Classifying Quadrics using Exact Arithmetic

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1 Introduction

A quadratic equation in three variables x_0, x_1, and x_2 is

$$a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_{12} + b_0x_0 + b_1x_1 + b_2x_2 + c = 0 \tag{1}$$

In matrix-vector form, the equation is

$$\mathbf{x}^{\mathrm{T}}A\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + c = 0 \tag{2}$$

where

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

and where the superscript T denotes the transpose operator. Observe that the matrix A is symmetric: $A^{\mathrm{T}} = A$.

The problem is to classify the set of solutions to the quadratic equation. In many applications, we are only interested in *quadric surfaces*. These include ellipsoids, hyperboloids, and paraboloids. However, a quadratic equation can define other surfaces including cylinder surfaces and planes. It is even possible to generate lines and a point via a quadratic. And it is possible that there is no solution to the equation!

A methodical approach is presented here for the classification. This is a mathematical construction assuming real numbers, but as we are all aware, implementations within a floating-point arithmetic system can cause problems due to round-off errors. If the coefficients of the quadratic equation are converted to their rational number equivalents, we may use exact arithmetic to classify the solution set of the quadratic equation without errors. The last part of this document provides the details.

2 Classification using Real Arithmetic

Since the matrix A is symmetric, it has an *eigendecomposition*

$$A = RDR^{\mathrm{T}} \tag{3}$$

where $R = [\mathbf{v}_0|\mathbf{v}_1|\mathbf{v}_2]$ is a rotation matrix whose real-valued columns $\mathbf{v}_i$ are linearly independent eigenvectors of A, and where $D = \mathrm{Diag}(d_0, d_1, d_2)$ is a real-valued diagonal matrix of eigenvalues of A. The eigenvector $\mathbf{v}_i$ corresponds to the eigenvalue d_i.

The eigenvalues are roots of a cubic polynomial called the *characteristic polynomial* of A, namely,

$$p(\lambda) = \det(\lambda I - A) = \lambda^3 - c_2\lambda^2 + c_1\lambda - c_0 = 0 \tag{4}$$

where I is the 3×3 identity matrix. The coefficients are

$$\begin{aligned} c_0 &= a_{00}(a_{11}a_{22} - a_{12}^2) - a_{01}(a_{01}a_{22} - a_{12}a_{02}) + a_{02}(a_{01}a_{12} - a_{02}a_{11}) \\ c_1 &= (a_{00}a_{11} - a_{01}^2) + (a_{00}a_{22} - a_{02}^2) + (a_{11}a_{22} - a_{12}^2) \\ c_2 &= a_{00} + a_{11} + a_{22} \end{aligned}$$

The value c_0 is the determinant of A, a sum of a single 3×3 determinant. The value c_2 is the trace of A, a sum of three 1×1 determinants. The value c_1 has no official name, but is a sum of three 2×2 determinants. All these determinants are referred to as the *principal minors* of the matrix A. Closed form equations exist for the roots of $p(\lambda) = 0$, but the resulting values are generally not rational numbers. This makes the classification using rational arithmetic somewhat challenging.

Define $\mathbf{y} = R^{\mathrm{T}}\mathbf{x}$ and $\mathbf{e} = R^{\mathrm{T}}\mathbf{b}$. Equation (2) may be rewritten as

$$\mathbf{y}^{\mathrm{T}} D\mathbf{y} + \mathbf{e}^{\mathrm{T}}\mathbf{y} + c = 0 \tag{5}$$

Let $\mathbf{y}$ have components labeled y_i and let $\mathbf{e}$ have components labeled e_i for $0 \le i \le 2$. Equation (5) becomes

$$d_0 y_0^2 + d_1 y_1^2 + d_2 y_2^2 + e_0 y_0 + e_1 y_1 + e_2 y_2 + c = 0 \tag{6}$$

