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Validating and Calibrating Agent-Based Models: A Case Study

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Abstract In this paper we deal with some validation and calibration experiments on a modified version of the Complex Adaptive Trivial System (CATS) model proposed in Gallegati et al. (2005 *Journal of Economic Behavior and Organization, 56*, 489–512). The CATS model has been extensively used to replicate a large number of scaling types stylized facts with a remarkable degree of precision. For such purposes, the simulation of the model has been performed entering ad hoc parameter values and using the same initial set up for all the agents involved in the experiments. Nowadays alternative robust and reliable validation techniques for determining whether the simulation model is an acceptable representation of the real system are available. Moreover many distributional and goodness-of-fit tests have been developed while several graphical tools have been proposed to give the researcher a quick comprehension of actual and simulated data. This paper discusses some validation experiments performed with the modified CATS model. In particular starting from a sample of Italian firms included in the CEBI database, we perform several ex-post validation experiments over the simulation period 1982–2000. In the experiments, the model parameters have been estimated using actual data and the initial set up consists of a sample of agents in 1982. The CATS model is then simulated over the period 1982–2000. Using

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alternative validation techniques, the simulations' results are ex-post validated with respect to the actual data. The results are promising in that they show the good capabilities of the CATS model in reproducing the observed reality. Finally we have performed a first calibration experiment via indirect inference, in order to ameliorate our estimates. Even in this case, the results are interesting.

Keywords Validation · Calibration · Agent-based models · Indirect inference · Size distribution · Tail analysis · EVT

1 Introduction

Agent-based models (ABM) have been developed to study the behaviour of several heterogeneous interacting agents. They are based on new microfoundations, according to a bottom-up approach: one builds a model starting from simple behavioural rules at the single agent level, then through interactions some aggregate statistical regularities emerge so that they cannot be inferred from the individual level. This emergent behaviour often feeds back to individual agents making their rules change (they may evolve in an adaptive way). Following this methodology, macroeconomics is no more a set of equations that occurs by summation and averaging of the individual decisions, but it is a self-organized criticality that rises from the micro-level.

Hence, in the last years, thanks to their flexibility, ABM have been widely used in many scientific fields, including economics (Axelrod 1997; Axtell 2000). However researchers have only recently started considering the issue of validation: that is whether a model and its results may be considered correct. As Sargent (1998) says: "This concern is addressed through model verification and validation. Model validation is usually defined to mean substantiation that a computerized model within its domain of applicability possesses a satisfactory range of accuracy consistent with the intended application of the model".

This paper deals with some validation experiments of a modified version of the Complex Adaptive Trivial System (CATS) model proposed in Gallegati et al. (2003b, 2005).

The CATS model has been extensively used (see, for example, Gallegati et al. 2003b, 2005) to replicate a great number of scaling type stylized facts with a nice degree of precision and, for these purposes, the simulation of the model has been performed entering ad hoc parameters' values and using the same initial set up for all the agents involved in the experiments. The above mentioned analysis have been performed following Kaldor's suggestion: "construct a hypothesis that could account for these stylized facts, without necessarily committing himself on the historical accuracy" (Kaldor 1965, p. 178).

In this paper our intentions are the following: using an initial set up of actual Italian data, we aim to verify if our model, simulated over a period for which actual data are fully available, is an acceptable representation of the actual system. In a nutshell, we intend to perform an ex-post validation of the model.

The model reproduces, in a medium term horizon, a good percentage of the actual capital data. The two samples (simulated and observed data) belong to the same distribution with a confidence interval of 95%. Moreover the model also reproduces the firms' growth dynamics at a micro level.

We have then performed a first simultaneous calibration/validation of the model via indirect inference, following the techniques presented in Gourieroux and Monfort (1996) and Gilli and Winker (2003). This procedure allows us to get results that are closet to actual data. For these reasons we believe indirect inference can be a powerful tool for validating ABM, even if some issues about complexity must be carefully analysed and solved.

The papers is organized as follows: Sect. 2 presents the state of the art for the validation of ABM; Sect. 3 introduces the model we have studied and validated; Sect 4 describes the database we used and the empirical evidence we aim to investigate; Sect. 5 shows the proceeding of the validation procedure; Sect. 6 deals with indirect inference for calibrating ABM models; while Sect. 7 concludes.

2 Empirical Validation of Agent-Based Models

The validation of ABM is becoming one of the major points in the agenda of those researchers, who work according to the agent-based approach.

In the literature Tesfatsion and Judd, 2006; Tesfatsion, 2007,[1] looking at the main methodological aspects, there are three alternative ways of validating computational models:

1. Descriptive output validation, matching computationally generated output against already available actual data. This kind of validation procedure is probably the most intuitive one and it represents a fundamental step towards a good model's calibration;
2. Predictive output validation, matching computationally generated data against yet-to-be-acquired system data. Obviously, the main problem concerning with this procedure is essentially due to the delay between the simulation results and the final comparison with actual data. This may cause some difficulties when trying to study long time phenomena. Anyway, since prediction should be the real aim of every model, predictive output validation must be considered an essential tool for an exhaustive analysis of a model meant to reproduce reality.
3. Input validation, ensuring that the fundamental structural, behavioral and institutional conditions incorporated in the model reproduce the main aspects of the actual system. This is what we prefer to call *ex ante* validation: the researcher, in fact, tries to introduce the correct parameters in the model before running it. The information about parameters can be obtained analysing actual data, thanks to the common empirical analysis. Input validation is obviously a necessary step one has to take before calibrating the model.

Since the empirical validation of ABM is still a brand new topic, at the moment there are only a limited number of contributions in the literature dealing with it, as summarized below.

[1] In her website, Leigh Tesfatsion maintains an entire section on the validation of ABM. See: http://www.econ.iastate.edu/tesfatsi/empvalid.htm

In his paper, Axtell (2000) develops the basic concepts and methods of an alignment process for ABM. In particular he shows how alignment can be used to test if two different computational models can be considered similar in terms of behaviour and output.

In Carley (1996), there's a first stress on model validation issues, even if the attention of the author is still focusing on computational modelling in general.

A very interesting experiment can be found in the papers by Gilli and Winker (2003) and Winker and Gilli (2001), in which the authors present an agent-based exchange market model and introduce a global optimisation algorithm, based on indirect inference, for calibrating the model's parameters via simulation.

In Troitzsch (2004), there is a comprehensive list of all the issues concerning the validation of simulation models to describe and predict real world phenomena.

In Fagiolo et al. (2007), one can finally find a very interesting discussion about the ways agent-based modellers have tried to face the empirical validation of their models. The authors briefly review the standard approaches to model validation employed by mainstream economists and then point out the main differences dealing with ABM validation. The paper concludes with some suggestions regarding the methodological aspects of validation.

Apart from the work of Gilli and Winker (2003), related to the use of indirect inference in ABM, and the surveys of Fagiolo et al. (2007) and Carley (1996), all the other papers we have cited mainly deal with descriptive validation, indicating that a lot of research has to be done as far the other kinds of validation are concerned.

