcal{J}(x) = J_0\right\},$$

where a, b, A, B, J_0 are given real numbers and $\mathcal{J}$ is an integral functional

$$\mathcal{J}(x) = \int_a^b \mathcal{K}(t, x(t), x'(t))\mathrm{d}t$$

defined on $\mathcal{F}_{a,b,A,B}$ with $K \in C^2([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

The problem of minimization of the functional $\mathcal{I}$ is called the isoperimetric problem. As in the previous subsection we present a necessary condition for a curve $x_0 \in \mathcal{F}_0$ to be extremal for the isoperimetric problem.

Theorem 3.2. *If $x_0 \in \mathcal{F}_0$ is a minimal curve for the isoperimetric problem, but it is not a stationary curve for the functional $\mathcal{J}$ then there exists $\lambda \in \mathbb{R}$ such that x_0 is a stationary curve for the functional $\mathcal{I}+\lambda\mathcal{J}$ defined on $\mathcal{F}_{a,b,A,B}$.*

Proof: We begin with the following observation. Since x_0 is not a stationary curve for the functional $\mathcal{J}$, it is not a solution of the Lagrange equation associated to $\mathcal{J}$:

$$\mathcal{K}_x(t, x, x') - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_v(t, x, x') = 0.$$

We denote the left hand side of this equation evaluated at x_0 by $\mathcal{A}(t)$:

$$\mathcal{A}(t) = \mathcal{K}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}_v(t, x_0(t), x_0'(t)).$$

Hence, there exists an internal point $t_1 \in (a, b)$ such that

$$\mathcal{A}(t_1) \neq 0.$$

For any internal point $t_2 \in (a, b)$ we consider a function

$$h(t) = \alpha_1 \varepsilon^2 h_0\left(\frac{t - t_1}{\sqrt{\varepsilon}}\right) + \alpha_2 \varepsilon^2 h_0\left(\frac{t - t_2}{\sqrt{\varepsilon}}\right),$$

where

$$h_0(t) = \begin{cases} \frac{(1-t)^3}{3}, & |t| \leq 1, \\ 0, & |t| > 1, \end{cases} \tag{47}$$

α_1, α_2 are real numbers and ε is a small positive parameter.

Notice that $h \in C^2([a, b], \mathbb{R})$, it's support lies in a small neighborhoods of the points $t_{1,2}$ and $h(a) = h(b) = 0$, provided ε to be sufficiently small. Hence, $x_0 + h \in \mathcal{F}_{a,b,A,B}$ and h is an admissible function for x_0 with respect to the functional $\mathcal{J}$. However, it may not be admissible with respect to $\mathcal{I}$. To be an admissible function with respect to $\mathcal{I}$, one needs

$$\mathcal{J}(x_0 + h) = \mathcal{J}(x_0) = J_0.$$

As the weak norm $\|h\|_w = O(\varepsilon^{3/2})$ is assumed to be small, we apply (4) to obtain

$$\begin{aligned} \Delta\mathcal{J}(x_0 + h, x_0) &= \mathcal{J}(x_0 + h) - \mathcal{J}(x_0) = \delta\mathcal{J}(x_0, h) + O(\varepsilon^{7/2}) = \\ &= \int\limits_{t_1-\varepsilon}^{t_1+\varepsilon} \mathcal{A}(t)h(t)\mathrm{d}t + \int\limits_{t_2-\varepsilon}^{t_2+\varepsilon} \mathcal{A}(t)h(t)\mathrm{d}t + O(\varepsilon^{7/2}) = \\ &= \left(\alpha_1\mathcal{A}(t_1) + \alpha_2\mathcal{A}(t_2)\right)\varepsilon^{5/2} \int\limits_{-1}^{1} h_0(s)\mathrm{d}s + O(\varepsilon^3). \end{aligned} \tag{48}$$

Consider an equation $\Delta\mathcal{J}(x_0 + h, x_0) = 0$ with respect to α_1. It can be rewritten as:

$$\left(\alpha_1\mathcal{A}(t_1) + \alpha_2\mathcal{A}(t_2)\right) \int\limits_{-1}^{1} h_0(s)\mathrm{d}s + O(\varepsilon^{1/2}) = 0.$$

Since $\mathcal{A}(t_1) \neq 0$ there exists a unique solution $\alpha_1 = \alpha_1(\alpha_2, \varepsilon)$, which admits an estimate

$$\alpha_1 = -\frac{\mathcal{A}(t_2)}{\mathcal{A}(t_1)}\alpha_2\big(1 + O(\varepsilon^{1/2})\big).$$

Thus, for such α_1 the function h becomes admissible for x_0 with respect to the functional $\mathcal{I}$. We define

$$\lambda = -\frac{\mathcal{B}(t_1)}{\mathcal{A}(t_1)},$$

where

$$\mathcal{B}(t) = \mathcal{L}_x(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t)).$$

Then similar to (48) one gets

$$\begin{aligned}\Delta\mathcal{I}(x_0 + h, x_0) &= \int\limits_{t_1-\varepsilon}^{t_1+\varepsilon} \mathcal{B}(t)h(t)\mathrm{d}t + \int\limits_{t_2-\varepsilon}^{t_2+\varepsilon} \mathcal{B}(t)h(t)\mathrm{d}t + O(\varepsilon^{7/2}) = \\ &= (\alpha_1\mathcal{B}(t_1) + \alpha_2\mathcal{B}(t_2))\,\varepsilon^{5/2}\int\limits_{-1}^{1} h_0(s)\mathrm{d}s + O(\varepsilon^3) = \\ &= (\lambda\mathcal{A}(t_2) + \mathcal{B}(t_2))\,\alpha_2\varepsilon^{5/2}\int\limits_{-1}^{1} h_0(s)\mathrm{d}s + O(\varepsilon^3).\end{aligned}$$

By assumptions of the Theorem x_0 is minimal. Hence $\Delta\mathcal{I}(x_0 + h, x_0) > 0$ for any α_2. This implies

$$\begin{aligned}&\lambda\mathcal{A}(t_2) + \mathcal{B}(t_2) = \\ &= \left[\frac{\partial(\mathcal{L} + \lambda\mathcal{K})}{\partial x}(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial(\mathcal{L} + \lambda\mathcal{K})}{\partial v}(t, x_0(t), x_0'(t))\right]\Bigg|_{t=t_2} = 0.\end{aligned}$$

Since t_2 is arbitrary we conclude that x_0 is a solution of the Lagrange equation for the functional $\mathcal{I} + \lambda\mathcal{J}$. $\square$

3.4 Lagrange problem

As in the previous subsection we consider an integral functional

$$\mathcal{I}(x) = \int\limits_{a_x}^{b_x} \mathcal{L}(t, x, x')\mathrm{d}t,$$

but the domain of definition will be a different subset of functions $\mathcal{F}_0$. Namely, we introduce

$$\mathcal{F}_0 = \left\{x \in C^2(I_x, \Omega) : \ a_x = a, \ b_x = b, \ x(a) = A, \ x(b) = B, \ \Phi(t, x) = 0\right\},$$

where a, b, A, B are given real numbers, Ω is a bounded domain in $\mathbb{R}^2$ and $\Phi \in C^2([a, b] \times \Omega, \mathbb{R})$.

The problem of minimization of the functional $\mathcal{I}$ defined on such set of fucntions is called the Lagrange problem. The following theorem gives a necessary condition for a curve $x_0 \in \mathcal{F}_0$ to be extremal. We denote the coordinates in $\Omega \subset \mathbb{R}^2$ by (y, z) and components of corresponding velocities by (u, w).

