Differential Geometry

Part III.

Introduction to Manifolds

2025

Chapter 9

Introduction to Manifolds

9.1 Remark on Smooth and Analytic Functions

Coordinates and Function Classes

Let $U \subseteq \mathbb{R}^n$ be an open set, with coordinates $x^1, \ldots, x^n$ and a point $p = (p^1, \ldots, p^n) \in U$. Following modern conventions, **coordinate indices are superscripts**.

Definition (Smoothness Classes)**.** A function $f : U \to \mathbb{R}$ is:

- C^k **at** p if all partial derivatives of order $\leq k$ exist and are continuous at p:

$$\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}} \quad (j \leq k).$$

- C^∞ **(smooth) at** p if it is C^k for all $k \geq 0$.

For vector-valued functions $f : U \to \mathbb{R}^m$, f is C^k (or C^∞) if each component $f^1, \ldots, f^m$ is.

Key Notes:

- C^∞ and "smooth" are synonymous.
- Smoothness is **local**: Defined pointwise but often assumed to hold on all of U.

Real-Analytic Functions

Definition. A function f is **real-analytic at** p if, in some neighborhood of p, it equals its Taylor series at p:

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(p)(x^i - p^i) + \frac{1}{2!}\sum_{i,j} \frac{\partial^2 f}{\partial x^i \partial x^j}(p)(x^i - p^i)(x^j - p^j) + \cdots$$

Properties:

- **Real-analytic** $\implies$ C^∞: Convergent power series can be differentiated term-by-term (e.g., $\sin x$, $\cos x$).
- **But** $C^\infty \not\Longrightarrow$ **real-analytic!**

Example (A C^∞ Non-Analytic Function). Define $f : \mathbb{R} \to \mathbb{R}$ by:

$$f(x) = \begin{cases} e^{-1/x} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

Key Observations:

1. f **is** C^∞: All derivatives $f^{(k)}(x)$ exist and are continuous, including at $x = 0$.
2. **Flat at 0**: $f^{(k)}(0) = 0$ for all $k \geq 0$ (proved by induction; see **Problem 1.2**).
3. **Taylor Series Fails**: The Taylor series at 0 is identically zero, but $f(x) \neq 0$ for $x > 0$. Thus, f is **not real-analytic** at 0.

Visual Aid:

Taylor's Theorem for C^∞ Functions

Definition (Star-Shaped Set)**.** A subset $S \subseteq \mathbb{R}^n$ is **star-shaped** with respect to $p \in S$ if for every $x \in S$, the line segment from p to x lies in S.

Lemma 9.1.1 (Taylor's Theorem with Remainder)**.** *Let $U \subseteq \mathbb{R}^n$ be* ***star-shaped*** *with respect to $p \in U$. If $f \in C^\infty(U)$, there exist smooth functions $g_i(x)$ on U such that:*

$$f(x) = f(p) + \sum_{i=1}^{n}(x^i - p^i)g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

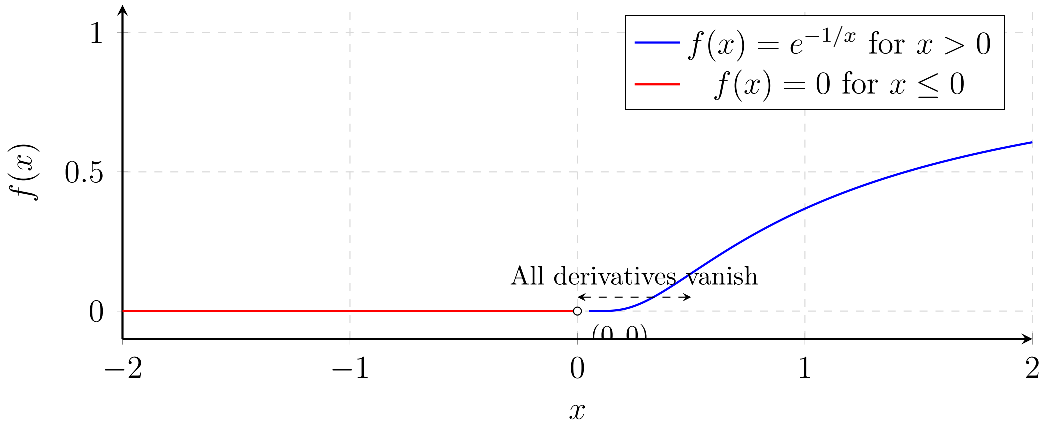


Figure 9.1. The C^∞ function $f(x)$ that is flat at $x = 0$ but not analytic. All derivatives $f^{(k)}(0) = 0$, yet $f(x) \neq 0$ for $x > 0$.

Remarks:

- **Star-shaped sets** generalize convexity (e.g., any convex set is star-shaped).
- This is the **first-order case**; higher-order generalizations exist but require more terms.

Intuition: Even if f doesn't equal its full Taylor series, it can be approximated by a polynomial with smooth remainder terms.

Summary of Key Points

- **Hierarchy:** Real-analytic $\implies C^\infty \implies C^k$ for any k. *(But the reverse implications are false!)*
- **Why C^∞?** Smooth functions are flexible enough for manifold theory but avoid the rigidity of analyticity.
- **Watch Out:**
  - **Not all C^∞ functions are analytic**.
  - **Taylor series may fail** to represent C^∞ functions globally.

9.2 Remark on Topological Spaces

The prototype of a topological space is the Euclidean space $\mathbb{R}^n$. However, Euclidean space comes with additional structures (e.g., metric, coordinates) that are not essential for defining continuity. The goal of topology is to distil the notion of continuity to its core by focusing on open sets.

Open Sets in $\mathbb{R}^n$

In $\mathbb{R}^n$, a set U is **open** if for every point $p \in U$, there exists an open ball $B(p, r) = \{x \in \mathbb{R}^n \mid d(x, p) < r\}$ entirely contained in U (Figure 9.2).

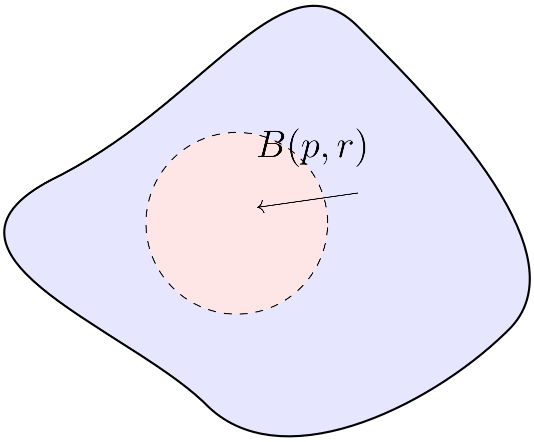


Figure 9.2. An open set U containing a point p and an open ball $B(p, r) \subset U$.

Key Properties

- **Arbitrary unions** of open sets are open.
- **Finite intersections** of open sets are open.

