3 Introduction to Sturm-Liouville Theory

3.1 Introduction

Between 1836 and 1837, Sturm and Liouville published a series of papers related to second order differential equations with special boundary conditions. Their work established a new area of mathematical analysis known as Sturm-Liouville theory.

A **regular Sturm-Liouville problem** is given by

$$
\begin{aligned}
\text{ODE} &: \quad (p(x)y')' + q(x)y + \lambda\sigma(x)y = 0, \quad a < x < b \\
\text{Regular} &: \quad p(x) > 0, \quad \sigma(x) > 0, \quad a \le x \le b, \\
\text{BC} &: \quad \alpha\, y(a) + \beta\, y'(a) = 0, \quad \gamma\, y(b) + \delta\, y'(b) = 0.
\end{aligned}
$$

We will assume that $\alpha^2 + \beta^2 > 0$ and that $\gamma^2 + \delta^2 > 0$. If $p(x)$ and/or $\sigma(x)$ vanish at one or both endpoints of the interval $[a, b]$, then such points are said to be **singular**, and we have a **singular Sturm-Liouville eigenvalue problem**. In this case we consider the following boundary value problem.

$$
\begin{aligned}
\text{ODE} &: \quad (p(x)y')' + q(x)y + \lambda\sigma(x)y = 0, \quad a < x < b \\
\text{Singular} &: \quad p(a) = \sigma(a) = 0, \quad \text{and/or} \quad p(b) = \sigma(b) = 0, \\
&\quad\;\; p(x) > 0, \quad \sigma(x) > 0, \quad a < x < b, \\
\text{BC} &: \quad |y(a)| < \infty \quad \text{if } a \text{ is singular}, \quad y(a) = 0 \text{ otherwise}, \\
&\quad\;\; |y(b)| < \infty \quad \text{if } b \text{ is singular}, \quad y(b) = 0 \text{ otherwise}.
\end{aligned}
$$

Finally, there is also a **periodic Sturm-Liouville problem**.

$$
\begin{aligned}
\text{ODE} &: \quad (p(x)y')' + q(x)y + \lambda\sigma(x)y = 0, \quad a < x < b \\
\text{Periodic} &: \quad p(a) = p(b), \quad p(x) > 0, \quad \sigma(x) > 0, \quad a \le x \le b, \\
\text{BC} &: \quad y(a) = y(b), \quad y'(a) = y'(b).
\end{aligned}
$$

For brevity, we will often write $p(x) = p, q(x) = q, y(x) = y$ and $\sigma(x) = \sigma$.

A number of well-known equations, such as Bessel's equation or Legendre's equation can be converted in the form of a Sturm-Liouville equation. Consider the general second order differential equation:

$$a(x)y'' + b(x)y' + c(x)y + \lambda d(x)y = 0. \tag{3.1}$$

A general way to convert this equation in the form of a Sturm-Liouville equation is to use an **integrating factor** $\nu(x)$. The procedure is as follows: Multiply (3.1) by $\nu(x)$ to obtain

$$a(x)\nu(x)y'' + b(x)\nu(x)y' + c(x)\nu(x)y + \lambda d(x)\nu(x)y = 0 \tag{3.2}$$

and find $\nu(x)$ from the condition that

$$(a(x)\nu(x))' = b(x)\nu(x).$$

By setting $p(x) = a(x)\nu(x)$ we obtain an ODE with respect to $p(x)$ with separable variables:

$$p' = p\frac{b(x)}{a(x)} \iff \frac{p'}{p} = \frac{b(x)}{a(x)} \iff \ln|p| = \int \frac{b(x)}{a(x)}dx.$$

which has solution

$$p(x) = \exp\Big(\int \frac{b(x)}{a(x)}dx\Big).$$

Hence, the integrating factor is given by $\nu(x) = \dfrac{p(x)}{a(x)}$, that is $\nu(x) = \dfrac{1}{a(x)}\exp\Big(\int \dfrac{b(x)}{a(x)}dx\Big)$ and (3.1) is a Sturm-Liouville equation

$$p(x)y'' + p'(x)y' + q(x)y + \lambda\sigma(x)y = 0$$

with

$$p(x) = \exp\Big(\int \frac{b(x)}{a(x)}dx\Big), \quad q(x) = \frac{c(x)}{a(x)}p(x), \quad \sigma(x) = \frac{d(x)}{a(x)}p(x).$$

Tasks: Sturm-Liouville form

Example 3.1 [Lecture example] *Convert the following equation in the form of a Sturm-Liouville equation:*

$$y'' + 2y' + y + \lambda y = 0.$$

✓ Solve exercise: Tutorial question 1.

