Intersection of a Triangle and a Cone

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Contents

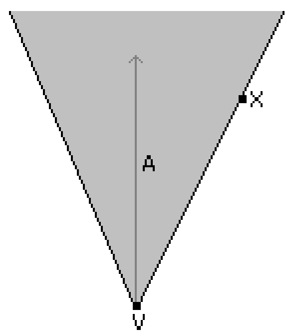
1 Introduction

Let the triangle have vertices $\mathbf{P}_i$ for $0 \le i \le 2$. The cone has vertex $\mathbf{V}$, axis direction vector $\mathbf{A}$, and angle θ between axis and outer edge. In most applications, the cone is *acute*, that is, $\theta \in (0, \pi/2)$. This document assumes that, in fact, the cone is acute, so $\cos\theta > 0$. The cone consists of those points $\mathbf{X}$ for which the angle between $\mathbf{X} - \mathbf{V}$ and $\mathbf{A}$ is θ. Algebraically the condition is

$$\mathbf{A} \cdot \left(\frac{\mathbf{X} - \mathbf{V}}{|\mathbf{X} - \mathbf{V}|} \right) = \cos\theta.$$

Figure 1.1 shows a 2D representation of the cone. The shaded portion indicates the *inside* of the cone, a region represented algebraically by replacing $=$ in the above equation with $\ge$.

Figure 1.1 An acute cone. The inside region is shaded.

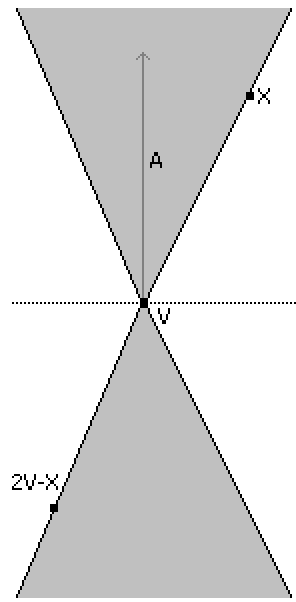


To avoid the square root calculation $|\mathbf{X} - \mathbf{V}|$, the cone equation may be squared to obtain the quadratic equation

$$(\mathbf{A} \cdot (\mathbf{X} - \mathbf{V}))^2 = (\cos^2\theta)|\mathbf{X} - \mathbf{V}|^2.$$

However, the set of point satisfying this equation is a *double cone*. The original cone is on the side of the plane $\mathbf{A} \cdot (\mathbf{X} - \mathbf{V}) = 0$ to which $\mathbf{A}$ points. The quadratic equation defines the original cone and its reflection through the plane. Specifically, if $\mathbf{X}$ is a solution to the quadratic equation, then its reflection through the vertex, $2\mathbf{V} - \mathbf{X}$, is also a solution. Figure 1.2 shows the double cone.

Figure 1.2 An acute double cone. The inside region is shaded.



To eliminate the reflected cone, any solutions to the quadratic equation must also satisfy $\mathbf{A}\cdot(\mathbf{X}-\mathbf{V}) \geq 0$. Also, the quadratic equation can be written as a quadratic form, $(\mathbf{X}-\mathbf{V})^{\mathrm{T}}M(\mathbf{X}-\mathbf{V}) = 0$ where $M = (\mathbf{A}\mathbf{A}^{\mathrm{T}} - \gamma^2 I)$ and $\gamma = \cos\theta$. Therefore, $\mathbf{X}$ is a point on the acute cone whenever

$$(\mathbf{X}-\mathbf{V})^{\mathrm{T}}M(\mathbf{X}-\mathbf{V}) = 0 \text{ and } \mathbf{A}\cdot(\mathbf{X}-\mathbf{V}) \geq 0.$$

2 Test Intersection

Testing if a triangle and cone intersect, and not having to compute points of intersection, is useful for a couple of graphics applications. For example, a spot light illuminates only those triangles in a scene that are within the cone of the light. It is useful to know if the vertex colors of a triangle's vertices need to be modified due to the effects of the light. In most graphics applications, if some of the triangle is illuminated, then all the vertex colors are calculated. It is not important to know the subregion of the triangle that is in the cone (a result determined by a *find* query). Another example is for culling of triangles from a view frustum that is bounded by a cone for the purposes of rapid culling.

