Sturm-Liouville Theory

We will now look a kind of ODE problem called a **Regular Sturm-Liouville Problem.**

Suppose we can rewrite our differential equation as

$$[p(x)y']' + q(x)y + \lambda\sigma(x)y = 0, \quad p(x)>0, \quad q(x)>0, \quad a \le x \le b$$

$$p(x)y'' + p'(x)y' + q(x)y + \lambda\sigma(x)y = 0$$

Boundary Condition: $\alpha y(a) + \beta y'(a) = 0$, $\gamma y(b) + \delta y'(b) = 0$.

$y'(a) = k_1 y(a)$ $\quad$ $y'(b) = k_2 y(b)$ $\quad$ $y(a) = y'(a) = 0$, $y(b) = y'(b) = 0$

Any solution y to this SLP is called an **Eigenfunction.**
Note that adjusting $\lambda \in \mathbb{R}$ will result in a different Eigenfunction.
So if y is an Eigenfunction, λ is its Eigenvalue.

Sturm-Liouville Theorem

If y_1 and y_2 are Eigenfunctions of the SLP

$$[p(x)y']' + q(x)y + \lambda\sigma(x)y = 0$$

And have distinct Eigenvalues $\lambda_1 \neq \lambda_2$, then

$$\int_a^b \sigma(x) y_1 y_2 \, dx = 0$$

Orthogonal with respect to $\sigma(x)$

Proof Let y_1 and y_2 be solns to SLP

$$[p(x)y_1']' + q(x)y_1 + \lambda_1\sigma(x)y_1 = 0 \quad \text{and} \quad [p(x)y_2']' + q(x)y_2 + \lambda_2\sigma(x)y_2 = 0$$

$\times y_2 \quad \Downarrow \quad \times y_1$

$$[p(x)y_1']'y_2 + q(x)y_1y_2 + \lambda_1\sigma(x)y_1y_2 = 0, \quad [p(x)y_2']'y_1 + q(x)y_1y_2 + \lambda_2\sigma(x)y_2y_1 = 0$$

$\Downarrow$ subtract

$$[p(x)y_1']'y_2 - [p(x)y_2']'y_1 + \lambda_1\sigma(x)y_1y_2 - \lambda_2\sigma(x)y_1y_2 = 0$$

$\Downarrow$ rearrange

$$(\lambda_1-\lambda_2)\sigma(x)y_1y_2 = [p(x)y_2']'y_1 - [p(x)y_1']'y_2$$

$$(\lambda_1-\lambda_2)\int_a^b \sigma(x)y_1y_2\,dx = \int_a^b [p(x)y_2']'y_1\,dx - \int_a^b [p(x)y_1']'y_2\,dx$$

$$\int_a^b f'g\,dx = [fg]_a^b - \int_a^b fg'\,dx$$

$$(\lambda_1-\lambda_2)\int_a^b \sigma(x)y_1y_2\,dx = [p(x)y_2'y_1]_a^b - \cancel{\int_a^b p(x)y_2'y_1'\,dx}$$

$$- [p(x)y_1'y_2]_a^b + \cancel{\int_a^b p(x)y_1'y_2'\,dx}$$

$$= [p(x)y_2'y_1]_a^b - [p(x)y_1'y_2]_a^b$$

$$= p(x)\left[y_2'(b)y_1(b) - y_2'(a)y_1(a) - y_1'(b)y_2(b) + y_1'(a)y_2(a)\right]$$

$$= p(x)\left[K_2\cancel{y_2(b)}y_1(b) - K_1\cancel{y_2(a)}y_1(a) - K_2\cancel{y_1(b)}y_2(b) + K_1\cancel{y_1(a)}y_2(a)\right]$$

$$= p(x)[0] = 0 \qquad (\lambda_1-\lambda_2)\int_a^b \sigma(x)y_1y_2\,dx = 0$$

So if $\lambda_1 \neq \lambda_2$ then $\int_a^b \sigma(x)y_1y_2\,dx = 0$

Facts

- There are infinitely many eigenvalues λ_n for each SLP and $\lim_{n\to\infty} \lambda_n = \infty$.

- If y_n and Y_n are eigenfunctions for λ_n, then $Y_n = K y_n$, $K \in \mathbb{R}$.

