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## 2.6.  BASIC EQUATIONS OF MATHEMATICAL PHYSICS

The subject of the theory of equations of mathematical physics is the study of differential, integral and functional equations describing natural phenomena. The construction of a mathematical model of the process begins with the establishment of values that are decisive for the process under study. Further, using physical laws (principles) expressing the relationship between these quantities, an equation (system of equations) in partial derivatives is constructed and additional conditions (initial and boundary) to the equation (system) are drawn up.

We will mainly study second-order partial differential equations with one unknown function, in particular the wave equation, the heat equation and the Laplace equation, commonly called the classical equations of mathematical physics.

### 2.6.1.    THE OSCILLATION EQUATION

Many problems of mechanics (vibrations of strings, rods, membranes and three-dimensional volumes) and physics (electromagnetic oscillation) lead to an oscillation equation of the form

$$\rho\frac{\partial^2 u}{\partial t^2} = \operatorname{div}(k\operatorname{grad}u) - qu + F(x,t),$$

where the unknown function $u = u(x,t)$ depends on $n$ $(n = 1,2,3)$ spatial variables $x = (x_1, x_2, ..., x_n)$ and time $t$, coefficients $\rho, k, q$ are determined by the properties of the medium, $F(x,t)$ is the density of the external disturbance:

$$\mathrm{div}(k\,\mathrm{grad}\,u) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(k\,\frac{\partial u}{\partial x_i}\right).$$

Consider a stretched string fixed at the ends. By string we mean a thin thread that does not exert any resistance to changing its shape, unrelated to changing its length. The tension force $T_0$ acting on the string is assumed to be significant, so the effect of gravity can be ignored.

Let the string be directed along the $x$ axis in the equilibrium position.

We will consider only the *transverse vibrations* of the string, assuming that the movement occurs in the same plane and that all points of the string move perpendicular to the $x$ axis.

Let's denote by $u(x,t)$ the displacement of the string points at time $t$ from the equilibrium position.

Considering further only *small vibrations* of the string, we will assume that the displacement $u(x,t)$, as well as the derivative $\frac{\partial u}{\partial x}$, are so small that their squares and products can be neglected compared to the quantities themselves.

For each fixed value of $t$, the graph of the function $u(x,t)$ obviously gives the shape of the string at this point in time (Fig. 1).

Denote by $F(x,t)$ the density of external forces acting on the string at point $x$ at time $t$ and directed perpendicular to the $x$ axis.
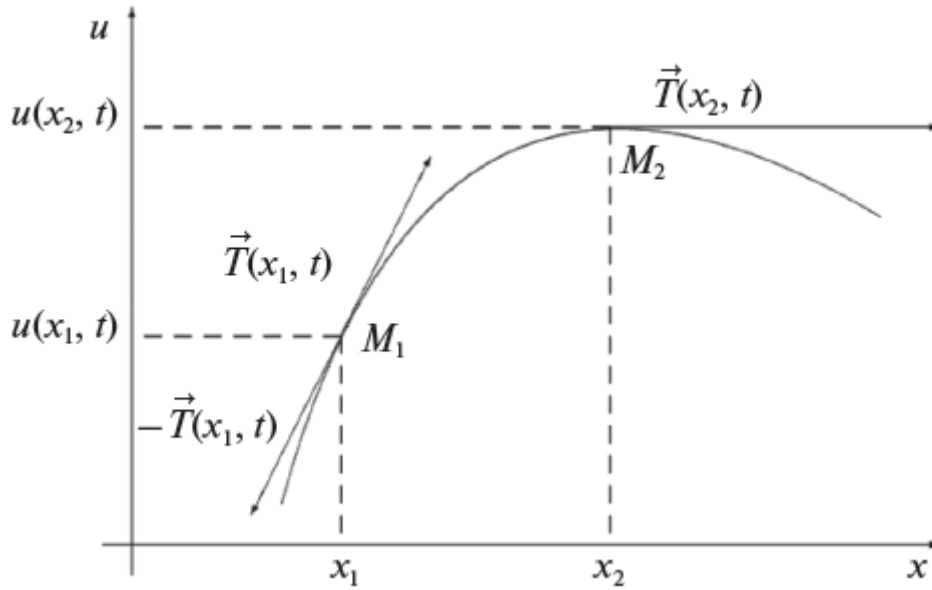
Fig. 1. Instantaneous profile of the string section $(x_1, x_2)$ at time $t$

