Largest Fixed-Aspect, Axis-Aligned Rectangle

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1 Formulation as a Linear Programming Problem

Let $\mathbf{V}_i = (x_i, y_i)$ for $0 \le i \le 3$ be the vertices of a convex quadrilateral. The vertices are assumed to be ordered in the counterclockwise direction. For the purposes of indexing, let $\mathbf{V}_4 = \mathbf{V}_0$. The edges of the quadrilateral have directions $\mathbf{E}_i = \mathbf{V}_{i+1} - \mathbf{V}_i$ for $0 \le i \le 3$. Inner pointing normal vectors are $\mathbf{N}_i = \mathbf{E}_i^{\perp}/|\mathbf{E}_i|$ where $(x, y)^{\perp} = (-y, x)$.

An axis-aligned rectangle is specified by its lower left corner $\mathbf{R}_0 = (u, v)$, a width w, and a height h. The vertices in counterclockwise order are $\mathbf{R}_0 = (u, v)$, $\mathbf{R}_1 = (u+w, v)$, $\mathbf{R}_2 = (u+w, v+h)$, and $\mathbf{R}_3 = (u, v+h)$. A rectangle has a fixed aspect ratio if $w/h = r$ for a specified aspect ratio $r > 0$.

The problem is to select $\mathbf{P} = (u, v, w)$ so that the corresponding axis-aligned rectangle with aspect ratio r is the largest such rectangle contained in the quadrilateral. The vertices of the rectangle must all lie in the quadrilateral, so $\mathbf{N}_i \cdot (\mathbf{R}_j - \mathbf{V}_i) \ge 0$ for all i and j, a total of 16 inequality constraints. The area of a rectangle is $A(w) = w^2/r$. We wish to maximize $A(w)$ subject to $w \ge 0$ and the previous 16 constraints. Notice that maximizing $A(w)$ for $w \ge 0$ is equivalent to maximizing $f(w) = w$ itself. Thus, we have the linear programming problem:

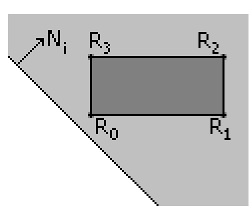
> Maximize $f(w) = w$ subject to the constraints $w \ge 0$ and $C_{ij}(u, v, w) = \mathbf{N}_i \cdot (\mathbf{R}_j - \mathbf{V}_i) \ge 0$ for $0 \le i \le 3$ and $0 \le j \le 3$.

If a linear programming solver requires $u \ge 0$ and $v \ge 0$, the quadrilateral may be translated into the first quadrant to satisfy these conditions.

2 Reduction of Constraints

The 16 inequality constraints have a lot of redundancy. For example, if $\mathbf{N}_i = (a_i, b_i)$ with $a_i > 0$ and $b_i > 0$, then $\mathbf{N}_i \cdot (\mathbf{R}_0 - \mathbf{V}_i) \ge 0$ automatically implies $\mathbf{N}_i \cdot (\mathbf{R}_j - \mathbf{V}_0) \ge 0$ for $1 \le j \le 3$. The implication is clearly shown by Figure 2.1.

Figure 2.1 Reduction of the four constraints $\mathbf{N}_i \cdot (\mathbf{R}_j - \mathbf{V}_i) \ge 0$ for $0 \le j \le 3$ to the single constraint $\mathbf{N}_i \cdot (\mathbf{R}_0 - \mathbf{V}_i) \ge 0$.



Note that the normal vector $\mathbf{N}_i$ is in quadrant 1 of the plane. If $\mathbf{N}_i$ is in quadrant 2, then $a_i < 0$ and $b_i > 0$. The constraint $\mathbf{N}_i \cdot (\mathbf{R}_1 - \mathbf{V}_i) \ge 0$ implies $\mathbf{N}_i \cdot (\mathbf{R}_j - \mathbf{V}_i) \ge 0$ for $j \in \{0, 2, 3\}$. Similar reductions apply when $\mathbf{N}_i$ is in quadrant 3 or in quadrant 4.