Finally, without any presumption of being complete and exhaustive, we cannot forget the mainly theoretical and methodological contributions by Axtell et al. (1996); Sargent (1998); Klevmarken (1998); Kleijnen (1998); Epstein (1999); Tesfatsion and Judd (2006) and all the other more general works about validation in econometrics.

3 The Modified CATS Model

The CATS model was first introduced in Gallegati et al. (2003b, 2005) to study financial fragility and power laws. Here, it's modified to better reproduce actual data, according to the input validation principle we have mentioned above.

In particular, even if the core structure is the same, the main differences of our version are:

1. A homogeneous market interest rate. This choice is due to the lack of micro-level data about interest rates. The main consequence is that the mean-field interaction of the original CATS model is not present here.
2. Heterogeneity in firms' capital productivity, according to data, while the version of Gallegati et al. (2005) assumes the same productivity among all the simulated firms.
3. Heterogeneity in the process for stochastic demand, since we divide firms into two groups, small and big firms, with different price generators.

Given these remarks, let us introduce the model.

Consider a sequential economy in the sense of Hahn (1982) populated by firms. In every period two markets are opened: one for a homogeneous produced good and one for credit. Following the levered aggregate supply class of models first presented in Greenwald and Stiglitz (1990, 1993), our model is fully supply-determined, i.e. the firms can sell all the output without demand constraints. Informational imperfections in the equity market permit firms to raise funds only from the credit market, apart from retained profits from previous periods. This assumption seems to reflect the examined reality in Italy, since new equity issues were rarely a financial option for Italian firms in the observed period. Moreover, the full distribution of dividends to shareholders was expensive, due to the fiscal system. Obviously, as demonstrated by the Modigliani-Miller theorem, in a perfect environment without taxes companies would not distinguish among these financial options.

Hence, in our model, the demand for credit is related to investment expenditure and it is completely satisfied at the fixed banks' interest rates. This hypothesis, introduced in Vagliasindi et al. (2006), helps us in identifying a suitable proxy of the individual interest rates (namely, the average interest rate), for the reason that we have no reliable information related to them.

At any time t, the economy consists of N_t firms, each belonging to two different sets (small firms and large ones), depending on their size, characterized by different levels of risk (price shocks). The size of a firm can be determined both using capital and the number of employees. In this division among small and big firms, we refer to employees, following the Italian fiscal law.[2] This assumption makes our model differ from the original one (Gallegati et al. 2005) where there is a single risk level.

Every firm $i \in N_t$ produces its output Y according to a linear production function, in which capital (K_{it}) is the only input and never depreciates:

$$Y_{it} = \phi_{it} K_{it}. \tag{1}$$

For each firm i the initial capital productivity ϕ_{it} in $t = 1$ corresponds to its actual productivity (estimated on data) and it evolves according to the following formula:

$$\phi_{it} = \phi_{it-1} + \varrho_{it}\phi^2_{it-1}, \quad \text{where } \varrho_{it} = \frac{M}{2}, \tag{2}$$

with M following an uniform distribution $U(-1, 1)$. All this reproduces the evidence from our database, reported in Fig. 1, where one can see the evolution of the average productivity for actual and simulated firms. It clearly emerges that the growth rates of productivity are decreasing in firms' size, contradicting Gibrat's law for means.

Each firm's demand for goods is affected by an iid idiosyncratic real shock. Since arbitrage opportunities are imperfect, the individual selling price is the random outcome of a market process around the average market price P_t of the output, according to the law $P_{it} = u_{it} P_t$, where $E(u_{it}) = \mu$ and $\sigma^2_{u_{it}} < +\infty$. Actual data suggest to split the price generator process into two different processes, depending on firms' size. To simplify the analysis we assume that u_{it} may come from two different uniform

[2] For the Italian fiscal Law a firm is considered: *"small"*, if it has less than 50 employees; *"medium"* if it has between 51 and 250 employees; *"large"* if it has more than 250 employees. In our sample, the percentage of firms is: ≈ 61% small, ≈ 30% medium, ≈ 9% large.

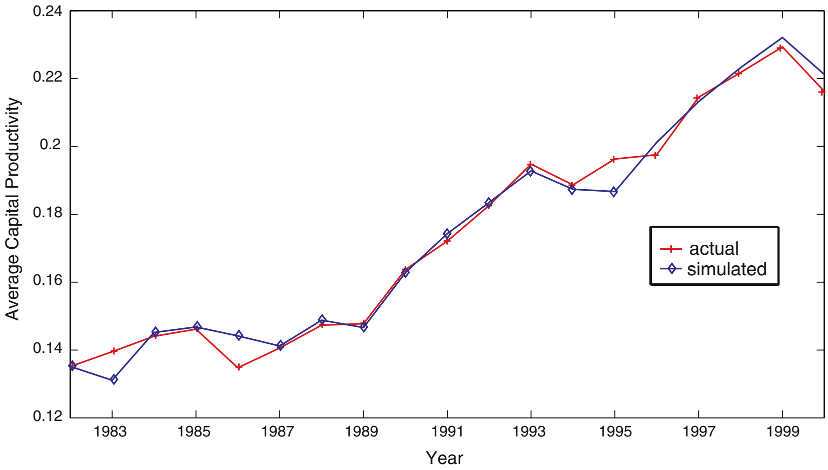

Fig. 1 Evolution of average capital productivity over time: actual (*red crosses*) and simulated (*blue diamonds*) data

distributions: small firms get a high average price and a stronger volatility, while big firms, probably because of their economies of scale, face more concentrated prices with a lower mean. In view of this hypothesis, that has a justification in actual data, we have that $\mu^{U_1} > \mu^{U_2}$ and $\sigma^2_{U_1} > \sigma^2_{U_2}$, where U_1 is the distribution of u_{it} when i is small and U_2 when i is large

Credit is the only external source of finance for firms, so they can finance their capital expenditure by recurring to net worth (A_{it}) or bank loans (L_{it}), i.e. $K_{it} = A_{it} + L_{it}$. Given the exogenous real interest rate $\overline{r_t}$, in every period debt commitments for every firm are equal to $\overline{r_t} L_{it}$. In view of the assumption that dividends are not distributed, financing costs equal debt commitments. Profit/loss (π_{it}) in real terms is:

$$\pi_{it} = u_{it} Y_{it} - \overline{r_t} L_{it} \tag{3}$$

In our model a firm fails if its net worth becomes negative, that is to say $A_{it} < 0$. The law of motion of A_{it} is, for hypothesis,

$$A_{it} = A_{it-1} + \pi_{it}. \tag{4}$$

As in Greenwald and Stiglitz (1993), we suppose that the probability of bankruptcy ($\Pr^b$) is directly incorporated into the firm's profit/loss function: bankruptcy is costly and increasing with the firm's size. For this purpose, we have used a standard quadratic cost function:

$$C^b_{it} = c Y^2_{it} \quad c > 0 \tag{5}$$

By maximizing its objective function Γ_{it}, every firm determines its optimal capital stock K^*_{it}:

$$\max_{K_{it}} \Gamma_{it} = \max_{K_{it}} \left[E(\pi_{it}) - E(C_{it}^b) \right]. \tag{6}$$

and hence the demand for credit.

