Fourier series

1. Decompose the function into a Fourier series $f(x) = \text{sign } x, -\pi < x < \pi$, and using the resulting decomposition, find the sum of the Leibniz series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Δ: $A(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{sign}x \, dx = 0$

$a_n(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{sign}x \cdot \cos nx \, dx = 0$

$b_n(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} \text{sign}x \cdot \sin nx = \frac{2}{n\pi}\left[(-1)^{n-1}+1\right]$

$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)x.$

let $x = \frac{\pi}{2}$. $\quad f(\frac{\pi}{2}) = \sum_{n=1}^{\infty} \frac{4 \cdot (-1)^{n-1}}{(2n-1)\pi} = \sum_{n=0}^{\infty} \frac{4 \cdot (-1)^n}{(2n+1)\pi} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$

since $f(\frac{\pi}{2}) = \text{sign}\frac{\pi}{2} = 1.$ $\quad$ thus $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$

Decompose the Fourier series of the function $f(x)$ on the specified interval, the length of the interval is the period (2-11).

2. $f(x) = \begin{cases} A, & 0 < x < l, \\ A/2, & x = l, \\ 0, & l < x < 2l, \end{cases}$ on the interval $(0.2l)$.

$A(f) = \frac{1}{2l} \int_0^{2l} f(x)\, dx = \frac{1}{2l} \cdot A \cdot l = \frac{A}{2}.$

$a_n(f) = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{\pi n x}{l} \, dx = \frac{A}{\pi n} \cdot \sin n\pi = 0$

$b_n(f) = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{\pi n x}{l} \, dx = \frac{A}{l} \int_0^{l} \sin \frac{\pi n x}{l} \, dx = \frac{A}{\pi n}(1 - \cos n\pi) = \frac{A}{\pi n}(1+(-1)^n)$

$f(x) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{2A}{\pi(2n-1)} \sin \frac{(2n-1)\pi x}{l}$

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3. $f(x) = |x|$ on the segment $[-1; 1]$.

Δ: $A(f) = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}$

$a_n(f) = \int_{-1}^{1} |x| \cos n\pi x\, dx = 2\int_{0}^{1} x \cos n\pi x \, dx = \frac{2}{n^2\pi^2}(\cos n\pi - 1)$

$b_n(f) = \int_{-1}^{1} |x| \sin n\pi x \, dx = 0$

$f(x) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2 \pi^2} \cos(2n-1)\pi x.$

4. $f(x) = \begin{cases} ax, & -\pi < x < 0, \\ bx, & 0 \leqslant x < \pi, \end{cases}$ in the interval $(-\pi, \pi)$.

Δ: $A(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left(\int_{0}^{\pi} bx\,dx + \int_{-\pi}^{0} ax\,dx\right) = \frac{\pi(b-a)}{4}$

$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\frac{b}{n^2}(\cos n\pi - 1) + \frac{a}{n^2}(1-\cos n\pi).$

$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{b}{\pi n^2}(-n\pi\cos n\pi) - \frac{a}{\pi n^2}(n\pi \cos n\pi)$

$= \frac{a+b}{n}(-1)^{n+1}$

$f(x) = \frac{\pi(b-a)}{4} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2\pi}(a-b)\cos(2n-1)x + \sum_{n=1}^{\infty}\frac{a+b}{n}(-1)^{n+1}\sin nx.$

5. $f(x) = \begin{cases} a, & -\pi/2 < x < \pi/2, \\ b, & \pi/2 \leqslant x < 3\pi/2, \end{cases}$ in the range $(-\pi/2, 3\pi/2)$.

$A(f) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\,dx = \frac{a+b}{2}$

$a_n(f) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\frac{2a}{n} - \frac{2b}{n}\right) = \frac{2}{n\pi}(a-b)$ ✗

$b_n(f) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\sin nx\,dx = 0$ $\quad \frac{1}{\pi} a \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos nx\,dx$

$f(x) = \frac{a+b}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}(a-b)\cos nx$ ✗ $\quad \frac{2(a-b)}{n\pi}\sin\frac{\pi}{2}n$

$\frac{2(a-b)}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos(2n-1)x.$

6. $f(x) = x + \operatorname{sign} x$ on the interval $(-\pi; \pi)$.

