Machine Learning week 5
Logistic Regression

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1 Likelihood function of logistic regression:

Consider the case has two classes. The posterior probability for C_1 can be written:

$$p(C_1|x) = \frac{p(x|C_1)p(C_1)}{p(x|C_1)p(C_1) + p(x|C_2)p(C_2)}$$

where we have defined:

$$a = log\frac{p(x|C_1)p(C_1)}{p(x|C_2)p(C_2)}$$

So the *logistic sigmoid function* defined by

$$\sigma(a) = \frac{1}{1 + e^{-a}}$$

The model logistic regression is defined as:

$$p(C_1|\phi) = y(\phi) = \sigma\left(w^T\phi\right)$$

$$p(C_2|\phi) = 1 - p(C_1|\phi)$$

For a data set ϕ_n, t_n where $t_n \in 0, 1$ and $\phi_n = \phi(x_n)$ with $n = 1, \cdots, N$, the likelihood function can be written:

$$p(t|w) = \prod_{n=1}^{N} y_n^{t_n}(1 - y_n)^{1-t_n}$$

where $t = (t_1, \cdots, t_N)^T$ and $y_n = p(C_1|\phi_n)$. So we can define an error function by taking the negative logarithm of the likelihood, which gives the cross-entropy error function in the form

$$L = -\log p(t|w) = -\sum_{n=1}^{N} (t_n \log y_n + (1 - t_n) \log(1 - y_n))$$

where $y_n = \sigma(z) = \sigma(w_0 + w_1\phi_1 + \cdots + w_D\phi_D)$. Now we want to minimize L by taking derivative of it with respect to w, we have:

$$\frac{\partial L}{\partial w_0} = \frac{\partial L}{\partial y_n} \frac{\partial y_n}{\partial z} \frac{\partial z}{\partial w_0}$$

We have

$$\frac{\partial L}{\partial y_n} = -\frac{t}{y} - \frac{1-t}{1-y} \qquad \frac{\partial y_n}{\partial z} = \sigma(z)(1 - \sigma(z)) \qquad \frac{\partial z}{\partial w_0} = 1 \tag{1}$$

So that:

$$\begin{aligned} \implies \frac{\partial L}{\partial w_0} &= y_n - t \\ \frac{\partial L}{\partial w_1} &= (y_n - t)\phi_1 \\ \implies \frac{\partial L}{\partial w} &= \sum_{n=1}^{N} (y_n - t_n)\phi_n \end{aligned} \tag{2}$$

2 Find $f(x)$ that $f'(x) = f(x)(1 - f(x))$

$$
\begin{aligned}
f'(x) &= f(x)(1 - f(x)) \\
\implies \frac{df(x)}{dx} &= f(x) - f^2(x) \\
dx &= \frac{df(x)}{f(x) - f^2(x)} \\
\implies \int dx &= \int \frac{df(x)}{f(x) - f^2(x)} \\
x &= \int \frac{df(x)}{f(x)(1 - f(x))} \\
x &= \int \left(\frac{1}{f(x)} - \frac{1}{f(x) - 1} \right) df(x) \\
x &= \int \frac{df(x)}{f(x)} - \int \frac{df(x)}{f(x) - 1} \\
x &= ln|f(x)| - ln|1 - f(x)| \\
x &= ln|\frac{f(x)}{f(x) - 1}| \\
\implies e^x &= \frac{f(x)}{f(x) - 1} \\
\implies \frac{1}{e^x} &= \frac{f(x) - 1}{f(x)} \\
\frac{1}{e^x} &= 1 - \frac{1}{f(x)} \\
\frac{1}{f(x)} &= 1 - \frac{1}{e^x} = \frac{e^x - 1}{e^x} \\
\implies f(x) &= \frac{e^x}{e^x - 1} \\
f(x) &= \frac{1}{1 - e^{-x}}
\end{aligned} \tag{3}
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