Quantum chaos

V

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1 Introduction

We want to understand quantum chaos, where chaos manifest itself in the statistical properties of our system. To do this, let us consider a quantum system whose classical counterpart is chaotic: the quantum billiard.

2 Quantum rectangular billiard

Let us start from the simplest geometry: the rectangle, that is we want to solve the Helmholtz equation

$$-\nabla^2\psi = E_n\psi, \quad \nabla = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \tag{1}$$

on a rectangular domain Ω of size $L_x \times L_y$ and Dirichlet boundary conditions $\psi|_{\partial\Omega} = 0$. We can then compare the numerical eigenvalues found with the analytical solution Fig. 1

$$E_{n,m} = \pi^2\left(\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2}\right), \quad n, m \in \mathbb{N}. \tag{2}$$

To perform the numerical computation, we first discretize the domain and then we build the Hamiltonian

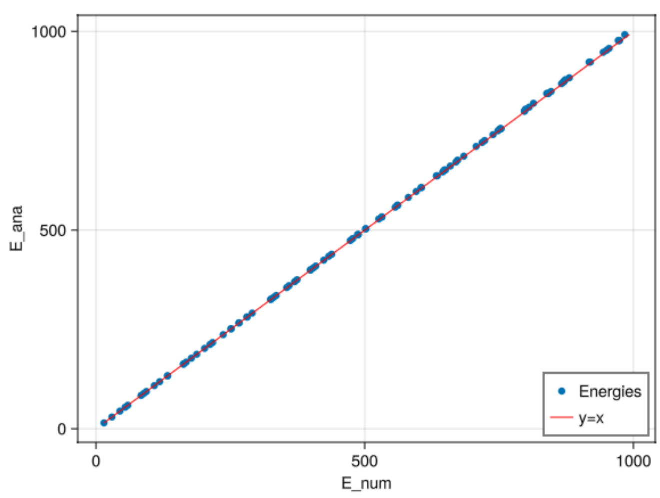


Figure 1: Comparison between numerical and analytical eigenvalues for the rectangular billiard. The red line is $y = x$.

matrix corresponding to the Laplacian operator using finite differences: we start from the $1D$ Laplacian with step h

$$u_i'' = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{3}$$

from which we can build the tridiagonal matrix

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -2 \end{pmatrix}. \tag{4}$$

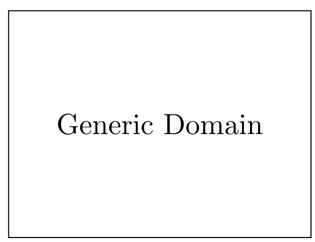
Then we obtain the $2D$ Laplacian using the Kronecker product

$$\nabla^2 = A_x \otimes I_y + I_x \otimes A_y, \tag{5}$$

where I is the identity matrix of appropriate size. Finally, the Hamiltonian is given by $H = -\nabla^2$.

3 The Expansion method

Now we want to generalize our approach to different geometries, for example the Bunimovich stadium. To do this, we will introduce a more general method, called the Expansion method (EM) [1], to construct the Hamiltonian that we want to diagonalize. The idea is to start from the rectangular billiard $L_x \times L_y$, of which we know the eigenvalues $E_{n,m}$ and the eigenfunctions $\phi_{n,m}(x)$, which circumscribe our domain Ω. The



Rectangle

$$\phi_{n,m} = \sqrt{\frac{2}{L_x}}\sin\left(\frac{\pi}{L_x}nx\right)\sqrt{\frac{2}{L_y}}\sin\left(\frac{\pi}{L_y}my\right) \tag{6}$$

are an orthonormal basis:

$$\int_{\text{Rect}} dx\, \phi_{n,m}(x)\phi_{h,k}(x) = \delta_{n,h}\delta_{m,h}, \tag{7}$$

that is we can expand any function $\psi(x)$ as

$$\psi(x) = \sum_{n,m} c_{n,m}\phi_{n,m}. \tag{8}$$

Let us now change the notation so to simplify it: instead of using two indices n, m from 1 to N, we will use a single index i from 1 to N^2. The expansion of ψ then reads

$$\psi(x) = \sum_i c_i\phi_i(x). \tag{9}$$

Suppose now $\psi(x)$ satisfies the Schrödinger equation over the domain Ω

$$H\psi = E\psi, \tag{10}$$

where the Hamiltonian H is the Laplacian plus the potential

$$V(x) = \begin{cases} 0, & x \in \Omega \\ \infty, & \text{otherwise.} \end{cases} \tag{11}$$

It is useful in this problem to slightly change the potential in the following way

$$V(x) = \begin{cases} 0, & x \in \Omega \\ V_0(\gg \max\{E_n\}), & x \in \text{Rect} \setminus \Omega \\ \infty, & \text{otherwise.} \end{cases} \tag{12}$$

The region Rect $\setminus \Omega$ will also be called region II.
Substituting (9) we get

$$\sum_{ij} \left(H_{ij} - E\delta_{ij}\right) c_j = 0, \tag{13}$$

where

$$H_{ij} = \int_{\text{Rect}} dx\, \phi_i(x) H \phi_j(x) = E_{ij}\delta_{ij} + V_0 v_{ij}, \tag{14}$$

and

$$v_{ij} = \int_{\text{II}} dx\, \phi_i(x)\phi_j(x). \tag{15}$$

So now the problem is to diagonalize the Hamiltonian.

4 The Bunimovich Stadium

Section to define the Bunimovich Stadium. Work in progress...

5 Results for the Stadium

References

[1] D. L. Kaufman, I. Kosztin, and K. Schulten, "Expansion method for stationary states of quantum billiards," *American Journal of Physics*, vol. 67, pp. 133–141, 02 1999.