Minimal counterexamples to Hendrickson's conjecture on globally rigid graphs

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In this paper we consider the class of graphs which are redundantly d-rigid and $(d+1)$-connected but not globally d-rigid, where d is the dimension. This class arises from counterexamples to a conjecture by Bruce Hendrickson. It seems that there are relatively few graphs in this class for a given number of vertices. Using computations we show that $K_{5,5}$ is indeed the smallest counterexample to the conjecture.

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1 Introduction

The basis of the class of graph we consider is the following theorem from Hendrickson.

Theorem 1 (Hendrickson [9]). *A globally d-rigid graph with at least $d+2$ vertices is $(d+1)$-connected and redundantly d-rigid.*

Hendrickson conjectured that the reverse direction of the theorem also holds. While this is true for $d \in \{1, 2\}$ [10], it was shown to be false in any higher dimension [1] using some bipartite graphs. Since then further counterexamples were found [3, 12]. Still very little is known on the set of graphs which are $(d+1)$-connected and redundantly d-rigid

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but not globally rigid. In this paper we computationally show that the well-known $K_{5,5}$ is indeed the smallest counterexample in dimension three.

Before starting the proof, we recall some definitions and results from rigidity theory. A framework (G, ρ) consisting of a graph G and a placement ρ of the vertices in d-dimensional space is *d-rigid* if edge-length preserving continuous motions can only arise from isometries in that space. A graph is d-rigid, if the framework with any generic placement is d-rigid. *Redundant d-rigidity* is obtained if the graph is d-rigid and remains so after removal of any of its edges. A graph is *globally d-rigid*, if all frameworks which yield the same edge-lengths, indeed also yield the same distances between all pairs of non-adjacent vertices. A graph that is redundantly d-rigid and $(d+1)$-connected but not globally d-rigid, i. e. a counterexample to Hendrickson's conjecture, we call an *H_d-graph* in this paper.

If the number of vertices is comparably low with respect to the dimension then Hendrickson's conjecture has been proven to hold.

Theorem 2 (Jordán [11, Thm. 3.2]). *A graph $G = (V, E)$ with $d + 2 \leq |V| \leq d + 4$ is globally d-rigid if and only if it is $(d + 1)$-connected and redundantly d-rigid.*

In this paper we extend the theorem to $|V| \leq d + 6$.

2 Computations

In this section we describe what computations we were able to do and what software we used for that.

Minimally rigid graphs with n vertices in dimension d do have $dn - \binom{d+1}{2}$ edges. We use Geng (a part of Nauty [15, 14]) to generate such graphs. With a plugin by Martin Larsson [13] this process can be sped up using sparsity properties of rigid graphs, i. e. edge counts on subgraphs. However, not all of these graphs are indeed d-rigid. We then use RigiComp [7], a Mathematica package, to check for rigidity. Hence, we obtain lists of all minimally rigid graphs with a certain number of vertices for dimension $d \leq 23$. We then can construct all d-rigid graphs by adding edges or we use Geng instead for getting graphs with at least $dn - \binom{d+1}{2}$ edges and check rigidity for all of them. From those we can select the $(d + 1)$-connected and redundantly d-rigid ones (see Table 1). Since redundantly rigid graphs have at least $dn - \binom{d+1}{2} + 1$ edges, we can ignore all minimally rigid ones. The set of redundantly d-rigid graphs is made available at [6].

We then compute, again with RigiComp, the globally d-rigid ones from these lists (see Table 2). The list of globally rigid graphs is available at [5]. Since checking redundant rigidity seems to be the slowest computation we may also first compute rigid, then the non-globally rigid ones. From those we check connectivity and only in the end we compute redundancy.

$\lvert V\rvert$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$d+2$	1	1	1	1	1	1
$d+3$	3	3	3	3	3	3
$d+4$	19	22	23	24	24	24
$d+5$	304	531	756	900	984	1 021
$d+6$	12 055	47 777	128 855	246 302	362 922	448 587
$d+7$	1 079 143	11 623 750	78 033 041			

$\lvert V\rvert$	$d=9$	$d=10$	$d=11$	$d=12$	$d=13$	$d=14$	$d=15$	$d=16$
$d+2$	1	1	1	1	1	1	1	1
$d+3$	3	3	3	3	3	3	3	3
$d+4$	24	24	24	24	24	24	24	24
$d+5$	1 040	1 047	1 051	1 052	1 053	1 053	1 053	1 053
$d+6$	498 232	522 517	533 092	537 457	539 203	539 912	540 194	540 313

$\lvert V\rvert$	$d=17$	$d=18$	$d=19$	$d=20$	$d=21$	$d=22$	$d=23$
$d+2$	1	1	1	1	1	1	1
$d+3$	3	3	3	3	3	3	3
$d+4$	24	24	24	24	24	24	24
$d+5$	1 053	1 053	1 053	1 053	1 053	1 053	1 053
$d+6$	540 360	540 381	540 389	540 393	540 394	540 395	540 395

Table 1: Number of redundantly d-rigid and $(d+1)$-connected graphs for different dimensions.

