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Analytic Geometry. Coordinate System

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Cartesian coordinate system I

- Previously we expanded a vector with a basis containing pair (on plane) or triplet (in space) of arbitrary vectors satisfying some conditions
- These vectors represent two (or three) axes governing two (three) **main directions**. These axes called **coordinate axes**.
- We call pair or triplet of numbers introduced with previous statements the **coordinates** if vector with respect to the **basis**
- It is natural to represent our basis with corresponding set of **unit** vectors and talk about coordinates with respect to these unit vectors
- This approach brought us non leak of generalization as we may represent any vector shaping the axis as product of the corresponding unit vector and arbitrary real number
- To make our reasoning more compact and clear we also will as usual demand these unit vectors to be orthogonal one to each other
- We call such basis **an orthonormal basis**

Cartesian coordinate system II

- Demand to the basis vectors to be right triplet is also usual condition. We call such coordinate system **Cartesian coordinate system**
- For proper operations in coordinate system we determine these basis vectors, and their common initial point called **origin**
- On plane we utilize as standard Cartesian system system with counterclockwise rotation from first coordinate to second
- We call **quadrant** each area of planar Cartesian coordinate system separated by two perpendicular half-axes
- Quadrants enumerated in counterclockwise order:

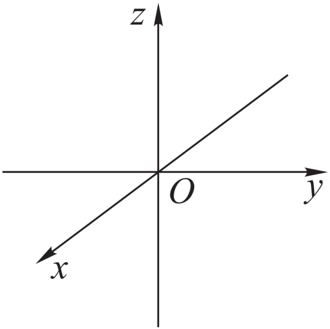
Sign of first coordinate	Sign of second coordinate	Quadrant
$+$	$+$	I
$-$	$+$	II
$-$	$-$	III
$+$	$-$	IV

- Each quadrant resembles quoter-plane

Cartesian coordinate system III

- Similar notation also introduced for Cartesian coordinates on space
- Each pair of axes shapes a plane. Totally three of them: xOy, xOz, and yOz
- Each of these three planes has Cartesian coordinate system corresponding with space system.
- We call **octant** zone in space separated with three pairwise perpendicular quadrants
- To address octans we utilize triplet of signs of corresponding half-axes: $(+ + +)$, $(+ - +)$
- Same addresses may be used for quadrants too of corresponding planar coordinate systems

Cartesian coordinate system IV



- Usual names of coordinate axes are x, y, z (sometimes Ox, Oy, Oz, or x_1, x_2, x_3, or x^1, x^2, x^3)
- Usual names of unit vectors directing these axes are $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ or $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$
- $\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{j} \cdot \boldsymbol{k} = 0$
- $\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$
- $\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = \boldsymbol{0}$
- Suppose that basis is right one
- $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$; $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$; $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$
- $\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$; $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$; $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$

- Order of coordinates may be changed while orientation is preserved
- Arbitrary vector may be expanded with respect to this basis:
  $\boldsymbol{v} = v_x\boldsymbol{i} + v_y\boldsymbol{j} + v_z\boldsymbol{k}$
  or
  $\boldsymbol{v} = v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2 + v^3\boldsymbol{e}_3$

Cartesian coordinate system V

- As our basis is fixed one, we may write for each vector just its coordinates: (v_x, v_y, v_z) or (v^1, v^2, v^3)
- Understanding of the coordinates (and vector itself) as $1 \times n$ (row) or $n \times 1$ (column) matrix, $n = 2$ or $n = 3$ appears more accurate for utilization of algebraic methods to study vectors:

$$\begin{aligned} \boldsymbol{a} &= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \\ \boldsymbol{b} &= \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} \end{aligned}$$

- Utilization of upper indices has wide application in many fields of study and allows easily operate with numerated objects
- We will use this notation only if it is necessary
- Now we can rewrite all product operations in terms of Cartesian coordinates

Dot Product and Coordinate System

- Consider orthonormal basis. Dot product for vectors expanded with respect to it has form for planar case