Let $\rho(x)$ be the linear density of the string, then the function $u(x,t)$ satisfies the differential *equation of string vibrations*:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + F(x,t).$$

If $\rho(x) = \rho = \text{const}$, that is, in the case of a homogeneous string, the equation is usually written as

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t), \ a = \sqrt{\frac{T_0}{\rho}}, \ f(x,t) = \frac{F(x,t)}{\rho}.$$

This equation will be called the *one-dimensional wave equation.*

If there is no external force, then we have: $F(x,t) = 0$ and get the *equation of free vibrations of the string*

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$$

The equation of small transverse vibrations of the membrane A =0 is similar:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T_0\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + F(x,t).$$

If the density $\rho$ is constant, then the membrane oscillation equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) + f(x,t), \quad a = \sqrt{\frac{T_0}{\rho}}, \quad f(x,t) = \frac{F(x,t)}{\rho}.$$

The last equation will be called the *two-dimensional wave equation*.

Three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) + f(x,t)$$

describes the processes of sound propagation in a homogeneous medium and electromagnetic waves in a homogeneous nonconducting medium. This equation is satisfied by the density of the gas, its pressure and velocity potential, as well as the components of the electric and magnetic field strengths and the corresponding potentials.

We will write the wave equations using the single formula

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u + f,$$

where $\Delta$ is the Laplace operator:

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

## 2.6.2.     THE EQUATION OF THERMAL CONDUCTIVITY (HEAT EQUATION)

The processes of heat propagation or particle diffusion in the medium are described by the heat equation

$$\rho \frac{\partial u}{\partial t} = \text{div}(k\,\text{grad}\,u) - qu + F(x,t).$$

Let's derive the equation of heat propagation.

Denote by $u(x,t)$ the temperature of the medium at point $x = (x_1, x_2, x_3)$ at time $t$, and by $\rho(x)$, $c(x)$ and $k(x)$, respectively, its density, specific density and thermal conductivity coefficient at point $x$.

Let $F(x,t)$ be the intensity of heat sources at point $x$ at time $t$.

Let's calculate the heat balance in an arbitrary volume $V$ over a period of time $(t, t+\Delta t)$. Denote by $S$ the boundary of $V$ and let $\vec{n}-$ be the external normal to it.

According to Fourier's law, through the surface $S$, an amount of heat

$$Q_1 = \iint_S k(x) \frac{\partial u}{\partial n} dS \Delta t = \Delta t \iint_S (k(x)\text{grad}\,u, \vec{n}) dS,$$

enters the volume $V$, equal, by virtue of the Gauss's-Ostrogradsky's formula (theorem):

$$Q_1 = \iiint_V \text{div}(k(x)\text{grad}\,u) dx \Delta t.$$

Due to the thermal sources in volume $V$ the amount of heat

$$Q_2 = \iiint_V F(x,t) dx \Delta t.$$

Since the temperature in volume $V$ has increased by

$$u(x,t+\Delta t)-u(x,t)\approx\frac{\partial u}{\partial t}\Delta t\ ,$$

over a period of time $(t,t+\Delta t)$, it is necessary to expend the amount of heat

$$Q_3=\iiint\limits_V c(x)\rho(x)\frac{\partial u}{\partial t}dx\Delta t\ .$$

On the other hand, $Q_3=Q_1+Q_2$ and therefore

$$\iiint\limits_V\left[\text{div}(k(x)\text{grad}u)+F-c(x)\rho(x)\frac{\partial u}{\partial t}\right]dx\Delta t=0\ ,$$

from where, due to the arbitrariness of the volume $V$, we obtain the equation of heat propagation

$$c(x)\rho(x)\frac{\partial u}{\partial t}=\text{div}(k(x)\text{grad}u)+F(x,t)\ . \tag{2.17}$$

