MM302 Differential Equations

Chapter 5

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5 Linear first order PDEs

- We start with linear first order PDEs of the form

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu = f,$$

  - $u = u(x, y)$ is the unknown function;
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  for some known constant c.

- The independent variable here is t to emphasise its physical meaning as time.

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- That is, it models a travelling wave which does not change its shape.

- We will therefore assume that, given $u(x,0) = F(x)$, the solution $u(x,t)$ is obtained by moving the graph of $F(x)$ with speed c.

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$$A = \int_a^b u(x, t)\, dx.$$

- As we are assuming that the shape of the wave does not change as time evolves, at a future time $t + \tau$ we also have

$$A = \int_{a+c\tau}^{b+c\tau} u(x, t + \tau)\, dx$$

where the integration limits have moved by a distance $c\tau$.

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- We now recall Leibniz's rule for differentiating an integral: if

$$I(\alpha) = \int_a^b f(x,\alpha)\,dx$$

where a and b are differentiable functions of α, then

$$\frac{dI}{d\alpha} = f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha} + \int_a^b \frac{\partial f}{\partial \alpha}\,dx.$$

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- Applying Leibniz's rule to our integral gives

$$\frac{dA}{d\tau} = cu(b+c\tau, t+\tau) - cu(a+c\tau, t+\tau) + \int_{a+c\tau}^{b+c\tau} \frac{\partial u}{\partial t}(x, t+\tau)\, dx = 0.$$

- Letting $\tau = 0$, this becomes

$$
\begin{aligned}
& c[u(b,t) - u(a,t)] + \int_a^b \frac{\partial u}{\partial t}(x,t)\,dx \\
= \quad & c \int_a^b \frac{\partial u}{\partial x}(x,t)\,dx + \int_a^b \frac{\partial u}{\partial t}(x,t)\,dx = 0
\end{aligned}
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so

$$\int_a^b \frac{\partial u}{\partial t}(x,t)\,dx + c\int_a^b \frac{\partial u}{\partial x}(x,t)\,dx = 0.$$

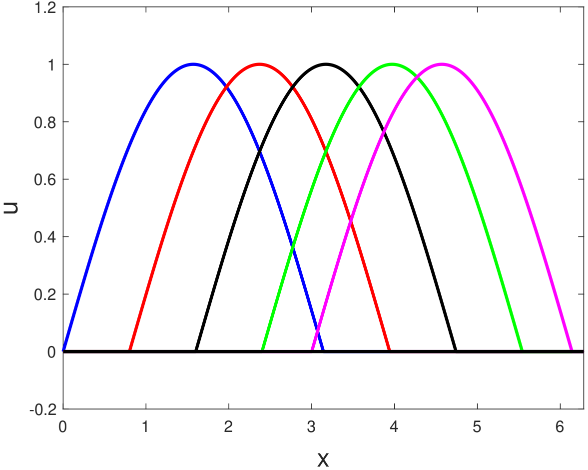
- This can only be true for all intervals $[a, b]$ if

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0,$$

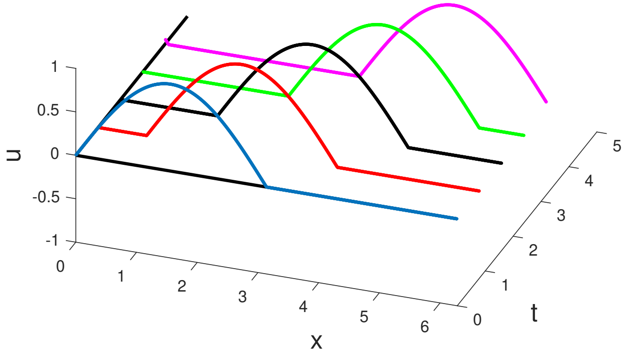
which is exactly the advection equation.

