

## 1.9. Electric Flux. Gauss' Law:

In this session, we study Gauss' law and its application:

- Gauss' law relates the electric fields at points on a closed Gaussian surface and the net charge enclosed by that surface and it is in fact equivalent to Coulomb's law but expressed in a different form with new concepts
- Gauss' law allows us to improve our knowledge of isolated conductors

### 1.9.1. Electric Flux:

(a) Flux:

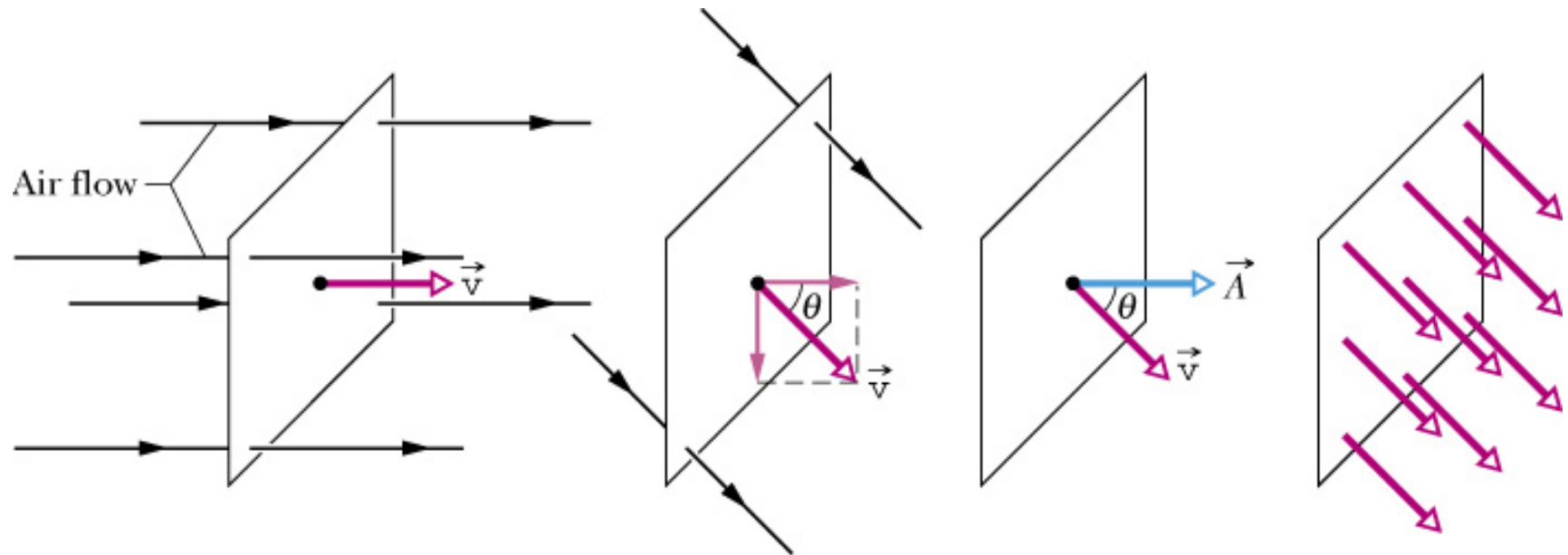
**Problem:** Consider an airstream of uniform velocity  $\vec{V}$  at a small square loop of area  $A$

$\Phi$ : volume flow rate (volume per time unit) of the airstream

$$\Phi = (v \cos \theta) A$$

- If we define an area vector  $\vec{A}$  whose magnitude is  $A$  and its direction is normal to the plane of the area:

$$\Phi = v A \cos \theta = \vec{v} \cdot \vec{A}$$



- If we assign a velocity vector to each point in the airstream passing through the loop:
  - the composite of all these vectors is a velocity field
  - so,  $\Phi$  is the flux of the velocity field through the loop

## (b) Electric flux:

**Aim:** Calculate the flux of an electric field

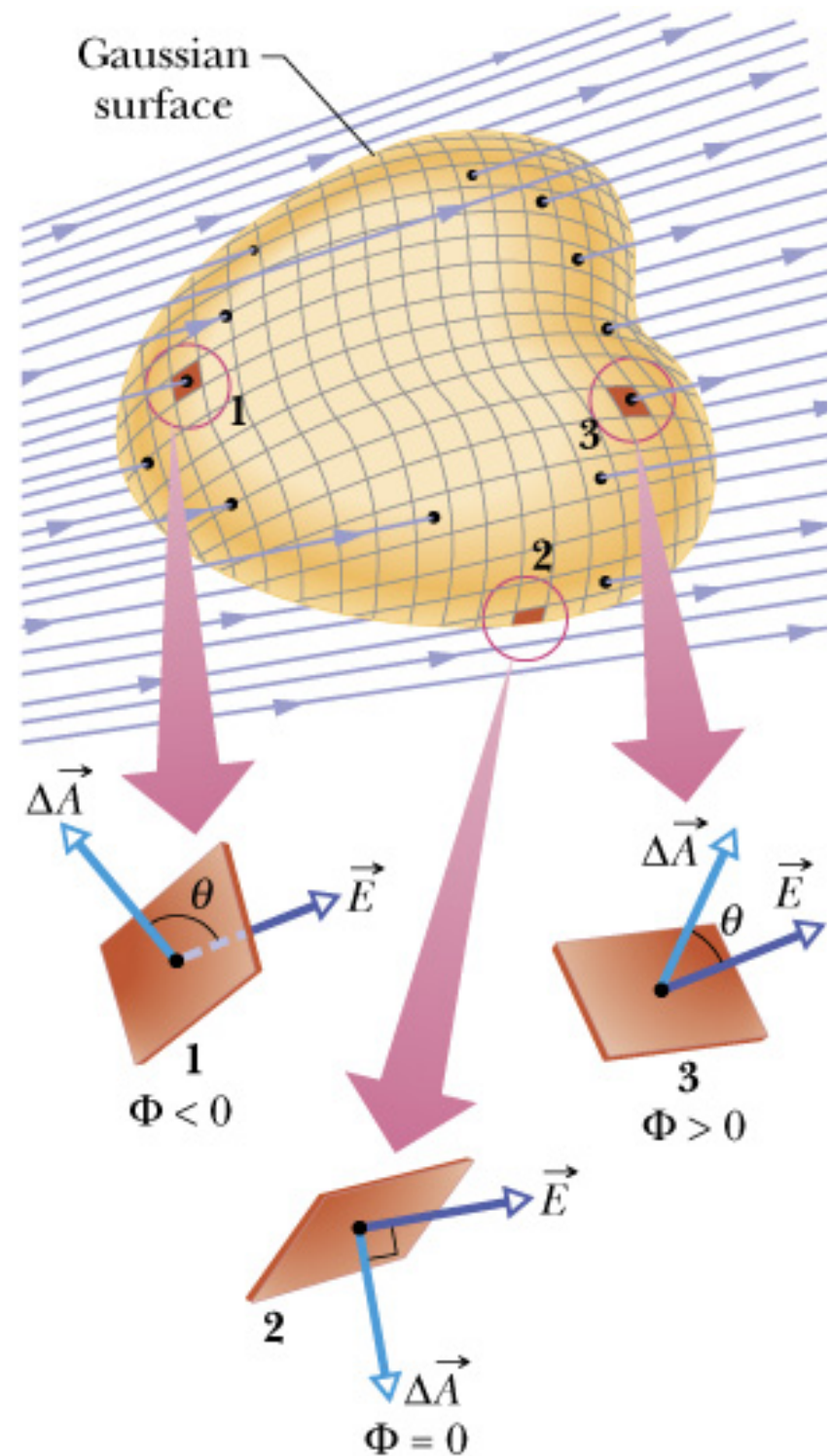
• Consider an arbitrary Gaussian surface in a nonuniform electric field

✓ the flux:  $\Phi = \sum \vec{E} \Delta \vec{A}$

$$\Phi = \oint \vec{E} d\vec{A}$$

**Unit:** N.m<sup>2</sup>/C

➤ The electric flux through a Gaussian surface is proportional to the net number of field lines passing through that surface



**Checkpoint 1 (page 607):** The figure shows a Gaussian cube of face area  $A$  immersed in a uniform electric field  $\vec{E}$  that has a positive direction of the  $z$  axis. In terms of  $E$  and  $A$ , what is the flux through (a) the front face (in the  $xy$  plane), (b) the rear face, (c) the top face, and (d) the whole cube?

