VIETNAM NATIONAL UNIVERSITY - HCMC
INTERNATIONAL UNIVERSITY

CHAPTER 3. APPLICATIONS OF DIFFERENTIATION

CALCULUS I

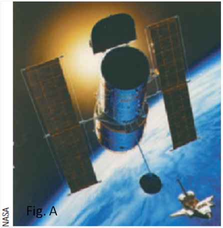
Lecturer: Associate Professor Nguyen Minh Quan

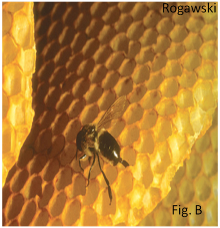
quannm@hcmiu.edu.vn

CONTENTS

Introduction

- Differentiation and integration can help us solve many types of real-world problems.
- Derivatives are met in many engineering and science problems, especially when modelling the behaviour of moving objects.
- We use the derivative to determine the maximum and minimum values of particular functions (e.g. cost, strength, profit, loss,...).





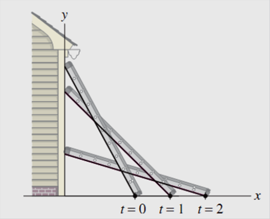
The Hubble Space Telescope was deployed by the space shuttle Discovery (Fig. A). The honeycomb structure is designed to minimize the amount of wax required.(Fig. B)

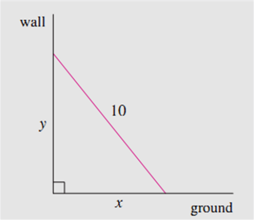
3.1 Related rates

Goal: calculate an unknown rate of change by relating it to other rates of change which are known.

Example: The sliding ladder problem

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall.



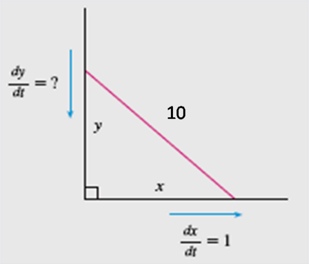


3.1 Related rates

Example: The sliding ladder problem

Solution:

Let x feet be the distance from the bottom of the ladder to the wall and y feet the distance from the top of the ladder to the ground.



By the Pythagorean Theorem: $x^2 + y^2 = 10^2 = 100$.

3.1 Related rates

Example: The sliding ladder problem

Solution (Cont.):

Differentiating each side with respect to t using the Chain Rule, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

When $x = 6$, the Pythagorean Theorem gives $y = 8$. Solving the above equation for the desired rate, we obtain

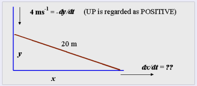
$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{6}{8}(1) = -\frac{3}{4} \quad ft/s.$$

The top of the ladder is sliding down the wall at a rate of 3/4 ft/s.

3.1 Related rates

Exercise

1. A $20m$ ladder leans against a wall. The top slides down at a rate of $4ms^{-1}$. How fast is the bottom of the ladder moving when it is $16m$ from the wall?



2. An earth satellite moves in a path that can be described by

$$\frac{x^2}{72.5} + \frac{y^2}{71.5} = 1$$

where x and y are in thousands of kilometres. If $dx/dt = 12900$ km/h for $x = 3200$ km and $y > 0$, find dy/dt.

3.1 Related rates

Example

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution:

The volume V of a sphere of radius r is defined by:

$$V = \frac{4}{3}\pi r^3$$

We differentiate both sides of this equation with respect to t.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{25\pi} \ cm/s$$

Strategy for solving related rates problems

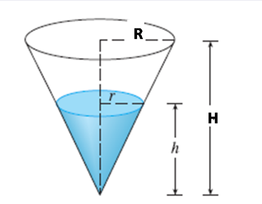
- Assign variables and restate the problem in terms of derivatives. Identify the rates of change that are known and the rate of change that is to be found.
- Find an equation relating those quantities whose rates are identified in the previous step.
- Differentiating this equation to obtain an equation involving the rates.
- Use the known equations and data to find the unknown derivative.