If the medium is homogeneous, that is, $c(x)$, $\rho(x)$ and $k(x)$ are constants, then equation (2.17) takes the form

$$\frac{\partial u}{\partial t}=a^2\Delta u+f\ , \tag{2.18}$$

where

$$a^2=\frac{k}{c\rho},\ f=\frac{F}{c\rho}\ .$$

Equation (2.18) is called the heat equation or the *diffusion equation.*

### 2.6.3.    THE STATIONARY EQUATION

For stationary processes $F(x,t) = F(x)$, $u(x,t) = u(x)$, and the equations of oscillations and heat take the form

$$-\text{div}(k\,\text{grad}\,u) + qu = F(x).\qquad\qquad(2.19)$$

For $k = const,\ q = 0$, equation (2.19) is called the Poisson equation:

$$\Delta u = -f,\ f = \frac{F}{k}.\qquad\qquad(2.20)$$

For $f = 0$, the equation (2.20) is called the Laplace equation:

$$\Delta u = 0.$$

Let us consider the potential flow of fluid without sources, namely: let inside a certain volume $V$ with a boundary $S$, which has a stationary flow of an incompressible fluid (density $\rho = const$), characterized by a velocity $\vec{v}(x_1, x_2, x_3)$. If the fluid flow is not vortex ($\text{rot}\,\vec{v} = 0$), then the velocity $\vec{v}$ is a potential vector, that is,

$$\vec{v} = \text{grad}\,u,\qquad\qquad(2.21)$$

where $u$ is a scalar function called the *velocity potential*.

If there are no sources, then

$$\text{div}\,\vec{v} = 0.\qquad\qquad(2.22)$$

Now from formulas (2.21) and (2.22) we get

$$\text{div}\,\text{grad}\,u = 0,$$

or

$$\Delta u = 0,$$

that is, the velocity potential satisfies the Laplace equation.

## 2.7. FORMULATION OF BASIC BOUNDARY VALUE PROBLEMS FOR A SECOND-ORDER DIFFERENTIAL EQUATION

### 2.7.1. CLASSIFICATION OF BOUNDARY VALUE PROBLEMS

As shown, the linear equation of the second order

$$\rho\frac{\partial^2 u}{\partial t^2} = \mathrm{div}(k\,\mathrm{grad}u) - qu + F(x,t) \qquad (2.23)$$

describes the processes of vibrations, equation

$$\rho\frac{\partial u}{\partial t} = \mathrm{div}(k\,\mathrm{grad}u) - qu + F(x,t) \qquad (2.24)$$

describes the processes of diffusion, and equation

$$-\mathrm{div}(k\,\mathrm{grad}u) + qu = F(x) \qquad (2.25)$$

describes stationary processes.

Let $G \subset R^n$ be the area where the process takes place and $S$ be its boundary. Thus, $G$ is the domain of setting equation (2.25). The domain of setting equations (2.23) and (2.24) will be considered cylinder $\Omega_T = G \times (0,T)$ height $T$ and base $G$. Its boundary consists of the lateral surface $S \times (0,T)$ and two bases: the lower $\bar{G} \times \{0\}$ and the upper $\bar{G} \times \{T\}$ (Fig. 2).
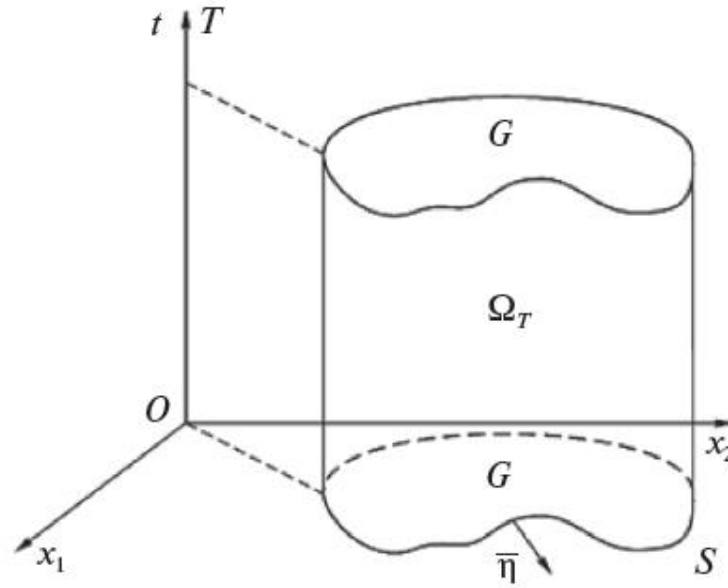
Fig. 2. The field of setting the equations of oscillations and diffusion

We will assume that the coefficients $\rho, k, q$ of equations (2.23)– (2.25) do not depend on time $t$; further, in accordance with their physical meaning, we will assume that $\rho(x) > 0,\ k(x) > 0,\ q(x) \geq 0, x \in \bar{G}$.

