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# A NOTE ON COMPUTING THE INTERSECTION OF SPHERES IN $\mathbb{R}^n$

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**Abstract**

Finding the intersection of $n$-dimensional spheres in $\mathbb{R}^n$ is an interesting problem with applications in trilateration, global positioning systems, multidimensional scaling and distance geometry. In this paper, we generalize some known results on finding the intersection of spheres, based on QR decomposition. Our main result describes the intersection of any number of $n$-dimensional spheres without the assumption that the centres of the spheres are affinely independent. A possible application in the interval distance geometry problem is also briefly discussed.

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*Keywords and phrases*: sphere intersection, $n$-dimensional spheres, QR decomposition.

## 1. Introduction

We consider the intersection of $m$ spheres in $\mathbb{R}^n$, presenting a theorem that shows how the points in the intersection are distributed in such a space. In general, we define this problem as follows.

Let $a_1, \dots, a_m$ be the centres of $m$ spheres in $\mathbb{R}^n$, and $d_1, \dots, d_m$ be their respective radii. The points $x$ at the intersection are given by the following equations:

$$\|x - a_i\|_2^2 = d_i^2, \quad i = 1, \dots, m. \tag{1.1}$$

There are methods for solving particular cases of problem (1.1). For example, Wu and Wu [22] show the intersection of four spheres in $\mathbb{R}^3$ when the centres of spheres are not in the same plane, that is, the centres are affinely independent. Gonçalves and Mucherino [11] presented a discussion on an intersection of three spheres in $\mathbb{R}^3$ by using an orthonormal basis, while Alves and Lavor et al. [1, 13] used Clifford algebra in calculating intersection of the spheres.

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Our study is based on the work of Coope [7], which considers $n$ spheres in $\mathbb{R}^n$ with affinely independent centres. In general, such intersection consists of at most two points. We generalize Coope's analysis by considering any number of spheres and centres which are not necessarily affinely independent.

Sphere intersection appears as a subproblem of important problems such as the distance geometry problem (DGP) [4, 8, 14, 15, 17, 19, 20]. The DGP has applications in several areas, for example in biochemistry [6], nanotechnology [2], robotics [21], sensor network localization [3], multidimensional scaling [5] and several others.

In Section 2, we present a theorem which states that the intersection of $m$ spheres in $\mathbb{R}^n$, where the centres generate an affine space of dimension $k$, is an empty set, a single point or an $(n-k)$-sphere. The proof is based on the QR decomposition [10] and also leads to a method for computing the intersection of spheres in the general case. Section 3 presents some computational experiments on random instances of the problem in order to illustrate the numerical stability of the proposed method. In Section 4, we present some conclusions and potential applications of the proposed approach.

## 2. Intersection of $n$-dimensional spheres in $\mathbb{R}^n$

An $i$-sphere is a generalization of the concept of a sphere. We say that an $i$-sphere in $\mathbb{R}^n$ is the intersection of a sphere with an affine subspace of dimension $i$. The following theorem describes the intersection of an arbitrary number of $n$-dimensional spheres in $\mathbb{R}^n$.

THEOREM 2.1. *Let $a_1, \ldots, a_m$ be the centres of $m$ spheres in $\mathbb{R}^n$, and $d_1, \ldots, d_m \in \mathbb{R}_+$ be their respective radii. If the affine hull of these $m$ centres has dimension $k \in \mathbb{N}$, then the possibilities for the sphere intersection are:*

(1) *the empty set;*
(2) *a single point;*
(3) *an $(n-k)$-sphere.*

PROOF. First, we translate the centres in such a way that one of them is translated to the origin. For convenience, we choose the last vector $a_m$ and subtract it from the other centres.

Let $\widehat{A}$ denote the $n \times (m-1)$ matrix of shifted centres, that is,

$$\widehat{A} = [a_1 - a_m, \ldots, a_{m-1} - a_m].$$

This matrix $\widehat{A}$ has rank $k$, because the affine hull of these $m$ centres has dimension $k$, and thus $\widehat{A}$ has $k$ linearly independent columns, where $k \leq \min\{n, m-1\}$.

