Assignment 2

Problem 1

Prove that if all the osculating planes of a curve pass through a common point, then the curve is a plane curve.

Problem 2

Find the envelope (caustic) of the family of light rays emanating from the origin (0,0) and reflecting off the circle given by $x^2 + y^2 = 2ax$. Assume the reflection follows the standard law of reflection.

Problem 3

Find the curvature and torsion of the curves:

1. $$r = \alpha\varphi$$

2. $$\boldsymbol{r} = (2t, \ln t, t^2)$$

HW2 Week 12th.

Problem 1

Prove that if all the osculating planes of a curve pass through a common point, then the curve is a plane curve.

Pf: Assume the curve is parametrised by vector function $\vec{r}(t)$. the common point is P.

osculating plane has normal vector $\vec{r}'(t) \times \vec{r}''(t)$. if this vector not dependent on t.

then the osculating plane is fixed. the plane is the curve plane at the same time.

i.e. it suffices to check $\left\| \frac{d(\vec{r}'(t) \times \vec{r}''(t))}{dt} \right\| = 0$

Consider an auxiliary function $f(t) = (\vec{r}(t) - P) \cdot (\vec{r}'(t) \times \vec{r}''(t))$

the curve pass through a common point $\Rightarrow f(t) \equiv 0$

$f'(t) = \vec{r}'(t) \cdot (\vec{r}'(t) \times \vec{r}''(t)) + (\vec{r}(t) - P) \cdot (\vec{r}'(t) \times \vec{r}''(t))'$ the first summand $= 0$ since they are perpendicular.

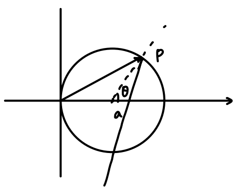
$f'(t) = 0 \Rightarrow (\vec{r}(t) - P) \cdot (\vec{r}'(t) \times \vec{r}''(t))' = 0$.

Since $\vec{r}(t) - P \not\equiv 0$. thus. $(\vec{r}'(t) \times \vec{r}''(t))' \equiv 0$.

Problem 2

Find the envelope (caustic) of the family of light rays emanating from the origin (0,0) and reflecting off the circle given by $x^2 + y^2 = 2ax$. Assume the reflection follows the standard law of reflection.

Sol:



the incidence: $P(\theta) = (a(1+\cos\theta), \sin\theta))$ $\|P(\theta)\| = \sqrt{2a^2(1+\cos\theta)} = 2a\cos\frac{\theta}{2} \rightarrow \theta \in [-\pi,\pi]$. always non-negative.

the direction of incident (normalized): $I(\theta) = \frac{P(\theta)}{\|P(\theta)\|} = (\cos\frac{\theta}{2}, \sin\frac{\theta}{2})$.

the external of normal vector $n(\theta) = (\cos\theta, \sin\theta)$.

by the law of reflection. $R(\theta) = I - 2[I \cdot n]n. = (\cos\frac{\theta}{2} - 2\cos\theta\cos\frac{\theta}{2}, \sin\frac{\theta}{2} - 2\sin\theta\cos\frac{\theta}{2})$

$\cos\frac{\theta}{2}(1 - 2\cos\theta) = \cos\frac{\theta}{2}(1 - 2(2\cos^2\frac{\theta}{2} - 1)) = 3\cos\frac{\theta}{2} - 4\cos^3\frac{\theta}{2}. = -\cos\frac{3\theta}{2}$.

$\sin\frac{\theta}{2} - 2\sin\theta\cos\frac{\theta}{2} = \sin\frac{\theta}{2} - 2\cos\frac{\theta}{2}(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}) = \sin\frac{\theta}{2}(1 - 4\cos^2\frac{\theta}{2}) = \sin\frac{\theta}{2}(-3 + 4\sin^2\frac{\theta}{2}) = -\sin\frac{3\theta}{2}$.