If any of the d_i are not zero, we can complete the square on the d_i terms:

$$d_i y_i^2 + e_i y_i = d_i \left(y_i + \frac{e_i}{2d_i} \right)^2 - \frac{e_i^2}{4d_i}$$

This is the basis for the classification, but requires us to analyze the signs of the d_i. When a value d_i is zero, we will then have to analyze the sign of the corresponding e_i value. In preparation for the classification, define the following quantities:

$$z_i = \left\{ \begin{array}{ll} y_i, & d_i = 0 \\ y_i + \frac{e_i}{2d_i}, & d_i \neq 0 \end{array} \right\}, \quad r = -c + \sum_{i=0, d_i \neq 0}^{2} \frac{e_i^2}{4d_i}$$

The classifications are presented next and depend on the signs of the d_i, the signs of the e_i, and the sign of r. To simply the discussion, the eigenvalues are assumed to be ordered as $d_0 \le d_1 \le d_2$.

2.1 The Case All $d_i \neq 0$

Equation (6) reduces to

$$d_0 z_0^2 + d_1 z_1^2 + d_2 z_2^2 = r$$

The subcases are

1. $r > 0$
   (a) $0 < d_0 \le d_1 \le d_2$ (ellipsoid)
   (b) $d_0 < 0 < d_1 \le d_2$ (hyperboloid of one sheet)
   (c) $d_0 \le d_1 < 0 < d_2$ (hyperboloid of two sheets)
   (d) $d_0 \le d_1 \le d_2 < 0$ (no solution)
2. $r < 0$
   (a) $0 < d_0 \le d_1 \le d_2$ (no solution)
   (b) $d_0 < 0 < d_1 \le d_2$ (hyperboloid of two sheets)

(c) $d_0 \le d_1 < 0 < d_2$ (hyperboloid of one sheet)
(d) $d_0 \le d_1 \le d_2 < 0$ (ellipsoid)

3. $r = 0$
   (a) $0 < d_0 \le d_1 \le d_2$ (point)
   (b) $d_0 < 0 < d_1 \le d_2$ (elliptic cone)
   (c) $d_0 \le d_1 < 0 < d_2$ (elliptic cone)
   (d) $d_0 \le d_1 \le d_2 < 0$ (point)

2.2 The Case $d_0 = 0 < d_1 \le d_2$

Equation (6) reduces to

$$d_1 z_1^2 + d_2 z_2^2 + e_0 z_0 = r$$

The subcases are

1. $e_0 \neq 0$ (elliptic paraboloid)
2. $e_0 = 0$
   (a) $r > 0$ (elliptic cylinder)
   (b) $r = 0$ (line)
   (c) $r < 0$ (no solution)

2.3 The Case $d_0 < 0 = d_1 < d_2$

Equation (6) reduces to

$$d_0 z_0^2 + d_2 z_2^2 + e_1 z_1 = r$$

The subcases are

1. $e_1 \neq 0$ (hyperbolic paraboloid)
2. $e_1 = 0$
   (a) $r \neq 0$ (hyperbolic cylinder)
   (b) $r = 0$ (two planes)

2.4 The Case $d_0 \le d_1 < d_2 = 0$

Equation (6) reduces to

$$d_0 z_0^2 + d_1 z_1^2 + e_2 z_2 = r$$

The subcases are

1. $e_2 \neq 0$ (elliptic paraboloid)
2. $e_2 = 0$
   (a) $r > 0$ (none)
   (b) $r = 0$ (line)
   (c) $r < 0$ (elliptic cylinder)

2.5 The Case $d_0 = d_1 = 0 < d_2$

Equation (6) reduces to

$$d_2 z_2^2 + e_0 z_0 + e_1 z_1 = r$$

The subcases are

1. $e_0 \neq 0$ or $e_1 \neq 0$ (parabolic cylinder)
2. $e_0 = e_1 = 0$
   (a) $r > 0$ (two planes)
   (b) $r = 0$ (one plane)
   (c) $r < 0$ (no solution)

