Distance Between Two Circles in 3D

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1 Discussion

A circle in 3D is represented by a center $\mathbf{C}$, a radius R, and a plane containing the circle, $\mathbf{N}\cdot(\mathbf{X}-\mathbf{C}) = 0$ where $\mathbf{N}$ is a unit length normal to the plane. If $\mathbf{U}$ and $\mathbf{V}$ are also unit length vectors so that $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{N}$ form a right-handed orthonormal coordinate system (the matrix with these vectors as columns is orthonormal with determinant 1), then the circle is parameterized as

$$\mathbf{X} = \mathbf{C} + R(\cos(\theta)\mathbf{U} + \sin(\theta)\mathbf{V}) =: \mathbf{C} + R\mathbf{W}(\theta)$$

for angles $\theta \in [0, 2\pi)$. Note that $|\mathbf{X}-\mathbf{C}| = R$, so the $\mathbf{X}$ values are all equidistant from $\mathbf{C}$. Moreover, $\mathbf{N}\cdot(\mathbf{X}-\mathbf{C}) = 0$ since $\mathbf{U}$ and $\mathbf{V}$ are perpendicular to $\mathbf{N}$, so the $\mathbf{X}$ lie in the plane.

Let the two circles be $\mathbf{C}_0 + R_0\mathbf{W}_0(\theta)$ for $\theta \in [0, 2\pi)$ and $\mathbf{C}_1 + R_1\mathbf{W}_1(\phi)$ for $\phi \in [0, 2\pi)$. The squared distance between any two points on the circles is

$$\begin{aligned} F(\theta,\phi) &= |\mathbf{C}_1 - \mathbf{C}_0 + R_1\mathbf{W}_1 - R_0\mathbf{W}_0|^2 \\ &= |\mathbf{D}|^2 + R_0^2 + R_1^2 + 2R_1\mathbf{D}\cdot\mathbf{W}_1 - 2R_0R_1\mathbf{W}_0\cdot\mathbf{W}_1 - 2R_0\mathbf{D}\cdot\mathbf{W}_0 \end{aligned}$$

where $\mathbf{D} = \mathbf{C}_1 - \mathbf{C}_0$. Since F is doubly periodic and continuously differentiable, its global minimum must occur when $\nabla F = (0, 0)$. The partial derivatives are

$$\frac{\partial F}{\partial \theta} = -2R_0\mathbf{D}\cdot\mathbf{W}_0' - 2R_0R_1\mathbf{W}_0'\cdot\mathbf{W}_1$$

and

$$\frac{\partial F}{\partial \phi} = 2R_1\mathbf{D}\cdot\mathbf{W}_1' - 2R_0R_1\mathbf{W}_0\cdot\mathbf{W}_1'.$$

Define $c_0 = \cos(\theta)$, $s_0 = \sin(\theta)$, $c_1 = \cos(\phi)$, and $s_1 = \sin(\phi)$. Then $\mathbf{W}_0 = c_0\mathbf{U}_0 + s_0\mathbf{V}_0$, $\mathbf{W}_1 = c_1\mathbf{U}_1 + s_1\mathbf{V}_1$, $\mathbf{W}_0' = -s_0\mathbf{U}_0 + c_0\mathbf{V}_0$, and $\mathbf{W}_1' = -s_1\mathbf{U}_1 + c_1\mathbf{V}_1$. Setting the partial derivatives equal to zero leads to

$$\begin{aligned} 0 &= s_0(a_0 + a_1c_1 + a_2s_1) + c_0(a_3 + a_4c_1 + a_5s_1) \\ 0 &= s_1(b_0 + b_1c_0 + b_2s_0) + c_1(b_3 + b_4c_0 + b_5s_0) \end{aligned}$$

