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CHAPTER 2. DIFFERENTIATION

CALCULUS I

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CONTENTS

2.1 Tangent, Velocity Problems. Rates of Change

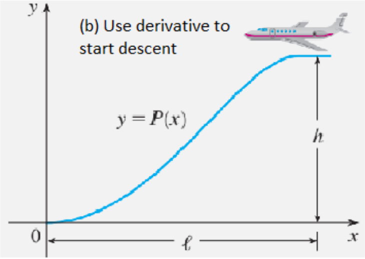
- A major application of Calculus is determining how one quantity varies with another. For example,
- How profit varies with amount spent on advertising,
- How the population of a colony of bacteria changes with time,
- How the energy loss of an electronic device changes with applied current, etc.

  $\rightarrow$ We need the concept "rates of change".

- References for this chapter: Chapter 2 and and Chapter 6 in the textbook.

Introduction

- What is the derivative of a function?
- This question has three equally important answers: A rate of change, the slope of a tangent line, and more formally, the limit of a difference quotient.
- In this chapter, we explore these three facets of the derivative and develop the basic techniques for computing derivatives.



2.1 Tangent, Velocity Problems. Rates of Change

Example

Suppose a car travels due north at a constant speed. After 3 hours the car has travelled 180 km.
(a) What is the speed of the car?
(b) Sketch a graph of displacement, s, as a function of time, t.

Hint:
(a)

$$\text{velocity} = \frac{\text{distance travelled}}{\text{time taken}} = \frac{180 \text{ km}}{3 \text{ h}} = 60\text{km/h}$$

(b) The graph of $s(t)$ is a straight line.
The velocity is the slope of the line:

$$\text{v} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}.$$

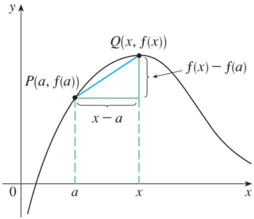
2.1 Tangent, Velocity Problems. Rates of Change

Can we measure the "rate of change" or "velocity" when the graph $y = f(x)$ is NOT a straight line? We need two important concepts:"average rate of change" and "instantaneous rate of change".

Definition

The average rate of change of y with respect to x on an interval $[x_1, x_2]$ is the slope of the secant line (i.e. the line through two points on a curve) joining $[x_1, y_1]$ and $[x_2, y_2]$:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

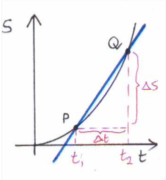


Average Velocity

$$\text{average velocity} = \frac{\text{change in displacement}}{\text{change in time}}$$

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

That is, average velocity = slope of secant line.



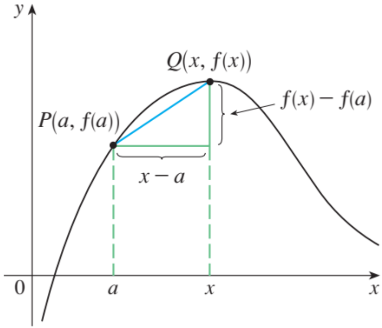
Note: With "distance" and "speed" we are not concerned with the direction in which the movement occurs. To be more precise, we use displacement (directed distance) and velocity, where *speed* $= |$*velocity*$|$.

Instantaneous Rate of Change

- Can we measure the velocity right at a certain time t or at a distance travel s?

- We need the concept "The instantaneous rate of change".

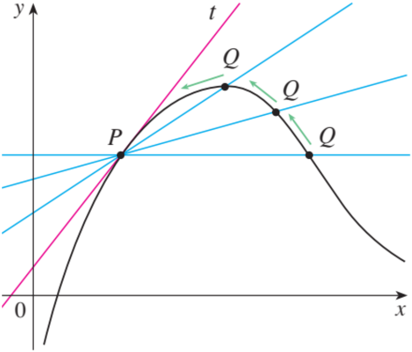
- The instantaneous rate of change of y at x_1 is the slope of the tangent to the curve at x_1.



