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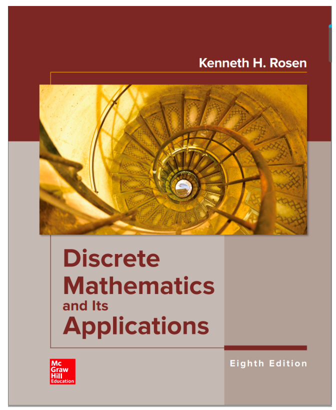
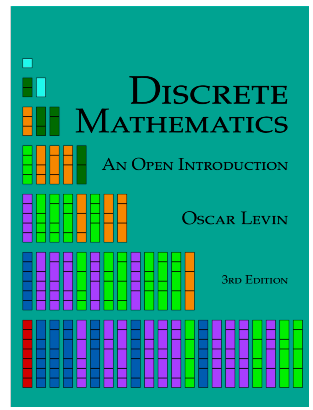
International University
School of Computer Science and Engineering




# Counting (part 1 & 2)

**Assoc. Prof. Dr. Nguyen Van Sinh**
**nvsinh@hcmiu.edu.vn**

# Introduction

- Part 1: The basic rules of counting
  - Product rules
  - Sum rule
  - The Inclusion-Exclusion rule
  - The Pigeonhole Principle
- Part 2: Permutations and Combinations
  - Permutation
  - Combination
  - Formulas of combinations
  - The extended combination principles

# Part 1: the principles rules

- The Product rule
- The Sum rule
- The Inclusion-Exclusion rule
- The Pigeonhole Principle

# The product rule

- Also called the multiplication rule
- If there are $n_1$ ways to do task 1, $n_2$ ways to do task 2
  - Then there are $n_1 n_2$ ways to do both tasks in sequence
  - This applies when doing the procedure in made up of separate task
  - We must make one choice AND second choice

# The product rule

- *An action can be implemented by k steps:*
- *Step 1 has* $n_1$ *ways, step 2 has* $n_2$ *ways,*
- *…,*
- *Step* k *has* $n_k$ *ways.*
- *Therefore, the total of action is*

$$n = n_1 . n_2 \ldots n_k$$

$$n = \prod_{i=1}^{k} n_i$$

# Example of product rule

- Sample question:
  - There are 20 math majors and 320 computer science majors
  - How many ways are there to pick one math major and one computer science major?

  → Solution is based on product rule:

  *the total is 20 * 320 = 6400 ways*

# Example of product rule

How many strings of 4 decimal digits…

a) Do not contain the same digit twice?

- We want to chose a digit, then another that is not the same, then another…
    - First digit: 10 possibilities
    - Second digit: 9 possibilities (all but first digit)
    - Third digit: 8 possibilities
    - Fourth digit: 7 possibilities
- Total = 10*9*8*7 = 5040

b) End with an even digit?

- First three digits have 10 possibilities
- Last digit has 5 possibilities
- Total = 10*10*10*5 = 5000

# Example of product rule

*Example:* Count the number of string with length 3 including letters in {A..E}

a. Any
b. No repeating the letter.

*Solution*: call $S = s_1\ s_2\ s_3$ is a string length 3, including letters in {A..E}.

a. $\forall i = 1..3$, $s_i$ has 5 ways to chose. Following the product rule, the number of above string is: 5 x 5 x 5 = 125.
b. $s_1$ has 5 choosing ways; after having $s_1$, $s_2$ has 4 choosing ways; after having $s_1s_2$, $s_3$ has 3 choosing ways. Following the product rule, the number of above string is: 5 x 4 x 3 = 60.

# Example of product rule

*Example:* Count the binary strings with length *n*.

*Solution:*

Let $b = b_1b_2...b_n$ is a binary string with length *n*.

$\forall i = 1..n$, $b_i$ has 2 ways to choose (0 or 1).

Base on the product rule, the number of binary strings with length n is:
$2x2x...x2=2^n$.

*Example:* Count the odd numbers including 2 numbers.

*Solution:*

Let n = *ab* is an odd number including 2 numbers.

Letter *a* has 9 choosing ways (1..9), Letter *b* has 5 choosing ways (1,3,5,7,9).

Base on the product rule, the number of odd numbers including 2 numbers is: 9x5=45.

