Equations of Line in Space

Assume that some straight line a in the space $\mathbb{E}$ is chosen and fixed. In order to study various equations determining this line we choose some coordinate system O, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in the space.

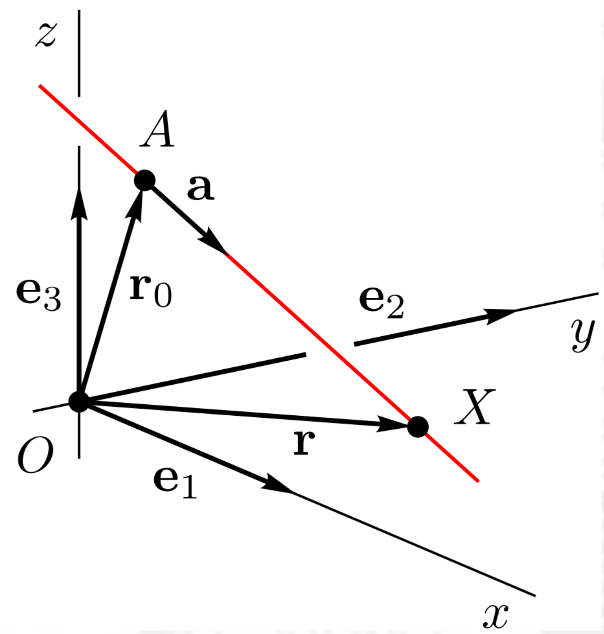

Figure 1: Derivation of vectorial parametric equation

Vectorial parametric equation of this line is:

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a} \cdot t \tag{1}$$

Coordinate parametric equations of this line are:

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \\ z = z_0 + a_z t \end{cases} \tag{2}$$

1 Non-parametric vectorial equation of a line

Suppose line expressed with its vectorial parametric equation (1).

Let us apply the *cross product* by the direction vector $\boldsymbol{a}$ to both sides of the this equation. As a result we get

$$\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{r}_0 \times \boldsymbol{a} + t(\boldsymbol{a} \times \boldsymbol{a}) \tag{3}$$

Cross product of the vector by itself is zero vector

For this reason the equation (3) actually does not contain the term with parameter t. This equation is usually written as follows:

$$(\boldsymbol{r} - \boldsymbol{r}_0) \times \boldsymbol{a} = \mathbf{0}. \tag{4}$$

The vector product of two constant vectors $\boldsymbol{r}_0$ and $\boldsymbol{a}$ is a constant vector. If we denote this vector $\boldsymbol{b} = \boldsymbol{r}_0 \times \boldsymbol{a}$, then the equation of the line (4) can be written as

$$\boldsymbol{r} \times \boldsymbol{a} = \boldsymbol{b}, \tag{5}$$

where $\boldsymbol{b} \perp \boldsymbol{a}$

Definition. Any one of the two equalities (4) or (5) is called the non-parametric **vectorial equation of a line in the space**.

The constant vector $\boldsymbol{b}$ in the equation (5) should be perpendicular to the directional vector $\boldsymbol{a}$, and expresses initial point of the line.

Problem 1

Derive non-parametric vectorial equation of the line crossing points $A(4, 12, 1)$, and $B(6, 6, 5)$. Basis is right orthonormal.

Solution

Vectorial parametric equation was derived on previous class:

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$$

Equation (4) for this line has form:

$$\left(r - \begin{pmatrix} 4 \\ 12 \\ 1 \end{pmatrix} \right) \times \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix} = \boldsymbol{0}$$

While we are informed about features of the basis, we can calculate $\boldsymbol{b}$:

$$\boldsymbol{b} = \boldsymbol{r}_0 \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 4 & 12 & 1 \\ -2 & 6 & -4 \end{vmatrix} = -54\boldsymbol{e}_1 + 14\boldsymbol{e}_2, +48\boldsymbol{e}_3$$

Hence, equation is

$$\boldsymbol{r} \times \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix} = \begin{pmatrix} -54 \\ 14 \\ 48 \end{pmatrix}$$

2 Canonical equation of a line in the space

Let us consider case then all $a_* \neq 0$ in coordinate parametric equations of the line in space (2):

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \\ z = z_0 + a_z t \end{cases}$$

Parameter t may be *explicitly* expressed with each equation:

$$t = \frac{x - x_0}{a_x}$$
$$t = \frac{y - y_0}{a_y}$$
$$t = \frac{z - z_0}{a_z}$$

Assuming the same value of parameter t, we derive equalities:

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z} \tag{6}$$

While a_x, a_y and a_z are proportional with direction cosines of the line, proportion $a_x : a_y : a_z$ must be preserved for all equivalent form of this equation.

