Differential Geometry. Home Assignments

Assignment 1

Problem 1

A circle with radius r that rolls without slipping along the outer circumference of the fixed circle with radius R is given. A point M is fixed on the circumference of the rolling circle.

Determine the equation describing the curve being traced by point M. Find singular points on this curve or justify that it has no singular points.

Problem 2

Prove the assertion below.

Suppose

$$\begin{cases} x = x(t), \\ y = y(t), \\ z = z(t) \end{cases}$$

is an arbitrary parametrisation of an elementary curve. Then any other parametrisation has the form

$$\begin{cases} x = x(\sigma(\tau)), \\ y = y(\sigma(\tau)), \\ z = z(\sigma(\tau)) \end{cases}$$

where $\sigma(\tau)$ is a continuous, strictly monotonic function.

Problem 3

Find the conditions for the existence of an asymptote to the space curve

$$\begin{cases} x = x(t), \\ y = y(t), \qquad t \in (a, b), \\ z = z(t), \end{cases}$$

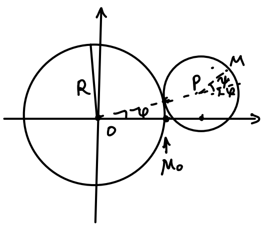
цhich tends to infinity as $t \to a$.

HW 1 Week 11 th.

Problem 1

A circle with radius r that rolls without slipping along the outer circumference of the fixed circle with radius R is given. A point M is fixed on the circumference of the rolling circle.

Determine the equation describing the curve being traced by point M. Find singular points on this curve or justify that it has no singular points.



consider M_0 at $(R+2r, 0)$ with counter-clockwise rolling

during the rolling process, denote the change of central angle of fixed circle by φ.

No slipping $\Rightarrow$ $R\varphi = r\psi$ the change of central angle of rolling circle by ψ.

denote $P: ((R+r)\cos\varphi,\ (R+r)\sin\varphi)$.

thus. $M: ((R+r)\cos\varphi + r\cos(\psi+\varphi),\ (R+r)\sin\varphi - r\sin(\psi+\varphi))$

then parametrize $\gamma_M : \begin{cases} x(\varphi) = (R+r)\cos\varphi + r\cos\left(\frac{R+r}{r}\varphi\right). \\ y(\varphi) = (R+r)\sin\varphi + r\sin\left(\frac{R+r}{r}\varphi\right). \end{cases}$

find singular point : $x'(\varphi) = -(R+r)\sin\varphi - (R+r)\sin\left(\frac{R+r}{r}\varphi\right)$

$y'(\varphi) = (R+r)\cos\varphi + (R+r)\cos\left(\frac{R+r}{r}\varphi\right).$

$[x'(\varphi)]^2 + [y'(\varphi)]^2 = 2(R+r) + 2(R+r)\left[\sin\varphi\sin\left(\frac{R+r}{r}\varphi\right) + \cos\varphi\cos\frac{R+r}{r}\varphi\right]$

$= 2(R+r) + 2(R+r)\left[\cos\left(\frac{R+r}{r}\varphi - \varphi\right)\right] = 2(R+r)\left(1 - \cos\frac{R}{r}\varphi\right).$

which $=0$ iff $\frac{R}{r}\varphi = \frac{\pi}{2} + 2k\pi \Rightarrow \varphi = \frac{\pi r}{2R} + \frac{2k\pi r}{R}.$

thus. when $\varphi = \frac{\pi r}{2R} + \frac{2k\pi r}{R}.$ γ_M has singular points.

$\left((R+r)\cos\left(\frac{\pi r}{2R} + \frac{2k\pi r}{R}\right) - r\sin\left(\frac{\pi r}{2R} + \frac{2k\pi r}{R}\right),\ (R+r)\sin\left(\frac{\pi r}{2R} + \frac{2k\pi r}{R}\right) + r\cos\left(\frac{\pi r}{2R} + \frac{2k\pi r}{R}\right)\right)$

Prove the assertion below.
Suppose

$$\gamma: \begin{cases} x = x(t), \\ y = y(t), \\ z = z(t) \end{cases}$$

is an arbitrary parametrisation of an elementary curve. Then any other parametrisation has the form

$$\begin{cases} x = x(\sigma(\tau)), \\ y = y(\sigma(\tau)), \\ z = z(\sigma(\tau)) \end{cases}$$

where $\sigma(\tau)$ is a continuous, strictly monotonic function.

Pf: Assume there exists another parametrization of γ. $\begin{cases} x = x_1(\sigma) \\ y = y_1(\sigma) \\ z = z_1(\sigma). \end{cases}$ since γ is elementary. $\sigma \in (\alpha, \beta)$

also since γ is elementary. $t \in (a, b)$. denote map. $\sigma(\tau): (a,b) \to (\alpha, \beta)$

(by elementary). x, y, z and $x_1^{-1}, y_1^{-1}, z_1^{-1}$ are t.p. map. $\sigma = x_1^{-1} \circ x = y_1^{-1} \circ y = z_1^{-1} \circ z$ is topological map.

