Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

# Triangulation of implicit surface with singularities

Master's thesis

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# Abstrakt

TODO Abstrakt po Slovensky

**Kľúčové slová:** TODO Kľúčové slová

# Abstract

TODO Abstract in English

**Keywords:** TODO Keywords

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TODO Introduction

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# Theoretical background

## 2.1 Implicit surfaces

Implicit functions are a tool for surface representation and manipulation. In computer graphics, they can be used for modelling of complex surfaces using boolean operations, realistic animations, rendering and other.

Implicit functions do not define the boundary explicitly, instead the surface is defined as a zero set of a function.

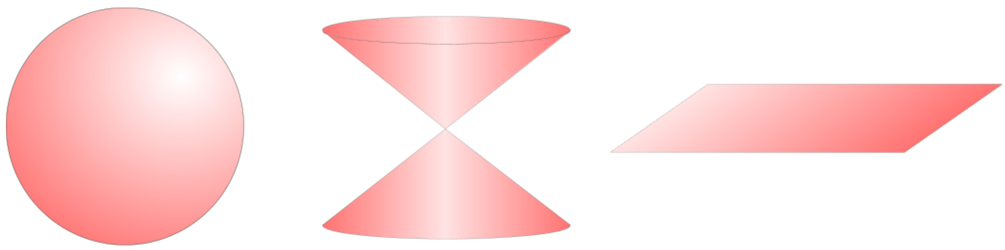
**Definition 1** *Given a function $F : \mathbb{R}^3 \to \mathbb{R}$, one can define an implicit surface as a set of points that fullfil $F(x, y, z) = 0$.*

Some examples of implicit surfaces and their equations can be seen on image 2.1.

Unit normal vector of the implicit surface in point $(x_0, y_0, z_0)$ is normalized gradient of the implicit function in that point.

**Definition 2** *Gradient vector of an implicit function $F : \mathbb{R}^3 \to \mathbb{R}$ is defined as*

$$\nabla F(x, y, z) = \left( \frac{\partial F(x,y,z)}{\partial x}, \frac{\partial F(x,y,z)}{\partial y}, \frac{\partial F(x,y,z)}{\partial z} \right).$$



$x^2 + y^2 + z^2 - 1 = 0$ | $x^2 + y^2 - z^2 = 0$ | $z = 0$

a) sphere | b) cone | c) plane

Figure 2.1: Implicit surfaces with corresponding equations.

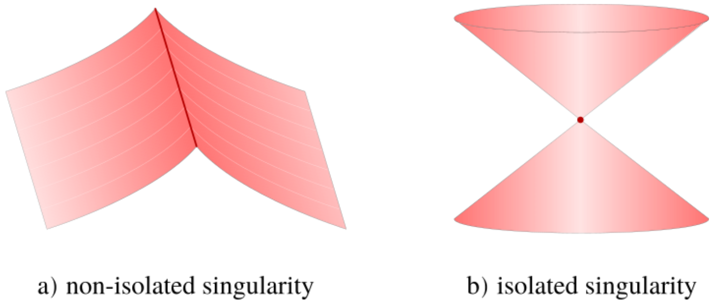



Figure 2.2: Isolated and non-isolated singularity.

*If $\nabla F(x, y, z) \neq 0$, we can define the unit normal vector of $F$ as a normalized gradient vector*

$$N(F(x, y, z)) = \frac{\nabla F(x, y, z)}{\|\nabla F(x, y, z)\|} \quad for \quad \nabla F(x, y, z) \neq 0.$$

Points lying on the implcit surface can be classified as regular or singular based on the value of the gradient vector in that point.

**Definition 3** *Point $P = (x, y, z)$ lying on the implicit surface is said to be regular, if $\nabla F(x, y, z) \neq 0$. On the contrary, point $P$ is said to be singular, if $\nabla F(x, y, z) = 0$.*

Singular points can be further classified as isolated or non-isolated based on their surroundings.

**Definition 4** *Singular point $P$ is said to be isolated, if there exists an open ball $B_\varepsilon(P)$, which does not contain any other singular point. Singular point $P$ is said to be non-isolated if it is not isolated.*

On the image 2.2 we can see example of isolated and non-isolated singularities.

## Curvature of a surface

Curvature is a fundamental concept in differential geometry of curves and surfaces. In case of curves, curvature is a measure of how much does the curve differ from a straight line. It is defined as the inverse of the radius of the osculating circle, which is the second order approximation of the curve.

For surfaces, curvature is a measure of how much does the surface differ from a plane. The definition of the curvature of a surface is not as straightforward as in the case of curves, as the curvature depends on the choice of the direction in which we measure the curvature.

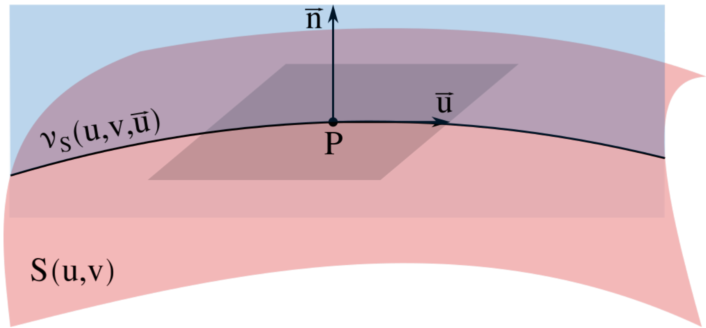



Figure 2.3: Normal cut of the parametric surface $S(u, v)$.

The idea of measuring the curvature of a surface has a long history in mathematics. One of the first contributors was a mathematician Carl Friedrich Gauss, who developed the idea of the Gaussian curvature of surfaces. In this subsection, we are drawing from the summary presented by Tiago Novello et al.[7].

### Normal curvature of the surface

Let $S$ be a parametric surface

$$S(u, v) : \mathbb{R}^2 \to \mathbb{R}^3$$

$$S(u, v) = (S_x(u, v), S_y(u, v), S_z(u, v)).$$

Let us denote the unit normal vector of the surface $S$ in the point $S(u, v)$ as $\overrightarrow{n(u, v)}$.

We define the normal curvature of the surface as a function of the location of the point on the surface given by parameters $u$ and $v$ and the unit tangent vector in that point $\overrightarrow{u}$.

**Definition 5** *Normal cut of a surface $S$ in the regular point $P$ in the direction of the unit tangent vector $\overrightarrow{u}$ is defined as an intersection of the surface $S$ and a plane given by the vectors $\overrightarrow{u}$ and $\overrightarrow{n(u, v)}$.*

The visualisation of the normal cut is shown on the image 2.3. It is clear, that the normal cut is a plane curve lying on the surface, we denote this normal cut as $\nu_S(u, v, \overrightarrow{u})$.