Instantaneous Rate of Change

The slope of the tangent line is said to be the limit of the slopes of the secant lines as Q approaches P.

$$m = \lim_{Q \to P} m_{PQ}$$



Instantaneous Rate of Change

Definition

The tangent line to the curve y = f(x) at the point P(a, f(a)) is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Definition

Then the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ has equation (in point-slope form)

$$y - f(a) = m(x - a)$$

(provided that m exists)

2.2 The Derivative

Definition

Derivative of a function f at a point a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If f is defined in an interval containing a and the limit exists, we say $f(x)$ is differentiable at a. The process of computing the derivative is called differentiation.

- One-sided limit is used for the endpoints of the interval. If $f(x)$ is differentiable at a point, it is also continuous at that point.
- Derivative = slope of tangent = instantaneous rate of change!

2.2 The Derivative

Every polynomial $P(x)$ is differentiable at every point. Every rational function $\frac{P(x)}{Q(x)}$ is also differentiable at *almost* every point, except where $Q(x) = 0$.

Example

If $f(x) = x^2 + x$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} (2x + 1 + h) = 2x + 1 \end{aligned}$$

2.2 The Derivative

Example

If $f(x) = \frac{x}{x-1}$, for $x \neq 1$, then

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} \\ &= \frac{-1}{(x-1)^2} \end{aligned}$$

2.2 The Derivative

Example

Show that the function $f(x) = |x|$ is not differentiable at $x = 0$.
Hint: Consider the limit from the left and the limit from the right as $x \to 0$.
Further question: Find $f'(x)$. Answer:

$$f'(x) = \begin{cases} 1, \text{if} x > 0 \\ DNE, \text{if} x = 0 \\ -1, \text{if} x < 0 \end{cases}$$

Recall the right derivative and the left derivative at x = a is defined by

$$f'_{+}(a) = \lim_{h \to 0^{+}} \frac{f(a+h) - f(a)}{h}, f'_{-}(a) = \lim_{h \to 0^{-}} \frac{f(a+h) - f(a)}{h}$$

Note that: $f'(a)$ exists $\Leftrightarrow$ $f'_{+}(a), f'_{-}(a)$ exist and equal.

Definition

Then the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ has equation (in point-slope form)

$$y - f(a) = f'(a)(x - a)$$

(provided that $f'(a)$ exists)

2.2 The Derivative

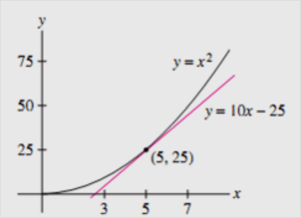
Example

Find an equation of the tangent line to the graph of $f(x) = x^2$ at $x = 5$.

We have

$$f'(5) = \lim_{x \to 5} \frac{f(x) - f(5)}{x - 5} = \lim_{x \to 5} \frac{x^2 - 5^2}{x - 5} = 10$$

The equation of the tangent line in point-slope form is
$y - 25 = 10(x - 5)$, or in slope-intercept form: $y = 10x - 25$.



2.2 The Derivative

Example

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x meters of this fabric is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What is its unit?

(b) What does it mean to say that $f'(1,000) = 9$?

2.2 The Derivative

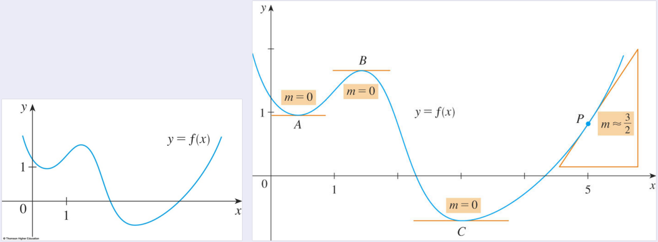
Example

It can be shown that for an object falling freely under gravity, taking $g = 10m/s^2$, the downward displacement s (in meters) after t seconds is given by $s(t) = 5t^2$. Suppose a ball is dropped from the top of a skyscraper. Find the velocity of the ball after 5 seconds.

2.2 The Derivative

Example

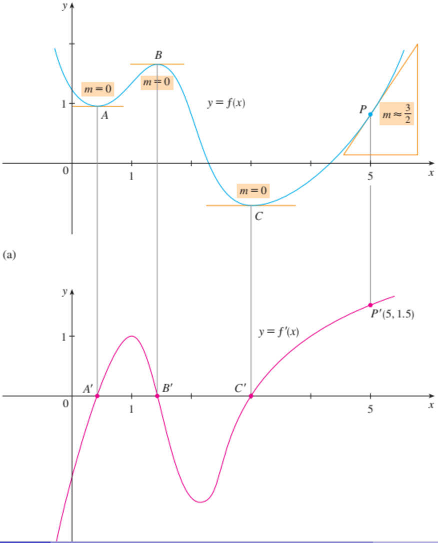
The graph of a function f is given in the figure. Use it to sketch the graph of the derivative f'.



