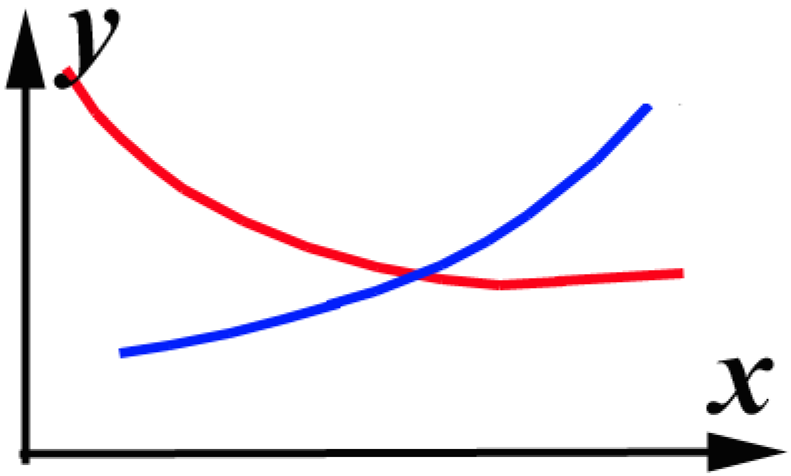
Chapter 2. Systems of nonlinear equations

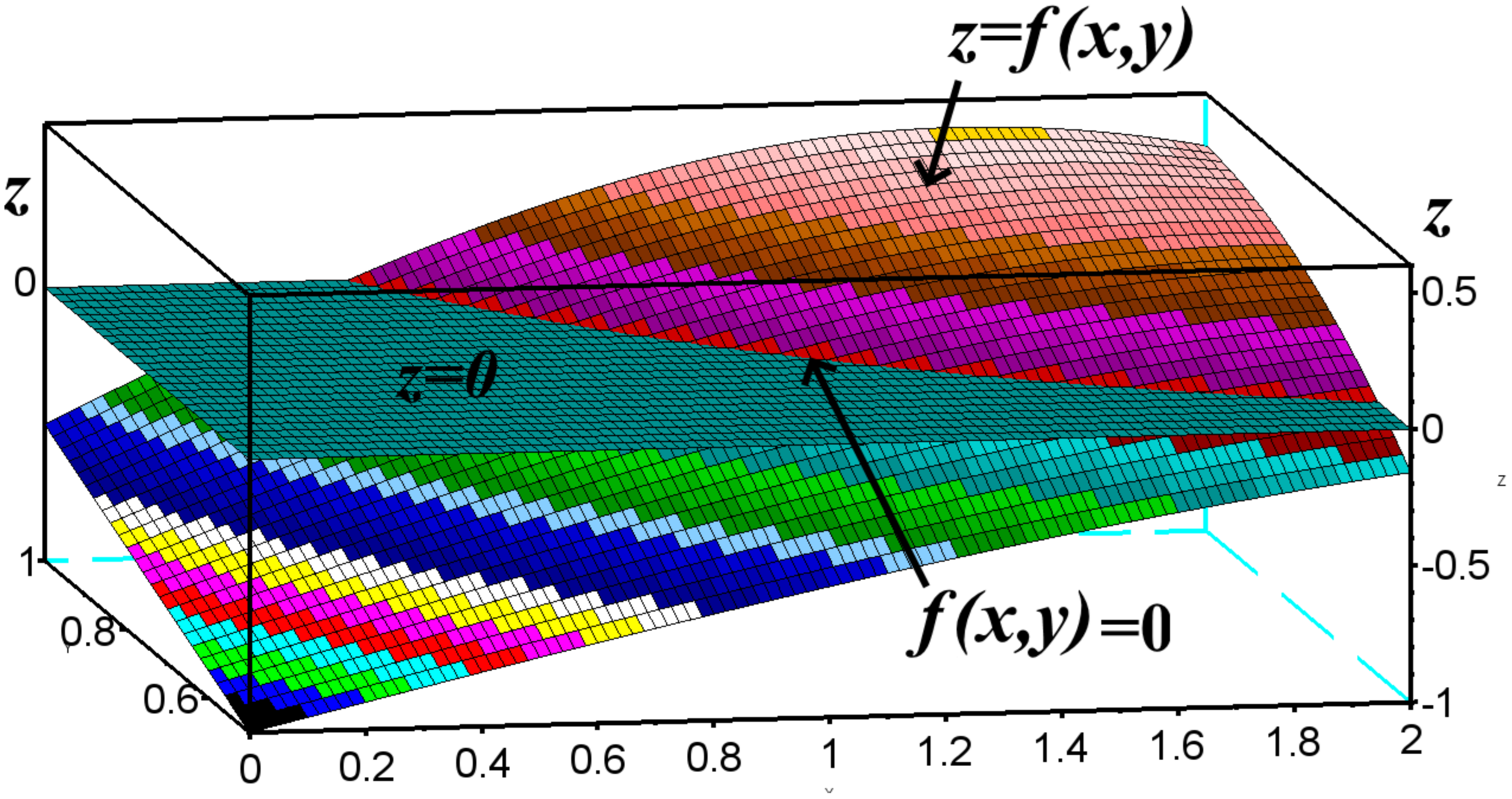
Now we turn to a system of 2 nonlinear equations with 2 unknowns x and y :

