Vietnam National University of HCMC

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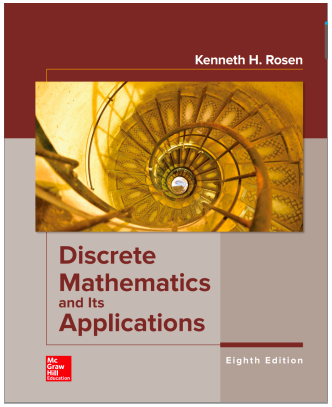

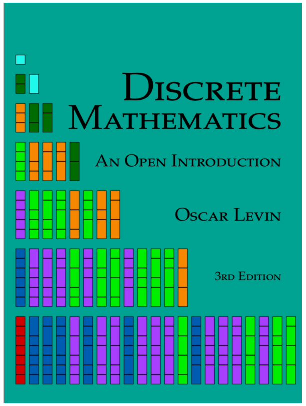

Session 3

Equivalences; Predicates and Quantifiers

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Propositional equivalences: Introduction

DEFINITION 1

A compound proposition that is always true, no matter what the truth values of the propositions that occurs in it, is called a *tautology*.
A compound proposition that is always false is called a *contradiction*.
A compound proposition that is neither a tautology or a contradiction is called a *contingency*.

Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical equivalences

DEFINITION 2

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- Example: Show that $\neg p \vee q$ and $p \rightarrow q$ are logically equivalent.

Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Propositional equivalences

- If we use a truth table to establish such a logical equivalence of two propositional variables, we need 4 rows (p, q: TT, TF, FT, FF).
- If we use a truth table to establish such a logical equivalence of three propositional variables, we need 8 rows (p, q, r: TTT, TTF, TFT, TFF , FTT, FTF , FFT, FFF).
- …Etc
- In general, 2^n rows are required if a compound proposition involves *n* propositional variables in order to get the combination of all truth values.

Example:

Using the truth table to show that p $\vee$ (q $\wedge$ r) and (p $\vee$ q) $\wedge$ (p $\vee$ r) are logically equivalent.

Solution:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

What happen if we have a series of variables?

Propositional equivalences

Constructing new logical equivalences

- Example: Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

  Solution:

  $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$ *by example on slide 3*

  $\equiv \neg(\neg p) \wedge \neg q$ *by the second De Morgan law*

  $\equiv p \wedge \neg q$ *by the double negation law*

- *Example: Show that* $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

  Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to T.

  $(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ *by example on slide 3*

  $\equiv (\neg p \vee \neg q) \vee (p \vee q)$ *by the first De Morgan law*

  $\equiv (\neg p \vee p) \vee (\neg q \vee q)$ *by the associative and communicative law for disjunction*

  $\equiv \mathrm{T} \vee \mathrm{T}$

  $\equiv \mathrm{T}$

- *Note: The above examples can also be done using truth tables*.

Logical equivalence

$$\boldsymbol{p} \leftrightarrow \boldsymbol{q}$$

- Compound propositions *p* and *q* are logically equivalent to each other (written $p \leftrightarrow q$,) **IFF** *p* and *q* contain the same truth values as each other in <u>all</u> rows of their truth tables.

Proving Equivalence via Truth Table

Ex. Prove that $p \vee q \leftrightarrow \neg(\neg p \wedge \neg q)$.

p	q	$p \vee q$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
F	F	F	T	T	T	F
F	T	T	T	F	F	T
T	F	T	F	T	F	T
T	T	T	F	F	F	T

At the end, the left hand side is the same as the right hand side

Equivalence laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.
- They provide a pattern or template that can be used to match all or part of a much more complicated proposition and to find an equivalence for it.