5.2 Method of characteristics for the advection equation

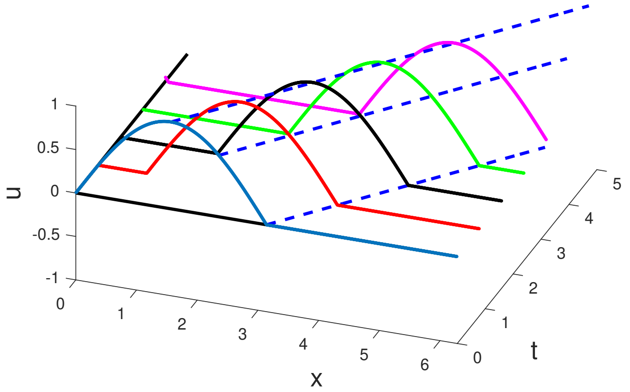
- Example of a travelling wave which does not change its shape.
- Moving from left to right (blue to pink) as $t \to \infty$..



- Same curves plotted on $x - t$ plane.



- Key idea: solution is constant along characteristics (shown by dashed lines)



- Suppose we have a moving observer whose position is given by $x = x(t)$, and consider $u(x(t), t)$ (i.e., the value of u the observer sees at their current position).

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- Using the chain rule, we have

$$\frac{d}{dt}u(x(t), t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt}.$$

- Comparing these, it follows that if $dx/dt = c$ then $du/dt = 0$.

- We have replaced the PDE by a system of two ODES:

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$$x(t) = ct + k$$

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where k is an arbitrary constant.

- If this line passes through $(x_0, 0)$, it will have equation

$$x(t) = ct + x_0$$

where $x_0 = x|_{t=0}$.

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Characteristics

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$$\frac{du}{dt} = 0$$

along these characteristics.

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- We can therefore write

$$u(x, t) = u(x_0, 0),$$

where

$$x_0 = x - ct.$$

- The initial shape $u(x, 0)$ therefore propagates with speed c, and the solution of the advection equation is always a travelling wave.

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- In other words, when an initial condition is given at $t = 0$ as

$$u(x,0) = F(x),$$

we obtain the solution of the advection equation in the form

$$u(x,t) = F(x - ct).$$

- We can differentiate to verify that, for any differentiable function F of one variable, $u(x,t) = F(x-ct)$ is a solution of the advection equation: as

$$\frac{\partial u}{\partial t} = -cF'(x-ct), \qquad \frac{\partial u}{\partial x} = F'(x-ct),$$

we see immediately that

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- When $t=0$, $u(x,t) = F(x)$, so $u(x,t) = F(x-ct)$ also satisfies the initial condition $u(x,0) = F(x)$.

Examples 5.1-5.2

5.3 Method of characteristics for linear first order PDEs

- We now generalise the method of characteristics for solving other linear first order PDEs.

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- We will consider general equation

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- We impose the initial condition

$$u(x,0) = F(x).$$

- As before, we consider a moving observer whose position is given by $x(t)$.

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- The system of two ODEs in this case is

$$\begin{aligned} \frac{dx}{dt} &= c(x,t), \\ \frac{d}{dt}u(x,t) + p(x,t)u(x,t) &= q(x,t). \end{aligned}$$

- The first ODE defines a family of characteristic curves along which the required function u is governed by the second ODE.

Method of characteristics: summary

- To find the solution $u(x,t)$ of a linear first order PDE with initial condition $u(x,0) = F(x)$:
  - Find the characteristic curve $x(t)$ which passes through the point $(x_0, 0)$ by solving the first ODE.
  - Solve the second ODE along this curve with initial condition $u(x_0, 0) = F(x_0)$.
  - Eliminate dependence on x_0 to obtain the final solution $u(x,t)$.
- Here $(x_0, 0)$ is the point where the characteristic through (x, t) crosses the x-axis.

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- Here $(x_0, 0)$ is the point where the characteristic through (x,t) crosses the x-axis.
- Note that this the method will only work if $x(t)$ intersects the x-axis.

Examples 5.3-5.5