If a triangle intersects a cone, it must do so either at a vertex, an edge point, or an interior triangle point. The algorithm described here is designed to provide early exits using a testing order of vertex-in-cone, edge-intersects-cone, and triangle-intersects cone. This order is a good one for an application where a lot of triangles tend to be fully inside the cone. Other orders may be used depending on how an application structures its world data.

To test if $\mathbf{P}_0$ is inside the cone, it is enough to test if the point is on the cone side of the plane $\mathbf{A}\cdot(\mathbf{X}-\mathbf{V}) \geq 0$

and if the point is inside the double cone. Although the test can be structured as

```
D0 = triangle.P0 - cone.V;
AdD0 = Dot(cone.A,D0);
D0dD0 = Dot(D0,D0);
if ( AdD0 >= 0 and AdD0*AdD0 >= cone.CosSqr*D0dD0 )
    triangle.P0 is inside cone;
```

if all the triangle vertices are outside the single cone, it will be important in the edge-cone intersection tests to know on which side of the plane $\mathbf{A} \cdot (\mathbf{X} - \mathbf{V}) = 0$ the vertices are. The vertex test is better structured as shown below. The term *outside cone* refers to the quantity being outside the single cone, not the double cone (a point could be outside the original single cone, but inside that cone's reflection).

```
D0 = triangle.P0 - cone.V;
AdD0 = Dot(cone.A,D0);
if ( AdD0 >= 0 )
{
    D0dD0 = Dot(D0,D0);
    if ( AdD0*AdD0 >= cone.CosSqr*D0dD0 )
    {
        triangle.P0 is inside cone;
    }
    else
    {
        triangle.P0 is outside cone, but on cone side of plane;
    }
}
else
{
    triangle.P0 is outside cone, but on opposite side of plane;
}
```

All three vertices of the triangle are tested in this manner.

If all three vertices are outside the cone, the next step is to test if the edges of the triangle intersect the cone. Consider the edge $\mathbf{X}(t) = \mathbf{P}_0 + t\mathbf{E}_0$ where $\mathbf{E}_0 = \mathbf{P}_1 - \mathbf{P}_0$ and $t \in [0, 1]$. The edge intersects the single cone if $\mathbf{A} \cdot (\mathbf{X}(t) - \mathbf{V}) \geq 0$ and $(\mathbf{A} \cdot (\mathbf{X}(t) - \mathbf{V}))^2 - \gamma^2 |\mathbf{X}(t) - \mathbf{V}|^2 = 0$ for some $t \in [0, 1]$. The second condition is a quadratic equation, $Q(t) = c_2 t^2 + 2c_1 t + c_0 = 0$ where $c_2 = (\mathbf{A} \cdot \mathbf{E}_0)^2 - \gamma^2 |\mathbf{E}_0|^2$, $c_1 = (\mathbf{A} \cdot \mathbf{E}_0)(\mathbf{A} \cdot \boldsymbol{\Delta}_0) - \gamma^2 \mathbf{E}_0 \cdot \boldsymbol{\Delta}_0$, and $c_0 = (\mathbf{A} \cdot \boldsymbol{\Delta}_0)^2 - \gamma^2 |\boldsymbol{\Delta}_0|^2$ where $\boldsymbol{\Delta}_0 = \mathbf{P}_0 - \mathbf{V}$. The domain of $Q(t)$ for which a root is sought depends on which side of the plane the vertices lie.

