Mathematical Logic

Lecture 4

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Let's start with examples:

• From $\forall x P(x)$ one can conclude $P(t)$ - formula, that results from $P(x)$ by substituting t for x

• From $P(t)$ - formula, that results from $P(x)$ by substituting t for x one can conclude $\exists x P(x)$

• From $P(t_1)$, $t_1 \equiv t_2$ formula that results from $P(x)$ by substituting t_1 for x, one can conclude $P(t_2)$ - formula that results from $P(x)$ by substituting t_2 for x.

But one has to be careful : Should we be allowed to draw an inference from

$$\forall x \exists y \; y < x \; - \text{true in the model of the real ordered field}$$

$$\text{to } \exists y \; y < y \; - \text{false in the model of the real ordered field}$$

by a substitution of y for x? No!

Why does this last substitution go wrong? In

$$\exists y \ y < x$$

the variable x occurs freely, but in

$$\exists y \ y < y$$

the second occurrence of y, which was substituted for x, is bound!

So, this corresponds to a change of meaning.

Problems like these can be avoided in the following manner: Draw an inference from $\forall x \exists y \ y < x$ convert into

$$\forall x \exists u \ u < x$$

$$\text{then : substitute } y \text{ for } x$$

to $\exists u \ u < y$ (this last formula is true in the model of the real ordered field - independent of which real number is chosen to be $s(y)$).

Wanted:
1.) An "intelligent" substitution function $\frac{t}{x}$ by which a term t is substituted for free occurrences of a variable x (substitutions for bound occurrences should be prohibited), such that problematic cases as the above one are avoided by automatic renaming of bound variables.

2.) It should be possible to substitute terms $t_0, \ldots, t_n$ "simultaneously" for pairwise distinct variables $x_0, \ldots, x_n$ (t_0 for x_0, t_1 for x_1, ...).

E.g.: $[P(x, y)]\frac{y,x}{x,y} = P(y, x)$ (y is substituted for x and simultaneously x is substituted for y.)

But: $[[P(x, y)\frac{y}{x}]\frac{x}{y}] = [P(y, y)]\frac{x}{y} = P(x, x)$ (So simultaneous substitution cannot be replaced by the iteration of simple substitutions.)

$\boxed{\textit{Definition } 1.}$ Let $\mathcal{S}$ be an arbitrary symbol set. Let $t_0, \ldots, t_n \in \mathcal{T}_\mathcal{S}$, let $x_0, \ldots, x_n$ be pairwise distinct variables. We define the substitution function $\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n}$ on $\mathcal{T}_\mathcal{S} \cup \mathcal{F}_\mathcal{S}$ as follows:

$$[x]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} = \begin{cases} t_i, & \text{for } x = x_i; \\ x, & \text{else.} \end{cases}$$

$$[c]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} = c$$

$$[f(t'_1,\ldots,t'_m)]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} := f([t'_1]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n},\ldots,[t'_m]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n})$$

$$[t'_1 \equiv t'_2]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} := [t'_1]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n} \equiv [t'_2]\frac{t_0,\ldots,t_n}{x_0,\ldots,x_n}$$

$$[P(t'_1, \ldots, t'_m)]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} \text{ is defined analogously to the case of } \phi$$

$$[\neg\phi]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} := \neg[\phi]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n}$$

$$[\phi \vee \psi]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} := ([\phi]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} \vee [\psi]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n})$$

Let $x_{i_1}, \ldots, x_{i_k}$ be those variables x_i among $x_0, \ldots, x_n$ for which it holds that: (i) $x_i \in \mathit{free}(\exists x\phi)$ (ii) $x_i \neq t_i$

Call these variables the relevant variables of the substitution.

$$[\exists x\phi]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} := \exists u[\phi]\frac{t_{i_1}, \ldots, t_{i_k}, u}{x_{i_1}, \ldots, x_{i_k}, x}$$

where $u := x$, if x does not occur in $t_{i_1}, \ldots, t_{i_k}$

else: let u be the first variable in $v_0, v_1, v_2, \ldots$ that does not occur in $\phi, t_{i_1}, \ldots, t_{i_k}$

Remark. Consider the substitution case for $\exists x \phi$:

• x is certainly distinct from any of $x_{i_1}, \ldots, x_{i_k}$, because $x \notin free(\exists x \phi)$