The normal vector $\mathbf{N}_i = (1,0)$ is on the boundary between quadrants 1 and 4. The four constraints associated with $\mathbf{N}_i$ still reduce to a single constraint involving $\mathbf{R}_0$. Similar observations apply to normals $(0,1)$, $(-1,0)$, and $(0,-1)$. Therefore, the linear programming problem does not need all 16 inequality constraints. It suffices to determine the quadrant that contains $\mathbf{N}_i$ and use the associated, single, inequality constraint. If $\theta_i \in [0, 2\pi)$ is the angle formed by $\mathbf{N}_i$ and the positive x-axis, then the associated constraint is

$$\mathbf{N}_i \cdot (\mathbf{R}_{\lfloor 2\theta_i/\pi \rfloor} - \mathbf{V}_i) \geq 0$$

where $\lfloor z \rfloor$ denotes the floor of z, the largest integer smaller or equal to z. The index of $\mathbf{R}$ is 0 if $\theta_i \in [0, \pi/2)$, is 1 if $\theta_i \in [\pi/2, \pi)$, is 2 if $\theta_i \in [\pi, 3\pi/2)$, or 3 if $\theta_i \in [3\pi/2, 2\pi)$.

The linear programming problem to maximize $f(w) = w$ now has only 5 constraints. One of them is $w \geq 0$. The other 4 are $\mathbf{N}_i \cdot (\mathbf{R}_{\lfloor 2\theta_i/\pi \rfloor} - \mathbf{V}_i) \geq 0$ for $0 \leq i \leq 3$.

3 Existence of Infinitely Many Solutions

The fact that multiple solutions exist is shown by the following example. Suppose the quadrilateral is the axis-aligned rectangle: $\mathbf{V}_0 = (0,0)$, $\mathbf{V}_1 = (1,0)$, $\mathbf{V}_2 = (1, 1/2)$, and $\mathbf{V}_3 = (0, 1/2)$. Let the aspect ratio be $r = 4/3$. The limiting factor for the maximum area axis-aligned rectangle with aspect ratio 4/3 is the height of the quadrilateral. The height of $h = 1/2$ for the inscribed rectangle leads to $w = rh = 2/3$. Many inscribed rectangles of these dimensions exist. The centers are $(c, 1/4)$ for $1/3 \leq c \leq 2/3$. The lower left corners are $(u, 0)$ for $0 \leq u \leq 1/3$.

4 Geometric Interpretation of the Constraints

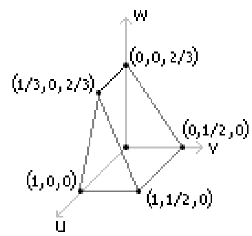
The last example may be analyzed by considering its convex polyhedral domain defined by the constraints. This domain is in (u, v, w) space. The maximum of $f(w) = w$ must occur at a vertex of the domain. We need only find a vertex of largest w. As the last example shows, there can be infinitely many points at which w is largest, in which case at least two vertices attain largest w, which in turn implies the convex hull of all such vertices attains largest w.

The normal vectors to the quadrilateral edges are $\mathbf{N}_0 = (0,1)$, $\mathbf{N}_1 = (-1,0)$, $\mathbf{N}_2 = (0,-1)$, and $\mathbf{N}_3 = (1,0)$. The corresponding constraints are $w \geq 0$ and

$$\begin{aligned}
0 &\leq \mathbf{N}_0 \cdot (\mathbf{R}_1 - \mathbf{V}_0) = v \\
0 &\leq \mathbf{N}_1 \cdot (\mathbf{R}_2 - \mathbf{V}_1) = -(u + w - 1) \\
0 &\leq \mathbf{N}_2 \cdot (\mathbf{R}_3 - \mathbf{V}_2) = -(v + 3w/4 - 1/2) \\
0 &\leq \mathbf{N}_3 \cdot (\mathbf{R}_0 - \mathbf{V}_3) = u
\end{aligned}$$

or $w \geq 0$, $u \geq 0$, $v \geq 0$, $u + w \leq 1$, and $4v + 3w \leq 2$. Figure 4.1 shows the convex polyhedral domain.

