BACKGROUND INFORMATION

If for each value $\alpha \in E \subset R$ the function $f(x;\alpha)$ is integrable by Riemann as a function of x on the segment $[a;b]$, then the integral

$$I(\alpha) = \int_a^b f(x;\alpha)dx \tag{1}$$

it is called a proper integral depending on the parameter α. Along with integrals of the form (1), integrals of a more general form are considered

$$\Phi(\alpha) = \int_{\varphi(\alpha)}^{\psi(\alpha)} f(x;\alpha)dx \tag{2}$$

depending on the parameter.

1. Continuity of the integral with respect to the parameter. If the function $f(x;\alpha)$ is continuous in a rectangle

$$K = \{(x;\alpha) : a \leqslant x \leqslant b, \alpha_1 \leqslant \alpha \leqslant \alpha_2\} \tag{3}$$

then the integral (1) is a continuous function of the parameter α on the segment $[\alpha_1;\alpha_2]$.

In particular, if the function $f(x;\alpha)$ is continuous in the rectangle K and $\alpha_0 \in [\alpha_1;\alpha_2]$, then

$$\lim_{\alpha\to\alpha_0} \int_a^b f(x;\alpha)dx = \int_a^b \lim_{\alpha\to\alpha_0} f(x;\alpha)dx \tag{4}$$

that is, a limit transition is possible under the sign of the integral (1).

2. Integration of integrals depending on the parameter. If the function $f(x;\alpha)$ is continuous in the rectangle (3), then the integral (1) is a function integrable on the segment $[\alpha_1;\alpha_2]$, and the equality is true

$$\int_{\alpha_1}^{\alpha_2} \left(\int_a^b f(x;\alpha)dx\right) d\alpha = \int_a^b \left(\int_{\alpha_1}^{\alpha_2} f(x;\alpha)d\alpha\right) dx \tag{5}$$

3. Differentiation of integrals depending on the parameter. If the functions are $f(x;\alpha)$ and $\frac{\partial f(x;\alpha)}{\partial\alpha}$ are continuous in rectangle (3), then integral (1) is a function continuously differentiable on the segment $[\alpha_1;\alpha_2]$, the derivative of which can be calculated by Leibniz's rule

$$I'(\alpha) = \int_a^b \frac{\partial f(x;\alpha)}{\partial\alpha} dx \tag{6}$$

If the functions are $f(x;\alpha)$ and $\frac{\partial f(x;\alpha)}{\partial\alpha}$ are continuous in rectangle (3), functions $\varphi(\alpha)$ and $\psi(\alpha)$ are differentiable on the segment $[\alpha;\alpha_2]$, and their values belong to the segment $[a;b]$, then integral (2) is a function differentiable on the segment $[\alpha_1;\alpha_2]$, and

$$\Phi'(\alpha) = f(\psi(\alpha);\alpha)\psi'(\alpha) - f(\varphi(\alpha);\alpha)\varphi'(\alpha) + \int_{\varphi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x;\alpha)}{\partial\alpha} dx \qquad (7)$$

Examples

Example 1. Find $\lim_{\alpha\to 0} \int_{-\pi}^{\pi} (x+\cos\alpha x)e^{x\sin\alpha}dx$. nick
\Since the integrand is continuous into a rectangle-

$$K = \{(x;\alpha) : -\pi \leqslant x \leqslant \pi, -1 \leqslant \alpha \leqslant 1\}$$

then the required limit A is $\int_{-\pi}^{\pi} f(x;0)dx$, where

$$f(x;0) = \lim_{\alpha\to 0}(x+\cos\alpha x)e^{x\sin\alpha} = x+1$$

Therefore,

$$A = \int_{-\pi}^{\pi}(x+1)dx = 2\pi$$

Example 2. Calculate the integral

$$I = \int_0^1 \frac{x^{\alpha_2}-x^{\alpha_1}}{\ln x}dx, \quad 0<\alpha_1 \leqslant \alpha_2$$

Δ Consider the function $f(x;\alpha) = x^\alpha$. This function is continuous in the rectangle $K = \{(x;\alpha) : 0 \leqslant x \leqslant 1, \alpha_1 \leqslant \alpha \leqslant \alpha_2\}$, where $\alpha_1 > 0$. Applying formula (5), we obtain

$$\int_{\alpha_1}^{\alpha_2}\left(\int_0^1 x^\alpha dx\right)d\alpha = \int_0^1\left(\int_{\alpha_1}^{\alpha_2} x^\alpha d\alpha\right)dx \qquad (8)$$