where

$$a_0 = -\mathbf{D}\cdot\mathbf{U}_0,\quad a_1 = -R_1\mathbf{U}_0\cdot\mathbf{U}_1,\quad a_2 = -R_1\mathbf{U}_0\cdot\mathbf{V}_1,\quad a_3 = \mathbf{D}\cdot\mathbf{V}_0,\quad a_4 = R_1\mathbf{U}_1\cdot\mathbf{V}_0,\quad a_5 = R_1\mathbf{V}_0\cdot\mathbf{V}_1,$$
$$b_0 = -\mathbf{D}\cdot\mathbf{U}_1,\quad b_1 = R_0\mathbf{U}_0\cdot\mathbf{U}_1,\quad b_2 = R_0\mathbf{U}_1\cdot\mathbf{V}_0,\quad b_3 = \mathbf{D}\cdot\mathbf{V}_1,\quad b_4 = -R_0\mathbf{U}_0\cdot\mathbf{V}_1,\quad b_5 = -R_0\mathbf{V}_0\cdot\mathbf{V}_1.$$

In matrix form we have

$$\begin{bmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{bmatrix}\begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} a_0 + a_1c_1 + a_2s_1 & a_3 + a_4c_1 + a_5s_1 \\ b_2s_1 + b_5c_1 & b_1s_1 + b_4c_1 \end{bmatrix}\begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ -(b_0s_1 + b_3c_1) \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}$$

Let M denote the 2×2 matrix on the right-hand side of the equation. Multiplying by the adjoint of M yields

$$\det(M)\begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} m_{11} & -m_{01} \\ -m_{10} & m_{00} \end{bmatrix}\begin{bmatrix} 0 \\ \lambda \end{bmatrix} = \begin{bmatrix} -m_{01}\lambda \\ m_{00}\lambda \end{bmatrix}. \tag{1}$$

Summing the squares of the vector components and using $s_0^2 + c_0^2 = 1$ yields

$$(m_{00}m_{11} - m_{01}m_{10})^2 = \lambda^2 \left(m_{00}^2 + m_{01}^2\right).$$

The above equation can be reduced to a polynomial of degree 8 whose roots $c_1 \in [-1, 1]$ are the candidates to provide the global minimum of F. Formally computing the determinant and using $s_1^2 = 1 - c_1^2$ leads to

$$m_{00}m_{11} - m_{01}m_{10} = p_0(c_1) + s_1 p_1(c_1)$$

where $p_0(z) = \sum_{i=0}^{2} p_{0i} z^i$ and $p_1(z) = \sum_{i=0}^{1} p_{1i} z$. The coefficients are

$$\begin{aligned}
p_{00} &= a_2 b_1 - a_5 b_2, \\
p_{01} &= a_0 b_4 - a_3 b_5, \\
p_{02} &= a_5 b_2 - a_2 b_1 + a_1 b_4 - a_4 b_5, \\
p_{10} &= a_0 b_1 - a_3 b_2, \\
p_{11} &= a_1 b_1 - a_5 b_5 + a_2 b_4 - a_4 b_2.
\end{aligned}$$

Similarly,

$$m_{00}^2 + m_{01}^2 = q_0(c_1) + s_1 q_1(c_1)$$

where $q_0(z) = \sum_{i=0}^{2} q_{0i} z^i$ and $q_1(z) = \sum_{i=0}^{1} q_{1i} z$. The coefficients are

$$\begin{aligned}
q_{00} &= a_0^2 + a_2^2 + a_3^2 + a_5^2, \\
q_{01} &= 2(a_0 a_1 + a_3 a_4), \\
q_{02} &= a_1^2 - a_2^2 + a_4^2 - a_5^2, \\
q_{10} &= 2(a_0 a_2 + a_3 a_5), \\
q_{11} &= 2(a_1 a_2 + a_4 a_5).
\end{aligned}$$

Finally,

$$\lambda^2 = r_0(c_1) + s_1 r_1(c_1)$$

where $r_0(z) = \sum_{i=0}^{2} r_{0i} z^i$ and $r_1(z) = \sum_{i=0}^{1} r_{1i} z$. The coefficients are

$$\begin{aligned}
r_{00} &= b_0^2, \\
r_{01} &= 0, \\
r_{02} &= b_3^2 - b_0^2, \\
r_{10} &= 0, \\
r_{11} &= 2 b_0 b_3.
\end{aligned}$$

