Phase plane analysis

Phase plane analysis

Consider $\dot{\underline{x}} = A\underline{x}$, and assume that $\underline{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a sol.

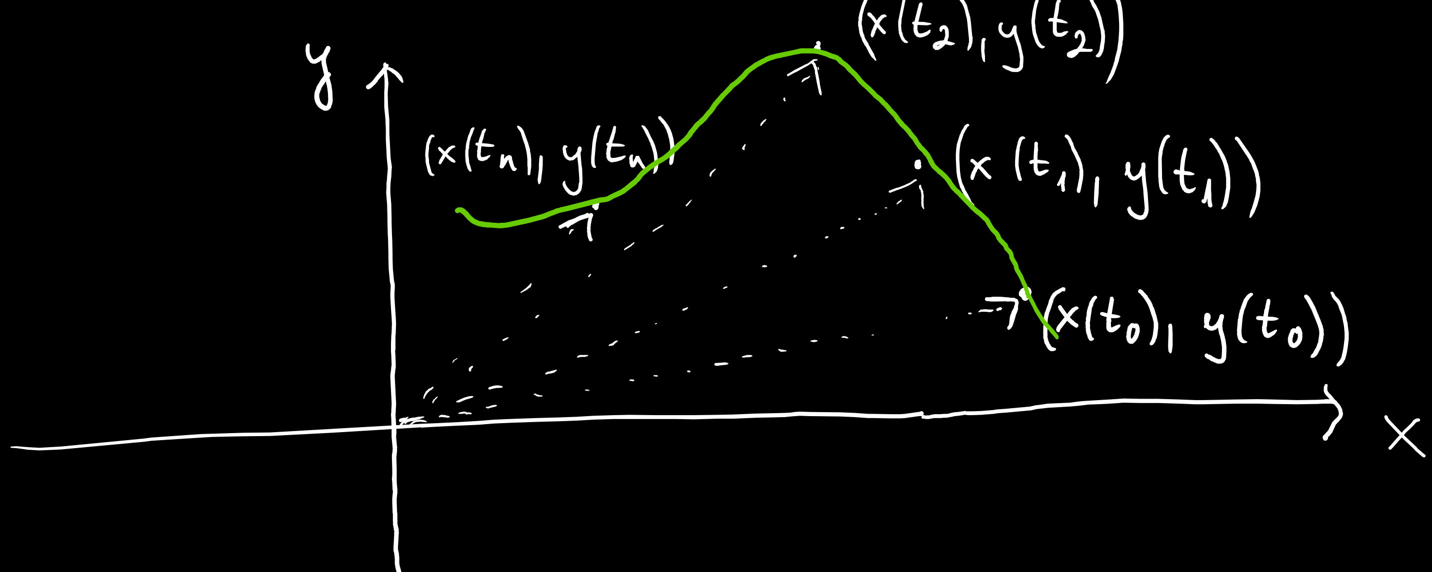
① Plot the graph of the solutions $x(t), y(t)$ over the t-axis

② Regard the solution as a parametric representation of a curve with parameter t in the x-y plane, called a phase plane

$\boxed{\text{Phase plane analysis}}$

Consider $\dot{\underline{x}} = A\,\underline{x}$, and assume that $\underline{x} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a solution.

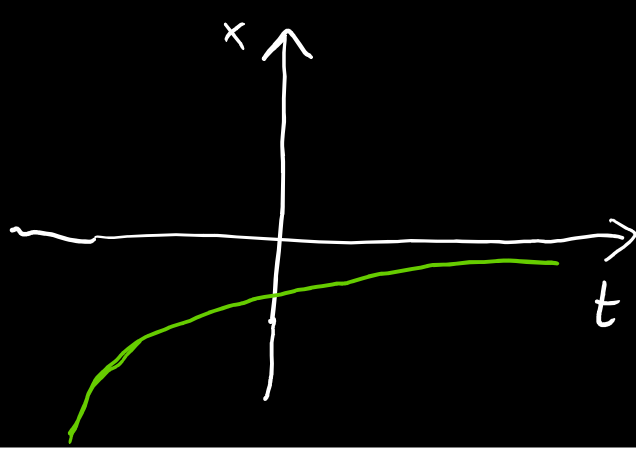
2.

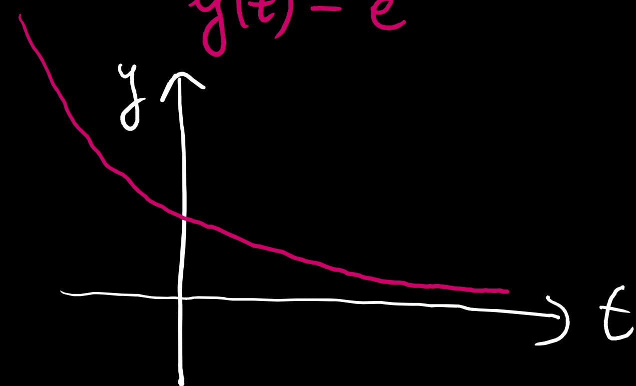


Phase plane analysis

$$\left.\begin{aligned}\dot{x} &= -3x + y\\ \dot{y} &= x - 3y\end{aligned}\right\} \qquad \begin{aligned}x(t) &= -c_1 e^{-4t} + c_2 e^{-2t}\\ y(t) &= c_1 e^{-4t} + c_2 e^{-2t}\end{aligned}$$

1. $c_1 = 1,\ c_2 = 0$ $\quad x(t) = -e^{-4t}$ $\quad y(t) = e^{-4t}$





Phase plane analysis

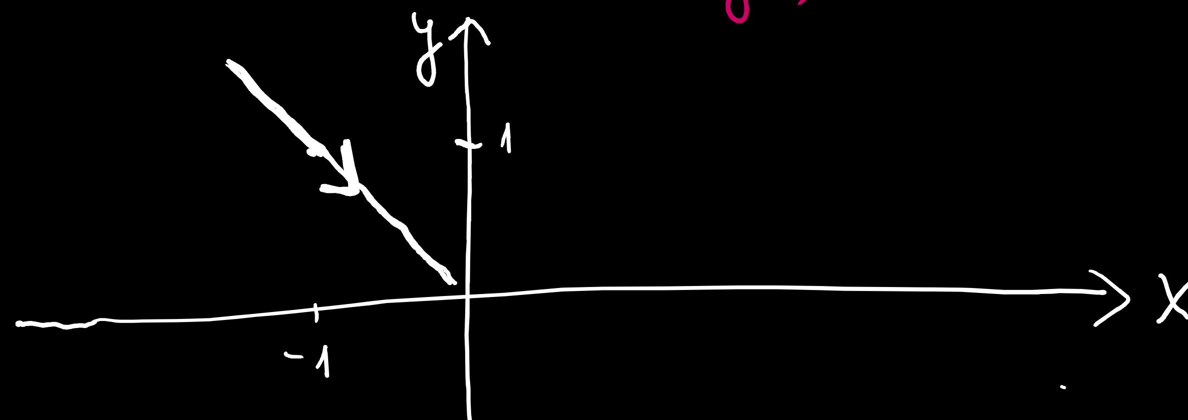
$$\left.\begin{aligned} \dot{x} &= -3x + y \\ \dot{y} &= x - 3y \end{aligned}\right\} \qquad \begin{aligned} x(t) &= -c_1 e^{-4t} + c_2 e^{-2t} \\ y(t) &= c_1 e^{-4t} + c_2 e^{-2t} \end{aligned}$$

1. $c_1 = 1,\ c_2 = 0$ $\qquad x(t) = -e^{-4t} \qquad\qquad y(t) = e^{-4t}$

$t=0:\ (-1, 1)$

$t=1:\ (-e^{-4}, e^{-4})$

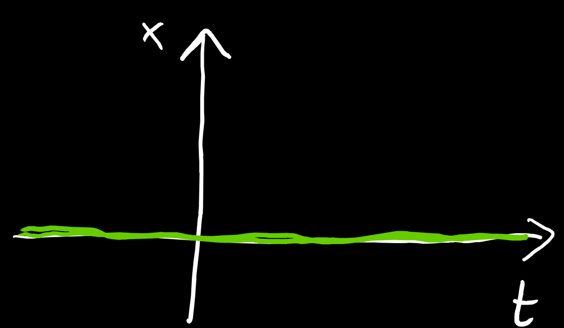
$\vdots$

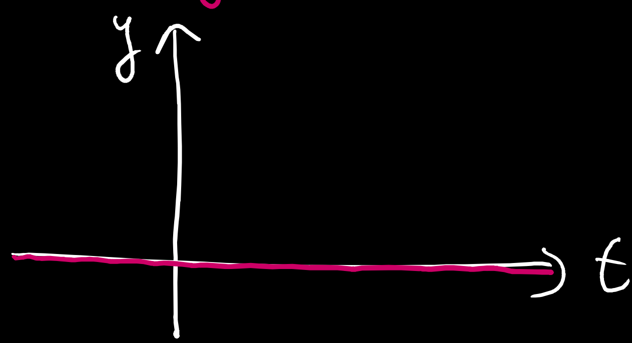


$$\boxed{\text{Phase plane analysis}}$$

$$\left.\begin{aligned} \dot{x} &= -3x + y \\ \dot{y} &= x - 3y \end{aligned}\right\} \qquad \begin{aligned} x(t) &= -c_1 e^{-4t} + c_2 e^{-2t} \\ y(t) &= c_1 e^{-4t} + c_2 e^{-2t} \end{aligned}$$