Example Find the Eigenvalues and Eigenfunctions for the SLP

$$y'' + \lambda y = 0, \quad y(0) = y(\pi) = 0, \quad \lambda > 0$$

Solution AE: $m^2 + \lambda = 0 \Rightarrow m = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\, i$

$$y = A\cos(\sqrt{\lambda}\, x) + B\sin(\sqrt{\lambda}\, x)$$

$y(0) = A\cos(0) + B\sin(0) = A = 0 \Rightarrow A = 0$

$y(\pi) = B\sin(\sqrt{\lambda}\,\pi) = 0$

Either $B = 0$ or $\sin(\sqrt{\lambda}\,\pi) = 0 \Leftrightarrow \sqrt{\lambda} \in \mathbb{Z}$

So Eigenvalues are squares of natural numbers.

$$1, 4, 9, \ldots \to \infty$$

And Eigen Functions are
$y_1 = \sin(x)$
$y_2 = \sin(\sqrt{4}\, x) = \sin(2x)$
$y_3 = \sin(\sqrt{9}\, x) = \sin(3x).$

Example Find the eigenvalues and eigenfunctions for the SLP

$$y'' + \lambda^2 y = 0, \quad y(0)+y'(0)=0, \quad y(1)+3y'(1)=0$$

If $\lambda=0$ $\quad y''=0, \; y'=A, \; y=Ax+B$

$$y(0)+y'(0) = A(0)+B+A = A+B=0 \Rightarrow B=-A$$
$$y(1)+3y'(1) = A+B+3A = 4A+B=0 \Rightarrow 3A=0 \Rightarrow A=0, \; B=0$$

If $\lambda \neq 0$ $\quad m^2+\lambda^2=0 \iff m=\pm\lambda i$

$$y = A\cos(\lambda x) + B\sin(\lambda x)$$
$$y' = -\lambda A\sin(\lambda x) + \lambda B\cos(\lambda x)$$

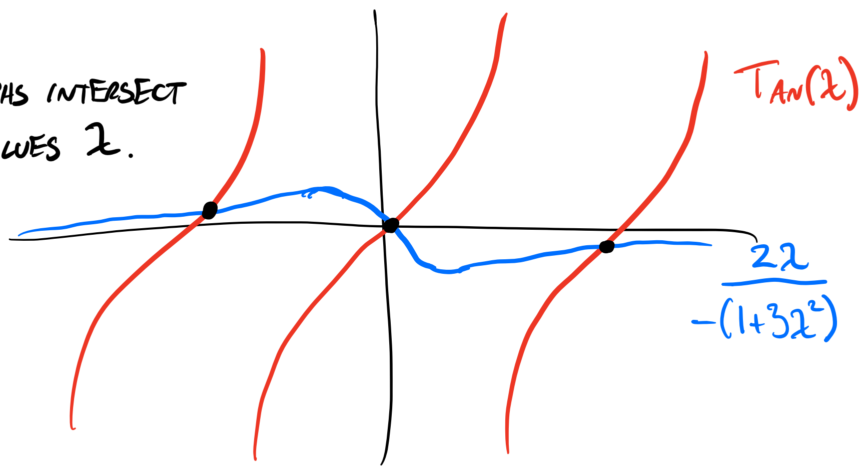
$$y(0)+y'(0) = A\cos(0)+B\sin(0)-\lambda A\sin(0)+\lambda B\cos(0)$$
$$= A+\lambda B = 0 \quad \Rightarrow A = -\lambda B$$

$$y(1)+3y'(1) = A\cos(\lambda)+B\sin(\lambda)+3[-\lambda A\sin(\lambda)+\lambda B\cos(\lambda)]$$
$$= -\lambda B\cos(\lambda)+B\sin(\lambda)+3\lambda^2 B\sin(\lambda)+3\lambda B\cos(\lambda)$$
$$= 2\lambda B\cos(\lambda)+B(1+3\lambda^2)\sin(\lambda) = 0$$

$B \neq 0$

$$\tan(\lambda) = \frac{2\lambda B}{-B(1+3\lambda^2)}$$
$$= \frac{2\lambda}{-(1+3\lambda^2)}$$

The two graphs intersect at eigenvalues λ.



