

# Combinatorics

## Lecture 6

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# Generating functions

We are going to discuss enumeration problems, and how to solve them using a powerful tool: generating functions.

A generating function is just a different way of writing a sequence of numbers. Here we will be dealing mainly with sequences of numbers  $(a_n)$  which represent the number of objects of size  $n$  for an enumeration problem.

The interest of this notation is that certain natural operations on generating functions lead to powerful methods for dealing with recurrences on  $a_n$ .

## Definition.

Let  $(a_n)_{n \geq 0}$  be a sequence of numbers. The generating function associated to this sequence is the series 
$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

# Basic operations

When dealing with generating functions  $F(x) = \sum_{n=0}^{\infty} a_n x^n$  and

$G(x) = \sum_{n=0}^{\infty} b_n x^n$ , we freely use the following operations:

- Differentiate  $F(x)$  term-wise

$$F'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

- Multiply  $F(x)$  by a scalar  $\lambda \in \mathbb{R}$  term-wise

$$\lambda F(x) = \sum_{n=0}^{\infty} \lambda a_n x^n$$

- $F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$

- $F(x) \cdot G(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) x^n$

## Example I.

- ① Consider  $(a_n)$  with  $a_n = 1$  for all  $n \in \mathbb{N}$ . The corresponding generating function is  $F(x) = 1 + x + x^2 + x^3 + \dots$

Note that  $(1 - x)F(x) = 1$

- ②  $F'(x) = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n$

So we found the generating function for  $(b_n)$  with  $b_n = n + 1$ .

- ③ Substituting  $x = -y$  in our series gives the generating function of the alternating sequence (i.e.  $a_n = (-1)^n$ ):

$$\frac{1}{1+y} = \sum_{n=0}^{\infty} (-1)^n y^n$$

Let  $a_n^{(k)}$  be the number of arrangements of  $n$  unlabeled balls in  $k$  labeled boxes and at least 1 ball per box. For each  $k$ , this gives a generating function:

$$B^{(k)}(x) = \sum_{n=0}^{\infty} a_n^{(k)} x^n.$$

Considering the case with only one box, we have

$$a_n^{(1)} = \begin{cases} 0, & \text{for } n = 0; \\ 1, & \text{for } n \geq 1. \end{cases}$$

So,

$$B^{(1)}(x) = 0 + x + x^2 + x^3 + \dots = x \cdot \sum_{n=0}^{\infty} x^n = \frac{x}{(1-x)}$$

**Key observation.** For two numbers of boxes  $s$  and  $t$ , we have the identity:

$$a_n^{(s+t)} = \sum_{l=0}^n a_l^{(s)} a_{n-l}^{(t)}$$

This is merely the observation that arrangements of  $n$  balls in  $s + t$  boxes are given by arranging some  $l$  balls in the first  $s$  boxes and the remaining  $n - l$  balls in the remaining  $t$  boxes. Looking at the right side of the equation, note that these numbers are exactly the coefficients of the product  $B^{(s)}(x) \cdot B^{(t)}(x)$ . So,  $B^{(s+t)}(x) = B^{(s)}(x) \cdot B^{(t)}(x)$  and therefore

$$B^{(k)}(x) = (B^{(1)}(x))^k = \left( \frac{x}{1-x} \right)^k$$

However, we want to write  $B^{(k)}(x)$  in the form  $\sum_{n=0}^{\infty} a_n^{(k)} x^n$ . To do this, note that deriving  $k - 1$  times the term  $(1 - x)^{-1}$  yields  $(k - 1)!(1 - x)^{-k}$ . Using this we obtain

$$\begin{aligned} B^{(k)}(x) &= \left( \frac{x}{(1-x)} \right)^k = \frac{x^k}{(k-1)!} \left( \frac{d^{k-1}}{dx^{k-1}} \frac{1}{(1-x)} \right) = \\ &= \frac{x^k}{(k-1)!} \left( \frac{d^{k-1}}{dx^{k-1}} \sum_{n=0}^{\infty} x^n \right) = \frac{x^k}{(k-1)!} \sum_{n=k-1}^{\infty} n(n-1) \dots (n-k+2) x^{n-k+1} \\ &= \sum_{n=k-1}^{\infty} C_n^{k-1} x^{n+1} = \sum_{n=0}^{\infty} C_{n+1}^{k-1} x^{n+1} \end{aligned}$$

## Example II.

We want to count integer solutions for  $a + b + c = n$  with a nonnegative even integer  $a$ , a non-negative integer  $b$  and  $c \in \{0, 1, 2\}$ .