2. $c_1 = c_2 = 0$: $x(t) = 0$ $\qquad y(t) = 0$





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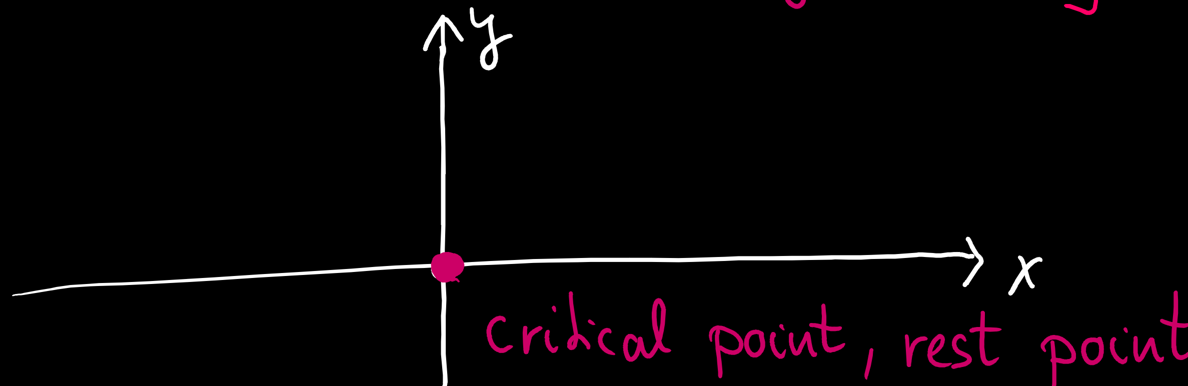
1. $c_1 = c_2 = 0$: $\left.\begin{aligned} x(t) &= 0 \\ y(t) &= 0 \end{aligned}\right\}$ equilibrium solution

$x(t) = y(t) = 0$



Equilibrium solutions

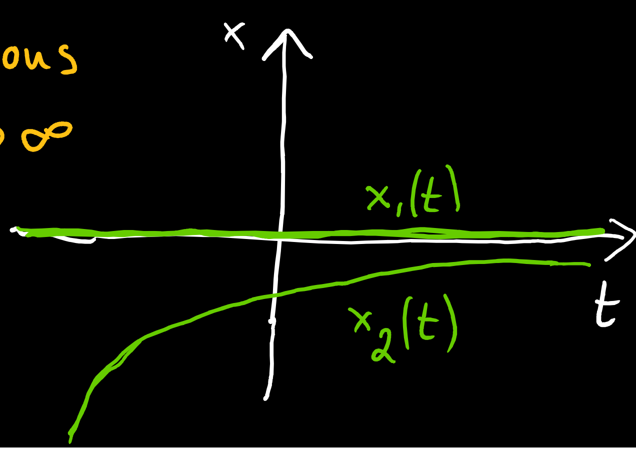
A constant solution $(x(t) = \alpha, y(t) = \beta)$ of $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ is called an **equilibrium** (steady state) **solution**. Such a solution does not define a trajectory in the phase-plane but a single point, called a **critical point** or **rest point**. Clearly, an equilibrium solution occurs when $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. Hence, if $\det \boldsymbol{A} \neq 0$, then $\boldsymbol{x} = \boldsymbol{0}$ is the only equilibrium solution; but if $\det \boldsymbol{A} = 0$ then there are many equilibrium solutions. Note that no trajectory can pass through a critical point because of uniqueness.

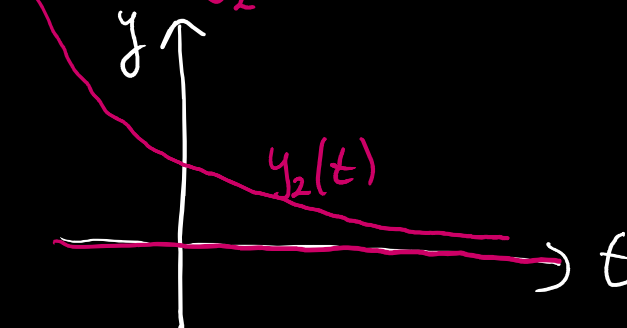
Phase plane analysis

$$\left.\begin{aligned} \dot{x} &= -3x + y \\ \dot{y} &= x - 3y \end{aligned}\right\} \qquad \begin{aligned} x(t) &= -c_1 e^{-4t} + c_2 e^{-2t} \\ y(t) &= c_1 e^{-4t} + c_2 e^{-2t} \end{aligned}$$

1. $c_1 = 1, \; c_2 = 0$ $\qquad x_2(t) = -e^{-4t}$ $\qquad y_2(t) = e^{-4t}$

How do solutions behave as $t \to \infty$ or $t \to -\infty$?



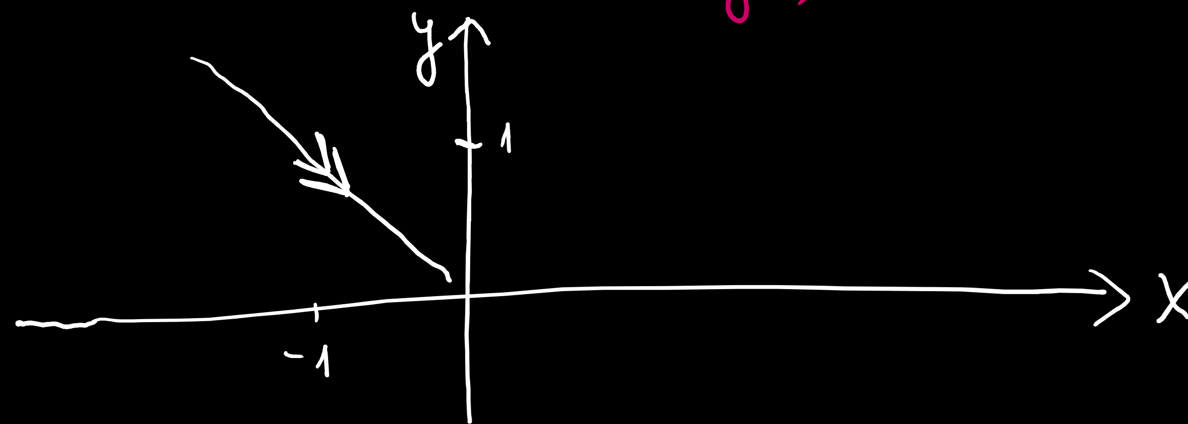


Phase plane analysis

$$\left.\begin{aligned} \dot{x} &= -3x + y \\ \dot{y} &= x - 3y \end{aligned}\right\} \qquad \begin{aligned} x(t) &= -c_1 e^{-4t} + c_2 e^{-2t} \\ y(t) &= c_1 e^{-4t} + c_2 e^{-2t} \end{aligned}$$

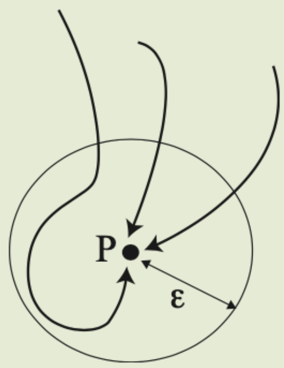
$c_1 = 1, \; c_2 = 0$ $\quad x(t) = -e^{-4t}$ $\quad y(t) = e^{-4t}$

How do trajectories behave near critical points?



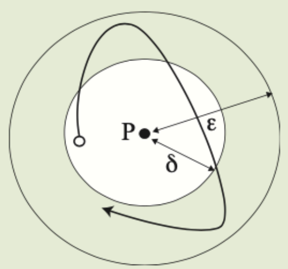
Definition 12: Stability of critical points

A critical point P is **attractive**, if there is a disk D_ϵ of radius $\epsilon > 0$ about P such that every path that has a point in D_ϵ approaches P as $t \to \infty$. Roughly speaking, a rest point P is attractive if it attracts all paths which at some instant are sufficiently close to it.



Attractive rest point P.

A critical point P is **stable**, if for every $\epsilon > 0$ there exists a $\delta > 0$ (which depends on ϵ) such that if a path has a point in the disk D_δ about P, then all subsequent points will lie in a D_ϵ disk about P. Roughly speaking, a rest point is stable, if all paths that become close remain close for all future time.



Stable rest point P.

A critical point P is called **locally asymptotically stable**, if for every sufficiently small $\epsilon > 0$, it is both stable and attractive. If ϵ can be taken to be arbitrarily large, we have **global asymptotic stability**. If a rest point is not stable, it is **unstable**.

Example 2.3

$$\dot{\underline{x}} = A\,\underline{x}$$

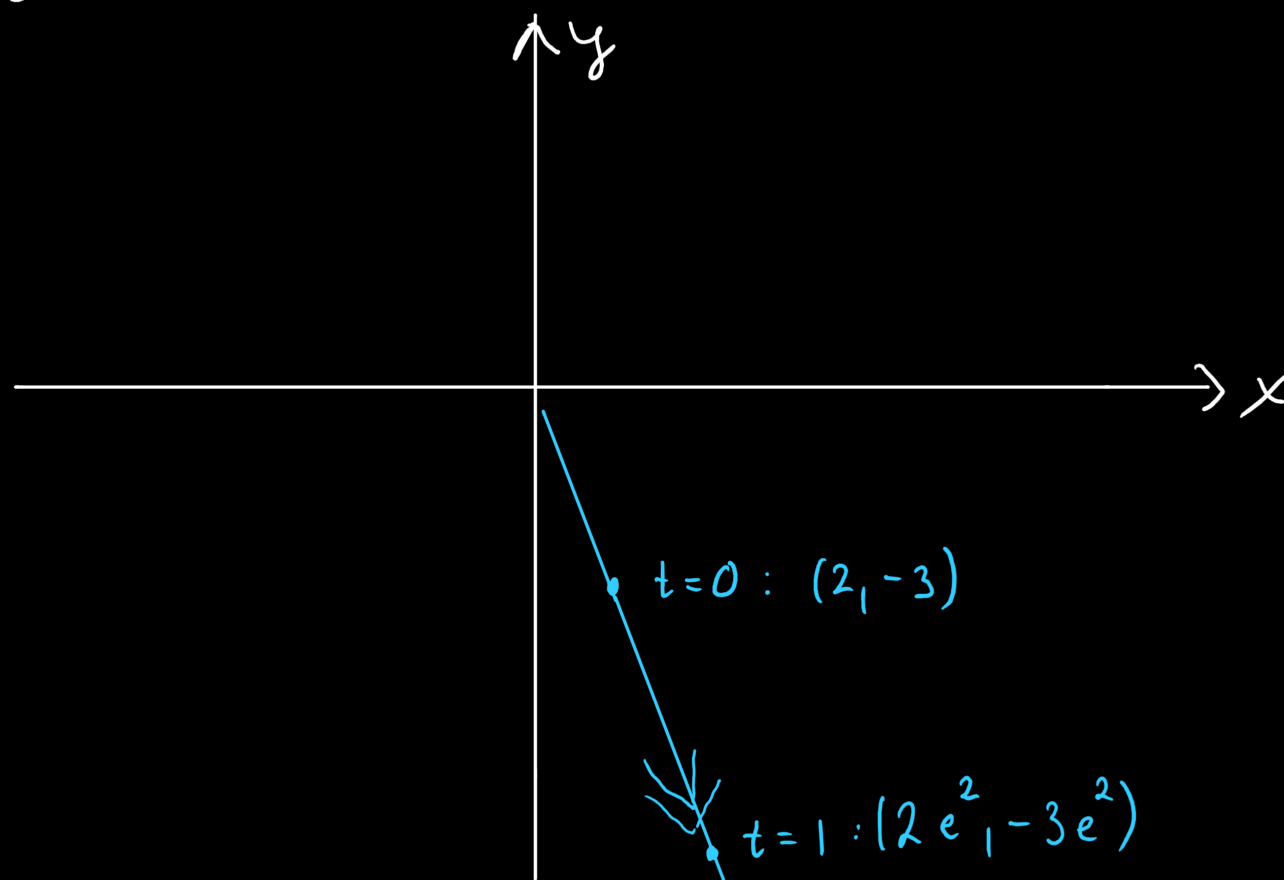
- Find and classify the critical points
- Sketch the phase portrait

a)

$$\left.\begin{aligned} \dot{x} &= 5x + 2y \\ \dot{y} &= 3x + 4y \end{aligned}\right\} \quad \text{G.S: } \underline{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$$