2.6 The Case $d_0 < 0 = d_1 = d_2$

Equation (6) reduces to

$$d_0 z_0^2 + e_1 z_1 + e_2 z_2 = r$$

The subcases are

1. $e_1 \neq 0$ or $e_2 \neq 0$ (parabolic cylinder)
2. $e_1 = e_2 = 0$
   (a) $r > 0$ (no solution)
   (b) $r = 0$ (one plane)
   (c) $r < 0$ (two planes)

2.7 The Case $d_0 = d_1 = d_2 = 0$

Equation (6) reduces to

$$b_0 x_0 + b_1 x_1 + b_2 x_2 + c = 0$$

The subcases are

1. $b_0 \neq 0$ or $b_1 \neq 0$ or $b_2 \neq 0$ (plane)
2. $b_0 = b_1 = b_2 = 0$ (no solution)

3 Classification using Rational Arithmetic

As mentioned previouslly, construction of the eigenvalues and eigenvectors must occur using real numbers (in theory, floating-point numbers in practice). However, the classifications were based on analysis of the signs of the eigenvalues and the signs of the components of the rotated vector $\mathbf{e}$. Assuming the coefficients of the quadratic equation are treated as rational numbers, we can actually avoid the real numbers in the classification.

3.1 Determining the Signs of the d_i

The characteristic polynomial is $p(\lambda) = \lambda^3 - c_2\lambda^2 + c_1\lambda - c_0$. The coefficients are rational numbers since the entries of matrix A are rational numbers. If $c_0 \neq 0$, all the eigenvalues are not zero. We may use Sturm sequences to count the number of positive roots. The first three Sturm polynomials are

$$\begin{aligned} p_0(\lambda) &= \lambda^3 - c_2\lambda^2 + c_1\lambda - c_0 \\ p_1(\lambda) &= 3\lambda^2 - 2c_2\lambda + c_1 \\ p_2(\lambda) &= \left(\tfrac{2}{9}c_2^2 - \tfrac{2}{3}c_1\right)\lambda + \left(c_0 - \tfrac{1}{9}c_1c_2\right) = c_3\lambda + c_4 \end{aligned}$$

where the last equality defines c_3 and c_4. If $c_3 = 0$, then p_0, p_1, and p_2 are the full Sturm sequence. If $c_3 = 0$, an additional polynomial is in the sequence, a constant one,

$$p_3(\lambda) = -\frac{c_3^2c_1 + (2c_2c_3 + 3c_4)c_4}{c_3^2} = c_5$$

where the last equality defines c_5. When $c_3 \neq 0$, the signs that determine the positive and negative root counts are

λ	sign p_0	sign p_1	sign p_2	sign p_3	sign changes
$-\infty$	$-$	$+$	sign $-c_3$	sign c_5	$n_{-\infty}$
0	sign $-c_0$	sign c_1	sign c_4	sign c_5	n_0
∞	$+$	$+$	sign c_3	sign c_5	n_∞

The number of positive roots is $n_0 - n_\infty$ and the number of negative roots is $n_{-\infty} - n_0$.

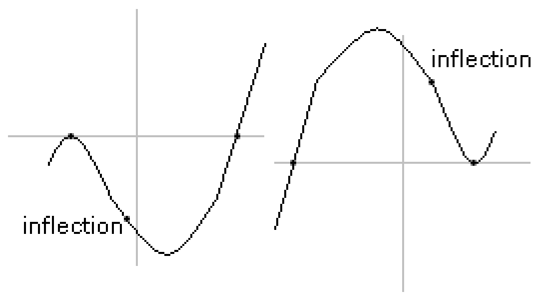
When $c_3 = 0$, the signs that determine the positive and negative root counts are

λ	sign p_0	sign p_1	sign p_2	sign changes
$-\infty$	$-$	$+$	sign c_4	$n_{-\infty}$
0	sign $-c_0$	sign c_1	sign c_4	n_0
∞	$+$	$+$	sign c_4	n_∞

The number of positive roots is $n_0 - n_\infty$ and the number of negative roots is $n_{-\infty} - n_0$.

