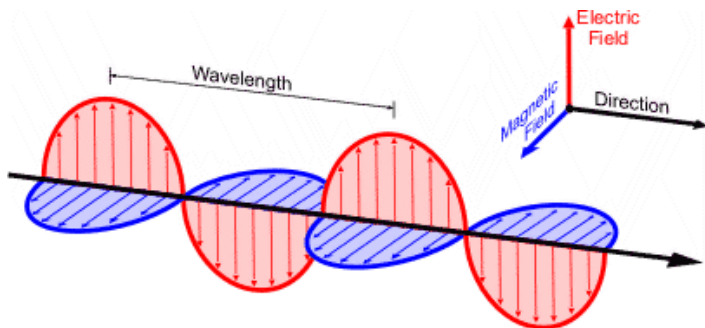


CHAPTER 2: ANALYTIC GEOMETRY OF SPACE, VECTOR FUNCTIONS

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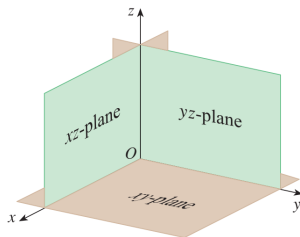


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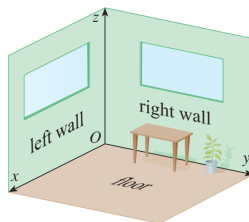
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- 5 Equations of Lines and Planes
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Three-Dimensional (3D) Coordinate

- We will introduce vectors and coordinate systems for 3D space. This will be the setting for our study of the calculus of functions of two variables
- We will see that vectors provide particularly simple descriptions of lines and planes in space.
- Reference for Chapter 2: Chapters 12-13 of the textbook by J. Stewart.



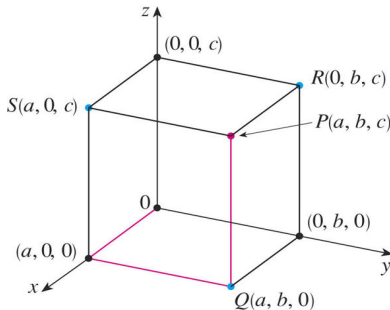
(a) Coordinate planes



(b)

Three-Dimensional (3D) Coordinate

- The Cartesian **coordinates** (a, b, c) of a point $P(a, b, c)$ in space are the numbers at which the planes through P perpendicular to the axes cut the axes. The value a is the x -coordinate, b is the y -coordinate, and c is the z -coordinate.
- If we drop a perpendicular from $P(a, b, c)$ to the xy -plane, we get a point $Q(a, b, 0)$ called the **projection** of P on the xy -plane.



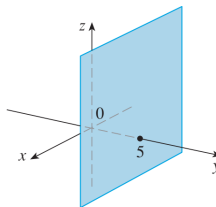
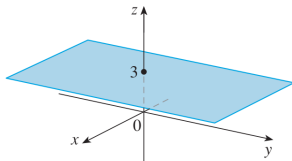
Three-Dimensional (3D) Coordinate

- The Cartesian product

$$\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

is denoted by \mathbb{R}^3 . It is called a **three-dimensional rectangular coordinate system**.

- In three-dimensional analytic geometry, an equation in x , y , and z represents a **surface** in \mathbb{R}^3 .
- The equation $z = 3$ represents the set of all points in \mathbb{R}^3 whose z -coordinate is 3. The right figure is the plane $y = 5$.

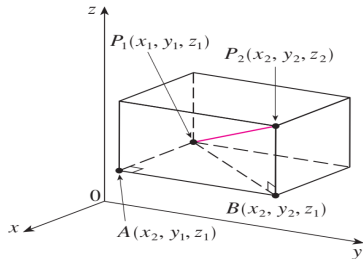


Three-Dimensional (3D) Coordinate

Distance between two points

The distance $|P_1P_2|$ between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Example The distance between $P_1(2, -1, 7)$ and $P_2(1, -3, 5)$ is

$$|P_1P_2| = \sqrt{(1 - 2)^2 + (-3 + 1)^2 + (5 - 7)^2} = 3.$$

Three-Dimensional (3D) Coordinate

Example An equation of a sphere with center $C(a, b, c)$ and radius r is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

In particular, if the center is the origin O , then an equation of the sphere is

$$x^2 + y^2 + z^2 = r^2$$

Example Show that

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

is the equation of a sphere, and find its center and radius.

Solution We have

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

$$\left(x + \frac{3}{2}\right)^2 + y^2 + (z - 2)^2 = \frac{21}{4}$$

It is the equation of a sphere with center $(-3/2, 0, 2)$ and radius $\sqrt{21}/2$.

Three-Dimensional (3D) Coordinate

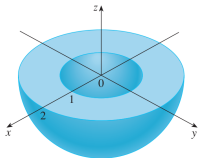
Example

Equations/inequalities

Description

- a) $x^2 + y^2 + z^2 \leq 4$ The solid ball bounded by the sphere $x^2 + y^2 + z^2 = 4$.
- b) $x^2 + y^2 + z^2 = 4$ The lower hemisphere cut from the sphere $x^2 + y^2 + z^2 = 4$ by the xy -plane.

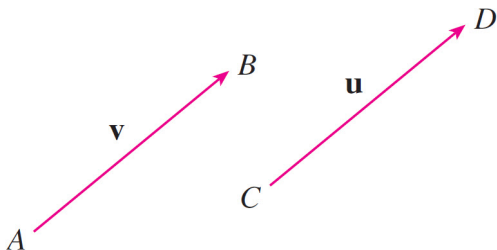
Example What region in \mathbb{R}^3 is represented by $1 \leq x^2 + y^2 + z^2 \leq 4$, $z \leq 0$?



Answer: Between (or on) the spheres and beneath (or on) the xy -plane.