Since $\int_0^1 x^\alpha dx = \frac{1}{\alpha+1}$, then the left part of formula (8) is equal to $\int_{\alpha_1}^{\alpha_2}\frac{d\alpha}{\alpha+1} =$ $= \ln\frac{1+\alpha_2}{1+\alpha_1}$. The right part of formula (8) is equal to I, since
Therefore,

$$\int_{\alpha_1}^{\alpha_2} x^\alpha d\alpha = \frac{x^{\alpha_2}-x^{\alpha_1}}{\ln x}$$

$$I = \ln\frac{1+\alpha_2}{1+\alpha_1}$$

Example 3. Find $I'(\alpha)$ if $I(\alpha) = \int_1^2 e^{\alpha x^2}\frac{dx}{x}$.
\Applying formula (6), we get

$$I'(\alpha)=\int_1^2 e^{\alpha x^2} x d x=\left.\frac{e^{\alpha x^2}}{2 \alpha}\right|_1^2=\frac{e^{4 \alpha}-e^\alpha}{2 \alpha}$$

Example 4. Calculate the integral

$$I(\alpha)=\int_0^{\pi / 2} \ln \left(\sin ^2 x+\alpha^2 \cos ^2 x\right) d x, \quad \alpha \neq 0$$

Δ Let $\alpha>0$ and $\alpha \neq 1$. Since the function

$$f(x ; \alpha)=\ln \left(\sin ^2 x+\alpha^2 \cos ^2 x\right)$$

is continuous and has a continuous derivative $\frac{\partial f(x ; \alpha)}{\partial \alpha}$ in a rectangle

$$K=\left\{(x ; \alpha): 0 \leqslant x \leqslant \pi / 2, \alpha_1 \leqslant \alpha \leqslant \alpha_2\right\}$$

where $\alpha_1>0$, then by formula (6) we get

$$I'(\alpha)=\int_0^{\pi / 2} \frac{2 \alpha \cos ^2 x}{\sin ^2 x+\alpha^2 \cos ^2 x} d x$$

Using the substitution $t=\operatorname{tg} x$, we find

$$\begin{aligned}
I'(\alpha)=2 \alpha \int_0^{+\infty} \frac{d t}{\left(t^2+1\right)\left(t^2+\alpha^2\right)} & =\frac{2 \alpha}{\alpha^2-1} \int_0^{+\infty}\left(\frac{1}{t^2+1}-\frac{1}{t^2+\alpha^2}\right) d t \\
& =\left.\frac{2 \alpha}{\alpha^2-1}\left(\operatorname{arctg} t-\frac{1}{\alpha} \operatorname{arctg} \frac{t}{\alpha}\right)\right|_0^{+\infty}=\frac{\pi}{\alpha+1}
\end{aligned}$$

where from

$$I(\alpha)=\pi \ln (\alpha+1)+C$$

Since $I(\alpha)$ is a continuous function for $\alpha>0$, and $I(1)=0$, then $C==-\pi \ln 2$. Hence, $I(\alpha)=\pi \ln ((\alpha+1) / 2)$ when $\alpha>0$. Given that $I(\alpha)$ is an even function, from here we get $I(\alpha)=\pi \ln ((|\alpha|+1) / 2)$ if $\alpha \neq 0$.

Example 5. Find $\Phi'(\alpha)$, if $\Phi(\alpha)=\int_{\cos \alpha}^{\sin \alpha} \operatorname{sh} \alpha x^2 d x$.

\By the formula (7) we find

$$\Phi'(\alpha)=\cos \alpha \cdot \operatorname{sh}\left(\alpha \sin ^2 \alpha\right)+\sin \alpha \cdot \operatorname{sh}\left(\alpha \cos ^2 \alpha\right)+\int_{\cos \alpha}^{\sin \alpha} x^2 \operatorname{ch} \alpha x^2 d x$$

