Chapter 3-2. Systems of linear algebraic equations

4) Iteration method $Ax=b$ $\det A \neq 0$

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ - - - & - - - & - - - & - - - & - - - \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{in} \\ - - - & - - - & - - - & - - - & - - - \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix}$$

$$x=\begin{pmatrix} x_1 \\ x_2 \\ - - \\ x_i \\ - - \\ x_n \end{pmatrix} \qquad b=\begin{pmatrix} b_1 \\ b_2 \\ - - \\ b_i \\ - - \\ b_n \end{pmatrix}$$

Iteration method defines a sequence of approximate solutions $x^{(k)}$ that converge to the exact solution $x^{(k)} \to x^*$ as $k \to \infty$

Let us transform $Ax=b$ to the form $x=Cx+d$, then choose some $x^{(1)}$ and organize iterations:

$$x^{(k+1)}=Cx^{(k)}+d\,, \qquad k=1,2,\ldots$$

Example:

$$1.01\,x_1 + 0.2\,x_2 = 3 \quad \Rightarrow \quad (1+0.01)\,x_1 + 0.2\,x_2 = 3$$

$$0.05\,x_1 + 1.08\,x_2 = 2 \quad \Rightarrow \quad 0.05\,x_1 + (1+0.08)\,x_2 = 2$$

$$x_1 = 3 - 0.01\,x_1 - 0.2\,x_2$$

$$x_2 = 2 - 0.05\,x_1 - 0.08\,x_2$$

choose $x_1^{(1)}=3\,,\ \ x_2^{(1)}=2$

$$x_1^{(k+1)} = 3 - 0.01\,x_1^{(k)} - 0.2\,x_2^{(k)}$$

$$x_2^{(k+1)} = 2 - 0.05\,x_1^{(k)} - 0.08\,x_2^{(k)}$$

A way of transforming $Ax=b$ to the form $x=Cx+d$ is <u>Jacobi iterative method</u>

Carl Jacobi 1804 - 1851

We illustrate it for the case n=3 :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Suppose the diagonal elements a_{ii} are non-zero and divide the first equation by a_{11}, the second by a_{22} and third by a_{33}:

$$x_1 = \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3)$$

$$x_2 = \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3)$$

$$x_3 = \frac{1}{a_{33}}(b_3 - a_{31}x_1 - a_{32}x_2)$$

We choose

$$x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}$$

and define iterations:

$$x_1^{(k+1)} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)})$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)})$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)})$$

$$\mathbf{x}^{(k+1)} = C\mathbf{x}^{(k)} + \beta \;, \qquad \beta_i = b_i / a_{ii}$$

diagonal elements of matrix *C* are zeros in the Jakobi method

$$x^{(k+1)} = Cx^{(k)} + \beta$$

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ - & - & - & - & - \\ c_{i1} & c_{i2} & c_{i3} & \cdots & c_{in} \\ - & - & - & - & - \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{pmatrix}$$

Theorem. A sufficient condition for convergence of $x^{(k)}$ to exact solution $x^{(k)} \to x^*$ is $\|C\| < 1$, where $\|C\|$ is a norm of matrix C.

(Proof is omitted).

The convergence means $x_i^{(k)} \to x_i^*$ for any i

For the properties of a norm, see textbook by S.Sastry.
There are 3+ types of <u>matrix</u> norms:

1) $$\|C\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |c_{ij}|$$ "column" norm

2) $$\|C\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |c_{ij}|$$ "row" norm

$$x_1 = 0.01\,x_1 - 0.2\,x_2 + 3$$
$$x_2 = -0.05\,x_1 + 0.08\,x_2 + 2$$

3) **Euclidean norm**:

$$\|C\|_e = \left(\sum_{i,j=1}^{n} |c_{ij}|^2 \right)^{1/2}$$

Column vector norms:

For the vector

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

some useful norms are

$$\| x \|_1 = |x_1| + |x_2| + \cdots + |x_n| = \sum_{i=1}^{m} |x_i|$$

$$\| x \|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2} = \left[\sum_{i=1}^{n} |x_i|^2 \right]^{1/2} = \| x \|_e$$

$$\| x \|_\infty = \max_i |x_i| .$$

The norm $\|\bullet\|_2$ is called the *Euclidean* norm since it is just the formula for distance in the Euclidean space.

$$A=\begin{pmatrix} 0 & 1 & 0 \\ 1.1 & -0.5 & 0 \\ 0 & 1 & 1.2 \end{pmatrix}$$

Scilab :

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norm(A,1)

norm(A,'inf')
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(Matlab : norm(A,'infty'))

Note: condition $\|C\| < 1$ is analogous to the condition $|\varphi'(x)| < 1$ in the section of Chapter 1 addressing the nonlinear equation $f(x)=0$ $x=\varphi(x)$

Theorem (on the accuracy of successive approximations $x^{(k)}$) : $x^{(k+1)}=Cx^{(k)}+\beta$

1)

$$\|x^{(k)} - x^*\| \le \frac{\|C\|^k}{1-\|C\|}\|\beta\|$$

2)

$$\|x^{(k)} - x^*\| \le \frac{\|C\|}{1-\|C\|}\|x^{(k+1)}-x^{(k)}\|$$

P. Seidel 1821-1896

A modified version of the Jacobi iteration method is Seidel method:

Seidel suggested to immediately insert the calculated $x_i^{(k+1)}$ into the right-hand sides of next equations. This accelerates the convergence $x^{(k)} \rightarrow x^*$

$$x_1^{(k+1)} = \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)})$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)})$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)})$$

The number of arithmetic operations per iteration is $\approx n^2$

Therefore, the effectiveness of iterative methods essentially depends on the required number of iterations k

Ill-conditioned systems of linear equations

Sometimes small changes in the coefficients of the system produce large changes in the solution. Such systems are called ill-conditioned. This can usually be expected when det(A) is small.

Example:

$$\left.\begin{aligned} 2.01x + y &= 4 \\ 2x + y &= 2 \end{aligned}\right\}$$

For ill-conditioned systems, the errors of approximate solutions $x^{(k)}$ are typically large.

Meanwhile the accuracy of approximate solutions $x^{(k)}$ can be improved using an iterative procedure and substitutions.

Notice:
If small changes in the coefficients of the system produce small changes in the solution, then the system is called well-conditioned.