

Geodesics on a Parameterized Surface

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1 Discussion

Consider a parameterized surface $\mathbf{x}(u, v)$. Let \mathbf{x}_u and \mathbf{x}_v denote the first-order partial derivatives. Let \mathbf{x}_{uu} , \mathbf{x}_{uv} , and \mathbf{x}_{vv} denote the second-order partial derivatives. From differential geometry, the metric for the surface is

$$G = [g_{ij}] = \begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_u \cdot \mathbf{x}_v & \mathbf{x}_v \cdot \mathbf{x}_v \end{bmatrix}$$

The inverse of the matrix is

$$G^{-1} = [g^{ij}] = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} \begin{bmatrix} \mathbf{x}_v \cdot \mathbf{x}_v & -\mathbf{x}_u \cdot \mathbf{x}_v \\ -\mathbf{x}_u \cdot \mathbf{x}_v & -\mathbf{x}_u \cdot \mathbf{x}_u \end{bmatrix}$$

The Christoffel symbols of the first kind are Γ_{ijk} and are symmetric in i and j . They are

$$\Gamma_{111} = \mathbf{x}_{uu} \cdot \mathbf{x}_u$$

$$\Gamma_{121} = \mathbf{x}_{uv} \cdot \mathbf{x}_u$$

$$\Gamma_{221} = \mathbf{x}_{vv} \cdot \mathbf{x}_u$$

$$\Gamma_{112} = \mathbf{x}_{uu} \cdot \mathbf{x}_v$$

$$\Gamma_{122} = \mathbf{x}_{uv} \cdot \mathbf{x}_v$$

$$\Gamma_{222} = \mathbf{x}_{vv} \cdot \mathbf{x}_v$$

The Christoffel symbols of the second kind are Γ_{jk}^i and are symmetric in i and j . They are

$$\begin{aligned} \Gamma_{11}^1 &= g^{11}\Gamma_{111} + g^{12}\Gamma_{112} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} [(\mathbf{x}_v \cdot \mathbf{x}_v)\mathbf{x}_{uu} \cdot \mathbf{x}_u - (\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uu} \cdot \mathbf{x}_v] \\ \Gamma_{12}^1 &= g^{11}\Gamma_{121} + g^{12}\Gamma_{122} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} [(\mathbf{x}_v \cdot \mathbf{x}_v)\mathbf{x}_{uv} \cdot \mathbf{x}_u - (\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uv} \cdot \mathbf{x}_v] \\ \Gamma_{22}^1 &= g^{11}\Gamma_{221} + g^{12}\Gamma_{222} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} [(\mathbf{x}_v \cdot \mathbf{x}_v)\mathbf{x}_{vv} \cdot \mathbf{x}_u - (\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{vv} \cdot \mathbf{x}_v] \\ \Gamma_{11}^2 &= g^{21}\Gamma_{111} + g^{22}\Gamma_{112} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} [-(\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uu} \cdot \mathbf{x}_u + (\mathbf{x}_u \cdot \mathbf{x}_u)\mathbf{x}_{uu} \cdot \mathbf{x}_v] \\ \Gamma_{12}^2 &= g^{21}\Gamma_{121} + g^{22}\Gamma_{122} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} [-(\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{uv} \cdot \mathbf{x}_u + (\mathbf{x}_u \cdot \mathbf{x}_u)\mathbf{x}_{uv} \cdot \mathbf{x}_v] \\ \Gamma_{22}^2 &= g^{21}\Gamma_{221} + g^{22}\Gamma_{222} = \frac{1}{|\mathbf{x}_u \times \mathbf{x}_v|^2} [-(\mathbf{x}_u \cdot \mathbf{x}_v)\mathbf{x}_{vv} \cdot \mathbf{x}_u + (\mathbf{x}_u \cdot \mathbf{x}_u)\mathbf{x}_{vv} \cdot \mathbf{x}_v] \end{aligned}$$

A geodesic curve must satisfy the following system of differential equations in parameter space where s is the arc length parameter,

$$\begin{aligned} \frac{d^2 u}{ds^2} + \Gamma_{11}^1 \left(\frac{du}{ds}\right)^2 + 2\Gamma_{12}^1 \frac{du}{ds} \frac{dv}{ds} + \Gamma_{22}^1 \left(\frac{dv}{ds}\right)^2 &= 0 \\ \frac{d^2 v}{ds^2} + \Gamma_{11}^2 \left(\frac{du}{ds}\right)^2 + 2\Gamma_{12}^2 \frac{du}{ds} \frac{dv}{ds} + \Gamma_{22}^2 \left(\frac{dv}{ds}\right)^2 &= 0 \end{aligned}$$

The curve itself is $\mathbf{y}(s) = \mathbf{x}(u(s), v(s))$. Initial conditions specified on the surface are $\mathbf{y}_0 = \mathbf{y}(0)$ and $\mathbf{y}_1 = \mathbf{y}'(0)$ with $|\mathbf{y}_1| = 1$. The Jacobian matrix is the 3×2 matrix whose columns are \mathbf{x}_u and \mathbf{x}_v . Note that $J^T J = G$. The derivative in parameter space is

$$\begin{bmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{bmatrix} = G^{-1} J^T \mathbf{y}'(s).$$

Given $\mathbf{y}'(0)$, the corresponding initial derivative in parameter space can be computed.

The system of equations is of the form

$$u'' = F(u, v, u', v')$$

$$v'' = G(u, v, u', v')$$

where the ' indicate derivatives. Setting $p = u'$ and $q = v'$, this becomes a system of four equations

$$u' = p$$

$$v' = q$$

$$p' = F(u, v, p, q)$$

$$q' = G(u, v, p, q)$$

The initial value problem can be solved using standard differential equation solvers such as Runge-Kutta 4th order.