Vietnam National University of HCMC

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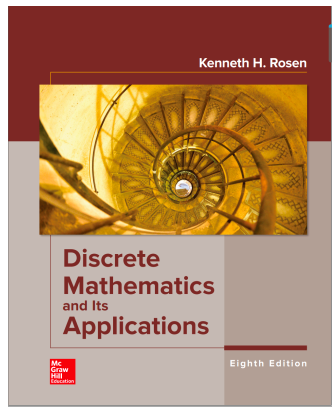
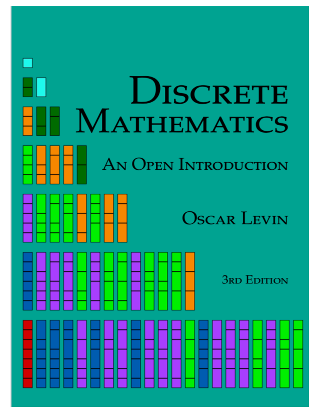
School of Computer Science and Engineering





Session 4

Nested Quantifiers & Methods of Proof

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Lecture 4

- Nested Quantifiers
- Methods of proof

Date: Sep 13th, 2022

Group 1: The monday class

Nested Quantifier

- A predicate can have more than one variables.
  - S(x, y, z): z is the sum of x and y
  - F(x, y): x and y are friends
- We can quantify individual variables in different ways
  - $\forall x, y, z\ (S(x, y, z) \rightarrow (x <= z \wedge y <= z))$
  - $\exists x\ \forall y\ \forall z\ (F(x, y) \wedge F(x, z) \wedge (y != z) \rightarrow \neg F(y, z))$

Nested Quantifier

- Exercise: translate the following English sentence into logical expression
  - "There is a rational number between every pair of distinct rational numbers"

- Use predicate Q(x), which is true when x is a rational number
  - $\forall x,y\ (Q(x) \wedge Q\ (y) \wedge (x < y) \rightarrow \exists u\ (Q(u) \wedge (x < u) \wedge (u < y)))$

Multiple quantifiers

- You can have multiple quantifiers on a statement
- $\forall x \exists y\ P(x, y)$
  - "For all x, there exists a y such that P(x,y)"
  - Example: $\forall x \exists y\ (x+y == 0)$
- $\exists x \forall y\ P(x,y)$
  - There exists an x such that for all y P(x,y) is true"
  - $\exists x \forall y\ (x*y == 0)$

Order of quantifiers

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!
- $\exists x \forall y \; P(x,y)$
  - $P(x,y) = (x+y == 0)$ is false
- $\forall x \exists y \; P(x,y)$
  - $P(x,y) = (x+y == 0)$ is true

Negating multiple quantifiers

- Recall negation rules for single quantifiers:
  - $\neg\forall x\ P(x) = \exists x\ \neg P(x)$
  - $\neg\exists x\ P(x) = \forall x\ \neg P(x)$
  - Essentially, you change the quantifier(s), and negate what it's quantifying
- Examples:
  - $\neg(\forall x \exists y\ P(x,y))$
    - $= \exists x\ \neg\exists y\ P(x,y)$
    - $= \exists x \forall y\ \neg P(x,y)$
  - $\neg(\forall x \exists y \forall z\ P(x,y,z))$
    - $= \exists x \neg\exists y \forall z\ P(x,y,z)$
    - $= \exists x \forall \neg y \forall z\ P(x,y,z)$
    - $= \exists x \forall y \exists z\ \neg P(x,y,z)$

Negating multiple quantifiers 2

- Consider $\neg(\forall x \exists y\ P(x,y)) = \exists x \forall y\ \neg P(x,y)$
  - The left side is saying "for all x, there exists a y such that P is true"
  - To disprove it (negate it), you need to show that "there exists an x such that for all y, P is false"
- Consider $\neg(\exists x \forall y\ P(x,y)) = \forall x \exists y\ \neg P(x,y)$
  - The left side is saying "there exists an x such that for all y, P is true"
  - To disprove it (negate it), you need to show that "for all x, there exists a y such that P is false"

Translating between English and quantifiers

- The product of two negative integers is positive
  - $\forall x \forall y\ ((x<0) \wedge (y<0) \rightarrow (xy > 0))$
  - Why conditional instead of and?
- The average of two positive integers is positive
  - $\forall x \forall y\ ((x>0) \wedge (y>0) \rightarrow ((x+y)/2 > 0))$
- The difference of two negative integers is not necessarily negative
  - $\exists x \exists y\ ((x<0) \wedge (y<0) \wedge (x-y \geq 0))$
  - Why and instead of conditional?
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers
  - $\forall x \forall y\ (|x+y| \leq |x| + |y|)$