For $a_x = 0$ we have system of equalities:

$$\begin{cases} x = x_0 \\ y = y_0 + a_y t \\ z = z_0 + a_z t, \end{cases}$$

and

$$\begin{cases} x = x_0 \\ \dfrac{y - y_0}{a_y} = \dfrac{z - z_0}{a_z}. \end{cases} \tag{7}$$

In coordinate system with right orthonormal basis this line perpendicular with Ox

For $a_y = 0$ we have system of equalities:

$$\begin{cases} y = y_0 \\ \dfrac{x - x_0}{a_x} = \dfrac{z - z_0}{a_z}. \end{cases} \tag{8}$$

In coordinate system with right orthonormal basis this line perpendicular with Oy

For $a_z = 0$ we have system of equalities:

$$\begin{cases} z = z_0 \\ \dfrac{x - x_0}{a_x} = \dfrac{y - y_0}{a_y}. \end{cases} \tag{9}$$

In coordinate system with right orthonormal basis this line perpendicular with Oz

If both $a_x = 0$ and $a_y = 0$, system parametric equations reduces to

$$\begin{cases} x = x_0 \\ y = y_0 \\ z = z_0 + a_z t \end{cases}$$

Hence z take all possible real values, there is no need to write third equation for applicate, and system reduces to

$$\begin{cases} x = x_0 \\ y = y_0. \end{cases} \tag{10}$$

This line is parallel with Oz.

For both $a_x = 0$ and $a_z = 0$ system reduces to

$$\begin{cases} x = x_0 \\ z = z_0. \end{cases} \tag{11}$$

This line is parallel with Oy.

For both $a_x = 0$ and $a_z = 0$ system reduces to

$$\begin{cases} y = y_0 \\ z = z_0. \end{cases} \tag{12}$$

This line is parallel with Ox.

Definition. Any one of the seven pairs of equations (6), (7), (8), (9), (10), (11), and (12) is called the **canonical equation of a line in the space**.

Problem 1

Write the canonical equations of the line passing through the point $A(1, -2, 2)$ with direction angles $\pi/3$, $2\pi/3$, $\pi/4$.

Basis is right orthonormal.

Solution

While we are given with direction angles in right orthonormal basis, we express direction vector of the line with direction cosines:

$$\boldsymbol{a} \mapsto \begin{pmatrix} \cos\dfrac{\pi}{3} \\ \cos\dfrac{2\pi}{3} \\ \cos\dfrac{\pi}{4} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} \\ -\dfrac{1}{2} \\ \dfrac{\sqrt{2}}{2} \end{pmatrix}$$

Canonical equation of this line is

$$\frac{x-1}{\frac{1}{2}} = \frac{y+2}{-\frac{1}{2}} = \frac{z-2}{\frac{\sqrt{2}}{2}}$$

or

$$\frac{x-1}{1} = \frac{y+2}{-1} = \frac{z-2}{\sqrt{2}}$$

Problem 2

Show that lines

$$\frac{x+1}{2} = \frac{y-5}{3} = \frac{z-7}{-1}$$

and

$$\frac{x+4}{5} = \frac{y-1}{-3} = \frac{2-3}{1}$$

expressed in a coordinate system with right orthonormal basis are perpendicular

Solution

Direction vector of first line has coordinates $(2, 3, -1)$, and direction vector of second line has coordinates $(5, -3, 1)$.

Dot product of these two vectors is

$$2 \cdot 5 + 3 \cdot (-3) + (-1) \cdot 1 = 10 - 9 - 1 = 0,$$

hence that vectors are perpendicular and lines are perpendicular too.

Problem 3

Find angle between lines

$$\begin{aligned} &\frac{x-1}{1} = \frac{y}{-4} = \frac{z+3}{1} \\ &\frac{x}{2} = \frac{y+2}{-2} = \frac{z}{-1} \end{aligned}$$

Basis is right orthonormal

Solution

Direction vector of first line has coordinates $(1, -4, 1)$, and direction vector of second line has coordinates $(2, -2, -1)$.