Combining these yields

$$0 = \left[(p_0^2 - r_0 q_0) + (1 - c_1^2)(p_1^2 - r_1 q_1)\right] + s_1 \left[2 p_0 p_1 - r_0 q_1 - r_1 q_0\right] = g_0(c_1) + s_1 g_1(c_1) \qquad (2)$$

where $g_0(z) = \sum_{i=0}^4 g_{0i} z^i$ and $g_1(z) = \sum_{i=0}^3 g_{1i} z^i$. The coefficients are

$$
\begin{aligned}
g_{00} &= p_{00}^2 + p_{10}^2 - q_{00} r_{00} \\
g_{01} &= 2(p_{00} p_{01} + p_{10} p_{11}) - q_{01} r_{00} - q_{10} r_{11} \\
g_{02} &= p_{01}^2 + 2 p_{00} p_{02} + p_{11}^2 - p_{10}^2 - q_{02} r_{00} - q_{00} r_{02} - q_{11} r_{11} \\
g_{03} &= 2(p_{01} p_{02} - p_{10} p_{11}) - q_{01} r_{02} + q_{10} r_{11} \\
g_{04} &= p_{02}^2 - p_{11}^2 - q_{02} r_{02} + q_{11} r_{11} \\
g_{10} &= 2 p_{00} p_{10} - q_{10} r_{00} \\
g_{11} &= 2(p_{01} p_{10} + p_{00} p_{11}) - q_{11} r_{00} - q_{00} r_{11} \\
g_{12} &= 2(p_{02} p_{10} + p_{01} p_{11}) - q_{10} r_{02} - q_{01} r_{11} \\
g_{13} &= 2 p_{02} p_{11} - q_{11} r_{02} - q_{02} r_{11}
\end{aligned}
$$

We can eliminate the s_1 term by solving $g_0 = -s_1 g_1$ and squaring to obtain

$$
0 = g_0^2 - (1 - c_1^2) g_1^2 = h(c_1)
$$

where $h(z) = \sum_{i=0}^8 h_i z^i$. The coefficients are

$$
\begin{aligned}
h_0 &= g_{00}^2 - g_{10}^2, \\
h_1 &= 2(g_{00} g_{01} - g_{10} g_{11}), \\
h_2 &= g_{01}^2 + g_{10}^2 - g_{11}^2 + 2(g_{00} g_{02} - g_{10} g_{12}), \\
h_3 &= 2(g_{01} g_{02} + g_{00} g_{03} + g_{10} g_{11} - g_{11} g_{12} - g_{10} g_{13}), \\
h_4 &= g_{02}^2 + g_{11}^2 - g_{12}^2 + 2(g_{01} g_{03} + g_{00} g_{04} + g_{10} g_{12} - g_{11} g_{13}), \\
h_5 &= 2(g_{02} g_{03} + g_{01} g_{04} + g_{11} g_{12} + g_{10} g_{13} - g_{12} g_{13}), \\
h_6 &= g_{03}^2 + g_{12}^2 - g_{13}^2 + 2(g_{02} g_{04} + g_{11} g_{13}), \\
h_7 &= 2(g_{03} g_{04} + g_{12} g_{13}), \\
h_8 &= g_{04}^2 + g_{13}^2.
\end{aligned}
$$

To find the minimum squared distance, compute all the real-valued roots of $h(c_1) = 0$. For each c_1, compute $s_1 = \pm\sqrt{1 - c_1^2}$ and choose either (or both) of these that satisfy equation (2). For each pair (c_1, s_1), solve for (c_0, s_0) in equation (1). The main numerical issue to deal with is how close to zero is $\det(M)$. (TO DO: Show that this case only occurs when circles are parallel and $\mathbf{D}$ is normal to both planes?)