CONTROL TASKS

Example 1.

1) Check if the following functions are originals and find their growth index:

a) $f(t)=2e^{3t}\sin at,\ a\in R$

b) $f(t)=e^{3+it^2}$

c) $f(t)=\dfrac{1}{t}$

Solution:

a) The function $f(t)=2e^{3t}\sin at, a\in R$ is continuous at any finite interval $[0,B], B>0$. Therefore, it is integrable on $[0,B]$. Since

$$|f(t)|=2e^{3t}|\sin at|\le 2e^{3t},$$

$M=2, \alpha_0=3$, then $f(t)$ is a function of bounded growth with a growth index of $\alpha_0=3$. Therefore, the $f(t)$ -function is the original.

b) The function $f(t)=e^{3+it^2}$ is continuous and, therefore, integrable on any finite interval $[0,B]$.

$$\left|e^{3+it^2}\right|=e^3\left|\cos t^2+i\sin t^2\right|=e^3\le e^3e^{0\cdot t},$$

where $M=e^3, \alpha_0=0$.

The function $f(t)$ is the original with the growth index $\alpha_0=0$.

c) Function $f(t)=\frac{1}{t}$ is not the original. Integral $\int\limits_0^B \frac{1}{t}\mathrm{d}t = \ln t\Big|_0^B = +\infty$ diverges. Function $f(t)$ is not integrable. The first condition for defining the original function has been violated.

Example 2.

2) The function $F(p)$ is given. Can it be an image of some original in some area? If so, specify this area.

a) $F(p)=1$,

b) $F(p)=\sin p$,

c) $F(p)=\frac{p}{p^2-2p+5}$

<u>Solution:</u>

a) Since

$$\lim_{\mathrm{Re}(p)\to+\infty} F(p)=1,$$

the necessary sign of the existence of an image is not fulfilled for $F(p)$ (Theorem 1). Function $F(p)$ is not an image.

b) Since

$$\lim_{\mathrm{Re}(p)\to+\infty} \sin p$$

does not exist, the necessary sign of the existence of the image is not fulfilled for $F(p)$. Function $F(p)$ is not an image.

c) The necessary indication of the existence of the image

$$\lim_{\mathrm{Re}(p)\to+\infty} \frac{p}{p^2-2p+5}=0$$

has been fulfilled. Function $F(p)$ is analytical in the entire domain except for the zeros of the denominator.

Solving the equation

$$p^2-2p+5=0,$$

we get the simple poles

$$p_{1,2}=1\pm 2i$$

of the function $F(p)$.

Therefore, $F(p)$ will be an image in the region $Re(p)>1$.

In order to verify the correctness of calculations, limiting ratios are used in operational calculus.

Theorem 3 (*on limiting ratios*).

If $f(t), f'(t)$ are originals and $f(t)\leftrightarrow F(p)$, then

$$\lim_{\mathrm{Re}\, p\to+\infty} pF(p)=\lim_{t\to+0} f(t)=f(0), \tag{2}$$

if there is a finite limit of $\lim\limits_{t\to+\infty} f(t)$, then

$$\lim_{p\to 0} pF(p)=\lim_{t\to+\infty} f(t). \tag{3}$$

Example 3.

3) Using the definition, find images of the following functions

a) $f(t)=\theta(t)$

b) $f(t)=e^{4t}$

c) $f(t)=sin t$

Solution:

a) Function $f(t)=\theta(t)$ is the original with a growth index of $\alpha_0=0$.

$$F(p)=\int_0^{+\infty} 1\cdot e^{-pt}dt=\lim_{B\to+\infty}\int_0^B 1\cdot e^{-pt}dt=\lim_{B\to+\infty}\left(-\frac{1}{p}\cdot e^{-pt}\Big|_0^B\right)=\frac{1}{p}.$$

b) The function $f(t)=e^{4t}$ is the original with the growth index $\alpha_0=4$

$$F(p)=\int_0^{+\infty} e^{4t}\cdot e^{-pt}dt=\lim_{B\to+\infty}\int_0^B e^{-(p-4)t}dt=-\lim_{B\to+\infty}\left(\frac{1}{p-4}e^{-(p-4)t}\Big|_0^B\right)=$$

$$=\lim_{B\to+\infty}\left(\frac{1}{p-4}-\frac{e^{-(p-4)B}}{p-4}\right)=\frac{1}{p-4}.$$

