Equations of Straight Line on Plane. Vector Forms.

1 Revisit definition of the line

Definition. Let A and B be a pair of points. We call **straight line** (AB) (or just line) union of all segments containing both points A and B.

We also proved that for every two distant points A and B there is single and only single line (AB) that contains them both and there is single and only single segment containing them as endpoints.

2 General Assumptions

Assume that some plane α in the space $\mathbb{E}$ is chosen and fixed.

On this plane we choose arbitrary coordinate system: origin O, and a pair of unit vectors.

Now we can define the points of the plane α and the points of a line on it by means of their radius-vectors in these coordinates.

Then the number of degrees of freedom of each point on plane immediately decreases from three to two.

We start with vectorial forms of equation of the line as they are more general and more close to definition of the line.

3 Equation of line in vectorial parametric form

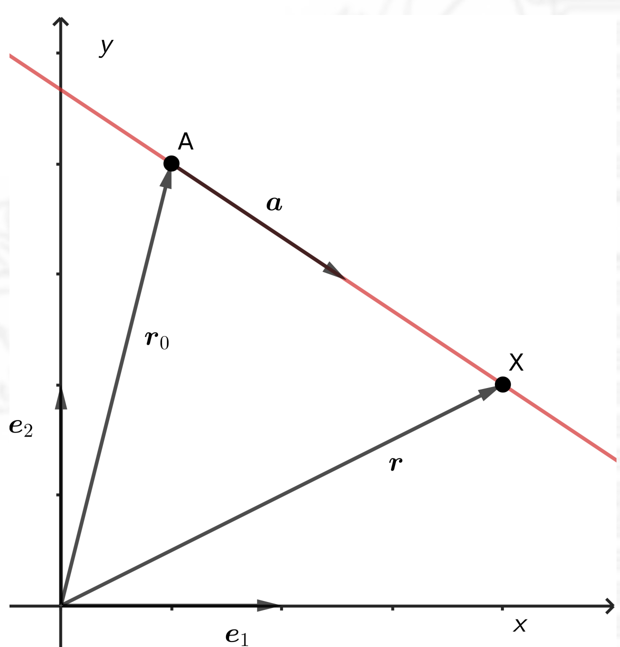

Figure 1: Derivation of vectorial parametric equation

Let's consider a line a on a plane α with some coordinate system origination in O and basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$.

Let A be some fixed point on this line, and X be arbitrary point of the line distant form A.

The position of the point X relative to the point A is marked by the vector $\overrightarrow{AX}$, while the position of the point A itself is determined by its radius vector $\boldsymbol{r}_0 = \overrightarrow{OA}$.

Therefore, position of X with respect to origin may be expressed with radius vector:

$$\boldsymbol{r} = \overrightarrow{OX} = \boldsymbol{r}_0 + \overrightarrow{AX}$$

By selecting and fixing arbitrary non-zero vector $\boldsymbol{a} || \overrightarrow{AX}$, $\boldsymbol{a} \neq \boldsymbol{0}$, we express $\overrightarrow{AX}$ as

$$\overrightarrow{AX} = \boldsymbol{a} \cdot t$$

t here is arbitrary number (positive, negative, or zero) explicitly defined with choice of $\boldsymbol{a}$. Modulus of t resembles length AX with respect to scale equal with $|\boldsymbol{a}|$, and sign means direction of displacement Thus, we established local coordinate system on the line a with origin in point A and basis vector $\boldsymbol{a}$.

From this two formulas we immediately derive

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a} \cdot t \tag{1}$$

This expression (1) is called **vectorial parametric equation of a line on a plane**.

Vector $\boldsymbol{a}$ is called a **directional vector of the line**, and point A called the **initial point**.

Real number t is **parameter** of this equation. Each particular value of the parameter t corresponds to some definite point on the line. The initial point A with the radius vector $\boldsymbol{r}_0$ is associated with the value $t = 0$.

This equation exactly corresponds with equation of uniform motion of the body on plane[1].

Parameter t corresponds with time of motion, and vector $\boldsymbol{a}$ with constant velocity vector of the body. Vector $\boldsymbol{r}_0$ means initial position of the body. Straight line is **trajectory** of the body.

Let us discuss typical problems involving this type of straight line equation.

[1]This interpretation contains strong idealization, as we overlook all possible resistance factors.

4 Problems involving vectorial parametric form of straight line equation

We further assume that our coordinate system is rectangular Cartesian.

4.1 Constructing the equation

Problem 1

Express vectorial parametric equation of a line crossing points $A(4, 12)$, and $B(6, 6)$

Solution

Suppose A be initial point. Vector $\overrightarrow{AB}$ has coordinates $\begin{pmatrix} 6-4 \\ 6-12 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$. Let it be directional vector.