We can find an approximate value for $f'(x)$ at any x by drawing a tangent to the graph $f(x)$ at that x and estimating its slope. We particularly notice that the slope is zero at three points: A, B and C.

2.2 The Derivative

The relation between $f(x)$ and $f'(x)$



2.2 The Derivative: Other Notations

- For a function $y = f(x)$, common notations for the derivative are:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

- The process of finding a derivative is called differentiation.
- The symbols D and d/dx are called differentiation operators.
- The Leibniz notation dy/dx is perhaps the most common.
- Note: dy/dx is not a normal ratio but a symbol for a derivative,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Differentiability and Continuity

Theorem

If f is differentiable at a then f is continuous at a.
Proof:

Differentiability and Continuity

Differentiability implies continuity. However, continuity does NOT imply differentiability. Consider the following example

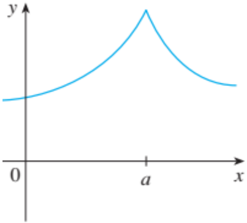
Example

In the previous example, we showed that the function $f(x) = |x|$ is not differentiable at $x = 0$. In the continuity section of Chapter 1, we already saw that this function is continuous.

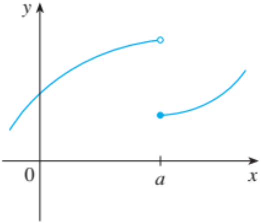
Differentiability and Continuity

When does a function FAIL to be differentiable?

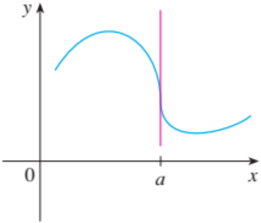
- Having a "corner" or "kink" (So the left and right hand limits are different, and the curve has no tangent at that point.)
- Having discontinuity (removable, jump or infinite).
- Having a vertical tangent (f is continuous, but $\lim_{x \to a} |f'(x)| = \infty$).



(a) A corner



(b) A discontinuity



(c) A vertical tangent

Differentiability and Continuity

Example

Determine whether $f'(0)$ exists.

$$\text{(a)} \quad f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\text{(b)} \quad f(x) = \begin{cases} x - 1, & \text{if } x < 0 \\ x^2 - 1, & \text{if } x \geqslant 0 \end{cases}$$

Hint: Consider $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ and show that
(a) $f'(0) = 0$; (b) DNE.

Higher Order Derivatives

- If f is a differentiable function, then f' is also a function. So, f' may have a derivative of its own, $(f')'$. This is called the second derivative of f and denoted f".
- In Leibniz notation, the second derivative of $y = f(x)$ is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

- The most familiar example is acceleration. If the displacement of a particle at time t is $s(t)$ Then it has velocity $v(t) = \frac{ds}{dt}$ and acceleration $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Higher Order Derivatives

- Similarly, the third derivative is:

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

- The nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times,

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

2.3 Rules of Differentiation

We now study the rules of differentiation.

- We start with the simplest function: $f(x) = c$. The graph is the horizontal line $y = c$, which has slope 0. So, we must have $f'(x) = 0$.
- This is also easily shown from the definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

- Therefore, we obtain the first rule of differentiation

$$\frac{d}{dx}(c) = 0$$

2.3 Rules of Differentiation: Power rule

Theorem

For all exponents n:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Note: The Power Rule is valid for all exponents.