Equivalently, we can think of this as counting arrangements of  $n$  unlabeled balls in boxes labeled  $a, b$  and  $c$  where box  $a$  should receive an even number of balls and  $c$  at most 2 balls. We first consider the more simple situations where only one of the variables/boxes exists:

- Solutions to  $a = n$  with even  $a$ . Clearly, there is a unique solution for even  $n$  and no solution for odd  $n$ . The corresponding generating function is:

$$A(x) = \sum_{n=0}^{\infty} x^{2n} = \sum_{n=0}^{\infty} (x^2)^n = \frac{1}{1 - x^2}$$

- Solutions to  $b = n$  where  $b$  is an integer. Clearly, there is exactly one solution for each  $n$ . The corresponding generating function is

$$B(x) = \sum_{n=0}^{\infty} x^n$$

- Solutions to  $c = n$  where  $c \in \{0, 1, 2\}$ . Clearly, there is exactly one solution if  $n \in \{0, 1, 2\}$  and no solution otherwise. The corresponding generating function is  $C(x) = 1 + x + x^2$ .
- With the same argument as in the previous example, multiplying the generating functions yields the generating function of the sequence we are interested in, so

$$\begin{aligned} F(x) &= A(x)B(x)C(x) = \frac{(1+x+x^2)}{(1+x)(1-x)^2} = \\ &= \frac{1}{4(1+x)} - \frac{3}{4(1-x)} + \frac{3}{2(1-x)^2} \end{aligned}$$

So the coefficient of  $x^n$ , and therefore the number of solutions to  $a + b + c = n$  we want to count is (Exercise!):

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2}$$

Example III (Catalan numbers). We want to count the number of well-formed parenthesis expressions with  $n$  pairs of parenthesis. For example  $((()))()$  is a well-formed expression with 4 pairs of parenthesis but  $)()())$  is not.

Every well-formed expression with  $n \geq 1$  pairs of parenthesis starts with "(" and there is a unique matching ")" such that the sequence in between and the sequence after is a well-formed (possibly empty) expression. For example:

$((()))()$     $((()))()$     $((()))()$

In other words, a well-formed expression with  $n$  pairs of parenthesis is obtained by putting a well-formed expression with  $k$  pairs in between "(" and ")" and then append a well-formed expression with  $n - k - 1$  pairs of parenthesis. This gives the equation:

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k-1}$$

So if  $F(x)$  is the generating function belonging for  $(a_n)$ , then we know:

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} a_k a_{n-k-1} \right) x^n = \\ &= 1 + \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k a_{n-k} \right) x^{n+1} = 1 + x \cdot \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k a_{n-k} \right) x^n = 1 + x F(x)^2 \end{aligned}$$

We can find a solution to this as (Exercise!):  $F(x) = \frac{1 - \sqrt{1 - 4x}}{2x}$ .

How to express this function "explicitly"?

Binomial formula shows that  $(1 + x)^n$  is the generating function for the sequence  $(a_k)$  with  $a_k = C_n^k$ .

**Definition (Binomial Coefficients for Real Numbers).** Recall that for  $n, k \in \mathbb{N}$  we have:  $C_n^k = \frac{n(n-1)\dots(n-k+1)}{k!}$ .

It does not make sense to talk about permutations of sets of size  $n \in \mathbb{R}$  and it is unclear what  $n!$  should be, but the formula above is well-defined for general  $n \in \mathbb{R}$ . With this in mind, we define  $p(n, k) := n(n-1)\dots(n-k+1)$  and  $C_n^k := \frac{p(n, k)}{k!}$ .