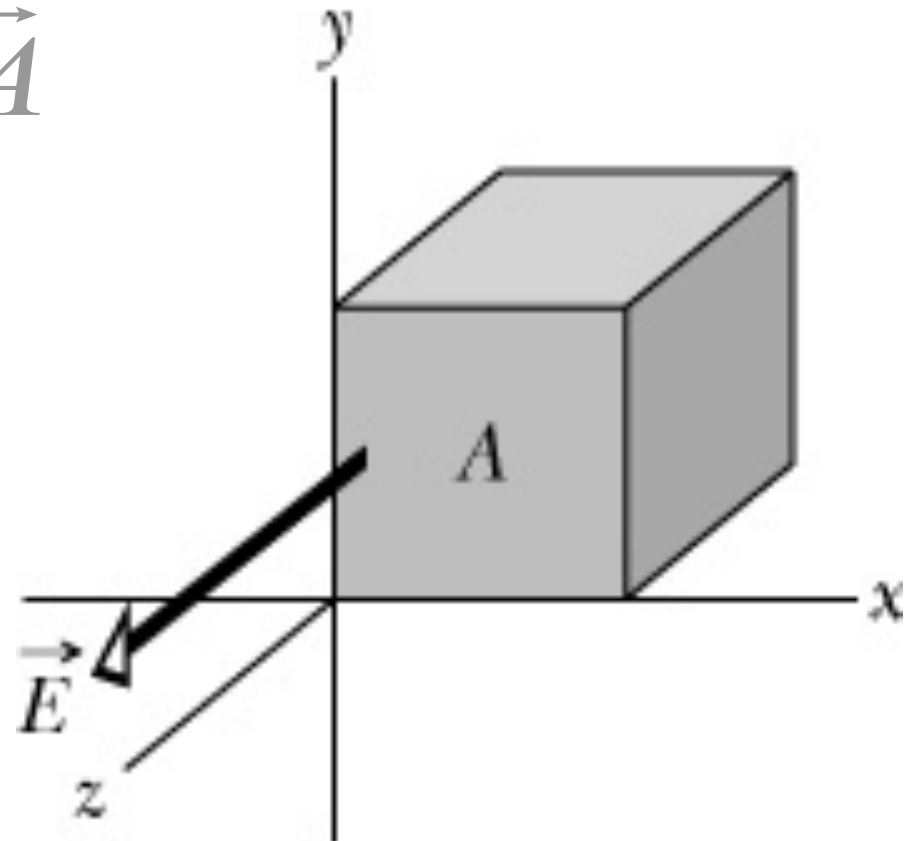
$$\Phi = \oint \vec{E} d\vec{A}$$

(a)  $\Phi = E \cdot A$

(b)  $\Phi = -E \cdot A$

(c)  $\Phi = 0$

(d)  $\Phi = 0$



### 1.9.2. Gauss' Law:

$$\epsilon_0 \Phi = q_{enc}$$

$q_{enc}$  : the net charge enclosed  
in the surface

or

$$\epsilon_0 \oint \vec{E} d\vec{A} = q_{enc}$$

$q_{enc} > 0$  : the net flux is outward

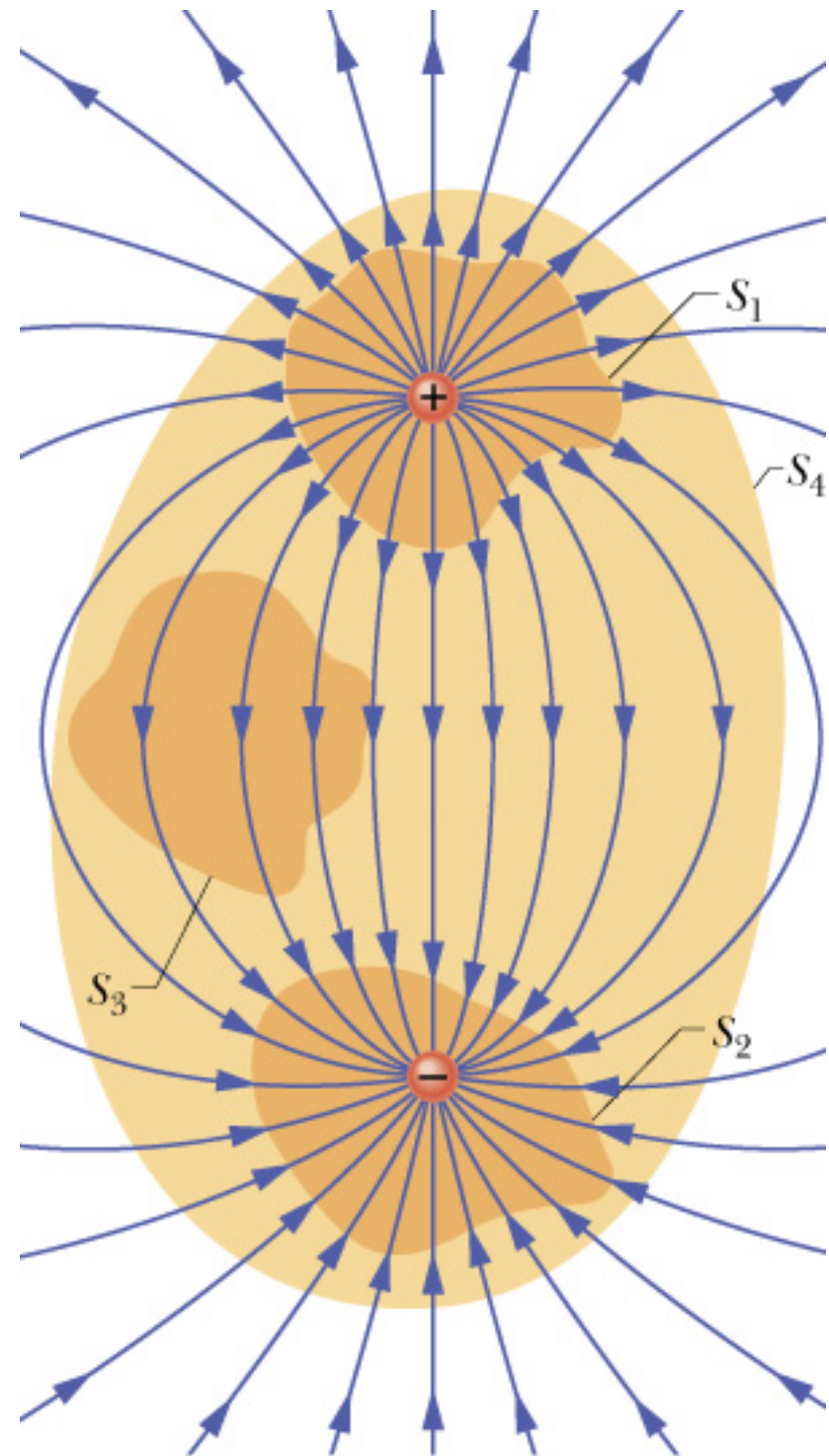
$q_{enc} < 0$  : the net flux is inward

The flux of electric field through a closed surface is proportional to the charge enclosed.

## Example of Gauss' Law:

Consider two point charges:

- Surface  $S_1$ : outward flux and positive charge
- $S_2$ : inward flux and negative charge
- $S_3$ : net flux is zero, no charge
- $S_4$ : net flux is zero, net charge is 0



**Sample Problem:** The cross section of a Gaussian surface  $S$  of five charged lumps of plastic and an electrically neutral coin is indicated. Calculate the net flux through  $S$  if  $q_1 = q_4 = +3.1 \text{ nC}$ ,  $q_2 = q_5 = -5.9 \text{ nC}$ , and  $q_3 = -3.1 \text{ nC}$ .