Equivalence laws

Logic Rules : p, q, r: propositional variables, 1 is a true value and 0 is a false value, we have the following equivalence laws:

1) Idempotent

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

Equivalence laws

2) Commutative

$$p \wedge q \equiv q \wedge p$$

$$p \vee q \equiv q \vee p$$

3) Associative

$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r) \leftrightarrow (p \wedge q \wedge r)$

$(p \vee q) \vee r \equiv p \vee (q \vee r)$

4) Distributive

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Equivalence laws

5) Implication

$$p \to q \equiv \neg p \vee q$$

6) Double negation

$$\neg(\neg p) \equiv p$$

7) Negation of De Mogan's

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\boldsymbol{\neg(p \to q) \equiv p \wedge \neg q}$$

Equivalence laws

8) Contra-positive

$$p \to q \equiv \neg q \to \neg p$$

9) Equivalence

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

10) Identity

$$p \wedge 1 \equiv p$$

and $\quad p \vee 0 \equiv p$

Equivalence laws

11) Complement

$$p \land \neg p \equiv 0$$

and $$p \lor \neg p \equiv 1$$

12) Domination

$$p \land 0 \equiv 0$$

and $$p \lor 1 \equiv 1$$

Equivalence laws

13) Absorption

$$p \wedge (p \vee q) \equiv p$$

and $$p \vee (p \wedge q) \equiv p$$

ex: (U v V) v (U ^ N ^ !Q ^ V)
= (U v V) v (U ^ V ^ N ^ !Q)
= ((U v V)

14) Simplification

$$(p \wedge q) \rightarrow p$$

15) Extension

$$p \rightarrow (p \vee q)$$

Equivalence laws

16) Reduction (shortening)

$$p \wedge q \rightarrow p \equiv 1$$

$$p \rightarrow (p \wedge q) \equiv p \rightarrow q$$

$$(p \vee q) \rightarrow q \equiv p \rightarrow q$$

$$p \rightarrow (p \vee q) \equiv 1$$

Equivalent laws - examples

- *Identity*: $p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$
- *Domination*: $p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$
- *Idempotent*: $p \vee p \equiv p$ $p \wedge p \equiv p$
- *Double negation:* $\neg\neg p \equiv p$
- *Commutative:* $p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
- *Associative:* $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

More equivalence laws

- *Distributive*: $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
  $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

- *De Morgan's*:
  $\neg(p \land q) \equiv \neg p \lor \neg q$
  $\neg(p \lor q) \equiv \neg p \land \neg q$

- *Trivial tautology/contradiction*:
  $p \lor \neg p \equiv \mathbf{T}$ $\quad$ $p \land \neg p \equiv \mathbf{F}$

Defining operators via equivalences

Using equivalences, we can *define* operators in terms of other operators:

- Exclusive or: $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$
  $p \oplus q \equiv (p \wedge \neg q) \vee (q \wedge \neg p)$
- Implies: $p \rightarrow q \equiv \neg p \vee q$
- Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
  $p \leftrightarrow q \equiv \neg(p \oplus q)$

Examples:

Check using a symbolic derivation whether:

$(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r.$

Solution:

$(p \wedge \neg q) \rightarrow (p \oplus r)$

$\equiv \neg(p \wedge \neg q) \vee (p \oplus r)$ [Expand definition of $\rightarrow$]

$\equiv \neg(p \wedge \neg q) \vee ((p \vee r) \wedge \neg(p \wedge r))$ [Definition of $\oplus$]

$\equiv (\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$ [De Morgan's]

$\equiv (q \vee \neg p) \vee ((p \vee r) \wedge \neg(p \wedge r))$ [Commutative]

$\equiv q \vee (\neg p \vee ((p \vee r) \wedge \neg(p \wedge r)))$ [Associative]

Examples:

$\equiv q \vee (((\neg p \vee (p \vee r)) \wedge (\neg p \vee \neg(p \wedge r)))$ [Distributive]

$\equiv q \vee (((\neg p \vee p) \vee r)) \wedge (\neg p \vee \neg(p \wedge r)))$ [Associative]

$\equiv q \vee ((\mathbf{T} \vee r) \wedge (\neg p \vee \neg(p \wedge r)))$ [Complement]

