Estimate of Ouptut Gap by non-stationary DFM

Kilian ANDRU, Pablo BARRIO

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1 Introduction

In this project, we estimate the output gap in both France and Germany by using a close method to the one introduced by Matteo Barigozzi and Matteo Luciani in "Measuring Ouput Gap in large datasets" (3) where they estimate Output Gap and Unemployment Gap through the use of a novel Dynamic Factor Model on a large non-stationary dataset. The model introduced in the paper corresponds to:

$$
\begin{aligned}
y_{i,t} &= \mathcal{D}_{i,t} + \lambda'(\mathbf{L})\mathbf{f_t} + \xi_i t \qquad &&, i = 1, ..., n, t = 1, ..., T, &(1)\\
\mathcal{D}_{i,t} &= b_{i,t-1} + \mathcal{D}_{i,t-1} + \epsilon_{i,t} \qquad &&, \epsilon_{i,t} \sim \mathcal{N}(0, \sigma^2_{\epsilon_i}) &(2)\\
b_{i,t} &= b_{i,t-1} + \eta_{i,t} \qquad &&, \eta_{i,t} \sim \mathcal{N}(0, \sigma^2_{\eta_i}) &(3)\\
\mathbf{f_t} &= \mathbf{A}(\mathbf{L})\mathbf{f_{t-1}} + \mathbf{u_t} \qquad &&, \mathbf{u_{i,t}} \sim \mathcal{N}(0, \Sigma_u) &(4)\\
\xi_{i,t} &= \rho_i \xi_{i,t-1} + e_{i,t} \qquad &&, e_{i,t} \sim \mathcal{N}(0, \Sigma_e) &(5)\\
& && &(6)
\end{aligned}
$$

Since the factors are not stationary, because data is not, they are decomposed into trends and cycles. In the paper, the factors admit a factor representation:

$$\mathbf{f_t} = \mathbf{\Psi_1}\tau_\mathbf{t} + \gamma_\mathbf{t} \tag{7}$$

The vector $\tau_\mathbf{t}$ represents the trends, and $\gamma_\mathbf{t}$ represent the stationary cycles. For this project anf or simplicity, we assume that the factors are static and we fix the number of trends to one.

2 Specifications of data

Before any estimations, we need to check the order of integration of variables. Indeed, even though we allow for non-stationarity, the model only allows for maximum integration of order 1, or I(1). Therefore, we only take the log of variables that are I(0) or I(1), and we do the first difference of the log of I(2) variables. Since the datasets we used for both countries already include the necessary transformations to obtain stationarity, we simply apply the lighter corresponding transformation to each series.

Another important specification that is worth discussing is the trend component of each series $\mathcal{D}_{i,t}$. We procede in the exact same way as Barigozzi and Luciani. There are two general modelization options for this component:

- No trend, only mean I_a: $b_{i,t} = 0 \quad , \epsilon_{i,t} = 0, \quad \eta_{i,t}$

$$\mathcal{D}_{i,t} = a_i$$

- Presence of trend and mean I_b: $b_{i,t} = b_i \neq 0 \quad , \epsilon_{i,t} = 0, \quad \eta_{i,t}$

$$\mathcal{D}_{i,t} = a_i + b_i t$$

To decide the specification of each variable we run simple regressions of each series on a vector of ones and a time vector, and we look whether they are significant or not.

There are two particular cases which are Gross Domestic Product (GDP) and Unemployment Rate (UR). Gross Domestic Income would also be a special case but it is not available in either of the datasets. Even though we do not estimate the Unemployement Gap, we still use the same modelization as in the paper in order to be consistent.

- GDP: $\mathcal{D}_{i,t} = b_{i,t-1} + D_{i,t-1}, \quad b_{i,t} = b_{i,t-1} + \eta_{i,t}, \quad \eta_{i,t} \sim \mathcal{N}(0, \sigma^2_{\eta_i})$
- UR: $\mathcal{D}_{i,t} = D_{i,t-1} + \epsilon_{i,t}$

Indeed, as in 3 we assume time varying trends or mean for these variables, so that we may capture the decline in the long-run growth, and hence, so that we may estimate the output gap.

