Comenius University in Bratislava
Faculty of Mathematics, Physics and Informatics

Triangulation of implicit surface with singularities

Master's thesis

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Study Programme:	Computer Graphics and Geometry
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Department:	Department of Algebra and Geometry
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Univerzita Komenského v Bratislave
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ZADANIE ZÁVEREČNEJ PRÁCE

Meno a priezvisko študenta:	Kristína Korecová
Študijný program:	matematika (Jednoodborové štúdium, bakalársky I. st., denná forma)
Študijný odbor:	matematika
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Sekundárny jazyk:	anglický

Názov: Algoritmy triangulácie implicitne definovanej plochy
Algorithms of implicit surface triangulation

Anotácia: Práca sa zaoberá oboznámením sa s algoritmami na získanie triangulácie regulárnej implicitne definovanej plochy. Navrhuje sa postup triangulácie, otestuje sa na regulárnych plochách a v prípade rýchleho postupu aj v niektorých typoch singulárnych bodov na ploche.

Cieľ: Získanie algoritmu triangulácie pre reagulárne implicitne definované plochy v ohraničenej časti euklidovského priestoru. Otestovanie na niektorých typoch algebraických aj transcendentných plôch. Ohodnotenie aproximácie pomocou vhodných charakteristík. Príprava na adaptivitu algoritmu. Príprava na spracovanie singularít na plochách.

Literatúra: B. R. de Araújo, Daniel S. Lopes, Pauline Jepp, Joaquim A. Jorge, and Brian Wyvill. 2015. A Survey on Implicit Surface Polygonization. ACM Comput. Surv. 47, 4, Article 60 (July 2015), 39 pages. DOI:https://doi.org/10.1145/2732197

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Kľúčové slová: implicitne definovaná plocha, triangulácia, aproximácia, adaptivita

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Abstrakt

TODO Abstrakt po Slovensky

Kľúčové slová: TODO Kľúčové slová

Abstract

TODO Abstract in English

Keywords: TODO Keywords

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Theoretical background

2.1 Implicit surfaces

Implicit functions are a tool for surface representation and manipulation. In computer graphics, they can be used for modelling of complex surfaces using boolean operations, realistic animations, rendering and other.

Implicit functions do not define the boundary explicitly, instead the surface is defined as a zero set of a function.

Definition 1 *Given a function $F : \mathbb{R}^3 \to \mathbb{R}$, one can define an implicit surface as a set of points that fullfil $F(x, y, z) = 0$.*

Some examples of implicit surfaces and their equations can be seen on image 2.1.

Normal vector of the implicit surface in point (x_0, y_0, z_0) is normalized gradient of the implicit function in that point.

Definition 2 *Gradient vector of an implicit function $F : \mathbb{R}^3 \to \mathbb{R}$ is defined as*

$$\nabla F(x, y, z) = \left(\frac{\partial F(x, y, z)}{\partial x}, \frac{\partial F(x, y, z)}{\partial y}, \frac{\partial F(x, y, z)}{\partial z} \right).$$

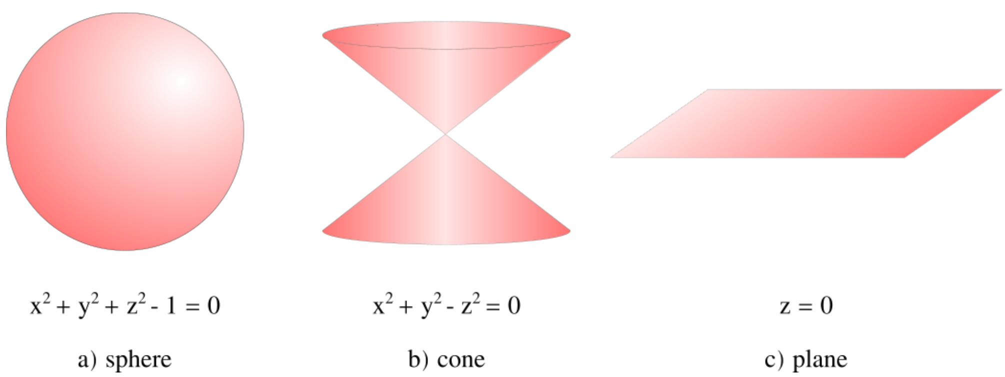

Figure 2.1: Implicit surfaces with corresponding equations.