Example

Find the derivatives of the following functions

(a) $f(x) = x^{7.2}$;

(b) $g(x) = \dfrac{1}{x^2}$,

(c) $x(t) = \dfrac{1}{t\sqrt{t}}$

2.3 Rules of Differentiation: Power rule

Example: Rain forest biodiversity

The number of tree species S in a given area A in the Pasoh Forest Reserve in Malaysia has been modeled by the power function

$$S(A) = 0.882A^{0.842}$$

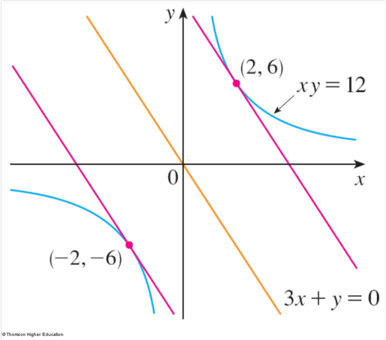
where A is measured in square meters. Find $S'(100)$ and interpret your answer.

2.3 Rules of Differentiation: Power rule

Example

Find the points on the hyperbola $y = \dfrac{12}{x}$ where the tangent is parallel to the line $3x + y = 0$.

Hint: $y = \dfrac{12}{x} \quad \to y' = -\dfrac{12}{x^2} = -3$. Hence $x = \pm 2$.



New Derivatives from Old

- Constant multiple rule:

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}f(x)$$

- Sum/difference rule: If f and g are both differentiable, then

$$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

Example

$$\frac{d}{dx}(2x^7 - 5x^4 - 8x) = 2\frac{d}{dx}(x^7) - 5\frac{d}{dx}(x^4) - 8\frac{d}{dx}(x)$$

$$= 2(7x^6) - 5(4x^3) - 8(1) = 14x^6 - 20x^3 - 8$$

Product Rule. Quotient Rule

- Product Rule: If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

  or $(fg)' = f.g' + f'.g$

- Quotient Rule: If f and g are both differentiable, then

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\,f' - f\,g'}{g^2}$$

Example

Find the derivatives of the given functions

(a) $f(t) = \sqrt{t}(1-3t)$

(b) $f(x) = \dfrac{x^2-1}{\sqrt{x}+2}$

Summary of Rules of Differentiation

Table of Differentiation Formulas

$$\frac{d}{dx}(c) = 0 \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

$$(cf)' = cf' \qquad (f+g)' = f' + g' \qquad (f-g)' = f' - g'$$

$$(fg)' = fg' + gf' \qquad \left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

The chain rule

How to calculate $F'(x)$ where $\mathrm{F}\,(\mathrm{x}) = \sqrt{x^2+1}$? Note that F is a composite function of the two simpler functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, that is, $F = f \circ g$. We already knew the derivatives of f and g, can we calculate the derivative of F? The following theorem give you the answer.

Theorem

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ (defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product:

$$F'(x) \;=\; f'(g(x))\; g'(x)$$

The chain rule

In Leibniz notation, if $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

or

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$$

The chain rule

Example

Find $F'(x)$ if $F(x) = \sqrt{x^2+1}$.

Solution

We have

$$F(x) = (f \circ g)(x) = f(g(x)),$$

where $f(u) = \sqrt{u}, g(x) = x^2 + 1$.

$$f'(u) = \frac{1}{2\sqrt{u}}, g'(x) = 2x.$$

By chain rule,

$$F'(x) = f'(g(x))\, g'(x) = \frac{1}{2\sqrt{x^2+1}}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

The chain rule

Example

Find $F'(x)$ if

$$(a)\quad F(x) = \cos\left(x^2\right), \quad (b)\quad F(x) = \sin\left(\frac{x}{x+1}\right)$$

$$\text{(c)}\quad F(x) = \sqrt{x + \sqrt{x^2+1}}$$

The chain rule

Corollary: General Power and Exponential Rules

- $\frac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$, or $\dfrac{d}{dx}\left([g(x)]^n\right) = n[g(x)]^{n-1} g'(x)$,
- $\frac{d}{dx} e^{g(x)} = g'(x)\, e^{g(x)}$,
- $\frac{d}{dx}(a^x) = a^x \ln a$.

Example

Find the derivatives:

(a) $f(x) = \left(x^3 + 9x + 2\right)^{-1/3}$,

(b) $f(x) = \dfrac{1}{1 + e^{-0.5x}}$,

(c) $f(x) = \left(x^2 + \sqrt{x}e^{\cos x}\right)^3$.