Lastly, for the idiosyncratic errors, we decide later if they are a random walk or a white noise, by studying the estimated idiosyncratic components estimated in the initialization step before the EM Algorithm.

3 Covid-Outliers treatment

The database we are using in this project comes from the EA-MD-QD collection (M. Barigozzi and C. Lissona, [1]) which is a large set of economics and financial times series for the euro area countries (both monthly and quarterly series). We chose to focus on two countries, France and Germany, and also to use all available periods, i.e. from January 2000 to December 2023, this means that we include the Covid period.

Before estimating our model with these data, we need to carry out some data processing, using the code supplied with the databases. We first aggregate monthly variables to obtain quarterly variables by taking their average over a quarter. We then proceed to transform the data, as done in the paper [3] ; We take the logs of all variables except indexes and variables in percentage points, we only differentiate the variables which are I(2), we keep I(1) variables in levels, as we use a model for non-stationary data.

We use the EM algorithm to impute missing data and outliers (Stock and Watson, 2002 [5]; McCracken and Ng, 2016 [4]). Concerning the Covid period (Quarter 1 of 2020- Quarter 4 of 2021), we use the EM algorithm coupled to a principal component analysis to impute them (see M. Barigozzi and C. Lissona, [1]).

4 Factors Estimation

Before estimating the model, we need to choose the number of factors. As it is said on the paper, a large number of papers estimate the output gap by imposing on trend and one cycle. Since we assumed that there is only one common trend in our model, and since we do not directly estimate the cycles but rather, only take the difference between factors and the trend, we start by estimating only two common factors.

To estimate the decomposition of the data into factors and trends, we do a Quasi-Maximum Likelihood maximization through an EM algorithm. But first, we need to initialize the values of parameters, factors and trends.

4.0.1 Estimation of the factors - EM algorithm

The EM is run on standardized data. We standardize it as in the paper:

$$y_{i,t} = \begin{cases} \frac{y_{i,t}-\tilde{a}_i}{\sigma_{\Delta y_i}} & \text{,if } \in [I_b, GDP] \\ 1\frac{y_{i,t}-\bar{y_i}}{\sigma_{\Delta y_{i,t}}} & \text{,if } \in [I_a, UR] \end{cases} \tag{8}$$

Before running the EM algorithm, we initialize it by using the Principal Components on the first difference of the dataset, so that we can obtain factor loadings. Then, we use these loadings to estimate the factors on level. For this we use data . Trends are first estimated by Least Squares, and the rest of parameters, such as variances of different components, are estimated based on these initialized values.

The E-step of the algorithm consists in applying a Kalman Filter and a Kalman Smoother to get better/smoother estimates of the factors, trends and idiosyncratic errors. On the M-step of the algorithm, the parameters of the model, such as the loadings, the parameters of the law of motion of trends, the covariance of I(1) idiosyncratic terms and the covariance of the prediction error.

After running the EM algorithm we obtain consistent estimates of the factors, which we decompose into a trend and cycle in the following part.

5 Factors decomposition and Output Gap estimation

We suppose that the components of the factors F_t are driven by a single common trend $\tau_t \sim I(1)$. It is therefore possible to proceed to a trend-cycle decomposition as follows, where ω_t is a stationary process corresponding to cycles or shocks :

$$\forall t \quad \begin{cases} F_t = \Psi \tau_t + \omega_t \\ \tau_t = \tau_{t-1} + \nu_t \qquad \nu_t \sim \mathcal{N}(0, \sigma_\nu) \end{cases} \tag{9}$$

With F_t, Ψ, ω_t vectors of dimensions $q \times 1$ and τ_t, ν_t scalars.

