

Distance from Linear Component to Tetrahedron

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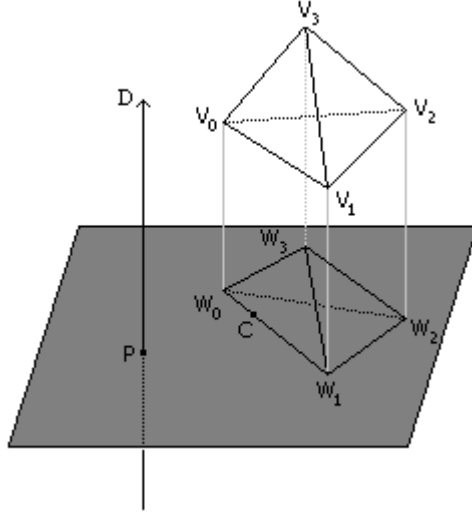
Let \mathbf{V}_i , $0 \leq i \leq 3$ be the vertices of the tetrahedron. The linear component is $\mathbf{P} + t\mathbf{D}$ where \mathbf{D} is a unit length vector and $t \in \mathbb{R}$ (line), $t \geq 0$ (ray), or $t \in [0, T]$ (segment). The construction can be modified slightly to handle \mathbf{D} that is not unit length. The tetrahedron can be parameterized by $\mathbf{V}_0 + s_1\mathbf{E}_1 + s_2\mathbf{E}_2 + s_3\mathbf{E}_3$ where $\mathbf{E}_i = \mathbf{V}_i - \mathbf{V}_0$, $s_i \geq 0$, and $s_1 + s_2 + s_3 \leq 1$.

1 Line and Tetrahedron

1.1 Distance

Translate the tetrahedron and line by subtracting \mathbf{P} . The tetrahedron vertices are now $\mathbf{U}_i = \mathbf{V}_i - \mathbf{P}$ for all i . The line becomes $t\mathbf{D}$. Project onto the plane containing the origin $\mathbf{0}$ and having normal \mathbf{D} . The projected line is the single point $\mathbf{0}$. The projected tetrahedron vertices are $\mathbf{W}_i = (I - \mathbf{D}\mathbf{D}^T)\mathbf{U}_i$ for all i . The boundary of the projected solid tetrahedron is a convex polygon, either a triangle or a quadrilateral. Figure 1.1 shows the line, tetrahedron, and projections.

Figure 1.1 Line, tetrahedron, and projections onto a plane perpendicular to the line.



If the convex polygon contains $\mathbf{0}$, the distance from the line to the tetrahedron is zero. Otherwise, the distance from the line to the tetrahedron is the distance from $\mathbf{0}$ to the convex polygon. The projected values are in a plane in 3D and can be projected into 2D with the standard technique of eliminating the coordinate corresponding to the maximum absolute component of \mathbf{D} . The distance between a point and convex polygon can be computed in 2D. This value must be adjusted to account for the 3D-to-2D projection. For example, if $\mathbf{D} = (d_0, d_1, d_2)$ with $|d_2| = \max_i\{|d_i|\}$ and r is the computed 2D distance, then the 3D distance is r/d_2 .

1.2 Closest Points

The set of tetrahedron points closest to the line in many cases consists of a single point. In other cases, the set can consist of a line segment of points. For example, consider the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. The line $(1/4, 1/4, 0) + t(0, 0, 1)$ intersects the tetrahedron for $t \in [0, 1/2]$, so the corresponding points are zero units of distance from the tetrahedron. The line $(-1, -1, 1/2) + t(0, 0, 1)$ is $\sqrt{2}$ units of distance from the tetrahedron. The closest points on the line are generated by $t \in [0, 1/2]$ and the closest points on the tetrahedron are $(0, 0, t)$ for the same interval of t values. The line $(1/2, -1/2, 0) + t(0, 0, 1)$ is $1/2$ units of distance from the tetrahedron. The closest points on the line are generated by $t \in [0, 1/2]$ and the closest points on the tetrahedron are $(1/2, 0, t)$ for the same interval of t values.