4 Data Description and Empirical Evidences

All our validation experiments, together with the subsequent empirical analysis, are based on firm-level observations from CEBI database, for the period 1982–2000. CEBI, formerly developed by Bank of Italy, is now maintained by Centrale dei Bilanci Srl and it collects balance sheets of thousands of Italian firms.

Thanks to several queries on the database, we have obtained a balanced panel of 21,702 Italian non-financial firms, all satisfying the following: (i) no missing data in each year; (ii) reliable data for capital, employees and costs. For each firm and year, we have data on equities, long term debts and loans, short term debts, total capital, gearing ratio, solvency ratio, debt ratio, number of employees, cost of employees and revenues.

Recent explorations (Gallegati et al. 2007) in industrial dynamics have detected two important empirical regularities, which are widespread across countries and persistent over time:

1. The distribution of firms' size is right skewed and can be described by a Zipf or power law probability density function (Gallegati et al. 2003a, 2004; Gaffeo et al. 2003; Axtell 2000, 2001; Ramsden and Kiss-Haypal 2000; Okuyama et al. 1999; Quandt 1966a,b; Mandelbrot 1960; Simon 1955);
2. Firms' growth rates are Laplace distributed, belonging to the Subbotin's Family (Stanley et al. 1996; Bottazzi and Secchi 2005)

Gallegati et al. (2005) have shown analytically that 1–2 (and other regularities we do not deal with) determine several industrial, financial and business cycle facts. A model should therefore be able to replicate the empirical evidence 1–2, and our validation exercise is focused on it.

The following section will present the validation exercise, i.e. if the mentioned model successfully reproduces evidences 1–2.

5 Simulation and Results

Our validation exercise is run with a sample of 21,702 firms over the period 1982–2000.

The validation procedure we have used is quite new for ABM, but it is based on some well-known results of extreme value theory (Embrechts et al. 1997), mainly as far as the analytical tests are concerned. In particular we here develop the approach first introduced in Bianchi et al. (2007).

In $t = 1$, each firm is initialised with its actual data from 1982: net worth, loans, productivity and so on. The market interest rate is exogenous and, each year, it is equal to the average historical interest rate, as in the Bank of Italy yearly Survey. A complete description of the procedure is available in Bianchi et al. (2007).

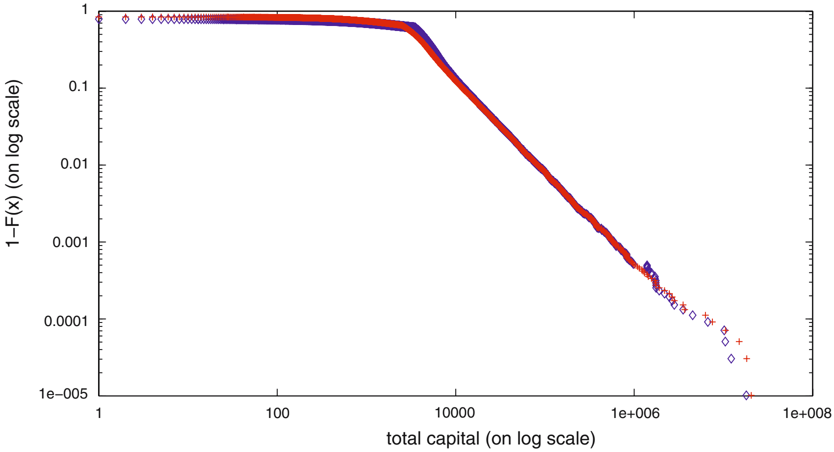

Fig. 2 Zipf's plot of total capital distributions: actual (*red crosses*) and simulated (*blue diamonds*) data

In each period actual data from the CEBI database are compared with the simulated data produced by the model. In particular our analysis can be divided into two different approaches: a pointwise analysis, meant to evaluate the evolution of the single firm, in order to study the predictive power of the model; and a distributional analysis, whose aim is to look for regularities.

Our experiments can be considered a detailed ex-post validation of the modified CATS model, that is to say a first step, necessary to develop all the subsequent analysis. In Cirillo (2007), the interested reader can find a complete analysis of all the data we have used, together with a detailed description of all the empirical evidences we will deal with.

5.1 Total Capital

Let us consider the total capital dynamics. Accepting a maximum deviation of ±15% between observed and simulated data in 2000 (that is a very low composite yearly deviation rate), we succeed in reproducing 17,144 firms over 21,702 (79%). Even if this is a good percentage of fitting, as Fig. 2 shows, the tails of the firms' size distribution are not adequately fitted. This can be interpreted as a signal of the naivity of our model, that probably extremely simplifies firms' behaviour. However, if we consider that this is the first time that our model has been validated and that a model simplifies reality by definition, 79% can be seen as a promising result. Similar percentages of fitting can be found in the previous years (in 1986, for example, the percentage of fitted firms is 78%, in 1990 it's 80% and 81% in 1996) and analysing the pooled distributions (78%).[3]

[3] As in Ijiri and Simon (1977), the use of pooled distribution is possible since the single distributions show similar slopes. In this paper, almost all the figures refer to year 2000.

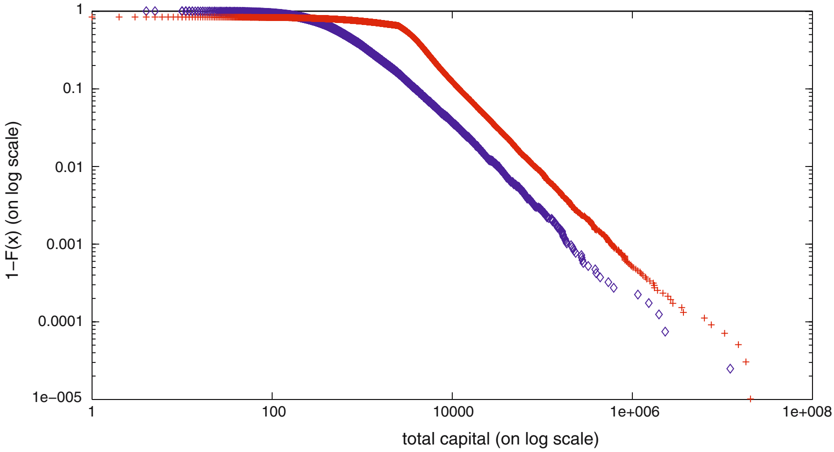

Fig. 3 Comparison between actual total capital in 1982 (*blue diamonds*) and 2000 (*red crosses*)