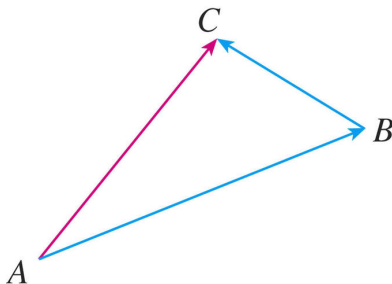
Combining Vectors

- The term **vector** is used by scientists to indicate a quantity (such as displacement or velocity or force) that has both **magnitude and direction**.
- A vector is often represented by an arrow or a directed line segment.
- For example, a particle moves along a line segment from point A to point B . One can describe this moving by the **displacement vector** $\mathbf{v} = \overrightarrow{AB}$.



Combining Vectors

- Two vectors are **equivalent** or **equal** if they have the same length and direction.
- The zero vector, denoted by **0**, has length 0. It is the only vector with no specific direction.
- Suppose a particle moves from A to B , and changes direction and moves from B to C . The resulting displacement vector $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

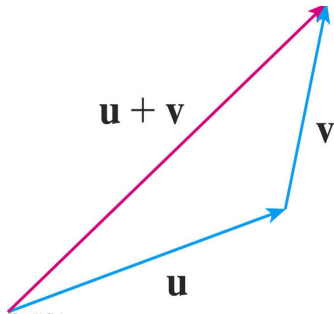


Combining Vectors

Definition

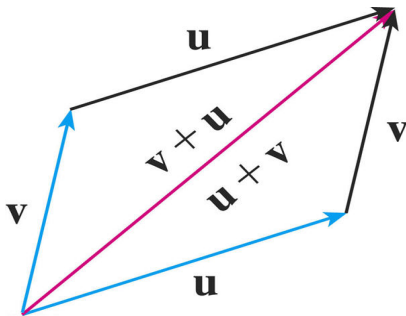
If \mathbf{u} and \mathbf{v} are vectors positioned so the initial point of \mathbf{v} is at the terminal point of \mathbf{u} , then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of \mathbf{u} to the terminal point of \mathbf{v} .

This definition is sometimes called the **Triangle Law**.



Combining Vectors

If we place \mathbf{u} and \mathbf{v} so they start at the same point, then $\mathbf{u} + \mathbf{v}$ lies along the diagonal of the parallelogram with \mathbf{u} and \mathbf{v} as sides. This is called the **Parallelogram Law**.

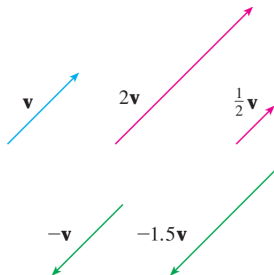


The Parallelogram Law

Combining Vectors

Definition

If c is a scalar (a real number) and \mathbf{v} is a vector, then scalar multiple $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of \mathbf{v} and whose direction is the same as \mathbf{v} if $c > 0$ and is opposite to \mathbf{v} if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.



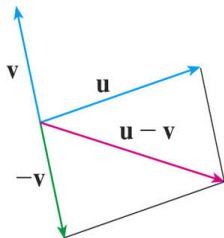
Note: Two nonzero vectors are **parallel** if they are scalar multiples of one another. Also, we call $-\mathbf{v}$ the **negative** of \mathbf{v} .

Combining Vectors

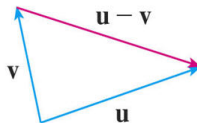
By the **difference** $\mathbf{u} - \mathbf{v}$ of two vectors we mean

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}).$$

So we can construct $\mathbf{u} - \mathbf{v}$ by first drawing the negative of \mathbf{v} , $-\mathbf{v}$, and then adding it to \mathbf{u} by the **Parallelogram Law** (Fig. (a) below). Alternatively, since $\mathbf{v} + (\mathbf{u} - \mathbf{v}) = \mathbf{u}$ the vector $\mathbf{u} - \mathbf{v}$, when added to \mathbf{v} , gives \mathbf{u} . So we could construct $\mathbf{u} - \mathbf{v}$ by means of the **Triangle Law** as in Fig. (b).



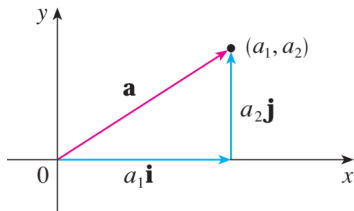
(a)



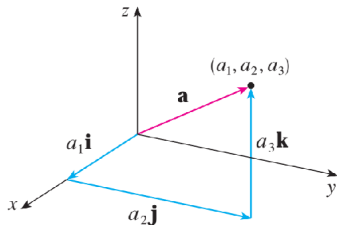
(b)

Components of a vector

- If we place the initial point of a vector \mathbf{a} at the origin, then the **terminal point** of \mathbf{a} has coordinates of the form $(a_1, a_2) \in \mathbb{R}^2$ or $(a_1, a_2, a_3) \in \mathbb{R}^3$.
- These coordinates are called the **components** of \mathbf{a} and we write
$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle.$$
- The vector $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, and $\mathbf{k} = \langle 0, 0, 1 \rangle$ are the **basic vectors**.

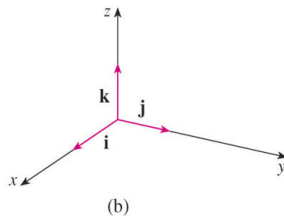
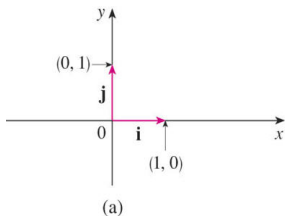


(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$

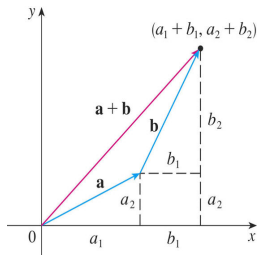


(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Components of a vector



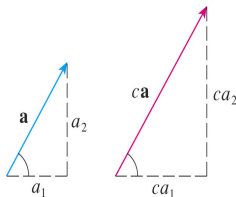
If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then the sum is $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$, at least for the case where the components are positive. So