If $x$ is a point in the intersection, then $\bar{x} = x - a_m$ is also at the intersection of the translated spheres. Then

$$\|\bar{x}\|_2^2 = d_m^2 \tag{2.1}$$

and

$$\|\bar{x} - (a_i - a_m)\|_2^2 = d_i^2, \quad i = 1, \ldots, m-1,$$

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or equivalently,

$$\|\bar{x}\|_2^2 - 2\bar{x}^\top(a_i - a_m) + \|a_i - a_m\|_2^2 = d_i^2, \quad i = 1, \ldots, m-1.$$

From equation (2.1),

$$d_m^2 - 2\bar{x}^\top(a_i - a_m) + \|a_i - a_m\|_2^2 = d_i^2, \quad i = 1, \ldots, m-1$$

and

$$(a_i - a_m)^\top \bar{x} = -\tfrac{1}{2}(d_i^2 - d_m^2 - \|a_i - a_m\|_2^2), \quad i = 1, \ldots, m-1. \tag{2.2}$$

Thus, in matrix form, equations (2.2) are given by $\widehat{A}^\top \bar{x} = c$, where

$$c_i = -\tfrac{1}{2}(d_i^2 - d_m^2 - \|a_i - a_m\|_2^2), \quad i = 1, \ldots, m-1.$$

Now, consider the QR decomposition of $\widehat{A}$:

$$\widehat{A} = QR = Q\begin{bmatrix}\widehat{R}\\ 0\end{bmatrix},$$

where $Q$ is an $n \times n$ orthogonal matrix and $R$ is an $n \times (m-1)$ matrix, with the last $n-k$ rows null. Also, $R$ has the same rank as $\widehat{A}$. We obtain

$$[\widehat{R}^\top \quad 0]Q^\top \bar{x} = c, \quad \text{and write} \quad Q^\top \bar{x} = \begin{bmatrix}y\\ z\end{bmatrix},$$

where $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{n-k}$, implying

$$[\widehat{R}^\top \quad 0]\begin{bmatrix}y\\ z\end{bmatrix} = c,$$

or equivalently,

$$\widehat{R}^\top y = c. \tag{2.3}$$

On the other hand,

$$d_m^2 = \|\bar{x}\|_2^2 = \left\|Q\begin{bmatrix}y\\ z\end{bmatrix}\right\|_2^2 = \left\|\begin{bmatrix}y\\ z\end{bmatrix}\right\|_2^2 = \|y\|_2^2 + \|z\|_2^2,$$

which implies that

$$\|z\|_2^2 = d_m^2 - \|y\|_2^2. \tag{2.4}$$

The intersection of spheres will be nonempty if and only if both the linear system (2.3) and equation (2.4) are consistent. Once the values of $y$ and $z$ are determined, the points in the intersection are given by

$$x = Q\begin{bmatrix}y\\ z\end{bmatrix} + a_m. \tag{2.5}$$

Since the linear system (2.3) is over-determined (that is, more equations than variables) and $\widehat{R}^\top$ has full rank, either this system has none or a unique solution. If the system (2.3) is inconsistent, then the intersection of $m$ spheres is empty. Whenever this linear system has a unique solution, equation (2.4) implies the following cases.

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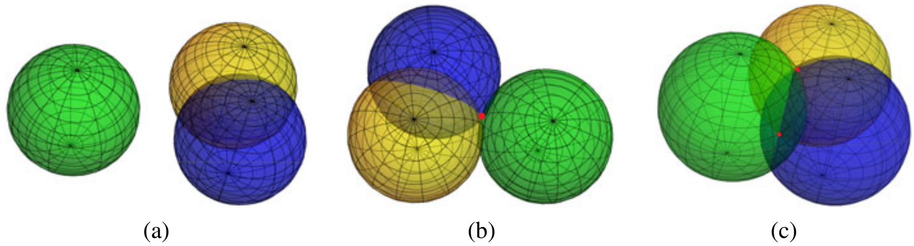

FIGURE 1. Three spheres in $\mathbb{R}^3$ with affinely independent centres. The intersection may be (a) empty, (b) a single point or (c) a 1-sphere.

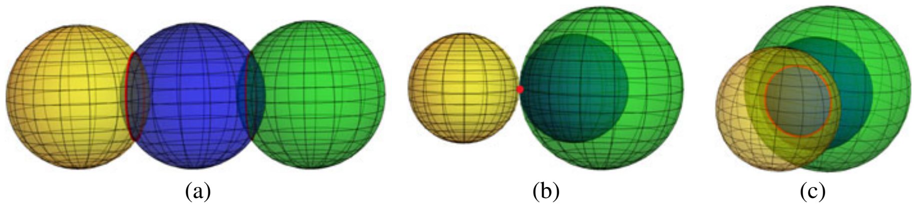

FIGURE 2. Three spheres in $\mathbb{R}^3$ with affinely dependent centres. The intersection may be (a) empty, (b) a single point, or (c) a 2-sphere.

- If $d_m^2 < \|y\|_2^2$, then the intersection is empty.
- If $d_m^2 = \|y\|_2^2$, then there is a single point in the intersection.
- If $d_m^2 > \|y\|_2^2$, then the intersection is a $(n-k)$-sphere.

