Rough Plane Analysis

David Eberly
Geometric Tools, LLC
http://www.geometrictools.com/
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Contents

Section 3.2.7, *Examples with Dissipative Forces*, of the *Game Physics* book is on motion of rigid bodies on planes that have *rough friction* modeled by $\mathbf{F} = -c\mathbf{V}/|\mathbf{V}|$ where $c > 0$ is the coefficient of friction and $\mathbf{V}$ is the object velocity. This type of friction is in contrast to *viscous friction*, which is modeled by $\mathbf{F} = -c\mathbf{V}$.

The examples in the section show you how to set up the Lagrangian equations of motion for various rigid bodies. However, there is still a big step from those equations to an implementation. This document provides additional details on how to do this. Each example in this document is implemented and occurs in the folder for the physics applications:

```
GeometricTools/WildMagic4/SamplePhysics/RoughPlaneParticle1
GeometricTools/WildMagic4/SamplePhysics/RoughPlaneParticle2
GeometricTools/WildMagic4/SamplePhysics/RoughPlaneThinRod1
GeometricTools/WildMagic4/SamplePhysics/RoughPlaneThinRod2
GeometricTools/WildMagic4/SamplePhysics/RoughPlaneFlatBoard
GeometricTools/WildMagic4/SamplePhysics/RoughPlaneSolidBox
```

1 One Particle on a Rough Plane

The equations of motion were shown to be

$$m\ddot{x} + \frac{c\dot{x}}{\sqrt{\dot{x}^2 + \dot{w}^2}} = 0, \quad m\ddot{w} + \frac{c\dot{w}}{\sqrt{\dot{x}^2 + \dot{w}^2}} + mg\sin\phi = 0$$

where $m > 0$ is the mass of the particle, ϕ is the angle formed by the xw-plane with the xy-plane, $c = \mu mg\cos\phi > 0$ is the coefficient of friction ($\mu > 0$ is a constant depending on the material), and $g > 0$ is the gravitational constant. The convention is that if $\sqrt{\dot{x}^2 + \dot{w}^2}$ is zero, the terms $\dot{x}/\sqrt{\dot{x}^2 + \dot{w}^2}$ and $\dot{w}/\sqrt{\dot{x}^2 + \dot{w}^2}$ are considered to be zero.

The application `RoughPlaneParticle1` is an implementation of this system and applies a fourth-order Runge-Kutta method to

$$\begin{array}{rcl} \dot{x} & = & u \\ \dot{u} & = & \begin{cases} -\alpha\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{w}^2}}, & \dot{x}^2 + \dot{w}^2 \neq 0 \\ 0, & \dot{x}^2 + \dot{w}^2 = 0 \end{cases} \\ \dot{w} & = & v \\ \dot{v} & = & \begin{cases} -\alpha\frac{\dot{w}}{\sqrt{\dot{x}^2+\dot{w}^2}} - \beta, & \dot{x}^2 + \dot{w}^2 \neq 0 \\ 0, & \dot{x}^2 + \dot{w}^2 = 0 \end{cases} \end{array}$$

where $\alpha = c/m$ and $\beta = g\sin\phi$. The output is a plot of the path of the particle in the xw-plane. Let us take a closer look, though, at the equations.

Rough friction, no slope. Suppose that $\phi = 0$. In this case the plane on which the particle travels is horizontal. If the velocity is $\mathbf{V} = (\dot{x}, \dot{w})$, then the equations of motion condense to

$$\frac{d\mathbf{V}}{dt} + \alpha\frac{\mathbf{V}}{|\mathbf{V}|} = \mathbf{0}$$

It is simple enough to verify that the solution to this differential equation is

$$\mathbf{V}(t) = \mathbf{V}_0 - \alpha t \frac{\mathbf{V}_0}{|\mathbf{V}_0|} = \left(1 - \frac{\alpha t}{|\mathbf{V}_0|}\right) \mathbf{V}_0$$

where $\mathbf{V}_0$ is the initial velocity of the particle. This equation is valid as long as the velocity remains nonzero. The velocity becomes zero at the time

$$t_{\max} = \frac{|\mathbf{V}_0|}{\alpha} = \frac{m|\mathbf{V}_0|}{c}$$

That is, the particle stops moving in finite time. This equation agrees with your physical intuition. The maximum time is proportional to the mass; the heavier the particle, the longer it takes to stop. The maximum time is also proportional to the initial speed; the faster the particle starts out, the longer it takes to stop. Finally, the maximum time is inversely proportional to the coefficient of friction (and, by implication, the magnitude of the frictional force). The larger the coefficient of friction, the greater the frictional force applied to the particle, so the smaller the time it takes to stop the particle. The position of the particle is obtained by one more integration,

$$\mathbf{X}(t) = \mathbf{X}_0 + \left(t - \frac{\alpha t^2}{2|\mathbf{V}_0|}\right)$$

The final position when the particle stops is

$$\mathbf{X}(t_{\max}) = \mathbf{X}_0 + \frac{|\mathbf{V}_0|}{\alpha} \mathbf{V}_0$$

The distance traveled during the time $t \in [0, t_{\max}]$ is

$$|\mathbf{X}(t_{\max}) - \mathbf{X}_0| = \frac{|\mathbf{V}_0|^2}{2\alpha}$$

Viscous friction, no slope. Suppose instead we assumed a viscous frictional force $\mathbf{F} = -c\mathbf{V}$ for some coefficient of friction $c > 0$. The equation of motion is

$$\frac{d\mathbf{V}}{dt} + \alpha \mathbf{V} = 0$$

which has the exact solution

$$\mathbf{V}(t) = \mathbf{V}_0 \exp(-\alpha t)$$

As a theoretical model, the equation has a problem in that the particle stops when the velocity becomes zero, but this does not happen until time t reaches infinity. In practice, we would consider the stopping time to be $t_{\max}$ at which the velocity vector is *nearly* zero. How close to zero is up to the application writer. For example, we might set the velocity to zero when $|\mathbf{V}(t)| \le \varepsilon$ for some user-selected absolute error $\varepsilon > 0$. In our current example, the time at which the absolute error is reached is determined by

$$\varepsilon = |\mathbf{V}(t_{\max})| = \exp(-\alpha t_{\max}) |\mathbf{V}_0|$$

or

$$t_{\max} = \frac{1}{\alpha} \ln\left(\frac{|\mathbf{V}_0|}{\varepsilon}\right) = \frac{m}{c} \ln\left(\frac{|\mathbf{V}_0|}{\varepsilon}\right)$$

where $\ln(z)$ is the natural logarithm function. Once again this agrees with your physical intuition. The maximum time is proportional to mass, directly proportional to the initial speed (actually, directly proportional to the logarithm of the initial speed), and inversely proportional to the coefficient of friction.

