Intersection of Infinite Cylinders

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Contents

1 Introduction

This document shows how to compute the intersection of two infinite cylinders, both treated as hollow shells rather than as solids. An infinite cylinder is defined by a center point $\mathbf{C}$, a unit-length direction $\mathbf{W}$ for its axis, and a positive radius r. The cylinder is parameterized by

$$\mathbf{X}(t,\theta) = \mathbf{C} + (r\cos\theta)\mathbf{U} + (r\sin\theta)\mathbf{V} + t\mathbf{W} \tag{1}$$

where $\{\mathbf{U}, \mathbf{V}, \mathbf{W}\}$ is a right-handed orthonormal set of vectors; that is, the vectors are unit length vectors, mutually perpendicular, and $\mathbf{W} = \mathbf{U} \times \mathbf{V}$. The parameter domains are $t \in (-\infty, \infty)$ and $\theta \in (-\pi, \pi]$. A quadric equation that implicitly defines the cylinder is

$$(\mathbf{X} - \mathbf{C})^{\mathrm{T}} M (\mathbf{X} - \mathbf{C}) = |M(\mathbf{X} - \mathbf{C})|^2 = |(\mathbf{X} - \mathbf{C}) - \mathbf{W} \cdot (\mathbf{X} - \mathbf{C})\mathbf{W}|^2 = r^2 \tag{2}$$

where I is the 3×3 identity matrix and where $M = I - \mathbf{W}\mathbf{W}^{\mathrm{T}}$. M is a projection matrix, so $M^2 = M$, which is why the left-most expression of equation (2) has only a single occurrence of M.

Consider two infinite cylinders with centers $\mathbf{C}_i$, unit-length axis directions $\mathbf{W}_i$, and radii r_i for $i = 0, 1$. The quadratic equations for the cylinders are $(\mathbf{X} - \mathbf{C}_i)^{\mathrm{T}} M_i (\mathbf{X} - \mathbf{C}_i) = r_i^2$.

2 Reduction to a Canonical Form

Rename the cylinders, if necessary, so that $r_0 \geq r_1$. If the last component of $\mathbf{W}_0$ is negative, change the axis direction by negating it. An implementation must remember whether each (or both) of these steps have been applied so that any computed intersection points may be adjusted to undo the steps.

Change variables so that the first cylinder has center at the origin $(0, 0, 0)$ and axis direction $\mathbf{k} = (0, 0, 1)$. This is accomplished by a translation of $\mathbf{C}_0$ and a rotation of $\mathbf{W}_0$ toward $\mathbf{k}$ about an axis perpendicular to both directions. The axis direction has been selected so that $\mathbf{W}_0 \cdot \mathbf{k} \geq 0$. If $\mathbf{W}_0 = \mathbf{k}$, there is nothing to do because the axis is already in the desired canonical direction. This configuration can be handled separately: The cylinders either do not intersect, intersect in a line, intersect in two lines, one is strictly contained by the other, or are the same cylinder.

Now consider when $\mathbf{W}_0 = (a, b, c)$ is not parallel to $\mathbf{k}$; the angle θ between $\mathbf{W}_0$ and $\mathbf{k}$ is acute and is the rotation angle. The dot product is $\cos\theta = \mathbf{W}_0 \cdot \mathbf{k} = c \geq 0$. We also know that $\sin\theta = |\mathbf{W}_0 \times \mathbf{k}| = |(b, -a, 0)| = \sqrt{a^2 + b^2}$. The Rodrigues' formula for the rotation matrix leads to

$$R_0 = I + \sin\theta \left(\frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{bmatrix} \right) + (1 - \cos\theta) \left(\frac{-1}{a^2+b^2} \begin{bmatrix} a^2 & ab & 0 \\ ab & b^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix} \right)$$