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} + t \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

We may let point B be the initial point of our line. Equation in this case took form

$$\boldsymbol{r} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} + t \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

Changing of the directional vector with it anticodrected "twin" yields two more equations:

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} + t \begin{pmatrix} -2 \\ 6 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} + t \begin{pmatrix} 2 \\ -6 \end{pmatrix}$$

Definition. Vectorial parametric equation of the line containong distant points $P_1(x_1, y_1)$, and $P_2(x_2, y_2)$ has form

$$\boldsymbol{r} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \pm t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

or

$$\boldsymbol{r} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \pm t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

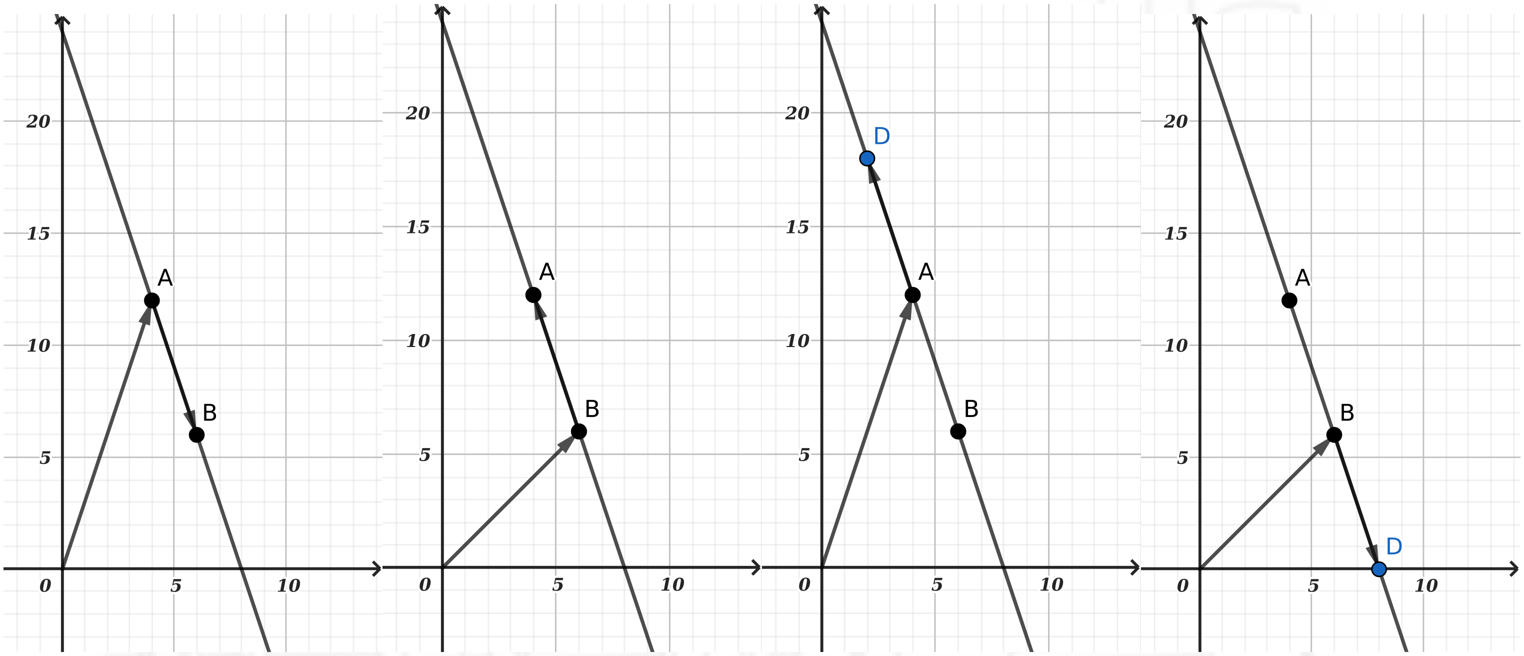

Figure 2: Various possibilities to initiate and direct direction vector

Problem 2

Remark: in physics and mechanics **point mass** is the concept of a physical object (typically matter) that has nonzero mass, and yet explicitly and specifically is (or is being thought of or modeled as) *infinitesimal* (infinitely small) in its volume or linear dimensions. Geometrical point with assigned mass is a good model for this concept.

Express as vectorial parametric equation a trajectory of point mass which started from the point $A(7,7)$ and moves uniformly shaping angle $\pi/6$ with first coordinate axis.

Solution

Described direction may be expressed with unit vectors $\pm\begin{pmatrix}\cos\dfrac{\pi}{6}\\ \sin\dfrac{\pi}{6}\end{pmatrix} = \pm\begin{pmatrix}\dfrac{\sqrt{3}}{2}\\ \dfrac{1}{2}\end{pmatrix} = \boldsymbol{a}$.

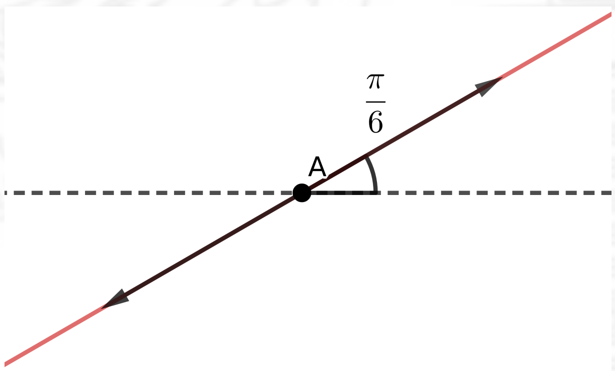

Figure 3: Two possibilities to establish direction vector

Initial point of the line is $A(7,7)$, thus $\boldsymbol{r}_0 = \begin{pmatrix}7\\ 7\end{pmatrix}$

$$\boldsymbol{r} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} \pm t \begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$

Problem 3

Find the vectorial parametric equation of the straight line whose slope is $\frac{-8}{3}$, and passes through the point $A(4, -9)$.