Additionally we have

$$p(n, k) = (n - k + 1)p(n, k - 1) = np(n - 1, k - 1)$$

**Lemma A.** For any integer  $n \geq 1$  we have

$$C_{\frac{1}{2}}^n = (-1)^{n+1} C_{2n-2}^{n-1} \frac{1}{2^{2n-1} \cdot n}$$

Proof. We do induction on  $n$ . For  $n = 1$  :  $C_{\frac{1}{2}}^1 = \frac{1}{2}$  - ok.

Induction step:

$$\begin{aligned} C_{\frac{1}{2}}^{n+1} &= \frac{p(1/2, n+1)}{(n+1)!} = \frac{n-1/2}{-n-1} C_{\frac{1}{2}}^n = \\ &= \frac{n-1/2}{-n-1} (-1)^{n+1} C_{2n-2}^{n-1} \frac{1}{2^{2n-1} \cdot n} = \\ &= \frac{2n(2n-1)}{2n \cdot 2n} (-1)^{n+2} C_{2n-2}^{n-1} \frac{1}{2^{2n-1} \cdot (n+1)} = (-1)^{n+2} C_{2n}^n \frac{1}{2^{2n+1} \cdot (n+1)} \end{aligned}$$

Corollary.

$$\sqrt{1+x} = \sum_{n=0}^{\infty} C_{\frac{1}{2}}^n x^n = 1 + \sum_{n=1}^{\infty} -2C_{2n-2}^{n-1}(-1)^n \frac{1}{2^{2n}n} x^n$$

We are now able to find the coefficients of the generating function for Catalan's numbers  $F(x)$ :

$$\begin{aligned} F(x) &= \frac{1 - \sqrt{1-4x}}{2x} = \frac{1}{2x} \sum_{n=1}^{\infty} 2C_{2n-2}^{n-1}(-1)^n \frac{1}{2^{2n}n} (-4x)^n = \\ &= \frac{1}{x} \sum_{n=1}^{\infty} C_{2n-2}^{n-1} \frac{1}{n} x^n = \sum_{n=0}^{\infty} C_{2n}^n \frac{1}{n+1} x^n \end{aligned}$$

Finally, we get the Catalan numbers  $\mathfrak{C}_n = C_{2n}^n \frac{1}{n+1}$ .

# Fibonacci numbers

The famous **Fibonacci sequence** is defined by its initial terms  $f_0 = f_1 = 1$  and the relation

$$f_{n+2} = f_{n+1} + f_n.$$

To derive the generating function formula

$$Fib(x) = f_0 + f_1x + f_2x^2 + f_3x^3 + \dots$$

multiply both sides of the equation by  $x + x^2$  obtain

$$\begin{aligned}(x+x^2)Fib(x) &= x+x^2+2x^3+3x^4+5x^5+\dots+x^2+x^3+2x^4+3x^5+\dots = \\ &= x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \dots\end{aligned}$$

In other words,

$$(x + x^2)Fib(x) = Fib(x) - 1$$

and

$$Fib(x) = \frac{1}{1 - x - x^2}$$

# Fibonacci numbers

$$\frac{1}{1-x-x^2} = \frac{1}{\sqrt{5}} \left( \frac{1}{x-x_1} + \frac{1}{x-x_2} \right) = \frac{1}{\sqrt{5}} \left( \frac{1}{x_1(1-\frac{x}{x_1})} + \frac{1}{x_2(1-\frac{x}{x_2})} \right)$$

$x_1 = \frac{-1+\sqrt{5}}{2}$ ,  $x_2 = \frac{-1-\sqrt{5}}{2}$  - roots of the equation  $1-x-x^2=0$ .

