Chapter 13. Introduction to numerical methods for Partial Differential Equations

Finite-Difference Method for Equation of Heat Transfer along a beam/rod of constant cross section

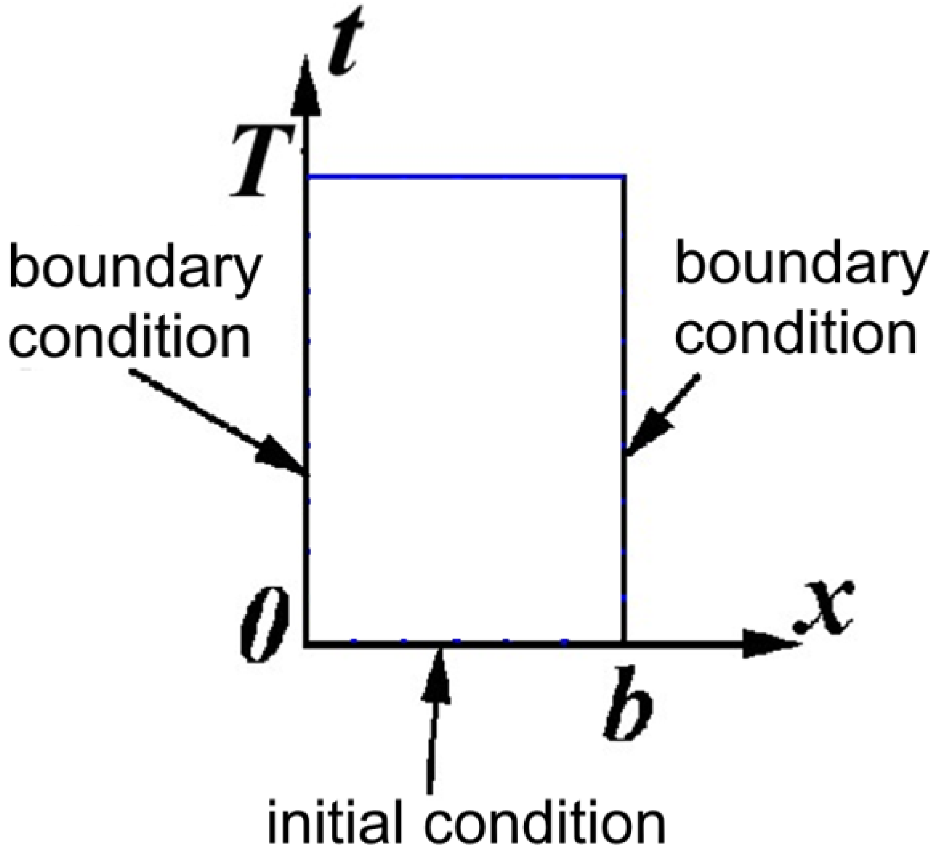
$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

a=const , $f(x,t)$ is given in a rectangle

$$0 < x < b, \qquad 0 < t < T$$

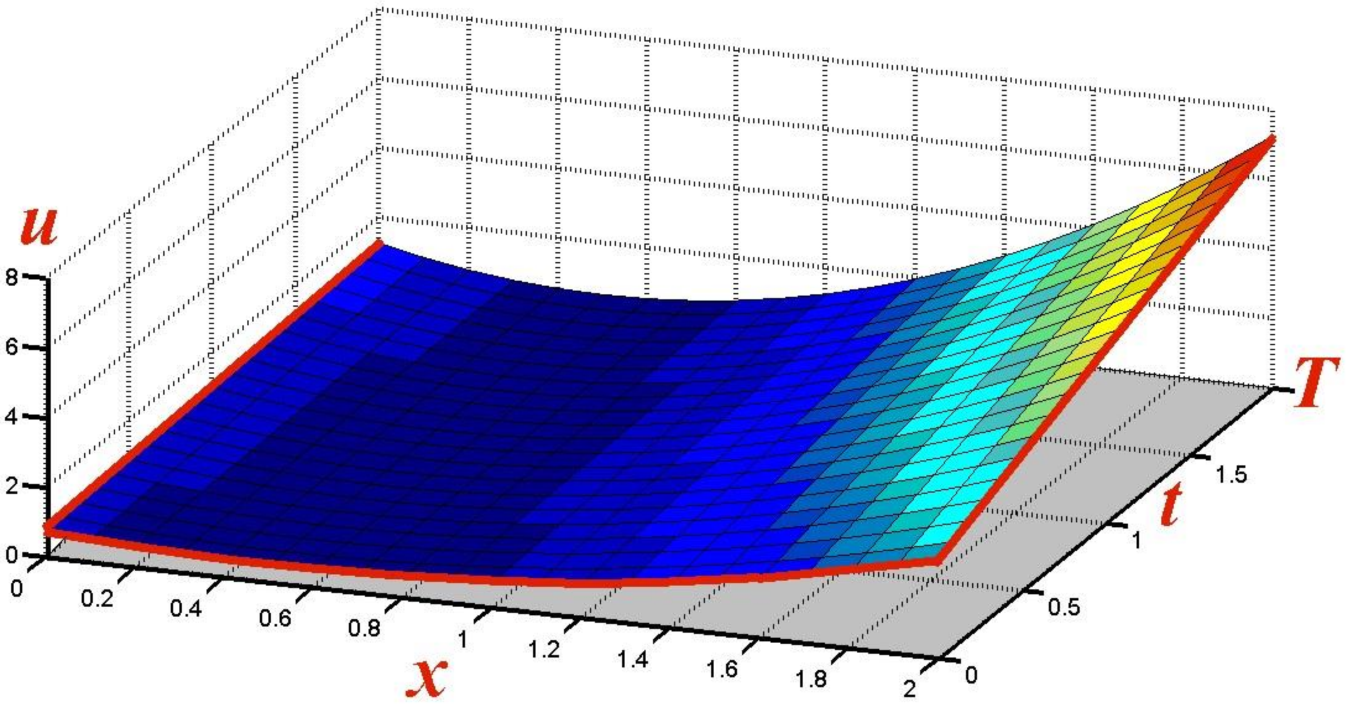
The boundary and initial conditions are as follows:

$u(0,t)=\psi_1(t)$



$u(b,t)=\psi_2(t)$

$u(x,0)=\varphi(x)$



If functions φ, ψ_1, ψ_2, f are continuous, and consistency conditions $\varphi(0)=\psi_1(t)$, $\varphi(b)=\psi_2(0)$ **are satisfied, then there exists a unique solution** ***$u(x, t)$ of the problem (see a course of Mathematical Physics).***

Let us split segment $0 \le x \le b$ into n parts

$x_0 = 0, \quad x_1 = h, \quad x_2 = 2h, \ldots, \quad x_n = b \qquad h = (b-0)/n$

$i = 0, \quad 1, \quad 2, \ldots, \quad n$

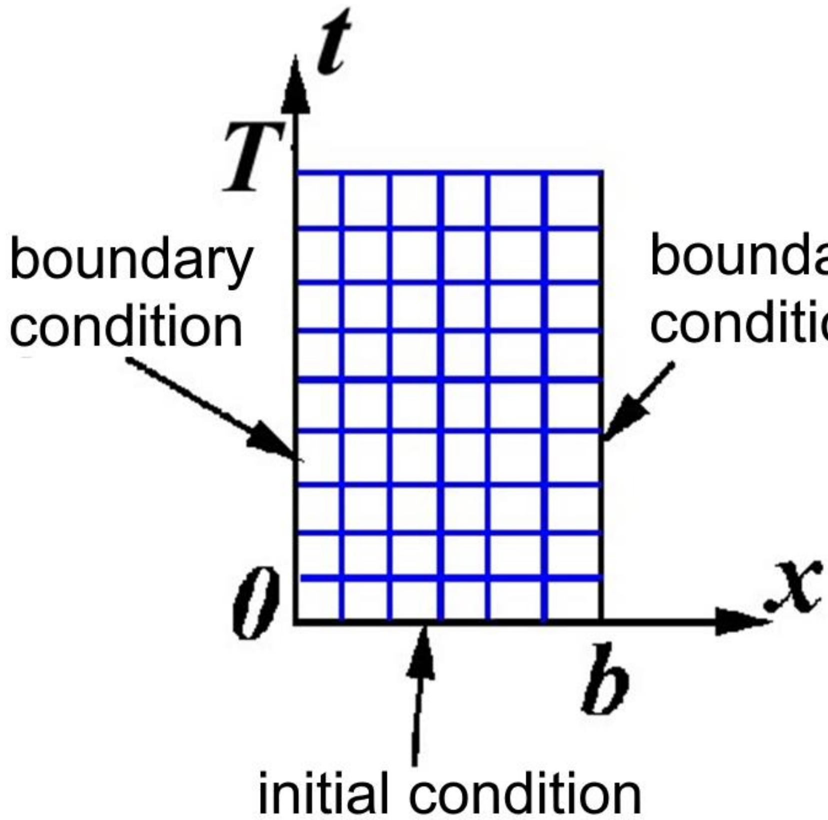
and we split $0 \le t \le T$ into m parts

$t_0 = 0, \quad t_1 = \tau, \quad t_2 = 2\tau, \quad \ldots, \quad t_m = T \qquad \tau = T/m$

$j = 0, \quad 1, \quad 2, \quad \ldots, \quad m$

By plotting horizontal and vertical lines, we obtain a mesh:

$u(0,t)=\psi_1(t)$



$u(b,t)=\psi_2(t)$

$u(x,0)=\varphi(x)$

mesh nodes: $(x_i,\ t_j)$

Values of the exact solution at nodes: $u(x_i, t_j)$

For calculation of an approximate solution we use formulas from Chapter 11:

$$\partial u/\partial t\,\Big|_{i,j} = [u(x_i, t_{j+1}) - u(x_i, t_j)]/\tau + O(\tau)$$

$$\partial^2 u/\partial x^2\,\Big|_{i,j} = [u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)]/h^2 + O(h^2)$$

Inserting this into equation

$$\frac{\partial u}{\partial t} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