Angle between lines is defined via cosines of angle between direction vectors, as orthonormal basis is specified

$$\cos\theta = \frac{1 \cdot 2 + (-4) \cdot (-2) + 1 \cdot (-1)}{\sqrt{1+16+1}\sqrt{4+4+1}} = \frac{9}{3\sqrt{3} \cdot 3} = \frac{1}{\sqrt{2}}$$

Hence, angle is $\frac{\pi}{4}$. Supplementary angle $\frac{3\pi}{4}$ may be yielded by putting minus before fraction.

Problem 4

Express canonical equation of the line passing through the point (a, b, c) and parallel with line

$$\frac{x - a'}{m} = \frac{y - b'}{n} = \frac{z - c'}{p}$$

Solution

Let desired equation be

$$\frac{x - a}{M} = \frac{y - b}{N} = \frac{z - c}{P}$$

While lines must be parallel, their direction vectors with coordinates (m, n, p) and (M, N, P) respectively must be collinear.

Hence

$$\frac{M}{m} = \frac{N}{n} = \frac{P}{p}$$

As particular case, we let $M = m$, $N = n$, $P = p$, and desired equation its

$$\frac{x - a}{m} = \frac{y - b}{n} = \frac{z - c}{p}$$

3 Relative position of line and plane

Suppose line is expressed with coordinate parametric equations (2), and plain expressed with general equation

$$Ax + By + Cz + D = 0$$

We are looking for conditions if line punches plane in arbitrary point.

Let us solve equations

$$\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t \\ z = z_0 + a_z t \\ Ax + By + Cz + D = 0 \end{cases}$$

simultaneously.

Three first equations explicitly express x, y and z with t, hence we substitute these values into last equation:

$$A(x_0 + a_x t) + B(y_0 + a_y t) + C(z_0 + a_z t) + D = 0$$

$$Ax_0 + By_0 + Cz_0 + D + t(Aa_x + Ba_y + Ca_z) = 0$$

$$t = -\frac{Ax_0 + By_0 + Cz_0 + D}{Aa_x + Ba_y + Ca_z}$$

t has finite value if $Aa_x + Ba_y + Ca_z \neq 0$. Meaning of this non-equality is that normal vector of plane with covariant coordinates (A, B, C) and direction vector of the line with coordinates (a_x, a_y, a_z) shape the angle different form right.

Substitution of this t into parametric equations yields coordinates of the piercing point.

If $Aa_x + Ba_y + Ca_z = 0$, two cases are possible

1. $Ax_0 + By_0 + Cz_0 + D = 0$, hence (x_0, y_0, z_0) lies on the plane, and all line lies on the plane
2. $Ax_0 + By_0 + Cz_0 + D \neq 0$, hence (x_0, y_0, z_0) lies in some non-zero distance from the plane, and line is parallel with plane.

4 Angle between line and plane

Definition. We call the **angle between line and plane** any of the pair of supplementary angles shaped by the line and its projection on plane.

We will operate with a sharp angle of this pair here.

Projection here is the line connecting piercing point of line and plane and base on plane of any segment perpendicular to plane, and connecting line and plane.

Hence, this projection is intersection line of given plane and plane perpendicular to it and containing given line.

Let equation of line be written in canonical form

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z},$$

and equation of plane be written in general form

$$Ax + By + Cz + D = 0$$

Direction vector of the line $\boldsymbol{a}$ has coordinates (a_x, a_y, a_z). Covariant (in general case) coordinates of normal vector of plane $\boldsymbol{n}$ are (A, B, C)

Sum of desired angle φ and the angle between direction vector of line and normal vector of plane $\angle(\boldsymbol{a}, \boldsymbol{n}) = \theta$ is a right angle.

Hence, $\sin\varphi = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Cosines of the angle between vectors has expression

$$\sin\varphi = \cos\theta = \frac{|\boldsymbol{a}\cdot\boldsymbol{n}|}{|\boldsymbol{a}||\boldsymbol{n}|} = \frac{|a_x A + a_y B + a_z C|}{|\boldsymbol{a}||\boldsymbol{n}|}.$$

We add modulus in numerator to guarantee we deal with sharp angle in a pair.