$\lvert V\rvert$	$d=3$	$d=4$	$d=5$	$d=6$	$d=7$	$d=8$
$d+2$	1	1	1	1	1	1
$d+3$	3	3	3	3	3	3
$d+4$	19	22	23	24	24	24
$d+5$	304	531	756	900	984	1 021
$d+6$	12 055	47 777	128 855	246 302	362 922	448 587
$d+7$	1 079 142	11 623 749	78 033 040			

Table 2: Number of globally d-rigid graphs for different dimensions. For $|V| \leq d+6$ this is the same as Table 1.

Note, that RigiComp provides both a probabilistic relatively fast computation which might give false negatives and a symbolic implementation (both based on [4]). For $|V| \leq d+5$ we can use the symbolic approach. For more vertices we used the probabilistic method. Nevertheless, for $|V| \leq d+6$ we get an exact answer due to the following.

Let us first assume we want to have all minimally rigid graphs. Every minimally rigid graph is tight which means it has $d|V| - \binom{d+1}{2}$ edges and every induced subgraph with $|V'| \geq d$ vertices has at most $d|V'| - \binom{d+1}{2}$ edges. These tight graphs are those that we get from initial computations from Geng with the plugin [13]. Not all of them, however, are rigid. The remaining ones are either flexible circuits or 0-extensions thereof or contain a non-tight circuit as a spanning subgraph. In [8] flexible circuits with at most $d+6$ vertices have been fully classified. Hence, we have a deterministic answer for minimal rigidity in that case. Computationally we may first use the probabilistic procedure and in case of a negative answer do check via flexible circuits. We can do similarly when checking d-rigidity (i. e. containing a spanning minimally rigid graph) and redundant rigidity by tracing back to minimal rigidity. Hence, for $|V| \leq d+6$ the results are deterministic. For $d=3$ also the computation for $|V| = d+7$ was done symbolically.

The computations show that all redundantly d-rigid graphs that are also $(d+1)$-connected are indeed globally d-rigid if $|V| \leq d+6$. For $|V| = d+7$ we see that $K_{5,5}$ and successive cones give the only H_d-graphs at least until $d=5$. The *cone* of a graph is obtained by adding a new vertex and edges from it to all the existing vertices (compare [16, 2]). This construction we also call *coning*.

3 Main Results

From the computations and some edge counting we can state the following main theorem.

Theorem 3. *A graph $G=(V,E)$ with $d+2 \leq |V| \leq d+6$ is globally d-rigid if and only if it is $(d+1)$-connected and redundantly d-rigid.*

For the proof we use two basic statements.

Lemma 4. *Let $G=(V,E)$ be a redundantly d-rigid graph with $d \geq 23$, $n=|V|=d+k$ and $k \leq 6$. Then G has a vertex of degree $n-1$.*

Proof. Since G is redundantly d-rigid it has $dn - \binom{d+1}{2} + \ell$ edges for some $\ell \geq 1$. We assume for contradiction that all vertices have degree less than $n-1$. Then

$$
\begin{aligned}
0 &= \sum_{v\in V} \deg(v) - 2|E| \\
&\leq n(n-2) - 2(dn - \binom{d+1}{2} + \ell) \\
&= (d+k)(d+k-2) - 2d(d+k) + d(d+1) - 2\ell \\
&= (d+k)(k-d-2) + d(d+1) - 2\ell = k^2 - d^2 - 2d - 2k + d^2 + d - 2\ell \\
&= k^2 - 2k - d - 2\ell \leq 24 - d - 2\ell
\end{aligned}
$$

But this cannot be true for $d \geq 23$ and therefore G has a vertex of degree $n-1$. $\square$

Let G be a graph and let G' be constructed from G by coning. Coning preserves several properties in rigidity while increasing the dimension. Rigidity is proven to be preserved by [16]. This means that a graph is d-rigid if and only if the cone is $(d+1)$-rigid. The same is true for global rigidity due to [2]. If the cone G' is redundantly $(d+1)$-rigid, then it is easy to see that G is redundantly d-rigid. From this we get the following Lemma.