$c_1 = 1$

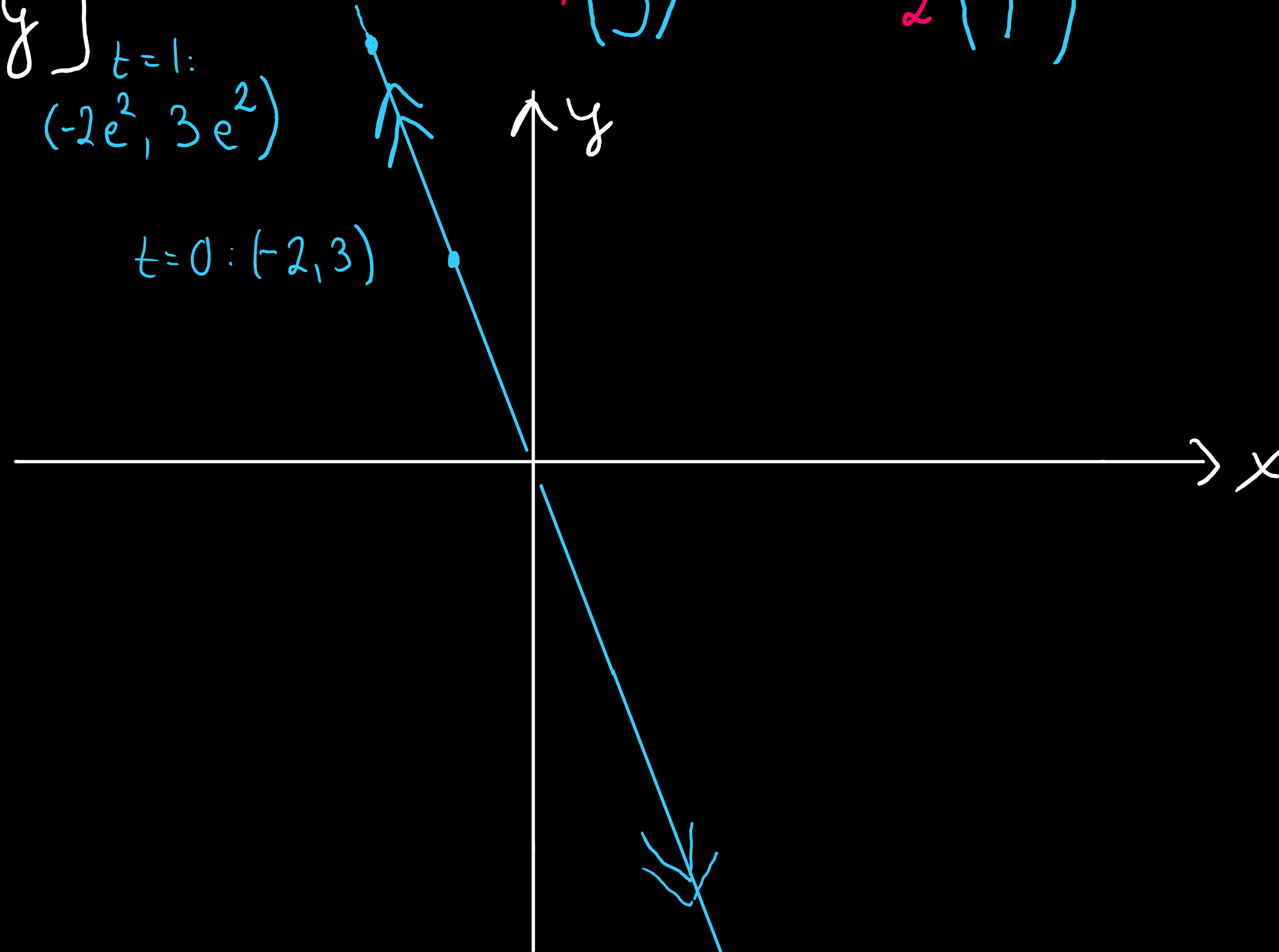
$c_2 = 0$



a)
$$\left.\begin{aligned} \dot{x} &= 5x + 2y \\ \dot{y} &= 3x + 4y \end{aligned}\right\} \quad \text{G.S: } \underline{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$$

$c_1 = -1$

$c_2 = 0$

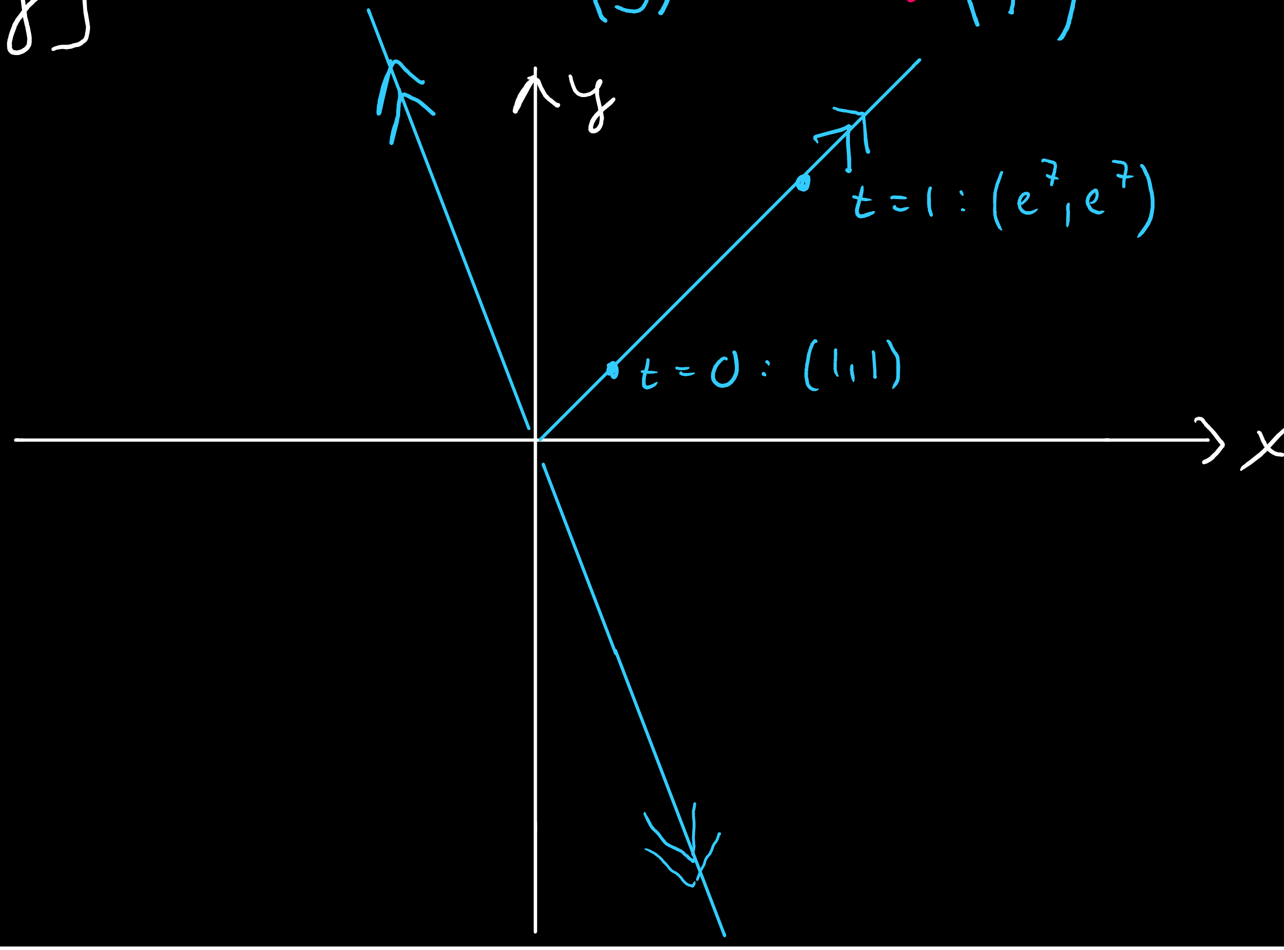


a)
$$\left.\begin{aligned}\dot{x} &= 5x + 2y \\ \dot{y} &= 3x + 4y\end{aligned}\right\}$$

G.S: $\underline{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$

$c_1 = 0$

$c_2 = 1$



a)