• Assume there are no variables x_i with $x_i \in free(\exists x \phi)$ and $x_i \neq t_i$, so $k = 0$, and there are no $t_{i_1}, \ldots, t_{i_k}$ to consider. Thus, x does not occur within $t_{i_1}, \ldots, t_{i_k} \Rightarrow \; u = x \Rightarrow [\exists x \phi]\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} = \exists u [\phi]\frac{u}{x} = \exists x [\phi]\frac{x}{x} = \exists x \phi$

• It follows from the definition of our substitution function that u does not occur within $t_{i_1}, \ldots, t_{i_k}$

Example. (For two variables x, y with $x \neq y$) :

• $[\exists y \; y < x]\frac{y}{x} = \exists u[y < x]\frac{y,u}{x,y} =$ since y occurs within t_{i_1} , i.e., within y, it follows from our definition that u must be distinct from $x, y =$)
$= \exists u[y]\frac{y,u}{x,y} < [x]\frac{y,u}{x,y} = \exists u \; u < y$

• $[\exists y \; y < x]\frac{x}{y}$ (since y is not free in $\exists y \; y < x$, only the substitution for u is going to remain)
$= \exists u \; [y < x]\frac{u}{y} =$ (since the number k of relevant variables is in this case 0, there are no $t_{i_1}, \ldots, t_{i_k}$ in which y could occur, thus it follows that $u = y$)
$= \exists y \; [y < x]\frac{y}{y} = \exists y \; y < x$. We see that nothing can be substituted for bound variables.

• $[\exists v_0 \; P(v_0, f(v_1, v_2))]\frac{v_0, v_2, v_4}{v_1, v_2, v_0} = \ldots$ (Exercise !)

Substitution Lemma.

Let $\mathfrak{M}$ be an $\mathcal{S}-$model:

1.) For all terms $t \in \mathcal{T}_\mathcal{S}$:

For all variable assignments s over $\mathfrak{M}$, for all terms $t_0, \ldots, t_n \in \mathcal{T}_\mathcal{S}$, for all pairwise distinct variables $x_0, \ldots, x_n$:

$$Val_{\mathfrak{M},s}(t\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n}) = Val_{\mathfrak{M},s\frac{Val_{\mathfrak{M},s}(t_0), \ldots, Val_{\mathfrak{M},s}(t_n)}{x_0, \ldots, x_n}}(t)$$

2.) For all formulas $\phi \in \mathcal{F}_\mathcal{S}$:

For all variable assignments s over $\mathfrak{M}$, for all terms $t_0, \ldots, t_n \in \mathcal{T}_\mathcal{S}$, for all pairwise distinct variables $x_0, \ldots, x_n$:

$$\mathfrak{M}, s \models \phi\frac{t_0, \ldots, t_n}{x_0, \ldots, x_n} \text{ iff } \mathfrak{M}, s\frac{Val_{\mathfrak{M},s}(t_0), \ldots, Val_{\mathfrak{M},s}(t_n)}{x_0, \ldots, x_n} \models \phi$$

Proof. Easy, but tedious.

Example. It is easy to express unique existence by means of substitution:

$$\exists ! x\phi := \exists x(\phi \wedge \forall y(\phi\frac{y}{x} \to y \equiv x))$$

By the substitution lemma:

$$\mathfrak{M}, s \models \exists ! x\phi \iff \text{ there is one and only one } d \in D \text{ such that: } \mathfrak{M}, s\frac{d}{x} \models \phi$$

Calculus

In the following we will make precise what we mean by a "mathematical proof"'.

Let $\mathcal{S}$ be an arbitrary symbol set. An $\mathcal{S}-$sequent is a finite sequence

$$\phi_1\phi_2\ldots\phi_n$$

of $\mathcal{S}-$formulas.

We call

- $\phi_1\phi_2\ldots\phi_{n-1}$ the antecedent of the sequent ("assumptions")
- ϕ_n the consequent of the sequent (i.e., the formula for which we want to claim that it follows from the assumptions)

While proving theorems informally, we implicitly "manipulate" sequents in various ways (we extend sequents, we drop auxiliary assumptions, ...).

Formally:

- Show that both $\phi_1\phi_2\ldots\phi_k\neg\psi\rho$ and $\phi_1\phi_2\ldots\phi_k\neg\psi\neg\rho$ are correct
- Conclude that $\phi_1\phi_2\ldots\phi_k\psi$ is correct

We have to make "show" and "conclude" precise. What we need is (i) rules which introduce sequents that are obviously correct, like $\phi\phi$ and
(ii) rules which lead from correct sequents to further correct sequents, e.g.,

$$\left.\begin{array}{r}\phi_1\phi_2\ldots\phi_k\neg\psi\rho\\ \phi_1\phi_2\ldots\phi_k\neg\psi\neg\rho\end{array}\right\}\text{Premises}$$

$$\left.\phi_1\phi_2\ldots\phi_k\psi\right\}\text{Conclusion}$$

The sequent calculus is a specific set of such rules (some have premises, some have not). We will see that its premise-free rules only lead to correct sequents and that its rules with premises lead from correct sequents to other correct sequent (so these rules are "correctness-preserving").