Although mathematically correct, a direct implementation will have potential numerical problems when (a, b) is nearly equal to $(0, 0)$, which is the case when $\mathbf{W}_0$ is nearly equal to $\mathbf{k}$. To avoid this, observe that $\sqrt{a^2 + b^2} = \sin\theta$ and $(1 - \cos\theta)/\sin^2\theta = 1/(1 + \cos\theta) = 1/(1 + c) \geq 1/2$, so

$$R_0 = I + \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ -a & -b & 0 \end{bmatrix} - \frac{1}{c} \begin{bmatrix} a^2 & ab & 0 \\ ab & b^2 & 0 \\ 0 & 0 & a^2+b^2 \end{bmatrix}$$

As (a, b) approaches $(0, 0)$, the matrix R_0 approaches I. The rotation matrix R_0 has the properties $R_0\mathbf{W}_0 = \mathbf{k}$ and $\mathbf{k} = R_0^{\mathrm{T}}\mathbf{W}_0$. We may use this to factor M_0,

$$M_0 = I - \mathbf{W}_0\mathbf{W}_0^{\mathrm{T}} = R_0^{\mathrm{T}}R_0 - R_0^{\mathrm{T}}\mathbf{k}\mathbf{k}^{\mathrm{T}}R_0 = R_0^{\mathrm{T}}\left(I - \mathbf{k}\mathbf{k}^{\mathrm{T}}\right)R_0$$

The change of variables $\bar{\mathbf{X}} = R_0(\mathbf{X} - \mathbf{C}_0)$ converts the quadratic equation $(\mathbf{X} - \mathbf{C}_0)^{\mathrm{T}}M_0(\mathbf{X} - \mathbf{C}_0) = r_0^2$ to

$$\bar{\mathbf{X}}^{\mathrm{T}}\left(I - \mathbf{k}\mathbf{k}^{\mathrm{T}}\right)\bar{\mathbf{X}} = r_0^2 \tag{3}$$

The second cylinder is similarly translated and rotated. Substituting $\mathbf{X} = \mathbf{C}_0 + R_0^{\mathrm{T}}\bar{\mathbf{X}}$ into the quadratic equation of the second cylinder leads to

$$\left(\bar{\mathbf{X}} - \bar{\mathbf{C}}\right)^{\mathrm{T}}\left(I - \bar{\mathbf{W}}\bar{\mathbf{W}}^{\mathrm{T}}\right)\left(\bar{\mathbf{X}} - \bar{\mathbf{C}}\right) = r_1^2 \tag{4}$$

where $\bar{\mathbf{C}} = R_0(\mathbf{C}_1 - \mathbf{C}_0)$ and $\bar{\mathbf{W}} = R_0\mathbf{W}_1$.

We may now rotate about the $\mathbf{k}$ axis by a rotation matrix R_1. Such a rotation leaves the vertical cylinder invariant. However, we may choose a particular rotation that transforms $\bar{\mathbf{W}}$ into a new vector $\hat{\mathbf{W}}$ such that $\mathbf{i}\cdot\hat{\mathbf{W}} = 0$; that is, the first component of the new vector is zero. If $\bar{\mathbf{W}} = (a, b, c)$, we know that $(a, b) \neq (0, 0)$, because $\bar{\mathbf{W}}$ is not parallel to $\mathbf{k}$. The rotation matrix is of the form

$$R_1 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To map (a, b, c) to $(*, 0, *)$, we need $a\sin\phi + b\cos\phi = 0$, which is accomplished by setting $(\sin\phi, \cos\phi) = (b, -a)/\sqrt{a^2 + b^2}$. Change variables by $\hat{\mathbf{X}} = R_1\bar{\mathbf{X}}$. Equation (3) becomes

$$\hat{\mathbf{X}}^{\mathrm{T}}\left(I - \mathbf{k}\mathbf{k}^{\mathrm{T}}\right)\hat{\mathbf{X}} = r_0^2 \tag{5}$$

and equation (4) becomes

$$(\hat{\mathbf{X}} - \hat{\mathbf{C}})^{\mathrm{T}}\left(I - \hat{\mathbf{W}}\hat{\mathbf{W}}^{\mathrm{T}}\right)(\hat{\mathbf{X}} - \hat{\mathbf{C}}) = r_1^2 \tag{6}$$

where $\hat{\mathbf{C}} = R_1\bar{\mathbf{C}}$ and $\hat{\mathbf{W}} = R_1\hat{\mathbf{W}}$. In fact, we may go one step further. If the first component of $\hat{\mathbf{C}}$ is negative, instead choose $(\sin\theta, \cos\theta) = (-b, a)/\sqrt{a^2 + b^2}$. This guarantees that $\mathbf{i}\cdot\hat{\mathbf{C}} \geq 0$.