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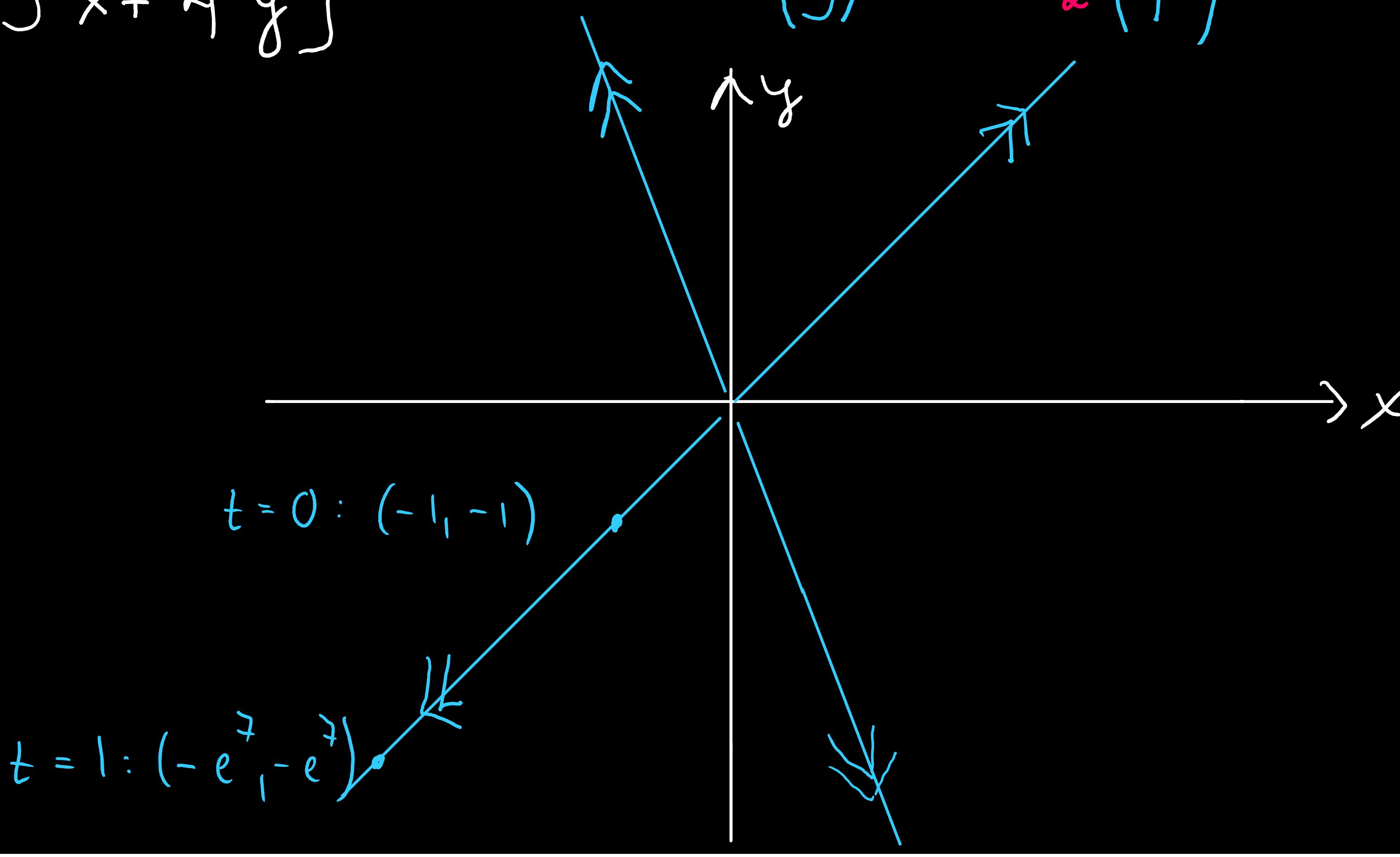
$c_1 = 0$

$c_2 = -1$



a)
$$\left.\begin{aligned}\dot{x} &= 5x + 2y \\ \dot{y} &= 3x + 4y\end{aligned}\right\}$$

G.S: $\underline{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$

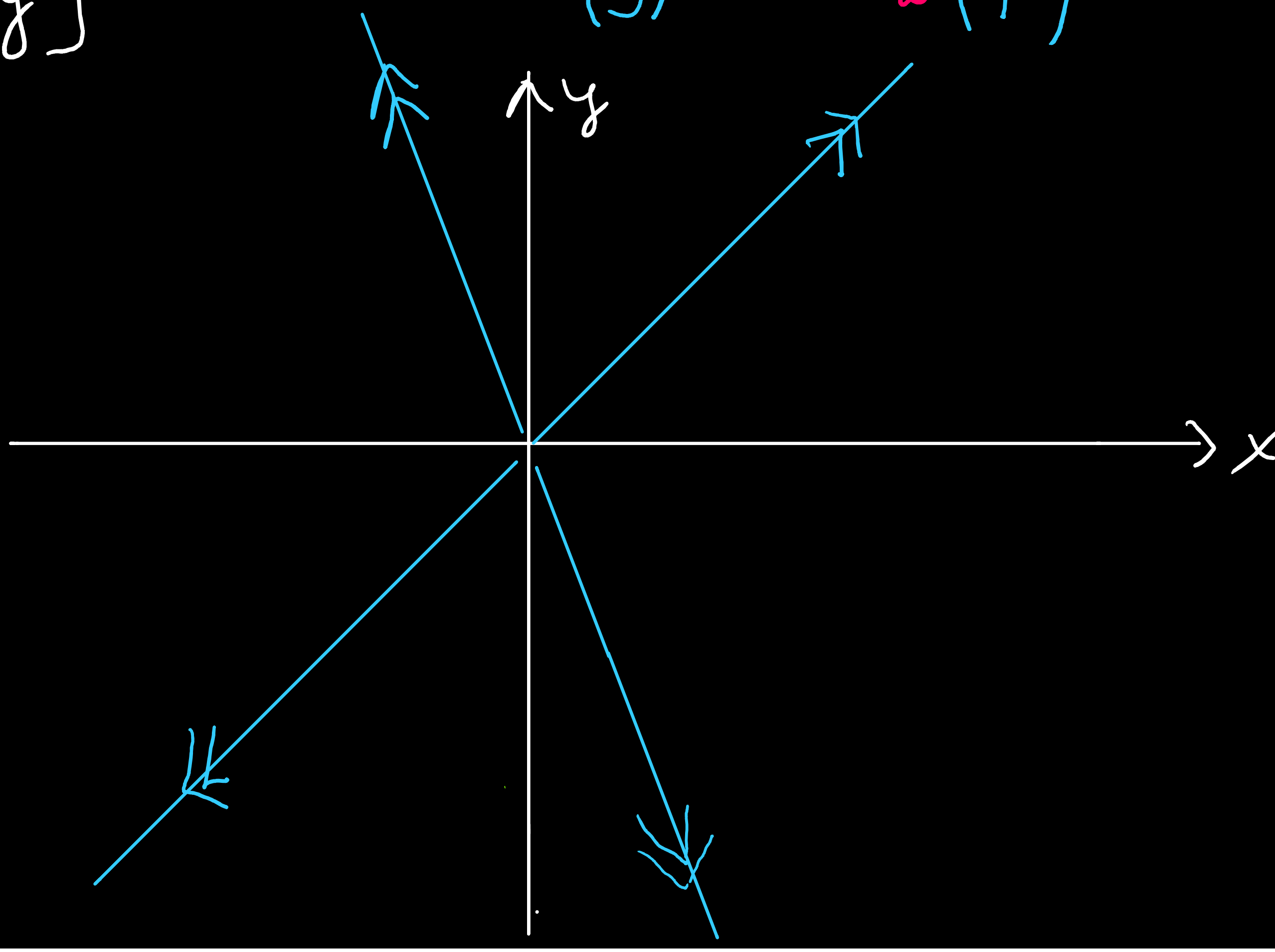
c_1, c_2 arbitrary:

$t \to \infty$: $c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$

is dominant

$t \to -\infty$: $c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t}$

is dominant



a) $\left.\begin{aligned} \dot{x} &= 5x + 2y \\ \dot{y} &= 3x + 4y \end{aligned}\right\}$ G.S: $\underline{x}(t) = c_1 \begin{pmatrix} 2 \\ -3 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t}$

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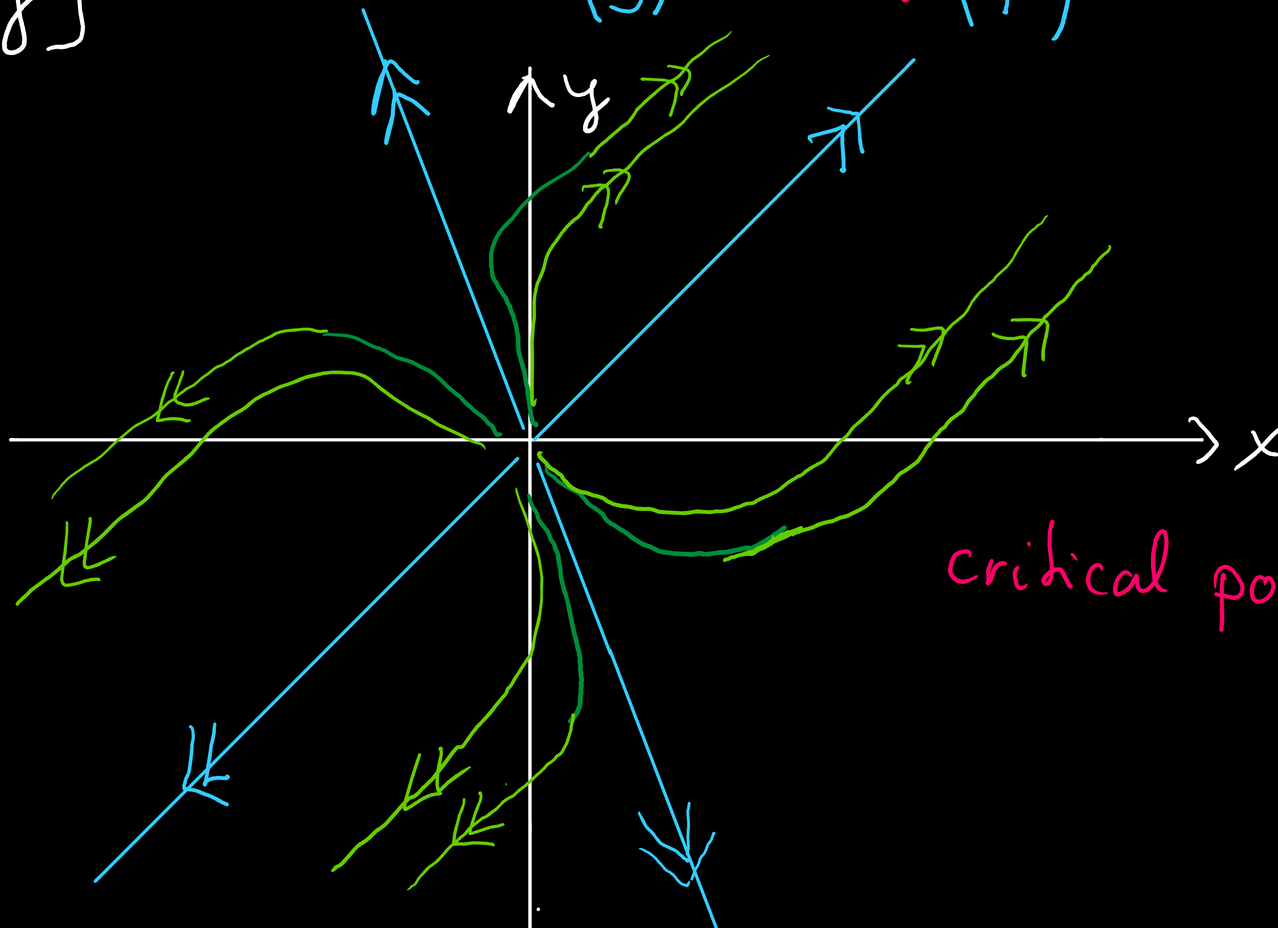


critical point

b)

$$\left.\begin{aligned} \dot{x} &= -3x + y \\ \dot{y} &= x - 3y \end{aligned}\right\}$$