$\equiv q \vee (\mathbf{T} \wedge (\neg p \vee \neg(p \wedge r)))$ [Domination]

$\equiv q \vee (\neg p \vee \neg(p \wedge r))$ [Identity]

$\equiv q \vee (\neg p \vee (\neg p \vee \neg r))$ [De Morgan's]

$\equiv q \vee ((\neg p \vee \neg p) \vee \neg r)$ [Associative]

$\equiv q \vee (\neg p \vee \neg r)$ [*Idempotent*]

$\equiv (q \vee \neg p) \vee \neg r$ [Associative]

$\equiv \neg p \vee q \vee \neg r$ [Commutative]

finish

Review: propositional logic

- Atomic propositions: p, q, r, …
- Boolean operators: $\neg \; \wedge \; \vee \; \oplus \; \rightarrow \; \leftrightarrow$
- Compound propositions: $s :\equiv (p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \equiv \neg(p \rightarrow q)$
- Proving equivalences using:
  - Truth tables.
  - Symbolic derivations. $p \equiv q \equiv r$ …
- Next: PREDICATES and QUANTIFIERS

Predicates: examples

Which statements are propositions:

- Minh loves ice cream. YES
- *X* loves ice cream. NO
- Everyone loves ice cream. YES
- Someone loves ice cream. YES

Examples

Which statements are propositions:

$3 + 4 = 7$ YES

$X + 4 = 5$ NO

$X + 2 = 5$ for any choice of X in $\{1, 2, 3\}$ YES

$X + 2 = 5$ for some X in $\{1, 2, 3\}$ YES

Proposition, YES or NO?

$12 > 7$ YES

$X > 4$ NO

$X > 4$ for any choice of X in $\{3, 4, 5\}$ YES

$X > 4$ for some X in $\{1, 2, 3\}$ YES

The efficiency of predicates.

Nam eats pizza at least once a week.
Viet eats pizza at least once a week.
John eats pizza at least once a week.
Minh eats pizza at least once a week.
Thu eats pizza at least once a week.
Hai eats pizza at least once a week.
Khoa eats pizza at least once a week.

The efficiency of predicates.

An eats pizza at least once a week.

………………………………………

………………………………………

Define:

$P(x)$ = "x eats pizza at least once a week."

Universe of Discourse - x is a student in Discrete Math class

- Note that $P(x)$ is not a proposition, $P(\text{Binh})$ is.

Predicates

Definition: A predicate, or propositional function, is a function defined on a set *U* and returns a proposition as its value.

The set *U* is called the ***universe of discourse***.

- We often denote a predicate by $P(x)$
- Note that $P(x)$ is not a proposition, but $P(a)$ where a is some fixed element of U is a proposition with well determined truth value

Subjects and predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the *subject* - the *object* or *entity* that the sentence is about.
  - The phrase "is sleeping" denotes the *predicate*- a property that is true **of** the subject.
- In predicate logic, a *predicate* is modeled as a *function* $P(\cdot)$ from objects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any object).

More about predicates

- Convention: Lowercase variables x, y, z... denote objects/entities; uppercase variables P, Q, R… denote predicates.
- Keep in mind that the *result of applying* a predicate P to an object x is the *proposition* $P(x)$. But the predicate P **itself** (*e.g.* P="is sleeping") is **not** a proposition (not a complete sentence).
  - *E.g.* if $P(x)$ = "x is a prime number", $P(3)$ is the *proposition* "3 is a prime number."

Example:

For instance, consider the statement "x = y + 3". We can denote this statement by Q(x, y), where x and y are variables and Q is the predicate. When values are assigned to the variables x and y, the statement Q(x, y) has a true value.

- Let Q(x, y) denote the statement "x = y + 3". What are the true values of the propositions Q(1, 2) and Q(3, 0)?
- Solution: To obtain Q(1, 2), set x = 1 and y = 2 in the statement Q(x, y). Hence, Q(1, 2) is the statement "1 = 2 + 3", which is false. The statement Q(3, 0) is the proposition "3 = 0 + 3", which is true.

