Computing a Point of Reflection on a Sphere

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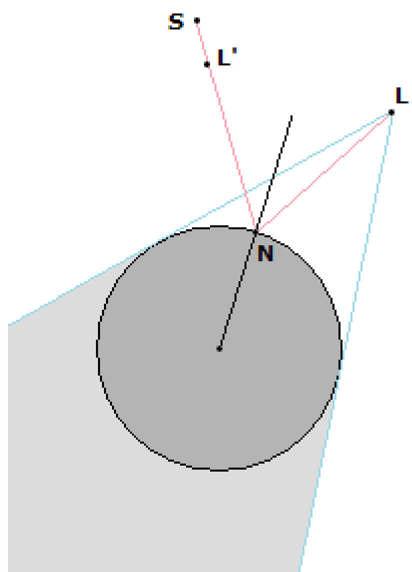
1 Introduction

A point light is located at position $\mathbf{L}$, which is more than one unit of distance from the origin. A sphere of radius 1 and centered at the origin will reflect light rays from the point light. A point $\mathbf{S}$ outside the sphere potentially receives a reflected ray of light. If it does, we wish to compute the point $\mathbf{N}$ on the sphere at which the light ray is reflected to reach the point $\mathbf{S}$.

Imagine the smallest-angled single-sided cone whose vertex is $\mathbf{L}$ and that contains the sphere. Figure 1.1 illustrates this.

Figure 1.1 The smallest-angled single-sided cone for the light position $\mathbf{L}$ and the sphere. The point $\mathbf{S}$ receives light as long as it is outside the sphere and not inside the shadowed area hidden from the light position. The point $\mathbf{L}'$ is the reflection of $\mathbf{L}$ through the ray whose origin is the sphere center and whose direction is $\mathbf{N}$.



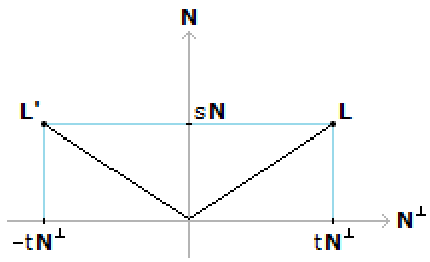
The point $\mathbf{S}$ will receive a reflected light ray as long as it is outside the sphere and not in the shadow of the sphere, as illustrated in Figure 1.1. The point $\mathbf{L}'$ is the reflection of $\mathbf{L}$ through the ray whose origin is the sphere center and whose direction is $\mathbf{N}$.

The key observation is that $\mathbf{N}$ must be chosen so that the vectors $\mathbf{S} - \mathbf{N}$ and $\mathbf{L}' - \mathbf{N}$ are parallel and in the same direction.

2 Reflection of a Vector

Figure 1.1 illustrates the reflection $\mathbf{L}'$ of the point $\mathbf{L}$ through the normal ray. We need to compute this reflection. Figure 2.1 helps to understand how to do this.

Figure 2.1 The reflection $\mathbf{L}'$ of $\mathbf{L}$ through a ray with direction $\mathbf{N}$.



The point $\mathbf{L}$ has a component in the $\mathbf{N}$ direction. After projecting out that component, the remainder is in some direction $\mathbf{N}^\perp$ that is perpendicular to $\mathbf{N}$. Thus, we may write

$$\mathbf{L} = s\mathbf{N} + t\mathbf{N}^\perp \tag{1}$$

where $s = \mathbf{N} \cdot \mathbf{L}$. The reflection through $\mathbf{N}$ amounts to changing sign on the perpendicular component,

$$\mathbf{L}' = s\mathbf{N} - t\mathbf{N}^\perp \tag{2}$$

The sum of the vectors in Equations (1) and (2) is

$$\mathbf{L} + \mathbf{L}' = 2s\mathbf{N} = 2(\mathbf{N} \cdot \mathbf{L})\mathbf{N} \tag{3}$$

We may solve for the reflected vector,

$$\mathbf{L}' = 2(\mathbf{N} \cdot \mathbf{L})\mathbf{N} - \mathbf{L} \tag{4}$$

3 Computing a Reflection Point

The assumption in this section is that $\mathbf{S}$ and $\mathbf{L}$ are not parallel vectors. For if they were, the point of reflection is $\mathbf{N} = \mathbf{L}/|\mathbf{L}|$; that is, the light is reflected in the opposite direction to reach $\mathbf{S}$. Our assumption has the consequence that $\mathbf{S} \times \mathbf{L} \neq \mathbf{0}$ (parallel vectors have a nonzero cross product).