3.2 Eigenvalues, eigenfunctions

In this section we are going to consider regular Sturm-Liouville problems. It will be convenient to introduce the following linear differential operator:

$$L[y] = (p(x)y')' + q(x)y.$$

Note that the Sturm-Liouville differential equation can be written as

$$L[y] + \lambda\sigma(x)y = 0.$$

Theorem 11: Lagrange's identity

Assume that $u = u(x)$ and $v = v(x)$ have continuous second derivatives on the interval $a \leq x \leq b$ and that these functions satisfy the boundary conditions of the regular Sturm-Liouville problem. Then

$$\int_a^b uL[v] - vL[u]dx = 0.$$

Proof (Outline): Use integration by parts to obtain that

$$\begin{aligned}
\int_a^b uL[v] - vL[u]dx &= \int_a^b u((pv')' + qv) - v((pu')' + qu)dx \\
&= \int_a^b u(pv')' - v(pu')'dx = \int_a^b u(pv')'dx - \int_a^b v(pu')'dx \\
&= \Big[upv'\Big]_a^b - \int_a^b u'pv'dx - \left(\Big[vpu'\Big]_a^b - \int_a^b v'pu'dx\right) \\
&= \Big[p\Big(uv' - u'v\Big)\Big]_a^b - \int_a^b p[u'v' - u'v']dx.
\end{aligned}$$

Clearly, the latter integral equals to zero. Use the boundary conditions to obtain that $\Big[p\Big(uv' - u'v\Big)\Big]_a^b = 0$ which completes the proof.

Definition 14: Eigenvalues and eigenfunctions of an S-L problem

Consider a Sturm-Liouville problem $L[y] + \lambda\sigma(x)y = 0$. The values of the parameter λ for which there exists nontrivial solutions of the problem are called **eigenvalues**. The corresponding nontrivial solutions $y(x)$ are called **eigenfunctions**.

Tasks: Eigenvalues and eigenfunctions

Example 3.2 *Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem*

$$y'' + \lambda y = 0, \ \ y(0) = 0, \ \ y(\pi) = 0.$$

✓ ~~Watch video: Example 3.2~~

✓ Solve exercise: Tutorial question 2.

Definition 15: Orthogonal eigenfunctions

Let λ_n and λ_m be **different** eigenvalues (i.e. $n \neq m$) with corresponding eigenfunctions $y_n(x)$ and $y_m(x)$. We say that these functions are **orthogonal** on $[a, b]$ with weight $\sigma(x)$ if

$$\int_a^b \sigma(x)\, y_n(x)\, y_m(x)\, dx = 0, \quad n \neq m.$$

The next theorem describes some of the basic properties of eigenvalues and eigenfunctions of a regular Sturm-Liouville problem.

Theorem 12: Properties of eigenvalues and eigenfunctions

Consider a regular Sturm-Liouville Problem.

1. All the eigenvalues are real and nonnegative and they form an infinite sequence $\{\lambda_n\}$ with $\lambda_n \to \infty$.
2. To each eigenvalue there corresponds only one independent eigenfunction.
3. The eigenfunctions are orthogonal with weight $\sigma(x)$, that is,

$$\int_a^b \sigma(x) y_1(x) y_2(x) dx = 0,$$

where $y_1(x)$ and $y_2(x)$ are eigenfunctions corresponding to distinct eigenvalues λ_1 and λ_2.

Proof. We are going to verify that (3) holds. The proofs of (1) and (2) are more advanced and we are going to omit it. Let $y_1(x)$ and $y_2(x)$ be eigenfunctions corresponding to distinct eigenvalues λ_1 and λ_2. Using Lagrange's identity (Theorem 11)

we obtain that

$$\int_a^b y_1 L[y_2] - y_2 L[y_1] dx = 0.$$

Using that

$$L[y_1] + \lambda_1\sigma(x)y_1 = 0 \quad \text{and} \quad L[y_2] + \lambda_2\sigma(x)y_2 = 0$$

we then obtain that

$$\int_a^b y_1(-\lambda_2\sigma y_2) - y_2(-\lambda_1\sigma y_1)dx = 0$$

and from this,

$$(\lambda_1 - \lambda_2)\int_a^b \sigma(x)y_1(x)y_2(x)dx = 0$$

Since $\lambda_1 \neq \lambda_2$, it follows that

$$\int_a^b \sigma(x)y_1(x)y_2(x)dx = 0$$

showing that $y_1(x)$ and $y_2(x)$ are indeed orthogonal with weight $\sigma(x)$.