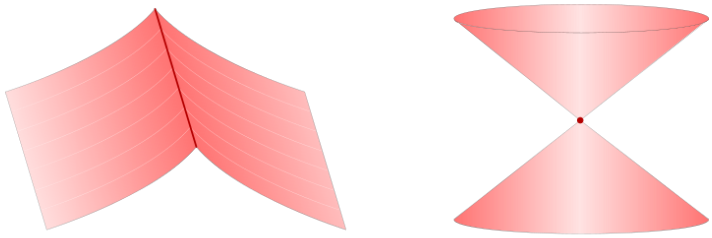

Figure 2.2: Isolated and non-isolated singularity.

If $\nabla F(x, y, z) \neq 0$, we can define normal vector of F as a normalized gradient vector

$$N(F(x, y, z)) = \frac{\nabla F(x, y, z)}{\|\nabla F(x, y, z)\|} \quad for \quad \nabla F(x, y, z) \neq 0.$$

Points lying on the implcit surface can be classified as regular or singular based on the value of the gradient vector in that point.

Definition 3 *Point $P = (x, y, z)$ lying on the implicit surface is said to be regular, if $\nabla F(x, y, z) \neq 0$. On the contrary, point P is said to be singular, if $\nabla F(x, y, z) = 0$.*

Singular points can be further classified as isolated or non-isolated based on their surroundings.

Definition 4 *Singular point P is said to be isolated, if there exists an open ball $B_\varepsilon(P)$, which does not contain any other singular point. Singular point P is said to be non-isolated if it is not isolated.*

On the image 2.2 we can see example of isolated and non-isolated singularities.

2.2 ADE singularities

ADE singularities, also reffered to as du Val singularities are a specific class of simple, isolated surface singularities. They were first TODO.

ADE classification and simply laced Dynkin diagrams

Definition 5 *[3] A vector space L over field F, with an operation $L \times L \to L$, denoted $(x, y) = [xy]$ and called the bracket or commutator of x and y, is called Lie algebra over F if the following axioms are satisfied:*

- *The bracket operation is bilinear.*
- $[xx] = 0$ *for all* x *in* L.
- $[x[yz]] + [y[zx]] + [z[xy]] = 0$ *for all* $x, y, z \in L$.

Simple Lie algebra is non-abelian Lie algebra, which contains no nonzero proper ideals.

Semisimple Lie algebra is a direct sum of simple Lie algebras.

There is a one-to-one Correspondence between Lie algebras and Lie groups.

Dynkin diagrams are graphs which classify semisimple Lie algebras (or equivalently semisimple Lie groups). Simply laced Dynkin diagrams are undirected diagrams with no multiple edges. Lie algebras which correspond to simply laced Dynkin diagrams are called simply laced Lie algebras.

ADE in ADE singularities reffers to ADE classification, which is used when some objects have a pattern that corresponds to simply laced Dynkin diagrams.

Simple Lie algebras over algebraically closed field (and their corresponding Lie groups) are classified based on their Dynkin diagrams as

- $A_n \quad n >= 1$,
- $B_n \quad n >= 2$,
- $C_n \quad n >= 3$,
- $D_n \quad n >= 4$,
- E_6, E_7, E_8, F_4, G_2.

The corresponding Dynkin diagrams can be seen on image 2.3.

Simply laced Dynkin diagrams are simple Dynkin diagrams with no directed and no multiple edges. A_n, D_n, E_6, E_7 and E_8 are therefore all simply laced Dynkin diagrams.

ADE singularities are in correspondence with simply laced Dynkin diagrams, each ADE singularity has its corresponding simply laced Dynkin diagram and equivalently, each simply laced Dynkin diagram corresponds to an ADE singularity. These singularities are denoted based on their corresponding Dynkin diagram.