The position of the particle is obtained by one more integration,

$$\mathbf{X}(t) = \mathbf{X}_0 + \left(\frac{1-\exp(-\alpha t)}{\alpha}\right)\mathbf{V}_0$$

The final position when the particle stops is

$$\mathbf{X}(t_{\max}) = \mathbf{X}_0 + \left(\frac{1-\exp(-\alpha t_{\max})}{\alpha}\right)\mathbf{V}_0 = \mathbf{X}_0 + \left(1 - \frac{\varepsilon}{|\mathbf{V}_0|}\right)\mathbf{V}_0$$

The distance traveled during the time $t \in [0, t_{\max}]$ is

$$|\mathbf{X}(t_{\max}) - \mathbf{X}_0| = \frac{|\mathbf{V}_0| - \varepsilon}{\alpha}$$

If we allow $\varepsilon = 0$ in which case $t_{\max} = \infty$, the velocity does not become zero during finite time, but the distance traveled is finite, namely $|\mathbf{V}_0|/\alpha$.

Comparing this to the distance traveled when rough friction is used, for a large enough initial speed, the particle travels farther than when viscous friction is used. Convince yourself that this does indeed agree with your physical intuition.

Rough friction, velocity has no x component. Suppose we do have an inclined plane ($\phi > 0$). If the initial velocity satisfies $\dot{x}(0) = 0$, then $\dot{x}(t) \equiv 0$ (the three bars means equal for all t). Clearly this function is a solution to the differential equation involving $\ddot{x}$. The differential equation involving $\ddot{w}$ becomes

$$\ddot{w} + \alpha\frac{\dot{w}}{|\dot{w}|} + g\sin\phi = 0$$

Suppose that $\dot{w}(0) > 0$. For at least a small interval of time, the derivative $\dot{w}(t)$ must also be positive and $\dot{w}/|\dot{w}| = 1$ on that interval. Thus,

$$\ddot{w} + \alpha + g\sin\phi = 0$$

which has the solution

$$\dot{w}(t) = \dot{w}_0 - (\alpha + g\sin\phi)t$$

The w-component of velocity becomes zero at time

$$t_0 = \frac{\dot{w}_0}{\alpha + g\sin\phi}$$

The particle moves in the upward direction, stops at time t_0, then moves in the downward direction. When the change in direction happens, it must be that $\dot{w}(t) < 0$ and $\dot{w}/|\dot{w}| = -1$. The differential equation becomes

$$\ddot{w} - \alpha + g\sin\phi = 0$$

which has the solution

$$\dot{w}(t) = (\alpha - g\sin\phi)(t - t_0)$$

for $t \geq t_0$. Now for the brain teaser. Your physical intuition is that once the particle stops moving upward, it must start moving downward. If you look at the last equation for the w-component of velocity, to move downward it is required that $c/m - g\sin\phi = \alpha - g\sin\phi < 0$. But what if c is large enough so that $c/m - g\sin\phi > 0$? Recall that $c = \mu mg\cos\phi$, so that c does depend on the angle of inclination, the mass, and the gravitational constant. To cause the particle to move downward, we need

$$c/m - g\sin\phi = g(\mu\cos\phi - \sin\phi) < 0$$

which implies the constraint

$$\mu < \tan\phi$$

Viscous friction, velocity has no x component. The w-equation of motion is

$$\ddot{w} + \alpha\dot{w} = -g\sin\phi$$

and has the solution

$$\dot{w}(t) = \frac{1}{\alpha}\left((\alpha\dot{w}_0 + g\sin\phi)\exp(-\alpha t) - g\sin\phi)\right)$$

The particle stops when $\dot{w}(t_0) = 0$. The stopping time is

$$t_0 = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha\dot{w}_0}{g\sin\phi}\right)$$

Once the particle changes direction and travels downward, notice that in infinite time that

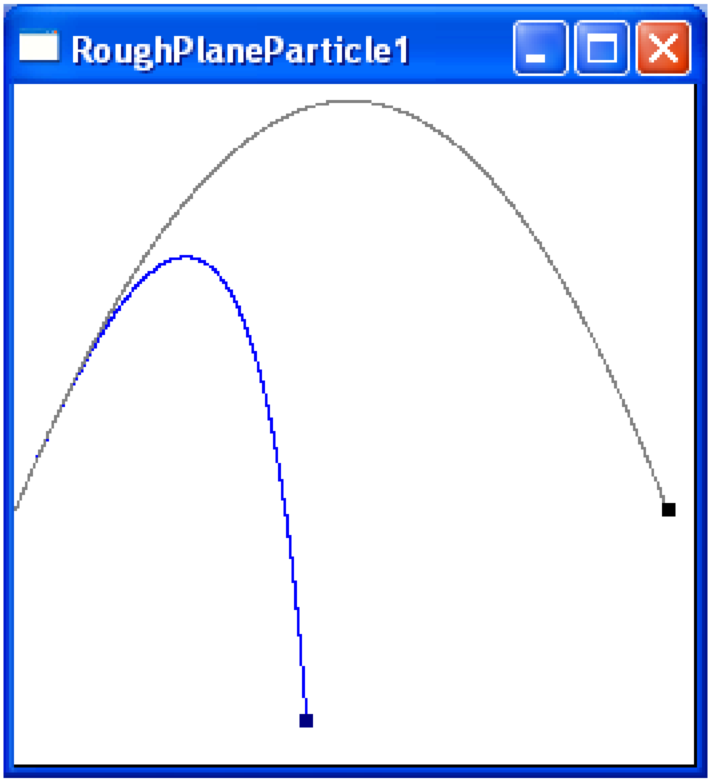
$$\dot{w}(\infty) = \lim_{t\to\infty}\dot{w}(t) = -\frac{g\sin\phi}{\alpha}$$

Thus, viscous friction causes the particle to reach a *steady-state* velocity that is finite. In contrast, rough friction velocity leads to

$$\dot{w}(\infty) = \lim_{t\to\infty}(\alpha - g\sin\phi)(t - t_0) = -\infty$$

The particle on a rough surface continues to greatly increase its speed over all time.

General case. To the best of my knowledge and skills, I do not see that there is a closed-form solution for the general equations of motion when the initial x-component of velocity is not zero. This is irrelevant, though, for the practical application since a numerical differential equation solver is employed. In the `RoughPlaneParticle1` application, notice that the solutions for both rough friction (gray curve) and viscous friction (blue curve) are plotted.



The x-axis is to the right; the w-axis is up. The simulation starts with $x(0) = w(0) = 0$, $\dot{x}_0 > 0$, and $\dot{w}_0 > 0$. It terminates when $w(t) = 0$ for the rough friction case. Notice that the viscous friction solution peaks at a smaller w than does the rough friction solution.

