Subdivision of a Parabolic Segment by Arc Length

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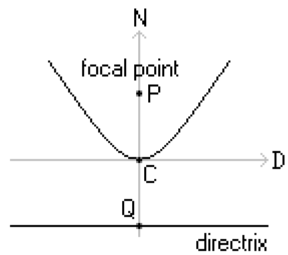
1 Discussion

A parabola is defined to be the set of points equidistant from a point $\mathbf{P}$ (the focal point) and a line $\mathbf{N} \cdot \mathbf{X} = d$ (the directrix). The normal vector to the line is chosen to be unit length, say $\mathbf{N} = (n_1, n_2)$ with $n_1^2 + n_2^2 = 1$. A direction vector to the line is $\mathbf{D} = (d_1, d_2) = (n_2, -n_1)$. Figure 1.1 shows a typical configuration.

Figure 1.1 A typical parabola. The vector $\mathbf{P} - \mathbf{Q}$ is perpendicular to the line direction $\mathbf{D}$. The point $\mathbf{C}$ is the midpoint of the line segment connecting $\mathbf{P}$ and $\mathbf{Q}$.



If $\mathbf{D} = (1, 0)$, $\mathbf{N} = (0, 1)$, and $2\ell = |\mathbf{P} - \mathbf{Q}|$, then the parabola is $4\ell y = x^2$ in Cartesian coordinates (x, y).

It is easily shown that

$$\mathbf{Q} = (\mathbf{D} \cdot \mathbf{P})\mathbf{D} + d\mathbf{N}$$

and

$$\mathbf{C} = (\mathbf{D} \cdot \mathbf{P})\mathbf{D} + \frac{1}{2}\left(\mathbf{N} \cdot \mathbf{P} + d\right)\mathbf{N}$$

Generally, the parabola is parameterized by

$$\mathbf{X}(t) = (t + \mathbf{D} \cdot \mathbf{P})\,\mathbf{D} + \left(\frac{t^2}{2(\mathbf{N} \cdot \mathbf{P} - d)} + \frac{\mathbf{N} \cdot \mathbf{P} + d}{2}\right)\mathbf{N}$$

for $t \in \mathbb{R}$. The derivative of the equation is

$$\mathbf{X}'(t) = \mathbf{D} + \left(\frac{t}{\mathbf{N} \cdot \mathbf{P} - d}\right)\mathbf{N}$$

A parabolic segment is specified by restricting t to an interval $[a, b]$. If instead two end points are provided, say $\mathbf{X}_0$ and $\mathbf{X}_1$, then the parameters are determined by $a = \mathbf{D} \cdot (\mathbf{X}_0 - \mathbf{P})$ and $b = \mathbf{D} \cdot (\mathbf{X}_1 - \mathbf{P})$.

The arc length L of the parabolic segment is

$$L = \int_a^b \left|\mathbf{X}'(t)\right|\, dt = \int_a^b \sqrt{1 + \frac{t^2}{(\mathbf{N} \cdot \mathbf{P} - d)^2}}\, dt = G(b) - G(a)$$

where

$$G(t) = \frac{t}{2}\sqrt{1 + \frac{t^2}{(\mathbf{N} \cdot \mathbf{P} - d)^2}} + \frac{\mathbf{N} \cdot \mathbf{P} - d}{2}\,\ln\left(\frac{t}{\mathbf{N} \cdot \mathbf{P} - d} + \sqrt{1 + \frac{t^2}{(\mathbf{N} \cdot \mathbf{P} - d)^2}}\right)$$

The midpoint of the parabolic segment is determined by the value t for which

$$G(t) - G(a) = \int_a^t \left|\mathbf{X}'(t)\right| \, dt = \int_t^b \left|\mathbf{X}'(t)\right| \, dt = G(b) - G(t)$$

Define

$$F(t) = 2G(t) - G(a) - G(b)$$

which has a derivative

$$F'(t) = 2G'(t) = 2\sqrt{1 + \frac{t^2}{(\mathbf{N} \cdot \mathbf{P} - d)^2}}$$

The midpoint of the parabolic segment is determined by the root t for $F(t) = 0$. This may be solved numerically using Newton's method. The initial guess is

$$t_0 = \frac{a+b}{2}$$

and the iterates are

$$t_{k+1} = t_k - \frac{F(t_k)}{F'(t_k)}, \quad k \geq 0$$