Example. The intervals $(-1/n, 1/n)$ in $\mathbb{R}^1$ are open, but their intersection $\bigcap_{n=1}^{\infty}(-1/n, 1/n)$ $\{0\}$ is **not** open. This motivates the need for finite intersections in axioms.

Axiomatic Definition of a Topology

Definition. A **topology** on a set S is a collection $\mathcal{T}$ of subsets (called **open sets**) satisfying:

1. $\varnothing$ and S are in $\mathcal{T}$.
2. $\mathcal{T}$ is closed under **arbitrary unions**.
3. $\mathcal{T}$ is closed under **finite intersections**.

The pair $(S, \mathcal{T})$ is called a **topological space**.

Examples

- **Standard topology on $\mathbb{R}^n$**: Open sets are unions of open balls.
- **Discrete topology**: Every subset is open (finest topology).
- **Trivial topology**: Only $\varnothing$ and S are open (coarsest topology).

Terminology

- A **neighborhood** of p is any open set containing p.
- $\mathcal{T}_1$ is **coarser** than $\mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ (fewer open sets).

Closed Sets and Alternative Topologies

A subset is **closed** if its complement is open. By de Morgan's laws:

- Arbitrary intersections of closed sets are closed.
- Finite unions of closed sets are closed.

Example (Finite-Complement Topology). Open sets are $\varnothing$, $\mathbb{R}^1$, and complements of finite sets.

$$\bigcup(\mathbb{R}^1 \setminus F_\alpha) = \mathbb{R}^1 \setminus (\bigcap F_\alpha)$$
$$\bigcap_{i=1}^{n}(\mathbb{R}^1 \setminus F_i) = \mathbb{R}^1 \setminus (\bigcup F_i)$$

Since arbitrary intersections and finite unions of finite sets are finite, this defines a topology.

Remark. The word "closed" in "closed under unions" (topology axioms) is unrelated to "closed subsets."

Local Criterion for Openness

Lemma 9.2.1 (Local Criterion for Openness)**.** *A subset $A \subset S$ is open **iff** for every $p \in A$, there exists an open set V_p such that $p \in V_p \subset A$.*

Proof. ($\Rightarrow$) Take $V_p = A$.
($\Leftarrow$) Then $A = \bigcup_{p\in A} V_p$ is a union of open sets. □

Subspace Topology (Relative Topology)

Let $(S, \mathcal{T})$ be a topological space and $A \subset S$ a subset. The **subspace topology** (or **relative topology**) on A is defined as:

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}.$$

This collection $\mathcal{T}_A$ satisfies the axioms of a topology on A:

1. **Arbitrary unions**: For any family $\{U_\alpha \cap A\} \subset \mathcal{T}_A$,

$$\bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A \in \mathcal{T}_A.$$

2. **Finite intersections**: For any finite family $\{U_i \cap A\} \subset \mathcal{T}_A$,

$$\bigcap_i (U_i \cap A) = \left(\bigcap_i U_i\right) \cap A \in \mathcal{T}_A.$$

3. **Contains $\varnothing$ and A**: $\varnothing = \varnothing \cap A$ and $A = S \cap A$ are in $\mathcal{T}_A$.

A subset $V \subset A$ is **open in A** (or **relatively open**) if $V = U \cap A$ for some U open in S. Importantly, V need not be open in S unless A itself is open in S.

Special Case: Open Subsets

If A is open in S, then $V \subset A$ is relatively open in A **if and only if** V is open in S.

9.3 Basis for a Topology

Definition (Basis for a Topology)**.** A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ on a space S is a *basis* for $\mathcal{T}$ if for every open set $U \subset S$ and point $p \in U$, there exists $B \in \mathcal{B}$ such that $p \in B \subset U$.

Example. The collection of all open balls $B(p, r)$ in $\mathbb{R}^n$ (with $r > 0$) is a basis for the standard topology.

Proposition. *A collection $\mathcal{B}$ of open sets is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.*

Proposition. *A collection $\mathcal{B}$ of subsets of S is a basis for* some *topology on S if and only if:*

1. $\bigcup_{B \in \mathcal{B}} B = S$ *(covers S),*
2. *For any $B_1, B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $p \in B_3 \subset B_1 \cap B_2$.*

Proposition. *If $\mathcal{B}$ is a basis for S and $A \subset S$, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a basis for the subspace topology on A.*

Second Countability

Definition (Second Countability)**.** A space is *second countable* if it has a countable basis.

Example.

- $\mathbb{R}^n$ with the standard topology is second countable.
- An uncountable set with the discrete topology is *not* second countable (every singleton must be in the basis).

Proposition. *The collection $\mathcal{B}_{rat}$ of open balls in $\mathbb{R}^n$ with* ***rational centres and radii*** *is a countable basis for $\mathbb{R}^n$.*

Proposition. *A subspace of a second-countable space is second countable.*

First Countability

Definition (First Countability)**.** A space is *first countable* if every point $p \in S$ has a countable *neighborhood basis*: a collection $\{B_i\}$ of neighborhoods of p such that for any neighborhood U of p, there exists $B_i \subset U$.

Example.

- $\mathbb{R}^n$ is first countable: $\{B(p, 1/n)\}_{n=1}^{\infty}$ is a neighborhood basis at p.
- An uncountable discrete space is first countable (singletons suffice) but not second countable.

Refinement: Any countable neighbourhood basis can be replaced with a descending sequence $U_1 \supset U_2 \supset \cdots$ of open sets.

Relation to second countability:

- Second countable $\implies$ first countable.
- The converse fails (e.g., discrete topology on $\mathbb{R}$).

9.4 Topological Manifolds

Definition (Locally Euclidean Space). A topological space M is **locally Euclidean of dimension** n if every point $p \in M$ has a neighborhood U such that there exists a homeomorphism $\phi\colon U \to \phi(U) \subseteq \mathbb{R}^n$, where $\phi(U)$ is open in $\mathbb{R}^n$.

Definition (Chart). The pair (U, ϕ) is called a **chart**, where:

- U is a **coordinate neighborhood** (or **coordinate open set**),
- ϕ is a **coordinate map** (or **coordinate system**).

The chart (U, ϕ) is **centered at** p if $\phi(p) = 0$.

Definition (Topological Manifold). A **topological manifold** is a Hausdorff, second-countable, locally Euclidean space. Its **dimension** n is the integer for which it is locally Euclidean.

Key Implications

1. **Invariance of Dimension**: For $n \neq m$, an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$. This ensures the dimension n is well-defined. While this is non-trivial to prove (it relies on deep results like Brouwer's invariance of domain), we will later see a simpler proof for smooth manifolds (Corollary 8.7).
2. **Disconnected Manifolds**: If a manifold has multiple connected components, each component may have a different dimension.

Examples and Non-Examples

Example (Euclidean Space). The simplest topological manifold is $\mathbb{R}^n$, covered by the single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n}$ is the identity map.

Key Observations:

- Every open subset $U \subseteq \mathbb{R}^n$ is also a topological manifold with chart $(U, \mathbb{1}_U)$.
- Subspaces of $\mathbb{R}^n$ inherit the **Hausdorff** and **second-countable** properties. Thus, to verify if a subspace is a topological manifold, we need only check local Euclideanity.

