Stability of Dynamical Systems

1. Research the dynamics of the one dimensional discrete system (25 or 50)

Let's do a little research and investigate discrete dynamics. Choose a parametric family which define discrete dynamical system.

- $\lambda x - x^3$
- $x^3 + \lambda$
- $\lambda \arctan(x)$
- $\lambda \sin(x)$
- $\lambda \sinh(x)$
- you can define your own parametric family

Simple version (25). For a chosen function (fix some λ) find fixed points, points of period two (if exists). Describe which of the fixed points are attracting/repelling. Find such a λ where the mapping is structurally unstable

Full version (50). For a chosen function (fix some λ) find fixed points, points of period two (if exists), *points of period more than two (if exists).* Describe which of the fixed points and *points of period two* are attracting/repelling. Find such a λ where the mapping is structurally unstable. *Draw the bifurcation diagram.*

Remark. It is not nessecary to calculate the value of fixed and periodic points. Just to refer to them later you can label them with the letter. To descibe the type of the points (attracting or repelling) it is not nessecary to calculate the value of the derivative at these points also. You just need to estimate the derivative from below/from above at the interval to which they belong. You can use some computer algebra systems or services like Desmos to draw graphs and make calculations. You can make iterative calculations (see week1, page 20) while investigate the dynamics to find trends.

Remember, that periodic-2 points is just a fixed points for $f \circ f$, and fixed points can be found as an intersection with the bissectrissa. Bifurcation diagram is a diagram which describes dependence between value of λ and values of fixed and periodic points.

2. Research the dynamics of multidimensional discrete system (20 for each)

Describe the dynamics of the linear maps with matrix representation, describe stable and unstable sets, indicate if they are non-hyperbolic.

1. $\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$
2. $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$
3. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -2 \end{pmatrix}$
4. $\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$
5. $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
6. $\begin{pmatrix} 2 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$
7. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$
8. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
9. $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$
10. $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
11. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
12. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$
13. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
14. $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

15. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & -2 & 2 \end{pmatrix}$

3. More problems (20 for each)

1. Let $T_{-1}(x) = x^3 + x$. Prove that T_{-1} is not structurally stable.
2. Let $T_\lambda(x) = x^3 - \lambda(x)$. Prove that T_λ is structurally stable if $-1 < \lambda < 0$.
3. Prove that T_{λ_0} is topologically conjugate to T_{λ_1} if $-1 < \lambda_0, \lambda_1 < 0$.
4. Prove that $F_4(x) = 4x(1-x)$ is not structually stable.
5. Prove that $S(x) = \sin(x)$ is not structurally stable.
6. Prove that, if $f \sim g$ via h and f has a local maximum at x_0, then g has either a local maximum or minimum at $h\left(x_0\right)$. (Here is $\sim$ denotes a conjugation.)
7. Give an example to show that we may have $f \sim g$ via h and x_0 a local maximum for f and $h\left(x_0\right)$ a local minimum for g.
8. Let $S_\lambda(x) = \lambda \sin(x)$. If $0 < \lambda_1 < \lambda_2 < 1$, prove that $S_{\lambda_1} \sim S_{\lambda_2}$.
9. Show, however, that neither S_{λ_1} nor S_{λ_2} is structurally stable.
10. (Sternberg's Theorem) Let p be a hyperbolic fixed point for f with $f'(p) = \lambda$ and $\lambda \neq 0$. Prove that f is locally topologically conjugate to its derivative map $x \to \lambda x$ as described in Theorem 9.8.
11. Let $f : [0,1] \to [0,1]$ be a diffeomorphism. Prove that, if $f'(x) > 0$, then f has only fixed points and no periodic points. Prove that, if $f'(x) < 0$, then f has a unique fixed point and all other periodic points have period two.
12. A diffeomorphism $f : [0,1] \to [0,1]$ is called Morse-Smale if f has only hyperbolic periodic points. (Note that, since f is onto, the endpoints of $[0,1]$ are necessarily periodic.) Prove that a Morse-Smale diffeomorphism has only finitely many periodic points.
13. Prove that a Morse-Smale diffeomorphism of $[0,1]$ is structurally stable.

14. A function $F : \mathbf{R}^n \to \mathbf{R}$ is called an integral for a linear map L if $F \circ L(\mathbf{x}) = F(\mathbf{x})$, i.e., F is constant along orbits of L. Show that

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$$

is an integral for

$$L(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{x}$$

15. Construct (non-trivial) integrals for each of the following linear maps.

a. $L(\mathbf{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$.

b. $L(\mathbf{x}) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \mathbf{x}$.