Derivatives of Logarithmic Functions

Theorem

$$\frac{d}{dx}(\log_a)\, x = \frac{1}{x \ln a}, \quad \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example

$$\frac{d}{dx}(\ln \sin x) = \cot x,$$

$$\frac{d}{dx}\left(\sqrt{\ln x}\right) = \frac{1}{2x\sqrt{\ln x}}.$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by the method of logarithmic differentiation

Example

Differentiate

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}, \quad (x > 0)$$

Logarithmic Differentiation

Solution: There are 3 steps

- Taking logarithms of both sides of the equation

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln\left(x^2+1\right) - 5\ln\left(3x+2\right).$$

- Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{2x}{x^2+1} - 5\frac{3}{3x+2}.$$

- Solving for $\dfrac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right).$$

Logarithmic Differentiation

Exercises

Differentiate

(a) $f(x) = \frac{(x+1)^2(2x^2+3)}{\sqrt{x^2+1}}$,

(b) $f(x) = x^{\sin x} \quad (x > 0)$,

(c) $f(x) = (\sin x)^{\ln x}, \quad (0 < x < \pi)$

Calculate limits by differentiation

Example

Prove that (a) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$, (b) $\lim\limits_{x\to 0}\dfrac{\tan(x)}{x}=1$, (c) $\lim\limits_{x\to 0}\dfrac{\arcsin(x)}{x}=1$,
(d) $\lim\limits_{x\to 0}\dfrac{\ln(x+1)}{x}=1$, (e) $\lim\limits_{x\to 0}\dfrac{e^x-1}{x}=1$.

Solution

(b) By the definition of derivative, we have

$$\lim_{x\to 0}\frac{\tan x}{x}=\lim_{x\to 0}\frac{\tan x-\tan 0}{x-0}=\tan'(0)=\frac{1}{\cos^2 0}=1.$$

One can prove other problems similarly.

Calculate limits by differentiation

Example

Find the limit
$$\lim_{x \to 2} \frac{2^x - 4}{x - 2}$$

Answer: $4 \ln 2$.

An application in Economics

Marginal Cost:

Example: A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x meters of fabric is $C = f(x)$ dollars. What is the meaning of $f'(x)$?

In this example, the function $C(x)$ is an example of a cost function. The derivative dC/dx is called the marginal cost, which can be approximated by

$$C'(x) \approx C(x+1) - C(x)$$

Marginal cost estimates the cost of producing one unit beyond the present production level.

An application in Economics

$$\text{Recall}: \quad Revenue = (number\ of\ units\ sold).(price\ per\ unit)$$

Suppose $r(x)$ is the revenue generated when x units of a particular commodity are produced, and $p(x)$ is the corresponding profit. When $x = a$ units are being produced, then

- The marginal revenue is $r'(a)$. It approximates $r(a+1) - r(a)$, the additional revenue generated by producing one more unit.

- The marginal profit is $p'(a)$. It approximates $p(a+1) - p(a)$, the additional profit generated by producing one more unit.

An application in Economics

Example

Suppose it costs $C(x) = x^3 - 6x^2 + 15x$ dollars to produce x stoves and your shop is currently producing 10 stoves a day. Find the marginal cost. About how much extra will it cost to produce one more stove a day?

Solution

The cost of producing one more stove a day when 10 are produced is about $C'(10)$. Since

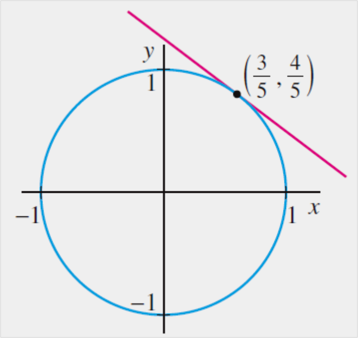
$$C'(x) = 3x^2 - 12x + 15$$

Thus, $C'(10) = 195 \quad (\$/\text{stove})$.
Thus, the additional cost will be about \$195 if you produce one more stove a day.

Implicit Differentiation

To differentiate using the methods covered thus far, we must have a formula for y in terms of x, for instance, $y = x^3 + 1$. But suppose that y is determined instead by an equation such as $x^2 + y^2 = 1$. In this case, we say that y is defined implicitly.

How can we find the slope of the tangent line at a point on the graph in this case?