Solution

We are told that the line passes through the point $A(4, -9)$ which has the radius vector $\begin{pmatrix} 4 \\ -9 \end{pmatrix}$, and, in the vectorial parametric equation of the line, this is our initial point with radius vector $\boldsymbol{r}_0$.

Therefore, to find the vectorial parametric equation of this line, we only need to find its direction vector.

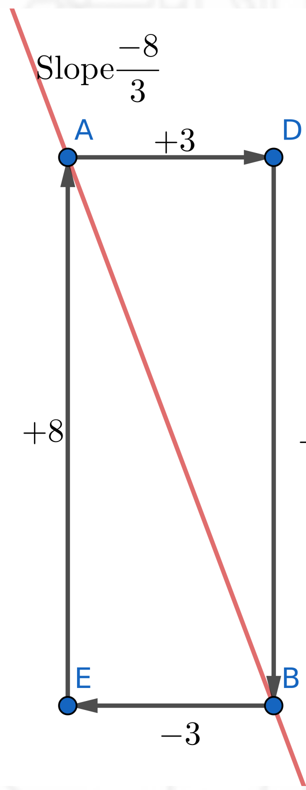

Figure 4: Vectorial interpretation for slope $\frac{-8}{3}$

We can do this by recalling what is meant by the slope of a line.

The slope of a line is the change in y divided by the change in x.

Therefore, if the slope of a line is $\frac{-8}{3}$, this means that for every three units we move horizontally, we must move 8 units vertically. There are two equivalent ways of writing this as a vector.

We can think of this as moving 3 units right and 8 units down or as 3 units left and then 8 units up. These are the two vectors, $\begin{pmatrix} 3 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 8 \end{pmatrix}$, respectively. In fact, these are equivalent; they are both direction vectors of the line.

Equation of the line is

$$\boldsymbol{r} = \begin{pmatrix} 4 \\ -9 \end{pmatrix} \pm t \begin{pmatrix} 3 \\ -8 \end{pmatrix}$$

4.2 Investigation of the features of line

Problem 4

Find a slope of a line expressed with vectorial parametric equation

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a}t$$

Solution

The slope of a line passing through two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Suppose we deal with points with radius vector $\boldsymbol{r}_0$ and $\boldsymbol{r}_0 + \boldsymbol{a}$ laying on the line. Corresponding values of parameter t are 0 and 1 respectively.

Let $\boldsymbol{r}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, $\boldsymbol{a} = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$.

$$\begin{aligned} x_1 &= x_0 \\ x_2 &= x_0 + x_a \\ y_1 &= y_0 \\ y_2 &= y_0 + y_a \end{aligned}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_a}{x_a} = \frac{\boldsymbol{a} \cdot \boldsymbol{e}_2}{\boldsymbol{a} \cdot \boldsymbol{e}_1}$$

Definition. The slope of the line expressed with vectorial parametric equation with direction vector $\boldsymbol{a}$

$$\boldsymbol{r} = \boldsymbol{r}_0 + t\boldsymbol{a}$$

has form

$$m = \frac{y_a}{x_a} = \frac{\boldsymbol{a} \cdot \boldsymbol{e}_2}{\boldsymbol{a} \cdot \boldsymbol{e}_1}$$

Definition. If slope of the line expressed as $\dfrac{p}{q}$, directional vector has coordinates $\alpha \begin{pmatrix} q \\ p \end{pmatrix}$, *alpha* is any non-zero real number.

Problem 5

Using the vectorial parametric equation of a line, identify whether the points $A(-7, 5)$, $B(-1, 2)$, and $C(5, -1)$ lie on the same line (are collinear).

Solution

There are a few ways of checking whether the three given points are collinear, one of which is to find the equation between one pair of points and then check if the third point satisfies the equation.

Let's do this by finding the vectorial parametric form of the equation of the line between $A(-7, 5)$ and $B(-1, 2)$.

We will take the equation in form

$$\boldsymbol{r} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + t \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$$

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} + t \begin{pmatrix} -1 - (-7) \\ 2 - 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} + t \begin{pmatrix} 6 \\ -3 \end{pmatrix}$$

Substitution of coordinates of C into equation yields for x:

$$\begin{aligned} &5 = -7 + 6t \\ &6t = 12 \\ &t = 2 \end{aligned}$$

If now we subsitute $t = 2$ into equation of line connecting A and B, we obtain:

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} + 2 \cdot \begin{pmatrix} 6 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 - 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix},$$

And $\boldsymbol{r}$ is radius vector for C. Thus, C lies on the line connecting A and B.