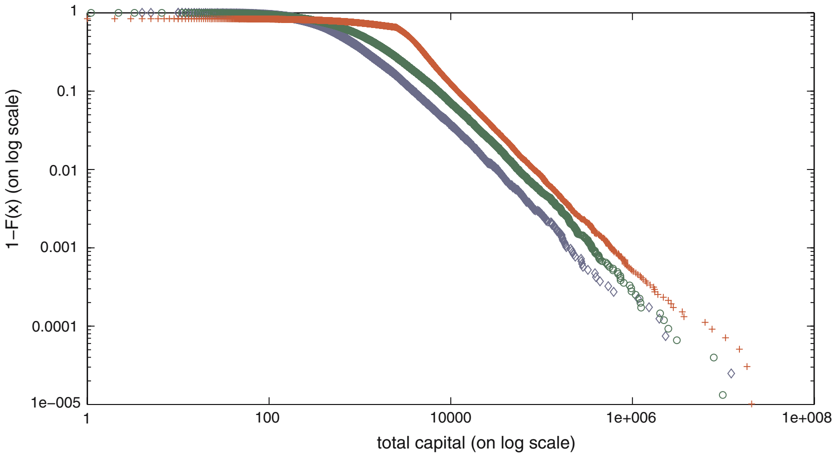

Fig. 4 Evolution of total capital over time: 1983 (*blue diamonds*), 1994 (*green circles*), 2000 (*red crosses*)

In order to verify the real goodness of these results, that's verifying if they are due to the goodness of the model rather than to the characteristics of data (few firms with respect to the universe and so on), we have performed an empirical analysis of actual data. In Fig. 3 one can observe the comparison, by the means of a Zipf's plot, between actual total capital in 1982 and 2000. The evidence is quite clear: there is a substantial difference between the two amounts of data. Accepting the usual 15% deviation, less than 10% of the firms (essentially the "average ones") can be considered as fitting the data. In fact, even if both distributions belong to the Paretian case, their shape and scale parameters are different. Several analytical tools, such as the Kolmogorov–Smirnov's statistics and the Kruskall–Wallis' test, confirm all this. Another informative picture is Fig. 4, that summarizes the evolution of total capital over time: all the distributions are rather different.

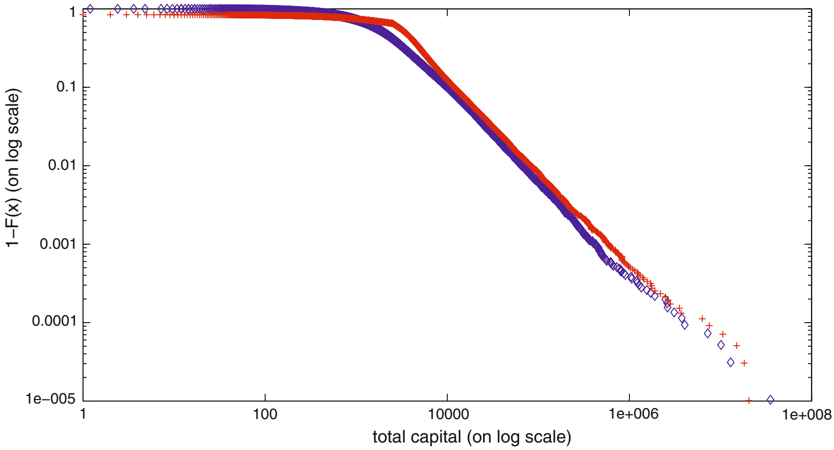

Fig. 5 Comparison between actual total capital in 2000 (*red crosses*) and naive forecast (*blue diamonds*)

Even if we calculate the average growth rate of firms from 1982 to 2000 and then we multiply the initial data in 1982 for this coefficient, in 2000 we succeed in fitting "only" 47% of the firms (see Fig. 5). Since our model can fit 79% of them, it must be considered as better performing.

Figure 2 also shows that both observed and simulated capital distributions are particularly skewed, with fat right tails (decreasing linear relationship in the plot). This reproduces a widely accepted result (Zipf 1932), according to which firms' size is power law distributed (Axtell 2000; Gaffeo et al. 2003; Gabaix et al. 2003) and shows regularly varying tails.

We have performed many graphical and analytical tests to check if our two samples (observed and simulated data) may be considered belonging to the same distribution.

For example, a first Quantile–Quantile plot (Fig. 6) supports the idea of a unique distribution for both samples, since there is a clear linear relationship between the observed and simulated data. Similar results are obtainable using boxplots.

The same results are supported by the Generalized Kolmogorov–Smirnov Test[4] (Prabhakar et al. 2003) with a confidence interval of 95%. Therefore, it's possible to say that our two samples belong to the same distribution or, to be more exact, to the same mixture.

In particular, excluding the right Paretian tails (top 10%), that we have separately analysed for computing Hill's parameters (we trimmed them after a threshold study, see Bianchi et al. 2007), we found out that our data (both actual and simulated) follow a lognormal distribution[5] (see also Cirillo 2007). Table 1 summarizes the maximum

[4] The Generalised or two Sample Kolmogorov–Smirnov test is a variation of the classical Kolmogorov–Smirnov test, based on the supremum metrics and the empirical cumulative density functions. See Shao (2003) for an introduction.

[5] Most of the firms follow a lognormal distribution and this characteristic is persistent over time, indicating a stable behavior in data (see Cirillo 2007).

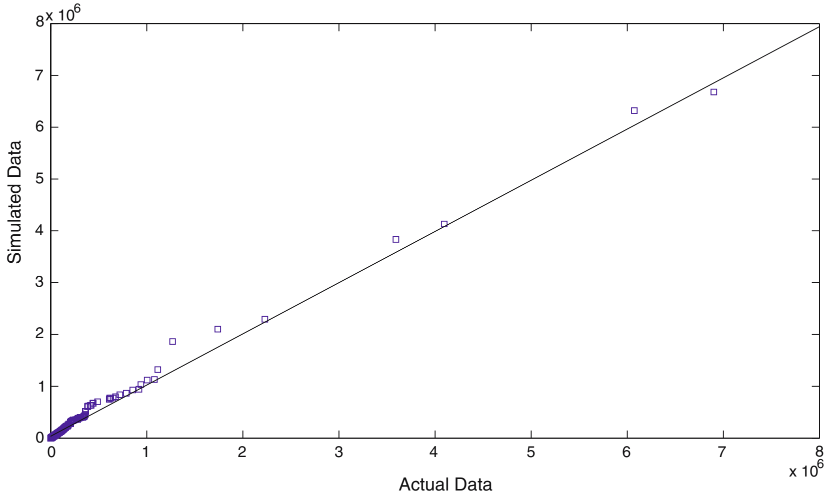

Fig. 6 Q–Q plot of the two capital distributions in 2000

Table 1 Estimated parameters of the two lognormal distributions in 2000

	Actual data	Simulated data
μ	5.5663 (5.5486–5.5840)	5.5712 (5.5535–5.5887)
σ	1.9979 (1.9854-2.0104)	2.0112 (1.9987-2.0237)

95% CI in brackets

likelihood estimates for the actual and simulated case in 2000. Similar estimates are available for all the other years.

