CALCULUS 1 (MA001IU) – MIDTERM EXAMINATION

Semester 2, 2023-24 • Duration: 90 minutes • Date: April 19, 2024

SUBJECT: CALCULUS 1	
Department of Mathematics	Lecturers
Prof. Pham Huu Anh Ngoc	Nguyen Dinh, Mai Duc Thanh, Nguyen Minh Quan

INSTRUCTIONS:

- Use of calculator is allowed. Each student is allowed one double-sided sheet of reference material (size A4 or similar). All other documents and electronic devices are forbidden.
- You must explain your answers in detail; no points will be given for the answer alone.
- There is a total of 10 (ten) questions. Each one carries 10 points.

1. Let $f(x) = \frac{x^2-3x+2}{|x-1|}$.
   (a) *[5 points]* Find $\lim\limits_{x\to 1^+} f(x)$ and $\lim\limits_{x\to 1^-} f(x)$.
   (b) *[5 points]* Does $\lim\limits_{x\to 1} f(x)$ exist?

2. Use the Intermediate Value Theorem to show that the equation $e^{-x} - x^2 + 4x - 2 = 0$ has at least three roots.

3. Find a formula for the inverse of the function $f(x) = 2 - \sqrt{3x-1}, \quad x \geq 1/3$.

4. In a certain country, income is taxed as follows. There is no tax on the income up to \$10,000. Any income over \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$ 20,000 is taxed 15%.
   (a) *[5 points]* How much tax is assessed on an income of \$25,000?
   (b) *[5 points]* Write the total tax assessed T as a function of income I.

5. Let $f(x) = \begin{cases} x^2 - 10 & x \leq c \\ 4x - 14 & x > c. \end{cases}$
   Find c such that the function $f(x)$ is continuous everywhere.

6. Let $f(x) = \dfrac{1}{(2x^2+\sqrt{x})^3} + \ln(x^4 + \sin(x^2) + 1)$. Find $f'(x)$.

7. Find the linear approximation of the function $y = f(x) = \sqrt{(x-1)^2+3}$ near $x_0 = 0$.

8. Let f and g be the differentiable functions and $h = f^{-1} \circ g$. Given that $g'(1) = 2$, $g(1) = 3$, $f(1) = 3$, and $f'(1) = 5$. Find $(f^{-1})'(3)$ and $h'(1)$.

9. The radius of a sphere was measured and found to be 30 cm with a possible error in measurement of at most 0.02 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

10. Consider the folium of Descartes given by $x^3 + y^3 = 6xy$.
    (a) *[5 points]* Consider y as a function of x. Find the derivative $\dfrac{dy}{dx}$.
    (b) *[5 points]* Write the equation of the tangent to the folium of Descartes at the point $(3, 3)$.

—END OF THE QUESTION PAPER—

ANSWERS

1. **(a)** *[5 points]* Note that $x^2-3x+2=(x-1)(x-2)$. We have

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-3x+2}{x-1} = \lim_{x\to 1^+}(x-2) = -1$$

and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-3x+2}{-(x-1)} = \lim_{x\to 1^-}(2-x) = 1.$$

(b) *[5 points]* According to the part a) , the one-sided limits

$$\lim_{x\to 1^+} f(x) = -1 \neq 1 = \lim_{x\to 1^-} f(x).$$

It implies that $\lim\limits_{x\to 1} f(x)$ does not exist.

2. Let $f(x)=e^{-x}-x^2+4x-2$. The function $f(x)$ is continuous in $(-\infty,\infty)$ [2pts]. Additionally, we observe that:

- $f(-4)=20.5982>0, \quad f(0)=-1<0,$
- $f(0)<0, \quad f(1)=1.3679>0,$
- $f(1)>0, \quad f(4)=-1.9817<0.$ [2pts+2pts+2pts]

Therefore, by the IVT, there are three roots c_1,c_2,c_3 of the equation $f(x)=0$, where $c_1\in(-4,0)$, $c_2\in(0,1)$, and $c_3\in(1,4)$, respectively. [2pts]

3. *[10 points]* Let $y=2-\sqrt{3x-1}$. We solve for x to obtain $3x-1=(2-y)^2$ or $x=\frac{(2-y)^2+1}{3}$. Thus, $f^{-1}(y)=\frac{(2-y)^2+1}{3}$. Interchange the role of x and y to get

$$y=f^{-1}(x)=\frac{(2-x)^2+1}{3}.$$

4. (a) *[5 points]* For an income of \$25,000, the total tax is $0\times \$10,000+10\%(\$20,000-\$10,000)+15\%(\$25,000-\$20,000)=\$0+\$1,000+\$750=\$1750$.

