SECOND-ORDER CURVES (QUADRIC CURVES)

1 On curves defined by the equations of the second degrees with two variables

Definition. *A* ***curve of the second order*** *is defined as the locus of points in the plane whose coordinates satisfy an equation of the form*

$$F(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0, \tag{1.1}$$

in which at least one of the coefficients a_{11}, a_{12}, a_{22} *is non-zero.*

Obviously, *this definition is invariant relative to the choice of the coordinate system.* We want to find a rectangular coordinate system in which the equation of the curve $F(x, y) = 0$ would take the simplest ("canonical") form.

We will assume the initial coordinate system to be rectangular (if it were not so, we would switch to a new rectangular coordinate system and thereby transform the initial polynomial $F(x, y)$ into a new polynomial, also of the second degree, from which we would begin our further considerations).

1.1 Canonical equation

Let us consider the reduction of the quadratic form of the equation with two variables to the canonical form using the transformations of the rectangular coordinate system.

The first step is to rotate the initial rectangular frame $O\mathbf{e}_1\mathbf{e}_2$ by some angle α to transform the quadratic form $\varphi(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2$ of the highest degree terms of the polynomial (1.1) to the canonical form

$$a'_{11}{x'}^2 + a'_{22}{y'}^2.$$

So, we do the following coordinate transformation

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha, \\ y &= x'\sin\alpha + y'\cos\alpha. \end{aligned} \tag{1.2}$$

We get

$$
\begin{aligned}
F(x,y) \equiv a_{11}({x'}^2 \cos^2\alpha - 2x'y'\cos\alpha\sin\alpha + {y'}^2\sin^2\alpha)+\\
+ 2a_{12}({x'}^2\cos\alpha\sin\alpha - x'y'\sin^2\alpha - {y'}^2\sin\alpha\cos\alpha + x'y'\cos^2\alpha)+\\
+ a_{22}({x'}^2\sin^2\alpha + 2x'y'\cos\alpha\sin\alpha + {y'}^2\cos^2\alpha)+\\
+ 2a_1x'\cos\alpha - 2a_1y'\sin\alpha + 2a_2x'\sin\alpha + 2a_2y'\cos\alpha + a_0 \equiv\\
\equiv F'(x',y') \equiv a'_{11}{x'}^2 + 2a'_{12}x'y' + a'_{22}{y'}^2 + 2a'_1x' + 2a'_2y' + a_0,
\end{aligned}
$$

where

$$
\begin{aligned}
a'_{11} &= a_{11}\cos^2\alpha + 2a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha,\\
a'_{12} &= -a_{11}\cos\alpha\sin\alpha + a_{12}(\cos^2\alpha - \sin^2\alpha) + a_{22}\cos\alpha\sin\alpha,\\
a'_{22} &= a_{11}\sin^2\alpha - 2a_{12}\cos\alpha\sin\alpha + a_{22}\cos^2\alpha,\\
a'_1 &= a_1\cos\alpha + a_2\sin\alpha,\\
a'_2 &= -a_1\sin\alpha + a_2\cos\alpha.
\end{aligned} \tag{1.3}
$$

Let us define the angle α by the requirement that $a'_{12} = 0$, i.e. by the requirement

$$
a_{12}\cos^2\alpha + (a_{22} - a_{11})\cos\alpha\sin\alpha - a_{12}\sin^2\alpha = 0 \tag{1.4}
$$

or

$$
a_{12}\tan^2\alpha + (a_{11} - a_{22})\tan\alpha - a_{12} = 0, \tag{1.5}
$$

moreover, it is natural to assume that $a_{12} \neq 0$ (if $a_{12} = 0$ there would be nothing to do, since $\varphi(x,y)$ would already have the form $a_{11}x^2 + a_{22}y^2$).

From (1.5) we get

$$
\tan\alpha = \frac{a_{22} - a_{11} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2}}{2a_{12}}. \tag{1.6}
$$

Since $(a_{11} - a_{22})^2 + 4a_{12}^2 > 0$, then by the formula (1.6), we can always determine the angle α we need.

Assuming $a'_{11} = \lambda_1$, $a'_{22} = \lambda_2$ for brevity, we formulate the result.

By rotating the coordinate system $O\mathbf{e}_1\mathbf{e}_2$ by the angle α determined from (1.6), one can transform the quadratic form

$$
\varphi(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2
$$

to the canonical form

$$
\varphi'(x',y') = \lambda_1{x'}^2 + \lambda_2{y'}^2,
$$

and the entire polynomial $F(x, y)$ to the following form

$$F'(x', y') = \lambda_1 x'^2 + \lambda_2 y'^2 + 2a_1' x' + 2a_2' y' + a_0. \tag{1.7}$$

Both coefficients λ_1 and λ_2 cannot be zero at the same time: if $\lambda_1 = \lambda_2 = 0$, then the second-degree polynomial $F(x, y)$ under transformation (1.2) would go into a polynomial of the first degree, which, as we know, is impossible.

So there are two main cases:

1. $\lambda_1 \neq 0$, $\lambda_2 \neq 0$.

2. One of the two coefficients λ_1, λ_2 is different from zero, the other is equal to zero.

1.2 First case: $\lambda_1 \neq 0$, $\lambda_2 \neq 0$

When moving the origin to some point $O' = (x_0', y_0')$, i.e. during conversion

$$x' = x'' + x_0',$$
$$y' = y'' + y_0',$$

the polynomial $F'(x', y')$ takes the form

$$F'(x', y') = F''(x'', y'') = \\ \lambda_1 x''^2 + \lambda_2 y''^2 + 2(\lambda_1 x_0' + a_1')x'' + 2(\lambda_2 y_0' + a_2')y'' + a_0', \tag{1.8}$$

where the free term a_0' is

$$a_0' = \lambda_1 {x_0'}^2 + \lambda_2 {y_0'}^2 + 2a_1' x_0' + 2a_2' y_0' + a_0 \equiv F'(x_0', y_0').$$

Let us now choose such coordinates x_0', y_0' of the new origin O'', so that the coefficients at x'' and y'' in (1.8) vanish. Thus we have

$$\lambda_1 x_0' + a_1' = 0, \qquad \lambda_2 y_0' + a_2' = 0. \tag{1.9}$$

Since $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, the equations (1.9) give the desired values for x_0', y_0'. So, in the new coordinate system, the original equation $F(x, y) = 0$ of our curve is transformed to the form

$$\boxed{\lambda_1 x''^2 + \lambda_2 y''^2 + a_0' = 0.} \tag{1.10}$$

Let's examine the equation (1.10). We have two cases:

A. hyperbolic: the coefficients λ_1 and λ_2 have different signs.

B. elliptic: coefficients λ_1 and λ_2 have the same sign.

A. hyperbolic case. First let $a_0' \neq 0$, one of the coefficients λ_1, λ_2 has the same sign as a_0'; let it be, for example, λ_2; then λ_1 and a_0' are opposite in sign.