Let's check the calculations using the limit ratios (2) and (3). In this case, $\lim\limits_{t\to+\infty} e^{4t}=+\infty$ (the final limit $f(t)$ does not exist), and condition (2) is fulfilled by

$$\lim_{\operatorname{Re} p\to+\infty} pF(p)=\lim_{\operatorname{Re} p\to+\infty}\frac{p}{p-4}=1=\lim_{t\to+0} e^{4t}=f(0).$$

c) The function $f(t)=\sin t$ is the original with the growth index $\alpha_0=0$

$$F(p)=\int_0^{+\infty}\sin t\, e^{-pt}dt=\begin{bmatrix} u=e^{-pt}, & dv=\sin t\,dt, \\ du=-pe^{-pt}dt, & v=-\cos t \end{bmatrix}=$$
$$=-e^{-pt}\cos t\Big|_0^{+\infty}-p\int_0^{+\infty}\cos t\; e^{-pt}dt=\begin{bmatrix} u=e^{-pt}, & dv=\cos t\,dt, \\ du=-pe^{-pt}dt, & v=\sin t \end{bmatrix}=$$
$$=1-p\left(pe^{-pt}\sin t\Big|_0^{+\infty}-p\int_0^{+\infty}\sin t\, e^{-pt}dt\right)=1-p^2\int_0^{+\infty}\sin t\, e^{-pt}dt.$$

From here

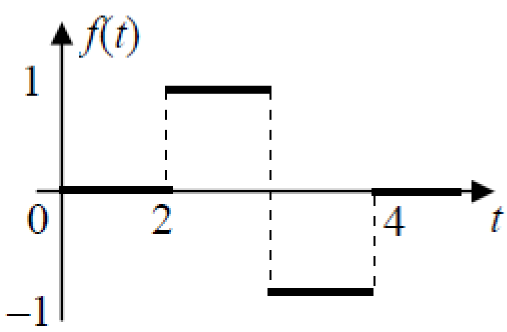
$$\int_0^{+\infty}\sin t\, e^{-pt}dt=1-p^2\int_0^{+\infty}\sin t\, e^{-pt}dt.$$

From the obtained equality, we express the desired integral:

$$F(p)=\int_0^{+\infty}\sin t\, e^{-pt}dt=\frac{1}{p^2+1}.$$

Example 4.

4) Find the image of the function given as follows:



Solution:

The function can be written in analytical form

$$f(t)=\begin{cases}0, & t\le 2,\quad t>4,\\ 1, & 2<t\le 3,\\ -1, & 3<t\le 4.\end{cases}$$

Using the Laplace transform formula:

$$F(p)=\int_0^{+\infty} e^{-pt}f(t)\mathrm{d}t=\int_2^3 e^{-pt}\mathrm{d}t-\int_3^4 e^{-pt}\mathrm{d}t=$$

$$=\frac{1}{p}\left(-e^{-3p}+e^{-2p}+e^{-4p}-e^{-3p}\right)=$$

$$=\frac{e^{-2p}}{p}\left(1-2e^{-p}+e^{-2p}\right)=\frac{\left(e^{-p}\left(1-e^{-p}\right)\right)^2}{p}=\frac{\left(e^{-p}-e^{-2p}\right)^2}{p}.$$

Example 5. (*Using tables and properties of the Laplace transform*)

5) Using the properties of *linearity* and *similarity*, find images of the following functions:

a) $f(t)=\cos t$

b) $f(t)=2-5\cos 2t$

<u>Solution:</u>

a) According to Euler's theorem, $\cos t=\dfrac{e^{it}+e^{-it}}{2}$. Since according to the image table $e^{it}\leftrightarrow\dfrac{1}{p-i}$, $e^{-it}\leftrightarrow\dfrac{1}{p+i}$, then according to the linearity property

$$f(t)=\cos t\leftrightarrow \frac{1}{2}\left(\frac{1}{p-i}+\frac{1}{p+i}\right)=\frac{p}{p^2+1}=F(p).$$

b) By the property of *linearity* and *similarity*

$$f(t) = 2 - 5\cos 2t \leftrightarrow \frac{2}{p} - 5\frac{p}{p^2+4} = F(p).$$

Example 6. (*The displacement property*)