In the last case, as $z \in \mathbb{R}^{n-k}$ if $d_m^2 > \|y\|_2^2$, we have that all the points of an $(n-k)$-sphere satisfy equation (2.4). So, when we apply the rotation and translation given by equation (2.5), the solution set remains an $(n-k)$-sphere. □

We observe that when the number of spheres is the same as the dimension of the space and the centres are affinely independent, the affine hull has dimension $k = n-1$. Thus, the intersection of the spheres will be empty, one point or an $(n-(n-1))$-sphere, that is, a 1-sphere, which is actually two points, as Coope has demonstrated [7].

To illustrate these results, we present figures for the particular case of three spheres intersecting in $\mathbb{R}^3$. In Figure 1, we see the case where the centres of the spheres are affinely independent, that is, the affine hull of the centres has dimension $n-1=2$. Theorem 2.1 shows that the intersection may be empty as in Figure 1(a), a single point as in Figure 1(b), or a $(2-1)$-sphere (two points) as in Figure 1(c).

If the three centres in $\mathbb{R}^3$ are affinely dependent and different, then the affine hull of these centres has dimension one. In this case, Theorem 2.1 implies that the intersection may be empty as in Figure 2(a), a single point as in Figure 2(b), or a $(3-1)$-sphere (2-sphere), that is, a circle, as in Figure 2(c).

The proof of Theorem 2.1 leads to a method for computing the intersection of $m$ spheres in $\mathbb{R}^n$, which is summarized in Algorithm 1.

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**Algorithm 1:** Sphere intersection in the general case.

**Input :** The centres $a_1, \ldots, a_m \in \mathbb{R}^n$ and the radii $d_1, \ldots, d_m \in \mathbb{R}_+$ of the $m$ spheres in $\mathbb{R}^n$.

**Output:** If the intersection is empty, the algorithm informs that. Otherwise, we have two cases:
(a) the intersection is a point: return the solution $x$;
(b) the intersection is an $(n-k)$-sphere: return $y$ and $Q$.

Define the matrix $\widehat{A} = [a_1 - a_m, \ldots, a_{m-1} - a_m]$;
Compute $\widehat{A} = QR$ and set $k = \mathrm{rank}(\widehat{A})$;
Compute the vector $c \in \mathbb{R}^{m-1}$ defined by $c_i = -\frac{1}{2}(d_i^2 - d_m^2 - \|a_i - a_m\|_2^2)$;
Obtain the matrix $\widehat{R}$ by removing the $n-k$ last rows of $R$;
**if** $c \notin \mathrm{Range}(\widehat{R}^\top)$ **then**
  Stop, the intersection is empty;
**end**
Solve the system $\widehat{R}^\top y = c$;
**if** $d_m^2 - \|y\|_2^2 < 0$ **then**
  Stop, the intersection is empty;
**end**
**if** $d_m^2 - \|y\|_2^2 = 0$ **then**
  Set $z = 0$, and return $x = Q\begin{bmatrix} y \\ z \end{bmatrix} + a_m$;
**end**
**if** $d_m^2 - \|y\|_2^2 > 0$ **then**
  Return $Q$ and $y$. Any point $x$ in the intersection can be obtained by
  $x = Q\begin{bmatrix} y \\ z \end{bmatrix} + a_m$, where $z$ is any vector satisfying $\|z\|_2^2 = d_m^2 - \|y\|_2^2$.
**end**

Thus, we have a method for computing the intersection of $n$-dimensional spheres in the general case. In the next section, we present some numerical experiments with Algorithm 1.

## 3. Numerical results

In this section, we present some computational experiments to illustrate the reliability and numerical stability of Algorithm 1 for computing the intersection of $m$ spheres in $\mathbb{R}^n$. We perform tests with dimension $n$ ranging in the interval $2 \le n \le 500$. As efficiency and stability measures, we considered: (a) the time spent by the algorithm on computing the intersection and (b) the error calculated through the mean distance error (MDE) [18], defined by

$$\mathrm{MDE}(x) = \frac{1}{m} \sum_{i=1}^{m} \frac{|\|x - a_i\|_2 - d_i|}{d_i},$$

where $x$ is the computed solution, $a_i$ is the $i$th column of the matrix $A$ (the centre of the $i$th sphere), $d_i$ is the radius of sphere $i$ and $m$ is the number of spheres. Note that if $x$ is an exact solution, then $\mathrm{MDE}(x) = 0$.

The instances are generated by the following steps.