we obtain at nodes

$$[u(x_i, t_{j+1}) - u(x_i, t_j)]/\tau + O(\tau) -$$

$$- a^2[u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)]/h^2 +$$

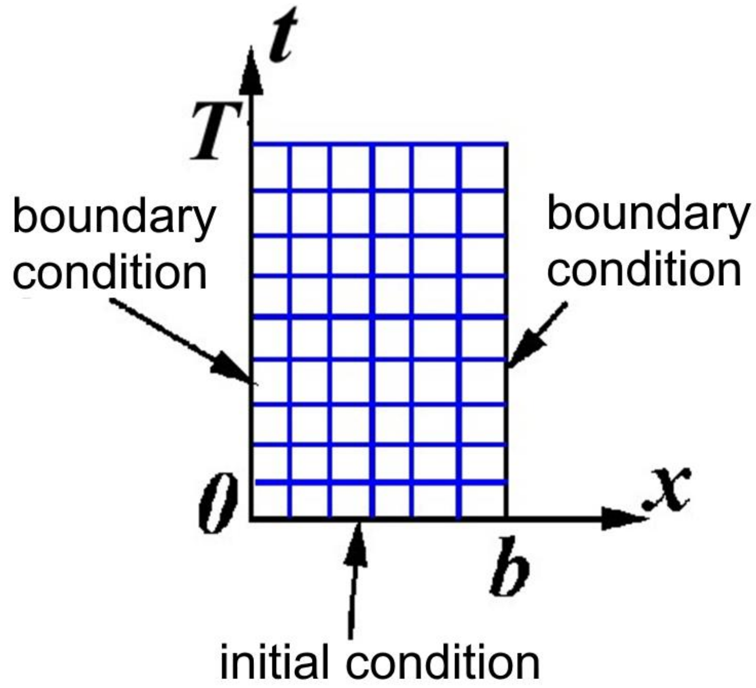
$$+ O(h^2) = f(x_i, t_j)$$

Omitting terms $O(\ )$, we pass to the approximate equation

$$u_{i,j+1} - u_{i,j} - \tau a^2[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]/h^2 =$$

$$= \tau f(x_i, t_j)$$

$$\boxed{u_{i,j+1} = u_{i,j} + [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]\,\tau a^2/h^2 + \tau f(x_i, t_j)}$$



At $j=0$ we know the initial distribution $u_{i,0}$, therefore it is possible to calculate $u_{i,1}$ for $i=1, 2, 3, \ldots, n-1$ on the layer $j=1$

$$\boxed{u_{i,j+1} = u_{i,j} + [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]\tau a^2/h^2 + \tau f(x_i, t_j)}$$

Then we find $u_{i,2}$ **for** $i=1, 2, 3, \ldots, n-1$ **and so on up to** $u_{i,m}$

Theorem **(On the stability and convergence of approximate solutions):**

If

$$\tau a^2/h^2 \le 1/2$$

then

$$\max_{i,j} | u(x_i, t_j) - u_{i,j} | = O(\tau + h^2)$$

Example:

```
tau=0.01
h=0.5
m=100
u= [50 60 35  5   15  40  60 40]
for i=1:8
    for j=1:m+1
        U(i,j)= u(i)     ;
     end
  end
  for  j= 1:m
      for i=2:7
     U(i,j+1)=U(i,j)+tau*(U(i+1,j)-2*U(i,j)+U(i-1,j))/(h*h)
+0.*tau*sin((i-1)*%pi/7)^2     ;
        end
        end
[t,x]=meshgrid(1:m+1,1:8)
surf(x,t,U)
```

Finite-Difference Method for Equation of Elastic String Oscillations

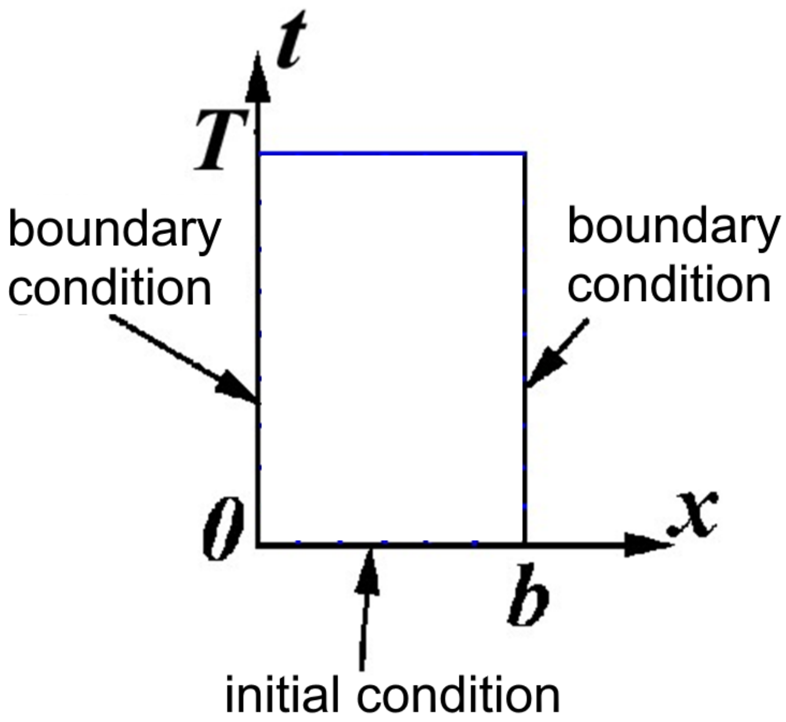
$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

where a=const , and $f(x,t)$ is given in the rectangle

$$0 < x < b, \quad 0 < t < T$$

$u(0,t) = \psi_1(t)$

$0 \le t \le T$



$u(b,t) = \psi_2(t)$

$u(x,0) = \varphi_1(x)\ , \quad 0 \le x \le b$

$\partial u/\partial t\,\Big|_{t=0} = \varphi_2(x)$

$u(0,t)=\psi_1(t) \qquad u(b,t)=\psi_2(t) \qquad 0 \le t \le T$

$u(x,0)=\varphi_1(x) \qquad 0 \le x \le b$

$\partial u/\partial t\big|_{t=0}=\varphi_2(x) \qquad 0 \le x \le b$

Consistency at corner points:

$$\varphi_1(0)=\psi_1(0), \qquad \varphi_1(b)=\psi_2(0), \qquad (*)$$

$$\varphi_2(0)=d\psi_1/dt\big|_{t=0}, \qquad \varphi_2(b)=d\psi_2/dt\big|_{t=0}, \qquad (**)$$

Theorem:

If φ_1 , φ_2 , f are continuous functions; ψ_1 , ψ_2 have continuous derivative, and consistency conditions are true, there exists a unique solution of the problem ***(see a course of Mathematical Physics)*.**

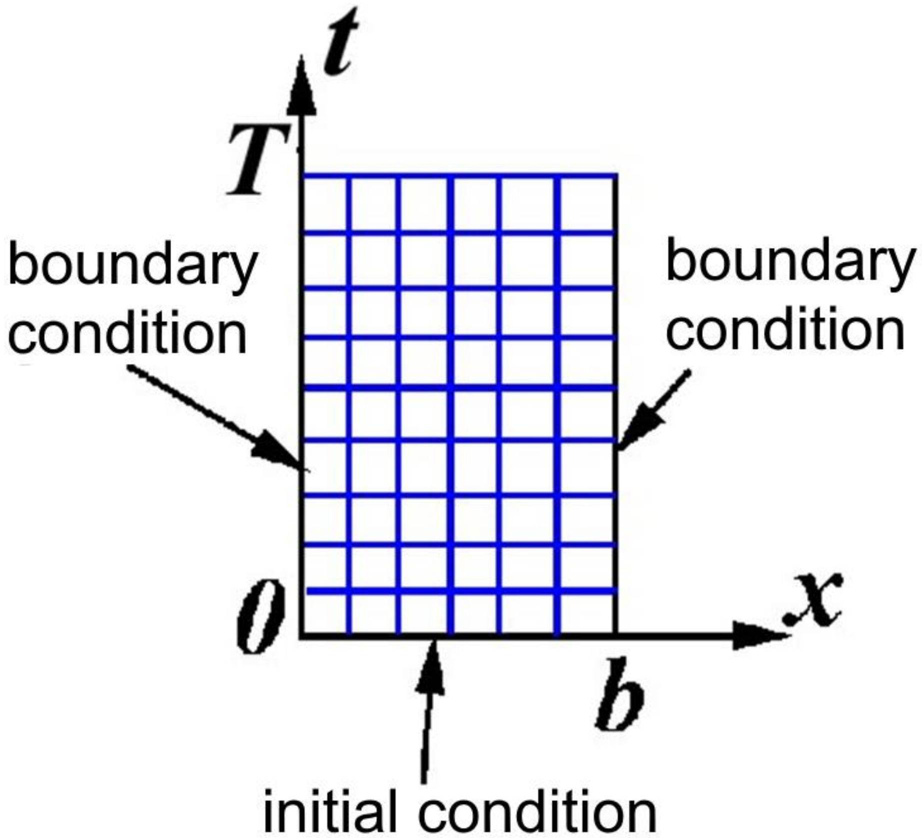
For calculation of an approximate solution we introduce a mesh

$x_0 = 0,\quad x_1=h,\quad x_2=2h, \ldots ,\quad x_n=b \qquad\qquad h= (b-0)/n$

$i = 0,\quad 1,\quad 2, \ldots ,\quad n$

$t_0 = 0,\quad t_1= \tau,\quad t_2=2\tau,\quad \ldots ,\quad t_m =T \qquad\qquad \tau= T/m$

$j = 0,\quad 1,\quad 2 ,\quad \ldots ,\quad m$



In order to calculate approximate solution at nodes (x_i, t_j), we write differential equation at nodes and then pass to algebraic equations.

True solution at nodes: $u(x_i, t_j)$

In order to obtain algebraic equations, we use formulas (see Chapter 11):

$$\partial^2 u/\partial t^2 \Big|_{i,j} = [u(x_i, t_{j+1}) - 2u(x_i, t_j) + u(x_i, t_{j-1})]/\tau^2 + O(\tau^2)$$

$$\partial^2 u/\partial x^2 \Big|_{i,j} = [u(x_{i+1}, t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)]/h^2 + O(h^2)$$

Substituting this into differential equation:

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

we obtain

$$[u(x_i, t_{j+1})-2u(x_i, t_j)+u(x_i, t_{j-1})]/\tau^2 + O(\tau^2) -$$

$$- a^2[u(x_{i+1},t_j) - 2u(x_i, t_j) + u(x_{i-1}, t_j)]/h^2 + O(h^2) = f(x_i, t_j)$$

Omitting $O(h^2)$, we arrive at algebraic equation for the approximate solution

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + [u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]\tau^2 a^2/h^2 + \tau^2 f(x_i, t_j) \qquad (***)$$

At $j=0$ we know initial values

$u_{i,0} = \varphi_1(x_i)$ and $\partial u/\partial t\,\big|_i = \varphi_2(x_i)$, therefore

$$(u_{i,1} - u_{i,0})/\tau \approx \varphi_2(x_i)$$

hence $\boxed{u_{i,1} = \varphi_1(x_i) + \tau\varphi_2(x_i)}$

for all $i=1, 2, 3, \ldots, n-1$ *on the layer* $j=1$

Now equation (***) gives:

$u_{i,2}$ at $i=1, 2, 3, \ldots, n-1$

Then $u_{i,3}$ at $i=1, 2, 3, \ldots, n-1$

and so on up to $u_{i,m}$

Theorem on the stability and convergence: if

$$\tau^2 a^2/h^2 \leq 1$$

then $\max_{i,j} \left| u(x_i, t_j) - u_{i,j} \right| = O(\tau^2 + h^2)$

Example: $f(x,t) \equiv 0$ $a \equiv 1$

Initial conditions:

$$u(x,0) = 50\sin(3\pi x/b)$$

$$\partial u/\partial t = 0 \qquad 0 \leq x \leq b, \quad b=6$$

Boundary conditions:

$$u(b,t) = 0$$

$$u(0,t) = 40\sin(1.5\,t) \qquad 0 \leq t \leq T, \quad T=15$$

```
clear // String oscillations
b=6
n=51
h= b/(n-1)
x=0: h : b
tau=0.1
for j=1:151
for i=1:n
U(i,j)= 50*sin(3*%pi * x(i) /b)
// 1st initial condition and boundary condition on the right
end
end
for j= 2: 150
U(1,j)= 40*sin(1.5*tau*(j-1) )
// boundary condition on the left
end
```

```
for i=1: n
fi2(i)= 0 // 2nd initial condition
end
for i=2:n-1 //
U(i,2)= U(i,1)+tau*fi2(i)
end
for j= 2: 150 //
for i=2: n-1   // all inner nodes
U(i,j+1) = 2*U(i,j)-U(i,j-1) +tau*tau*+(U(i+1,j)-2*U(i,j)+U(i-
1,j)))/(h*h)
end
for i=1:n
u(i)=U(i,j+1) // for a graph
end
xgrid
plot(0,60,x',u,'r')
```

```
sleep(10)
plot(x',u,'b')
end //  j
// [xx,tt]=meshgrid(1:151,1:n)
// surf(xx,tt,U)
```

Finite-Difference Method for Solving Poisson's Equation

$$\frac{\partial^2 u}{\partial t^2} + a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t) \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

Function $f(x,y)$ **is given in the receangle**

$$0 < x < b, \quad 0 < y < T$$

Practical applications of the Poisson's equation: It governs

1) Potential of electrostatic field, where $f(x,y)$ is density of distributed charges.

2) Velocity potential of fluid or gas stream (at low velocities, if viscosity is neglected), in this case $f(x,y) \equiv 0$.

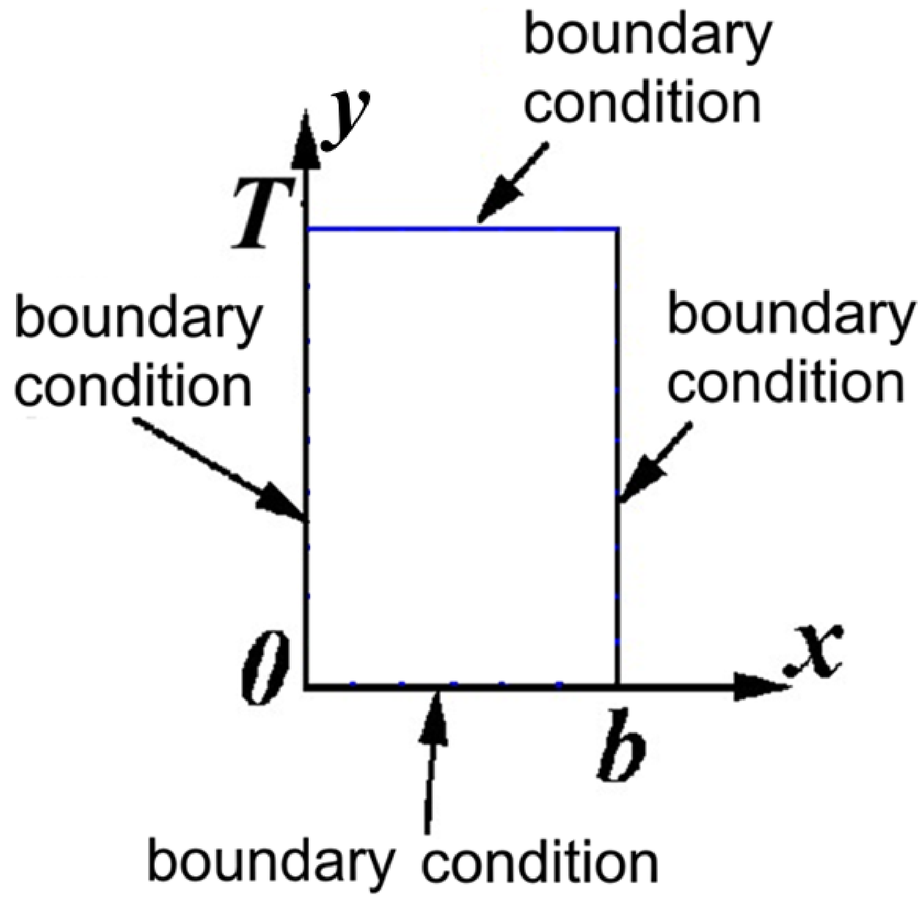
3) Temperature of 2D-dimensional media in steady state, where $f(x,y)$ is distribution of heat sources in the plane (x,y)

Boundary conditions:

$$u(x,T)=\varphi_2(x) \qquad 0 \le x \le b$$

$u(0,y)=\psi_1(y)$

$0 \le y \le T$



$u(b,y)=\psi_2(y)$

$$u(x,0)=\varphi_1(x) \qquad 0 \le x \le b$$

Problem: find a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

which equals to given boundary conditions on the sides of rectangle

Consistency conditions:

$\varphi_1(0) = \psi_1(0)$, $\quad \varphi_1(b) = \psi_2(0)$, $\quad$ (*)

$\varphi_2(0) = \psi_1(T)$, $\quad \varphi_2(b) = \psi_2(T)$, $\quad$ (**)

Theorem:

If functions φ_1 , φ_2 , ψ_1 , ψ_2 , f are continuous, and consistency conditions (*), (**) are satisfied, then there exists a unique solution of this problem (course of Math Physics).