Thus $R(\theta) = (\cos\frac{3\theta}{2}, \sin\frac{3\theta}{2})$ $R^{\perp} = (-\sin\frac{3\theta}{2}, \cos\frac{3\theta}{2})$

For given R. the implicit function $\varphi(x,y,\theta) = R^{\perp} \cdot ((x,y) - P(\theta)) = (-\sin\frac{3\theta}{2})(x - a(1+\cos\theta)) + (\cos\frac{3\theta}{2}) \cdot (y - a\sin\theta)$

$$\varphi(x,y,\theta) = \cos\frac{3\theta}{2}y - \sin\frac{3\theta}{2}x + a\{\sin\frac{3\theta}{2}(1+\cos\theta) - \cos\frac{3\theta}{2}\sin\theta]$$

$$= \cos\frac{3\theta}{2}y - \sin\frac{3\theta}{2}x + a[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}]. = 0.$$

check: 1) $\Delta\varphi = \varphi_x^2 + \varphi_y^2 = 1 \neq 0$ $\varphi_\theta \not\equiv 0$.

2) $\det\begin{pmatrix} -\sin\frac{3\theta}{2} & \cos\frac{3\theta}{2} \\ -\frac{3}{2}\cos\frac{3\theta}{2} & -\frac{3}{2}\sin\frac{3\theta}{2} \end{pmatrix} = \frac{3}{2} \neq 0$.

Solve:

$$\begin{cases} \varphi(x,y,\theta) = 0 \\ \varphi_\theta(x,y,\theta) = 0 \end{cases} \Rightarrow \begin{cases} \cos\frac{3\theta}{2}y - \sin\frac{3\theta}{2}x + a[\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}]. = 0 & (1) \\ \sin\frac{3\theta}{2}y + \cos\frac{3\theta}{2}x - a[\cos\frac{3\theta}{2} + \frac{1}{3}\cos\frac{\theta}{2}] = 0 & (2) \end{cases}$$

$(1)^2+(2)^2 \Rightarrow \cos^2\frac{3\theta}{2}\, y^2 + \sin^2\frac{3\theta}{2}\, x^2 + a^2\left[\sin\frac{3\theta}{2}+\sin\frac{\theta}{2}\right]^2 + 2\cos\frac{3\theta}{2}\, ya\left[\sin\frac{3\theta}{2}+\sin\frac{\theta}{2}\right] - 2\sin\frac{3\theta}{2}\, x a\left[\sin\frac{3\theta}{2}+\sin\frac{\theta}{2}\right].$

$-2xy\cos\frac{3\theta}{2}\sin\frac{3\theta}{2} + \sin^2\frac{3\theta}{2}\, y^2 + \cos^2\frac{3\theta}{2}\, x^2 + a^2\left[\cos\frac{3\theta}{2}+\frac13\cos\frac{\theta}{2}\right]^2 - 2\sin\frac{3\theta}{2}\, y a\left[\cos\frac{3\theta}{2}+\frac13\cos\frac{\theta}{2}\right] - 2\cos\frac{3\theta}{2}\, x a\left[\cos\frac{3\theta}{2}+\frac13\cos\frac{\theta}{2}\right]$

$+2xy\cos\frac{3\theta}{2}\sin\frac{3\theta}{2} = 0$

$\Rightarrow x^2+y^2+a^2+a^2\left[2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac23\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\frac19\cos^2\frac{\theta}{2}\right] + 2ay\left[\cos\frac{3\theta}{2}\sin\frac{\theta}{2}-\frac13\sin\frac{3\theta}{2}\cos\frac{\theta}{2}\right] - 2ax$

$-2ax\left(\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac13\cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right).$

square

$\Rightarrow (x^2+y^2-2ax)^2 = a^4\left[2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac23\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\frac19\cos^2\frac{\theta}{2}\right]^2 + a^4 + 4a^2x^2\left(\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac13\cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right)^2.$

