VIETNAM NATIONAL UNIVERSITY - HCMC
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CHAPTER 1. FUNCTIONS, LIMITS AND CONTINUITY

(CALCULUS 1. References: Textbook by J. Stewart)

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CONTENTS

1.1. An introduction to Calculus

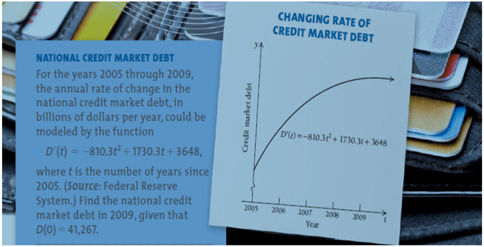
Calculus generally considered to have been founded in the 17th century by I. Newton and G. Leibniz. The fundamental objects that we deal with in calculus are functions.

The electrical power produced by a wind turbine can be estimated by a function. We will explore this function in this Chapter.

1.1. An introduction to Calculus

- Calculus has two major branches, differential calculus (concerning rates of change and slopes of curves), and integral calculus (concerning accumulation of quantities and the areas under curves). Both are based on the concept of limits.

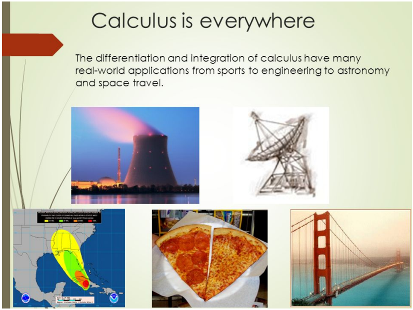
- Calculus are used in many fields: Mathematics, Physics, Engineering, Biotechnology, Computing science, Data science, and other sciences.



1.1. An introduction to Calculus

- Calculus give us a way to construct quantitative models in practice, and to deduce the predictions of such models.

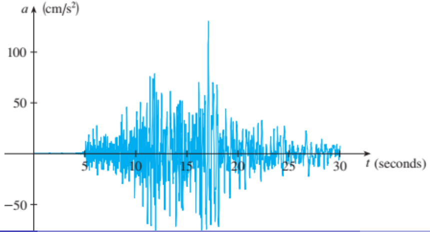
- This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming, combining them, and applications.



1.2 Functions and graphs: Straight Lines.

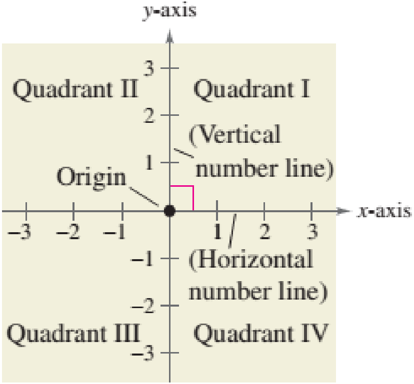
Functions arise whenever one quantity depends on another. There are four ways to represent a function:

- By a formula (algebraically). Ex: The area of a circle is $A(r) = \pi r^2$, where r is the radius.
- By a table of data (numerically). Ex: A data table representing the population $P(t)$ vs. t for some years t.
- By a description in words (verbally).
- By a figure/graph (visually). Ex: The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Below shows a graph generated by seismic activity during the Northridge earthquake (Los Angeles, 1994).



1.2 Functions and graphs: Straight Lines.

- Coordinates and Graphs: O is the origin, Ox is the x-axis, Oy is the y-axis (x, y) are the x- and y-coordinates.

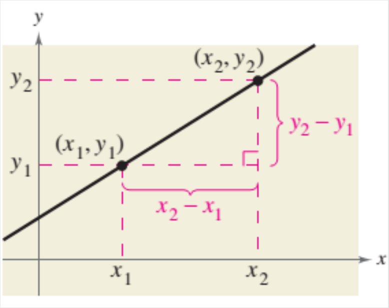


1.2 Functions and graphs: Straight Lines.

- Consider any two points (x_1, y_1) and (x_2, y_2) on a straight line. On the interval $[x_1, x_2]$. We call $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ the change in x and y, respectively.
- The slope (or gradient) of the line through (x_1, y_1) and (x_2, y_2) $(x_1 \neq x_2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope m tell us the rate of change of y with respect to x.



1.2 Functions and graphs: Straight Lines.

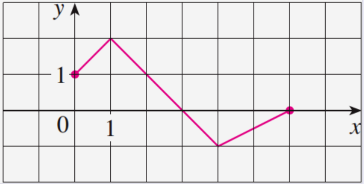
Example

For the graph shown below, state the slope

(a) for $0 < x < 1$

(b) for $1 < x < 4$

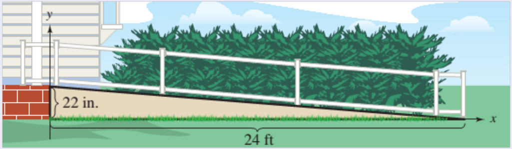
(c) for $4 < x < 6$



1.2 Functions and graphs: Straight Lines.

Exercise

The maximum recommended slope of a wheelchair ramp is $1/12$. A business is installing a wheelchair ramp that rises 22 inches over a horizontal length of 24 feet. Is the ramp steeper than recommended? Given 1 foot=12 inches.



Solution:

The horizontal length of the ramp is 24 feet or 288 inches:

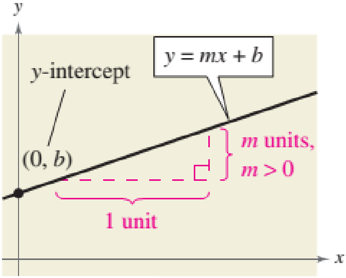
$$|Slope| = \frac{|vertical\ \ change|}{|horizontal\ \ change|} = \frac{22}{288} \approx 0.076 < \frac{1}{12}.$$

1.2 Functions and graphs: Straight Lines.

- Suppose a straight line crosses the y -axis at $y = b$. We call b the y-intercept.
- For any point (x, y) on the line, the slope is defined by

$$m = \frac{y - b}{x - 0}$$

- This leads to the *slope-intercept form*: $y = mx + b$

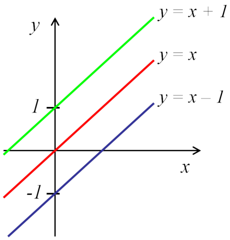


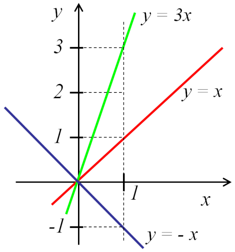
Positive slope, line rises.