**Definition 6** *Oriented normal curvature of the surface in the regular point $P$ in the direction of the unit tangent vector $\overrightarrow{u}$ is defined as the curvature of the normal cut $\nu_S(u, v, \overrightarrow{u})$. Non-oriented normal curvature is defined as an absolute value of the oriented normal curvature.*

**Definition 7** *Minimal and maximal curvature in the point $P = S(u, v)$ are defined as*

$$\kappa_{min}(u, v) = \min_{\overrightarrow{u} \in T_P(u,v)} \nu_S(u, v, \overrightarrow{u}),$$

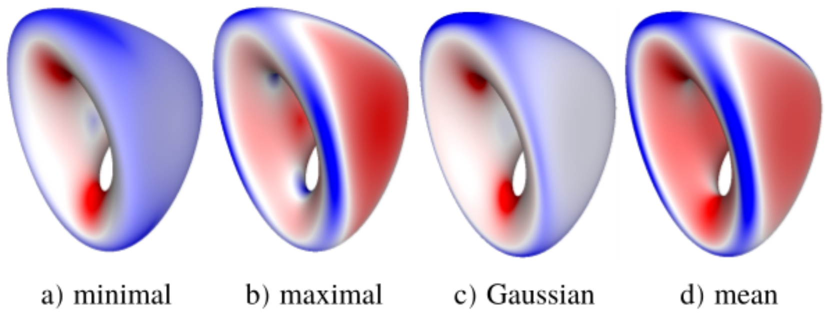



Figure 2.4: Visualisation of the curvature of the double-torus.

$$\kappa_{max}(u,v) = \max_{\vec{u} \in T_P(u,v)} \nu_S(u,v,\vec{u}),$$

*where $T_S(u,v)$ is a tangent plane of the surface $S$ in the point $P$.*

*Minimal and maximal curvature are called principal curvatures.*

**Definition 8** *Gaussian curvature is defined as a product of principal curvatures:*

$$\kappa_G(u,v) = \kappa_{min}(u,v)\kappa_{max}(u,v).$$

Gaussian curvature describes the shape of the suface in the local neighborhood of the point. The points where Gaussian curvature is positive are called eliptic points. The points where Gaussian curvature is negative are called hyperbolic points. The points where only one of $\kappa_{min}, \kappa_{max}$ is zero are called parabolic and the points where both $\kappa_{min}$ and $\kappa_{max}$ are zero are called planar. The shape of the surface in the local neigborhoods of the points is as follows:

- elliptic points $\longrightarrow$ sufrace is curved like a sphere,
- hyperbolic points $\longrightarrow$ surface is curved like a saddle,
- parabolic points $\longrightarrow$ surface is curved like a parabolic cylinder,
- planar points $\longrightarrow$ surface is flat.

Gaussian curvature is an instrinsic propery, which means that it is also independent of the placement of the surface in the space.

**Definition 9** *Mean curvature is defined as an arithmetic mean of principal curvatures:*

$$\kappa_M(u,v) = \frac{\kappa_{min}(u,v) + \kappa_{max}(u,v)}{2}.$$

Minimal, maximal, Gaussian and mean curvature are visualized on the image 2.4.

### Curvature formulas for implicit surface

A version of curvature formulas for implicit surfaces appeared in [10] and were reformulated, summarized and prooved by Ron Goldman [4]. In this subsection we point out these formulas.

Let $F : \mathbb{R}^3 \to R$ be an implicit function which defines surface by the equation $F(x, y, z) = 0$. Let us denote $F_t = \frac{\partial F}{\partial t}$ and $F_{ts} = \frac{\partial^2 F}{\partial t \partial s}$. Hessian matrix - the matrix of second derivatives is defined as

$$H(F) = \begin{pmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{pmatrix},$$

and the adjoint of the Hessian is defined as

$$H^*(F) = \begin{pmatrix} F_{yy}F_{zz} - F_{yz}F_{zy} & F_{yz}F_{zx} - F_{yx}F_{zz} & F_{yx}F_{zy} - F_{yy}F_{zx} \\ F_{xz}F_{zy} - F_{xy}F_{zz} & F_{xx}F_{zz} - F_{xz}F_{zx} & F_{xy}F_{zx} - F_{xx}F_{zy} \\ F_{xy}F_{yz} - F_{yx}F_{zy} & F_{yx}F_{xz} - F_{xx}F_{yz} & F_{xx}F_{yy} - F_{xy}F_{yx} \end{pmatrix}.$$

We can now formulate the formulas of Gaussian, mean, minimal and maximal curvature.

Gaussian curvature of the implicit surface defined by function $F$ is given by

$$\kappa_G = \frac{\nabla F * H^*(F) * \nabla F^T}{|\nabla F|^4}.$$

Mean curvature of the implicit surface defined by function $F$ is given by

$$\kappa_M = \frac{\nabla F * H^*(F) * \nabla F^T - |\nabla F|^2 Trace(H)}{2|\nabla F|^3}.$$

The principal curvatures $\kappa_{min}$ and $\kappa_{max}$ can be calculated from Gaussian curvature and mean curvature as

$$\kappa_{min}, \kappa_{max} = \kappa_M \pm \sqrt{\kappa_M^2 - \kappa_G}.$$

## 2.2 ADE singularities

ADE singularities, also reffered to as du Val singularities are a specific class of simple, isolated surface singularities. They were classified by Arnold's [1] according to ADE classification [5] based on correspondence of these singualrities to simply laced Dynkin diagrams [2]. We know infinitely many $A$ singularities – $A_1, A_2, \ldots$, infinitely many $D$ singularities – $D_4, D_5, \ldots$ and three $E$ singularities – $E_6, E_7$ and $E_8$. ADE singularities are specified by their normal forms. When working in complex space, each singularity has a single normal form:

- $A_n$ $F(x,y,z) = x^{n+1} + y^2 + z^2$,
- $D_n$ $F(x,y,z) = yx^2 + y^{n-1} + z^2$,
- $E_6$ $F(x,y,z) = x^3 + y^4 + z^2$,
- $E_7$ $F(x,y,z) = x^3 + xy^3 + z^2$,
- $E_8$ $F(x,y,z) = x^3 + y^5 + z^2$.

Each ADE singularity on a surface can be locally expressed by their normal form.

In the real case, changing the signs in these equations produces different surfaces and therefore, ADE singualrities can be further classified by their signature.

**Definition 10** *Let's mark real surface singularities based on their signature as follows:*

- $A_{n\pm\pm}$ $F(x,y,z) = x^{n+1} \pm y^2 \pm z^2$,
- $D_{n\pm\pm}$ $F(x,y,z) = yx^2 \pm y^{n-1} \pm z^2$,
- $E_{6\pm\pm}$ $F(x,y,z) = x^3 \pm y^4 \pm z^2$,
- $E_{7\pm\pm}$ $F(x,y,z) = x^3 \pm xy^3 \pm z^2$,
- $E_{8\pm\pm}$ $F(x,y,z) = x^3 \pm y^5 \pm z^2$.

The most common example of a surface with ADE singularity is an ordinary cone. Given as the zero set of the function $F(x,y,z) = x^2 - y^2 - z^2$, cone has a singular point $P = (0,0,0)$. This singular point is an example of $A_{1--}$ singularity.

## Correspondence between $SO(3,\mathbb{R})$ group and ADE singularities

$SO(3,\mathbb{R})$ is special orthogonal group over the field of real numbers in three dimensions. It is also called $3D$ rotation group, as it is a group of all rotations about the origin in $R^3$.

**Definition 11** $SO(3,\mathbb{R})$ *is a group of* $3\times 3$ *orthogonal matrices of real numbers with determinant* $1$.

$$SO(3,\mathbb{R}) = \Big\{ A \in \mathbb{R}^{3\times 3} \, | AA^T = I, \; det(A) = 1 \Big\}.$$

Simply laced Dynkin diagrams correspond to all finite subgroups of $SO(3,\mathbb{R})$. Finite subgroups of $SO(3,\mathbb{R})$ are the rotational symmetry groups of

- pyramid with $n$ vertices (cyclic subgroup $\overline{C}_n$),
- double pyramid with $n$ vertices (dihedral subgroup $\overline{D}_n$),

- platonic solids
  - tetrahedron (tetrahedral subgroup $\overline{T}$)
  - octahedron (octahedral subgroup $\overline{O}$)
  - icosahedron (icosahedral subgroup $\overline{I}$)

The correspondence is as follows:

- $A_n \iff \overline{C}_{n+1}$,
- $D_n \iff \overline{D}_{n+2}$,
- $E_6 \iff \overline{T}$,
- $E_7 \iff \overline{O}$,
- $E_8 \iff \overline{I}$.