Figure 1.1 shows that $\mathbf{N}$ must bisect the rays whose common origin is $\mathbf{N}$ and whose directions are $\mathbf{S} - \mathbf{N}$ and $\mathbf{L} - \mathbf{N}$. Therefore, we may represent

$$\mathbf{N} = x\mathbf{S} + y\mathbf{L} \tag{5}$$

for some scalars $x > 0$ and $y > 0$. Observe that

$$\mathbf{S} \times \mathbf{N} = y\mathbf{S} \times \mathbf{L}, \quad \mathbf{N} \times \mathbf{L} = x\mathbf{S} \times \mathbf{L}, \quad \mathbf{N} \cdot \mathbf{L} = x\mathbf{S} \cdot \mathbf{L} + y\mathbf{L} \cdot \mathbf{L} \tag{6}$$

Because $\mathbf{N}$ is a unit-length vector, we also know that

$$1 = \mathbf{N} \cdot \mathbf{N} = x^2\mathbf{S} \cdot \mathbf{S} + 2xy\mathbf{S} \cdot \mathbf{L} + y^2\mathbf{L} \cdot \mathbf{L} \tag{7}$$

As noted in the introduction, $\mathbf{N}$ must be chosen so that the vectors $\mathbf{S} - \mathbf{N}$ and $\mathbf{L}' - \mathbf{N}$ are parallel. Their cross product must be the zero vector,

$$\begin{aligned} \mathbf{0} &= (\mathbf{S} - \mathbf{N}) \times (\mathbf{L}' - \mathbf{N}) \\ &= (\mathbf{S} - \mathbf{N}) \times [(2\mathbf{N} \cdot \mathbf{L} - 1)\mathbf{N} - \mathbf{L}] \\ &= (2\mathbf{N} \cdot \mathbf{L} - 1)\mathbf{S} \times \mathbf{N} - \mathbf{S} \times \mathbf{L} + \mathbf{N} \times \mathbf{L} \\ &= [(2\mathbf{N} \cdot \mathbf{L} - 1)y - 1 + x]\mathbf{S} \times \mathbf{L} \\ &= [(2x\mathbf{S} \cdot \mathbf{L} + 2y\mathbf{L} \cdot \mathbf{L} - 1)y - 1 + x]\mathbf{S} \times \mathbf{L} \end{aligned} \tag{8}$$

where we have used Equations (2), (5), and (6). By assumption, $\mathbf{S} \times \mathbf{L} \neq \mathbf{0}$, so Equation (8) implies

$$(2x\mathbf{S} \cdot \mathbf{L} + 2y\mathbf{L} \cdot \mathbf{L} - 1)y - 1 + x = 0 \tag{9}$$

Equations (7) and (9) are two quadratic equations in two unknowns x and y,

$$p(x, y) = ax^2 + 2bxy + cy^2 - 1 = 0, \;\; q(x, y) = 2bxy + 2cy^2 + x - y - 1 = 0 \tag{10}$$

where $a = \mathbf{S} \cdot \mathbf{S}$, $b = \mathbf{S} \cdot \mathbf{L}$, and $c = \mathbf{L} \cdot \mathbf{L}$. We may solve $q(x, y) = 0$ for x in terms of y,

$$x = \frac{-2cy^2 + y + 1}{2by + 1} \tag{11}$$

Substituting into the equation $p(x, y) = 0$, we have

$$\frac{a(-2cy^2 + y + 1)^2 + 2by(2by + 1)(-2cy^2 + y + 1) + (cy^2 - 1)(2by + 1)^2}{(2by + 1)^2} = 0 \tag{12}$$

The numerator of Equation (12) is the quartic polynomial,

$$r(y) = 4c(ac - b^2)y^4 - 4(ac - b^2)y^3 + (a + 2b + c - 4ac)y^2 + 2(a - b)y + a - 1 = 0 \tag{13}$$

Observe that

$$ac - b^2 = (\mathbf{S} \cdot \mathbf{S})(\mathbf{L} \cdot \mathbf{L}) - (\mathbf{S} \cdot \mathbf{L})^2 = |\mathbf{S} \times \mathbf{L}|^2 \neq 0 \tag{14}$$

so the coefficient of y^4 is not zero, which means $r(y)$ really has degree 4.

Now compute the real-valued roots of $r(y) = 0$. For each root $\bar{y} > 0$, compute $\bar{x} = (-2c\bar{y}^2 + \bar{y} + 1)/(2b\bar{y} + 1)$ from Equation (11). Of all the pairs $(\bar{x}, \bar{y})$, select that pair for which $\bar{x} > 0$ and $\bar{y} > 0$. The point of reflection is $\mathbf{N} = \bar{x}\mathbf{S} + \bar{y}\mathbf{L}$.