Solving Systems of Polynomial Equations

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Contents

1 Introduction

It is, of course, well known how to solve systems of linear equations. Given n equations in m unknowns, $\sum_{j=0}^{m} a_{ij}x_j = b_i$ for $0 \leq i < n$, let the system be represented in matrix form by $A\mathbf{x} = \mathbf{b}$ where $A = [a_{ij}]$ is $n \times m$, $\mathbf{x} = [x_j]$ is $m \times 1$, and $\mathbf{b} = [b_i]$ is $n \times 1$. The $n \times (m+1)$ augmented matrix $[A|\mathbf{b}]$ is constructed and row-reduced to $[E|\mathbf{c}]$. The augmented matrix has the properties:

- The first nonzero entry in each row is 1.
- If the first nonzero entry in row r is in column c, then all other entries in column c are 0.
- All zero rows occur last in the matrix.
- If the first nonzero entries in rows 1 through r occur in columns c_1 through c_r, then $c_1 < \ldots < c_r$.

If there is a row whose first m entries are zero, but the last entry is not zero, then the system of equations has no solution. If there is no such row, let $\rho = \mathrm{rank}([E|\mathbf{c}])$ denote the number of nonzero rows of the augmented matrix. If $\rho = m$, the system has exactly one solution. In this case $E = I_m$, the $m \times m$ identity matrix, and the solution is $\mathbf{x} = \mathbf{c}$. If $\rho < m$, the system has infinitely many solutions, the solution set having dimension $m - \rho$. In this case, the zero rows can be omitted to obtain the $\rho \times (m+1)$ matrix $[I_\rho|F|\mathbf{c}_+]$ where I_ρ is the $\rho \times \rho$ identity matrix, F is $\rho \times (m-\rho)$, and $\mathbf{c}_+$ consists of the first ρ entries of $\mathbf{c}$. Let $\mathbf{x}$ be partitioned into its first ρ components $\mathbf{x}_+$ and its remaining $m - \rho$ components $\mathbf{x}_-$. The general solution to the system is $\mathbf{x}_+ = \mathbf{c}_+ - F\mathbf{x}_-$ where the $\mathbf{x}_-$ are the free parameters in the system.

Generic numerical linear system solvers for square systems ($n = m$) use row-reduction methods so that (1) the order of time for the algorithm is small, in this case $O(n^3)$, and (2) the calculations are robust in the presence of a floating point number system. It is possible to solve a linear system using cofactor expansions, but the order of time for the algorithm is $O(n!)$ which makes this an expensive method for large n. However, $n = 3$ for many computer graphics applications. The overhead for a generic row-reduction solver normally uses more cycles than a simple cofactor expansion, and the matrix of coefficients for the application are usually not singular (or nearly singular) so that robustness is not an issue, so for this size system the cofactor expansion is a better choice.

Systems of polynomial equations also arise regularly in computer graphics applications. For example, determining the intersection points of two circles in 2D is equivalent to solving two quadratic equations in two unknowns. Determining if two ellipsoids in 3D intersect is equivalent to showing that a system of three quadratic equations in three unknowns does not have any real-valued solutions. Computing the intersection points between a line and a polynomial patch involves setting up and solving systems of polynomial equations. A method for solving such systems involves eliminating variables in much the same way that you do for linear systems. However, the formal calculations have a flavor of cofactor expansions rather than row-reductions.

2 Linear Equations in One Formal Variable

To motivate the general idea, consider a single equation $a_0 + a_1 x = 0$ in the variable x. If $a_1 \neq 0$, there is a unique solution $x = -a_0/a_1$. If $a_1 = 0$ and $a_0 \neq 0$, there are no solutions. If $a_0 = a_1 = 0$, any x is a solution.