As far as the right tails of the two distributions, the Mean Excess Function versus Threshold Plot (MEPLOT) clearly shows a Paretian behaviour. An upward sloping mean excess function, as in Fig. 7, indicates a heavy tail in the sample distribution. That is why, thanks to the semiparametric Hill's method,[6] we have decided to estimate the shape parameters of the two samples, in order to see if data have a similar behaviour in the right tails.

Figure 8 reports the Hill's estimates of the shape parameter for the simulated and actual capital. In the first case $\alpha = 1.62$, while in the second one $\alpha = 1, 68$. Hence,

[6] The well-known Hill's Estimator ξ, together with the Pickard's one, is the most used way to determine the shape parameter $\alpha = \frac{1}{\xi}$ of a distribution belonging to the GEV family.
In particular

$$\xi = \frac{1}{k-1} \sum_{i=1}^{k-1} \ln x_{i,N} - \ln x_{k,N} \quad \text{for } k \geq 2, \tag{7}$$

where k is the upper order statistics and N the sample size.

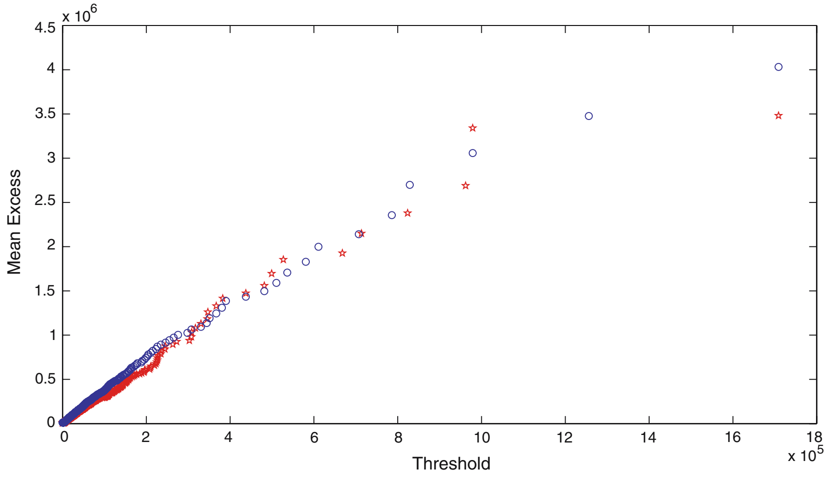

Fig. 7 Meplot of actual (*blue circles*) and simulated (*red stars*) capital

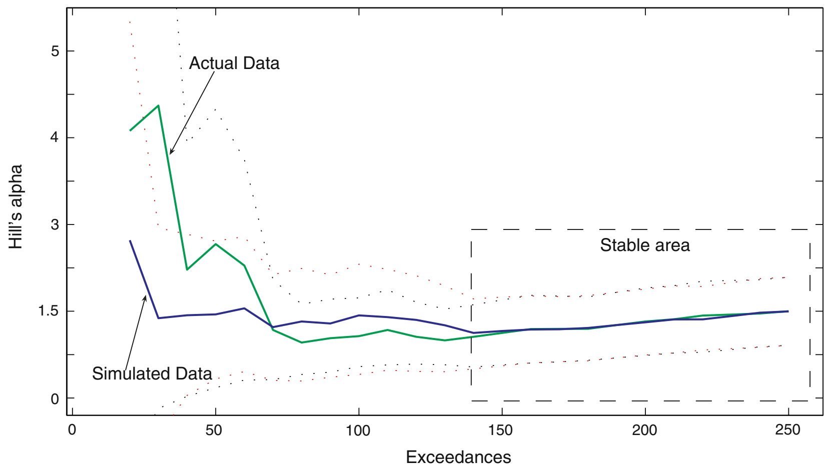

Fig. 8 Hill's plot of actual and simulated total capital

the two parameters are quite similar and belong to the Paretian field ($0.8 < \alpha < 2$),[7] but we cannot claim that the two tails behave in the same way. Simulated capital, in fact, shows a slightly heavier tail (since its alpha is lower),[8] demonstrating that we slightly overestimated the observed values.

[7] Once again the results concerning the pooled distributions are very similar. The reason can be found in the words of Ijiri and Simon (1977): "We conclude that when two or more Pareto distributions are pooled together, the resulting distribution is Pareto if and only if all the distributions have similar slopes [...]. This result is important in dealing with the aggregation of empirical firm size distributions."

[8] As clearly showed in Kleiber (2003), the Pareto density has a polynomial right tail that varies at infinity with index ($-\alpha$–1), implying that the right tail is heavier as α is smaller.

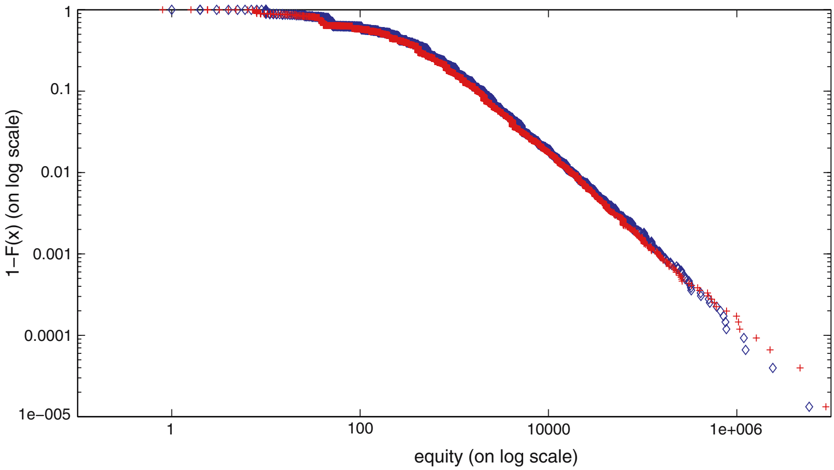

Fig. 9 Zipf's plot of actual and simulated net worth distributions

5.2 Net Worth and Loans

As far as net worth is concerned, accepting a maximum deviation of $\pm 15\%$ between actual and simulated data in 2000, we succeed in reproducing 74% of the firms. This number is lower than that of total capital, indicating some more problems of fitting.[9]

Other positive results, see Fig. 9, are the skewness of the two distributions and the presence of a clear Paretian behaviour in both actual and simulated net worth. Hill's estimates of the shape parameters both show heavy right tails: actual data present $\alpha = 1.61$, while the simulation produces $\alpha = 1.54$.

As far as the possibility of a unique distribution for the two samples, the two-sided generalized Kolmogorov–Smirnov test rejects such a null hypothesis. On the contrary the one-sided right version of the test[10] is not rejected, indicating that we get a better fitting of medium and big firms, but we fail in forecasting the smallest one (see in Fig. 9), as in Bianchi et al. (2007).

The results we get about loans are very similar to those of the total capital: we succeed in fitting 80% of the firms.

Moreover, similarly for total capital and net worth, both graphical and analytical tests support the idea of a unique distribution for both actual and simulated debt data.