Suppose, $\theta = 2t$. This change of variable yields equivalent equation:

$$\boldsymbol{r} = \begin{pmatrix} -7 \\ 5 \end{pmatrix} + \tau \begin{pmatrix} 12 \\ -6 \end{pmatrix}$$

Direction vector for this equivalent form of equation is $\overrightarrow{AC} = \begin{pmatrix} 5-(-7) \\ -1-5 \end{pmatrix} = \begin{pmatrix} 12 \\ -6 \end{pmatrix}$.

Proposition. *Three points lay on the same line if each pair of them yields equivalent equation of line. From the vectorial parametric equation of line it means that all vectors connecting these two points are collinear*

Note: this also makes legal to use term "collinear" for such triplet of points

Problem 6
Find x-intercept and y-intercept of line expressed with vectorial parametric equation:

$$\boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} x_a \\ y_a \end{pmatrix}$$

Solution
Let $\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$.
Putting $x = 0$ now yields for first coordinate:

$$\begin{aligned} x_0 + t_x x_a &= 0 \\ t_x x_a &= -x_0 \\ t_x &= -\frac{x_0}{x_a}. \end{aligned}$$

Calculation of second coordinate yields y-intercept:

$$y = y_0 - \frac{x_0 y_a}{x_a}$$

or

$$\boldsymbol{r}_y = \begin{pmatrix} 0 \\ y_0 - \dfrac{x_0 y_a}{x_a} \end{pmatrix}.$$

Putting $y = 0$ yields

$$t_y = -\frac{y_0}{y_a};$$

and x-intercept

$$x = x_0 - \frac{y_0 x_a}{y_a}$$

or

$$\boldsymbol{r}_x = \begin{pmatrix} x_0 - \dfrac{y_0 x_a}{y_a} \\ 0 \end{pmatrix}.$$

Problem 7

Find angle between two lines expressed with vectorial parametric equations:

$$\boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} x \\ y \end{pmatrix};$$
$$\boldsymbol{r}' = \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} + \tau \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Solution

Slope of first line has expression:

$$m_1 = \frac{y}{x}$$

Slope of the second line is

$$m_2 = \frac{y'}{x'}$$

Tangent of the angle between two lines has expression with their slopes:

$$\tan\alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \left(\frac{y'}{x'} - \frac{y}{x}\right) / \left(1 + \frac{y}{x}\frac{y'}{x'}\right) = \frac{xy' - x'y}{xx'} / \frac{xx' + yy'}{xx'} = \frac{xy' - x'y}{xx' + yy'}$$

Definition. Coordinates of direction vectors $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{a}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ of two parallel lines connected with condition:

$$xy' = x'y$$

Definition. Coordinates of direction vectors $\boldsymbol{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\boldsymbol{a}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ of two perpendicular lines connected with condition:

$$xx' = -yy'$$

Problem 8

Find intersection point of two lines expressed with vectorial parametric equations:

$$\boldsymbol{r} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} x \\ y \end{pmatrix};$$

$$\boldsymbol{r}' = \begin{pmatrix} x_0' \\ y_0' \end{pmatrix} + \tau \begin{pmatrix} x \\ y \end{pmatrix}$$

Solution

Suppose lines have different initial points. In opposite case solution is trivial.
Equality $\boldsymbol{r} = \boldsymbol{r}'$ may be expanded as

$$\begin{cases} x_0 + tx = x_0' + \tau x' \\ y_0 + ty = y_0' + \tau y' \end{cases}$$

x_0, y_0, x_0', y_0', x, y, x', y' are here known parameters, and we are looking for t and τ.
Reordering yields:

$$\begin{cases} tx - \tau x' = x_0' - x_0 = \Delta x \\ ty - \tau y' = y_0' - y_0 = \Delta y \end{cases}$$

Application of Cramer's rule yields:

$$\Delta = \begin{vmatrix} x & x' \\ y & y' \end{vmatrix} = xy' - x'y$$

$$\Delta_1 = \begin{vmatrix} \Delta x & x' \\ \Delta y & y' \end{vmatrix} = \Delta x y' - x' \Delta y$$

$$\Delta_2 = \begin{vmatrix} x & \Delta x \\ y & \Delta y \end{vmatrix} = x \Delta y - \Delta x y$$

Therefore,

$$t = \frac{\Delta_1}{\Delta} = \frac{\Delta x y' - x' \Delta y}{xy' - x'y}$$

$$\tau = \frac{\Delta_2}{\Delta} = \frac{x \Delta y - \Delta x y}{xy' - x'y}$$

4.3 Special cases of the lines

Problem 9

Derive vectorial parametric equation of lines parallel with one of coordinate axes and passing through point $A(x, y)$.

Solution

For horizontal line we choose direction vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and for vertical line we choose direction vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Horizontal line:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Vertical line:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is important to underline that these direction vectors may be replaced with any non-zero multiply.

Note coordinate axes may be expresses with vectorial parametric equations as:

$$Ox \text{ axis: } \boldsymbol{r} = t\boldsymbol{e}_1$$
$$Oy \text{ axis: } \boldsymbol{r} = t\boldsymbol{e}_2$$

5 Normal vectorial equation of a line on a plane

Let $\boldsymbol{n} \neq \boldsymbol{0}$ be a vector lying on the plane α in question and being perpendicular to the line a in question.