2 Two Particles on a Rough Plane

The general set of equations are as complicated for that of one particle on a rough plane.

$$\mu_0\ddot{x} = F_x(\dot{x}, \dot{y}, \theta, \dot{\theta}) = -\frac{c_1(\dot{x}-L_1\dot{\theta}\sin\theta)}{\sqrt{(\dot{x}-L_1\dot{\theta}\sin\theta)^2+(\dot{y}+L_1\dot{\theta}\cos\theta)^2}} - \frac{c_2(\dot{x}+L_2\dot{\theta}\sin\theta)}{\sqrt{(\dot{x}+L_2\dot{\theta}\sin\theta)^2+(\dot{y}-L_2\dot{\theta}\cos\theta)^2}}$$

$$\mu_0\ddot{y} = F_y(\dot{x}, \dot{y}, \theta, \dot{\theta}) = -\frac{c_1(\dot{y}+L_1\dot{\theta}\cos\theta)}{\sqrt{(\dot{x}-L_1\dot{\theta}\sin\theta)^2+(\dot{y}+L_1\dot{\theta}\cos\theta)^2}} - \frac{c_2(\dot{y}-L_2\dot{\theta}\cos\theta)}{\sqrt{(\dot{x}+L_2\dot{\theta}\sin\theta)^2+(\dot{y}-L_2\dot{\theta}\cos\theta)^2}}$$

$$\mu_2\ddot{\theta} = F_\theta(\dot{x}, \dot{y}, \theta, \dot{\theta}) = -\frac{c_1L_1(-\dot{x}\sin\theta+\dot{y}\cos\theta+L_1\dot{\theta})}{\sqrt{(\dot{x}-L_1\dot{\theta}\sin\theta)^2+(\dot{y}+L_1\dot{\theta}\cos\theta)^2}} - \frac{c_2L_2(\dot{x}\sin\theta-\dot{y}\cos\theta+L_2\dot{\theta})}{\sqrt{(\dot{x}+L_2\dot{\theta}\sin\theta)^2+(\dot{y}-L_2\dot{\theta}\cos\theta)^2}}$$

The application `RoughPlaneParticle2` has an implementation on the CD-ROM that shows a rod with angular speed moving across a plane.

Let us assume that the masses are made of the same material. In this case, the coefficients of friction are $c_i = \mu g m_i$ where m_i are the masses, g is the gravitational constant, and μ is a constant that depends on the material itself. By the construction shown in the book, we know that the first moments about the center of mass are zero,

$$\begin{aligned}(0,0) &= m_1(x_1 - x, y_1 - y) + m_2(x_2 - x, y_2 - y) \\ &= m_1(L_1\cos\theta, L_1\sin\theta) - m_2(L_2\cos\theta, L_2\sin\theta) \\ &= (m_1L_1 - m_2L_2)(\cos\theta, \sin\theta)\end{aligned}$$

where (x, y) is the center of mass and θ is the orientation of the particles relative to the center of mass and the positive x-axis. Consequently,

$$m_1L_1 - m_2L_2 = 0$$

In fact this may be used to construct the distances from the center of mass,

$$L_1 = \frac{m_2}{m_1 + m_2}L, \quad L_2 = \frac{m_1}{m_1 + m_2}L$$

where L is the distance between the points, $L = |(x_2 - x_1, y_2 - y_1)|$.

No Angular Speed. In the event that the initial angular speed of the system is zero, we can reduce the equations to those that were derived for the one-particle system. The generalized force corresponding to θ is formally written as a function of four parameters, $F_\theta(\dot{x}, \dot{y}, \theta, \dot{\theta})$. The associated differential equation is $\mu_2\ddot{\theta} = F_\theta(\dot{x}, \dot{y}, \theta, \dot{\theta})$. Notice that

$$\begin{aligned}F_\theta(\dot{x}, \dot{y}, \theta, 0) &= -\tfrac{c_1L_1(-\dot{x}\sin\theta + \dot{y}\cos\theta)}{\sqrt{\dot{x}^2+\dot{y}^2}} - \tfrac{c_2L_2(\dot{x}\sin\theta - \dot{y}\cos\theta)}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ &= \tfrac{\mu g(m_1L_1 - m_2L_2)(\dot{x}\sin\theta - \dot{y}\cos\theta)}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ &= 0\end{aligned}$$

The last equality is true since we know $m_1L_1 - m_2L_2 = 0$. This means that if $\dot{\theta}(0) = 0$, then $\dot{\theta}(t) = 0$ for all time t. Consequently, the other two generalized forces are

$$F_x = -\frac{c_1\dot{x} - c_2\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -\frac{\mu g\mu_0\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad F_y = -\frac{c_1\dot{y} - c_2\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -\frac{\mu g\mu_0\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

The equations of motion for the center of mass are then

$$\ddot{x} = -\frac{\mu g\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \quad \ddot{y} = -\frac{\mu g\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

which is what we derived for a single particle traveling on a rough plane with no incline.

No Linear Velocity. Suppose that at an instant of time, the center of mass stops, so $(\dot{x}, \dot{y}) = (0, 0)$ at that instant. The generalized forces at that instant of time may be shown to be

$$\begin{aligned}F_x(0, 0, \theta, \dot{\theta}) &= \tfrac{\mu g(m_1 - m_2)\dot{\theta}\sin\theta}{|\dot{\theta}|} \\ F_y(0, 0, \theta, \dot{\theta}) &= \tfrac{-\mu g(m_1 - m_2)\dot{\theta}\cos\theta}{|\dot{\theta}|} \\ F_\theta(0, 0, \theta, \dot{\theta}) &= \tfrac{-\mu g(m_1L_1 + m_2L_2)\dot{\theta}}{|\dot{\theta}|}\end{aligned}$$

If the masses are different ($m_1 \neq m_2$), then the generalized force on the center of mass is not zero, so the center of mass must start to move again. Moreover, that force is

$$(F_x, F_y) = \frac{\mu g(m_1 - m_2)\dot{\theta}}{|\dot{\theta}|}(\sin\theta, -\cos\theta)$$

which is in the direction perpendicular to the line segment connecting the mass positions. On the other hand, if the masses are the same ($m_1 = m_2$, in which case $L_1 = L_2$), then generalized force on the center of mass is zero and the center remains stationary from that time on. If the rod is spinning about its center of mass ($\dot{\theta} \neq 0$), then the equation of motion for the angular speed is

$$\mu_2 \ddot{\theta} = -2\mu g m_1 L_1 \frac{\dot{\theta}}{|\dot{\theta}|}$$

which is of the same form as the differential equation for the x-component of the one-particle system. Based on our analysis of that system, we know that the two-particle system must stop spinning in a finite amount of time.

Viscous Friction. The use of viscous friction instead of rough friction leads to equations that can be solved in closed form. The frictional forces applied to the particles are $\mathbf{F}_i = -c_i \mathbf{V}_i = -c_i(\dot{x}_i, \dot{y}_i)$ for $i = 1, 2$. The generalized forces are reduced using these relationships. First,

$$\begin{aligned}
F_x &= \textstyle\sum_{i=1}^{2} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial x} \\
&= \textstyle\sum_{i=1}^{2} -c_i \mathbf{V}_i \cdot (1, 0) \\
&= -c_1(\dot{x} - L_1 \dot{\theta} \sin\theta) - c_2(\dot{x} + L_2 \dot{\theta} \sin\theta) \\
&= -\mu g m_1(\dot{x} - L_1 \dot{\theta} \sin\theta) - \mu g m_2(\dot{x} + L_2 \dot{\theta} \sin\theta) \\
&= -\mu g \mu_0 \dot{x} + \mu g(m_1 L_1 - m_2 L_2)\dot{\theta} \sin\theta \\
&= -\mu g \mu_0 \dot{x}
\end{aligned}$$