Now consider two equations in the same variable, $a_0 + a_1 x = 0$ and $b_0 + b_1 x = 0$ where $a_1 \neq 0$ and $b_1 \neq 0$. The first equation is multiplied by b_1, the second equation is multiplied by a_1, and the two equations are subtracted to obtain $a_0 b_1 - a_1 b_0 = 0$. This is a necessary condition that a value x be a solution to both equations. If the condition is satisfied, then solving the first equation yields $x = -a_0/a_1$. In terms of the row-reduction method for linear systems discussed in the last section, $n = 2$, $m = 1$, and the augmented matrix is listed below with its reduction steps:

$$\left[\begin{array}{c|c} a_1 & -a_0 \\ b_1 & -b_0 \end{array}\right] \sim \left[\begin{array}{c|c} a_1 b_1 & -a_0 b_1 \\ a_1 b_1 & -a_1 b_0 \end{array}\right] \sim \left[\begin{array}{c|c} a_1 b_1 & -a_0 b_1 \\ 0 & a_0 b_1 - a_1 b_0 \end{array}\right] \sim \left[\begin{array}{c|c} 1 & -a_0/a_1 \\ 0 & a_0 b_1 - a_1 b_0 \end{array}\right]$$

The condition $a_0 b_1 - a_1 b_0$ is exactly the one mentioned in the previous section to guarantee that there is at least one solution.

The row-reduction presented here is a formal construction. The existence of solutions and the solution x itself are obtained as functions of the parameters a_0, a_1, b_0, and b_1 of the system. These parameters are not necessarily known scalars and can themselves depend on other variables. Suppose that $a_0 = c_0 + c_1 y$ and $b_0 = d_0 + d_1 y$. The original two equations are $a_1 x + c_1 y + c_0 = 0$ and $b_1 x + d_1 y + d_0 = 0$, a system of two equations in two unknowns. The condition for existence of solutions is $0 = a_0 b_1 - a_1 b_0 = (c_0 + c_1 y) b_1 - a_1(d_0 + d_1 y) = (b_1 c_0 - a_1 d_0) + (b_1 c_1 - a_1 d_1) y$. This condition is the result of starting with two equations in unknowns x and y and eliminating x to obtain a single equation for y. The y-equation has a unique solution as long as $b_1 c_1 - a_1 d_1 \neq 0$. Once y is computed, then $a_0 = c_0 + c_1 y$ is computed and $x = -a_0/a_1$ is computed.

Let us modify the problem once more and additionally set $a_1 = e_0 + e_1 y$ and $b_1 = f_0 + f_1 y$. The two equations are

$$e_1 xy + e_0 x + c_1 y + c_0 = 0$$
$$f_1 xy + f_0 x + d_1 y + d_0 = 0$$

This is a system of two *quadratic* equations in two unknowns. The condition for existence of solutions is

$$\begin{aligned} 0 &= a_0 b_1 - a_1 b_0 \\ &= (c_0 + c_1 y)(f_0 + f_1 y) - (e_0 + e_1 y)(d_0 + d_1 y) \\ &= (c_0 f_0 - e_0 d_0) + ((c_0 f_1 - e_0 d_0) + (c_1 f_0 - e_1 d_0))y + (c_0 f_1 - e_1 d_1) y^2. \end{aligned}$$

This equation has at most two real-valued solutions for y. Each solution leads to a value for $x = -a_0/a_1 = -(c_0 + c_1 y)/(e_0 + e_1 y)$. The two equations define hyperbolas in the plane whose asymptotes are axis-aligned. Geometrically the two hyperbolas can only intersect in at most two points.

Similar constructions arise when there are additional linear equations. For example, if $a_0 + a_1 x = 0$, $b_0 + b_1 x = 0$, and $c_0 + c_1 x = 0$, then solving pairwise leads to the conditions for existence: $a_0 b_1 - a_1 b_0 = 0$ and $a_0 c_1 - a_1 c_0 = 0$. If both are satisfied, then a solution is $x = -a_0/a_1$. Allowing $a_0 = a_{00} + a_{10} y + a_{01} z$, $b_0 = b_{00} + b_{10} y + b_{01} z$, and $c_0 = c_{00} + c_{10} y + c_{01} z$ leads to three linear equations in three unknowns. The two conditions for existence are two linear equations in y and z, an elimination of the variable x. These two equations can be further reduced by eliminating y in the same manner. Note that in using this approach, there are many quantites of the form $AB - CD$. This is where my earlier comment comes in about the method having a "flavor of cofactor expansions". These terms are essentially determinants of 2×2 submatrices of the augmented matrix.