The estimation of the trend is made through an EM algorithm, we use the following equations to run a Kalman Filter and a Kalman Smoother at each iteration of the algorithm :

$$\forall t, \quad F_t = \Psi \tau_t + \upsilon_t \qquad \upsilon_t \sim \mathcal{N}(0, \sigma_\upsilon) \qquad \textit{(Measure equation)}$$

$$\forall t, \quad \tau_t = \tau_{t-1} + \nu_t \qquad \nu_t \sim \mathcal{N}(0, \sigma_\nu) \qquad \textit{(State equation)}$$

To initialise the states and parameters, we start by performing an principal component analysis on the standardised factors to get $\tilde{\Psi}$, we then compute the estimated trend $\tilde{\tau}_t = \tilde{\Psi}' F_t$. The initial states are $\hat{\tau}_0^0 = \tilde{\tau}_0$ and $P_{0,\tau}^0 = \widehat{\mathrm{Var}}(\tilde{\tau}_t)$. The covariance of υ_t is obtained by computing $\tilde{\sigma}_\upsilon = \widehat{Cov}(F_t - \tilde{\Psi}\tilde{\tau}_t)$. Concerning ν_t, its initial variance is $\tilde{\sigma}_\nu = \widehat{\mathrm{Var}}(\tilde{\tau}_t - \tilde{\tau}_{t-1})$. We also choose to normalize Ψ such as $\Psi'\Psi = q$ the number of factors as it is equivalent to normalise τ_t with regards to T and a more common choice.

At each iteration k of the EM algorithm, we start the Expectation step by running a Kalman filter and Kalman smoother on $\hat{\tau}_0^{k-1}$ and $P_{0,\tau}^{k-1}$ to get a new estimation of the states $\hat{\tau}_{t|T}^k$ and $P_{t|T,\tau}^k$.

We then update the parameters through the Maximisation step :

$$\hat{\Psi}^k = (\sum_{t=1}^{T} F_t \hat{\tau}^k_{t|T})(\sum_{t=1}^{T}(\hat{\tau}^k_{t|T})^2 \hat{P}^k_{t|T})^{-1} \tag{10}$$

$$\hat{\sigma}^k_{\nu} = \frac{1}{T}(\sum_{t=2}^{T}(\hat{\tau}^k_{t|T})^2 \hat{P}^k_{t|T}) - \sum_{t=2}^{T} \hat{\tau}^k_{t|T}\hat{\tau}^k_{t-1|T}\hat{P}^k_{t,t-1|T})) \tag{11}$$

$$\hat{\sigma}^k_{\upsilon,j} = \frac{1}{T}\sum_{t=1}^{T}[(F_{j,t} - \hat{\Psi}^k_j \hat{\tau}^{k-1}_{t|T})^2 + \hat{\Psi}^k_j \hat{P}^{k-1}_{t|T}\hat{\Psi}^k_j] \quad , j = 1, ..., q \tag{12}$$

We stop this algorithm when the difference of log-likelihood between two iterations is less than a set threshold (10^{-2} in our estimations). As in the reference paper we suppose that υ_t and ν_t are Gaussian processes with zero-mean, but these assumptions are not necessary for the convergence of the algorithm and the consistency of the estimated parameters, as proved by Barigozzi and Luciani, 2019 (2).

Finally, we can compute the cycles by using the formula $\hat{\omega}_t = F_t - \hat{\Psi}\hat{\tau}_t$, we can then rewrite equation (1) as :

$$y_{i,t} = \mathcal{D}_{i,t} + \lambda'_i \Psi \tau_t + \lambda'_{\mathbf{i}} \omega_{\mathbf{t}} + \xi_{it} \qquad , i = 1, ..., n, t = 1, ..., T, \tag{13}$$

Let i_0 be the index of the variable corresponding to the GDO, we then have the output gap defined by $\lambda'_{\mathbf{i_0}} \omega_{\mathbf{t}}$.

6 Results

To estimate the confidence bands, we use a bootstrap procedure, which we run over 1000 iterations. The shaded areas in the following figures correspond to the 68% and 84% confidence bands.

6.1 France

In this section we present the different results obtained for the estimation of the output gap in France between 2002 and beginning of 2024.

As mentioned previously, since we assume that there is only one common trend to the whole economy, we first estimate the model with two factors. One to account for the common trend in the economy, and another to capture the nominal cycles. It is important to notice that we do not directly estimate the cycles, as is done in ([3]). The common cycles we estimate are the difference between the factors and the common trend (τ_t) multiplied by its corresponding coefficient (Ψ_j). This means that we obtain as many common cycles as factors. The study of spectral densities in the paper by Barigozzi and Luciani, suggest that there should only be one common trend that drives the long run dynamics of the economy, and two or three cycles.