Translating between English and quantifiers

- $\exists x \forall y\ (x+y = y)$
  - There exists an additive identity for all real numbers
- $\forall x \forall y\ (((x \geq 0) \wedge (y<0)) \rightarrow (x-y > 0))$
  - A non-negative number minus a negative number is greater than zero
- $\exists x \exists y\ (((x \leq 0) \wedge (y \leq 0)) \wedge (x-y > 0))$
  - The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)
- $\forall x \forall y\ (((x \neq 0) \wedge (y \neq 0)) \leftrightarrow (xy \neq 0))$
  - The product of two non-zero numbers is non-zero if and only if both factors are non-zero

Example 1

- Rewrite these statements so that the negations only appear within the predicates

a) $\neg\exists y \exists x\ P(x,y)$
   - $\forall y \neg\exists x\ P(x,y)$
   - $\forall y \forall x\ \neg P(x,y)$

b) $\neg\forall x \exists y\ P(x,y)$
   - $\exists x \neg\exists y\ P(x,y)$
   - $\exists x \forall y\ \neg P(x,y)$

c) $\neg\exists y\ (Q(y) \wedge \forall x\ \neg R(x,y))$
   - $\forall y\ \neg(Q(y) \wedge \forall x\ \neg R(x,y))$
   - $\forall y\ (\neg Q(y) \vee \neg(\forall x\ \neg R(x,y)))$
   - $\forall y\ (\neg Q(y) \vee \exists x\ R(x,y))$

Example 2

- Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z\, T(x,y,z)$
- $\neg(\forall x \exists y \forall z\, T(x,y,z))$
- $\neg \forall x \exists y \forall z\, T(x,y,z)$
- $\exists x \neg \exists y \forall z\, T(x,y,z)$
- $\exists x \forall y \neg \forall z\, T(x,y,z)$
- $\exists x \forall y \exists z\, \neg T(x,y,z)$

b) $\forall x \exists y\, P(x,y) \vee \forall x \exists y\, Q(x,y)$
- $\neg(\forall x \exists y\, P(x,y) \vee \forall x \exists y\, Q(x,y))$
- $\neg \forall x \exists y\, P(x,y) \wedge \neg \forall x \exists y\, Q(x,y)$
- $\exists x \neg \exists y\, P(x,y) \wedge \exists x \neg \exists y\, Q(x,y)$
- $\exists x \forall y\, \neg P(x,y) \wedge \exists x \forall y\, \neg Q(x,y)$

Methods of proof

Mathematical reasoning

Mathematical Reasoning

We need mathematical reasoning to

- determine whether a mathematical argument is correct or incorrect and
- construct mathematical arguments.

Mathematical reasoning is not only important for conducting **proofs** and **program verification**, but also for **artificial intelligence** systems (drawing logical inferences from knowledge and facts).

We focus on **deductive** proofs

Terminology

An **axiom** is a basic assumption about mathematical structure that needs no proof.

- Things known to be true (facts or proven theorems)
- Things believed to be true but cannot be proved

We can use a **proof** to demonstrate that a particular statement is true. A proof consists of a sequence of statements that form an argument.

The steps that connect the statements in such a sequence are the **rules of inference**.

Cases of incorrect reasoning are called **fallacies**.

Terminology

A **theorem** is a statement that can be shown to be true.

A **lemma** is a simple theorem used as an intermediate result in the proof of another theorem.

A **conjecture** is a proposition that follows directly from a theorem that has been proved.

A **conjecture** is a statement whose truth value is unknown. Once it is proven, it becomes a theorem.

Proofs

A **theorem** often has two parts

- Conditions (premises, hypotheses)
- conclusion

A **correct (deductive) proof** is to establish that

- If the conditions are true then the conclusion is true
- I.e., Conditions $\rightarrow$ conclusion is a tautology

Often there are missing pieces between conditions and conclusion. Fill it by an argument

- Using conditions and axioms
- Statements in the argument connected by proper rules of inference

Rules of Inference

Rules of inference provide the justification of the steps used in a proof.

