2nd-order. Surfaces.

orthogonal transformation $\begin{pmatrix} \cos\alpha_{11} & \cos\alpha_{12} & \cos\alpha_{13} \\ \cos\alpha_{21} & \cos\alpha_{22} & \cos\alpha_{23} \\ \cos\alpha_{22} & \cos\alpha_{32} & \cos\alpha_{33} \end{pmatrix}$ α_{ij} is angle between e_i and e_j'

Characteristic equation:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0. \quad \text{or} \quad \lambda^3 - S\lambda^2 + K_1\lambda - \delta = 0.$$

Invariants. $\delta.\ \Delta.\ S.\ K_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$

semi-invariant. $K_2 = \begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{33} & a_3 \\ a_3 & a_0 \end{vmatrix}$ $K_3 = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} & a_1 \\ a_{31} & a_{33} & a_3 \\ a_1 & a_3 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} & a_2 \\ a_{32} & a_{33} & a_3 \\ a_2 & a_3 & a_0 \end{vmatrix}.$

type: I: $\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + \frac{\Delta}{\delta} = 0$

II: $\lambda_1 X^2 + \lambda_2 Y^2 \pm \sqrt{-\frac{\Delta}{K_1}}\, Z = 0.$

III: $\lambda_1 X^2 + \lambda_2 Y^2 + \frac{K_3}{K_1} = 0$

IV: $SX^2 \pm \sqrt{-\frac{K_3}{S}}\, Y = 0.$

V: $SX^2 + \frac{K_2}{S} = 0.$

$\delta \xrightarrow{\neq 0}$ type I. (ellipsoid/hyperboloid/cone). central surface. (single center).

1) have asymptotic cone (real/imaginary cone). $\frac{X^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 0.$

2) no special direction.

$\xrightarrow{\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq 0}$ not revolution. 3-principal direction.

$\xrightarrow{\lambda_1 = \lambda_2 \neq \lambda_3 = 0}$ revolution: 1 p.d + any direction $\perp$ the p.d.

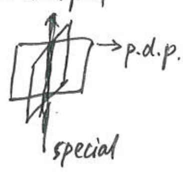
$\xrightarrow{=0}$ Δ $(\lambda_3 = 0)$. $\xrightarrow{\neq 0}$ type II 1) no center. rank $A = 2$. rank $A^* = 3$. no asymptotic cone. but have cones of asymptotic direction.

2). have one special direct (Oz').

$\xrightarrow{\lambda_1 = \lambda_2 \neq 0}$ revolution · any direction perpendicular to vector λ_3. is principal

not principal but special

$\xrightarrow{\lambda_1 \neq \lambda_2 \neq 0}$ not revolution. (hyperbolic). 2-p.d. mutually $\perp$.



$\xrightarrow{=0}$ cylinder/plane. $K_1 \xrightarrow{\neq 0}$ type III. (line center). rank A = rank $A^* = 2$.

1). 1 special direct (Oz').

2). have asymptotic cone. (intersecting plane)

$\xrightarrow{=0}$ $K_3 \xrightarrow{\neq 0}$ type IV. (parabolic cylinder only).

1) no center. 2). special direction (any direct on $Y'O'Z$).

3) cone of asymptotic direction: $y=0$ (plane)

$\xrightarrow{=0}$ type V. (plane center).

1) special direction ($Y'O'Z$).

2). asymptotic cone: central plane.

common. $\lambda_1 \neq 0$. one principal direction.

* $\lambda_1 = \lambda_2 = \lambda_3$. sphere. (any direction is principal direction)

Quadratic curves.

general equation.

$a=\sqrt{-\frac{\Delta}{\delta\lambda_1}}$ $b=\sqrt{-\frac{\Delta}{\delta\lambda_2}}$

$p=\sqrt{-\frac{\Delta}{S^3}}$

orthogonal transformation. $\begin{cases} x = c_{11}x' + c_{12}y' + c_1 \\ y = \cancel{c_{21}}y' + c_{22}y' + c_2 \end{cases}$ $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ is orthogonal.

if $c_1 = c_2 = 0$. homogenous orthogonal transformation.

Invariant   semi-invariant

$S = a_{11} + a_{22}$ $\quad \Delta = \begin{vmatrix} a_{11} & a_{12} & a_1 \\ a_{21} & a_{22} & a_2 \\ a_1 & a_2 & a_0 \end{vmatrix}$ $\quad \delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ $\quad K = \begin{vmatrix} a_{11} & a_1 \\ a_1 & a_0 \end{vmatrix} + \begin{vmatrix} a_{22} & a_2 \\ a_2 & a_0 \end{vmatrix}$

$\Delta = 0$. lines.

characteristic root $\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0 \iff \lambda^2 - S\lambda + \delta = 0.$

$\delta \xrightarrow{\neq 0}$ single center $\xrightarrow{>0}$ elliptical type. no asymptotic direction.

$\lambda_1 x^2 + \lambda_2 y^2 + \frac{\Delta}{\delta} = 0$ $\xrightarrow{<0}$ hyperbolical type 2 asympototic direction

$\boxed{com}$ non-asymptotic direction cross center $\Leftrightarrow$ diameter.