We obtain the following results when estimating the model with two factors:

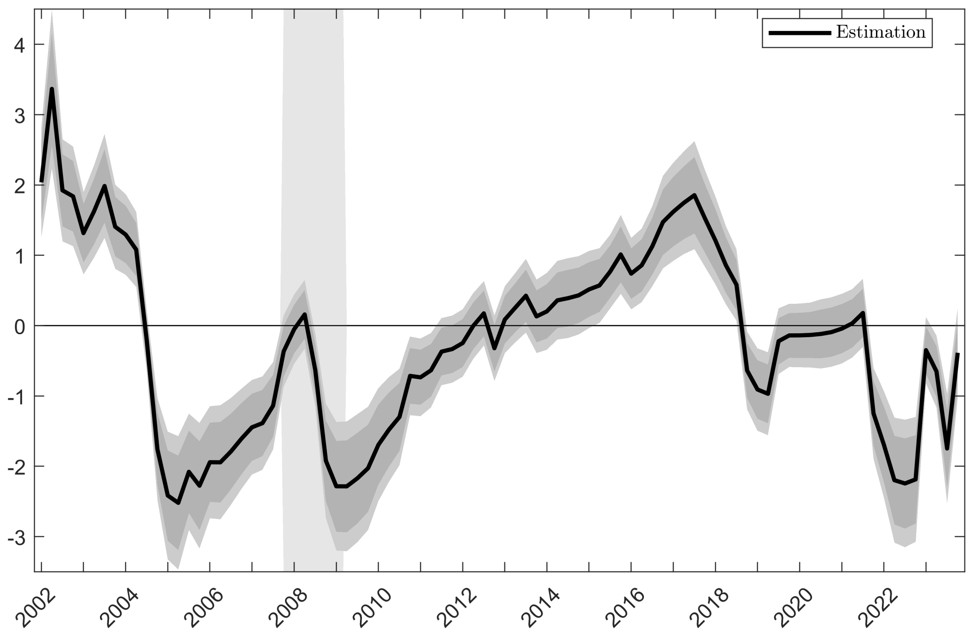


Figure 1: Estimated Output Gap in France with the two factors model

This output gap captures well the 2008 Financial crisis but is less precise when we look at the COVID-period. Indeed, during the COVID period (2020-2021), the output gap is slightly negative. We could expect it to be more decreasing during this period, but since we imputed outliers, COVID values included, before estimating the model, this is normal.

We also estimate the model with 3 and 4 factors. The plots are available in the appendix 5a, 5b. We can already observe without further study, when estimating the output gap with 3 or 4 factors, that the confidence bands of 68 % and 84 % contain always contain zero. We therefore keep the one factor model as our best estimate of the output gap. However, it could be the case that there is a mistake in the code of the boostrap method that estimates the confidence bands because we think these bands are unusual compared to the one factor case.

To further compare different results with different number of factors we look at the variance of idiosyncratic components across different variables:

	GDP	PCE	TEMP	UR	LTIRT
q=2	0.0001	0.0007	0.0	0.1458	0.0002
q = 3	0.0001	0.0008	0.0	0.0997	0.0005
q = 4	0.0001	0.0008	0.0	0.1454	0.0002

Figure 2: Table: Variance of idiosyncratic components for different number of factors q

We can observe that variances are very similar for the selected variables across models. More importantly, the model with only two factors seems to perform better overall, apart from the variance of Unemployment rate idiosyncratic component.

Therefore, it seems like the better estimator of output gap in France is the one associated to the two factors model.

6.2 Germany

In this section we present the different results obtained for the estimation of the output gap in Germany between 2002 and beginning of 2024.