G.S: $\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-4t}$

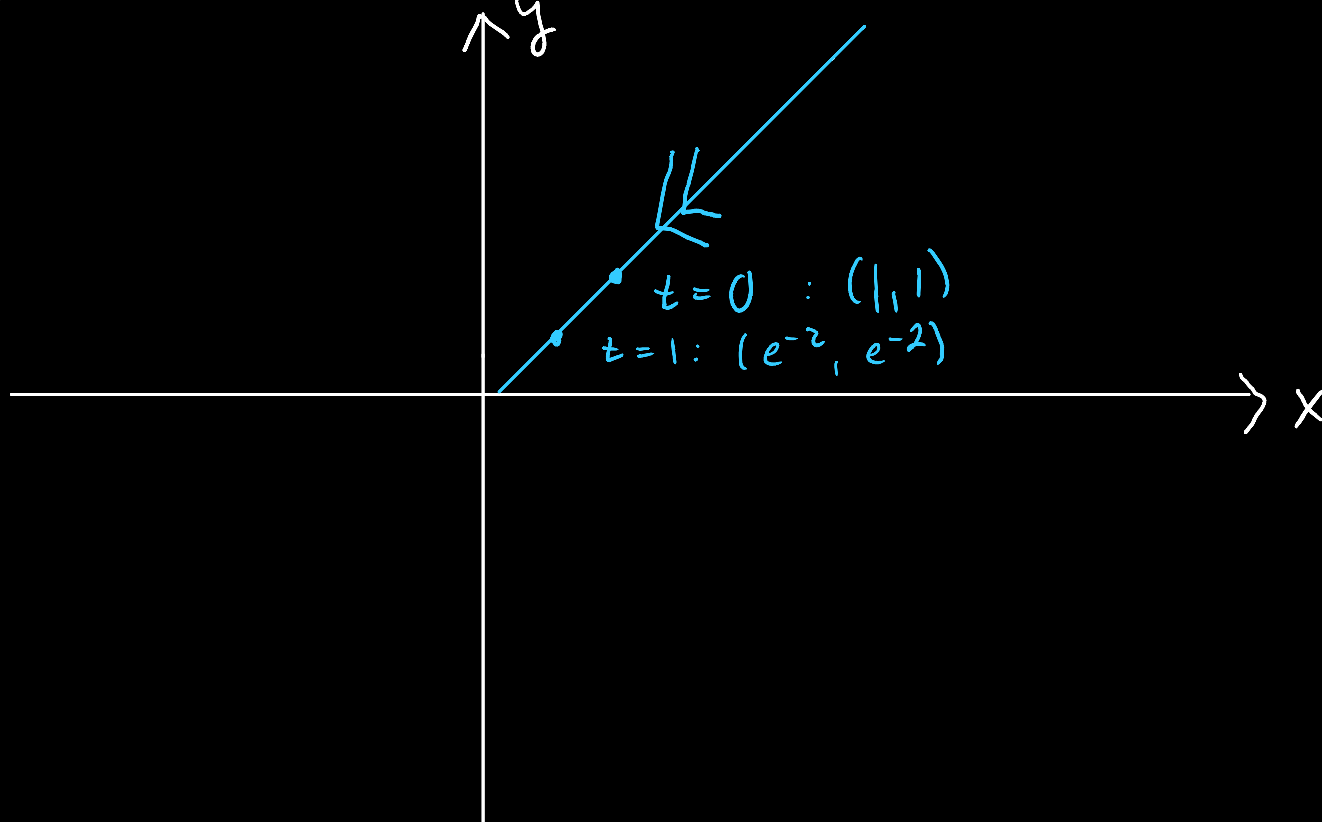
Equilibrium solution : $x(t) = 0$, $y(t) = 0$

Critical point : $(0, 0)$

b)
$$\left.\begin{aligned}\dot{x} &= -3x + y \\ \dot{y} &= x - 3y\end{aligned}\right\}$$

G.S: $\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-4t}$

$c_1 = 1$, $c_2 = 0$



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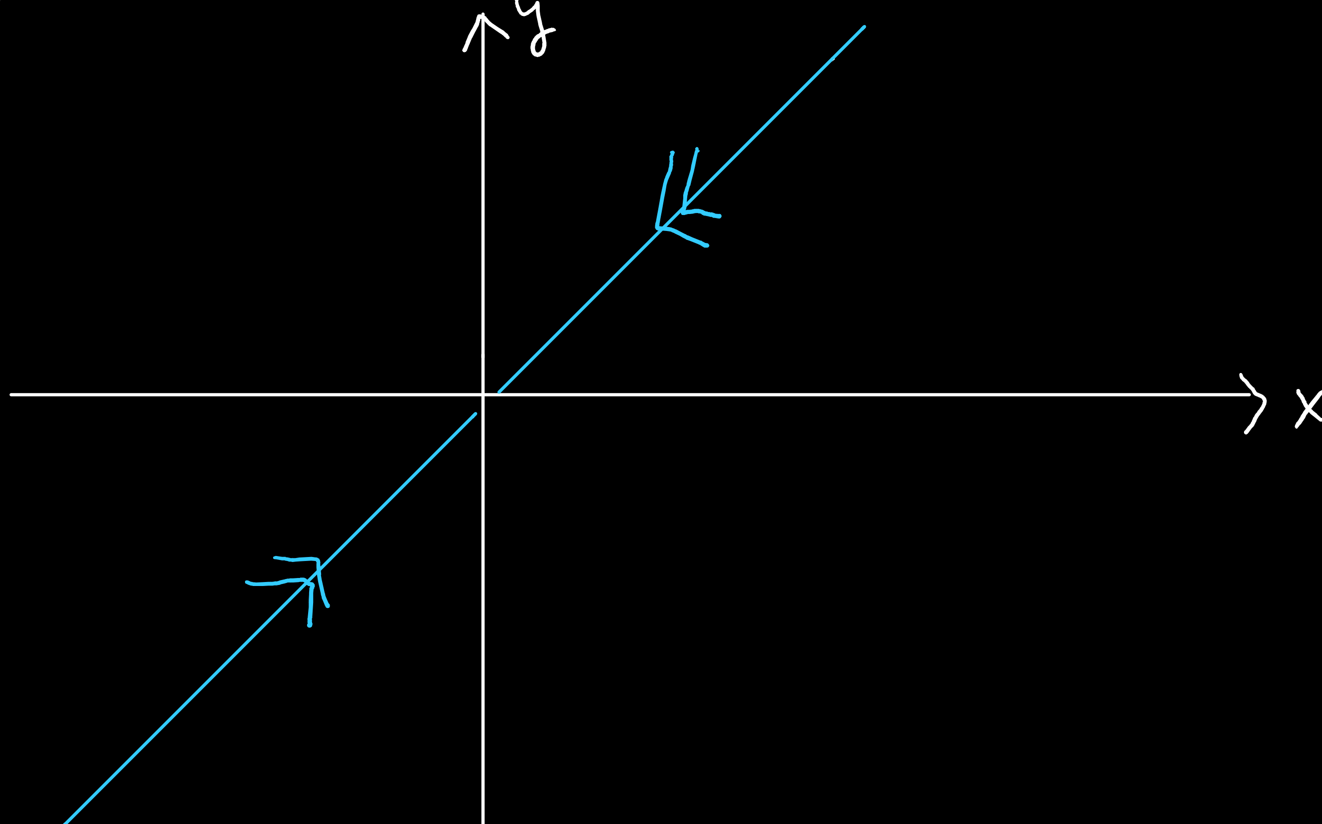
$c_1 = 0, \quad c_2 = 1$