6) Find images of the following functions:

a) $f(t) = e^{-3t}\mathrm{ch}2t$,

b) $f(t) = e^{2t}\cos nt$

Solution:

a) According to the image table, we have $ch2t \leftrightarrow \frac{p}{p^2-4}$. The presence of a multiplier e^{-3t} implies the use of the displacement theorem (*displacement property*). Therefore:

$$e^{-3t}\,ch2t \leftrightarrow \frac{p+3}{(p+3)^2-4} = F(p)$$

b) Since $\cos nt \leftrightarrow \frac{p}{p^2+n^2}$, then

$$f(t) = e^{2t}\cos nt \leftrightarrow \frac{p-2}{(p-2)^2-n^2} = F(p)$$

Example 7. (*Image differentiation.)*

7) Find images of the following functions

a) $f(t) = te^{at}$,

b) $f(t)=te^{t}\cos t$

c) $f(t)=t^{2}\sin t$

<u>Solution:</u>

a) The presence of a multiplier t indicates the need to apply the image differentiation theorem:

Since $e^{at}\leftrightarrow\dfrac{1}{p-a}$, then

$$f(t)=te^{at}\leftrightarrow(-1)^{1}\left(\frac{1}{p-a}\right)'=\frac{1}{(p-a)^{2}}=F(p).$$

b) To find the image, we apply the theorems of image differentiation and displacement

$$\cos t\leftrightarrow\frac{p}{p^{2}+1},$$

$$t\cos t\leftrightarrow-\left(\frac{p}{p^{2}+1}\right)'=-\frac{p^{2}+1-2p^{2}}{(p^{2}+1)^{2}}=\frac{p^{2}-1}{(p^{2}+1)^{2}}.$$

$$f(t)=te^{t}\cos t\leftrightarrow\frac{(p-1)^{2}-1}{((p-1)^{2}+1)^{2}}=\frac{p^{2}-2p}{(p^{2}-2p+2)^{2}}=F(p)$$

c) The presence of a multiplier t^{2} indicates the need to apply the image differentiation theorem

$$sin\,t \leftrightarrow \frac{1}{p^2+1},$$

$$t^2\,sin\,t \leftrightarrow (-1)^2\left(\frac{1}{p^2+1}\right)''$$

$$\left(\frac{1}{p^2+1}\right)' = \frac{-2p}{(p^2+1)^2},$$

$$\left(\frac{1}{p^2+1}\right)'' = \left(\frac{-2p}{(p^2+1)^2}\right)' = \frac{6p^2-2}{(p^2+1)^3}.$$

$$f(t) = t^2\,sin\,t \leftrightarrow \frac{6p^2-2}{(p^2+1)^3} = F(p).$$

Example 8. (*Image integration.)*

8) Find images of the following functions:

a) $f(t) = \frac{e^t - 1}{t}$,

b) $f(t) = \frac{1-\cos t}{t}$

<u>Solution:</u>

a) The function $f(t)$ is continuous for all $t>0$ and is bounded in the vicinity of zero (according to L'hopital's rule $\lim\limits_{t\to+0}\frac{e^t-1}{t}=1$).

Since

$$e^t - 1 \leftrightarrow \frac{1}{p-1} - \frac{1}{p},$$

then by the image integration theorem we obtain

$$f(t)=\frac{e^t-1}{t}\leftrightarrow\int_p^\infty\left(\frac{1}{z-1}-\frac{1}{z}\right)dz=\left(ln|z-1|-ln|z|\right)\Big|_p^\infty=$$

$$=ln\left|\frac{z-1}{z}\right|_p^\infty=ln\frac{p}{p-1}=F(p).$$

b) Since $\lim\limits_{t\to+0}\frac{1-\cos t}{t}=\lim\limits_{t\to+0}\frac{2\sin^2\frac{t}{2}}{t}=\lim\limits_{t\to+0}\sin\frac{t}{2}=0$, then $f(t)$ is continuous and bounded at $t>0$. Let's apply the image integration theorem. Since $1-\cos t\leftrightarrow\frac{1}{p}-\frac{p}{p^2+1}$, then

$$f(t)=\frac{1-\cos t}{t}\leftrightarrow\int_p^\infty\left(\frac{1}{z}-\frac{z}{z^2+1}\right)dz=\left(ln\,z-\frac{1}{2}ln\left(z^2+1\right)\right)\Big|_p^\infty=$$

$$=ln\frac{z}{\sqrt{z^2+1}}\Big|_p^\infty=ln\frac{\sqrt{p^2+1}}{p}=F(p).$$

Example 9. *(Differentiation of the original.)*

9) Find images of the following functions:

a) $f(t)=\sin^2 t$,

b) $f(t)=te^t$.