If both $\mathbf{P}_0$ and $\mathbf{P}_1$ are on the opposite side of the plane, then the edge cannot intersect the single cone. If both $\mathbf{P}_0$ and $\mathbf{P}_1$ are on the cone side of the plane, then the full edge must be considered, so we need to determine if $Q(t) = 0$ for some $t \in [0, 1]$. Moreover, the test should be fast since we do not need to know *where* the intersection occurs, just that there is one. Since the two vertices are outside the cone and occur when $t = 0$ and $t = 1$, we already know that $Q(0) < 0$ and $Q(1) < 0$. In order for the quadratic to have a root somewhere in $[0, 1]$, it is necessary that the graph be concave. For if it were convex, the graph would lie below the line segment connecting the points $(0, Q(0))$ and $(1, Q(1))$. This line segment never intersects the axis $Q = 0$. Thus, the concavity condition is $c_2 < 0$. Additionally, the t-value for the local maximum must occur in $[0,1]$. This value is $\hat{t} = -c_1/c_2$. We could compute $\hat{t}$ directly by doing the division. However, the division can be avoided. The test $0 \le \hat{t} \le 1$ is equivalent to the test $0 \le c_1 \le -c_2$ since $c_2 < 0$. The final condition for there to be a root is that $Q(\hat{t}) \ge 0$. This happens when the discriminant for the quadratic is nonnegative: $c_1^2 - c_0 c_2 \ge 0$. In summary, when $\mathbf{P}_0$ and $\mathbf{P}_1$ are both on the cone side of the plane, the corresponding edge intersects the cone when

$$c_2 < 0 \text{ and } 0 \le c_1 \le -c_2 \text{ and } c_1^2 \ge c_0 c_2.$$

If $\mathbf{P}_0$ is on the cone side and $\mathbf{P}_1$ is on the opposite side, the domain of Q can be reduced to $[0, \tilde{t}]$ where $\mathbf{P}_0 + \tilde{t}\mathbf{E}_0$ is the point of intersection between the edge and the plane. The parameter value is $\tilde{t} = -(\mathbf{A} \cdot \mathbf{\Delta}_0)/(\mathbf{A} \cdot \mathbf{E}_0)$. If this point is $\mathbf{V}$ and it is the only intersection of the edge with the cone, at first glance the algorithm given here does not appear to handle this case because it assumes that $Q < 0$ at the end points of the edge segment corresponding to $[0, \tilde{t}]$. It appears that $Q(\tilde{t}) = 0$ and $c_2 \ge 0$ are consistent to allow an intersection. However, the geometry of the situation indicates the line containing the edge never intersects the cone. This can only happen if $Q(t) \le 0$, so it must be the case that $c_2 < 0$ occurs. Now we analyze when Q has roots on the interval $[0, \tilde{t}]$. As before, $c_2 < 0$ is a necessary condition since $Q(0) < 0$ and $Q(\tilde{t}) < 0$. The t-value for the local maximum must be in the domain, $0 \le \hat{t} \le \tilde{t}$. To avoid the divisions, this is rewritten as $0 \le c_1$ and $c_2(\mathbf{A} \cdot \mathbf{\Delta}_0) \le c_1(\mathbf{A} \cdot \mathbf{E}_0)$. The condition that the discriminant of the quadratic be nonnegative still holds. In summary, when $\mathbf{P}_0$ is on the cone side and $\mathbf{P}_1$ is on the opposite side, the corresponding edge intersects the cone when

$$c_2 < 0 \text{ and } 0 \le c_1 \text{ and } c_2(\mathbf{A} \cdot \mathbf{\Delta}_0) \le c_1(\mathbf{A} \cdot \mathbf{E}_0) \text{ and } c_1^2 \ge c_0 c_2.$$

Finally, if $\mathbf{P}_1$ is on the cone side and $\mathbf{P}_0$ is on the opposite side, the domain for Q is reduced to $[\tilde{t}, 1]$. Once again the graph must be concave, the discriminant of the quadratic must be nonnegative, and $\hat{t} \in [\tilde{t}, 1]$. The edge intersects the cone when

$$c_2 < 0 \text{ and } c_1 \le -c_2 \text{ and } c_2(\mathbf{A} \cdot \mathbf{\Delta}_0) \le c_1(\mathbf{A} \cdot \mathbf{E}_0) \text{ and } c_1^2 \ge c_0 c_2.$$

All three edges of the triangle are tested in this manner.

If all three edges are outside the cone, it is still possible that the triangle and cone intersect. If they do, the curve of intersection is an ellipse that is interior to the triangle. Moreover, the axis of the cone must intersect the triangle at the center of that ellipse. It is enough to show this intersection occurs by computing the intersection of the cone axis with the plane of the triangle and showing that point is inside the triangle. Of course this test does not need to be applied when all three vertices are on the opposite side of the plane, another early exit since it is known by this time on which side of the plane the vertices lie.