For calculation of an approximate solution we introduce a mesh

$x_0 = 0, \quad x_1 = h, \quad x_2 = 2h, \; \ldots, \; x_n = b \qquad h = (b-0)/n$

$i = 0, \quad 1, \; 2, \; 3, \; \ldots, \qquad n$

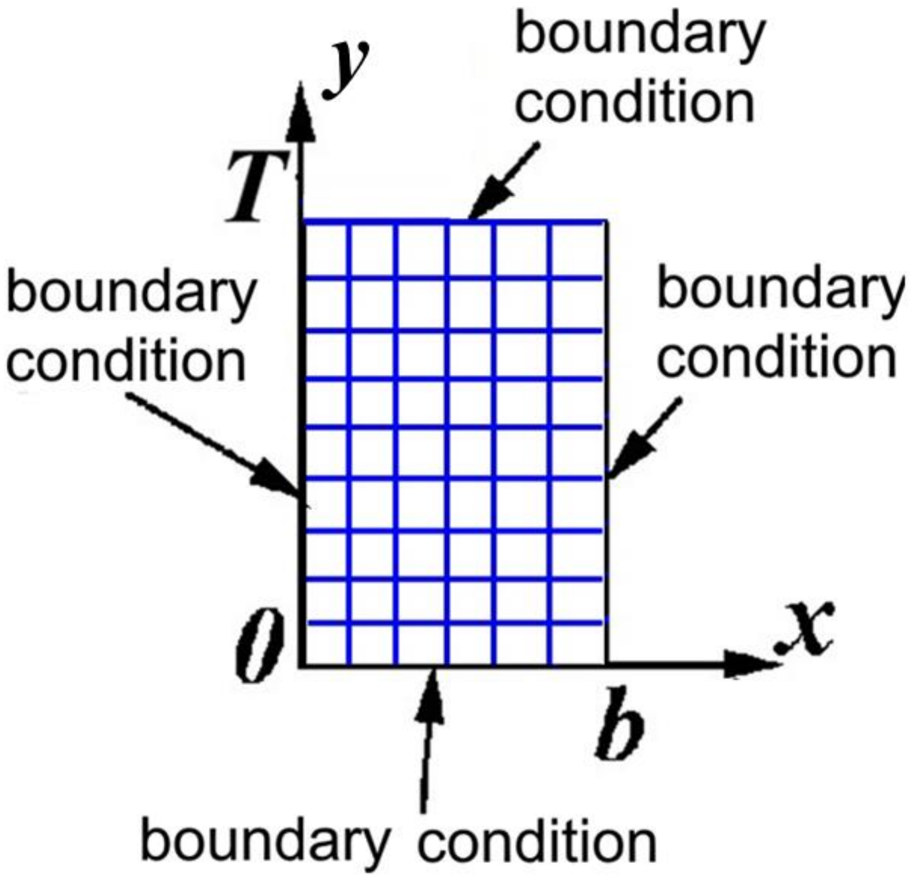
$y_0 = 0, \; y_1 = \tau, \; y_2 = 2\tau, \; \ldots, \; y_m = T \qquad \tau = (T-0)/m$

$j = 0, \quad 1, \; 2, \ldots, \qquad m$

$u(x,T)=\varphi_2(x) \qquad 0 \leq x \leq b$

$u(0,y)=\psi_1(y)$

$0 \leq y \leq T$



$u(b,y)=\psi_2(y)$

$u(x,0)=\varphi_1(x) \qquad 0 \leq x \leq b$

Evidently,

$$\partial^2 u/\partial x^2 \big|_{i,j} \approx [u(x_{i+1}, x_j) - 2u(x_i, x_j) + u(x_{i-1}, x_j)]/h^2$$

$$\partial^2 u/\partial y^2 \big|_{i,j} \approx [u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})]/\tau^2$$

Let us insert this into equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

then we obtain at nodes (x_i, y_j)

$$[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}]/h_2 +$$

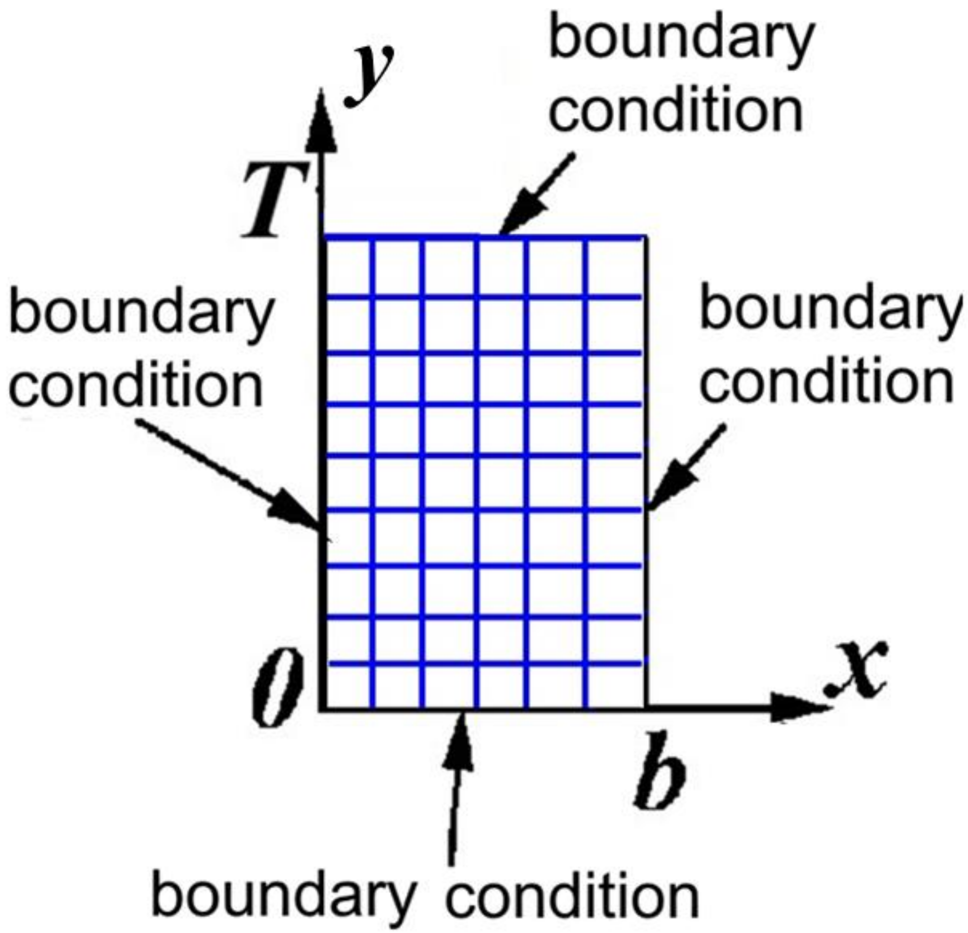
$$+[u_{i,j+1} - 2u_{i,j} + u_{i,j-1}]/\tau^2 = f(x_i, y_j)$$