(b) *[5 points]* The total tax function $T(I)$ is

$$T(I)=\begin{cases} 0 & \text{if } I\le 10,000 \\ 0.10\times(I-10,000) & \text{if } 10,000<I\le 20,000 \\ 1,000+0.15\times(I-20,000) & \text{if } I>20,000 \end{cases}$$

5. The function $f(x)$ is continuous on the open intervals $(-\infty,c)$ and (c,∞).
The function is continuous at $x=c$ if and only if $\lim\limits_{x\to c} f(x)=f(c)$. This means that the limit from the left must equal the limit from the right at $x=c$, and both must also equal $f(c)$. *[4 pts]*
We have $\lim\limits_{x\to c^-} f(x)=c^2-10$, $\lim\limits_{x\to c^+} f(x)=4c-14$, and $f(c)=c^2-10$. Solve $c^2-10=4c-14$ to arrive at $c=2$. *[6 points]*

6. *[10 points]* By the chain rule,

$$f'(x)=-\frac{12x+\dfrac{3}{2\sqrt{x}}}{(2x^2+\sqrt{x})^4}+\frac{4x^3+2x\cos(x^2)}{x^4+\sin(x^2)+1}.$$

[5pts for each term]

7. The linear approximation of $f(x)$ near x_0 is given by

$$f(x) \approx f'(x_0)(x-x_0)+f(x_0). \quad [3pts]$$

Note that $f'(x) = \dfrac{x-1}{\sqrt{(x-1)^2+3}}$ and $f'(0) = -\dfrac{1}{2}$ [4pts]. Moreover, $f(0)=2$. So the linear approximation of f near $x_0 = 0$ is $f(x) \approx -\frac{1}{2}x+2$. [3pts]

8. Note that as $f(1)=3$, so $f^{-1}(3)=1$. By the derivative of inverse function, one has

$$\left(f^{-1}\right)'\left(3\right) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(1)} = \frac{1}{5}. \quad [5pts] \tag{1}$$

By the chain rule, we have

$$h'(x) = \left(f^{-1}\right)'(g(x))\,g'(x). \tag{2}$$

Therefore,

$$h'(1) = \left(f^{-1}\right)'\left(g(1)\right)g'(1) = \left(f^{-1}\right)'\left(3\right)g'(1).$$

Combine with $\left(f^{-1}\right)'\left(3\right) = \frac{1}{5}$ from Eq. (1), one obtains $h'(1) = \frac{1}{5}g'(1) = \frac{2}{5}$. [5pts]

9. If the radius of the sphere is r, then its volume is $V(r) = \dfrac{4}{3}\pi r^3$ [3pts]. The error in the calculated value of V is ΔV, which can be approximated by the differential:

$$dV = V'(r)dr = 4\pi r^2 dr \quad [5pts]$$

When $r = 30$ cm, $dr = 0.02$ cm, then $\Delta V \approx dV = 4\pi \times 30^2 \times 0.02 = 72\pi = 226.08\ \text{cm}^3$. [2pts]

10. **(a)** *[5 points]* We use implicit differentiation. Differentiating both sides the given equation with respect to x:

$$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$$

Then solve for $\dfrac{dy}{dx}$ to obtain

$$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x} = \frac{2y-x^2}{y^2-2x}.$$

(b) *[5 points]* To find the equation of the tangent line at the point $(3,3)$, we substitute $x=3$ and $y=3$ into the equation for $\dfrac{dy}{dx}$ to obtain the slope of the tangent line at $(3,3)$ of $\dfrac{dy}{dx}|_{(3,3)} = -1$. The equation of the tangent line is $y-3 = -1(x-3)$ or $y = -x+6$.

—THE END —