3.1 Related rates

Filling a tank: State the problem

Suppose that water is pouring down through a conical filter. Its volume V , height h and radius r are functions of the elapsed time t, and they are related by the equation

$$V = \frac{\pi}{3} r^2 h$$



3.1 Related rates

Filling a tank: State the problem

The relation of the three rates is obtained by differentiating this equation

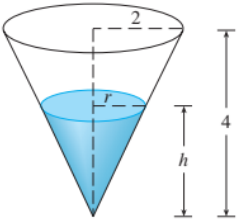
$$\frac{dV}{dt} = \frac{\pi}{3}\left(r^2\frac{dh}{dt} + 2rh\frac{dr}{dt}\right)$$

Given $\frac{dh}{dt}$ and $\frac{dr}{dt}$, can we find $\frac{dV}{dt}$?

3.1 Related rates

Example: Filling a tank

A water tank has the shape of an inverted circular cone with base radius $R = 2$ m and height $H = 4$ m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.



3.1 Related rates

Example: Filling a tank

Solution:

Let V, r, and h be the volume of the water, the radius of the surface, and the height **of the water** at time t (min). The quantities V and h are related by the equation:

$$V = \frac{\pi}{3} r^2 h$$

In order to eliminate r, we use the similar triangles:

$$\frac{r}{h} = \frac{R}{H} = \frac{2}{4} \Rightarrow r = \frac{h}{2}. \text{ Thus}$$

$$V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

3.1 Related rates

Example: Filling a tank

Solution (Cont.): Differentiating both sides:

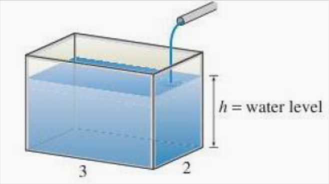
$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}. \qquad \text{So}$$

$$\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt} = \frac{4}{\pi 3^2} \cdot 2 = \frac{8}{9\pi} \approx 0.28 \text{ m/min}$$

3.1 Related rates

Exercise

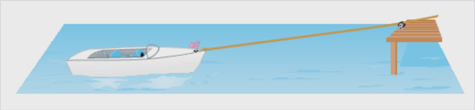
Water pours into a fish tank at the rate of $3ft^3/min$. The base of the tank is a rectangle of dimensions 2×3 (ft^2). How fast is the water level rising?



3.1 Related rates

Exercise

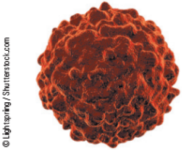
A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



3.1 Related rates

Exercise: Growth of a tumor

When the diameter of a spherical tumor is 16 mm it is growing at a rate of 0.4 mm a day. How fast is the volume of the tumor changing at that time?



Hint: $\dfrac{dV}{dt} = 4\pi r^2 \dfrac{dr}{dt} = 4\pi 8^2\,(0.2) \approx 161$ mm^3/day.

3.1 Related rates

Exercise

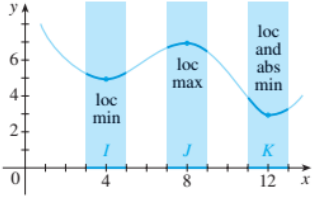
At noon, ship A is 150 km west of ship B. Ship A is sailing east at 35 km/h and ship B is sailing north at 25 km/h. How fast is the distance between the ships changing at 4:00 PM?

3.2 Maxima, Minima, and Optimization Problems

Optimization problems are problems in which we are required to find the optimal (i.e. the best) way of doing something. These problems can be reduced to **calculating the maximum or minimum values** of a function.

Definition

- A function f has a local maximum at a if there exists $\delta > 0$ such that $f(x) \le f(a)$ whenever $|a - x| < \delta$. A local minimum is defined similarly.
- A function f has an absolute maximum (or global maximum) at x_0 if $f(x) \le f(x_0)$; for all $x \in D(f)$. A absolute (global) minimum is defined similarly.



3.2 Maxima, Minima, and Optimization Problems

Weierstrass' Theorem

Suppose f is a continuous function on a closed interval $I = [a, b]$, then $f(I) = \{f(x) | x \in [a, b]\}$ is a closed interval $J = [c, d]$ (c is (global) minimum and d is (global) maximum).

Exercise

(a) Show that the function $f(x) = \frac{1}{x}$ has no maximum on $(0, 1)$.
(b) Find a function g that is continuous on $(0, 1)$ but has no minimum.