# The sum rule

- Also called the addition rule
- If there are $n_1$ ways to do task 1, and $n_2$ ways to do task 2:
  - ✓ If these tasks can be done at the same time,
  - ✓ Then there are $n_1 + n_2$ ways to do one of the two tasks
  - ✓ We must make one choice OR a second choice.

Other saying:

- *If a task can be done in m ways and a second task in n ways, and if these two tasks cannot be done at the same time, then there are m + n ways to do either task.*

# The sum rule

An action is implemented by 1 in k distinguished steps:

Step 1 has $n_1$ ways, step 2 has $n_2$ ways, ..., step k has $n_k$ ways. Therefore, the total of actions is:

$$n = n_1 + n_2 + \ldots + n_k = \sum_{i=1}^{k} n_i$$

# The sum rule

The set style:

- Simple case: given two sets A, B separately, we have:

  $|A \cup B| = |A| + |B|$

- General case: given k set separately by each of pair:

  $A_1, A_2, \ldots, A_k$ we have:

  $|A_1 \cup A_2 \cup \ldots \cup A_k| = |A_1| + |A_2| + \ldots + |A_k|$

$$\left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k} |A_i|$$

# Example of sum rule

- Sample question
  - There are 18 math majors and 325 CS majors
  - How many ways are there to pick one math major or one CS major?

**Solution:** the total is 18 + 325 = 343

# Example of sum rule

How many strings of 4 decimal digits…

- Have exactly three digits that are 9s?
  - The string can have:
    - The non-9 as the first digit
    - OR the non-9 as the second digit
    - OR the non-9 as the third digit
    - OR the non-9 as the fourth digit
    - Thus, we use the sum rule
  - For each of those cases, there are 9 possibilities for the non-9 digit (any number other than 9)
  - Thus, the answer is 9+9+9+9 = 36

# Example of sum rule

Example: Count the number of 1 byte has two first bits are 00 or 11.

Solution:

Let A, B are sets of byte (binary string with length 8) which have two first bits are 00 or 11 respectively.

Hence, $|A| = |B| = 2^6 = 64$.

$A \cap B = \emptyset$. Therefore, base on the sum rule, the byte has two first bits are 00 or 11 is: $|A| + |B| = 64 + 64 = 128$.

# Wedding picture example

- Consider a wedding picture of 6 people
  - There are 10 people, including the bride and groom

a) How many possibilities are there if the bride must be in the picture

- Product rule: place the bride AND then place the rest of the party
- First place the bride
  - She can be in one of 6 positions
- Next, place the other five people via the product rule
  - There are 9 people to choose for the second person, 8 for the third, etc.
  - Total = 9*8*7*6*5 = 15120
- Product rule yields 6 * 15120 = 90,720 possibilities 8

# Counting subsets of a finite set

Let S be a finite set. Use product rule to show that the number of different subsets of S is $2^{|S|}$

**Solution:**

- Let S be a finite set. List the elements of S in arbitrary order.
- Recall from Section 2.2 that there is a one-to-one correspondence between subsets of S and bit strings of length |S|.
- Namely, a subset of S is associated with the bit string with:
  - a1 in the $i^{th}$ position if the $i^{th}$ element in the list is in the subset,
  - a0 in this position otherwise.
- By the product rule, there are $2^{|S|}$ bit strings of length |S|. Hence, $|P(S)| = 2^{|S|}$

*see page 338 (example 10) for more detail.*

*Example: find the subset of string “abc” ?*

# Counting loops

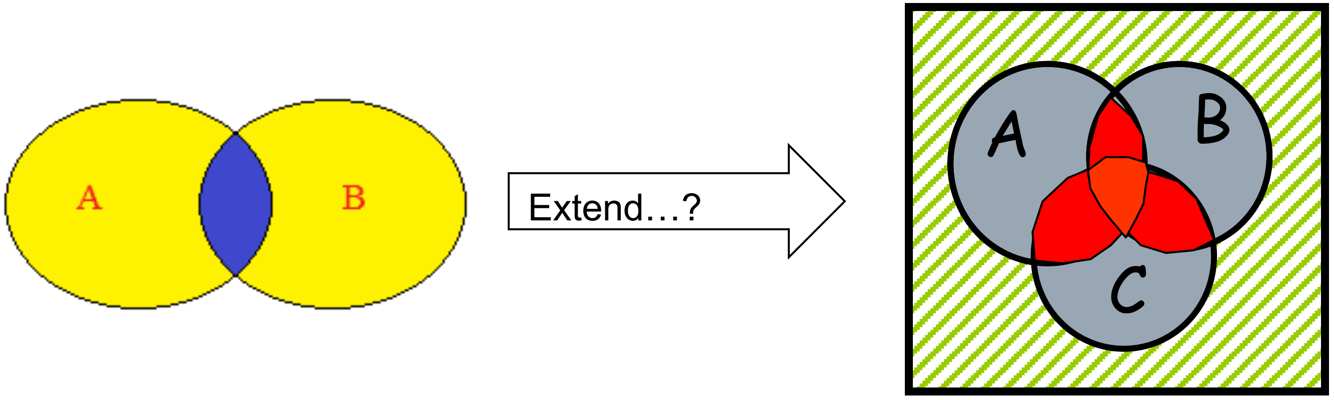
How many times will the following program loop iterate before the final solution is generated? What is the final value of K?

```
K:=0
for i1: = 1 to n1
        for i2 := 1 to n2
                for i3:= 1 to n3
                        K:= K+1
```