3 Any Degree Equations in One Formal Variable

Consider the polynomial equation in x, $f(x) = \sum_{i=0}^{n} a_i x^i = 0$. The roots to this equation can be found either by closed form solutions when $n \leq 4$ or by numerical methods for any degree. How you go about computing polynomial roots is not discussed in this document. If you have a second polynomial equation in the same variable, $g(x) = \sum_{j=0}^{m} b_j x^j = 0$, the problem is to determine conditions for existence of a solution, just like we did in the last section. The assumption is that $a_n \neq 0$ and $b_m \neq 0$. The last section handled the case when $n = m = 1$.

3.1 Case $n = 2$ and $m = 1$

The equations are $f(x) = a_2x^2 + a_1x + a_0 = 0$ and $g(x) = b_1x + b_0 = 0$ where $a_2 \neq 0$ and $b_1 \neq 0$. It must also be the case that

$$0 = b_1 f(x) - a_2 x g(x) = (a_1b_1 - a_2b_0)x + a_0b_1 =: c_1x + c_0$$

where the coefficients c_0 and c_1 are defined by the last equality in the displayed equation. The two equations are now reduced to two linear equations, $b_1x + b_0 = 0$ and $c_1x + c_0 = 0$.

A bit more work must be done as compared to the last section. In that section the assumption was made that the leading coefficients were nonzero ($b_1 \neq 0$ and $c_1 \neq 0$). In the current construction, c_1 is derived from previously specified information, so we need to deal with the case when it is zero. If $c_1 = 0$, then $c_0 = 0$ is necessary for there to be a solution. Since $b_1 \neq 0$ by assumption, $c_0 = 0$ implies $a_0 = 0$. The condition $c_1 = 0$ implies $a_1b_1 = a_2b_0$. When $a_0 = 0$, a solution to the quadratic is $x = 0$. To be also a solution of $g(x) = 0$, we need $0 = g(0) = b_0$ which in turn implies $0 = a_2b_0 = a_1b_1$, or $a_1 = 0$ since $b_1 \neq 0$. In summary, this is the case $f(x) = a_2x^2$ and $g(x) = b_1x$. Also when $a_0 = 0$, another root of the quadratic is determined by $a_2x + a_1 = 0$. This equation and $b_1x + b_0 = 0$ are the case discussed in the last section and can be reduced appropriately.

One could also directly solve for $x = -b_0/b_1$, substitute into the quadratic, and multiply by b_1^2 to obtain the existence condition $a_2b_0^2 - a_1b_0b_1 + a_0b_1^2 = 0$.

3.2 Case $n = 2$ and $m = 2$

The equations are $a_2x^2 + a_1x + a_0 = 0$ and $b_2x^2 + b_1x + b_0 = 0$ where $a_2 \neq 0$ and $b_2 \neq 0$. It must also be the case that

$$0 = b_2 f(x) - a_2 g(x) = (a_1b_2 - a_2b_1)x + (a_0b_2 - a_2b_0) =: c_1x + c_0.$$

The two quadratic equations are reduced to a single linear equation whose coefficients c_0 and c_1 are defined by the last equality in the displayed equation. If $c_1 = 0$, then for there to be solutions it is also necessary that $c_0 = 0$. In this case, consider that

$$0 = b_0 f(x) - a_0 g(x) = (a_2b_0 - a_0b_2)x^2 + (a_1b_0 - a_0b_1)x = -c_0x^2 + (a_1b_0 - a_0b_1)x = (a_1b_0 - a_0b_1)x.$$

If $a_1b_0 - a_0b_1 \neq 0$, then the solution must be $x = 0$ and the consequences are $0 = f(0) = a_0$ and $0 = g(0) = b_0$. But this contradicts $a_1b_0 - a_0b_1 \neq 0$. Therefore, if $a_1b_2 - a_2b_1 = 0$ and $a_0b_2 - a_2b_0 = 0$, then $a_1b_0 - a_0b_1 = 0$ must follow. These three conditions imply that $(a_0, a_1, a_2) \times (b_0, b_1, b_2) = (0, 0, 0)$, so (b_0, b_1, b_2) is a multiple of (a_0, a_1, a_2) and the two quadratic equations were really only one equation. Now if $c_1 \neq 0$, we have reduced the problem to the case $n = 2$ and $m = 1$. This was discussed in the previous subsection.