$\lambda_1\lambda_2 \neq 0$. two principal direction mutually ~~per~~ perpendicular. ($\lambda_1 = \lambda_2$. no asymptotic. any is principal).

(direction of principal diameter also principal)

$\lambda_1 x^2 + 2\sqrt{-\frac{\Delta}{S}}\, y = 0$

$\lambda_1 x^2 + \frac{K}{S} = 0.$

By solving $\begin{cases} (a_{11}-\lambda)\alpha + a_{12}\beta = 0. \\ a_{21}\alpha + (a_{22}-\lambda)\beta = 0. \end{cases} \Rightarrow \begin{matrix} \lambda_1 \to \{\alpha_1, \beta_1\} \\ \lambda_2 \to \{\alpha_2, \beta_2\} \end{matrix}$ canonical axes $\xrightarrow[\text{divide by length}]{} \begin{matrix} e_1' \\ e_2' \end{matrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e_1' & e_2' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$

$\xrightarrow{=0.}$ $\Delta \xrightarrow{\neq 0}$ parabola   1-asymptotic direction. (axis/central line).

(no center).

$\xrightarrow{=0} K$ $\xrightarrow{<0}$ parallel line. $\xrightarrow{>0}$ imaginary ~ $\xrightarrow{=0}$ coincide line.

(line center).

com. asymptotic line $\iff$ diameter. (line center. only diameter - central line).

$\lambda_1 = 0$. $\lambda_2 \neq 0$.

principal direction. $\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}$ or $\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$. asymptotic direction $\begin{pmatrix} -a_{12} \\ a_{11} \end{pmatrix} \overset{or}{=} \begin{pmatrix} -a_{22} \\ a_{21} \end{pmatrix}$.

1. Asymptotic direction $\{\alpha, \beta\}$. $a_{11}\alpha^2 + 2a_{12}\alpha\beta + a_{22}\beta^2 = 0.$

2. Diameter $\{\alpha', \beta'\}$. $\alpha F_x' + \beta F_y' = 0.$ or $(a_{11}x + a_{12}y + a_1)\alpha + (a_{21}x + a_{22}y + a_2)\beta = 0.$ ①

$\begin{cases} \alpha' = -(a_{12}\alpha + a_{22}\beta) \\ \beta' = a_{11}\alpha + a_{21}\beta \end{cases}$ ($\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ is the direction of chords. non-asymptotic direction).

3. transform: rotation: $\tan\alpha = \frac{\lambda_1 - a_{11}}{a_{12}}$ translation: $(x, y) \to (x_0, y_0)$ (center)

4. tangent line: at (x_0, y_0). $(a_{11}x_0 + a_{12}y_0 + a_1)x + (a_{21}x_0 + a_{22}y_0 + a_2)y + a_1x_0 + a_2y_0 + a_0 = 0.$

the diameter ① intersect the line at non-singular $M(x_0, y_0)$.

tangent line at M has direction of chords. $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

Asymptotic direction $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} \to \varphi(x,y,z) = a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz = 0.$

(the generatrices of the cone are lines have a.d. of the surface).

只是叫 "cone", 不一定是锥.

"asymptotic cone": $\varphi(x,y,z) = 0$. vertex is center.

Diameter plane. chords $\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$. plane: $F'_x\alpha + F'_y\beta + F'_z\gamma = 0.$

Special direction, (平行于所有径面的方向). all have except type I.

$$\begin{pmatrix}\alpha_0\\ \beta_0\\ \gamma_0\end{pmatrix} \Leftarrow \begin{cases} a_{11}\alpha_0 + a_{12}\beta_0 + a_{13}\gamma_0 = 0\\ a_{21}\alpha_0 + a_{22}\beta_0 + a_{23}\gamma_0 = 0\\ a_{31}\alpha_0 + a_{32}\beta_0 + a_{33}\gamma_0 = 0. \end{cases}$$

Tagent plane. (all tangent line $\xrightarrow{\text{consist}}$ plane).

$$F'_x(x_0,y_0,z_0)(x-x_0) + F'_y(x_0,y_0,z_0)(y-y_0) + F'_z(x_0,y_0,z_0)(z-z_0) = 0.$$

$$\begin{cases} \dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} + \dfrac{z_0z}{c^2} = 1.\\ \dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} - \dfrac{z_0z}{c} = \pm 1.\\ \dfrac{x_0x}{a^2} + \dfrac{y_0y}{b^2} - \dfrac{z_0z}{c^2} = 0\\ \dfrac{x_0x}{p} \pm \dfrac{y_0y}{q} = z+z_0. \end{cases}$$

point (real. non-decomposable surface).

tangent plane × the surface.

pair of imaginary intersecting lines ($\Delta<0$). elliptic. (ellipsoid. two-sheet hy-. elliptical. para-).

— real ———— ($\Delta>0$). hyperbolic. (one-sheet. hy.. hyper.. para-)

— coinciding ———— ($\Delta=0$) parabolic (real cone (vertex ×) elliptic. parabolic. hyperbolic cylinder).

Principal direction. (主轴的方向). p.d.p. (径面的一种).

Simplest equation

$x^2+y^2+z^2=1.$

$x^2+y^2+z^2=\pm1.$

$x^2\pm y^2=z.$

$x^2+y^2-z^2=0$ (real. non-decomposing cone).