# The inclusion-exclusion principle

- When counting the possibilities, we can't include a given outcome more than once!

- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - Let $A_1$ have 5 elements, $A_2$ have 3 elements, and 1 element be both in $A_1$ and $A_2$
  - Total in the union is 5+3-1 = 7, not 8



# The inclusion-exclusion principle

- How may bit strings of length eight start with 1 or end with 00?

- Count bit strings that start with 1
  - Rest of bits can be anything: $2^7 = 128$
  - This is $|A_1|$
- Count bit strings that end with 00
  - Rest of bits can be anything: $2^6 = 64$
  - This is $|A_2|$
- Count bit strings that both start with 1 and end with 00
  - Rest of the bits can be anything: $2^5 = 32$
  - This is $|A_1 \cap A_2|$

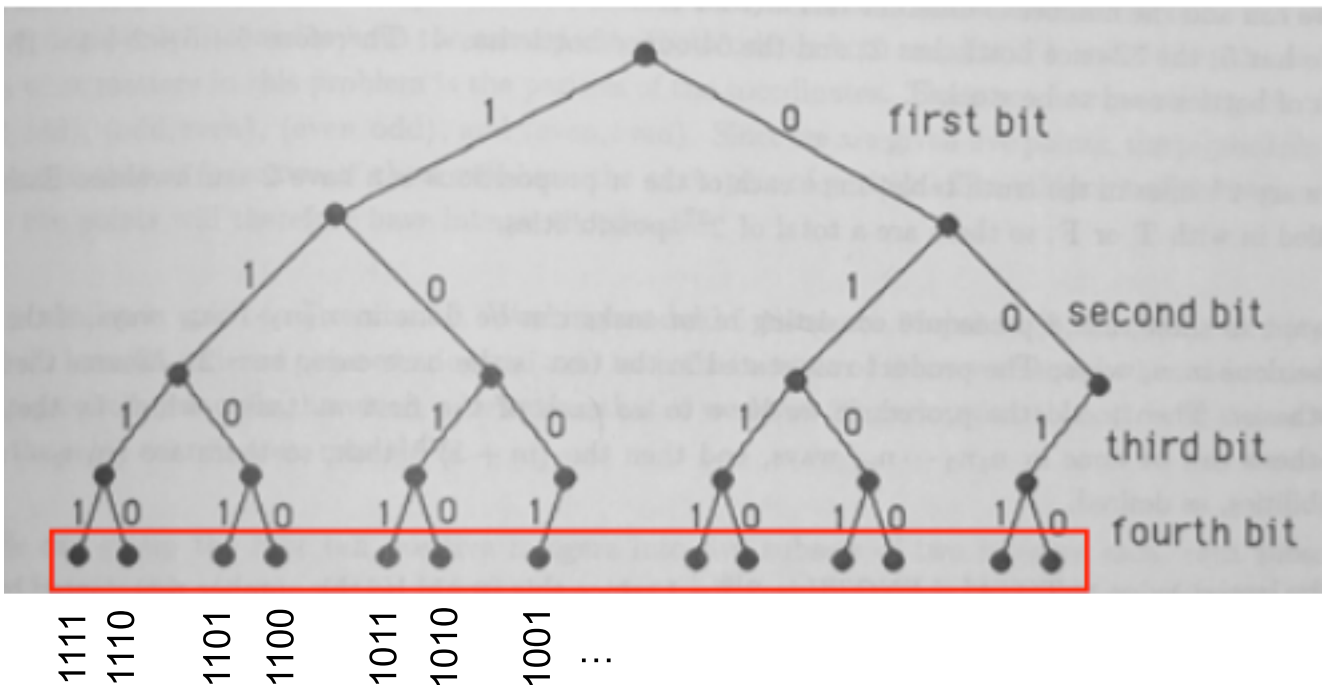
- Use formula $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
- Total is $128 + 64 – 32 = 160$

