Intro to Random processes - Markov Chain

January 3, 2019

Random Processes

Introduction

- A *Random process* (RP) is a collection of random variables
- Denote: $\{X_t, t \in T\}$
- For each value of t, X_t is one random variable
- Think of t as time, when time changes the random variable also change

Examples

$\{X_t\}$ could be:

- The revenue of a store before time t
- The number of people in a store at time t
- Price of a stock at time t
- Position of a bug that traveling randomly on a flat surface

Idea

- $\{X_t\}$ is like function of t, but the outcome is random
- Very useful for modeling real life quantities b/c in real life nothing is certain
- Use random process to predict future value

Types of RP

- T is the index set
- if T is countable: $\{X_t\}$ is discrete-time RP
- Ex: $\{X_t, t = 0, 1, \ldots\}$
- if T is uncountable: $\{X_t\}$ is continuous-time RP
- Ex: $\{X_t : t \geq 0\}$

States

- For each t, X_t is called a state at time t of the process
- The state space is the set of all possible values of X_t

Markov Chain

Definition

- A random process $\{X_n, n = 0, 1, 2 \ldots\}$
- State space $= \{0, 1, 2, \ldots\}$
- If $X_n = i$ say that the process is in state i at time n
- Suppose that if $X_n = i$ then there is a probability P_{ij} that $X_{n+1} = j$ for all i and j.

$$P(X_{n+1} = j | X_n = i) = P_{ij}$$

Definition

- Probability that next state is j only depends on the present state, not on the past states $X_0, X_1, \ldots, X_{n-1}$
- $\{X_n\}$ is called a Markov chain
- P_{ij} is probability the process will change from state i to state j
- $P_{ij} \geq 0, \ \sum_{j=0}^{\infty} P_{ij} = 1, \ i = 0, 1, 2, \ldots$

Transition matrix

$$\mathbf{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

Example: Weather forecast

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β. Find a Markov chain that modeling the system.

Solution

- State: 0 if rain, 1 if no rain
- Transition matrix

$$\mathbf{P} = \begin{pmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{pmatrix}$$

Example: Market share

Suppose that two competing television channels, channel 1 and channel 2, each have 50% of the viewer market at some initial point in time. Assume that over each one-year period channel 1 captures 10% of channel 2's share, and channel 2 captures 20% of channel 1's share. Find a Markov chain that modeling the system.

Solution

- Consider one customer:
- $X_n = 0$ if he watches channel 1
- $X_n = 1$ if he watches channel 2
- Transition matrix

$$\mathbf{P} = \begin{pmatrix} .8 & .2 \\ .1 & .9 \end{pmatrix}$$

n-steps transition

- P_{ij}: one step transition probability
- Given process in state i at time m, want to know probability that it will be in state j after n steps
- Want to find $P_{ij}^n = P(X_{k+n} = j | X_k = i)$
- P_{ij}^n doesn't depend on k

Chapman-Kolmogorov equation

$$P_{ij}^{m+n} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}$$

Proof

$$
\begin{aligned}
P_{ij}^{n+m} &= P\{X_{n+m} = j \mid X_0 = i\} \\
&= \sum_{k=0}^{\infty} P\{X_{n+m} = j, X_n = k \mid X_0 = i\} \\
&= \sum_{k=0}^{\infty} P\{X_{n+m} = j \mid X_n = k, X_0 = i\} P\{X_n = k \mid X_0 = i\} \\
&= \sum_{k=0}^{\infty} P_{kj}^{m} P_{ik}^{n}
\end{aligned}
$$

n-step transition matrix

$$\mathbf{P}^{(n)} = \begin{pmatrix} P^n_{00} & P^n_{01} & P^n_{02} & \cdots \\ P^n_{10} & P^n_{11} & P^n_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ P^n_{i0} & P^n_{i1} & P^n_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

then

$$P^{(n+m)} = P^{(n)}.P^{(m)} \text{ or } P^{(n)} = P^n$$

Example

Back to the weather forecast example, let $\alpha = .7$ and $\beta = .4$. If it rains today, calculate the probability that it will rain 4 days from now.

Solution

- Transition matrix

$$\mathbf{P} = \begin{pmatrix} .7 & .3 \\ .4 & .6 \end{pmatrix}$$

- Want to find P_{00}^4
- Need to calculate $\mathbf{P}^{(4)} = \mathbf{P}^4$

$$\mathbf{P}^{(2)} = \mathbf{P}^2 = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix} \cdot \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$$

$$= \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix},$$

$$\mathbf{P}^{(4)} = (\mathbf{P}^2)^2 = \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix} \cdot \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix}$$

$$= \begin{Vmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{Vmatrix}$$

So $P^4_{00} = 0.5749$

Unconditional probability

$$P(X_n = j) = \sum_{i=1}^{\infty} P(X_n = j | X_0 = i) P(X_0 = i)$$
$$= \sum_{i=1}^{\infty} P_{ij}^{n} P(X_0 = i)$$

Example

Weather example: Suppose probability rain today is .4, what is the probability that it will rain 4 days from now?

Solution

$$
\begin{aligned}
P(X_4 = 0) &= P_{00}^4(.4) + P_{10}^4(.6) \\
&= 0.5700
\end{aligned}
$$

Types of state

- State j is accessible from state i if $P^n_{ij} > 0$ for some $n \geq 0$
- Two states that are accessible from each other are said to *communicate*
- If i communicates with j and j communicates with k then i communicates with k.
- Markov chain is *irreducible* if all states communicate with each other.

Limiting distribution

- Suppose X_0 has distribution $q_i = P(X_0 = i)$
- Distribution of X_n:

$$P(X_n = j) = \sum_{i=1}^{\infty} P_{ij}^n q_i$$

- What happen when n goes to infinity?

Theorem

Let $\{X_n\}$ be an irreducible ergodic Markov chain. then there exists the limit of P_{ij}^n as $n \to \infty$ and this limit does not depend on i:

$$\lim_{n\to\infty} P_{ij}^n = \pi_j$$

π_j are the unique nonnegative solution of

$$\pi_j = \sum_{i=1}^{\infty} \pi_i P_{ij}, \ \sum_{j=1}^{n} \pi_j = 1$$

Meaning

- $\{\pi_j\}$ is the distribution of X_n when n is large enough
- After some steps the distribution of X_n is approximately $\{\pi_j\}$ and will not change much

Example

Back to Market share example: Initial market share of each channel is 50%. What will be the market share after a long time?

Solution

Limiting distribution

$$\pi_0 = .8\pi_0 + .1\pi_1$$

$$\pi_1 = .2\pi_0 + .9\pi_1$$

so $\pi_0 = 1/3$ and $\pi_1 = 2/3$

Practice

Book: Introduction to Probability Models

Chapter 4: 5, 10, 11, 29, 35