<u>Solution:</u>

a) Let $f(t) \leftrightarrow F(p)$. Since $f(0)=0$, then $f'(t) \leftrightarrow pF(p) - f(0) = pF(p)$. Calculate the derivative of the function $f(t)$ and find the image for $f'(t)$

$$f'(t) = (sin^2 t)' = 2 sin t cos t = sin 2t \leftrightarrow \frac{2}{p^2+4}$$

Thus, according to the original differentiation theorem, to

determine the image $F(p)$ we have the equation $pF(p) = \frac{2}{p^2+4}$,

solving which we get $F(p) = \frac{2}{p(p^2+4)}$.

b) Let $f(t) \leftrightarrow F(p)$. Since $f(0)=0$, then

$f'(t) \leftrightarrow pF(p) - f(0) = pF(p)$.

Let's find the image for the derivative:

$$f'(t) = \left(te^t\right)' = e^t + te^t \leftrightarrow = \frac{1}{p-1} + F(p)$$

Thus, to determine $F(p)$, we have the equation

$$\frac{1}{p-1} + F(p) = pF(p)$$

Therefore

$$F(p) = \frac{1}{(p-1)^2}$$

Example 10. *(Integrating the original).*

10) Find images of the following functions:

a) $f(t)=\int_0^t \sin\tau d\tau$,

b) $f(t)=\int_0^t \tau^2 e^{-\tau} d\tau$

Solution:

a) Since $\sin t \leftrightarrow \frac{1}{p^2+1}$, then by the original integration theorem

$$\int_0^t \sin\tau\, d\tau \leftrightarrow \frac{1}{p}\cdot\frac{1}{p^2+1}=\frac{1}{p(p^2+1)}$$

b) By the delay theorem $t^2 e^{-t} \leftrightarrow \frac{2!}{(p+1)^3}\cdot$ Then we get

$$\int_0^t \tau^2 e^{-\tau} d\tau \leftrightarrow \frac{1}{p}\cdot\frac{2}{(p+1)^3}=\frac{2}{p(p+1)^3}$$

Theorem *(on convolution, Borel's theorem).*

The convolution of originals

$$(f_1\cdot f_2)(t)=\int_0^t f_1(\tau)f_2(t-\tau)d\tau=\int_0^t f_1(t-\tau)f_2(\tau)d\tau$$

corresponds to the product of images

$$(f_1 \cdot f_2)(t) \leftrightarrow F_1(p)F_2(p).$$

Notation for examples (The convolution of functions):

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,\mathrm{d}\tau = \int_0^t f_1(t-\tau) f_2(\tau)\,\mathrm{d}\tau$$

Example 11.

11) Find the convolution and the image of the convolution (by the properties of the Laplace transform and by the convolution theorem).

a) $t * e^t$

b) $\sin t * t$.

<u>Solution:</u>

a) Let's find the convolution using the formula

$$t * e^t = \int_0^t (t-\tau)e^\tau\,\mathrm{d}\tau = t\left(e^t - 1\right) - \int_0^t \tau e^\tau\,\mathrm{d}\tau = \begin{vmatrix} u = \tau & \mathrm{d}v = e^\tau \mathrm{d}\tau \\ \mathrm{d}u = \mathrm{d}\tau & v = e^\tau \end{vmatrix} =$$

$$= t\left(e^t - 1\right) - \tau e^\tau \Big|_0^t + \int_0^t e^\tau\,\mathrm{d}\tau = t\left(e^t - 1\right) - \left(\tau \cdot e^\tau - e^\tau\right)\Big|_0^t =$$

$$= te^t - t - te^t + e^t - 1 = e^t - t - 1.$$

Let's find the convolution image using the properties of linearity, displacement

$$t * e^t = e^t - t - 1 \leftrightarrow \frac{1}{p-1} - \frac{1}{p^2} - \frac{1}{p} = \frac{1}{p^2(p-1)}$$

Let's find the convolution image according to Borel's theorem

$$t \leftrightarrow \frac{1}{p^2}, \quad e^t \leftrightarrow \frac{1}{p-1}, \quad t * e^t \leftrightarrow \frac{1}{p^2(p-1)}$$

b) <u>HOMEWORK №1.</u>

Example 12.

12) Find images of the following functions:

a) $f(t) = \int_0^t \cos(t-\tau) e^{2\tau} d\tau$,

b) $f(t) = \int_0^t e^{2(\tau - t)} \tau^2 d\tau$

<u>Solution:</u>

a) The function $f(t)$ is a convolution of $f(t) = f_1(t) * f_2(t)$, where $f_1(t) = \cos t$, $f_2(t) = e^{2t}$.