Theorem 3.3. *If a curve $x_0 = (y_0, z_0) \in \mathcal{F}_0$ is a minimal curve for the Lagrange problem and*

$$\left|\Phi'_y(t, y_0(t), z_0(t))\right| + \left|\Phi'_z(t, y_0(t), z_0(t))\right| \neq 0, \quad \forall\, t \in [a, b],$$

then there exists a function $\lambda \in C^1([a, b], \mathbb{R})$ such that x_0 satisfies the Lagrange equations assocoated to the Lagrangian $\mathcal{M} = \mathcal{L} + \lambda\Phi$, i.e.

$$\mathcal{M}_y - \frac{\mathrm{d}}{\mathrm{d}t} M_u = 0, \quad \mathcal{M}_z - \frac{\mathrm{d}}{\mathrm{d}t} M_w = 0.$$

Proof: We begin with construction of admissible function $h \in \mathcal{G}_{x_0}$. One notes that an arbitrary function $h \in C^2([a, b], \mathbb{R}^2)$ satisfying $h(a) = h(b) = 0$ is not admissible for x_0, in general. Denote the components of h by (ξ, η) and assuming that $\|h\|_w \ll 1$. Expanding equality $\Phi(t, x_0 + h) = 0$ with respect to h, we obtain

$$\Phi'_y(t, y_0(t), z_0(t))\xi(t) + \Phi'_z(t, y_0(t), z_0(t))\eta(t) = O\big(\|h\|_w^2\big).$$

Thus, one sees that components ξ and η are not indepenedent.

Take an internal points $x_0 \in (a, b)$ and suppose that

$$\Phi'_z(t, y_0(t), z_0(t))\Big|_{t=t_0} \neq 0.$$

We set

$$\xi(t) = \alpha\varepsilon^2 h_0\left(\frac{t - t_0}{\sqrt{\varepsilon}}\right), \tag{49}$$

where α is a real number, ε is a small positive parameter and h_0 is defined by (47). Hence,

$$\eta(t) = -\frac{\Phi'_y(t, y_0(t), z_0(t))}{\Phi'_z(t, y_0(t), z_0(t))}\xi(t) + O\big(\|h\|_w^2\big) =$$

$$= -\frac{\Phi'_y(t, y_0(t), z_0(t))}{\Phi'_z(t, y_0(t), z_0(t))}\xi(t) + O\big(\varepsilon^3\big)$$

In the case $\Phi'_z(t, y_0(t), z_0(t))\big|_{t=t_0} = 0$ the components ξ and η change their roles.

By construction the function h is admissible for x_0 and it is possible to consider $\Delta\mathcal{I}(x_0 + h, x_0)$. Introduce the following notations:

$$\mathcal{A}(t) = \mathcal{L}_y(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_u(t, x_0(t), x_0'(t)),$$
$$\mathcal{B}(t) = \mathcal{L}_z(t, x_0(t), x_0'(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_w(t, x_0(t), x_0'(t)).$$

The standard arguments show that

$$\Delta\mathcal{I}(x_0 + h, x_0) = \int\limits_a^b \mathcal{A}(t)\xi(t)\mathrm{d}t + \int\limits_a^b \mathcal{B}(t)\eta(t)\mathrm{d}t + O\big(\varepsilon^{7/2}\big) =$$
$$\int\limits_a^b \left(\mathcal{A}(t) - \frac{\Phi'_y(t, y_0(t), z_0(t))}{\Phi'_z(t, y_0(t), z_0(t))}\mathcal{B}(t)\right)\xi(t)\mathrm{d}t + O\big(\varepsilon^3\big). \quad (50)$$

Substitute (49) and (47) into (50) to get

$$\Delta\mathcal{I}(x_0 + h, x_0) =$$
$$= \left(\mathcal{A}(t) - \frac{\Phi'_y(t, y_0(t), z_0(t))}{\Phi'_z(t, y_0(t), z_0(t))}\mathcal{B}(t)\right)\Bigg|_{t=t_0} \alpha\varepsilon^{5/2} \int\limits_{-1}^{1} h_0(s)\mathrm{d}s + O\big(\varepsilon^3\big) \geq 0.$$

Since ε is arbitrary, we obtain that for any $t_0 \in (a, b)$

$$\Big(\Phi'_z(t, y_0(t), z_0(t))\mathcal{A}(t) - \Phi'_y(t, y_0(t), z_0(t))\mathcal{B}(t)\Big)\Big|_{t=t_0} = 0. \quad (51)$$

We set

$$\lambda(t_0) = -\frac{\mathcal{A}(t)}{\Phi'_y(t, y_0(t), z_0(t))}\bigg|_{t=t_0} = -\frac{\mathcal{B}(t)}{\Phi'_z(t, y_0(t), z_0(t))}\bigg|_{t=t_0}. \quad (52)$$

We emphasize that due to the theorem assumptions at least one of the derivatives $\Phi'_y(t, y_0(t), z_0(t))$, $\Phi'_z(t, y_0(t), z_0(t))$ is non-zero, hence, at least one of the fractions in (52) is correctly defined. On the other hand, if both derivatives $\Phi'_y(t, y_0(t), z_0(t))$, $\Phi'_z(t, y_0(t), z_0(t))$ are non-zero, then both fractions in (52) coincide. This implies

$$\mathcal{A}(t) + \lambda(t)\Phi'_y(t, y_0(t), z_0(t))\bigg|_{t=t_0} = 0, \quad (53)$$

$$\mathcal{B}(t) + \lambda(t)\Phi'_z(t, y_0(t), z_0(t))\bigg|_{t=t_0} = 0. \quad (54)$$

Note that (53) are the Lagrange equations associated to the Lagrangian $\mathcal{M} = \mathcal{L} + \lambda\Phi$. This finishes the proof. $\square$

3.5 Functionals depending high order derivatives

In this subsection we investigate another generalization of the simplest variational problem. This generalization relates not to a modification of the subset of functions $\mathcal{F}_0$ (being considered as a domain of definition for an integral functional), but to a modification of the integral functional itself. To be more precise, we will suppose that the Lagrangian of an integral functional $\mathcal{I}$ may depend on derivatives of higher orders. For simplicity of exposition we consider the one-dimensional case. Thus, we will study a finctional

$$\mathcal{I}(x) = \int_a^b \mathcal{L}(t, x, x', \dots, x^{(n)}) \mathrm{d}t,$$

where $\mathcal{L} \in C^2([a,b] \times \mathbb{R} \times \mathbb{R}^n)$. We denote the third variable of the function $\mathcal{L}$ by $v = (v_1, \dots, v_n)^{tr} \in \mathbb{R}^n$.

The functional $\mathcal{I}$ is assumed to be defined on a subset of functions

$$\mathcal{F}_0 = \left\{ x \in C^{2n}([a,b], \mathbb{R}) : \; x^{(k)}(a) = A_k, \; x^{(k)}(b) = B_k, \; k = 0, \dots, n-1 \right\}$$

for some given real constants $A_k, B_k, \; k = 0, \dots, n-1$.

We modify the definition of the weak distance. Particularly, for any $x, y \in \mathcal{F}_0$ one sets

$$d_w(x, y) = \sum_{k=0}^{n} \max_{t \in [a,b]} |x^{(k)}(t) - y^{(k)}(t)|.$$

As in the case of variational problem with fixed ends, for any $x \in \mathcal{F}_0$ one defines the set of admissible functions

$$\mathcal{G}_x = \left\{ h \in C^{2n}([a,b], \mathbb{R}) : \; h^{(k)}(a) = 0, \; h^{(k)}(b) = 0, \; k = 0, \dots, n-1 \right\}.$$

Note that $\mathcal{G}_x$ does not depend on x, so we skip the subscript x.