Example (Cusp - Topological Manifold). The graph $C = \{(x, y) \in \mathbb{R}^2 \mid y = x^{2/3}\}$ is a topological manifold.

Proof:

- As a subspace of $\mathbb{R}^2$, C is Hausdorff and second-countable.
- The map $\phi\colon C \to \mathbb{R}$ defined by $\phi(x, x^{2/3}) = x$ is a homeomorphism, showing C is locally Euclidean of dimension 1.

Example (The Cross - Not a Manifold). Let X be the union of the lines $x = 0$ and $y = 0$ in $\mathbb{R}^2$ (a "cross" centered at $p = (0, 0)$). Then X is **not** locally Euclidean at p, hence not a topological manifold.

Proof by Contradiction:

1. Suppose X is locally Euclidean of dimension n at p. Then there exists a neighborhood U of p and a homeomorphism $\psi\colon U \to B(0, \varepsilon) \subseteq \mathbb{R}^n$ with $\psi(p) = 0$.
2. Restricting to $U \setminus \{p\}$, we get a homeomorphism $U \setminus \{p\} \to B(0, \varepsilon) \setminus \{0\}$.
3. **Case Analysis**:
   - If $n = 1$, $B(0, \varepsilon) \setminus \{0\}$ has **2 connected components**.
   - If $n \geq 2$, $B(0, \varepsilon) \setminus \{0\}$ is **connected**.
4. However, $U \setminus \{p\}$ has **4 connected components** (the four "arms" of the cross). This is a contradiction, since homeomorphisms preserve connectedness.

Conclusion: No such n exists, so X is not locally Euclidean at p.

Key Takeaways

- A topological manifold is a space that "looks like" $\mathbb{R}^n$ locally, while being Hausdorff and second-countable.
- Subspaces of $\mathbb{R}^n$ inherit the Hausdorff and second-countable properties, simplifying verification.
- **Local Euclideanity** is the critical condition to check. Tools like connectedness can help disprove it (as in the cross example).

9.5 Atlases on Topological Manifolds

Definition (C^∞-Compatible Charts)**.** Let M be a topological manifold. Two charts (U, ϕ) and (V, ψ), where:

- $U, V \subset M$ are open sets,
- $\phi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ are homeomorphisms onto open subsets of $\mathbb{R}^n$,

are called C^∞**-compatible** if the **transition functions**:

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are smooth (C^∞). If $U \cap V = \varnothing$, the charts are automatically compatible.

Notation:

- $U_{\alpha\beta} = U_\alpha \cap U_\beta$ (intersection of two chart domains)
- $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ (triple intersection)

2. C^∞ Atlas

Definition (C^∞ Atlas)**.** A C^∞ **atlas** on M is a collection $\mathcal{A} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts such that $M = \bigcup_\alpha U_\alpha$ (the charts cover M).

Key Properties:

- **Reflexive & Symmetric:** If (U, ϕ) is compatible with (V, ψ), then (V, ψ) is compatible with (U, ϕ).
- **Not Transitive:** If (U_1, ϕ_1) is compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is compatible with (U_3, ϕ_3), this **does not** guarantee that (U_1, ϕ_1) and (U_3, ϕ_3) are compatible.

Reason: The composite $\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$ is only guaranteed to be smooth on $\phi_1(U_{123})$, not necessarily on all of $\phi_1(U_{13})$.

Example: C^∞ Atlas on the Circle S^1

Let $S^1 = \{e^{it} \in \mathbb{C} \mid t \in [0, 2\pi)\}$. Define two charts:

$$U_1 = \{e^{it} \mid -\pi < t < \pi\}, \quad \phi_1(e^{it}) = t$$
$$U_2 = \{e^{it} \mid 0 < t < 2\pi\}, \quad \phi_2(e^{it}) = t$$

Intersection: $U_1 \cap U_2 = A \sqcup B$, where:

$$A = \{e^{it} \mid -\pi < t < 0\}$$
$$B = \{e^{it} \mid 0 < t < \pi\}$$

Transition Functions:

- On $\phi_1(A) = (-\pi, 0)$:
$$\phi_2 \circ \phi_1^{-1}(t) = t + 2\pi$$
- On $\phi_1(B) = (0, \pi)$:
$$\phi_2 \circ \phi_1^{-1}(t) = t$$

Both cases are smooth, so (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible and form an atlas.

Compatibility of Charts in an Atlas

Lemma 9.5.1 (Compatibility of Charts in an Atlas)**.** *If two charts (V, ψ) and (W, σ) are both compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.*

Proof:

- Let $p \in V \cap W$. Since $\{(U_\alpha, \phi_\alpha)\}$ covers M, $p \in U_\alpha$ for some α.
- On $\psi(V \cap W \cap U_\alpha)$, the transition function decomposes as:
$$\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$$
Both factors are C^∞ (by compatibility with the atlas), so the composition is C^∞ at $\psi(p)$.
- Since smoothness is local, $\sigma \circ \psi^{-1}$ is smooth on all of $\psi(V \cap W)$.
- Similarly, $\psi \circ \sigma^{-1}$ is smooth on $\sigma(V \cap W)$.

Thus, (V, ψ) and (W, σ) are compatible.

9.6 Smooth Manifold

Definition (Maximal Atlas)**.** An **atlas** on a locally Euclidean space is said to be **maximal** if it is not contained in any strictly larger atlas. In other words, if $\mathcal{M}$ is a maximal atlas and $\mathcal{A}$ is any other atlas containing $\mathcal{M}$, then $\mathcal{A} = \mathcal{M}$.

Theorem 9.6.1. *Any atlas $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.*

Proof.

- Adjoin all charts (V_i, ψ_i) compatible with $\mathcal{U}$. By compatibility (Proposition 5.8), the enlarged collection remains an atlas.
- Any chart compatible with the new atlas must also be compatible with $\mathcal{U}$, hence already included. Thus, the atlas is maximal.
- Uniqueness follows since any other maximal atlas $\mathcal{M}'$ containing $\mathcal{U}$ must satisfy $\mathcal{M}' \subseteq \mathcal{M}$, implying $\mathcal{M}' = \mathcal{M}$.

□

Definition (Smooth Manifold)**.** A **smooth** (or C^∞) **manifold** is a topological manifold M equipped with a maximal atlas, called a **differentiable structure** on M.

- The **dimension** of M is n if all connected components of M are locally homeomorphic to $\mathbb{R}^n$.
- A 1-dimensional manifold is a **curve**, a 2-dimensional manifold is a **surface**, and an n-dimensional manifold is an n**-manifold**.

Practical Note: To verify that a topological space M is a smooth manifold, it suffices to:

1. Show M is Hausdorff and second countable.
2. Exhibit **any** C^∞ atlas (not necessarily maximal).

Examples of Smooth Manifolds

Example (Euclidean Space). $\mathbb{R}^n$ is a smooth manifold with the trivial atlas $\{(\mathbb{R}^n, \phi)\}$, where $\phi = (r^1, \ldots, r^n)$ are the standard coordinates.