WHAT IF THE ODE IS NOT IN

$$[p(x)y']' + q(x)y + \lambda\sigma(x)y = 0$$

EXAMPLE $\quad y'' + 2y' + \lambda y = 0$

DEFINE $p(x) > 0 \qquad p(x)y'' + 2p(x)y' + \lambda p(x)y = 0$

IF $[p(x)y']' = p(x)y'' + 2p(x)y'$

↓ PRODUCT RULE

$$p(x)y'' + p'(x)y' = p(x)y'' + 2p(x)y'$$

$$\Rightarrow \quad p'(x) = 2p(x)$$

$$p(x) = e^{2x}$$

$$e^{2x}y'' + 2e^{2x}y' + \lambda e^{2x}y = 0$$

$$[e^{2x}y']' + \lambda e^{2x}y = 0 \qquad\qquad p(x) = \sigma(x)e^{2x}$$

$$\Updownarrow$$

$$e^{2x}y'' + 2e^{2x}y' + \lambda e^{2x}y = 0$$

$$\Updownarrow$$

$$y'' + 2y' + \lambda y = 0$$

Rayleigh Quotient

Consider the SLP $(py')' + qy + \lambda\sigma y = 0.$

If we multiply by y, integrate, and rearrange, we can arrive at the following formula for λ.

$$\lambda = \frac{\int_a^b [p(y')^2 - qy^2]\,dx - [pyy']_a^b}{\int_a^b \sigma y^2\,dx}$$

This is called the Rayleigh Quotient. Occasionally it can give us valuable information about λ.

Harmonic Equation.

Example Let $y'' + \lambda y = 0$

$y(0)=0$ or $y'(0)=0$ and $y(a)=0$ or $y'(a)=0$

① When is $\lambda = 0$?

② When is $\lambda > 0$?

$$\lambda = \frac{\int_0^a (y')^2\,dx - [yy']_0^a}{\int_0^a y^2\,dx}$$

$p=1$
$\sigma=1$
$q=0$

$$[yy']_0^a = y(a)y'(a) - y(0)y'(0) = 0 - 0 = 0$$

$$\lambda = \frac{\int_0^a (y')^2\,dx}{\int_0^a y^2\,dx}.$$

① $\lambda = 0 \Rightarrow \int_0^a (y')^2\,dx = 0 \Leftrightarrow y' = 0 \Leftrightarrow y = \text{Constant}$

② $\lambda > 0 \Rightarrow$ Always.

SOLVE THE SLP $\quad y''+2y'+y+\lambda y=0, \quad y(0)=y(3)=0$

$\times e^{2x} \quad \Rightarrow \quad e^{2x}y''+2e^{2x}y'+e^{2x}y+\lambda e^{2x}y=0$

$$[e^{2x}y']'+e^{2x}y+\lambda e^{2x}y=0 \quad \checkmark$$

$y''+2y'+(1+\lambda)y=0$

AE: $\quad m^2+2m+(1+\lambda)=0 \qquad m=\dfrac{-2\pm\sqrt{2^2-4(1+\lambda)}}{2}$

$$=\frac{-2\pm\sqrt{4-4-4\lambda}}{2}$$

$$=\frac{-2\pm\sqrt{-4\lambda}}{2}=\frac{-2\pm2\sqrt{-\lambda}}{2}$$

$$=-1\pm\sqrt{-\lambda}$$

IF $\lambda<0 \quad y=Ae^{(-1-\sqrt{-\lambda})x}+Be^{(-1+\sqrt{-\lambda})x}$

$y(0)=Ae^0+Be^0=0 \quad \Rightarrow \quad B=-A$

$y(3)=Ae^{(-1-\sqrt{-\lambda})3}-Ae^{(-1+\sqrt{-\lambda})3}=0 \qquad A=0$ IS POSSIBLE.

$\quad =Ae^{-3}[e^{-3\sqrt{-\lambda}}-e^{+3\sqrt{-\lambda}}]=0$

IF $A\neq0 \qquad e^{-3\sqrt{-\lambda}}=e^{3\sqrt{-\lambda}} \Leftrightarrow 1=e^{6\sqrt{-\lambda}} \Leftrightarrow \lambda=0$

CONTRADICTION.

$\lambda=0 \quad \Rightarrow \quad m=-1$ REPEATED.

$$y=(Ax+B)e^{-x}$$

$y(0)=Be^0=0 \quad \Rightarrow \quad B=0$

$y(3)=(3A)e^{-3}=0 \quad \Rightarrow \quad A=0 \qquad y=0$ IS A SOLN.