The conclusion is that ADE singularities correspond to finite subgroups of $SO(3, \mathbb{R})$, which represent certain types of symmetries in $\mathbb{R}^3$.

## 2.3 Non-isolated translation surface singularities

## 2.4 Tringulation of regular implicit srufaces

## 2.5 Data structures for triangulation algorithm

## 2.6 CSG modelling for implcit surfaces

In this section we will discuss how constructive solid geometry can be used for modelling complex implicit surfaces. First major publication regarding constructive solid geometry was published in 1977 by Requicha and Voelcker [9]. More detailed mathematical foundations were published a year later, in 1978 by Requicha and Tilove [8].

### Constructive solid geometry (CSG)

Constructive solid geometry is a technique used for modelling complex geometric objects using boolean operations on sets of points – union, intersection and difference. It gained populatity in 1980s as a powerful tool to create complex shapes from sets of geometrical primitives, such as cylinders, spheres and cones. The resulting model can be represented as a CSG tree, which contains geometrical primitives in its leaves and boolean operations in its internal nodes [3]. An example of such CSG tree can be seen on figure 2.5.

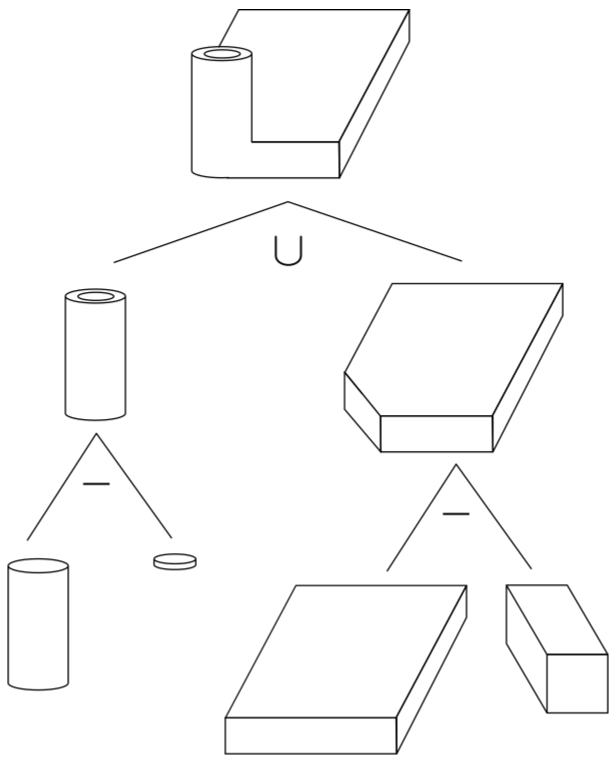



Figure 2.5: Example of CSG tree [3].

### CSG for implcit surfaces

As the implicit surface divides space into inside ($F(x, y, z) < 0$) and outside ($F(x, y, z) > 0$), one can easily combine the idea of implicit surfaces and CSG modelling. Given two implicit surfaces represented by implicit functions $F$ and $G$, the surface of the intersection of the interiors can be defined by implicit fuction $H = min(F, G)$. Similarly, the surface of the union of the interiors can be defined by implicit function $H = max(F, G)$ and lastly, the surface of the difference of the interiors can be defined by implicit function $H = max(F, -G)$. Two dimensional example of this idea can be seen on figure 2.6.

**Theorem 1** *The minimum function $min(F, G)$ of two functions $F : \mathbb{R}^3 \to \mathbb{R}$ and $G : \mathbb{R}^3 \to \mathbb{R}$ is defined as*

$$min(F, G) = F + G - \sqrt{F^2 + G^2}$$

*. The maximum function $max(F, G)$ of two functions $F : \mathbb{R}^3 \to \mathbb{R}$ and $G : \mathbb{R}^3 \to \mathbb{R}$ is defined as*

$$max(F, G) = F + G + \sqrt{F^2 + G^2}$$

.

These formulas allow us to model implicit surfaces which are the result of performing finite number of operations union, intersection and difference on arbitrary implicit functions.

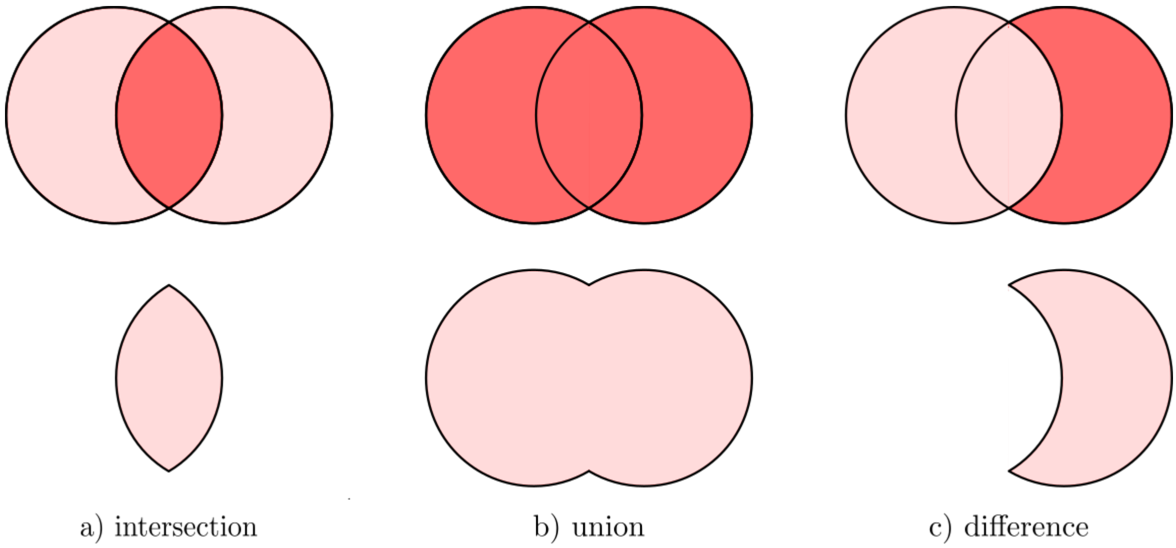



Figure 2.6: Boolean operations on implicit curves.

An easy example is creating an implicit equation representing a half of a sphere. Let us have an implicit function $F(x,y,z) = x^2 + y^2 + z^2 - 1$, which represents a unit sphere and another implicit function $G(x,y,z) = z$ which represents the $xy$ plane. The minimum function of these two functions

$$min(F,G) = x^2 + y^2 + y^2 - 1 + z - \sqrt{(x^2 + y^2 + y^2 - 1)^2 + z^2}$$

represents the surface of a half sphere. The visualisation of these surfaces can be seen on figure 2.7.

