MM302 Differential Equations

Chapter 4

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4.1 Background and notation

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- These methods have led to huge advances in understanding physical problems.
- Many other types of PDEs cannot be solved in this way, but instead the behaviour of their solutions can be simulated using numerical methods.
- These techniques are taught elsewhere (e.g. in MM306, MM402 and MM406).

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- To be well posed, a problem must satisfy three criteria:
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  - Stability. A small change in the equation, initial or boundary conditions leads a small change in the solution.
- If any of these conditions do not hold, we say that the problem is ill posed.
- In this class, we will assume that our problems are well posed.

Notation

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- We will assume that u is smooth enough so $u_{tx} = u_{xt}$.

- We can now define a general PDE for $u(x, y, z, \ldots, t)$ as

$$F(u, x, y, \ldots, t, u_x, u_y, \ldots, u_t, u_{xx}, u_{yx}, \ldots, u_{tx}, \ldots) = 0.$$

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- For a PDE specified on a region of space Ω, we need boundary conditions (BCs) on the boundary $\partial\Omega$: this leads to a Boundary Value Problem (BVP) .

- If one of the independent variables is time, we need initial conditions (ICs) at time $t = 0$: this leads to an Initial Value Problem (IVP).

- The advection equation describes the motion of a scalar quantity $u(x, t)$ as it is carried along (or *advected*) by a known constant velocity field c:

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- The wave equation governs all small amplitude waves, such as water waves, sound and radio waves, propagation of light, etc:

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

where c is the (known) speed at which the wave travels.

- The heat equation (or diffusion equation) represents any diffusive phenomenon such as the flow of heat in a metallic rod, with u representing temperature and k the conductivity:

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- Laplace's equation has physical significance in the modelling of, e.g., electrostatic potentials, or the temperature distribution in a conducting plate:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Differential operators

- All of these PDEs can be written using a differential operator, usually denoted by $\mathcal{L}$, which is a mapping between sets of differentiable functions.

$$\mathcal{L}(u) = f$$

$$\mathcal{L} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}, \qquad \text{advection equation}$$

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}, \qquad \text{wave equation}$$

$$\mathcal{L} = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2}, \qquad \text{heat equation}$$

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \text{Laplace's equation}$$

- The order of a PDE is the highest order of the partial derivatives it contains.
- The advection equation is first order (as it contains only first derivatives).
- The other three equations are second order (as they contain second derivatives).
- This continues for higher order derivatives. . .

Linearity

- The PDE $\mathcal{L}(u) = f$ is said to be linear if the differential operator $\mathcal{L}$ satisfies

$$\mathcal{L}(u + v) = \mathcal{L}(u) + \mathcal{L}(v)$$

$$\mathcal{L}(cu) = c\mathcal{L}(u)$$

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- An example of a nonlinear equation might be

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + 4u^2 - \sin(u) = x^2.$$

- Nonlinear PDEs are NOT considered in this class.

- The linearity test can be condensed to checking that

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2).$$

where c_1 and c_2 are real constants.

Examples 4.1-4.2

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- Suppose that u_1 and u_2 are solutions of the heat equation, i.e., $\mathcal{L}(u_1) = 0$ and $\mathcal{L}(u_2) = 0$ where

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- For linearity, we need to show that

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2)$$

for any real constants c_1 and c_2.

Example 4.1

Here we have

$$
\begin{aligned}
\mathcal{L}(c_1u_1 + c_2u_2) &= \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) \\
&= c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} - kc_1\frac{\partial^2 u_1}{\partial x^2} - kc_2\frac{\partial^2 u_2}{\partial x^2} \\
&= c_1\left(\frac{\partial u_1}{\partial t} - k\frac{\partial^2 u_1}{\partial x^2}\right) + c_2\left(\frac{\partial u_2}{\partial t} - k\frac{\partial^2 u_2}{\partial x^2}\right) \\
&= c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2)
\end{aligned}
$$

so the heat equation is linear.

Example 4.2

Show that the equation

$$u^2\frac{\partial^2 w}{\partial u^2} + \sin(v)\frac{\partial^2 w}{\partial v^2} = 0$$

for the function $w(u, v)$ is linear.

Example 4.2

Show that the equation

$$u^2\frac{\partial^2 w}{\partial u^2} + \sin(v)\frac{\partial^2 w}{\partial v^2} = 0$$

for the function $w(u, v)$ is linear.