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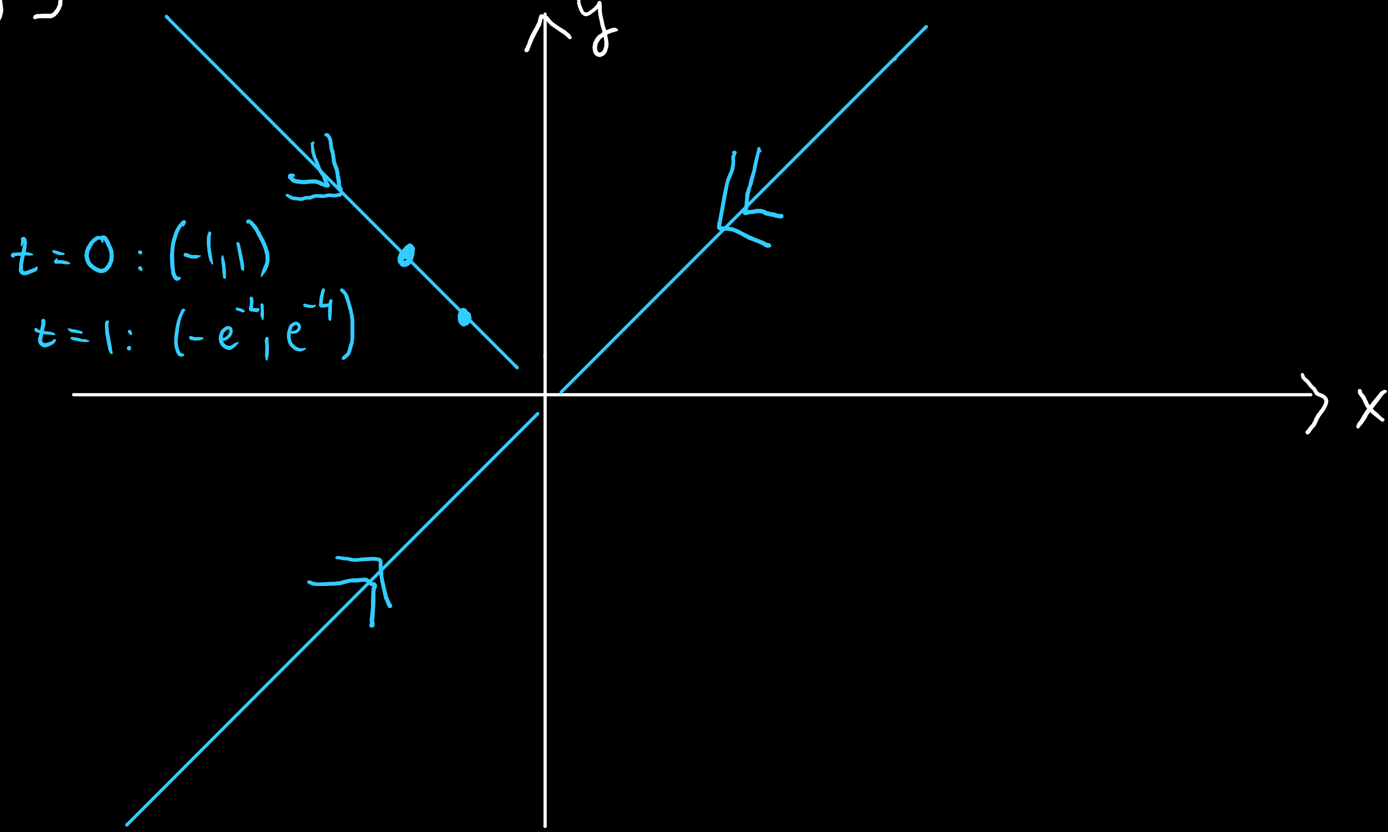
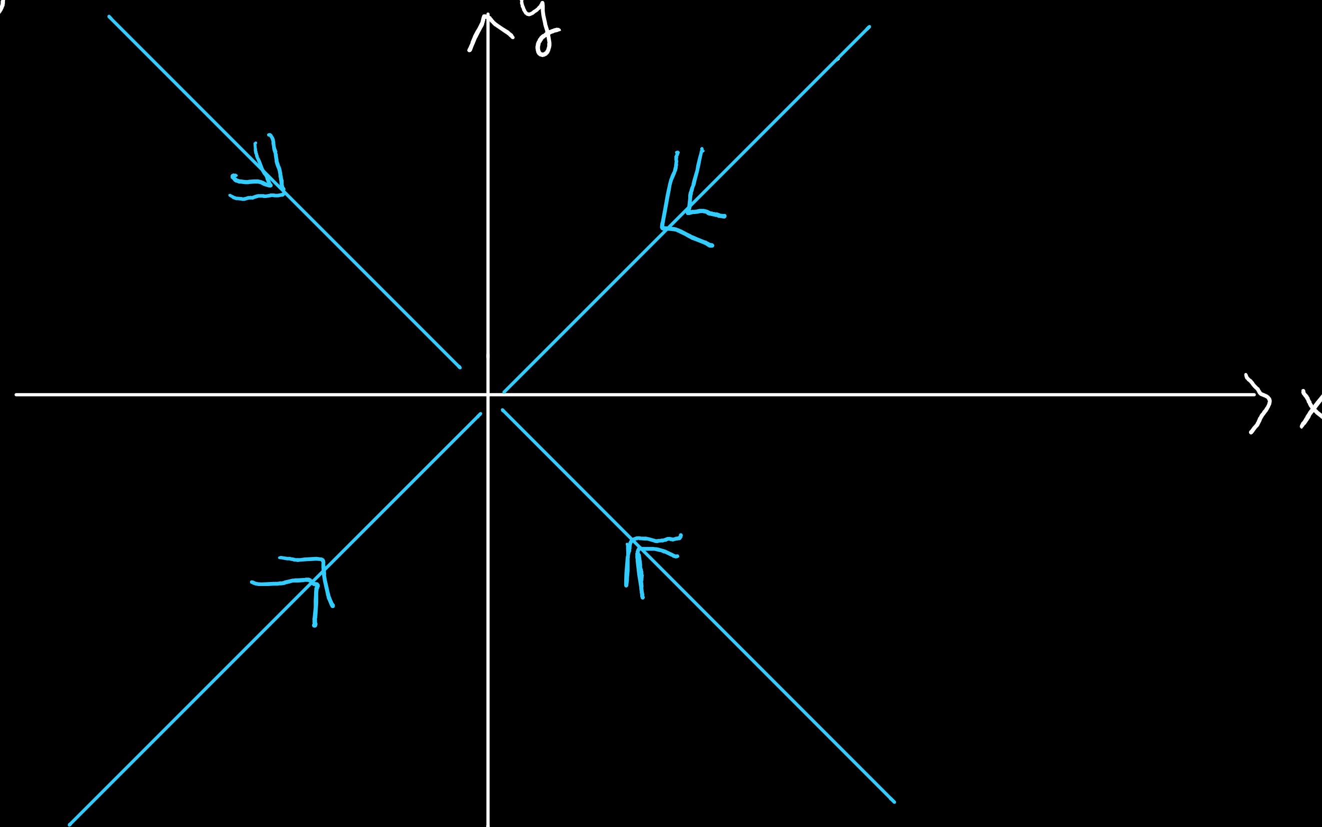
$c_1 = 0, \quad c_2 = 1$



b) $\left.\begin{aligned}\dot{x} &= -3x + y \\ \dot{y} &= x - 3y\end{aligned}\right\}$

$$G.S: \ \underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-4t}$$

c_1, c_2 are arbitrary



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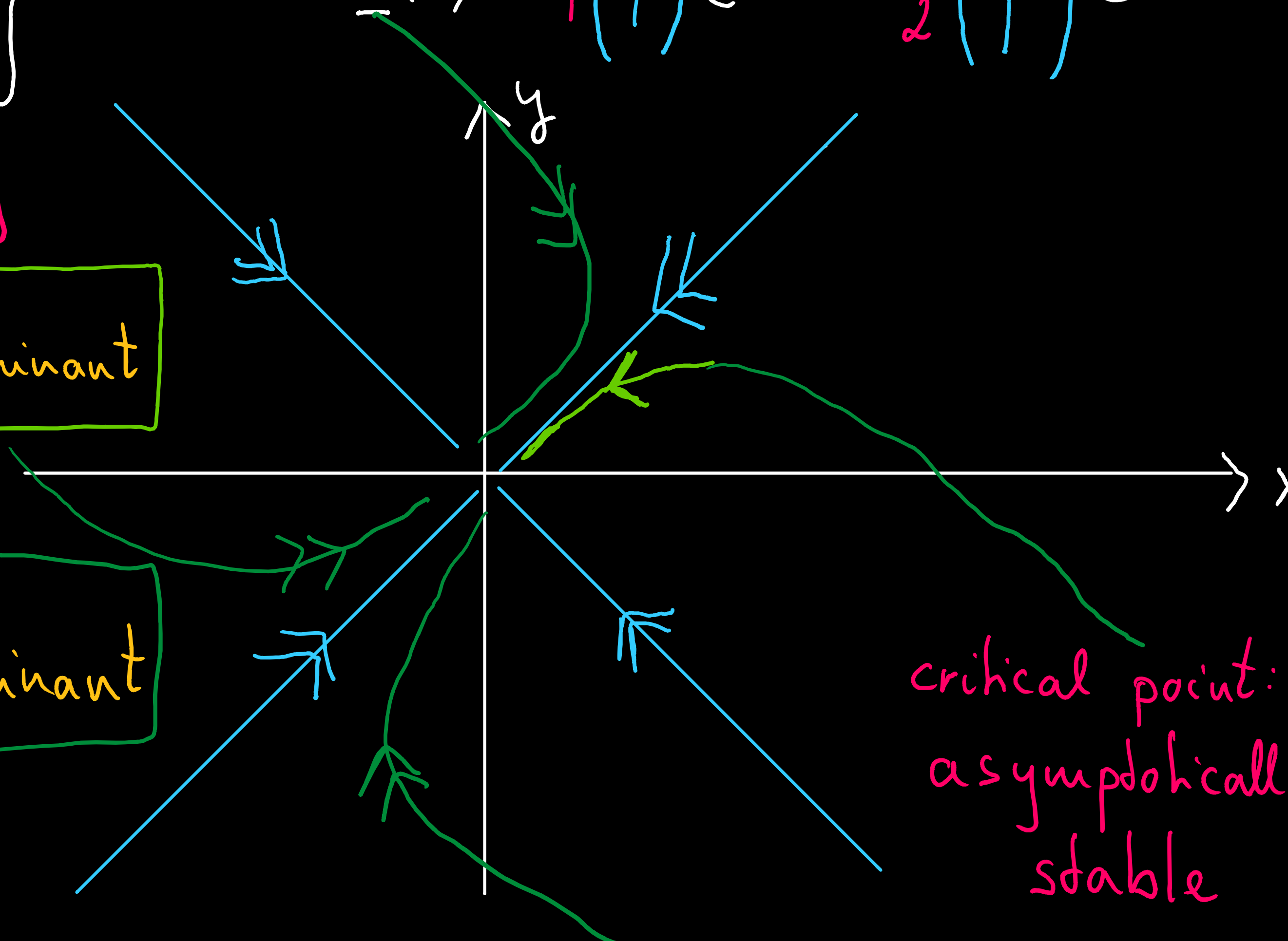
c_1, c_2 are arbitrary

$$\boxed{t \to \infty : \ c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} \text{ dominant}}$$

$$\boxed{t \to -\infty : \ c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-4t} \text{ dominant}}$$



critical point: node
asymptotically
stable

c)

$$\left.\begin{aligned}\dot{x} &= x + 2y \\ \dot{y} &= 2x + y\end{aligned}\right\} \quad \text{G.S: } \underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