Components of a vector

Similarly, *to subtract vectors we subtract components.*

From the similar triangles, we see that the components of $c\mathbf{a}$ are ca_1 and ca_2 . So *to multiply a vector by a scalar we multiply each component by that scalar.*



If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} \pm \mathbf{b} = \langle a_1 \pm b_1, a_2 \pm b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle.$$

Since $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, we have $\langle a_1, a_2 \rangle = a_1\mathbf{i} + a_2\mathbf{j}$

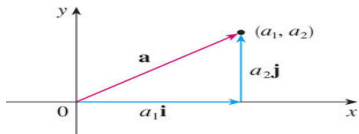
Components of a vector

Similarly, for three-dimensional vectors,

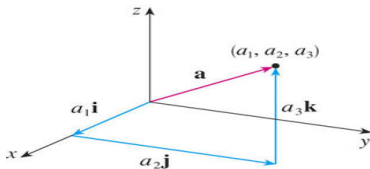
$$\langle a_1, a_2, a_3 \rangle \pm \langle b_1, b_2, b_3 \rangle = \langle a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Since $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, we have $\langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Components of a vector

- Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ in \mathbb{R}^3 , then

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

- Similarly, in two dimensions, the vector from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$$

- The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}.$$

- The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Example The length of $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ is $\sqrt{14}$.

Components of a vector

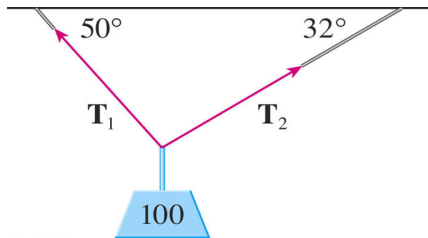
- Any vector whose length is 1 is a **unit vector**.
- For instance, the vector **i**, **j**, and **k** are unit vectors.
- If $\mathbf{v} \neq \mathbf{0}$, $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector, called the **direction** of \mathbf{v} or the **unit vector in the direction of \mathbf{v}** .
- Any nonzero vector can be expressed as a product of its length and direction:

$$\mathbf{v} = |\mathbf{v}| \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (\text{length of } \mathbf{v}) \cdot (\text{direction of } \mathbf{v})$$

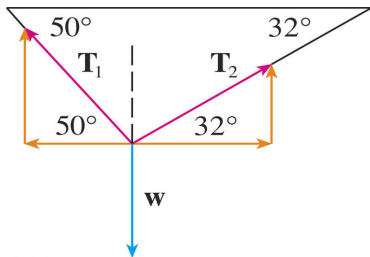
- A force is represented by a vector because it has both a magnitude and a direction. If several forces are acting on an object, the **resultant force** experienced by the object is the vector sum of these forces.

Components of a vector

Example A 100-lb weight hangs from two wires as shown in the figure below. Find the tensions (forces) \mathbf{T}_1 and \mathbf{T}_2 in both wires and their magnitudes.



Components of a vector



Solution We see that

$$\mathbf{T}_1 = -|T_1| \cos 50^\circ \mathbf{i} + |T_1| \sin 50^\circ \mathbf{j}$$

$$\mathbf{T}_2 = |T_2| \cos 32^\circ \mathbf{i} + |T_2| \sin 32^\circ \mathbf{j}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight \mathbf{w} and so we must have $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\mathbf{j}$:

$$\begin{aligned} & (-|T_1| \cos 50^\circ + |T_2| \cos 32^\circ) \mathbf{i} \\ & + (|T_1| \sin 50^\circ + |T_2| \sin 32^\circ) \mathbf{j} = 100\mathbf{j}. \end{aligned}$$

Components of a vector

Equating components, we get

$$\begin{aligned}-|T_1| \cos 50^\circ + |T_2| \cos 32^\circ &= 0 \\ |T_1| \sin 50^\circ + |T_2| \sin 32^\circ &= 100.\end{aligned}$$

Solving gives

$$|T_1| \sin 50^\circ + \frac{|T_1| \cos 50^\circ}{\cos 32^\circ} \sin 32^\circ = 100.$$

So

$$\begin{aligned}|T_1| &= \frac{100}{\sin 50^\circ + \tan 32^\circ \cos 50^\circ} \approx 85.64 \text{ lb} \\ |T_2| &= \frac{|T_1| \cos 50^\circ}{\cos 32^\circ} \approx 64.91 \text{ lb}\end{aligned}$$

Hence the tension vectors are

$$T_1 \approx -55.05\mathbf{i} + 65.60\mathbf{j} \quad \text{and} \quad T_2 \approx 55.05\mathbf{i} + 34.40\mathbf{j}.$$

The dot product

Definition

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \mathbf{a} and \mathbf{b} is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Similarly, if $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2.$$

The dot product is sometimes called the **scalar product** (or **inner product**).

The dot product

Theorem

If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and λ is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$;
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$;
4. $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda \mathbf{b})$;
5. $\mathbf{0} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{0} = 0$.

Theorem

If θ is the angle between the vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cdot \cos \theta$$

The dot product

Corollary

If θ is the angle between the nonzero vectors \mathbf{a} and \mathbf{b} , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example Find the angle between the vectors $\mathbf{a} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. **Solution**

$$\mathbf{a} \cdot \mathbf{b} = 1 \times 6 + (-2) \times 3 + (-2) \times 2 = -4$$

$$|\mathbf{a}| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{b}| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} \right) = \cos^{-1} \left(\frac{-4}{3 \times 7} \right) \approx 1.76 \text{ rad}.$$

The dot product

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Thus,

$$\text{Two vectors } \mathbf{a} \text{ and } \mathbf{b} \text{ are orthogonal} \iff \mathbf{a} \cdot \mathbf{b} = 0$$

