Distance Between Point and Circle or Disk in 3D

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1 Point and Circle

A circle in 3D is represented by a center $\mathbf{C}$, a radius R, and a plane containing the circle, $\mathbf{N} \cdot (\mathbf{X} - \mathbf{C}) = 0$ where $\mathbf{N}$ is a unit length normal to the plane. If $\mathbf{U}$ and $\mathbf{V}$ are also unit length vectors so that $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{N}$ form a right-handed orthonormal coordinate system (the matrix with these vectors as columns is orthonormal with determinant 1), then the circle is parameterized as

$$\mathbf{X} = \mathbf{C} + R(\cos(\theta)\mathbf{U} + \sin(\theta)\mathbf{V}) =: \mathbf{C} + R\mathbf{W}(\theta)$$

for angles $\theta \in [0, 2\pi)$. Note that $|\mathbf{X} - \mathbf{C}| = R$, so the $\mathbf{X}$ values are all equidistant from $\mathbf{C}$. Moreover, $\mathbf{N} \cdot (\mathbf{X} - \mathbf{C}) = 0$ since $\mathbf{U}$ and $\mathbf{V}$ are perpendicular to $\mathbf{N}$, so the $\mathbf{X}$ lie in the plane.

For each angle $\theta \in [0, 2\pi)$, the squared distance from a specified point $\mathbf{P}$ to the corresponding circle point is

$$F(\theta) = |\mathbf{C} + R\mathbf{W}(\theta) - \mathbf{P}|^2 = R^2 + |\mathbf{C} - \mathbf{P}|^2 + 2R(\mathbf{C} - \mathbf{P}) \cdot \mathbf{W}.$$

The problem is to minimize $F(\theta)$ by finding θ_0 such that $F(\theta_0) \leq F(\theta)$ for all $\theta \in [0, 2\pi)$. Since F is a periodic and differentiable function, the minimum must occur when $F'(\theta) = 0$. Also, note that $(\mathbf{C} - \mathbf{P}) \cdot \mathbf{W}$ should be negative and as large in magnitude as possible to reduce the right-hand side in the definition of F. The derivative is

$$F'(\theta) = 2R(\mathbf{C} - \mathbf{P}) \cdot \mathbf{W}'(\theta)$$

where $\mathbf{W} \cdot \mathbf{W}' = 0$ since $\mathbf{W} \cdot \mathbf{W} = 1$ for all θ. The vector $\mathbf{W}'$ is unit length vector since $\mathbf{W}'' = -\mathbf{W}$ and $0 = \mathbf{W} \cdot \mathbf{W}'$ implies $0 = \mathbf{W} \cdot \mathbf{W}'' + \mathbf{W}' \cdot \mathbf{W}' = -1 + \mathbf{W}' \cdot \mathbf{W}'$. Finally, $\mathbf{W}'$ is perpendicular to $\mathbf{N}$ since $\mathbf{N} \cdot \mathbf{W} = 0$ implies $0 = \mathbf{N} \cdot \mathbf{W}'$. All conditions imply that $\mathbf{W}$ is parallel to the projection of $\mathbf{P} - \mathbf{C}$ onto the plane and points in the same direction.

Let $\mathbf{Q}$ be the projection of $\mathbf{P}$ onto the plane. Then

$$\mathbf{Q} - \mathbf{C} = \mathbf{P} - \mathbf{C} - (\mathbf{N} \cdot (\mathbf{P} - \mathbf{C}))\,\mathbf{N}.$$

The vector $\mathbf{W}(\theta)$ must be the unitized projection $(\mathbf{Q} - \mathbf{C})/|\mathbf{Q} - \mathbf{C}|$. The closest point on the circle to $\mathbf{P}$ is

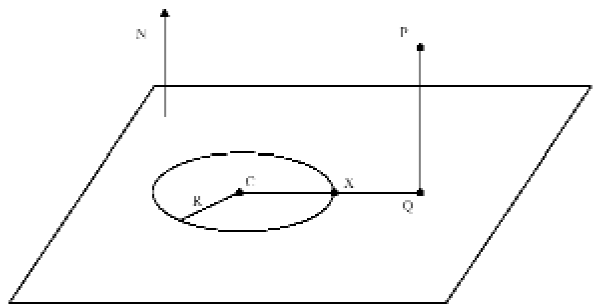
$$\mathbf{X} = \mathbf{C} + R\frac{\mathbf{Q} - \mathbf{C}}{|\mathbf{Q} - \mathbf{C}|}$$

assuming that $\mathbf{Q} \neq \mathbf{C}$. The distance from point to circle is then $|\mathbf{P} - \mathbf{X}|$.

If the projection of $\mathbf{P}$ is exactly the circle center $\mathbf{C}$, then all points on the circle are equidistant from $\mathbf{C}$. The distance from point to circle is the length of the hypotenuse of any triangle whose vertices are $\mathbf{C}$, $\mathbf{P}$, and any circle point. The lengths of the adjacent and opposite triangle sides are R and $|\mathbf{P} - \mathbf{C}|$, so the distance from point to circle is $\sqrt{R^2 + |\mathbf{P} - \mathbf{C}|^2}$.

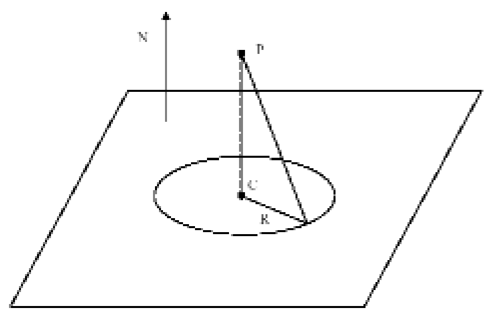
The typical case where $\mathbf{P}$ does not project to circle center is shown in Figure 1.1.

Figure 1.1 Typical case, closest point to circle.



The case when $\mathbf{P}$ does project to circle center is shown in Figure 1.2.

Figure 1.2 Typical case, closest point to circle.



2 Point and Disk

This requires a minor modification of the point and circle algorithm. The disk is the set of all points $\mathbf{X} = \mathbf{C} + \rho \mathbf{W}(\theta)$ where $0 \leq \rho \leq R$. If the projection of $\mathbf{P}$ is contained in the disk, then the projection is already the closest point to $\mathbf{P}$. If the projection is outside the disk, then the closest point to $\mathbf{P}$ is the closest point on the disk boundary, a circle.

Figure 2.1 shows the case when $\mathbf{P}$ projects inside the disk.

Figure 2.1 Closest point when **P** projects inside the disk.

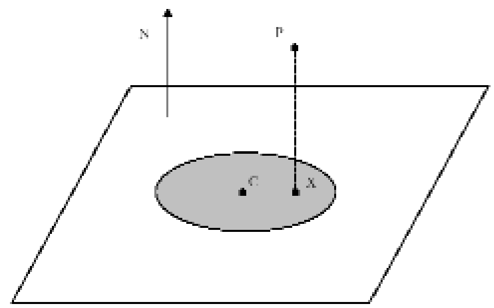


Figure 2.2 shows the case when **P** projects outside the disk.

Figure 2.2 Closest point when **P** projects outside the disk.