Implicit Differentiation

To compute $\frac{dy}{dx}$, first take the derivative of both sides of the equation and evaluate:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(1)$$

$$2x + \frac{d}{dx}\left(y^2\right) = 0$$

To calculate $\frac{d}{dx}\left(y^2\right)$, we think of y as a function $y = f(x)$, then $y^2 = f(x)^2$, and apply the Chain Rule: $\frac{d}{dx}\left(y^2\right) = 2y\frac{dy}{dx}$. Hence,

$$2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

Implicit Differentiation

Example

Find the slope of the tangent line at the point $P = (3/5, 4/5)$.

Solution

Substitute $x = \frac{3}{5}, y = \frac{4}{5}$ into the equation $\frac{dy}{dx} = -\frac{x}{y}$, we obtain the slope

$$\left.\frac{dy}{dx}\right|_P = -\frac{3}{4}$$

Implicit Differentiation

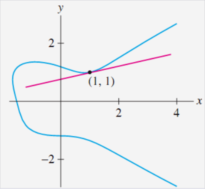
How to find the implicit derivative?

Method

Step 1. Differentiate both sides of the equation with respect to x.
Step 2. Solve for y'.

Example

Find an equation of the tangent line at the point $P = (1, 1)$ on the curve $y^4 + xy = x^3 - x + 2$.



Implicit Differentiation

Solution

Differentiate both sides of the equation with respect to x

$$4y^3y' + \left(xy' + y\right) = 3x^2 - 1$$

Then factor out y'

$$y'\left(4y^3 + x\right) = 3x^2 - 1 - y$$

$$y' = \frac{3x^2 - 1 - y}{4y^3 + x}. \text{ Thus, } \left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1}{5}$$

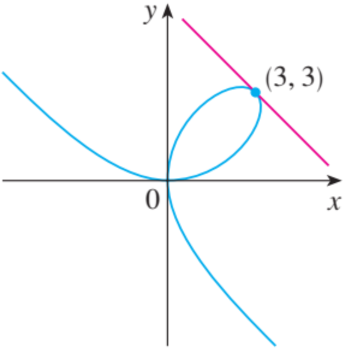
The equation of the tangent line can be written

$$y - 1 = \frac{1}{5}(x - 1) \text{ or } y = \frac{1}{5}x + \frac{4}{5}$$

Implicit Differentiation

Example

(a) Find y' if $x^3 + y^3 = 6xy$.
(b) Find the tangent to the curve (which is called folium of Descartes) $x^3 + y^3 = 6xy$ at the point $(3, 3)$.
(c) At what points in the first quadrant is the tangent line horizontal?



Implicit Differentiation

Exercises

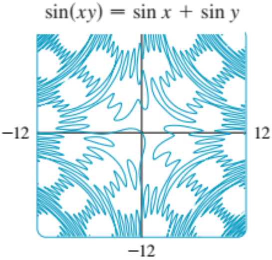
Find y' if

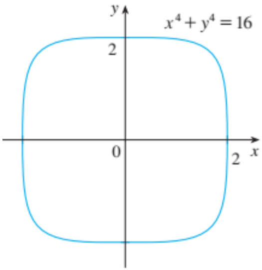
(a) $2x^3 + x^2y - xy^3 = 2$,

(b) $y^5 + x^2y^3 = 1 + ye^{x^2}$,

(c) $\sin(xy) = \sin x + \sin y$,

(d) $x^4 + y^4 = 16$. Also, find y'' for (d).





Differentiation of Inverse Functions

Theorem: Derivative of the Inverse

Assume that $f(x)$ is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If b belongs to the domain of $g(x)$ and $f'(g(b)) \neq 0$, then $g'(b)$ exists and

$$g'(b) = \frac{1}{f'(g(b))}$$

Differentiation of Inverse Functions

Example

Differentiate $g(x) = f^{-1}(x)$ where $g(x)$ is the inverse of

$$f(x) = x^2 + 4$$

on the domain $\{x : x \geqslant 0\}$.

Solution

By technique of finding the inverse function, we obtain

$$g(x) = \sqrt{x - 4}$$

By the derivative of the inverse theorem

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{2g(x)} = \frac{1}{2\sqrt{x - 4}}$$

Differentiation of Inverse Functions

Example: Calculating $g'(x)$ without solving for $g(x)$

Calculate $g'(1)$, where $g(x)$ is the inverse of $f(x) = x + e^x$.