# Tree diagrams

- We can use tree diagrams to enumerate the possible choices
- Once the tree is laid out, the result is the number of (valid) leaves

# Tree diagrams

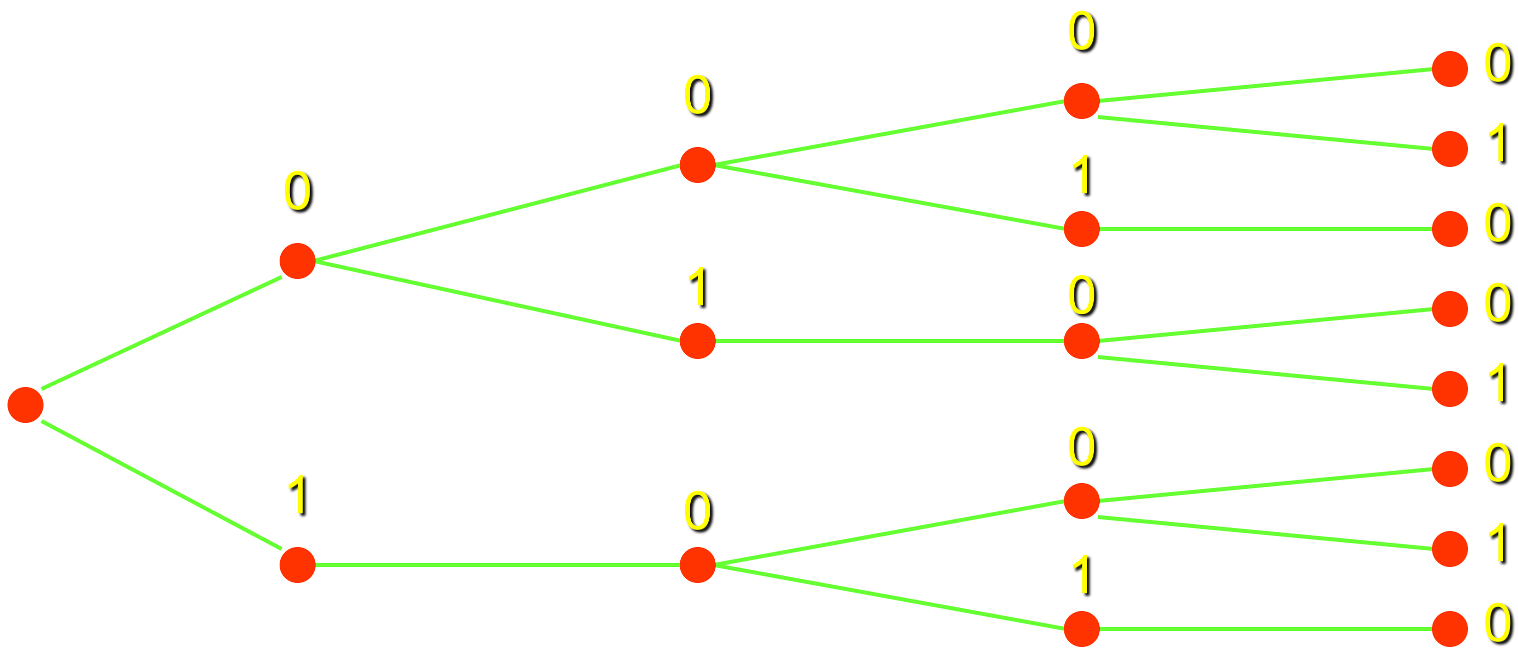
- Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s



# Tree diagrams

How many bit strings of length four do not have two consecutive 1s?

Task 1 (1st bit) | Task 2 (2nd bit) | Task 3 (3rd bit) | Task 4 (4th bit)




There are 8 strings.

# Examples: Inc-Exc principle

*Example:* There are 50 students in a class: There are 30 female students, and 35 CS students. Prove that there are at least 15 female students with major in CS.