Similarly it is shown that

$$F_y = -\mu g \mu_0 \dot{y}$$

Finally,

$$\begin{aligned}
F_\theta &= -c_1 L_1(-\dot{x}\sin\theta + \dot{y}\cos\theta + L_1\dot{\theta}) - c_2 L_2(\dot{x}\sin\theta - \dot{y}\cos\theta + L_2\dot{\theta}) \\
&= \mu g(m_1 L_1 - m_2 L_2)\dot{x}\sin\theta - \mu g(m_1 L_1 - m_2 L_2)\dot{y}\cos\theta - \mu g(m_1 L_1^2 + m_2 L_2^2)\dot{\theta} \\
&= -\mu g \mu_2 \dot{\theta}
\end{aligned}$$

The equations of motion, $\mu_0 \ddot{x} = F_x$, $\mu_0 \ddot{y} = F_y$, and $\mu_2 \ddot{\theta} = F_\theta$ reduce to

$$\ddot{x} = -\mu g \dot{x}, \quad \ddot{y} = -\mu g \dot{y}, \quad \ddot{\theta} = -\mu g \dot{\theta}$$

These are uncoupled, linear, second-order equations that have closed form solutions. Let the initial data be $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$, $\theta(0) = \theta_0$, and $\dot{\theta}(0) = \dot{\theta}_0$. The solutions are

$$\begin{aligned}
x(t) &= x_0 + \tfrac{\dot{x}_0}{\mu g}\,(1 - \exp(-\mu g t)) \\
y(t) &= y_0 + \tfrac{\dot{y}_0}{\mu g}\,(1 - \exp(-\mu g t)) \\
\theta(t) &= \theta_0 + \tfrac{\dot{\theta}_0}{\mu g}\,(1 - \exp(-\mu g t))
\end{aligned}$$

The same problem occurs as with one particle. The two-particle system stops moving, but in infinite time. An implementation will decide that the system has stopped when the linear velocity and angular speed are *nearly* zero. In infinite time, the final position and orientation are $x(\infty) = x_0 + \dot{x}_0/(\mu g)$, $y(\infty) = y_0 + \dot{y}_0/(\mu g)$, and $\theta(\infty) = \theta_0 + \dot{\theta}_0/(\mu g)$.

If you are willing to model your frictional forces as viscous friction instead of rough friction, there is no need to use a differential equation solver. The previous closed-form expressions may be used instead.

3 Multiple Particles on a Rough Plane

Once again, the general set of equations are as complicated for that of one or two particles on a rough plane. I leave it as an exercise for the reader to implement `RoughPlaneParticleN`. The source code for the application `RoughPlaneParticle2` may be used as a starting point.

No Angular Speed. The generalized force for θ is written as $F_\theta(\dot{x}, \dot{y}, \theta, \dot{\theta})$, just as in the case for two particles. Notice that

$$\begin{aligned} F_\theta(\dot{x}, \dot{y}, \theta, 0) &= -\sum_{i=1}^{p} \frac{c_i L_i(-\dot{x}\sin(\theta+\phi_i)+\dot{y}\cos(\theta+\phi_i))}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ &= \frac{\mu g}{\sqrt{\dot{x}^2+\dot{y}^2}} \left(\dot{x}\sum_{i=1}^{p} m_i L_i \sin(\theta+\phi_i) - \dot{y}\sum_{i=1}^{p} m_i L_i \cos(\theta+\phi_i)\right) \\ &= 0 \end{aligned}$$

where the last equality is true based on our observation in the book that the first moments about the center of mass are zero. If $\dot{\theta}(0) = 0$, then $\mu_2 \ddot{\theta} = F_\theta$ and $F_\theta(\dot{x}, \dot{y}, \theta, 0) = 0$ imply that $\dot{\theta}(t) = 0$ for all time. Just as in the two-particle case, the equations of motion for the center of mass reduce to

$$\ddot{x} = -\frac{\mu g \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \quad \ddot{y} = -\frac{\mu g \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

No Linear Velocity. Suppose that at an instant of time, the center of mass stops, so $(\dot{x}, \dot{y}) = (0, 0)$ at that instant. The generalized forces at that instant of time may be shown to be

$$\begin{aligned} F_x(0, 0, \theta, \dot{\theta}) &= \mu g \frac{\dot{\theta}}{|\dot{\theta}|} \sum_{i=1}^{p} m_i \sin(\theta+\phi_i) \\ F_y(0, 0, \theta, \dot{\theta}) &= -\mu g \frac{\dot{\theta}}{|\dot{\theta}|} \sum_{i=1}^{p} m_i \cos(\theta+\phi_i) \\ F_\theta(0, 0, \theta, \dot{\theta}) &= -\mu g \frac{\dot{\theta}}{|\dot{\theta}|} \sum_{i=1}^{p} m_i L_i \end{aligned}$$

In order for the center of mass to remain at rest, the particle system must be balanced in the sense that $\sum_{i=1}^{p} m_i \sin(\theta+\phi_i) = 0$ and $\sum_{i=1}^{p} m_i \cos(\theta+\phi_i) = 0$. Using the double angle identities and applying a few algebraic steps, the θ dependence is removed to obtain equivalent conditions,

$$\sum_{i=1}^{p} m_i \cos\phi_i = 0, \quad \sum_{i=1}^{p} m_i \sin\phi_i = 0.$$

Assuming these balancing conditions are satisfied, the center of mass remains stationary from the current time on. The particle system may spin about its center of mass ($\dot{\theta} \neq 0$) and is governed by the equation of motion for angular speed,

$$\mu_2 \ddot{\theta} = -\left(\mu g \sum_{i=1}^{p} m_i L_i\right) \frac{\dot{\theta}}{|\dot{\theta}|}$$

which is of the same form as the differential equation for the x-component of the one-particle system. Based on our analysis of that system, we know that the particle system must stop spinning in a finite amount of time.

Viscous Friction. The use of viscous friction instead of rough friction leads to the same closed-form equations for the center of mass and the orientation, regardless of the number of particles in the system. The differential equations are

$$\ddot{x} = -\mu g \dot{x}, \;\; \ddot{y} = -\mu g \dot{y}, \;\; \ddot{\theta} = -\mu g \dot{\theta}$$

Let the initial data be $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, $y(0) = y_0$, $\dot{y}(0) = \dot{y}_0$, $\theta(0) = \theta_0$, and $\dot{\theta}(0) = \dot{\theta}_0$. The solutions are

$$\begin{aligned} x(t) &= x_0 + \tfrac{\dot{x}_0}{\mu g}\left(1 - \exp(-\mu g t)\right) \\ y(t) &= y_0 + \tfrac{\dot{y}_0}{\mu g}\left(1 - \exp(-\mu g t)\right) \\ \theta(t) &= \theta_0 + \tfrac{\dot{\theta}_0}{\mu g}\left(1 - \exp(-\mu g t)\right) \end{aligned}$$