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y$$

  and for space case

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$

  Here a_* and b_* are coordinates of vector in given basis
- Proof of planar case:
  - Let $\boldsymbol{i}$, and $\boldsymbol{j}$ be these unit vectors
  - $|\boldsymbol{i}|^2 = \boldsymbol{i}^2 = |\boldsymbol{j}|^2 = \boldsymbol{j}^2 = 1$, $\boldsymbol{i} \cdot \boldsymbol{j} = 0$
  - $\boldsymbol{a} \cdot \boldsymbol{b} = (a_x \boldsymbol{i} + a_y \boldsymbol{j}) \cdot (b_x \boldsymbol{i} + b_y \boldsymbol{j}) = a_x b_x \boldsymbol{i}^2 + a_x b_y \boldsymbol{i} \cdot \boldsymbol{j} + a_y b_x \boldsymbol{i} \cdot \boldsymbol{j} + a_y b_y \boldsymbol{i}^2 = a_x b_x + a_y b_y$ $\square$
- Reproduce this proof for space case as **home assignment**

Dot Product. Matrix Form

- Now we write dot product in more strict form
- Suppose coordinates

$$\begin{aligned} \boldsymbol{a} &= \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \\ \boldsymbol{b} &= \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} \end{aligned}$$

- Now we can write our dot product as a matrix expression:

$$\boldsymbol{a} \cdot \boldsymbol{b} \mapsto \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} = \sum_{i=1}^{3} a_i b^i = a_i b^i$$

- Final writing is called **Einstein notation**: we overlook summation symbol if each term formed with upper and lover index and limit of summation is known or obvious
- $\boldsymbol{a}^2 = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}^T$

Cross Product and Coordinate System

- Consider right coordinate system in space
- $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$; $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$; $\boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}$
- $\boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$; $\boldsymbol{k} \times \boldsymbol{j} = -\boldsymbol{i}$; $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$
- We calculate cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ with coordinates (a_x, a_y, a_z) and (b_x, b_y, b_z)
- $\boldsymbol{a} \times \boldsymbol{b} = (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) \times (b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k})$
- Expand the expression
- $$\boldsymbol{a} \times \boldsymbol{b} =$$
  $$\begin{aligned} = \quad & a_x b_x \boldsymbol{i} \times \boldsymbol{i} + a_x b_y \boldsymbol{i} \times \boldsymbol{j} + a_x b_z \boldsymbol{i} \times \boldsymbol{k} \quad + \\ + \quad & a_y b_x \boldsymbol{j} \times \boldsymbol{i} + a_y b_y \boldsymbol{j} \times \boldsymbol{j} + a_y b_z \boldsymbol{j} \times \boldsymbol{k} \quad + \\ + \quad & a_z b_x \boldsymbol{k} \times \boldsymbol{i} + a_z b_y \boldsymbol{k} \times \boldsymbol{j} + a_z b_z \boldsymbol{k} \times \boldsymbol{k} \end{aligned}$$
- $\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - b_y a_z)\boldsymbol{i} + (a_z b_x - b_z a_x)\boldsymbol{j} + (a_x b_y - b_x a_y)\boldsymbol{k}$

Cross Product. Strict Form. Levi-Civita Symbols I

- Suppose we have arbitrary set of enumerated objects. We may replace these objects with their indices for more compact explanation
- We call **permutation** any ordered combination of these objects (or their indices)
- We say that permutation is **even** if it takes the even number of pairwise transpositions to transform natural order into this permutation
- For tree elements permutations (1, 2, 3), (2, 3, 1), (3, 1, 2) are even
- Res of permutations we call **odd**
- We call **Levi-Civita symbol** element of array (27 elements) formed by following rule:

$$\varepsilon^{ijk} = \varepsilon_{ijk} = \begin{cases} 1 & (i, j, k) \text{ is even permutation of } (1,\ 2,\ 3) \\ -1 & (i, j, k) \text{ is odd permutation of } (1,\ 2,\ 3) \\ 0 & \text{there are coinciding values of the indices} \end{cases}$$

Cross Product. Strict Form. Levi-Civita Symbols II

- We rewrite product of two basis unit vectors first:

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = C_{ij}^1 \boldsymbol{e}_1 + C_{ij}^2 \boldsymbol{e}_2 + C_{ij}^2 \boldsymbol{e}_2 = \sum_{k=1}^{3} C_{ij}^k \boldsymbol{e}_k$$

- As we remember our definition for orthonormal right basis, these coefficients C_{ij}^k are exactly Levi-Civita symbols
- $C_{ij}^k = \varepsilon_{ijk}$

Cross Product. Strict Form. Levi-Civita Symbols III

- Now we rearrange our formula:

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \sum_{k=1}^{3} C_{ij}^k \boldsymbol{e}_k$$

-

$$\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} = \\
= a^1 b^1 \boldsymbol{e}_1 \times \boldsymbol{e}_1 + a^1 b^2 \boldsymbol{e}_1 \times \boldsymbol{e}_2 + a^1 b^3 \boldsymbol{e}_1 \times \boldsymbol{e}_3 \quad + \\
+ a^3 b^1 \boldsymbol{e}_2 \times \boldsymbol{e}_1 + a^2 b^2 \boldsymbol{e}_2 \times \boldsymbol{e}_2 + a^2 b^3 \boldsymbol{e}_2 \times \boldsymbol{e}_3 \quad + \\
+ a^3 b^1 \boldsymbol{e}_3 \times \boldsymbol{e}_1 + a^3 b^2 \boldsymbol{e}_3 \times \boldsymbol{e}_2 + a^3 b^3 \boldsymbol{e}_3 \times \boldsymbol{e}_3 \quad = \\
= \sum_{i=1}^{3} \sum_{j=1}^{3} a^i b^j (\boldsymbol{e}_i \times \boldsymbol{e}_j) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a^i b^j C_{ij}^k \boldsymbol{e}_k = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a^i b^j \varepsilon_{ijk} \boldsymbol{e}_k
\end{aligned}$$

Cross Product. Strict Form. Determinants

$$
\begin{aligned}
\boldsymbol{a} \times \boldsymbol{b} &= (a_y b_z - b_y a_z)\boldsymbol{i} + (a_z b_x - b_z a_x)\boldsymbol{j} + (a_x b_y - b_x a_y)\boldsymbol{k} = \\
&= \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} a_x & a_y \\ b_y & b_y \end{vmatrix} \boldsymbol{k} = \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}
\end{aligned}
$$

Mixed Product and Coordinate System

- $(\boldsymbol{abc}) = \boldsymbol{a}(\boldsymbol{b} \times \boldsymbol{c})$. Let (a_x, a_y, a_z), (b_x, b_y, b_z) and (c_x, c_y, c_z) be their coordinates
- Now we just combine approaches:
- $\boldsymbol{a} \cdot \boldsymbol{p} = a_x p_x + a_y p_y + a_z p_z$ (here $\boldsymbol{p}$ is arbitrary vector, $\boldsymbol{b} \times \boldsymbol{c}$ as particular case)
- Coordinates of $\boldsymbol{b} \times \boldsymbol{c}$ are $(b_y c_z - c_y b_z, b_z c_x - c_z b_x, b_x c_y - c_x b_y)$
- $(\boldsymbol{abc}) = a_x(b_y c_z - c_y b_z) + a_y(b_z c_x - c_z b_x) + a_z(b_x c_y - c_x b_y)$
- This expression is just one of possible approaches to write **determinant** of the matrix

$$(\boldsymbol{abc}) = \det \begin{pmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{pmatrix} = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

- In terms of Levi-Civita symbols:

$$(\boldsymbol{abc}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} a^i b^j c^k \varepsilon_{ijk}$$

Coordinates of Point and Directed Segment I

- Suppose, there is arbitrary point A and coordinate system with origin O different from that point
- There is one and only one segment OA connecting these points
- Suppose, we assign to it direction $\overrightarrow{OA} = \boldsymbol{a}$
- Uniqueness of that segment establishes the fact that coordinates of corresponding vector describe point A explicitly
- Thus, we explicitly assign coordinates of vector $\boldsymbol{a}$ to point A and call that vector
- This may be explained as displacement of origin into locus of point A **radius vector** of point A

Coordinates of Point and Directed Segment II

- Now we add distant from A and O point B with radius vector $\boldsymbol{b}$.
- Now we are left to wonder, what are coordinates of $\overrightarrow{AB}$?
- We commit agreement that coordinates of any directed segments are coordinates of corresponding abstract vector
- $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \boldsymbol{b} - \boldsymbol{a}$
- In terms of coordinates:

$$\boldsymbol{b} - \boldsymbol{a} \mapsto \begin{pmatrix} b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \end{pmatrix}$$

Building the Point

Let us revisit procedure of restoration of coordinates of point (radius vector) or to building the point (radius vector) for given coordinates