Since $\cos t \leftrightarrow \frac{p}{p^2+1}$, $e^{2t} \leftrightarrow \frac{1}{p-2}$, then

$$\int_0^t \cos(t-\tau) e^{2\tau} d\tau = \cos t * e^{2t} \leftrightarrow \frac{p}{p^2+1} \cdot \frac{1}{p-2} = \frac{p}{(p^2+1)(p-2)}.$$

b) <u>HOMEWORK №2.</u>

Delay theorem *(delay property, Remark).*

If $f(t)\theta(t) \leftrightarrow F(p)$ and $\tau > 0$, then

$$f(t-\tau)\theta(t-\tau) \leftrightarrow e^{-p\tau} F(p)$$

Example 13.

13) Find images of the following functions:

a) $f(t) = e^{t-3}\,\theta(t-3)$

b) $f(t) = (t-1)^2\,\theta(t-1)$

Solution:

a) For the function $e^t\,\theta(t) \leftrightarrow \dfrac{1}{p-1}$. By the delay theorem

$$e^{t-3}\,\theta(t-3) \leftrightarrow \frac{e^{-3p}}{p-1}.$$

It should be noted that $e^{t-3}\,\theta(t) = e^{-3}\,e^t\,\theta(t) \leftrightarrow \dfrac{e^{-3}}{p-1}$

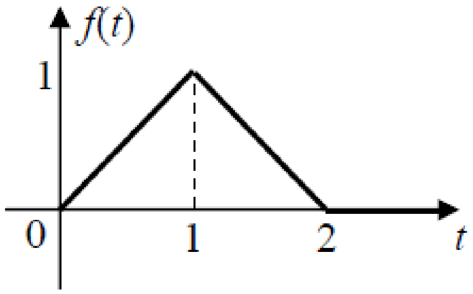
b) For the function $t^2\,\theta(t) \leftrightarrow \dfrac{2}{p^3}$. By the delay theorem

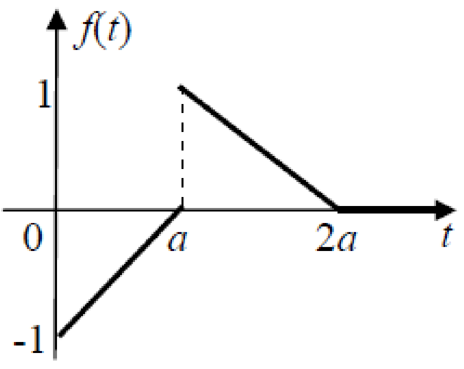
$$(t-1)^2\,\theta(t-1) \leftrightarrow \frac{2e^{-p}}{p^3}.$$

It should be noted that $(t-1)^2\,\theta(t) = (t^2-2t+1)\theta(t) \leftrightarrow \frac{2}{p^3} - \frac{2}{p^2} + \frac{1}{p}$.

Example 14.

14) Find images of the following functions defined graphically:

a) 

b) 

Solution:

a) The function can be written in analytical form

$$f(t) = \begin{cases} 0, & t \le 0, \quad t \ge 2, \\ t, & 0 < t \le 1, \\ 2-t, & 1 < t \le 2. \end{cases}$$

Since

$$f_1(t)=\begin{cases} t, & t\in[0,1] \\ 0, & t\notin[0,1] \end{cases}$$

$$f_1(t)=t\,\theta(t)-t\,\theta(t-1),$$

$$f_2(t)=\begin{cases} 2-t, & t\in[1,2] \\ 0, & t\notin[1,2] \end{cases}$$

$$f_2(t)=(2-t)\theta(t-1)-(2-t)\theta(t-2),$$

the composite function $f(t)=f_1(t)+f_2(t)$ is represented by one analytical expression in the form:

$$f(t)=t\,\theta(t)-t\,\theta(t-1)+(2-t)\theta(t-1)-(2-t)\theta(t-2)=$$

$$=t\,\theta(t)-2(t-1)\theta(t-1)+(t-2)\theta(t-2).$$

Applying the delay theorem, we find the image of the function

$$f(t)\leftrightarrow\frac{1}{p^2}-\frac{2e^{-p}}{p^2}+\frac{e^{-2p}}{p^2}=\frac{1}{p^2}(1-e^{-p})^2.$$

b) <u>HOMEWORK №3.</u>