Problem set 10

[Revised Feb 06, 9:21 PM]

Solving Ordinary Differential Equations

1. Consider the following ODE

$$\frac{dy}{dt} = -2y + \cos(4t), \qquad t \in [0, 8]$$

   subject to the initial condition

$$y(0) = 3.$$

   The exact solution to this IVP is

$$y(t) = 0.1\cos(4t) + 0.2\sin(4t) + 2.9\exp(-2t).$$

   In `euler.py` an empty code implementing the Forward Euler method is provided. Study the function `def ivp_solve_euler(t0, t1, y0, dt)`

   - Implement the RHS required for this ODE.
   - Run Euler's method like this

```
t, y, y_exact = ivp_solve_euler(0.0, 8.0, 3.0, dt)
```

   The script provided will report the error (difference between the exact and numerical solution) at the final time.

   - Use timesteps `dt` with values `0.1, 0.05, 0.025, 0.0125, 0.00625` and verify that the error at the final time is $O(h)$ accurate.

2. Consider the same problem in Q1, however this time we you will solve it using backward Euler's method. To do this, make a copy of `euler.py` called `beuler.py`. Rename `ivp_solve_euler()` to `ivp_solve_beuler()`. Change the definition of how the update for `y` is performed such that it implements the backward Euler method.

- Use timesteps `dt` with values `0.1, 0.05, 0.025, 0.0125, 0.00625` and verify that the error at the final time is $O(h)$ accurate.
- Compare the errors obtained with backward Euler to forward Euler. Are the errors obtained with backward Euler larger or smaller than those of forward Euler?

3. Consider the same problem in Q1, however this time you will solve it using the fourth order Runge Kutta method (RK4). To do this, make a copy of `euler.py` and call it `rk4.py`. Rename `ivp_solve_euler()` to `ivp_solve_rk4()`. Change the definition of how the update for `y` is performed such that it implements RK4.

   - Use timesteps `dt` with values `0.1, 0.05, 0.025, 0.0125, 0.00625` and verify that the error at the final time is $O(h^4)$ accurate.
   - Compare the errors obtained with RK4 to forward Euler and backward Euler. Are the errors obtained with RK4 larger or smaller than those of forward Euler and backward Euler?

4. For this question you will examine the stability of Euler's method and backward Euler. You will solve the ODE

$$\frac{dy}{dt} = -20y, \quad t \in [0, 2], \quad y(0) = 3$$

   which has the exact solution

$$y(t) = 3\exp(-20t).$$

   - Make a copy of `euler.py` and call it `euler_stiff.py`. Within `euler_stiff.py`:
     - Modify the definition of the exact solution `y_exact` with $y(t) = 3\exp(-20t)$.
     - Modify the RHS update.
     - The conditional stability criterion is $\Delta t \leq 2/20 = \Delta t_{\text{stable}}$. Plot the solution obtained using $\Delta t = 1.1\Delta t_{\text{stable}}$ and $\Delta t = 0.1\Delta t_{\text{stable}}$.
   - Make a copy of `beuler.py` and call it `beuler_stiff.py`. Within `beuler_stiff.py`:
     - Modify the definition of the exact solution `y_exact` with $y(t) = 3\exp(-20t)$.
     - Modify the update of `y`.
     - Plot the solution obtained using $\Delta t = 1.1\Delta t_{\text{stable}}$ and $\Delta t = 0.1\Delta t_{\text{stable}}$, where $\Delta t_{\text{stable}} = 2/20$.