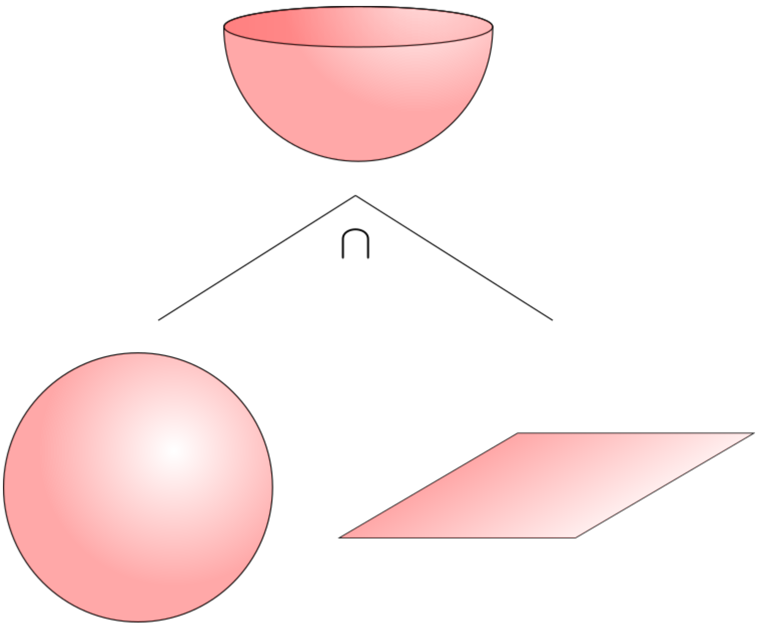



Figure 2.7: Intersection of a sphere and a plane.

# Chapter 3

# Our contributions

## 3.1 Triangulation adaptive to the local curvature

As we explained in the begining of chapeter 2.1, curvature of the surface is a measure of how much the surface bends.

The triangulation of the surface should be accurate enough, but also memory efficient. This can be achieved by creating a triangulation which is locally adaptive to the curvature of the surface. Therefore having smaller triangles in the places where the surface is curved and having bigger triangles where surface is flatter.

In this section we present our implementation of the triangulation adaptive to the local curvature.

In the original algorithm, the height of the triangle which is projected to the surface is set to the constant value $\frac{\sqrt{3}}{2}e$, where $e$ is the required length of the side of the triangle. To achieve the adaptivity of the triangles size, we set the height of the triangle to depend on the curvature in the given point, as shown in the image 3.1.

To identify the curved areas, we decided to use the maximal curvature. As the minimal and maximal curvature are both signed, we want to identify the areas depending on the absolute value of these curvatures.

We do not allow arbitrary height of the triangle to avoid edge-cases. We decided to restrict the allowed height to $\frac{1}{4}\frac{\sqrt{3}}{2}e > h > 4\frac{\sqrt{3}}{2}e$. We create a variable $m-$multiplicator, depending on $\kappa_T$ and set the height of the new triangle to $h = m\frac{\sqrt{3}}{2}e$.

As $\kappa_T$ has values in range $\langle 0, \infty \rangle$.

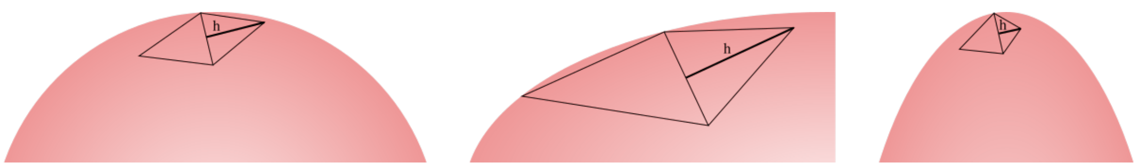



Figure 3.1: Adaptive height of the new triangle.

**Definition 12** *Let us define triangulation curvature of the surface $S$ in the point $P = S(u, v)$ as $\kappa_T(u, v) = max(|\kappa_{min}|, |\kappa_{max}|)$.*

NOT FINISHED YET

## 3.2 Triangulation of ADE singularities

### Analysis of the geometry of ADE singularities

ADE singularities are simple, isolated surface singularities, which can be expressed by corresponding implicit equations.

We already know, that $A_{1--}$ singularity is locally represented as a cone. In this section we discuss geometric structure of other ADE surface singularities.

**Definition 13** *(TODO rewrite) Let us define branch of ADE singularity as the part of the surface, which is connected to the rest only by the singular point.*

For our needs, we pick one triangulation vector for each branch of each ADE singularity. This triangulation vector is normalized vector either in the direction each ADE singularity. This triangulation vector is normalized vector either in the direction of rotation symmetry axis or an intersection of reflection symmetry planes of the corresponding branch. If the branch has only one reflection symmetry plane, the triangulation vector is picked to lie in the reflection symmetry plane. of the corresponding branch. If the branch has only one reflection symmetry plane, the triangulation vector is picked to lie in the reflection symmetry plane.

In the general case, triangulation vectors serve us as a partial information about the orientation of a singularity with respect to its normal form. In the general case, triangulation vectors serve us as a partial information about the orientation of a singularity with respect to its normal form.

### $A_n$ singularities

As we can see from the equations $F(x, y, z) = x^{n+1} \pm y^2 \pm z^2$, $A_{n-+}$ singularities are just rotated $A_{n+-}$ singularities and $A_{n++}$ singularities are a single point if $n$ is odd and reflected $A_{n--}$ singularities if $n$ is even. are a single point if $n$ is odd and reflected $A_{n--}$ singularities if $n$ is even. We therefore only discuss geometry of $A_{n--}$ and $A_{n+-}$ singularities.

$A_{n--}$ singularities are topologically equivalent to a cone if $n$ is odd, therefore they have two branches. If $n$ is even, they are topologically equivalent to a half cone or a plane, therefore they have a single branch. As $n$ gets bigger, the tip of the cone gets sharper. As $A_{n--}$ singularities are rotationally symmetrical, we pick the direction of

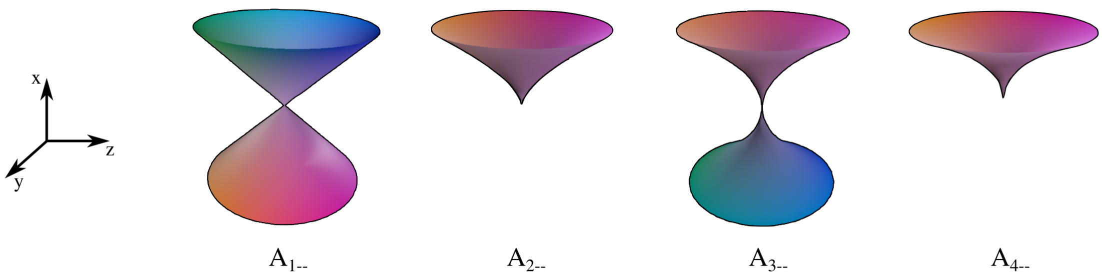



Figure 3.2: $A_{n--}$ singularities. [6]

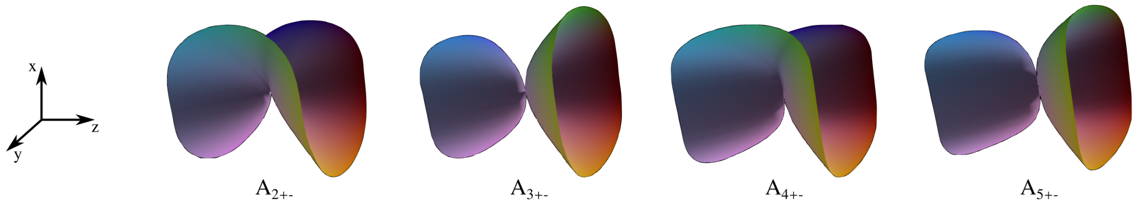



Figure 3.3: $A_{n+-}$ singularities. [6]

axis of symmetry as triangulation vector. For a normal form, the triangulation vectors are $(1,0,0)$ (and $(-1,0,0)$ if $n$ is odd). are $(1,0,0)$ (and $(-1,0,0)$ if $n$ is odd). First four $A_{n--}$ singularities can be seen on image 3.2.