As in Fujiwara (2004), the distribution of loans is also power law. The Hill's estimates of the shape parameters of the Paretian right tails are $\alpha = 1.71$ for the actual data and $\alpha = 1.69$ for the simulated ones, demonstrating an overestimate of biggest firms.

[9] In particular this may depend on the hypothesis of the model that: (1) firms cannot raise funds on the equity market, (2) profits are entirely retained in the firm and (3), as suggested by one of the referees, that all firms face the very same interest rate. However, these simplifying hypothesis, typical of CATS model, do not seems to affect too much the robustness of our validation results.

[10] $H_0 : F_1^+(x) = F_2^+(x)$. $H_1 : F_1^+(x) > F_2^+(x)$.

Table 2 Estimated Subbotin's parameters

	Observed data	Simulated data
μ	0.0011 (0.0006)	0.0023 (0.0015)
a	0.0592 (0.0195)	0.0627 (0.0261)
b	1.0223 (0.4153)	1.0247 (0.4402)
$-loglik$	3.0314	3.2392

Standard errors in brackets

5.2.1 Growth Rates

Looking at growth rates, several studies (Axtell 2000; Bottazzi and Secchi 2005; Hall 1987) find a tent-shape behaviour. In particular, the Laplace and Lévy distributions seem to provide the best fitting (Bottazzi and Secchi 2005; Gabaix et al. 2003).

Having the results of Bianchi et al. (2007) in mind, we have investigated if the empirical distributions of growth rates (in terms of capital) belong to the well-known Subbotin's family (Subbotin 1923), which represents a generalization of several particular cases, such as Laplace and Gaussian distributions. The functional form of Subbotin's family is:

$$f(x, a, b) = \frac{1}{2ab^{\frac{1}{b}}\Gamma\left(1 + \frac{1}{b}\right)} e^{-\frac{1}{b}\left|\frac{x-\mu}{a}\right|^{b}}, \tag{8}$$

where μ is the mean, b and a two different shape parameters and Γ is the standard Gamma. If $b \to 1$ the Subbotin distribution becomes a Laplace, a Gaussian for $b \to 2$.

Using maximum likelihood, we have estimated the three Subbotin's parameters on our data. Table 2 contains the results.

At a first glace, observed and simulated growth rates show several similarities:

1. The two means are very close to zero;
2. Since b is very near to 1, both distributions are in the field of attraction of the Laplacian case;[11]
3. The values of a, the Laplacian shape parameter, are not very different in both cases, even if simulated data show slightly fatter tails ($0.062 > 0.059$);

Overall, the CATS model is able to mimic firms' growth dynamic, once again with some discrepancies as far as the tails are concerned.

6 Indirect Inference and ABM

The use of indirect inference for calibrating ABM was proposed by Gilli and Winker (2003). In their work, the two authors achieve very good results, even if they stress some problems about the use of indirect inference with ABM. In particular, they show

[11] Some authors prefer a truncated Lévy distribution. The *querelle* is open. See Kleiber (2003).

that, even when applied to very simple multi-agent models, indirect inference may return an objective function with a complex landscape which is difficult to minimize. To solve this problem they suggest to combine indirect inference together with standard grid methods.

In what follows we develop the methodology proposed by Gourieroux and Monfort (1996) and Gilli and Winker (2003), introducing a new way for calibrating ABM looking at the distributional properties of data.

Anyway, let us first give some theoretical explanations of indirect inference as estimation and calibration tool.

6.1 An Introduction to Indirect Inference

Indirect inference (Gourieroux and Monfort 1996) can be understood as a generalization of simulated GMM. Quite often in economics, since the actual model has a complicated structure (in our situation the real industrial model), direct inference is intractable for analytical reasons.[12] If the structural (true) model can be easily simulated for any fixed parameter value in the parameter space, indirect inference is one of the viable estimation procedures to use. According to indirect inference, estimation of the structural parameters consists of two steps. The first one involves the use of an easy-to-estimate auxiliary model in order to achieve consistent and asymptotically normal estimates for some auxiliary parameters θ (pseudo true value). In the second step, simulations are used in order to minimize the distance between $\hat{\theta}$ and $\tilde{\theta}$, where $\hat{\theta}$ is the vector of the auxiliary parameters estimated using the observations, while $\tilde{\theta}$ is the vector of auxiliary parameters estimated using simulation values.

Using the notation of Gourieroux and Monfort (1996), indirect inference can be introduced as follows. Given a parameter ϕ, let

$$\widetilde{y}^h(\phi) = \{\widetilde{y}_0^h, \widetilde{y}_1^h, \ldots, \widetilde{y}_T^h\} \tag{9}$$

be data simulated from the true model where h $= 1, \ldots,$ H with H being the number of simulated paths.[13] Now, let's match various functions of the simulated data with those of actual data in order to get estimates of the parameters.

Suppose Q_T is the objective function of a certain estimation method applied to an auxiliary model indexed by the parameter ϕ. Define the related estimator based on the observed data by

$$\widetilde{\theta}_T = \arg\min_{\phi \in \Theta} Q_T(y). \tag{10}$$

[12] It is useful when the moments and the likelihood function of the true model are difficult to deal with, but the true model is amenable to data simulation.

[13] The number of simulated observations must be the same as the number of actual observations for the purpose of the bias calibration.

The corresponding estimator based on the h-th simulated path is defined by

$$\widetilde{\theta}_T^h(\phi) = \arg\min_{\phi\in\Theta} Q_T(\widetilde{y}^h(\phi)), \tag{11}$$

where Θ is a compact set.

The indirect inference estimator is set as

$$\phi_{T,H}^{II} = \arg\min_{\phi\in\Phi} ||\widehat{\theta}_T - \frac{1}{H}\sum_{h=1}^{H} \widetilde{\theta}_T^h(\phi)||, \tag{12}$$

with Φ compact set. If H tends to infinity, the indirect inference estimation becomes

$$\phi_{T,H}^{II} = \arg\min_{\phi\in\Phi} ||\widehat{\theta}_T - E(\widetilde{\theta}_T^h(\phi))||. \tag{13}$$

Let's define $b_T(\phi) = E(\widetilde{\theta}_T^h(\phi))$ as the *binding function*. When the number of parameters in the auxiliary model is the same as that in the true model and the function b is invertible, the indirect inference estimator is equal to

$$\phi_T^{II} = b_T^{-1}(\phi). \tag{14}$$

As far as all the properties of indirect inference (unbiasedness, efficiency and so on), we refer to the original works of Gourieroux and Monfort (1996).

6.2 Calibration

For our analysis, the structural model is the modified version of the CATS model we have introduced. The auxiliary model, on the contrary, is represented by the parameters of capital distributions.

As we have seen before, the distributions of actual and simulated data are very similar. In particular they show very close moments and scale parameters, while the greater differences are related to the shape parameters α's, that's to say the two right tails. Our aim is to calibrate the shape parameters, in order to improve our capability of reproducing empirical data. Since the two distributions are in the Paretian field of attraction, if we reduce the distance between the shape parameters of actual and simulated data, we can reach our goals. Figure 10 gives a graphical representation of our method.