1.2 Functions and graphs: Straight Lines.

The slope-intercept form is very convenient for graph-sketching.

Q: (a) What if $m_1 = m_2$? (b) What if $m_1 m_2 = -1$?[1]





Other forms

- Point-Slope Form: $y = y_1 + m(x - x_1)$.
- Two Point Form: For a line passing through points (x_1, y_1) and (x_2, y_2):

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad (= m)$$

[1](a) Parallel; (b) Perpendicular.

1.2 Functions and graphs: Straight Lines.

Exercise

The equation of the line passing through the point $(-2, 0)$ and $(3, 1)$.

Hint: Slope $m = 1/5$.

Exercise

Find the slope-intercept forms of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.

Answer: (a) $y = 2/3x - 7/3$, (b) $y = -3/2x + 2$.

1.2 Functions and graphs: Straight Lines.

Exercise

A kitchen appliance manufacturing company determines that the total cost in dollars of producing x units of a blender is

$$C = 25x + 3500$$

Describe the practical significance of the y-intercept and slope of this line.

Solution:

The y-intercept tells you that the cost of producing zero units is \$3500. This is the fixed cost of production.

The slope of tells you that the cost of producing each unit is \$25. [Economists call the cost per unit the marginal cost.] So, the cost increases at a rate of \$25 per unit.

1.2 Functions and graphs: Linear model

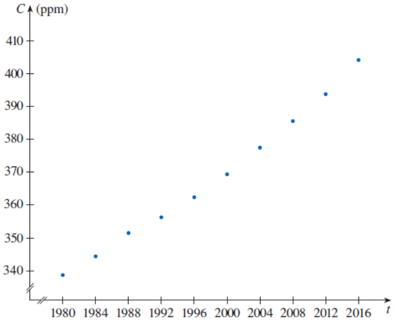
Example

Table 1 lists the average carbon dioxide level in the atmosphere, measured in parts per million at Mauna Loa Observatory from 1980 to 2016. Use the data in Table 1 to find a model for the carbon dioxide level.

Table 1

Year	CO_2 level (in ppm)	Year	CO_2 level (in ppm)
1980	338.7	2000	369.4
1984	344.4	2004	377.5
1988	351.5	2008	385.6
1992	356.3	2012	393.8
1996	362.4	2016	404.2

1.2 Functions and graphs: Linear model



Solution

We use the data in Table 1 to make the scatter plot as in the figure above where t represents time (in years) and C represents the CO_2 level. We find the equation of the line that passes through the first and last data points.

The slope is $m = \dfrac{404.2 - 338.7}{2016 - 1980} = 1.819.$

Thus, $C - 338.7 = 1.819\,(t - 1980)$ or $C = 1.819t - 3262.92.$

1.2 Functions and graphs: Linear model

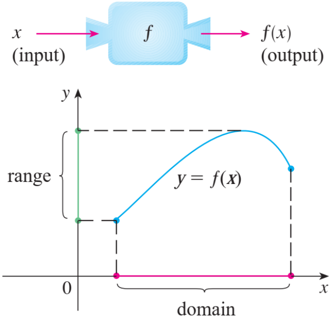
Exercises

At a certain place, the CO_2 concentration in the atmosphere was measured to be 338.7 ppm in the year 1980 and 404.2 ppm in 2016. Assume a linear model. Find an equation for the CO_2 concentration C (in ppm) as a function of time t (in years). Use your equation to predict the CO_2 concentration in 2024.

Functions: Domain and Range

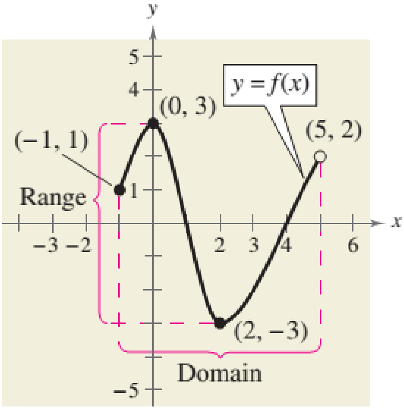
Definition

A function of a variable x is a rule f that assigns to each value of x in a set D a **unique** number $f(x)$ in a set E, called the value of the function at x. [We read "f(x)" or "f of x".]



The set D is called the domain and the range is the set of all possible values of $f(x)$ as x varies throughout the domain.

Functions: Domain and Range



Example

The domain of the function $y = f(x) = \sqrt{x}$ is the set $D = \{x \in \mathbb{R} : x \geqslant 0\}$, and the range of this function is $[0, \infty)$ (why?).

1.2. Functions and graphs

Example: Piecewise Defined Functions

The functions in the following example is defined by different formulas in different parts of their domains. Such function is called piecewise defined function.

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq 1 \\ x^2 & \text{if } x > 1 \end{cases}$$

Exercises

The logistic function (or sigmoid function) is used in Machine Learning, Deep Learning/AI, Biology, etc. It is given by $f(x) = \dfrac{1}{1 + e^{-(a+bx)}}$, where $a, b \in \mathbb{R}$. Find the domain and the range of f.

1.2. Functions and graphs

Exercises

Express the area A of a rectangle as a function of the length x if the length of the rectangle is twice its width.

Exercises

A Boeing 747 crosses the Atlantic Ocean (3000 miles) with an airspeed of 500 miles per hour. The cost C (in dollars) per passenger is given by

$$C(x) = 100 + \frac{x}{10} + \frac{36,000}{x}$$

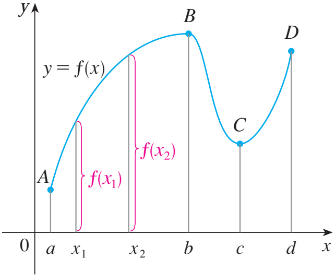
where x is the ground speed (airspeed $\pm$ wind).
(a) What is the cost per passenger for quiescent (no wind) conditions?
(b) What is the cost per passenger with a head wind of 50 miles per hour?

