Hypothesis testing

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What is hypothesis testing?

Estimator

- Suppose we want to investigate a random variable X
- θ is an unknown parameter of X (mean, variance...)
- T is a statistic, calculated from sample
- If $E(T) = \theta$ then can use T to estimate θ

Hypothesis

- Now we want to have some statement about θ
- Ex: $0 < \theta < 1$ or $\theta = 1$
- This statement is called *hypothesis* and we don't know if it is true or not.
- Want to test if we can accept the hypothesis or reject it

Setting

- Call the hypothesis H_0 (null hypothesis)
- $X_1, \ldots, X_n$: sample of X
- Find condition C for $T(X_1, \ldots, X_n)$ so that :
  - if $T \in C$ then accept H_0
  - If $T \notin C$ then reject H_0

Idea

- We try to see if H_0 is *consistent* with the data
- Only reject H_0 if the data is very unlikely if H_0 is true
- Fix $0 < \alpha < 1$ (usually 5%)
- Want to find C so that $P(T \notin C) = \alpha$
- α: level of significance

Error

- 2 types of errors
- Type 1: Reject H_0 when it is correct
- Type 2: Accept H_0 when it is wrong

Test for mean of normal population with known variance

Setting

- $X \sim \text{N}(\mu, \sigma^2)$ where μ is unknown, σ^2 is known
- Null hypothesis: $H_0 : \mu = \mu_0$
- Alternative hypothesis: $H_1 : \mu \neq \mu_0$
- If H_0 is true then $\bar{X}$ is close to μ_0
- Accept H_0 if $\bar{X}$ is not too far from μ_0

Significance level

- Choose the significance level to be α
- Want $P($ type 1 error$)= \alpha$
- If H_0 is true then $X \sim \mathrm{N}(\mu_0, \sigma^2)$, then $\bar{X} \sim \mathrm{N}(\mu_0, \sigma^2/n)$
- want to find c so that $P(|\bar{X} - \mu_0| > c) = \alpha$

- Equivalent to
$$P\left(\left|\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}\right| > \frac{c}{\sigma/\sqrt{n}}\right) = \alpha$$
- $$2P\left(\frac{\bar{X}-\mu_0}{\sigma/\sqrt{n}} > \frac{c}{\sigma/\sqrt{n}}\right) = \alpha$$
- so $\dfrac{c}{\sigma/\sqrt{n}} = z_{\alpha/2}$
- or $c = \dfrac{z_{\alpha/2}\sigma}{\sqrt{n}}$

The test

- Significance level α
- Test statistic: $T = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$
- $t =$ value of T calculated from sample
- Reject H_0 if $|t| > z_{\alpha/2}$

Example

It is known that if a signal of value μ is sent from location A, then the value received at location B is normally distributed with mean μ and standard deviation 2. The people at location B guest that the signal value $\mu = 8$ will be sent today. Test this hypothesis with 5% level of significance if the same signal value is independently sent 5 times and the average value received at location B is $X = 9.5$.

Solution

- $X \sim \mathrm{N}(\mu, 4), n = 5, \bar{x} = 9.5$
- H_0: $\mu = \mu_0 = 8$
- $t = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \dfrac{-1.5}{2/\sqrt{5}} = -1.68$
- $z_{\alpha/2} = z_{.025} = 1.96$
- $|t| < z_{.025} \rightarrow \text{accept } H_0$

Different level

- Larger level $\alpha = 10\%$
- $z_{.05} = 1.645$
- then $|t| > z_{.05} \to$ reject H_0

Choose level

- If we're confidence that H_0 is true then choose high level
- If rejecting H_0 would cause bad problem (large cost) then choose low level (1-5%)

p-value

- Suppose t is the observed value of T, then $P(|T| \geq |t| \big| H_0 \text{ is true})$ is called the p-value of the test.
- H_0 accepted for any significance level less than p-value

Example

- Same transmitting problem
- Case 1: $\bar{x} = 8.5$
- p-value

$$P(\left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq \frac{0.5}{2/\sqrt{5}}) = .576$$

- Accept H_0 for any level $\alpha < .576$.

- Case 2: $\bar{x} = 11.5$
- p-value

$$P(\left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| \geq \frac{3.5}{2/\sqrt{5}}) = .00005$$

- Need α to be very small if accepting H_0
- Always reject H_0

One-sided test

- $H_0 : \mu = \mu_0$ or $\mu \leq \mu_0$
  vs
  $H_1 : \mu > \mu_0$
- Reject H_0 if $\bar{X}$ is much larger than μ_0

$$\frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} > z_\alpha$$

One-sided test

- $H_0 : \mu = \mu_0$ or $\mu \geq \mu_0$
  vs
  $H_1 : \mu < \mu_0$
- Reject H_0 if $\bar{X}$ is much smaller than μ_0

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} < -z_\alpha$$

Example

All cigarettes have an average nicotine content of at least 1.6 mg per cigarette. A firm claims that the amount of its cigarette is less than 1.6 mg. To test this claim, a sample of 20 of the firm's cigarettes were analyzed and the average nicotine content is 1.54. If the standard deviation of a cigarette's nicotine content is .8 mg, what conclusions can be drawn at the 5 percent level of significance?

- $H_0 : \mu \geq 1.6$ vs $H_1 : \mu < 1.6$
- $t = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{1.54-1.6}{0.8/\sqrt{20}} = -.336$
- p-value:
  $P(T < -.336) = 0.368 > 0.05$
- Can not reject H_0
- Can not be 95% sure that the claim is true.
- Data supports H_1 but not strong enough to reject H_0.

Relation with C.I.