Rewrite the equation (1.10) as

$$\frac{x''^2}{-\dfrac{a_0'}{\lambda_1}} + \frac{y''^2}{-\dfrac{a_0'}{\lambda_2}} = 1. \tag{1.11}$$

The denominator $-\dfrac{a_0'}{\lambda_1}$ in the first term is a positive number; denote it by a^2; the denominator of $-\dfrac{a_0'}{\lambda_2}$ is negative; denote it by $-b^2$. Equation (1.11), i.e. equation (1.10), becomes

$$\boxed{\frac{x''^2}{a^2} - \frac{y''^2}{b^2} = 1.}$$

This is the **canonical equation of a hyperbola**.

If $a_0' = 0$ in the hyperbolic case, then we can assume without limiting the generality that $\lambda_1 > 0$, $\lambda_2 < 0$; introduce the notation $\lambda_1 = a^2$, $\lambda_2 = -b^2$; equation (1.10) is rewritten as

$$\boxed{a^2x''^2 - b^2y''^2 = 0 \text{ or } (ax'' + by'')(ax'' - by'') = 0.} \tag{1.12}$$

This is **the equation of a pair of intersecting lines**. We consider the equation (1.12) to be the canonical equation of a curve decomposing into a pair of real intersecting lines.

So, in the hyperbolic case, the equation (1.10) defines a hyperbola, or a pair of intersecting lines.

B. elliptic case. Now λ_1 and λ_2 have the same sign. Again, we first assume that $a_0' \neq 0$. If the common sign of the numbers λ_1 and λ_2 is opposite to the sign of a_0', then rewriting the equation (1.10) as (1.11)

$$\frac{x''^2}{-\dfrac{a_0'}{\lambda_1}} + \frac{y''^2}{-\dfrac{a_0'}{\lambda_2}} = 1,$$

we see that both denominators $-\dfrac{a_0'}{\lambda_1}$ and $-\dfrac{a_0'}{\lambda_2}$ are positive; denoting them respectively by a^2 and b^2, we obtain **the canonical equation of an ellipse** with semiaxes a, b:

$$\boxed{\frac{x''^2}{a^2}+\frac{y''^2}{b^2}=1.}$$

If λ_1 and λ_2 have the same sign as a_0', then the denominators in (1.11) are negative, and we get the equation

$$\boxed{\frac{x''^2}{a^2}+\frac{y''^2}{b^2}=-1.} \tag{1.13}$$

This is **the equation of an "imaginary ellipse"**, or an ellipse with imaginary semi-axes ai and bi; there is not a single real point of the plane that would satisfy this equation.

Let now $a_0'=0$ in the elliptic case. Equation (1.10) becomes

$$\lambda_1 x''^2+\lambda_2 y''^2=0.$$

Since λ_1 and λ_2 have the same sign, this equation can be rewritten as

$$\boxed{a^2x''^2+b^2y''^2=0 \text{ or } (ax''+biy'')(ax''-biy'')=0.} \tag{1.14}$$

This is **the canonical equation of a curve decomposing into a pair of intersecting imaginary conjugate lines**; it is satisfied by a single real point O', which is the intersection point of two imaginary conjugate lines

$$ax''+iby''=0, \qquad ax''-iby''=0.$$

So, in the elliptic case, the equation (1.10) — and hence the initial equation (1.8) — defines either real or imaginary ellipse, or a pair of imaginary conjugate lines with one common real point.

1.3 Second case: $\lambda_1\lambda_2=0$

Let one of the coefficients λ_1, λ_2 in the equation (1.7), for example λ_1 be different from zero, and $\lambda_2=0$. Then, in the $O\mathbf{e}_1'\mathbf{e}_2'$ coordinate system, the equation

$$F(x,y)=0$$

would take the form

$$F'(x', y') \equiv \lambda_1 x'^2 + 2a_1' x' + 2a_2' y' + a_0 = 0. \tag{1.15}$$

There are two further possibilities:

A. $a_2' \neq 0$. Then the equation (1.15) can be solved with respect to y', i.e. can be represented in the form

$$\boxed{y' = px'^2 + qx' + r,}$$

Thus, our curve is a graph of a trinomial of the second degree, i.e. **parabola**.

B. $a_2' = 0$. Then the equation (1.15) is

$$\lambda_1 x'^2 + 2a_1' x' + a_0 = 0. \tag{1.16}$$

This is a quadratic equation for x'; it has two solutions:

$$\boxed{x' = x_1', \qquad x' = x_2'} \tag{1.17}$$

we have **a pair of parallel lines** (1.17) — **real** if the roots x_1' and x_2' of the quadratic equation (1.16) are real, **imaginary and conjugate**, if such are the roots x_1' and x_2' of equation (1.16). Finally, if $x_1' = x_2'$, then the equation (1.15), and hence the equation 1.1), defines a **pair of merged** (or **coinciding**) real lines.

Let's summarize. Any second-order curve can be

1. an ellipse (real or imaginary),
2. a hyperbola,
3. a parabola,
4. a pair of straight lines:
   (a) intersecting lines (real or imaginary conjugate),
   (b) parallel lines (real or imaginary conjugate),
   (c) coinciding lines (real).

2 Theorem on quadratic form transformation

2.1 Matrix of quadratic form

Definition. *A* ***quadratic form*** *is a homogeneous polynomial of the second degree. If there are two variables, the form is called binary, if there are three, it is triadic, and so on.*

Examples of binary quadratic form:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

and triadic quadratic form:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}xz + 2a_{23}yz + a_{33}z^2.$$

The **matrices** of these **forms** are called, respectively, the matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $a_{ij} = a_{ji}$ for all indices i and j. Thus, *matrices of quadratic forms are symmetric (with respect to the main diagonal).*

2.2 Quadratic form transformation

Theorem*. *With a linear transformation of a quadratic form, the matrix A' of the transformed form is obtained from the matrix A of the original form, if the latter is multiplied on the right by the transformation matrix S, and on the left by the transposed matrix S^T:*

$$A' = S^T AS.$$

Remark. Note that the transformation matrix S is, generally speaking, not symmetric, since an arbitrary linear homogeneous transformation is considered. Thus, $S^T \neq S$.

Proof. Let the binary quadratic form be given

$$Q(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2, \tag{2.1}$$

and a linear homogeneous transformation of the variables x, y to the variables x', y' is performed with the transition matrix S:

$$\begin{aligned} x &= c_{11}x' + c_{12}y', \\ y &= c_{21}x' + c_{22}y', \end{aligned} \tag{2.2}$$

hence,

$$S = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Let us write the form Q as

$$Q(x, y) = (a_{11}x + a_{12}y)x + (a_{12}x + a_{22}y)y$$

and substitute (2.2) first only instead of x and y, which are inside brackets. We get:

$$\begin{aligned} &[a_{11}(c_{11}x' + c_{12}y') + a_{12}(c_{21}x' + c_{22}y')]x+ \\ &\qquad + [a_{12}(c_{11}x' + c_{12}y') + a_{22}(c_{21}x' + c_{22}y')]y = \\ &\qquad\qquad = (b_{11}x' + b_{12}y')x + (b_{21}x' + b_{22}y')y, \end{aligned} \tag{2.3}$$

where

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = AS.$$

But expression (2.3) can also be rewritten as

$$(b_{11}x + b_{21}y)x' + (b_{12}x + b_{22}y)y'.$$

Substitute (2.2) in the remaining x and y, we get:

$$\begin{aligned} &[b_{11}(c_{11}x' + c_{12}y') + b_{21}(c_{21}x' + c_{22}y')]x'+ \\ &\qquad + [b_{12}(c_{11}x' + c_{12}y') + b_{22}(c_{21}x' + c_{22}y')]y' = \\ &\qquad\qquad = (a'_{11}x' + a'_{21}y')x' + (a'_{12}x' + a'_{22}y')y', \end{aligned}$$

where

$$\begin{pmatrix} a'_{11} & a'_{21} \\ a'_{12} & a'_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = (AS)^T S,$$

whence

$$\begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix} = \left((AS)^T S\right)^T = S^T \left((AS)^T\right)^T = S^T AS.$$