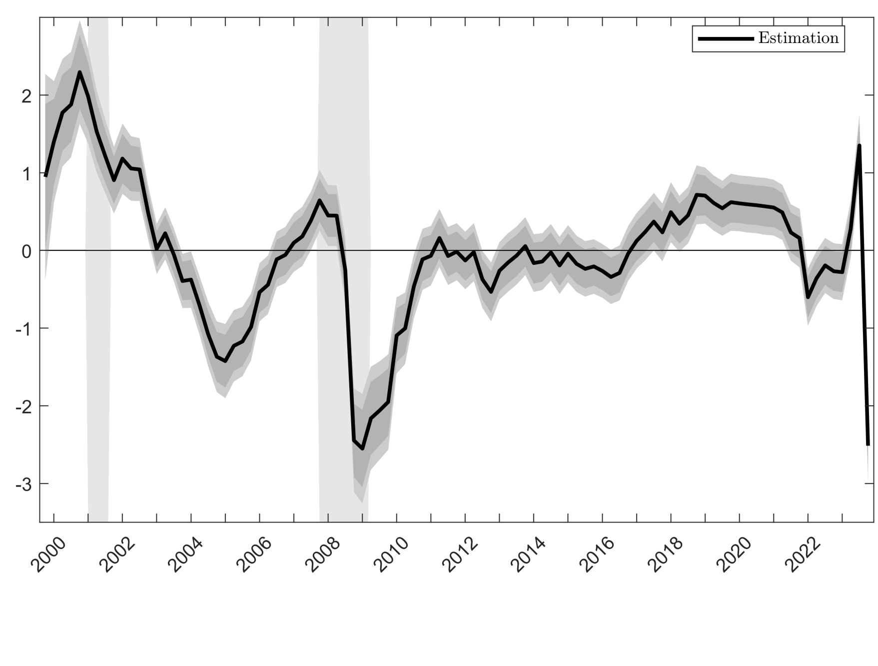


Figure 3: Estimated Output Gap in Germany with the three factors model

As said earlier, we estimated the model for only one trend and two or three cycles corresponding to the number of factors. We estimated the model for three factors and obtained the figure 3 for the output gap.

We also estimated the model for 2 and 3 factors, the plots are available in the appendix 6a, 6b. Concerning the model with only 2 factors, we can observe that the Output gap is not significantly different from zero based on the 68% and 84% confidence bands. The model for 4 factors is significantly different from zero, confidence bands are fairly narrow but the estimated output gap we obtain here is very similar to the one found with the model for 3 factors, adding an extra factor does not provide any additional information concerning cycles. It is therefore preferable to keep the model with 3 factors and thus 3 cycles.

We can also look to the variance of the idiosyncratic components across different variables for each model :

	GDP	PCE	TEMP	UR	LTIRT
q=2	0.0002	0.0002	0.0001	0.1755	0.1373
q = 3	0.0	0.0002	0.0001	0.0895	0.1357
q = 4	0.0	0.0001	0.0001	0.0518	0.1202

Figure 4: Table: Variance of idiosyncratic components for different number of factors q

We can conclude from this table, that the models with 3 and 4 factors seem to perform better that the one with only 2 factors. But we can also observe than the model with 4 factors does not appear to perform much better than the model with 3 factors. The model with a three-cycle approach therefore still seems better.

7 Conclusion

In this project, we have used a very close method to the one proposed by Barigozzi and Luciani in (3) to estimate the output gap on a large non-stationary dataset in both France and Germany. In both cases, we find results consistent with the performance of these economies in the last 20 years. To decompose the factors between trends and cycles, we used a different version than the one presented in the paper and assumed that there was only one common trend that drives long-term dynamics in the economy. To estimate this trend we used an EM-algorithm with Kalman Filter and Smoother, initialized by principal components analysis.

References

[1] Matteo Barigozzi and Claudio Lissona. Ea-md-qd: Large euro area and euro member countries datasets for macroeconomic research. *Zedono*, 2024.

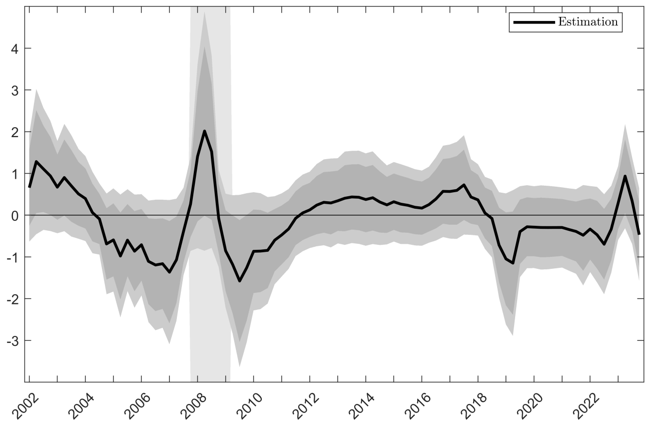
[2] Matteo Barigozzi and Matteo Luciani. Quasi maximum likelihood estimation of non-stationary large approximate dynamic factor models. 2019.