But what is the correctness of a sequent? (we will use Γ, Δ,... as variables for sequents).

$\boxed{\textit{Definition 2.}}$ For all $\mathcal{S}-$sequents $\Gamma\ \phi$:
$\Gamma\ \phi$ is correct iff $\{\psi | \psi$ is sequence member of $\Gamma\} \models \phi$

Once we are given the rules of the sequent calculus, derivability of sequents in the sequent calculus can be defined analogously to derivability of terms in the terms calculus.

Finally, we can define the derivability of formulas from other formulas on basis of the derivability of sequents:

$\boxed{\textit{Definition 3.}}$ Let Φ be a set of $\mathcal{S}-$formulas, let ϕ be an $\mathcal{S}-$formula: ϕ is derivable from Φ (briefly $\Phi \vdash \phi$) if and only if there are $\phi_1, \ldots, \phi_n \in \Phi$, such that $\phi_1, \ldots, \phi_n \phi$ is derivable in the sequent calculus.

Lemma 1.

$\Phi \vdash \phi \iff$ there is a finite set $\Phi' \subseteq \Phi$ such that $\Phi' \vdash \phi$.

Proof. Obvious. ■

Now we are going to introduce the rules of the sequent calculus. These rules are divided into the following groups: basic rules; rules for propositional connectives; rules for quantifiers; rules for equality.

Rules

Antecedent rule :

$$\frac{\Gamma\phi}{\Gamma'\phi} \qquad \text{for} \quad \Gamma \subseteq \Gamma'$$

So, we are always allowed to add assumptions and we are always allowed to permute them.

Proof. Assume that $\Gamma\phi$ is correct, i.e., $\Gamma \models \phi$. Let Γ' be a set of $\mathcal{S}-$formulas, such that $\Gamma \subseteq \Gamma'$. Let $\mathfrak{M}, s$ be chosen arbitrarily, such that $\mathfrak{M}, s \models \Gamma' \Rightarrow \mathfrak{M}, s \models \Gamma \Rightarrow \mathfrak{M}, s \models \phi$ (by assumption). It follows that $\Gamma' \models \phi$, i.e., $\Gamma'\phi$ is correct. ■

Rules

Assumption rule:

$$\frac{}{\Gamma\phi} \qquad \text{for } \phi \text{ being a sequence member of } \Gamma$$

So, we are always allowed to conclude assumptions from themselves.

Proof. If $\phi \in \Gamma$, then certainly $\Gamma \models \phi$; hence, $\Gamma\phi$ is correct. ■

Proof by cases:

$$\frac{\begin{array}{c}\Gamma\psi\phi \\ \Gamma\neg\psi\phi\end{array}}{\Gamma\phi}$$

Thus, if we can show ϕ both under the assumption ψ and under the assumption $\neg\psi$ (and since one of these two assumptions must actually be the case), we are allowed to conclude ϕ without assuming anything about ψ or $\neg\psi$.

Rules

Proof. Assume that $\Gamma\psi\phi$, $\Gamma\neg\psi\phi$ are correct, i.e., $\Gamma \cup \{\psi\} \models \phi$ and $\Gamma \cup \{\neg\psi\} \models \phi$. Let $\mathfrak{M}, s$ be chosen arbitrarily such that $\mathfrak{M}, s \models \Gamma$.
There are two possible cases:
Case 1: $\mathfrak{M}, s \models \psi \Rightarrow \mathfrak{M}, s \models \Gamma \cup \{\psi\} \Rightarrow \mathfrak{M}, s \models \phi$
Case 2: $\mathfrak{M}, s \not\models \psi \Rightarrow \mathfrak{M}, s \models \neg\psi \Rightarrow \mathfrak{M}, s \models \Gamma \cup \{\neg\psi\} \Rightarrow \mathfrak{M}, s \models \phi \Rightarrow$
$\Gamma \models \phi \Rightarrow \Gamma\phi$ is correct. ■

Contradiction:

$$\frac{\begin{array}{c}\Gamma\neg\psi\rho\\ \Gamma\neg\psi\neg\rho\end{array}}{\Gamma\psi}$$