But the product $S^T AS$ is a symmetric matrix. Indeed, transposing it, we get:

$$\left(S^T AS\right)^T = S^T A^T \left(S^T\right)^T = S^T AS,$$

since $\left(S^T\right)^T = S$, and $A^T = A$ as a matrix of a quadratic form; the coincidence of the matrix S^TAS with the result of its transposition means that this matrix is symmetrical.

So, as a result of the transformation, we have obtained a quadratic form

$$Q'(x', y') = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2$$

from new variables x', y' with a matrix, which is the product S^TAS.

2.3 Variation of the quadratic form determinant

Definition. *The* ***determinant*** *(or* ***discriminant****) of a quadratic form is the determinant of a matrix of this form.*

Conclusion of Theorem*. *The determinant $|A'|$ of the transformed quadratic form is obtained from the determinant $|A|$ of the original form by multiplying by the square of transition matrix determinant $|S|^2$.*

When multiplying matrices, their determinants are also multiplied. Therefore, from the matrix equality

$$A' = S^TAS$$

implies the equality in the determinants

$$|A'| = |S^T||A||S| = |A||S||S^T|.$$

On the other hand, transposing a matrix does not change its determinant, so $|S^T| = |S|$. Thus, we get:

$$|A'| = |A||S|^2.$$

3 Invariants of the second degree polynomial

Let some the second degree polynomial of variables x, y be given:

$$F(x, y) \equiv \varphi(x, y) + 2l(x, y) + a_0, \tag{3.1}$$

$$\varphi(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2, \qquad l(x, y) = a_1x + a_2y. \tag{3.2}$$

Denote by δ the determinant

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \text{ here } a_{12} = a_{21},$$

i.e. discriminant of the quadratic form $\varphi(x, y)$.

When passing from the rectangular coordinate system Oxy to the new rectangular coordinate system $O'x'y'$, the polynomial $F(x, y)$ passes into the polynomial

$$F'(x', y') \equiv \varphi'(x', y') + 2l'(x', y') + a'_0. \tag{3.3}$$

Since the general transformation of coordinates is reduced to the transfer of the origin of coordinates and to the transition to a new coordinate system with the same origin, we will consider separately both of these special cases. As we know, when transferring the origin, i.e. when transforming $x = x' + x_0$, $y' = y + y_0$, the coefficients a_{11}, a_{12}, a_{22} at the leading terms of the polynomial $F(x, y)$ remain unchanged. In other words, the matrix of the quadratic form $\varphi(x, y)$ remains unchanged, and hence its discriminant δ.

If the new coordinate system has the same origin $O' = O$ as the old one, then

$$\varphi'(x', y') \equiv a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2, \qquad l'(x', y') \equiv a'_1x' + a'_2y',$$

where we have

$$\delta' = \begin{vmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \delta.$$

Home assignement: check that the transformation (1.2) preserves the discriminant δ.

Let us now formulate two definitions.

Definition 1. General transformation

$$\begin{aligned} x &= c_{11}x' + c_{12}y' + c_1, \\ y &= c_{21}x' + c_{22}y' + c_2 \end{aligned} \tag{3.4}$$

is called **orthogonal** if its matrix is orthogonal

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

i.e.

$$c_{11}^2 + c_{12}^2 = 1, \quad c_{21}^2 + c_{22}^2 = 1, \quad c_{11}c_{21} + c_{12}c_{22} = 0.$$

Definition 2. Let a rational function[1] $J(a_{11}, a_{12}, a_{22}, a_1, a_2, a_0)$ of the coefficients of the polynomial

$$F(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0.$$

[1] *A rational function* (of some variables ξ, η, ζ, ...) is simply a polynomial in these variables. In this case, $\xi = a_{11}$, $\eta = a_{12}$, and so on.

is given. Under an arbitrary orthogonal transformation (3.4), the polynomial $F(x, y)$ goes identically to

$$F'(x', y') \equiv a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_1x' + 2a'_2y' + a'_0.$$

If, at the same time, for any set of values a_{11}, a_{12}, a_{22}, a_1, a_2, a_0

$$J(a'_{11}, a'_{12}, a'_{22}, a'_1, a'_2, a_0) = J(a_{11}, a_{12}, a_{22}, a_1, a_2, a_0), \tag{3.5}$$

then the function J is called the **orthogonal invariant** of the polynomial $F(x, y)$.

An example of an orthogonal invariant of a polynomial is

$$\delta = \delta(a_{11}, a_{12}, a_{22}, a_1, a_2, a_0) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Similarly, the orthogonal invariant is the function

$$S = S(a_{11}, a_{12}, a_{22}, a_1, a_2, a_0) = a_{11} + a_{22}.$$

Indeed, if the transformation (3.4) is a rotation of the coordinate frame (by some angle α), then from (1.1) and (1.3) follows that

$$S' = S.$$

But the function S obviously does not change under reflection

$$x = x', \qquad y = -y',$$

and also when moving the origin of coordinates, and therefore, for any orthogonal transformation.

Let us finally prove the invariance of the function

$$\Delta = \Delta(a_{11}, a_{12}, a_{22}, a_1, a_2, a_0) = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix}.$$

To do this, along with the polynomial $F(x, y)$, consider the quadratic form

$$\Phi(x, y, t) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1xt + 2a_2yt + a_0t^2,$$

and along with the transformation (3.4), consider the transformation

$$\begin{aligned} x &= c_{11}x' + c_{12}y' + c_1t', \\ y &= c_{21}x' + c_{22}y' + c_2t', \\ t &= 0 \cdot x' + 0 \cdot y' + 1 \cdot t', \end{aligned} \tag{3.6}$$

Under this transformation, the quadratic form $\Phi(x,y,t)$ goes over to the quadratic form

$$\Phi'(x',y',t') = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_1x't' + 2a'_2y't' + a'_0t'^2$$

(where the coefficients are a'_{11}, a'_{12}, etc. are the same as in the polynomial $F'(x',y')$). The discriminant of the quadratic form $\Phi(x,y,t)$ is our determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix}.$$

During the transformation (3.6), it is multiplied by the square of the determinant of this transformation, i.e. by

$$\begin{vmatrix} c_{11} & c_{12} & c_1 \\ c_{21} & c_{22} & c_2 \\ 0 & 0 & 1 \end{vmatrix}^2 = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}^2 = 1,$$

whence it follows that $\Delta(a'_{11}, a'_{12}, a'_{22}, a'_1, a'_2, a'_0) = \Delta(a_{11}, a_{12}, a_{22}, a_1, a_2, a_0)$. We have proved the following theorem.