For right orthonormal basis we can expand length:

$$\sin\varphi = \frac{|a_x A + a_y B + a_z C|}{\sqrt{a_x^2 + a_y^2 + a_z^2}\sqrt{A^2 + B^2 + C^2}}$$

5 Lines parallel and perpendicular with plane

Line is perpendicular with plane if direction vector of this line and normal vector of plane are collinear, hence

$$\frac{A}{a_x} = \frac{B}{a_y} = \frac{C}{a_z}$$

Line is parallel with plane if $\sin\varphi = 0$, hence

$$a_x A + a_y B + a_z C = 0.$$

This condition has vectored form

$$\boldsymbol{a}\cdot\boldsymbol{n} = 0.$$

Problem

Write equation for the locus of all line passing through point $X(a, b, c)$ and parallel with given plane

$$Ax + By + Cz = 0$$

Solution

We utilize vectorial parametric equation:

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a}.$$

As initial point with radius vector $\boldsymbol{r}_0$we use given common point of lines X.

Since all lines are perpendicular with desired vector $\boldsymbol{n}$ with covariant coordinates (A, B, C),

$$\boldsymbol{a}\cdot\boldsymbol{n} = 0.$$

Dot product of the line equation by $\boldsymbol{n}$ yields

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r}_0 \cdot \boldsymbol{n}$$

or

$$(\boldsymbol{r} - \boldsymbol{r}_0) \cdot \boldsymbol{n} = 0$$

Rewriting this equation in coordinates yields

$$A(x - z) + B(y - b) + C(z - c) = 0$$

Therefore, locus of all lines parallel to give plane and passing thought fixed point is a plane parallel to given and containing given point.

6 Two lines laying in the same plane

Suppose two lines are governed with canonical equations

$$\frac{x - x_0}{a_x} = \frac{y - y_0}{a_y} = \frac{z - z_0}{a_z}$$
$$\frac{x - x_0'}{a_x'} = \frac{y - y_0'}{a_y'} = \frac{z - z_0'}{a_z'}.$$

In what case that lines lay in the same plane?

Let direction vectors of that lines be $\boldsymbol{a}$ with coordinates (a_x, a_y, a_z) and $\boldsymbol{a}'$ with coordinates (a_x', a_y', a_z') respectively.

Radius vector connecting initial points of lines (x_0, y_0, z_0) and (x_0', y_0', z_0') is $\boldsymbol{r}' - \boldsymbol{r}$, where $\boldsymbol{r}$ and $\boldsymbol{r}'$ are radius vectors of these points. Coordinates of this vector are $(x_0' - x_0, y_0' - y_0, z_0' - z_0)$.

Lines lay in the same plane if vectors $\boldsymbol{a}$, $\boldsymbol{a}'$ and $\boldsymbol{r}' - \boldsymbol{r}$ are coplanar.

Hence, their mixed product is zero:

$$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{r}' - \boldsymbol{r}) = 0$$

In coordinates:

$$(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}) \begin{vmatrix} x_0' - x_0 & y_0' - y_0 & z_0' - z_0 \\ a_x & a_y & a_z \\ a_x' & a_y' & a_z' \end{vmatrix} = 0$$

Hence oriented volume of basis $(\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3})$ is never zero, we yield

$$\begin{vmatrix} x_0' - x_0 & y_0' - y_0 & z_0' - z_0 \\ a_x & a_y & a_z \\ a_x' & a_y' & a_z' \end{vmatrix} = 0$$

Problem 1
Yield equation of line crossing point (1,1,1) and two lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{4}$$

Solution
Initial point of this line is $(1,1,1)$. Hence, equation of this line

$$\frac{x-1}{a_x} = \frac{y-1}{a_y} = \frac{z-1}{a_z}$$

Two crossing planes shape a plane, hence condition of intersection with planes has form:

$$\begin{cases} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a_x & a_y & a_z \end{vmatrix} = 0 \\ \begin{vmatrix} 0 & 1 & 2 \\ 2 & 1 & 4 \\ a_x & a_y & a_z \end{vmatrix} = 0 \end{cases}$$

$$\begin{cases} a_x - 2a_y + a_z = 0 \\ 2a_x + 4a_y - 2a_z = 0 \end{cases}$$
$$\begin{cases} a_x - 2a_y + a_z = 0 \\ a_x + 2a_y - a_z = 0 \end{cases}$$

We are looking for proportion $a_x : a_{=}y : a_z$, hence we divide equations by a_z:

$$\begin{cases} \frac{a_x}{a_z} - 2\frac{a_y}{a_z} = -1 \\ \frac{a_x}{a_z} + 2\frac{a_y}{a_z} = 1, \end{cases}$$

hence

$$\begin{cases} \frac{a_x}{a_z} = 0 \\ \frac{a_x}{a_z} = \frac{1}{2}, \end{cases}$$

We took 0, 1 and 2 as a_x, a_y, a_z.
Therefore, equation of the line are

$$\begin{cases} x = 1 \\ \dfrac{y-1}{1} = \dfrac{z-1}{2} \end{cases}$$

Problem 2
Yield equation of line crossing point (1,1,1), crossing line and two lines

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3},$$

and perpendicular with line

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{4}$$

.
Basis is right orthonormal.

