Geometric Invariance

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Contents

1 Vector Fields

Let $\mathbb{R}^n$ denote n-tuples of real numbers. A *vector field* is a function $V : \mathbb{R}^n \to \mathbb{R}^n$. The k^{th} component of V is a function $\xi_k : \mathbb{R}^n \to \mathbb{R}$. The vector field can be written as an n-tuple

$$V = (\xi_1(x), \dots, \xi_n(x)), \;\; x \in \mathbb{R}$$

or can be thought of as an $n \times 1$ column vector (when used in matrix calculations). We also can write the vector field as a linear combination

$$V = \sum_{k=1}^{n} \xi_k(x) \frac{\partial}{\partial x_k}$$

where the symbols $\partial/\partial x_k$ are placekeepers for the k^{th} component. In this form V can be thought of as a directional derivative operator $V \bullet \nabla$ which can be applied to functions $f : \mathbb{R}^n \to \mathbb{R}$.

Although the general notation uses indexed variables $x = (x_1, \dots, x_n)$, for small dimensions we might use different names. For example, if $n = 2$, then we might use (x, y) for the position vector; for $n = 3$ we might use (x, y, z) for the position.

Example. Let $V = (xy, x^2)$ be a vector field on $\mathbb{R}^2$. In directional derivative form,

$$V = xy\frac{\partial}{\partial x} + x^2\frac{\partial}{\partial y}.$$

Let $f(x, y) = \sin(x + y^2)$. The derivative of f in the direction V is

$$Vf = (V \bullet \nabla)f = xy\cos(x + y^2) + 2x^2 y\cos(x + y^2).$$

Another notation we use for directional derivative is $D_V f$. $\bowtie$

As a directional derivative operator, V measures an *infinitesimal* rate of change in the direction indicated. That is, V is a spatially varying tangent vector to certain curves. To determine the curves themselves, you need to solve the following system of ordinary differential equations:

$$\frac{dx}{dt} = V(x(t)), \, t > 0; \;\; x(0) = x_0$$

where x_0 is the starting point on the curve. The curve represents the *global action* of the vector field.

Example. Let $V = (x, 0)$ with initial point $(2, -1)$. The system of equations is

$$\begin{aligned} \tfrac{dx}{dt} &= x, \quad x(0) = 2 \\ \tfrac{dy}{dt} &= 0, \quad y(0) = -1 \end{aligned}$$

The second equation says y is constant along the curve, so $y(t) \equiv -1$. The first equation has solution $x(t) = 2\exp(t)$. If you think of V as velocity of a particle initially at $(2, -1)$, then the particle flows along the straight line $(x(t), y(t)) = (2\exp(t), -1)$, but has increasing speed. $\bowtie$

Example. This example shows that solving the system may not always be as easy as you would like. Let $V = (-y, x)$ with initial point (x_0, y_0). The system of equations is

$$\begin{aligned} \tfrac{dx}{dt} &= -y, \quad x(0) = x_0 \\ \tfrac{dy}{dt} &= x, \quad\;\; y(0) = y_0 \end{aligned}$$

The equations are coupled. However, this is a linear problem and can be solved by methods learned in a standard ordinary differential equations course. In matrix form the system is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

A matrix system of the form $x'(t) = Ax(t)$ where A is a constant $n \times n$ matrix and $x \in \mathbb{R}^n$, with initial data $x(0) = x_0$, has solution $x(t) = \exp(tA)x_0$ where the matrix $\exp(tA)$ is formally obtained by taking the Taylor series for $\exp(z)$ and replacing z by the matrix tA. For our example,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \exp\left(t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Note that the end result is a rotation of the initial data, so the particle trajectories are circles. $\bowtie$

2 Prolongations

Let $x, u \in \mathbb{R}$ and think of $u = u(x)$ (u is a function of x). If we make a change of variables $g(x, y) = (\bar{x}, \bar{u})$ such that $\bar{u} = \bar{u}(\bar{x})$, then how is $d\bar{u}/d\bar{x}$ related to du/dx?

EXAMPLE. Let the change of variables be

$$\begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$ for some $\theta > 0$. As long as du/dx is finite on the domain of u, then there is a small enough angle θ so that $\bar{u}$ is a function of $\bar{x}$.