$A_{n+-}$ singularities are topologically equivalent to a cone if $n$ is odd, therefore they have two branches. In the contrary with the previous singularities, as $n$ gets bigger, the tip of the cone gets less sharp and flatter. Branches of these singularities have reflection symmetry planes $x = 0$ and $y = 0$, therefore we pick the vectors $(0,0,1)$ and $(0,0,-1)$ as the triangulation vectors.

If $n$ is even, $A_{n+-}$ singularities are topologically equivalent to a plane with shape similar to hyperbolic paraboloid, therefore they have a single branch. First four $A_{n+-}$ singularities can be seen on image 3.3. For this case, we pick the vector $(1,0,0)$ as a triangulation vector as these singularities have reflection symmetry planes $y = 0$ and $z = 0$.

### $D_n$ singularities

Given by equations $F(x,y,z) = yx^2 \pm y^{n-1} \pm z^2$, we consider 8 categories. For given sign combination and parity of n, the singularities are topologically equivalent, with sharper(or flatter) features around the singularities for increasing value of $n$ similar to $A_n$ singularities.

We can therefore say that $D_n$ singularities can be classified into 8 categories locally represented by the following equations:

- $D_{4++}$ $\quad yx^2 + y^3 + z^2$

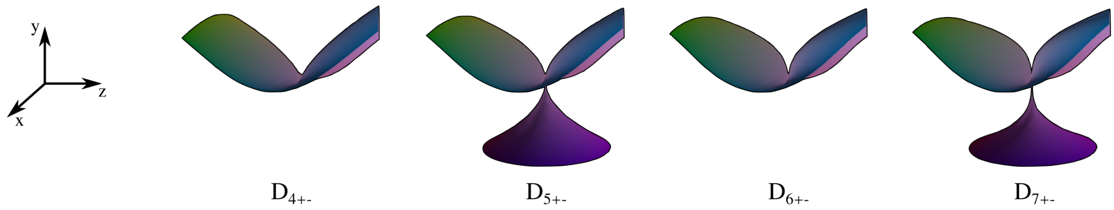



Figure 3.4: $D_{n+-}$ singularities. [6]

- $D_{5++}$ $yx^2 + y^4 + z^2$
- $D_{4+-}$ $yx^2 + y^3 - z^2$
- $D_{5+-}$ $yx^2 + y^4 - z^2$
- $D_{4-+}$ $yx^2 - y^3 + z^2$
- $D_{5-+}$ $yx^2 - y^4 + z^2$
- $D_{4--}$ $yx^2 - y^3 - z^2$
- $D_{5--}$ $yx^2 - y^4 - z^2$.

Now we look at some equivalences between these 8 categories. $D_{4++}$ singularity is reflected $D_{4+-}$ singularity. $D_{5++}$ singularity is reflected $D_{5--}$ singularity. $D_{5-+}$ singularity is reflected $D_{5+-}$ singularity. $D_{4-+}$ singularity is reflected $D_{4--}$ singularity.

We therefore only analyze geometry of $D_{n+-}$ singularities and $D_{n--}$ singularities.

$D_{n+-}$ singularities are topologically equivalent to a plane when $n$ is even and to a cone when $n$ is odd. Again, as $n$ gets bigger, the features around singularities get sharper. Symmetry planes of these singularities are $x = 0$ and $z = 0$, therefore we pick $(0, 1, 0)$ (and $(0, -1, 0)$ when $n$ is odd) as trinauglation vectors. First four $D_{n+-}$ singularities can be seen on image 3.4.

$D_{n--}$ singularities are topologically equivalent to a cone when $n$ is odd and to a 3 halfcones connected in the singular point when $n$ is even. First four $D_{n--}$ singularities can be seen on image 3.5.

Symmetry plane for all branches of these singularities is $z = 0$. the intersection of the surface and plane $z = 0$ is displayed on image 3.6.

For $D_{n--}$ singularity, the intersections of the two branches where $y \geq 0$ are bounded by curves $y = 0$ and $x^2 = y^{n-2}$. For given $r$, we pick the triangulation vectors as $(r, \frac{1}{2}r^{\frac{2}{n-2}}, 0)$ and $(-r, \frac{1}{2}r^{\frac{2}{n-2}}, 0)$. The resulting vectors are displayed on image 3.7 by blue arrow. Parameter $r$ is changed based on the length of the edge of triangulation triangle. displayed on image 3.7 by blue arrow. Parameter $r$ is changed based on the length of the edge of triangulation triangle.

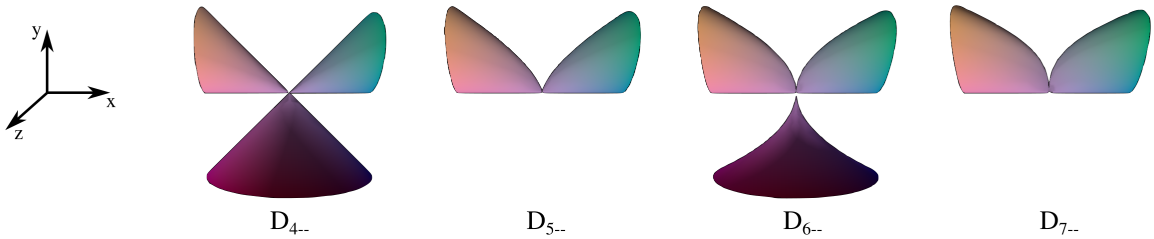



Figure 3.5: $D_{n--}$ singularities. [6]

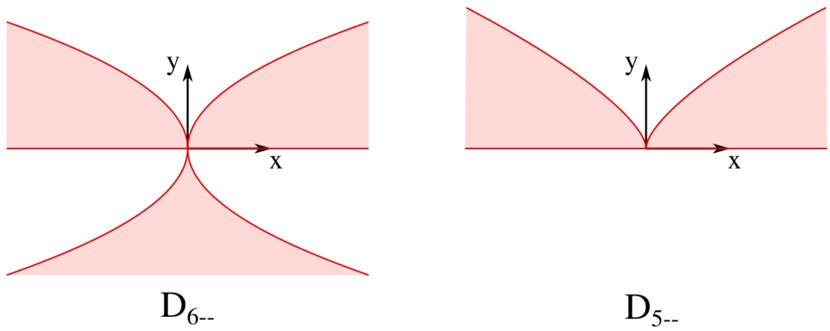



Figure 3.6: Intersection of $D_{n--}$ singularities with plane $z = 0$.

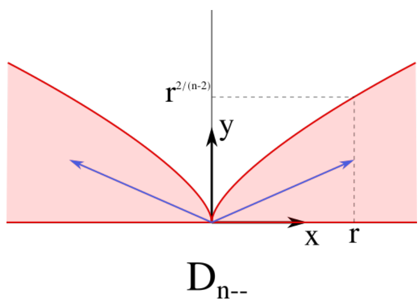



Figure 3.7: Triangulation vectors for two branches of $D_{n--}$ singularities.