*Solution:* Let A be a set of female students, B be a set of CS student. Then

$|A \cap B| = |A| + |B| - |A \cup B| = 30 + 35 - |A \cup B| \geq 15$

because $|A \cup B| \leq 50$

# Pigeonhole principle

If there are more pigeons

than pigeonholes

then some hole must contain two or more pigeons

# Pigeonhole principle

If 20 pigeon flies into 19 pigeonholes, then at least one of the pigeonholes must have at least two pigeons in it. Such observations lead to the *pigeonhole principle*.

THE PIGEONHOLE PRINCIPLE. Let k be a positive integer. If more than k objects are placed into k boxes, then at least one box will contain two or more objects.

# Application of pigeonhole principle

An exam is graded on a scale 0-100. How many students should be there in the class so that at least two students get the same score?

More than 101.

# Generalized pigeonhole principle

If N objects are placed in k boxes, then there is at least one box containing at least N/k objects.

Application 1.
In a class of 73 students, there are at least 73/12=7 who are born in the same month.
Application 2.

- How many students in a class must there be to ensure that 6 students get the same grade (one of A, B, C, D, or F)?
  - The “boxes” are the grades. Thus, $k = 5$
  - Thus, we set $\lceil N/5 \rceil = 6$
  - Lowest possible value for $N$ is 26

# More applications of pigeonhole principle

- A bowl contains 10 red and 10 yellow balls

a) How many balls must be selected to ensure 3 balls of the same color?
   - One solution: consider the "worst" case
     - Consider 2 balls of each color
     - You can't take another ball without hitting 3
     - Thus, the answer is 5
   - Via generalized pigeonhole principle
     - How many balls are required if there are 2 colors, and one color must have 3 balls?
     - How many pigeons are required if there are 2 pigeon holes, and one must have 3 pigeons?
     - number of boxes: $k = 2$
     - We want $\lceil N/k \rceil = 3$
     - What is the minimum $N$?
     - $N = 5$

# More applications of pigeonhole principle

- A bowl contains 10 red and 10 yellow balls

b) How many balls must be selected to ensure 3 yellow balls?

  - Consider the "worst" case
    - Consider 10 red balls and 2 yellow balls
    - You can't take another ball without hitting 3 yellow balls
    - Thus, the answer is 13

**Doing some exercises as your homework (without submission)**

➢ 4, 6,14, 16, 20, 22 (page 417)

➢ Use a tree diagram to find the number of bit strings of length 4 with no 3 consecutive 1s.

# Part 2: Permutations and Combinations

- Permutation
- Combination
- Formulas of combinations
- The extended combination principles

# Permutation

- A permutation is an ordered arrangement of the elements of some set *S*
  - Let *S* = {a, b, c}
  - c, b, a is a permutation of S
  - b, c, a is a *different* permutation of S
- An *r*-permutation is an ordered arrangement of *r* elements of the set
  - A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards
- The notation for the number of *r*-permutations: $P(n,r)$
  - The poker hand is one of P(52,5) permutations

# Permutation

*r*-permutation notation: $P(n,r)$

– The poker hand is one of P(52,5) permutations

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \boxed{\frac{n!}{(n-r)!}}$$

$$= \prod_{i=n-r+1}^{n} i$$

There are $n$ ways to choose the first element

- $n$-1 ways to choose the second
- $n$-2 ways to choose the third
- …
- $n$-$r$+1 ways to choose the $r^{th}$ element

Note that P(n,n) = n!

# Example of permutation

- How many ways are there for 5 people in this class to give presentations?

- There are 27 students in the class
  - P(27,5) = 27*26*25*24*23 = 9,687,600
  - Note that the order they go in does matter in this example!

# Exercise

- How many permutations of {a, b, c, d, e, f, g} end with a?
  - Note that the set has 7 elements

- The last character must be a
  - The rest can be in any order
- Thus, we want a 6-permutation on the set {b, c, d, e, f, g}
- $P(6,6) = 6! = 720$

- Why is it not $P(7,6)$?

# Combination

## In permutation, *order matters*.

- What if order *doesn't* matter?
- In poker, the following two hands are equivalent:
  - A♦, 5♥, 7♣, 10♠, K♠
  - K♠, 10♠, 7♣, 5♥, A♦

- The number of *r*-combinations of a set with *n* elements, where *n* is non-negative and $0 \le r \le n$ is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

**Note: some notations are the same** C(n, r) = $C_n^r = \binom{n}{r}$

# Example of combination

In how many bit strings of length 10, there are exactly four 1's?

