MATH 5378
Differential Geometry Paper

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Fall 2018

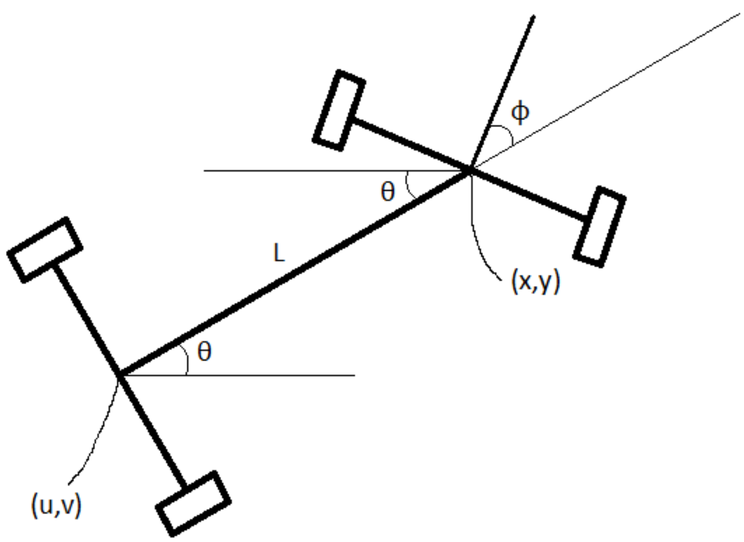
Overview

We will be creating a geometric model of a car and analyzing how it behaves.

Model

The simplest way to think about a car is to think about it as an autonomous front axle that moves around, and a rear axle that trails behind it. This isn't the mechanism by which the car actually behaves, so we'll need to place some restrictions on it, but for now let's see where this takes us.

The following image is a representation of what this model would look like



In this model θ represents the angle of the car relative to some chosen $\mathbb{R}^2$ coordinate system, ϕ represents the angle of steering relative to the angle of the car, x and y represent the horizontal and vertical positions of the center of the front axle in the previously mentioned coordinate system, u and v similarly represent the position of the center of the rear axle, and L represents the distance between the centers of the axles.

On a real car it would be the front tires themselves that rotate, and not the entire front axle, but the behavior is the same in both cases and this is easier to depict and visualize. If necessary, you can imagine a mechanism that allows the front tires to freely conterrotate even while applying a net torque (driving forward), to allow for the movement of the tires as the front axle rotates.

We will be assuming infinite friction between the wheels and the pavement, so that the tires don't skid when the car is traveling at speed.

As previously implied, we will be representing the car as being a purely front wheel drive car, for simplicity. Extending this to a rear wheel drive is relatively simple as long as $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and will be discussed later, but front wheel drive is easier to work with for now. The front wheel drive model allows the car to behave strangely at extreme values of ϕ, but none of the behavior we'll be looking at relies on this.

In practice, we can represent everything about the car at any moment in time with ϕ, x, y, and θ. L doesn't chanve and (u, v) can always be expressed as $(x - L\cos\theta, y - L\sin\theta)$, so we don't need to keep track of what it's doing as long as we keep track of x, y, and θ.

We can think of our 4 main parameters ϕ, x, y, and θ as being axes of a 4 dimensional parameters space P.

Basic Actions

Next we should consider the ways in which the car is allowed to move. The first is turning the steering wheel, and the second is pressing the gas pedal (or braking, which is the same thing). We can consider these behaviors as being vector spaces in P. Let's name these vector spaces "Steer" and "Drive" for intuition, or "A" and "B" for short.

Finding Steer is easy enough, turning the wheel just changes ϕ so

$$A = \frac{\partial}{\partial \phi}$$

Our car is a front wheel drive car, so lets call the path length traveled by (x, y) s, and call $B = \frac{\partial}{\partial s}$. We'd like this in terms of the coordinates we're already using though, so let's look at how they change as s changes. $\frac{\partial x}{\partial s}$ and $\frac{\partial y}{\partial s}$ are simple enough, $\frac{\partial x}{\partial s} = \cos(\theta + \phi)$ and $\frac{\partial y}{\partial s} = \sin(\theta + \phi)$. $\frac{\partial \phi}{\partial s}$ is trivial, it equals 0. To find $\frac{\partial \theta}{\partial s}$ we need to consider how the rear axle is trailing behind.

For a moment, let's ignore what the front axle is doing, just remove it from the car and pick up the end of the L stick at the point where the front axle was. Now we can walk around holding the front of the car and let the rear axle do what it will. Since the rear wheels are unpowered, we are basically describing a wheelbarrow.

There are basically 2 types of movement, walking parallel to the stick (let's call it parallel movement), and walking perpendicular to the stick (let's call it perpendicular movement). Parallel movement does not cause any change in θ, perpenicular movement does. Specifically, a velocity v perpendicular to the stick (and in the $+\theta$ direction) will create an angular velocity $\frac{\partial\theta}{\partial t} = \frac{v}{L}$. Any velocity can be broken down into its parallel and perpendicular components, and a velocity of unit speed in a direction ϕ relative to the stick will cause the velocity in the perpendicular direction to equal $\sin(\phi)$. Thus $\frac{\partial\theta}{\partial t} = \frac{\sin(\phi)}{L}$, or, replacing the front axle, $\frac{\partial\theta}{\partial s} = \frac{\sin(\phi)}{L}$.