$x^2\pm y^2=1.$ $x^2-y^2=1$ $y^2=x$ (real. non-decomposing cylinder)

1. Axioms of geometry and basic theorems derived from them (without proofs) .

2. Vectors: directed segments and abstract (free) vectors. **Addition of vectors. Multiply vector by number.**

3. Product of vectors. **Dot product** of vectors, definition and basic properties. Left and right triplets of vectors. **Cross product** of vectors, definition and basic properties. **Mixed product** of vectors. definition and basic properties.

4. Expansion of vectors on plane and in space. **Coordinates of vector**. **Basis vectors**.

5. Coordinate system with right orthonormal basis on plane and in space. **Formulas for dot, cross and mixed products**. Radius vector. **Coordinates of point**.

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\vec{e_1} + (a_z b_x - a_x b_z)\vec{e_2} + (a_x b_y - a_y b_x)\vec{e_3}$$

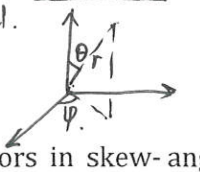
$$(\vec{a}\vec{b}\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c})$$

圆柱 球

6. Polar, cylindrical and spherical coordinates.

ellipse 椭圆.
hyperbola 双曲
parabola ~~双曲~~抛物

spherical.



$$\begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases}$$

7. Vectors in skew-angular basis. Linearly dependent and independent systems of vectors. Basis vectors as linearly independent system. Example: **analytic geometry on a single line**.

8. Dot product in skew-angular basis. **Gram matrix**.

$$\vec{a}\cdot\vec{b} = (a_1\ a_2\ a_3)\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

9. Cross and mixed products in skew-angular basis. Oriented volume of basis. **Effectivised formulas for mixed and cross products**

$$(abc) = \pm\sqrt{\det G}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} a^i b^j c^k \varepsilon_{ijk} = \pm\sqrt{\det G}\begin{vmatrix} a^1 & a^2 & a^3 \\ b^1 & b^2 & b^3 \\ c^1 & c^2 & c^3 \end{vmatrix}$$

+ right basis - left basis

$$a\times b = \pm\sqrt{\det G}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\sum_{q=1}^{3} a^i b^j \varepsilon_{ijq} g^{qk} e_k.$$

Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd.} \\ 0 & i=j \text{ or } i=k \text{ or } k=j. \end{cases}$$

$$C_{ijk} = \pm\sqrt{\det G}\ \varepsilon_{ijk}$$

$$C_{ij}^{k} = \pm\sqrt{\det G}\sum_{q=1}^{3}\varepsilon_{ijq} g^{qk}$$

10. The product of two mixed products

$$(abc)(xyz) = \begin{vmatrix} ax & ay & az \\ bx & by & bz \\ cx & cy & cz \end{vmatrix}$$ (dot product).

11. **Triple product. Jacobi identity** .

$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$. $(\vec{a}\times\vec{b})\times\vec{c} = \vec{b}\cdot(\vec{a}\cdot\vec{c}) - \vec{a}\cdot(\vec{b}\cdot\vec{c})$

$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$.

12. Analytic geometry of points on plane and in space: **distance form the origin, distance between two points.**

13. Analytic geometry of the segment on plane and in space: **division ratio, inclination** 倾角. **and slope on plane. Direction cosines in space.**

$$\tan\alpha = \frac{m_2 - m_1}{1 - m_1 m_2}$$

14. **Area of polygon**

15. Equation of the locus: Degrees of freedom. (similar as dimension)

16. Steps to discuss and plot locus. Intersection of curves.

$$\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases}$$

17. **Vectorial equations of straight line on plane in parametric and normal form** .

$\vec{r} = \vec{r}_0 + \vec{a}t$.

18. Coordinate parametric equations.

$\cancel{\vec{r} = \vec{r_0} + a}$ $\begin{cases} x = x_0 + a_x t \\ y = y_0 + a_y t. \end{cases}$

19. **Canonical equation of line and its forms** : line passing through two given points, line passing through the origin:

$\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$. / $x = x_1$ / $y = y_1$.

20. Explicit equations of the line in slope- point and slope- intercept form.

$y = m(x - x_0) + y_0$. $y = mx + b$.
(slope-point) (slope intercept)

21. Double intercept equation of the line

$\frac{x}{a} + \frac{y}{b} = 1$ (~~slope~~ double intercept)

22. **General equation of the line in orthonormal basis** . Explanation of coefficients. **Angle between lines**. Parallel and perpendicular lines

$Ax+By+C=0.$

23. Normal form of the equation. **Measure of distances from the line.**

$x\cos w+y\sin w-p=0.$

$$d=x_1\cos w+y_1\sin w-p=\frac{|Ax_1+By_1+C|}{\sqrt{A^2+B^2}}$$

24. **Bundles (beams) of lines** . Intersection of three lines in a point.

proper

improper $=$

have solution (ALS). $\Leftrightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=0.$

25. Line in polar coordinates.

normal equation: $r\cos\varphi\cos w+r\sin\varphi\sin w-p=0.$

$\Rightarrow r\cos(\varphi-w)=p.$

$\boxed{26}$. General equation of the line in arbitrary basis. Dual basis and covariant coordinates on plane. Transformation of bases on plane. Explanation of the coefficients in general equation of line.