Vectorial parametric equation of this line is

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{a} \cdot t$$

Let's apply the dot product by the vector $\boldsymbol{n}$ to both sides of the equation this equation:

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r_0} \cdot \boldsymbol{n} + t\boldsymbol{a} \cdot \boldsymbol{n}.$$

As $\boldsymbol{a} \parallel a$, and $\boldsymbol{n} \perp a$, $\boldsymbol{a} \cdot \boldsymbol{n} = 0$.

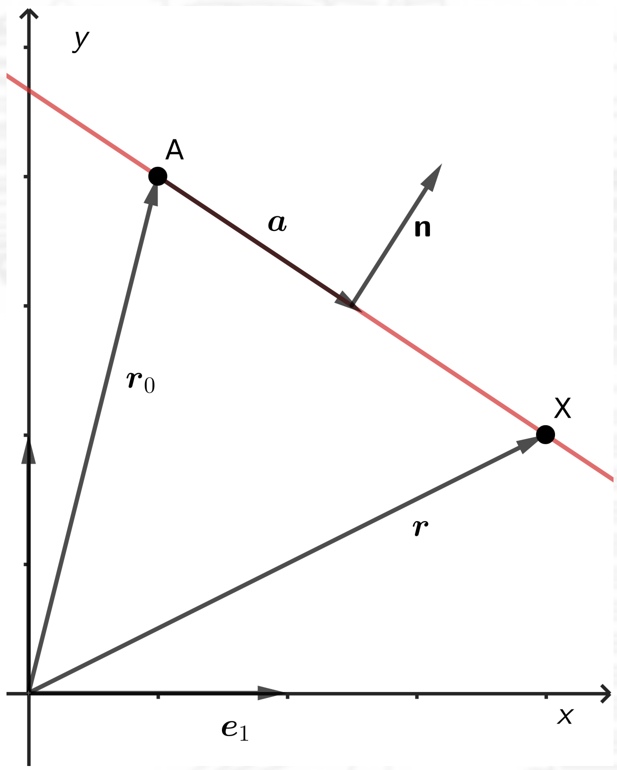

Figure 5: Derivation of vectorial normal equation

Thus, equation has form:

$$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{r_0} \cdot \boldsymbol{n}.$$

Tracking vectors $\boldsymbol{r}_0$ and $\boldsymbol{n}$ as constant, we obtain equation:

$$\boldsymbol{r} \cdot \boldsymbol{n} = D.$$

D here scalar constant having meaning of dot product of initial point radius vector and vector $\boldsymbol{n}$.

Equivalent form of this equation is

$$(\boldsymbol{r} - \boldsymbol{r_0}) \cdot \boldsymbol{n} = 0.$$

Definition. Non-zero constant $\boldsymbol{n}$ is called **normal vector of the line**

Definition. These two equations are called **normal vectorial equation of the line on plane**

6 Problems involving vectorial normal form of straight line equation

6.1 Constructing the normal vector and line equation

Problem 1

Find normal vector of line passing through point A and B with known coordinates in arbitrary rectangular Cartesian coordinate system.

Solution

To express any perpendicular with the line vector we let its dot product with vector $\overrightarrow{AB}$ (or $\overrightarrow{BA} = -\overrightarrow{AB}$, or any other non-zero multiply) equal with zero.

Additional condition is non-zero length on $\boldsymbol{n}$.

Letting its coordinates be $\boldsymbol{n} = \begin{pmatrix} x \\ y \end{pmatrix}$ and coordinates of direction vector $\overrightarrow{AB} = \begin{pmatrix} x_a \\ y_a \end{pmatrix}$, we obtain system of equations:

$$\begin{cases} xx_a + yy_a = 0 \\ x^2 + y^2 = C, C \neq 0. \end{cases}$$

It must be noted that points A and B are different, thus at least $x_a \neq 0$, or $y_a \neq 0$

For non-zero x_a first equation yields:

$$x = -y\frac{y_a}{x_a} \tag{2}$$

$y_a = 0$ yields $x = 0$ for any value of y. This is a vector parallel with Oy axis. As we demand only direction, we are free to fix $\boldsymbol{n} = \begin{pmatrix} 0 \\ -x_a \end{pmatrix} = \begin{pmatrix} y_a \\ -x_a \end{pmatrix}$.

For $x_a = 0$ and $y_a \neq 0$, we write (2) in reversed form:

$$y = -x\frac{y_a}{x_a}.$$

This yields $y = 0$ for any value of x. This is a vector parallel with Oy axis. As we demand only direction, we are free to fix $\boldsymbol{n} = \begin{pmatrix} -y_a \\ 0 \end{pmatrix} = \begin{pmatrix} -y_a \\ x_a \end{pmatrix}$.

Non-zero x_a and y_a reveal dependence for length of the vector:

$$y^2\left(1 + \left(\frac{y_a}{x_a}\right)^2\right) = C,$$

thus $y \neq 0$.

As C is our free choice (we demand only direction, but not length of $\boldsymbol{n}$), we may overlook solving this equation and rewrite expression (2):

$$\frac{x}{y} = -\frac{y_a}{x_a}, \tag{3}$$

which is exactly expression of connection between slopes of perpendicular lines.

