

BACKGROUND INFORMATION

If for each value $\alpha \in E \subset R$ the function $f(x; \alpha)$ is integrable by Riemann as a function of x on the segment $[a; b]$, then the integral

$$I(\alpha) = \int_a^b f(x; \alpha) dx \quad (1)$$

it is called a proper integral depending on the parameter α . Along with integrals of the form (1), integrals of a more general form are considered

$$\Phi(\alpha) = \int_{\varphi(\alpha)}^{\psi(\alpha)} f(x; \alpha) dx \quad (2)$$

depending on the parameter.

1. Continuity of the integral with respect to the parameter. If the function $f(x; \alpha)$ is continuous in a rectangle

$$K = \{(x; \alpha) : a \leq x \leq b, \alpha_1 \leq \alpha \leq \alpha_2\} \quad (3)$$

then the integral (1) is a continuous function of the parameter α on the segment $[\alpha_1; \alpha_2]$.

In particular, if the function $f(x; \alpha)$ is continuous in the rectangle K and $\alpha_0 \in [\alpha_1; \alpha_2]$, then

$$\lim_{\alpha \rightarrow \alpha_0} \int_a^b f(x; \alpha) dx = \int_a^b \lim_{\alpha \rightarrow \alpha_0} f(x; \alpha) dx \quad (4)$$

that is, a limit transition is possible under the sign of the integral (1).

2. Integration of integrals depending on the parameter. If the function $f(x; \alpha)$ is continuous in the rectangle (3), then the integral (1) is a function integrable on the segment $[\alpha_1; \alpha_2]$, and the equality is true

$$\int_{\alpha_1}^{\alpha_2} \left(\int_a^b f(x; \alpha) dx \right) d\alpha = \int_a^b \left(\int_{\alpha_1}^{\alpha_2} f(x; \alpha) d\alpha \right) dx \quad (5)$$

3. Differentiation of integrals depending on the parameter. If the functions are $f(x; \alpha)$ and $\frac{\partial f(x; \alpha)}{\partial \alpha}$ are continuous in rectangle (3), then integral (1) is a function continuously differentiable on the segment $[\alpha_1; \alpha_2]$, the derivative of which can be calculated by Leibniz's rule

$$I'(\alpha) = \int_a^b \frac{\partial f(x; \alpha)}{\partial \alpha} dx \quad (6)$$

If the functions are $f(x; \alpha)$ and $\frac{\partial f(x; \alpha)}{\partial \alpha}$ are continuous in rectangle (3), functions $\varphi(\alpha)$ and $\psi(\alpha)$ are differentiable on the segment $[\alpha; \alpha_2]$, and their values belong to the segment $[a; b]$, then integral (2) is a function differentiable on the segment $[\alpha_1; \alpha_2]$, and

$$\Phi'(\alpha) = f(\psi(\alpha); \alpha)\psi'(\alpha) - f(\varphi(\alpha); \alpha)\varphi'(\alpha) + \int_{\varphi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x; \alpha)}{\partial \alpha} dx \quad (7)$$

Examples

Example 1. Find $\lim_{\alpha \rightarrow 0} \int_{-\pi}^{\pi} (x + \cos \alpha x) e^{x \sin \alpha} dx$. nick

\ Since the integrand is continuous into a rectangle-

$$K = \{(x; \alpha) : -\pi \leq x \leq \pi, -1 \leq \alpha \leq 1\}$$

then the required limit A is $\int_{-\pi}^{\pi} f(x; 0) dx$, where

$$f(x; 0) = \lim_{\alpha \rightarrow 0} (x + \cos \alpha x) e^{x \sin \alpha} = x + 1$$

Therefore,

$$A = \int_{-\pi}^{\pi} (x + 1) dx = 2\pi$$

Example 2. Calculate the integral

$$I = \int_0^1 \frac{x^{\alpha_2} - x^{\alpha_1}}{\ln x} dx, \quad 0 < \alpha_1 \leq \alpha_2$$

Δ Consider the function $f(x; \alpha) = x^\alpha$. This function is continuous in the rectangle $K = \{(x; \alpha) : 0 \leq x \leq 1, \alpha_1 \leq \alpha \leq \alpha_2\}$, where $\alpha_1 > 0$. Applying formula (5), we obtain

$$\int_{\alpha_1}^{\alpha_2} \left(\int_0^1 x^\alpha dx \right) d\alpha = \int_0^1 \left(\int_{\alpha_1}^{\alpha_2} x^\alpha d\alpha \right) dx \quad (8)$$

Since $\int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$, then the left part of formula (8) is equal to $\int_{\alpha_1}^{\alpha_2} \frac{d\alpha}{\alpha+1} = \ln \frac{1+\alpha_2}{1+\alpha_1}$. The right part of formula (8) is equal to I , since

Therefore,

$$\int_{\alpha_1}^{\alpha_2} x^\alpha d\alpha = \frac{x^{\alpha_2} - x^{\alpha_1}}{\ln x}$$

$$I = \ln \frac{1 + \alpha_2}{1 + \alpha_1}$$

Example 3. Find $I'(\alpha)$ if $I(\alpha) = \int_1^2 e^{\alpha x^2} \frac{dx}{x}$.