1.2. Functions and graphs

Increasing and Decreasing Functions

A function is called **increasing** on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.
A function is called **decreasing** on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.

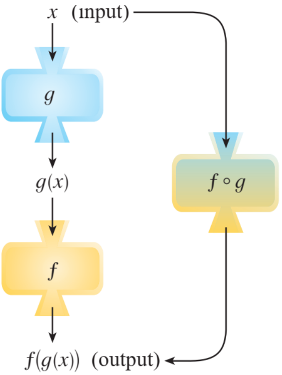
The following function is said to be increasing on the interval (a, b), decreasing on (b, c), and increasing again on (c, d).



Composite functions

Definition

Given two functions f and g, the composite function (also called the composition of f and g) is $f \circ g$ defined by $(f \circ g)(x) = f(g(x))$.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Composite functions

Example

Given $f(x) = 3x^2, g(x) = x - 1$ find each of the following:
(a) $(f \circ g)(x)$.
(b) $(g \circ f)(x)$.

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(x-1) = 3(x-1)^2$
(b) $(g \circ f)(x) = g(f(x)) = g\left(3x^2\right) = 3x^2 - 1$.

In general, $f \circ g \neq g \circ f$. Note that the notation $(f \circ g)(x)$ means that the function g is applied first and then f is applied second.

Composite functions

Exercises

1. Find $f \circ g \circ h$ where $f(x) = \dfrac{1}{x}$, $g(x) = x + 1$, and $h(x) = e^{-x}$.

2. If $T(x) = \frac{1}{\sqrt{1+\sqrt{x}}}$, give an example for f, g, and h such that $f \circ g \circ h = T$.

Composite functions

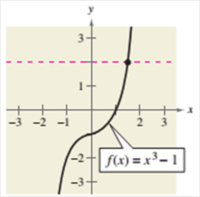
Exercises

3. The spread of a contaminant is increasing in a circular pattern on the surface of a lake. The radius of the contaminant can be modeled by $r(t) = 5.25\sqrt{t}$, where is the radius in meters and is the time in hours since contamination.
(a) Find a function that gives the area of the circular leak in terms of the time since the spread began.
(b) Find the size of the contaminated area after 36 hours.
(c) Find when the size of the contaminated area is 6250 square meters.

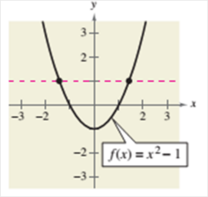
One-to-one functions

Definition

A function f is injective (or one-to-one, or an injection) if for every $y \in E$, there is **at most** one $x \in D$ such that $f(x) = y$.



One-to-one



Not one-to-one

Example The function $f(x) = x^2, x \in \mathbb{R}$ is not one-to-one because both $f(-2) = 4$ and $f(2) = 4$. However, we can turn $f(x) = x^2$ into a one-to-one function if we restrict ourselves to $0 \leqslant x < \infty$.

Inverse functions

Definition

Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(f \circ g)(x) = (g \circ f)(x) = x$$

then we say that $f(x)$ and $g(x)$ are inverses of each other.

- More specifically we will say that $g(x)$ is the inverse of $f(x)$ and denote it by $g = f^{-1}$, that is,

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$$

- Given $f(x)$, how to find $f^{-1}(x)$?
- Method: (1) From $y = f(x)$, we solve for x to find $f^{-1}(y)$. (2) Then, interchange x and y.

Inverse functions

Example

Given $f(x) = 3x - 2$. Find $f^{-1}(x)$.

Solution

We have

$$y = f(x) = 3x - 2.$$

We solve for x:

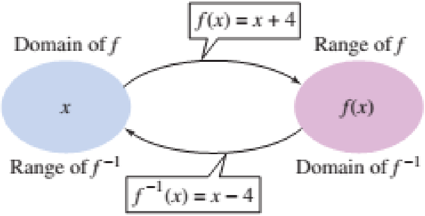
$$x = \frac{1}{3}(y + 2).$$

Thus, $f^{-1}(y) = \frac{1}{3}(y + 2)$. We then interchange x and y to obtain

$$f^{-1}(x) = \frac{1}{3}(x + 2).$$

Re-check: $f\left(f^{-1}(x)\right) = f\left(\frac{1}{3}(x+2)\right) = 3\left[\frac{1}{3}(x+2)\right] - 2 = x.$

Inverse functions



Exercises

1. Let $f(x) = x + 4$. Find $f^{-1}(x)$.
2. Let $f(x) = \sqrt{x-3}$. Find $f^{-1}(x)$ and the domain of f^{-1}.

Graph of a function

The graph of a function

The graph of a function $f(x)$ consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. Namely, it is the set $\{(x, f(x)) : x \in D\}$.

The graph of a function f is a useful picture of its behavior. If (x, y) is a point on the graph, then $f(x)$ is the height of the graph above the point x. The height may be positive or negative, depending on the sign of $f(x)$.

1.2. Functions and graphs

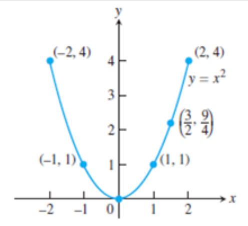
Example: the graph of a function

Graph the function $y = x^2$ on $[-2, 2]$.