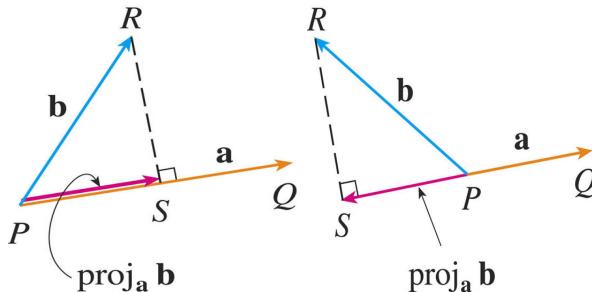
Example

a = $\langle 3, -2, 1 \rangle$ and **b** = $\langle 0, 2, 4 \rangle$ are orthogonal because

$$\mathbf{a} \cdot \mathbf{b} = (3)(0) + (-2)(2) + (1)(4) = 0.$$

The dot product

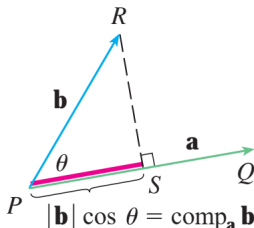
Projection Suppose that $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{PR}$. If S is the foot of the perpendicular from R to the line containing \overrightarrow{PQ} , then the vector with representation \overrightarrow{PS} is called the **vector projection** of \mathbf{b} onto \mathbf{a} and is denoted by $\text{proj}_{\mathbf{a}} \mathbf{b}$.



The vector projection of \mathbf{b} onto \mathbf{a}

The dot product

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the number $|\mathbf{b}| \cos \theta$, where θ is the angle between **a** and **b**. This is denoted by $\text{comp}_a \mathbf{b}$.



$$\begin{aligned}\text{comp}_a \mathbf{b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \\ \text{proj}_a \mathbf{b} &= \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right) \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}\end{aligned}$$

The dot product

Example

Find the scalar projection and vector projection of $\mathbf{b} = \langle 1, 1, 2 \rangle$ onto $\mathbf{a} = \langle -2, 3, 1 \rangle$.

Solution Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$,

$$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{(-2) \times 1 + 3 \times 1 + 1 \times 2}{\sqrt{14}} = \frac{3}{\sqrt{14}}.$$

Thus

$$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{3}{\sqrt{14}} \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14} \mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14} \right\rangle.$$

The dot product

One use of projections occurs in physics in calculating work. If the force moves the object from P to Q , then the **displacement vector** is \overrightarrow{PQ} .

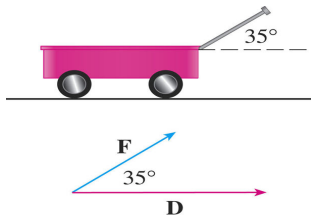
Definition

The **work** done by a constant force \mathbf{F} acting through a displacement \overrightarrow{PQ} is

$$\text{Work} = \mathbf{F} \cdot \overrightarrow{PQ} = |\mathbf{F}| |\overrightarrow{PQ}| \cos \theta.$$

The dot product

Example A wagon is pulled a distance of 100 m along a horizontal path by a constant force of 70 N. The handle of the wagon is held at an angle of 35° above the horizontal. Find the work done by the force.



Solution If \mathbf{F} and \mathbf{D} are the force and displacement vectors, then the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{D} = |\mathbf{F}| |\mathbf{D}| \cos 35^\circ \\ &= (70)(100) \cos 35^\circ \approx 5734 \text{ N} \cdot \text{m} = 5734 \text{ J.} \end{aligned}$$

The cross product

Definition

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** $\mathbf{a} \times \mathbf{b}$ of \mathbf{a} and \mathbf{b} is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Cross product is also called the **vector product**.

$\mathbf{a} \times \mathbf{b}$ is defined only when \mathbf{a} and \mathbf{b} are **three-dimensional vectors**.

The cross product

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

The cross product

Then the cross product of the vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example

Show that $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector \mathbf{a} in \mathbb{R}^3 .

Solution If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, then

$$\begin{aligned} \mathbf{a} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\ &= (a_2a_3 - a_3a_2)\mathbf{i} + (a_3a_1 - a_1a_3)\mathbf{j} + (a_1a_2 - a_2a_1)\mathbf{k} = \mathbf{0}. \end{aligned}$$

The cross product

Example Show that

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j}. \end{array}$$

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is *orthogonal* to both \mathbf{a} and \mathbf{b} .

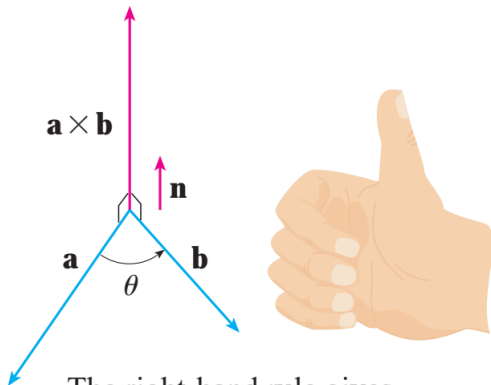
Proof Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Then

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= (a_2 b_3 - a_3 b_2) a_1 + (a_3 b_1 - a_1 b_3) a_2 \\ &\quad + (a_1 b_2 - a_2 b_1) a_3 \\ &= 0 \end{aligned}$$

A similar computation shows that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.

The cross product

The direction of $\mathbf{a} \times \mathbf{b}$ is given by the right-hand rule: If the curled fingers of the right hand are rotated from the direction of \mathbf{a} to the direction of \mathbf{b} , **the thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.**



The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

The cross product

Example Find a vector perpendicular to the plane of $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane because it is perpendicular to both vectors. In terms of components,

$$\overrightarrow{PQ} = (2 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (-1 - 0)\mathbf{k} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$\overrightarrow{PR} = (-1 - 1)\mathbf{i} + (1 + 1)\mathbf{j} + (2 - 0)\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}.$$

Thus,

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \mathbf{k} \\ &= 6\mathbf{i} + 6\mathbf{k}.\end{aligned}$$

The cross product

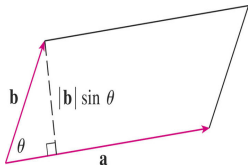
Theorem

If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Thus,

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b} .



$$|\mathbf{a} \times \mathbf{b}| = \text{area of parallelogram}$$

The cross product

Corollary

Two nonzero vectors \mathbf{a} and \mathbf{b} are *parallel* if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Example Find the area of the triangle with vertices $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$.