- Does the order of these positions matter?
  - No
  - Positions 2,3,5,7 is the same as positions 7,5,3,2
- Thus, the answer is C(10,4) = 210

# Proof of combination formula

- Let $C(n,r)$ be the number of ways to generate unordered combinations
- The number of ordered combinations (i.e. $r$-permutations) is $P(n,r)$
- The number of ways to order a single one of those $r$-permutations $P(r,r)$
- The total number of unordered combinations is the total number of ordered combinations (i.e. $r$-permutations) divided by the number of ways to order each combination
- Thus, $C(n,r) = P(n,r)/P(r,r)$

# Proof of combination formula

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

- An alternative (and more common) way to denote an *r*-combination:

$$C(n,r) = \binom{n}{r}$$

# Circular seating

- How many ways are there to sit 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

- First, place the first person in the north-most chair
  - Only one possibility
- Then place the other 5 people
  - There are P(5,5) = 5! = 120 ways to do that
- By the product rule, we get 1*120 =120

- Alternative means to answer this:
- There are P(6,6)=720 ways to seat the 6 people around the table
- For each seating, there are 6 “rotations” of the seating
- Thus, the final answer is 720/6 = 120

# Other applications

How many different non-negative integer solutions for the variables x1, x2, x3, x4 with x1 + x2 + x3 + x4 = 10?

A solution like x1 = 1, x2 = 0, x3 = 4, x4 = 5 divides 10 into four parts: (1, 0, 4, 5) or "X | | XXXX | XXXXX".

We need three dividers ("|") to divide 10 boxes ("X") into four parts. The number of ways of choosing three slots out of 10+3 slots is
C(10+4-1, 3) = C(13, 3) = 286.

# Book-shelf problem

In how many ways can you put n different books on k different shelves? (shelves can hold all books).

Solution 2: There are n! ways to put them into a sequence. For each sequence, we need to cut the sequence into k subsequences using k-1 dividers.

→ how many bit-strings are there with k-1 "|" (dividers) and n "X" (books): C(n+k-1,k-1).

Total: C(n+k-1,k-1) n! = (n+k-1)!/(k-1)!

# Pascal's identity

If n, k are positive integers and $n \geq k$, then

$$C(n+1, k) = C(n, k) + C(n, k-1)$$

# Pascal's triangle

With the help of Pascal's triangle, the Binomial Formula can considerably simplify the process of expanding powers of binomial expressions

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

… … … … … …

**In Pascal's triangle, each number is:**

- ✓ **the sum of the numbers to its upper left**
- ✓ **and upper right:**

Since we have $C(n + 1, k) = C(n, k - 1) + C(n, k)$ and $C(0, 0) = 1$, we can use Pascal's triangle to simplify the computation of $C(n, k)$:

$k$ →, $n$ ↓

$C(0, 0) = 1$

$C(1, 0) = 1 \quad C(1, 1) = 1$

$C(2, 0) = 1 \quad C(2, 1) = 2 \quad C(2, 2) = 1$

$C(3, 0) = 1 \quad C(3, 1) = 3 \quad C(3, 2) = 3 \quad C(3, 3) = 1$

$C(4, 0) = 1 \quad C(4, 1) = 4 \quad C(4, 2) = 6 \quad C(4, 3) = 4 \quad C(4, 4) = 1$

**E.g.** the 4<sup>th</sup> row 1 4 6 4 1 gives the coefficients in the expansion of $(x + y)^4$

# Binomial coefficients

- $C(n, r)$ is also called a **binomial coefficient** and denoted by

$$C(n,r) = \binom{n}{r}$$

- Recall that a **binomial expression** is the sum of two terms, such as $(x + y)$. Consider $(x + y)^2 = (x + y)(x + y)$.
- When expanding such an expression, we have to form all possible products of a term in the first factor and a term in the second factor:

$$(x + y)^2 = x\,x + x\,y + y\,x + y\,y$$

- Then we can sum identical terms:

$$(x + y)^2 = x^2 + 2\,x\,y + y^2$$

# Binomial coefficients

**Now expanding $(x + y)^3 = (x + y)(x + y)(x + y)$ we have**

$$(x + y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

**Notice that**

- There is only one term $x^3$, because there is only one possibility: choose $x$ from all three factors: C(3, 3) = 1.
- There is three times the term $x^2y$, because we have to choose $x$ in two out of the three factors: C(3, 2) = 3.
- Similarly, there is three times the term $xy^2$ (C(3, 1) = 3) and only one term $y^3$ (C(3, 0) = 1).