From this we immediately obtain

$$Fib(x) = \frac{1}{\sqrt{5}x_1} \left( 1 + \frac{x}{x_1} + \frac{x^2}{x_1^2} + \dots \right) - \frac{1}{\sqrt{5}x_2} \left( 1 + \frac{x}{x_2} + \frac{x^2}{x_2^2} + \dots \right)$$

Therefore

$$\begin{aligned} f_n &= \frac{1}{\sqrt{5}} (x_1^{-1-n} - x_2^{-1-n}) = \frac{(-1)^n}{\sqrt{5}} (x_1^{n+1} - x_2^{n+1}) = \\ &= \frac{(-1)^n}{\sqrt{5}} \left( \left( \frac{-1+\sqrt{5}}{2} \right)^{n+1} - \left( \frac{-1-\sqrt{5}}{2} \right)^{n+1} \right) \end{aligned}$$

# More algebraic operations

Let  $A(s) = a_0 + a_1s + a_2s^2 + \dots$ ,  $B(t) = b_0 + b_1t + b_2t^2 + \dots$  - two generating functions, and  $B(0) = b_0 = 0$ .

Definition. A **substitution** of a generating function  $B$  into a generating function  $A$  is a generating function

$$\begin{aligned} A(B(t)) &= a_0 + a_1B(t) + a_2B(t)^2 + a_3B(t)^3 + \dots = \\ &= a_0 + a_1b_1t + (a_1b_2 + a_2b_1)t^2 + (a_1b_3 + 2a_2b_1b_2 + a_3b_1^3)t^3 + \dots \end{aligned}$$

## Theorem (Inverse function).

Let  $B(t) = b_1t + b_2t^2 + \dots$ , and  $b_1 \neq 0$ , then there exists  $A(s) = a_1s + a_2s^2 + \dots$  and  $C(u) = c_1u + c_2u^2 + \dots$  such that

$$A(B(t)) = t \quad \text{and} \quad B(C(u)) = u.$$

Moreover, the functions  $A$  and  $C$  are unique.

Function  $A$  is called left inverse to function  $B$ , and function  $C$  is called right inverse to function  $B$ .

Proof. Let us prove the existence and uniqueness of the left inverse function (The proof for the right inverse is similar.) We will determine the coefficients of the function  $A$  successively. The coefficient  $a_1$  is determined from the condition  $a_1b_1$ , whence

$$a_1 = \frac{1}{b_1}$$

Let us now assume that the coefficients  $a_1, a_2, \dots, a_n$  have already been determined. The coefficient  $a_{n+1}$  is determined from the condition

$$a_{n+1}b_1^{n+1} + \dots + a_1b_{n+1} = 0,$$

where dots denote some polynomial in  $a_1, \dots, a_n, b_1, \dots, b_n$ .

Thus, the condition is a linear equation on  $a_{n+1}$ , and the coefficient  $b_1^{n+1}$  of  $a_{n+1}$  is nonzero. Clearly, this equation has a unique solution. ■

So, we learned how to add and multiply power series and substitute them for each other. We would also like to learn how to divide them into each other. The last operation is not always well-defined. However, it is possible to divide by a power series whose value at zero is different from zero.

### Proposition.

$A(s) = \sum_{i=0}^{\infty} a_i x^i$  - generating function, and  $A(0) = a_0 \neq 0$ . Then there is a unique  $B(s) = \sum_{i=0}^{\infty} b_i x^i$ , such that  $A(s)B(s) = 1$ .

Proof. Let's do the proof again by induction. The value of the coefficient  $b_0$  is easy to find :  $b_0 = \frac{1}{a_0}$ . Let now all the coefficients of the series  $B$  up to the degree  $n - 1$  be uniquely determined. The coefficient at  $s^n$  is determined from the condition

$$a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0 = 0$$

This is a linear equation on  $b_n$ , and the coefficient  $a_0$  of  $b_n$  is nonzero. Therefore, the equation has a unique solution. ■

Sometimes it is convenient to expand generating functions in powers  $\{\frac{x^n}{n!}\}$ .

**Some ideas.** Say the numbers  $(a_n)$  and  $(b_n)$  count arrangements of type  $A$  and  $B$ , respectively, using  $n$  labeled objects.

Exercise. Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence defined recursively as follows:

- (i)  $a_0 = 0, a_1 = 1$
- (ii)  $a_{n+2} = -a_n + 2a_{n+1}$  for all  $n \geq 0$

Using the generating function technique, find the formula for  $a_n$ .