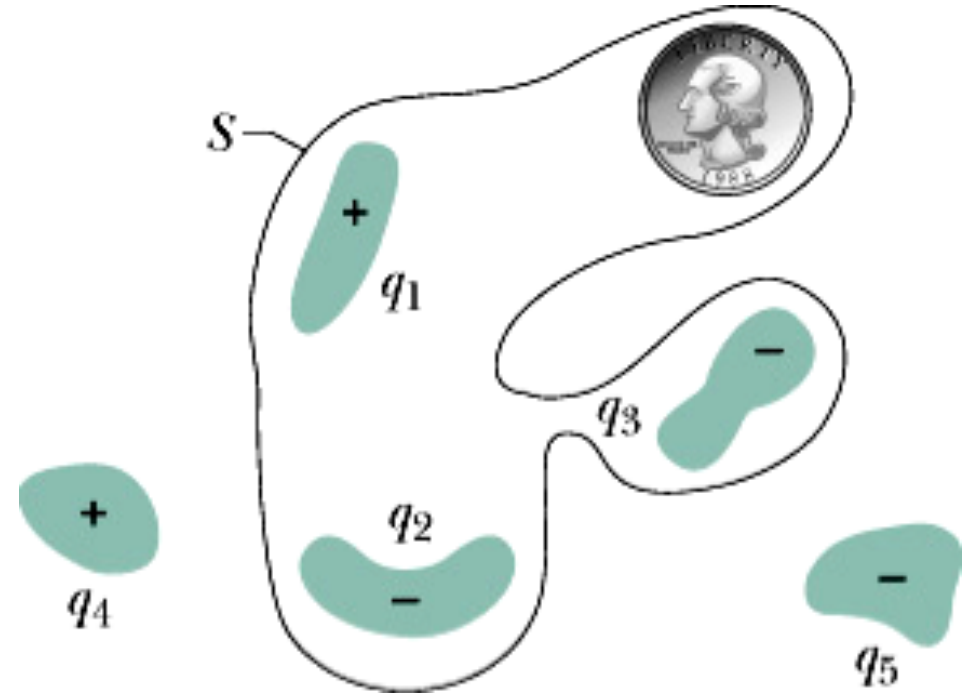
$$\epsilon_0 \Phi = q_{enc}$$

$$\Phi = \frac{q_{enc}}{\epsilon_0} = \frac{q_1 + q_2 + q_3}{\epsilon_0}$$

$$1 \text{ nC} = 10^{-9} \text{ C}$$

$$\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/(\text{Nm}^2)$$

$$\Phi = -670 \text{ Nm}^2/\text{C}$$





### 1.9.3. Gauss' Law and Coulomb's Law:

**Aim:** derive Coulomb's law from Gauss' law

- Divide the surface into differential areas  $dA$ :

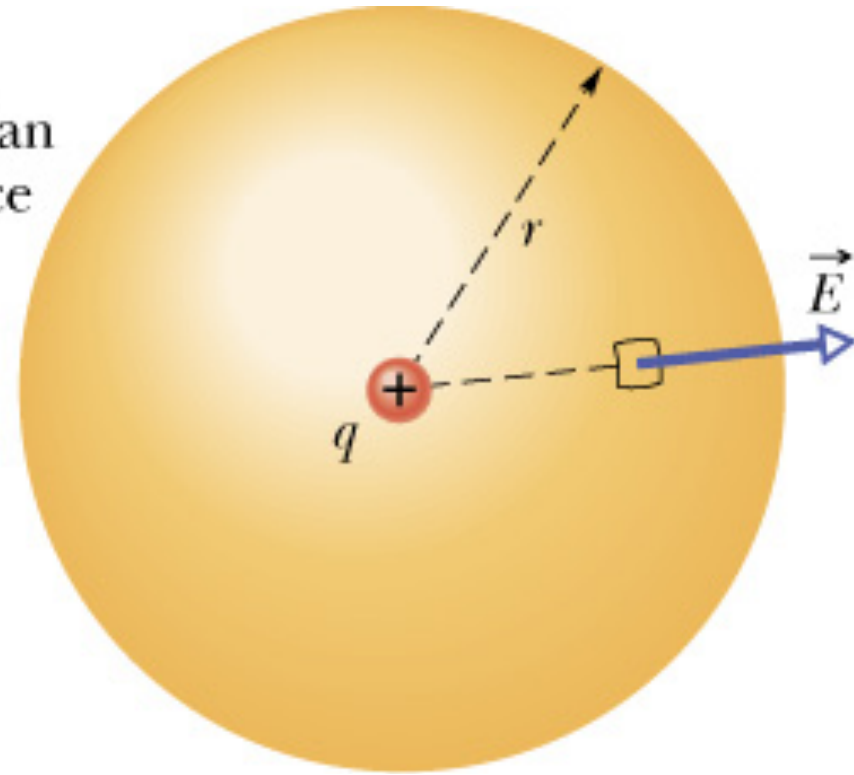
$$\epsilon_0 \oint \vec{E} d\vec{A} = \epsilon_0 \oint E dA = q$$

$E$  is constant at the surface

$$\epsilon_0 E \oint dA = \epsilon_0 E (4\pi r^2) = q$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} : \text{Coulomb's law}$$

Gaussian  
surface





## 1.10. Conductors in Electrostatic Equilibrium:

Concepts: When no net motion of charge occurs, this is called **electrostatic equilibrium**.

- net motion: Though the individual free electrons in the conductor are constantly in motion, in an isolated conductor there is no net motion of charge

### 1.10.1. Properties of isolated conductors:

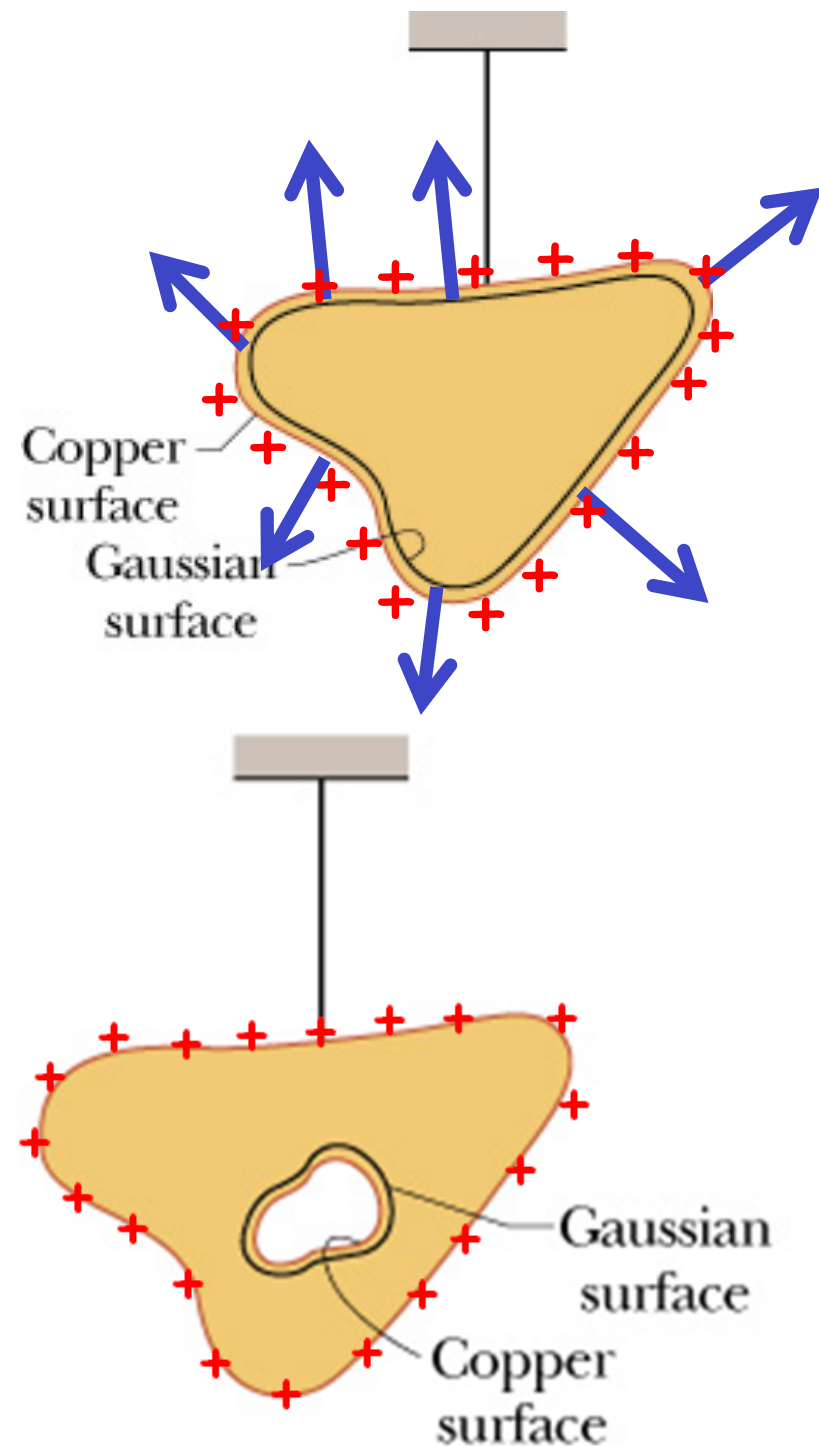
1. The electric field is zero everywhere inside the conductor
2. The electric field just outside a charged conductor is perpendicular its surface
3. Any excess charge on an isolated conductor must be entirely on its surface (derived from property 1)
4. On irregularly shaped conductors, charge accumulates at sharp curvatures