3.2 Maxima, Minima, and Optimization Problems

Fermat's Theorem

Let f be defined on $[a, b]$; if f has a local maximum (or minimum) at a point $c \in (a, b)$ and if $f'(c)$ exists, then $f'(c) = 0$.

Example

Find the extreme values on $[0, 6]$ of

$$f(x) = 2x^3 - 15x^2 + 24x + 7.$$

Solution:
Solve

$$f'(x) = 6x^2 - 30x + 24 = 6(x - 1)(x - 4) = 0$$

in $[0, 6]$ and get the critical points $x = 1$ and $x = 4$. Now, $f(0) = 7, f(1) = 18, f(4) = -9(min), f(6) = 43(max)$.

3.2 Maxima, Minima, and Optimization Problems

The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

- Step 1. Find the values of f at the critical points of in (a, b).
- Step 2. Find the values of f at the endpoints of the interval.
- Step 3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Remark: a critical point is any value in the domain where either the function is not differentiable or its derivative is 0.

3.2 Maxima, Minima, and Optimization Problems

Example

Find the extreme values on $[-1, 2]$ of $f(x) = 1 - (x-1)^{2/3}$.

Solution:

Solving

$$f'(x) = -\frac{2}{3(x-1)^{1/3}} = 0$$

in $[-1, 2]$ we get no root but the critical point is $x = 1$ where f is not differentiable.

Extreme values are $f(1) = 1$ (max) and $f(-1) = 1 - \sqrt[3]{4}$ (min).

3.2 Maxima, Minima, and Optimization Problems

Exercise

(a) Find the extreme values on $[1, 4]$ of

$$f(x) = x^2 - 8 \ln x$$

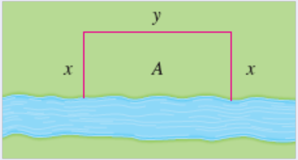
(b) Find the extreme values on $[0, 3]$ of

$$x^3 - 3x + 1$$

3.2 Maxima, Minima, and Optimization Problems

Example

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



3.2 Maxima, Minima, and Optimization Problems

Solution

We want to maximize the area A of the rectangle. Let x and y be the depth and width of the rectangle (in feet). Then $A = xy$.
We use the given information $2x + y = 2400$ to eliminate one variable: $y = 2400 - 2x$. Thus, it leads to maximize the function

$$A(x) = x(2400 - 2x), 0 \leqslant x \leqslant 1200$$

$$A'(x) = 2400 - 4x = 0 \Rightarrow x = 600$$

Make a comparison of $A(0), A(600)$, and $A(1200)$ to get the max.

$$A(0) = 0, A(600) = 720,000(\max), A(1200) = 0$$

Thus the rectangular field should be 600 ft deep and 1200 ft wide.

3.2 Maxima, Minima, and Optimization Problems

Economists call $C(x)$ the cost function, $R(x) = xp(x)$- the revenue function ($p(x)$-the demand function or function of price per sold unit), and $P(x)$-the profit function where the profit is $P(x) := R(x) - C(x)$. $P'(x), R'(x)$ and $C'(x)$ are said to be the marginal profit, marginal revenue and marginal cost, respectively. Note that:

$$C(x + \Delta x) \approx C(x) + C'(x) \Delta x$$

If $\Delta x = 1$, then $C'(x) \approx C(x+1) - C(x)$. Similarly,

$$R'(x) \approx R(x+1) - R(x)$$

Economists interpret these quantities as **the additional** profit, revenue, and cost that result from producing and selling one additional unit of the product when the production and sales levels are at x units.

3.2 Maxima, Minima, and Optimization Problems

Example

Suppose it costs $C(x) = x^3 - 6x^2 + 15x$ dollars to produce x items of a certain product and your shop is currently producing 10 items a day. About how much extra will it cost to produce one more item a day?

Solution

The cost of producing one more stove a day when 10 are produced is about $C'(10)$. Since

$$C'(x) = 3x^2 - 12x + 15$$

Thus, $C'(10) = 195$.

That is, the additional cost will be about \$195 if you produce 11 items a day.