We can determine the relationship between the derivatives with a little bit of calculus (using the chain rule). We have $\bar{x} = cx - su$, so taking derivatives with respect to $\bar{x}$, we get

$$1 = \frac{d\bar{x}}{d\bar{x}} = c\frac{dx}{d\bar{x}} - s\frac{du}{d\bar{x}} = c\frac{dx}{d\bar{x}} - \frac{du}{dx}\frac{dx}{d\bar{x}}$$

which implies that

$$\frac{dx}{d\bar{x}} = \frac{1}{c - su_x}$$

where $u_x = du/dx$. We also have $\bar{u} = sx + cu$, so taking derivatives again with respect to $\bar{x}$ gives us

$$\frac{d\bar{u}}{d\bar{x}} = s\frac{dx}{d\bar{x}} + c\frac{du}{d\bar{x}} = s\frac{dx}{d\bar{x}} + c\frac{du}{dx}\frac{dx}{d\bar{x}} = \frac{s + cu_x}{c - su_x}.$$

The group action $g(x, u) = (\bar{x}, \bar{u})$ and condition $u = u(x)$ induce an action on the variables (x, u, u_x), called the first-order *prolongation* of g. We denote it by $\mathrm{pr}^{(1)}g(x, u, u_x)$. For our example,

$$\mathrm{pr}^{(1)}g(x, u, u_x) = \left(cx - su, sx + cu, \frac{s + cu_x}{c - su_x}\right).$$

Note that when $\theta = 0$, the mapping is the identity. Here we are given the group action, but we can compute the infinitesimal action by taking the derivative with respect to θ at $\theta = 0$. [In the last section we had V and found g. Now we do the reverse process.]

The vector field V corresponding to g is

$$V = \left(\frac{d\bar{x}}{d\theta}\bigg|_{\theta=0} \right) \frac{\partial}{\partial x} + \left(\frac{d\bar{u}}{d\theta}\bigg|_{\theta=0} \right) \frac{\partial}{\partial u} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}.$$

The vector field corresponding to $\mathrm{pr}^{(1)}g$ is given by

$$\mathrm{pr}^{(1)}V = V + \left(\frac{d\bar{u}_{\bar{x}}}{d\theta}\bigg|_{\theta=0} \right) \frac{\partial}{\partial u_x} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x}.$$

I'll leave the derivative details up to you.

The same idea can be applied to determining the relationships between higher-order derivatives. In our example,

$$\bar{u}_{\bar{x}\bar{x}} = \frac{d}{d\bar{x}}\left(\frac{s + cu_x}{c - su_x} \right) = \frac{u_{xx}}{(c - su_x)^3}$$

and

$$\frac{d\bar{u}_{\bar{x}\bar{x}}}{d\theta}\bigg|_{\theta=0} = 3u_x u_{xx}.$$

The second-order group prolongation is

$$\mathrm{pr}^{(2)}g(x, u, u_x, u_{xx}) = \left(cx - su, sx + cu, \frac{s + cu_x}{c - su_x}, \frac{u_{xx}}{(c - su_x)^3} \right)$$

and the corresponding vector field is

$$\mathrm{pr}^{(2)}V = \mathrm{pr}^{(1)}V + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}} = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1+u_x^2)\frac{\partial}{\partial u_x} + 3u_x u_{xx}\frac{\partial}{\partial u_{xx}}.$$

An important point to note is that the prolongation vectors have place keepers x, u, u_x, and u_{xx}. These are simply *names* for the components of the vector field and no analytic relationship between x and u is used.

3 Invariants

Given a vector field $V : \mathbb{R}^n \to \mathbb{R}^n$, an *invariant* is a function $f : \mathbb{R}^n \to \mathbb{R}$ such that the directional derivative satisfies $Vf = 0$. That is, f remains constant as you walk in the direction of V. Consequently the level sets of f are the solution curves to the system of differential equations $dx/dt = V(x)$.

Example. Consider the vector field

$$V = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

An invariant is $f(x, y) = x^2 + y^2$ since $Vf = V \bullet \nabla f = (-y, x) \bullet (2x, 2y) \equiv 0$. The level curves of f are circles, which is intuitive since as we saw before, V is the infinitesimal generator for rotation. $\bowtie$

For most vector fields we may not be able to guess at the invariants f, so we need a constructive method. As indicated earlier, the level sets of an invariant f are solutions to a system of differential equations.

