Exercises (10 tasks, 5 points in total)

Each correctly solved task gives 0.5 points.

1. **Show that if the nonzero vectors $p_0, p_1, \ldots, p_\ell$ satisfy**

$$p_i^T A p_j = 0 \quad \textbf{for all } i \neq j,$$

where A is symmetric and positive definite, then these vectors are linearly independent. (This result implies that A has at most n conjugate directions.)

2. **Verify the formula for the step length in the conjugate gradient method (see page 5, LectureNotes5.pdf):**

$$\alpha_k = -\frac{r_k^T p_k}{p_k^T A p_k}.$$

3. **Show that if $f(x)$ is a strictly convex quadratic, then the function**

$$h(\sigma) \overset{\textbf{def}}{=} f(x_0 + \sigma_0 p_0 + \cdots + \sigma_{k-1} p_{k-1}),$$

is also a strictly convex quadratic in the variable $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{k-1})^T$.

4. **Verify from the formulae (for a given x_0)**

$$r_0 = Ax_0 - b, \quad p_0 \leftarrow -r_0, \quad \alpha_0 = -\frac{r_0^T p_0}{p_0^T A p_0},$$
$$x_1 = x_0 + \alpha_0 p_0, \quad r_1 = Ax_1 - b, \quad p_1 \leftarrow -r_1 + \frac{r_1^T A p_0}{p_0^T A p_0} p_0$$

that the following relations hold:

$$\textbf{span}\{r_0, r_1\} = \textbf{span}\{r_0, Ar_0\}$$

$$\textbf{span}\{p_0, p_1\} = \textbf{span}\{r_0, Ar_0\}.$$

5. **Let $\{(\lambda_i, v_i)\}_{i=1}^n$ be the eigenpairs of the symmetric matrix A. Show that the eigenvalues and eigenvectors of**

$$[I + P_k(A)A]^T A [I + P_k(A)A]$$

are $\lambda_i[1 + \lambda_i P_k(\lambda_i)]^2$ and v_i, respectively.

6. **Consider the problem**

$$\min_x \; x^2 \quad \textbf{s.t. } c(x) = 0,$$

where

$$c(x) = \begin{cases} x^6 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   (a) **Show that the constraint function is twice continuously differentiable at all x (including $x = 0$) and that the feasible points are $x = 0$ and $x = 1/(k\pi)$ for all nonzero integers k.**

   (b) **Verify that each feasible point except $x = 0$ is an isolated local solution (see page 5, LectureNotes6.pdf).**

(c) **Verify that $x = 0$ is a global solution and a strict local solution, but not an isolated local solution.**

7. **Consider the constrained optimization problem**

$$\min\ (x_2 + 100)^2 + 0.01x_1^2 \quad \textbf{s.t.}\ \ x_2 - \cos x_1 \geq 0.$$

Does this problem have a finite or infinite number of local solutions? Use the first-order optimality (KKT) conditions (see Theorem 21 (page 2, LectureNotes7.pdf))

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, && && (1)\\ c_i(x^*) &= 0, && \textbf{for all } i \in \mathcal{E}, && (2)\\ c_i(x^*) &\geq 0, && \textbf{for all } i \in \mathcal{I}, && (3)\\ \lambda_i^* &\geq 0, && \textbf{for all } i \in \mathcal{I}, && (4)\\ \lambda_i^* c_i(x^*) &= 0, && \textbf{for all } i \in \mathcal{E} \cup \mathcal{I}. && (5) \end{aligned}$$

to justify your answer.

8. **If f is convex and the feasible region**

$$\Omega = \{x \mid c_i(x) = 0,\ i \in \mathcal{E};\ c_i(x) \geq 0,\ i \in \mathcal{I}\}$$

is convex, show that local solutions of

$$\min_{x \in \Omega} f(x)$$

are also global solutions. Moreover, show that the set of global solutions is convex.

9. **Let $v : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth vector function. Consider the (generally nonsmooth) unconstrained problems**

$$f(x) = \|v(x)\|_\infty = \max_{i=1,\ldots,m} |v_i(x)|, \qquad \textbf{and} \qquad f(x) = \max_{i=1,\ldots,m} v_i(x).$$

Reformulate these (generally nonsmooth) unconstrained optimization problems as smooth constrained optimization problems.

10. **For Example 1 (page 7, LectureNotes6.pdf) show that the vector**

$$d = -\left(I - \frac{\nabla c_1(x) \nabla c_1(x)^T}{\|\nabla c_1(x)\|^2}\right) \nabla f(x) \Big/ \left\| \left(I - \frac{\nabla c_1(x) \nabla c_1(x)^T}{\|\nabla c_1(x)\|^2}\right) \nabla f(x) \right\|$$

satisfies

$$\nabla c_1(x)^T d = 0, \qquad \nabla f(x)^T d < 0,$$

whenever the first-order condition

$$\nabla f(x^*) = \lambda^* \nabla c_1(x^*)$$

is not satisfied.