Definition. Normal vector of the line passing through points A and B has coordinates

$$\begin{pmatrix} -y_a \\ x_a \end{pmatrix} \text{ or } \begin{pmatrix} y_a \\ -x_a \end{pmatrix},$$

where x_a and x_b are coordinates if the vector $\overrightarrow{AB}$ (or $\overrightarrow{BA}$).

Any non-zero multiply of this vector is also normal vector.

Problem 2

Express vectorial normal equation of a line crossing points $A(4, 12)$, and $B(6, 6)$ in rectangular Cartesian coordinates

Solution

As initial point we are free to select A or B

Vector $\overrightarrow{AB}$ has coordinates $\begin{pmatrix} 6-4 \\ 6-12 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$

As coordinates of $\boldsymbol{n}$ we took $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

For initial point A parameter $D = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 4 \cdot 6 + 2 \cdot 2 = 28$, and the equation is:

$$\boldsymbol{r} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 28$$

For initial point B parameter $D = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 6 \cdot 6 + 6 \cdot 2 = 48$

$$\boldsymbol{r} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 48$$

With taking opposite-directed normal vector $\begin{pmatrix} -6 \\ -2 \end{pmatrix}$, we yield following two forms of equation.

For initial point A parameter $D = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 4 \cdot (-6) + 2 \cdot (-2) = -28$, and the equation is:

$$\boldsymbol{r} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = -28$$

For initial point B parameter $D = \begin{pmatrix} 6 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = 6 \cdot 6 + 6 \cdot 2 = -48$

$$\boldsymbol{r} \cdot \begin{pmatrix} 6 \\ 2 \end{pmatrix} = -48$$

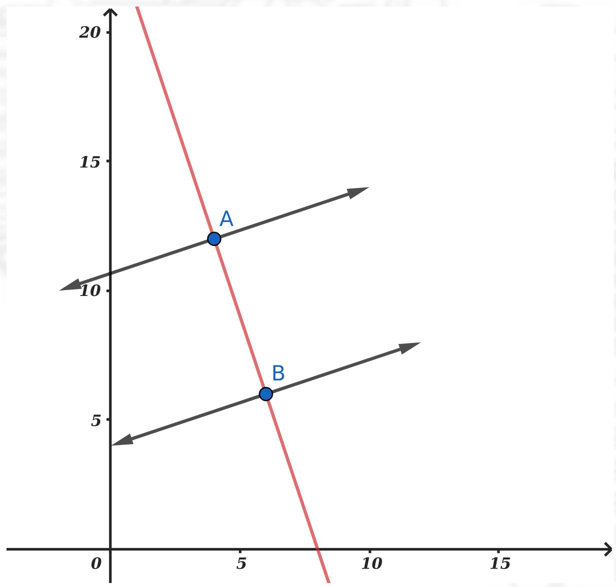

Figure 6: Direction of normal and initial points to derive normal vectorial equation of line passing through points A and B

Problem 3

Express as vectorial normal equation a trajectory of point mass which started from the point $A(7,7)$ and moves uniformly shaping angle $\pi/6$ with first coordinate axis.

Solution

Normal direction for specified "shaping angle $\pi/6$ with first coordinate axis" will have angle with that axis

$\varphi_1 = \frac{\pi}{6} + \frac{\pi}{2} = \frac{\pi}{6} + \frac{3\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$, or

$\varphi_2 = \frac{\pi}{6} - \frac{\pi}{2} = \frac{\pi}{6} - \frac{3\pi}{6} = -\frac{2\pi}{6} = -\frac{\pi}{3}$

To express this normal we took corresponding unit vector $\boldsymbol{n} = \begin{pmatrix} \cos\varphi_1 \\ \sin\varphi_1 \end{pmatrix}$ or $\boldsymbol{n} = \begin{pmatrix} -\cos\varphi_1 \\ -\sin\varphi_1 \end{pmatrix}$

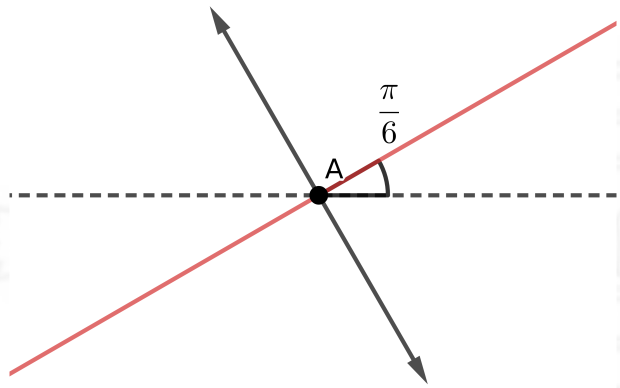

Figure 7: Two possibilities to establish normal vector

Initial point of the line is $A(7,7)$, thus $\boldsymbol{r}_0 = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$, and

$$D = 7\cos\frac{2\pi}{3} + 7\sin\frac{2\pi}{3} = \frac{7}{2}(-1+\sqrt{3}) \text{ for first variant of normal vector}$$
$$D = 7\cos\left(-\frac{\pi}{3}\right) + 7\sin\left(-\frac{\pi}{3}\right) = \frac{7}{2}(1-\sqrt{3}) \text{ for second variant of normal vector}$$

Finally, this yields two equivalent equations:

$$\boldsymbol{r} \cdot \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{7}{2}(-1+\sqrt{3}) \text{ or}$$
$$\boldsymbol{r} \cdot \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} = \frac{7}{2}(1-\sqrt{3})$$

6.2 Investigation of the features of line

Problem 4

Find a slope of a line expressed with vectorial normal equation

$$\boldsymbol{r} \cdot \boldsymbol{n} = D.$$

Solution

Suppose coordinates of $\boldsymbol{n}$ are $\big(x_n, y_n\big)$.