Solution
Initial point of this line is $(1,1,1)$. Hence, equation of this line

$$\frac{x-1}{a_x} = \frac{y-1}{a_y} = \frac{z-1}{a_z}$$

Line in question crosses $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$, hence first condition on direction vector is

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ a_x & a_y & a_z \end{vmatrix} = 0$$
$$a_x - 2a_y + a_z = 0$$

Line perpendicular with $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{4}$, hence second condition is perpendicularity of their direction vectors in right orthonormal basis:

$$2a_x + a_y + 4a_z = 0$$

System to be solved is

$$\begin{cases} a_x - 2a_y + a_z = 0 \\ 2a_x + a_y + 4a_z = 0 \end{cases}$$

We look for the proportion again, and it is $9 : 2 : (-5)$, hence desired equation of line is

$$\frac{x-1}{9} = \frac{y-1}{9} = \frac{z-1}{-5}.$$

7 The equation of a line passing through two given points in the space

Assume that two distinct points $A \neq B$ in the space are given. We write their coordinates

$$A(x_0, y_0, z_0), \quad B(x_1, y_1, z_1).$$

The vector $\boldsymbol{a} = \overrightarrow{AB}$ can be used for the directional vector of the line passing through the points A and B. Then from expression for their coordinates we derive the coordinates of $\boldsymbol{a}$:

$$\boldsymbol{a} = \begin{pmatrix} x_1 - x_0 \\ y_1 - y_0 \\ y_1 - y_0 \end{pmatrix}$$

We rewrite canonical equation of the line using this relation:

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0} \tag{13}$$

The equations (13) correspond to the case where the inequalities $x_1 \neq x_0$, $y_1 \neq y_0$, and $z_1 \neq z_0$ are fulfilled.

If one or pair of that inequalities is not fulfilled, we rewrite equation in forms (7), (8), (9), (10), (11), or (12)

For $x_0 = x_1$ we have system of equalities:

$$\begin{cases} x = x_0 = x_1 \\ \dfrac{y - y_0}{y_1 - y_0} = \dfrac{z - z_0}{z_1 - z_0}. \end{cases} \tag{14}$$

In coordinate system with right orthonormal basis this line perpendicular with Ox

For $y_0 = y_1$ we have system of equalities:

$$\begin{cases} y = y_0 = y_1 \\ \dfrac{x - x_0}{x_1 - x_0} = \dfrac{z - z_0}{z_1 - z_0}. \end{cases} \tag{15}$$

In coordinate system with right orthonormal basis this line perpendicular with Oy

For $z_0 = z_1$ we have system of equalities:

$$\begin{cases} z = z_0 = z_1 \\ \dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0}. \end{cases} \tag{16}$$

In coordinate system with right orthonormal basis this line perpendicular with Oz

If both $x_0 = x_1$ and $y_0 = y_1$, system parametric equations reduces to

$$\begin{cases} x = x_0 = x - 1 \\ y = y_0 = y_1. \end{cases} \tag{17}$$

This line is parallel with Oz.

For both $x_0 = x_1$ and $z_0 = z_1$ system reduces to

$$\begin{cases} x = x_0 = x_1 \\ z = z_0 = z_1. \end{cases} \tag{18}$$

This line is parallel with Oy.

For both $y_0 = y_1$ and $y_0 = y_1$ system reduces to

$$\begin{cases} y = y_0 = y_1 \\ z = z_0 = z_1. \end{cases} \tag{19}$$

This line is parallel with Ox.

The conditions $x_0 = x_1$, $y_0 = y_1$ and $z_0 = z_1$ cannot be fulfilled simultaneously since $A \neq B$.

Definition. Any one of the seven pairs of equalities (13), (14), (15), (16), (17), (19), and (19) is called the equation of a line passing through two given points $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$.