(1) A random integer $m$ is generated such that $m \geq 2$ (number of spheres).
(2) A random integer $k$ is generated such that $1 \leq k \leq m - 1$ (dimension of the affine hull of the $m$ centres).
(3) A matrix $A$ of dimension $n \times m$ and rank $k$ is then created (its columns are the centres of the spheres).
In order to create a matrix $A$ with rank $k$, we randomly generate nonzero values $\lambda_i$ independently and uniformly distributed in the interval [0,10] and vectors $\mathbf{u}_i \in \mathbb{R}^n$, $\mathbf{v}_i \in \mathbb{R}^m$ with $1 \leq i \leq k$, whose entries are sampled from a uniform distribution on $[0, 1]$. Then we define

$$A = \sum_{i=1}^{k} \lambda_i \mathbf{u}_i \mathbf{v}_i^\top.$$

(4) Finally, we generate a random point $x^* \in \mathbb{R}^n$, and calculate the distances $d_i = \|x^* - a_i\|_2$ for $1 \leq i \leq m$ (the entries of vector $d$ correspond to the sphere radii).

We observe that the instances are generated so that the intersection will always be nonempty. If the intersection of the $m$ spheres is just one point $x$, then the error of the algorithm is calculated by $\mathrm{MDE}(x)$. However, we know by Theorem 2.1 that the intersection can have infinitely many solutions, defining an $(n - k)$-sphere. To calculate the error, we choose a random solution $z$ of the equation (2.4) to obtain a point $x$ at the intersection of the $m$ spheres, using equation (2.5), and calculate the value of $\mathrm{MDE}(x)$.

Ten instances were generated for each dimension $n$. Table 1 shows the average of the time (in seconds) and errors over ten instances, for every dimension, in addition to the smallest and largest time and errors. The plots of Figures 3 and 4 are on a logarithmic scale and respectively represent the average of the time and errors (MDE).

Algorithm 1 was implemented in MATLAB R2013A, and all the experiments were carried out on an INTEL CORE I5-2410M CPU (2.3 GHz, 4 GB RAM, Windows 7, 64 bits).

Note that from Table 1 and Figure 4, Algorithm 1 was stable for this set of random instances: for all considered problems, the average error was between the orders $10^{-16}$ and $10^{-15}$. Figure 3 shows that the average time has not exceeded 0.3 s.

## 4. Conclusions and potential applications

We presented a method for solving the general case of sphere intersections in $\mathbb{R}^n$, extending Coope's method [7]. The algorithm was accurate and numerically stable in all the cases we tested. In the literature, we found only methods for calculating the intersection of spheres in $\mathbb{R}^n$, when the number of spheres is equal to or greater

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TABLE 1. Numerical results of Algorithm 1 on randomly generated instances.

| Dimension | Smallest error | Largest error | Average error | Smallest time | Largest time | Average time |
|---|---|---|---|---|---|---|
| 3 | 0.00E+00 | 7.23E-16 | 1.57E-16 | 7.63E-05 | 5.72E-03 | 1.05E-03 |
| 4 | 4.69E-17 | 2.58E-16 | 1.47E-16 | 5.09E-05 | 1.66E-04 | 8.33E-05 |
| 5 | 5.47E-17 | 5.76E-16 | 2.00E-16 | 6.56E-05 | 1.16E-04 | 9.22E-05 |
| 10 | 2.69E-17 | 4.05E-16 | 1.56E-16 | 7.23E-05 | 2.22E-04 | 1.06E-04 |
| 20 | 4.86E-17 | 7.59E-16 | 2.27E-16 | 1.44E-04 | 4.47E-04 | 2.67E-04 |
| 30 | 7.04E-17 | 2.82E-15 | 4.48E-16 | 2.55E-04 | 6.35E-04 | 3.60E-04 |
| 40 | 8.91E-17 | 2.65E-16 | 1.69E-16 | 4.23E-04 | 1.10E-03 | 5.94E-04 |
| 50 | 9.51E-17 | 3.12E-16 | 1.52E-16 | 9.87E-04 | 1.57E-03 | 1.14E-03 |
| 100 | 1.18E-16 | 2.77E-16 | 1.90E-16 | 2.33E-03 | 2.75E-03 | 2.55E-03 |
| 150 | 1.54E-16 | 5.59E-16 | 3.52E-16 | 5.15E-03 | 9.60E-03 | 6.31E-03 |
| 200 | 1.88E-16 | 1.10E-15 | 3.43E-16 | 1.66E-02 | 2.07E-02 | 1.84E-02 |
| 250 | 2.00E-16 | 5.81E-16 | 2.93E-16 | 1.76E-02 | 3.33E-02 | 2.34E-02 |
| 300 | 1.86E-16 | 6.62E-16 | 3.31E-16 | 2.51E-02 | 5.62E-02 | 3.54E-02 |
| 350 | 1.96E-16 | 6.33E-16 | 3.71E-16 | 3.88E-02 | 7.60E-02 | 6.24E-02 |
| 400 | 2.02E-16 | 1.06E-15 | 3.96E-16 | 6.82E-02 | 1.27E-01 | 1.00E-01 |
| 450 | 2.20E-16 | 1.17E-15 | 5.46E-16 | 8.49E-02 | 1.82E-01 | 1.49E-01 |
| 500 | 3.61E-16 | 8.76E-16 | 4.70E-16 | 9.43E-02 | 2.69E-01 | 2.11E-01 |

