Numerical Methods for Partial Differential Equations

David Eberly
Geometric Tools, LLC
http://www.geometrictools.com/

Created: March 2, 1999
Last Modified: March 2, 2008

Contents

Second-order linear partial differential equations arise naturally in modeling physical phenomena. They are characterized as parabolic, hyperbolic, or elliptic. Let $x \in \mathbb{R}$, $t \geq 0$, and $u = u(x,t) \in \mathbb{R}$ in the following examples.

1. Heat Transfer, Population Dynamics (parabolic).
   Diffusion of heat $u(x,t)$ in a rod of length L and with heat source $f(x)$ is modeled by

$$
\begin{aligned}
&u_t(x,t) = u_{xx}(x,t) + f(x), && x \in (0,L),\ t > 0, && \text{(from conservation laws)}\\
&u(x,0) = g(x), && x \in [0,L], && \text{(initial heat distribution)}\\
&u(0,t) = a(t),\ u(L,t) = b(t), && t \geq 0, && \text{(temperature known at boundaries)}\\
&\text{or} && && \\
&u_x(0,t) = u_x(L,t) = 0, && t \geq 0, && \text{(insulated boundaries)}
\end{aligned}
$$

2. Wave and Shock Phenomena (hyperbolic).
   Displacement $u(x,t)$ of an elastic string is modeled by

$$
\begin{aligned}
&u_{tt}(x,t) = u_{xx}(x,t), && x \in (0,L),\ t > 0, && \text{(from conservation laws)}\\
&u(x,0) = f(x),\ u_t(x,0) = g(x), && x \in [0,L], && \text{(initial displacement and speed)}\\
&u(0,t) = a(t),\ u(L,t) = b(t), && t \geq 0, && \text{(location of string ends)}
\end{aligned}
$$

3. Steady-State Heat Flow, Potential Theory (elliptic).
   Steady-state distribution of heat $u(x)$ in a bar of length L with heat source $f(x)$ is modeled by

$$
\begin{aligned}
&u_{xx}(x) = -f(x), && x \in (0,L), && (t \to \infty \text{ in the heat equation})\\
&u(0) = A,\ u(L) = B, && && \text{(boundary conditions)}
\end{aligned}
$$

1 Numerical Solution by Finite Differences

1.1 Heat Equation

Consider the heat equation with no source and constant temperature at the rod ends:

$$
\begin{aligned}
&u_t(x,t) = u_{xx}(x,t), && x \in (0,L),\ t > 0,\\
&u(x,0) = g(x), && x \in [0,L],\\
&u(0,t) = u(L,t) = 0, && t \geq 0.
\end{aligned}
$$

Numerical solution is as follows:

- Select $m+1$ spatial locations uniformly sampled as

$$x_i = i\Delta x,\ \ 0 \leq i \leq m,\ \ \Delta x = L/m.$$

- Select temporal samples as
$$t_j = j\Delta t, \;\; j \geq 0, , \;\; \Delta t > 0.$$
- The estimates of temperature are
$$u_i^{(j)} \doteq u(x_i, t_j), \;\; 0 \leq i \leq m, \; j \geq 0.$$
- The sampled initial temperature is
$$g_i = g(x_i), \;\; 0 \leq i \leq m.$$
- Approximate time derivative by forward difference
$$u_t(x,t) \doteq \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}.$$
  Approximate spatial derivatives by central difference
$$u_{xx}(x,t) \doteq \frac{u(x+\Delta x, t) - 2u(x,t) + u(x - \Delta x, t)}{(\Delta x)^2}.$$
  Replace in heat equation to obtain
$$\frac{u_i^{(j+1)} - u_i^{(j)}}{\Delta t} = \frac{u_{i+1}^{(j)} - 2u_i^{(j)} + u_{i-1}^{(j)}}{(\Delta x)^2}.$$
- The boundary conditions are
$$u_0^{(j)} = u_m^{(j)} = 0, \;\; j \geq 0.$$

The numerical algorithm is implemented as

$$\begin{aligned}
&u_i^{(0)} = g_i, && 0 \leq i \leq m \\
&u_0^{(j)} = u_m^{(j)} = 0, && j \geq 0 \\
&u_i^{(j+1)} = u_i^{(j)} + \tfrac{\Delta t}{(\Delta x)^2}\left(u_{i+1}^{(j)} - 2u_i^{(j)} + u_{i-1}^{(j)}\right), && 1 \leq i \leq m-1, \; j \geq 0.
\end{aligned}$$

For this to be *stable*, you need $\Delta t < (\Delta x)^2/2$.

An alternate scheme is Crank-Nicholson method

$$u_i^{(j+1)} = u_i^{(j)} + \frac{\Delta t}{(\Delta x)^2}\left(\frac{u_{i+1}^{(j)} - 2u_i^{(j)} + u_{i-1}^{(j)}}{2} + \frac{u_{i+1}^{(j+1)} - 2u_i^{(j+1)} + u_{i-1}^{(j+1)}}{2}\right).$$

Method is stable for all $\Delta t > 0$, but is harder to solve since $u_i^{(j+1)}$ is implicitly defined.