Problem
Write equation of the line passing through the origin and point $(1, 1, 1)$

Solution
$x_0 = y_0 = z_0 = 0$, $x_1 = y_1 = z_1 = 1$.
Hence

$$x = y = z$$

8 The equation of a line in the space as the intersection of two planes

In the vectorial form the equations of two intersecting planes can be written as:

$$\begin{cases} \boldsymbol{r} \cdot \boldsymbol{n}_1 = S_1 \\ \boldsymbol{r} \cdot \boldsymbol{n}_2 = S_2 \end{cases} \tag{20}$$

For the planes given by these equations do actually intersect their normal vectors should be non-parallel: $\boldsymbol{n}1 \nparallel \boldsymbol{n}_2$. In the coordinate form the equations of two intersecting planes can be written as:

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases} \tag{21}$$

Definition. Any one of the two pairs of equalities (20) and (20) is called the **equation of a line in the space obtained as the intersection of two planes**.

Planes involved into this equation actually shape a *proper beam* of planes.

Problem 1
Find algorithm to restore initial point and direction vector of the line by the equations

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}$$

Solution

1. To pick proper initial point we fix coordinate z_0 with arbitrary value, and solve system of equations:

$$\begin{cases} A_1x + B_1y = -(C_1z_0 + D_1) \\ A_2x + B_2y = -(C_2z_0 + D_2) \end{cases}$$

   Hence, initial point is obtained.

2. To express direction vector of the line we must note that normal vectors of both planes $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are normal with any direction vector of the line. One of such common-perpendicular vectors is cross product $\boldsymbol{n}_1 \times \boldsymbol{n}_2$. If we are provided with key features of basis, we calculate it without any difficulty.

Problem 2
Given the equations

$$2x - y + z = 6$$
$$x + 4y - 2z = 8.$$

Find

1. the point of the line for $z = 1$,
2. the points in which the line pierces the coordinate planes,
3. the direction vector of the line.

Basis is right orthonormal

Solution

1. Letting $z = 1$ yields system of two equations:

$$\begin{cases} 2x - y = 5 \\ x + 4y = 10. \end{cases}$$

   Solving, we obtain $x = \dfrac{10}{3}$, $y = \dfrac{5}{3}$

2. To find pierces we must let $x = 0$, $y = 0$, $z = 0$ and find two other coordinates. Systems of equations are

$$\begin{cases} -y + z = 6 \\ 4y - 2z = 8. \end{cases}$$

   Solution $(0, 10, 16)$.

$$\begin{cases} 2x + z = 6 \\ x - 2z = 8. \end{cases}$$

   Solution $(4, 0, -2)$.

$$\begin{cases} 2x - y = 6 \\ x + 4y = 8. \end{cases}$$

   Solution $\left(\dfrac{32}{9}, \dfrac{10}{9}, 0\right)$.

3. To find direction vector we calculate cross product:

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 2 & -1 & 1 \\ 1 & 4 & -2 \end{vmatrix} = -2\boldsymbol{e}_1 + 5\boldsymbol{e}_2 + 9\boldsymbol{e}_3$$

   Hence, direction vector has coordinates $(-2, 5, 9)$.

Problem 3

Find the point where the line

$$\begin{aligned} x + 2y - z - 6 &= 0 \\ 2x - y + 3z + 13 &= 0 \end{aligned}$$

pierces the plane

$$3x - 2y + 3z + 16 = 0$$

Solution

Problem in provided setup is actually a problem of looking for center of proper bundle of planes where two planes shaping line and third plane are *main planes*.

Hence, we are investigating the system

$$\begin{cases} x + 2y - z - 6 = 0 \\ 2x - y + 3z + 13 = 0 \\ 3x - 2y + 3z + 16 = 0 \end{cases}$$

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 3 & -2 & 3 \end{vmatrix} = 10,$$

and system has only one solution

Sterilizing z from equations yields

$$\begin{aligned} 3x + 2y - 1 = 0 \\ x - y + 3 = 0. \end{aligned}$$

Solving these two equations, we obtain $x = -1$, $y = 2$. Substituting these values in $x + 2y - z - 6 = 0$, we find $z = -3$.

Hence, the line pierces the plane in point $(-1, 2, -3)$.

