Bicubic Bézier Exact Interpolation

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1 Introduction

Given a 4×4 grid of points, a bicubic Bézier surface patch may be constructed to pass through the 16 points, thus leading to an exact interpolation of the points.

2 The Algorithm

Let the data points be $\mathbf{P}_{ij}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 3$. A bicubic Bézier patch $\mathbf{X}(s,t)$ is of the form shown below where the coefficient of $(1-s)^{3-i}s^i$ is the combinatorial symbol for 3 *choose* i and has value $3!/((3-i)!i!)$. Similarly, the coefficient of $(1-t)^{3-j}t^j$ has value $3!/((3-j)!j!)$. The domain is $(s,t) \in [0,1]^2$.

$$
\begin{array}{rcllllllllll}
\mathbf{X}(s,t) & = & \multicolumn{10}{l}{\sum_{i=0}^{3}\sum_{j=0}^{3}\begin{pmatrix} 3 \\ i \end{pmatrix}(1-s)^{3-i}s^i\begin{pmatrix} 3 \\ j \end{pmatrix}(1-t)^{3-j}t^j\mathbf{C}_{ij}} \\
& = & & (1-s)^3 & [& (1-t)^3\mathbf{C}_{00} & + & 3(1-t)^2t\mathbf{C}_{01} & + & 3(1-t)t^2\mathbf{C}_{02} & + & t^3\mathbf{C}_{03} \;] \\
& & + & 3(1-s)^2s & [& (1-t)^3\mathbf{C}_{10} & + & 3(1-t)^2t\mathbf{C}_{11} & + & 3(1-t)t^2\mathbf{C}_{12} & + & t^3\mathbf{C}_{13} \;] \\
& & + & 3(1-s)s^2 & [& (1-t)^3\mathbf{C}_{20} & + & 3(1-t)^2t\mathbf{C}_{21} & + & 3(1-t)t^2\mathbf{C}_{22} & + & t^3\mathbf{C}_{23} \;] \\
& & + & s^3 & [& (1-t)^3\mathbf{C}_{30} & + & 3(1-t)^2t\mathbf{C}_{31} & + & 3(1-t)t^2\mathbf{C}_{32} & + & t^3\mathbf{C}_{33} \;]
\end{array}
\tag{1}
$$

We want to choose the $\mathbf{C}_{ij}$ so that $\mathbf{X}(i/3, j/3) = \mathbf{P}_{ij}$ for $0 \leq i \leq 3$ and $0 \leq j \leq 3$. This gives us a linear system of 16 vector-valued equations in 16 unknowns, The matrix of coefficients is invertible, leading to the solution

$$
\begin{aligned}
\mathbf{C}_{00} &= \mathbf{X}_{00} \\
\mathbf{C}_{01} &= \left(-5\mathbf{X}_{00} + 18\mathbf{X}_{01} - 9\mathbf{X}_{02} + 2\mathbf{X}_{03}\right)/6 \\
\mathbf{C}_{02} &= \left(2\mathbf{X}_{00} - 9\mathbf{X}_{01} + 18\mathbf{X}_{02} - 5\mathbf{X}_{03}\right)/6 \\
\mathbf{C}_{03} &= \mathbf{X}_{03} \\
\mathbf{C}_{10} &= \left(-5\mathbf{X}_{00} + 18\mathbf{X}_{10} - 9\mathbf{X}_{20} + 2\mathbf{X}_{30}\right)/6 \\
\mathbf{C}_{11} &= (25\mathbf{X}_{00} - 90\mathbf{X}_{01} + 45\mathbf{X}_{02} - 10\mathbf{X}_{03} \\
&\quad - 90\mathbf{X}_{10} + 324\mathbf{X}_{11} - 162\mathbf{X}_{12} + 36\mathbf{X}_{13} \\
&\quad + 45\mathbf{X}_{20} - 162\mathbf{X}_{21} + 81\mathbf{X}_{22} - 18\mathbf{X}_{23} \\
&\quad - 10\mathbf{X}_{30} + 36\mathbf{X}_{31} - 18\mathbf{X}_{32} + 4\mathbf{X}_{33})/36 \\
\mathbf{C}_{12} &= (-10\mathbf{X}_{00} + 45\mathbf{X}_{01} - 90\mathbf{X}_{02} + 25\mathbf{X}_{03} \\
&\quad + 36\mathbf{X}_{10} - 162\mathbf{X}_{11} + 324\mathbf{X}_{12} - 90\mathbf{X}_{12} \\
&\quad - 18\mathbf{X}_{20} + 81\mathbf{X}_{21} - 162\mathbf{X}_{22} + 45\mathbf{X}_{23} \\
&\quad +4\mathbf{X}_{30} - 18\mathbf{X}_{31} + 36\mathbf{X}_{32} - 10\mathbf{X}_{33})/36 \\
\mathbf{C}_{13} &= \left(-5\mathbf{X}_{03} + 18\mathbf{X}_{13} - 9\mathbf{X}_{23} + 2\mathbf{X}_{33}\right)/6 \\
\mathbf{C}_{20} &= \left(2\mathbf{X}_{00} - 9\mathbf{X}_{10} + 18\mathbf{X}_{20} - 5\mathbf{X}_{30}\right)/6 \\
\mathbf{C}_{21} &= (-10\mathbf{X}_{00} + 36\mathbf{X}_{01} - 18\mathbf{X}_{02} + 4\mathbf{X}_{03} \\
&\quad + 45\mathbf{X}_{10} - 162\mathbf{X}_{11} + 81\mathbf{X}_{12} - 18\mathbf{X}_{13} \\
&\quad - 90\mathbf{X}_{20} + 324\mathbf{X}_{21} - 162\mathbf{X}_{22} + 36\mathbf{X}_{23} \\
&\quad + 25\mathbf{X}_{30} - 90\mathbf{X}_{31} + 45\mathbf{X}_{32} - 10\mathbf{X}_{33})/36 \\
\mathbf{C}_{22} &= (4\mathbf{X}_{00} - 18\mathbf{X}_{01} + 36\mathbf{X}_{02} - 10\mathbf{X}_{03} \\
&\quad - 18\mathbf{X}_{10} + 81\mathbf{X}_{11} - 162\mathbf{X}_{12} + 45\mathbf{X}_{13} \\
&\quad + 36\mathbf{X}_{20} - 162\mathbf{X}_{21} + 324\mathbf{X}_{22} - 90\mathbf{X}_{23} \\
&\quad - 10\mathbf{X}_{30} + 45\mathbf{X}_{31} - 90\mathbf{X}_{32} + 25\mathbf{X}_{33})/36 \\
\mathbf{C}_{23} &= \left(2\mathbf{X}_{03} - 9\mathbf{X}_{13} + 18\mathbf{X}_{23} - 5\mathbf{X}_{33}\right)/6 \\
\mathbf{C}_{30} &= \mathbf{X}_{30} \\
\mathbf{C}_{31} &= \left(-5\mathbf{X}_{30} + 18\mathbf{X}_{31} - 9\mathbf{X}_{32} + 2\mathbf{X}_{33}\right)/6 \\
\mathbf{C}_{32} &= \left(2\mathbf{X}_{30} - 9\mathbf{X}_{31} + 18\mathbf{X}_{32} - 5\mathbf{X}_{33}\right)/6 \\
\mathbf{C}_{33} &= \mathbf{X}_{33}
\end{aligned}
\tag{2}
$$