The axis of the second cylinder is parameterized by $\hat{\mathbf{C}} + s\hat{\mathbf{W}}$. We know that the first component of $\hat{\mathbf{W}}$ is zero (by construction). The second component cannot also be zero; otherwise, we would have an axis parallel to $\mathbf{k}$. Thus, we can choose $s = -\mathbf{j}\cdot\hat{\mathbf{C}}/\mathbf{j}\cdot\hat{\mathbf{W}}$ and redefine the center of the second cylinder to be $\hat{\mathbf{C}} - (\mathbf{j}\cdot\hat{\mathbf{C}}/\mathbf{j}\cdot\hat{\mathbf{W}})\hat{\mathbf{W}}$. This center has second component zero because $\mathbf{j}\cdot\tilde{\mathbf{C}} = 0$. Thus, equation (6) is still valid when we instead define $\hat{\mathbf{C}} = R_1\bar{\mathbf{C}} - (\mathbf{j}\cdot R_1\bar{\mathbf{C}}/\mathbf{j}\cdot\hat{\mathbf{W}})\hat{\mathbf{W}}$.

We may now translate both cylinders in the $\mathbf{k}$ direction so that the second cylinder's center is on the $\mathbf{i}$ axis. This amounts to the change of variables $\tilde{\mathbf{X}} = \hat{\mathbf{X}} - (\mathbf{k}\cdot\hat{\mathbf{C}})\mathbf{k}$ and $\tilde{\mathbf{C}} = \hat{\mathbf{C}} - (\mathbf{k}\cdot\hat{\mathbf{C}})\mathbf{k}$. Equation (5) becomes

$$\tilde{\mathbf{X}}^{\mathrm{T}}\left(I - \mathbf{k}\mathbf{k}^{\mathrm{T}}\right)\tilde{\mathbf{X}} = r_0^2 \tag{7}$$

The translation portion of the change of variables is in the $\mathbf{k}$ direction and so is annihilated by the projection matrix $I - \mathbf{k}\mathbf{k}^{\mathrm{T}}$. Equation (6) becomes

$$(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})^{\mathrm{T}} \left(I - \tilde{\mathbf{W}}\tilde{\mathbf{W}}^{\mathrm{T}} \right) (\tilde{\mathbf{X}} - \tilde{\mathbf{C}}) = r_1^2 \tag{8}$$

where $\tilde{\mathbf{W}} = \hat{\mathbf{W}}$.

Here is the summary of our final configuration, which is the *canonical form* of the problem. The symbols over the final variables are dropped for simplicity. After all the change of variables, we have a vertical cylinder with radius r_0, say,

$$\mathbf{X}^{\mathrm{T}} \left(I - \mathbf{k}\mathbf{k} \right) \mathbf{X} = r_0^2 \tag{9}$$

and a nonvertical cylinder

$$(\mathbf{X} - \mathbf{C})^{\mathrm{T}} \left(I - \mathbf{W}\mathbf{W}^{\mathrm{T}} \right) (\mathbf{X} - \mathbf{C}) = r_1^2 \tag{10}$$

where $r_1 \leq r_0$, $\mathbf{C} = (c_0, 0, 0)$ with $c_0 \geq 0$, and $\mathbf{W} = (0, w_1, w_2)$ with $w_1 \neq 0$. We wish to compute intersections, if they exist, for the two cylinders. Naturally, all the changes of variables must be tracked in the application so that intersection points may be computed in the final coordinate system but then transformed back to the original space.