We assume that $x \in \mathcal{F}_0$, $h \in \mathcal{G}$ and $\|h\|_w \equiv d_w(h, 0) \ll 1$. For such x and h we consider $\Delta\mathcal{I}(x+h, x) = \mathcal{I}(x+h) - \mathcal{I}(x)$:

$$\Delta\mathcal{I}(x+h, x) = \int_a^b \left(\mathcal{L}(t, x+h, x'+h', \dots, x^{(n)} + h^{(n)}) - \mathcal{L}(t, x, x', \dots, x^{(n)}) \right) \mathrm{d}t.$$

Using the Taylor formula and integrating by parts leads to

$$\Delta\mathcal{I}(x+h,x) = \int\limits_a^b \Big(\mathcal{L}_x h + \mathcal{L}_{v_1} h' + \ldots \mathcal{L}_{v_n} h^{(n)}\Big) \mathrm{d}t + O\Big(\|h\|_w^2\Big) =$$

$$= \int\limits_a^b \Big(\mathcal{L}_x - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{v_1} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{v_1} + \ldots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathcal{L}_{v_n}\Big) h \mathrm{d}t + O\Big(\|h\|_w^2\Big). \quad (55)$$

It is to be emphasized that after integration by parts all non-integral terms vanish due to conditions $h^{(k)}(t)|_{t=a,b} = 0, k = 0, \ldots, n-1$. We also notice that all partial derivatives $\mathcal{L}_x, \mathcal{L}_{v_k}$ in (55) are evaluated at the point $(t, x, x', \ldots, x^{(n)})$.

The first term in the right hand side of (55) defines the first variation of the functional in this case

$$\delta\mathcal{I}(x,h) = \int\limits_a^b \Big(\mathcal{L}_x - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{v_1} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{v_1} + \ldots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathcal{L}_{v_n}\Big) h \mathrm{d}t.$$

Moreover, due to the Dubois-Raymond lemma we arive at the following necessary condition for a curve $x_0 \in \mathcal{F}_0$ to be extremal.

Theorem 3.4. *If a curve x_0 is extremal for the integral functional $\mathcal{I}$ depending on high-order derivatives, then it satisfies an equation*

$$\mathcal{L}_x - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_{v_1} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{L}_{v_1} + \ldots + (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathcal{L}_{v_n} = 0. \quad (56)$$

Equation (56) is called the Poisson-Euler equation. It is a differential equation of order $2n$, hence, it's general solution depends on $2n$ arbitrary constants.

3.6 Practice

Problem 1. *Find the distance between the point* $(4, 0)$ *and a curve*

$$9x^2 + 4t^2 = 36.$$

Solution. We may formulate this problem as a variational problem with movable end. Indeed, let us consider the functional, which gives us the length of a curve connecting the point $(4, 0)$ with some point (t_0, x_0) on $9x^2 + 4t^2 = 36$:

$$\mathcal{I}(x) = \int_4^{t_0} \sqrt{1 + (x')^2} dt.$$

We have to minimize this functional inside the set of functions

$$\mathcal{F}_0 = \left\{ x \in C^2(I_x, \mathbb{R}) : \ a_x = 4, \ x(a_x) = 0, \ 9x^2(b_x) + 4b_x^2 = 36 \right\}.$$

We already know that the general solution of the Lagrange equation for this functional is

$$x(t) = C_1 t + C_2,$$

where C_1, C_2 are arbitrary constants. Substituting this expression into the boundary condition at the left end, we obtain

$$4C_1 + C_2 = 0.$$

Let us consider the transversality condition at the right end

$$\Big(\mathcal{L}_v(t, x, x')\varphi_{B,t}(t, x) - \big(\mathcal{L}(t, x, x') - x'\mathcal{L}_v(t, x, x')\big)\varphi_{B,x}(t, x)\Big)\Big|_{t=b_x} = 0,$$

where the function $\varphi_B(t, x)$ is

$$\varphi_B(t, x) = 9x^2 + 4t^2 - 36.$$

We also note that

$$\varphi_B(b_x, x(b_x)) = 0.$$

Taking this into account, the transversality condition reads

$$\left[\frac{x'}{\sqrt{1 + (x')^2}} 8t - \left(\sqrt{1 + (x')^2} - \frac{x'^2}{\sqrt{1 + (x')^2}} \right) 18x \right] \Big|_{t=t_0} = 0$$

or, equivalently,

$$C_1 4t_0 - 9(C_1 t_0 + C_2) = 0.$$

Thus, we have a system of three algebraic equations, which define the constants C_1, C_2 and t_0:

$$\begin{aligned}
&4C_1 + C_2 = 0, \\
&9(C_1 t_0 + C_2)^2 + 4t_0^2 = 36, \\
&C_1 4t_0 - 9(C_1 t_0 + C_2) = 0.
\end{aligned}$$

This system has two solutions

$$C_1 = 0,\ C_2 = 0,\ t_0 = 3; \quad C_1 = 0,\ C_2 = 0,\ t_0 = -3.$$

Obviously, the first solution gives the minimum $\mathcal{I}_{\min} = 1$.

Problem 2. *Find the distance between point (-1,5) and the curve*

$$x^2 - t = 0.$$

Solution. Let us consider the functional which gives us the length of the curve

$$\mathcal{I}(x) = \int\limits_{1}^{t_0} \sqrt{1 + (x')^2}\mathrm{d}t.$$

Substituting the general solution of the Lagrange equation

$$x(t) = C_1 t + C_2$$

into the boundary condition at $t = -1$, one obtains

$$-C_1 + C_2 = 5.$$

Consider the transversality condition at $t = t_0$ with a function

$$\varphi_B(t, x) = x^2 - t.$$

This condition takes the form

$$\left[\frac{x'}{\sqrt{1 + x'^2}}(-1) - \frac{1}{\sqrt{1 + x'^2}} 2x \right] \bigg|_{t=t_0} = 0$$

Thus, we have a system of three equations defining C_1, C_2, t_0:

$$\begin{aligned} &- C_1 + C_2 = 5, \\ &(C_1 t_0 + C_2)^2 - t_0 = 0, \\ &- C_1 - 2(C_1 t_0 + C_2) = 0. \end{aligned}$$

It has a unique solution $C_1 = -2, C_2 = 3, t_0 = 1$.

Problem 3. *Write the transversality conditions for the functional if* $\varphi_A(t_1, x(t_1)) = 0$ *and* $\varphi_B(t_2, x(t_2)) = 0$

$$\mathcal{I}(x) = \int\limits_{t_1}^{t_2} h(t, x)\sqrt{1 + (x')^2}\mathrm{d}t.$$

Solution. We consider the transversality condition only at the left end. The case of the right end can be considered in a similar way.

The transversality condition at the point t_1 has the following form

$$\left(\frac{h(t,x)x'}{\sqrt{1+x'^2}}\varphi_{A,t}(t,x) - \frac{h(t,x)}{\sqrt{1+x'^2}}\varphi_{A,x}(t,x)\right)\Big|_{t=t_1} = 0$$

or equivalently

$$(x'\varphi_{A,t}(t,x) - \varphi_{A,x}(t,x))\Big|_{t=t_1} = 0.$$

Introduce two vectors:

$$\mathbf{y}_1 = \begin{pmatrix} 1 \\ x' \end{pmatrix}, \quad \mathbf{y}_2 = \begin{pmatrix} -\varphi_{A,x}(t,x) \\ \varphi_{A,t}(t,x)' \end{pmatrix}.$$

Then $\mathbf{y}_1$ is tangent to the curve $x = x(t)$ and $\mathbf{y}_2$ is tangent to the boundary curve $\varphi_A(t,x) = 0$. Hence, we can rewrite the transversality condition in the form

$$(\mathbf{y}_1, \mathbf{y}_2) = 0,$$

where $(\cdot,\cdot)$ is a standard scalar product.