Normal vector $\boldsymbol{n}$ is perpendicular with direction vector of to the line a in question, thus it is direction vector for any line perpendicular with a. Slope of this line is

$$m_n = \frac{y_n}{x_n},$$

and slopes of two perpendicular lines (m_1, m_2) are connected as

$$m_1 = -\frac{1}{m_2}.$$

Hence, slope of line in question is:

$$m = -\frac{x_n}{y_n}$$

Problem 5

Find the vectorial normal equation of the straight line whose slope is $\frac{-8}{3}$, and passes through the point $A(4, -9)$.

Solution

We are told that the line passes through the point $A(4, -9)$ which has the radius vector $\begin{pmatrix} 4 \\ -9 \end{pmatrix}$, and, in the vectorial normal equation of the line, this is our initial point with radius vector $\boldsymbol{r}_0$.

Therefore, to find the vectorial normal equation of this line, we only need to find its normal vector.

Slope of the line is a tangent of the angle shaped by the line and positive direction of the first coordinate axis, and normal vector adds or subtracts from this angle $\pi/2$.

Let us express this transformation of tangent:

$$\tan\left(\varphi + \frac{\pi}{2}\right) = \frac{\sin\left(\varphi + \frac{\pi}{2}\right)}{\cos\left(\varphi + \frac{\pi}{2}\right)} = \frac{\cos\varphi}{-\sin\varphi}$$

$$\tan\left(\varphi - \frac{\pi}{2}\right) = \frac{\sin\left(\varphi - \frac{\pi}{2}\right)}{\cos\left(\varphi - \frac{\pi}{2}\right)} = \frac{-\cos\varphi}{\sin\varphi} = \tan\left(\varphi + \frac{\pi}{2}\right) = -\frac{1}{\tan\varphi}$$

This yields expression for coordinates of normal vector $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$, or $\begin{pmatrix} -3 \\ -8 \end{pmatrix}$.

In first case $D = 12 - 72 = -60$, for the second case $D = -12 + 72 = 60$.

Two equal forms of equation are:

$$\boldsymbol{r} \cdot \begin{pmatrix} 3 \\ 8 \end{pmatrix} = -60, \text{ or}$$

$$\boldsymbol{r} \cdot \begin{pmatrix} -3 \\ -8 \end{pmatrix} = 60$$

Definition. The slope of the line expressed with vectorial normal equation with normal vector $\boldsymbol{n}$

$$\boldsymbol{r} \cdot \boldsymbol{n} = D$$

has form

$$m = -\frac{x_n}{y_n} = -\frac{\boldsymbol{n} \cdot \boldsymbol{e}_1}{\boldsymbol{n} \cdot \boldsymbol{e}_2}$$

Definition. If slope of the line expressed as $\frac{p}{q}$, normal vector has coordinates $\begin{pmatrix} -p \\ q \end{pmatrix}$, or $\begin{pmatrix} p \\ -q \end{pmatrix}$.

Problem 6

Using the vectorial norma equation of a line, identify whether the points $A(-7, 5)$, $B(-1, 2)$, and $C(5, -1)$ lie on the same line (are collinear).

Solution

There are a few ways of checking whether the three given points are collinear, one of which is to find the equation between one pair of points and then check if the third point satisfies the equation.

Let's do this by finding the vectorial normal form of the equation of the line between $A(-7, 5)$ and $B(-1, 2)$.

We will take the equation in form

$$\boldsymbol{r} \cdot \boldsymbol{n} = D$$

Vector $\overrightarrow{AB}$ has coordinates $\begin{pmatrix} -1-(-7) \\ 2-5 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$, thus as normal we took vector $\boldsymbol{n} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$.

$$D = -7 \cdot 3 + 5 \cdot 6 = 30 - 21 = 9.$$

Hence, equation is

$$\boldsymbol{r} \cdot \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 9$$

With putting radius vector of C into it ve obtain:

$$\begin{pmatrix} 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 15 - 6 = 9.$$

Threfore, C satisfies the equation of line and lies on this line.

