Real Analysis 2024. Homework 5.

1. Let $p \in X$ and δ_p be Dirac measure. Prove that

$$\int_X f d\delta_p = f(p)$$

for every function $f : X \to \mathbb{R}$.

Proof. First, consider the case when $f = \chi_E$. Then, by the definition of the integral

$$\int_X f d\delta_p = 1 \cdot \delta_p(E) + 0 \cdot \delta_p(X \setminus E) = \begin{cases} 0, & p \notin E; \\ 1, & p \in E \end{cases} = \chi_E(p) = f(p).$$

Then, by linearity (or by definition) this statement is true for simple functions. Let $f \geq 0$ be a measurable (actually, any) function $f : X \to \mathbb{R}$. If ϕ is simple and $\phi \leq f$ then $\phi(p) \leq f(p)$ and

$$\int_X \phi d\delta_p = \phi(p) \leq f(p).$$

So

$$\int_X f d\delta_p = \sup \left\{ \int_X \phi d\delta_p : \phi \text{ is simple and } \phi \leq f \right\} \leq f(p).$$

Although let $\phi_0(p) = f(p)\chi_{\{p\}}$. Then

$$\int_X \phi_0 d\delta_p = f(p)$$

and

$$\int_X f d\delta_p = \sup \left\{ \int_X \phi d\delta_p : \phi \text{ is simple and } \phi \leq f \right\} = f(p).$$

$\square$

2. Let $n \in \mathbb{N}$, $0 \leq k \leq 2^n - 1$. Consider an interval

$$\Delta_{k,n} = \left[\frac{k}{2^n}, \frac{k+1}{2^n} \right)$$

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and let $f_{n,k}(x) = \chi_{\Delta_{k,n}}$ be a characteristic function of this interval. This defines a countable family of functions that can be considered as a sequence. For example,

$$g_1 = f_{1,0}, \; g_2 = f_{1,1}, \; g_3 = f_{2,0}, \; g_4 = f_{2,1}, \; g_5 = f_{2,2}, \; g_6 = f_{2,3}, \; g_7 = f_{3,0}, \ldots.$$

Let $m_n = 1 + 2 + \ldots + 2^{n-1} = 2^n - 1$ and for $m_n \le j < m_{n+1}$ we let $k = j - m_n$. Then the above sequnce is defined as follows:

$$g_j = f_{n,j-m_n}, \quad m_n \le j < m_{n+1}.$$

(a) Prove that a sequence g_j converges in $L^1[0,1]$.

(b) Indicate a subsequence that converges to the limit function a.e.

Proof. (a) $\int_0^1 g_j dx = \int_0^1 f_{n,j-m_n} dx = \frac{1}{2^n} \to 0$.

(b) First, note that for every $x \in [0,1]$ and for every $J \in \mathbb{N}$ we can find $j_1 > J$ such that $g_{j_1}(x) = 1$. and $j_2 > J$ such that $g_{j_2}(x) = 0$. Hence, g_j doesn' converge for every $x \in [0,1]$. At the same time

$$g_{m_n} = f_{n,0} = \chi_{[0,1/2^n]} \to 0, \quad x \ne 0.$$

$\square$

3. Let $\mu(X) < +\infty$, $f_n \in L^1(X, \mu)$, and $f_n \rightrightarrows f$ on X. Prove that $f \in L^1(X, \mu)$ and $\|f_n - f\|_1 \xrightarrow[n \to \infty]{} 0$.

Proof. By monotonicity of the integral we have

$$\|f_n - f\|_1 \le \mu(X) \sup_{x \in X} |f_n(x) - f(x)| \to 0, \; n \to +\infty.$$

$\square$

4. Provide an example of a set and a measure such that $\mu(X) = +\infty$, $f_n, f \in L^1(X, \mu))$, and $f_n \rightrightarrows f$ on X but $\|f_n - f\|_1 \nrightarrow 0$.

Let $X = [0, \infty)$, μ be Lebesgue measure,

$$f_n = \frac{1}{n}\chi_{[n,2n)}.$$

Then $|f_n| < 1/n$ so $f_n \rightrightarrows 0$ on X. At the same time

$$\|f_n\|_1 = 1 \nrightarrow 0.$$