Solution

Make a table of xy-pairs that satisfy the equation $y = x^2$. Plot the points (x, y) whose coordinates appear in the table, and draw a smooth curve through the plotted points.

x	$y = x^2$
-2	4
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

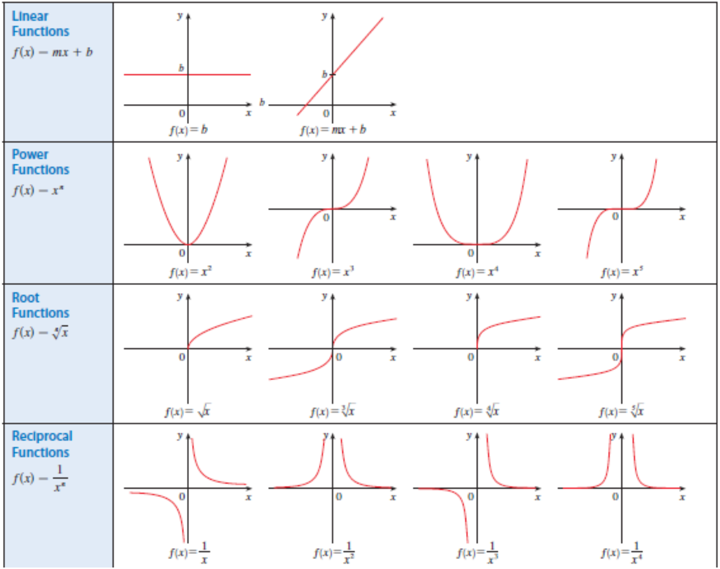


Some essential functions

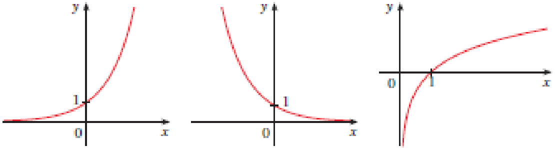
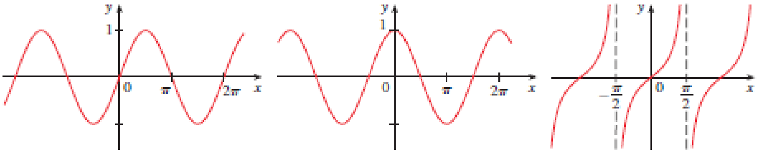
- Polynomials: $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$.
- Power function: $y = x^a$.
- Trigonometric functions: *sine* (sin), *cosine* (cos), *tangent* (tan), ...
  And
  $$\operatorname{cosec} x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x} ...$$
- Exponential function: $y = a^x$ (a is the base). The most common exponential function (often called the exponential function) is $f(x) = e^x$. e is an irrational number called the exponential constant, $e = 2.7182818$. We will study e in detail later on.
- If $x = a^y$ then $y = log_a x$. This is a logarithmic function. a is again called the base.
  Note: $\ln x = log_e x$.

Some essential functions

Table 3 Families of Essential Functions and Their Graphs

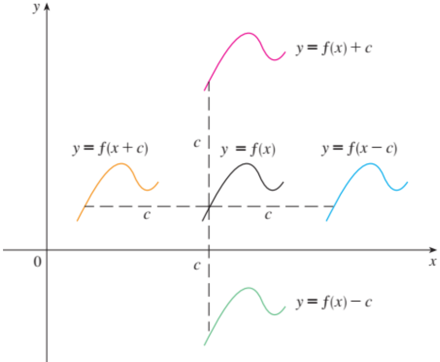


Some essential functions

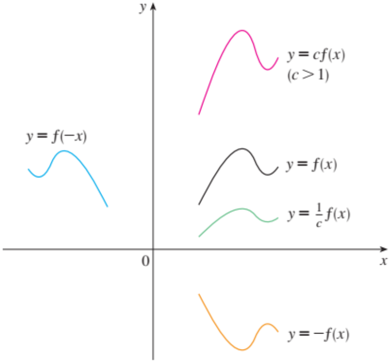
Exponential and Logarithmic Functions $f(x) = b^x$ $f(x) = \log_b x$	 $f(x) = b^x$ $(b > 1)$ $f(x) = b^x$ $(b < 1)$ $f(x) = \log_b x$ $(b > 1)$
Trigonometric Functions $f(x) = \sin x$ $f(x) = \cos x$ $f(x) = \tan x$	 $f(x) = \sin x$ $f(x) = \cos x$ $f(x) = \tan x$

New Functions from Old Functions

Suppose we know the graph of a certain function. By some simple transformations, we can quickly obtain the graphs of some related functions.



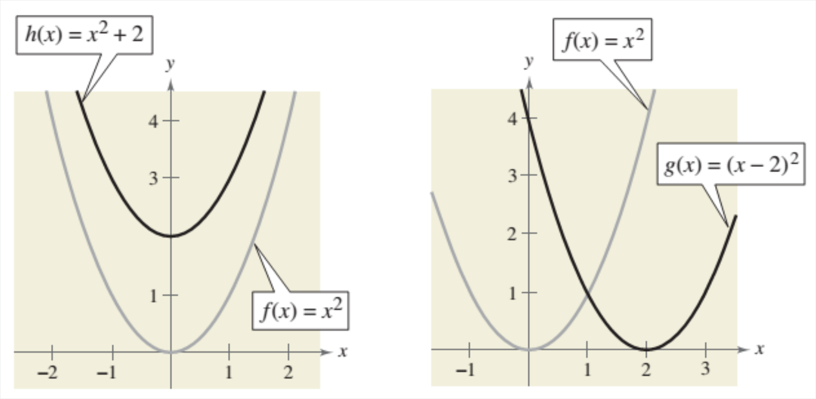
Translating the graph of f



Stretching and reflecting the graph of f

New Functions from Old Functions

- $y = f(x) + c$, shift the graph of $y = f(x)$ up by c units.
- $y = f(x) - c$, shift the graph of $y = f(x)$ down c units.
- $y = f(x + c)$, shift the graph of $y = f(x)$ left c units.
- $y = f(x - c)$, shift the graph of $y = f(x)$ right c units



New Functions from Old Functions

Example

Sketch the graphs
(a) $y = x^2$
(b) $y = x^2 - 1$
(c) $y = (x - 1)^2$
(d) $y = (x - 1)^2 - 3$.

New Functions from Old Functions

To obtain the graph of

- $y = cf(x)$, stretch $y = f(x)$ vertically by a factor c,
- $y = f(cx)$, compress $y = f(x)$ horizontally by a factor c.
- $y = -f(x)$ reflect the graph of $y = f(x)$ about the x-axis.
- $y = f(-x)$ reflect the graph of $y = f(x)$ about the y-axis.

Example

Sketch the graphs
(a) $y = 2\sin x$,
(b) $y = \sin(\pi x)$,
(c) $y = 2e^{-x}$.