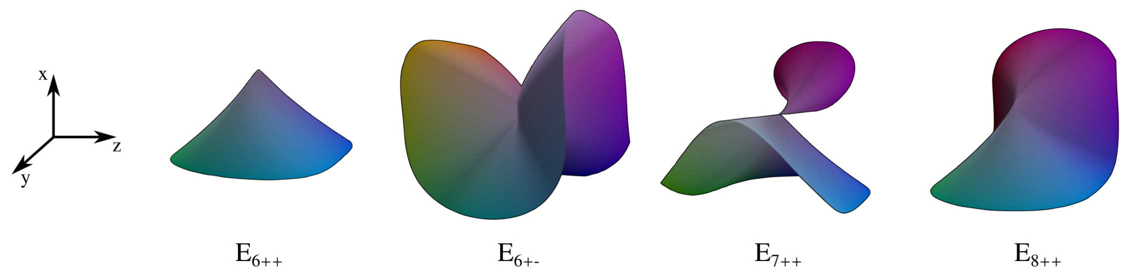



Figure 3.8: $E_n$ singularities. [6]

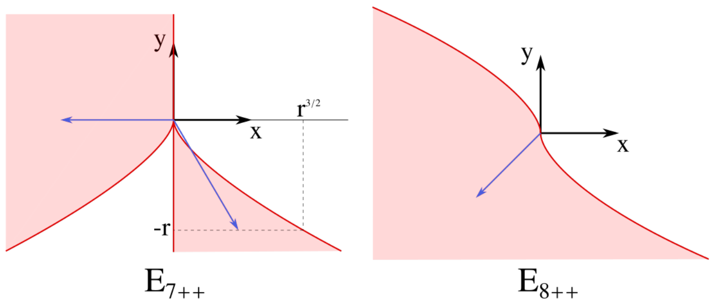



Figure 3.9: Intersection of $E_{7++}$ and $E_{8++}$ singularities with plane $z = 0$.

The third branch where $y \leq 0$ has has another plane of symmetry $x = 0$, therefore triangulation vector for this branch is chosen as $(0, -1, 0)$.

### $E_6, E_7$ and $E_8$ singularities

Given by equations $F(x, y, z) = x^3 \pm y^4 \pm z^2$, $F(x, y, z) = x^3 \pm xy^3 \pm z^2$ and $F(x, y, z) = x^3 \pm y^5 \pm z^2$, we can see the following equivalences: $E_{6++}$ singularity is reflected $E_{6--}$ singularity. $E_{6+-}$ singularity is reflected $E_{6-+}$ singularity. $E_{7+-}, E_{7-+}$ and $E_{7--}$ are all reflected $E_{7++}$ singularity. $E_{8+-}, E_{8-+}$ and $E_{8--}$ are all reflected $E_{8++}$ singularity.

We only analyze geometry of $E_{6++}$, $E_{6+-}$, $E_{7++}$ and $E_{8++}$ singularities. These singularities are displayed on the image 3.8.

Both $E_{6++}$ and $E_{6+-}$ are topologically equivalent to a plane, thus they each have only one branch. The planes of symmetry of both of these branches are $y = 0$ and $z = 0$, therefore we pick $(-1, 0, 0)$ as the triangulation vector.

$E_{7++}$ singularity is topologically equivalent to a cone, therefore it has $E_{7++}$ singularity is topologically equivalent to a cone, therefore it has two branches. The plane of symmetry of this singularity is $z = 0$.

$E_{8++}$ singularity is also topologically equivalent to a plane, therefore it has only one branch. This branch has only one plane of symmetry $z = 0$.

We again look at the intersection of the surfaces with the plane of symmetry, this is displayed on image 3.9.

For $E_{7++}$ singularity, we pick $(-1, 0, 0)$ and $(\frac{1}{2}r^{\frac{3}{2}}, -r, 0)$ as triangulation vectors.

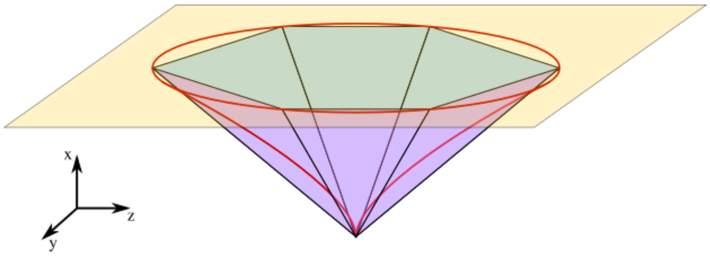



Figure 3.10: Triangulation of $A_{n--}$ singularity.

For $E_{8++}$ singularity, we pick $(-1,-1,0)$ as a triangulation vector. These vectors are displayed on the image 3.9 as blue arrows.

## Analytical calculation of local triangulation of some ADE singularities

For given edge size $e$, we want to calculate the local triangulation of ADE singularities, such that edges on the border of the local triangulation have length $e$.

### $A_{n--}$ singularities

For $A_{n--}$ singularities, we create a disc of 6 isosceles triangles with vertex in the singular point. The bases of these triangles create regular hexagon in the plane $P$ parallel to the plane $x = 0$, as showed on the image 3.10. 3.10. Given by equation $x^{n+1} - y^2 - z^2 = 0$, we find the distance of the plane $P$ from the plane $x = 0$ for the given length $e$ of the sides of the hexagon.

Let $e$ be the length of the side of the hexagon, then the circumscribed circle has radius $e$. This circle is identical with the intersection of the surface and the plane $x = h$. The equation of the intersecting circle is $y^2 + z^2 = h^{n+1}$ therefore, the radius can be also expressed as $r = h^{\frac{n+1}{2}}$, which emerges $h = e^{\frac{2}{n+1}}$. Knowing the distance of the plane, one can easily calculate the length of the arms of the triangles using Pythagorean theorem:

$$a^2 = h^2 + e^2 \implies a = \sqrt{e^{\frac{4}{n+1}} + e^2}$$

### $D_n$ singualrities

Some $D_n$ singularities have branches with elliptical intersection with a plane parallel to the plane $y = 0$. As ellipses have 2 axes of symmetry, we create 8 triangles for these branches.

Let us have an ellipse $E$ with semi-major axis $a$ and semi-minor axis $b$. We are going to create eight triangles with apex in the singular point. The other points of the triangles lie on the ellipse and they have the same length of the base.

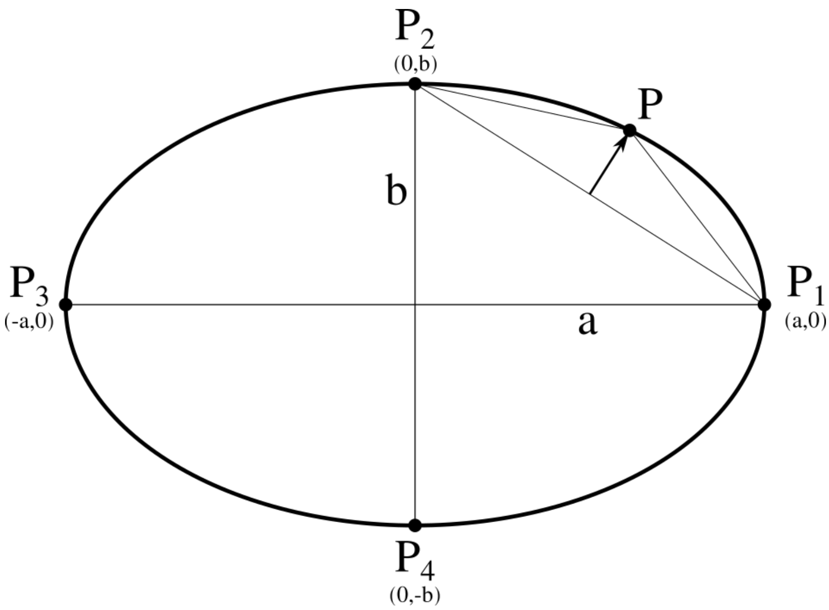



Figure 3.11: Equidistant points on ellipse.