Unfortunately, given our model, we cannot directly calibrate the shape parameters. Anyway, we can make them change, influencing firms' size distributions and acting on other parameters of the auxiliary model, as required by indirect inference. Since our model is very sensitive to the price generator processes, we have decided to calibrate the supports of the two generators. To be more exact, starting from the original values of the CATS model (0.5–1.5 for both small and big firms), we let them vary until the distance between the two right tails is minimized. The only requirement are: (1) the minimum of each support cannot be smaller than 0; (2) small firms' generator must have a mean value equal to one, acting as a numeraire.

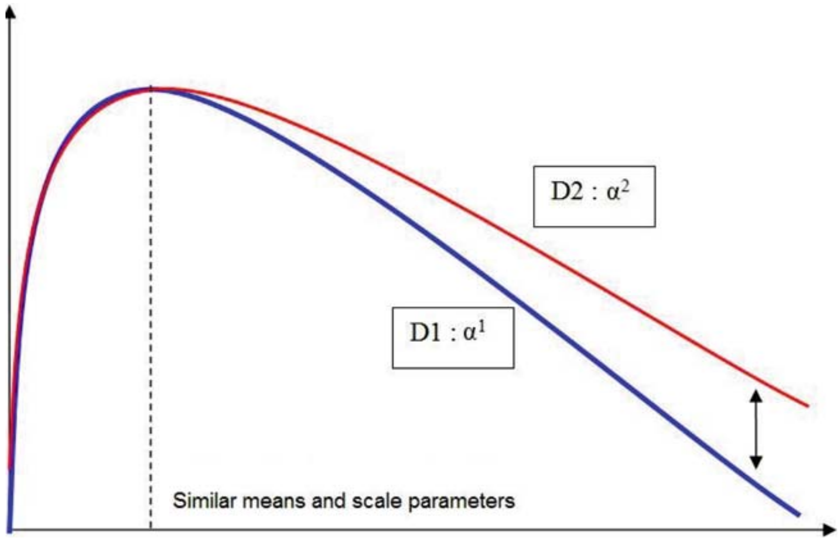

Fig. 10 Graphical explanation of our calibration method

The procedure we have used is the one by Gourieroux and Monfort (1996), which makes use of a quadratic loss function. In order to have perfect identification, besides shape parameters, we calibrate also the scale ones, even if they are already similar.

It is important to stress that our indirect inference procedure is run year by year, minimizing the distance between the actual and the simulated distributions. Moreover, in simulations, every year, apart from 1982 which is initialised with actual data, starts with the final results of the previous period. All this makes our estimates conditionally dependent on the simulation path. For all these reasons, we have run the procedure several times and then considered the average values, according to the Montecarlo methodology.

The results we achieve are quite promising, even if very preliminary. In particular, after calibration via indirect inference, we succeed in reproducing the total capital of 18,230 firms in 2000, so we ameliorate our fitting passing from 79% to 84%. Similar results are available for the other years, as shown by Table 3. Figure 11 shows the new comparison between actual and simulated total capital in 2000 after calibration via indirect inference.

Table 3 also contains the Montecarlo estimates we have obtained via indirect inference for the price generators' supports in different years. It is very interesting to see how small firms show a higher volatility, while the prices of big firms are more concentrated, as expected in our model.

So, modifying the two supports of the price generators, indirect inference allows us to sensibly reduce the differences between the two distributions.

Unfortunately, at the moment, even using grid methods, as suggested by Gilli and Winker (2003), we are not able to ameliorate our estimates, since our objective function is very flat. Probably the only solution is to develop a new and less naive model, avoiding most of the simplifications we have assumed in this paper. However, we

Table 3 Price generators' supports and fitting of total capital

	Small firms	Big firms	Before calibr. (%)	After calibr. (%)
1982	0.50–1.50	0.50–1.50	100	100
1986	0.62*–1.38*	0.70*–1.20*	78	86
1990	0.39*–1.61*	0.68**–1.25*	80	86
1994	0.38*–1.62**	0.66*–1.26*	79	83
1996	0.32*–1.68*	0.68**–1.23**	81	85
2000	0.31*–1.69*	0.68**–1.23**	79	84

*95% CI, **98% CI

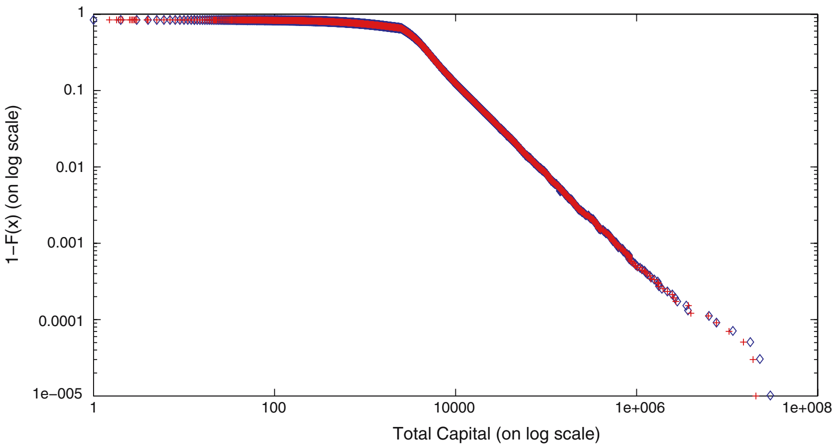

Fig. 11 Comparison between actual and simulated total capital in 2000 after calibration

believe that the results we have achieved can be a good starting point for future studies.[14]

7 Conclusions and Future Research

Even if the results we have presented are preliminary they shows that, in the interval 1982–2000, our modified CATS model has good capabilities in replicating empirical evidence, with few exceptions.

More reliable results could be obtained improving the specification of the model, and then defining new parameters to be calibrated.

In future validation experiments, we intend to modify the model specification, endogenizing the banking sector (see Vagliasindi et al. 2006) and the price generator process and including a labour market module. Moreover, we hope to match our

[14] In particular, we believe that it could be interesting to analyse small and big firms separately. We have run some tests but, at the moment, we do not find them particularly conclusive and more time is needed. We aim to deepen this point in the future.

database with other dataset in order to increase the available information, which is a fundamental aspects for correctly calibrating simulation models.

Finally the use of indirect inference in ABM should be further investigated and developed.

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References

Axelrod, R. (1997). Advancing the art of simulation in the social sciences. In R. Conte, R. Hegselmann, & P. Terna (Eds.), *Simulating social phenomena* (pp. 21–40). Berlin: Springer-Verlag.

Axtell, R. (2000). *Why agents? On the varied motivations for agent computing in the social sciences*. Center on Social and Economic Dynamics, Working Paper 17.

Axtell, R. (2001). Zipf's distribution of US firms sizes. *Sciences, 293*, 1818–1820.