Symmetry. Even Functions. Odd Functions

Definition

If f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function.

For example, the function $f(x) = x^2$ is even.
The graph of an even function is symmetric with respect to the y-axis.

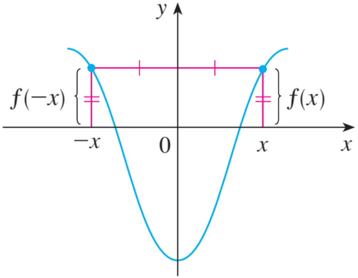
Definition

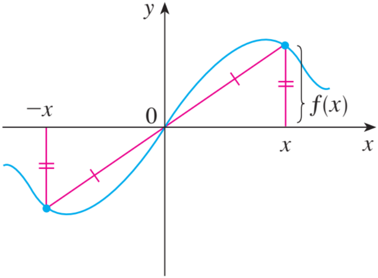
If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function.

For example, the function $f(x) = x^3$ is odd.
The graph of an odd function is symmetric about the origin.

Symmetry. Even Functions. Odd Functions

An even function (left) and an odd function (right):





1.2. Functions and graphs

Not every curve in the coordinate plane can be the graph of a function!

The Vertical Line Test for a Function

A curve in the $xy-$plane is the graph of a function iff no vertical line intersects the curve more than once.

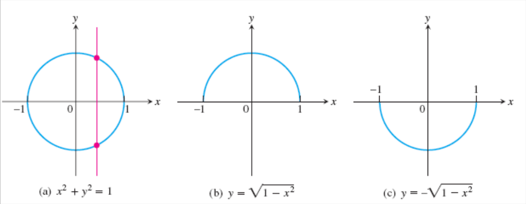
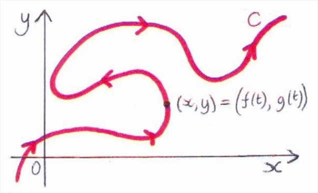


Figure: (a) The circle is not the graph of a function. (b) The upper semicircle is the graph of a function $y = \sqrt{1 - x^2}$ (c) The lower semicircle is the graph of a function $y = -\sqrt{1 - x^2}$.

Parametric Curves

Imagine that a particle moves along the curve C. C might not be described by an equation of the form $y = f(x)$ (why not?).



But the $x-$ and $y-$ coordinates of the particle are both functions of time: $x = f(t)$ and $y = g(t)$. t is called a parameter. C is called a parametric curve. $x = f(t)$ and $y = g(t)$ are the parametric equations of C. We can also write $c(t) = (f(t), g(t))$.

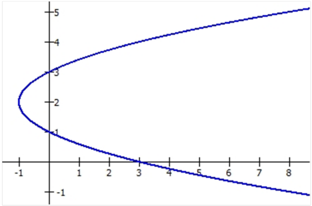
Parametric Curves

Example

Sketch the curve defined by $x = t^2 - 2t$, $y = t + 1$.

We construct a table of values and thus plot the curve:

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5



Parametric Curves

Example

What curve is represented by the following parametric equations?

$$x = \cos t, y = \sin t, 0 \leq t \leq \pi.$$

Hint: Use $\sin^2 t + \cos^2 t = 1$. Think about the equation of the unit circle.

Example

Eliminate the parameter to find a Cartesian equation of the following parametric equations $x = \sqrt[3]{t} - 1, y = 2t^2 + t + 1$.

2.1. Limits. Definitions. One-sided Limits

In this section, we define limits and study them using numerical and graphical techniques. We begin with the following question: How do the values of a function $f(x)$ behave when x approaches a number c, whether or not $f(c)$ is defined?

Example: Numerical and Graphical Approach

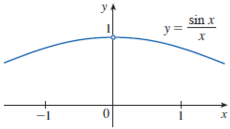
Consider the function $f(x) = \frac{\sin x}{x}$.
Note that $f(0)$ is not defined (undefined) but $f(x)$ can be computed for values of x close to 0.
We use the phrases "x approaches 0" or "x tends to 0" to indicate that x takes on values (both positive and negative) that get closer and closer to 0. Notation: $x \to 0$.

2.1. Limits. Definitions. One-sided Limits

Example (Cont.)

The following table gives the impression that $f(x)$ gets closer and closer to 1 as x approaches 0 through positive and negative values (from both sides: the left or the right).

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983



We say that $f(x)$ approaches or converges to 1 as $x \to 0$ and write:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

2.1. Limits. Definitions. One-sided Limits

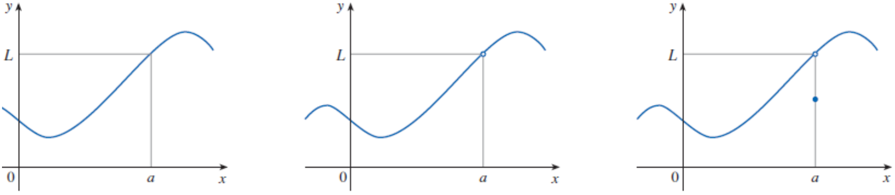
We first study the intuitive definition of a limit by a graphical approach.

Definition: Intuitive Definition of a Limit

Suppose $f(x)$ is defined when x is near the number a. We write

$$\lim_{x \to a} f(x) = L$$

(and say "the limit of $f(x)$, as x approaches a, equals L")
if we can make the values of $f(x)$ arbitrarily close to L as x by restricting x to be sufficiently close to a (on either side of a) but not equal to a.



(a) (b) (c)

2.1. Limits. Definitions. One-sided Limits

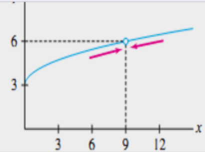
Example: Numerical and Graphical Approach

Investigate graphically and numerically

$$\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$$

Solution

The graph of $f(x)$ has a gap at $x = 9$ since $f(9)$ is NOT defined.



x	$\frac{x-9}{\sqrt{x}-3}$	x	$\frac{x-9}{\sqrt{x}-3}$
8.9	**5.98329**	9.1	**6.01662**
8.99	**5.99833**	9.01	**6.001666**
8.999	**5.99983**	9.001	**6.000167**
8.9999	**5.9999833**	9.0001	**6.0000167**

Numerical evidence and the graph indicate that $f(x) \to 6$ as $x \to 9$.

