Chapter 15. Fourier series and transform

Let *f(x)* be continuous or have finite number of first-type discontinuities, and it is periodic with period *2l* . Then the Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos\omega_n x+b_n\sin\omega_n x), \qquad \textbf{(1)}$$

$$a_0=\frac{1}{\ell}\int_{-\ell}^{\ell}f(x)dx, \qquad \textbf{(2)}$$

$$a_n=\frac{1}{\ell}\cdot\int_{-\ell}^{\ell}f(x)\cdot\cos\omega_n x dx \quad (n=1,2,3,\ldots), \qquad \textbf{(3)}$$

$$b_n=\frac{1}{\ell}\cdot\int_{-\ell}^{\ell}f(x)\cdot\sin\omega_n x dx \quad (n=1,2,3,\ldots). \qquad \textbf{(4)}$$

$$\omega=n\pi/l$$

converges at any x , and its sum

$S(x)=f(x)$ at points of continuity,

$S(x)=[\,f(x-0)+f(x+0)\,]/2$ at discontinuities.

If $f(x)$ is <u>even</u>, then Fourier series simplifies:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos \omega_n x \,,$$

$$a_n = \frac{2}{\ell}\cdot\int_0^{\ell} f(x)\cdot\cos\omega_n x dx \quad (n=0,1,2,3,\ldots)|, \qquad \mathbf{(5)}$$

<u>Example</u>

l=1, f(x)=x **(even at $-1\le x\le 1$)** ***$\omega=n\pi/l=n\pi$***

Scilab:

```
clear
I=1:1:11
for n=1:11
nn=n-1
I(n)=2*integrate('x*cos(%pi*x*nn)','x', 0, 1)
end
//disp(I)
//plot(I,'r')
//xgrid
for i=1:30
    xx(i)=0.1*(i-1)
    x=xx(i)
    yy(i)=0.5*I(1)+I(2)*cos(%pi*x)...
    +I(3)*cos(%pi*x*2)...
```

```
    +I(4)*cos(%pi*x*3)...
    +I(5)*cos(%pi*x*4)...
    +I(6)*cos(%pi*x*5)
end
plot(xx,yy)
xgrid()
```

Fouries transform for non-periodic functions

Fourier transform is a mathematical model that decomposes a function or signal into its constituent frequencies. It helps to transform the signals between two different domains like **transforming the frequency domain to the time domain**. It is a powerful tool **used in many fields, such as signal processing, physics, and engineering**, to analyze the frequency content of signals or functions that vary over time or space.

$$F(\omega) = \int_{-\infty}^{+\infty} f(x) \exp(-i\omega x)\, dx$$

$$A(\omega) = |F(\omega)|, \quad \operatorname{tg} \alpha(\omega) = \arg F(\omega)$$

Inverse Fourier transform:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \exp(i\omega x)\, d\omega$$

Example

1D FFT (of a vector):

$$f(t) = \sin(2\pi 50t) + \sin(2\pi 200t + \pi/4) + 2\sin(2\pi 400t) + \text{noise}$$

```
clear
//Frequency components of a signal
// build a noised signal sampled at 1000hz  containing  pure frequencies at 50
// and 70 Hz
sample_rate = 1000;
t = 0:1/sample_rate:0.6;
N = size(t,'*'); //number of samples
f = sin(2*%pi*50*t) + sin(2*%pi*200*t+%pi/4) +...
2*sin(2*%pi*400*t)+ grand(1,N,'nor',0,1);
F=fft(f);  //  !!!

// f is real so the fft response is conjugate symmetric and we
retain only the first N/2 points
ff = sample_rate*(0:(N/2))/N; //associated frequency vector
n = size(ff,'*')
clf()
plot(ff, abs(F(1:n)))
```