The Sturm sequence approach only lets you know the number of *distinct roots*, $n_{\mathrm{distinct}} = n_{-\infty} - n_{\infty}$. If this number is 3, then the root signs are all known. However, if it is 1 or 2, then more work must be done. If $n_{\mathrm{distinct}} = 1$, then the distinct root has multiplicity 3, and the positive or negative root count is increased from 1 to 3 accordingly. If $n_{\mathrm{distinct}} = 2$, three cases occur. First, if the negative root count is 0, then the positive root count was 2 and can be increased to 3. One positive root has multiplicity 2. Second, if the positive root count is 0, then the negative root count was 2 and can be increased to 3. One negative root has multiplicity 2. Third, and most complicated, is when the positive and negative root counts are both 1. Figure 3.1 illustrates the possibilities for the graph of the polynomial.

Figure 3.1 The positive root has multiplicity 2 when the inflection point has a positive polynomial value. The negative root has multiplicity 2 when the inflection point has a negative polynomial value.



A simple way to distinguish which of these cases applies is to notice that in one case the inflection point of the graph has a positive function value and in the other case the inflection point has a negative function value. The inflection point occurs when $p''(\lambda) = 0$, at $\lambda' = c_2/3$, and the polynomial value to test is $p(\lambda')$.

If $c_0 = 0$ and $c_1 \neq 0$, the characteristic polynomial is $p(\lambda) = \lambda(\lambda^2 - c_2\lambda + c_1)$. Zero is a root of multiplicity one. The other roots are not zero. A Sturm sequence may be used to count the number of positive roots. The polynomials are

$$
\begin{aligned}
q_0(\lambda) &= \lambda^2 - c_2\lambda + c_1 \\
q_1(\lambda) &= 2\lambda - c_2 \\
q_2(\lambda) &= \tfrac{1}{4}c_2^2 - c_1 = c_3
\end{aligned}
$$

where the last equality defines c_3. The signs that determine the positive and negative root counts are

λ	sign q_0	sign q_1	sign q_2	sign changes
$-\infty$	$+$	$-$	sign c_3	$n_{-\infty}$
0	sign c_1	sign $-c_2$	sign c_3	n_0
∞	$+$	$+$	sign c_3	n_{∞}

The number of positive roots is $n_0 - n_{\infty}$ and the number of negative roots is $n_{-\infty} - n_0$. The total number of distinct roots is $n_{\mathrm{distinct}} = n_{-\infty} - n_{\infty}$. If this number is 1, the only possibility is that a positive root occurs with multiplicity 2.

If $c_0 = c_1 = 0$ and $c_2 \neq 0$, the characteristic polynomial is $p(\lambda) = \lambda^2(\lambda - c_2)$. The third root is $\lambda = c_2 \neq 0$.

If $c_0 = c_1 = c_2 = 0$, then $p(\lambda) = \lambda^3$ and the only root is zero. In this case, the matrix A is the zero matrix and the classification is trivial.

3.2 The Case All $d_i \neq 0$

The Sturm sequence approach allows us to compute the signs of the d_i, but we also need to know the sign of $r = -c + \sum_{i=0}^{2} e_i^2/(4d_i)$. The e_i values come from $\mathbf{e} = R^{\mathrm{T}}\mathbf{b}$. But to compute R requires solving the eigensystem for A, something that requires real number arithmetic. As it turns out, a slightly different formulation of the problem allows us to circumvent this issue.