2.1. Limits. Definitions. One-sided Limits

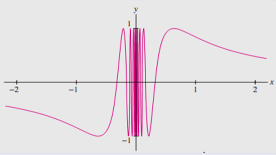
Example: A limit that does not exist

Investigate graphically and numerically

$$\lim_{x \to 0} \sin \frac{1}{x}$$

Solution

The function $f(x)$ is NOT defined at $x = 0$.



x	$\sin \frac{1}{x}$	x	$\sin \frac{1}{x}$
0.1	0.5440	−0.1	−0.5440
0.05	0.9129	−0.05	−0.9129
0.01	−0.5064	−0.01	0.5064
0.001	0.8269	−0.001	−0.8269
0.0005	0.9300	−0.0005	−0.9300
0.00002	−0.9998	−0.00002	0.9998
0.00001	0.0357	−0.00001	−0.0357

Numerical evidence and the graph suggests that the values of $f(x)$ bounce around and do not tend toward any limit L as $x \to 0$.

2.1. Limits. Definitions. One-sided Limits

Properties

We assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then,

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

$$\lim_{x\to a}(f(x)g(x)) = (\lim_{x\to a} f(x))(\lim_{x\to a} g(x))$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{ if } \lim_{x\to a} g(x) \neq 0$$

Note: a can be ∞.

2.1. Limits. Definitions. One-sided Limits

Example

Show that $\lim_{t\to 2}(3t-5)=1$.

Solution

$$\begin{aligned}\lim_{t\to 2}(3t-5) &= \lim_{t\to 2}\{(3)(t)+(-5)\}\\ &= (\lim_{t\to 2} 3)(\lim_{t\to 2} t)+(\lim_{t\to 2}(-5))\\ &= 3\cdot 2-5=1.\end{aligned}$$

Example

Evaluate $\lim_{x\to -3}\frac{x^2+x-6}{x+3}$.

$$\lim_{x\to -3}\frac{x^2+x-6}{x+3}=\lim_{x\to -3}\frac{(x-2)(x+3)}{x+3}=\lim_{x\to -3}(x-2)=-5$$

2.1. Limits. Definitions. One-sided Limits

Definition: Limit at infinity

If f is a function and L_1 and L_2 are real numbers, the statements

$$\lim_{x \to -\infty} f(x) = L_1$$

and

$$\lim_{x \to \infty} f(x) = L_2$$

denote the limits at infinity. The first statement is read "the limit of $f(x)$ as approaches $-\infty$ is L_1" and the second is read "the limit of $f(x)$ as approaches is L_2".

In other words, $\lim_{x \to \infty} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large (x tends to ∞).

2.1. Limits. Definitions. One-sided Limits

Example: Limit at infinity

Show that $\lim\limits_{x\to\infty} \frac{3x-1}{2x+5} = 3/2$.

Solution: Applying properties of limits

$$\lim_{x\to\infty} \frac{3x-1}{2x+5} = \lim_{x\to\infty} \frac{3-\frac{1}{x}}{2+\frac{5}{x}} = \frac{3}{2}$$

Exercise

Evaluate $\lim\limits_{x \to -\infty} \frac{x}{\sqrt{x^2+1}}$.

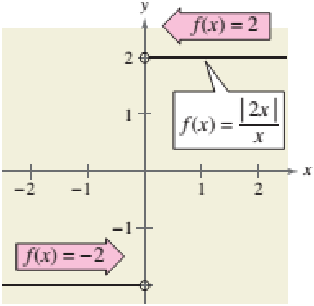
Solution

$$\lim_{x \to -\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x \to -\infty} \frac{x/|x|}{\sqrt{1+\frac{1}{x^2}}} = -1$$

2.1. Limits. Definitions. One-sided Limits

Example: Limit at one side

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for $f(x) = \frac{|2x|}{x}$.



$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2, \quad \lim_{x \to 0^+} \frac{|2x|}{x} = 2.$$

2.1. Limits. Definitions. One-sided Limits

Theorem: Existence of a Limit

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

That is, if f is a function and a and L are real numbers, then

$$\lim_{x \to a} f(x) = L$$

if and only if both the left and right limits exist and are equal to L.

2.1. Limits. Definitions. One-sided Limits

Example

Find the limit of $f(x)$ as x approaches 1.

$$f(x) = \begin{cases} 4 - x, & \text{if } x < 1 \\ 4x - x^2, & \text{if } x > 1 \end{cases}$$

Solution

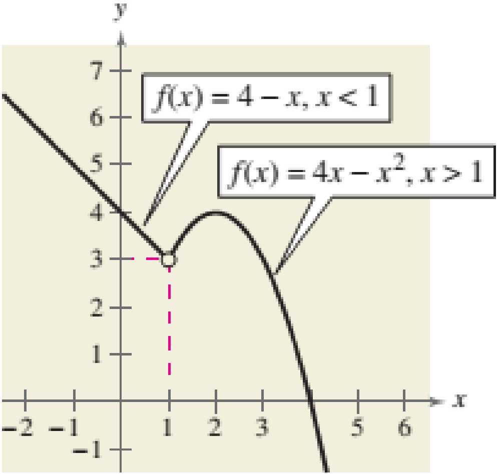
Remember that you are concerned about the value of near $x = 1$ rather at $x = 1$. We have

$$\lim_{x \to 1^-} f(x) = 4 - 1 = 3; \quad \lim_{x \to 1^+} f(x) = 4(1) - 1^2 = 3.$$

Because the one-sided limits both exist and are equal to 3, it follows that

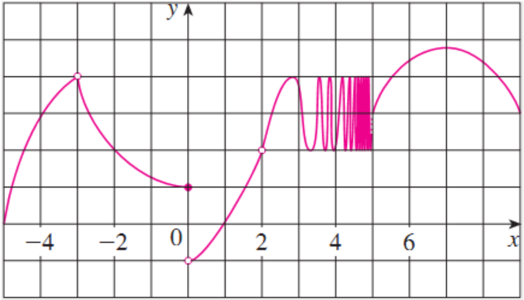
$$\lim_{x \to 1} f(x) = 3.$$

2.1. Limits. Definitions. One-sided Limits



Example

For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why?
(a) $\lim\limits_{x \to -3^-} h(x)$, (b) $\lim\limits_{x \to -3^+} h(x)$, (c) $\lim\limits_{x \to -3} h(x)$, (d) $\lim\limits_{x \to 0^-} h(x)$, (e) $\lim\limits_{x \to 0^+} h(x)$, (f) $\lim\limits_{x \to 0} h(x)$.



