Counting technique

February 21, 2019

Equally likely outcomes

Probability space (Ω, P) has equally likely outcomes if for any $x \neq y$ in Ω

$$P(\{x\}) = P(\{y\}).$$

If so then

$$P(E) = \frac{|E|}{|\Omega|}.$$

Counting

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Basic principle

- Suppose there is a job that has 2 steps
- There are m ways to do step 1
- There are n ways to do step 2
- There are $m \times n$ ways to do the job.

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Example

Two balls are ”randomly drawn” from a bowl containing 6 white and 5 black balls. What is the probability that one of the drawn balls is white and the other black?

Size of sample space

How many ways to pick 2 balls out ot 11 balls? (regardless of color)

- 11 ways for 1st ball
- 10 ways for 2nd ball
- $11 \times 10 = 110$ ways total

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Size of event 1 black 1 white

- the colors could be BW or WB
- If BW: $5 \times 6 = 30$ ways
- If WB: $6 \times 5 = 30$ ways
- Total $30 + 30 = 60$ ways
- Probability = 60/110.

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Ordering

There are n balls of different colors. How many ways to arrange them on a straight line?

Solution

- Think of a line with n positions
- n ways to choose ball for 1st position
- $(n-1)$ ways for 2nd position, (n-2) for 3rd ...
- Total $n \cdot (n-1) \cdot (n-2) \ldots 2 \cdot 1$ ways

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Similar questions

- How many ways to distribute n distinct presents to n children?
- How many possible results of a competition of n teams?
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- One ordering of n distinct objects is called a n - permutation.
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Example

Mr. Jones has 9 books that he is going to put on his bookshelf. Of these, 4 are mathematics books, 3 are chemistry books and 2 are history books. He wants to arrange his books so that all the books of the same subject are together on the shelf. How many different arrangements are possible?

Solution

- 3 blocks of books of same subject: 3! ways to arrange the blocks
- 4! ways to arrange the mathematics block
- 3! ways to arrange the chemistry block
- 2! ways to arrange the history block
- Total: 3!4!3!2!=1728 ways

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How many ways to choose a group of 2 out of 5 people if the order of the group is not important.

Solution

- $5 \times 4 = 20$ ways to choose 2 people
- Any group AB was counted 2! times (AB, BA)
- If the order is not important then there are

$$\frac{5 \times 4}{2!} = \frac{5!}{3!2!}$$

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Combination

The number of ways to choose an unordered group of k objects out of n distinct objects is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Read ”n choose k”
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Example

A committee of size 5 is to be selected randomly from a group of 6 men and 9 women. What is the probability that the committee consists of 3 men and 2 women?

Solution

- Size of sample space: $\binom{15}{5}$
- Size of event: $\binom{6}{3}$ ways to choose the men, $\binom{9}{2}$ ways to choose the women
- The probability is

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Practice

A woman has n keys, of which one will open her door. If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try?

Practice

How many different letter arrangements can be made from the letters

(a) Fluke?

(b) Propose?

(c) Mississippi?

(d) Arrange?

Practice

A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if

(a) both books are to be on the same subject?

(b) the books are to be on different subjects?

Conditional probability

- The probability of an event changes when we know more information about the experiment
- The sample space becomes smaller, we have to compute probability with respect to new sample space.

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Example

Roll 2 dice.
Event $E = \{$ sum of 2 dice = 7$\}$.

- Sample space $\Omega = \{(x, y) : x, y = 1, ..6\}$
- $P(E) = \frac{6}{36} = 1/6.$

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Suppose we know $x = 2$.

- New sample space $F = \{(2, y) :, y = 1, ..6\}$
- In F, only the elements in EF satisfy E
- New probability for E

$$P(E|F) = \frac{|EF|}{|F|} = \frac{1}{6}$$

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Conditional probability

- $P(E|F)$ is called the conditional probability of E given F, or probability of E conditioned on F.
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Example

A box contains 5 defective (that immediately fail when put in use), 10 partially defective (that fail after a couple of hours of use), and 25 acceptable transistors. A transistor is chosen at random from the box and put into use. If it does not immediately fail, what is the probability it is acceptable?

Solution

- New sample space = 10 partially defective + 25 acceptable
- $P(\text{acceptable}|\text{not defective}) = 25/35$

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General formula

$$
\begin{aligned}
P(E|F) &= \frac{|EF|}{|F|} \\
&= \frac{|EF|/|\Omega|}{|F|/|\Omega|} \\
&= \frac{P(EF)}{P(F)}
\end{aligned}
$$

Previous example

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\begin{aligned}
&P(\text{acceptable}|\text{not defective}) \\
&\quad = \frac{P(\text{acceptable and not defective})}{P(\text{not acceptable})} \\
&\quad = \frac{25/40}{35/40} = \frac{25}{35}
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Backward formula

$$P(EF) = P(E|F)P(F).$$

Example

Ms. Perez figures that there is a 30 percent chance that her company will set up a branch office in Phoenix. If it does, she is 60 percent certain that she will be made manager of this new operation. What is the probability that Perez will be a Phoenix branch office manager?

Solution

$E =$\{ Perez will be manager of new branch\}

$F =$\{ New branch will be opened\}

$E \subset F \to E = EF$

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$E \subset F \rightarrow E = EF$

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$$E = \{\text{ Perez will be manager of new branch}\}$$
$$F = \{\text{ New branch will be opened}\}$$
$$E \subset F \to E = EF$$

- $P(F) = 0.3$
- $P(E|F) = 0.6$
- $P(E) = P(EF) = P(F)P(E|F) = 0.18.$

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Example

A student is taking a one-hour examination. Suppose the probability that the student will finish the exam in less than x hours is $x/2$, for all $0 \leq x \leq 1$. Then, given that the student is still working after .75 hour, what is the conditional probability that he will use the full hour ?

Solution

- L_x: finish less than x hours
- $L_x \subset L_y$ if $y \geq x$
- F: use the full hour
- $P(F) = P(L_1^c) = 1 - P(L_1) = .5$

- E : still working after .75 hour
- $E = L^c_{.75}$

then

$$P(F|E) = \frac{P(FL^c_{.75})}{P(L^c_{.75})} = \frac{P(F)}{1 - P(L_{.75})}$$
$$= \frac{.5}{1 - .375} = .8$$

Example

A fair coin is flipped twice. what is the conditional probability that both flips land on heads, given that

(a) the first flip lands on heads?

(b) at least one flip lands on heads?

Solution for (a)

- $A = \{HH\}$ (both head)
- $F = \{HH, HT\}$ (first is head)

$$P(A|F) = \frac{P(AF)}{P(F)} = \frac{P(\{HH\})}{P(\{HH, HT\})}$$
$$= \frac{1/4}{2/4} = \frac{1}{2}$$

Solution for (b)

- $B = \{HH, HT, TH\}$ (at least one head)

$$P(A|B) = \frac{P(AB)}{P(B)}$$
$$= \frac{P(\{HH\})}{P(\{HH, HT, TH\})} = \frac{1}{3}$$

Practice

Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red? At least 1 ball is red?

Homework 2

Chapter 3: 17, 18, 20, 22, 25, 27