3 Computing Intersection Curves

Equation (9) is a quadratic equation that represents a vertical cylinder. If $\mathbf{X} = (x_0, x_1, x_2)$, the equation is simply $x_0^2 + x_1^2 = r_0^2$. Using equation (1), we may parameterize the cylinder of equation (10) by

$$\mathbf{X}(t, \theta) = \mathbf{C} + (r_1 \cos\theta)\mathbf{U} + (r_1 \sin\theta)\mathbf{V} + t\mathbf{W}$$

where $\mathbf{U} = (1, 0, 0)$, $\mathbf{V} = (0, w_2, -w_1)$, and $\mathbf{W} = (0, w_1, w_2)$. The three vectors form a right-handed orthonormal set. We may substitute the first two components of $\mathbf{X}(t, \theta)$ into equation (9) to obtain

$$f(t, \theta) = (c_0 + r_1 \cos\theta)^2 + (r_1 w_2 \sin\theta + t w_1)^2 - r_0^2 = 0 \tag{11}$$

That is, the intersection points are represented in parameter space as the zero-valued level set of $f(t, \theta)$. Equation (11) is a quadratic polynomial in t with roots formally defined by

$$t = \frac{-r_1 w_2 \sin\theta \pm \sqrt{r_0^2 - (c_0 + r_1 \cos\theta)^2}}{w_1} \tag{12}$$

The discriminant is

$$\Delta = r_0^2 - (c_0 + r_1 \cos\theta)^2 \in [r_0^2 - (c_0 + r_1)^2, r_0^2 - (c_0 - r_1)^2]$$

and is an even function of $\theta \in (-\pi, \pi]$. When $\Delta > 0$, the equation defines two functions of t that depend on θ. When $\Delta < 0$, there are no real-valued roots. The separation between the cases is $\Delta = 0$, which we now analyze.

If $r_0^2 - (c_0 + r_1)^2 \geq 0$, then $\Delta \geq 0$ and there are real-valued roots $t(\theta)$ to the quadratic equation for all $\theta \in (-\pi, \pi]$. If $r_0^2 - (c_0 - r_1)^2 < 0$, then there are no real-valued roots for any θ. The special case of $r_0^2 - (c_0 - r_1)^2 = 0$ leads to a single point of intersection, but we have already handled that case previously.

The remaining case is $0 \in (r_0^2 - (c_0 + r_1)^2, r_0^2 - (c_0 - r_1)^2)$. The inequality $r_0^2 - (c_0 + r_1)^2 < 0$ implies $r_0 - r_1 < c_0$. The inequality $r_0^2 - (c_0 - r_1)^2 > 0$ implies $c_0 < r_0 + r_1$. Therefore, we know that $r_0 - r_1 < c_0 < r_0 + r_1$. Setting $\Delta = 0$, we obtain $\cos\theta = (-c_0 \pm r_0)/r_1$ that determines the possible roots for θ. There are real-valued roots for $t(\theta)$ when θ is between two consecutive θ-roots and $\Delta(\theta) > 0$ between those roots.

4 Examples

Example 4.1 Let $r_0 = 3$, $r_1 = 2$, $\mathbf{C} = (4, 0, 0)$, and $\mathbf{W} = (0, 3/5, 4/5)$. From the geometry, we expect a single closed loop of intersection. The discriminant is $\Delta(\theta) = 9 - (4 + 2\cos\theta)^2$. It is zero when $\cos\theta = -7/2$ or $\cos\theta = -1/2$. The first equation has no real-valued roots because $-7/2$ is larger than 1 in magnitude. Thus, $\cos\theta = -1/2$ is the only constraint, which leads to $\theta = \pm 2\pi/3$. Notice that $\Delta(0) = -27$, $\Delta(\pm\pi) = 5$, and $\Delta'(\pm\pi) = 0$. It must be that $\Delta(\theta) \geq 0$ for $|\theta| \geq 2\pi/3$. Equivalently, $\Delta(\theta) \geq 0$ for $\theta \in [2\pi/3, 4\pi/3]$. One half of the loop is generated parametrically by

$$\mathbf{X}(\theta) = \begin{bmatrix} 4 + 2\cos\theta \\ +\frac{32}{25}\sin\theta + \frac{3}{5}\left(\frac{-8\sin\theta - 5\sqrt{9-(4+2\cos\theta)^2}}{3}\right) \\ -\frac{24}{25}\sin\theta + \frac{4}{5}\left(\frac{-8\sin\theta - 5\sqrt{9-(4+2\cos\theta)^2}}{3}\right) \end{bmatrix}$$

