Complex Analysis 2024. Homework 5.

1. Assume that f is holomorphic in domain D and $\arg f(z)$ is constant. Prove that f is constant in D.

   Proof. $\square$

2. Assume that f is holomorphic in domain D and

$$A \operatorname{Im} f(z) + B \operatorname{Re} f(z) + C = 0, z \in D,$$

   for some real constants A, B, C. Prove that f is constant.

   Proof. Consider a function

$$g = (B - iA) f(z) + C.$$

   Then g is holomorphic

$$\operatorname{Re} g = A \operatorname{Im} f(z) + B \operatorname{Re} f(z) + C = 0$$

   and, consequently, $g = iC_1$ in domain D for some real constant C_1. Hence,

$$f(z) = \frac{iC_1 - C}{B - iA}.$$

   $\square$

3. Calculate the integral $\int_\gamma \frac{dz}{z}$ along the following paths:

   (a) $\gamma_1 = e^{it}, \quad t \in [0, 4\pi]$;

   (b) $\gamma_2 = e^{-it}, \quad t \in [0, 2\pi]$;

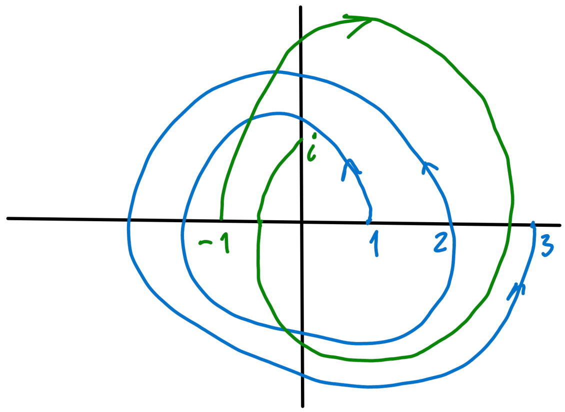
   (c) See paths γ_3 (green) γ_4 (blue) in the picture below.

   Answers.

   (a) $4\pi i$;

   (b) $-2\pi i$;

   (c) $\int_{\gamma_3} \frac{dz}{z} = -\frac{5\pi}{2} i$;



(d) $\int_{\gamma_4} \frac{dz}{z} = \ln 3 + 4\pi i$.

4. Calculate $\int_\gamma dz$, where γ is the left half of the ellipse $\frac{1}{36}x^2 + \frac{1}{4}y^2 = 1$ from $z = 2i$ to $z = -2i$.

   Solution.

$$\int_\gamma dz = z|_{2i}^{-2i} = -4i.$$

5. Calculate $\int_\gamma \left(z + \frac{1}{z}\right) dz$, where γ is a circle $|z| = 2$ oriented counter-clockwise.

   Solution.

$$\int_\gamma z dz = 0$$

since $f(z) = z$ is holomorphic in $\mathbb{C}$ and γ is closed path;

$$\int_\gamma \frac{dz}{z} = 2\pi i.$$

Hence,

$$\int_\gamma \left(z + \frac{1}{z}\right) dz = 2\pi i.$$

6. Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial with $c_k \in \mathbb{R}$. Show that
$$\int_{-1}^{1} f(x)^2 dx \leq \pi \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \pi \sum_{k=0}^{n} c_k^2.$$
Hint. For the first inequality, apply Cauchy-Goursat's theorem to the function $f(z)^2$ separately on the top half and the bottom half of the unit disk.

Proof.
$$\int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} = \frac{1}{2} \sum_{k,j=1}^{n} \int_0^{2\pi} c_k c_j e^{i(k-j)\theta} d\theta = \pi \sum_{k=0}^{n} c_k^2.$$
Since $\int_0^{2\pi} c_k c_j e^{i(k-j)\theta} d\theta = 2\pi \delta_{k,j}$.

Then integral over the top half is equal to 0 and
$$\int_{-1}^{1} f(x)^2 dx + \int_0^{\pi} f(z)^2 dz = 0.$$
Then integral over the lower half is equal to 0 and
$$-\int_{-1}^{1} f(x)^2 dx + \int_{\pi}^{2\pi} f(z)^2 dz = 0.$$
Hence,
$$2\int_{-1}^{1} f(x)^2 dx \leq \int_0^{2\pi} |f(z)|^2 |dz| \, 2\pi \int_0^{2\pi} \left|f\left(e^{i\theta}\right)\right|^2 d\theta.$$
□

7. Show that an analytic function $f(z)$ has a primitive in D if and only if $\int_\gamma f(z)dz = 0$ for every closed path γ in D.

Proof. The sufficiency is obvious. Assume now that $\int_\gamma f(z)dz = 0$ for every closed path γ in D. Fix $z_0 \in D$ and for $z \in D$
$$F(z) = \int_\gamma f(z)dz$$

for some path γ that connects z_0 and z.

The definition doesn't depend on path γ. Assume that γ_1 and γ_2 connect z_0 and z. Then the compund path $\lambda = \gamma_1 \cup \gamma_2^{-1}$ is closed and

$$\int_{\gamma_1} f dz - \int_{\gamma_2} f dz = \int_{\gamma_1 \cup \gamma_2} f dz = 0.$$

To prove that F is differentiable consider z and $\delta > 0$ such that $B(z, \delta) \subset D$. Then

$$\frac{F(z+w) - F(z)}{w} = \frac{1}{w} \int_z^{z+w} f(\xi) d\xi \to f(z), \; w \to 0,$$

where the integral $\int_z^{z+w}$ is consedred over a segment that connects z and $z + w$. □

8. Show that
$$\left| \oint_{|z|=R} \frac{\log z}{z^2} dz \right| \leq 2\sqrt{2}\pi \frac{\log R}{R}, \quad R > e^{\pi}.$$

Proof. Consider a parametrization $z = Re^{it}$ $-\pi \leq t \leq \pi$. Then

$$|\log z| = \sqrt{\log^2 R + t^2} \leq \sqrt{2} \log R, \quad R > e^{\pi}$$

and
$$\left| \oint_{|z|=R} \frac{\log z}{z^2} dz \right| \leq 2\pi R \frac{\sqrt{2} \log R}{R^2}.$$

□

9*. Show that if D is a bounded domain with smooth boundary, then

$$\int_{\partial D} \bar{z} dz = 2i \, \mathrm{Area}(D).$$

Proof. To prove this apply Green's formula. □