- Point M to coordinates
  1. Establish segment parallel with third axis with POI and pint M_alpha on plane α shaped with two other coordinates as endpoints
  2. Establish perpendicular to third coordinate axis MM_3. Length of OM_3 ("height") is modulus of third coordinate. Sign we select with respect if M_3 lies on positive or negative half-axis
  3. For point M_alpha we build first and second coordinates in plane: we establish perpendiculars $M_\alpha M_1$ and $M_\alpha M_2$ from point M_α to corresponding coordinate axes. Lengths of M_1 and M_2 are moduli of corresponding coordinates. Sign depends on half-axes
- Coordinates to point M
  1. We start with building point M_alpha laying in plane α shaped with two first coordinate axes
  2. We establish lines parallel with coordinate axes and crossing points M_1 and M_2. M_α is exactly cross point of these lines
  3. After this we establish line perpendicular with α and crossing it in point M_α
  4. We establish on this line segment MM_α with length equal to modulus of third coordinate. Half-space is selected with respect to sign of this coordinate

Operations with Vectors in Coordinates. Problems Corner I

Problem 1 Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have coordinates $(-10, 20)$ and $(17, 46)$ with respect to orthonormal basis on plane.

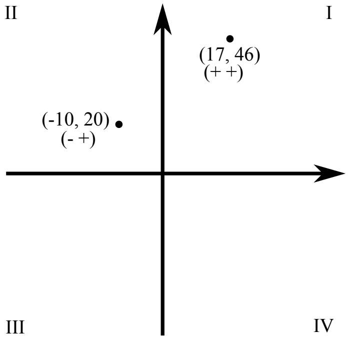
- (1) Plot these points. Name quadrants points lay in. Write addresses of these quadrants.
- (2) Find their dot product. Find angle between vectors
- (3) Suppose that in space third coordinate of these vectors is 0 and basis is right. Find length of their cross product. How it is directed?

We discuss each point (1) (2) (3) separately
Now point (1)

Operations with Vectors in Coordinates. Problems Corner II

Two vectors ***a*** and ***b*** have coordinates $(-10, 20)$ and $(17, 46)$ with respect to orthonormal basis on plane.

- (1) Plot these points. Name quadrants points lay in. Write addresses of these quadrants.



Operations with Vectors in Coordinates. Problems Corner III

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have coordinates $(-10, 20)$ and $(17, 46)$ with respect to orthonormal basis on plane.

- (2) Find their dot product. Find angle between vectors

As we have proven, dot product of these vectors is just sum of product of corresponding coordinates
$\boldsymbol{ab} = -10 \cdot 17 + 20 \cdot 46 = 750$
Length of vector is exactly square root or it's dot product by itself
$|\boldsymbol{a}| = \sqrt{(-10)^2 + 20^2} = \sqrt{500} = 10\sqrt{5} \approx 22.4$
$|\boldsymbol{b}| = \sqrt{17^2 + 46^4} = \sqrt{2405} \approx 49$
$\cos\angle(\boldsymbol{a}, \boldsymbol{b}) = \dfrac{\boldsymbol{ab}}{|\boldsymbol{a}||\boldsymbol{b}|}$
$\cos\angle(\boldsymbol{a}, \boldsymbol{b}) = \dfrac{750}{22.4 \cdot 49} \approx 0.68$
$\angle(\boldsymbol{a}, \boldsymbol{b}) \approx 47^\circ$

Operations with Vectors in Coordinates. Problems Corner IV

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have coordinates $(-10, 20)$ and $(17, 46)$ with respect to orthonormal basis on plane.

- (3) Suppose that in space third coordinate of these vectors is 0 and basis is right. Find length of their cross product. How it is directed?

Operations with Vectors in Coordinates. Problems Corner V

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ have coordinates $(-10, 20)$ and $(17, 46)$ with respect to orthonormal basis on plane.

- (3) Suppose that in space third coordinate of these vectors is 0 and basis is right. Find length of their cross product. How it is directed?