Lemma 5. *Let $d \geq 23$ and let $G=(V,E)$ be a redundantly d-rigid and $(d+1)$-connected graph with $n=|V|=d+k$ and $k \leq 6$. If G is not globally d-rigid then there exists an H_{d-1}-graph with $n-1$ vertices.*

Proof. By Lemma 4 we know that G has a vertex of degree $n-1$. Hence, removing this vertex gives a graph G' that is redundantly $(d-1)$-rigid by and d-connected but not globally $(d-1)$-rigid on $n-1$ vertices. $\square$

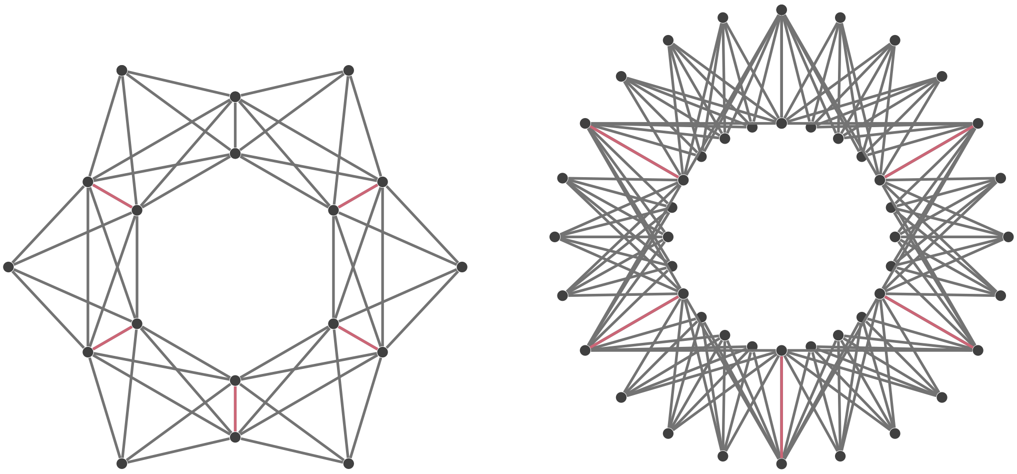
Together with the computational results we can finally proof the main theorem.

Proof of Theorem 3. Let G be a redundantly d-rigid and $(d+1)$-connected graph with $|V| \leq d+6$. If $d \leq 22$, then we have seen from the computations that G is globally d-rigid. Assuming $d \geq 23$ we know from Lemma 5 that if G was not globally d-rigid then the removal of a vertex with degree $n-1$, which does exist, gives a non-globally $(d-1)$-rigid graph that is d-connected and redundantly $(d-1)$-rigid graph. If $d-1=22$ we know from the computations that such a graph does not exist. If $d-1 \geq 23$ we continue inductively. This proves that G is globally d-rigid. $\square$

4 Examples

It is known from [1] that $K_{5,5}$ is an H_3-graph. By successive coning we get H_d-graphs for all $d \geq 3$ by [2]. More generally, it is known from [1] that $K_{m,n}$ is an H_d-graph if $m+n = \binom{d+2}{2}$ and $m,n \geq d+2$. This means that there are two H_4-graphs $K_{6,9}$ and $K_{7,8}$ with 15 vertices each, which is out of the computational scope of this paper.

For $d=3$ a construction of six globally 3-rigid graphs, glued in a cycle (see for example Figure 1a) gives an H_3-graph due to [12]. In Figure 1b we show an example of such a gluing with $K_{5,5}$, i. e., a non-globally 3-rigid base graph, which also gives an H_3-graph as shown by computations. Both these graphs are not minimal in terms of edge counts with respect to the H_3 property. Indeed we can find five edges (indicated in red in the figures) that we can remove such that the result is still an H_3-graph.



(a) Obtained from gluing six K_5 ([12]). (b) Obtained from gluing six $K_{5,5}$.

Figure 1: Two H_3-graphs.

5 Conclusion

While the computations were handy to show that there are no other small counterexamples to Hendrickson's conjecture other than the known ones, they are insufficient for a full classification. Nevertheless, we think that computational experiments might reveal new counterexamples. For this, however either some more theory or improved software/hardware is needed.

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