EXAMPLE. For the vector field in the last example, the level curves of invariants are given by

$$\begin{aligned}\tfrac{dx}{dt} &= -y, \quad x(0) = x_0 \\ \tfrac{dy}{dt} &= x, \quad\;\; y(0) = y_0\end{aligned}$$

We saw earlier that this system has solution $x = x_0 \cos(t) - y_0 \sin(t)$ and $y = x_0 \sin(t) + y_0 \cos(t)$. It can be shown that $x^2 + y^2 = x_0^2 + y_0^2$. Another way of determining the level curves (for a system of 2 equations) is to take the ratio of the two equations to get a single differential equation

$$\frac{dy}{dx} = \frac{x}{-y}.$$

This equation is separable: $xdx + ydy = 0$. Integrate to get $\frac{x^2}{2} + \frac{y^2}{2} = c$ for constant c. Substituting in the initial data gives us $x^2 + y^2 = x_0^2 + y_0^2$. $\bowtie$

We can also look for invariants with respect to more than one vector field. Let $V_k : \mathbb{R}^n \to \mathbb{R}^n$ for $k = 1, \ldots, m$ be vector fields. A function $f : \mathbb{R}^n \to \mathbb{R}$ is an *invariant* for the vector fields if $V_k f = 0$ for all k.

EXAMPLE. Consider the vector fields on $\mathbb{R}^3$,

$$V_1 = 2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y}, \;\; V_2 = 1\frac{\partial}{\partial x} + 1\frac{\partial}{\partial z}.$$

A function $f(x, y, z)$ is an invariant if $0 = V_1 f = (2, -3, 0) \bullet \nabla f$ and $0 = V_2 f = (1, 0, 1) \bullet \nabla f$. The gradient of f must be a vector which is orthogonal to both $(2, -3, 0)$ and $(1, 0, 1)$. Consequently ∇f is parallel to $(-3, -2, 3) = (2, -3, 0) \times (1, 0, 1)$. An invariant is $f(x, y, z) = -3x - 2y + 3z$. $\bowtie$

3.1 Independent Functions

One of the questions you should ask is: How many invariants are there for a vector field (or vector fields)? For example, an invariant for the rotation vector field $V = -y\partial/\partial x + x\partial/\partial y$ is $f(x, y) = x^2 + y^2$. Another invariant is $\sin(x^2 + y^2)$. In fact for any differentiable function $g : \mathbb{R} \to \mathbb{R}$, $g(f(x, y))$ is an invariant. But in some sense these all *depend* on the quantity $x^2 + y^2$.

Let $f_k : \mathbb{R}^n \to \mathbb{R}$ for $k = 1, \ldots, m$ where $m \leq n$ be m differentiable functions. These are said to be *functionally independent at* $x \in \mathbb{R}^n$ if the $m \times n$ matrix of first derivatives $[\partial f_i/\partial x_j]$ has full rank m.

EXAMPLE. Let $f_1(x, y) = x^2 + y^2$ and $f_2(x, y) = \sin(x^2 + y^2)$. The first derivative matrix is

$$\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x\cos(x^2 + y^2) & 2y\cos(x^2 + y^2) \end{bmatrix}.$$

The second row is just a scalar times the first row, so the matrix has rank 1 (not full rank), so the two functions are dependent. $\bowtie$

EXAMPLE. Let $f_1(x, y) = x^2 + y^2$ and $f_2(x, y) = x$. The first derivative matrix is

$$\begin{bmatrix} 2x & 2y \\ 1 & 0 \end{bmatrix}.$$

For $(x, y) \neq (0, 0)$, the matrix can be row-reduced to the identity, so the matrix has full rank. Thus, f_1 and f_2 are functionally independent. $\bowtie$

EXAMPLE. Consider only the vector field on $\mathbb{R}^3$,

$$V = 2\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} + 0\frac{\partial}{\partial z}.$$

Two invariants are $f_1(x, y, z) = -3x - 2y$ and $f_2(x, y, z) = z$. They are functionally independent since the second derivative matrix

$$\begin{bmatrix} -3 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has full rank (2).

3.2 Lie Algebras

Given vector fields $V_k : \mathbb{R}^n \to \mathbb{R}^n$ for $k = 1, \ldots, m$, with $m \leq n$, how many functionally independent invariants are there? To answer this question we need a little background in *Lie algebras*.