Example (Open Submanifold). Any open subset V of a manifold M is a manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V})\}$ is an atlas for V.

Example (Zero-Dimensional Manifolds). A zero-dimensional manifold is a countable discrete space, as each point is homeomorphic to $\mathbb{R}^0$ (a singleton). Second countability implies countability.

Example (Graph of a Smooth Function). For $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ a C^∞ function, its **graph** $\Gamma(f) = \{(x, f(x)) \mid x \in U\}$ is a manifold. The single chart $(\Gamma(f), \phi)$, where $\phi(x, f(x)) = x$, defines a C^∞ structure.

Particular case: Elliptic paraboloids and hyperbolic paraboloids are smooth manifolds.

Example (Unit Circle S^1). The circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ has a C^∞ atlas with four charts:

1. **Upper/lower semicircles** $U_{1,2}$: $\phi_i(x, y) = x$ (projection to x-axis).
2. **Right/left semicircles** $U_{3,4}$: $\phi_i(x, y) = y$ (projection to y-axis).

Transition maps are smooth, e.g., on $U_1 \cap U_3$:

$$\phi_3 \circ \phi_1^{-1}(x) = \sqrt{1 - x^2} \quad \text{(smooth for } x \in (-1, 1)).$$

9.7 Remark on Derivation and Germs of Functions

Recall that a secant plane to a surface in $\mathbb{R}^3$ is a plane determined by three points of the surface. As the three points approach a point p on the surface, if the corresponding secant planes approach a limiting position, then the plane that is the limiting position of the secant planes is called *the tangent plane* to the surface at p.

Intuitively, the tangent plane to a surface at p is the plane in $\mathbb{R}^3$ that just "touches" the surface at p. A vector at p is tangent to a surface in $\mathbb{R}^3$ if it lies in the tangent plane at p.

Such a definition of a tangent vector to a surface presupposes that the surface is embedded in a *Euclidean space*, and so would not apply to the projective plane, for example, which does not sit inside an $\mathbb{R}^n$ in any natural way.

We have to find a characterisation of tangent vectors in $\mathbb{R}^n$ that will generalise to manifolds.

Directional Derivatives as Tangent Vectors

In elementary calculus, a tangent vector v at a point $p \in \mathbb{R}^3$ is represented as an arrow:

$$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

Given a smooth curve $c(t) = (p^1 + tv^1, \ldots, p^n + tv^n)$ through p, with direction v. The **directional derivative** of a function f along v is:

$$D_v f = \frac{d}{dt}\bigg|_{t=0} f(c(t)) = \sum_{i=1}^{n} v^i \frac{\partial f}{\partial x^i}(p).$$

This suggests identifying v with the operator

$$D_v = \sum v^i \left. \frac{\partial}{\partial x^i} \right|_p$$

.

Key Idea: Tangent vectors at p are derivations—linear operators on functions satisfying the Leibniz rule.

Germs of Functions

Definition (Agreement of Functions at a Point)**.** Let f and g be functions defined on topological spaces (e.g., open subsets of $\mathbb{R}^n$). We say f and g **agree at a point** p if:

1. There exists an open neighborhood U of p such that f and g are both defined on U.
2. $f(x) = g(x)$ for all $x \in U$.

This defines an equivalence relation $\sim$ on the set of functions defined near p, where $f \sim g$ if they agree at p.

Definition (Germ of Function)**.** The equivalence class $[f]$ is called the **germ** of f at p.

The set of all germs of smooth (C^∞) functions at p is denoted by C_p^∞.

Example. Two functions

- $f(x) = \frac{1}{1-x}$ (defined on $\mathbb{R} \setminus \{1\}$)

- $g(x) = 1 + x + x^2 + \cdots$ (defined on $|x| < 1$)

have the same germ at any $p \in (-1, 1)$.

1. Germ Condition

For f and g to define the same germ at p, there must exist an open neighborhood U of p where $f(x) = g(x)$ for all $x \in U$.

2. Series Convergence

The function $g(x)$ is the Taylor series for $f(x)$:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x| < 1.$$

This series converges absolutely on $(-1, 1)$.

3. Local Agreement

For any $p \in (-1, 1)$, choose $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset (-1, 1)$. On this interval:

- $f(x)$ is defined (since $x \neq 1$),
- $g(x)$ converges to $f(x)$.

Thus, $f(x) = g(x)$ on $(p - \epsilon, p + \epsilon)$, proving they have the same germ at p.

4. Failure Outside $(-1, 1)$

For $|x| \geq 1$, $g(x)$ diverges, while $f(x)$ remains defined (except at $x = 1$). Hence, the germs differ outside $(-1, 1)$.

Algebra Structure: Germs form an $\mathbb{R}$-algebra C_p^∞ under pointwise operations.

An algebra over a field K is a vector space A over K with a multiplication map

$$\mu : A \times A \to A$$

usually written $\mu(a, b) = a \cdot b$, such that for all a, b, $c \in A$ and $r \in K$,

(i) (associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,

(ii) (distributivity) $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c$,

(iii) (homogeneity) $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$.

Equivalently, an algebra over a field K is a ring A (with or without multiplicative identity) that is also a vector space over K such that the ring multiplication satisfies the homogeneity condition (iii). Thus, an algebra has three operations: the addition and multiplication of a ring and the scalar multiplication of a vector space. Usually, we omit the multiplication sign and write ab instead of $a \cdot b$.

Derivations at a Point

A map $L : V \to W$ between vector spaces over a field K is called a *linear map* or a *linear operator* if for any $r \in K$ and $u, v \in V$,

$$\begin{aligned} &\text{(i)} \quad L(u+v) = L(u) + L(v); \\ &\text{(ii)} \quad L(rv) = rL(v). \end{aligned}$$

To emphasise the fact that the scalars are in the field K, such a map is also said to be K-linear.

If A and A' are algebras over a field K, then an algebra homomorphism is a linear map $L : A \to A'$ that preserves the algebra multiplication: $L(ab) = L(a)L(b)$ for all $a, b \in A$.

A **derivation at** p is a linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g).$$

Key Properties

- **Annihilation of Constants**: For any constant function $c(x) = c$, $D(c) = 0$.

  Proof. Apply the Leibniz rule to $1 \cdot 1$:

  $$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1) \implies D(1) = 0.$$

  Linearity then gives $D(c) = cD(1) = 0$. □

- **Locality**: Derivations depend only on the local behaviour of functions (their germs). If $f = g$ near p, then $D(f) = D(g)$.

Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is, in fact, a real vector space, since the sum of two derivations at p and a scalar multiple of a derivation at p is again a derivation at p.

- The tangent space $T_p(\mathbb{R}^n)$ is the set of all possible tangent vectors at $p \in \mathbb{R}^n$.

- Formally, it is the space of directional derivatives at p, or equivalently, the space of vectors "attached" to p.
- It is an n-dimensional real vector space isomorphic to $\mathbb{R}^n$.