For simplicity assume $\tau = h$, then

$$(u_{i+1,\,j} - 4u_{i,j} + u_{i-1,\,j} +$$

$$+u_{i,\,j+1} + u_{i,\,j-1})/h^2 - f(x_i, y_j) = 0 \qquad (***)$$

The number of unknowns is equal to number of inner nodes (n-1)*(m-1)



Gaussian’s method of elimination of unknowns is inefficient if (n-1)*(m-1) is large.

Therefore, is makes sense to use an iterative method:

$$\delta\,[(u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})/h^2 - f(x_i, y_j)] + u_{i,j} = u_{i,j}$$

$$\boxed{\delta\,[(u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1})/h^2 - f(x_i, y_j)]^{(k)} + u_{i,j}^{(k)} = u_{i,j}^{(k+1)}}$$

[**This is analogous to the iteration method for systems of linear algebraic equations** $x^{(k+1)} = Cx^{(k)} + d$]

Clarification: Let us add derivative with respect to time into Poisson's equation

$$-\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$$

From Math Physics it is known that time-dependent solution tends to steady $\hat{u}(x,y)$:

$u(x,y,t) \rightarrow \hat{u}(x,y)$ as $t \rightarrow \infty$.

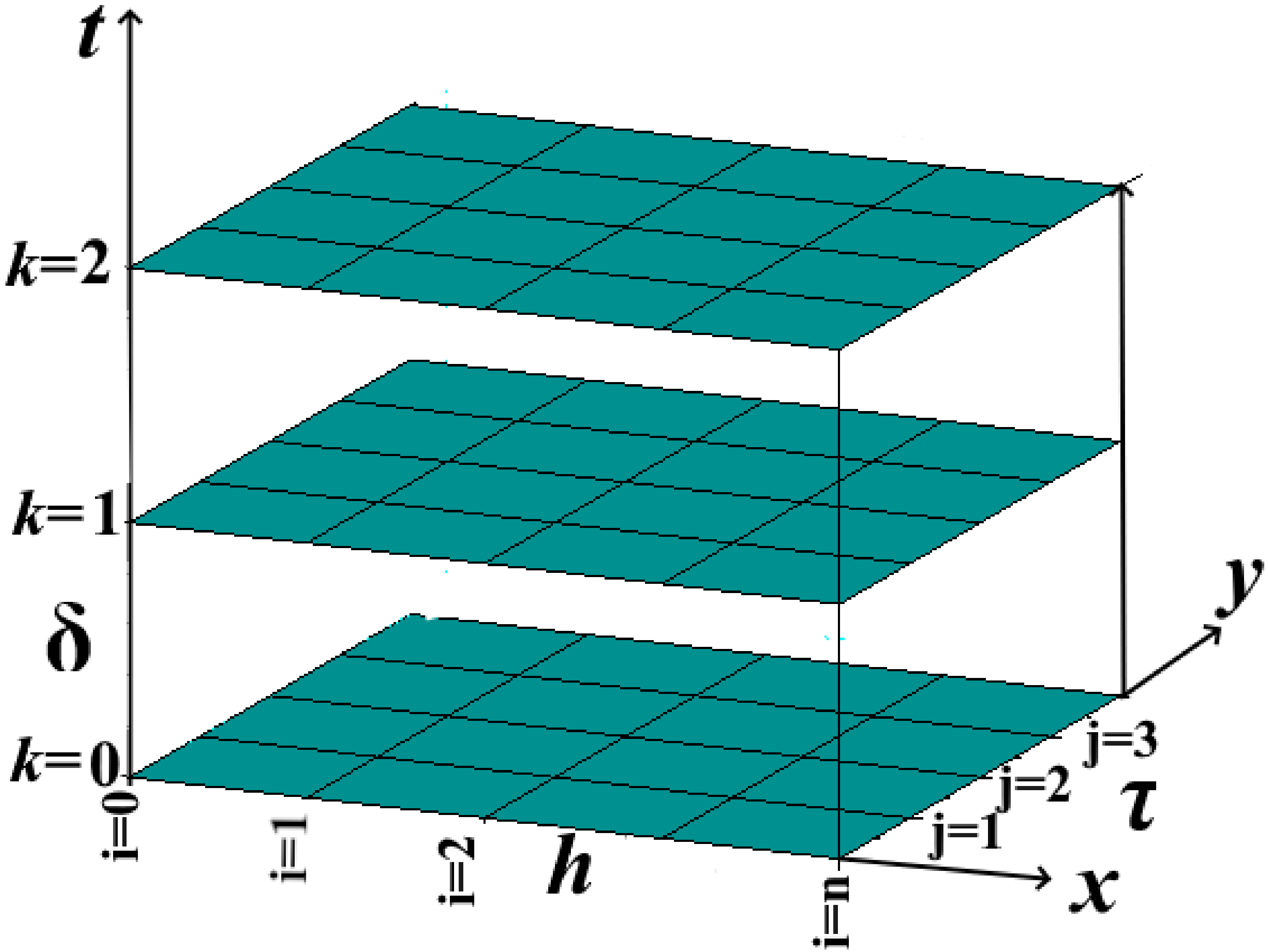
Due to $\partial\hat{u}(x,y)/\partial t = 0$, the function $\hat{u}(x,y)$ is solution of the Poisson's equation.