A variation is to compute $a_2 g(x) - b_2 f(x) = (a_2 b_1 - a_1 b_2)x + (a_2 b_0 - a_0 b_2) = 0$ and $b_1 f(x) - a_1 g(x) = (a_2 b_1 - a_1 b_2)x^2 + (a_0 b_1 - a_1 b_0) = 0$. Solve for x in the first equation, $x = (a_0 b_2 - a_2 b_0)/(a_2 b_1 - a_1 b_2)$ and replace in the second equation and multiply by the denominator term to obtain

$$(a_2 b_1 - a_1 b_2)(a_1 b_0 - a_0 b_1) - (a_2 b_0 - a_0 b_2)^2 = 0.$$

3.3 General Case $n \geq m$

The elimination process is recursive. Given that the elimination process has already been established for the cases with degrees smaller than n, we just need to reduce the current case $f(x)$ of degree n and $g(x)$ of degree $m \leq n$ to one with smaller degrees. It is assumed here that $a_n \neq 0$ and $b_m \neq 0$.

Define $h(x) = b_m f(x) - a_n x^{n-m}$. The conditions $f(x) = 0$ and $g(x) = 0$ imply that

$$\begin{aligned} 0 &= h(x) \\ &= b_m f(x) - a_n x^{n-m} g(x) \\ &= b_m \sum_{i=0}^{n} a_i x^i - a_n x^{n-m} \sum_{i=0}^{m} b_i x^i \\ &= \sum_{i=0}^{n} a_i b_m x^i - \sum_{i=0}^{m} a_n b_i x^{n-m+i} \\ &= \sum_{i=0}^{n-m-1} a_i b_m x^i + \sum_{i=n-m}^{n-1} (a_i b_m - a_n b_{i-(n-m)}) x^i \end{aligned}$$

where it is understood that $\sum_{i=0}^{-1}(*) = 0$ (summations are zero whenever the upper index is smaller than the lower index). The polynomials $h(x)$ has degree at most $n-1$. Therefore, the polynomials $g(x)$ and $h(x)$ both have degrees smaller than n, so the smaller degree algorithms already exist to solve them.

4 Any Degree Equations in Any Formal Variables

A general system of polynomial equations can always be written formally as a system of polynomial equations in one of the variables. The conditions for existence, as constructed formally in the last section, are new polynomial equations in the remaining variables. Morever, these equations typically have higher degree than the original equations. As variables are eliminated, the degree of the reduced equations increase. Eventually the system is reduced to a single (high-degree) polynomial equation in one variable. Given solutions to this equation, they can be substituted into the previous conditions of existence to solve for other variables. This is similar to the "back substitution" that is used in linear system solvers.

5 Two Variables, One Quadratic Equation, One Linear Equation

The equations are $Q(x, y) = \alpha_{00} + \alpha 10x + \alpha_{01} y + \alpha_{20} x^2 + \alpha_{11} xy + \alpha_{02} y^2 = 0$ and $L(x, y) = b_{00} + b_{10} x + b_{01} y = 0$. These can be written formally as polynomials in x,

$$f(x) = (\alpha_{20})x^2 + (\alpha_{11} y + \alpha_{10})x + (\alpha_{02} y^2 + \alpha_{01} y + \alpha_{00}) = a_2 x^2 + a_1 x + a_0$$

and

$$g(x) = (\beta_{10})x + (\beta_{01}y + \beta_{00}) = b_1 x + b_0.$$

The condition for existence of $f(x) = 0$ and $g(x) = 0$ is $h(x) = h_0 + h_1 x + h_2 x^2 = 0$ where

$$\begin{aligned} h_0 &= \alpha_{02}\beta_{00}^2 - \alpha_{01}\beta_{00}\beta_{01} + \alpha_{00}\beta_{01}^2 \\ h_1 &= \alpha_{10}\beta_{01}^2 + 2\alpha_{02}\beta_{00}\beta_{10} - \alpha_{11}\beta_{00}\beta_{01} - \alpha_{01}\beta_{01}\beta_{10} \\ h_2 &= \alpha_{20}\beta_{01}^2 - \alpha_{11}\beta_{01}\beta_{10} + \alpha_{02}\beta_{10}^2. \end{aligned}$$

Given a root x to $h(x) = 0$, the formal value of y is obtained from $L(x, y) = 0$ as $y = -(b_{00} + b_{10}x)/b_{01}$.