3.3 Rayleigh quotient

Consider the Sturm-Liouville ODE

$$(py')' + qy + \lambda\sigma y = 0.$$

Multiplying by y and integrating from a to b we have that

$$\int_a^b y(py')'dx + \int_a^b qy^2dx + \lambda\int_a^b \sigma y^2dx = 0.$$

We apply integration by parts to the first integral by setting $u = y$ and $v = py'$ to obtain

$$[pyy']_a^b - \int_a^b p(y')^2dx + \int_a^b qy^2dx + \lambda\int_a^b \sigma y^2dx = 0$$

and so

$$\lambda\int_a^b \sigma y^2dx = \int_a^b [p(y')^2 - qy^2]dx - [pyy']_a^b$$

giving

$$\lambda = \frac{\displaystyle\int_a^b [p(y')^2 - qy^2]dx - [pyy']_a^b}{\displaystyle\int_a^b \sigma y^2dx}$$

Given an eigenfunction $y(x)$ we can calculate the corresponding eigenvalue λ using the above formula. This formula is called the **Rayleigh quotient**.

3.4 Harmonic equation

A very common eigenvalue problem is

$$\begin{gathered} y'' + \lambda y = 0, \quad 0 < x < a \\ y(0) = 0 \ \text{ or } \ y'(0) = 0 \quad \text{and} \quad y(a) = 0 \ \text{ or } \ y'(a) = 0. \end{gathered} \tag{3.3}$$

The ODE $y'' + \lambda y = 0$ is called **harmonic equation**.

Tasks: Rayleigh quotient for harmonic equation

Example 3.3 *Verify that the Rayleigh quotient of*

$$\begin{gathered} y'' + \lambda y = 0, \quad 0 < x < a \\ y(0) = 0 \ \textit{ or } \ y'(0) = 0 \quad \textit{and} \quad y(a) = 0 \ \textit{ or } \ y'(a) = 0 \end{gathered}$$

is

$$\lambda = \frac{\displaystyle\int_0^a (y')^2 dx}{\displaystyle\int_0^a y^2 dx}. \tag{3.4}$$

✓ ~~Watch video: Example 3.3~~

✓ Solve exercise: Tutorial question 3.

The identity (3.4) shows that all eigenvalues of (3.3) are non-negative. Moreover, we can only have $\lambda = 0$ if $y' = 0$, that is, $y =$ constant. This may only happen when the boundary conditions are $y'(0) = 0$ and $y(a) = 0$.

To close this Chapter, we summarise below the eigenvalues and eigenfunctions for the four possible combinations of the boundary conditions:

Example 12: Eigenvalues and eigenfunctions for the harmonic equation

(i) $y(0) = 0,\ y(a) = 0$

$$\lambda_n = \frac{n^2\pi^2}{a^2}\,,\ y_n(x) = \sin\left(\frac{n\pi x}{a}\right) \quad n = 1, 2, 3, \ldots$$

(ii) $y'(0) = 0,\ y'(a) = 0$

$$\lambda_0 = 0,\ y_0(x) = 1, \quad \lambda_n = \frac{n^2\pi^2}{a^2}\,,\ y_n(x) = \cos\left(\frac{n\pi x}{a}\right) \quad n = 1, 2, 3, \ldots$$

which can be written more simply as

$$\lambda_n = \frac{n^2\pi^2}{a^2}\,, \quad y_n(x) = \cos\left(\frac{n\pi x}{a}\right), \quad n = 0, 1, 2, \ldots$$

noting that n starts at 0 in this case.

(iii) $y(0) = 0,\ y'(a) = 0$

$$\lambda_n = \left(n - \frac{1}{2}\right)^2 \frac{\pi^2}{a^2}\,, \quad y_n(x) = \sin\left[\left(n - \frac{1}{2}\right)\frac{\pi x}{a}\right] \quad n = 1, 2, 3, \ldots$$

(iv) $y'(0) = 0,\ y(a) = 0$

$$\lambda_n = \left(n - \frac{1}{2}\right)^2 \frac{\pi^2}{a^2}\,,\ y_n(x) = \cos\left[\left(n - \frac{1}{2}\right)\frac{\pi x}{a}\right] \quad n = 1, 2, 3, \ldots$$