Distance from Point to a General Quadratic Curve or a General Quadric Surface

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This document describes an algorithm for computing the distance from a point in 2D to a general quadratic curve defined implicitly by a second-degree quadratic equation in two variables or from a point in 3D to a general quadric surface defined implicitly by a second-degree quadratic equation in three variables.

1 The Problem

The general quadratic equation is

$$Q(\mathbf{x}) = \mathbf{x}^{\mathrm{T}} A\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} + c = 0$$

where A is a symmetric $N \times N$ matrix ($N = 2$ or $N = 3$ not necessarily invertible, for example in the case of a cylinder or paraboloid), $\mathbf{b}$ is an $N \times 1$ vector, and c is a scalar. The parameter is $\mathbf{x}$, a $N \times 1$ vector. Given the surface $Q(\mathbf{x}) = 0$ and a point $\mathbf{y}$, find the distance from $\mathbf{y}$ to the surface and compute a closest point $\mathbf{x}$.

Geometrically, the closest point $\mathbf{x}$ on the surface to $\mathbf{y}$ must satisfy the condition that $\mathbf{y} - \mathbf{x}$ is normal to the surface. Since the surface gradient $\nabla Q(\mathbf{x})$ is normal to the surface, the algebraic condition for the closest point is

$$\mathbf{y} - \mathbf{x} = t\nabla Q(\mathbf{x}) = t(2A\mathbf{x} + \mathbf{b})$$

for some scalar t. Therefore,

$$\mathbf{x} = (I + 2tA)^{-1}(\mathbf{y} - t\mathbf{b})$$

where I is the identity matrix. One could replace this equation for $\mathbf{x}$ into the general quadratic equation to obtain a polynomial in t of at most sixth degree.

2 Reduction to a Polynomial Equation

Instead of immediately replacing $\mathbf{x}$ in the quadratic equation, we can reduce the problem to something simpler to code. Factor A using an eigendecomposition to obtain $A = RDR^{\mathrm{T}}$ where R is an orthonormal matrix whose columns are eigenvectors of A and where D is a diagonal matrix whose diagonal entries are the eigenvalues of A. Then

$$\begin{aligned}
\mathbf{x} &= (I + 2tA)^{-1}(\mathbf{y} - t\mathbf{b}) \\
&= (RR^{\mathrm{T}} + 2tRDR^{\mathrm{T}})^{-1}(\mathbf{y} - t\mathbf{b}) \\
&= [R(I + 2tD)R^{\mathrm{T}}]^{-1}(\mathbf{y} - t\mathbf{b}) \\
&= R(I + 2tD)^{-1}R^{\mathrm{T}}(\mathbf{y} - t\mathbf{b}) \\
&= R(I + 2tD)^{-1}(\boldsymbol{\alpha} - t\boldsymbol{\beta})
\end{aligned}$$

where the last equation defines $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Replacing in the quadratic equation and simplifying yields

$$0 = (\boldsymbol{\alpha} - t\boldsymbol{\beta})^{\mathrm{T}}(I + 2tD)^{-1}D(I + 2tD)^{-1}(\boldsymbol{\alpha} - t\boldsymbol{\beta}) + \boldsymbol{\beta}^{\mathrm{T}}(I + 2tD)^{-1}(\boldsymbol{\alpha} - t\boldsymbol{\beta}) + c.$$

The inverse diagonal matrix is

$$(I + 2tD)^{-1} = \mathrm{Diag}\{1/(1 + 2td_0), 1/(1 + 2td_1)\}.$$

for 2D or

$$(I + 2tD)^{-1} = \mathrm{Diag}\{1/(1 + 2td_0), 1/(1 + 2td_1), 1/(1 + 2td_2)\}.$$

for 3D. Multiplying through by $((1+2td_0)(1+2td_1))^2$ in 2D leads to a polynomial of at most fourth degree. Multiplying through by $((1+2td_0)(1+2td_1)(1+2td_2))^2$ in 3D leads to a polynomial equation of at most sixth degree.

The roots of the polynomial are computed and $\mathbf{x} = (I + 2tA)^{-1}(\mathbf{y} - t\mathbf{b})$ is computed for each root t. The distances between $\mathbf{x}$ and $\mathbf{y}$ are computed and the minimum distance is returned by the `PointQuadDistanceN` for $N = 2$ or $N = 3$.