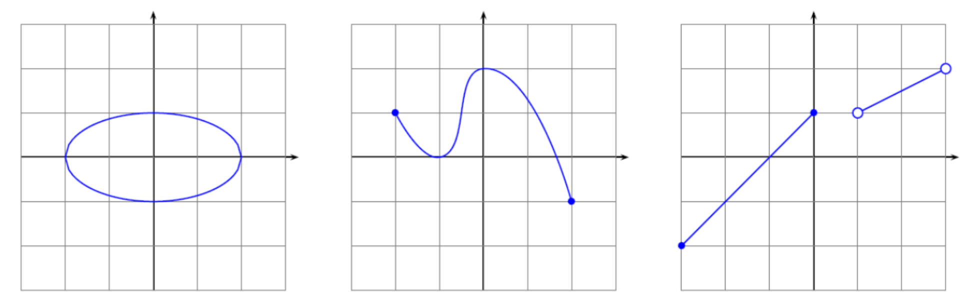
Due date: 04:30 PM, Wed October 09, 2024.
Instructions:

- The solutions **must be** in the order as listed, that is, Q1, Q2, Q3, ..., Q10. You can temporarily leave a blank for a question without a solution (and can go back to resolve it later).
- Submit the **solution papers (hard copy)** in the class on October 09, 2024. For any **late** submissions, please submit your papers in my P.O. mailbox in front of O2.610 (outside the office and nearby the office door) and also a single file (scanned) on Blackboard (for late submission only).

Chapter 1: Functions, limit and continuity.

1. A cell phone plan has a basic charge of \$35 a month. The plan includes 400 free minutes and charges 10 cents for each additional minute of usage. Write the monthly cost as a function of x the number of minutes used and graph as a function of for $0 \le x \le 600$.

2. (a) Find the domain of $f(x) = \sqrt{4 - x|x|}$.
Hint: Consider two cases: $x < 0$ and $x > 0$. Take the union to obtain the answer: $D = (-\infty, 2]$.
(b) Find the domain and range of the function $f(x) = \sqrt{1 - \cos x}$.
(c) Find the domain, range and sketch the graphs of the functions $f(x) = \sqrt{4 - x^2}$ and $g(x) = \sqrt{x - 5}$.
(d) Find the domain and range of the relations whose graphs are shown bellow. Which of those graphs are graphs of functions?



(a) (b) (c)

3. Find a formula for the inverse of the function:
(a) $y = f(x) = \sqrt{x+1}$, $x \ge -1$.
(b) $y = f(x) = \ln(x+3)$, $x > -3$.
(c) $y = f(x) = \dfrac{x}{2x+1}$, $x \ne -1/2$.
(d) Let $f(x) = 2x + \ln x$, $x > 0$. Suppose $f(x)$ has the inverse $f^{-1}(x)$ on $[1, \infty)$. Find $f^{-1}(2)$.
(e) Let $f(x) = \dfrac{x-2}{x-1}$. Find the inverse of $f(x)$ and determine its domain and range.

4. Find the following limits
(a) $\displaystyle\lim_{x\to 4} \frac{\sqrt{1+2\text{x}} - 3}{\sqrt{x} - 2}$, (b) $\displaystyle\lim_{x\to 4^-} \frac{x+1}{x-4}$, (c) $\displaystyle\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^3}\right)$,
(d) $\displaystyle\lim_{x\to 0} xe^{-\cos\left(\frac{1}{x^2}\right)}$, (e) $\displaystyle\lim_{x\to\infty} \left(-\frac{2e^{ax}}{e^{3x}} + be^{-cx}\right)$, where a, b, c are constants, $a < 3$, and $c > 0$.

5. According to the Theory of Relativity, the length L observed by an observer in relative motion with respect to the object, is a function of its velocity v with respect to an observer (Lorentz contraction). For an object whose length at rest is 10 m, the function is given by

$$L(v) = 10\sqrt{1 - \frac{v^2}{c^2}}$$

where c is the speed of light (300,000 km/s).
(a) Find $L(0.5c)$, $L(0.9c)$.
(b) How does the length of an object change as its velocity increases?
(c) Find $\lim_{v \to c^-} L(v)$.

6. The toll T charged for driving on a certain stretch of a toll road is $5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is $7.
(a) Sketch a graph of T as a function of the time t, measured in hours past midnight.
(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

7. Find the values of a and b that make f continuous everywhere:

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } \ x < 2 \\ ax^2 - bx + 3 & \text{if } \ 2 \leqslant x < 3 \\ 2x - a + b & \text{if } \ x \geqslant 3 \end{cases}$$

8. Show that there is a root of the equation in the given interval (for (a) and (b) only). Use the Intermediate Value Theorem (IVT).
(a) $e^{-x^2} = x, \ (0, 1)$.
(b) $x^{10} - x^9 - 1 = 0, \ (0, \infty)$.
(c) Show that the equation $x^4 - 10x^3 - 25x^2 - x - 1 = 0$ has at least two distinct real roots.

9. The population of a certain species is defined by the following function

$$P(t) = \frac{1,000}{1 + 9e^{-t}},$$

where t is measured in years.
(a) Find all horizontal asymptotes.
(b) Estimate how long it takes for the population to reach 900.
(c) Find the inverse of this function in form of $t = f(P)$ and explain its meaning.
(d) Use the inverse function to find the time required for the population to reach 900. Re-check with the result of part (b).

10. A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Use the IVT to show that there is a point on the path that the monk will cross at exactly the same time of day on both days.

-THE END-