Practical applications of predicate logic

- It is the basis for clearly expressed formal specifications for any complex system.
- It is the basis for *automatic theorem provers* and many other Artificial Intelligence systems.
  - *E.g.* automatic program verification systems.
- Predicate-logic like statements are supported by some of the more sophisticated *database query engines* and *container class libraries*
  - these are types of programming tools.

Universes of Discourse (U.D.s)

- The power of distinguishing objects from predicates is that it lets you state things about *many* objects at once.
- E.g., let $P(x)$="$x+1>x$". We can then say, "For *any* number x, $P(x)$ is true" instead of $(\mathbf{0}+1>\mathbf{0}) \wedge (\mathbf{1}+1>\mathbf{1}) \wedge (\mathbf{2}+1>\mathbf{2}) \wedge \ldots$
- The collection of values that a variable x can take is called x's *universe of discourse*.

Quantifier expressions

- *Quantifiers* provide a notation that allows us to *quantify* (count) *how many* objects in the universal of discourse (ud). satisfy a given predicate.
- "$\forall$" is the FOR ALL or *universal* quantifier. $\forall x\, P(x)$ means *for all* x in the u.d., *P* holds.
- "$\exists$" is the EXISTS or *existential* quantifier. $\exists x\, P(x)$ means <u>there *exists*</u> an *x* in the ud (that is, 1 or more) <u>such that</u> $P(x)$ is true.

The universal quantifier

Let $P(x)$ be a predicate on some universe of discourse U.

Consider the statement

"$P(x)$ is true for all x in the universe of discourse."

We write it $\forall x\, P(x)$, and say "for all x, $P(x)$ "

The proposition $\forall x\, P(x)$ is

- TRUE if $P(a)$ is true when we substitute x by any element a in U
- FALSE if there is an element a in U for which $P(a)$ is false.

Examples:

- Let $P(x)$ be the predicate $x + 1 > x$, where the universe of discourse are the real numbers.

Then $\forall x\, P(x)$ is a TRUE proposition because for all real number x , $x + 1$ is always greater than x

- Let $Q(x)$ be the predicate $x < 1$, where the universe of discourse are the real numbers.

Then $\forall x\, Q(x)$ is a FALSE proposition because we can find a real number, say 3 such that $3 < 1$ is a FALSE proposition.

Examples:

In the special case that the universe of discourse, U, is finite, say $U = \{x_1, x_2, x_3, \ldots, x_n\}$. Then

$\forall x\ P(x)$ corresponds to the proposition:

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

We can write a program to loop through the elements in the universe and check each for truthfulness. If all are true, then the proposition is true. Otherwise it is false!

The existential quantifier

The *existential quantifier* of P(x) is the proposition:

"There exists an element *x* in the universe of discourse such that P(x) is true."

We write it ∃x P(x), and say "for some x, P(x)"

$\exists x\, P(x)$ is FALSE if $P(x)$ is false for every single x.

$\exists x\, P(x)$ is TRUE if there is an x for which $P(x)$ is true.

Examples:

- Let $P(x)$ be the predicate $x > 3$, where the universe of discourse are the real numbers.

Then $\exists\, x\, P(x)$ is a TRUE proposition because we can find a real number, say 4 such that $4 > 3$ is a TRUE proposition

- Let $Q(x)$ be the predicate $x = x + 1$, where the universe of discourse are the real numbers.

Then $\exists\, x\, Q(x)$ is a FALSE proposition because for all real number x , x and $x + 1$ are distinct real numbers.

Examples:

In the special case that the universe of discourse, U, is finite, say $U = \{x_1, x_2, x_3, \ldots, x_n\}$. Then

$\exists x\, P(x)$ corresponds to the proposition:

$P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$

We can write a program to loop through the elements in the universe and check each for truthfulness.