$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - b_y a_z)\boldsymbol{i} + (a_z b_x - b_z a_x)\boldsymbol{j} + (a_x b_y - b_x a_y)\boldsymbol{k}$
Because $a_z = 0$ and $b_z = 0$, $\boldsymbol{a} \times \boldsymbol{b} = (a_x b_y - b_x a_y)\boldsymbol{k} = (-10 \cdot 46 - 20 \cdot 17)\boldsymbol{k} = -800\boldsymbol{k}$
Coordinates of the cross product are (0, 0, -800). It lies on negative half of third coordinate axis

Operations with Vectors in Coordinates. Problems Corner VI

Problem 2
There are points A B and C having coordinates with respect to arbitrary Cartesian coordinate system (1,1,1), (1,3,2), and (2,3.-7).
Find area and angles of this triangle.

Operations with Vectors in Coordinates. Problems Corner VII

There are points A B and C having coordinates with respect to arbitrary Cartesian coordinate system (1,1,1), (1,3,2), and (2,3,-7).
Find area and angles of this triangle.
We will take $\angle BAC$ as reference.
First, we obtain coordinates of abstract vectors corresponding with directed segments shaping angle:
$\boldsymbol{b} = \overrightarrow{AB} \mapsto (B_x - A_x, B_y - A_y, B_z - A_z) = (0, 2, 1)$, and
$\boldsymbol{c} = \overrightarrow{AC} \mapsto (C_x - A_x, C_y - A_y, C_z - A_z) = (1, 2, -8)$.
Now we calculate cross product and looking for its length which is exactly area of parallelogram shaped by vectors:
$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - b_y a_z)\boldsymbol{i} + (a_z b_x - b_z a_x)\boldsymbol{j} + (a_x b_y - b_x a_y)\boldsymbol{k} \mapsto (-18, 1, -2)$
$S = \frac{1}{2}|\boldsymbol{a} \times \boldsymbol{b}| = \frac{1}{2}\sqrt{(\boldsymbol{a} \times \boldsymbol{b})^2} = \frac{1}{2}\sqrt{18^2 + 1 + 2^2} \approx 9.07$

Operations with Vectors in Coordinates. Problems Corner VIII

To find cosines of the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$ we use their dot product:
$\boldsymbol{b} \cdot \boldsymbol{c} = 0 \cdot 1 + 2 \cdot 2 - 8 \cdot 1 = -4$
$\cos \angle(\boldsymbol{ab}) = \dfrac{\boldsymbol{b} \cdot \boldsymbol{c}}{|\boldsymbol{b}||\boldsymbol{c}|} \approx -0.22$ As **home assignment** find cosines for two other angels and check if area computation gives the same result if we take $\angle ABC$ or $\angle ACB$ as reference

Operations with Vectors in Coordinates. Problems Corner IX

Problem 3
Consider a triangle prism having triangle $\triangle ABC$ as base. Coordinates of these points are: (1,1,1), (1,3,2), and (2,3,-7).
Corresponding with A point of opposite base A' has coordinates (7, 5, 6). Find volume of this prism

Operations with Vectors in Coordinates. Problems Corner X

Consider a triangle prism having triangle $\triangle ABC$ as base. Coordinates of these points are: (1,1,1), (1,3,2), and (2,3,-7).
Corresponding with A point of opposite base A' has coordinates (7, 5, 6). Find volume of this prism
First, we obtain coordinates of abstract vectors corresponding with directed segments shaping angle:
$\boldsymbol{b} = \overrightarrow{AB} \mapsto (B_x - A_x, B_y - A_y, B_z - A_z) = (0, 2, 1)$,
$\boldsymbol{c} = \overrightarrow{AC} \mapsto (C_x - A_x, C_y - A_y, C_z - A_z) = (1, 2, -8)$, and
$\boldsymbol{d} = \overrightarrow{AA'} \mapsto (A'_x - A_x, A'_y - A_y, A'_z - A_z) = (6, 4, 5)$

Operations with Vectors in Coordinates. Problems Corner XI

For any prism we know that its volume is product of height and area of its base.
In other hand, area of the base triangle is exactly half of the area of corresponding parallelogram involved into cross product
Thus, out volume is exactly half the volume parallelepiped involved into calculation of mixed product of vectors:

$$V = \frac{1}{2}|(\boldsymbol{bcd})| = \frac{1}{2}\left|\begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{vmatrix}\right| = \frac{1}{2}|(\boldsymbol{bcd})| = \frac{1}{2}\left|\begin{vmatrix} 0 & 2 & 1 \\ 1 & 2 & -8 \\ 6 & 4 & 5 \end{vmatrix}\right| = \frac{1}{2}|-114| = 57$$