Thus, for this functional the transversality condition is equivalent to the condition of orthogonality.

Problem 4. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int\limits_{t_0}^{5} \frac{\sqrt{1+(x')^2}}{x}\mathrm{d}t, \quad x(5) = 5, \quad x(t_0) = t_0 - 5.$$

Solution. The Lagrange equation for this functional has the following form

$$\frac{\sqrt{1+(x')^2}}{x} - \frac{(x')^2}{x\sqrt{1+(x')^2}} = const$$

or

$$\frac{1}{x\sqrt{1+(x')^2}} = const.$$

Solutions of this differential equation are circles

$$(t - C_1)^2 + x^2 = C_2^2.$$

Substituting this expression to the boundary condition at the right end, one obtains

$$C_2 = C_1^2 - 10C_1.$$

We already know that the transversality condition for this functional is equivalent to the condition of orthogonality. Hence, the circle correspondent to the stationary curve must be orthogonal to the line $x = t - 5$. Taking this into account, we obtain that the stationary curve is

$$(t-5)^2 + x^2 = 25.$$

Problem 5. *Find all stationary curves of the isoperimetric problem*

$$\mathcal{I}(x) = \int_0^1 (x')^2 \mathrm{d}t, \quad x(0) = 0, \quad x(1) = 1, \quad \mathcal{J}(x) = \int_0^1 x \mathrm{d}t = 0.$$

Solution. Constitute the Lagrangian $\mathcal{M} = \mathcal{L} + \lambda\mathcal{K}$

$$\mathcal{M} = v^2 + \lambda x$$

and the corresponding Lagrange equation

$$\lambda - 2x'' = 0.$$

The general solution of this differential equation is

$$x(t) = \lambda t^2/4 + C_1 t + C_2.$$

Substituting this expression into the boundary conditions and to the relation $\mathcal{J}(x) = 0$, we obtain

$$C_2 = 0, \quad \lambda/4 + C_1 = 1,$$
$$\int_0^1 x \mathrm{d}t = \int_0^1 (\lambda t^2/4 + C_1 t + C_2)\mathrm{d}t = \lambda/12 + C_1/2 + C_2 = 0.$$

Solving this system, one obtains

$$C_1 = -2, \quad C_2 = 0, \quad \lambda = 12.$$

Thus, the unique stationary curve is

$$x(t) = 3t^2 - 2t.$$

Problem 6. *Find all stationary curves of the isoperimetric problem*

$$\mathcal{I}(x) = \int_0^1 (x')^2 \mathrm{d}t, \quad x(0) = 0, \quad x(1) = 0, \quad \mathcal{J}(x) = \int_0^1 x e^{-t} \mathrm{d}t = 1.$$

Solution. Constitute the Lagrangian $\mathcal{M} = \mathcal{L} + \lambda\mathcal{K}$

$$\mathcal{M} = v^2 + \lambda x e^{-t}$$

and the corresponding Lagrange equation

$$\lambda e^{-t} - 2x'' = 0.$$

The general solution of this differential equation is

$$x(t) = \lambda e^{-t}/2 + C_1 t + C_2.$$

The boundary conditions and relation $\mathcal{J}(x) = 1$ imply

$$\lambda/2 + C_2 = 0, \quad e^{-1}\lambda/2 + C_1 + C_2 = 0,$$
$$\int_0^1 x\mathrm{d}t = \int_0^1 (\lambda e^{-t}/2 + C_1 t + C_2)e^{-t}\mathrm{d}t = 1.$$

This system of equations defines the coefficients C_1, C_2 and λ.

Problem 7. *Among curves x connecting two points $(-a, 0)$ and $(a, 0)$ and having fixed length $J(x) = J_0$ ($J_0 > 2a$) find those curve $x = x(t)$ that gives the maximum area of the domain bounded by the curve and the t- axis.*

Solution. The area $\mathcal{I}(x)$ of the domain bounded by the curve $x(t)$ and the t-axis is

$$\mathcal{I}(x) = \int_{-a}^{a} x(t)\mathrm{d}t.$$

On the other hand, the length $\mathcal{J}(x)$ is

$$\mathcal{J}(x) = \int_{-a}^{a} \sqrt{1 + (x')^2})\mathrm{d}t.$$

Hence, one has to analyze the Lagrange equation associated to the Lagrangian $\mathcal{M} = x + \lambda\sqrt{1 + v^2}$. Note that $\mathcal{M}$ doesn't depend on t. Thus, there exists an integral of motion:

$$-x'\frac{\lambda x'}{\sqrt{1 + (x')^2}} + x + \lambda\sqrt{1 + (x')^2} = Const,$$

or

$$x - C_1 = -\frac{\lambda}{\sqrt{1 + (x')^2}},$$

where C_1 is a constant. Let us introduce the parameter

$$\tan s = x',$$

then

$$x - C = -\lambda \cos s,$$

and

$$\mathrm{d}t = \frac{\mathrm{d}x}{\tan s} = \frac{\lambda \sin s \mathrm{d}s}{\tan s} = \lambda \cos s \mathrm{d}s.$$

So,

$$t - C_2 = \lambda \sin s.$$

Thus,

$$(x - C_1)^2 + (t - C_2)^2 = \lambda^2.$$

and we get an equation of the circle. Using the boundary conditions

$$x(-a) = x(a) = 0$$

and relation

$$\int_{-a}^{a} \sqrt{1 + (x')^2)}\mathrm{d}t = J_0$$

one can find the constants C_1, C_2 and λ.

Problem 8. *Find all stationary curves of the isoperimetric problem*

$$\mathcal{I}(x) = \int_0^\pi (x')^2 \mathrm{d}t, \quad x(0) = 0, \quad x(\pi) = 0, \quad \mathcal{J}(x) = \int_0^\pi x^2 \mathrm{d}t = 1.$$

Solution. Constitute the new Lagrangian $\mathcal{M} = \mathcal{L} + \lambda\mathcal{K}$ and consider the corresponding Lagrange equation. It is of the form

$$\lambda x + x'' = 0.$$

The general solution of this equation has different expression for different λ:

$$\begin{aligned}
&x(t) = C_1 \sinh(\sqrt{|\lambda|}t) + C_2 \cosh(\sqrt{|\lambda|}t), \quad \lambda < 0,\\
&x(t) = C_1 t + C_2, \quad \lambda = 0,\\
&x(t) = C_1 \sin(\sqrt{\lambda}t) + C_2 \cos(\sqrt{\lambda}t), \quad \lambda > 0.
\end{aligned}$$

Substituting these expressions into the boundary conditions, one sees that

$$C_1 = C_2 = 0$$

provided $\lambda \leq 0$. But the trivial solution does not satisfy the integral condition $\mathcal{J}(0) \neq 1$