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases}$$



Geometric interpretation in 2D:
typically each function determines a curve in (x, y) plane.
Intersection of the curves gives a solution.

Interpretation in 3D:



another surface illustrates function $z=g(x,y)$

Method of tangents (Newton's method)

Let us recall the method of tangent lines for a single equation

$$f(x)=0 \qquad\qquad a \le x \le b\ .$$

Algorithm for calculation of successive approximations to a solution:

$$c_{k+1}=c_k - f(c_k)/f'(c_k) \qquad k=1,2,\ldots$$

This was obtained by plotting a line which is tangent to curve $y=f(x)$ at an initial point $x=c_1$, $y=f(c_1)$ on the curve.

Method of tangents (Newton's method)

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Similarly, in the case of 2 equations, we should obtain 2 planes which are tangent to surfaces $z=f(x, y)$, $z=g(x, y)$ at initial points $x_1, y_1, z_{1f}=f(x_1, y_1)$ and $x_1, y_1, z_{1g}=g(x_1,y_1)$.

Then we should:

1) **find intersections of the tangent planes with plane z=0; such intersections are 2 straight lines,**

2) **find intersection of the 2 lines in the plane z=0. It gives us a point x_2, y_2 which is the next approximation to the root.**

Formula for calculation of x_2, y_2 :

[in case of 1 equation it was $c_2 = c_1 - f(c_1)/f'(c_1)$]

$$\begin{pmatrix} x_2 \\ \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \\ y_1 \end{pmatrix} - \underbrace{\mathrm{inv}(F'_1)}_{\text{2 x 2 matrix}} \times \underbrace{\begin{pmatrix} f(x_1,y_1) \\ \\ g(x_1,y_1) \end{pmatrix}}_{\text{vector}}$$

column vectors

$$F'_1 = \underbrace{\begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}}_{\text{matrix}} \Bigg|_{\text{at } x_1, y_1}$$

inv – inverse of the matrix

$$\mathbf{F}' = \begin{pmatrix} \partial f/\partial x & \partial f/\partial y \\ \partial g/\partial x & \partial g/\partial y \end{pmatrix}$$

$$\text{inv } \mathbf{F}' = \frac{1}{f_x g_y - g_x f_y} \begin{pmatrix} \partial g/\partial y & -\partial f/\partial y \\ -\partial g/\partial x & \partial f/\partial x \end{pmatrix}$$

see Algebra

$$f_x = \partial f/\partial x, \quad f_y = \partial f/\partial y, \ldots$$

Proof of the formula.