Since the d_i are all not zero, the matrix A is invertible. Define $\mathbf{u} = -A^{-1}\mathbf{b}/2$. The quadratic equation factors into

$$(\mathbf{x} - \mathbf{u})^{\mathrm{T}} A (\mathbf{x} - \mathbf{u}) = \mathbf{u}^{\mathrm{T}} A \mathbf{u} - c$$

Using the eigendecomposition for $A = RDR^{\mathrm{T}}$ and defining $\mathbf{y} = R^{\mathrm{T}}(\mathbf{x} - \mathbf{u})$, the equation is further modified to

$$d_0 y_0^2 + d_1 y_1^2 + d_2 y_2^2 = \mathbf{y}^{\mathrm{T}} D \mathbf{y} = \frac{1}{4}\mathbf{b}^{\mathrm{T}} A^{-1} \mathbf{b} - c = r$$

The last equality defines a new r, which is a quantity whose sign we can compute from the original quadratic equation coefficients. The analysis of the signs of d_i and r is exactly as described in Section 2.1.

3.3 The Case Exactly One $d_i = 0$

In the case when $d_0 = 0 < d_1 \leq d_2$, the quadratic equation reduces to

$$d_1 z_1^2 + d_2 z_2^2 + e_0 z_0 = -c + e_1^2/(4d_1) + e_2^2/(4d_2) = r$$

The categorization of the solution set required us to analyze the sign of e_0 and r, as well as those of d_1 and d_2. If $e_0 \neq 0$, the solution is an elliptic paraboloid. Notice that

$$\begin{bmatrix} e_0 \\ e_1 \\ e_2 \end{bmatrix} = \mathbf{e} = R^{\mathrm{T}}\mathbf{b} = \begin{bmatrix} \mathbf{v}_0 \cdot \mathbf{b} \\ \mathbf{v}_1 \cdot \mathbf{b} \\ \mathbf{v}_2 \cdot \mathbf{b} \end{bmatrix}$$

where $\mathbf{v}_i$ are unit-length eigenvectors for A. At first glance it appears that we need to actually compute the e_i for the sign test on e_0 and the construction of r. In fact, we do not have to do this.

Since A has an eigenvalue zero of multiplicity 1, the rank of A is 2. We need only determine the two rows of A that are linearly independent, say, $\mathbf{w}_0$ and $\mathbf{w}_1$. An eigenvector (not necessarily unit length) is the cross product, $\mathbf{p}_0 = \mathbf{w}_0 \times \mathbf{w}_1$. Since the intent is to use only rational arithmetic, the normalization is not done. We may choose any two vectors $\mathbf{p}_1$ and $\mathbf{p}_2$ such that the $\mathbf{p}_i$ are mutually orthogonal and have rational components. We may as well choose $\mathbf{p}_1 = \mathbf{w}_0$ and $\mathbf{p}_2 = \mathbf{p}_0 \times \mathbf{p}_1$.

The value of $e_0 = \mathbf{v}_0 \cdot \mathbf{b}$ is zero exactly when the rational value $e'_0 = \mathbf{p}_0 \cdot \mathbf{b}$ is zero. The latter quantity is rational and is used to determine whether or not e_0 is zero. If it is not zero, then we know the solution set is an elliptic paraboloid.

If $e'_0 = 0$, then define the matrix $P = [\mathbf{p}_0 \mid \mathbf{p}_1 \mid \mathbf{p}_2]$ in which case

$$P^{\mathrm{T}} A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{p}_1^{\mathrm{T}} A \mathbf{p}_1 & \mathbf{p}_1^{\mathrm{T}} A \mathbf{p}_2 \\ 0 & \mathbf{p}_2^{\mathrm{T}} A \mathbf{p}_1 & \mathbf{p}_2^{\mathrm{T}} A \mathbf{p}_2 \end{bmatrix}, \quad P^{\mathrm{T}} \mathbf{b} = \begin{bmatrix} 0 \\ \mathbf{p}_1 \cdot \mathbf{b} \\ \mathbf{p}_2 \cdot \mathbf{b} \end{bmatrix}$$