Solution The area of the parallelogram determined by P , Q , and R is

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6\mathbf{i} + 6\mathbf{k}| = 6\sqrt{2}.$$

The triangle's area is half of this, $3\sqrt{2}$.

The cross product. Properties

Theorem

If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and λ is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(\lambda \mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\lambda \mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

The product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ that occurs in Property 6 is called the **vector triple product** of \mathbf{a} , \mathbf{b} , and \mathbf{c} .

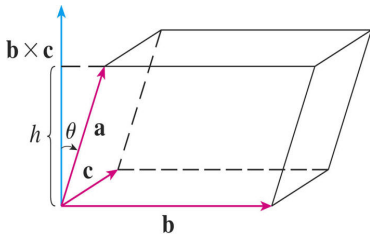
The cross product. Triple Products

- The product that occurs in Property 5 is called the **scalar triple product** of the vectors **a**, **b**, and **c**. It can be shown that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

- The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$



The cross product

Example

Find the volume of the box (parallelepiped) determined by $\mathbf{a} = \langle 1, 2, -1 \rangle$, $\mathbf{b} = \langle -2, 0, 3 \rangle$, and $\mathbf{c} = \langle 0, 7, -4 \rangle$.

Solution

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = \begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix} - \begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} \\ &= -21 - 16 + 14 = -23.\end{aligned}$$

The volume is $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = 23$.

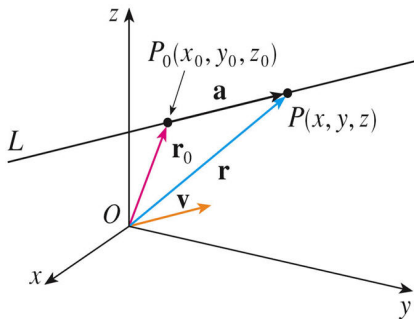
Note that if $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, then the vectors must lie in the same plane; that is, they are **coplanar**.

Equations of Lines and Planes

Equations for Lines Suppose L is a line in three-dimensional space that passes a point $P_0(x_0, y_0, z_0)$. Let \mathbf{v} be a vector parallel to L , $P(x, y, z)$ be an arbitrary point on L and let \mathbf{r}_0 and \mathbf{r} be the position vectors of $P_0(x_0, y_0, z_0)$ and $P(x, y, z)$, respectively. Then

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

which is a **vector equation** of L .



Equations of Lines

Suppose $\mathbf{v} = \langle a, b, c \rangle$, then we have the three scalar equations:

$$\boxed{x = x_0 + ta, \quad y = y_0 + tb \quad z = z_0 + tc, \quad t \in \mathbb{R}} \quad (1)$$

These equations are called **parametric equations** of the line through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$.

Note The vector equation and parametric equations of a line are not unique.

Equations of Lines

Example

- (a) Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
- (b) Find two other points on the line.

Solution (a) The vector equation is

$$\begin{aligned}\mathbf{r} &= (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}) \\ &= (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}.\end{aligned}$$

Parametric equations are

$$x = 5 + t, \quad y = 1 + 4t, \quad z = 3 - 2t, \quad t \in \mathbb{R}.$$

- (b) Choosing the parameter value $t = 1$ gives $x = 6$, $y = 5$, and $z = 1$, so $(6, 5, 1)$ is a point on the line. Similarly, $t = -1$ gives the point $(4, -3, 5)$.

Equations of Lines

If none of a , b , or c is 0, we can solve each of Equations (1) for t , equate the results, and obtain

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These equations are called **symmetric equations** of L . If $a = 0$, we can write the equations of L as

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Equations for Line segments
given by the vector equation

The line segment from \mathbf{r}_0 to \mathbf{r}_1 is

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1, \quad 0 \leq t \leq 1}$$

Equations of Lines

Example Show that the lines L_1 and L_2 with parametric equations

$$\begin{array}{lll} x = 1 + t & y = -2 + 3t & z = 4 - t \\ x = 2s & y = 3 + s & z = -3 + 4s \end{array}$$

are **skew lines**; that is, they do not intersect and are not parallel (and therefore do not lie in the same plane).

Solution The lines are not parallel because the corresponding vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. If L_1 and L_2 had a point of intersection, there would be values of t and s such that

$$\begin{aligned} 1 + t &= 2s \\ -2 + 3t &= 3 + s \\ 4 - t &= -3 + 4s \end{aligned}$$

These equations have no solution, so L_1 and L_2 do not intersect. Thus L_1 and L_2 are skew lines.

Equations of Lines

Example

Show that the midpoint of the line segment joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

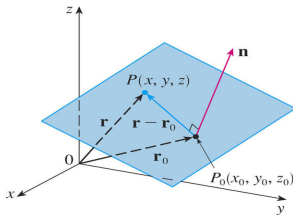
$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right).$$

Solution

$$\begin{aligned}\overrightarrow{OM} &= \overrightarrow{OP_1} + \overrightarrow{P_1M} = \overrightarrow{OP_1} + \frac{1}{2}\overrightarrow{P_1P_2} \\ &= \overrightarrow{OP_1} + \frac{1}{2}(\overrightarrow{OP_2} - \overrightarrow{OP_1}) = \frac{1}{2}(\overrightarrow{OP_1} + \overrightarrow{OP_2}) \\ &= \frac{x_1 + x_2}{2}\mathbf{i} + \frac{y_1 + y_2}{2}\mathbf{j} + \frac{z_1 + z_2}{2}\mathbf{k}.\end{aligned}$$

Equations of Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector \mathbf{n} that is orthogonal to the plane. This orthogonal vector \mathbf{n} is called a **normal vector**. The plane consists of all points $P(x, y, z)$ for which $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0 \rangle$ is orthogonal to \mathbf{n} .