And so we have, from $B = \frac{\partial x}{\partial s}\frac{\partial}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial}{\partial y} + \frac{\partial\theta}{\partial s}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial s}\frac{\partial}{\partial\phi}$,

$$B = \cos(\theta+\phi)\frac{\partial}{\partial x} + \sin(\theta+\phi)\frac{\partial}{\partial y} + \frac{\sin(\phi)}{L}\frac{\partial}{\partial\theta}$$

Any action can be expressed by a linear combination of A and B.

Less Basic Actions

To start off, we can execute behaviors simultaneously by adding their vector fields together, so let's define "Turn", or "C" as being equal to A + B.

$$C = \cos(\theta+\phi)\frac{\partial}{\partial x} + \sin(\theta+\phi)\frac{\partial}{\partial y} + \frac{\sin(\phi)}{L}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}$$

As turns go, Turn is rather sharp in most unit systems, turning a full radian as it travels one unit of distance, but that's fine. If we wanted to, we could have created a more gradual turn by multiplying B by some constant. We won't be using Turn for anything else, but it's one of the most common actions of a car, and it demonstrates the idea of combining actions by simultaneous execution. We could also multipy A or B by any non-constant function, most reasonably functions of time, but functions of the current values of various parameters are also valid.

Now let's do something a bit more interesting. Without worrying yet about what intuitive name to call it, let's define "D" as being equal to the Lie bracket of A and B.

$$D = [A,B] = -\sin(\theta+\phi)\frac{\partial}{\partial x} + \cos(\theta+\phi)\frac{\partial}{\partial y} + \frac{\cos(\phi)}{L}\frac{\partial}{\partial\theta}$$

So what is D? Taking this Lie bracket is the same thing as taking the derivative of B along the flow of A. None of this matters if we can't do it with a car though, so a more physical way of looking at it is that D is what you get if you follow A for some small distance ϵ, then follow B for ϵ, then go back along A for ϵ, then go back along B for ϵ, and continue to cycle through. As ϵ approaches 0 this procedure causes the car to behave as though we applied D. Naturally, you cannot truly drive your car an infinitesimally small distance, so D can never be truly followed, but we can get arbitrarily close to D by making ϵ very small.

Let's examine the physical implications of D. First off, D is equivalent to B with ϕ replaced by $\phi + \frac{\pi}{2}$, which means that applying D is essentially the same as driving along the front axle. This means, among other things, that a real car, with ϕ restricted to within $\frac{\pi}{2}$ of forward, can more or less behave the same as our modeled car, which has no such restriction. Swell. Let's call D "Wiggle".

We won't be naming this next step, but let's take the Lie bracket [A,D] to verify something interesting.

$$[A, D] = -\cos(\theta + \phi)\frac{\partial}{\partial x} - \sin(\theta + \phi)\frac{\partial}{\partial y} - \frac{\sin(\phi)}{L}\frac{\partial}{\partial \theta} = -B$$

So [A,D]=-B. This is a little strange, but it makes sense. As mentioned, D is basically just a rotated B, so having it interact with A in the same way again will just apply that same offset again, replacing $\phi + \frac{\pi}{2}$ with $\phi + \pi$. Since B consists of multiples of single trig functions of ϕ, this is equivalent to -B.

On to another thing we're naming. We'll refer the Lie bracket of B and D as "E".

$$E = [B, D] = \frac{\sin(\theta)}{L}\frac{\partial}{\partial x} - \frac{\cos(\theta)}{L}\frac{\partial}{\partial y}$$

The same qualifying statements apply as for D, but E ignores ϕ entirely and shifts the point (x,y) to the left of the car's orientation, withoug changing θ at all, which means that (u,v) is also shifted to the left. In essence the entire car moves sideways, which is wild. We'll call it "Slide"

To finish out this set of interesting behaviors, let's try to find a way to make the tail of the car swing around without changing x or y. Inspection of the vector fields we've already created reveals that parts of E closely resemble D when $\phi = 0$. So let's assume $\phi = 0$ and define "F" = D+LE. F isn't interesting when $\phi \neq 0$, so we'll only be defining it for when it is.

$$F = \frac{1}{L}\frac{\partial}{\partial \theta}$$

This allows us to fishtail around in place despite having infinite friction, which is nice. We'll call F "Drift".

Through combining all of these actions, we can basically move our car however we want.

Rear Wheel Drive

If we want to model our car more like real life, speed should come from the rear tires. Since we have infinite friction, a car being propelled in this way will not move when $\phi = \pm\frac{\pi}{2}$, so we'll restrict ϕ to the open interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. This places restrictions on what functions can apply A, which is a large part of why we didn't use it, but for the most part this doesn't matter. In order to get a drive operator that assumes unit speed for the rear axle, we'll need the relationship between the speed of the front axle and the speed of the rear axle. Calling r the path length of the rear axle, $\frac{\partial s}{\partial t} \cos\phi = \frac{\partial r}{\partial t}$, so the real wheel drive vector field "R" = B$\cos\phi$. You can do everything that we did with front wheel drive with this vector field instead, but a lot more corrections are needed to see the same behaviors.