The ADE surface singualrities were classified by Arnold's [2] and they are specified by their normal forms. When working in complex space, each singularity has a single normal form:

- $A_n \qquad F(x, y, z) = x^{n+1} + y^2 + z^2$,
- $D_n \qquad F(x, y, z) = yx^2 + y^{n-1} + z^2$,

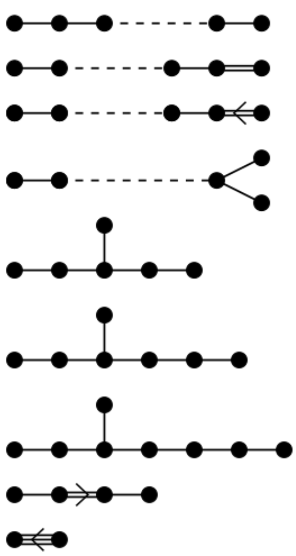

Figure 2.3: Finite Dynkin diagrams[1].

- E_6 $\quad F(x,y,z) = x^3 + y^4 + z^2$,
- E_7 $\quad F(x,y,z) = x^3 + xy^3 + z^2$,
- E_8 $\quad F(x,y,z) = x^3 + y^5 + z^2$.

Each ADE singularity on a surface can be locally expressed by their normal form.

In the real case, changing the signs in these equations produces different surfaces and therefore, ADE singualrities can be further classified by their signature.

Definition 6 *Let's mark real surface singularities based on their signature as follows:*

- $A_{n\pm\pm}$ $\quad F(x,y,z) = x^{n+1} \pm y^2 \pm z^2$,
- $D_{n\pm\pm}$ $\quad F(x,y,z) = yx^2 \pm y^{n-1} \pm z^2$,
- $E_{6\pm\pm}$ $\quad F(x,y,z) = x^3 \pm y^4 \pm z^2$,
- $E_{7\pm\pm}$ $\quad F(x,y,z) = x^3 \pm xy^3 \pm z^2$,
- $E_{8\pm\pm}$ $\quad F(x,y,z) = x^3 \pm y^5 \pm z^2$.

The most common example of a surface with ADE singularity is an ordinary cone. Given as the zero set of the function $F(x,y,z) = x^2 - y^2 - z^2$, cone has a singular point $P = (0,0,0)$. This singular point is an example of A_{1--} singularity.

Correspondence between $SL(2, \mathbb{C})$ group and ADE singularities

$SL(2, \mathbb{C})$ is special linear group of degree two over complex numbers.

Definition 7 *$SL(2, \mathbb{C})$ is a group of unimodular (determinant is 1) matrices of complex numbers.*

$$SL(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{C}, ad - bc = 1 \right\}.$$

Simply laced Dynkin diagrams correspond to all finite subgroups of $S0(3, \mathbb{R})$. Finite subgroups of $SO(3, \mathbb{R})$ are the rotational symmetry groups of

- pyramid with n vertices (cyclic subgroup $\overline{C}_n$),
- double pyramid with n vertices (dihedral subgroup $\overline{D}_n$),
- platonic solids
  - tetrahedron (tetrahedral subgroup $\overline{T}$)
  - octagedron (octahedral subgroup $\overline{O}$)
  - icosahedron (icosahedral subgroup $\overline{I}$)

These correspond to simply laced Dynkin diagrams:

- $A_n \iff \overline{C}_{n+1}$,
- $D_n \iff \overline{D}_{n+2}$,
- $E_6 \iff \overline{T}$,
- $E_7 \iff \overline{O}$,
- $E_8 \iff \overline{I}$.

The conclusion is that ADE singularities correspond to finite subgroups of $SO(3, \mathbb{R})$, which represent certain types of symmetries in $\mathbb{R}^3$.

2.3 Non-isolated translation surface singularities

2.4 Non-isolated translation surface singularities

2.5 Tringulation of regular implicit srufaces

2.6 Data structures for triangulation algorithm

Chapter 3

Our contributions

3.1 Triangulation adaptive to the local curvature

3.2 Triangulation of ADE singularities

Analysis of the geometry of ADE singularities

ADE singularities are simple, isolated surface singularities, which can be expressed by corresponding implicit equations.

We already know, that A_{1--} singularity is locally represented as a cone. In this section we will discuss geometric structure of other ADE surface singularities.