A topological map in $\mathbb{R}$ is continuous and monotonic.

by the elementaryness of γ. $\nexists\ \tau_1, \tau_2 \in (a,b)$ s.t. $\sigma(\tau_1) = \sigma(\tau_2)$. thus. $\sigma(\tau)$ is strictly monotonic.

Problem 3

Find the conditions for the existence of an asymptote to the space curve

$$\begin{cases} x = x(t), \\ y = y(t), \qquad t \in (a, b), \\ z = z(t), \end{cases}$$

ɰhich tends to infinity as $t \to a$.

Sol: the asymptote (straight line in space.). has expression : $\dfrac{x - x_0}{d_x} = \dfrac{y - y_0}{d_y} = \dfrac{z - z_0}{d_z} = k.$

parametrize : $\begin{cases} x = d_x t + x_0 \\ y = d_y t + y_0 \\ z = d_z t + z_0 \end{cases}$ we need both direction vector (d_x, d_y, d_z) and initial point exists.

that is, $\begin{cases} \lim\limits_{t \to a} \dfrac{x(t)}{t} = d_x < \infty \\ \lim\limits_{t \to a} \dfrac{y(t)}{t} = d_y < \infty \\ \lim\limits_{t \to a} \dfrac{z(t)}{t} = d_z < \infty \end{cases}$ and $\begin{cases} \lim\limits_{t \to a} (x(t) - d_x t) = x_0 < \infty \\ \lim\limits_{t \to a} (y(t) - d_y (t)) = y_0 < \infty \\ \lim\limits_{t \to a} (z(t) - d_z (t)) = z_0 < \infty \end{cases}$

Prove the assertion below.
Suppose

$$\gamma : \begin{cases} x = x(t), \\ y = y(t), \\ z = z(t) \end{cases}$$

is an arbitrary parametrisation of an elementary curve. Then any other parametrisation has the form

$$\begin{cases} x = x(\sigma(\tau)), \\ y = y(\sigma(\tau)), \\ z = z(\sigma(\tau)) \end{cases}$$

where $\sigma(\tau)$ is a continuous, strictly monotonic function.

Pf: denote $t = \sigma(\tau)$. by the definition of elementary function. σ should be a topological mapping.

That is, σ is bijection. and σ, σ^{-1} is continuous

It suffices to check σ is strictly monotonic.

assume the converse. $\exists\ \tau_1 < \tau_2 < \tau_3$. s.t. $\sigma(\tau_2) \geq \max(\sigma(\tau_1), \sigma(\tau_3))$. or $\leq \min(\sigma(\tau_1), \sigma(\tau_3))$.

w.l.g. $\sigma(\tau_2) \geq \max(\sigma(\tau_1), \sigma(\tau_3))$

by the continuity and mean value theorem, $\exists\ \tau_1' \in (\tau_1, \tau_2)$. $\tau_2' \in (\tau_2, \tau_3)$ s.t. $\max(\sigma(\tau_1), \sigma(\tau_3)) \leq \sigma(\tau_1') = \sigma(\tau_2') \leq \sigma(\tau_2)$.

which contradicts with the injective of σ.

Thus, σ is cont. and strictly monotonic. $\blacksquare$.

Problem 3
Find the conditions for the existence of an asymptote to the space curve

$$\begin{cases} x = x(t), \\ y = y(t), \qquad t \in (a, b), \\ z = z(t), \end{cases}$$

ɥhich tends to infinity as $t \to a$.

Sol: denote $\vec{r}(t) = (x(t), y(t), z(t))$.

Assume there exist a asymptote with direction vector $\vec{\alpha}$ and initial point $\vec{b}$.

then the asymptote could be parametrised: $\vec{l} = \vec{\alpha} u + \vec{b} \Rightarrow \begin{cases} x(u) = x_0 + \alpha_x u \\ y(u) = y_0 + \alpha_y u \\ z(u) = z_0 + \alpha_z u. \end{cases}$

$$(*) \begin{cases} \lim\limits_{t \to a} \dfrac{r(t)}{\|r(t)\|} = \vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z). \quad \text{- finite} \\ \lim\limits_{t \to a} [\vec{r}(t) - (\vec{r}(t) \cdot \vec{\alpha}) \cdot \vec{\alpha}] = \vec{b} = (x_0, y_0, z_0) \quad \text{- finite point} \end{cases}$$

That is if the condition $(*)$ holds. (i.e. the direction limit exists and orthogonal projection converges to finite point).

the curve exists asymptote.