A Collinearity Test Independent of Input Point Order

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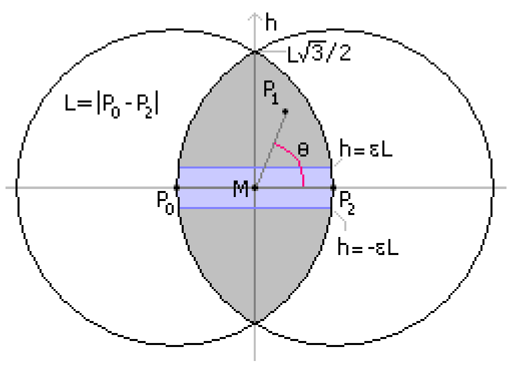
1 Discussion

Given three points $\mathbf{Q}_i$, $0 \le i \le 2$, construct an algorithm for determining collinearity that is order-independent when implemented in a floating point number system. Within that system the points can be labeled as collinear when they are "nearly" collinear with a suitable definition for what means "nearly".

Let i_0 and i_2 be the indices of those points that are farthest apart. Let i_1 be the other index. Define $\mathbf{P}_j = \mathbf{Q}_{i_j}$. Points $\mathbf{P}_0$ and $\mathbf{P}_2$ are farthest apart. Figure 1.1 shows the region that must contain $\mathbf{P}_1$. This region is the intersection of two circles centered at $\mathbf{P}_0$ and $\mathbf{P}_1$, each of radius $L = |\mathbf{P}_0 - \mathbf{P}_1|$.

Figure 1.1 The region that must contain $\mathbf{P}_1$.



The violet region in the figure contains those points within distance εL, $\varepsilon \in [0, \sqrt{3}/2]$, from the line segment connecting $\mathbf{P}_0$ and $\mathbf{P}_2$. For a user-selected small ε, if $\mathbf{P}_1$ is in the violet region we will say that the points are (nearly) collinear.

Mathematically it is sufficient to calculate the length of the projection of $\mathbf{P}_1 - \mathbf{P}_0$ onto the orthogonal complement of the line through $\mathbf{P}_0$ and $\mathbf{P}_2$, then compare that value to εL. However, this has the potential to be order-dependent since swapping the roles $\mathbf{P}_0$ and $\mathbf{P}_2$ could lead to some numerical significance between the projections of $\mathbf{P}_1 - \mathbf{P}_0$ and $\mathbf{P}_1 - \mathbf{P}_2$. Instead define $\mathbf{M} = (\mathbf{P}_0 + \mathbf{P}_2)/2$ and project $\mathbf{\Delta} = \mathbf{P}_1 - \mathbf{M}$ onto the orthogonal complement. That distance is $|\mathbf{\Delta} - (\mathbf{U} \cdot \mathbf{\Delta})\mathbf{U}|$ where $\mathbf{U} = (\mathbf{P}_2 - \mathbf{P}_0)/L$. In squared terms,

$$|\mathbf{\Delta}|^2 - (\mathbf{U} \cdot \mathbf{\Delta})^2 \le \varepsilon^2 L^2 \ \text{ or } \ |\mathbf{\Delta}|^2 \le (\mathbf{U} \cdot \mathbf{\Delta})^2 + (\varepsilon L)^2$$

Without vector normalization, the test is

$$L^2 |\mathbf{\Delta}|^2 \le ((\mathbf{P}_2 - \mathbf{P}_0) \cdot \mathbf{\Delta})^2 + \varepsilon^2 L^4.$$

If $\mathbf{V} = \mathbf{\Delta}/L$, then the test is also equivalent to

$$|\mathbf{V}| \sin\theta \le \varepsilon$$

where θ is the angle between $\mathbf{U}$ and $\mathbf{V}$. The left-hand side, when multiplied by L, is just the projection of $\mathbf{P}_1 - \mathbf{M}$ onto the vertical axis. In geometric terms this requires either the length of $\mathbf{P}_1 - \mathbf{M}$ to be small compared to that of $\mathbf{P}_2 - \mathbf{P}_0$ or the angle between these two vectors to be small.