So, if assuming $\neg\psi$ leads to a contradiction, then we are allowed to infer ψ

Proof. Assume that $\Gamma\neg\psi\rho$, $\Gamma\neg\psi\neg\rho$ are correct, i.e., $\Gamma \cup \{\neg\psi\} \models \rho$ and $\Gamma \cup \{\neg\psi\} \models \neg\rho$.
So for all $\mathfrak{M}, s$ with $\mathfrak{M}, s \models \Gamma \cup \{\neg\psi\}$ it must hold that:

$$\mathfrak{M}, s \models \rho \text{ and } \mathfrak{M}, s \models \neg\rho$$

Thus, there are no $\mathfrak{M}, s$ such that $\mathfrak{M}, s \models \Gamma \cup \{\neg\psi\} \Rightarrow$ for all $\mathfrak{M}, s$ with $\mathfrak{M}, s \models \Gamma$ holds: $\mathfrak{M}, s \models \psi \Rightarrow \Gamma \models \psi \Rightarrow \Gamma\psi$ is correct. ■

$\boxed{\vee\text{- Introduction in the antecedent:}}$

$$\frac{\Gamma\phi\rho \qquad \Gamma\psi\rho}{\Gamma(\phi \vee \psi)\rho}$$

Disjunctions $\phi \vee \psi$ in the antecedent allow for being treated in terms of two cases - case ϕ on the one hand and case ψ on the other.

$\vee$- Introduction in the consequent:

1.) $\frac{\Gamma\phi}{\Gamma(\phi\vee\psi)}$

2.) $\frac{\Gamma\psi}{\Gamma(\phi\vee\psi)}$

So, we are always allowed to weaken consequents by introducing disjunctions.

Excluded middle

$$\frac{}{\phi \vee \neg\phi}$$

Triviality

$$\frac{\Gamma\phi \qquad \Gamma\neg\phi}{\Gamma\psi}$$

Chain syllogism

$$\Gamma \phi \psi$$

$$\frac{\Gamma \phi}{\Gamma \psi}$$

Contraposition

1.) $\frac{\Gamma \phi \psi}{\Gamma \neg \psi \neg \phi}$

2.) $\frac{\Gamma \phi \neg \psi}{\Gamma \psi \neg \phi}$

3.) $\frac{\Gamma \neg \phi \psi}{\Gamma \neg \psi \phi}$

4.) $\frac{\Gamma \neg \phi \neg \psi}{\Gamma \psi \phi}$

Disjunctive syllogism

$$\frac{\begin{array}{c}\Gamma(\phi \vee \psi)\\ \Gamma\neg\phi\end{array}}{\Gamma\psi}$$

∃ - Introduction in the consequent

$$\frac{\Gamma\phi\frac{t}{x}}{\Gamma\ \exists x\phi}$$

Here x and y are arbitrary variables.
Thus, if we can conclude from Γ that t has the property expressed by the formula ϕ, then we are also allowed to conclude from Γ that there exists something which has the property expressed by ϕ.

$\exists$ - Introduction in the antecedent:

$$\frac{\Gamma \phi\frac{y}{x}\psi}{\Gamma\ \exists x\phi\psi}$$

Reflexivity

$$\frac{}{t \equiv t}$$

So, the equality relation on D is reflexive (independent of the model).

Substitution rule:

$$\frac{\Gamma \phi\frac{t}{x}}{\Gamma t \equiv t' \phi\frac{t'}{x}}$$

Symmetry

$$\frac{\Gamma \quad t_1 \equiv t_2}{\Gamma \quad t_2 \equiv t_1}$$

Contradiction:

$$\frac{\begin{array}{c}\Gamma \quad t_1 \equiv t_2 \\ \Gamma \quad t_2 \equiv t_3\end{array}}{\Gamma \quad t_1 \equiv t_3}$$

Exercises.

Exercise 1. Let $t_0, t_1, \ldots, t_n$ be $\mathcal{S}-$terms, $x_0, x_1, \ldots, x_n$ pairwise distinct variables, ϕ an $\mathcal{S}-$formula. Let π is a permutation of the elements $0, \ldots, n$. Prove:

$$\phi\frac{t_0, t_1, \ldots, t_n}{x_0, x_1, \ldots, x_n} = \phi\frac{t_{\pi(0)}, t_{\pi(1)}, \ldots, t_{\pi(n)}}{x_{\pi(0)}, x_{\pi(1)}, \ldots, x_{\pi(n)}}$$