2.2. Evaluating Limits. The Squeeze Theorem

Example: Multiplying by the Conjugate

Evaluate

$$\lim_{x \to 10} \frac{\sqrt{x-6}-2}{x-10}.$$

$$\begin{aligned}
\lim_{x \to 10} \frac{\sqrt{x-6}-2}{x-10} &= \lim_{x \to 10} \frac{(\sqrt{x-6}-2)(\sqrt{x-6}+2)}{(x-10)(\sqrt{x-6}+2)} \\
&= \lim_{x \to 10} \frac{(x-6)-4}{(x-10)(\sqrt{x-6}+2)} \\
&= \lim_{x \to 10} \frac{1}{\sqrt{x-6}+2} \\
&= \frac{1}{\sqrt{10-6}+2} = \frac{1}{4}.
\end{aligned}$$

2.2. Evaluating Limits. The Squeeze Theorem

Exercises

1. Evaluate
$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}.$$

2. Assume
$$\lim_{x \to -4} f(x) = 2, \lim_{x \to -4} g(x) = 3$$
Evaluate
$$(a) \quad \lim_{x \to -4} f(x)g(x), \qquad (b) \quad \lim_{x \to -4} (2f(x) + 3g(x),$$
$$(c) \quad \lim_{x \to -4} \frac{g(x)}{x^2}, \qquad (d) \quad \lim_{x \to -4} \frac{f(x)+1}{3g(x)-2}$$

2.2. Evaluating Limits. The Squeeze Theorem

Exercises

3. Find the following limits

(a) $\lim\limits_{x\to 1} \frac{\sqrt{x}-1}{x^3-1}$,

(b) $\lim\limits_{x\to\infty} (\sqrt{x+1} - \sqrt{x-1})$,

(c) $\lim\limits_{x\to\infty} \left(-\frac{2e^{ax}}{e^{3x}} + be^{-cx}\right)$, where a, b, c are constants, $0 < a < 3$, and $c > 0$.

4. Let

$$f(x) = \begin{cases} x^2 - 2 \text{ for } x < 0 \\ 2 - x^2 \text{ for } x \geqslant 0 \end{cases}$$

Find $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to -2} f(x)$.

2.2. Evaluating Limits. The Squeeze Theorem

Sandwich Theorem or Squeeze Theorem

If for x is near a:

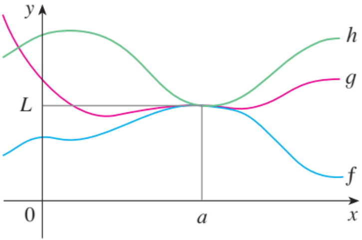
$$f(x) \leqslant g(x) \leqslant h(x)$$

and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x).$$

Then

$$\lim_{x \to a} g(x) = L.$$



2.2. Evaluating Limits. The Squeeze Theorem

Example

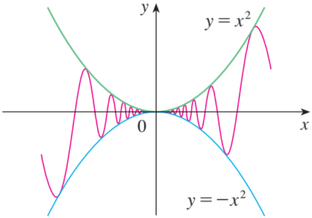
Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

Solution

Applying the Sandwich Theorem, note that:

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

and $\lim_{x \to 0} x^2 = \lim_{x \to 0} -x^2 = 0$. Therefore $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.



2.2. Evaluating Limits. The Squeeze Theorem

Exercise

Find the following limits

1. $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x^3}\right)$.
2. $\lim\limits_{x \to 0} x^2 e^{-\cos\left(\frac{1}{x}\right)}$.

3.1. Continuity: Definitions and properties

Definition

Suppose f is defined in an open interval I that contains a, then f is continuous at a if and only if

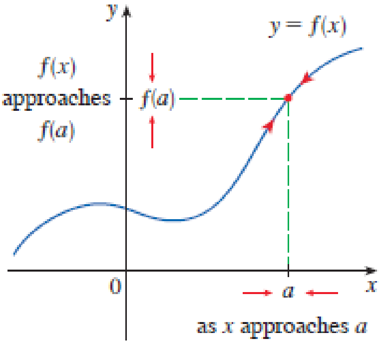
$$\lim_{x \to a} f(x) = f(a).$$

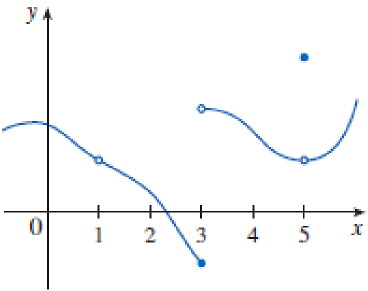
The definition implicitly requires three things:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

[Precise definition (additional reading): We say that f is continuous at a if, given any number $\epsilon > 0$, there exists $\delta > 0$ such that if x is any point of I satisfying $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.]

3.1. Continuity: Definitions and properties





Left figure: $f(x)$ is continuous at $x = a$; Right figure: A discontinuous function at $x = 1$, $x = 3$, and $x = 5$.

3.1. Continuity: Definitions and properties

Example

Show that $f(x) = x$ and $g(x) = k$ (constant) are continuous everywhere.

Solution:

At any point a

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a = f(a)$$

Therefore f is continuous everywhere.

$$\lim_{x \to a} g(x) = \lim_{x \to a} k = k = g(a)$$

Therefore g is continuous everywhere.

3.1. Continuity: Definitions and properties

Example

Show that

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is discontinuous at 0.

Solution:

Note that $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = 0$, thus $\lim_{x \to 0} f(x)$ does NOT exist! Therefore $f(x)$ is discontinuous at 0.