If a surface in space *(x,y,z)* is given by expression *G(x,y,z)*=0, then equation of the tangent plane is

$$G_x\cdot(x-x_0) + G_y\cdot(y-y_0) + G_z\cdot(z-z_0)=0$$

(partial derivatives)

In the case under consideration $z=f(x,y) \Rightarrow f(x,y)-z=0$:

$$f_x\cdot(x-x_1) + f_y\cdot(y-y_1) - (z- z_{1f})=0$$

$g(x,y)-z=0$:

$$g_x\cdot(x-x_1) + g_y\cdot(y-y_1) - (z- z_{1g})=0$$

Find intersections with plane $z=0$**:**

$$f_x\cdot(x-x_1) + f_y\cdot(y-y_1) + z_{1f}=0$$

$$g_x\cdot(x-x_1) + g_y\cdot(y-y_1) + z_{1g}=0$$

Now find intersection of the lines:

$$\left.\begin{array}{l} f_x\cdot(x_2-x_1) + f_y\cdot(y_2-y_1) + z_{1f}=0 \\ g_x\cdot(x_2-x_1) + g_y\cdot(y_2-y_1) + z_{1g}=0 \end{array}\right\}$$

$$\left.\begin{aligned} f_x \cdot (x_2 - x_1) + f_y \cdot (y_2 - y_1) + z_{1f} &= 0 \\ g_x \cdot (x_2 - x_1) + g_y \cdot (y_2 - y_1) + z_{1g} &= 0 \end{aligned}\right\}$$

All derivatives are calculated at $x = x_1,\ y = y_1$

In matrix form:

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \times \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} + \begin{pmatrix} z_{1f} \\ z_{1g} \end{pmatrix} = 0$$

$$F'_1 \times \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} + \begin{pmatrix} z_{1f} \\ z_{1g} \end{pmatrix} = 0$$

multiply by **inv(F′$_1$):**

$$\begin{pmatrix} x_2 - x_1 \\ \\ y_2 - y_1 \end{pmatrix} + \mathrm{inv}(F'_1) \times \begin{bmatrix} z_{1f} \\ \\ z_{1g} \end{bmatrix} = 0$$

$$\begin{bmatrix} x_2 \\ \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \\ y_1 \end{bmatrix} - \mathrm{inv}(F'_1) \times \begin{pmatrix} f(x_1,y_1) \\ \\ g(x_1,y_1) \end{pmatrix}$$

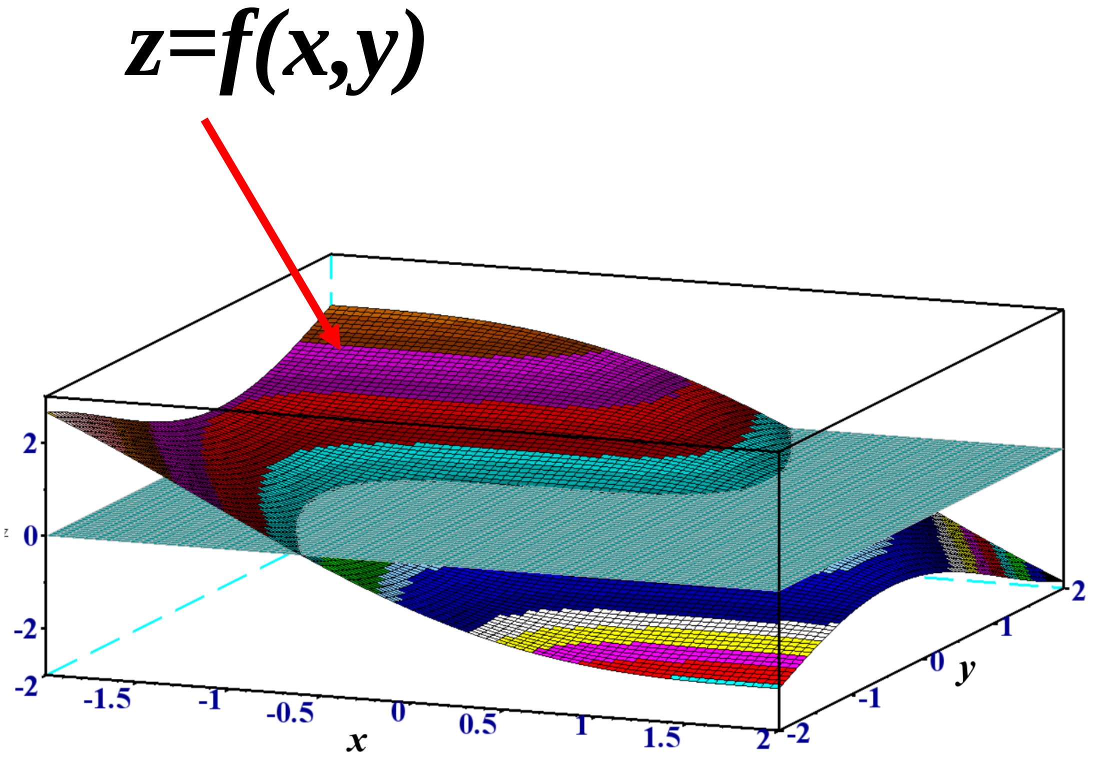
General formula:

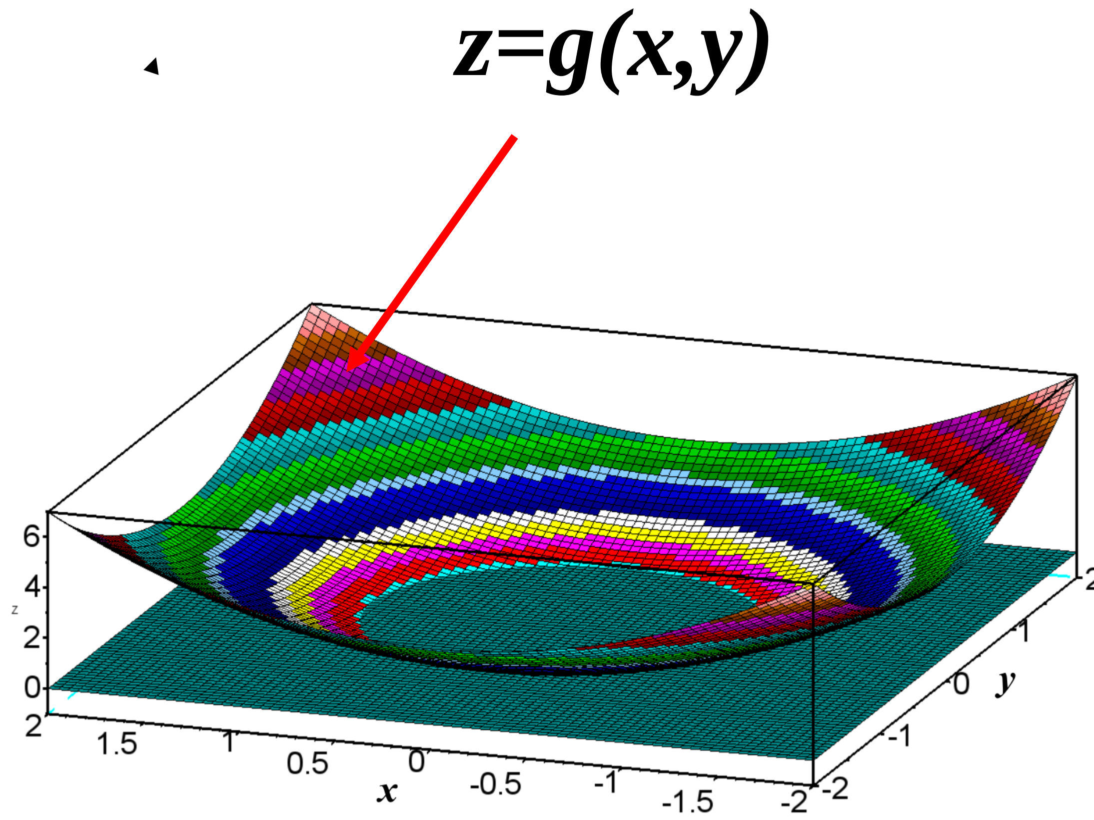
$$\underbrace{\begin{pmatrix} x_{k+1} \\ \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ \\ y_k \end{pmatrix}}_{\text{column vectors}} - \underbrace{\mathrm{inv}(F'_k)}_{\text{matrix}} \times \underbrace{\begin{pmatrix} f(x_k, y_k) \\ \\ g(x_k, y_k) \end{pmatrix}}_{\text{column vector}}$$