For $\lambda > 0$ we obtain from the boundary conditions that

$$C_2 = 0, \quad \lambda = n^2, \ n \in \mathbb{N}.$$

The integral condition $\mathcal{J}(x) = 1$ implies

$$C_1^2 \int\limits_0^\pi \sin^2(nt)\mathrm{d}t = 1$$

and

$$C_1 = \sqrt{\frac{2}{\pi}}.$$

Thus, there exists many stationary curves in this isoperimetric problem

$$x(t) = \sqrt{\frac{2}{\pi}}\sin(nt), \quad n \in \mathbb{N}.$$

Problem 9. *Find all stationary curves of the isoperimetric problem*

$$\begin{aligned}
&\mathcal{I}(x,y) = \int\limits_0^1 ((x')^2 + (y')^2 - 4ty' - 4y)\mathrm{d}t,\\
&x(0) = 0, \quad y(0) = 0, \quad x(1) = 1, \quad y(1) = 1,\\
&\mathcal{J}(x,y) = \int\limits_0^1 ((x')^2 - tx' - (y')^2)\mathrm{d}t = 2.
\end{aligned}$$

Solution. We construct the modified Lagrangian

$$\mathcal{M} = u^2 + w^2 - 4tw - 4y + \lambda(u^2 - tu - w^2)$$

and associated system of Lagrange equations

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}(2x' + 2\lambda x' - \lambda t) = 0,\\
&-4 - \frac{\mathrm{d}}{\mathrm{d}t}(2y' - 4t - 2\lambda y') = 0.
\end{aligned}$$

Solving this system, we get

$$
\begin{aligned}
x(t) &= \frac{\lambda t^2}{4(1+\lambda)} + C_1 t + C_2,\\
y(t) &= C_3 t + C_4.
\end{aligned}
$$

Substituting these expressions into the boundary conditions, one gets

$$
\frac{\lambda}{4(1+\lambda)} + C_1 = 1, \quad C_2 = 0, \quad C_3 = 1, \quad C_4 = 0.
$$

Denote by

$$
\mu = \frac{\lambda}{4(1+\lambda)}
$$

and use the integral condition condition $\mathcal{J}(x) = 2$:

$$
\int_0^1 ((2\mu t + C_1)^2 - (2\mu t + C_1)t - 1)\mathrm{d}t = 2.
$$

Integrating and taking into account that $\mu + C_1 = 1$ yields

$$
\mu = 6,\ C_1 = -5 \quad \text{or} \quad \mu = -5,\ C_1 = 6.
$$

Thus, there are two stationary curves:

$$
\begin{aligned}
x(t) &= 6t^2 - 5t, \quad y(t) = t,\\
x(t) &= -5t^2 + 6t, \quad y(t) = t.
\end{aligned}
$$

Problem 10. *Consider a homogeneous thin string with a constant linear density ρ being subjected to the constant gravity force. The string is fixed at the points*

$$
t = -a, x = 0, \quad \text{and} \quad t = a, x = 0.
$$

Find the form of the string if it's length is $2l,\ l > a$,

Solution. The potential energy ΔU of the small element Δl of the string is

$$
\Delta U \simeq g\rho x \Delta l \simeq g\rho x\sqrt{1 + (x')^2}\Delta t.
$$

Hence,

$$
U(x) = \rho g \int_{-a}^{a} x\sqrt{1 + (x')^2}\mathrm{d}t.
$$

The string's length is

$$L(x) = \int_{-a}^{a} \sqrt{1 + (x')^2} \mathrm{d}t.$$

We need to find a curve, which minimizes the potential energy U provided that $L(x) = 2l$.

Consider the modified Lagrangian

$$\mathcal{M} = x\sqrt{1 + v^2} + \lambda\sqrt{1 + v^2})$$

and the corresponding Lagrange equation (we can see that the Lagrangian doesn't depend on t)

$$(x + \lambda)[\sqrt{1 + (x')^2} - \frac{(x')^2}{\sqrt{1 + (x')^2}}] = C_1,$$

or

$$x + \lambda = C_1\sqrt{1 + (x')^2}.$$

Introduce a parameter s such that

$$x' = \sinh s.$$

This leads to

$$x = -\lambda + C_1 \cosh s$$

and

$$\mathrm{d}t = \frac{\mathrm{d}x}{x'} = \frac{C_1 \sinh s}{\sinh s} \mathrm{d}s = C_1 \mathrm{d}s.$$

Hence

$$t = C_1 s + C_2$$

and

$$x(t) = \lambda + C_1 \cosh \frac{(t - C_2)}{C_1}.$$

Substituting this into the boundary conditions and taking into account that $L(x) = 2l$, we obtain

$$C_2 = 0, \quad \lambda = -C_1 \cosh \frac{a}{C_1}, \quad C_1 \sinh \frac{a}{C_1} = l.$$

Problem 11. *Find all stationary curves of the Lagrange problem*

$$\mathcal{I}(x, y) = \int_{1}^{2} ((x')^2 + (y')^2)\mathrm{d}t,$$

$$x(1) = 1, \quad y(1) = -1, \quad x(2) = 1, \quad y(2) = -2,$$

$$t + x + y = 1.$$

Solution. One can define the function Φ to be

$$\Phi(t, x, y) = t + x + y - 1.$$

Then we constitute the modified Lagrangian $\mathcal{M} = \mathcal{L} + \lambda\Phi$

$$\mathcal{M} = u^2 + w^2 + \lambda(t)\Big(t + x + y - 1\Big)$$

and corresponding Lagrange equations

$$\lambda(t) - 2x'' = 0, \quad \lambda(t) - 2y'' = 0.$$

Together with this system of differential equations we consider equation $t + x + y = 1$. Expressing λ from the first equation and substituting it to the second one, we obtain

$$x'' = y'', \quad t + x + y = 1.$$

The general solution of this system is

$$x(t) = C_1 t + C_2, \quad y(t) = -(C_1 + 1)t + 1 - C_2.$$

Substituting to the boundary conditions yields $C_1 = 0, C_2 = 1$. Hence, there exists a unique stationary curve

$$x_0(t) = 1, \quad y_0(t) = -t.$$

Problem 12. *Find all stationary curves of the Lagrange problem*

$$\mathcal{I}(x, y) = \int_0^1 (x'y - xy' - (x')^2)\mathrm{d}t,$$
$$x(0) = 0, \quad y(0) = 0, \quad x(0) = \pi/4, \quad y(1) = 0,$$
$$x + y = \arctan(t).$$

Solution. Define function Φ as

$$\Phi(t, x, y) = x + y - \arctan(t).$$

Then the modified Lagrangian $\mathcal{M} = \mathcal{L} + \lambda\Phi$ is

$$\mathcal{M} = uy - xw - u^2 + \lambda(t)\Big(x + y - \arctan(t)\Big)$$

and corresponding Lagrange equations are

$$-y' + \lambda(t) + 2x'' - y' = 0, \quad x' + \lambda(t) + x' = 0.$$

We add to this system of differential equations the constraint $x + y = \arctan(t)$. Expressing λ from the second equation and substituting it to the first one, we obtain

$$x'' - x' = y', \quad x + y = \arctan(t).$$

The general solution of this system is

$$\begin{aligned} x(t) &= t\arctan(t) - \frac{1}{2}\ln(1+t^2) + C_1 t + C_2, \\ y(t) &= (1-t)\arctan(t) + \frac{1}{2}\ln(1+t^2) - C_1 t - C_2. \end{aligned}$$

Substituting to the boundary conditions yields $C_1 = \frac{1}{2}\ln 2, C_2 = 0$. Hence, there exists a unique stationary curve

$$\begin{aligned} x_0(t) &= t\arctan(t) - \frac{1}{2}\ln(1+t^2) + \frac{t}{2}\ln 2, \\ y_0(t) &= (1-t)\arctan(t) + \frac{1}{2}\ln(1+t^2) - \frac{t}{2}\ln 2. \end{aligned}$$