A tangent vector $v \in T_p(\mathbb{R}^n)$ can be expressed as:

$$v = \sum_{i=1}^{n} v^i \left. \frac{\partial}{\partial x^i} \right|_p,$$

where:

- v^i are the components of v in the standard basis,
- $\left. \frac{\partial}{\partial x^i} \right|_p$ are the partial derivative operators at p, forming a basis for $T_p(\mathbb{R}^n)$.

Theorem 9.7.1 (Derivations and Tangent Space)**.** *The map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ defined by*

$$\phi(v) = D_v = \sum_{i=1}^{n} v^i \left. \frac{\partial}{\partial x^i} \right|_p$$

is a vector space isomorphism. This means:

- *Every tangent vector v defines a derivation D_v (via directional derivatives).*
- *Every derivation D comes from a unique tangent vector $v = (D(x^1), \ldots, D(x^n))$.*

Proof. We justify injectivity and surjectivity.

- **Injectivity**: If $D_v = 0$, then applying D_v to the coordinate functions x^j gives:

$$0 = D_v(x^j) = \sum_i v^i \frac{\partial x^j}{\partial x^i}(p) = v^j \implies v = 0.$$

- **Surjectivity**: For any derivation D, define $v^i = D(x^i)$. Using the Taylor expansion of f near p:

$$f(x) = f(p) + \sum_i (x^i - p^i) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p),$$

apply D to both sides:

$$D(f) = \sum_i D(x^i) g_i(p) = \sum_i v^i \frac{\partial f}{\partial x^i}(p) = D_v(f).$$

Thus, $D = D_v$.

$\square$

Corollary 9.7.2. *Tangent vectors can be written as:*

$$v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p .$$

- A derivation D generalizes the notion of a directional derivative:
  - $D(f)$ measures how f changes in the "direction" specified by D.
  - The Leibniz rule encodes the product rule from calculus.
- The basis $\left.\frac{\partial}{\partial x^i}\right|_p$ corresponds to the standard basis vectors e_i in $\mathbb{R}^n$.

9.8 Vector Fields

A **vector field** X on an open subset $U \subseteq \mathbb{R}^n$ assigns to each point $p \in U$ a tangent vector $X_p \in T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\left\{\left.\frac{\partial}{\partial x^i}\right|_p\right\}_{i=1}^n$, we write:

$$X_p = \sum_{i=1}^n a^i(p) \left.\frac{\partial}{\partial x^i}\right|_p, \quad a^i(p) \in \mathbb{R}.$$

Omitting p, the field is expressed globally as $X = \sum a^i \frac{\partial}{\partial x^i}$, where $a^i : U \to \mathbb{R}$ are **coefficient functions**.

Smoothness

X is C^∞ (smooth) if all a^i are smooth on U. *This ensures compatibility with calculus operations (e.g., differentiation, flows).*

Examples and Visual Intuition

Example (Rotational Field). On $\mathbb{R}^2 \setminus \{0\}$, let $p = (x, y)$. The field:

$$X = \frac{-y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$

represents **counterclockwise rotation** around the origin, with arrows tangent to concentric circles (see Figure 9.3).

Example (Hyperbolic Field). The field $Y = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ **expands along the x-axis and contracts along the y-axis**, resembling a saddle point.

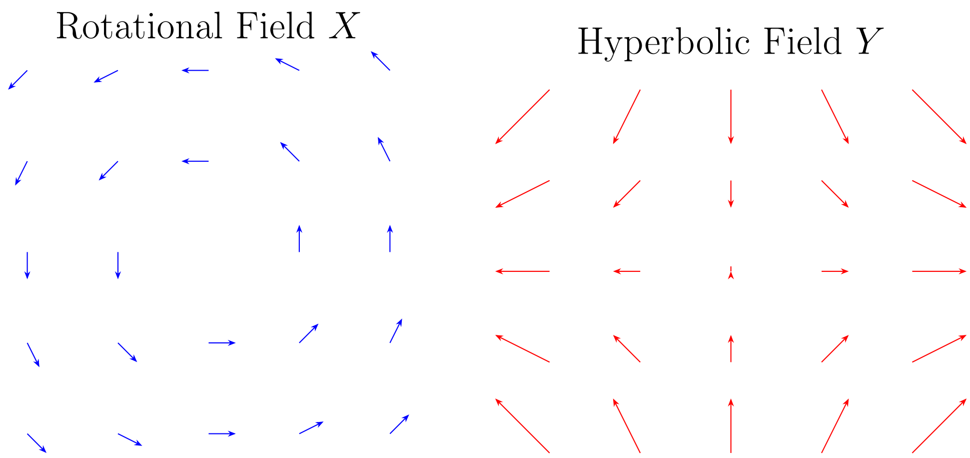


Figure 9.3. Visualization of vector fields X (rotational) and Y (hyperbolic).

Algebraic Structure: Vector Fields as Modules

Representation as Column Vectors

Smooth vector fields $X = \sum a^i \frac{\partial}{\partial x^i}$ **correspond bijectively** to column vectors of smooth functions:

$$X \longleftrightarrow \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}, \quad a^i \in C^\infty(U).$$

Module over $C^\infty(U)$

The set $\mathfrak{X}(U)$ of smooth vector fields is:

- A **vector space** over $\mathbb{R}$.
- A **module** over the ring $C^\infty(U)$, with scalar multiplication defined pointwise:
$$(fX)_p = f(p)X_p \quad \text{for } f \in C^\infty(U),\, X \in \mathfrak{X}(U).$$

Key properties:

- **Closed under multiplication**: $fX = \sum (fa^i)\frac{\partial}{\partial x^i}$ is smooth.
- **Local-global compatibility**: Module structure reflects coordinate-free operations on manifolds.

Modules: Generalizing Vector Spaces

Definition. An R-module A is an abelian group with scalar multiplication $R \times A \to A$ satisfying:

1. **Associativity**: $(rs)a = r(sa)$.
2. **Identity**: $1 \cdot a = a$.
3. **Distributivity**: $(r + s)a = ra + sa$, $r(a + b) = ra + rb$.

Key Insight:

- If R is a field, A is a **vector space**.
- Here, $R = C^\infty(U)$ (not a field), so $\mathfrak{X}(U)$ is a module.

Definition. An R-module homomorphism $f\colon A \to A'$ preserves:

1. **Addition**: $f(a + b) = f(a) + f(b)$.
2. **Scalar multiplication**: $f(ra) = rf(a)$.

9.9 Vector Fields as Derivations

In differential geometry, **vector fields** on a manifold M are often interpreted as **derivations** on the space of smooth functions $C^\infty(M)$. This perspective provides a deep algebraic understanding of vector fields and connects geometric intuition with abstract algebraic structures.

- **Traditional Definition of a Vector Field**: A vector field X on an open subset M assigns to each point $p \in M$ a tangent vector $X_p \in T_pM$. Locally, in coordinates $(x^1, \ldots, x^n)$, X_p can be expressed as:

$$X_p = \Sigma a^i(p) \left. \frac{\partial}{\partial x^i} \right|_p ,$$

  where $a^i(p)$ are smooth functions on M. We say that the vector field X is C^∞ on M if the coefficient functions $a^i(p)$ are all C^∞ on M.