Definition 8 *TODO rewrite Let's define branch of ADE singularity as the part of surface, which is connected to the rest only by the singular point.*

For our needs, we will pick one triangulation vector for each branch of each ADE singularity. This trinauglation vector is either in the direction of rotation symmetry axis or an intersection of reflection symmetry planes of the corresponding branch.

In the general case, triangulation vectors will serve as an orientation of a singularity with respect to its normal form.

A_n singularities

As we can see from the equations $F(x, y, z) = x^{n+1} \pm y^2 \pm z^2$, A_{n-+} singularities are just rotated A_{n+-} singularities and A_{n++} singularities are a single point if n is odd and reflected A_{n--} singularities if n is even. We will therefore only discuss geometry of A_{n--} and A_{n+-} singularities.

A_{n--} singularities are topologically equivalent to a cone if n is odd, therefore they have two branches. If n is even, they are topologically equivalent to a half cone or a plane, therefore they have a single branch. As n gets bigger, the tip of the cone gets

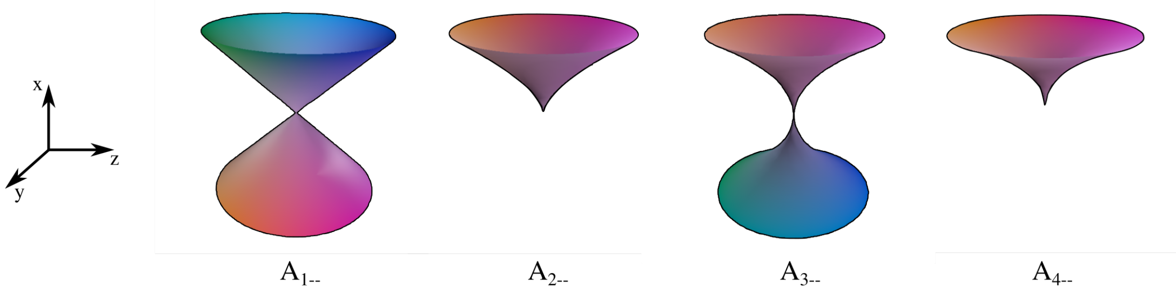

Figure 3.1: A_{n--} singularities. [4]

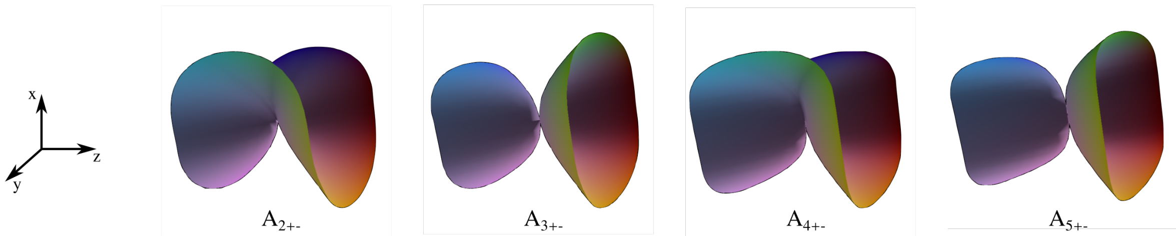

Figure 3.2: A_{n+-} singularities. [4]

sharper. As A_{n--} singularities are rotationally symmetrical, we will pick the direction of axis of symmetry as triangulation vector. For a normal form, the triangulation vectors are $(1, 0, 0)$ (and $(-1, 0, 0)$ if n is odd). First four A_{n--} singularities can be seen on image 3.1.

A_{n+-} singularities are topologically equivalent to a cone if n is odd, therefore they have two branches. In the contrary with the previous singularities, as n gets bigger, the tip of the cone gets less sharp and flatter. Branches of these singualrities have reflection symmetry planes $x = 0$ and $y = 0$, therefore we will pick the vectors $(0, 0, 1)$ and $(0, 0, -1)$ as the triangulation vectors.

If n is even, A_{n+-} singualrities are topologically equivalent to a plane with shape similar to hyperbolic paraboloid, therefore they have a single branch. First four A_{n+-} singularities can be seen on image 3.2. For this case, we will pick the vector $(1, 0, 0)$ as a triangulation vector as these singularities have reflection symmetry planes $y = 0$ and $z = 0$.