\ Applying formula (6), we get

$$I'(\alpha) = \int_1^2 e^{\alpha x^2} x dx = \frac{e^{\alpha x^2}}{2\alpha} \Big|_1^2 = \frac{e^{4\alpha} - e^{\alpha}}{2\alpha}$$

Example 4. Calculate the integral

$$I(\alpha) = \int_0^{\pi/2} \ln(\sin^2 x + \alpha^2 \cos^2 x) dx, \quad \alpha \neq 0$$

Δ Let $\alpha > 0$ and $\alpha \neq 1$. Since the function

$$f(x; \alpha) = \ln(\sin^2 x + \alpha^2 \cos^2 x)$$

is continuous and has a continuous derivative $\frac{\partial f(x; \alpha)}{\partial \alpha}$ in a rectangle

$$K = \{(x; \alpha) : 0 \leq x \leq \pi/2, \alpha_1 \leq \alpha \leq \alpha_2\}$$

where $\alpha_1 > 0$, then by formula (6) we get

$$I'(\alpha) = \int_0^{\pi/2} \frac{2\alpha \cos^2 x}{\sin^2 x + \alpha^2 \cos^2 x} dx$$

Using the substitution $t = \operatorname{tg} x$, we find

$$\begin{aligned} I'(\alpha) &= 2\alpha \int_0^{+\infty} \frac{dt}{(t^2 + 1)(t^2 + \alpha^2)} = \frac{2\alpha}{\alpha^2 - 1} \int_0^{+\infty} \left(\frac{1}{t^2 + 1} - \frac{1}{t^2 + \alpha^2} \right) dt \\ &= \frac{2\alpha}{\alpha^2 - 1} \left(\operatorname{arctg} t - \frac{1}{\alpha} \operatorname{arctg} \frac{t}{\alpha} \right) \Big|_0^{+\infty} = \frac{\pi}{\alpha + 1} \end{aligned}$$

where from

$$I(\alpha) = \pi \ln(\alpha + 1) + C$$

Since $I(\alpha)$ is a continuous function for $\alpha > 0$, and $I(1) = 0$, then $C = -\pi \ln 2$. Hence, $I(\alpha) = \pi \ln((\alpha + 1)/2)$ when $\alpha > 0$. Given that $I(\alpha)$ is an even function, from here we get $I(\alpha) = \pi \ln((|\alpha| + 1)/2)$ if $\alpha \neq 0$.

Example 5. Find $\Phi'(\alpha)$, if $\Phi(\alpha) = \int_{\cos \alpha}^{\sin \alpha} \operatorname{sh} \alpha x^2 dx$.

\By the formula (7) we find

$$\Phi'(\alpha) = \cos \alpha \cdot \operatorname{sh}(\alpha \sin^2 \alpha) + \sin \alpha \cdot \operatorname{sh}(\alpha \cos^2 \alpha) + \int_{\cos \alpha}^{\sin \alpha} x^2 \operatorname{ch} \alpha x^2 dx$$

Tasks

2. Find the limit:

$$1. \lim_{\alpha \rightarrow 0} \int_0^1 \sqrt{1 + \alpha^2 x^4} dx;$$

$$2. \lim_{\alpha \rightarrow 0} \int_{-1}^1 \sqrt{x^2 + \alpha^2} dx;$$

$$3. \lim_{\alpha \rightarrow 1} \int_2^4 \frac{x dx}{1 + x^2 + \alpha^6};$$

$$4. \lim_{\alpha \rightarrow 1} \int_0^1 x^2 e^{\alpha x^3} dx;$$

$$5. \lim_{\alpha \rightarrow 0} \int_0^\pi x \cos(1 + \alpha)x dx$$

13. Calculate $I'(\alpha)$, if:

$$1. I(\alpha) = \int_0^1 \sin(\alpha x) dx;$$

$$2. I(\alpha) = \int_1^3 \frac{\cos(\alpha x^3)}{x} dx;$$

$$3. I(\alpha) = \int_1^2 e^{\alpha x^2} \frac{dx}{x};$$

$$4. I(\alpha) = \int_2^3 \operatorname{ch}(\alpha^4 x^2) \frac{dx}{x}.$$

14. Calculate $\Phi'(\alpha)$, If:

$$1. \Phi(\alpha) = \int_0^\alpha \frac{\ln(1 + \alpha x)}{x} dx;$$

$$2. \Phi(\alpha) = \int_\alpha^{2\alpha} \frac{\sin \alpha x}{x} dx$$

$$3. \Phi(\alpha) = \int_{\sin \alpha}^{\cos \alpha} e^{\alpha \sqrt{1 - x^2}} dx;$$

$$4. \Phi(\alpha) = \int_{3\alpha}^{\alpha^2} e^{\alpha x^2} dx;$$

$$5. \Phi(\alpha) = \int_{\cos \alpha}^{\sin \alpha} e^{\alpha^4 x^2} dx;$$

$$6. \Phi(\alpha) = \int_{e^{-\alpha}}^{e^{\alpha}} \ln(1 + \alpha^2 x^2) \frac{dx}{x};$$

$$7. \Phi(\alpha) = \int_{\alpha e^{-\alpha}}^{\alpha e^{\alpha}} \ln(1 + \alpha^2 x^2) dx$$

$$8. \Phi(\alpha) = \int_{\operatorname{ch} \alpha}^{\operatorname{sh} \alpha} \ln(1 + x^2 + \alpha^2) dx.$$