Predicates - not so boring examples

Universe of discourse is all creatures, define:

$L(x)$ = "x is a lion."
$F(x)$ = "x is fierce."
$C(x)$ = "x drinks coffee."

All lions are fierce. $\forall x\,(L(x) \rightarrow F(x))$

Some lions don't drink coffee. $\exists x\,(L(x) \wedge \neg C(x))$

Some fierce creatures don't drink coffee. $\exists x\,(F(x) \wedge \neg C(x))$

Negations

Example:

B(x) = "x is a hummingbird."

L(x) = "x is a large bird."

H(x) = "x lives on honey."

R(x) = "x is richly colored."

All hummingbirds are richly colored. $\forall x\, (B(x) \rightarrow R(x))$

No large birds live on honey. $\neg\exists x\, (L(x) \wedge H(x))$

Birds that do not live on honey are dully colored. $\forall x\, (\neg H(x) \rightarrow \neg R(x))$

Negations

No large birds live on honey. $\neg\exists x\,(L(x) \wedge H(x))$

$\forall x\, P(x)$ means "$P(x)$ is true for every x."

What about $\neg\forall x\, P(x)$?

Not["$P(x)$ is true for every x."]

"There is an x for which $P(x)$ is not true."

$\exists x\, \neg P(x)$

So, $\neg\forall x\, P(x)$ is the same as $\exists x\, \neg P(x)$.

Predicates - quantifier negation

No large birds live on honey. $\neg\exists x\,(\mathrm{L}(x) \wedge \mathrm{H}(x))$

$\exists x\ P(x)$ means "$P(x)$ is true for some x."

What about $\neg\exists x\ P(x)$?

Not["$P(x)$ is true for some x."]

"$P(x)$ is not true for all x."

$\forall x\ \neg P(x)$

So, $\neg\exists x\ P(x)$ is the same as $\forall x\ \neg P(x)$.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\ (L(x) \wedge H(x))$

- $\neg\forall x\ P(x)$ is the same as $\exists x\ \neg P(x)$.
- $\neg\exists x\ P(x)$ is the same as $\forall x\ \neg P(x)$.

General rule: to negate a quantifier, move negation to the right, changing quantifiers as you go.

Predicates - quantifier negation

No large birds live on honey.

$\neg\exists x\, (L(x) \wedge H(x)) \equiv \forall x\, \neg(L(x) \wedge H(x))$ Negation rule

$\equiv \forall x\, (\neg L(x) \vee \neg H(x))$ DeMorgan's

$\equiv \forall x\, (L(x) \rightarrow \neg H(x))$ Subst for $\rightarrow$

Large birds do not live on honey.

More Example: Negating Quantified Expressions

What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution:

The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten $\exists x(x^2 \le x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \ne 2)$. The truth values of these statements depend on the domain.

Biding variables

A variable is *bound* if it is known or quantified. Otherwise, it is *free*.

Examples:

- $P(x)$ x is free
- $P(5)$ x is bound to 5
- $\forall x\ P(x)$ x is bound by quantifier

Reminder: in a proposition, all variables must be bound.

Predicates - multiple quantifiers

To bind many variables, use many quantifiers.

Example: $P(x,y)$ = "$x > y$"

- $\forall x\, P(x,y)$ NOT a proposition
- $\forall x \forall y\, P(x,y)$ FALSE proposition
- $\forall x \exists y\, P(x,y)$ TRUE proposition
- $\forall x\, P(x, 3)$ FALSE proposition

Predicates - the meaning of multiple quantifiers

<u>Example</u>: $N(x,y)$ = "*x* is sitting next to *y*"

$\forall x \forall y\, N(x,y)$ - everyone is sitting next to everyone else. **False**

$\exists x \exists y\, N(x,y)$ - there are two people sitting next to each other. **True?**

$\forall x \exists y\, N(x,y)$ - every person is sitting next to somebody. **True?**

$\exists x \forall y\, N(x,y)$ - a particular person is sitting next to everyone else. **False**

