Elements of Probablity

February 14, 2019

3 parts of this course

- Probability: Theory of the randomness
- Statistics: the art of learning from data
- Random process: Probability with time line

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Statistics in IT

- Data Science: use Stats and computer science to analyze data
- Data: big and scattered
- Stats helps to draw conclusion from data and predict what's next

Statistics in IT

- Statistical Machine Learning: use Stats to help the bots understand large data with random noise
- Use probability to help making decision

Statistics in EE

- Quality control: testing defective components
- Average lifetime of component
- System design: It's unlikely that all customers of an electric network use their maximum rated current at the same time, but what capacity do I need to deliver enough power 99.9% of the time?

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Frequency interpretation

If we have a large amount of chips, roughly 5% of those are defective.

Measure of belief

- If we have one chip, the chance that it is defective is 5%.
- We're 95% sure that the chip is not defective

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Main concepts

- Sample space
- Events
- Axioms of Probability

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Outcome of experiment

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- If we toss a coin, how likely it will land on its head?
- If we toss a dice, how likely it will land on face number 1?

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- The coin could land on head or tail.
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Examples

- Gender of a unborn baby

$$\Omega = \{\text{male}, \text{female}\}$$

- Roll a dice

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- Roll 2 dice

$$\Omega = \{(x, y) : x, y = 1, \ldots, 6\}$$

- Choose a real number at random between 0 and 1

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Events

- A *subset* A of Ω is a set whose elements are elements of Ω
- Subset of sample space Ω are called **event**.
- Two events are *mutually exclusive* if they have no common element.

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- Roll 1 dice, event = having an odd face

$$A = \{1, 3, 5\}$$

- Roll 2 dice, event = sum of 2 faces is 6

$$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

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Intersection

Let A, B are 2 events in Ω. Define new event AB to be the subset of all elements that are in both A and B

$$AB = A \cap B$$

Union and difference

- $A \cup B$ is the set of all elements that are in A or in B.
- $A \setminus B$ is the set of all elements that are in A but not in B.

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- $AB = \{1, 3\}$
- $A \cup B = \{1, 2, 3, 5\}$
- $A \setminus B = \{5\}$
- $B \setminus A = \{2\}$

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Complement

The complement A^c of event A is the subset containing all the elements of Ω that are not in A.

$$A \cup A^c = \Omega$$

$$AA^c = \emptyset.$$

Properties of events

- $AB = BA$
- $A \cup B = B \cup A$
- $A(BC) = (AB)C$
- $AB \cup C = (A \cup C)(B \cup C)$
- $(A \cup B)C = AC \cup BC$

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De Morgan's Law

- $(A \cup B)^c = A^c B^c$
- $(AB)^c = A^c \cup B^c$

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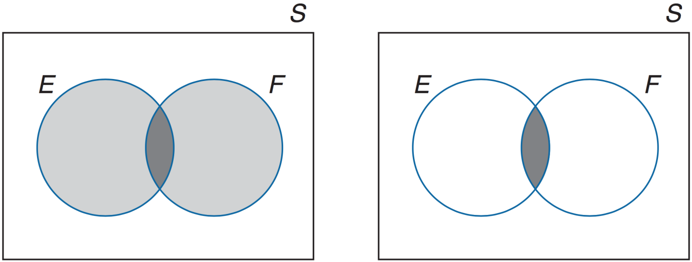
Venn's diagram

We can use circles to present events and see their relation from the picture.



(a) Shaded region: $E \cup F$ (b) Shaded region: EF

Axioms of Probability

If A is an event in Ω, define the **Probablity of A** to be number $P(A)$

- $0 \leq P(A) \leq 1$
- $P(\Omega) = 1$
- If $A_1, A_2, \ldots$ are mutually exclusive events then

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

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Example

Suppose there is a coin for which the chance to show head is twice more likely than the chance to show tail.

- $\Omega = \{H, T\}$
- $P(\{H\}) = 2/3, P(\{T\}) = 1/3$

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Idea

- Think of Ω as everything that can happen
- The total probablity of Ω is 1
- Distribute this 1 to all elements of Ω, each will have a fraction of 1
- Think of this fraction as probability of the element

Example

Roll 1 dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$

- Distribute evenly: $P(\{i\}) = 1/6$ for all $i = 1, \ldots, 6$
- Favor 1 face: $P(\{1\}) = 1/2$, $P(\{i\}) = 1/12,\ i = 2 \ldots 6$
- $P(\{1\}) = P(\{2\}) = P(\{3\}) = 1/4, P(\{4\}) = P(\{5\}) = P(\{6\}) = 1/12$

Properties

- $P(A^c) = 1 - P(A)$
- $P(A \cup B) = P(A) + P(B) - P(AB)$

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Equally likely outcomes

If $P(\{x\})$ is the same for all x in Ω then we say that Ω has equally likely outcomes.

If Ω has equally likely outcomes then

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega} = \frac{|A|}{|\Omega|}.$$

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Example

A group has 20 students. Among those, 12 students play basketball, 15 students play soccer and 4 students don't play any sport. If we choose one student at random from the group, what is the probability that the student plays only soccer?

Sample space

- Experiment: Choose 1 student out of 20 at random
- Each student has the same chance to be chosen $\rightarrow$ Equally likely outcomes
- Sample space:
  Ω = set of 20 students.

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Event

- S = the chosen student plays soccer = set of all students who play soccer.
- B = the chosen student plays basketball = set of all students who play basketball.

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Event

- The chosen student doesn't play sport $= (S \cup B)^c$
- Want to find $P(S \setminus B)$.

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Solution

- $P(S \setminus B) = P(S) - P(SB)$
- $P(SB) = P(S) + P(B) - P(S \cup B)$
- $P(S \cup B) = 1 - P((S \cup B)^c)$

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Solution

- $P(B) = 12/20$
- $P((S \cup B)^c) = 4/20$
- $P(S \setminus B) = 1 - P((S \cup B)^c) - P(B) = 4/20$

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Homework 1

Chapter 3: 2, 4, 5, 6, 8, 10