Consider two vector fields

$$V(x) = \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} \quad \text{and} \quad W(x) = \sum_{j=1}^{n} b_j(x)\frac{\partial}{\partial x_j}.$$

Each vector field can be applied as a directional derivative to functions from $\mathbb{R}^n \to \mathbb{R}$. In particular, the compositions $V(Wf)$ and $W(Vf)$ are well-defined. Thus,

$$\begin{array}{rcl} W(Vf) & = & \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}\right) \\ & = & \sum_{j=1}^{n} b_j \left[\sum_{i=1}^{n}\left(a_i \frac{\partial^2 f}{\partial x_j \partial x_i} + \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i}\right)\right] \\ & = & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_i \frac{\partial^2 f}{\partial x_i \partial x_j} b_j + b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i}\right) \end{array}$$

and

$$\begin{array}{rcl} V(Wf) & = & \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} b_j \frac{\partial f}{\partial x_j}\right) \\ & = & \sum_{i=1}^{n} a_i \left[\sum_{j=1}^{n}\left(b_j \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j}\right)\right] \\ & = & \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_i \frac{\partial^2 f}{\partial x_i \partial x_j} b_j + a_j \frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i}\right) \end{array}$$

where we assume that f is sufficiently smooth to guarantee that its mixed partial derivatives are the same. The difference of the above quantities is

$$W(Vf) - V(Wf) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial a_i}{\partial x_j} b_j - \frac{\partial b_i}{\partial x_j} a_j\right)\frac{\partial f}{\partial x_i} =: \sum_{i=1}^{n} c_i(x)\frac{\partial f}{\partial x_i}.$$

This motivates the *Lie product* of two vector fields. Define the vector field

$$[V, W] = \sum_{i=1}^{n} c_i(x)\frac{\partial}{\partial x_i}$$

where $c_i(x)$ was defined earlier. In terms of matrix operations, if DV and DW are the first derivative matrices of the coefficient functions of V and W, then $[V, W] = (DV)W - (DW)V$. Thus, the difference of the mixed directional derivatives is

$$W(Vf) - V(Wf) = [V, W]f.$$

NOTE: Given two directions $V(x)$ and $W(x)$, the order of directional differentiation at x is irrelevant if and only if $[V(x), W(x)]f(x) = 0$.

Given a collection of vector fields $V_k : \mathbb{R}^n \to \mathbb{R}^n$ for $k = 1, \ldots, m$ with $m \leq n$, the *Lie algebra* of the vector fields is obtained by constructing the smallest vector space which contains all sums, scalar multiples, and Lie products of the V_k. We will call this vector space $L(V_1, \ldots, V_m)$.

3.3 Independent Invariants

Now we get back to the question of how many functionally independent invariants are there for a collection of vector fields. First note that if f is an invariant, then by definition $V_k f = 0$ for all $1 \leq k \leq m$. Intuitively if the V_k are linearly independent, then the actions of V_k should require you to move around on a k-dimensional surface. That gives you $n - k$ degrees of freedom in which to "stack up" the level surfaces for invariants. So you might expect to find $n - k$ functionally independent invariants, one corresponding to each degree of freedom.

However, the situation is slightly more complicated. Since $V_k f = 0$ for all k, it is easily seen that $[V_i, V_j]f = (DV_i)V_j f - (DV_j)V_i f = 0$, so f is also an invariant for the vector field $[V_i, V_j]$. It is possible that the Lie product of two linearly independent vector fields is another vector field independent of the first two. In this case, you lose a degree of freedom for the invariants.

Example. Let $V = (x_1, x_2, x_3)$ and $W = (x_1, x_2, 0)$. The Lie product is

$$[V, W] = (DV)W - (DW)V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -x_3 \end{bmatrix}$$

The vectors V, W, and $[V, W]$ are linearly independent.

The last argument shows that the number of degrees of freedom is not $n - k$, but rather it should be $n - \dim(L)$ where $\dim(L)$ is the dimension of the Lie algebra.

Example. Consider the example where $x, u \in \mathbb{R}$, $u = u(x)$, and

$$V_1 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \quad V_2 = 1\frac{\partial}{\partial x}, \quad V_3 = 1\frac{\partial}{\partial u}.$$

These correspond to rotation in (x, u), translation in x, and translation in u. We want to find some low-order invariants of all three.

(1) Zero-order invariants: We want functions f such that $V_k f = 0$ for $k = 1, 2, 3$. The Lie algebra generated by the vector fields is all of $\mathbb{R}^2$, so $n - \dim(L) = 2 - 2 = 0$, so there are no zero-order invariants.