(a)  $E$  inside the conductor must be zero (property 1): if this were not so, the free  $e^-$  (always present in a conductor) would move in response and thus current would always exist (**not Gauss' law**)

→ If  $E$  is zero, the flux through the Gaussian surface (see figures), then the net charge inside the surface must be zero (**Gauss' law**)

→ All of the charge must be on the surface (property 3)

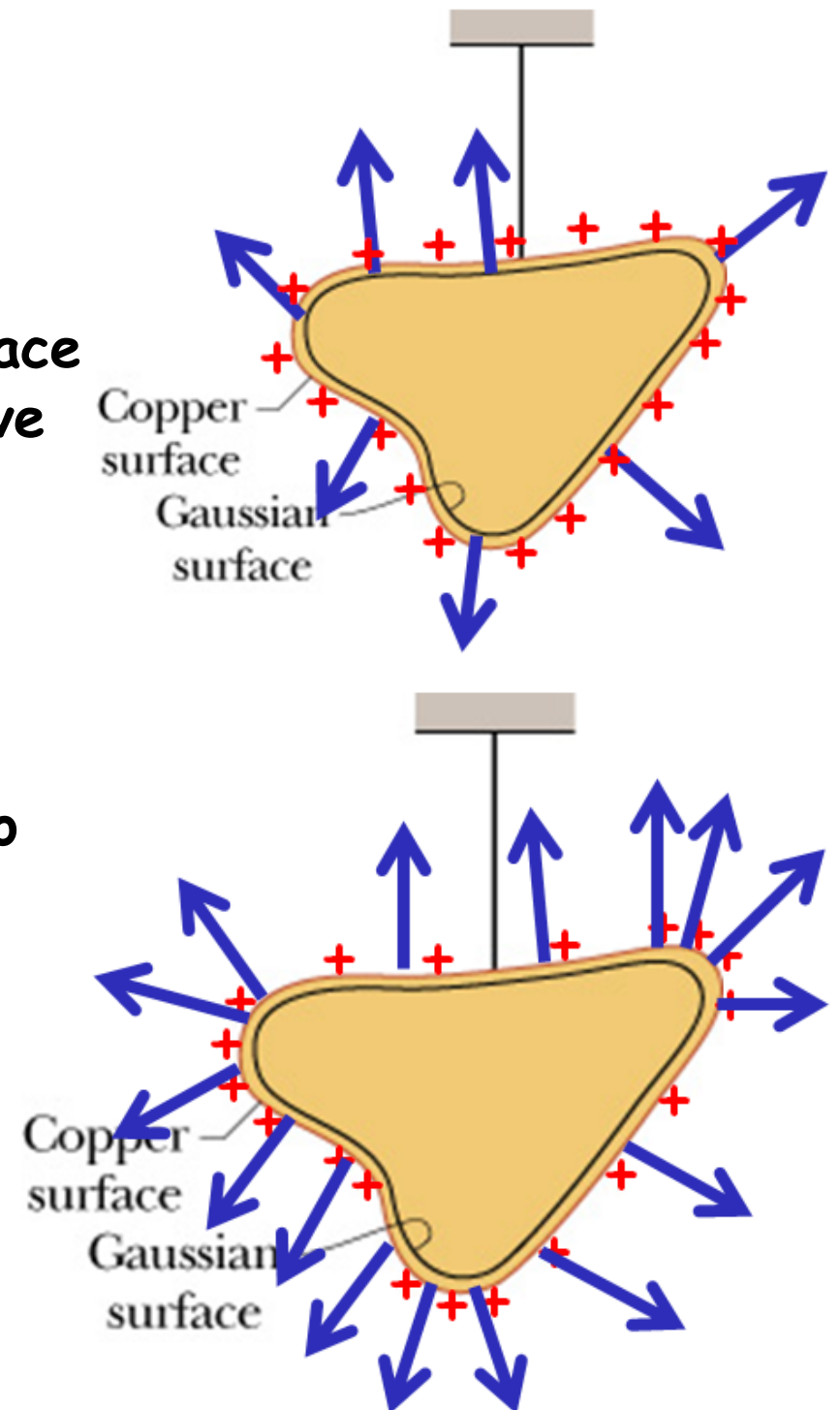
- For an isolated conductor with a cavity:
  - there is no net charge on the cavity walls
  - the charge remains on the outer surface



(b)  $E$  is perpendicular to the conductor's surface (property 2):  
if it were not then it would have a component parallel to the surface. This would exert forces on the surface charges, thus the charges would move

(c) Charges tend to accumulate at sharp curvatures (small radii) (property 4):

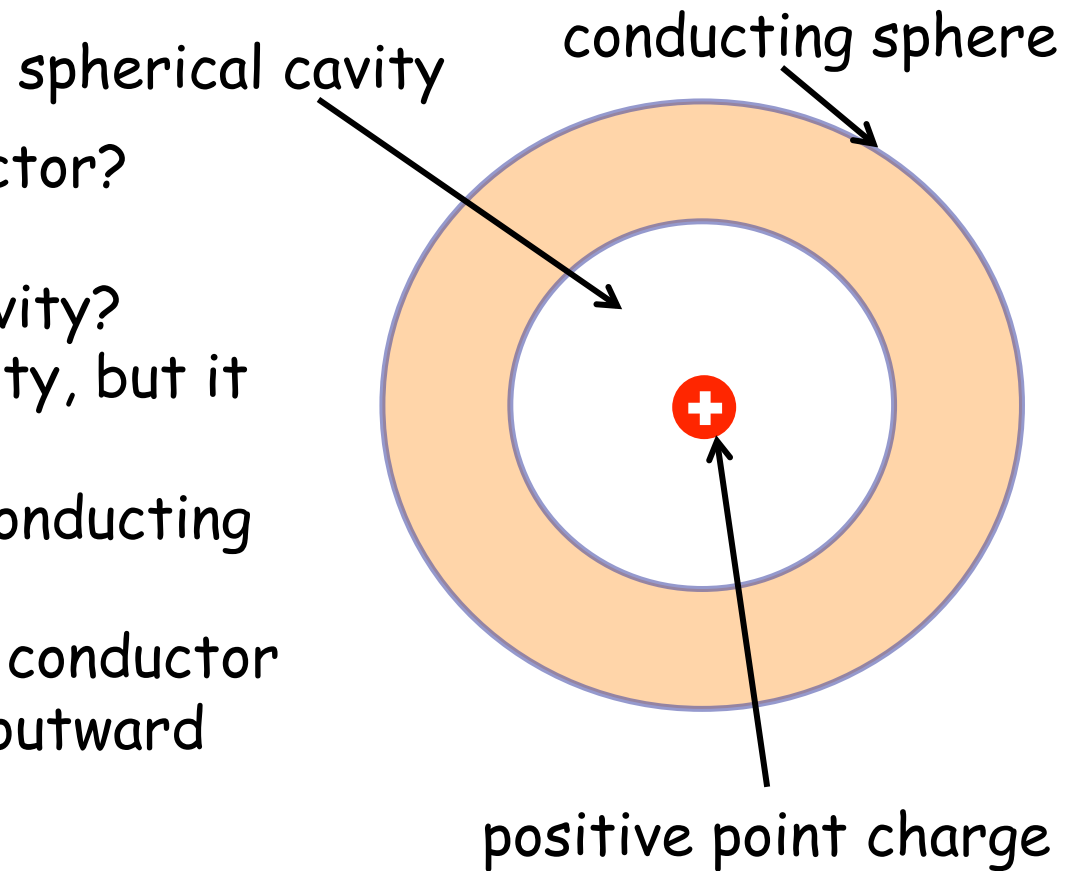
The sharper curvatures need more charges to flow into these regions to ensure that the parallel components of the surface are totally cancelled.