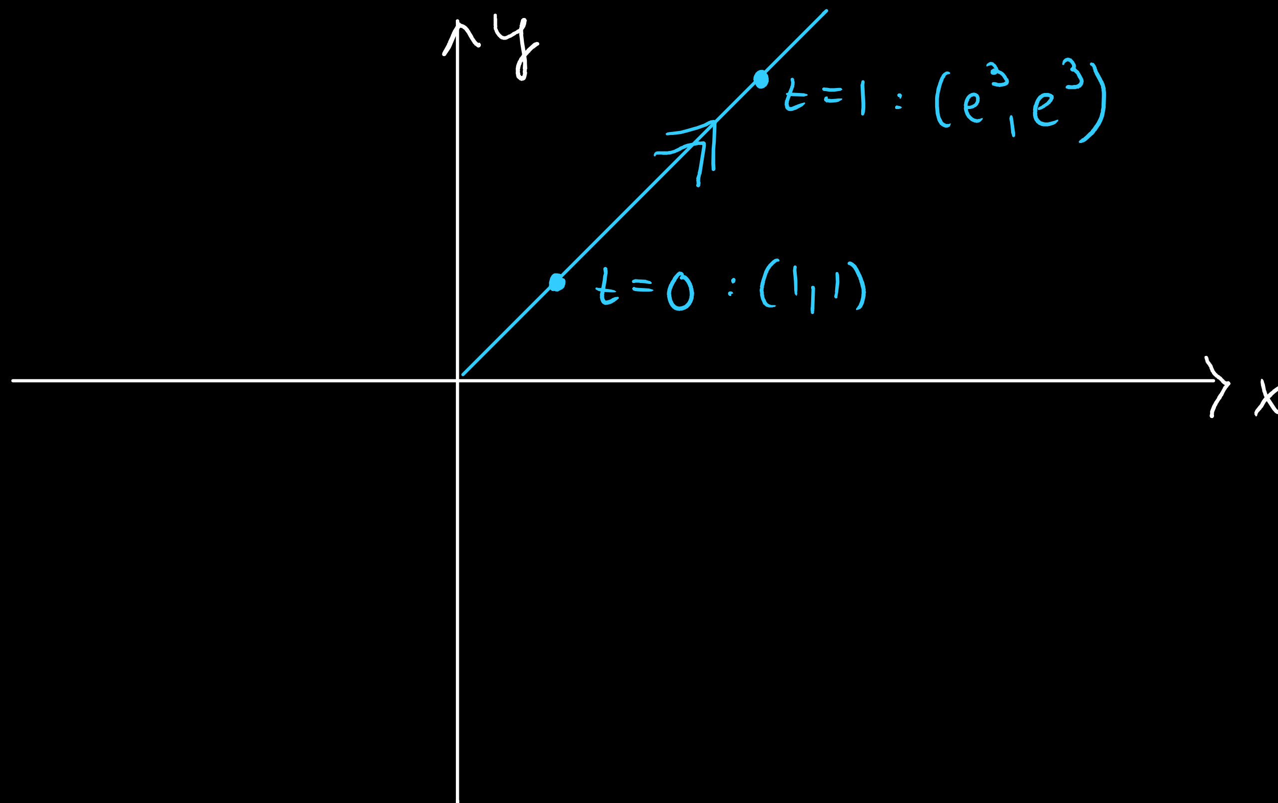
Equilibrium solution : $x(t) = 0$, $y(t) = 0$

Critical point : $(0, 0)$

eg $\left.\begin{aligned} \dot{x} &= x + 2y \\ \dot{y} &= 2x + y \end{aligned}\right\}$

G.S: $\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

$c_1 = 1, \quad c_2 = 0$



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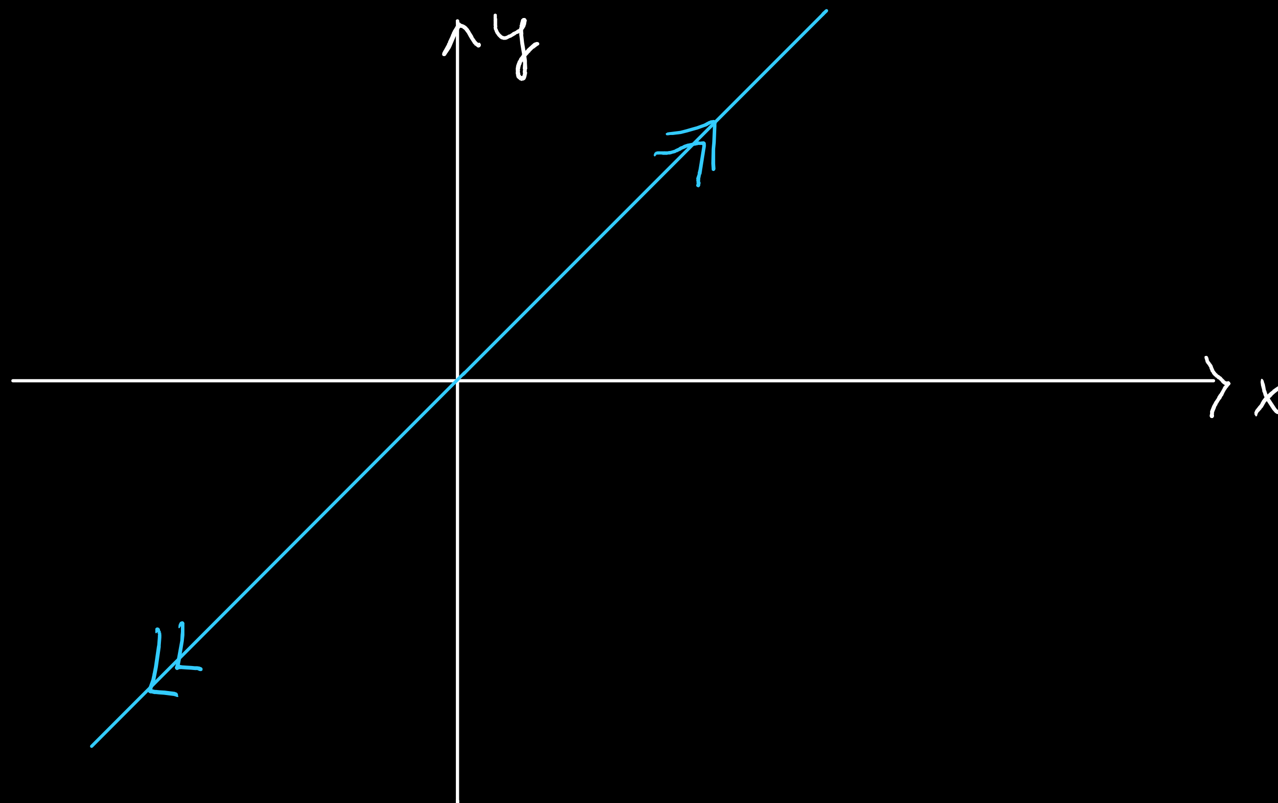
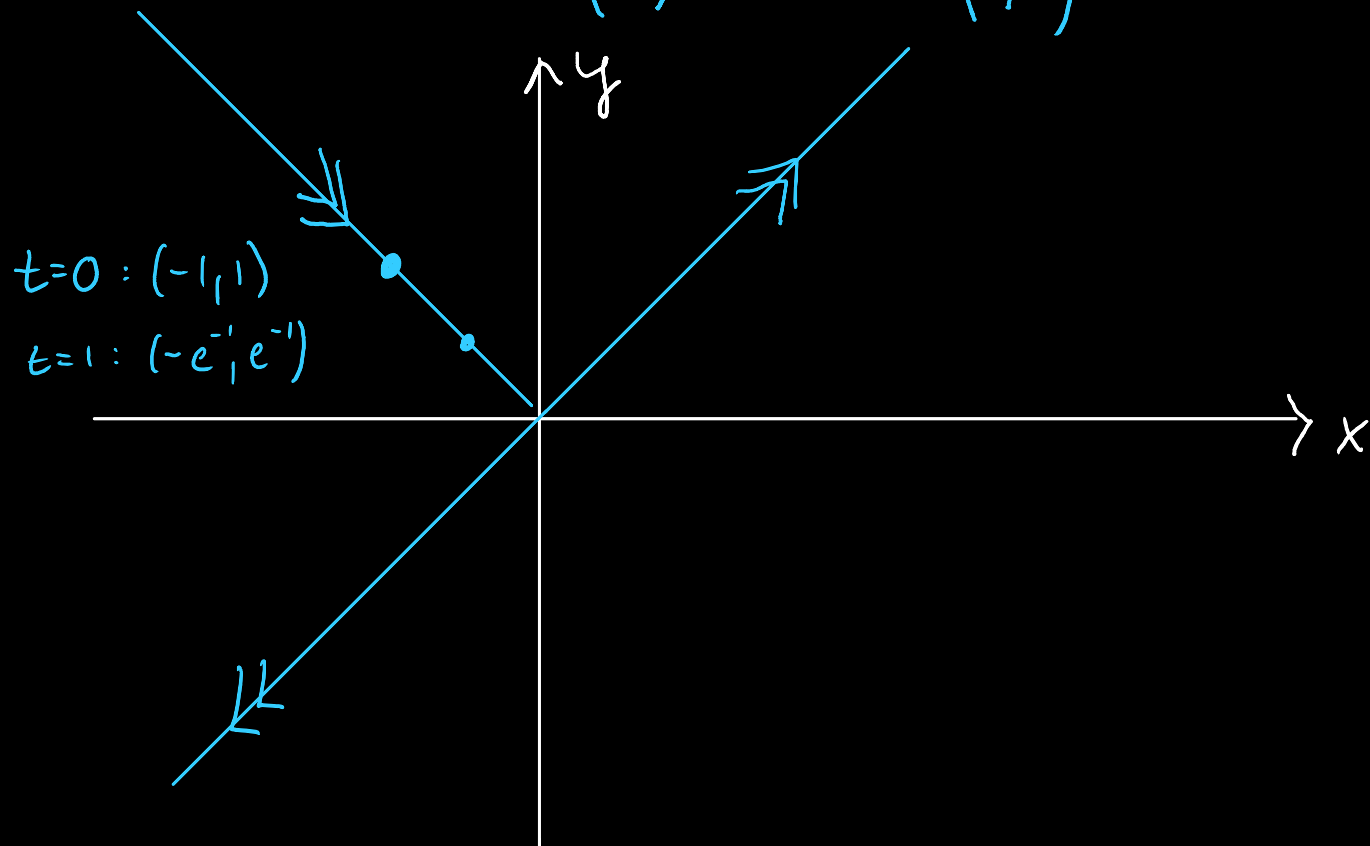
$c_1 = -1, \; c_2 = 0$



eg $\left.\begin{aligned} \dot{x} &= x + 2y \\ \dot{y} &= 2x + y \end{aligned}\right\}$

