Complex Analysis 2024. Homework 2.

1. In assumption that $f = u + iv$ is $\mathbb{C}-$differentiable prove that

$$f'(z_0) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial f}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Proof. A $\mathbb{C}-$differentiable function f satisfies Cauchy-Riemann identities

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

In this case

$$f'_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

□

2. Prove the fundamental property of exponent

$$e^{w+z} = e^w e^z, \; w, z \in \mathbb{C}.$$

Proof. Let $w = u + iv$ and $z = x + iy$, $x, v \in \mathbb{R}$. Then

$$\begin{aligned} e^{x+iy}e^{u+iv} &= e^x(\cos y + i\sin x)e^u(\cos v + i\sin v) = \\ & e^{x+u}(\cos y\cos v - \sin x\sin v + i(\sin y\cos v + \cos y\sin v)) = \\ & \qquad e^{x+u}(\cos y + v + i\sin(y+v)) = e^{x+u+i(y+v)} = e^{w+z}. \end{aligned}$$

□

3. Prove that

$$\overline{e^z} = e^{\bar{z}}.$$

Proof. If $z = x + iy$ then $\bar{z} = x - iy$ and

$$\overline{e^z} = \overline{e^x\cos y + e^x\sin y} = e^x\cos y - e^x\sin y = e^{x-iy} = e^{\bar{z}}.$$

□

4. Find all points at which the function $f(z) = |z|^2$ is differentiable. Find partial derivatives $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar{z}}$.

Proof. First, $f(z) = x^2 + y^2$ is $\mathbb{R}-$differentiable on $\mathbb{C}$,

$$f'_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = x - iy = \bar{z};$$

$$f'_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = x + iy = z.$$

$f'_{\bar{z}} = 0$ only if $z = 0$. Consequently, it f is $\mathbb{C}-$differentiable only at 0 (but not holomorphic!).

□

5. Prove that a function $f(z) = \bar{z}$ is not complex differentiable at any point.

   Proof. First, $f(z) = x - iy$ is $\mathbb{R}-$differentiable on $\mathbb{C}$.

   $$f'_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = 0;$$

   $$f'_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 1 \neq 0.$$

   Consequently, Cauchy-Riemann condition is not satisfied at any point of $\mathbb{C}$. □

6. Calculate

   $$z_1 = (1 + \sqrt{3}i)^9; \quad z_2 = (3 - 3i)^5; \quad z_3 = e^{(1+i)\frac{\pi}{2}};$$

   $$z_1 = -2^9; \quad z_2 = -cdot3^5(1 - i); \quad z_3 = ie^{\pi/2}.$$

7. How are numbers z_1 and z_2 related if $arg(z_1) = arg(z_2)$?

   Solution. This equation implies that $z_1 = az_2$ for some $a > 0$.