**Sample problem:** Consider a charge inside a conductor

What will happen when  
we add a charge inside a conductor?  
(using Gauss' law to figure out)

- a. Is  $E$  field still zero in the cavity?
- b.  $E$  field is not zero in the cavity, but it is zero in the conductor?
- c. Is  $E$  field zero outside the conducting sphere?
- d. Is  $E$  field the same as if the conductor were not there (i.e. radial outward everywhere)?

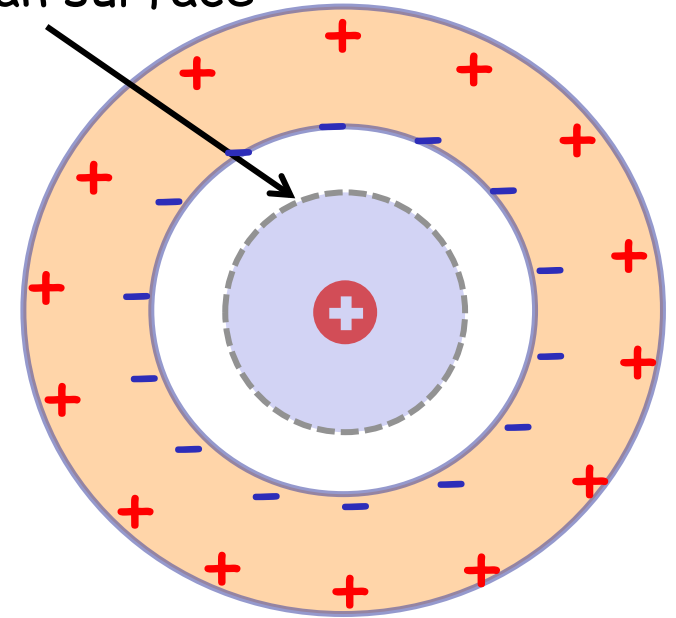


*a. Is  $E$  field still zero in the cavity?*

No, there is charge enclosed

$$\epsilon_0 \oint \vec{E} d\vec{A} = q_{enc}$$

Gaussian surface



*b.  $E$  field is not zero in the cavity, but it is zero in the conductor?*

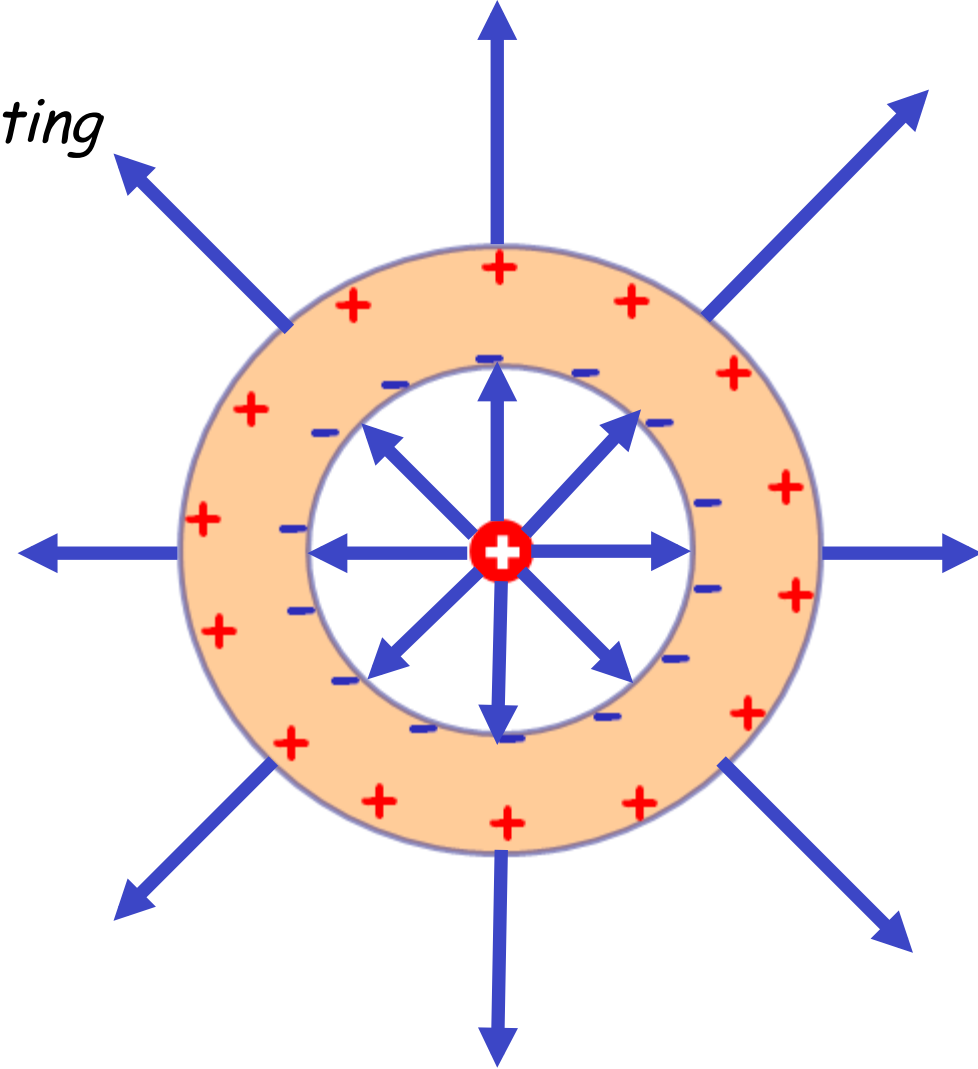
Yes (electrostatic equilibrium), if we enlarge the Gaussian surface to be inside the conductor the enclosed net charge is zero  $\rightarrow$  an equal and opposite charge induced on the inner surface and a positive charge induced on the outer surface