1.2 Wave Equation

Consider the wave equation where the string ends are clamped (no displacement):

$$\begin{aligned}
&u_{tt}(x,t) = u_{xx}(x,t), && x \in (0,L),\ t > 0,\\
&u(x,0) = f(x),\ u_t(x,0) = g(x), && x \in [0,L],\\
&u(0,t) = u(L,t) = 0, && t \geq 0.
\end{aligned}$$

Using similar notation as in heat equation, and using centralized differences both in space and time, the numerical method is

$$\begin{aligned}
&u_i^{(0)} = f_i, && 0 \leq i \leq m\\
&u_i^{(1)} = u_i^{(0)} + (\Delta t) g_i, && 0 \leq i \leq m\\
&u_0^{(j)} = u_m^{(j)} = 0, && j \geq 0\\
&\tfrac{u_i^{(j+1)} - 2u_i^{(j)} + u_i^{(j-1)}}{(\Delta t)^2} = \tfrac{u_{i+1}^{(j)} - 2u_i^{(j)} + u_{i-1}^{(j)}}{(\Delta x)^2}, && 1 \leq i \leq m-1,\ j \geq 1.
\end{aligned}$$

The method is stable when $\Delta t < \Delta h$. If right-hand side is modified as in Crank-Nicholson, then method is stable for all $\Delta t > 0$.

1.3 Potential Equation

Consider steady-state heat flow for the constant temperature boundary problem:

$$\begin{aligned}
&u_{xx}(x) = -f(x), \quad x \in (0,L),\\
&u(0) = u(L) = 0
\end{aligned}$$

The numerical method is

$$\begin{aligned}
&u_0 = 0,\ u_m = 0,\\
&\tfrac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta x)^2} = -f_i, \quad 1 \leq i \leq m-1
\end{aligned}$$

Define the $(m-1) \times 1$ vectors $u = [u_i]$ where $1 \leq i \leq m-1$ and $b = [-(\Delta x)^2 f_i]$. This vector is the unknown in a linear system

$$Au = b$$

where A is tridiagonal with main diagonal -2 and sub- and super-diagonals 1. Such systems are solved robustly in $O(m)$ time.

2 Extension to Higher Dimensions

Consider $u(x,y,t)$ for two dimensional problem. The heat equation is

$$u_t = u_{xx} + u_{yy},$$

the wave equation is

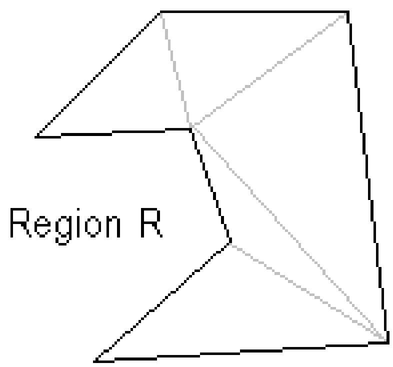
$$u_{tt} = u_{xx} + u_{yy},$$

and the potential equation is

$$u_{xx} + u_{yy} = f(x, y).$$

If domain for (x, y) is a rectangle, then discretizations such as in the one dimensional problems extend easily.

If domain is not rectangular, then use *finite elements*. For example, consider $u_{xx} + u_{yy} = 0$ where R is nonrectangular. Let $u(x, y)$ be specified on ∂R (boundary of R). Decompose region R into triangles.



On each triangle approximate the true solution $u(x, y)$ by linear function $v(x, y)$ which interpolates the triangle vertices. That is, $v(x, y)$ is determined by

$$\begin{aligned}
&N \bullet (x, y, v) = c,\\
&N = [(x_2, y_2, v_2) - (x_1, y_1, v_1)] \times [(x_3, y_3, v_3) - (x_1, y_1, v_1)],\\
&c = N \bullet (x_1, y_1, v_1).
\end{aligned}$$

The boundary v_i are known, but the interior v_i must be determined.

Solving the potential equation on R is equivalent to finding function u which minimizes

$$I = \int\int_R u_x^2 + u_y^2 \, dxdy$$

subject to the boundary conditions. Define $\tilde{I}$ to be the approximate integral where u is replaced by v. For triangle T, let $v_T(x, y) = \alpha_T x + \beta_T y + \gamma_T$. Then

$$\tilde{I} = \sum_T \left(\alpha_T^2 + \beta_T^2\right) \text{area}(T).$$

Since α_T and β_T are linear in the interior v_i, $\tilde{I}$ is quadratic in v_i. Minimizing a quadratic function can be done by solving a linear system (set derivatives equal to zero) or by conjugate gradient method (equivalent to solving the linear system, but uses root finding techniques).