Reconstructing a Point from Distances

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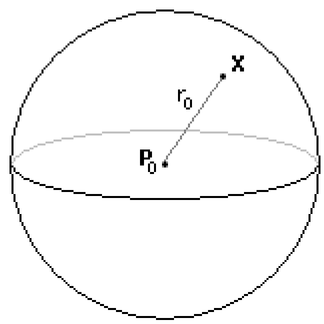
Contents

Given a collection of n known points in three dimensions, call them $\mathbf{P}_i$ for $0 \le i < n$, if another point $\mathbf{X}$ is known to be at a distance r_i from point $\mathbf{P}_i$ for all i, we would like to determine $\mathbf{X}$.

1 Case $n = 1$

If there is only a single known point $\mathbf{P}_0$, there are infinitely many points $\mathbf{X}$ which are a distance r_0 from $\mathbf{P}_0$. These points lie on a sphere centered at $\mathbf{P}_0$ and having radius r_0.

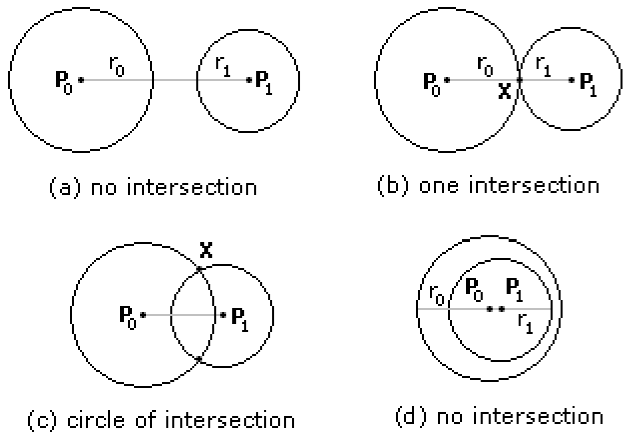
Figure 1.1 The set of all points a known distance r_0 from a fixed point $\mathbf{P}_0$ is a sphere.



2 Case $n = 2$

Given two distinct points $\mathbf{P}_0$ and $\mathbf{P}_1$, the set of points $\mathbf{X}$ for which $|\mathbf{X} - \mathbf{P}_0| = r_0$ and $|\mathbf{X} - \mathbf{P}_1| = r_1$ is either the empty set, a single point, or a circle. The set of points is the intersection of two spheres. The distance between sphere centers is $d = |\mathbf{P}_1 - \mathbf{P}_0|$. If $r_0 + r_1 < d$, the spheres do not intersect. If $r_0 + r_1 = d$, the spheres intersect in a single point. If $r_0 + r_1 > d$, the spheres intersect in a circle.

Figure 2.1 The set of all points a known distance r_0 from a fixed point $\mathbf{P}_0$ and a known distance r_1 from a fixed point $\mathbf{P}_1$ is either (a) empty (and separated), (b) a single point, (c) a circle, or (d) empty (and contained).



When there is a single point of intersection, the point is

$$\mathbf{X} = \mathbf{P}_0 + \frac{r_0}{r_0 + r_1} (\mathbf{P}_1 - \mathbf{P}_0) \tag{1}$$

When there is a circle of intersection, the points are of the form

$$\mathbf{X} = (1 - s)\mathbf{P}_0 + s\mathbf{P}_1 + \alpha\mathbf{U} + \beta\mathbf{V} \tag{2}$$

where $(1 - s)\mathbf{P}_0 + s\mathbf{P}_1$ is a point on the segment connecting the sphere centers for some $s \in [0, 1]$. The vectors $\mathbf{U}$ and $\mathbf{V}$ are unit length, mutually perpendicular, and both perpendicular to the segment direction $\mathbf{P}_1 - \mathbf{P}_0$. Replacing this equation in $|\mathbf{X} - \mathbf{P}_0| = r_0$, $|\mathbf{X} - \mathbf{P}_1| = r_1$, and squaring both equations leads to

$$r_0^2 = s^2 d^2 + \alpha^2 + \beta^2, \;\; r_1^2 = (s - 1)^2 d^2 + \alpha^2 + \beta^2 \tag{3}$$

where $d = |\mathbf{P}_1 - \mathbf{P}_0|$ is the distance between the sphere centers. Subtracting the two equations,

$$r_1^2 - r_0^2 = [(s - 1)^2 - s^2]d^2 = (1 - 2s)d^2 \tag{4}$$

This may be solved for s, namely,

$$s = \frac{1}{2}\left(1 - \frac{r_1^2 - r_0^2}{d^2}\right) \tag{5}$$

The circle has center $\mathbf{K} = (1 - s)\mathbf{P}_0 + s\mathbf{P}$, lies in the plane $(\mathbf{P}_1 - \mathbf{P}_0) \cdot (\mathbf{X} - \mathbf{K}) = 0$, and has radius $r = \sqrt{\alpha^2 + \beta^2} = \sqrt{r_0^2 - s^2 d^2}$. This is represented parametrically by

$$\mathbf{X}(\theta) = \mathbf{K} + r\left((\cos\theta)\mathbf{U} + (\sin\theta)\mathbf{V}\right) \tag{6}$$

for $\theta \in [0, 2\pi)$.

3 Case $n = 3$

Given three distinct points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$ and three distances r_0, r_1, and r_2, the set of points $\mathbf{X}$ which are the specified distances from the fixed points is either empty, a single point, two points, or a circle. The set S of common points is the set of intersection of three spheres. The possibilities are described geometrically,

1. Two spheres do not intersect. $S = \emptyset$ (the empty set).
2. Two spheres intersect in a point $\mathbf{A}$.
   (a) The third sphere does not contain $\mathbf{A}$. $S = \emptyset$.
   (b) The third sphere contains $\mathbf{A}$. $S = \{\mathbf{A}\}$.
3. Two spheres intersect in a circle C.
   (a) The third sphere does not intersect C. $S = \emptyset$.
   (b) The third sphere intersects C in a single point $\mathbf{A}$. $S = \emptyset$.
   (c) The third sphere intersects C in two points $\mathbf{A}$ and $\mathbf{B}$. $S = \{\mathbf{A}, \mathbf{B}\}$.
   (d) The third sphere contains all of C. $S = C$.

Item 1 occurs when

$$r_i + r_j < |\mathbf{P}_i - \mathbf{P}_j| \text{ for some pair } (i, j) \text{ with } i \neq j \tag{7}$$

In item 2, suppose that sphere i and sphere j intersect in a single point, where $i \neq j$. This point is

$$\mathbf{A} = \mathbf{P}_i + \frac{r_i}{r_i + r_j}(\mathbf{P}_j - \mathbf{P}_i) \tag{8}$$

If k is the other index ($k \neq i$ and $k \neq j$), then item 2a occurs when

$$r_k \neq |\mathbf{A} - \mathbf{P}_k| \tag{9}$$

Item 2b occurs when

$$r_k = |\mathbf{A} - \mathbf{P}_k| \tag{10}$$