Curved integrals

BACKGROUND INFORMATION

1. **Green's formula.**

   Let the boundary G of a flat bounded domain G consist of a finite set of piecewise smooth curves. Then if the functions $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are continuous on $\bar{G}$, then Green's formula is valid

$$\iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_\Gamma Pdx + Qdy \tag{1}$$

where the contour Γ is oriented so that when traversing it, the area G remains on the left.

From formula (1) for $Q = x, P = -y$ we get

$$S = \frac{1}{2}\int_\Gamma xdy - ydx \tag{2}$$

where $S = \iint_G dxdy$ is the area of the G area bounded by the Γ contour (when traversing the Γ contour, the G area remains on the left).

2. Conditions for the independence of a curved integral from the integration path. If the functions $P(x; y)$ and $Q(x; y)$ are continuous in the plane domain G, then the curvilinear integral

$$\int_{\Gamma_{AB}} Pdx + Qdy \tag{3}$$

does not depend on the integration path Γ_{AB} (the curve Γ_{AB} lies in the domain G, A is its beginning, B is its end) if and only if the expression $Pdx+Qdy$ is a complete differential some function $u = u(x; y)$, i.e. in the domain of G the condition is satisfied

$$du = Pdx + Qdy \quad \text{or} \quad \frac{\partial u}{\partial x} = P, \quad \frac{\partial u}{\partial y} = Q. \tag{4}$$

At the same time

$$\int_{\Gamma_{AB}} Pdx + Qdy = u(B) - u(A) \tag{5}$$

Here

$$u(x; y) = \int_{\Gamma_{M_0M}} Pdx + Qdy \tag{6}$$

where Γ_{M_0M} is some curve with the beginning at a fixed point $M_0\,(x_0; y_0)$ and the end at the point $M(x; y)$ lying in the region G.

Let the functions $P, Q, \frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ be continuous in the flat domain G. Then in order for the curvilinear integral (3) to be independent of the integration path, it is necessary, and in the case when G is a simply connected domain, then it is sufficient that the condition is fulfilled in the domain of G

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{7}$$

3. **The Gauss-Ostrogradsky theorem.**

   Let $G \in R^3$ be an elementary domain bounded by a piecewise smooth surface, and let the functions $P(x;y;z), Q(x;y;z), R(x;y;z)$ together with its derivatives $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ are continuous in $\bar{G}$. Then

$$\iint_S Pdydz + Qdzdx + Rdxdy = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz \tag{8}$$

where S is the outer side of the surface bounding the area of G.

Formula (8) is called the Gauss-Ostrogradsky formula. Sometimes it is written as

$$\iint_S (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dxdydz \tag{9}$$

where $\cos\alpha, \cos\beta, \cos\gamma$ are the guiding cosines of the external normal to the surface of S. The Gauss-Ostrogradsky formula can be written in vector form.

4. **The Stokes theorem.**

Let S be an oriented piecewise smooth surface bounded by a correspondingly oriented contour L. Let the functions $P(x;y;z), Q(x;y;z), R(x;y;z)$ be continuously differentiable in some domain $G \supset S$. Then

$$\int_L Pdx+Qdy+Rdz \stackrel{L}{=} \int_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial z}\right) dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy. \tag{10}$$

Formula (10) is called the Stokes formula. This formula can be written as follows:

$$\begin{aligned}\int_L Pdx + Qdy + Rdz = \iint_S \Bigg(&\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cos\alpha + \\ &+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\cos\beta + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\cos\gamma\Bigg) dS\end{aligned} \tag{11}$$

where $(\cos\alpha;\cos\beta;\cos\gamma)$ is the vector of the unit normal to the surface S, directed accordingly to the direction of the contour L. Formula (11) is sometimes written in symbolic form

$$\int_L Pdx + Qdy + Rdz = \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS \tag{12}$$

The Stokes formula can be written in vector form.

Let's say that a closed curve is oriented positively relative to some vector **a** if the direction on the curve (from the side to which vector **a** is directed) is opposite to the direction of the clockwise movement, and is oriented negatively relative to the vector **a** if the direction on the curve coincides with the direction of the clockwise movement.

EXAMPLES

Example 1. Calculate the curvilinear integral using Green's formula

$$I = \int_G x^2 y dx - xy^2 dy$$

where Γ is a circle $x^2 + y^2 = R^2$, run counterclockwise.