- Suppose that w_1 and w_2 are solutions of the given equation, i.e., $\mathcal{L}(w_1) = 0$ and $\mathcal{L}(w_2) = 0$ where

$$\mathcal{L} \equiv u^2\frac{\partial^2}{\partial u^2} + \sin(v)\frac{\partial^2}{\partial v^2}.$$

- For any real constants c_1 and c_2, we have

$$
\begin{aligned}
& \mathcal{L}(c_1 w_1 + c_2 w_2) \\
= \; & u^2 \frac{\partial^2}{\partial u^2} (c_1 w_1 + c_2 w_2) + \sin(v) \frac{\partial^2}{\partial v^2} (c_1 w_1 + c_2 w_2) \\
= \; & c_1 \, u^2 \frac{\partial^2 w_1}{\partial u^2} + c_2 \, u^2 \frac{\partial^2 w_2}{\partial u^2} + c_1 \sin(v) \frac{\partial^2 w_1}{\partial v^2} + c_2 \sin(v) \frac{\partial^2 w_2}{\partial v^2} \\
= \; & c_1 \left(u^2 \frac{\partial^2 w_1}{\partial u^2} + \sin(v) \frac{\partial^2 w_1}{\partial v^2} \right) + c_2 \left(u^2 \frac{\partial^2 w_2}{\partial u^2} + \sin(v) \frac{\partial^2 w_2}{\partial v^2} \right) \\
= \; & c_1 \mathcal{L}(w_1) + c_2 \mathcal{L}(w_2)
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so the equation is linear.

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- Linearity and homogeneity also apply to boundary conditions:

$$u(0,t) + \frac{\partial u}{\partial x}(0,t) = g(t) \qquad \text{is a linear inhomogeneous BC,}$$

$$\frac{\partial u}{\partial x}(a,t) - u^2(a,t) = 0 \qquad \text{is a nonlinear homogeneous BC,}$$

$$u(a,t) + \kappa \frac{\partial u}{\partial x}(a,t) = 0 \qquad \text{is a linear homogeneous BC .}$$

- A problem is homogeneous if both the differential equation and its boundary conditions are homogeneous.

4.4 The principle of superposition

- If $u = u_1$ and $u = u_2$ are solutions of the linear homogeneous PDE $\mathcal{L}(u) = 0$, then $u = c_1 u_1 + c_2 u_2$ is also a solution, for arbitrary constants c_1 and c_2.

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- As $\mathcal{L}(u_1) = 0$ and $\mathcal{L}(u_2) = 0$, the linearity of $\mathcal{L}$ gives

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$$c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2) = \mathcal{L}(c_1 u_1 + c_2 u_2) = 0.$$

- This result is called the principle of superposition since one solution can be superposed (or added) to another.

- More generally, if $u_1, u_2, \ldots, u_N$ are solutions of $\mathcal{L}(u) = 0$, then

$$u = \sum_{n=1}^{N} c_n \, \mathcal{L}(u_n)$$

is also a solution.

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- This also holds as $N \to \infty$, so that the sum of the resulting infinite series is also a solution of $\mathcal{L}(u) = 0$.

- For an inhomogeneous linear equation $\mathcal{L}(u) = f$, the RHS vectors associated with different solutions can be added together in an analogous way: if u_1 and u_2 are solutions of $\mathcal{L}(u) = f_1$ and $\mathcal{L}(u) = f_2$, respectively, then $u_1 + u_2$ is a solution of $\mathcal{L}(u) = f_1 + f_2$.

- This also extends to N solutions in the obvious way.

- We can also apply this principle when $f_1 = 0$ and $f_2 \neq 0$, which helps us to find the general solution of a inhomogeneous PDE.

> The general solution of an inhomogeneous PDE is
>
> general solution of the homogeneous equation +
> particular solution of the inhomogeneous equation.

- That is, if u_G is the general solution of $\mathcal{L}(u) = 0$, and u_P is a particular solution of $\mathcal{L}(u) = f$, then $u_G + u_P$ is the general solution of $\mathcal{L}(u) = f$.

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Show that

$$u = f(2x + y^2) + g(2x - y^2)$$

satisfies the PDE

$$y^2 \frac{\partial^2 u}{\partial x^2} + \frac{1}{y}\frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

for arbitrary functions f and g.

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- As f and g are each functions of only one variable, we can use $'$ to denote differentiation of these functions (without specifying the variable).

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- First derivatives:

$$\frac{\partial u}{\partial x} = 2f'(2x + y^2) + 2g'(2x - y^2)$$

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- Second derivatives:

$$\frac{\partial^2 u}{\partial x^2} = 4f''(2x + y^2) + 4g''(2x - y^2)$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= 2f'(2x + y^2) + 2yf''(2x + y^2) \cdot 2y \\ &- (2g'(2x - y^2) + 2yg''(2x - y^2) \cdot (-2y)) \\ &= 2\left(f'(2x + y^2) - g'(2x - y^2)\right) \\ &+ 4y^2\left(f''(2x + y^2) + g''(2x - y^2)\right)\end{aligned}$$

$$y^2\frac{\partial^2 u}{\partial x^2}+\frac{1}{y}\frac{\partial u}{\partial y}-\frac{\partial^2 u}{\partial y^2}=0$$

- Substituting these expressions into the PDE gives

$$\begin{aligned}
& y^2\left[4f''(2x+y^2)+4g''(2x-y^2)\right] \\
+\quad & \frac{1}{y}\left[2yf'(2x+y^2)-2yg'(2x-y^2)\right] \\
-\quad & \left[2\left(f'(2x+y^2)-g'(2x-y^2)\right)\right. \\
+\quad & \left.4y^2\left(f''(2x+y^2)+g''(2x-y^2)\right)\right] \\
=\quad & 0
\end{aligned}$$

as required.