4 Thin Rod on a Rough Plane

Assuming the thin rod is a homogeneous material of constant density $\delta(L) \equiv \delta_0$, the coefficient of friction is $c = \mu g \delta_0$. The rough friction case has the equations of motions:

$$\mu_0 \ddot{x} = F_x(\dot{x}, \dot{y}, \theta, \dot{\theta}) = \int_{-L_2}^{L_1} \tfrac{-c(\dot{x} - L\dot{\theta}\sin\theta)}{\sqrt{(\dot{x} - L\dot{\theta}\sin\theta)^2 + (\dot{y} + L\dot{\theta}\cos\theta)^2}}\, dL$$

$$\mu_0 \ddot{y} = F_y(\dot{x}, \dot{y}, \theta, \dot{\theta}) = \int_{-L_2}^{L_1} \tfrac{-c(\dot{y} + L\dot{\theta}\cos\theta)}{\sqrt{(\dot{x} - L\dot{\theta}\sin\theta)^2 + (\dot{y} + L\dot{\theta}\cos\theta)^2}}\, dL$$

$$\mu_2 \ddot{\theta} = F_\theta(\dot{x}, \dot{y}, \theta, \dot{\theta}) = \int_{-L_2}^{L_1} \tfrac{-cL(-\dot{x}\sin\theta + \dot{y}\cos\theta + L\dot{\theta})}{\sqrt{(\dot{x} - L\dot{\theta}\sin\theta)^2 + (\dot{y} + L\dot{\theta}\cos\theta)^2}}\, dL$$

Recall that

$$\mu_0 = \int_{-L_2}^{L_1} \delta_0 \, dL, \;\; \mu_1 = \int_{-L_2}^{L_1} \delta_0 \, L \, dL = 0, \;\; \mu_2 = \int_{-L_2}^{L_1} \delta_0 \, L^2 \, dL$$

No Angular Speed. Once again it is simple to show that $F_\theta(\dot{x}, \dot{y}, \theta, 0) = 0$, namely

$$\begin{aligned} F_\theta(\dot{x}, \dot{y}, \theta, 0) &= \int_{-L_2}^{L_1} \tfrac{-cL(-\dot{x}\sin\theta + \dot{y}\cos\theta)}{\sqrt{\dot{x}^2 + \dot{y}^2}}\, dL \\ &= \tfrac{\mu g}{\sqrt{\dot{x}^2 + \dot{y}^2}}\left(\dot{x}\sin\theta - \dot{y}\cos\theta\right) \int_{-L_2}^{L_1} \delta_0 \, L \, dL \\ &= 0 \end{aligned}$$

The equation $\mu_2 \ddot{\theta} = F_\theta$ and $\dot{\theta}(0) = 0$ imply $\dot{\theta}(t) = 0$ for all t. Once again, the equations of motion for the center of mass reduce to

$$\ddot{x} = -\frac{\mu g \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \;\; \ddot{y} = -\frac{\mu g \dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$$

No Linear Velocity. Suppose that at an instant of time, the center of mass stops, so $(\dot{x}, \dot{y}) = (0, 0)$ at that instant. The generalized forces at that instant of time may be shown to be

$$\begin{aligned} F_x(0,0,\theta,\dot{\theta}) &= \mu g \tfrac{\dot{\theta}\sin\theta}{|\dot{\theta}|} \int_{-L_2}^{L_1} \delta_0 \tfrac{L}{|L|}\, dL \\ F_y(0,0,\theta,\dot{\theta}) &= -\mu g \tfrac{\dot{\theta}\cos\theta}{|\dot{\theta}|} \int_{-L_2}^{L_1} \delta_0 \tfrac{L}{|L|}\, dL \\ F_\theta(0,0,\theta,\dot{\theta}) &= -\mu g \tfrac{\dot{\theta}}{|\dot{\theta}|} \int_{-L_2}^{L_1} \delta_0\, |L|\, dL \end{aligned}$$

In order for the center of mass to remain at rest, the particle system must be balanced in the sense that

$$\int_{-L_2}^{L_1} \delta_0 \frac{L}{|L|}\, dL = 0$$

In particular, this condition is satisfied for constant density material, $\delta(L) = \delta_0$ for all L, in which case $L_2 = L_1$. The integral is then

$$\begin{aligned} \int_{-L_2}^{L_1} \delta_0 \tfrac{L}{|L|}\, dL &= \delta_0 \int_{-L_1}^{L_1} \tfrac{L}{|L|}\, dL \\ &= \delta_0 \left(\int_0^{L_1} 1\, dL + \int_{-L_1}^{0} -1\, dL \right) \\ &= \delta_0 (L_1 + (-L_1)) \\ &= 0 \end{aligned}$$

Assuming this balancing condition is satisfied, the center of mass remains stationary from the current time on. The particle system may spin about its center of mass ($\dot{\theta} \neq 0$) and is governed by the equation of motion for angular speed,

$$\mu_2 \ddot{\theta} = -\left(\mu g \int_{-L_2}^{L_1} \delta_0\, |L|\, dL \right) \frac{\dot{\theta}}{|\dot{\theta}|}$$

which is of the same form as the differential equation for the x-component of the one-particle system. Based on our analysis of that system, we know that the rod must stop spinning in a finite amount of time.

Integration of the Generalized Forces. The equations of motion for the thin rod are referred to as *integro-differential equations*. That is, the differential equations have terms that are integrals. The numerical differential equation solver can handle this new twist by using a numerical integration algorithm to evaluate the generalized forces. The expense of doing so is dependent on the integration scheme. The more accurate you want the approximation to the integral, the more time you will spend in computing it. The sample application `RoughPlaneThinRod1` is an implementation that uses a numerical integrator applied directly to the integrals in the generalized forces.

In our particular problem, we can obtain closed-form expressions for the integrals. First, notice that

$$\begin{aligned} (\dot{x} - L\dot{\theta}\sin\theta)^2 + (\dot{y} + L\dot{\theta}\cos\theta)^2 &= (\dot{x}^2 + \dot{y}^2) + 2(-\dot{x}\sin\theta + \dot{y}\cos\theta)L\dot{\theta} + (L\dot{\theta})^2 \\ &= (\dot{x}\cos\theta + \dot{y}\sin\theta)^2 + [(-\dot{x}\sin\theta + \dot{y}\cos\theta) + L\dot{\theta}]^2 \\ &= \lambda_1^2 + (\lambda_2 + L\dot{\theta})^2 \end{aligned}$$

where $\lambda_1 = \dot{x}\cos\theta + \dot{y}\sin\theta = (\dot{x},\dot{y})\cdot(\cos\theta,\sin\theta)$ and $\lambda_2 = -\dot{x}\sin\theta + \dot{y}\cos\theta = (\dot{x},\dot{y})\cdot(-\sin\theta,\cos\theta)$. L is the variable of integration in the generalized force expressions. Moreover, the integrals factor to three basic ones:

$$\begin{aligned}
I_0 &= \int_{-L_2}^{L_1} \frac{1}{\sqrt{\lambda_1^2+(\lambda_2+L\dot{\theta})^2}}\, dL \\
I_1 &= \int_{-L_2}^{L_1} \frac{L}{\sqrt{\lambda_1^2+(\lambda_2+L\dot{\theta})^2}}\, dL \\
I_2 &= \int_{-L_2}^{L_1} \frac{L^2}{\sqrt{\lambda_1^2+(\lambda_2+L\dot{\theta})^2}}\, dL
\end{aligned}$$