3.1. Continuity: Definitions and properties

Exercises

1. Show that
$$f(x) = \begin{cases} -x+1 & x < 0 \\ x^2+1 & x \geq 0 \end{cases}$$
is continuous everywhere.

2. Show that
$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$
is continuous at 0.

3.1. Continuity: Definitions and properties

Exercises

3. Find all values of a such that $f(x)$ is continuous on $\mathbb{R}$:

$$f(x) = \begin{cases} x+1 \text{ if } x \leqslant a \\ x^2 \text{ if } x > a \end{cases}$$

4. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if} \quad x < 0 \\ 3-x & \text{if} \quad 0 \leqslant x < 3 \\ (x-3)^2 & \text{if} \quad x \geqslant 3 \end{cases}$$

Where f is discontinuous?

3.1. Continuity: Definitions and properties

Theorem

Suppose f, g are continuous at a. Then

$$f+g, f-g, fg$$

are continuous at a. So are the functions kf, if k is a constant, and f/g if $g(a) \neq 0$.

In particular, every polynomial $P(x)$ is defined and continuous at every point. If $Q(x)$ is another polynomial and $Q(a) \neq 0$, then $\dfrac{P(x)}{Q(x)}$ is also continuous at a.

3.1. Continuity: Definitions and properties

Theorem

If f is a continuous bijection from an interval I onto an interval J, then f^{-1} is continuous on J.

Example

The functions x^n are continuous bijections from $[0, \infty)$ onto itself. Therefore $\sqrt{x}$, $\sqrt[n]{x}$ are defined and continuous at every non-negative points. If n is odd, $\sqrt[n]{x}$ is continuous at every point.

3.1. Continuity: Definitions and properties

Theorem

The exponential functions a^x and their inverses $\log_a x$ are continuous at every point of their domains.

Theorem

Let $F(x) = f(g(x))$ be a composite function. If g is continuous at a and f is continuous at $g(a)$, then $F(x)$ is continuous at a.

Example

$\sqrt{x^2+1}$ and $\sqrt[3]{x^5+4x^2-7x+3}$ are composite functions of continuous functions and therefore defined and continuous *everywhere*.

3.1. Continuity: Definitions and properties

Theorem

The following types of functions are continuous at every number **in their domains**: polynomials, rational functions, root functions, trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions.

Example

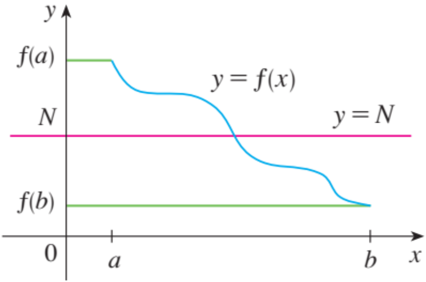
As a result, the following functions are continuous in their domain:

1. $f(x) = \sin(x^2 + x - 1)$.
2. $f(x) = e^{x^2-1}$.
3. $f(x) = \dfrac{\ln(\sqrt{x^4+1})}{\sin(x^2+x-1)}$.

3.2. The Intermediate Value Theorem

The Intermediate Value Theorem

Suppose f is continuous on an interval that contains two points a, b and $f(a) \neq f(b)$. Then for every value N between $f(a)$ and $f(b)$, there exists c between a and b such that $f(c) = N$.



Note: If f is continuous on $[a, b]$ and $f(a)f(b) < 0$, equation $f(x) = 0$ has a real root $c \in (a, b)$.

3.2. The Intermediate Value Theorem

Example: Using the IVT to show the existence of a root

Show that there is a root of the equation $f(x) = 5^x - 6x = 0$ between 0 and 1.

Solution

f is continuous on $[0, 1]$, $f(0) = 1 > 0$ and $f(1) = -1 < 0$. Hence, by the IVT, there exists a real root $c \in (0, 1)$ of the equation $5^x - 6x = 0$.

Example: Using the IVT to show the existence of a root

Show that there is a root of the equation $x^4 + x^2 - x - 3 = 0$ on $(1, 2)$.

3.2. The Intermediate Value Theorem

Exercises

1. Show that the equation

$$x^2 - x - 1 = \frac{1}{x+1}$$

   has a real root in $(1, 2)$.

2. Show that the equation $x^4 + x^2 - x - 3 = 0$ has at least two real roots.

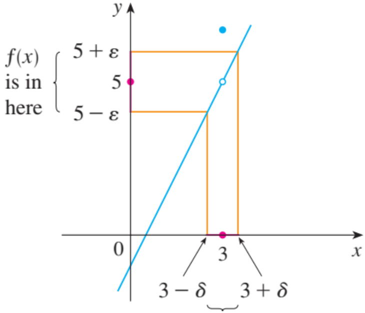
3. Show that there exists a value c between 1 and 2 such that $\sqrt{c} + \sqrt{c-1} = 2$.

The precise definition of a limit

Let

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$$

Q: How close to 3 does x have to be so that $f(x)$ differs from 5 by less than 0.01?

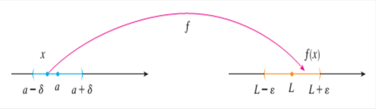


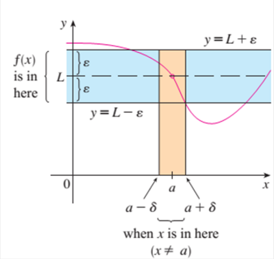
A: As $x \to 3$, if $|x - 3| < 0.005$ then $|f(x) - 5| = 2|x - 3| < 0.01$.

The precise definition of a limit

Precise Definition of a Limit

We have
$$\lim_{x \to a} f(x) = L$$
if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that:
if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.



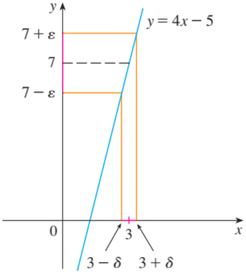


The precise definition of a limit

Exercise

Show that $\lim_{x \to 3} (4x - 5) = 7$.

Hint: Show that there is a number δ (by pointing out δ appropriately) such that: if $0 < |x - 3| < \delta$ then $|f(x) - 7| < \epsilon$, for every $\epsilon > 0$.



–END OF CHAPTER 1. THANK YOU!–