Taylors series

Using the definition of the functions of the complex variable $\cos z$ and $\sin z$, prove that:

1. $\sin z \cdot \cos z = \frac{1}{2} \sin 2z$
2. $\sin^2 z + \cos^2 z = 1$.

Pf: by def. of the functions. $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ $\quad \cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

and $e^{iz} = \dfrac{e^{iz} - e^{-iz}}{2i} \cdot i + \dfrac{e^{iz} + e^{-iz}}{2} = \cos z + i \sin z$

1. $\sin z \cdot \cos z = \dfrac{e^{iz} - e^{-iz}}{2i} \cdot \dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{e^{i2z} - e^{-i2z}}{4i} = \frac{1}{2} \sin 2z.$

2. $\sin^2 z + \cos^2 z = (\cos z + i \sin z)(\cos z - i \sin z) = e^{iz} \cdot e^{-iz} = 1.$

Using the definition of the functions of the complex variable $\operatorname{sh} z$ and $\operatorname{ch} z$, prove that:

3. $\operatorname{ch}^2 z - \operatorname{sh}^2 z = 1$
4. $\operatorname{ch} 2z = \operatorname{ch}^2 z + \operatorname{sh}^2 z$.

Pf: by def. $\operatorname{ch} z = \dfrac{e^{z} + e^{-z}}{2}$ $\quad \operatorname{sh} z = \dfrac{e^{z} - e^{-z}}{2}$

3. $\operatorname{ch}^2 z - \operatorname{sh}^2 z = \left(\dfrac{e^{z} + e^{-z}}{2}\right)^2 - \left(\dfrac{e^{z} - e^{-z}}{2}\right)^2 = \dfrac{e^{2z} + 2 + e^{-2z}}{4} - \dfrac{e^{2z} - 2 + e^{-2z}}{4} = \dfrac{4}{4} = 1.$

4. $\operatorname{ch}^2 z + \operatorname{sh}^2 z = \left(\dfrac{e^{z} + e^{-z}}{2}\right)^2 + \left(\dfrac{e^{z} - e^{-z}}{2}\right)^2 = \dfrac{e^{2z} + 2 + e^{-2z}}{4} + \dfrac{e^{2z} - 2 + e^{-2z}}{4} = \dfrac{e^{2z} + e^{-2z}}{2} = \operatorname{ch} 2z$

∘ Decompose the z function in a series of degrees:

5. $e^z \sin z$ $\quad = \displaystyle\sum_{n=0}^{\infty} \frac{2^{\frac{n}{2}} \sin\left(\frac{\pi n}{4}\right)}{n!} z^n$
6. $\operatorname{ch} z \cos z$ $\quad = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{2^{2n}}{(4n)!} z^{4n}$
7. $e^{z \operatorname{ctg} \alpha} \cos z, \quad \sin \alpha \neq 0$ $\quad = \displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\alpha)}{\sin^n \alpha} \frac{z^n}{n!}$
8. $e^{z \cos \alpha} \cos(z \sin \alpha)$. $\quad = \displaystyle\sum_{n=0}^{\infty} \frac{\cos(n\alpha)}{n!} z^n$

5. $f(z) = e^z \sin z = e^z \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right) = \dfrac{1}{2i}\left[e^{z(1+i)} - e^{z(1-i)}\right]$ $\quad f^{(n)}(z) = \dfrac{1}{2i}\left[(1+i)^n e^{z(1+i)} - (1-i)^n e^{z(1-i)}\right]$

$f^{(n)}(z) = (1+i)^{n-2} e^{z(1+i)} + (1-i)^{n-2} e^{z(1-i)}$

$f^{(n)}(0) = \dfrac{1}{2i}\left[(\sqrt{2})^n \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)^n - (\sqrt{2})^n \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)^n \right]$

$= \dfrac{2^{\frac{n}{2}}}{2i} \left(e^{\frac{n\pi}{4} i} - e^{-\frac{n\pi}{4} i}\right) = 2^{\frac{n}{2}} \sin \dfrac{\pi n}{4}$

$e^z \sin z = \displaystyle\sum_{n=0}^{\infty} \frac{(1+i)^{n-2} + (1-i)^{n-2}}{n!} z^n$

6. $f(z) = \dfrac{e^{z} + e^{-z}}{2} \cdot \dfrac{e^{iz} + e^{-iz}}{2} = \dfrac{1}{4}\left(e^{(1+i)z} + e^{(1-i)z} + e^{(-1+i)z} + e^{(-1-i)z}\right).$

$\operatorname{ch} z \cos z = \dfrac{1}{4} \displaystyle\sum_{n=0}^{\infty} \frac{(1+i)^n + (1-i)^n + (-1+i)^n + (-1-i)^n}{n!} z^n$

7. $f(z) = e^{z\cot\alpha}\cos z.$ $\qquad f^{(n)}(z) = (e^{z\cot\alpha}\cdot\cos z)^{(n)} = \sum_{k=0}^{n} C_n^k (\cot\alpha)^k e^{z\cot\alpha}\cdot(\cos z)^{n-k}$

$$f^{(n)}(0) = \sum_{k=0}^{n} C_n^k (\cot\alpha)^k \cdot \cos\frac{\pi(n-k)}{2}.$$

$$f(z) = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} C_n^k (\cot\alpha)^k \cdot \cos\frac{\pi(n-k)}{2}}{n!} z^n$$

8. $f(z) = e^{z\cos\alpha}\cos(z\sin\alpha).$ $\qquad f^{(n)}(0) = \sum_{k=0}^{n} C_n^k (\cos\alpha)^k (\sin\alpha)^{n-k} \cos\frac{\pi(n-k)}{2}.$

$$f(z) = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} C_n^k (\cos\alpha)^k (\sin\alpha)^{n-k} \cos\frac{\pi(n-k)}{2}}{n!}$$

$$\operatorname{ch} z \cos z = \frac{1}{4}\left(e^{(1+i)z} + e^{(1-i)z} + e^{(-1+i)z} + e^{(-1-i)z}\right).$$

$$(\operatorname{ch} z \cos z)^{(n)}(0) = \frac{1}{4}\left[(1+i)^n + (1-i)^n + (-1+i)^n + (-1-i)^n\right] = \frac{2^{\frac{n}{2}}}{4}\left[e^{\frac{n\pi}{4}i} + e^{-\frac{n\pi}{4}i} + e^{\frac{3\pi}{4}i} + e^{-\frac{3\pi}{4}i}\right]$$

$$= \frac{2^{\frac{n}{2}}}{2}\left[\cos\frac{n\pi}{4} + \cos\frac{3n}{4}\pi\right] = \frac{2^{\frac{n}{2}}}{2}\cdot 2\cos\frac{n\pi}{2}\cdot\cos\frac{n\pi}{4} = 2^{\frac{n}{2}}\cos\frac{n\pi}{2}\cos\frac{n\pi}{4} \overset{n'=\frac{n}{4}}{=} 2^{2n'}\cos n'\pi\cos 2n'\pi = 2^{2n'}(-1)^{n'}$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{2^{2n}}{(4n)!} z^{4n}.$$