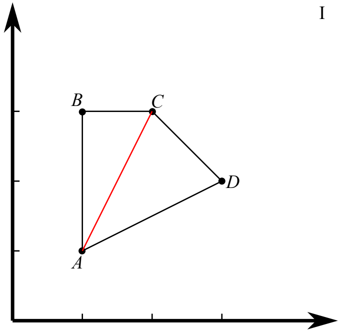
Operations with Vectors in Coordinates. Problems Corner XII

Problem 4

There is a quadrilateral $ABCD$ on plane. Coordinates of it's edges with respect to arbitrary Cartesian coordinate system are $A \mapsto (1,1)$, $B \mapsto (1,3)$, $C \mapsto (2,3)$, $D \mapsto (3,4)$.
Find cosines of the angles of this figure.

Operations with Vectors in Coordinates. Problems Corner XIII

There is a quadrilateral $ABCD$ on plane. Coordinates of it's edges with respect to arbitrary Cartesian coordinate system are $A \mapsto (1,1)$, $B \mapsto (1,3)$, $C \mapsto (2,3)$, $D \mapsto (3,4)$.
Find cosines of the angles of this figure, and area of this figure.
Here is a sketch to solution. Complete it as a home assignment. We draw our quadrilateral first.



1. For each angle we are looking for cosines in the same maner as in problem 1
2. Now we establish segment AC and look for area of $\triangle ABC$ and $\triangle ACD$. Their sum is answer.
3. Most direct approach to find these areas is calculation of the lenght of cross product of corresponding vectors
4. As we know cosines of $\angle B$ and $\angle D$, we use equation $\sin^2 \alpha = 1 - \cos^2 \alpha$ and have our answer

Trigonometry and Cartesian Coordinates I

Problem 5 There is vector $\boldsymbol{a} \mapsto (1/2, \sqrt{3}/2)$. Find angel this vector shapes with positive direction of first coordinate axis

Trigonometry and Cartesian Coordinates II

We tare unit vector governing first coordinate axis, $\boldsymbol{i} -> (1,0)$

$$|\boldsymbol{a}| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1, \; |\boldsymbol{i}| = 1$$

$$\boldsymbol{a} \cdot \boldsymbol{i} = |\boldsymbol{a}||\boldsymbol{i}| \cos \angle(\boldsymbol{a}, \boldsymbol{i}) = \cos \angle(\boldsymbol{a}, \boldsymbol{i}) == a_x \cdot 1 + a_y c\dot{0} = \frac{1}{2}$$

- For any unit vector with respect to right Cartesian coordinates we say that its first coordinate is **cosines**, and its second coordinate is **sines** of the angle shaped by this vector and first unit vector of the basis.
- This definition is more general and applicable for all angels greater than straight
- Moduli of these sines and cosines may correspond in various quadrants, bu their signs usually form unique combination
- Locus of terminal points of all unit vectors is unit circle around the origin of coordinate system

Trigonometry and Cartesian Coordinates III

- Consider Cartesian coordinate system in space with unit vectors $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, $\boldsymbol{e_3}$, and arbitrary vector $\boldsymbol{a}$ laying on an axis shaped with unit vector $\boldsymbol{e}$
- $\boldsymbol{e} \cdot \boldsymbol{e_i} = \cos\angle(\boldsymbol{e}, \boldsymbol{e_i}) = \cos\alpha_i,\ i = 1, 2, 3$
- These angles α_i define direction of the axis with unit vector $\boldsymbol{e}$
- We call them **direction cosines**
- Coordinates of the vector $\boldsymbol{e}$ are $\cos\alpha_1$, $\cos\alpha_2$, $\cos\alpha_3$
- $\boldsymbol{e}^2 = 1 = \cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3$
- $\boldsymbol{a} = |a|\boldsymbol{e} = |a|(\cos\alpha_1\boldsymbol{e_i} + \cos\alpha_2\boldsymbol{e_i} + \cos\alpha_3\boldsymbol{e_i})$

Polar Coordinates I

- This definitions give us a clue to introduce some alternatives for our Cartesian coordinate system
- Suppose we deal with plane coordinates
- Each vector may be described with its length r and angle φ it shaping with positive direction of first coordinate axis
- For $r = 0$ φ is not defined and may be any reasonable real number
- We use segment of lenght 2π on the axis of real numbers as domain for φ. Usual segments are $[0, 2\pi)$, or $(-\pi, \pi]$
- Suppose we figured out these parameters. Now Cartesian coordinates of vector are