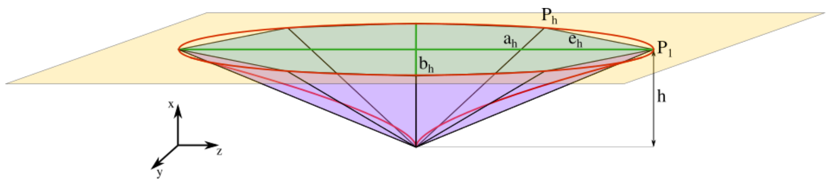



Figure 3.12: TODO.

As displayed on image 3.11, we pick the leftmpost, the rightmost, the top and the bottom points. Then we can calculate the point $P$ on ellipse equidistant from points $P_1$ and $P_2$.

$$\frac{1}{2}(a,b) + \frac{t}{2}(b,a) \in E \implies \frac{(a+tb)^2}{4a^2} + \frac{(b+ta)^2}{4b^2} = 1$$

$$4b^2(a^2 + 2atb + t^2b^2) + 4a^2(b^2 + 2atb + t^2a^2) - a^2b^2 = 0$$

$$4(b^4 + a^4)t^2 + 8ab(b^2 + a^2)t + 7a^2b^2 = 0$$

$$t = \frac{ab(\sqrt{3a^4 + 2a^2b^2 + 3b^4} - a^2 - b^2)}{a^4 + b^4}$$

$$P = \frac{1}{2}(a,b) + \frac{ab(\sqrt{3a^4 + 2a^2b^2 + 3b^4} - a^2 - b^2)}{2(a^4 + b^4)}(b,a)$$

TODO binary search for height

Given edge length $e$, we are not able to calculate the height in which the distance between points $P1$ and $P$ is $e$. The visualization showing this is on the image 3.12.

We use binary search to find such height. Given the height and the singularity class, we can calculate the semi-major axis and semi-minor axis as

$$D_{n+-} \quad : \quad -hx^2 + h^{n-1} - z^2 = 0 \quad h > 0$$
$$x^2 + \frac{z^2}{h} = h^{n-2}$$
$$\frac{x^2}{h^{n-2}} + \frac{z^2}{h^{n-1}} = 1 \implies a_h = max(h^{\frac{n-2}{2}}, h^{\frac{n-1}{2}}) \wedge b_h = min(h^{\frac{n-2}{2}}, h^{\frac{n-1}{2}}).$$

As we can see, we get the same ellipse for $D_{n--}$ singularities:

$$D_{n--} \quad : \quad -hx^2 - h^{n-1} - z^2 = 0 \quad h > 0$$
$$2|n \wedge x^2 + \frac{z^2}{h} = -h^{n-2} \implies x^2 + \frac{z^2}{h} = h^{n-2}.$$

Then

$$P_h = \frac{1}{2}(h^{\frac{n-2}{2}}, h^{\frac{n-1}{2}}) + \frac{h^{\frac{2n-3}{2}}(\sqrt{3h^{2n-4} + 2h^{2n-3} + 3h^{2n-2}} - h^{n-2} - h^{n-1})}{2(h^{2n-4} + h^{2n-2})}(h^{\frac{n-1}{2}}, h^{\frac{n-2}{2}})$$

$$P_h = \frac{1}{2}(h^{\frac{n-2}{2}}, h^{\frac{n-1}{2}}) + \frac{h^{\frac{1}{2}}(\sqrt{3 + 2h + 3h^2} - 1 - h)}{2(1 + h^2)}(h^{\frac{n-1}{2}}, h^{\frac{n-2}{2}})$$

and we can calculate $e_h = ||P_h - P_1||$.

As $e \leq a_h$, we can start the binary search on the interval $\langle 0, a^{\frac{2}{n-2}} \rangle$ or $\langle 0, a^{\frac{2}{n-1}} \rangle$ and finish, when required precision is reached.

TODO want to try to prove that $||P_h - P_1||$ is monotone in h.

#### $E_6, E_7$ and $E_8$ singualrities

## Triangulation of a plane with multiple $A_{n--}$ singularities

In this section, we present an approach for creating an implicit equation of a surface which consists of a plane and arbitrary many $A_{n--}$ singularities $C^1$ smoothly connected to this plane.

### Input and output

In this section, the following data are provided on the input:

1. the number of singularities - $m$,
2. $m$ discrete points on a plane - $(x_1, y_1), ..., (x_m, y_m)$,
3. $m$ degrees of the singularities - $n_1, ..., n_m$,
4. $m$ heights at which each singularity is connected - $h_1, ..., h_m$.

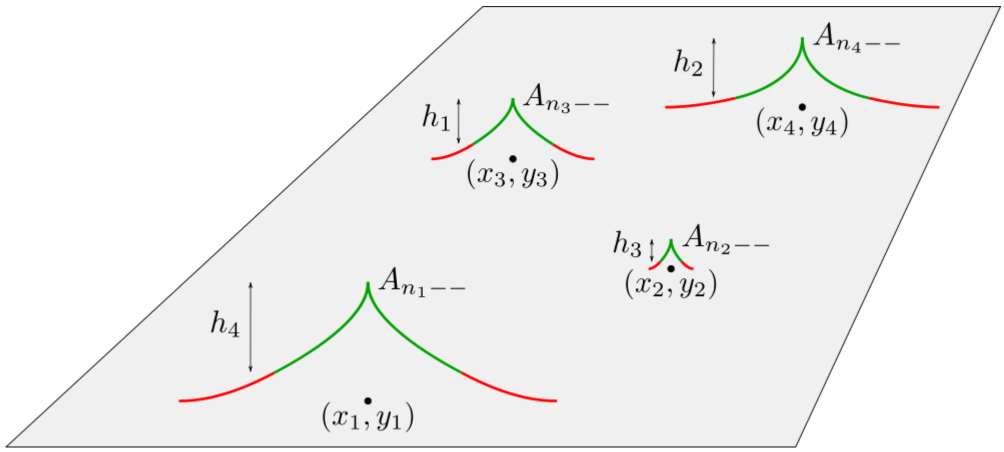



Figure 3.13: Plane with singularities.

The visualisation of desired output function can be seen on the figure 3.13. On this figure, the singularity is displayed by green color, the red color is used to display the function which connects the singularity to a plane - the bump function. There are some limitations on the input data. As we do not want the singularities or the bump functions to intsect, we require that each pair of input points is distanced $d_{ij}$ from each other. We specify the value of $d_{ij}$ in the section TODO.

### Bump function

**Definition 14** *The support supp$(f)$ of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a set of points where $f$ is not zero:*

$$supp(f) = \{x \in \mathbb{R}^n : f(x) \neq 0\}.$$

*The closed support of the function $f$ is defined as a closure of supp$(f)$.*

Bump function if a function $f : \mathbb{R}^n \to \mathbb{R}$ which is smooth ($C^\infty$) and compactly supported (the closed support of the function $f$ is a compact subset of $\mathbb{R}^n$).

The most common example of such bump function is the function

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & x \in (-1, 1) \\ 0, & otherwise, \end{cases}$$

which is both $C^\infty$ and compactly supported.

The bump function can be used to smoothly connect a curve to a line or a surface to a plane in higher dimension. If we only need to connect a curve and a line $C^n$ smoothly, we only need a bump function which connects $C^n$ smoothly to a line.