We have calculated the Fourier series of function $f_1(x) = \operatorname{sign} x$ on $(-\pi, \pi)$

$$f_1(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)x.$$

then calculate the Fourier series of function $f_2(x) = x$.

$$A(f_2) = 0$$

(odd function in symmetric domain)

$$a_n(f_2) = 0$$

$$b_n(f_2) = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi n}(-\pi \cos n\pi) = -\frac{2}{n}\cos n\pi$$

$$f_2(x) = \frac{2}{n}\sum_{n=1}^{\infty}(-1)^{n-1}$$

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1} + \frac{2}{n}\sum_{n=1}^{\infty}(-1)^{n-1} \sin nx = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}(1+\pi)}{n} \sin nx.$$

7. $f(x) = \pi^2 - x^2$ on the interval $(-\pi; \pi)$.

$$A(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\,dx = \pi^2 - \frac{1}{3}\pi^2 = \frac{2}{3}\pi^2$$

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\cos nx \, dx = -\frac{2}{\pi}\int_0^{\pi} x^2 \cos nx = \frac{4}{n^2}(-1)^{n-1}$$

$$b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\sin nx \, dx = 0$$

$$f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n-1}\cos nx$$

8. $f(x) = x^3$ on the interval $(-\pi; \pi)$.

$$A(f) = 0 \qquad a_n(f) = 0.$$

$$b_n(f) = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx \, dx = \frac{2}{\pi n^4}\left(-(n\pi)^3(-1)^n + 6n\pi\cdot(-1)^n\right)$$

$$= \frac{2\pi^2}{n}(-1)^{n-1} + \frac{12}{n^3}(-1)^n$$

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{2\pi^2}{n}(-1)^{n-1} + \frac{12}{n^3}(-1)^n\right]\sin nx$$

9. $f(x)=e^{ax}, a\neq 0$, in the interval $(-\pi;\pi)$.

$$A(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\,dx=\frac{1}{2\pi a}\left(e^{a\pi}-e^{-a\pi}\right)$$

$$a_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx.=\frac{(-1)^n a\left(e^{a\pi}-e^{-a\pi}\right)}{\pi(a^2+n^2)}$$

$$b_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx=\frac{(-1)^{n-1} n\left(e^{a\pi}-e^{-a\pi}\right)}{\pi(a^2+n^2)}$$

$$f(x)=\frac{1}{2\pi a}\left(e^{a\pi}-e^{-a\pi}\right)+\sum_{n=1}^{\infty}\frac{(-1)^n a\left(e^{a\pi}-e^{-a\pi}\right)}{\pi(a^2+n^2)}\cos nx+\sum_{n=1}^{\infty}\frac{(-1)^{n-1} n\left(e^{a\pi}-e^{-a\pi}\right)}{\pi(a^2+n^2)}\sin nx.$$

10. $f(x)=e^{2|x|}$ in the interval $(-\pi;\pi)$.

$$A(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2|x|}\,dx=\frac{1}{2\pi}\cdot 2\int_{0}^{\pi} e^{2x}\,dx=\frac{e^{2\pi}-1}{2\pi}$$

$$a_n(f)=\frac{2}{\pi}\int_{0}^{\pi} e^{2x}\cos nx=\frac{4e^{2\pi}(-1)^n-4}{\pi(n^2+4)}$$

$$b_n(f)=0$$

$$f(x)=\frac{e^{2\pi-1}}{2\pi}+\sum_{n=1}^{\infty}\frac{4e^{2\pi}(-1)^n-4}{\pi(n^2+4)}\cos nx$$

11. $f(x)=\sin ax, a\in Z$ in the interval $(-\pi;\pi)$.

$$A(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin ax\,dx=0 \qquad a_n(f)=0.$$

i) $a=0$ $\quad f\equiv 0$

ii) $a<0$. by the orthogonality of triangular function

$a_n(f)=b_n(f)=0$

iii) $a>0$. $\quad a_n(f)=0$

$$b_n(f)=\begin{cases}0 & a\neq n\\ \frac{1}{\pi}\int_{-\pi}^{\pi}\sin nx\sin nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-\cos 2nx}{2}=\frac{1}{2\pi}2\pi=1. & a=n.\end{cases}$$

$$f(x)=\sum_{n=1}^{\infty} b_n(f)\sin nx.=\sin ax.$$

$$\frac{2\sin\pi a}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n\sin nx}{n^2-a^2}$$