Quantum chaos

V

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1 Introduction

We want to understand quantum chaos, where chaos manifest itself in the statistical properties of our system. To do this, let us consider a quantum system whose classical counterpart is chaotic: the quantum billiard.

2 Quantum rectangular billiard

Let us start from the simplest geometry: the rectangle, that is we want to solve the Helmholtz equation

$$-\nabla^2\psi = E_n\psi, \quad \nabla = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{1}$$

on a rectangular domain Ω of size $L_x \times L_y$ and Dirichlet boundary conditions $\psi|_{\partial\Omega} = 0$. We can then compare the numerical eigenvalues found with the analytical solution Fig. **??**

$$E_{n,m} = \pi^2\left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2}\right), \quad n, m \in \mathbb{N}. \tag{2}$$

To perform the numerical computation, we first discretize the domain and then we build the Hamiltonian

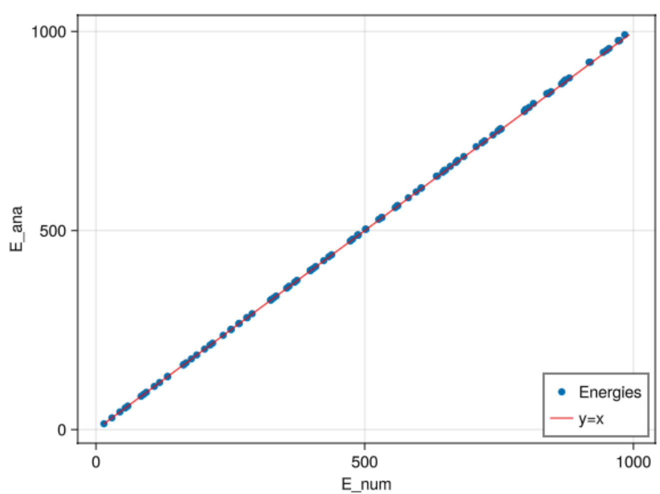

Figure 1: Comparison between numerical and analytical eigenvalues for the rectangular billiard. The red line is $y = x$.

matrix corresponding to the Laplacian operator using finite differences: we start from the $1D$ Laplacian with step h

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{3}$$

from which we can build the tridiagonal matrix

$$A = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix}. \tag{4}$$

Then we obtain the $2D$ Laplacian using the Kronecker product

$$\nabla^2 = A_x \otimes I_y + I_x \otimes A_y, \tag{5}$$

where I is the identity matrix of appropriate size. Finally, the Hamiltonian is given by $H = -\nabla^2$.