One important rule is called **modus ponens** or the **law of detachment**. It is based on the tautology
$(p \wedge (p \rightarrow q)) \rightarrow q$. We write it in the following way:

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Rules of Inference

The general form of a rule of inference is:

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore\ q \end{array}$$

The rule states that if p_1 **and** p_2 **and** … **and** p_n are all true, then q is true as well.

Each rule is an established tautology of

$p_1 \to p_2 \to \ldots \to p_n \to q$

These rules of inference can be used in any mathematical argument and do not require any proof.

Rules of Inference

$$\begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array}$$

Addition

$$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$$

Simplification

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

Conjunction

$$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore\ \neg p \end{array}$$

Modus tollens

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

Hypothetical syllogism (chaining)

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array}$$

Disjunctive syllogism (resolution)

Arguments

Just like a rule of inference, an **argument** consists of one or more hypotheses (or premises) and a conclusion.

We say that an argument is **valid**, if whenever all its hypotheses are true, its conclusion is also true.

However, if any hypothesis is false, even a valid argument can lead to an incorrect conclusion.

Proof: show that hypotheses $\rightarrow$ conclusion is true using rules of inference

Arguments

Example:

"If 101 is divisible by 3, then 101^2 is divisible by 9. 101 is divisible by 3. Consequently, 101^2 is divisible by 9."

Although the argument is **valid**, its conclusion is **incorrect**, because one of the hypotheses is false ("101 is divisible by 3.").

If in the above argument we replace 101 with 102, we could correctly conclude that 102^2 is divisible by 9.

Arguments

Which rule of inference was used in the last argument?

p: "101 is divisible by 3."
q: "101^2 is divisible by 9."

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

Modus ponens

Unfortunately, one of the hypotheses (p) is false. Therefore, the conclusion q is incorrect.

Arguments

Another example:

"If it rains today, then we will not have a barbeque today. If we do not have a barbeque today, then we will have a barbeque tomorrow. Therefore, if it rains today, then we will have a barbeque tomorrow."

This is a **valid** argument: If its hypotheses are true, then its conclusion is also true.

Arguments

Let us formalize the previous argument:

p: “It is raining today.”

q: “We will not have a barbecue today.”

r: “We will have a barbecue tomorrow.”

So the argument is of the following form:

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Hypothetical syllogism

Arguments

Another example:

Gary is either intelligent or a good actor.
If Gary is intelligent, then he can count from 1 to 10.
Gary can only count from 1 to 3.
Therefore, Gary is a good actor.

i: “Gary is intelligent.”
a: “Gary is a good actor.”
c: “Gary can count from 1 to 10.”

Arguments

i: "Gary is intelligent."
a: "Gary is a good actor."
c: "Gary can count from 1 to 10."

Step 1:	$\neg c$	Hypothesis
Step 2:	$i \rightarrow c$	Hypothesis
Step 3:	$\neg i$	Modus tollens Steps 1 & 2
Step 4:	$a \vee i$	Hypothesis
Step 5:	a	Disjunctive Syllogism Steps 3 & 4

Conclusion: **a** ("Gary is a good actor.")

Arguments

Yet another example:

If you listen to me, you will pass the exam.
You passed the exam.
Therefore, you have listened to me.

Is this argument valid?

No, it assumes $((p \rightarrow q) \wedge q) \rightarrow p$.

This statement is not a tautology. It is false if p is false and q is true.

Rules of Inference for Quantified Statements

$$\frac{\forall x\, P(x)}{\therefore P(c) \text{ if } c \in U}$$

Universal instantiation

$$\frac{P(c) \text{ for an arbitrary } c \in U}{\therefore \forall x\, P(x)}$$

Universal generalization

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some element } c \in U}$$

Existential instantiation

$$\frac{P(c) \text{ for some element } c \in U}{\therefore \exists x\, P(x)}$$

Existential generalization

Rules of Inference for Quantified Statements

Example:

Every IU student is a genius.
George is an IU student.
Therefore, George is a genius.

U(x): “x is an IU student.”
G(x): “x is a genius.”

Rules of Inference for Quantified Statements

The following steps are used in the argument:

Step 1: $\forall x\,(U(x) \rightarrow G(x))$ Hypothesis
Step 2: $U(George) \rightarrow G(George)$ Univ. instantiation using Step 1

Step 3: $U(George)$ Hypothesis
Step 4: $G(George)$ Modus ponens using Steps 2 & 3

$$\frac{\forall x\, P(x)}{\therefore P(c) \text{ if } c \in U}$$

Universal instantiation

Proving Theorems

Direct proof:

An implication $p \rightarrow q$ can be proved by showing that if p is true, then q is also true.