Theorem 1. *Functions of the coefficients of the polynomial (3.1)*

$$S = a_{11} + a_{22}, \qquad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix}$$

*are **orthogonal invariants** of this polynomial.*

Remark 1. It follows from our reasoning that if the polynomial $F(x,y)$ satisfies any of the conditions $\delta \gtreqless 0$, $\Delta \gtreqless 0$, then when passing to any affine coordinate system $O'x'y'$ it transforms into a polynomial $F'(x',y')$ satisfying the same condition (because the determinants δ', Δ' constructed for $F'(x',y')$ are obtained respectively from δ and Δ by multiplying by a positive number — the square of the transformation determinant).

Conclusion of Theorem 1. *If by any orthogonal transformation*

$$x = c_{11}x' + c_{12}y',$$
$$y = c_{21}x' + c_{22}y',$$

we reduce the form

$$\varphi(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

to the canonical form

$$\varphi'(x', y') = a'_{11}{x'}^2 + a'_{22}{y'}^2,$$

then the coefficients a'_{11} *and* a'_{22} *are necessarily the roots of the quadratic equation*

$$\lambda^2 - S\lambda + \delta = 0. \tag{3.7}$$

Indeed, it follows from the invariance of S and δ that

$$S = a_{11} + a_{22} = a'_{11} + a'_{22}, \qquad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a'_{11} & 0 \\ 0 & a'_{22} \end{vmatrix} = a'_{11}a'_{22}, \tag{3.8}$$

i.e. the sum of the numbers a'_{11} and a'_{22} is equal to S, and their product is equal to δ. And this means that these numbers themselves are the roots of the equation (3.7). The equation (3.7) is called **characteristic equation of quadratic form** $\varphi(x, y)$. It always has real roots, which immediately follows from the fact that the discriminant of the equation (3.7) is

$$S^2 - 4\delta = (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}) = (a_{11} - a_{22})^2 + 4a_{12}^2 \geqslant 0. \tag{3.9}$$

Remark 2. This discriminant is equal to zero if and only if

$$a_{11} = a_{22}, \qquad a_{12} = 0 \tag{3.10}$$

at the same time. Equalities (3.10) express a necessary and sufficient condition for the roots of the characteristic equation to be equal to each other.

We know that by rotating through the angle α (determined from (1.6)) the quadratic form

$$\varphi(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2$$

is transformed into $\varphi'(x', y') = a'_{11}{x'}^2 + a'_{22}{y'}^2$, where a'_{11} and a'_{22} are always roots of the characteristic equation (3.7) . But if the roots of this equation are λ_1 and λ_2, then we do not know which of them is the coefficient at ${x'}^2$ (i.e. a'_{11}), and what is the coefficient at ${y'}^2$.

Assuming that the roots λ_1 and λ_2 of the characteristic equation are given, we find the angle α by which the coordinate system must be rotated so that the form $\varphi(x, y)$ goes exactly to $\lambda_1 x'^2 + \lambda_2 y'^2$ (not in $\lambda_2 x'^2 + \lambda_1 y'^2$). This angle will also be the angle of inclination of the new Ox' axis to the old Ox. To do this, we rewrite the first two equalities (1.3):

$$\begin{aligned} a'_{11} &= \lambda_1 = a_{11}\cos^2\alpha + 2a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha, \\ a'_{12} &= 0 = -a_{11}\cos\alpha\sin\alpha + a_{12}\cos^2\alpha - a_{22}\sin^2\alpha + a_{11}\cos\alpha\sin\alpha. \end{aligned}$$

Multiply the first of these equalities by $\cos\alpha$, the second by $-\sin\alpha$ and add these equalities together. We get

$$\begin{aligned} \lambda_1\cos\alpha = a_{11}(\cos^3\alpha + \cos\alpha\sin^2\alpha) + 2a_{12}\cos^2\alpha\sin\alpha - a_{12}\cos^2\alpha\sin\alpha + \\ + a_{12}\sin^3\alpha + a_{22}(\sin^2\alpha\cos\alpha - \cos\alpha\sin^2\alpha), \end{aligned}$$

i.e.

$$\lambda_1\cos\alpha = a_{11}\cos\alpha + a_{12}\sin\alpha,$$

whence it follows that

$$\tan\alpha = \frac{\lambda_1 - a_{11}}{a_{12}}. \tag{3.11}$$

This is the slope of the new x-axis. Note that if $a_{12} = 0$, then the form $\varphi(x, y)$ already has a canonical form and there is no need for any rotation of the coordinate system.

4 Semi-invariant

Theorem. *The function*

$$K = \begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix} \tag{4.1}$$

is an invariant of a homogeneous orthogonal transformation; if the function

$$F(x, y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0$$

by homogeneous orthogonal transformation can be reduced to the form

$$F'(x', y') = a'_{11}x'^2 + 2a'_1x' + a'_0$$

then K is also an orthogonal invariant.

Remark. For parallel translation of axis K generally changes. Thus, K is not an invariant, but only the so-called **semi-invariant**.

Proof. Let us consider the function

$$f = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 - \lambda(x^2 + y^2),$$

producing a homogeneous orthogonal transformation

$$x = c_{11}x' + c_{12}y',$$
$$y = c_{21}x' + c_{22}y',$$

we get[2]

$$f' = a'_{11}x'^2 + 2a'_{12}x'y' + a'_{22}y'^2 + 2a'_1x + 2a'_2y + a'_0 - \lambda(x'^2 + y'^2).$$

Earlier it was proved that Δ is an orthogonal invariant. Using this with respect to the function f, we get

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_1 \\ a_{21} & a_{22}-\lambda & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix} = \begin{vmatrix} a'_{11}-\lambda & a'_{12} & a'_1 \\ a'_{21} & a'_{22}-\lambda & a'_2 \\ a'_1 & a'_2 & a'_0 \end{vmatrix}.$$

Equating the coefficients of λ of the first degree in the left and right sides of this equality, we obtain

$$\begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix} = \begin{vmatrix} a'_{11} & a'_1 \\ a'_1 & a'_0 \end{vmatrix} + \begin{vmatrix} a'_{22} & a'_2 \\ a'_2 & a'_0 \end{vmatrix}.$$

Let us now assume that there exists a homogeneous orthogonal transformation ω_1 for which the function F takes the form

$$F'(x', y') = a'_{11}x'^2 + 2a'_1x' + a'_0.$$

Let us prove that then the function K is an orthogonal invariant. Indeed, let ω be an arbitrary orthogonal transformation. Consider an orthogonal transformation

$$\omega' = \omega\omega_1^{-1};$$

then

$$\omega = \omega'\omega_1.$$

[2]Since matrix c_{ik} is orthogonal: $x^2 + y^2 = (c_{11}x' + c_{12}y')^2 + (c_{21}x' + c_{22}y')^2 = (c_{11}^2 + c_{21}^2)x'^2 + 2(c_{11}c_{12} - c_{21}c_{22})x'y' + (c_{12}^2 + c_{22}^2)y'^2 = x'^2 + y'^2$, $c_{11}^2 + c_{21}^2 = 1$, $c_{11}c_{12} + c_{21}c_{22} = 0$, $c_{12}^2 + c_{22}^2 = 1$.