And let's use the formula (1), where

$$P = x^2 y, \quad Q = -xy^2, \quad \frac{\partial Q}{\partial x} = -y^2, \quad \frac{\partial P}{\partial y} = x^2$$

Then

$$I = -\int_D \left(x^2 + y^2\right) dxdy$$

where D is a circle of radius R centered at $(0;0)$. Turning to the polar coordinates, we get

$$I = -\int_0^{2\pi} d\varphi \int_0^R r^3 dr = -\frac{\pi R^4}{2}$$

Example 2. Using formula (2), find the area S bounded by the astroid

$$x = a\cos^3 t, \quad y = a\sin^3 t, \quad 0 \leqslant t \leqslant 2\pi.$$

Δ Applying the formulas (2) and Properties of a curved integral of the second kind, we obtain

$$S = \tfrac{1}{2}\int_0^{2\pi} \left(x(t)y'(t) - y(t)x'(t)\right) dt = \tfrac{3a^2}{2}\int_0^{2\pi} \left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right) dt =$$
$$= \tfrac{3a^2}{8}\int_0^{2\pi} \sin^2 2tdt = \tfrac{3a^2}{16}\int_0^{2\pi} (1 - \cos 4t)dt = \tfrac{3\pi a^2}{8}.$$

From formula (1) for $Q = x, P = -y$ we get

$$S = \frac{1}{2}\int_\Gamma xdy - ydx \tag{2}$$

where $S = \iint_G dxdy$ is the area of the G area bounded by the Γ contour (when traversing the Γ contour, the G area remains on the left).

Example 3. Show that the curvilinear integral

$$I=\int_{AB}\left(3x^2y+y\right)dx+\left(x^3+x\right)dy$$

where $A(1;-2), b(2;3)$, does not depend on the integration path, and calculate this integral.

Δ Since the functions $P=3x^2y+y, Q=x^3+x$, $\frac{\partial P}{\partial x}$ and $\frac{\partial Q}{\partial y}$ are continuous in R^2 and the condition (7) is satisfied, then the integral does not depend on the integration path and is expressed by the formula (5).

The function $u(x;y)$ can be found by formula (6). Note, however, that the integrand is a complete differential, since

$$\begin{aligned}&\left(3x^2y+y\right)dx+\left(x^3+x\right)dy=\left(3x^2ydx+x^3dy\right)+(ydx+xdy)=\\&=d\left(x^3y\right)+d(xy)=d\left(x^3y+xy\right)=du\end{aligned}$$

Therefore, $u=x^3y+xy$, and by formula (5) we find

$$I=u(B)-u(A)=30-(-4)=34$$

Example 4. Calculate the integral

$$I=\iint_S x^3dydz+y^3dzdx+z^3dxdy$$

where S is the outer side of the lateral surface of the cone $G: x^2+y^2\leqslant z^2$, $0\leqslant z\leqslant 1$.

Δ Let's denote by I_1 the integral along the outer side of the full surface of the cone S_1, by I_2 the integral along the upper side of its base S_2. Then $I=I_1-I_2$. We apply the Gauss-Ostrogradsky formula to the integral I_1

$$I_1=3\iiint_G\left(x^2+y^2+z^2\right)dxdydz$$

Turning to cylindrical coordinates, we calculate the resulting triple integral

$$I_1=3\int_0^1 dz\int_0^{2\pi}d\varphi\int_0^z\left(r^2+z^2\right)rdr=\frac{9}{10}\pi$$

Let's calculate the integral along the base of the cone:

$$I_2=\iint_{S_2}x^3dydz+y^3dzdx+z^3dxdy=\iint_{S_2}dxdy=\pi$$

Therefore, $I=-\pi/10$.

Example 5. Calculate the integral

$$A=\int_L\left(y^2-z^2\right)dx+\left(z^2-x^2\right)dy+\left(x^2-y^2\right)dz$$

where L is the curve of the intersection of the paraboloid $x^2 + y^2 + z = 3$ with the plane $x + y + z = 2$, oriented positively relative to the vector $(1; 0; 0)$.

Δ Let's apply the Stokes formula. For the surface S bounded by the curve L, we take a part of the secant plane $x+y+z = 2$ lying inside the paraboloid. The unit vector of the normal to S, directed according to the direction of the curve L, is the vector $(1/\sqrt{3}; 1/\sqrt{3}; 1/\sqrt{3})$. Since $P = y^2 - z^2, Q = z^2 - x^2, R = x^2 - -y^2$, then

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = -2(z + y), \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = -2(x + z)$$
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -2(y + x)$$

Applying the formula (11), we obtain

$$A = -\frac{4}{\sqrt{3}} \iint_S (x + y + z)dS = -\frac{8}{\sqrt{3}} \iint_S dS$$