Problem 13. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_0^1 (x^2 + 2(x')^2 + (x'')^2)\mathrm{d}t,$$

$$x(0) = -1, \quad x'(0) = 2, \quad x(1) = 0, \quad x'(1) = \frac{1}{e}.$$

Solution. Note that the Lagrangian depends on the second derivative of unknown function x. Thus, $\mathcal{I}$ is a functional depending on higher derivatives. Taking this into account constitute the Euler-Poisson equation

$$2x - 4\frac{\mathrm{d}}{\mathrm{d}t}(x') + 2\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x'') = 0$$

or, equivalently,

$$x^{(4)} - 2x'' + x = 0.$$

This is a linear equation of the fourth order. It's characteristic equation

$$\lambda^4 - 2\lambda^2 + 1 = 0$$

has two double roots $\lambda = \pm 1$. Hence, the general condition of the Euler-Poisson equation is

$$x(t) = (C_1 t + C_2)\mathrm{e}^t + (C_3 t + C_4)\mathrm{e}^{-t}.$$

Substitute this expression into the boundary conditions to obtain

$$\begin{aligned}
&C_2 + C_4 = -1,\\
&C_1 + C_2 + C_3 - C_4 = 2,\\
&(C_1 + C_2)\mathrm{e} + (C_3 + C_4)\mathrm{e}^{-1} = 0,\\
&(2C_1 + C_2)\mathrm{e} - C_4\mathrm{e}^{-1} = \mathrm{e}^{-1}.
\end{aligned}$$

Solving this system gives

$$C_1 = C_2 = 0, \quad C_3 = 1, \quad c_4 = -1.$$

Thus, there exists a unique stationary curve

$$x(t) = (x - 1)\mathrm{e}^{-t}.$$

Problem 14. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_{-a}^{a} (\rho x + \frac{\mu}{2}(x'')^2)\mathrm{d}t,$$
$$x(-a) = 0, \quad x'(-a) = 0, \quad x(a) = a, \quad x'(a) = 0,$$

where a, ρ and μ are positive constants.

Solution. The Euler-Poisson equation for this functional is

$$x^{(4)} = -\rho/\mu.$$

It's general solution has the form

$$x(t) = -\frac{\rho}{24\mu}t^4 + C_1 t^3 + C_2 t^2 + C_3 t + C_4.$$

The boundary conditions yield

$$C_1 = C_3 = 0, \quad C_2 = \frac{\rho}{12\mu}a^2, \quad C_4 = -\frac{\rho}{24\mu}a^4.$$

Hence, there exists a unique stationary curve

$$x(t) = -\frac{\rho}{24\mu}(x^2 - a^2)^2.$$

Problems for self-control:

Problem 15. *Find the distance between point (-1,3) and the curve*

$$x = 1 - 3t.$$

Problem 16. *Find the distance between the point* $(2,3)$ *and the curve*

$$x = 5 - t.$$

Problem 17. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int\limits_0^{t_0} (x^2 + (x')^2)\mathrm{d}t, \quad x(0) = 0, \quad x(t_0) = \pi/4.$$

Problem 18. *Find the distance between the two curves*

$$x^2 + t^2 = 1, \quad x + t = 4.$$

Problem 19. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int\limits_{t_0}^{2} (x + (x')^2)\mathrm{d}t, \quad x(t_0) = t_0^2, \quad x(2) = 1.$$

Problem 20. *Find all stationary curves of the isoperimetric problem*

$$\mathcal{I}(x) = \int\limits_0^1 ((x')^2)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = \frac{1}{4}, \quad \mathcal{J}(x) = \int\limits_0^1 (x - (x')^2)\mathrm{d}t = \frac{1}{12}.$$

Problem 21. *Find all stationary curves of the isoperimetric problem*

$$\mathcal{I}(x) = \int\limits_0^1 (t^2 + (x')^2)\mathrm{d}t, \quad x(0) = 0, \quad x(1) = 0, \quad \mathcal{J}(x) = \int\limits_0^1 x^2\mathrm{d}t = 2.$$

Problem 22. *Find all stationary curves of the isoperimetric problem*

$$\mathcal{I}(x) = \int_0^1 (x')^2 \mathrm{d}t, \quad x(0) = 1, \quad x(1) = 6, \quad \mathcal{J}(x) = \int_0^1 x \mathrm{d}t = 3.$$

Problem 23. *Find all stationary curves of the isoperimetric problem*

$$\begin{aligned}
&\mathcal{I}(x) = \int_0^1 ((x')^2 - x'y' - (y')^2)\mathrm{d}t,\\
&x(0) = 1, \quad y(0) = 2, \quad x(1) = 3, \quad y(1) = 2,\\
&\mathcal{J}(x) = \int_0^1 (x' + 2ty + \frac{1}{5}(y')^2)\mathrm{d}t = \frac{20}{3}.
\end{aligned}$$

Problem 24. *Among the closed curves (on the plane) of length $2l$, find those one which limits the largest area.*

Problem 25. *Find all stationary curves of the Lagrange problem*

$$\begin{aligned}
&\mathcal{I}(x) = \int_2^4 ((x')^2 - x'y' - (y')^2)\mathrm{d}t,\\
&x(2) = 1, \quad y(2) = 3, \quad x(4) = -3, \quad y(4) = 1,\\
&t + x = y.
\end{aligned}$$

Problem 26. *Find all stationary curves of the Lagrange problem*

$$\begin{aligned}
&\mathcal{I}(x) = \int_0^1 \left(x'y + ty' + \frac{1}{2}(x')^2\right) \mathrm{d}t,\\
&x(0) = 0, \quad y(0) = 0, \quad x(1) = -1/2, \quad y(1) = 3/4,\\
&y + 2tx + x^2 = 0.
\end{aligned}$$

Problem 27. *Find all stationary curves of the functional*

$$\begin{aligned}
&\mathcal{I}(x) = \int_0^1 (x'\mathrm{e}^{-t} + x^2 + 2(x')^2 + (x'')^2)\mathrm{d}t,\\
&x(0) = x'(0) = 0, \quad x(1) = x'(1) = -\frac{1}{16e}.
\end{aligned}$$

Problem 28. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_0^{\pi/4} (t^2 + 16x^2 - (x'')^2)\mathrm{d}t,$$
$$x(0) = 0, \quad x'(0) = 2, \quad x(\pi/4) = 1, \quad x'(\pi/4) = 0.$$

Problem 29. *Find all stationary curves of the functional*

$$\mathcal{I}(x) = \int_0^{\pi/2} (2x'\mathrm{e}^{-t} - 4x^2 - 3(x')^2 + (x'')^2)\mathrm{d}t,$$
$$x(0) = -\frac{\pi}{20}\mathrm{e}^{\pi}, \quad x'(0) = \frac{1}{20}\Big(\pi\mathrm{e}^{\pi} + 2\pi\mathrm{e}^{\pi/2} + 2\Big),$$
$$x(\pi/2) = 0, \quad x'(\pi/2) = \frac{1}{10}\mathrm{e}^{\pi/2}.$$

4 Hamiltonian formalism

In this section we derive a canonical form of the Lagrange equation and consider its application to mechanical systems.