The other half of the loop is generated parametrically by

$$\mathbf{X}(\theta) = \begin{bmatrix} 4 + 2\cos\theta \\ +\frac{32}{25}\sin\theta + \frac{3}{5}\left(\frac{-8\sin\theta + 5\sqrt{9-(4+2\cos\theta)^2}}{3}\right) \\ -\frac{24}{25}\sin\theta + \frac{4}{5}\left(\frac{-8\sin\theta + 5\sqrt{9-(4+2\cos\theta)^2}}{3}\right) \end{bmatrix}$$

where $\theta \in [2\pi/3, 4\pi/3]$. If necessary, the θ samples for a polyline approximation may be chosen so that the samples are separated by constant arc length. $\bowtie$

Example 4.2 Let $r_0 = 3$, $r_1 = 2$, $\mathbf{C} = (1, 0, 0)$, and $\mathbf{W} = (0, 3/5, 4/5)$. From the geometry, we expect two closed loops of intersection. The loops touch at the point $(3, 0, 0)$. The discriminant is $\Delta(\theta) = 9 - (1 + 2\cos\theta)^2 \geq 0$. The function is zero only at $\theta = 0$, and it is verifiable that $\Delta'(0) = 0$. One loop is generated by

$$\mathbf{X}(\theta) = \begin{bmatrix} 1 + 2\cos\theta \\ +\frac{32}{25}\sin\theta + \frac{3}{5}\left(\frac{-8\sin\theta - 5\sqrt{9-(1+2\cos\theta)^2}}{3}\right) \\ -\frac{24}{25}\sin\theta + \frac{4}{5}\left(\frac{-8\sin\theta - 5\sqrt{9-(1+2\cos\theta)^2}}{3}\right) \end{bmatrix}$$

The other loop is generated by

$$\mathbf{X}(\theta) = \begin{bmatrix} 1 + 2\cos\theta \\ +\frac{32}{25}\sin\theta + \frac{3}{5}\left(\frac{-8\sin\theta + 5\sqrt{9-(1+2\cos\theta)^2}}{3}\right) \\ -\frac{24}{25}\sin\theta + \frac{4}{5}\left(\frac{-8\sin\theta + 5\sqrt{9-(1+2\cos\theta)^2}}{3}\right) \end{bmatrix}$$

where $\theta \in (-\pi, \pi]$. $\bowtie$

Example 4.3 Let $r_0 = 3$, $r_1 = 2$, $\mathbf{C} = (0,0,0)$, and $\mathbf{W} = (0, 3/5, 4/5)$. From the geometry, we expect two closed loops of intersection. The loops are disjoint. The discriminant is $\Delta(\theta) = 9 - 4\cos^2\theta \geq 5 > 0$. The function is never zero. One loop is generated by

$$\mathbf{X}(\theta) = \begin{bmatrix} 2\cos\theta \\ +\frac{32}{25}\sin\theta + \frac{3}{5}\left(\frac{-8\sin\theta - 5\sqrt{9-4\cos^2\theta}}{3}\right) \\ -\frac{24}{25}\sin\theta + \frac{4}{5}\left(\frac{-8\sin\theta - 5\sqrt{9-4\cos^2\theta}}{3}\right) \end{bmatrix}$$

The other loop is generated by

$$\mathbf{X}(\theta) = \begin{bmatrix} 2\cos\theta \\ +\frac{32}{25}\sin\theta + \frac{3}{5}\left(\frac{-8\sin\theta + 5\sqrt{9-4\cos^2\theta}}{3}\right) \\ -\frac{24}{25}\sin\theta + \frac{4}{5}\left(\frac{-8\sin\theta + 5\sqrt{9-4\cos^2\theta}}{3}\right) \end{bmatrix}$$

where $\theta \in (-\pi, \pi]$. $\bowtie$