6 Two Variables, Two Quadratic Equations

Consider two quadratic equations $F(x, y) = \alpha_{00} + \alpha 10x + \alpha_{01}y + \alpha_{20}x^2 + \alpha_{11}xy + \alpha_{02}y^2 = 0$ and $G(x, y) = \beta_{00} + \beta_{10}x + \beta_{01}y + \beta_{20}x^2 + \beta_{11}xy + \beta_{02}y^2 = 0$. These can be written formally as polynomials in x,

$$f(x) = (\alpha_{20})x^2 + (\alpha_{11}y + \alpha_{10})x + (\alpha_{02}y^2 + \alpha_{01}y + \alpha_{00}) = a_2x^2 + a_1x + a_0$$

and

$$g(x) = (\beta_{20})x^2 + (\beta_{11}y + \beta_{10})x + (\beta_{02}y^2 + \beta_{01}y + \beta_{00}) = b_2x^2 + b_1x + b_0.$$

The condition for existence is

$$0 = (a_2b_1 - a_1b_2)(a_1b_0 - a_0b_1) - (a_2b_0 - a_0b_2)^2 = \sum_{i=0}^{4} h_i y^i =: h(y)$$

where

$$\begin{aligned} h_0 &= d_{00}d_{10} - d_{20}^2 \\ h_1 &= d_{01}d_{10} + d_{00}d_{11} - 2d_{20}d_{21} \\ h_2 &= d_{01}d_{11} + d_{00}d_{12} - d_{21}^2 - 2d_{20}d_{22} \\ h_3 &= d_{01}d_{12} + d_{00}d_{13} - 2d_{21}d_{22} \\ h_4 &= d_{01}d_{13} - d_{22}^2 \end{aligned}$$

with

$$\begin{aligned} d_{00} &= \alpha_{20}\beta_{10} - \beta_{20}\alpha_{10} \\ d_{01} &= \alpha_{20}\beta_{11} - \beta_{20}\alpha_{11} \\ d_{10} &= \alpha_{10}\beta_{00} - \beta_{10}\alpha_{00} \\ d_{11} &= \alpha_{11}\beta_{00} + \alpha_{10}\beta_{01} - \beta_{11}\alpha_{00} - \beta_{10}\alpha_{01} \\ d_{12} &= \alpha_{11}\beta_{01} + \alpha_{10}\beta_{02} - \beta_{11}\alpha_{01} - \beta_{10}\alpha_{02} \\ d_{13} &= \alpha_{11}\beta_{02} - \beta_{11}\alpha_{02} \\ d_{20} &= \alpha_{20}\beta_{00} - \beta_{20}\alpha_{00} \\ d_{21} &= \alpha_{20}\beta_{01} - \beta_{20}\alpha_{01} \\ d_{22} &= \alpha_{20}\beta_{02} - \beta_{20}\alpha_{02} \end{aligned}$$

For each root $\bar{y}$ to $h(y) = 0$, the quadratic $F(x, \bar{y}) = 0$ can be solved for values $\bar{x}$. To make sure you have a solution to both equations, test that $G(\bar{x}, \bar{y}) = 0$.

7 Three Variables, One Quadratic Equation, Two Linear Equations

Let the three equations be $F(x, y, z) = \sum_{0 \le i+j+k \le 2} \alpha_{ijk} x^i y^j z^k$, $G(x, y, z) = \sum_{0 \le i+j+k \le 1} \beta_{ijk} x^i y^j z^k$, and $H(x, y, z) = \sum_{0 \le i+j+k \le 1} \gamma_{ijk} x^i y^j z^k$. As polynomial equations in x, these are written as $f(x) = a_2 x^2 + a_1 x + a_0 = 0$, $g(x) = b_1 x + b_0 = 0$, and $h(x) = c_1 x + c_0 = 0$ where

$$\begin{aligned}
a_0 &= \textstyle\sum_{0 \le j+k \le 2} \alpha_{0jk} y^j z^k \\
a_1 &= \textstyle\sum_{0 \le j+k \le 1} \alpha_{1jk} y^j z^k \\
a_2 &= \alpha_{200} \\
b_0 &= \beta_{010} y + \beta_{001} z + \beta_{000} \\
b_1 &= \beta_{100} \\
c_0 &= \gamma_{010} y + \gamma_{001} z + \gamma_{000} \\
c_1 &= \gamma_{100}
\end{aligned}$$