c. *Is  $E$  field zero outside the conducting sphere?*

No, the positive charge on the outer surface generates  $E$

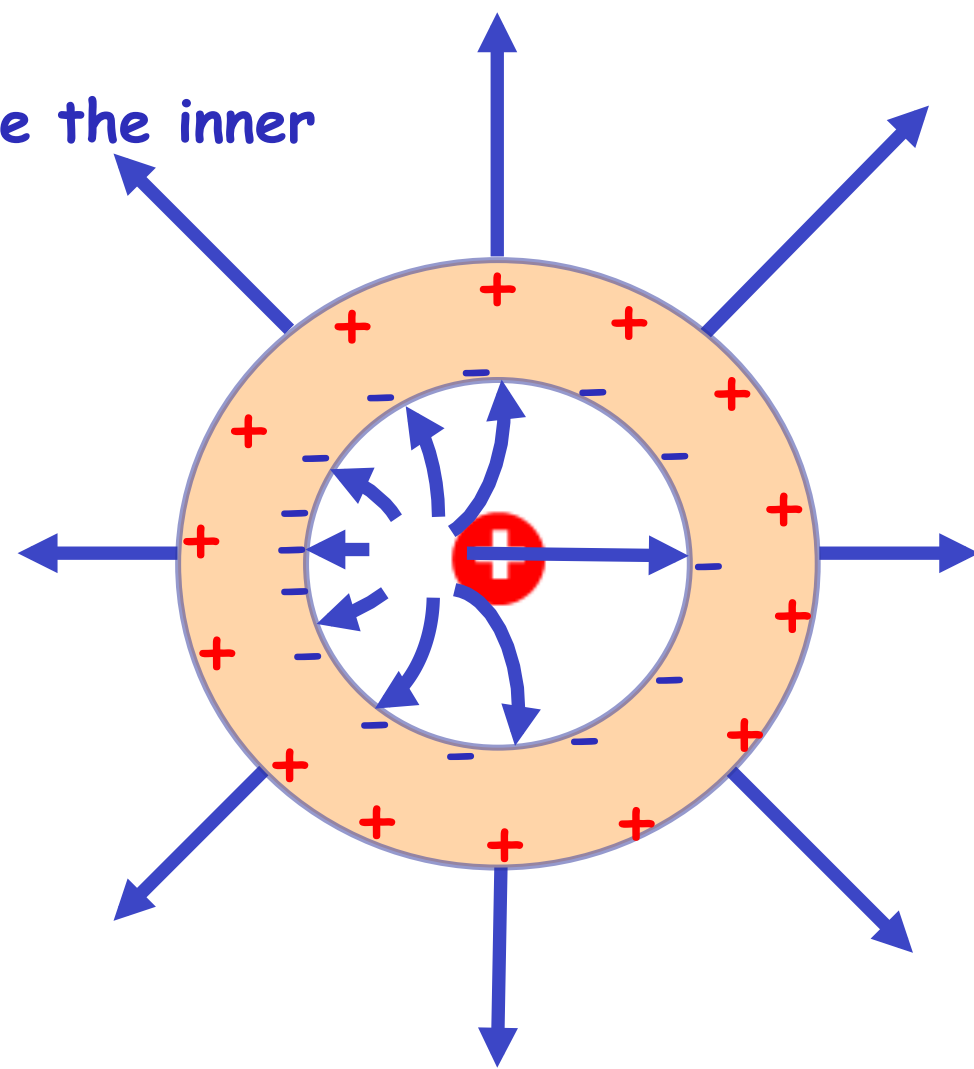
d. *Is  $E$  field the same as if the conductor were not there (i.e. radial outward everywhere)?*

Yes, the positive charge on the symmetric surface acts like a point charge at the center, so the field is the same



**Question:** What happens if we move the inner charge off the center?

- The induced negative will be redistributed on the inner wall (skewed distribution)
- The field lines are distorted but they remain perpendicular to the wall
- $E = 0$  inside the conductor, so the distribution of positive charge is uniform (the inner positive charge is shielded by the induced negative charge)





### 1.10.2. The external electric field due to a charged conductor:

**Goal:** Calculate  $E$  just outside the surface

**Method:** We consider a small section, we then consider a tiny cylindrical Gaussian surface to be embedded in the section (see the figure)

There is no flux through the internal end cap and the curved surface:

$$\epsilon_0 \oint \vec{E} d\vec{A} = q_{enc}$$

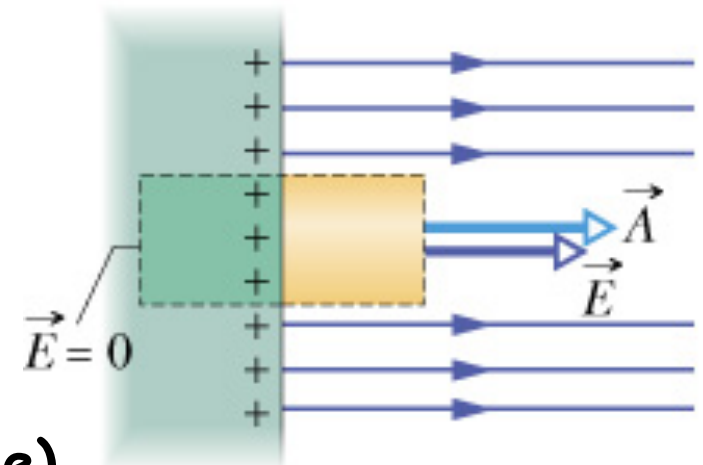
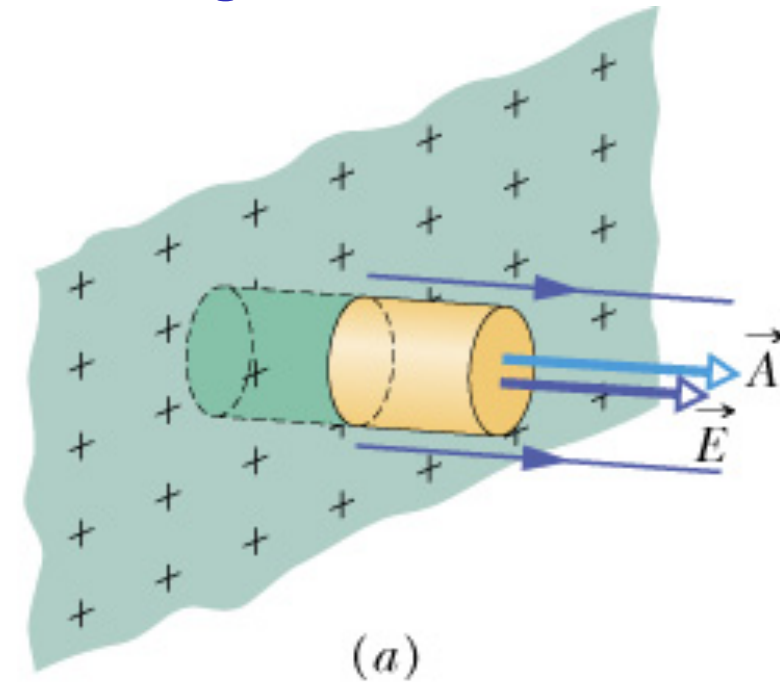
$E$  is perpendicular to the section and constant over the area  $A$ :

$$\epsilon_0 EA = q_{enc} = \sigma A$$

$$E = \frac{\sigma}{\epsilon_0}$$

(conducting surface)

$\sigma$ : surface charge density



### 1.10.3. Electric fields due to other charge geometries:

In this section, we use Gauss' law to determine  $E$  of other charge geometries

#### (a) Cylindrical Symmetry:

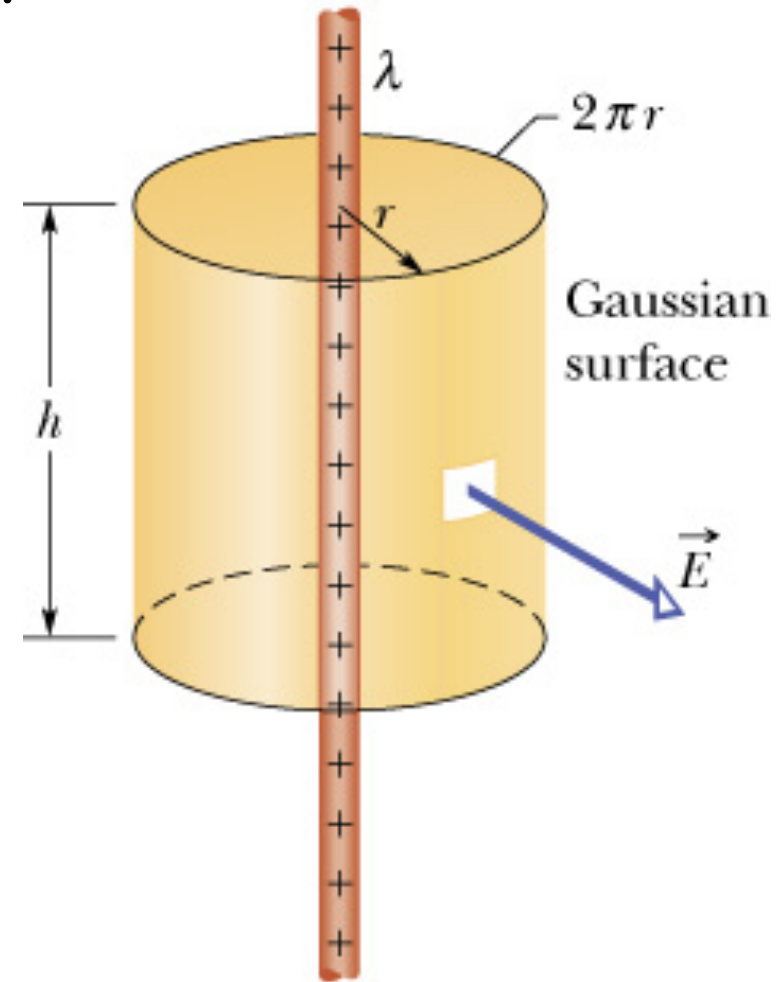
**Problem:** Consider a very long, uniformly charged, cylindrical plastic rod. Calculate  $E$  at a distance  $r$  from the axis.