Since $z = 2 - x - y$ on the surface of S, then

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$$

By formula for the surface integral of a first kind we find

$$A = -8 \iint_D dxdy$$

where D is the projection of S onto the plane xOy. Excluding z from the equations

$$x^2 + y^2 + z = 3, \quad x + y + z = 2,$$

we get

$$(x - 1/2)^2 + (y - 1/2)^2 = 3/2,$$

that is, $D-$ is a circle of radius $\sqrt{3/2}$. Therefore,

$$\iint_D dxdy = \frac{3}{2}\pi, \quad A = -12\pi$$

TASKS

Using Green's formula, calculate the curvilinear integral over a closed curve Γ, traversed so that its interior remains on the left $(1 - 11)$.

1. $\int_\Gamma (xy + x + y)dx + (xy + x - y)dy$, if:

    (a) Γ - ellipse $x^2/a^2 + y^2/b^2 = 1$;

(b) Γ - circle $x^2+y^2=ax$.

2. $\int_\Gamma (2xy-y)dx+x^2dy, \Gamma$ - ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3. $\int_\Gamma \frac{xdy+ydx}{x^2+y^2}, \Gamma$ - circle $(x-1)^2+(y-1)^2=1$.

4. $\int_\Gamma (x+y)^2dx-\left(x^2+y^2\right)dy, \Gamma-$ the boundary of a triangle with vertices $(1;1),(3;2),(2;5)$.

5. $\int_\Gamma \left(y-x^2\right)dx+\left(x+y^2\right)dy, \quad \Gamma-$ the boundary of the circular sector is $0<r<R, 0<\varphi<\alpha\leqslant\pi/2$, where $(r;\varphi)$ are the polar coordinates.

6. $\int_\Gamma e^x[(1-\cos y)dx+(\sin y-y)dy], \Gamma-$ the border of the area $0<x<\pi$, $0<y<\sin x$.

7. $\int_\Gamma e^{y^2-x^2}(\cos 2xydx+\sin 2xydy, \quad \Gamma$ - circle $x^2+y^2==R^2$.

8. $\int_\Gamma \left(e^x\sin y-y\right)dx+\left(e^x\cos y-1\right)dy, \Gamma$ - the border of the area $x^2+y^2<ax$, $y>0$.

9. $\int_\Gamma \frac{dx-dy}{x+y}, \Gamma$ - the border of a square with vertices $(1;0),(0;1),(-1;0),(0;-1)$.

10. $\int_\Gamma \sqrt{x^2+y^2}dx+y\left(xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right)dx, \Gamma$ - circle $x^2+y^2=R^2$.

11. $\int_\Gamma (x+y)^2dx-(x-y)^2dy, \Gamma-$ the boundary of the region formed by the segment AB, where $A(1;1), B(2;6)$, and the arc of the parabola $y=ax^2+bx+c$ passing through the points $A, B, O(0;0)$.

After making sure that the integral expression is a complete differential, calculate the curved integral along the curve Γ with the beginning at point A and the end at point $B(12-24)$.

12. $\int_\Gamma xdy+ydx, A(-1;3), B(2;2)$.

13. $\int_\Gamma xdx+ydy, A(-1;0), B(-3;4)$.

14. $\int_\Gamma (x+y)dx+(x-y)dy, A(2;-1), B(1;0)$.

15. $\int_\Gamma 2xydx+x^2dy, A(0;0), B(-2;-1)$.

16. $\int_\Gamma \left(x^4+4xy^3\right)dx+\left(6x^2y^2-5y^4\right)dy, A(-2;-1), B(0;3)$.

17. $\int_\Gamma \left(x^2+2xy-y^2\right)dx+\left(x^2-2xy-y^2\right)dy, \quad A(3;0), \quad A(0;-3)$.

18. $\int_\Gamma \left(3x^2-2xy+y^2\right)dx+\left(2xy-x^2-3y^2\right)dy,\ A(-1;2), B(1;-2)$.

19. $\int_\Gamma f(x+y)(dx+dy), f(t)$ - continuous function, $A(0;0),\ B\left(x_0;y_0\right)$.

20. $\int_\Gamma \varphi(x)dx+\psi(y)dy, \quad \varphi(t), \quad \psi(t)$ - continuous functions, $A\left(x_1;y_1\right), B\left(x_2;y_2\right)$.

21. $\int_\Gamma e^x\cos ydx-e^x\sin ydy, A(0;0), B\left(x_0;y_0\right)$.

22. $\int_\Gamma xdx + y^2dy - z^3dz, A(-1;0;2), B(0;1;-2)$.