Still more conventions

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x{>}0\ P(x)$ is shorthand for "For all x that are greater than zero, $P(x)$."
    $=\forall x\ (x{>}0 \rightarrow P(x))$
  - $\exists x{>}0\ P(x)$ is shorthand for "There is an x greater than zero such that $P(x)$."
    $=\exists x\ (x{>}0 \wedge P(x))$

More to know about binding

- $\forall x\ \exists x\ P(x)$ - x is not a free variable in $\exists x\ P(x)$, therefore the $\forall x$ binding <u>isn't used</u>.
- $(\forall x\ P(x)) \wedge Q(x)$ - The variable x is outside of the *scope* of the $\forall x$ quantifier, and is therefore free. Not a complete proposition!
- $(\forall x\ P(x)) \wedge (\exists x\ Q(x))$ – This is legal, because there are 2 <u>different</u> x's!

Quantifier equivalence laws

- Definitions of quantifiers: If u.d.=a,b,c,…
  $\forall x\ P(x) \equiv P(\mathrm{a}) \wedge P(\mathrm{b}) \wedge P(\mathrm{c}) \wedge \ldots$
  $\exists x\ P(x) \equiv P(\mathrm{a}) \vee P(\mathrm{b}) \vee P(\mathrm{c}) \vee \ldots$
- From those, we can prove the laws:
  $\forall x\ P(x) \equiv \neg\exists x\ \neg P(x)$
  $\exists x\ P(x) \equiv \neg\forall x\ \neg P(x)$
- Which *propositional* equivalence laws can be used to prove this?

More equivalence laws

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$
  $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$
- $\forall x\ \mathbf{(}P(x) \wedge Q(x)\mathbf{)} \equiv \mathbf{(}\forall x\ P(x)\mathbf{)} \wedge \mathbf{(}\forall x\ Q(x)\mathbf{)}$
  $\exists x\ \mathbf{(}P(x) \vee Q(x)\mathbf{)} \equiv \mathbf{(}\exists x\ P(x)\mathbf{)} \vee \mathbf{(}\exists x\ Q(x)\mathbf{)}$
- Exercise:
  See if you can prove these yourself.
  - What propositional equivalences did you use?

Example

- $\forall x(\ C(x) \lor \exists y(\ C(y) \land F(x,y)\ )\ )$ where
  - C(x): “x has a computer”
  - F(x,y): “x and y are friend”
  - U.d. for x and y are all students in school
- $\forall x(\ C(x) \lor \exists y(\ C(y) \land F(x,y)\ )\ )$
  - For all x, x has a computer OR
  - There exist a y who has a computer AND x and y are friends

Another example

- If <u>a person is female</u> and <u>is a parent</u>, then <u>this person is someone's mother</u>.
- F(x): "x is female", P(x): "x is a parent"
- M(x,y): x is mother of y
- For every person x, if x is a parent, then x is someone's mother.
- $\forall x \; (\; (\; F(x) \wedge P(x) \;) \rightarrow \exists y M(x,y) \;)$

Translating from English

- Consider “For every student in this class, that student has studied calculus”
- Rephrased: “For every student x in this class, x has studied calculus”
  - Let C(x) be “x has studied calculus”
  - Let S(x) be “x is a student”
- $\forall x\ C(x)$
  - True if the universe of discourse is all students in this class

Translating from English 2

- What about if the universe of discourse is all students (or all people?)
  - $\forall x\ (S(x) \wedge C(x))$
    - This is wrong! Why?
  - $\forall x\ (S(x) \rightarrow C(x))$
- Another option:
  - Let Q(x,y) be "x has studied y"
  - $\forall x\ (S(x) \rightarrow Q(x, \text{calculus}))$

Translating from English 3

- Consider:
  - “Some students have visited Mexico”
  - “Every students in this class has visited Canada or Mexico”
- Let:
  - $S(x)$ be “x is a student in this class”
  - $M(x)$ be “x has visited Mexico”
  - $C(x)$ be “x has visited Canada”