- **Derivation Definition**: A **derivation** on the algebra $C^\infty(M)$ is a linear map $D : C^\infty(M) \to C^\infty(M)$ that satisfies the **Leibniz rule**:

$$D(fg) = D(f)g + f\,D(g), \quad \text{for all } f, g \in C^\infty(M).$$

Every smooth vector field X on M defines a derivation D_X on $C^\infty(M)$:

$$D_X(f) = X(f) = X^i \frac{\partial f}{\partial x^i},$$

where $X(f)$ is the directional derivative of f along X.

Conversely, every derivation D on $C^\infty(M)$ arises from a unique smooth vector field X on M. This establishes a one-to-one correspondence between:

$$\text{Vector Fields on } M \longleftrightarrow \text{Derivations on } C^\infty(M).$$

Algebraic Properties of Derivations

The space of derivations on $C^\infty(M)$ forms a **Lie algebra** under the **Lie bracket**:

$$[D_1, D_2](f) = D_1(D_2(f)) - D_2(D_1(f)).$$

This Lie bracket corresponds to the **Lie bracket of vector fields**:

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

Practical Interpretation

- **Geometric Perspective**: A vector field X describes infinitesimal motion along curves on M.
- **Algebraic Perspective**: A derivation D_X describes infinitesimal changes in smooth functions along X.

Both perspectives are equivalent, and the choice of interpretation depends on the context of the problem.

Examples

1. **Coordinate Vector Fields**: The coordinate vector field $\frac{\partial}{\partial x^i}$ acts as a derivation:

$$\frac{\partial}{\partial x^i}(f) = \frac{\partial f}{\partial x^i}.$$

2. **Linear Vector Fields on** $\mathbb{R}^n$: A vector field $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ acts as:

$$X(f) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

3. **Non-Coordinate Vector Fields**: On S^2, the vector field $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ acts as:

$$X(f) = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

Applications

- **Flow of Vector Fields**: The derivation D_X describes the infinitesimal generator of the flow ϕ_t associated with X:

$$D_X(f) = \lim_{t \to 0} \frac{f(\phi_t(p)) - f(p)}{t}.$$

- **Differential Equations**: Solving $\dot{\gamma}(t) = X_{\gamma(t)}$ is equivalent to finding the integral curves of X, which correspond to paths where γ follows the derivation D_X.

Summary and Intuition

Concept	**Description**	**Example**
Tangent vector	Directional derivative operator	$v = 2\left.\frac{\partial}{\partial x}\right\vert_p$
Germ	Local equivalence class of functions	$[x^2]$ at $p = 0$
Vector field	Smooth assignment of tangent vectors	$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

Why This Matters:

- Provides a coordinate-free way to define tangent vectors on manifolds.
- Unifies calculus and algebra via derivations.

9.10 Tangent Space in a Point

Definition. The **germ** of a smooth (C^∞) function at a point p in a manifold M is its equivalence class where two functions are equivalent if they agree on some neighborhood of p. The set of all germs at p is denoted $C_p^\infty(M)$.

Key Properties:

- Forms an $\mathbb{R}$-algebra under pointwise operations.

- Captures *local* behavior of functions near p while ignoring differences away from p.

Definition. A **derivation at** p is a linear map $D : C_p^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule:

$$D(fg) = D(f)g(p) + f(p)D(g).$$

A **tangent vector at** p is a derivation at p. The set of all tangent vectors forms the **tangent space** T_pM.

Geometric Interpretation:

- Tangent vectors generalize directional derivatives from $\mathbb{R}^n$ to manifolds.
- Physically, they represent possible "velocities" of curves passing through p.

Example (Velocity Vector). Let $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth curve with $\gamma(0) = p$. Then:

$$D_\gamma(f) := \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)(t)$$

is a tangent vector representing the derivative of f along γ at p.

Coordinate Basis for T_pM

Given a chart $(U, \phi) = (U, x^1, \dots, x^n)$ near p, define basis vectors:

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} (f \circ \phi^{-1}),$$

where r^i are standard coordinates on $\mathbb{R}^n$.

Key Facts:

1. $\left\{ \left.\frac{\partial}{\partial x^i}\right|_p \right\}$ forms a basis for T_pM.
2. In 1D, we write $\left.\frac{d}{dt}\right|_p$ instead of $\left.\frac{\partial}{\partial t}\right|_p$.

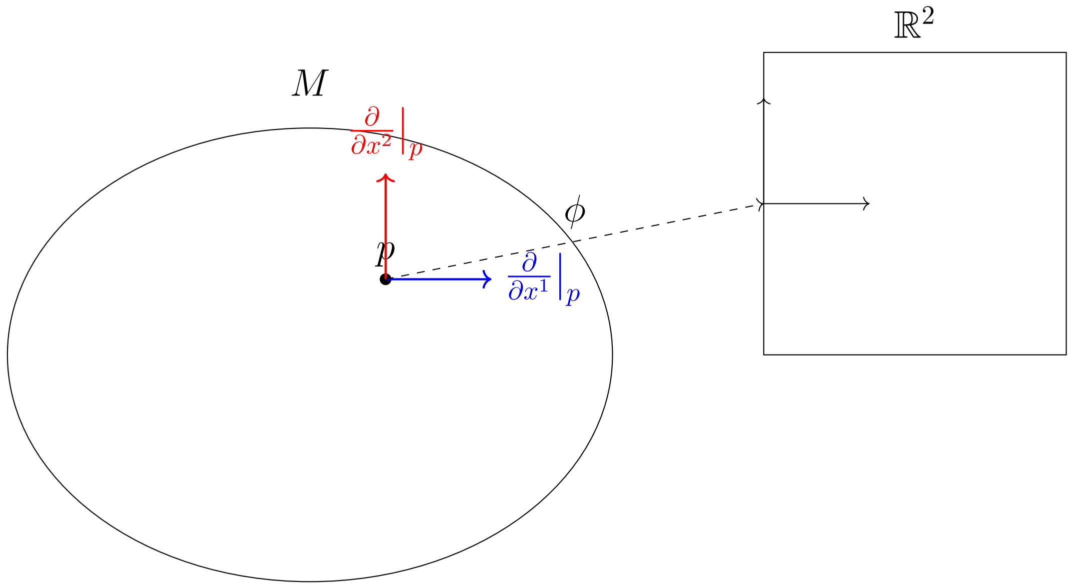


Figure 9.4. Basis vectors in T_pM induced by a coordinate chart ϕ.

Important Remarks

1. **Local Nature of T_pM:** For any open $U \subset M$ containing p, $T_pU = T_pM$ because germs depend only on arbitrarily small neighborhoods.

