Intersection of Ellipses

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Contents

1 Introduction

This article describes how to compute the points of intersection of two ellipses, a geometric query labeled *find intersections*. It also shows how to determine if two ellipses intersect without computing the points of intersection, a geometric query labeled *test intersection*. Specifically, the geometric queries for the ellipses E_0 and E_1 are:

- *Find Intersections.* If E_0 and E_1 intersect, find the points of intersection.
- *Test Intersection.* Determine if
  - E_0 and E_1 are separated (there exists a line for which the ellipses are on opposite sides),
  - E_0 properly contains E_1 or E_1 properly contains E_0, or
  - E_0 and E_1 intersect.

An implementation of the *find* query, in the event of no intersections, might not necessarily determine if one ellipse is contained in the other or if the two ellipses are separated. Let the ellipses E_i be defined by the quadratic equations

$$
\begin{aligned}
Q_i(\mathbf{X}) &= \mathbf{X}^{\mathrm{T}} A_i \mathbf{X} + \mathbf{B}_i^{\mathrm{T}} \mathbf{X} + C_i \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{00}^{(i)} & a_{01}^{(i)} \\ a_{01}^{(i)} & a_{11}^{(i)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_0^{(i)} & b_1^{(i)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + c^{(i)} \\
&= 0
\end{aligned}
$$

for $i = 0, 1$. It is assumed that the A_i are positive definite. In this case, $Q_i(\mathbf{X}) < 0$ defines the inside of the ellipse and $Q_i(\mathbf{X}) > 0$ defines the outside.

2 Find Intersection

The two polynomials $f(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2$ and $g(x) = \beta_0 + \beta_1 x + \beta_2 x^2$ have a common root if and only if the Bézout determinant is zero,

$$(\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_1\beta_0 - \alpha_0\beta_1) - (\alpha_2\beta_0 - \alpha_0\beta_2)^2 = 0.$$

This is constructed by the combinations

$$0 = \alpha_2 g(x) - \beta_2 f(x) = (\alpha_2\beta_1 - \alpha_1\beta_2)x + (\alpha_2\beta_0 - \alpha_0\beta_2)$$

and

$$0 = \beta_1 f(x) - \alpha_1 g(x) = (\alpha_2\beta_1 - \alpha_1\beta_2)x^2 + (\alpha_0\beta_1 - \alpha_1\beta_0),$$

solving the first equation for x and substituting it into the second equation. When the Bézout determinant is zero, the common root of $f(x)$ and $g(x)$ is

$$\bar{x} = \frac{\alpha_2\beta_0 - \alpha_0\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}.$$

The common root to $f(x) = 0$ and $g(x) = 0$ is obtained from the linear equation $\alpha_2 g(x) - \beta_2 f(x) = 0$ by solving for x.

The ellipse equations can be written as quadratics in x whose coefficients are polynomials in y,

$$Q_i(x, y) = \left(a_{11}^{(i)} y^2 + b_1^{(i)} y + c^{(i)}\right) + \left(2a_{01}^{(i)} y + b_0^{(i)}\right) x + \left(a_{00}^{(i)}\right) x^2.$$

Using the notation of the previous paragraph with f corresponding to Q_0 and g corresponding to Q_1,

$$\begin{array}{l} \alpha_0 = a_{11}^{(0)} y^2 + b_1^{(0)} y + c^{(0)},\ \ \alpha_1 = 2a_{01}^{(0)} y + b_0^{(0)},\ \ \alpha_2 = a_{00}^{(0)}, \\ \beta_0 = a_{11}^{(1)} y^2 + b_1^{(1)} y + c^{(1)},\ \ \beta_1 = 2a_{01}^{(1)} y + b_0^{(1)},\ \ \beta_2 = a_{00}^{(1)}. \end{array}$$