D_n singularities

Given by equations $F(x, y, z) = yx^2 \pm y^{n-1} \pm z^2$, we will consider 8 categories. For given sign combination and parity of n, the singularities are topologically equivalent, with sharper(or flatter) features around the singualrities for increasing value of n similar to A_n singualrities.

We can therefore say that D_n singularities can be classified into 8 categories represented by the following singularities:

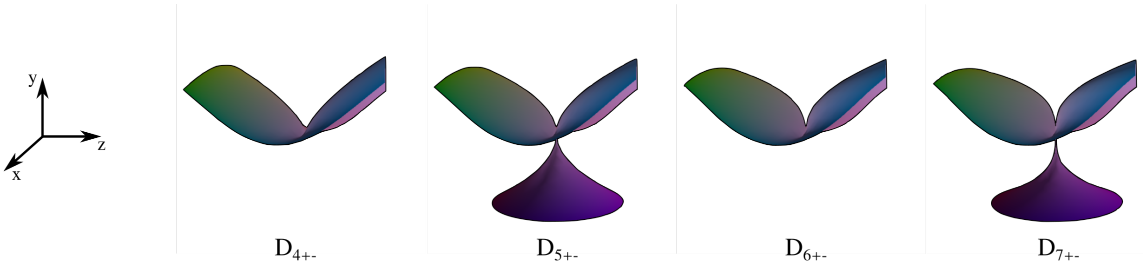

Figure 3.3: D_{n+-} singularities. [4]

- D_{4++} $yx^2 + y^3 + z^2$
- D_{5++} $yx^2 + y^4 + z^2$
- D_{4+-} $yx^2 + y^3 - z^2$
- D_{5+-} $yx^2 + y^4 - z^2$
- D_{4-+} $yx^2 - y^3 + z^2$
- D_{5-+} $yx^2 - y^4 + z^2$
- D_{4--} $yx^2 - y^3 - z^2$
- D_{5--} $yx^2 - y^4 - z^2$.

Now we will look at some equivalences between these 8 categories. D_{4++} singularity is reflected D_{4+-} singularity. D_{5++} singularity is reflected D_{5--} singularity. D_{5-+} singularity is reflected D_{5+-} singularity. D_{4-+} singularity is reflected D_{4--} singularity.

We will therefore only analyze geometry of D_{n+-} singualrities and D_{n--} singularities.

D_{n+-} singularities are topologically equivalent to a plane when n is even and to a cone when n is odd. Again, as n gets bigger, the features around singularities get sharper. Symmetry planes of these singularities are $x = 0$ and $z = 0$, therefore we pick $(0, 1, 0)$ (and $(0, -1, 0)$ when n is odd) as trinauglation vectors. First four D_{n+-} singularities can be seen on image 3.3.

D_{n--} singularities are topologically equivalent to a cone when n is odd and to a 3 halfcones connected in the singular point when n is even. First four D_{n--} singularities can be seen on image 3.4.

Symmetry plane for all branches of these singularities is $z = 0$. the intersection of the surface and plane $z = 0$ is displayed on image 3.5.

For D_{n--} singularity, the intersections of the two branches where $y \geq 0$ are bounded by curves $y = 0$ and $x^2 = y^{n-2}$. For given r, we will pick the triangulation vectors as $(r, \frac{1}{2}r^{\frac{2}{n-2}}, 0)$ and $(-r, \frac{1}{2}r^{\frac{2}{n-2}}, 0)$. The resulting vectors are displayed on image 3.6 by blue arrow.

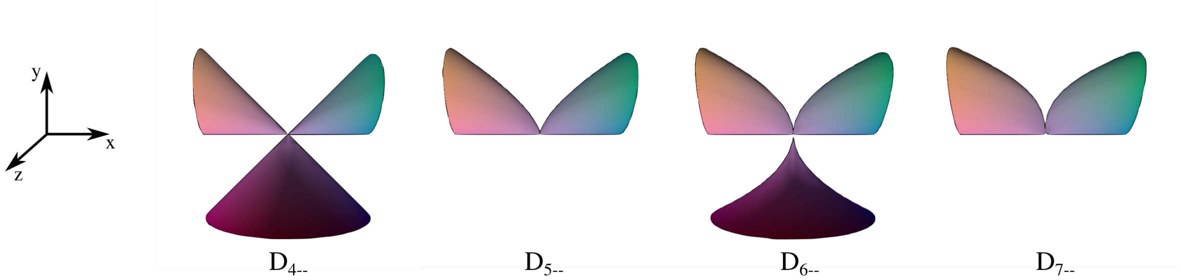

Figure 3.4: D_{n--} singularities. [4]