3.2 Maxima, Minima, and Optimization Problems

Since $P'(x) = 0$ means that $C'(x) = R'(x)$, the maximum profit must occur where the marginal revenue is equal to the marginal cost. Thus, we arrive at the principle:

A Basic Principle in Economics

The maximum profit occurs where the cost of manufacturing and selling an additional unit of a product is approximately equal to the revenue generated by the additional unit.

3.2 Maxima, Minima, and Optimization Problems

Example

A liquid form of penicillin manufactured by a pharmaceutical firm is sold in bulk at a price of \$200 per unit. If the total production cost (in dollars) for x units is

$$C(x) = 500,000 + 80x + 0.003x^2$$

and if the production capacity of the firm is at most $30,000$ units in a specified time, how many units of penicillin must be manufactured and sold in that time to maximize the profit?

3.2 Maxima, Minima, and Optimization Problems

Example: Solution

First we find a formula for the quantity to be maximized, i.e., the profit. The total revenue for selling x units is $R(x) = 200x$. Hence the profit $P(x)$ is

$$P(x) = R(x) - C(x) = 200x - \left[500,000 + 80x + 0.003x^2\right]$$

Since the capacity of production is at most $30,000$ units, x must lie in the interval $[0, 30,000]$.

$$\frac{dP}{dx} = 120 - 0.006x = 0 \Rightarrow x = 20,000$$

Comparing $P(20,000) = 700,000$ with the values of P at the endpoints $P(0) = -500,000$ and $P(30,000) = 400,000$ yields the maximum profit $P = \$\ 700,000$, which occurs when $x = 20,000$ units are manufactured and sold.

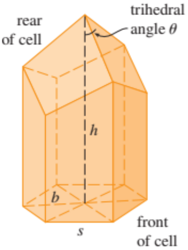
3.2 Maxima, Minima, and Optimization Problems

Exercise*: Honeycomb problem

Bees build honeycomb structures out of cells with a hexagonal base and three rhombus-shaped faces on top as in the following figure. Using geometry, we can show that the surface area of this cell is

$$A(\theta) = 6hs + \frac{3}{2}s^2\left(\sqrt{3}\csc\theta - \cot\theta\right)$$

where h, s,and θ are as indicated in the figure.



3.2 Maxima, Minima, and Optimization Problems

Honeycomb problem (cont.)

(a) Show that this angle is approximately 54.7^o by finding the critical point of $A(\theta)$ for $0 < \theta < \pi/2$ (assume h and s are constant).
(b) Confirm, by graphing

$$f(\theta) = \sqrt{3}\csc\theta - \cot\theta,$$

that the critical point indeed minimizes the surface area.

3.3 The Mean Value Theorem. Derivative Tests.

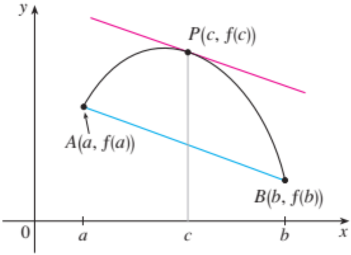
The Mean Value Theorem (MVT)

If f is a continuous function on the interval $[a, b]$ and differentiable on (a, b), then there exists a number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$



3.3 The Mean Value Theorem. Derivative Tests.

A Corollary of the MVT: Rolle's Theorem

If f is a continuous function on the interval $[a, b]$ and differentiable on (a, b). If, in addition $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Example: An application of Rolle's Thm

Show that $f(x) = x^3 + 9x - 4 = 0$ has at most one real root.

Solution:

Proof by contradiction. If $f(x) = 0$ had two real roots a and b, then $f(a) = f(b) = 0$ and Rolle's Theorem would imply that $f'(c) = 0$ for some $c \in (a, b)$. However $f'(c) = 3c^2 + 9 > 0$. Contradiction!

Homework: Show that $f(x) = 0$ has a unique real root. Hint: Show the existence of real roots by using the fact that $f(0) < 0, f(1) > 0$.

3.3 The Mean Value Theorem. Derivative Tests.

Example: An application of Rolle's Thm

Let $f(x) = x(x-1)(x-2)(x-3)(x-4)$. Show that the equation $f'(x) = 0$ has 4 real roots.

