Machine Learning week 1

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1 Exercise 1:

1.1 Problems:

To evaluate a new test for detecting Hansen's disease, a group of people 5% of which are known to have Hansen's disease are tested. The test finds Hansen's disease among 98% of those with the disease and 3% of those who don't. What is the probability that someone testing positive for Hansen's disease under this new test actually has it?

1.2 Solve:

H: Predicted by the test (0:non-infected, 1:infected)
G: Given (0:non-infected, 1:infected)

P(G=1) = 0.05
P(G=1) = 0.95
P(P=1 $\mid G = 1) = 0.98$
$P(P = 1 \mid G = 0) = 0.03$

$$P(G = 1 \mid H = 1) = \frac{P(H = 1 \mid G = 1) * P(G = 1)}{P(H = 1)}$$

$$P(G = 1 \mid H = 1) = \frac{P(H = 1 \mid G = 1) * P(G = 1)}{P(H = 1 \mid G = 1) * P(G = 1) + P(H = 1 \mid G = 0) * P(G = 0)}$$

$$P(G = 1 \mid H = 1) = \frac{0.98 * 0.05}{0.98 * 0.05 + 0.03 * 0.95}$$

$$P(G = 1 \mid H = 1) = 0.632$$

Consider the case when $n = 2$, and where the covariance matrix Σ is diagonal

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \tag{1}$$

In this case, the multivariate Gaussian density has the form:

$$\begin{aligned}
p(x, \mu, \Sigma) &= \frac{1}{2\pi \begin{vmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{vmatrix}} exp\left(-\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x2 - \mu_2 \end{bmatrix}^T \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}^{-1} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix} \right) \\
&= \frac{1}{2\pi \left(\sigma_1^2 \cdot \sigma_2^2 - 0 \cdot 0\right)^{\frac{1}{2}}} exp\left(-\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 \\ x2 - \mu_2 \end{bmatrix} \right) \\
&= \frac{1}{2\pi\sigma_1\sigma_2} exp\left(-\frac{1}{2} \begin{bmatrix} x_1 - \mu_1 \\ x2 - \mu_2 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\sigma_1^2} \left(x1 - \mu_1\right) \\ \frac{1}{\sigma_2^2} \left(x2 - \mu_2\right) \end{bmatrix} \right) \\
&= \frac{1}{2\pi\sigma_1\sigma_2} exp\left(-\frac{1}{2\sigma_1^2} \left(x_1 - \mu_1\right)^2 - \frac{1}{2\sigma_2^2} \left(x_2 - \mu_2\right)^2 \right) \\
&= \frac{1}{\sqrt{2\pi}\sigma_1} exp\left(-\frac{1}{2\sigma_1^2} \left(x_1 - \mu_1\right)^2 \right) \cdot \frac{1}{\sqrt{2\pi}\sigma_2} exp\left(-\frac{1}{2\sigma_2^2} \left(x_2 - \mu_2\right)^2 \right)
\end{aligned} \tag{2}$$

We can see that the last equation is just the product of two independent Gaussian densities, one with μ_1 and variance σ_1^2 and the other with mean μ_2 and variance σ_2^2. To prove that the above expression is normalized, we have to show that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma_1^2} \left(x_1 - \mu_1\right)^2 - \frac{1}{2\sigma_2^2} \left(x_2 - \mu_2\right)^2 \right) d\sigma_1 d\sigma_2 = 2\pi\sigma_1\sigma_2$$

Let

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma_1^2} \left(x_1 - \mu_1\right)^2 - \frac{1}{2\sigma_2^2} \left(x_2 - \mu_2\right)^2 \right) d\sigma_1 d\sigma_2 \\
&= \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} exp\left(-\frac{1}{2\sigma_1^2\sigma_2^2} \left(x_1 - \mu_1\right)^2 \sigma_2^2 - \frac{1}{2\sigma_1^2\sigma_2^2} \left(x_2 - \mu_2\right)^2 \sigma_1^2 \right) d\sigma_1 d\sigma_2
\end{aligned} \tag{3}$$

Set

$$
\begin{aligned}
x &= (x_1 - \mu_1)^2 \sigma_2^2 = r\cos\theta \\
y &= (x_2 - \mu_2)^2 \sigma_1^2 = r\sin\theta
\end{aligned} \tag{4}
$$

Using trigonometric identity $\cos\theta^2 + \sin\theta^2 = 1$, we have $x^2 + y^2 = r^2$. Also we have:

$$
\begin{aligned}
\frac{\partial (x, y)}{\partial (r, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r\cos^2\theta + r\sin^2\theta \\
&= r
\end{aligned} \tag{5}
$$

From equation (4) we have:

$$
\begin{aligned}
I &= \int_0^{2\pi} \int_0^{\infty} exp\left(-\frac{r^2}{2\sigma_1^2\sigma_2^2}\right) r dr d\theta \\
&= 2\pi \int_0^{\infty} exp\left(-\frac{r^2}{2\sigma_1^2\sigma_2^2}\right) r dr
\end{aligned} \tag{6}
$$

Replace $r^2 = u$:

$$
\begin{aligned}
I &= 2\pi \int_0^{\infty} exp\left(-\frac{u}{2\sigma_1^2\sigma_2^2}\right) \frac{1}{2} du \\
&= \pi \left[exp\left(-\frac{u}{2\sigma_1^2\sigma_2^2}\right) \left(-2\sigma_1^2\sigma_2^2\right) \right] \\
&= 2\pi\sigma_1^2\sigma_2^2
\end{aligned} \tag{7}
$$