**Each correctly solved task gives 0.3 points. In total, you can get 3 points for 10 exercises.**

## EXERCISES 1

**1** Compute the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$ of the Rosenbrock function

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Show that $x^* = (1, 1)^T$ is the only local minimizer of this function, and that the Hessian matrix at that point is positive definite.

**2** Show that the function $f(x) = 8x_1 + 12x_2 + x_1{}^2 - 2x_2{}^2$ has only one stationary point, and that it is neither a maximum or minimum, but a saddle point. Sketch the contour lines of $f$.

**3** Let $a$ be a given $n$-vector, and $A$ be a given $n \times n$ symmetric matrix. Compute the gradient and Hessian of $f_1(x) = a^T x$ and $f_2(x) = x^T Ax$.

**4** Write the second-order Taylor expansion

$$f(x + p) = f(x) + \nabla f(x)^T p + \tfrac{1}{2} p^T \nabla^2 f(x + tp)p,$$

for the function $\cos(1/x)$ around a nonzero point $x$, and the third-order Taylor expansion of $\cos(x)$ around any point $x$.
Evaluate the second expansion for the specific case of $x = 1$.

**5** Consider the function $f: \mathrm{R}^2 \to \mathrm{R}$ defined by $f(x) = \|x\|^2$. Show that the sequence of iterates $\{x_k\}$ defined by

$$x_k = \left(1 + \frac{1}{2^k}\right) \begin{bmatrix} \cos k \\ \sin k \end{bmatrix}$$

satisfies $f(x_{k+1}) < f(x_k)$ for $k = 0, 1, 2, \ldots$. Show that every point on the unit circle $\{x \mid \|x\|^2 = 1\}$ is a limit point for $\{x_k\}$.
*Hint: Every value $\theta \in [0, 2\pi]$ is a limit point of the subsequence $\{\xi_k\}$ defined by*

$$\xi_k = k (\mathrm{mod}\ 2\pi) = k - 2\pi \left\lfloor \frac{k}{2\pi} \right\rfloor,$$

*where the operator $\lfloor \cdot \rfloor$ denotes rounding down to the next integer.*

**6** Prove that all isolated local minimizers are strict. (*Hint: Take an isolated local minimizer $x^*$ and a neighborhood $\mathcal{N}$. Show that for any $x \in \mathcal{N}$, $x \neq x^*$ we must have* $f(x) > f(x^*)$.)

**7** Suppose that $f(x) = x^T Qx$, where $Q$ is an $n \times n$ symmetric positive semidefinite matrix. Show using the definition that $f(x)$ is convex on the domain $\mathrm{R}^n$.

*Hint: It may be convenient to prove the following equivalent inequality:*

$$f(y + \alpha(x - y)) - \alpha f(x) - (1 - \alpha)f(y) \leq 0,$$

*for all $\alpha \in [0, 1]$ and all $x, y \in \mathbb{R}^n$.*

**8** Suppose that $f$ is a convex function. Show that the set of global minimizers of $f$ is a convex set.

**9** Consider the function $f(x_1, x_2) = \left(x_1 + x_2^2\right)^2$. At the point $x^T = (1, 0)$ we consider the search direction $p^T = (-1, 1)$. Show that $p$ is a descent direction and find all minimizers of the problem:

$$\min_{\alpha > 0} f(x_k + \alpha p_k).$$

**10** Consider the sequence $\{x_k\}$ defined by

$$x_k = \begin{cases} \left(\frac{1}{4}\right)^{2^k}, & k \text{ even}, \\ (x_{k-1})/k, & k \text{ odd}. \end{cases}$$

Is this sequence Q-superlinearly convergent? Q-quadratically convergent? R-quadratically convergent?