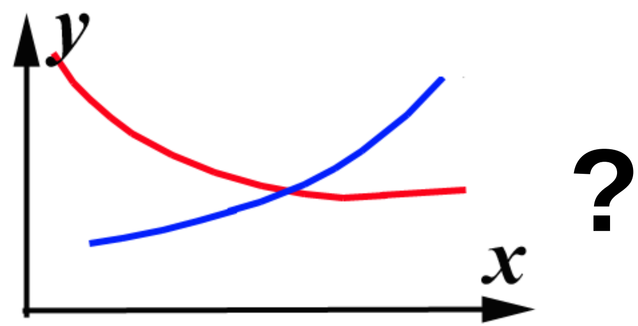
$k = 1, 2, 3, \ldots$

Example: $\sin(x+y) - x - 0.1 = 0$

$$x^2 + y^2 - 1 = 0$$







Scilab:

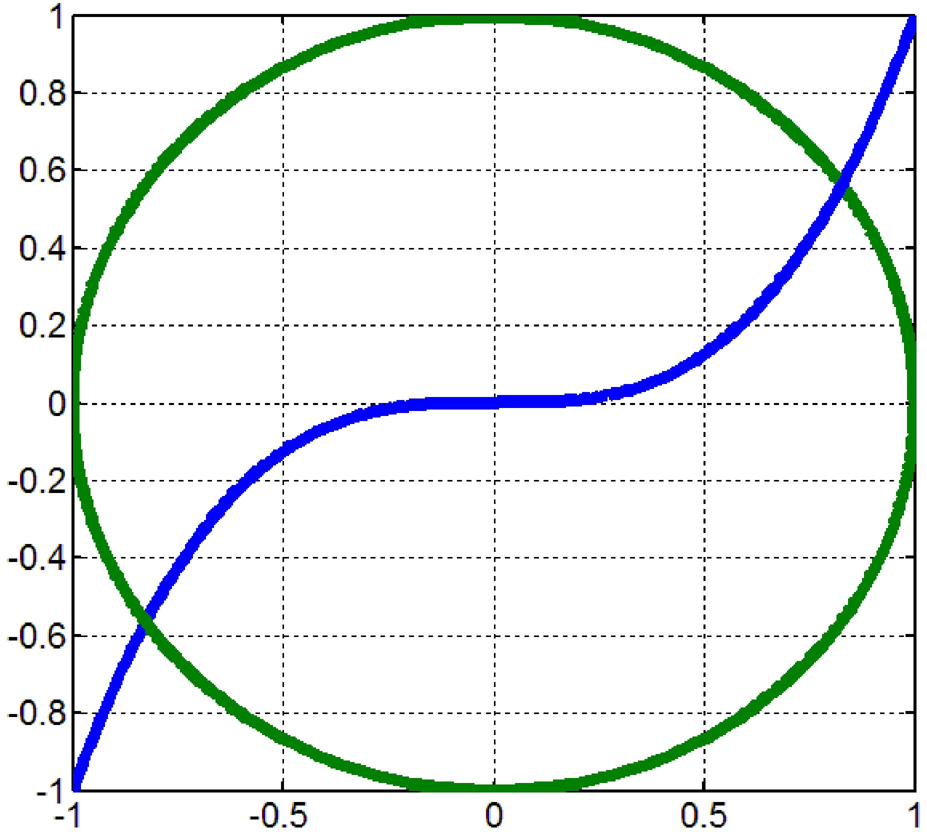
```
clear
 x=0
 y=1
 X=[x y]'
 for i=1:50
     f=sin(x+y)-x-0.1
     g=x*x+y*y-1
     F=[f  g]'
     dfdx=cos(x+y)-1
     dfdy=cos(x+y)
     dgdx=2*x
     dgdy=2*y
     Fderivat=[dfdx  dfdy;  dgdx  dgdy]
     X=X-inv(Fderivat)*F
     x=X(1)
     y=X(2)
     disp(X)
 end
```