The condition for existence of x-solutions to $f = 0$ and $g = 0$ is

$$0 = a_2 b_0^2 - a_1 b_0 b_1 + a_0 b_1^2 = \sum_{0 \le i+j \le 2} d_{ij} y^i z^j =: D(y, z)$$

where

$$\begin{aligned}
d_{20} &= \alpha_{200} \beta_{010}^2 - \beta_{100} \alpha_{110} \beta_{010} + \beta_{100}^2 \alpha_{020} \\
d_{11} &= 2\alpha_{200} \beta_{010} \beta_{001} - \beta_{100}(\alpha_{110} \beta_{001} + \alpha_{101} \beta_{010}) + \beta_{100}^2 \alpha_{011} \\
d_{02} &= \alpha_{200} \beta_{001}^2 - \beta_{100} \alpha_{101} \beta_{001} + \beta_{100}^2 \alpha_{002} \\
d_{10} &= 2\alpha_{200} \beta_{010} \beta_{000} - \beta_{100}(\alpha_{110} \beta_{000} + \alpha_{100} \beta_{010}) + \beta_{100}^2 \alpha_{010} \\
d_{01} &= 2\alpha_{200} \beta_{001} \beta_{000} - \beta_{100}(\alpha_{101} \beta_{000} + \alpha_{100} \beta_{001}) + \beta_{100}^2 \alpha_{001} \\
d_{00} &= \alpha_{200} \beta_{000}^2 - \beta_{100} \alpha_{100} \beta_{000} + \beta_{100}^2 \alpha_{000}
\end{aligned}$$

The condition for existence of x-solutions to $g = 0$ and $h = 0$ is

$$0 = b_0 c_1 - b_1 c_0 = e_{10} y + e_{01} z + e_{00} =: E(y, z)$$

where

$$\begin{aligned}
e_{10} &= \beta_{010} \gamma_{100} - \gamma_{010} \beta_{100} \\
e_{01} &= \beta_{001} \gamma_{100} - \gamma_{001} \beta_{100} \\
e_{00} &= \beta_{000} \gamma_{100} - \gamma_{000} \beta_{100}
\end{aligned}$$

We now have two equations in two unknowns, a quadratic equation $D(y,z) = 0$ and a linear equation $E(y,z) = 0$. This case was handled in an earlier section. For each solution $(\bar{y}, \bar{z})$, a corresponding x value is computed by solving either $G(x,\bar{y},\bar{z}) = 0$ or $H(x,\bar{y},\bar{z}) = 0$ for $\bar{x}$.

8 Three Variables, Two Quadratic Equations, One Linear Equation

Let the three equations be $F(x,y,z) = \sum_{0 \le i+j+k \le 2} \alpha_{ijk} x^i y^j z^k$, $G(x,y,z) = \sum_{0 \le i+j+k \le 2} \beta_{ijk} x^i y^j z^k$, and $H(x,y,z) = \sum_{0 \le i+j+k \le 1} \gamma_{ijk} x^i y^j z^k$. As polynomial equations in x, these are written as $f(x) = a_2 x^2 + a_1 x + a_0 = 0$, $g(x) = b_2 x^2 + b_1 x + b_0 = 0$, and $h(x) = c_1 x + c_0 = 0$ where

$$
\begin{aligned}
a_0 &= \textstyle\sum_{0 \le j+k \le 2} \alpha_{0jk} y^j z^k \\
a_1 &= \textstyle\sum_{0 \le j+k \le 1} \alpha_{1jk} y^j z^k \\
a_2 &= \alpha_{200} \\
b_0 &= \textstyle\sum_{0 \le j+k \le 2} \beta_{0jk} y^j z^k \\
b_1 &= \textstyle\sum_{0 \le j+k \le 1} \beta_{1jk} y^j z^k \\
b_2 &= \beta_{200} \\
c_0 &= \gamma_{010} y + \gamma_{001} z + \gamma_{000} \\
c_1 &= \gamma_{100}
\end{aligned}
$$