In our work, we connect two surfaces with $C^1$ continuity and for this purpose we use the function

$$f(x) = \begin{cases} -q \cdot cos(k \cdot x) - q, & x \in (-\frac{\pi}{k}, \frac{\pi}{k}) \\ 0, & otherwise, \end{cases}$$

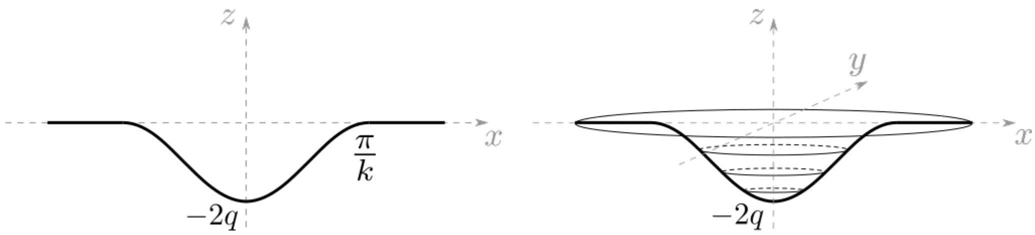



Figure 3.14: $C^1$ cosine bump function.

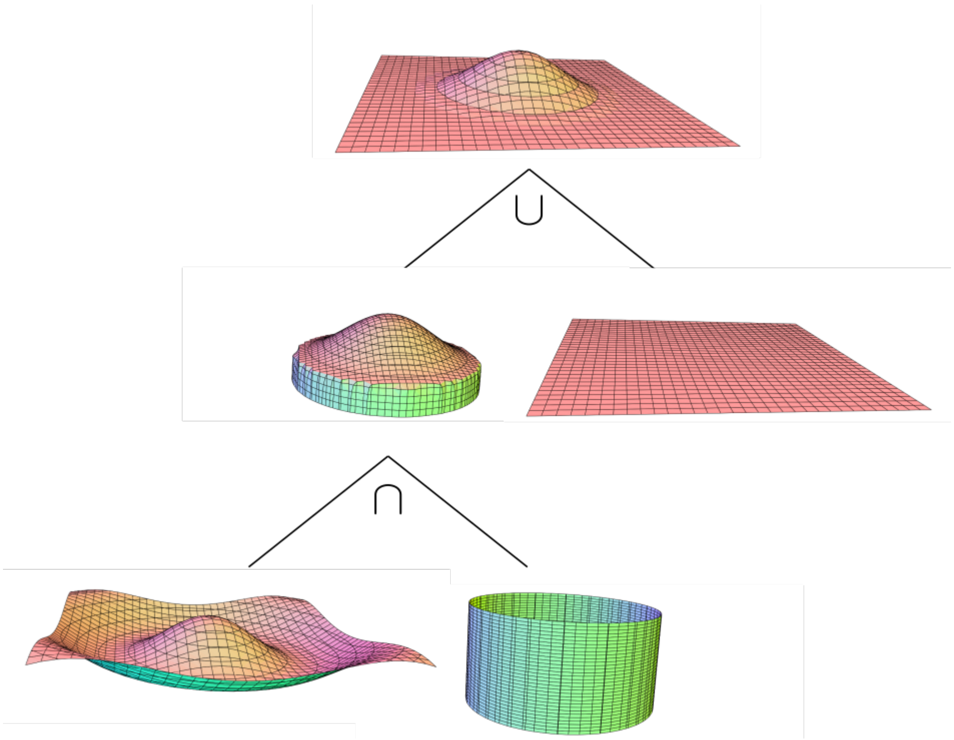



Figure 3.15: Construction of the cosine bump function using CSG.

rotated about $z$-axis. The result of the rotation is the following function:

$$f(x,y) = \begin{cases} -q \cdot cos(k \cdot (x^2+y^2)) - q, & x^2+y^2 <= (\frac{\pi}{k})^2 \\ 0, & otherwise. \end{cases}$$

Both of these functions can be seen on the figure 3.14.

### Implicit equation of the cosine bump function

To construct the implicit equation of the cosine bump function, we use CSG - constructive solid geometry, which is described in the section 2.6.

First, we cut out the part of the rotated cosine, where $x^2+y^2 < (\frac{\pi}{k})^2$ using cylinder and intersection operation. Next, we use a plane and the union operation to *glue* the bump to the plane. The described process is displayed on the figure 3.15 in the form of CSG tree. We use the following equations of the surfaces to model the cosine bump function:

| Funcion name | Implicit equation |
|---|---|
| Rotated cosine function | $x + q \cdot cos(k \cdot \sqrt{(y-p_y)^2 + (z-p_z)^2}) + q = 0$ |
| Cylinder | $(y-p_y)^2 + (z-p_z)^2 - (\frac{\pi}{k})^2 = 0$ |
| Plane | x=0 |

Table 3.1: Implicit equations for bump function modeling.

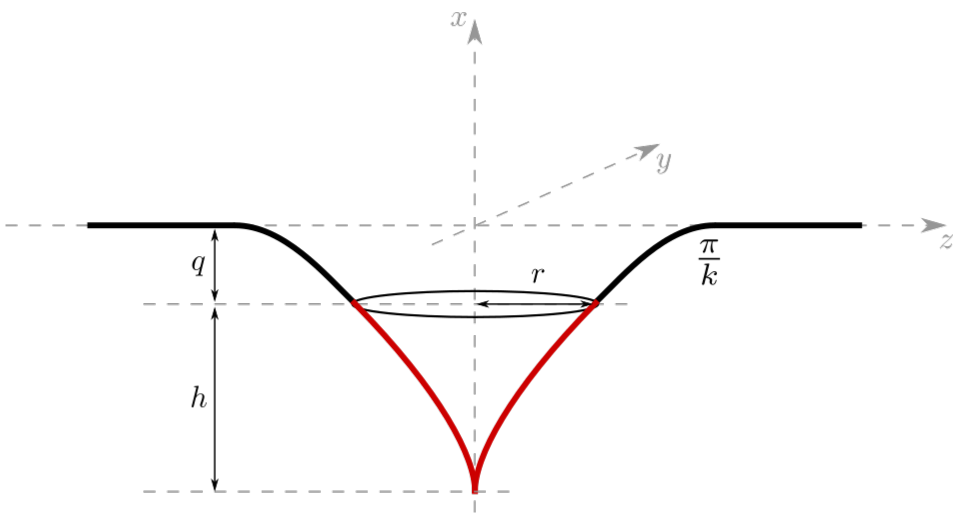



Figure 3.16: Attaching the singularity to a plane using the cosine bump function.

Parameters $q$ and $k$ allow us to change the amplitude and the frequency of the cosine function, parameters $p_y$ and $p_z$ are used to move the bump function to the given point $(p_y, p_z)$.

#### Attaching singularities to the plane using the cosine bump function

Given the type of the singularity - $n$ and given height - $h$, we calculate the constants of the cosine bump function to connect $C^1$ smoothly to the given singularity.

The singularity given by the implicit equation $x^{n+1} - y^2 - z^2$ intersected with the plane $x = h$ produces a circle with the radius $r = \sqrt{h^{n+1}}$. The cosine bump function is scaled using $q$ and $k$ to smoothly connect the singularity in the middle of the cosine bump function. This approach is displayed on the figure 3.16. As the singularity is attached in the middle of the bump function, we get the equality $r = \frac{\pi}{2k}$ and therefore $\sqrt{h^{n+1}} = \frac{\pi}{2k} \implies k = \frac{\pi}{2\sqrt{h^{n+1}}}$. The parameter $q$ is calculated from $C^1$ continuity requirement.

**Numerical calculation of local triangulation of ADE singularities**

## 3.3 Triangulation of non-isolated singularities of translation surfaces

# Chapter 4

# Results

## 4.1 Quality criteria

## 4.2 Comparison with TODO

# Chapter 5

# Future work

# Conclusion

TODO Conclusion

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# Appendix A

# Appendix B