3.3 The Mean Value Theorem. Derivative Tests.

Applications of the MVT

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

Example

Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

Hints:

$$f'(x) = 12x(x-2)(x+1)$$

How to use the derivatives to find the extrema?

First Derivative Test

Suppose that $f(x)$ is continuous on an interval I where $x = c$ is the only critical number. Then,

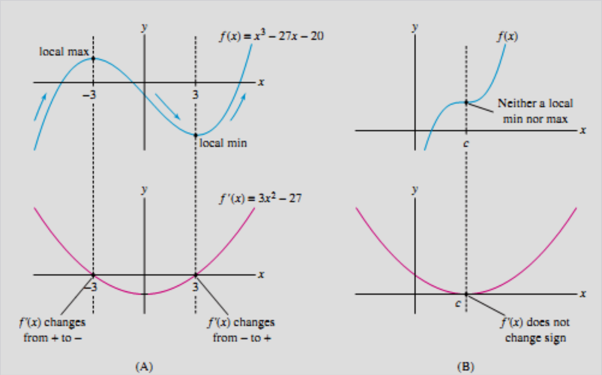
(a) If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, the absolute maximum of $f(x)$ on I is $f(c)$.

(b) If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, the absolute minimum of $f(x)$ on I is $f(c)$.

(c) If f' does not change sign as x moves through c, then f does not have a local extremum at c.

First Derivative Test

Example



First Derivative Test

Example

Find the extrema of

$$f(x) = \frac{1}{x^2 + 1}.$$

Solution:

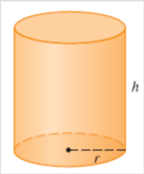
$$f'(x) = -\frac{2x}{(x^2+1)^2} = 0 \Leftrightarrow x = 0$$

By First Derivative Test (case (a)), f has maximum value 1 at $x = 0$.

First Derivative Test and Optimization Problems

Example : Minimizing cost

A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.



Solution: The surface area is $A = 2\pi r^2 + 2\pi rh$.

$$\pi r^2 h = 1000 \Rightarrow h = \frac{1000}{\pi r^2} \Rightarrow A = 2\pi r^2 + \frac{2000}{r}.$$

We minimize the function $A(r) = 2\pi r^2 + \frac{2000}{r},\ r \in (0, \infty)$.

First Derivative Test and Optimization Problems

Solution(Cont.)

$$A'(r) = 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow r = \sqrt[3]{500/\pi}$$

By applying the first derivative test one can see $r = \sqrt[3]{500/\pi}$ is an absolute minimum. And

$$h = 2\sqrt[3]{500/\pi}$$

Q: Can we apply the closed interval method?

Second Derivative Test

Suppose that $f(x)$ is continuous on an interval I where $x = c$ is the only critical number and that $f'(c) = 0$. Then,

If $f''(c) > 0$, the absolute minimum of $f(x)$ on I is $f(c)$

If $f''(c) < 0$, the absolute maximum of $f(x)$ on I is $f(c)$

Second Derivative Test

Example

Find the extrema of

$$f(x) = xe^{-x^2}, x > 0$$

Solution: Applying second derivative test:

$$f'(x) = (1 - 2x^2)e^{-x^2} = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$$

$$f''(x) = (4x^3 - 6x)e^{-x^2} \Rightarrow f''(\frac{1}{\sqrt{2}}) < 0$$

thus by the second derivative test, we obtain
$Max = f(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2e}}$.

Second Derivative Test

Example

Use Second Derivative Test and find extremum values of

$$(a)\ \ f(x) = e^x + e^{-2x} \text{ and } (b)\ \ g(x) = \frac{x}{x^2+1}, x > 0$$

Solution:

(a) Solve $f'(x) = e^x - 2e^{-2x} = 0$ and obtain $x = \frac{\ln 2}{3}$. Since $f''(x) = e^x + 4e^{-2x} > 0$. By the Second Derivative Test, f has absolute minimum at $x = \frac{\ln 2}{3}$.