23. $\int_\Gamma yzdx + xzdy + xydz, A(2;-1;0), B(1;2;3)$.

24. $\int_\Gamma \frac{xdx+ydy+zdz}{\sqrt{x^2+y^2+z^2}}, A \in S_1, B \in S_2$, where S_1 - sphere $x^2+y^2+z^2 = R_1^2$, S_2- sphere $x^2+y^2+z^2 = R_2^2\,(R_1 > 0, R_2 > 0)$.

Using the Gauss-Ostrogradsky theorem, calculate the integrals (25-29).

25. $\iint_S (1+2x)dydz + (2x+3y)dzdx + (3y+4z)dxdy$, where S :
    - (a) the outer side of the pyramid surface $x/a + y/b + z/c \leqslant 1,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0$
    - (b) the inner side of the surface $|x-y+z|+|y-z+x|+ +|z-x+y| = a$.

26. $\iint_S zdxdy + (5x+y)dydz$, where S :
    - (a) the outer side of the full cone surface $x^2+y^2 \leqslant z^2,\ 0 \leqslant z \leqslant 4$
    - (b) the inner side of the ellipsoid $x^2/4 + y^2/9 + z^2 = 1$;
    - (c) the outer side of the area boundary $1 < x^2+y^2+z^2 < 4$.

27. $\iint_S x^2dydz + y^2dzdx + z^2dxdy$, where S :
    - (a) the inner side of the parallelepiped surface $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ 0 \leqslant z \leqslant c$
    - (b) the outside of the full surface $x^2/a^2+y^2/b^2 \leqslant z^2/c^2, 0 \leqslant z \leqslant c$ (cone).

28. $\iint_S x^3dydz + y^3dzdx + z^3dxdy$, where S :
    - (a) the outer side of the tetrahedron surface $x+y+z \leqslant a, x \geqslant 0,\ y \geqslant 0, z \geqslant 0$
    - (b) the inner side of the sphere $x^2+y^2+z^2 = R^2$.

29. $\iint_S x^4dydz + y^4dzdx + z^4dxdy$, where S :
    - (a) sphere $x^2+y^2+z^2 = R^2$;
    - (b) the outer side of the full surface of the hemisphere $x^2+y^2+z^2 \leqslant R^2$, $z \geqslant 0$.

Using the Stokes formula, calculate the integrals (30-37).

30. $\int_L (x+z)dx + (x-y)dy + xdz, L$ - ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = c$, oriented negatively with respect to the vector $(0;0;1)$.

31. $\int_L y^2dx + z^2dy + x^2dz, L-$ the boundary of a triangle with vertices at points $(a;0;0), (0;a;0), (0;0;a)$, oriented positively with respect to the vector $(0;1;0)$.

32. (a) $\int_L ydx + zdy + xdz$

    (b) $\int_L \frac{xdy-ydx}{x^2+y^2} + zdz$; where $L-$ circle $x^2+y^2+z^2=R^2$, $x+y+z=0$, oriented positively with respect to the vector $(0;0;1)$.

33. $\int_L \left(y^2-z^2\right)dx + \left(z^2-x^2\right)dy + \left(x^2-y^2\right)dz, L$ - the intersection curve of the cube surface $|x| \leqslant a, |y| \leqslant a, |z| \leqslant a$ the plane $x++y+z=3a/2$, oriented positively relative to the vector $(1;0;0)$.

34. $\int_L (y-z)dx + (z-x)dy + (x-y)dz$, where:

    (a) L - the circle $x^2+y^2+z^2=R^2, y=x\operatorname{tg}\varphi, \varphi \in (0;\pi)$, oriented positively relative to the vector $(1;0;0)$;

    (b) L - ellipse $x^2+y^2=a^2, x/a+z/c=1, a>0, c>0$, oriented negatively relative to the vector $(1;0;0)$.

35. $\int_L ydx - zdy + xdz, L$ - curve $x^2+y^2+2z^2=2a^2, y-x=0$, oriented positively with respect to the vector $(1;0;0)$.

36. $\int_L \left(y^2+z^2\right)dx + \left(z^2+x^2\right)dy + \left(x^2+y^2\right)dz, L$ - the curve $x^2+y^2+z^2=2ax$, $x^2+y^2=2bx$, $z>0$, $0<b<a$, oriented positively relative to the vector $(0;0;1)$.

37. $\int_L z^3dx + x^3dy + y^3dz, L$ - the curve $2x^2-y^2+z^2=a^2, x+y=0$, oriented positively relative to the vector $(1;0;0)$.