4.1 Legendre's transformation

Let $\mathcal{L}$ be a real-valued C^2-function of a variable $v \in \mathbb{R}^n$. Then it's derivative $\mathcal{L}'$ is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. We denote a variable in the image of $\mathcal{L}'$ by p:

$$p = \mathcal{L}'(v), \quad v \in \mathbb{R}^n \tag{57}$$

and assume that $\mathcal{L}''$ is non-degenerate in some domain $\Sigma \in \mathbb{R}^n$, i.e.

$$\det\left(\frac{\partial^2\mathcal{L}}{\partial v_k \partial v_j}\right) \neq 0, \quad \forall\, v \in \Sigma.$$

Then by the Implicit Function Theorem equation (57) can locally be resolved with respect to v. Note that if one supposes in addition $\mathcal{L}'$ to be one-to-one map then v can be expressed in terms of p globally in Σ. Further we will assume that $\mathcal{L}'$ is bijective between Σ and Σ' and $v = V(p)$ is a unique solution of (57).

Definition 4.1. *A real-valued function*

$$\mathcal{H}(p) = \big(pv - \mathcal{L}(v)\big)|_{v=V(p)}, \quad p \in \Sigma'$$

is called the Legendre transform of the function $\mathcal{L}$.

Theorem 4.1. *The function $\mathcal{H}$ is a C^2-function, which satisfies*

$$\mathcal{H}'(p) = V(p), \quad \mathcal{H}''(p)\mathcal{L}''(v)|_{v=V(p)} = 1.$$

The Legendre transform of $\mathcal{H}$ is $\mathcal{L}$.

Proof: First we note that due to relation (57)

$$\mathrm{d}\mathcal{H}(p) = \mathcal{H}'(p)\mathrm{d}p = (p\mathrm{d}v + v\mathrm{d}p - \mathcal{L}'(v)\mathrm{d}v)\Big|_{v=V(p)} = V(p)\mathrm{d}p.$$

Hence, $\mathcal{H}'(p) = V(p)$. Moreover, since the Implicit Function Theorem guarantees the C^1-smothness of $V(p)$, this implies $\mathcal{H} \in C^2(\Sigma', \Sigma)$.

Differentiating the identity $\mathcal{L}'(V(p)) = p$ with respect to p yields

$$\mathcal{L}''(V(p))\mathcal{H}''(p) = 1.$$

Finally, denote the inverse function to $p \mapsto V(p)$ by $P(v)$. Since $V(p) = \mathcal{H}'(p)$ one has $\mathcal{H}'(P(v)) = v$. Taking this into account, the Legendre transform of H takes the form

$$(vp - \mathcal{H}(p))|_{p=P(v)} = \big(vp - pv + \mathcal{L}(V(p))\big)|_{p=P(v)} = \mathcal{L}(v).$$

This finishes the proof. $\square$

4.2 Hamilton equations

In this subsection we consider the variational problem with fixed ends, i.e. a problem of minimization of an integral functional

$$\mathcal{I}(x) = \int_a^b \mathcal{L}(t, x, x')\mathrm{d}t,$$

defined on a set of functions

$$\mathcal{F}_0 = \left\{x \in C^2([a,b], \Omega) : \ x(a) = A, \ x(b) = B\right\},$$

where a, b, A, B are given real numbers, Ω is a domain in $\mathbb{R}^n$ and the Lagrangian $\mathcal{L} \in C^2([a,b] \times \Omega \times \mathbb{R}^n, \mathbb{R})$.

We assume that for any $(t, x) \in [a, b] \times \Omega$ the map

$$v \mapsto p = \mathcal{L}_v(t, x, v)$$

is a C^1-invertible map between domains Σ and Σ' and $v = V(t, x, p)$ is a unique solution of $p = \mathcal{L}_v(t, x, v)$. The variable p in the image of this map is called the *momentum*.

We define a function $\mathcal{H}$, which will be called the *Hamiltonian*, to be the Legendre transform of $\mathcal{L}$ with respect to the variable v:

$$\mathcal{H}(t, x, p) = (pv - \mathcal{L}(t, x, v))\,|_{v=V(t,x,p)}.$$

Hence, the Hamiltonian $\mathcal{H} \in C^2([a, b] \times \Omega \times \Sigma', \mathbb{R})$.

Theorem 4.2. *A curve $x_0 \in \mathcal{F}_0$ is a stationary curve for the functional $\mathcal{I}$ if and only if x_0 and the function p_0 defined by*

$$p_0 = \mathcal{L}_v(t, x_0(t), x_0'(t))$$

satisfy equations

$$x'(t) = \mathcal{H}_p(t, x(t), p(t)), \quad p'(t) = -\mathcal{H}_x(t, x(t), p(t)). \tag{58}$$

The system (58) is called the *canonical system oif Hamilton equations.*
Proof: Assume first that x_0 is a stationary point of the functional $\mathcal{I}$. Then by Theorem 1.2 x_0 is a solution of the Lagrange equations $\mathcal{L}_x - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v = 0$. Taking this into account together with the relation $x_0'(t) = V(p_0(t))$, one obtains

$$p_0'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0(t), x_0'(t)) = \mathcal{L}_x(t, x_0(t), x_0'(t)) = -\mathcal{H}_x(t, x_0(t), p_0(t)).$$

Thus, we proved that x_0, p_0 satisfy the second equation (58). The first equation (58) is a consequence of the first relation of Theorem 4.1.

Conversally, assume that x_0, p_0 satisfy the Hamilton equations. Since the right hand side of the first equation (58) is continuously differentiable, one obatins that $x_0 \in C^2([a, b], \Omega)$. Moreover, due to the first equation (58) and a relation $\mathcal{H}_p(t, x, p) = v$, we have that $V(p_0(t)) = x_0'(t)$. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{L}_v(t, x_0, x_0'(t)) = p_0'(t) = -\mathcal{H}_x(t, x_0(t), p_0(t)) = \mathcal{L}_x(t, x_0(t), x_0'(t)).$$

Thus, we conclude that x_0 satisfies the Lagrange equation and, hence, is a stationary curve for the functional $\mathcal{I}$. $\square$

Theorem 4.3. *If the Hamiltonian does not depend on time t, then $\mathcal{H}$ is an integral of motion, i.e. it preserves it's value along any solution of the Hamilton equations.*

Proof: Note that it is sufficient to prove that for any solution $x(t), p(t)$ of the Hamilton equations (58)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t), p(t)) = 0.$$

Indeed, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t), p(t)) = \mathcal{H}_x(x(t), p(t))x'(t) + \mathcal{H}_p(x(t), p(t))p'(t) =$$
$$= \mathcal{H}_x(x(t), p(t))\mathcal{H}_p(x(t), p(t)) + \mathcal{H}_p(x(t), p(t))(-\mathcal{H}_x(x(t), p(t))) = 0.$$

Thus, $\mathcal{H}$ is an integral of motion. $\square$

Consider a function $F \in C^1(\Omega \times \Sigma', \mathbb{R})$ and investigate a question when this function is an integral of motion.

Definition 4.2. *For any functions $F, G \in C^1(\Omega \times \Sigma', \mathbb{R})$ an expression*

$$\{F, G\} = \sum_{k=1}^{n} \frac{\partial F}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial x}$$

is called the Poisson bracket.