- Due to the symmetry,  $E$  must have the same magnitude at the same  $r$
- Consider a Gaussian surface (closed)

coaxial with the rod:  $\epsilon_0 \oint \vec{E} d\vec{A} = q_{enc}$

$$\epsilon_0 E(2\pi r h) = \lambda h \Rightarrow$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$



(b) Planar Symmetry:

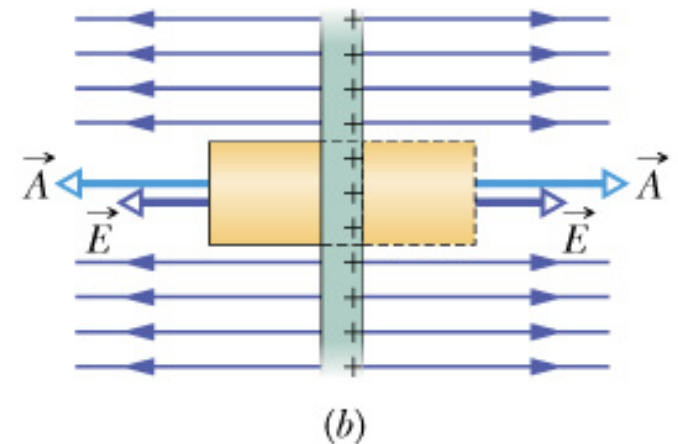
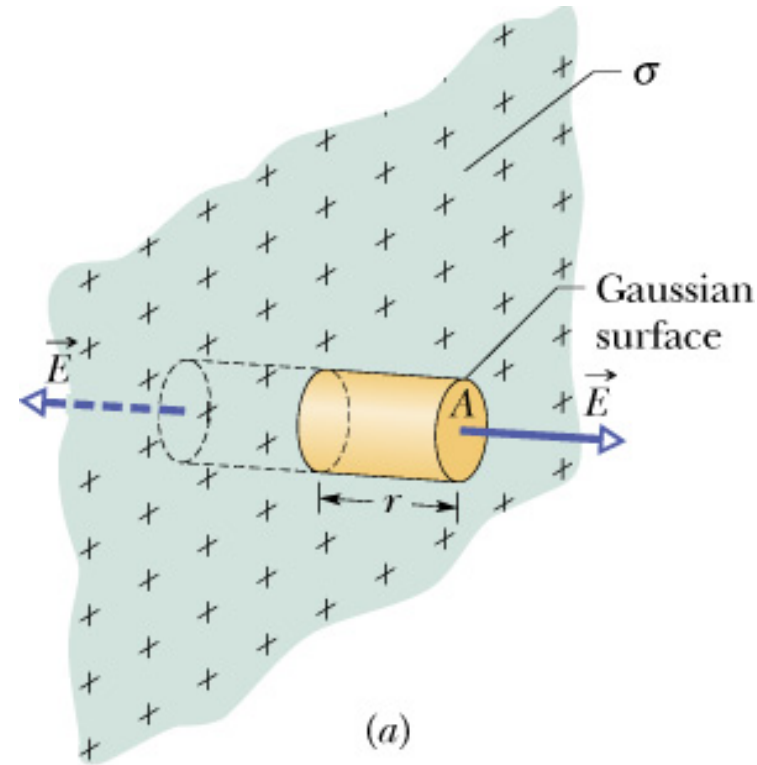
b1. Non-conducting sheet:

$$\epsilon_0 \oint \vec{E} d\vec{A} = q_{enc}$$

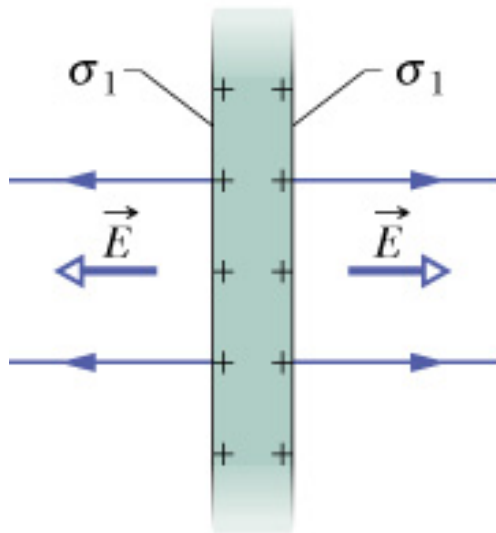
The electric field lines pierce the two Gaussian end caps:

$$\epsilon_0 (EA + EA) = \sigma A$$

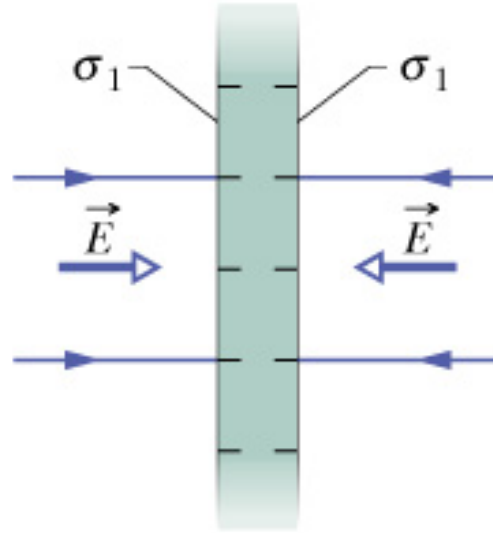
$$E = \frac{\sigma}{2\epsilon_0}$$



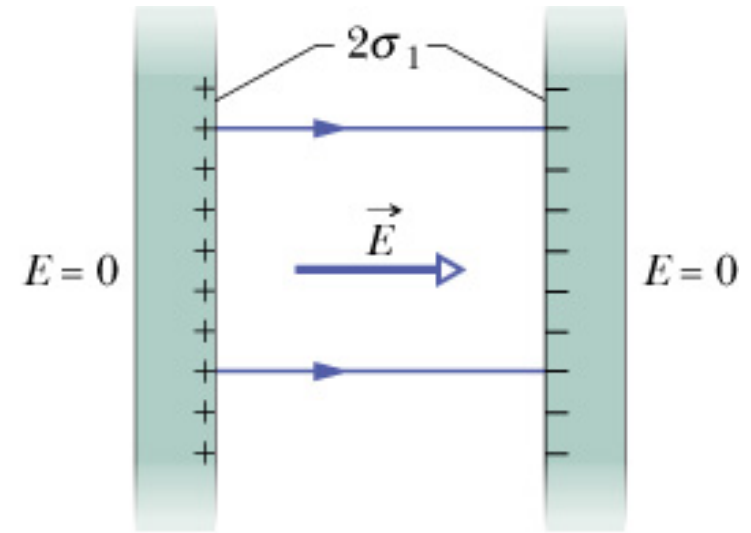
## b2. Two conducting plates:



(a)



(b)



(c)