Problem 7
Find x-intercept and y-intercept of line expressed with vectorial normal equation:

$$\boldsymbol{r} \cdot \boldsymbol{n} = D$$

Solution
Suppose normal vector has coordinates $\boldsymbol{n} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$
Let $\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$.
Putting $x = 0$ now yields:

$$\begin{aligned} \begin{pmatrix} 0 \\ y \end{pmatrix} \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= D \\ y y_n &= D \\ y &= \frac{D}{y_n} \end{aligned}$$

or

$$\boldsymbol{r}_y = \begin{pmatrix} 0 \\ \dfrac{D}{y_n} \end{pmatrix}.$$

Putting $y = 0$ now yields:

$$\begin{aligned} \begin{pmatrix} x \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= D \\ x x_n &= D \\ x &= \frac{D}{x_n} \end{aligned}$$

or

$$\boldsymbol{r}_x = \begin{pmatrix} \dfrac{D}{x_n} \\ 0 \end{pmatrix}.$$

Problem 8
Find angle between two lines expressed with vectorial normal equations:

$$\boldsymbol{r} \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix} = D$$

$$\boldsymbol{r}' \cdot \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = D'$$

Solution

Slope of first line has expression:

$$m_1 = -\frac{x_n}{y_n}$$

Slope of the second line is

$$m_2 = -\frac{x'_n}{y'_n}$$

Tangent of the angle between two lines has expression with their slopes:

$$\tan\alpha = \frac{m_2 - m_1}{1 + m_1 m_2} = \left(-\frac{x'_n}{y'_n} - \left(-\frac{x_n}{y_n}\right)\right) / \left(1 + \frac{x_n}{y_n}\frac{x'_n}{y'_n}\right) =$$
$$\frac{x_n y'_n - x'_n y_n}{y_n y'_n} / \frac{x_n x'_n + y_n y'_n}{y_n y'_n} = \frac{x_n y'_n - x'_n y_n}{x_n x'_n + y_n y'_n}$$

Definition. Coordinates of normal vectors $\boldsymbol{n} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and $\boldsymbol{n}' = \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$ of two parallel lines connected with condition:

$$x_n y'_n = x'_n y_n$$

Definition. Coordinates of normal vectors $\boldsymbol{n} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and $\boldsymbol{n}' = \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$ of two perpendicular lines connected with condition:

$$x_n x'_n = -y_n y'_n$$

This formula and two corollary match with corresponding results for the direction vector because we just rotated two vectors in question by the angle $\pi/2$

Problem 10

Find intersection point of two lines expressed with vectorial normal equations:

$$\boldsymbol{r} \cdot \begin{pmatrix} x_n \\ y_n \end{pmatrix} = D$$

$$\boldsymbol{r}' \cdot \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = D'$$

Solution

Suppose that radius vector of intersection point has coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$, and substitute them into both equations:

$$\begin{cases} xx_n + yy_n = D \\ xx'_n + yy'_n = D' \end{cases}$$

Application of Cramer's rule yields:

$$\Delta = \begin{vmatrix} x_n & y_n \\ x'_n & y'_n \end{vmatrix} = x_n y'_n - x'_n y_n$$

$$\Delta_1 = \begin{vmatrix} D & y_n \\ D' & y_n \end{vmatrix} = Dy'_n - D'y_n$$

$$\Delta_2 = \begin{vmatrix} x_n & D \\ x'_n & D' \end{vmatrix} = x_n D' - x'_n D$$

Therefore,

$$x = \frac{\Delta_1}{\Delta} = \frac{Dy'_n - D'y_n}{x_n y'_n - x'_n y_n}$$
$$y = \frac{\Delta_2}{\Delta} = \frac{x_n D' - x'_n D}{x_n y'_n - x'_n y_n}$$

6.3 Special cases of the lines

Problem 10

Derive vectorial normal equation of lines parallel with one of coordinate axes and passing through point $A(x_0, y_0)$.

Solution

For horizontal line we choose normal vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and for vertical line we choose normal vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Horizontal line:

$$\boldsymbol{r} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = y_0$$

Vertical line:

$$\boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_0$$

It is important to underline that these normal vectors may be replaced with any non-zero multiply. Selection of unit normal vector in this case makes equations most illustrative.

Substitution of coordinates $\boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ automatically yields coordinate forms of these liens equations:

$$x = x_0$$
$$y = y_0$$

Note coordinate axes may be expresses with vectorial normal equations as:

$$Ox \text{ axis: } \boldsymbol{r} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$
$$Oy \text{ axis: } \boldsymbol{r} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

Problem 11

For vectorial normal equation of straight line

$$\overrightarrow{r} \cdot \overrightarrow{n} = D$$

restore corresponding vectorial parametric equation in rectangular Cartesian coordinates.

Solution

We will use y-intercept

$$\boldsymbol{r}_y = \begin{pmatrix} 0 \\ \dfrac{D}{y_n} \end{pmatrix}$$

as initial point.

Coordinates of direction b=vector also have known expression with coordinates of normal vector (we took only on form now):

$$x_a = -y_n = -\boldsymbol{n} \cdot \boldsymbol{e}_2$$
$$y_a = x_n = \boldsymbol{n} \cdot \boldsymbol{e}_1$$

Answer is

$$\boldsymbol{r} = \begin{pmatrix} 0 \\ \dfrac{D}{y_n} \end{pmatrix} + t \begin{pmatrix} -\boldsymbol{n} \cdot \boldsymbol{e}_2 \\ \boldsymbol{n} \cdot \boldsymbol{e}_1 \end{pmatrix}$$