Figure 4.1 The convex polyhedral domain defined by $w \geq 0$, $u \geq 0$, $v \geq 0$, $u + w \leq 1$, and $4v + 3w \leq 2$. The 6 vertices are shown as black dots. The points of maximum w are $(u, 0, 2/3)$ for $0 \leq u \leq 1/3$.



The base of the polyhedron is the original quadrilateral and occurs when $w = 0$. This is generally the case since the rectangle corner (u, v) is always a point in the quadrilateral. The polyhedron has four other faces, each face defined by a constraint associated with a quadrilateral edge.

Generally, the convex polyhedral domain is pyramidal with 5 faces, the base being the original quadrilateral. The other 4 faces must slant "inwards" to form the pyramid. The geometric shape of the pyramid implies that is has either a single vertex attaining maximal w (a unique inscribed axis-aligned rectangle with specified aspect ratio) or a line segment attaining maximal w (infinitely many inscribed axis-aligned rectangles with specified aspect ratio). This observation allows us to construct the inscribed rectangle in a simple manner that does not require use of a general linear programming solver.

5 An Algorithm to Construct the Inscribed Rectangle

The geometry of the convex polyhedral domain indicates the following construction. If the pyramid has a single vertex attaining maximal w, that vertex is the common point of four faces and can be found by solving either (1) three linear equations in three unknowns or (2) four linear equations in three unknowns, the system necessarily having rank 3. If the pyramid has a line segment attaining maximal w, the segment lies on the line of intersection of two planes corresponding to the equality constraints for two opposite edges of the quadrilateral. All cases can be handled by computing the line of intersection of the planes represented by two linear equations, then clipping that line against the planes for two other linear equations.

Specifically, the four inequality constraints defining the faces of the pyramid (not the base) are of the form $\mathbf{M}_i \cdot \mathbf{P} + d_i \geq 0$ for $0 \leq i \leq 3$. The line of intersection of constraints $i = 0$ and $i = 2$ is

$$\mathbf{P}(t) = t(\mathbf{M}_0 \times \mathbf{M}_2) + \mathbf{K}_0$$

for some point $\mathbf{K}_0$ on the line and for all real t. The clipping is achieved by substituting this line equation into the other constraints,

$$0 \leq \mathbf{M}_1 \cdot \mathbf{P}(t) + d_1 = t(\mathbf{M}_1 \cdot \mathbf{M}_0 \times \mathbf{M}_2) + (\mathbf{M}_1 \cdot \mathbf{K}_0 + d_1) = \alpha_1 t + \beta_1$$
$$0 \leq \mathbf{M}_3 \cdot \mathbf{P}(t) + d_3 = t(\mathbf{M}_3 \cdot \mathbf{M}_0 \times \mathbf{M}_2) + (\mathbf{M}_3 \cdot \mathbf{K}_0 + d_3) = \alpha_3 t + \beta_3$$

The geometry of the convex polyhedral domain implies that neither α_1 nor α_3 is zero. The inequality $\alpha_1 t + \beta_1 \geq 0$ defines a semiinfinite t-interval I_1 which is $[-\beta_1/\alpha_1, +\infty)$ if $\alpha_1 > 0$ or $(-\infty, -\beta_1/\alpha_1]$ if $\alpha_1 < 0$.

A semiinfinite interval is similary defined by $\alpha_3 t + \beta_3 \geq 0$, call it I_3. The intersection $I = I_1 \cap I_3$ is either empty, a singleton, or a interval of positive and finite length. The geometry of the polyhedral domain rules out I being a semiinfinite interval.

I is a single point when $-\beta_1/\alpha_1 = -\beta_3/\alpha_3$, in which case $t = -\beta_1/\alpha_1$ and $\mathbf{P}(t) = (u(t), v(t), w(t))$ provides the inscribed rectangle. If I is an interval of positive and finite length, call it $I = [t_0, t_1]$, then choose $t = t_0$ or $t = t_1$, whichever one produces the largest $w(t)$ value. Once again, $\mathbf{P}(t) = (u(t), v(t), w(t))$ provides the inscribed rectangle.

If I is empty, then compute the line of intersection of constraints $i = 1$ and $i = 3$, and repeat the above construction. This time, the t-interval is named I' and cannot be empty, and the appropriate value of t is constructed to lead to maximum $w(t)$.