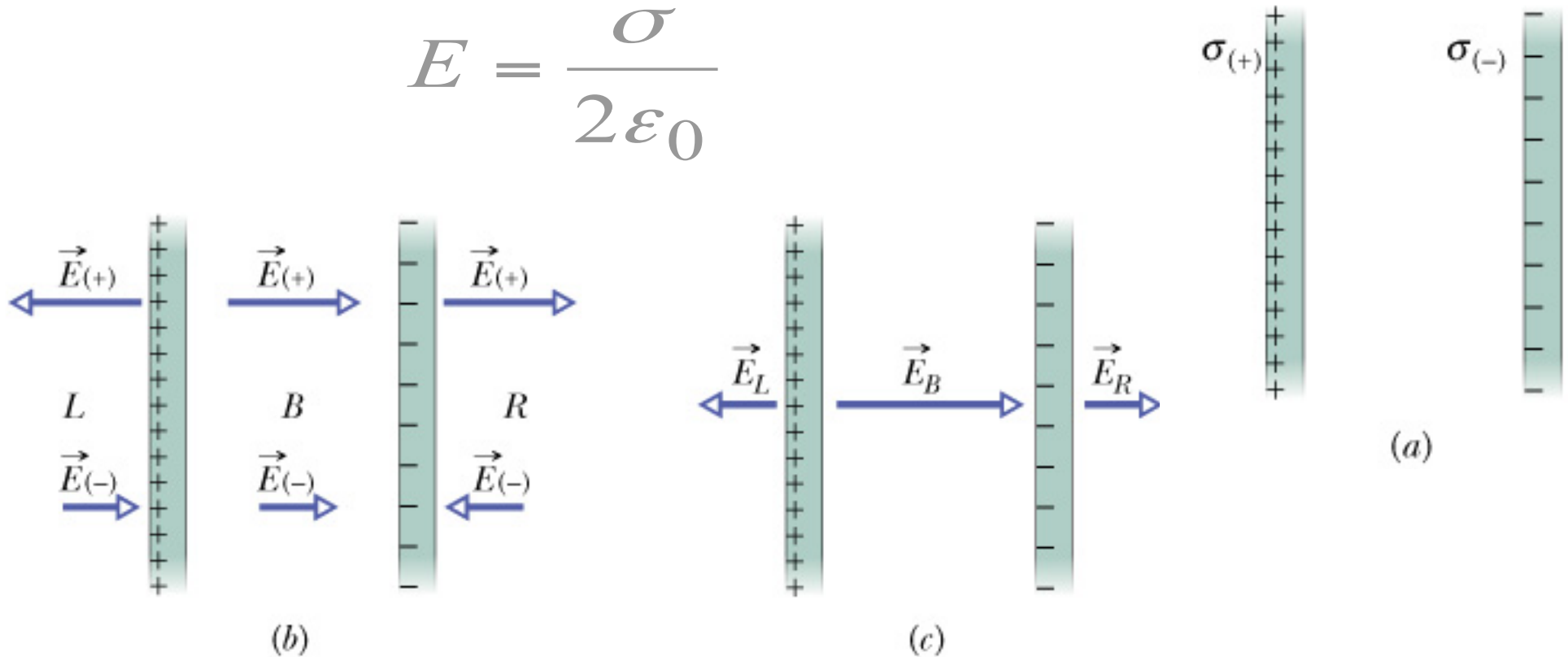
For a conducting plate, (a) or (b):  $E = \frac{\sigma_1}{\epsilon_0}$

For configuration (c), two plates:  
 $\sigma$ : the new surface charge density

$$E = 2 \frac{\sigma_1}{\epsilon_0} = \frac{\sigma}{\epsilon_0}$$

**Checkpoint:** Find the electric field of two large, parallel, nonconducting sheets: (a) to the left of the sheets, (b) between the sheets, and (c) to the right of the sheets if  $\sigma(+)>\sigma(-)$

$$E = \frac{\sigma}{2\epsilon_0}$$



(a):  $E_{(+)}-E_{(-)}$ ; (b):  $E_{(+)}+E_{(-)}$ ; (c)  $E_{(+)}-E_{(-)}$

The direction of electric field for three cases shown in Fig. (c)

### (c) Spherical Symmetry:

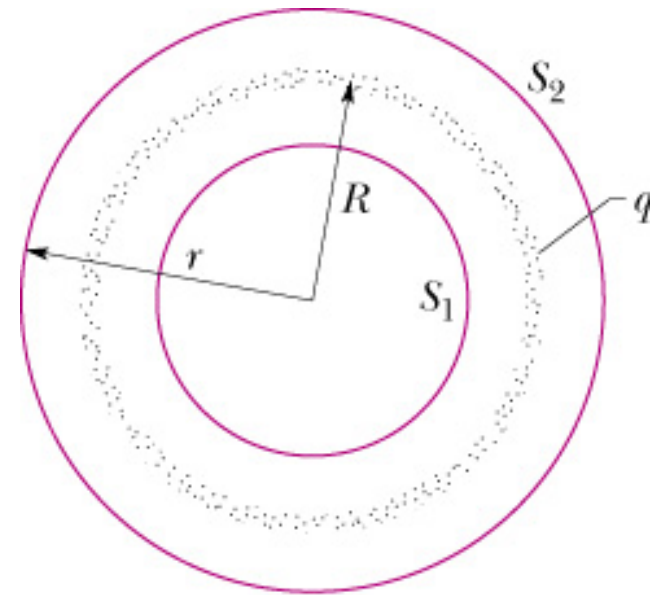
Recall the shell theorem (see lecture 1):

- ✚ A shell of uniform charge attracts or repels a charged particle that is outside the shell as if all the shell's charge were concentrated at its center
- ✚ If a charged particle is located inside a shell of uniform charge, there is no net electrostatic force on the particle from the shell

b1. A thin, uniformly charged spherical shell:

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad (r \geq R)$$

$$E = 0 \quad (r < R)$$



### b1. A uniformly charged sphere:

The charges are evenly distributed throughout the volume, its density:

$$\rho = \frac{q}{\frac{4}{3}\pi R^3}$$

- The electric field at a distance  $r \leq R$ :

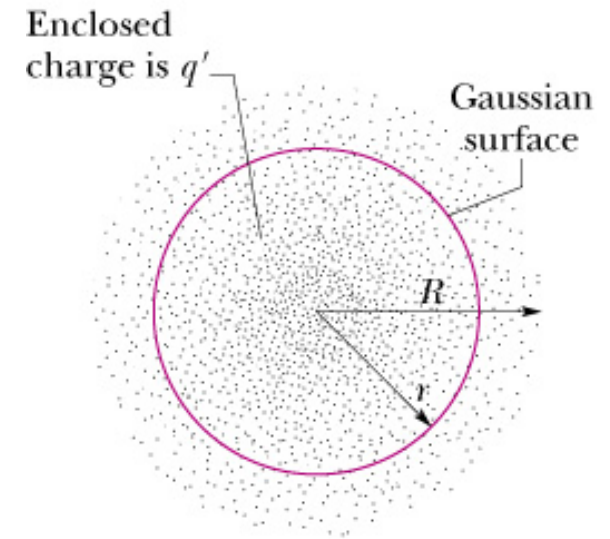
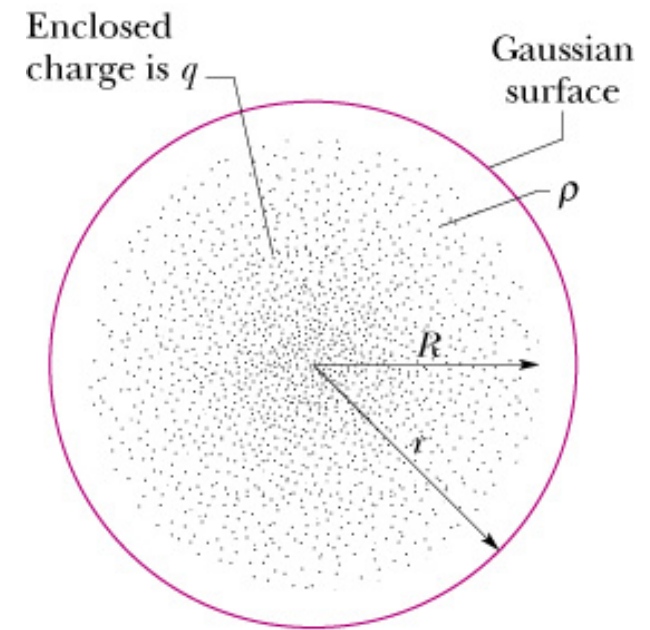
$$E = \frac{1}{4\pi\epsilon_0} \frac{q'}{r^2} \quad (r \leq R)$$

$$q' = \rho \frac{4}{3}\pi r^3 = q \frac{r^3}{R^3}$$

$$E = \left( \frac{q}{4\pi\epsilon_0 R^3} \right) r \quad (r \leq R)$$

- The electric field at a distance  $r > R$ : the charge sphere acts like a point charge at the center

$$E = \frac{q}{4\pi\epsilon_0 r^2} \quad (r > R)$$





**Homework: 1, 7, 12, 13, 14, 17, 21, 22, 24, 35, 36, 39,  
43, 44, 51, 52 (page 622-626)**