Next, represent ω' as a product of a homogeneous orthogonal transformation ω_3 and a translation of axes ω_2; then

$$\omega = \omega_3\omega_2\omega_1.$$

After a homogeneous orthogonal transformation ω_1, the function F becomes the function

$$F_1 = a'_{11}x'^2 + 2a'_1x' + a'_0$$

and, as has been shown, K does not change. Let's perform the translation ω_2:

$$x' = x'' + x_0,$$
$$y' = y'' + y_0.$$

Function F_1 will go to function

$$F_2 = a'_{11}(x'' + x_0)^2 + 2a'_1(x'' + x_0) + a'_0 = a'_{11}x''^2 + 2(a'_{11}x_0 + a'_1)x'' + \\ + a'_{11}x_0^2 + 2a'_1x_0 + a'_0.$$

Calculating the function K, for the functions F_1 and F_2 we have

$$K' = \begin{vmatrix} a'_{11} & a'_1 \\ a'_1 & a'_0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & a'_0 \end{vmatrix} = \begin{vmatrix} a'_{11} & a'_1 \\ a'_1 & a'_0 \end{vmatrix},$$

$$K'' = \begin{vmatrix} a'_{11} & a'_{11}x_0 + a'_1 \\ a'_{11}x_0 + a'_1 & a'_{11}x_0^2 + 2a'_1x_0 + a'_0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & a'_{11}x_0^2 + 2a'_1x_0 + a'_0 \end{vmatrix} =$$
$$= a'^2_{11}x_0^2 + 2a'_1a'_{11}x_0 + a'_{11}a'_0 - a'^2_{11}x_0^2 - 2a'_1a'_{11}x_0 - a'^2_1 = a'_{11}a'_0 - a'^2_1 =$$
$$= \begin{vmatrix} a'_{11} & a'_1 \\ a'_1 & a'_0 \end{vmatrix} = K'.$$

Thus, the function K does not change during translation. Finally, the homogeneous orthogonal transformation of ω_3 again does not change K. Hence, K will not change as a result of the transformation of $\omega_3\omega_2\omega_1$, which is equal to ω.

5 Central case

So, by rotating the original rectangular coordinate system by the angle α, determined from the formula (3.11), we reduce the polynomial

$$F(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 \tag{5.1}$$

to the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + 2a_1' x' + 2a_2' y' + a_0'. \tag{5.2}$$

Further study of the curve $F(x, y) = 0$ consisted in the analysis of two cases: central (when $\lambda_1 \neq 0$, $\lambda_2 \neq 0$) and parabolic (when only one of the two numbers λ_1, λ_2 is non-zero). Since $\lambda_1\lambda_2 = \delta$, the central case is $\delta \neq 0$, and the parabolic case is $\delta = 0$.

Suppose $\delta \neq 0$. Let us prove that in this case, before any rotation of the Oxy coordinate system, we can transfer the origin according to

$$\begin{aligned} x &= \xi + x_0, \\ y &= \eta + y_0 \end{aligned} \tag{5.3}$$

and transform the polynomial $F(x, y) = 0$ into

$$F^*(\xi, \eta) = a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + a_0^*. \tag{5.4}$$

At the same time x_0, y_0 in (5.3), i.e. the coordinates of the new origin $O' = (x_0, y_0)$ are uniquely determined. Indeed, let's substitute (5.3) into (5.1). We will get

$$\begin{aligned} F(x, y) \equiv a_{11}\xi^2 + 2a_{12}\xi\eta + a_{22}\eta^2 + \\ + 2(a_{11}x_0 + a_{12}y_0 + a_1)\xi + 2(a_{21}x_0 + a_{22}y_0 + a_2)\eta + F(x_0, y_0). \end{aligned}$$

Now we define x_0 and y_0 so that the coefficients at ξ and η vanish, i.e.

$$\begin{aligned} a_{11}x_0 + a_{12}y_0 + a_1 &= 0, \\ a_{21}x_0 + a_{22}y_0 + a_2 &= 0. \end{aligned} \tag{5.5}$$

Since by assumption

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0,$$

the equations (5.5) are solved uniquely and give us the required x_0, y_0.

Now, having the roots λ_1, λ_2 of the characteristic equation

$$\lambda^2 - S\lambda + \delta = 0,$$

we only need to determine the angle α from the formula (3.11)

$$\tan\alpha = \frac{\lambda_1 - a_{11}}{a_{12}}$$

and — by rotating the coordinate system $O'\xi\eta$ through this angle α — transform the polynomial (5.1) into

$$F'(x', y') = \lambda_1 x'^2 + \lambda_2 y'^2 + a_0^*. \tag{5.6}$$

Let's define the free term a^*. To do this, we use the invariance of the determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix} = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & a_0^* \end{vmatrix} = \lambda_1 \lambda_2 a_0^* = \delta a_0^*,$$

whence it follows that

$$a_0^* = \frac{\Delta}{\delta}.$$

Thus, when passing from the original coordinate system Oxy to the new system $O'x'y'$, the polynomial $F(x, y)$ is identically transformed into

$$F(x', y') = \lambda_1 {x'}^2 + \lambda_2 {y'}^2 + \frac{\Delta}{\delta}.$$

We have proved the following theorem.

Theorem 2. *Let a second-order curve be given in an arbitrary rectangular coordinate system Oxy by an equation*

$$F(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0,$$

for which

$$\delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0.$$

Take a new coordinate system $O'x'y'$, the origin of which is the point $O' = (x_0, y_0)$, defined by the equations (5.5), and the axis $O'x'$ is tilted to the axis Ox at the angle α defined by the equation (3.11). In the $O'x'y'$ coordinate system, the curve $F(x, y) = 0$ will have the equation

$$\lambda_1 {x'}^2 + \lambda_2 {y'}^2 + \frac{\Delta}{\delta} = 0. \tag{5.7}$$

Remark 1. As can be seen from the equation (5.7), the point $O' = (x_0, y_0)$, which is the beginning of the new coordinate system $O'x'y'$, is the center of symmetry of our curve. We will see below that in the case $\delta \neq 0$ the curve $F(x, y) = 0$ has a single center of symmetry. Therefore, the curve $F(x, y) = 0$ is called **central** in this case.

Remark 2. It is essential to note that both the new coordinate system $O'x'y'$ (i.e. its origin and the inclination of the axis $O'x'$ to the old abscissa axis Ox), and all three coefficients λ_1, λ_2, Δ/δ in the equation (5.7) we directly, without any complicated calculations, determine by the coefficients of the original equation

$$F(x, y) = 0$$

of our curve[3]. The equation (5.7) is called the **reduced equation of central curve**.

For $\Delta = 0$, the equation (5.7) has the form

$$\lambda_1 {x'}^2 + \lambda_2 {y'}^2 = 0. \tag{5.8}$$

and defines a pair of lines intersecting at the origin O' (i.e. at the center of the curve). These lines are real for $\delta = \lambda_1\lambda_2 < 0$, imaginary (conjugate) for $\delta = \lambda_1\lambda_2 > 0$.