(b) Solve $g'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = 0$ and obtain $x = 1$. It is an absolute maximum because at $x = 1$

$$g''(x) = \frac{-2x(x^2+1)^2 - (1-x^2)2(x^2+1)2x}{(x^2+1)^4} < 0.$$

3.5 Curve Sketching

Definition of Concavity

A function (or its graph) is called concave upward on an interval I if f' is an increasing function on I. It is called concave downward on I if is decreasing on I. A point where a curve changes its direction of concavity is called an *inflection point*.

Concavity Test

(a) If $f''(x) > 0$ for all $x \in I$, then the graph of is concave upward on I.
(b) If $f''(x) < 0$ for all $x \in I$, then the graph of is concave downward on I.

3.5 Curve Sketching

Example: Analyzing a curve using derivatives

Discuss the curve $y = f(x) = x^4 - 4x^3$ with respect to concavity, points of inflection, and local maxima and minima. Use this information to sketch the curve.

Solution:

Critical points:

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \Rightarrow x = 0; x = 3$$

$$f''(x) = 12x(x-2) = 0 \Rightarrow x = 0; x = 2 \ (IP)$$

To use the Second Derivative Test we evaluate at these critical numbers:

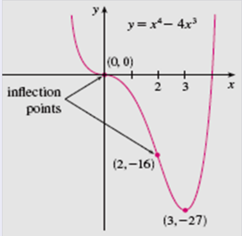
$$f''(0) = 0, f''(3) = 36 > 0$$

3.5 Curve Sketching

Example: Analyzing a curve using derivatives

Solution (Cont.):

f(3)=-27 is a local minimum. The Second Derivative Test gives no information about the critical number 0. However, the First Derivative Test tell us that f does not have a local maximum or minimum at 0.



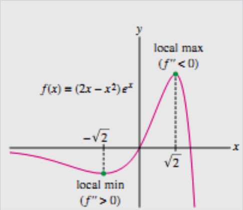
Interval	$f''(x) = 12x(x-2)$	Concavity
$(-\infty, 0)$	+	upward
$(0, 2)$	−	downward
$(2, \infty)$	+	upward

3.5 Curve Sketching

Exercise

Analyze the critical points of $f(x) = \left(2x - x^2\right) e^x$

Answer:



3.5 Curve Sketching

Review on Asymptotes

To describe the behavior of a curve at infinite distances from the origin we have the concept of asymptote.

The line $x = a$ is a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true: $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^+} f(x) = -\infty$ or $\lim_{x\to a^-} f(x) = \infty$, $\lim_{x\to a^-} f(x) = -\infty$.

The line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$

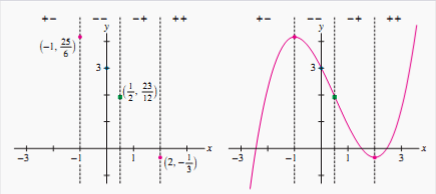
3.5 Curve Sketching

Example

Sketch the graph of

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 3$$

Answer:



3.6 Indefinite Forms and l'Hospital's Rules

l'Hospital's Rules: form of type 0/0

Let $f(x), g(x)$ be continuous functions on an open interval I containing a and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = 0$. Suppose f, g are differentiable on $I \setminus \{a\}$, $g'(x) \neq 0, \forall x \in I \setminus \{a\}$, and $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$. Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = L \left(= \lim_{x \to a} \frac{f'(x)}{g'(x)} \right)$$

Example

Find $\lim_{x\to 0}\dfrac{e^x - e^{-x} - 2\sin x}{x - \sin x}$

Solution:

Applying the L'Hospital's rule:

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2\sin x}{x - \sin x} = \lim_{x\to 0}\frac{e^x + e^{-x} - 2\cos x}{1 - \cos x}$$

$$= \lim_{x\to 0}\frac{e^x - e^{-x} + 2\sin x}{\sin x} = \lim_{x\to 0}\frac{e^x + e^{-x} + 2\cos x}{\cos x} = 4$$

3.6 l'Hospital's Rules:

3.6 l'Hospital's Rules: form of type ∞/∞

Let $f(x), g(x)$ be continuous functions on an open interval I containing a and $\lim_{x\to a}|f(x)| = \lim_{x\to a}|g(x)| = +\infty$. Suppose f, g are differentiable on $I\setminus\{a\}$, $g'(x) \neq 0, \forall x \in I\setminus\{a\}$, and $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$. Then:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = L\left(= \lim_{x\to a}\frac{f'(x)}{g'(x)}\right)$$