$\lambda > 0$, $m = -1 \pm \sqrt{-1}\sqrt{\lambda} = -1 \pm i\sqrt{\lambda}$

$$y = e^{-x}\left(A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)\right)$$

$$y(0) = e^{0}(A\cos(0) + B\sin(0)) = 0 \qquad A = 0$$

$$y(3) = e^{-3}(B\sin(3\sqrt{\lambda})) = 0$$

EITHER $B = 0$ OR $\sin(3\sqrt{\lambda}) = 0$

$$3\sqrt{\lambda} = K\pi \qquad K \in \mathbb{Z}$$

$$\sqrt{\lambda} = \frac{K\pi}{3}$$

$$\lambda = \frac{K^2\pi^2}{9}$$

$$\lambda_0 = 0, \ \lambda_1 = \frac{\pi^2}{9}, \ \lambda_2 = \frac{4\pi^2}{9}$$

$$y_1 = e^{-x}B\sin\left(3\frac{\pi}{3}x\right) = Be^{-x}\sin(\pi x) \qquad B \in \mathbb{R}.$$

2019 Q1

$$y'' - 2xy' + \alpha y = 0 \qquad \alpha \in \mathbb{R}$$

(i) USE POWER SERIES TO FIND A SOLUTION IN THE FORM $y = \sum_{n=0}^{\infty} a_n x^n$ NEAR $x_0 = 0$.

(ii) SHOW THE GENERAL SOLN CONSISTS OF AN ODD SERIES AND EVEN SERIES.

Soln

EQUATION IS IN NORMALISED FORM. SINCE BOTH $-2x$ AND α ARE POLYNOMIALS, THEY ARE ANALYTIC EVERYWHERE.

SO $x_0 = 0$ IS AN ORDINARY POINT.

I CAN ASSUME $y = \sum_{n=0}^{\infty} a_n x^n$, $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$,

$$y'' = \sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} (n-1)n\, a_n x^{n-2} - 2x \sum_{n=1}^{\infty} n a_n x^{n-1} + \alpha \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 2\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha a_n x^n = 0$$

WHEN $n = 0$ $\quad (0+1)(0+2) a_{0+2} + \alpha a_0 = 0$

$$2a_2 + \alpha a_0 = 0$$

$$a_2 = -\frac{\alpha}{2} a_0$$

WHEN $n \geq 1$ $\quad (n+1)(n+2) a_{n+2} - 2n a_n + \alpha a_n = 0$

$$(n+1)(n+2) a_{n+2} = (2n - \alpha) a_n$$

$$a_{n+2} = \frac{(2n-\alpha)}{(n+2)(n+1)} a_n$$

$a_4 = ?$ $\qquad a_4 = \frac{(2(2)-\alpha)}{4\cdot 3} a_2 = \frac{(4-\alpha)(-\alpha)}{4\cdot 3\cdot 2} a_0$

$a_6 = ?$ $\quad a_6 = \frac{(2(4)-\alpha)}{6 \cdot 5} a_4 = \frac{(8-\alpha)(4-\alpha)(-\alpha)}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} a_0$

$a_8 = ?$ $\quad a_8 = \frac{(2(6)-\alpha)}{8 \cdot 7} a_6 = \frac{(12-\alpha)(8-\alpha)(4-\alpha)(-\alpha)}{8!} a_0$

$$a_{2n} = \frac{(4n-4-\alpha)(4(n-1)-4-\alpha)\cdots(4-4-\alpha)}{(2n)!} a_0$$

$$= \frac{(4(n-1)-\alpha)(4(n-2)-\alpha)\cdots(4(n-n)-\alpha)}{(2n)!} a_0$$

So $y = \sum_{n=0}^{\infty} a_{2n} x^{2n}$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(4(n-1)-\alpha)\cdots(4-\alpha)(-\alpha)}{(2n)!} x^{2n}$$

$a_{n+2} = \frac{(2n-\alpha)}{(n+2)(n+1)} a_n$ $\quad a_1 = a_1$

$$a_3 = \frac{(2-\alpha)}{3 \cdot 2} a_1$$

$$a_5 = \frac{(6-\alpha)}{5 \cdot 4} a_3 = \frac{(6-\alpha)(2-\alpha)}{5!} a_1$$

$2n+1$

$n=2$

$$a_{2n+1} = \frac{((4n-2)-\alpha)(4(n-1)-2)\cdots(6-\alpha)(2-\alpha)}{(2n+1)!} a_1$$

$$y = a_1 \sum_{n=0}^{\infty} \frac{(4n-2-\alpha)\cdots(2-\alpha)}{(2n+1)!} x^{2n+1}$$

(iii) DEDUCE THAT IF α IS EVEN THEN ONE OF THE SOLUTIONS IS A POLYNOMIAL.

SOLN $(2k-\alpha)$ IS A FACTOR OF THE COEFFICIENT OF ALL BUT FINITELY MANY TERMS IN ONE OF THE SOLUTIONS FOR y. SO IF $\alpha = 2k$, THEN THIS SOLUTION IS A POLYNOMIAL.