Minimum-Area Rectangle Containing a Convex Polygon

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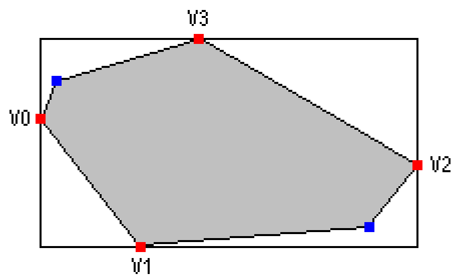
1 Introduction

Given a convex polygon with ordered vertices $\mathbf{P}_i$ for $0 \leq i < N$, the problem is to construct the minimum-area rectangle that contains the polygon. The rectangle is not required to be axis-aligned with the coordinate system axes. It is the case that at least one of the edges of the convex polygon must be contained by an edge of the minimum-area rectangle. Given this is so, an algorithm for computing the minimum-area rectangle need only compute the tightest fitting bounding rectangles whose orientations are determined by the polygon edges.

2 Proof of Edge Containment

The proof is by contradiction. Suppose that in fact no edge of the convex polygon is contained by an edge of the minimum-area rectangle. The rectangle must be supported by four vertices of the convex polygon, as illustrated by Figure 2.1

Figure 2.1 Minimum-area rectangle that has no coincident polygon edges.



The supporting vertices are drawn in red and labeled $\mathbf{V}_0$ through $\mathbf{V}_3$. Other polygon vertices are drawn in blue. For the sake of the argument, rotate the convex polygon so that the axes of this rectangle are $(1, 0)$ and $(0, 1)$ as shown in the figure.

Define $\mathbf{U}_0(\theta) = (\cos\theta, \sin\theta)$ and $\mathbf{U}_1(\theta) = (-\sin\theta, \cos\theta)$. There exists a value $\varepsilon > 0$ such that the $\mathbf{V}_i$ are always the supporting vertices of the bounding rectangle with axes $\mathbf{U}_0(\theta)$ and $\mathbf{U}_1(\theta)$ for all angles θ satisfying the condition $|\theta| <= \varepsilon$. To compute the bounding rectangle area, the supporting vertices are projected onto the axis lines $\mathbf{V}_0 + s\mathbf{U}_0(\theta)$ and $\mathbf{V}_0 + t\mathbf{U}_1(\theta)$. The intervals of projection are $[0, s_1]$ and $[t_0, t_1]$ where $s_1 = \mathbf{U}_0(\theta) \cdot (\mathbf{V}_2 - \mathbf{V}_0)$, $t_0 = \mathbf{U}_1(\theta) \cdot (\mathbf{V}_1 - \mathbf{V}_0)$, and $t_1 = \mathbf{U}_1(\theta) \cdot (\mathbf{V}_3 - \mathbf{V}_0)$.

Define $\mathbf{K}_0 = (x_0, y_0) = \mathbf{V}_2 - \mathbf{V}_0$ and $\mathbf{K}_1 = (x_1, y_1) = \mathbf{V}_3 - \mathbf{V}_1$. From Figure 1 it is clear that $x_0 > 0$ and $y_1 > 0$. The area of the rectangle for $|\theta| \leq \varepsilon$ is

$$A(\theta) = s_1(t_1 - t_0) = [\mathbf{K}_0 \cdot \mathbf{U}_0(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1(\theta)].$$

In particular, $A(0) = x_0 y_1 > 0$.

Since $A(\theta)$ is differentiable on its domain and since $A(0)$ is assumed to be the global minimum, it must be that $A'(0) = 0$. Generally,

$$\begin{aligned} A'(\theta) &= [\mathbf{K}_0 \cdot \mathbf{U}_0(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1'(\theta)] + [\mathbf{K}_0 \cdot \mathbf{U}_0'(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1(\theta)] \\ &= -[\mathbf{K}_0 \cdot \mathbf{U}_0(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_0(\theta)] + [\mathbf{K}_0 \cdot \mathbf{U}_1(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1(\theta)] \end{aligned}$$

Therefore, $0 = A'(0) = -x_0x_1 + y_0y_1$, or $x_0x_1 = y_0y_1$. Since $x_0 > 0$ and $y_1 > 0$, it must be that $\mathrm{Sign}(x_1) = \mathrm{Sign}(y_0)$. Moreover, since $A(0)$ is assumed to be the global minimum, it must be that $A''(0) \geq 0$. Generally,

$$\begin{aligned} A''(\theta) &= -[\mathbf{K}_0 \cdot \mathbf{U}_0(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_0'(\theta)] - [\mathbf{K}_0 \cdot \mathbf{U}_0'(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_0(\theta)] \\ &\quad +[\mathbf{K}_0 \cdot \mathbf{U}_1(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1'(\theta)] + [\mathbf{K}_0 \cdot \mathbf{U}_1'(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1(\theta)] \\ &= -[\mathbf{K}_0 \cdot \mathbf{U}_0(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1(\theta)] - [\mathbf{K}_0 \cdot \mathbf{U}_1(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_0(\theta)] \\ &\quad -[\mathbf{K}_0 \cdot \mathbf{U}_1(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_0(\theta)] - [\mathbf{K}_0 \cdot \mathbf{U}_0(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1(\theta)] \\ &= -2\left\{[\mathbf{K}_0 \cdot \mathbf{U}_0(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_1(\theta)] + [\mathbf{K}_0 \cdot \mathbf{U}_1(\theta)][\mathbf{K}_1 \cdot \mathbf{U}_0(\theta)]\right\} \end{aligned}$$

In particular, $A''(0) = -2(x_0y_1 + x_1y_0) \geq 0$. However, note that $x_0y_1 > 0$ since $A(0) > 0$ and $x_1y_0 > 0$ since $\mathrm{Sign}(x_1) = \mathrm{Sign}(y_0)$, which implies that $A''(0) < 0$, a contradiction.

3 The Algorithm

Pseudocode for the algorithm is given below.

```
ordered vertices P[0] through P[N-1];
define P[N] = P[0];

minimumArea = infinity;
for (i = 1; i <= N; i++)
{
    U0 = P[i] - P[i-1];
    U0 /= U0.Length();
    U1 = (-U0.y,U0.x);
    s0 = t0 = s1 = t1 = 0;
    for (j = 1; j < N; j++)
    {
        D = P[j] - P[0];
        test = Dot(U0,D);
        if ( test < s0 ) s0 = test; else if ( test > s1 ) s1 = test;
        test = Dot(U1,D);
        if ( test < t0 ) t0 = test; else if ( test > t1 ) t1 = test;
    }
    area = (s1-s0)*(t1-t0);
    if ( area < minimumArea )
        minimumArea = area;
}
```