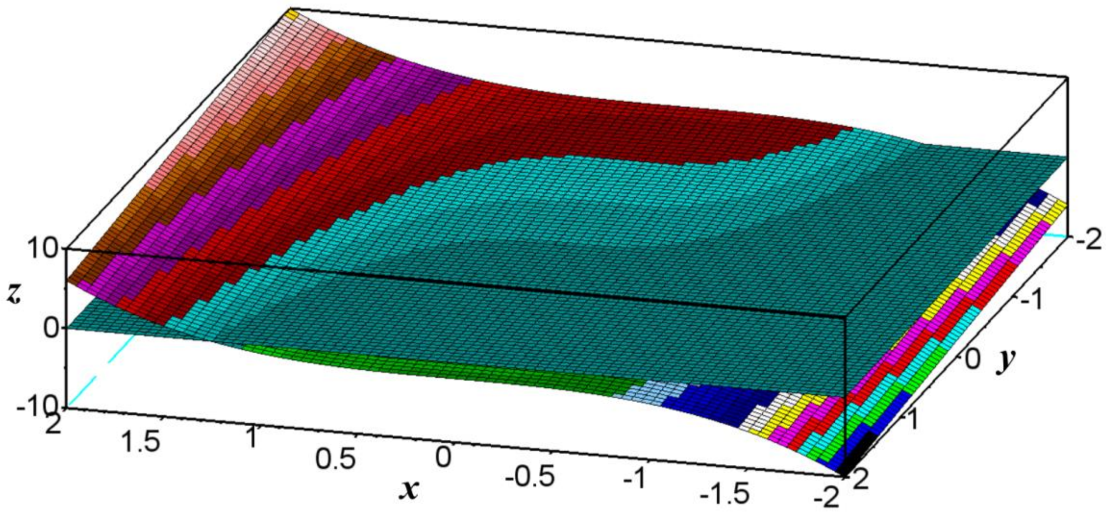
Answer: 0.8772863 0.4799675

Example of non-unique solution:

$$x^2 + y^2 - 1 = 0$$

$$x^3 - y = 0$$





Method of solving a system of equations by a transition to a <u>minimization problem</u>:

$$\begin{cases} f(x,y)=0 \\ g(x,y)=0 \end{cases} \qquad F(x,y) = f^2 + g^2 \rightarrow \text{min}$$

Example:

$$x^2 + y^2 - 1 = 0$$

$$x^3 - y = 0$$

$$0.5<x<1, \quad 0.4<y<0.6$$

Scilab:

```
clear     // Method of transition to minimization problem
x=0.5 : 0.001 : 1
y=0.4 : 0.001 : 0.6
min=10
for i=1:501
for j=1:201
f1= x(i)^2 + y(j)^2 -1
f2= x(i)^3 -y(j)
F=f1*f1 + f2*f2
if (F<min)
min=F
xmin=x(i)
ymin=y(j)
end
end
end
disp(min, xmin, ymin)
```

Same example using EXCEL: $x^2 + y^2 - 1 = 0$

$$x^3 - y = 0$$

	A	B	C
1	0.5	0.5	
2	=A1*A1+B1*B1-1	=A1^3-B1	=A2*A2+B2*B2

Solve (the extension of EXCEL)

Target cell: C2

Search of minimum

Changing cells: A1; B1

Answer:

1) A1= 0. 82603144 B1= 0. 56362407

2) A1=−0. 82603144 B1=−0. 56362407

Using EXCEL, solve the example already solved with Scilab:

$$\sin(x+y) - x - 0.1 = 0$$

$$x^2 + y^2 - 1 = 0$$

Three equations

$$x^2 + y^2 + z^2 - 1 = 0$$

$$2x^2 + y^2 - 4z = 0$$

$$3x^2 - 4y + z^2 = 0$$

```
x= 1
y= 1
z= 1
X=[x y z]'
for i=1:15
    f=x*x+y*y+z*z-1
    g=2*x*x +y*y -4*z
    h=3*x*x-4*y+z*z
    F=[f  g  h]'
    dfdx=2*x
    dfdy=2*y
    dfdz=2*z
    dgdx=4*x
    dgdy=2*y
    dgdz=-4
    dhdx=6*x
    dhdy=-4
    dhdz=2*z
    Fderivat=[dfdx  dfdy dfdz;  dgdx  dgdy dgdz; dhdx dhdy dhdz]
    X=X-inv(Fderivat)*F
    x=X(1)
    y=X(2)
    z=X(3)
    disp(X)
end
```