A triangle normal is $\mathbf{N} = \mathbf{E}_0 \times \mathbf{E}_1$. The point of intersection between cone axis $\mathbf{V} + s\mathbf{A}$ and plane $\mathbf{N} \cdot (\mathbf{X} - \mathbf{P}_0) = 0$, if it exists, occurs when $s = (\mathbf{N} \cdot \mathbf{\Delta}_0)/(\mathbf{N} \cdot \mathbf{A})$. The point of intersection can be written in planar coordinates as

$$\mathbf{V} + s\mathbf{A} = \mathbf{P}_0 + t_0\mathbf{E}_0 + t_1\mathbf{E}_1$$

or

$$(\mathbf{N} \cdot \mathbf{\Delta}_0)\mathbf{A} - (\mathbf{N} \cdot \mathbf{A})\mathbf{\Delta}_0 = t_0(\mathbf{N} \cdot \mathbf{A})\mathbf{E}_0 + t_1(\mathbf{N} \cdot \mathbf{A})\mathbf{E}_1.$$

Define $\mathbf{U} = (\mathbf{N} \cdot \mathbf{\Delta}_0)\mathbf{A} - (\mathbf{N} \cdot \mathbf{A})\mathbf{\Delta}_0$. To solve for t_0, cross the equation on the right with $\mathbf{E}_1$, then dot with $\mathbf{N}$. Similarly solve for t_1 by crossing on the right with $\mathbf{E}_0$ and dotting with $\mathbf{N}$. The result is

$$t_0(\mathbf{N} \cdot \mathbf{A})|\mathbf{N}|^2 = \mathbf{N} \cdot \mathbf{U} \times \mathbf{E}_1 \text{ and } t_1(\mathbf{N} \cdot \mathbf{A})|\mathbf{N}|^2 = -\mathbf{N} \cdot \mathbf{U} \times \mathbf{E}_0.$$

To be inside the triangle it is necessary that $t_0 \geq 0$, $t_1 \geq 0$, and $t_0 + t_1 \leq 1$. The comparisons can be performed without the divisions, but require two cases depending on sign of $\mathbf{N} \cdot \mathbf{A}$. In the code, the quantities $\mathbf{N}$, $\mathbf{N} \cdot \mathbf{A}$, $\mathbf{N} \cdot \boldsymbol{\Delta}_0$, $\mathbf{U}$, and $\mathbf{N} \times \mathbf{U}$ are computed. If $\mathbf{N} \cdot \mathbf{A} \geq 0$, then the point is inside the triangle when $\mathbf{N} \times \mathbf{U} \cdot \mathbf{E}_0 \leq 0$, $\mathbf{N} \times \mathbf{U} \cdot \mathbf{E}_1 \geq 0$, and $\mathbf{N} \times \mathbf{U} \mathbf{E}_2 \leq (\mathbf{N} \cdot \mathbf{A})|\mathbf{N}|^2$. The inequalities in these three tests are reversed in the case $\mathbf{N} \cdot \mathbf{A} \leq 0$.

3 Find Intersection

The analysis in the previous section can be extended to actually partition the triangle into the component inside the cone and the component outside. The curve of separation will be a quadratic curve, possibly a line segment. If the triangle is represented as $\mathbf{X}(s,t) = \mathbf{P}_0 + s\mathbf{E}_0 + t\mathbf{E}_1$ for $s \geq 0$, $t \geq 0$, and $s + t \leq 1$, the points of intersetion of the single cone and triangle are determined by

$$\mathbf{A} \cdot (\mathbf{X}(s,t) - \mathbf{V}) \geq 0 \text{ and } (\mathbf{A} \cdot (\mathbf{X}(s,t) - \mathbf{V}))^2 - \gamma^2 |\mathbf{X}(s,t)|^2 = 0.$$

If any portion of the triangle satisifes the linear inequality, this trims down the triangle domain to a subset: the entire triangle, a subtriangle, or a subquadrilateral. On that subdomain the problem is to determine where the quadratic function is zero. Thus, the problem reduces to finding the intersection in 2D of a triangle or quadrilateral with a quadratic object. Locating the zeros amounts to actually finding the roots of $Q(t)$ for the edges of the triangle discussed in the previous section, and/or determining the ellipse of intersection if the cone passes through the triangle interior.