The Bézout determinant is a quartic polynomial $R(y) = u_0 + u_1 y + u_2 y^2 + u_3 y^3 + u_4 y^4$ where

$$\begin{array}{rcl} u_0 & = & v_2 v_{10} - v_4^2 \\ u_1 & = & v_0 v_{10} + v_2(v_7 + v_9) - 2v_3 v_4 \\ u_2 & = & v_0(v_7 + v_9) + v_2(v_6 - v_8) - v_3^2 - 2v_1 v_4 \\ u_3 & = & v_0(v_6 - v_8) + v_2 v_5 - 2v_1 v_3 \\ u_4 & = & v_0 v_5 - v_1^2 \end{array}$$

with

$$\begin{array}{rcl} v_0 & = & 2\left(a_{00}^{(0)} a_{01}^{(1)} - a_{00}^{(1)} a_{01}^{(0)}\right) \\ v_1 & = & a_{00}^{(0)} a_{11}^{(1)} - a_{00}^{(1)} a_{11}^{(0)} \\ v_2 & = & a_{00}^{(0)} b_0^{(1)} - a_{00}^{(1)} b_0^{(0)} \\ v_3 & = & a_{00}^{(0)} b_1^{(1)} - a_{00}^{(1)} b_1^{(0)} \\ v_4 & = & a_{00}^{(0)} c^{(1)} - a_{00}^{(1)} c^{(0)} \\ v_5 & = & 2\left(a_{01}^{(0)} a_{11}^{(1)} - a_{01}^{(1)} a_{11}^{(0)}\right) \\ v_6 & = & 2\left(a_{01}^{(0)} b_1^{(1)} - a_{01}^{(1)} b_1^{(0)}\right) \\ v_7 & = & 2\left(a_{01}^{(0)} c^{(1)} - a_{01}^{(1)} c^{(0)}\right) \\ v_8 & = & a_{11}^{(0)} b_0^{(1)} - a_{11}^{(1)} b_0^{(0)} \\ v_9 & = & b_0^{(0)} b_1^{(1)} - b_0^{(1)} b_1^{(0)} \\ v_{10} & = & b_0^{(0)} c^{(1)} - b_0^{(1)} c^{(0)} \end{array}$$

For each $\bar{y}$ solving $R(\bar{y}) = 0$, solve $Q_0(x, \bar{y}) = 0$ for up to two values $\bar{x}$. Keep only those $(\bar{x}, \bar{y})$ for which both $Q_0(\bar{x}, \bar{y}) = 0$ and $Q_1(\bar{x}, \bar{y}) = 0$.

3 Test Intersection

3.1 Variation 1

All level curves defined by $Q_0(x, y) = \lambda$ are ellipses, except for the minimum (negative) value λ for which the equation defines a single point, the center of every level curve ellipse. The ellipse defined by $Q_1(x, y) = 0$ is a curve that generally intersects many level curves of Q_0. The problem is to find the minimum level value λ_0 and maximum level value λ_1 attained by any (x, y) on the ellipse E_1. If $\lambda_1 < 0$, then E_1 is properly contained in E_0. If $\lambda_0 > 0$, then E_0 and E_1 are separated. Otherwise, $0 \in [\lambda_0, \lambda_1]$ and the two ellipses intersect.

This can be formulated as a constrained minimization that can be solved by the method of Lagrange multipliers: Minimize $Q_0(\mathbf{X})$ subject to the constraint $Q_1(\mathbf{X}) = 0$. Define $F(\mathbf{X}, t) = Q_0(\mathbf{X}) + tQ_1(\mathbf{X})$. Differentiating yields $\nabla F = \nabla Q_0 + t\nabla Q_1$ where the gradient indicates the derivatives in $\mathbf{X}$. Also, $\partial F/\partial t = Q_1$. Setting the t-derivative equal to zero reproduces the constraint $Q_1 =$. Setting the $\mathbf{X}$-derivative equal to zero yields $\nabla Q_0 + t\nabla Q_1 = \mathbf{0}$ for some t. Geometrically this means that the gradients are parallel.