$e^1=\begin{pmatrix}\frac{g_{22}}{\det G}\\ -\frac{g_{12}}{\det G}\end{pmatrix}$ $e^2=\begin{pmatrix}\frac{-g_{12}}{\det G}\\ \frac{g_{11}}{\det G}\end{pmatrix}$

Gram matrix. reciprocal.

$\vec{a}=(a_1,a_2)$

$\vec{b}=(b^1,b^2)$

$\vec{a}\cdot\vec{b}=a_1\cdot b^1+a_2\cdot b^2$

27. Generalization of the explicit equations of line for any skew-angular basis. **General formula for angle coefficient (slope)**. **General formula to measure angle between lines**. Parallel and perpendicular lines.

motion: New axes $\begin{cases} a_1x+b_1y+C_1=0 \\ a_2x+b_2y+C_2=0 \end{cases}$ $x'=\frac{a_1x+b_1y+C_1}{\sqrt{a_1^2+b_1^2}}$ $y'=\frac{a_2x+b_2y+C_2}{\sqrt{a_2^2+b_2^2}}$

$\frac{\sin\varphi}{\sin(w-\varphi)}$

28. Transformation of coordinate system. General definitions. Example: Rotation matrix. **Example: Transformation to coordinates expressed with equations of perpendicular lines** .

$e_1'=x_1e_1+x_2e_2+x_3e_3$

$e_2'=y_1e_1+y_2e_2+y_3e_3$

$e_3'=z_1e_1+z_2e_2+z_3e_3.$

$S=\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}.$

$(e_1'\ e_2'\ e_3')=(e_1\ e_2\ e_3)\}S$

$(X_1\ X_2\ X_3)^T=S\begin{pmatrix}X_1'\\X_2'\\X_3'\end{pmatrix}.$

29. Plane in space. Vectorial equatins: parametric and normal.

$r_0=r_0+at+b\tau$

$\begin{cases} x=x_0+at+b\tau \\ y=y_0+at+b\tau \\ z=z_0+at+b\tau. \end{cases}$

30. Plane in space. **Canonic equation in vector and coordinate forms** .

$(r,a,b)=D.$ $(r,a,b)=(e_1,e_2\ e_3)\begin{vmatrix} x & y & z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

$\vec{r}\cdot\vec{n}=D$

$\vec{r}=\begin{pmatrix}r^1\\r^2\\r^3\end{pmatrix}$ $\vec{n}=\begin{pmatrix}n^1\\n^2\\n^3\end{pmatrix}$

31. **General equation of plane**. Explanation of coefficients. Covariant coordinates and dual basis in space. **Angle between planes**.

$n_i=\sum_{j=1}^{3}n^jg_{ij}$

(normal vector)

$Ax+By+Cz+D=0.$

$$\cos\theta=\frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{}\cdot\sqrt{}}$$

32. **Equation of the plane passing through three points** .

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

33. Normal equation of the plane in right orthonormal basis. **Measuring distances.**

$\vec{r} \cdot \vec{n} = \vec{r_0} \cdot \vec{n}$

scalar form

$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0$

34. **Triple intercept equation of the plane.**

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

distance $d = \left| \dfrac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}} \right|$

35. Beams and bundles of planes. **Equation of the plane from proper beam. Equation of the plane from proper bundle.**

bundle proper same point

improper

beam proper

improper

No. 李升洺

Date 20221124

Week12. 5,10

P1. Let those point satisfied the condition be (x_n, y_n).

$$\begin{cases} y = k(x - x_n) + y_n & \text{(the tangent line)} \\ y^2 = 2px. & \text{(the parabola)} \end{cases}$$

k must exist otherwise we have the unique tangent line $x=0$.

$$\Rightarrow \quad k^2(x-x_n)^2 + 2ky_n(x-x_n) + y_n^2 = 2px$$

$$k^2x^2 + (2ky_n - 2k^2x_n - 2p)x + k^2x_n^2 + y_n^2 - 2kx_ny_n = 0.$$

has unique solution.

i.e. $(2ky_n - 2k^2x_n - 2p)^2 - 4k^2(k^2x_n^2 + y_n^2 - 2kx_ny_n) = 0.$

$$2x_npk^2 - 2y_npk + p^2 = 0.$$

the tangent line ~~are~~ mutually perpendicular. $k_1k_2 = -1$.

$$\frac{p^2}{2x_np} = -1 \qquad x_n = -\frac{p}{2}.$$

thus the locus is $x = -\frac{p}{2}$.