The condition for existence of x-solutions to $f = 0$ and $h = 0$ is

$$0 = a_2 c_0^2 - a_1 c_0 c_1 + a_0 c_1^2 = \sum_{0 \le i+j \le 2} d_{ij} y^i z^j =: D(y,z)$$

where

$$
\begin{aligned}
d_{20} &= \alpha_{200}\gamma_{010}^2 - \gamma_{100}\alpha_{110}\gamma_{010} + \gamma_{100}^2\alpha_{020} \\
d_{11} &= 2\alpha_{200}\gamma_{010}\gamma_{001} - \gamma_{100}(\alpha_{110}\gamma_{001} + \alpha_{101}\gamma_{010}) + \gamma_{100}^2\alpha_{011} \\
d_{02} &= \alpha_{200}\gamma_{001}^2 - \gamma_{100}\alpha_{101}\gamma_{001} + \gamma_{100}^2\alpha_{002} \\
d_{10} &= 2\alpha_{200}\gamma_{010}\gamma_{000} - \gamma_{100}(\alpha_{110}\gamma_{000} + \alpha_{100}\gamma_{010}) + \gamma_{100}^2\alpha_{010} \\
d_{01} &= 2\alpha_{200}\gamma_{001}\gamma_{000} - \gamma_{100}(\alpha_{101}\gamma_{000} + \alpha_{100}\gamma_{001}) + \gamma_{100}^2\alpha_{001} \\
d_{00} &= \alpha_{200}\gamma_{000}^2 - \gamma_{100}\alpha_{100}\gamma_{000} + \gamma_{100}^2\alpha_{000}
\end{aligned}
$$

The condition for existence of x-solutions to $g = 0$ and $h = 0$ is

$$0 = b_2 c_0^2 - b_1 c_0 c_1 + b_0 c_1^2 = \sum_{0 \le i+j \le 2} e_{ij} y^i z^j =: E(y,z)$$

where

$$
\begin{aligned}
e_{20} &= \beta_{200}\gamma_{010}^2 - \gamma_{100}\beta_{110}\gamma_{010} + \gamma_{100}^2\beta_{020} \\
e_{11} &= 2\beta_{200}\gamma_{010}\gamma_{001} - \gamma_{100}(\beta_{110}\gamma_{001} + \beta_{101}\gamma_{010}) + \gamma_{100}^2\beta_{011} \\
e_{02} &= \beta_{200}\gamma_{001}^2 - \gamma_{100}\beta_{101}\gamma_{001} + \gamma_{100}^2\beta_{002} \\
e_{10} &= 2\beta_{200}\gamma_{010}\gamma_{000} - \gamma_{100}(\beta_{110}\gamma_{000} + \beta_{100}\gamma_{010}) + \gamma_{100}^2\beta_{010} \\
e_{01} &= 2\beta_{200}\gamma_{001}\gamma_{000} - \gamma_{100}(\beta_{101}\gamma_{000} + \beta_{100}\gamma_{001}) + \gamma_{100}^2\beta_{001} \\
e_{00} &= \beta_{200}\gamma_{000}^2 - \gamma_{100}\beta_{100}\gamma_{000} + \gamma_{100}^2\beta_{000}
\end{aligned}
$$

We now have two equations in two unknowns, quadratic equations $D(y,z) = 0$ and $E(y,z) = 0$. This case was handled in an earlier section. For each solution $(\bar{y}, \bar{z})$, a corresponding x value is computed by solving $F(x, \bar{y}, \bar{z}) = 0$ for values $\bar{x}$. It should be verified that $G(\bar{x}, \bar{y}, \bar{z}) = 0$ and $G(\bar{x}, \bar{y}, \bar{z}) = 0$.