Note that $\nabla Q_i = 2A_i\mathbf{X} + \mathbf{B}_i$, so

$$\mathbf{0} = \nabla Q_0 + t\nabla Q_1 = 2(A_0 + tA_1)\mathbf{X} + (\mathbf{B}_0 + t\mathbf{B}_1).$$

Formally solving for $\mathbf{X}$ yields

$$\mathbf{X} = -(A_0 + tA_1)^{-1}(\mathbf{B}_0 + t\mathbf{B}_1)/2 = \frac{1}{\delta(t)}\mathbf{Y}(t)$$

where $\delta(t)$ is the determinant of $(A_0 + tA_1)$, a quadratic polynomial in t, and $\mathbf{Y}(t)$ has components quadratic in t. Replacing this in $Q_1(\mathbf{X}) = 0$ yields

$$\mathbf{Y}(t)^{\mathrm{T}}A_1\mathbf{Y}(t) + \delta(t)\mathbf{B}_1^{\mathrm{T}}\mathbf{Y}(t) + \delta(t)^2C_1 = 0,$$

a quartic polynomial in t. The roots can be computed, the corresponding values of $\mathbf{X}$ computed, and $Q_0(\mathbf{X})$ evaluated. The minimum and maximum values are stored as λ_0 and λ_1, and the earlier comparisons with zero are applied.

This method leads to a quartic polynomial, just as the *find* query did. But this query does answer questions about the relative positions of the ellipses (separated or proper containment) when the *find* query indicates that there is no intersection.

3.2 Variation 2

An iterative method can be set up that attempts to find a separating line between the two ellipses. This does not directly handle proper containment of one ellipse by the other, but a similar algorithm can be derived for the containment case. Let the ellipses be in factored form, $(\mathbf{X} - \mathbf{C}_i)^{\mathrm{T}}M_i(\mathbf{X} - \mathbf{C}_i) = 1$ where M_i is positive definite and $\mathbf{C}_i$ is the center of the ellipse, $i = 0, 1$. A potential separating axis (not to be confused with a separating line that is perpendicular to a separating axis) is $\mathbf{C}_0 + t\mathbf{N}$ where $\mathbf{N}$ is a unit length vector. The t-interval of projection of E_0 onto the axis is $I_0(\mathbf{N}) = [-r_0, r_0]$ where $r_0 = \sqrt{\mathbf{N}^{\mathrm{T}}M_0^{-1}\mathbf{N}}$. The t-interval of projection of E_1 onto the axis is $I_1(\mathbf{N}) = [\mathbf{N}\cdot\mathbf{\Delta} - r_1, \mathbf{N}\cdot\mathbf{\Delta} + r_1]$ where $\mathbf{\Delta} = \mathbf{C}_1 - \mathbf{C}_0$ and $r_1 = \sqrt{\mathbf{N}^{\mathrm{T}}M_1^{-1}\mathbf{N}}$.

Select an initial $\mathbf{N}$. If the intersection $F(\mathbf{N}) := I_0(\mathbf{N}) \cap I_1(\mathbf{N}) = \emptyset$, then the ellipses are separated. If $F(\mathbf{N}) \neq \emptyset$, then the given axis does not separate the ellipses. When the intervals overlap, $F(\mathbf{N}) = [f_0, f_1]$ where $f_0 = \max\{\mathbf{N} \cdot \mathbf{\Delta} - r_1, -r_0\}$ and $f_1 = \min\{\mathbf{N} \cdot \mathbf{\Delta} + r_1, r_0\}$. The function $D(\mathbf{N}) = f_1 - f_0 > 0$ when there is overlap. If the two intervals have a single point of intersection, then $f_0 = f_1$. If the intervals are disjoint, then $f_1 < f_0$ and $D(\mathbf{N}) < 0$. The problem now is to search the space of unit length vectors, starting at the initial $\mathbf{N}$, to determine if there is such a vector that makes $D < 0$. It is enough to determine if $D = 0$ and the graph of D has a transverse crossing at that location.