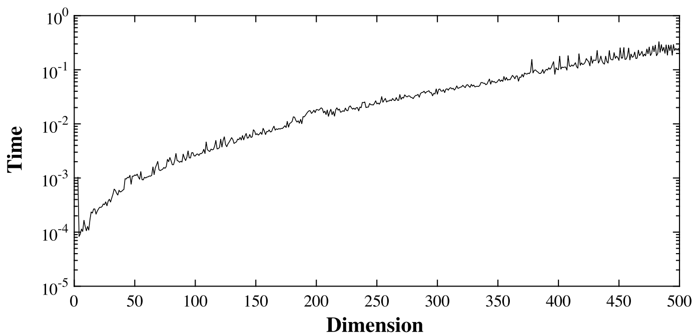


FIGURE 3. Average time (in seconds) of Algorithm 1.

than the dimension of the space, and when the centres are affinely independent. The proposed method allows us to calculate the intersection of any number of spheres in $\mathbb{R}^n$ without the affine independence assumption.

One of the possible applications of our results is on the generalization of methods that were initially developed to solve the specific classes of the DGP as the algorithms branch-and-prune [18] and geometric build-up [9].

For example, in the *interval* discretizable distance geometry problem (*i*DDGP) [12, 16], some distances are allowed to be inexact, represented by an interval, let

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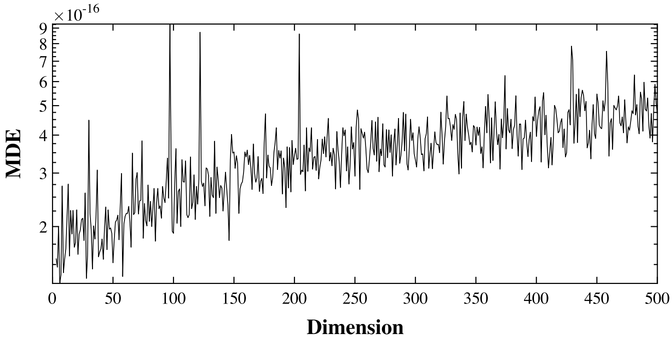


FIGURE 4. Average errors (MDE) of Algorithm 1.

us say $d \in [\underline{d}, \bar{d}]$. More specifically, in the application of *i*BP (*interval* branch-and-prune) to an *i*DDGP instance, a common sub-problem to be solved is determining the intersection of $m-1$ spheres with one spherical shell (that is, $\underline{d}_m \le d_m \le \bar{d}_m$) in $\mathbb{R}^n$. By using Algorithm 1, we may obtain the vector $y$ and matrix $Q$ that describe the intersection of the $m-1$ spheres as

$$x(z) = Q\begin{bmatrix} y \\ z \end{bmatrix} + a_{m-1}, \tag{4.1}$$

and perform its intersection with the spherical shell defined by

$$\underline{d}_m \le \|x(z) - a_m\|_2 \le \bar{d}_m.$$

Due to equation (4.1) and the discussion presented in Section 2, it follows that the desired intersection may be obtained by finding a vector $z$ such that

$$\|z\|_2^2 = d_{m-1}^2 - \|y\|_2^2, \quad \text{and} \quad \alpha \le w_2^T z \le \beta,$$

where

$$\alpha = \tfrac{1}{2}(\underline{d}_m^2 - \gamma), \quad \beta = \tfrac{1}{2}(\bar{d}_m^2 - \gamma) \quad \text{and} \quad \gamma = d_{m-1}^2 + \|a_{m-1} - a_m\|_2^2 + 2w_1^T y$$

with $[w_1^T \ w_2^T] = (a_{m-1} - a_m)^T Q$. Therefore, we can characterize the intersection of $m-1$ spheres with one spherical shell by a quadratic equation and two linear inequalities. Application of this strategy in *i*BP implementations is a subject for future research.

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