# Binomial theorem

- Theorem: Given any numbers *a* and *b* and any nonnegative integer *n*,

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

**Example.**

$$(a+b)^4 = \sum_{i=0}^{n} \binom{4}{i} a^{4-i} b^i$$

$$= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4$$

$$(a + b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$$

# Proof of binomial theorem

- Proof: Use induction on *n*.
- Base case: Let $n = 0$. Then
  - $(a + b)^0 = 1$ and

$$\sum_{i=0}^{0}\binom{0}{i}a^{0-i}b^{i} = \binom{0}{0}a^{0-0}b^{0} = 1.$$

  - Therefore, the statement is true when $n = 0$.

# Proof of binomial theorem

- Inductive step
  - Suppose the statement is true when $n = k$ for some $k \geq 0$.
  - Then

$$(a+b)^{k+1} = (a+b)(a+b)^k$$

$$= (a+b)\sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^i$$

$$= \sum_{i=0}^{k}\binom{k}{i}a^{k-i+1}b^i + \sum_{i=0}^{k}\binom{k}{i}a^{k-i}b^{i+1}$$

# Proof of binomial theorem

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k-i+1} b^i + \sum_{i=0}^{k-1} \binom{k}{i} a^{k-i} b^{i+1} + b^{k+1}$$

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k}{i} a^{k-i+1} b^i + \sum_{i=1}^{k} \binom{k}{i-1} a^{k-i+1} b^i + b^{k+1}$$

$$= a^{k+1} + \sum_{i=1}^{k} \left( \binom{k}{i} + \binom{k}{i-1} \right) a^{k-i+1} b^i + b^{k+1}$$

# Proof of binomial theorem

$$= a^{k+1} + \sum_{i=1}^{k} \binom{k+1}{i} a^{k-i+1} b^i + b^{k+1}$$

$$= \sum_{i=0}^{k+1} \binom{k+1}{i} a^{k-i+1} b^i.$$

- Therefore, the statement is true when $n = k + 1$.

- Thus, the statement is true for all $n \geq 0$.

# Example of binomial theorem

- Expand $(a + b)^8$.
  - $C(8, 0) = C(8, 8) = 1$.
  - $C(8, 1) = C(8, 7) = 8$.
  - $C(8, 2) = C(8, 6) = 28$.
  - $C(8, 3) = C(8, 5) = 56$.
  - $C(8, 4) = 70$.

# Example of binomial theorem

- Therefore,

  $(a + b)^8 = a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3$
  $+ 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8.$

# Example: Approximating $(1+x)^n$

- Theorem: For small values of *x*,

$$(1+x)^n \approx 1 + nx.$$

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2.$$

$$(1+x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3.$$

and so on.

The number of terms to be included will depend on the desired accuracy.

# Formulas of combination

$C(n, 0) = C(n, n) = 1$

$C(n, n\text{-}k) = C(n, k)$

$C(n, k) = C(n\text{-}1, k\text{-}1) + C(n\text{-}1, k)$

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^{n-k} y^k$$

# Formulas of combination

*Example:*

Count the subset of X, |X|=n.

*Solution.*

Replace x = y = 1 in binomial Newton

Which has subset of X is $2^n$.

# Formulas of combination

*Other solution:*

Call $X=\{x_1, x_2, \ldots, x_n\}$, for each subset A of X is set to string of bit $b = b_1b_2\ldots b_n$ such that $b_i=1$ if and only if $x_i \in A$. We have $2^n$ string b, so there are $2^n$ subsets A.

# Formulas of combination

*Example:* given X, |X| = n.
Prove that the subsets have odd number equal even number.
*Solution.*
Replace x = 1 and y = -1 into Newton binomial $\rightarrow$ result.

# The extended combination principles

A. **Repeated permutation**
B. **Repeated combination**

# Repeated permutation

*Given* **n** *elements with* k *kinds: kind* 1 *has* $n_1$ *elements, kind* 2 *has* $n_2$ *elements, ..., kind* k has $n_k$ ***elements****.* *The way to arrange* n *this elements is called "repeated permutation".*

# Repeated permutation

Let $S=s_1s_2...s_n$ is a repeated permutation.