A further reduction uses the change of variables

$$\begin{aligned}
\lambda_1 z &= \lambda_2 + L\dot{\theta} \\
\lambda_1 dz &= \dot{\theta}\, dL \\
\lambda_1 z_0 &= \lambda_2 - L_2\dot{\theta} \\
\lambda_1 z_1 &= \lambda_2 + L_1\dot{\theta}
\end{aligned}$$

and more factoring to obtain three basic integrals

$$\begin{aligned}
J_0 &= \int_{z_0}^{z_1} \frac{1}{\sqrt{1+z^2}}\, dz \\
&= \left[\ln(z+\sqrt{1+z^2})\right]_{z_0}^{z_1} \\
&= \ln(z_1+\sqrt{1+z_1^2}) - \ln(z_0+\sqrt{1+z_0^2})
\end{aligned}$$

$$\begin{aligned}
J_1 &= \int_{z_0}^{z_1} \frac{z}{\sqrt{1+z^2}}\, dz \\
&= \left[\sqrt{1+z^2}\right]_{z_0}^{z_1} \\
&= \sqrt{1+z_1^2} - \sqrt{1+z_0^2}
\end{aligned}$$

$$\begin{aligned}
J_2 &= \int_{z_0}^{z_1} \frac{z^2}{\sqrt{1+z^2}}\, dz \\
&= \left[(z/2)\sqrt{1+z^2} - (1/2)\ln(z+\sqrt{1+z^2})\right]_{z_0}^{z_1} \\
&= (z_1/2)\sqrt{1+z_1^2} - (1/2)\ln(z_1+\sqrt{1+z_1^2}) - (z_0/2)\sqrt{1+z_0^2} + (1/2)\ln(z_0+\sqrt{1+z_0^2})
\end{aligned}$$

The exact relationship between the integrals is obtained by working through the details of the change of variables:

$$\begin{aligned}
I_0 &= [\lambda_1/(|\lambda_1|\dot{\theta})]J_0 \\
I_1 &= [\lambda_1/(|\lambda_1|\dot{\theta}^2)][\lambda_1 J_1 - \lambda_2 J_0] \\
I_2 &= [\lambda_1/(|\lambda_1|\dot{\theta}^3)][\lambda_1^2 J_2 - 2\lambda_1\lambda_2 J_1 + \lambda_2^2 J_0]
\end{aligned}$$

and

$$
\begin{aligned}
F_x &= -c[\dot{x}I_0 - (\dot{\theta}\sin\theta)I_1] \\
F_y &= -c[\dot{y}I_0 + (\dot{\theta}\sin\theta)I_1] \\
F_\theta &= -c[(-\dot{x}\sin\theta + \dot{y}\cos\theta)I_1 + \dot{\theta}I_2]
\end{aligned}
$$

The numerical differential equation solver evaluates F_x, F_y, and F_θ by first computing J_0, J_1, and J_2, then computing I_0, I_1, and I_2, followed by the evaluation of the last set of displayed equations. Care must be given to detecting and handling the degenerate cases that occur, namely when $\lambda_1 = 0$ or $\lambda_2 = 0$ or $\dot{\theta} = 0$.

Well, as it turns out, the details are quite tedious and the resulting numerical method is expensive to compute. I suggest searching for a physics hack.

One Physics Hack Coming Up! Using our analyses for no angular speed and no linear velocity, I suggest that the model for rough friction use the decoupled system

$$\ddot{x} = -\frac{\mu g \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$$\ddot{y} = -\frac{\mu g \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$$\ddot{\theta} = \left(\frac{\int_{-L_2}^{L_1} -\mu g \delta_0 \,|L|\, dL}{\int_{-L_2}^{L_1} \delta_0 L^2 \, dL}\right)\frac{\dot{\theta}}{|\dot{\theta}|} = \left(\frac{-\mu g(L_1^2+L_2^2)/2}{(L_1^3+L_2^3)/3}\right)\frac{\dot{\theta}}{|\dot{\theta}|}$$

With this model, no numerical differential equation solver is needed because the solutions may be written in closed form (see the earlier discussion in this section). The center of mass is $\mathbf{p}(t) = (x(t), y(t))$ and its linear velocity is $\mathbf{v}(t) = (\dot{x}(t), \dot{y}(t))$. The velocity is

$$\mathbf{v}(t) = \left(1 - \frac{\mu g t}{|\mathbf{v}_0|}\right)\mathbf{v}_0$$

as long as $\mu g t \le |\mathbf{v}_0|$. For times when this inequality is false, the center of mass is stopped and its linear velocity is zero. The position is

$$\mathbf{p}(t) = \mathbf{p}_0 + t\left(1 - \frac{\mu g t^2/2}{|\mathbf{v}_0|}\right)\mathbf{v}_0$$

as long as $\mu g t \le |\mathbf{v}_0|$. For times when this inequality is false, the center of mass is stopped and its position is that when time satisfies $\mu g t = |\mathbf{v}_0|$.

The angular speed is

$$\dot{\theta}(t) = \left(1 - \frac{Kt}{|\dot{\theta}_0|}\right)\dot{\theta}_0$$

where $K = -(3\mu g/2)(L_1^2 + L_2^2)/(L_1^3 + L_2^3)$. The equation is valid as long as $Kt \le |\dot{\theta}_0|$. For times when this inequality is false, the rod stopped spinning about the center of mass and its angular speed is zero. The orientation angle is

$$\theta(t) = \theta_0 + t\left(1 - \frac{Kt^2/2}{|\dot{\theta}_0|}\right)\dot{\theta}_0$$

as long as $Kt \le |\dot{\theta}_0|$. For times when this inequality is false, the orientation is fixed and the angle is that when time satisfies $Kt = |\dot{\theta}_0|$.

The application `RoughPlaneThinRod2` is an implementation that uses the physics hack.

5 Flat Board on a Rough Plane

Assuming the flat board is a homogeneous material of constant density $\delta(L) \equiv \delta_0$, the coefficient of friction is $c = \mu g \delta_0$. The region R has local coordinates (α, β) that satisfy $|\alpha| \le \alpha_0$ and $|\beta| \le \beta_0$ for some choice of $\alpha_0 > 0$ and $\beta_0 > 0$. The rough friction case has the equations of motions:

$$\mu_0 \ddot{x} = F_x(\dot{x}, \dot{y}, \theta, \dot{\theta}) = \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \frac{-c(\dot{x}-\dot{\theta}(\alpha\sin\theta+\beta\cos\theta))}{\sqrt{(\dot{x}-\dot{\theta}(\alpha\sin\theta+\beta\cos\theta))^2+(\dot{y}+\dot{\theta}(\alpha\cos\theta-\beta\sin\theta))^2}}\,d\alpha\,d\beta$$

$$\mu_0 \ddot{y} = F_y(\dot{x}, \dot{y}, \theta, \dot{\theta}) = \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \frac{-c(\dot{y}+\dot{\theta}(\alpha\cos\theta-\beta\sin\theta))}{\sqrt{(\dot{x}-\dot{\theta}(\alpha\sin\theta+\beta\cos\theta))^2+(\dot{y}+\dot{\theta}(\alpha\cos\theta-\beta\sin\theta))^2}}\,d\alpha\,d\beta$$

$$\mu_2 \ddot{\theta} = F_\theta(\dot{x}, \dot{y}, \theta, \dot{\theta}) = \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \frac{-c(-\dot{x}(\alpha\sin\theta+\beta\cos\theta)+\dot{y}(\alpha\cos\theta-\beta\sin\theta)+\dot{\theta}(\alpha^2+\beta^2))}{\sqrt{(\dot{x}-\dot{\theta}(\alpha\sin\theta+\beta\cos\theta))^2+(\dot{y}+\dot{\theta}(\alpha\cos\theta-\beta\sin\theta))^2}}\,d\alpha\,d\beta$$