- $(1-\alpha)$ Confidence interval of μ is

$$I = (\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$$

- If $\mu = \mu_0$ then $P(\mu_0 \in I) = 1 - \alpha$
- then reject H_0 with level α if $\mu_0 \notin I$.

Test of mean of normal population when variance is unknown

Test statistic

- Use $T = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$, t is observed value of T
- If H_0 is true then T has t-distribution with $n - 1$ degree of freedom

$$P(|T| > t_{\alpha/2,n-1}) = \alpha$$

- α significance level test: Reject H_0 if $|t| > t_{\alpha/2,n-1}$, accept otherwise
- p-value test: p-value$= P(|T| \geq |t|)$, accept if $\alpha < p$-value.

Example

A public health official claims that the mean home water use is 350 gallons a day. 20 randomly selected homes was investigated with the average daily water uses as follows:

340	344	362	375
356	386	354	364
332	402	340	355
362	322	372	324
318	360	338	370

Do the data contradict the claim?

Solution

- $\bar{x} = 353.8,\ s = 21.8478,\ \mu_0 = 350,\ n = 20$
- $t = \frac{\sqrt{20}3.8}{21.8478} = .7778$
- $T \sim$ t-distribution 19 degree freedom
- $t < t_{0.05,19} = 1.730$ so accept H_0 for 10% level of significance
- p-value: $P(|T| > t) = .4462$
  Accept H_0 for any $\alpha < .4462$

Test of variance of normal population

Setting

- $X \sim \mathrm{N}(\mu, \sigma^2)$, μ, σ unknown
- Want to test
    - H_0: $\sigma = \sigma_0$
    - H_1: $\sigma \neq \sigma_0$

Test statistic

- If H_0 is true then $\dfrac{(n-1)S^2}{\sigma_0^2} \sim \chi^2_{n-1}$
- Use test statistic $T = \dfrac{(n-1)S^2}{\sigma_0^2}$

$$P(\chi^2_{1-\alpha/2,n-1} < T < \chi^2_{\alpha/2,n-1}) = 1-\alpha$$

The test

- Significance level α
- t is observed value of T
- Accept H_0 if $\chi^2_{1-\alpha/2,n-1} < t < \chi^2_{\alpha/2,n-1}$
- Reject H_0 otherwise
- p-value
  $= 2\min\{P(T < t), 1 - P(T > t)\}$

One-sided test

- H_0: $\sigma \leq \sigma_0$ vs
- H_1: $\sigma > \sigma_0$
- Reject H_0 if t is too large
- p-value $= P(T > t | H_0 \text{ is true})$

Example

A machine that automatically controls the amount of ribbon on a tape has recently been installed. This machine will be judged to be effective if the standard deviation σ of the amount of ribbon on a tape is less than .15 cm. If a sample of 20 tapes yields a sample variance of $S^2 = .025\ cm^2$, are we justified in concluding that the machine is ineffective?

Solution

- $H_0: \ \sigma^2 \leq .0225$ vs $H_1: \ \sigma^2 > .0225$
- $s^2 = .025,$
  $t = 19(.025)/.0225 = 21.111, n = 20$
- $\chi^2_{19,0.025} = 32.852, \chi^2_{19,0.975} = 8.907$
- $\chi^2_{19,0.025} \geq t \geq \chi^2_{19,0.975}$: Accept H_0
- p-value: $P(T > 21.111) = .3307$
- for $\alpha < .3307$ accept H_0

Solution

- $H_0: \ \sigma^2 \leq .0225$ vs $H_1: \ \sigma^2 > .0225$
- $s^2 = .025$,
  $t = 19(.025)/.0225 = 21.111, n = 20$
- $\chi^2_{19,0.025} = 32.852, \chi^2_{19,0.975} = 8.907$
- $\chi^2_{19,0.025} \geq t \geq \chi^2_{19,0.975}$: Accept H_0
- p-value: $P(T > 21.111) = .3307$
- for $\alpha < .3307$ accept H_0

Test of equality of means of two normal populations

Known variance

- $X \sim \mathrm{N}(\mu_x, \sigma_x^2)$, $Y \sim \mathrm{N}(\mu_y, \sigma_y^2)$
- Know σ_x, σ_y, don't know μ_x, μ_y
- $H_0 : \mu_x = \mu_y$ or $\mu_x - \mu_y = 0$
- $H_1 : \mu_x \neq \mu_y$

Distribution

- $\bar{X} - \bar{Y}$ is estimator for $\mu_x - \mu_y$
- $\bar{X} - \bar{Y} \sim \mathrm{N}(\mu_x - \mu_y, \dfrac{\sigma_x^2}{n} + \dfrac{\sigma_y^2}{m})$
- Test statistic: $T = \dfrac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_x^2}{n} + \frac{\sigma_y^2}{m}}}$
- $T \sim \mathrm{N}(0, 1)$

Two-sided test

- Significance level α
- t is observed value of T
- Reject H_0 if $|t| > z_{\alpha/2}$
- p-value = $P(|T| > |t|)$

Example

A company produces a sample of 10 tires using one method and a sample of 8 using another method. They want to show that there is no difference in the average life time of tires . The first tire set is tested at location A where the standard deviation is known to be 4000 km, and second set is tested at location B where sd is 6000. What conclusion can be drawn with 5% level of significance from the following data?

TABLE 8.3 *Tire Lives in Units of 100 Kilometers*

Tires Tested at A	Tires Tested at B
61.1	62.2
58.2	56.6
62.3	66.4
64	56.2
59.7	57.4
66.2	58.4
57.8	57.6
61.4	65.4
62.2	
63.6	

Homework 11

Chapter 8: 3, 4, 10, 12, 13, 21, 27, 47