If we define the 2×2 symmetric matrix $F = [f_{ij}]$, where $f_{ij} = \mathbf{p}_i^{\mathrm{T}} A \mathbf{p}_j$ for $i, j \in \{1, 2\}$, and the 2×1 vector $\mathbf{g} = [g_i]$, where $g_i = \mathbf{p}_i \cdot \mathbf{b}$ for $i \in \{1, 2\}$, and $\mathbf{y} = [y_i]$, where $y_i = \mathbf{p}_i \cdot \mathbf{x}$ for $i \in \{1, 2\}$, then the quadratic equation (2) becomes

$$\mathbf{y}^{\mathrm{T}} F \mathbf{y} + \mathbf{g}^{\mathrm{T}} \mathbf{y} + c = 0$$

This is a quadratic equation in two variables.

The eigenvalues of F are d_1 and d_2, both nonzero, so we may use the centered factorization just as we did in the three-variable case when all eigenvalues were nonzero. Define $\mathbf{u} = -F^{-1}\mathbf{g}/2$, so that

$$(\mathbf{y} - \mathbf{u})^{\mathrm{T}} F (\mathbf{y} - \mathbf{u}) = \mathbf{u}^{\mathrm{T}} F \mathbf{u} - c = \mathbf{g}^{\mathrm{T}} F^{-1} \mathbf{g}/4 - c = r$$

The newly define r value is computable using rational arithmetic, and the sign tests on d_1, d_2, and r proceed as shown in Section 2.2.

The cases for when only d_1 is zero or only d_2 is zero are similarly handled.

3.4 The Case Exactly Two $d_i = 0$

The rank of A is one, so A must have one nonzero row and the other two rows are scalar multiples of it (possibly zero rows). If $\mathbf{p}_2$ is a nonzero row, we may choose two orthogonal vectors $\mathbf{p}_0$ and $\mathbf{p}_1$ that are also orthogonal to $\mathbf{p}_2$. Moreover, these vectors may be chosen with rational components, and they are linearly independent eigenvectors corresponding to the zero eigenvalue. And it is also the case that $\mathbf{p}_2$ is an eigenvector for the nonzero eigenvalue $\lambda = c_2$.

For $i \in \{0, 1\}$, the value $e_i = \mathbf{v}_i \cdot \mathbf{b}$ is zero exactly when $e'_i = \mathbf{p}_i \cdot \mathbf{b}$ is zero. If either e'_0 or e'_1 is not zero, the solution set is a parabolic cylinder. If both are zero, then we need to test the sign of r. This involves e_2, which we cannot compute using rational numbers. The technique shown in Section 3.3 is used and effectively scales e_2 to a rational number e'_2.

Define the matrix $P = [\mathbf{p}_0 \mid \mathbf{p}_1 \mid \mathbf{p}_2]$ in which case

$$P^{\mathrm{T}} A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{p}_2^{\mathrm{T}} A \mathbf{p}_2 \end{bmatrix}, \quad P^{\mathrm{T}} \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{p}_2 \cdot \mathbf{b} \end{bmatrix}$$

Define $f_2 = \mathbf{p}_2^{\mathrm{T}} A \mathbf{p}_2 = c_2 |\mathbf{p}_2|^2$, where we have used the fact that $\mathbf{p}_2$ is an eigenvector for eigenvalue c_2; that is, $A\mathbf{p}_2 = c_2 \mathbf{p}_2$. Define $e'_2 = \mathbf{p}_2 \cdot \mathbf{b}$. The quadratic equation (2) becomes

$$f_2 y_2^2 + e'_2 y_2 + c = 0$$

This is a quadratic equation in one variable that factors into

$$f_2 \left(y_2 + \frac{e_2'}{2f_2} \right)^2 = -c + (e_2')^2/(4f_2) = r$$

The signs of f_2 and r are analyzed as in Sections 2.5 and 2.6.

The class `QuadricSurface` in the files `WmlQuadricSurface.*` has the implementation using the exact rational arithmetic support in the library.