There are $C(n, n_1)$ ways to choose $n_1$ positions for elements of kind 1.

After setup elements of kind1, there are $C(n-n_1, n_2)$ ways to choose $n_2$ positions for elements of kinds 2.

...

After setup kind1...k-1 there are $C(n-n_1-...-n_{k-1}, n_k)$ ways to choose $n_k$ positions to setup elements $k$.

# Repeated permutation

Base on the product rule, the number of repeated permutation as follows:

$C(n, n_1), C(n-n_1, n_2), \ldots, C(n-n_{1-}\ldots-n_{k-1}, n_k)$

$$P(n;n_1,n_2,\ldots,n_k) = n!/(n_1!n_2!\ldots n_k!)$$

# Repeated permutation

- Examples:
- Compute the ways to arrange the following string:
  a) MISSISSIPPI. RS: 11!/(1!4!4!2!)
  b) SUCCESS. RS: 7!/(3!1!2!1!)
- Count the ways to arrange 9 balls including 2 green balls, 3 red balls and 4 yellow balls in a line.

  RS: 9!/(2!3!4!)

# Repeated combination

Given $n$ kinds of elements, each kind has not less than $k$ elements. One way to choose $k$ elements (may be repeated) from $n$ kinds of elements is called repeated combination.

# Repeated combination

Other way, *given* $|X|=n$, a repeated combination k of X is an unordered selection k elements of X, in which the elements can be repeated.

# Repeated combination

- Suppose each repeated combination $S=s_1s_2..s_k$ is arranged as follows:
- The first is all elements of kind1, the next is all elements of kind2, …, the last is all elements of kindn.
- Perform $k$ elements by $k$ sign "x" and use $n-1$ sign"|" to divide $n$ kinds.

# Repeated combination

Therefore, each repeated combination is equivalent to one selection *n-1* positions in *k+n-1* positions to set n-1 sign "|".

There are C(k+n-1, n-1) ways to choose. So, the number of repetition is:

$$C(k+n-1, n-1) = C(k+n-1, k)$$

# Repeated combination

*Example:* count the ways to buy 10 fruits with 3 kinds: orange, tangerine, mango.
RS: C(10+2, 3)=C(12, 3)=220

*Example. Count the number of integer results*
$x+y+z = 12$, where $x \geq 0$, $y \geq 0$, $z \geq 0$.
RS: C(12+2, 3)=C(14, 3)=392

# Repeated combination

*Example.* Count the number of integer results:
$x + y + z = 12$ with $x \geq 1$, $y \geq -2$, $z \geq 3$.

Let $x' = x-1$, $y' = y+2$, $z' = z-3$
Equivalent
$x' + y' + z' = 10$ with $x' \geq 0$, $y' \geq 0$, $z' \geq 0$.
RS: $C(10+2, 3)=C(12, 3)=220$

# Repeated combination

*Example.* Count the integer results

$x + y + z \leq 12$ with $x \geq 0$, $y \geq 0$, $z \geq 0$.

Adding $t = 12 - (x + y + z) \geq 0$

Equivalent

$x + y + z + t = 12$ with $x \geq 0$, $y \geq 0$, $z \geq 0$, $t \geq 0$.

RS: $C(12+3, 3)=C(15, 3)=455$

# Repeated combination

*Example.* Count *the integer results*

$x + y + z = 11,\ 0 \le x \le 3,\ 0 \le y \le 4,\ 0 \le z \le 6$

Let U is a set of all results (non-negative) of equation. N=C(11+2,2)

Let $A_1$ a set of results with $x \ge 4,\ y \ge 0,\ z \ge 0$

$A_2$ a set of results with $y \ge 5,\ x \ge 0,\ z \ge 0$

$A_3$ a set of results with $z \ge 7,\ x \ge 0,\ y \ge 0$

# Repetition combination

Base on inclusion-exclusion rule:

We have $N^* = N-|A1|-|A2|-|A3|$

$+ |A1 \cap A2| +|A1 \cap A3| +|A2 \cap A3|$

$-|A1 \cap A2 \cap A3|$

$N=78$, $|A1|+|A2|+|A3|=79$,

$|A1 \cap A2| +|A1 \cap A3| +|A2 \cap A3|=7$

$|A1 \cap A2 \cap A3|=0$, therefore $N^*=6$