Problem 4

Show that the lines represented by each of the following pairs of planes intersect:

$$\begin{aligned} x - y - z - 7 = 0 \\ 3x - 4y - 11 = 0, \end{aligned}$$

and

$$\begin{aligned} x + 2y - z - 1 = 0, \\ x + y + 1 = 0. \end{aligned}$$

Find intersection points.

Solution

Problem actually is again a problem of looking for center of proper bundle of planes.

We start with checking that planes shape proper bundle:

$$\begin{vmatrix} 1 & -1 & -1 & -7 \\ 3 & -4 & 0 & -11 \\ 1 & 2 & -1 & -1 \\ 1 & 1 & 0 & 1 \end{vmatrix} = 0$$

Hence, we take any three planes and solve corresponding system.

$$\begin{cases} x-y-z-7=0 \\ x+2y-z-1=0, \\ x+y+1=0. \end{cases}$$

$$\begin{cases} y=-2 \\ x=1 \\ z=-4 \end{cases}$$

Answer: Lines intersect in point $(1,-2,-4)$.

9 Equations of line in projections

Consider canonical equations of a line

$$\frac{x-x_0}{a_x}=\frac{y-y_0}{a_y}=\frac{z-z_0}{a_z}.$$

We can express explicitly x and y az linear functions of z:

$$\begin{cases} \frac{x-x_0}{a_x}=\frac{z-z_0}{a_z} \\ \frac{y-y_0}{a_y}=\frac{z-z_0}{a_z} \end{cases} \tag{22}$$

$$\begin{cases} x=\frac{a_x}{a_z}z+(x_0-\frac{a_x}{a_z}z_0) \\ y=\frac{a_y}{a_z}z+(y_0-\frac{a_y}{a_z}z_0) \end{cases} \tag{23}$$

$$\begin{cases} x=Mz+b_x \\ y=Nz+b_y \end{cases} \tag{24}$$

Definition. We call equations (23) and (24) **equations of line in projections**

M has meaning of the angular coefficient (slope in orthonormal basis of the projection) of line on plane xOz.

N has meaning of the angular coefficient (slope in orthonormal basis of the projection) of line on plane yOz.

Definition. Parameters M and N from equations (24) are called **angular coefficients of the line in space**.

Backward expression of z for equations (24) yields

$$z = \frac{x - b_x}{M}$$
$$z = \frac{y - b_y}{N}.$$

Hence, canonical equation of the line may be restored:

$$\frac{x - b_x}{M} = \frac{y - b_y}{N} = \frac{z}{1}$$

Problem
Write canonical equation for the line:

$$2x + y - z + 1 = 0$$
$$3x - y + 2z - 3 = 0$$

Solution
While we sum the equations, we yield

$$5x + z - 2 = 0,$$

hence

$$x = -\frac{1}{5}z + \frac{2}{5}$$

While we multiply first equation by -3, second by 2 and sum, we yield

$$-5y + 7z - 9 = 0,$$

hence

$$t = \frac{7}{5}z - \frac{9}{5}$$

Therefore,

$$z = \frac{x - \frac{2}{5}}{-\frac{1}{5}}, \quad z = \frac{y + \frac{9}{5}}{\frac{7}{5}}$$

Canonical equation is

$$\frac{x - \frac{2}{5}}{-\frac{1}{5}} = \frac{y + \frac{9}{5}}{\frac{7}{5}} = \frac{z}{1}$$

10 Projection planes

Suppose line expressed with canonical equation

$$\frac{x-x_0}{a_x}=\frac{y-y_0}{a_y}=\frac{z-z_0}{a_z}$$

Each identity here represents a plane containing line itself and its projection on coordinate plane:

$$\begin{aligned}\frac{x-x_0}{a_x}&=\frac{y-y_0}{a_y}\\ \frac{x-x_0}{a_x}&=\frac{z-z_0}{a_z}\\ \frac{y-y_0}{a_y}&=\frac{z-z_0}{a_z}.\end{aligned}$$

In coordinate system with right orthonormal basis these planes are perpendicular with corresponding coordinate planes.

Problem
Find the equations of the projection planes of the line of intersection of the planes

$$\begin{aligned}2x+3y-5z+6=0\\ 3x-2y+z-8=0.\end{aligned}$$

Solution
To find the projection planes, eliminate z, y, and x in turn between the two equations to obtain

$$\begin{aligned}17x-7y-34=0\\ 13x-7z-12=0\\ 13y-172+34=0,\end{aligned}$$

as the projection planes of the line on the xOy, xOz and yOz planes respectively.