9 Three Variables, Three Quadratic Equations

Let the three equations be $F(x,y,z) = \sum_{0 \le i+j+k \le 2} \alpha_{ijk} x^i y^j z^k$, $G(x,y,z) = \sum_{0 \le i+j+k \le 2} \beta_{ijk} x^i y^j z^k$, and $H(x,y,z) = \sum_{0 \le i+j+k \le 2} \gamma_{ijk} x^i y^j z^k$. As polynomial equations in x, these are written as $f(x) = a_2 x^2 + a_1 x + a_0 = 0$, $g(x) = b_2 x^2 + b_1 x + b_0 = 0$, and $h(x) = c_2 x^2 + c_1 x + c_0 = 0$ where

$$
\begin{aligned}
a_0 &= \textstyle\sum_{0 \le j+k \le 2} \alpha_{0jk} y^j z^k \\
a_1 &= \textstyle\sum_{0 \le j+k \le 1} \alpha_{1jk} y^j z^k \\
a_2 &= \alpha_{200} \\
b_0 &= \textstyle\sum_{0 \le j+k \le 2} \beta_{0jk} y^j z^k \\
b_1 &= \textstyle\sum_{0 \le j+k \le 1} \beta_{1jk} y^j z^k \\
b_2 &= \beta_{200} \\
c_0 &= \textstyle\sum_{0 \le j+k \le 2} \gamma_{0jk} y^j z^k \\
c_1 &= \textstyle\sum_{0 \le j+k \le 1} \gamma_{1jk} y^j z^k \\
c_2 &= \gamma_{200}
\end{aligned}
$$

The condition for existence of x-solutions to $f = 0$ and $g = 0$ is

$$0 = (a_2 b_1 - a_1 b_2)(a_1 b_0 - a_0 b_1) - (a_2 b_0 - a_0 b_2)^2 = \sum_{0 \le i+j \le 4} d_{ij} y^i z^k =: D(y,z)$$

where

The condition for existence of x-solutions to $f = 0$ and $h = 0$ is

$$0 = (a_2 c_1 - a_1 c_2)(a_1 c_0 - a_0 c_1) - (a_2 c_0 - a_0 c_2)^2 = \sum_{0 \le i+j \le 4} e_{ij} y^i z^k =: E(y,z)$$

where

The two polynomials $D(y,z)$ and $E(y,z)$ are fourth degree. The equations $D(y,z) = 0$ and $E(y,z) = 0$ can be written formally as polynomials equations in y, $d(y) = \sum_{i=0}^{4} \delta_i y^i$ and $e(y) = \sum_{i=0}^{4} \epsilon_i y^i$ where the coefficients are polynomials in z with degree$(d_i(z)) = 4 - i$ and degree$(e_i(z)) = 4 - i$. The construction for eliminating y results in a polynomial in z obtained by computing the determinant of the Bézout matrix for d and e, the 4×4 matrix $M = [M_{ij}]$ with

$$M_{ij} = \sum_{k=\max(4-j,4-i)}^{\min(4,7-i-j)} w_{k,7-i-j-k}$$

for $0 \le i \le 3$ and $0 \le j \le 3$, with $w_{i,j} = \delta_i \gamma_j - \delta_j \gamma_i$ for $0 \le i \le 4$ and $0 \le j \le 4$. In expanded form,

$$M = \begin{bmatrix} w_{4,3} & w_{4,2} & w_{4,1} & w_{4,0} \\ w_{4,2} & w_{3,2} + w_{4,1} & w_{3,1} + w_{4,0} & w_{3,0} \\ w_{4,1} & w_{3,1} + w_{4,0} & w_{2,1} + w_{3,0} & w_{2,0} \\ w_{4,0} & w_{3,0} & w_{2,0} & w_{1,0} \end{bmatrix}.$$

The degree of $w_{i,j}$ is $8-i-j$. The Bézout determinant $\det(M(z))$ is a polynomial of degree 16 in z. For each solution $\bar{z}$ to $\det(M(z)) = 0$, corresponding values $\bar{y}$ are obtained by solving the quartic equation $D(y, \bar{z}) = 0$. Finally, corresponding values $\bar{x}$ are obtained by solving the quadratic equation $F(x, \bar{y}, \bar{z}) = 0$. Any potential solution $(\bar{x}, \bar{y}, \bar{z})$ should be tested if $G(\bar{x}, \bar{y}, \bar{z}) = 0$ and $H(\bar{x}, \bar{y}, \bar{z}) = 0$.