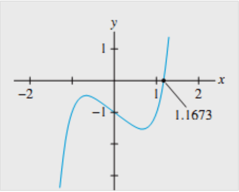
Example

Applying l'Hospital's Rules, we can calculate the following limit

$$\lim_{x\to\infty}\frac{x^2}{e^{2x}} = \lim_{x\to\infty}\frac{2x}{2e^{2x}} = \lim_{x\to\infty}\frac{2}{4e^{2x}} = 0.$$

3.7 Newton's Method

- Reminder: A "zero" or "root" of a function $f(x)$ is a solution to the equation $f(x) = 0$.
- Newton's Method is a procedure for finding numerical approximations to zeros of functions (a root-finding method).
- Numerical approximations are important because it is often impossible to find the zeros exactly.
- For example, the polynomial $f(x) = x^5 - x - 1$ has one real root **but there is no algebraic formula for this root**. Newton's Method shows that $c \approx 1.1673$.



3.7 Newton's Method

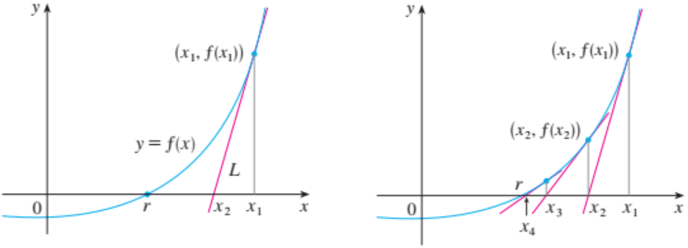
Newton's Method

To find a numerical approximation to a root of $f(x) = 0$:

Step 1. Choose initial guess x_0 (close to the desired root if possible).

Step 2. Generate successive approximations $x_1, x_2, \ldots$ where

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



3.7 Newton's Method

Example: Newton's Method

Starting with $x_0 = 2$, find the second approximation x_2 to the root of the equation. Round x_2 to 4 decimal places.

$$x^3 - 2x - 5 = 0$$

Solution:

$$f(x) = x^3 - 2x - 5 \Rightarrow f'(x) = 3x^2 - 2$$

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$$\Rightarrow x_1 = 2.1; x_2 = 2.0946$$

3.7 Newton's Method

Example: Newton's Method

Starting with $x_0 = 2$, find the approximation to the root of the equation $x^3 - 2x - 5 = 0$ correct to six decimal places.

Solution:

$$f(x) = x^3 - 2x - 5 \Rightarrow f'(x) = 3x^2 - 2$$

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2}$$

$$\Rightarrow x_1 = 2.1; x_2 = 2.09468121; x_3 = 2.0945514; x_4 = 2.094551;$$

The desired approximation to the root is $x_4 = 2.094551$.

3.7 Newton's Method

Example: Approximating $\sqrt{5}$

Calculate the first three approximations x_1, x_2, x_3 to find a root of $f(x) = x^2 - 5$ using the initial guess $x_0 = 2$.

Solution:

$$f(x) = x^2 - 5 \Rightarrow f'(x) = 2x$$

$$x_{n+1} = x_n - \frac{x^2 - 5}{2x_n}$$

$$\Rightarrow x_1 = 2.25; x_2 = 2.23611; x_3 = 2.23606797789$$

Note that $\sqrt{5} = 2.236067977499 \Rightarrow x_3$ is accurate to within an error of less than 10^{-9}. Impressive accuracy!

3.8 Anti-derivatives and Indefinite Integrals

Definition

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem

If $F(x)$ is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

Example

The antiderivative of $f(x) = x^2$ is $x^3/3 + C$ where C is an arbitrary constant.

3.8 Anti-derivatives and Indefinite Integrals

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sec^2 x$	$\tan x$
$x^n \quad (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$1/x$	$\ln\lvert x\rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
$\cos x$	$\sin x$		

3.8 Anti-derivatives and Indefinite Integrals

Example. A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = 6$ cm/s and its initial displacement is $s(0) = 9$ cm. Find its position function.

-End of Chapter 3-