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

Polar Coordinates II

- For Cartesian coordinates we build corresponding polar coordinates using following rules:
  - $r = \sqrt{x^2 + y^2}$
  - Suppose $\varphi_0 = \arctan \dfrac{y}{x}$
  - For segment $[0, 2\pi)$: $\varphi = \begin{cases} \varphi_0, & x > 0, y \geqslant 0 \\ \varphi_0 + 2\pi, & x > 0, y < 0 \\ \varphi_0 + \pi, & x < 0 \\ \dfrac{\pi}{2}, & x = 0, y > 0 \\ \dfrac{3\pi}{2}, & x = 0, y < 0 \\ \tilde{}, & x = 0, y = 0 \end{cases}$

Polar Coordinates III

- For segment $(-\pi, \pi]$: $\varphi = \begin{cases} \varphi_0, & x > 0 \\ \varphi_0 + \pi, & x < 0, y \geqslant 0 \\ \varphi_0 - \pi, & x < 0, y < 0 \\ \frac{\pi}{2}, & x = 0, y > 0 \\ -\frac{\pi}{2}, & x = 0, y < 0 \\ \tilde{}, & x = 0, y = 0 \end{cases}$
- It should be noted that φ is defined with both $\arctan \frac{y}{x}$ and sign of x or y
- This leads developers of software packages for geometry computations to introduce function 'atan2' calculating this polar angle by both x and y

Cylindrical Coordinates I

- This definitions give us a clue to introduce some alternatives for our Cartesian coordinate system
- Suppose we deal with plane coordinates
- Each vector may be described with its length r and angle φ it shaping with positive direction of first coordinate axis
- For $r = 0$ φ is not defined and may be any reasonable real number
- We use segment of lenght 2π on the axis of real numbers as domain for φ. Usual segments are $[0, 2\pi)$, or $(-\pi, \pi]$
- Suppose we figured out these parameters. Now Cartesian coordinates of vector are

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

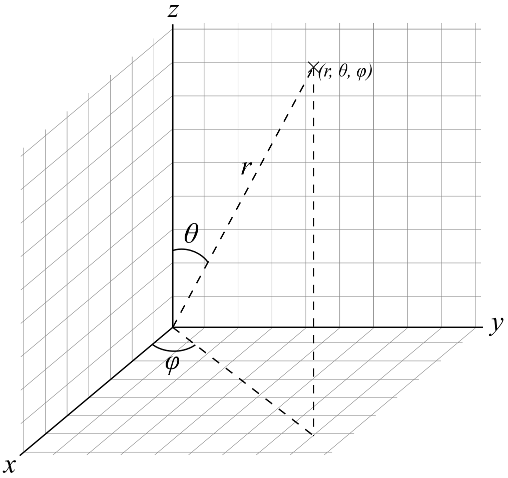
Cylindrical Coordinates II

- For Cartesian coordinates we build corresponding polar coordinates using following rules:
  - If we revisit our procedure of building point by its coordinates and vice-versa, we notice that third coordinate resembles height of the point above (or below) coordinate plane xOy
  - Thus, we can extend polar coordinates with "height' defined in the same manner
  - This height corresponds with Cartesian coordinates with rule: $h = z$

Spherical Coordinates I

- Second approach to space-generalization of polar coordinate leads to introduction of spherical Coordinates
- Suppose we figured out projection of our vector on plane xOy and find out that polar angle of this projection is φ
- Consider plane shaped with this projection and vector itself. It also contains perpendicular connecting terminal points ov vector and its projection
- Thus, third coordinate axis is contained in this plane and is perpendicular with vector projection
- Consider planar coordinates shaped on this plane with axis z and vector projection as main directions.
- We assume z as first axis of this coordinate system and build polar coordinates for it.
- We denote this second polar angle as θ
- There is no need to consider this angle greater than straight angle

Spherical Coordinates II



- $x = r \cos\varphi \sin\theta$
- $y = r \sin\varphi \sin\theta$
- $z = r \cos\theta$
- $r = \sqrt{x^2 + y^2 + z^2}$
- $\theta = \arctan \dfrac{\sqrt{x^2 + y^2}}{z}$
- φ defined as we do it for polar coordinates