Axtell, R., Axelrdod, R., Epstein, J. M., & Cohen, M. D. (1996). Aligning simulation models: A case study and results. *Computational and Mathematical Organization Theory, 1*, 123–141.

Bianchi, C., Cirillo, P., Gallegati, M., & Vagliasindi, P. (2007). Validation in agent-based models: An investigation on the CATS model. *Journal of Economic Behaviour and Organization*, forthcoming.

Bottazzi, G., & Secchi, A. (2005). Explaining the distribution of firm growth rates. *Rand Journal of Economics, 37*, 234–263.

Carley, K. (1996). *Validating computational models*. Working Paper: www.econ.iastate.edu/tesfatsi/EmpVal/EmpVal.Carley.pdf

Cirillo, P. (2007). Some considerations about Gibrat's law in Italy. *Economics Letters*, forthcoming.

Embrechts, P., Mikosch, T., & Kluppelberg, C. (1997). *Modelling extremal events*. Berlin and New York: Springer-Verlag.

Epstein, J. (1999). Agent-based computational models and generative social sciences. *Complexity, 4*, 41–60.

Fagiolo, G., Moneta, A., & Windrum, P. (2007). Empirical validation of agent-based models: Alternatives and Prospects. *Journal of Artificial Societies and Social Simulation*, *10*(2), 8.

Fujiwara, Y. (2004). Zipf law in firms bankruptcy. *Physica A, 337*, 219–230.

Gabaix, X., Gopikrishnan, P., Plerou, V., & Stanley, H. E. (2003). A theory of power law distributions in financial markets fluctuations. *Nature, 423*, 267–270.

Gaffeo, E., Di Guilmi, C., & Gallegati, M. (2003). Power law scaling in the world income distribution. *Economics Bullettin, 15*, 1–7.

Gallegati, M., Giulioni, G., Palestrini, A., & Delli Gatti, D. (2003a). Financial fragility, patterns of firms' entry and exit and aggregate dynamics. *Journal of Economic Behavior and Organization, 51*, 79–97.

Gallegati, M., Giulioni, G., & Kichiji, N. (2003b). Complex dynamics and financial fragility in an agent-based model. *Advances in Complex Systems, 6*, 770–779.

Gallegati, M., Delli Gatti, D., Di Guilmi, C., Gaffeo, E., Giulioni, G., & Palestrini, A. (2004). Business cycles fluctuations and firms' size distribution dynamics. *Advances in Complex Systems, 7*, 1–18.

Gallegati, M., Delli Gatti, D., Di Guilmi, C., Gaffeo, E., Giulioni, G., & Palestrini, A. (2005). A new approach to business fluctuations: Heterogeneous interacting agents, scaling laws and financial fragility. *Journal of Economic Behavior and Organization, 56*, 489–512.

Gallegati, M., Delli Gatti, D., Gaffeo, E., Giulioni, G., Kirman, A., Palestrini, A., & Russo, A. (2007). Complex dynamics and empirical evidence. *Information Science*, *177*, 1202–1221.

Gilli, M., & Winker, P. (2003). A global optimization heuristic for estimating agent-based models. *Computational Statistics and Data Analysis, 42*, 299–312.

Gourieroux, C., & Monfort, A. (1996). *Simulation-based econometric methods*. Oxford: Oxford University Press.

Greenwald, B. C., & Stiglitz, J. E. (1990). Macroeconomic models with equity and credit rationing. In R. Hubbard (Ed.), *Information, capital markets and investment*. Chicago: Chicago University Press.

Greenwald, B. C., & Stiglitz, J. E. (1993). Financial market imperfections and business cycles. *The Quarterly Journal of Economics, 108*, 77–114.

Hahn, F. (1982). *Money and inflation*. Oxford: Blackwell Publishing.

Hall, B. E. (1987). The relationship between firm size and growth. *Journal of Industrial Economics, 35*, 583–606.

Ijiri, Y., & Simon, H. A. (1977). *Skew distributions and the size of business firms*. Amsterdam: North Holland.

Kaldor, N. (1965). Capital accumulation and economic growth. In F. A. Lutz, & D. C. Hague (Eds.), *The theory of capital. Proceedings of a Conference held by the International Economic Association*. London: MacMillan.

Kleiber, C., & Kotz, S. (2003). *Statistical size distributions in economics and actuarial sciences*. New York: Wiley.

Kleijnen, J. P. C. (1998). Experimental design for sensitivity analysis, optimization and validation of simulation models. In J. Banks (Ed.), *Handbook of simulation* (Chap. 6). New York: Wiley.

Klevmarken, N. A. (1998). Statistical inference in microsimulation models: Incorporating external information. Working Paper of Uppsala University, Department of Economics.

Mandelbrot, B. (1960). The Pareto-Lévy law and the distribution of income. *International Economic Review, 1*, 79–106.

Okuyama, K., Takayasu, H., & Takayasu, M. (1999). Zipf's law in income distribution of companies. *Physica A, 269*, 125–131.

Prabhakar, M. D. N., Xie, M., & Jiang, R. (2003). *Weibull models*. New York: Wiley.

Quandt, R. E. (1966a). On the size distribution of firms. *American Economic Review, 56*, 416–432.

Quandt, R. E. (1966b). Old and new methods of estimation and the pareto distribution. *Metrika, 10*, 55–82.

Ramsden, J., & Kiss-Haypal, G. (2000). Company size distribution in different countries. *Physica A, 277*, 220–227.

Sargent, T. J. (1998). Verification and validation in simulation models. *Proceedings of 1998 Winter Simulation Conference*, pp. 52–64.

Shao, J. (2003). *Mathematical statistics*. New York: Springer-Verlag.

Simon, H. A. (1955). On a class of skew distribution functions. *Biometrika, 42*, 425–440.

Stanley, M., Amaral, L., Buldyrev, S., Havling, S., Leshorn, H., Maas, P., Salinger, M., & Stanley, E. (1996). Scaling behavior in the growth of companies. *Nature, 379*, 804–806.

Subbotin, M. T. (1923). The law of frequency of error. *Mathematicheskii Sbornik, 31*, 296–301.

Tesfatsion, L. (2007). Website on Validation of ACE: http://www.econ.iastate.edu/tesfatsi/empvalid.htm

Tesfatsion, L., & Judd, K. (2006). *Handbook of computational economics 2*. Amsterdam: North Holland.

Troitzsch, K. (2004). Validating simulation models. *Proceedings of the 18th European Simulation Multiconference*, pp. 98–106.

Vagliasindi, P., Cirillo, P., & Verga, G. (2006). Imprese e mercato del credito in un modello agent-based. *Rivista Internazionale di Scienze Sociali, 114*, 459–486.

Winker, P., & Gilli, M. (2001). *Indirect estimation of parameters of agent based models of financial markets*. Working Paper presented at the 2001 International Conference on Computing in Economics and Finance of the Society for Computational Economics.

Zipf, G. K. (1932). *Selective studies and the principle of relative frequency in language*. Cambridge: Cambridge Press.