P2. let the tangent point (x_0, y_0).

the tangent line $\frac{x_0x}{a^2} + \frac{y_0y}{a^2-9} = 1.$

$$\Rightarrow \quad (a^2-9)x_0x + a^2y_0y - a^2(a^2-9) = 0.$$

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{a^2-9} = 1. & \text{(ellipse with foci } (\pm 3, 0)) \\ x + y - 5 = 0. \end{cases}$$

$$\Rightarrow \quad (2a^2-9)x^2 - 10a^2x + 34a^2 - a^4 = 0$$

unique solution. $(-10a^2)^2 - 4(2a^2-9)(34a^2-a^4) = 0.$ $\Rightarrow a=0$ or $a = \pm\sqrt{17}$ or $a = \pm 3$.

since $a > 0$ and $a > 3$. we have $a = +\sqrt{17}$.

the ellipse $\frac{x^2}{17} + \frac{y^2}{8} = 1.$

P3. by the condition $\begin{cases} 2 \cdot \frac{a}{e} = 12.8. \\ 2 \cdot \frac{b}{e} = 7.2. \end{cases} \Rightarrow \begin{cases} a = \frac{8}{\sqrt{10}} \\ b = \frac{6}{\sqrt{10}} \\ c = 10 \end{cases}$

$$\Rightarrow \begin{cases} \frac{x^2}{6.4} - \frac{y^2}{3.6} = 1 \\ \frac{y^2}{6.4} - \frac{x^2}{3.6} = 1. \end{cases} \text{ or } \begin{cases} \frac{x^2}{3.6} - \frac{y^2}{6.4} = 1. \\ \frac{y^2}{3.6} - \frac{x^2}{6.4} = 1. \end{cases}$$

conjugate hyperbolas.

P4. For arbitrary point $P(x_0, y_0)$ on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

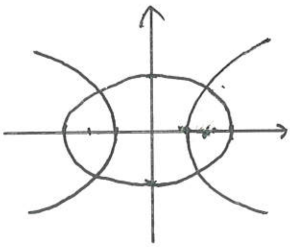
the asymptote. $y = \pm \frac{b}{a} x$. $\Rightarrow bx \pm ay = 0$.

$$d_1 \cdot d_2 = \frac{|bx_0 + ay_0|}{\sqrt{b^2+a^2}} \cdot \frac{|bx_0 - ay_0|}{\sqrt{b^2+a^2}} = |b^2x_0^2 - a^2y_0^2| \cdot \frac{1}{\sqrt{b^2+a^2}}$$

since $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$. $\Rightarrow b^2x_0^2 - a^2y_0^2 = a^2b^2$

$\Rightarrow d_1 \cdot d_2 = \frac{a^2 \cdot b^2}{\sqrt{b^2+a^2}}$ does not depend on x_0, y_0

P5.



Let the intersection point be (x_0, y_0).

the tangent of ellipse $\frac{x_0 x}{a_1^2} + \frac{y_0 y}{b_1^2} = 1$.

the tangent of hyperbola. $\frac{x_0 x}{a_2^2} - \frac{y_0 y}{b_2^2} = 1$.

the slope $m_1 = -\frac{x_0}{y_0} \cdot \frac{b_1^2}{a_1^2}$

$m_2 = \frac{x_0}{y_0} \cdot \frac{b_2^2}{a_2^2}$

$$m_1 \cdot m_2 = -\frac{x_0^2}{y_0^2} \cdot \frac{b_1^2 b_2^2}{a_1^2 a_2^2}$$

since $\frac{x_0^2}{a_1^2} + \frac{y_0^2}{b_1^2} = 1$ and $\frac{x_0^2}{a_2^2} - \frac{y_0^2}{b_2^2} = 1$ holds. simultaneously.

we have $\frac{x_0^2}{y_0^2} = \frac{\frac{1}{b_1^2} + \frac{1}{b_2^2}}{\frac{1}{a_2^2} - \frac{1}{a_1^2}}$

thus $m_1 \cdot m_2 = -\frac{b_1^2 + b_2^2}{a_1^2 - a_2^2} = -\frac{(a_1^2 - c^2) + (c^2 - a_2^2)}{a_1^2 - a_2^2} = -1$.

two tangent lines are perpendicular. i.e. ~~insect~~ intersect orthogonally.

Length.

1. $|h(u,v)| \le \|u\| \cdot \|v\|$
2. $\|u+v\| \le \|u\| + \|v\|$.

Least square.

distance (between vectors). $dist(u,v) = \|u-v\|$.

if $dist(v, u_0) = \min_{u \in U} dist(v, u)$ $u_0 = Proj_U(v)$.

$Ax = b$ if $\|Ax_0 - b\| \le \|Ax - b\|$ for any $x \in \mathbb{R}^n$. x_0 is the least squares solution

~~to~~ all least square solution is $Ax = b_0$. $b_0 = Proj_{ImA}(b)$.

Norm

$\sup_{\|x\|=1} \|L(x)\| = \|L\|_2$

Conditional Number

$K(L) = \|L\|_2 \cdot \|L^{-1}\|_2$. $K(L) = \frac{\sigma_1(L)}{\sigma_n(L)}$

Angle.

v. U. smallest angle.

$\cos\theta_{min} = \max_{u_1 \in U_1,\, u_2 \in U_2} \frac{(u_1, u_2)_V}{\|u_1\| \cdot \|u_2\|}$

$\cos\theta_{min} = \|P_1 P_2\|_2$

principal angle. $\theta_1 \le \cdots \le \theta_k$. $\cos\theta_1 \ge \cos\theta_2 \cdots \ge \cos\theta_k$ singular value

$\|P_2 P_1\|_2 = \|P_2|_{U_1}\|_2$

Canonical form. (for operator).

self-adjoint $D = Q^{-1} A Q$.

U consist of orthonormal basis.

normal $A = U^{-1} D' U$. D' bbock diag. real-eigenvalue. 1×1. complex-eigenvalue $a \pm bi$.
U consist of orthonormal basis 2×2. $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

symmetric matrix.