We have **vector equation** of the plane:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad (2)$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \quad (3)$$

Equations of Planes

- Suppose $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. Then the vector equation (2) becomes

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (4)$$

Equation (4) is the **scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$** .

- We can rewrite the equation of a plane as

$$ax + by + cz + d = 0 \quad (5)$$

where $d = -(ax_0 + by_0 + cz_0)$. Equation (5) is called a **linear equation** in x , y , and z .

Equations of Planes

Example Find an equation of the plane that passes through the points $P(1, 3, 2)$, $Q(3, -1, 6)$, and $R(5, 2, 0)$.

Solution Since both $\overrightarrow{PQ} = \langle 2, -4, 4 \rangle$ and $\overrightarrow{PR} = \langle 4, -1, -2 \rangle$ lie in the plane, $\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$ is a normal vector of the plane. Thus

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = 12\mathbf{i} + 20\mathbf{j} + 14\mathbf{k}.$$

An equation of the plane is

$$12(x - 1) + 20(y - 3) + 14(z - 2) = 0$$

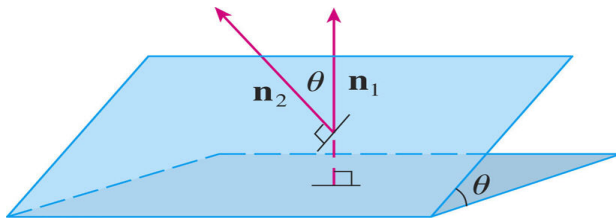
or

$$6x + 10y + 7z = 50.$$

Equations of Planes

Angles Between Planes Two planes are parallel if their normal vectors are parallel.

If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the **acute angle** between their normal vectors.



The angle between planes

Equations of Planes

Example Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$.

Solution The normal vectors of these planes are $\mathbf{n}_1 = \langle 1, 1, 1 \rangle$ and $\mathbf{n}_2 = \langle 1, -2, 3 \rangle$ and so, if θ is the angle between the planes, then

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{1 \cdot 1 + 1(-2) + 1 \cdot 3}{\sqrt{1+1+1}\sqrt{1+4+9}} = \frac{2}{\sqrt{42}}$$
$$\theta = \cos^{-1} \left(\frac{2}{\sqrt{42}} \right) \approx 72^\circ.$$

Equations of Planes

Distance from a Point to a Plane

Example Find a formula for the distance from a point $P_1(x_1, y_1, z_1)$ to the plane

$$ax + by + cz + d = 0.$$

Solution Let $P_0(x_0, y_0, z_0)$ be any point in the given plane and let $\mathbf{b} = \overrightarrow{P_0P_1}$. Then $\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$. The distance from P_1 to the plane is equal to the absolute value of the scalar projection of \mathbf{b} onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$.

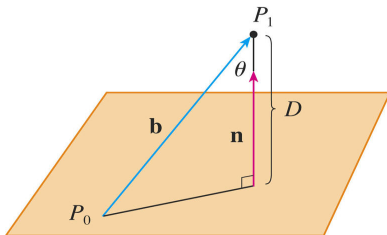
$$\begin{aligned} D &= |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

Equations of Planes

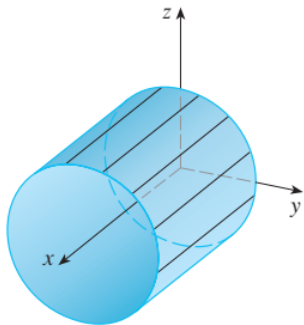
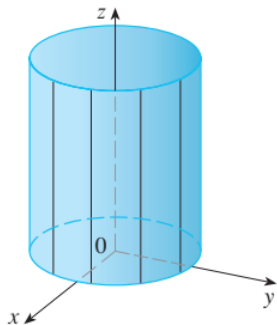
$$D = \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Since P_0 lies in the plane, $ax_0 + by_0 + cz_0 + d = 0$. Thus

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$



Equations of Cylinders



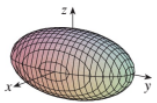
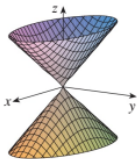
When you are dealing with **surfaces**, it is important to recognize that an equation like $x^2 + y^2 = 1$ (left) or $y^2 + z^2 = 1$ (right) represents a cylinder and not a circle.

Equations of Quadric Surfaces

Quadric Surfaces

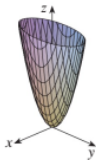
A **quadric** surface is the graph of a **second-degree** equation in three variables x , y , and z . The most general such equation is

$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ where A, B, C, \dots, J are constants.

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <p>All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.</p>	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <p>Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.</p>

Equations of Quadric Surfaces

Elliptic Paraboloid



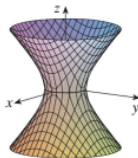
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.

Vertical traces are parabolas.

The variable raised to the first power indicates the axis of the paraboloid.

Hyperboloid of One Sheet



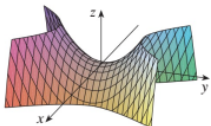
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses.

Vertical traces are hyperbolas.

The axis of symmetry corresponds to the variable whose coefficient is negative.

Hyperbolic Paraboloid



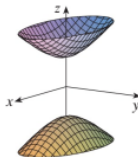
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas.

Vertical traces are parabolas.

The case where $c < 0$ is illustrated.

Hyperboloid of Two Sheets



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$.

Vertical traces are hyperbolas.

The two minus signs indicate two sheets.

Vector Functions and Space Curves

Definitions

When a particle moves through space during a time interval I , we think of the particle's coordinates as functions defined on I :

$$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I. \quad (6)$$

The points $(x, y, z) = (f(t), g(t), h(t))$, $t \in I$, make up the **curve** in space that we call the particle's **path**. The equation and interval in (6) **parametrize** the curve.