Translating from English 4

- Consider: “Some students have visited Mexico”
  - Rephrasing: “There exists a student who has visited Mexico”
- $\exists x\ M(x)$
  - True if the universe of discourse is all students
- What about if the universe of discourse is all people?
  - $\exists x\ (S(x) \rightarrow M(x))$
    - This is wrong! Why?
  - $\exists x\ (S(x) \wedge M(x))$

Translating from English 5

- Consider: “Every student in this class has visited Canada or Mexico”
- $\forall x\ (M(x) \vee C(x))$
  - When the universe of discourse is all students
- $\forall x\ (S(x) \rightarrow (M(x) \vee C(x))$
  - When the universe of discourse is all people
- Why isn’t $\forall x\ (S(x) \wedge (M(x) \vee C(x)))$ correct?

Translating from English 6

- Note that it would be easier to define
  $V(x, y)$ as "x has visited y"
  - $\forall x\, (S(x) \wedge V(x,\text{Mexico}))$
  - $\forall x\, (S(x) \rightarrow (V(x,\text{Mexico}) \vee V(x,\text{Canada}))$

Translating from English 7

- Translate the statements:
  - "All hummingbirds are richly colored"
  - "No large birds live on honey"
  - "Birds that do not live on honey are dull in color"
  - "Hummingbirds are small"
- Assign our propositional functions
  - Let $P(x)$ be "x is a hummingbird"
  - Let $Q(x)$ be "x is large"
  - Let $R(x)$ be "x lives on honey"
  - Let $S(x)$ be "x is richly colored"
- Let our universe of discourse be all birds

Translating from English 8

- Our propositional functions
  - Let $P(x)$ be "x is a hummingbird"
  - Let $Q(x)$ be "x is large"
  - Let $R(x)$ be "x lives on honey"
  - Let $S(x)$ be "x is richly colored"
- Translate the statements:
  - "All hummingbirds are richly colored"
    - $\forall x\ (P(x) \rightarrow S(x))$
  - "No large birds live on honey"
    - $\neg\exists x\ (Q(x) \wedge R(x))$
    - Alternatively: $\forall x\ (\neg Q(x) \vee \neg R(x))$
  - "Birds that do not live on honey are dull in color"
    - $\forall x\ (\neg R(x) \rightarrow \neg S(x))$
  - "Hummingbirds are small"
    - $\forall x\ (P(x) \rightarrow \neg Q(x))$

References: .. equivalence

Equivalence	**Name**
$p \wedge T \equiv p$; $p \vee F \equiv p$	Identity Laws
$p \wedge F \equiv F$; $p \vee T \equiv T$	Domination Laws
$p \wedge p \equiv p$; $p \vee p \equiv p$	Idempotent Laws
$\neg(\neg p) \equiv p$	Double Negation Law
$p \wedge q \equiv q \wedge p$; $p \vee q \equiv q \vee p$	Commutative Laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$ $(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative Laws [no need for parens]
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive Laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws [generalizes]
$p \vee (p \wedge q) \equiv p$; $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \vee \neg p \equiv T$; $p \wedge \neg p \equiv F$	Negation Laws

References: .. equivalence

$p \rightarrow q \equiv \neg p \vee q$

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$

$p \wedge q \equiv \neg(p \rightarrow \neg q)$ [Transform]

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

References: .. equivalence

Rules of Inference

Modus Ponens

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline q \end{array}$$

Modus Tollens

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \neg p \end{array}$$

Hypothetical Syllogism

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline p \rightarrow r \end{array}$$

Addition

$$\begin{array}{l} p \\ \hline p \vee q \end{array}$$

Resolution

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline q \vee r \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

Conjunction

$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$

Homework 2:

Doing the following exercises: 6, 8,10, 12, 16,18, 24, 32, 34, 36 (start from pages: 38)

Submit your solutions to the blackboard on 12/9/2022 before 10:00 PM.