Let $\Delta \neq 0$, the equation (5.7) (i.e. the equation (1.10) with $a_0' = \Delta/\delta$) then rewritten in the form

$$\frac{{x'}^2}{-\dfrac{\Delta}{\delta\lambda_1}} + \frac{{y'}^2}{-\dfrac{\Delta}{\delta\lambda_2}} = 1. \tag{5.9}$$

We have two cases:

A. **hyperbolic case.** $\delta = \lambda_1\lambda_2 < 0$; denoting by λ_1 that of the two roots of the characteristic equation whose sign coincides with the sign of Δ, we set

$$a^2 = -\frac{\Delta}{\delta\lambda_1} > 0, \qquad -b^2 = -\frac{\Delta}{\delta\lambda_2} < 0, \tag{5.10}$$

and get the equation of the hyperbola

$$\frac{{x'}^2}{a^2} - \frac{{y'}^2}{b^2} = 1. \tag{5.11}$$

B. **elliptic case.** $\delta = \lambda_1\lambda_2 > 0$, the values λ_1 and λ_2 have the same sign, and this sign coincides with the sign of their sum S.

If this sign of S is opposite to the sign of Δ, then we can put (denoting by λ_1 that of the two roots λ_1, λ_2 of the characteristic equation for which $|\lambda_1| \leqslant |\lambda_2|$)

$$a^2 = -\frac{\Delta}{\delta\lambda_1} > 0, \qquad b^2 = -\frac{\Delta}{\delta\lambda_2} > 0, \tag{5.12}$$

and get the ellipse equation

$$\frac{{x'}^2}{a^2} + \frac{{y'}^2}{b^2} = 1; \tag{5.13}$$

[3]Therefore, it was necessary first to carry out the transfer of the beginning, i.e. solve equations (5.5), and then rotate the coordinate system: otherwise, the coordinates of the new beginning would be determined by the formulas (1.9), which include the coefficients a_1', a_2', for the calculation of which the last two formulas (1.3) are needed.

if the sign of S (that is, the sign of λ_1 and λ_2) coincides with the sign of Δ, then we set

$$a^2 = \frac{\Delta}{\delta\lambda_1} > 0, \qquad b^2 = \frac{\Delta}{\delta\lambda_2} > 0, \tag{5.14}$$

the equation (5.9) becomes the equation of an imaginary ellipse:

$$-\frac{x'^2}{a^2} - \frac{y'^2}{b^2} = 1. \tag{5.15}$$

Let's consider the special case $\lambda_1 = \lambda_2$. Then in (5.13), respectively in (5.15), we have $a^2 = b^2$, and we get the equation of:

- the circle with real radius a

$$x'^2 + y'^2 = a^2$$

- or the imaginary circle with imaginary radius ia

$$x'^2 + y'^2 = -a^2.$$

As mentioned above, the roots λ_1 and λ_2 of the characteristic equation are equal if and only if simultaneously

$$a_{11} = a_{22}, \qquad a_{12} = 0.$$

Since in this case the initial coordinate system is a completely arbitrary rectangular system, the *circle equation in any rectangular coordinate system has the form*

$$A(x^2 + y^2) + 2a_1x + 2a_2y + a_0 = 0 \tag{5.16}$$

or (abbreviated by $A = a_{11} = a_{22} \neq 0$)

$$x^2 + y^2 + 2a'_1x + 2a'_2y + a'_0 = 0. \tag{5.17}$$

Conversely, *if in some rectangular coordinate system a curve is defined by the equation (5.16), then this curve is a circle*; to obtain its canonical equation, it suffices to move the origin to the point $O' = (x_0, y_0)$ defined by the equations (5.5), which now (for the curve (5.17)) look like

$$x_0 + a'_1 = 0,$$
$$y_0 + a'_2 = 0.$$

And indeed, putting in (5.17)

$$x = x' - a'_1,$$
$$y = y' + a'_2,$$

we get (after reduction of similar members)

$$x'^2 + y'^2 = {a'_1}^2 + {a'_2}^2 - a'_0,$$

which gives a real circle for ${a'_1}^2 + {a'_2}^2 > a'_0$, imaginary for ${a'_1}^2 + {a'_2}^2 < a'_0$. For ${a'_1}^2 + {a'_2}^2 = a'_0$ we get

$$x'^2 + y'^2 = 0,$$

i.e.

$$(x' + iy')(x' - iy') = 0$$

which is a pair of imaginary lines with slopes $\pm i$. So, a circle of zero radius is a pair of imaginary lines intersecting at one real point (the center of the circle).

Let's summarize.

In the central case, $\delta \neq 0$, we have the following possibilities:

<table>
<tr><td></td><td>$\Delta = 0$,
degeneracy case</td><td colspan="2">$\Delta \neq 0$</td></tr>
<tr><td>hyperbolic case,
$\delta < 0$</td><td>pair of intersecting
real lines</td><td colspan="2">hyperbola</td></tr>
<tr><td rowspan="2">elliptic case,
$\delta > 0$</td><td rowspan="2">pair of imaginary conjugate
intersecting lines</td><td rowspan="2">ellipse:</td><td>real if
S and Δ have
different signs</td></tr>
<tr><td>imaginary if
S and Δ have
the same sign</td></tr>
</table>

Remark 3. We have seen (Remark 1 in Section 3) that if the polynomial $F(x, y)$ satisfies any of the conditions $\delta = 0$, $\delta \neq 0$, $\delta > 0$, $\delta < 0$, $\Delta = 0$, $\Delta \neq 0$, $\Delta > 0$, $\Delta < 0$, then the polynomial $F'(x', y')$ satisfies the same condition ($F'(x', y')$ is the polynomial to which $F(x, y)$ has passed when transforming from the Oxy coordinate system to an arbitrary affine coordinate system $O'x'y'$). A similar assertion is also true for the invariant S (in the case of a certain quadratic form $\varphi(x, y)$. Therefore, the table just given, which decides whether we are in the central or non-central (parabolic) case, as well as in the elliptic or hyperbolic, degenerate or non-degenerate case, remains valid for an arbitrarily chosen affine coordinate system.

6 Parabolic case: $\delta = 0$

In any case, including the parabolic one, by rotating the coordinate system through the angle α, determined from the equality (3.11), one can transform the equation of the curve under study

$$F(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0 \tag{6.1}$$

to the form

$$F'(x', y') = \lambda_1 x'^2 + \lambda_2 y'^2 + 2a_1'x' + 2a_2'y' + a_0 = 0. \tag{6.2}$$

In this case $\delta = \lambda_1\lambda_2$.

Since now $\delta = 0$, one of the roots of the characteristic equation is equal to zero.

Let $\lambda_1 = 0$, $\lambda_2 \neq 0$. Then $S = \lambda_1 + \lambda_2 = \lambda_2$, and the equation (6.2) can be written as

$$F'(x', y') \equiv Sy'^2 + 2a_1'x' + 2a_2'y' + a_0 = 0. \tag{6.3}$$

In the equation (3.11) one has to put $\lambda_1 = 0$, so we get a particularly simple formula for determining the angle α:

$$\tan\alpha = -\frac{a_{11}}{a_{12}} = -\frac{a_{12}}{a_{22}} \tag{6.4}$$

(the last equality follows from $\delta = a_{11}a_{22} - a_{12}^2$, i.e. $a_{11}/a_{12} = a_{12}/a_{22}$).