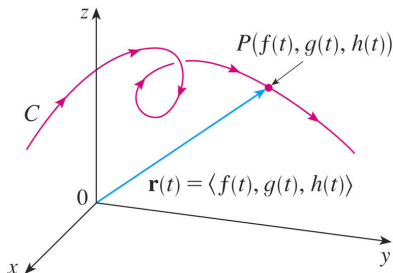
The vector $\mathbf{r}(t) = \overrightarrow{OP} = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ from the origin to the particle's **position** $P = (f(t), g(t), h(t))$ at time t is the particle's **position vector**. The functions f , g , and h are the **components** or **coordinate functions** of the position vector.

Vector Functions and Space Curves

Definitions (cont'd)

More generally, a **vector-valued function** or **vector function** is a function whose range is a set of vectors. The vector function's domain to be the intersection of the domains of its component functions.

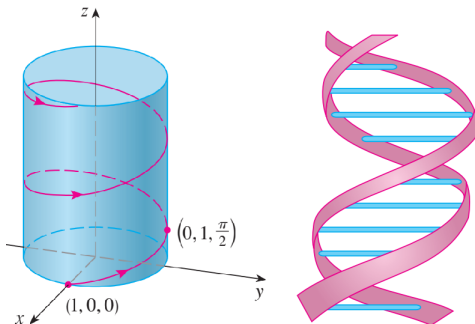
When we need to distinguish real-valued functions from vector functions, we refer to real-valued functions as **scalar functions**.



Vector Functions and Space Curves

Example: Space Curves Sketch the curve whose vector equation is $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$. **Solution** We have

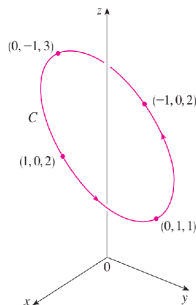
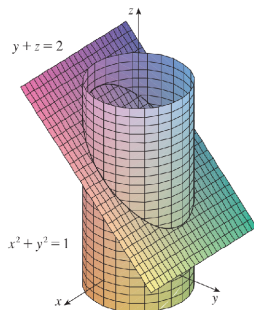
$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Thus, the curve must lie on the circular cylinder $x^2 + y^2 = 1$. The curve spirals **upward** around the cylinder as $z = t$ increases. Each time t increases by 2π , the curve completes one turn around the cylinder. The curve is called a **helix**.



Vector Functions and Space Curves

Example

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$



Answer: $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2 - \sin t)\mathbf{k}, 0 \leq t \leq 2\pi.$

Vector Functions. Limits and Continuity

Definition

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Example If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\begin{aligned} \lim_{t \rightarrow \pi/4} \mathbf{r}(t) &= \left(\lim_{t \rightarrow \pi/4} \cos t \right) \mathbf{i} + \left(\lim_{t \rightarrow \pi/4} \sin t \right) \mathbf{j} + \left(\lim_{t \rightarrow \pi/4} t \right) \mathbf{k} \\ &= \frac{\sqrt{2}}{2} \mathbf{i} + \frac{\sqrt{2}}{2} \mathbf{j} + \frac{\pi}{4} \mathbf{k}. \end{aligned}$$

Vector Functions. Continuity

Definition

A vector function $\mathbf{r}(t)$ is **continuous** at a if

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \mathbf{r}(a).$$

The function is **continuous** if it is continuous at every point in its domain.

A vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is continuous at a if and only if its component functions $f(t)$, $g(t)$, and $h(t)$ are continuous at a .

Example The function $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ is continuous.

Derivatives of vector functions

Definition

The derivative of $\mathbf{r}(t)$ is the limit of the difference quotient

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

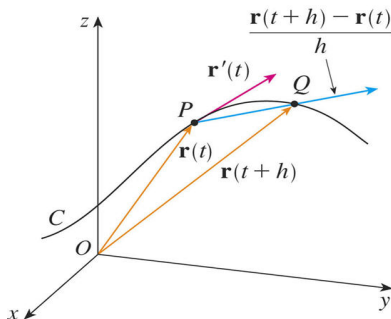
if this limit exists.

The vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}(t)$ at the point P , provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

The **tangent line** to C at P is defined to be the line through P parallel to the tangent vector $\mathbf{r}'(t)$. The **unit tangent vector** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Derivatives of vector functions



Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Derivatives of vector functions

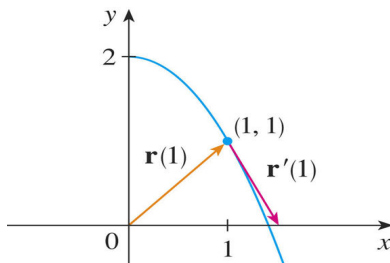
Example For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2 - t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vector $\mathbf{r}(1)$ and the tangent vector $\mathbf{r}'(1)$. Find the corresponding unit tangent vector.

Solution

$$\mathbf{r}'(t) = \frac{1}{2\sqrt{t}}\mathbf{i} - \mathbf{j} \quad \text{and} \quad \mathbf{r}'(1) = \frac{1}{2}\mathbf{i} - \mathbf{j}.$$

The unit tangent vector at the point where $t = 1$ is

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\frac{1}{2}\mathbf{i} - \mathbf{j}}{\sqrt{5}/2} = \frac{1}{\sqrt{5}}\mathbf{i} - \frac{2}{\sqrt{5}}\mathbf{j}.$$



Derivatives of vector functions

Definition

The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is **differentiable** at $t = a$ if f , g , and h are differentiable at a . Also, \mathbf{r} is said to be **differentiable** if it is differentiable at every point of its domain. The curve traced by \mathbf{r} is **smooth** if $d\mathbf{r}/dt$ is continuous and never equal to $\mathbf{0}$, i.e., if f , g , and h have first derivatives that are not simultaneously 0.

- A curve that is made up of a finite number of smooth curves pieced together in a continuous fashion so that the initial point of one curve is the terminal point of the immediately preceding one is called **piecewise smooth**.
- The second derivative of a vector function \mathbf{r} is the derivative of \mathbf{r}' , that is, $\mathbf{r}'' = (\mathbf{r}')'$.