Therefore the idea was to solve numerically the equation

$$-\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$$

and find limit $u(x,y,t) \rightarrow \hat{u}(x,y)$

We introduce 3D mesh



and will seek $u_{i,j,k}$

$$(u_{i+1,j,k} - 4u_{i,j,k} + u_{i-1,j,k} + u_{i,j+1,k} +$$

$$+ u_{i,j-1,k})/h^2 - f(x_i, y_j) = \partial u(x,y)/\partial t \,\Big|_{i,j,k}$$

$$(u_{i+1,j,k} - 4u_{i,j,k} + u_{i-1,j,k} + u_{i,j+1,k} +$$

$$+ u_{i,j-1,k})/h^2 - f(x_i, y_j) = (u_{i,j,k+1} - u_{i,j,k})/\delta$$

δ is the time step, distance between layers

$$\delta[(u_{i+1,j,k} - 4u_{i,j,k} + u_{i-1,j,k} + u_{i,j+1,k} +$$

$$+ u_{i,j-1,k})/h^2 - f(x_i, y_j)] + u_{i,j,k} = u_{i,j,k+1}$$

compare with the equation shown above:

$$\boxed{\begin{aligned}&\delta[(u_{i+1,j} - 4u_{i,j} + u_{i-1,j} + u_{i,j+1} + \\ &+ u_{i,j-1})/h^2 - f(x_i, y_j)]^{(k)} + u_{i,j}^{(k)} = u_{i,j}^{(k+1)}\end{aligned}}$$

If δ is chosen properly, then iterations $u_{i,j}^{(k)}$ converge to a solution $u_{i,j}$ of the algebraic system $\quad u_{i,j}^{(k)} \to u_{i,j}$

In its turn, solutions $u_{i,j}$ of the algebraic system converge to a solution of the differential equation at $\tau, h \to 0$:

$$\max_{i,j} | u(x_i, y_j) - u_{i,j} | = O(\tau^2 + h^2) \to 0$$

Notice on possible types of boundary conditions:

In practice, instead of the condition

$u(0,y)=\psi_1(y)$ at $0 \le y \le T$,

the condition of given derivative can be in need:

$$\partial u/\partial x \,\big|_{x=0} = \psi_1(y)$$

Then value $u_{0,j}$ in the algebraic system (***) is not known, i.e., value of solution on the left side of rectangle is not known.

In this case, at each *k*-step we must make an adjustment of $u_{0,j}^{(k+1)}$ after calculation of $u_{i,j}^{(k+1)}$ at inner nodes.

For the adjustment, we must use the condition of given derivative $\partial u/\partial x \,\big|_{x=0} =\psi_1(y)$

The simplest formula:

$$u_{1,j} - u_{0,j} = h\,\psi_1(y_j) \Rightarrow u_{0,j}^{(k+1)} = u_{1,j}^{(k+1)} - h\,\psi_1(y_j)$$

More accurate formula:

$$-3u_{0,j} + 4u_{1,j} - u_{2,j} = 2h\,\psi_1(y_j)$$

$$3u_{0,j} = 4u_{1,j} - u_{2,j} - 2h\,\psi_1(y_j)$$

$$u_{0,j}^{(k+1)} = [\,4u_{1,j}^{(k+1)} - u_{2,j}^{(k+1)} - 2h\,\psi_1(y_j)\,]/3$$

Example: $f(x,y) = -5*[\sin(x\pi/b)\sin(y\pi/b)]^4$,

in the square $0<x<6$, $0<y<6$

$\partial u/\partial x=2$ at $x=0$!

```
clear
delta=0.008
n=31
b=6
h= b/(n-1)
for j=1:n
for i=1:n
u(i,j)= 20 // zero approximation
end
end
// Boundary conditions on upper, lower and right sides
// we  retain u=20
unew=u
```

```
for k= 1: 300 // simple iterations
for j= 2: n-1 //
for i=2: n-1
f = -5*(sin(h*(i-1)*%pi/b)*sin(h*(j-1)*%pi/b))^4 ;
unew(i,j)=u(i,j)+delta*((u(i,j+1)-4*u(i,j) +u(i,j-1) +...
u(i+1,j) +u(i-1,j) ) /(h*h) -f ) ;
end // по i
// On the left boundary we prescribe heat flux
//  du/dx=2 :
unew(1,j)=unew(2,j) - 2*h
end //  j
u=unew
end //  k
[xx,yy]=meshgrid(1:n,1:n)
surf(xx,yy,u)
xgrid
```

Shift of the heater to window:

$$f(x,y) = -35/[2+x^2+(y-3)^2]$$