[3] Matteo Barigozzi and Matteo Luciani. Measuring the ouput gap using large datasets. *The Review of Economics and Statistics*, 2023.

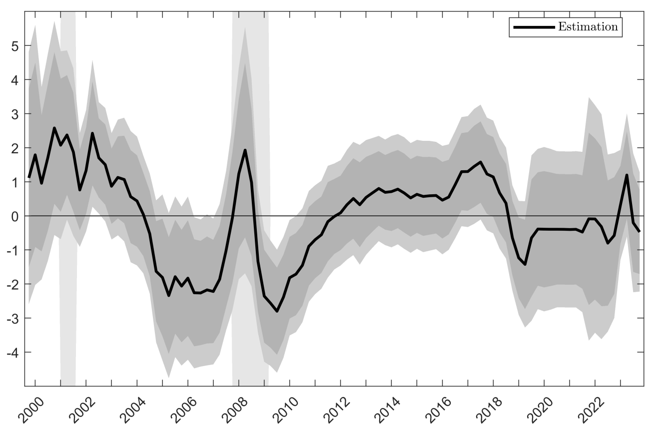
[4] Michael W. McCracken and Serena Ng. Fred-md: A monthly database for macroeconomic research. *Journal of Business and Economic Statistics*, 2016.

[5] James H. STOCK and Mark W. WATSON. Forecasting using principal components from a large number of predictors. *Journal of the American Statistical Association*, 2002.

8 Appendix

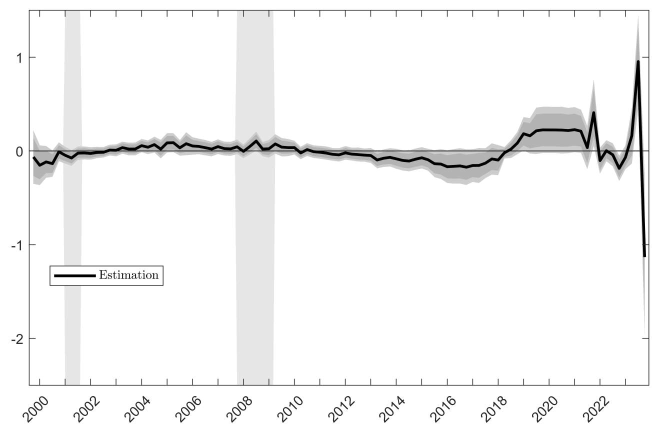


(a) Estimated Output Gap in France with the three factors model

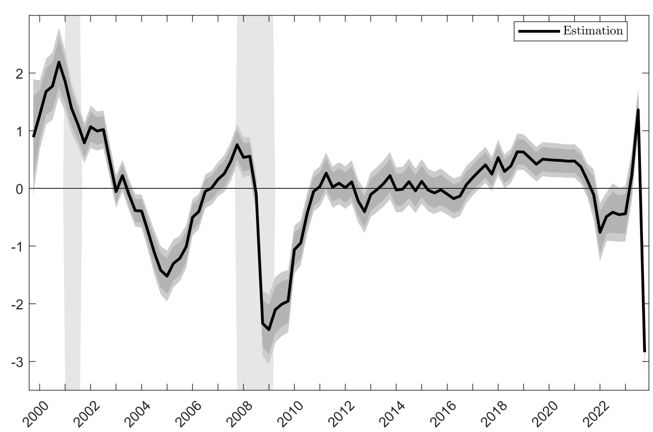


(b) Estimated Output Gap in France with the four factors model

Figure 5



(a) Estimated Output Gap in Germany with the two factors model



(b) Estimated Output Gap in Germany with the four factors model

Figure 6