G.S: $\underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$

$c_1 = 0, \; c_2 = 1$



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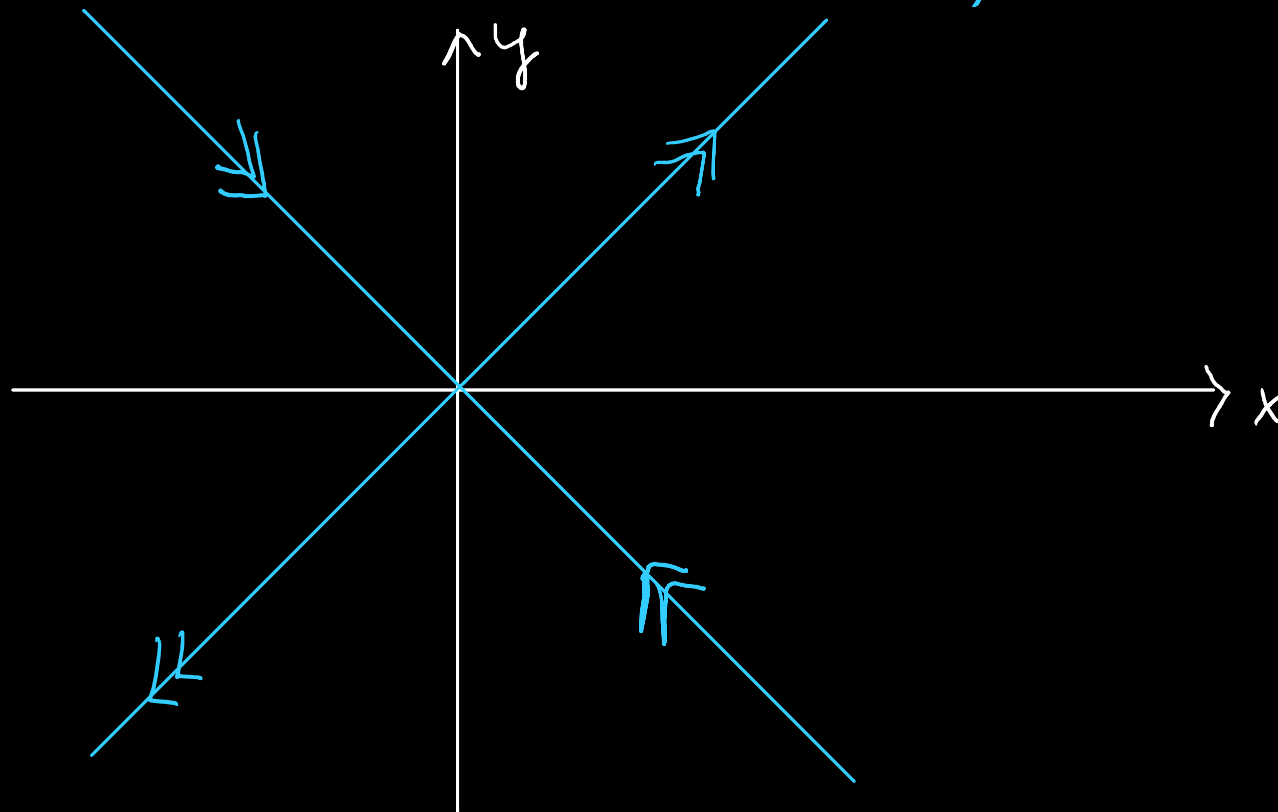
$c_1 = 0, \ c_2 = -1$



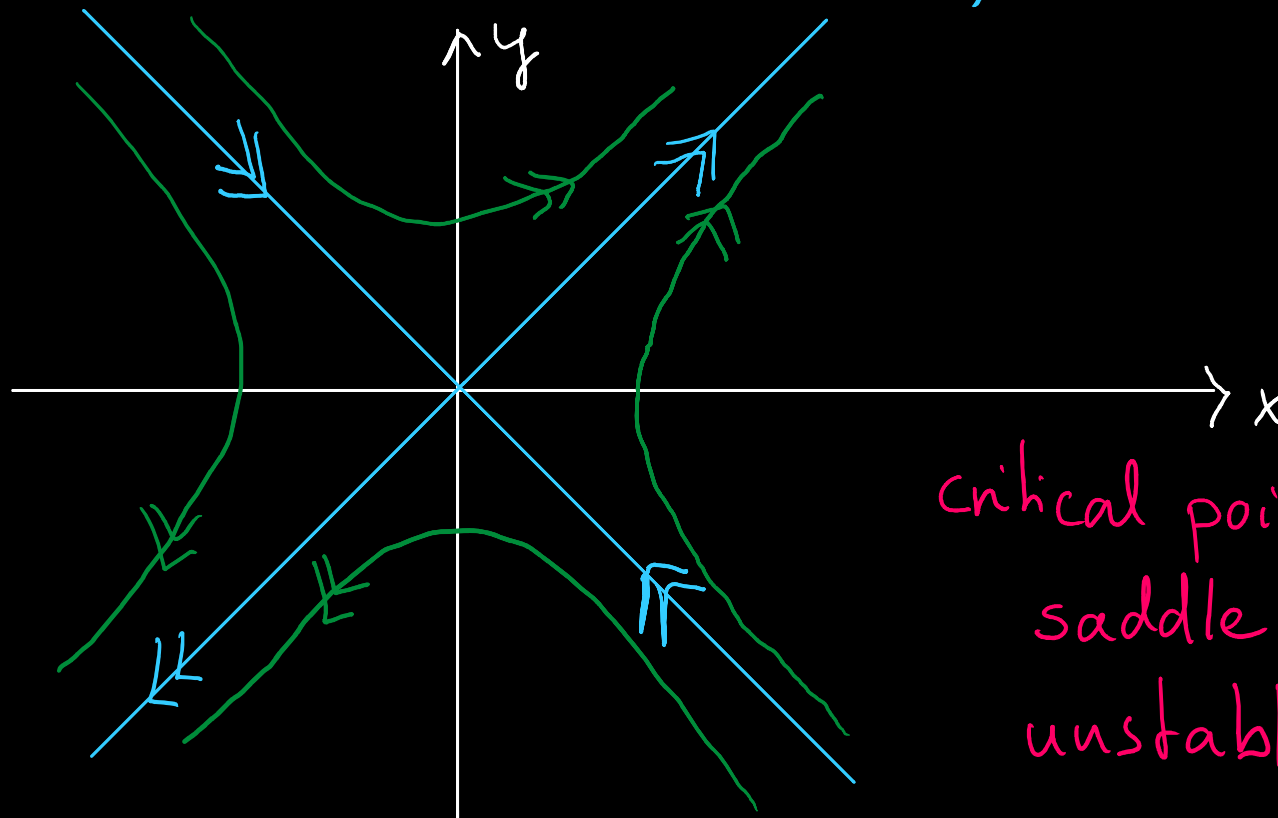
eg $\left.\begin{aligned}\dot{x} &= x + 2y \\ \dot{y} &= 2x + y\end{aligned}\right\}$

$$\text{G.S: } \underline{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$$

c_1, c_2 arbitrary:

$t \to \infty$: $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t}$ is dominant

$t \to -\infty$: $c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t}$ is dominant



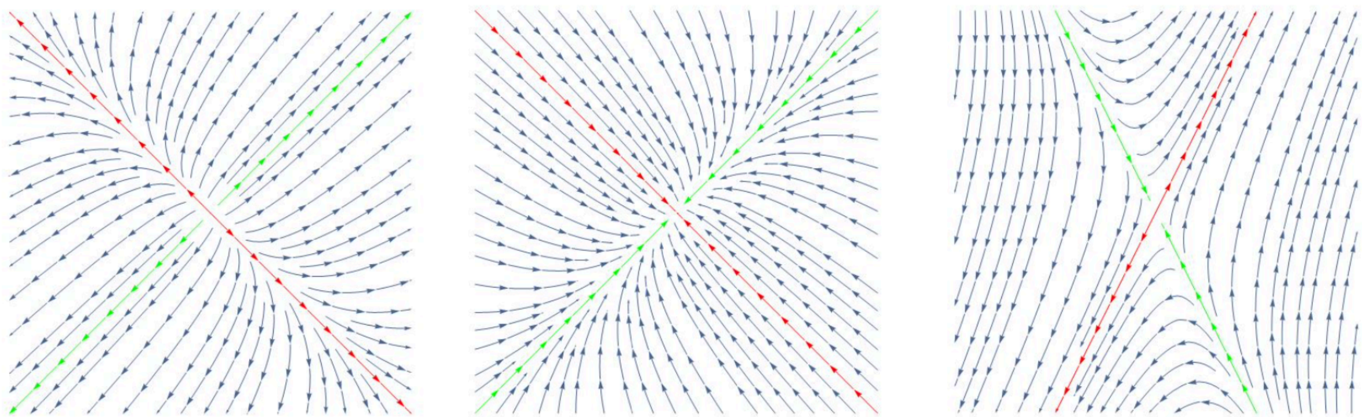
critical point: saddle unstable

Definition 13: Naturure/Type of critical points

Consider the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$. A **node** is a critical point P_0 at which every trajectory is either directed towards P_0 (nodal sink) or away from P_0 (nodal source).
A **saddle point** is a critical point P_0 at which two trajectories are incoming at P_0, and two are outcoming (corresponding to straight lines defined by eigenvectors); all the other trajectories in the neighbourhood of P_0 bypass P_0.

Summary of phase portraits
(depending on the sign and nature of the eigenvalues):

Eigenvalues	Type of Rest Point	Stability
$\lambda_1 > \lambda_2 > 0$	node	unstable
$\lambda_1 < \lambda_2 < 0$	node	asympt. stable
$\lambda_1 < 0 < \lambda_2$	saddle	unstable



Node $0 < \lambda_1 < \lambda_2$
Unstable

Node: $\lambda_1 < \lambda_2 < 0$
Asympt. stable

Saddle: $\lambda_1 < 0 < \lambda_2$
Unstable