(2) First-order invariants: We want functions f such that $\mathrm{pr}^{(1)}V_k f = 0$ for $k = 1, 2, 3$. The prolongations are

$$W_1 = \mathrm{pr}^{(1)}V_1 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + u_x^2)\frac{\partial}{\partial u_x},\ W_2 = \mathrm{pr}^{(1)}V_2 = V_2,\ \ W_3 = \mathrm{pr}^{(1)}V_3 = V_3.$$

Some computations will show you that

$$[W_1, W_2] = W_3,\ \ [W_1, W_2] = -W_1,\ \ [W_2, W_3] = 0.$$

So we get nothing new from the pairwise products. Since the W_k act on $\mathbb{R}^3$ and are linearly independent, $n - \dim(L) = 3 - 3 = 0$, so there are no first-order invariants.

(3) Second-order invariants: We want functions f such that $\mathrm{pr}^{(2)}V_k f = 0$ for $k = 1, 2, 3$. The prolongations are

$$W_1 = \mathrm{pr}^{(2)}V_1 = -u\frac{\partial}{\partial x} + x\frac{\partial}{\partial u} + (1 + u_x^2)\frac{\partial}{\partial u_x} + (3u_x u_{xx})\frac{\partial}{\partial u_{xx}},\ W_2 = \mathrm{pr}^{(2)}V_2 = V_2,\ \ W_3 = \mathrm{pr}^{(2)}V_3 = V_3.$$

Some computations will show you that

$$[W_1, W_2] = W_3,\ \ [W_1, W_2] = -W_1,\ \ [W_2, W_3] = 0.$$

So we get nothing new from the pairwise products. Since the W_k act on $\mathbb{R}^4$ and are linearly independent, $n - \dim(L) = 4 - 3 = 0$, so there is exactly one functionally independent second-order invariant.

To construct an invariant $f = f(x, u, u_x, u_{xx})$, we need $W_k f = 0$ for $k = 1, 2, 3$. Note that $W_2 f = 0$ implies $f_x = 0$ and $W_3 f = 0$ implies $f_u = 0$. Thus, $f = f(u_x, u_{xx})$ only. To avoid confusion we will rename u_x, u_{xx} to v and w. Then $f = f(v, w)$ and $W_3 f = 0$ implies $0 = (1 + v^2)f_v + (3vw)f_w = (1 + v^2, 3vw) \bullet \nabla f$. The level curves of f must be solutions to the system of equations

$$\begin{aligned} \tfrac{dv}{dt} &= 1 + v^2 \\ \tfrac{dw}{dt} &= 3vw \end{aligned}$$

Taking the ratio, we get $dv/dw = (1 + v^2)/(3vw)$. Separating variables leads to

$$\frac{3v}{1 + v^2}dv = \frac{1}{w}dw$$

and integrating gives us

$$\frac{3}{2}\ln(1 + v^2) = \ln(w) + \ln(c)$$

for positive constant c. We can solve for c and replace v and w to get

$$f(u_x, u_{xx}) = \frac{u_{xx}}{(1 + u_x^2)^{3/2}} = c$$

so f is the invariant. Notice that this is the curvature of the graph of $u(x)$. If you translate or rotate the graph of u, the curvature is preserved.

Higher-order invariants can be constructed similarly. For a function $u(x, y)$, the most important vector fields on $\mathbb{R}^3$ for us to consider are

$$\begin{array}{ll}
1\frac{\partial}{\partial x} & \text{translation in } x \\
1\frac{\partial}{\partial y} & \text{translation in } y \\
-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} & \text{rotation in } (x, y) \\
x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} & \text{uniform magnification in } (x, y) \\
(a + bu)\frac{\partial}{\partial u} & \text{affine transformation of } u
\end{array}$$

The corresponding global actions obtained by solving the system $d(\bar{x}, \bar{y}, \bar{u})/dt = V(\bar{x}, \bar{y}, \bar{u})$ with initial data $(\bar{x}(0), \bar{y}(0), \bar{u}(0)) = (x, y, u)$ are respectively

$$\begin{aligned}
(\bar{x}, \bar{y}, \bar{u}) &= (x + t, y, u) \\
(\bar{x}, \bar{y}, \bar{u}) &= (x, y + t, u) \\
(\bar{x}, \bar{y}, \bar{u}) &= (x\cos(t) - y\sin(t), x\sin(t) + y\cos(t), u) \\
(\bar{x}, \bar{y}, \bar{u}) &= (x\exp(t), y\exp(t), u) \\
(\bar{x}, \bar{y}, \bar{u}) &= (x, y, u\exp(bt) + at)
\end{aligned}$$

The ideas of invariants also apply when you have more than one dependent variable.