Recall that

$$\mu_0 = \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0\,d\alpha\,d\beta$$

$$\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0\alpha\,d\alpha\,d\beta = 0$$

$$\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0\beta\,d\alpha\,d\beta = 0$$

$$\mu_2 = \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0(\alpha^2+\beta^2)\,d\alpha\,d\beta$$

No Angular Speed. Once again it is simple to show that $F_\theta(\dot{x}, \dot{y}, \theta, 0) = 0$, namely

$$\begin{aligned}
F_\theta(\dot{x}, \dot{y}, \theta, 0) &= \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \frac{-c(-\dot{x}(\alpha\sin\theta+\beta\cos\theta)+\dot{y}(\alpha\cos\theta-\beta\sin\theta))}{\sqrt{\dot{x}^2+\dot{y}^2}}\,d\alpha\,d\beta \\
&= \frac{\mu g}{\sqrt{\dot{x}^2+\dot{y}^2}} \left[(\dot{x}\sin\theta - \dot{y}\cos\theta)\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0\alpha\,d\alpha\,d\beta + (\dot{x}\cos\theta + \dot{y}\sin\theta)\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0\beta\,d\alpha\,d\beta\right] \\
&= 0
\end{aligned}$$

The equation $\mu_2\ddot{\theta} = F_\theta$ and $\dot{\theta}(0) = 0$ imply $\dot{\theta}(t) = 0$ for all t. Once again, the equations of motion for the center of mass reduce to

$$\ddot{x} = -\frac{\mu g \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}, \quad \ddot{y} = -\frac{\mu g \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

No Linear Velocity. Suppose that at an instant of time, the center of mass stops, so $(\dot{x}, \dot{y}) = (0, 0)$ at that instant. The generalized forces at that instant of time may be shown to be

$$\begin{aligned} F_x(0,0,\theta,\dot{\theta}) &= \mu g \tfrac{\dot{\theta}}{|\dot{\theta}|}\left[\sin\theta \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \tfrac{\delta_0 \alpha}{\sqrt{\alpha^2+\beta^2}}\, d\alpha\, d\beta + \cos\theta \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \tfrac{\delta_0 \beta}{\sqrt{\alpha^2+\beta^2}}\, d\alpha\, d\beta\right] = 0 \\ F_y(0,0,\theta,\dot{\theta}) &= \mu g \tfrac{\dot{\theta}}{|\dot{\theta}|}\left[\cos\theta \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \tfrac{\delta_0 \alpha}{\sqrt{\alpha^2+\beta^2}}\, d\alpha\, d\beta - \sin\theta \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \tfrac{\delta_0 \beta}{\sqrt{\alpha^2+\beta^2}}\, d\alpha\, d\beta\right] = 0 \\ F_\theta(0,0,\theta,\dot{\theta}) &= -\mu g \tfrac{\dot{\theta}}{|\dot{\theta}|} \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0 \sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta \end{aligned}$$

The fact that F_x and F_y are zero is based on the rectangle being symmetric in α and β. Thus, the center of mass remains stationary from the current time on. The particle system may spin about its center of mass ($\dot{\theta} \neq 0$) and is governed by the equation of motion for angular speed,

$$\mu_2 \ddot{\theta} = -\left(\mu g \int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \delta_0 \sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta\right) \frac{\dot{\theta}}{|\dot{\theta}|}$$

which is of the same form as the differential equation for the x-component of the one-particle system. Based on our analysis of that system, we know that the rod must stop spinning in a finite amount of time.

Integration of the Generalized Forces. It is possible to obtain closed form expressions for the generalized forces, but the details are quite tedious and the resulting numerical method is expensive to compute. I suggest the same physics hack used for the thin rod.

The Physics Hack Using our analyses for no angular speed and no linear velocity, I suggest that the model for rough friction use the decoupled system

$$\begin{aligned} \ddot{x} &= -\tfrac{\mu g \dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ \ddot{y} &= -\tfrac{\mu g \dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} \\ \ddot{\theta} &= -\mu g \left(\tfrac{\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta}{\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} (\alpha^2+\beta^2)\, d\alpha\, d\beta}\right) \tfrac{\dot{\theta}}{|\dot{\theta}|} \end{aligned}$$

Although the integral in the numerator may be computed using a numerical integrator, it can be expressed in closed form as

$$\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} \sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta = \frac{4}{3}\,\alpha_0\beta_0\sqrt{\alpha_0^2+\beta_0^2} + \frac{\alpha_0^3}{2}\,\ln\left(\frac{\sqrt{\alpha_0^2+\beta_0^2}+\beta_0}{\sqrt{\alpha_0^2+\beta_0^2}-\beta_0}\right) + \frac{\beta_0^3}{2}\,\ln\left(\frac{\sqrt{\alpha_0^2+\beta_0^2}+\alpha_0}{\sqrt{\alpha_0^2+\beta_0^2}-\alpha_0}\right)$$

The denominator is

$$\int_{-\beta_0}^{\beta_0}\int_{-\alpha_0}^{\alpha_0} (\alpha^2+\beta^2)\, d\alpha\, d\beta = \frac{4}{3}\,\alpha_0\beta_0\left(\alpha_0^2+\beta_0^2\right)$$

The application `RoughPlaneFlatBoard` uses this hack.