Under these assumptions, the oscillation equation (2.23) is of the hyperbolic type, the diffusion (heat equation) equation (2.24) is of the parabolic type, and the stationary equation (2.25) is of the elliptical type.

Further, in order to fully describe the physical process, it is necessary, in addition to the equation describing this process, to specify the initial state of this process (initial conditions) and the regime at the boundary of the region in which the process occurs (boundary conditions).

There are three types of problems for differential equations.

1) The Cauchy problem for hyperbolic and parabolic equations: initial conditions are specifying, the region $G$ coincides with the entire space $R^n$, there are no boundary conditions.

2) Boundary value problem for elliptic type equations: boundary conditions are specifying at the boundary $S$, the initial conditions, of course, are absent.

3) A mixed problem for hyperbolic and parabolic equations: both initial and boundary conditions are specifying, $G \neq R^n$.

Let us describe in more detail each of the listed boundary value problems for the equations (2.23)– (2.25) under consideration.

## 2.7.2.THE CAUCHY PROBLEM

For the oscillation equation (2.23), the Cauchy problem is posed as follows: find a function $u(x,t)$ of class $C^2 (t>0) \cap C^1 (t \geq 0)$ satisfying equation (2.23) in the half-space $t > 0$ and the initial conditions at $t = 0$:

$$u\big|_{t=0} = u_0(x), \quad \frac{\partial u}{\partial t}\bigg|_{t=0} = u_1(x). \qquad (2.26)$$

At the same time, it is necessary:

$$F \in C(t>0), \ u_0 \in C^1\left(R^n\right), \ u_1 \in C\left(R^n\right).$$

For the thermal conductivity equation (heat equation) (2.24), the Cauchy problem is posed as follows: find a function $u(x,t)$ of class $C^2 (t>0) \cap C^1 (t \geq 0)$ satisfying equation (2.24) in the half-space $t > 0$ and the initial conditions at $t = 0$:

$$u\big|_{t=0} = u_0(x).$$

(2.27)

At the same time, it is necessary that

$$F \in C(t > 0), \quad u_0 \in C(R^n).$$

The above statement of the Cauchy problem admits the following generalization. Let the differential equations of the 2nd order be given:

$$\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_{i0} \frac{\partial^2 u}{\partial x_i \partial t} + \Phi\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial u}{\partial t}\right), \quad (2.28)$$

piecewise smooth surface $\Sigma$: $t = \sigma(x)$ and functions $u_0$ and $u_1$ on $\Sigma$. The Cauchy problem for equation (2.28) consists in finding, in some part of the domain $t > \sigma(x)$ adjacent to the surface $\Sigma$, a solution $u(x,t)$ satisfying the boundary conditions on $\Sigma$

$$u\big|_{\Sigma} = u_0, \quad \frac{\partial u}{\partial n}\bigg|_{\Sigma} = u_1,$$

where $\vec{n}$ is the normal to $\Sigma$ directed towards increasing $t$.

## 2.7.3. BOUNDARY VALUE PROBLEM FOR ELLIPTIC TYPE EQUATIONS. A MIXED TASK

The boundary value problem for equation (2.25) consists in finding a function $u(x)$ of class $C^2(G) \cap C^1(\bar{G})$ satisfying in the domain $G$ equation (2.25) and a boundary condition on $S$ of the form

$$\alpha u + \beta \frac{\partial u}{\partial n}\bigg|_S = v, \ (2.29)$$

where $\alpha, \beta, v$ — are given continuous functions on $S$, and

$$\alpha \geq 0, \ \beta \geq 0, \ \alpha + \beta > 0.$$

The following types of boundary conditions are distinguished (2.29).