1). all its upper left minor non-zero. $LAL^T = D$. (L is low triangular)

2). $Q^{-1} A Q = D$. Q is orthogonal (unitary, in complex case).

isometries

orthogonal operator. $A = U^{-1} D' U$. U transition of standard to orthonormal basis.

D' bbock diag. $[\pm 1]$. $\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$.

Find: compute characteristic value. (we have write the canonical form straightly)

find eigenvector. $Q = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}$. $D' = Q^T A Q$.

Any operator orthonormal basis. $[L]_B$. upper triangular

Decomposition:

Polar. $L = SQ$. (positive * isometry) $= QS_1$ (invertible)

$A = SQ$ (symmetric positive definite.* orthogonal/unitary).

Singular Value.

$A \in M_{m,n}(\mathbb{R})$. $Q_1 A Q_2 = \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}$.

$A = Q_1^{-1} D Q_2^{-1} = \underbrace{(Q_1^{-1} D Q_1)}_{S} \underbrace{(Q_1^{-1} Q_2^{-1})}_{Q}$. $\Rightarrow$ polar decomposition.

$$e_n = V_n - \sum_{k=1}^{n-1} \frac{h(V_n, e_k)}{h(e_k, e_k)} e_k$$

$$\mathrm{Proj}_U(V) = \sum_{i=1}^{k} \frac{h(V, u_i)}{h(u_i, u_i)} u_i$$

李开言 2022111221

3 | 2 | 2.5

1 (1). ~~Firstly~~ show it's a subspace in real vector space.

$\forall A, A' \in \mathcal{H}_2$. s.t. $A^T = \bar{A}$ $A'^T = \bar{A'}$

$(A+A')^T = A^T + A'^T = \bar{A} + \bar{A'} = \overline{A+A'}$

$\forall a \in \mathbb{R}$.

$(aA)^T = a \cdot A^T = a \cdot \bar{A} = \overline{aA}$.

(2) $\forall t \in \mathbb{C}$. $\forall A \in \mathcal{H}_2$ s.t. $A^T = \bar{A}$.

$(tA)^T = t \cdot A^T = t\bar{A} \neq \overline{tA}$

if t has a non-zero imaginary part. $t \neq \bar{t}$. $\mathcal{H}_2$ not satisfy the scalar multiplication.

thus. $\mathcal{H}_2$ is not a subspace in complex vector space $M_2(\mathbb{C})$.

(3). $\forall A = \begin{pmatrix} a_{11}+b_{11}i & a_{12}+b_{12}i \\ a_{21}+b_{21}i & a_{22}+b_{22}i \end{pmatrix} \in \mathcal{H}_2$. $a_{ij}, b_{ij} \in \mathbb{R}$ $1 \le i, j \le 2$.

$A^T = \bar{A}$ implies. $\begin{cases} b_{11} = 0 \\ b_{22} = 0 \\ a_{21} + b_{21} i = a_{12} - b_{12} i \\ a_{12} + b_{12} i = a_{21} - b_{21} i \end{cases} \Rightarrow \begin{cases} b_{11} = b_{22} = 0 \\ b_{21} + b_{12} = 0 \\ a_{21} = a_{12} \end{cases}$

thus A can be rewrite as. $A = \begin{pmatrix} a_{11} & a_{12}+b_{12}i \\ a_{12} - b_{12}i & a_{22} \end{pmatrix}$.

For any $A \in \mathcal{H}_2$. have the form above, we can find the standard basis. s.t.

$A = a_{11}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b_{12}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$.

thus. $\dim \mathcal{H}_2 = 4$.

2. Denote. $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

we find that. ~~$v_1 = v_2 + v$~~ $v_2 = v_1 - v_3$. v_1, v_3 are linear independent. v_1, v_2, v_3 are l.d.

Thus. $\dim U = 2$. the standard dot product is non-degenerate. thus we have $\mathbb{R}^4 = U \perp U^\perp$.

$\dim U^\perp = \dim \mathbb{R}^4 - \dim U = 2$.

3.(1) Since all entries in A.B are real number, its sufficient to prove the operator is orthogonal. i.e. the matrix of the operators is orthogonal.

$$AA^T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = E_3.$$

✓

$$BB^T = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} = E_3.$$

Thus. we have the matrix are orthogonal. as well as the operator.

(2) consider the $\mathbb{R}^3$ with standard basis $\{e_1, e_2, e_3\}$.

$\det A = 1 \Rightarrow L_A$ is a rotation.

$\forall \vec{x} \in \mathbb{R}^3$ with coordinate column $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$L_A(\vec{x}) = \begin{pmatrix} y \\ z \\ x \end{pmatrix}. \qquad L_B(\vec{x}) = \begin{pmatrix} -z \\ y \\ -x \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}.$$

1/2

L_A is a rotation, L_B is a reflection. Why?

(3) the rotation matrix $\begin{pmatrix} 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \\ 1 & 0 & 0 \end{pmatrix}$

— it is not a canonical form $\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos\varphi & \sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{array}\right)$

the angle $\alpha = \frac{\pi}{2}$.

the axis : $x = y = z$.