Let's start studying the equation (6.3) by calculating the invariant Δ. We have

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & a_1' \\ 0 & S & a_2' \\ a_1' & a_2' & a_0 \end{vmatrix} = -{a_1'}^2 S,$$

wherefrom

$${a_1'}^2 = -\frac{\Delta}{S}, \qquad |a_1'| = \sqrt{-\frac{\Delta}{S}}, \tag{6.5}$$

so a_1' goes to zero if and only if $\Delta = 0$[4].

Let's consider at first the case $\Delta = 0$, i.e. $a_1' = 0$, then the equation (6.3) looks like

$$Sy'^2 + 2a_2'y' + a_0 = 0. \tag{6.6}$$

This is a quadratic equation; denoting its roots as y_1', y_2', we see that the line (6.6) is a pair of lines parallel to the Ox' axis:

$$y' = y_1', \qquad y' = y_2'.$$

[4]Since a_1' is a real number, it follows from the formula (6.5) that in the parabolic case the determinant Δ is either opposite in sign to the number S, or is equal to zero

Their slope with respect to the original coordinate system Oxy is

$$k = \tan\alpha = -\frac{a_{11}}{a_{12}} = -\frac{a_{12}}{a_{22}}.$$

To completely determine these lines, it is enough to find their intersection points with one of the old coordinate axes Ox or Oy, for which it is enough to solve the original equation $F(x, y) = 0$ together with $x = 0$ or $y = 0$.

However, it is more convenient to think this way.

The equation (6.3) can be rewritten as

$$F'(x', y') \equiv Sy'^2 + 2a_1'x' + 2a_2'y' + a_0 \equiv S\left(y' + \frac{a_2'}{S}\right)^2 + a_0' = 0, \tag{6.7}$$

where

$$a_0' = a_0 - \frac{{a_2'}^2}{S}.$$

By shifting the coordinate system

$$x'' = x',$$
$$y'' = y' + a_2'/S$$

we transform the equation (6.7) to the form

$$Sy''^2 + a_0' = 0. \tag{6.8}$$

Let's put

$$\left|\frac{a_0'}{S}\right| = b^2.$$

Now three cases are possible:

1. $\dfrac{a_0'}{S} > 0$, $\dfrac{a_0'}{S} = b^2$, equation (6.8) is written as

   $$y'' = \pm bi,$$

   we have a pair of parallel imaginary conjugate lines.

2. $\dfrac{a_0'}{S} < 0$, $\dfrac{a_0'}{S} = -b^2$, equation (6.8) is written as

   $$y'' = \pm b,$$

   and defines a pair of different real parallel lines.

3. $\dfrac{a_0'}{S} = 0$, equation (6.8) takes the form

$$y''^2 = 0$$

and defines a pair of merged lines.

Let's move on to the second case: $\Delta \neq 0$, i.e. $a_1' \neq 0$. The curve $F(x, y) = 0$ has the equation (6.3) in the $O'x'y'$ coordinate system

$$F'(x', y') \equiv Sy'^2 + 2a_1'x' + 2a_2'y' + a_0 = 0,$$

i.e. it is (since $a_1' \neq 0$) a parabola (which we already know from Section 1.3).

Let's find its parameter p. To do this, we will move the origin of coordinates

$$\begin{aligned} x' &= \xi + x_0, \\ y' &= \eta + y_0. \end{aligned} \tag{6.9}$$

Inserting (6.9) into (6.3), we get

$$\begin{aligned} F'(x', y') &\equiv S(\eta + y_0)^2 + 2a_1'(\xi + x_0) + 2a_2'(\eta + y_0) + a_0 \equiv \\ &\equiv S\eta^2 + 2a_1'\xi + 2(Sy_0 + a_2')\eta + Sy_0{}^2 + 2a_1'x_0 + 2a_2'y_0 + a_0 = 0. \end{aligned}$$

Since $S = \lambda_2 \neq 0$, then, by equating the coefficient at η to zero, we obtain the equation

$$Sy_0 + a_2' = 0,$$

from which we determine y_0:

$$y_0 = -\frac{a_2'}{S}. \tag{6.10}$$

After that, we equate the following expression to zero

$$Sy_0{}^2 + 2a_1'x_0 + 2a_2'y_0 + a_0.$$

Since $a_1' \neq 0$, we get the equation for x_0:

$$Sy_0{}^2 + 2a_1'x_0 + 2a_2'y_0 + a_0 = 0, \tag{6.11}$$

from which we determine x_0.

In the $O'\xi\eta$ coordinate system, the equation $F(x, y) = 0$ takes the form $S\eta^2 + 2a_1'\xi = 0$ or

$$\eta^2 = -2\frac{a_1'}{S}\xi. \tag{6.12}$$

By changing, if necessary, the positive direction of the $O'\xi$ axis to the opposite one, one can always achieve that the number

$$p = -\frac{a_1'}{S}$$

positive. We finally write the equation (6.12) as

$$\eta^2 = 2p\xi, \qquad p > 0, \tag{6.13}$$

where (based on the formula (6.5))

$$p = \sqrt{-\frac{\Delta}{S^3}} \tag{6.14}$$

(the root is taken, of course, positive).

The direction of the axis of the parabola is (up to sign) the direction of the axis $O'\xi$, i.e. the direction of the Ox' axis. Its slope (with respect to the old Oxy coordinate system) is

$$\tan\alpha = -\frac{a_{11}}{a_{12}} = -\frac{a_{12}}{a_{22}}.$$

To fully determine the location of the parabola, you also need to know the coordinates of the vertex[5] $O' = (x_0, y_0)$, as well as in which direction the parabola is facing the concavity (i.e., what should be the positive direction of the $O'\xi$ axis so that the numbers $a_1' = a_1\cos\alpha + a_2\sin\alpha$ and S had opposite signs).

7 Affine classification of curves of the second order

We will show that the second-order affine classification of curves is given by the names of the curves themselves, i.e. that the *affine classes of second-order curves are the classes:*

1. *real ellipses;*
2. *imaginary ellipses;*
3. *hyperbolas;*
4. *pairs of real intersecting lines;*

[5]Calculation of which by formulas (6.10) and (6.11) is difficult (because of the coefficients a_1' and a_2').

5. *pairs of imaginary (conjugate) intersecting lines;*
6. *parabolas;*
7. *pairs of parallel real lines;*
8. *pairs of parallel imaginary conjugate lines;*
9. *pairs of coinciding real lines.*

We need to prove two statements:

A. *All curves of the same name (i.e. all ellipses, all hyperbolas, etc.) are affinely equivalent to each other.*

B. *Two curves of different names are never affine equivalent.*

Proof of statement A.
With affine transformation

$$x' = \frac{x}{a}, \qquad y' = \frac{y}{b}$$

ellipse given by equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

goes to the circle $x^2 + y^2 = 1$. Similarly, one can show that any hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

is affine equivalent to an equilateral hyperbola

$$x^2 - y^2 = 1.$$

Hence, all ellipses, respectively, all hyperbolas are affinely equivalent to each other. All imaginary ellipses, being affinely equivalent to the circle $x^2+y^2 = -1$ of radius i, are also affinely equivalent to each other.