Solution By the derivative of the inverse theorem

$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(c)} = \frac{1}{1 + e^c}$$

where $c = g(1) = f^{-1}(1)$. On the other hand $f(0)=1$, thus $c = f^{-1}(1) = 0$. Therefore, $g'(1) = \frac{1}{2}$.

Differentiation of Inverse Functions

Table of Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Remarks

The notations $\sin^{-1} x$,$\cos^{-1} x$ here mean $\arcsin x$, $\arccos x$, etc., respectively. They are respectively the inverse functions of $y = \sin x$, $y = \cos x$,...[while $(\sin x)^{-1} = \dfrac{1}{\sin x}$, be careful, $\sin^{-1} x$ and $(\sin x)^{-1}$ are two different functions].

Differentiation of Inverse Functions

Example

Differentiate

$$(a)\quad f(x) = \arcsin\sqrt{x}, \qquad (b)\ f(x) = \arctan(3x+1)$$

Solution

By the chain rule:

(a)

$$\left(\arcsin\sqrt{x}\right)' = \frac{\left(\sqrt{x}\right)'}{\sqrt{1-\left(\sqrt{x}\right)^2}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

(b)

$$\left(\arctan(3x+1)\right)' = \frac{(3x+1)'}{1+(3x+1)^2} = \frac{3}{1+(3x+1)^2}$$

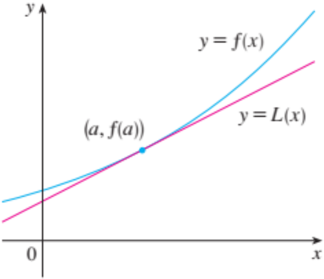
Linear Approximations

Definition

The approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at $x = a$, and the function $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at $x = a$ (when x is near a).



Linear Approximations

Example

(a) Find the linearization of the function $f(x) = e^x$ at $a = 0$ and use it to approximate the number $e^{0.01}$.
(b) Find the linearization of the function $f(x) = \sqrt{x}$ at $a = 1$ and use it to approximate the number 1.001.

Solution
(a) We have $a = 0, f(x) = e^x \Rightarrow f'(x) = e^x, f'(0) = 1$.
By linear approximation,

$$f(x) \approx 1 + 1(x - 0) = x + 1$$

Thus, $e^{0.01} = f(0.01) \approx 1.01$.

Exercises

1. Find the linearization of the function at $a = 0$

(a) $f(x) = \sin x$.

(b) $f(x) = \cos x$.

2. Use the linearization to estimate $\tan\left(\frac{\pi}{4} + 0.02\right)$.

3. Use the linearization to estimate $\sqrt{3.98}$.

Approximations

Example

Let $D(t)$ be the US national debt at time t. The table below gives approximate values of this function by providing end of year estimates, in billions of dollars, from 1985 to 2010. Interpret and estimate the value of $D'(2000)$.

t	$D(t)$
1985	1945.9
1990	3364.8
1995	4988.7
2000	5662.2
2005	8170.4
2010	14,025.2

Approximations

Solution

$$D'(2000) \approx \frac{D(t) - D(2000)}{t - 2000}$$

t	Time interval	Average rate of change $= \dfrac{D(t) - D(2000)}{t - 2000}$
1985	[1985, 2000]	247.75
1990	[1990, 2000]	229.74
1995	[1995, 2000]	134.70
2005	[2000, 2005]	501.64
2010	[2000, 2010]	836.30

$D'(2000)$ lies somewhere between 134.7 and 501.64 billion dollars per year. We can estimate $D'(2000) \approx (134.7 + 501.64)/2 \approx 318$ billion dollars per year.

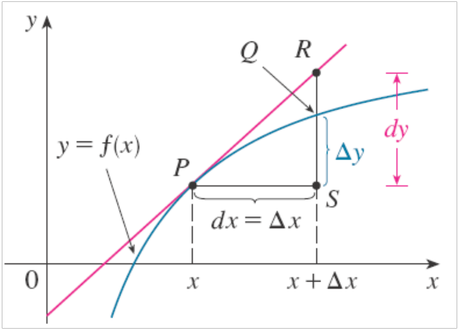
Differentials

Let

$$\Delta y = f(x) - f(a) \quad \text{and} \quad \Delta x = x - a.$$

Graphically, we see that letting $dx = \Delta x$, the corresponding change in the function is Δy. The corresponding change in the tangent line is $dy := f'(x)dx$.