6 Solid Box on a Rough Plane

The center of mass is

$$\mathbf{p} = (x, w\cos\phi - h\sin\phi, w\sin\phi + h\cos\phi)$$

The position of a point in the box represented in the coordinate system with center of mass as origin and coordinate axes dependent on the orientation is

$$\mathbf{r} = \mathbf{p} + \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3$$

where

$$\mathbf{u}_1 = (\cos\theta, -\sin\theta\cos\phi, -\sin\theta\sin\phi), \quad \mathbf{u}_2 = (\sin\theta, \cos\theta\cos\phi, \cos\theta\sin\phi), \quad \mathbf{u}_3 = (0, -\sin\phi, \cos\phi)$$

For the point to be inside the box we need $|\alpha| \le a$, $|\beta| \le b$, and $|\gamma| \le h$. Recall that ϕ is the angle of inclination of the plane and is a constant through time. We will need the derivatives of the first two coordinate vectors with respect to θ,

$$\frac{\partial \mathbf{u}_1}{\partial\theta} = (-\sin\theta, -\cos\theta\cos\phi, -\cos\theta\sin\phi), \quad \frac{\partial \mathbf{u}_2}{\partial\theta} = (\cos\theta, -\sin\theta\cos\phi, -\sin\theta\sin\phi)$$

The partial derivatives of $\mathbf{r}$ that are needed for generalized force calculations are

$$\frac{\partial\mathbf{r}}{\partial x} = (1,0,0), \quad \frac{\partial\mathbf{r}}{\partial w} = (0, \cos\phi, \sin\phi),$$

and

$$\frac{\partial\mathbf{r}}{\partial\theta} = \alpha\frac{\partial\mathbf{u}_1}{\partial\theta} + \beta\frac{\partial\mathbf{u}_2}{\partial\theta} = (-\alpha\sin\theta + \beta\cos\theta, -\cos\phi(\alpha\cos\theta + \beta\sin\theta), -\sin\phi(\alpha\cos\theta + \beta\sin\theta))$$

The velocity of a point in the box is

$$\begin{aligned}
\mathbf{v} &= \dot{\mathbf{r}} = \dot{\mathbf{p}} + \alpha\dot{\mathbf{u}}_1 + \beta\dot{\mathbf{u}}_2 = \dot{\mathbf{p}} + \dot\theta\left(\alpha\tfrac{\partial\mathbf{u}_1}{\partial\theta} + \beta\tfrac{\partial\mathbf{u}_2}{\partial\theta}\right) \\
&= (\dot x + \dot\theta(-\alpha\sin\theta + \beta\cos\theta), \cos\phi(\dot w - \dot\theta(\alpha\cos\theta + \beta\sin\theta)), \sin\phi(\dot w - \dot\theta(\alpha\cos\theta + \beta\sin\theta)))
\end{aligned}$$

Generalized Force for Gravity. Let $\mathcal{V}$ be the volumetric region of the box. The gravitational force on the center of mass of the box is $\mathbf{F}_{\text{grav}} = -mg\mathbf{k} = -mg(0,0,1)$. The generalized forces are

$$\begin{aligned}
F_x &= \textstyle\int_{\mathcal{V}} \mathbf{F}_{\text{grav}} \cdot \frac{\partial\mathbf{r}}{\partial x}\, d\mathcal{V} = \int_{\mathcal{V}} -mg\boldsymbol{k}\cdot(1,0,0)\, d\mathcal{V} = 0 \\
F_w &= \textstyle\int_{\mathcal{V}} \mathbf{F}_{\text{grav}} \cdot \frac{\partial\mathbf{r}}{\partial w}\, d\mathcal{V} = \int_{\mathcal{V}} -mg\boldsymbol{k}\cdot(0,\cos\phi,\sin\phi)\, d\mathcal{V} = -(mg\sin\phi)(8abh) \\
F_\theta &= \textstyle\int_{\mathcal{V}} \mathbf{F}_{\text{grav}} \cdot \frac{\partial\mathbf{r}}{\partial \theta}\, d\mathcal{V} = \int_{\mathcal{V}} -mg\boldsymbol{k}\cdot\left(\alpha\frac{\partial\mathbf{u}_1}{\partial\theta} + \beta\frac{\partial\mathbf{u}_2}{\partial\theta}\right)\, d\mathcal{V} = mg\sin\phi\int_{\mathcal{V}}(\alpha\cos\theta + \beta\sin\theta)\, d\mathcal{V} = 0
\end{aligned}$$

Generalized Force for Rough Friction. Let R be the surface region of the face of the box that lies on the inclined plane. In local coordinates, this is the region $|\alpha| \le a$, $|\beta| \le b$, and $\gamma = -h$. The rough friction force at a point $\mathbf{r}(\alpha, \beta, -h)$ is

$$\mathbf{F}_{\text{fric}} = -c\,\frac{\mathbf{v}(\alpha,\beta,-h)}{|\mathbf{v}(\alpha,\beta,-h)|}$$

The generalized forces are

$$
\begin{aligned}
F_x &= \int_R \mathbf{F}_{\text{fric}} \cdot \frac{\partial \mathbf{r}}{\partial x}\, dR \\
&= \int_R \frac{-c(\dot{x}+\dot{\theta}(-\alpha\sin\theta+\beta\cos\theta))}{\sqrt{(\dot{x}+\dot{\theta}(-\alpha\sin\theta+\beta\cos\theta))^2+(\dot{w}-\dot{\theta}(\alpha\cos\theta+\beta\sin\theta))^2}}\, dR
\end{aligned}
$$

$$
\begin{aligned}
F_w &= \int_R \mathbf{F}_{\text{fric}} \cdot \frac{\partial \mathbf{r}}{\partial w}\, dR \\
&= \int_R \frac{-c(\dot{w}-\dot{\theta}(\alpha\cos\theta+\beta\sin\theta))}{\sqrt{(\dot{x}+\dot{\theta}(-\alpha\sin\theta+\beta\cos\theta))^2+(\dot{w}-\dot{\theta}(\alpha\cos\theta+\beta\sin\theta))^2}}\, dR
\end{aligned}
$$

$$
\begin{aligned}
F_\theta &= \int_R \mathbf{F}_{\text{fric}} \cdot \frac{\partial \mathbf{r}}{\partial \theta}\, dR \\
&= \int_R \frac{-c(\dot{x}(-\alpha\sin\theta+\beta\cos\theta)+\dot{w}(\alpha\cos\theta+\beta\sin\theta)+\dot{\theta}(\alpha^2+\beta^2))}{\sqrt{(\dot{x}+\dot{\theta}(-\alpha\sin\theta+\beta\cos\theta))^2+(\dot{w}-\dot{\theta}(\alpha\cos\theta+\beta\sin\theta))^2}}\, dR
\end{aligned}
$$

As advertised, these generalized forces are exactly those for a flat board on a rough plane.

The analyses for no angular speed or no linear velocity are the same as those for the flat board on a rough plane. Just as for a flat board, my advice is to use the physics hack that decouples the equations to

$$\ddot{x} = -\frac{\mu g \dot{x}}{\sqrt{\dot{x}^2+\dot{w}^2}}$$

$$\ddot{w} = -\frac{\mu g \dot{w}}{\sqrt{\dot{x}^2+\dot{w}^2}} - g\sin\phi$$

$$\ddot{\theta} = -\mu g \left(\frac{\int_{-b}^{b}\int_{-a}^{a} \sqrt{\alpha^2+\beta^2}\, d\alpha\, d\beta}{\int_{-b}^{b}\int_{-a}^{a} (\alpha^2+\beta^2)\, d\alpha\, d\beta} \right) \frac{\dot{\theta}}{|\dot{\theta}|}$$

The extra term in the w-equation is due to having the box on an inclined plane, whereas the flat board was assumed to be an a horizontal plane. The integrals in the θ-equation are computed just as in the flat board example. The application `RoughPlaneSolidBox` uses this hack.