0

$\chi_A(t) = -(t-1)(t^2+t+1)$

$\lambda_1 = 1 \quad \lambda_2 = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$

canonical $\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$\frac{2\pi}{3}$

(4). $L_B(\vec{x}) = \begin{pmatrix} -z \\ y \\ -x \end{pmatrix}$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{x-z}{2} + \frac{x+z}{2} \\ y \\ \frac{z-x}{2} + \frac{x+z}{2} \end{pmatrix} = \begin{pmatrix} \frac{x-z}{2} \\ y \\ \frac{z-x}{2} \end{pmatrix} + \begin{pmatrix} \frac{x+z}{2} \\ 0 \\ \frac{x+z}{2} \end{pmatrix}$$

we can find that. $L_B(\vec{x}) = \begin{pmatrix} \frac{x-z}{2} \\ y \\ \frac{z-x}{2} \end{pmatrix} - \begin{pmatrix} \frac{x+z}{2} \\ 0 \\ \frac{x+z}{2} \end{pmatrix} = \begin{pmatrix} -z \\ y \\ -x \end{pmatrix}.$

and $\begin{pmatrix} \frac{x-z}{2} \\ y \\ \frac{z-x}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{x+z}{2} \\ 0 \\ \frac{x+z}{2} \end{pmatrix} = \frac{x^2-z^2}{4} + \frac{z^2-x^2}{4} = 0.$

thus. the mirror is span $\left\{ v = \begin{pmatrix} \frac{x-z}{2} \\ y \\ \frac{z-x}{2} \end{pmatrix} \right\}$ ✓

the mirror = span $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ equation : $x + z = 0.$

Newton - Leibnize.

primise. $[ab]$ ① F is primitive.

② F is primitive except a finite number and f is continuous.

Euler. formula.

$$e^{ix} = \cos x + i\sin x.$$

$$\sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right). \qquad \sum_{k=1}^{n} \sin kx = \frac{\sin\frac{nx}{2}\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}.$$

$$\cos kx = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right) \qquad \sum_{k=1}^{n} \cos kx = \frac{\cos\frac{nx}{2}\sin\frac{n+1}{2}x}{\cos\frac{x}{2}}.$$

Wallis Formula.

$$I = \int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \begin{cases} \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2} & n=2k. \\ \frac{(2k)!!}{(2k+1)!!} & n=2k+1. \end{cases}$$

$$\lim_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 \cdot \frac{1}{2n+1} = \frac{\pi}{2}.$$

Abel Transformation

$$A_k = \sum_{i=1}^{k} a_i. \qquad \sum_{k=1}^{n} a_k b_k = A_n b_n + \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}).$$

$$\sum_{k=n+1}^{m} a_k b_k = A_m b_m - A_n b_{n+1} - \sum_{k=n+1}^{m-1} A_k (b_{k+1} - b_k).$$

$$= (A_m - A_n) b_{n+1} - \sum_{k=n+1}^{m-1} (A_k - A_m)(b_{k+1} - b_k).$$

Hölder's inequality $f, g \in C[a,b]$, $\frac{1}{p} + \frac{1}{q} = 1$.

$$\left|\int_a^b fg\right| \le \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left(\int_a^b |g|^q\right)^{\frac{1}{q}}.$$

$p=q=2$. Cauchy's inequality $\left|\int_a^b fg\right| \le \left|\int_a^b |f|^2 \cdot \int_a^b |g|^2\right|^{\frac{1}{2}}$

Minkowski's inequality. $f, g \in C[a,b]$, $p \ge 1$. $\left(\int_a^b |f+g|^p\right)^{\frac{1}{p}} \le \left(\int_a^b |f|^p\right)^{\frac{1}{p}} \left(\int_a^b |g|^p\right)^{\frac{1}{p}}$

Chebyshev's inequality f increase g decrease.

$$\frac{1}{b-a}\int_a^b fg \le \left(\frac{1}{b-a}\int_a^b f\right)\left(\frac{1}{b-a}\int_a^b g\right).$$

For sum. $\{a_k\}$ non decrease $\{b_k\}$ non-increase.

$$\frac{1}{n}\sum_{k=1}^n a_k b_k \le \left(\frac{1}{n}\sum_{k=1}^n a_k\right)\left(\frac{1}{n}\sum_{k=1}^n b_k\right).$$

Average of a function. $f\in R[a,b]$ for any $[a,b]\subset\mathbb{R}$.

$$F_\delta(x)=\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} f(t)\,dt.$$

1) $F_\delta\in C(\mathbb{R})$. 2) $f\in C^k(\mathbb{R}) \to F\in C^{k+1}(\mathbb{R})$

3) $f\in C(\mathbb{R})$. $\lim\limits_{\delta\to 0^+} F_\delta(x)=f(x)$.

camparisioon test. $f(x)=O(g(x))$. $\exists k.\ \exists c.\ \forall x\in[c,b).\ f(x)\le k\,g(x)$

$a_k=O((b_k))$ $\exists K.\ \forall k>K$ $a_k\le K b_k$.

Stirling's formula.

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}} \quad \theta_n\in(0,1)$$

Asympototic Formula. (~~eo~~ estimate the error)

$$H_n=\log n+\gamma+\delta_n.\quad \delta_n\to 0.$$

$$\lim_{n\to\infty}\sum_{k=1}^n f(k)-\int_1^{n+1} f=\gamma$$

Check uni. conv.