The differential dy is then defined by $df = dy = f'(a)\,dx$.



Differentials

Note: the linear approximation can be rewritten as $\Delta y \approx dy$.

Example

If $y = x^3$ then $dy = 3x^2 dx$.

Example

The radius of a sphere was measured and found to be 21 cm with a possible error in measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?
Hint: $dV = 4\pi r^2 dr = 4\pi 21^2 (0.05) \approx 277 \text{ cm}^3$.

More applications: An application in Physics

Example: Moving objects

The position of a particle is given by the equation

$$s = f(t) = t^3 - 6t^2 + 9t$$

where is measured in seconds and in meters.
(a) Find the velocity at time t.
(b) What is the velocity after 2 s? After 4 s?
(c) When is the particle at rest?
(d) When is the particle moving forward (that is, in the positive direction)?
(e) Find the total distance traveled by the particle during the first five seconds.
(f) Find the acceleration at time t and after 4 s.

Example: Moving objects

Hint:

(a) $v(t) = \dfrac{ds}{dt} = 3t^2 - 12t + 9$

(b) $v(2) = -3\ m/s, v(4) = 9\ m/s$

(c) $v(t) = 0 \Leftrightarrow t = 1, t = 3$

(d) $v(t) > 0 \Leftrightarrow t < 1,$ or $t > 3$

(e) $S_{total} = |f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| = 28$ m

(f) $a(t) = 6t - 12,\ a(4) = 12\ m/s^2$.

Exercise: Moving objects

A particle moves along the x-axis, its position at time t given by $x(t) = \dfrac{t}{1+t^2}$, $t \geqslant 0$, where t is measured in seconds and x in meters.

(a) Find the velocity at time.

(b) When is the particle moving to the right? When is it moving to the left?

(c) Find the total distance traveled during the first 4 s.

Hint: (c) $|x(1) - x(0)| + |x(4) - x(1)| = 1/2 + (1/2 - 4/17) = 13/17$ m.

Exponential Growth and Decay

In many natural phenomena, quantities y grow or decay at a rate proportional to their size. That is,

$$\frac{dy}{dt} = ky$$

It is called a differential equation.

Theorem

The only solutions of the differential equation $dy/dt = ky$ are the exponential functions $y(t) = y(0)e^{kt}$.

In the context of population growth, where $P(t)$ is the size of a population at time, we can write $dP/dt = kP$, where k is called the relative growth rate.

Exponential Growth and Decay

Example

Use the fact that the world population was 2560 million in 1950 and 3040 million in 1960 to model the population of the world in the second half of the 20th century. (Assume that the growth rate is proportional to the population size.) What is the relative growth rate? Use the model to predict the population in the year 2020.

Solution

We measure the time t in years and let $t = 0$ in the year 1950. We measure the population $P(t)$ in millions of people. We have

$$P(10) = 2560e^{10k} = 3040 \Rightarrow k = \frac{1}{10} \ln \frac{3040}{2560} \simeq 0.0172$$

the population in 2020 will be $P(70) \simeq 8524$ millions.

Continuously Compounded Interest

Suppose an amount of $\$A_0$ is invested in the bank, with continuous compounding of interest at interest rate r, the amount after t years is

$$A(t) = A_0 e^{rt}$$

If we differentiate this function, we get $\dfrac{dA}{dt} = rA(t)$, which implies that, with continuous compounding of interest, the rate of increase of an investment is proportional to its size.

Example

Suppose \$1000 is invested at 6% interest, compounded continuously. Find the the value of the investment after 36 months.

Answer: With $r = 0.06, t = 3$ (years), then

$$A(3) = \$A_0 e^{rt} = \$1000 e^{(0.06)3} = \$1197.22$$

Exercise

How long will it take an investment to double in value if the interest rate is 6% compounded continuously?

Hint: Find t such that $A(t) = 2A_0$.

End of Chapter 2-Thank you!.