Derivatives of vector functions

Definition

If $\mathbf{r}(t)$ is the position vector of a particle moving along a smooth curve in space, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}$$

is the particle's **velocity vector**. At any time t , the direction of \mathbf{v} is the **direction of motion**, the magnitude of \mathbf{v} is the particle's **speed**, and the derivative

$$\mathbf{a} = d\mathbf{v}/dt,$$

when it exists, is the particle's **acceleration vector**.

Note

$$\text{Velocity} = |\mathbf{v}| \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = (\text{Speed}) \cdot (\text{Direction})$$

Derivatives of vector functions

Example The vector $\mathbf{r} = (3 \cos t)\mathbf{i} + (3 \sin t)\mathbf{j} + t^2\mathbf{k}$ gives the position of a moving body at time t . Find the body's speed and direction when $t = 2$. At what times, if any, are the body's velocity and acceleration orthogonal?

Solution

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -(3 \sin t)\mathbf{i} + (3 \cos t)\mathbf{j} + 2t\mathbf{k},$$

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = -(3 \cos t)\mathbf{i} - (3 \sin t)\mathbf{j} + 2\mathbf{k}.$$

At $t = 2$, the body's speed and direction are $|\mathbf{v}(2)| = 5$ and

$$\frac{\mathbf{v}(2)}{|\mathbf{v}(2)|} = \left(-\frac{3}{5} \sin 2\right)\mathbf{i} + \left(\frac{3}{5} \cos 2\right)\mathbf{j} + \frac{4}{5}\mathbf{k},$$

The body's velocity and acceleration are orthogonal when

$$\mathbf{v} \cdot \mathbf{a} = 9 \sin t \cos t - 9 \cos t \sin t + 4t = 4t = 0.$$

Therefore, $t = 0$.

Derivatives of vector functions

Theorem

Suppose \mathbf{u} and \mathbf{v} are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

$$1. \quad \frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$$

$$2. \quad \frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$$

$$3. \quad \frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t).$$

$$4. \quad \frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$$

$$5. \quad \frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$6. \quad \frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t)).$$

Derivatives of vector functions

Example If $\mathbf{r}(t)$ is a differentiable vector function of constant length, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$:

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = 0.$$

Solution Since $\mathbf{r} \cdot \mathbf{r} = |\mathbf{r}|^2$ is constant,

$$0 = \frac{d}{dt}(\mathbf{r} \cdot \mathbf{r}) = \mathbf{r}' \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{r}' = 2\mathbf{r} \cdot \mathbf{r}'.$$

Thus, $\mathbf{r} \cdot \mathbf{r}' = 0$.

Integrals of vector functions

Definition

If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over the interval $a \leq t \leq b$, then \mathbf{r} is **integrable** over $[a, b]$ and the **definite integral** of \mathbf{r} from a to b is

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}.$$

For example,

$$\int_0^\pi \langle 1, t, \sin t \rangle dt = \left\langle \int_0^\pi 1 dt, \int_0^\pi t dt, \int_0^\pi \sin t dt \right\rangle = \left\langle \pi, \frac{1}{2}\pi^2, 2 \right\rangle.$$

Integrals of vector functions

- An **antiderivative** of $\mathbf{r}(t)$ on an interval I is a vector function $\mathbf{R}(t)$ such that $\mathbf{R}'(t) = \mathbf{r}(t)$ at each point of I .
- If $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$ on I , it can be shown that every antiderivative of $\mathbf{r}(t)$ on I has the form $\mathbf{R}(t) + \mathbf{C}$ for some constant \mathbf{C} .
- The set of all antiderivatives of \mathbf{r} on I is the **indefinite integral** of \mathbf{r} on I and denoted by $\int \mathbf{r}(t)dt$.
- Thus, if $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, then

$$\int \mathbf{r}(t)dt = \mathbf{R}(t) + \mathbf{C}$$

Integrals of vector functions

Example The velocity of a particle moving in the space is

$$\frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} - (\sin t)\mathbf{j} + \mathbf{k}.$$

Find the particle's position as a function of t if $\mathbf{r} = 2\mathbf{i} + \mathbf{k}$ when $t = 0$.

Solution

$$\mathbf{r}(t) = \int \mathbf{r}'(t) dt = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k} + \mathbf{C}.$$

To determine \mathbf{C} , we use the initial condition $\mathbf{r}(0) = 2\mathbf{i} + \mathbf{k}$:

$$\begin{aligned}(\sin 0)\mathbf{i} + (\cos 0)\mathbf{j} + 0\mathbf{k} + \mathbf{C} &= 2\mathbf{i} + \mathbf{k} \\ \mathbf{C} &= 2\mathbf{i} - \mathbf{j} + \mathbf{k}.\end{aligned}$$

The particle's position as a function of t is

$$\mathbf{r}(t) = (\sin t + 2)\mathbf{i} + (\cos t - 1)\mathbf{j} + (t + 1)\mathbf{k}.$$

Length of space curves

Arc Length Suppose $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f' , g' , and h' are continuous.

If the curve is traversed exactly once as t increases from $t = a$ to $t = b$, then it can be shown that its **length** is

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \end{aligned}$$

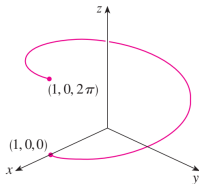
That is,

$$L = \int_a^b |\mathbf{r}'(t)| dt$$

Length of space curves

Example

Find the length of the arc of the helix with vector equation $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.



Solution We have $\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j} + \mathbf{k}$, so $|\mathbf{r}'(t)| = \sqrt{2}$. The arc length is

$$L = \int_0^{2\pi} |\mathbf{r}'(t)| dt = \int_0^{2\pi} \sqrt{2} dt = 2\sqrt{2}\pi.$$

—END OF CHAPTER 2—