1. $f_n \rightrightarrows f$ on E.
2. $\forall\varepsilon>0\ \exists N\in\mathbb{N}, \forall n>N.$ $\sup\limits_{x\in E}|f_n(x)-f(x)|<\varepsilon$.
3. $\lim\limits_{n\to\infty}\sup\limits_{x\in E}|f_n(x)-f(x)|=0$.
4. $\exists\{\varepsilon_n\}$. nonnegative numbers. s.t. $\forall x\in E.$ $|f_n(x)-f(x)|\le\varepsilon_n$. $\varepsilon_n\to 0$.

$f_n\rightrightarrows f$. g bounded. ~~onE~~, $f_n g\rightrightarrows fg$ on X.

permutation of limit. $\lim\limits_{n\to\infty}\lim\limits_{x\to p} f_n(x)=\lim\limits_{x\to p}\lim\limits_{n\to\infty} f_n(x)$

(pre. $f_n\rightrightarrows f$. $\exists A_n=\lim\limits_{x\to p} f_n(x)$

For continuous.

(1). pointwise. ① $f_n \rightrightarrows f$ ② $\forall n$. f_n continuous on p. $\Rightarrow f$ continuous in p

(2) uni. ① $f_n \rightrightarrows f$ ② $\forall n$. f_n continuous in E $\Rightarrow f$ continuous on E

Dini. f is pointwise limit of f_n. [$u_k \geq 0$. $\sum u_k$ conv. to continuous sum $\Rightarrow$ uni. conv.]

for every x, $\{f_n(x)\}$ increasing (w.r.t. n) $f_n \rightrightarrows f$

For integrable.

(1) squeeze thm. (NS)

(2) $f_n \in R[a,b]$. $f \in R[a,b]$. $f_n \rightrightarrows f$. $\Rightarrow \int_a^b f_n \to \int_a^b f$.

(3). $f_n \in C[a,b]$, $f_n \rightrightarrows f$. Then $\int_a^b f_n \to \int_a^b f$

$\to f \in C[a,b]$ (previous thm.)

For differentiation.

(1). f_n differentiable $\not\Rightarrow$ f differentiable ($f_n = \sqrt{\frac{nx^2+1}{n}} \rightrightarrows f(x) = |x|$

(2). f_n differentiable. if $\{f_n'\}$ uni. conv. to φ. $\exists c \in [a,b]$. $f_n(c)$ conv.

$\Rightarrow$ ① f_n uni conv. to some f. ② f is differentiable ③ $f' = \varphi$.

$$\sum_{k=0}^{n} e^{ikx} = \frac{1 - e^{i(n+1)x}}{1 - e^{ix}}.$$

$[a,b] \subset (2\pi m, 2\pi(m+1))$ b_k monotone. $\to 0$. $\sum b_k \sin kx \to$ conv.

$p \in (0,1]$. $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$ not conv. uniformly on $(0, 2\pi)$ [证不一致. x_n 与 n 有关.

Power series.

For given R. $\bar{B}_r$ $r \in (0,R)$. conv. disc.

on the disc. continuous.

Abel thm. $S(R) = \sum_{n=1}^{\infty} a_n R^n$ conv.

then the series conv. uni. on $[0,R]$. $S(R) = \lim_{x \to R-} S(x)$

f is smooth. $f^{(m)}(x) = \sum_{k=m}^{+\infty} \frac{k!}{(k-m)!} a_k x^{k-m}$

$x \in (-R,R)$. $\int_0^x f = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$

$\sum_{k=0}^{\infty} \frac{a_k}{k+1} R^{k+1}$ conv. $\int_0^R f = \sum_{k=0}^{\infty} \frac{a_k}{k+1} R^{k+1}$ (improper. maybe)

$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$. $|x|<1$. (denote $\binom{\alpha}{0} = 1$.)

麦克劳林. $|x|<1$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $x \in \mathbb{R}$.

$\sin x = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ $x \in \mathbb{R}$

$\cos x = \sum_{n=0} \frac{(-1) x^{2n}}{(2n)!}$ $x \in \mathbb{R}$

$\ln(1+x) = \sum_{n=1} (-1)^{n-1} \cdot \frac{x^n}{n}$.

$\frac{1}{1-x} = \sum_{n=0} x^n$

$(1+x)^m = \sum_{n=0} \binom{m}{n} x^n$.

$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1}$ $\arcsin x = \sum_{n=1} \frac{(2n-1)!!}{(2n+1)(2n)!!} x^{2n+1}$

证. $|f_n(x) - f(x)| \to 0$

① $f_n(x)$ 泰勒展开

② $f_n'(x)$ 求 $f_n(x)$ 关于 x 的最大值

③ $\sup\limits_{x\in(0,1)}$ $\geq$ 可取 $x=\frac{1}{n}$.

④ $\left|\sum\limits_{k=n+1}^{n+p} u_k(x)\right| \geq \varepsilon$. 常考虑 $\left|\sum\limits_{k=n+1}^{2n} u_k(x)\right|$.

$f_n(x)$ 连续 若 $f_n \rightrightarrows f$. 必有 f 连续.
(若作出 f 不连续(点点收敛时). 则有 f_n 不 $\rightrightarrows f$).