Theorem 4.4. *A function $F \in C^1(\Omega \times \Sigma', \mathbb{R})$ is an integral of motion if and only if the Poisson bracket of F with the Hamiltonian $\mathcal{H}$ is zero:*

$$\{F, \mathcal{H}\} = 0.$$

Proof: Indeed, for any solution $x(t), p(t)$ of the Hamilton equations

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x(t), p(t)) = F_x(x(t), p(t))x'(t) + F_p(x(t), p(t))p'(t) =$$
$$= F_x(x(t), p(t))\mathcal{H}_p(x(t), p(t)) - F_p(x(t), p(t))\mathcal{H}_x(x(t), p(t)) =$$
$$= \{F, \mathcal{H}\}(x(t), p(t)).$$

Thus, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}F(x(t), p(t)) = 0 \iff \{F, \mathcal{H}\}(x(t), p(t)) = 0.$$

$\square$

4.3 Practice

Problem 1. Calculate the Legendre transform of the function $\mathcal{L}(v) = e^v$.

Solution. Consider the map

$$v \to p = \mathcal{L}'(v) = e^v.$$

The inverse map is

$$V(p) = (\mathcal{L}')^{-1}(p) = \ln p.$$

So, the Legendre transform is

$$\mathcal{H}(p) = pV(p) - \mathcal{L}(V(p)) = p \ln p - \exp(\ln p) = p \ln p - p.$$

Problem 2. *Calculate the Legendre transformation of the function $\mathcal{L}(v) = v^\alpha$, where α is a positive parameter.*

Solution. Consider the map

$$v \to p = \mathcal{L}'(v) = \alpha v^{\alpha-1}.$$

The inverse map is

$$V(p) = (\mathcal{L}')^{-1}(p) = \left(\frac{p}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

So, the Legendre transformation is

$$\mathcal{H}(p) = pV(p) - \mathcal{L}(V(p)) = p\left(\frac{p}{\alpha}\right)^{\frac{1}{\alpha-1}} - \left(\frac{p}{\alpha}\right)^{\frac{\alpha}{\alpha-1}}.$$

Problem 3. *Calculate the Legendre transform of the function $\mathcal{L}(v_1, v_2) = v_1^2 + v_2^2$.*

Solution. First we note that it is a two-dimensional problem. As in the previous case we consider the map

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \mathcal{L}'(v) = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}.$$

The inverse map is

$$\begin{pmatrix} V_1(p_1, p_2) \\ V_2(p_1, p_2) \end{pmatrix} = (\mathcal{L}')^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} p_1/2 \\ p_2/2 \end{pmatrix}.$$

Taking this into account, one obtains the Legendre transformation

$$\mathcal{H}(p_1,p_2) = p_1V_1(p_1,p_2)+p_2V_2(p_1,p_2)-\mathcal{L}(V_1(p_1,p_2),V_2(p_1,p_2)) = \frac{(p_1)^2}{4}+\frac{(p_2)^2}{4}.$$

Problem 4. *Write and solve the Hamilton's canonical equations for the functional*

$$\mathcal{I}(x) = \int_a^b tx(x')^2\mathrm{d}t.$$

Solution. The Lagrangian has the following form

$$\mathcal{L} = txv^2.$$

Define the momentum

$$p = \frac{\partial \mathcal{L}}{\partial v} = 2txv.$$

Solving this equation with respect to v, one obtains

$$V(p) = \frac{p}{2tx}.$$

Thus, performing the Legendre transformation of $\mathcal{L}$ with respect to v gives

$$\mathcal{H} = pV(p) - txV^2(p) = \frac{p^2}{4tx}.$$

Let's write the Hamilton's canonical equations. Since

$$H_p = \frac{p}{2tx}, \quad H_x = -\frac{p^2}{4tx^2},$$

one gets

$$x' = \frac{p}{2tx}, \quad p' = \frac{p^2}{4tx^2}.$$

To solve this system of differential equation, we divide the second equation over the first one to get

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{p}{2x}$$

or equivalently

$$\frac{\mathrm{d}p}{p} = \frac{\mathrm{d}x}{2x}.$$

The genral solution of this equation is

$$p = C_1\sqrt{x}.$$

We substitute this expression into the first Hamilton equation:

$$x' = \frac{C_1}{2t\sqrt{x}}.$$

Hence,

$$x(t) = \left(\frac{C_1}{2}\ln t + C_2\right)^{2/3}, \quad p(t) = C_1\left(\frac{C_1}{2}\ln t + C_2\right)^{1/3}.$$

Problem 5. *Write and solve the Hamilton's canonical equations for the functional*

$$\mathcal{I}(x) = \int\limits_a^b ((x_1')^2 + x_2^2 + (x_2')^2)\mathrm{d}t.$$

Solution. Note that the Lagrangian is of the form

$$\mathcal{L} = (v_1)^2 + x_2^2 + (v_2)^2.$$

Hence, the momenta can be expressed as

$$p_1 = 2x_1', \quad p_1 = 2x_2'.$$

Solving these equations with respect to velocities, we construct the inverse map

$$V_1(p_1, p_2) = \frac{p_1}{2}, \quad V_2(p_1, p_2) = \frac{p_2}{2}.$$

Thus, the Hamiltonian reads

$$\mathcal{H} = p_1V_1 + p_2V_2 - V_1^2 - x_2^2 - V_2^2 = \frac{p_1^2}{4} + \frac{p_2^2}{4} - x_2^2.$$

Taking into account that

$$H_{p_1} = \frac{p_1}{2}, \quad H_{p_2} = \frac{p_2}{2}, \quad H_{x_1} = 0, \quad H_{x_2} = -2x_2,$$

we write the Hamilton's canonical equations

$$x_1' = \frac{p_1}{2}, \quad x_2' = \frac{p_2}{2},$$
$$p_1' = 0; \quad p_2' = 2x_2.$$

Clearly,

$$p_1(t) = C_1, \quad x_1(t) = \frac{C_1}{2}t + C_2.$$

To obtain the other two components of a solution we differentiate the second Hamilton's equation and substitute p_2' expressed from the second one. This leads to

$$x_2'' = x_2.$$

The general solution of this equation is

$$x_2(t) = C_3 \sinh(t) + C_4 \cosh(t).$$

Hence,

$$p_2(t) = 2C_3 \sinh(t) + 2C_4 \cosh(t).$$

Problems for self-control:

Problem 6. *Calculate the Legendre transformation of the function* $\mathcal{L}(v) = \sqrt{1 + v^2}$.

Problem 7. *Calculate the Legendre transformation of the function* $\mathcal{L}(v_1, v_2) = v_1^2 + v_2^3$.

Problem 8. *Calculate the Legendre transformation of the function* $\mathcal{L}(v_1, v_2, v_3) = v_1^2 + 2v_2^2 + 3v_3^2$.

Problem 9. *Write and solve the Hamilton's canonical equations for the functional*

$$\mathcal{I}(x) = \int_a^b tx\sqrt{x'}\mathrm{d}t.$$

Problem 10. *Write and solve the Hamilton's canonical equations for the functional*

$$\mathcal{I}(x) = \int_a^b \sqrt{t^2 + x^2}\sqrt{1 + (x')^2}\mathrm{d}t.$$

Problem 11. *Write and solve the Hamilton's canonical equations for the functional*

$$\mathcal{I}(x) = \int_a^b (2x_1x_2 - 2x_1^2 + (x_1')^2 - (x_2')^2)\mathrm{d}t.$$

Problem 12. *Write and solve the Hamilton's canonical equations for the functional*

$$\mathcal{I}(x0 = \int_a^b (t^2 + x_1(x_1')^2 + x_2(x_2')^2)\mathrm{d}t.$$

Problem 13. *Write and solve the Hamilton's canonical equations for the functional*

$$\mathcal{I}(x) = \int_a^b \left(2tx_1 - (x_1')^2 + \frac{(x_2')^3}{3}\right)\mathrm{d}t.$$