$+4a^2y^2\left[\cos\frac{3\theta}{2}\sin\frac{\theta}{2}-\frac13\sin\frac{3\theta}{2}\cos\frac{\theta}{2}\right]^2 + 2a^4\left[2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac23\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\frac19\cos^2\frac{\theta}{2}\right]$

$+4a^3y\left[\cos\frac{3\theta}{2}\sin\frac{\theta}{2}-\frac13\sin\frac{3\theta}{2}\cos\frac{\theta}{2}\right] - 4a^3x\left(\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac13\cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right).$

$-8a^2xy\left[\cos\frac{3\theta}{2}\sin\frac{\theta}{2}-\frac13\sin\frac{3\theta}{2}\cos\frac{\theta}{2}\right]\left(\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac13\cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right).$

$+4a^3y\left[\cos\frac{3\theta}{2}\sin\frac{\theta}{2}-\frac13\sin\frac{3\theta}{2}\cos\frac{\theta}{2}\right]\left[2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac23\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\frac19\cos^2\frac{\theta}{2}\right]$

$-4a^3y\left[\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac13\cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right]\left[2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac23\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\frac19\cos^2\frac{\theta}{2}\right]$

$= 4a^2x^2\left(\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac13\cos\frac{3\theta}{2}\cos\frac{\theta}{2}\right)^2 + 4a^2y^2\left[\cos\frac{3\theta}{2}\sin\frac{\theta}{2}-\frac13\sin\frac{3\theta}{2}\cos\frac{\theta}{2}\right]^2$

$+a^4 + a^4\left[2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac23\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\frac19\cos^2\frac{\theta}{2}\right]^2 + 2a^4\left[2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}+\frac23\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+\sin^2\frac{\theta}{2}+\frac19\cos^2\frac{\theta}{2}\right]$

$= 4a^2(x^2+y^2)$

$\Rightarrow (x^2+y^2-2ax)^2 = 4a^2(x^2+y^2)$

Problem 3

Find the curvature and torsion of the curves:

1.
$$r = \alpha\varphi$$

2.
$$\boldsymbol{r} = (2t, \ln t, t^2)$$

1. 2-dim. planar. $k_2 = 0$.

use the polar coordinates. as parametrization. $r(\varphi)=\alpha\varphi$. $r'=\alpha$. $r''=0$

$$k_1 = \frac{\left|r^2+2\left(\frac{dr}{d\varphi}\right)^2 - r\frac{d^2r}{d\varphi^2}\right|}{\left(r^2+\left(\frac{dr}{d\varphi}\right)^2\right)^{3/2}} = \frac{|\alpha^2\varphi^2+2\alpha^2|}{((\alpha\varphi)^2+\alpha^2)^{3/2}} = \frac{|\varphi+2|}{|\alpha|(\varphi^2+1)^{3/2}}$$

2. $\vec r = (2t, \ln t, t^2)$

$\vec r' = (2, \frac1t, 2t)$

$\vec r'' = (0, -\frac{1}{t^2}, 2)$

$\vec r''' = (0, \frac{2}{t^3}, 0)$.

$$k_1 = \frac{\|r'(t)\times r''(t)\|}{\|r'(t)\|^3} = \frac{\|(\frac4t, -4, -\frac{2}{t^2})\|}{\left(\sqrt{4+\frac{1}{t^2}+4t^2}\right)^3} = \frac{|4+\frac{2}{t^2}|}{|2t+\frac1t|^3}$$

$$k_2 = \frac{(r', r'', r''')}{|r'\times r''|^2} = \frac{\begin{vmatrix} 2 & \frac1t & 2t \\ 0 & -\frac{1}{t^2} & 2 \\ 0 & \frac{2}{t^3} & 0 \end{vmatrix}}{(4+\frac{2}{t^2})^2} = \frac{-2\cdot\frac{4}{t^3}}{4(2+\frac{1}{t^2})^2} = -\frac{2t}{(2t+1)^2}$$