Boundary condition of the first kind $(\alpha = 1, \ \beta = 0)$:

$$u\big|_S = u_0.$$

The boundary condition of the second kind $(\alpha = 0, \ \beta = 1)$:

$$\frac{\partial u}{\partial n}\bigg|_S = u_1.$$

Boundary condition of the third kind $(\alpha \geq 0, \ \beta = 1)$:

$$\alpha u + \frac{\partial u}{\partial n}\bigg|_S = u_2.$$

The corresponding boundary value problems are called problems of *the I, II and III kind.* For the Laplace and Poisson equations, the boundary value problem of the first kind:

$$\Delta u = -f \ , \ u|_S = u_0$$

is called the *Dirichlet problem*; the boundary value problem of the second kind:

$$\Delta u = -f \ , \ \left.\frac{\partial u}{\partial n}\right|_S = u_1$$

is called the *Neumann problem.*

For the oscillation equation (2.23), the mixed problem is posed as follows: find a function $u(x,t)$ of class $C^2(\Omega_\infty) \cap C^1(\bar{\Omega}_\infty)$ satisfying equation (2.23) in the cylinder $\Omega_\infty$, the initial conditions (2.26) at $t = 0$ and the boundary condition (2.29) at $x \in S$, $t \geq 0$.

Similarly, for the diffusion equation (2.24), the mixed problem is posed as follows: find a function $u(x,t)$ of class $C^2(\Omega_\infty) \cap C^1(\bar{\Omega}_\infty)$ satisfying equation (2.24) in the cylinder $\Omega_\infty$, the initial condition (2.27) at $t = 0$ and the boundary condition (2.29) at $x \in S$, $t \geq 0$.

## 2.7.4. THE CORRECTNESS OF THE FORMULATION OF MATHEMATICAL PHYSICS PROBLEMS

Since the problems of mathematical physics describe real physical processes, the mathematical formulation of these problems must meet the following requirements:

1) the solution exists in some class of M1 functions;

2) the solution is the only one in a certain class of M2 functions;

3) the solution continuously depends on the data of the problem (initial and boundary data, free term, coefficients of the equation, and so on).

The continuous dependence of the solution $u$ on the data of the problem $\tilde{u}$ means the following: let the sequence $\tilde{u}_k$, $k = 1,2,...$ , in some sense tends to $\tilde{u}$ and $\tilde{u}_k$, $k = 1,2,...$, $u$ are the corresponding solutions to the problem; then $u_k \to u$, $k \to \infty$ in the sense of convergence, appropriately chosen.

The requirement of continuous dependence of the solution is due to the fact that the data of a physical problem, as a rule, are determined from experiment approximately, and therefore one must be sure that the solution of the problem will not significantly depend on measurement errors.

A problem satisfying the listed requirements 1)-3) is called *correctly posed*, and the corresponding set of functions $M_1 \cap M_2$ is a *correctness class*.

Consider the following system of differential equations with $N$ unknown functions $u_1, u_2,...,u_N$:

$$\frac{\partial^{k_i} u_i}{\partial t^{k_k}} = \Phi_i\left(x, t, u_1, u_2, \dots, u_N, \dots, \frac{\partial^{\alpha_0 + \alpha_1 + \dots + \alpha_n} u_j}{\partial t^{\alpha_0} \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \dots\right), \qquad (2.30)$$

where $i = 1, 2, \dots, N$.

Here, the right-hand sides do not contain derivatives of order higher than $k_i$ and derivatives with respect to $t$ of order higher than $k_i - 1$, that is

$$\alpha_0 + \alpha_1 + \dots + \alpha_n \le k_i, \quad \alpha_0 \le k_i - 1.$$

For system (2.30), we set the following Cauchy problem: to find a solution $u_1, u_2, \dots, u_N$ of this system satisfying the initial conditions at $t = t_0$:

$$\left.\frac{\partial^k u_i}{\partial t^k}\right|_{t=t_0} = \varphi_{ik}(x), \quad k = 0, 1, \dots, k_i - 1, \quad i = 1, 2, \dots, N, \qquad (2.31)$$

where $\varphi_{ik}(x)$ are the given functions in some domain $G \subset R^n$.