Let us prove the affine equivalence of all parabolas. We will prove even more, namely, that all parabolas are similar to each other. It suffices to prove that the parabola given in some coordinate system by its canonical equation

$$y^2 = 2px,$$

is similar to

$$y^2 = 2x,$$

which is obvious if we subject the plane to a similarity transformation

$$\xi = \frac{x}{p}, \qquad \eta = \frac{y}{p}.$$

Then $x = p\xi$, $y = p\eta$, so that under our transformation the curve $y^2 = 2px$ becomes the curve

$$p^2\eta^2 = 2p \cdot p\xi,$$

i.e. into a parabola

$$\eta^2 = 2\xi,$$

which was to be proved.

Let's move on to decomposing curves. Above (formulas (1.12) and (1.14)) it was proved that a curve decomposing into a pair of intersecting lines in some (even rectangular) coordinate system has the equation

$$a^2x^2 - b^2y^2 = 0, \text{ if it is real,}$$
$$a^2x^2 + b^2y^2 = 0, \text{ if it is imaginary.}$$

Doing an additional coordinate transformation

$$x = ax', \qquad y = by',$$

we see that any curve decomposing into a pair of intersecting real lines

$$x^2 - y^2 = 0,$$

or imaginary conjugate lines

$$x^2 + y^2 = 0.$$

In case of curves that decomposing into a pair of parallel lines, each of them can be (even in some rectangular coordinate system) given by the equation

$$y^2 - b^2 = 0$$

for real lines, and

$$y^2 + b^2 = 0$$

for imaginary lines, respectively. The transformation of coordinates $x = bx'$, $y = y'$ allows us to set $b = 1$ in these equations (or $b = 0$ for coinciding lines). This implies the affine equivalence of all decomposing second-order curves that have the same name.

Proof of statement B.

First of all, we note that under an affine transformation of a plane, the order of an algebraic curve remains unchanged. So, any decomposing curve of the second order is a pair of lines, and under an affine transformation a line becomes a line, a pair of intersecting lines becomes a pair of intersecting ones, and a pair of parallel lines becomes a pair of parallel ones; moreover, the real lines become real, and the imaginary lines become imaginary. This follows from the fact that all the coefficients c_{ik} in the formulas defining the affine transformation are real numbers.

It follows from what has been said that a line that is affinely equivalent to a given decomposing second-order curve is a decomposing curve of the same name.

We pass to non-decomposing curves. Again, with an affine transformation, a real curve cannot go into an imaginary one, and vice versa. Therefore, the class of imaginary ellipses is affinely invariant.

Consider classes of real non-decomposing curves: ellipses, hyperbolas, parabolas.

Among all curves of the second order only an ellipse lies in some rectangle, while parabolas and hyperbolas (as well as all decomposing curves) extend to infinity. Under the affine transformation, the rectangle $ABCD$ containing the given ellipse will become a parallelogram containing the transformed curve, which, therefore, cannot go to infinity and, therefore, is an ellipse.

Thus, a curve affinely equivalent to an ellipse is necessarily an ellipse. It only remains to prove that under an affine transformation of a plane, a hyperbola cannot pass into a parabola, and vice versa.

Lemma. *If a parabola has common points with each of the two half-planes defined in the plane of the given line d, then it has at least one point in common with the line d.*

Indeed, we have seen that there is a coordinate system in which the given parabola has the equation

$$y^2 = x.$$

Let, relative to this coordinate system, the line d have the equation

$$Ax + By + C = 0. \tag{7.1}$$

By assumption, the parabola $y^2 = x$ has two points $M_1 = (x_1, y_1)$ and $M_2 = (x_2, y_2)$, one of which M_1 lies in the positive half-plane and M_2 lies in negative half-plane with respect to the equation (7.1). Therefore, remembering that $x_1 = y_1^2$, $x_2 = y_2^2$, we can write

$$Ay_1^2 + By_1 + C > 0, \qquad Ay_2^2 + By_2 + C < 0,$$

so that the polynomial $Ay^2 + By + C$ at the two ends of the segment $[y_1, y_2]$ of the real line takes values opposite in sign. But then there is a value $y = y_0$ lying between y_1 and y_2 for which the polynomial $Ay^2 + By + C$ takes the value zero:

$$Ay_0^2 + By_0 + C = 0.$$

The point $M(x_0, y_0)$, where $x_0 = y_0^2$, lies on the parabola $y^2 = x$ and on the line $Ax + By + C = 0$. The lemma is proven. $\square$

Let, under some affine transformation $\mathcal{A}$, the hyperbola K goes over to the curve K'; let us prove that K' cannot be a parabola. For this we denote by d the second axis of the hyperbola K. Under the transformation of $\mathbb{A}$, the line d passes into some line d', and the half-planes defined by the line d pass into the half-planes determined by the line d'. The hyperbola K does not have any points in common with the line d, but it does have points in common with each of the two half-planes into which the line d divides the plane; the curve K' has the same properties with respect to the straight line d'. Therefore, due to the lemma just proved, the curve K' cannot be a parabola — and the assertion about the affine non-equivalence of a hyperbola and a parabola is proved.

8 Table for determining the type of curve given by a second-order equation with two variables in rectangular coordinates

Equation:

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_1x + 2a_2y + a_0 = 0$$

Characteristic equation:

$$\lambda^2 - S\lambda + \delta = 0$$

Invariants:

$$S = a_{11} + a_{22}, \qquad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix}$$

Semi-invariant:

$$K = \begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix}$$

Type attribute	Class	Class attribute	Curve	Reduced Equation	Canonical Equation
$\delta \neq 0$	1	$\delta > 0,\ S\Delta < 0$	ellipse	$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
	2	$\delta > 0,\ S\Delta > 0$	imaginary ellipse		$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$
	3	$\delta > 0,\ \Delta = 0$	point		$a^2x^2 + b^2y^2 = 0$
	4	$\delta < 0,\ \Delta \neq 0$	hyperbola		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
	5	$\delta < 0,\ \Delta = 0$	pair of intersecting lines		$a^2x^2 - b^2y^2 = 0$
$\delta = 0,\ \Delta \neq 0$	6	$\delta = 0,\ \Delta \neq 0$	parabola	$Sy^2 + 2\sqrt{-\frac{\Delta}{S}}x = 0$	$y^2 = 2px$
$\delta = 0,\ \Delta = 0$	7	$\delta = 0,\ \Delta = 0,\ K < 0$	pair of parallel lines	$Sx^2 + \frac{K}{S} = 0$	$x^2 - a^2 = 0$
	8	$\delta = 0,\ \Delta = 0,\ K > 0$	pair of imaginary parallel lines		$x^2 + a^2 = 0$
	9	$\delta = 0,\ \Delta = 0,\ K = 0$	pair of coinciding lines		$x^2 = 0$