2. **Change of Coordinates:** If $(V, \psi) = (V, y^1, \ldots, y^n)$ is another chart at p, the basis vectors transform via:

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i}.$$

3. **Physical Interpretation:** In physics, $\frac{\partial}{\partial x^i}\Big|_p$ represents a "direction" in which one can differentiate functions at p.

Comparison to Euclidean Space

When $M = \mathbb{R}^n$, $T_p\mathbb{R}^n$ is naturally identified with $\mathbb{R}^n$ itself via:

$$v = (v_1, \ldots, v_n) \leftrightarrow D_v := \sum_{i=1}^{n} v_i \left.\frac{\partial}{\partial r^i}\right|_p.$$

9.11 Problems Corner

Problem 1

Justify that the shere S^n expresed with equation

$$x_0^2 + x_1^2 + x_2^2 + ...x_n^2 = 1$$

is a smooth manifold.

Solution

We cover S^n with **stereographic projection charts**, which are smooth and overlap smoothly.

Step 1: Define the Charts

Let $U_i^+ = \{(x_0, \ldots, x_n) \in S^n \mid x_i > 0\}$ and $U_i^- = \{(x_0, \ldots, x_n) \in S^n \mid x_i < 0\}$.

For each U_i^+ and U_i^-, define a chart $\phi_i^\pm$ that projects the sphere onto $\mathbb{R}^n$ by omitting the i-th coordinate:

$$\phi_i^\pm(x_0, \ldots, x_n) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i} \right).$$

This is well-defined since $x_i \neq 0$ on $U_i^\pm$.

Step 2: Check Smoothness of Transition Maps

The transition maps between ϕ_i^+ and ϕ_j^+ (for $i \neq j$) are given by:

$$\phi_j^+ \circ (\phi_i^+)^{-1}(u_1, \ldots, u_n) = \left(\frac{u_1}{u_j}, \ldots, \frac{u_{i-1}}{u_j}, \frac{1}{u_j}, \frac{u_{i+1}}{u_j}, \ldots, \frac{u_n}{u_j} \right),$$

where $u_j \neq 0$. These are smooth since they are rational functions with non-zero denominators.

Problem 2

For the sphere S^n:

1. Construct an arbitrary atlas.
2. Construct a minimal atlas (i.e., an atlas containing the minimal number of charts).
3. Construct a minimal atlas where each chart is homeomorphic to a disk.
4. Construct a minimal atlas where:

- Each chart is homeomorphic to a disk, and
- Any non-empty intersection of charts is homeomorphic to a disk.

Solution

1. Construct an Arbitrary Atlas

We use **stereographic projections** from multiple points to cover S^n.

- **Charts**: For each $i = 0, 1, \dots, n$, define:

  $$U_i^+ = \{(x_0, \dots, x_n) \in S^n \mid x_i > 0\}, \quad U_i^- = \{(x_0, \dots, x_n) \in S^n \mid x_i < 0\}$$

  with maps $\phi_i^\pm : U_i^\pm \to \mathbb{R}^n$:

  $$\phi_i^\pm(x_0, \dots, x_n) = \left(\frac{x_0}{x_i}, \dots, \frac{\widehat{x_i}}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

  (the hat denotes omission).

- **Transition Maps**: For $U_i^+ \cap U_j^+$:

  $$\phi_j^+ \circ (\phi_i^+)^{-1}(u_1, \dots, u_n) = \left(\frac{u_1}{u_j}, \dots, \frac{1}{u_j}, \dots, \frac{u_n}{u_j}\right)$$

  which is smooth on $\phi_i^+(U_i^+ \cap U_j^+)$.

This gives an atlas with $2(n+1)$ charts.

2. Construct a Minimal Atlas

Theorem 9.11.1. *The minimal number of charts for S^n is 2.*

- **Charts**: Use stereographic projections from north and south poles:

  $$U_+ = S^n \setminus \{(0, \dots, 0, 1)\}, \quad \phi_+(x) = \left(\frac{x_0}{1 - x_n}, \dots, \frac{x_{n-1}}{1 - x_n}\right)$$

  $$U_- = S^n \setminus \{(0, \dots, 0, -1)\}, \quad \phi_-(x) = \left(\frac{x_0}{1 + x_n}, \dots, \frac{x_{n-1}}{1 + x_n}\right)$$

- **Transition Map**:

  $$\phi_- \circ \phi_+^{-1}(u) = \frac{u}{\|u\|^2}, \quad u \in \mathbb{R}^n \setminus \{0\}$$

3. Minimal Atlas with Disk Charts

- **Charts**: Use hemispherical charts:

$$U_i = \{x \in S^n \mid x_i > 0\}, \quad \phi_i(x) = (x_0, \dots, \widehat{x_i}, \dots, x_n)$$

  mapping to the open unit disk $D^n \subset \mathbb{R}^n$.

- **Minimality**: 2 charts suffice (e.g., U_1 and U_2).

4. Minimal Atlas with Disk Charts and Disk Intersections

- **Challenge**: For S^1, $U_+ \cap U_-$ is two disjoint arcs (not a disk).
- **Solution**:
  - For S^1: Use 3 charts, each covering $240°$ arcs.
  - For S^2: Use 4 hemispherical charts with enlarged overlap regions.
  - General S^n: Requires $n+2$ charts (by the nerve theorem).
- **Example for** S^2:

$$\begin{cases} U_1 = \text{northern hemisphere} \cup \text{thin southern band} \\ U_2 = \text{southern hemisphere} \cup \text{thin northern band} \\ U_3, U_4 = \text{eastern/western ``side-hemispheres''} \end{cases}$$

Summary

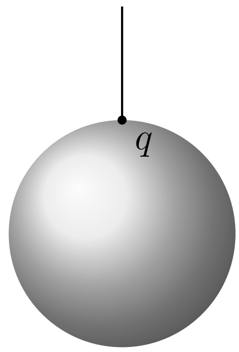
Part	Requirement	Min Charts	Construction
1	Arbitrary atlas	$2(n+1)$	Stereographic projections
2	Minimal atlas	2	North/south poles
3	Minimal disk charts	2	Hemispheres
4	Disk charts + disk intersections	$n+2$	Enlarged caps

Problem 3

Prove that the sphere with a hair at q in $\mathbb{R}^3$ is not locally Euclidean at q. Hence, it cannot be a topological manifold.

Let $X \subset \mathbb{R}^3$ be the union of the unit sphere S^2 and a line segment ℓ connecting a point $q \in S^2$ to $(0,0,2)$ ("the sphere with a hair"). Prove that:

1. X is not locally Euclidean at q.
2. Consequently, X cannot be a topological manifold.



$X = S^2 \cup \ell \subset \mathbb{R}^3$

Figure 9.5. The sphere with a hair

Solution

Part 1: Not Locally Euclidean at q

We match the definition:

Definition. A space is *locally Euclidean of dimension* n at a point p if there exists an open neighbourhood $U \subset X$ containing p that is homeomorphic to an open subset of $\mathbb{R}^n$.