CASE 1. Let $\mathbf{0}$ be strictly inside the convex polygon. In this case, the line intersects the tetrahedron in an interval of points. Let $E = [\mathbf{E}_1 \ \mathbf{E}_2 \ \mathbf{E}_3]$ be the matrix whose columns are the specified edge vectors of the tetrahedron. Let \mathbf{s} be the 3×1 vector whose components are the s_i parameters. The line segment of intersection is $t\mathbf{D} + \mathbf{P} = E\mathbf{s} + \mathbf{V}_0$ for $t \in [t_{\min}, t_{\max}]$. The problem now is to compute the t -interval. The edge vectors of the tetrahedron are linearly independent, so E is invertible. Multiplying the vector equation by the inverse and solving for the tetrahedron parameters yields

$$\mathbf{s} = E^{-1}(t\mathbf{D} + \mathbf{P} - \mathbf{V}_0) = \mathbf{A}t + \mathbf{B}$$

where $\mathbf{A} = (a_1, a_2, a_3) = E^{-1}\mathbf{D}$ and $\mathbf{B} = (b_1, b_2, b_3) = E^{-1}(\mathbf{P} - \mathbf{V}_0)$. The parameters \mathbf{s} must satisfy the inequality constraints for the tetrahedron. The parameter t is therefore constrained by the four inequalities:

$$a_1t + b_1 \geq 0, \ a_2t + b_2 \geq 0, \ a_3t + b_3 \geq 0, \ (a_1 + a_2 + a_3)t + (b_1 + b_2 + b_3) \leq 1.$$

Each of these inequalities defines a semiinfinite interval of the form $[\bar{t}, \infty)$ or $(-\infty, \bar{t}]$. In this particular case, we know the intersection of the four intervals must be nonempty and of the form $[t_{\min}, t_{\max}]$.

The division required to compute E^{-1} can be avoided. Let us assume that the tetrahedron is oriented so that $\det(E) > 0$. Multiply by the adjoint E^{adj} to obtain

$$\det(E)\mathbf{s} = E^{\text{adj}}(t\mathbf{D} + \mathbf{P} - \mathbf{V}_0) = \boldsymbol{\alpha}t + \boldsymbol{\beta}.$$

The four t -inequalities are of the same form as earlier, but where a_i refers to the components of $\boldsymbol{\alpha}$, b_i refers to the components of $\boldsymbol{\beta}$, and the last inequality becomes a comparison to $\det(E)$ instead of to 1.

CASE 2. Let $\mathbf{0}$ be on the convex polygon boundary or outside the polygon. Let \mathbf{C} be the closest polygon point (in 3D) to $\mathbf{0}$. The line $t\mathbf{D} + \mathbf{C}$ intersects the tetrahedron with \mathbf{U}_i vertices either in a single point or in an interval of points. The method in case 1 may be used again, but now you need to be careful with the interval construction when using floating point arithmetic. If the intersection is a single point, theoretically $t_{\min} = t_{\max}$, but numerically you might wind up with an empty intersection. It is not difficult to trap this and handle appropriately. Observe that cases 1 and 2 are handled by the same code since in case 1 you can choose $\mathbf{C} = \mathbf{0}$.

2 Ray and Tetrahedron

Use the line-tetrahedron algorithm for computing the closest line points with parameters $I = [t_{\min}, t_{\max}]$ (with possibly $t_{\min} = t_{\max}$). Define $J = I \cap [0, \infty)$. If $J \neq \emptyset$, the ray-tetrahedron distance is the same as the line-tetrahedron distance. The closest ray points are determined by J . If $J = \emptyset$, the ray origin \mathbf{P} is closest to the tetrahedron.

3 Segment and Tetrahedron

Use the line-tetrahedron algorithm for computing the closest line points with parameters $[t_{\min}, t_{\max}]$ (with possibly $t_{\min} = t_{\max}$). Define $J = I \cap [0, T]$. If $J \neq \emptyset$, the segment-tetrahedron distance is the same as the line-tetrahedron distance. The closest segment points are determined by J . If $J = \emptyset$, the closest segment point is \mathbf{P} when $t_{\max} < 0$ or $\mathbf{P} + T\mathbf{D}$ when $t_{\min} > T$.