Example: Give a direct proof of the theorem "If n is odd, then n^2 is odd."

Idea: Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems of math to show that q must also be true (n^2 is odd).

Proving Theorems

n is odd.

Then $n = 2k + 1$, where k is an integer.

Consequently, $n^2 = (2k + 1)^2$.

$$= 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

Since n^2 can be written in this form, it is odd.

Proving Theorems

Indirect proof:

An implication $p \to q$ is equivalent to its **contra-positive** $\neg q \to \neg p$. Therefore, we can prove $p \to q$ by showing that whenever q is false, then p is also false.

Example: Give an indirect proof of the theorem "If 3n + 2 is odd, then n is odd."

Idea: Assume that the conclusion of this implication is false (n is even). Then use rules of inference and known theorems to show that p must also be false (3n + 2 is even).

Proving Theorems

n is even.

Then n = 2k, where k is an integer.

It follows that

$$
\begin{aligned}
3n + 2 &= 3(2k) + 2 \\
&= 6k + 2 \\
&= 2(3k + 1)
\end{aligned}
$$

Therefore, 3n + 2 is even.

We have shown that the contrapositive of the implication is true, so the implication itself is also true (If 3n + 2 is odd, then n is odd).

Proving Theorems

Indirect Proof is a special case of proof by contradiction

Suppose n is even (negation of the conclusion).

Then $n = 2k$, where k is an integer.

It follows that
$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

Therefore, $3n + 2$ is even.

However, this is a contradiction since $3n + 2$ is given to be odd, so the conclusion (n is odd) holds.

Another Example on Proof

Anyone performs well is either intelligent or a good actor.
If someone is intelligent, then he/she can count from 1 to 10.
Gary performs well.
Gary can only count from 1 to 3.
Therefore, not everyone is both intelligent and a good actor

- P(x): x performs well
- I(x): x is intelligent
- A(x): x is a good actor
- C(x): x can count from 1 to 10

Another Example on Proof

Hypotheses:

1. Anyone performs well is either intelligent or a good actor.
   $\forall x\ (P(x) \rightarrow I(x) \lor A(x))$
2. If someone is intelligent, then he/she can count from 1 to 10.
   $\forall x\ (I(x) \rightarrow C(x)\ )$
3. Gary performs well.
   $P(G)$
4. Gary can only count from 1 to 3.
   $\neg C(G)$

Conclusion: not everyone is both intelligent and a good actor

$\neg \forall x (I(x) \land A(x))$

Another Example on Proof

Direct proof:

Step 1: $\forall x\,(P(x) \rightarrow I(x) \vee A(x))$	Hypothesis
Step 2: $P(G) \rightarrow I(G) \vee A(G)$	Univ. Inst. Step 1
Step 3: $P(G)$	Hypothesis
Step 4: $I(G) \vee A(G)$	Modus ponens Steps 2 & 3
Step 5: $\forall x\,(I(x) \rightarrow C(x))$	Hypothesis
Step 6: $I(G) \rightarrow C(G)$	Univ. inst. Step5
Step 7: $\neg C(G)$	Hypothesis
Step 8: $\neg I(G)$	Modus tollens Steps 6 & 7
Step 9: $\neg I(G) \vee \neg A(G)$	Addition Step 8
Step 10: $\neg(I(G) \wedge A(G))$	Equivalence Step 9
Step 11: $\exists x \neg(I(x) \wedge A(x))$	Exist. general. Step 10
Step 12: $\neg\forall x\,(I(x) \wedge A(x))$	Equivalence Step 11

Conclusion: $\neg\forall x\,(I(x) \wedge A(x))$, not everyone is both intelligent and a good actor.

Summary, Section 1.5

- Terminology (axiom, theorem, conjecture, argument, etc.)
- Rules of inference (Tables 1 and 2)
- Valid argument (hypotheses and conclusion)
- Construction of valid argument using rules of inference
  - For each rule used, write down and the statements involved in the proof
- Direct and indirect proofs
  - Other proof methods (e.g., induction, pigeon hole) will be introduced in later chapters

Homework3:

- 4, 8, 12, 14, 16, 18, 22, 26, 30, 32. Starting from page 68
- Reading chapter 4: Induction and Recursion

Submit your solution to the backboard before 10:00 PM 19/9/2022: YourFullName_ID.docx