We analyze two cases for X at q:

Case 1: Dimension $n = 2$

- Suppose $\exists$ open $U \subset X$ containing q and a homeomorphism $\phi : U \to V \subset \mathbb{R}^2$.
- Consider $U \setminus \{q\}$:
  - In X: Disconnected (contains separate sphere and hair components)
  - In $\mathbb{R}^2$: $V \setminus \{\phi(q)\}$ is always connected
- Contradiction: Homeomorphisms preserve connectedness.

Case 2: Dimension $n = 3$

- No neighborhood of q in X is open in $\mathbb{R}^3$ due to the hair
- The hair is "too thin" to have a 3D neighbourhood

Part 2: Not a Topological Manifold

We match the assertion:

Theorem 9.11.2. *A topological manifold requires every point to have a locally Euclidean neighbourhood.*

Since X fails this at q, it cannot be a manifold.

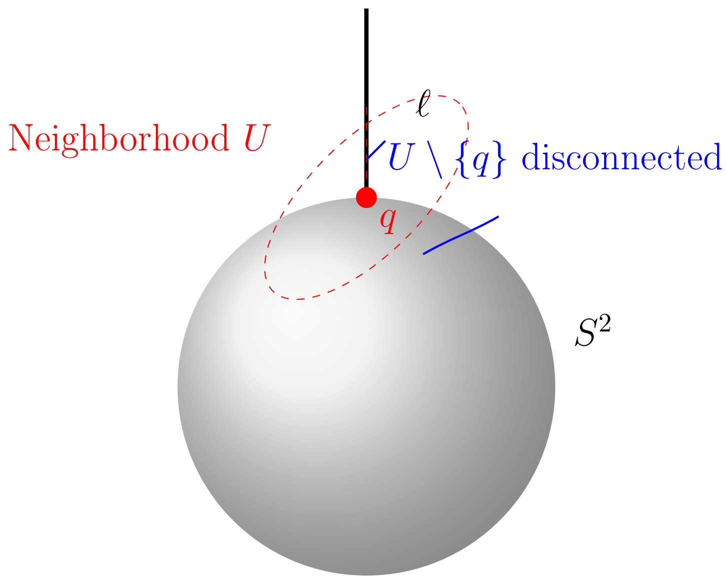


Figure 9.6. Problem 3 visualisation

Conclusion

The space X:

- Fails to be locally Euclidean at q (all dimensions)
- Cannot be a topological manifold

Problem 4

Let X be the vector field

$$x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},$$

and $f(x,y,z)$ the function $x^2+y^2+z^2$ on $\mathbb{R}^3$. Compute Xf.

Solution

Given:

- The vector field X on $\mathbb{R}^3$:

$$X=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}.$$

- The function $f(x,y,z)$:

$$f(x,y,z)=x^2+y^2+z^2.$$

Compute Xf:

The action of the vector field X on f is given by:

$$Xf=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f.$$

Compute the partial derivatives:

$$\frac{\partial f}{\partial x}=2x,\quad \frac{\partial f}{\partial y}=2y.$$

Now apply X to f:

$$Xf=x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}=x\cdot 2x+y\cdot 2y=2x^2+2y^2.$$

Final Answer:

$$Xf=2x^2+2y^2$$

Problem 5
Let D and D' be derivations at p in $\mathbb{R}^n$, and let $c\in\mathbb{R}$. Prove that:

- the sum $D + D'$ is a derivation at p,
- the scalar multiple cD is a derivation at p.

Solution

Recall:

A derivation at $p \in \mathbb{R}^n$ is a linear map $D : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ satisfying the Leibniz rule:

$$D(fg) = f(p)Dg + g(p)Df \quad \text{for all } f, g \in C^\infty(\mathbb{R}^n).$$

1. Sum of Derivations:

Let D and D' be derivations at p. We show $D + D'$ is a derivation:

- **Linearity**: $D + D'$ is linear because sums of linear maps are linear.
- **Leibniz Rule**:

$$\begin{aligned}
(D + D')(fg) &= D(fg) + D'(fg) \\
&= \big(f(p)Dg + g(p)Df\big) + \big(f(p)D'g + g(p)D'f\big) \\
&= f(p)(Dg + D'g) + g(p)(Df + D'f) \\
&= f(p)(D + D')g + g(p)(D + D')f.
\end{aligned}$$

Thus, $D + D'$ satisfies the Leibniz rule.

2. Scalar Multiple of a Derivation:

Let $c \in \mathbb{R}$ and D be a derivation at p. We show cD is a derivation:

- **Linearity**: cD is linear because scalar multiples of linear maps are linear.
- **Leibniz Rule**:

$$\begin{aligned}
(cD)(fg) &= c \cdot D(fg) \\
&= c\big(f(p)Dg + g(p)Df\big) \\
&= f(p)(cD)g + g(p)(cD)f.
\end{aligned}$$

Thus, cD satisfies the Leibniz rule.

Conclusion:

Both $D + D'$ and cD are derivations at p.

Problem 6
Prove the theorem:

Theorem 9.11.3. *Every second-countable topological space is first-countable.*

(Recall that manifolds are often assumed to be second-countable. This problem shows that such manifolds are automatically first-countable, which is useful for constructions involving sequences or local bases.)

Solution

Proof. We proceed with a detailed proof using the definitions of second-countability and first-countability.

Definitions:

Definition (Second-countable space)**.** A topological space (X, τ) is called *second-countable* if it has a countable basis $\mathcal{B}$ for its topology. That is, there exists a countable collection $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$ of open sets such that every open set in X can be written as a union of elements of $\mathcal{B}$.

Definition (First-countable space)**.** A topological space (X, τ) is called *first-countable* if every point $x \in X$ has a countable neighborhood basis. A *neighborhood basis* at x is a collection $\mathcal{N}_x$ of open neighborhoods of x such that every open set containing x contains some element of $\mathcal{N}_x$.

Proof:

Let X be a second-countable space with a countable basis $\mathcal{B} = \{B_n\}_{n\in\mathbb{N}}$. We will show that X is first-countable by constructing a countable neighborhood basis at each point.

1. Fix an arbitrary point $x \in X$.
2. Define the collection:
$$\mathcal{B}_x = \{B \in \mathcal{B} \mid x \in B\}.$$
   - Since $\mathcal{B}$ is countable, $\mathcal{B}_x$ is also countable (being a subset of a countable set).
   - Each $B \in \mathcal{B}_x$ is an open set containing x, so $\mathcal{B}_x$ is a collection of open neighborhoods of x.

3. We verify that $\mathcal{B}_x$ is a neighborhood basis at x:

    - Let U be an arbitrary open set containing x.
    - Since $\mathcal{B}$ is a basis, there exists some $B \in \mathcal{B}$ such that:

    $$x \in B \subseteq U.$$

    - By construction, this B belongs to $\mathcal{B}_x$.

4. Thus, for every open U containing x, there exists $B \in \mathcal{B}_x$ such that $B \subseteq U$.

5. Therefore, $\mathcal{B}_x$ is a countable neighborhood basis at x.

Since x was arbitrary, every point in X has a countable neighborhood basis. Hence, X is first-countable. $\square$