Tasks

2. Find the limit:

1. $\lim_{\alpha\to 0}\int_0^1 \sqrt{1+\alpha^2x^4}dx;$

2. $\lim_{\alpha\to 0}\int_{-1}^1 \sqrt{x^2+\alpha^2}dx;$

3. $\lim_{\alpha\to 1}\int_2^4 \frac{xdx}{1+x^2+\alpha^6};$

4. $\lim_{\alpha\to 1}\int_0^1 x^2e^{\alpha x^3}dx;$

5. $\lim_{\alpha\to 0}\int_0^\pi x\cos(1+\alpha)xdx$

13. Calculate $I'(\alpha)$, if:

1. $I(\alpha)=\int_0^1 \sin(\alpha x)dx;$

2. $I(\alpha)=\int_1^3 \frac{\cos(\alpha x^3)}{x}dx;$

3. $I(\alpha)=\int_1^2 e^{\alpha x^2}\frac{dx}{x};$

4. $I(\alpha)=\int_2^3 \operatorname{ch}\left(\alpha^4x^2\right)\frac{dx}{x}.$

14. Calculate $\Phi'(\alpha)$, If:

1. $\Phi(\alpha)=\int_0^\alpha \frac{\ln(1+\alpha x)}{x}dx;$

2. $\Phi(\alpha)=\int_\alpha^{2\alpha} \frac{\sin\alpha x}{x}dx$

3. $\Phi(\alpha)=\int_{\sin\alpha}^{\cos\alpha} e^{\alpha\sqrt{1-x^2}}dx;$

4. $\Phi(\alpha)=\int_{3\alpha}^{\alpha^2} e^{\alpha x^2}dx;$

5. $\Phi(\alpha)=\int_{\cos\alpha}^{\sin\alpha} e^{\alpha^4x^2}dx;$

6. $\Phi(\alpha)=\int_{e^{-\alpha}}^{e^\alpha} \ln\left(1+\alpha^2x^2\right)\frac{dx}{x};$

7. $\Phi(\alpha)=\int_{\alpha e^{-\alpha}}^{\alpha e^\alpha} \ln\left(1+\alpha^2x^2\right)dx$

8. $\Phi(\alpha)=\int_{\operatorname{ch}\alpha}^{\operatorname{sh}\alpha} \ln\left(1+x^2+\alpha^2\right)dx.$

HW19.

1. $\lim_{\alpha\to 0}\int_0^1 \sqrt{1+\alpha^2x^4}dx$;

$$= \int_0^1 \lim_{\alpha\to 0} \sqrt{1+\alpha^2x^4}\, dx$$

$$= \int_0^1 \lim_{\alpha\to 0} 1+\frac{1}{2}\alpha^2x^4\, dx.$$

$$= \int_0^1 1\, dx = 1.$$

3. $\lim_{\alpha\to 1}\int_2^4 \frac{xdx}{1+x^2+\alpha^6}$;

$$= \frac{1}{2}\lim_{\alpha\to 1}\int_2^4 \frac{d(x^2)}{1+x^2+\alpha^6}$$

$$= \frac{1}{2}\lim_{\alpha\to 1} \ln|1+x^2+\alpha^6|\Big|_2^4.$$

$$= \frac{1}{2}\lim_{\alpha\to 1} \ln\left|\frac{17+\alpha^6}{5+\alpha^6}\right| = \frac{1}{2}\ln 3.$$

5. $\lim_{\alpha\to 0}\int_0^\pi x\cos(1+\alpha)xdx$

$$\int x\cos(1+\alpha)x\,dx = \frac{x\sin(1+\alpha)x}{1+\alpha} - \int\frac{\sin(1+\alpha)x}{1+\alpha}dx.$$

$$= \frac{x\sin(1+\alpha)x}{1+\alpha} + \frac{\cos(1+\alpha)x}{(1+\alpha)^2} + C$$

$$\lim_{\alpha\to 0}\int_0^\pi x\cos(1+\alpha)x\,dx = \lim_{\alpha\to 0}\frac{\cos(1+\alpha)\pi}{(1+\alpha)^2} - \frac{1}{(1+\alpha)^2} = -2$$

1. $I(\alpha)=\int_0^1 \sin(\alpha x)dx$;

$$I'(\alpha) = \int_0^1 x\cos\alpha x\, dx = \int_0^1 \frac{x\sin\alpha x}{\alpha} + \frac{\cos\alpha x}{\alpha^2}\, dx$$

$$= \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2}$$

3. $I(\alpha)=\int_1^2 e^{\alpha x^2}\frac{dx}{x}$;

$$I'(\alpha) = \int_1^2 \frac{1}{x}\cdot x^2\cdot e^{\alpha x^2}dx = \frac{1}{2\alpha}\int_1^2 e^{\alpha x^2} d(\alpha x^2).$$

$$= \frac{1}{2\alpha}\left(e^{4\alpha} - e^{2\alpha}\right).$$