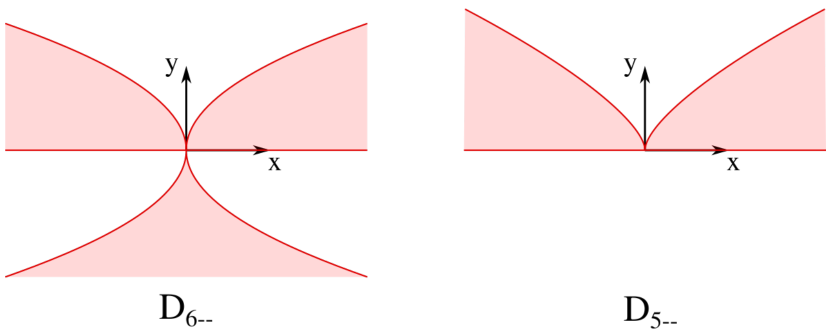

Figure 3.5: Intersection of D_{n--} singularities with plane $z = 0$.

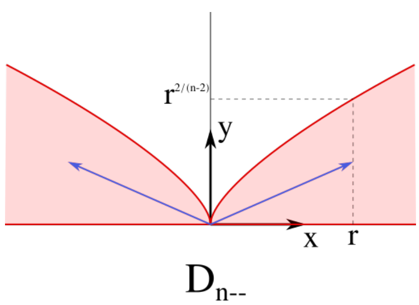

Figure 3.6: Triangulation vectors for two branches of D_{n--} singularities.

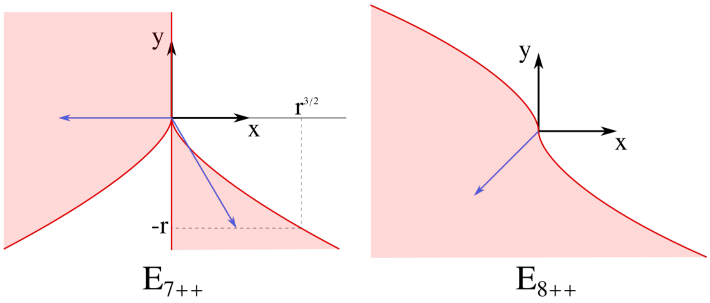

Figure 3.7: Intersection of E_{7++} and E_{8++} singularities with plane $z = 0$.

The third branch where $y \leq 0$ has has another plane of symmetry $x = 0$, therefore triangulation vector for this branch is chosen as $(0, -1, 0)$.

E_6, E_7 and E_8 singularities

Given by equations $F(x, y, z) = x^3 \pm y^4 \pm z^2$, $F(x, y, z) = x^3 \pm xy^3 \pm z^2$ and $F(x, y, z) = x^3 \pm y^5 \pm z^2$, we can see the following equivalences: E_{6++} singularity is reflected E_{6--} singularity. E_{6+-} singularity is reflected E_{6-+} singularity. E_{7+-}, E_{7-+} and E_{7--} are all reflected E_{7++} singularity. E_{8+-}, E_{8-+} and E_{8--} are all reflected E_{8++} singularity.

We will only analyze geometry of E_{6++}, E_{6+-}, E_{7++} and E_{8++} singularities.

Both E_{6++} and E_{6+-} are topologically equivalent to a plane, thus they each have only one branch. The planes of symmetry of both of these branches are $y = 0$ and $z = 0$, therefore we pick $(-1, 0, 0)$ as the triangulation vector.

E_{7++} singularity is topologically equival to a cone, therefore it has two branches. The plane of symmetry of this singularity is $z = 0$.

E_{8++} singularity is also topologically equivalent to a plane, therefore it has only one branch. This branch has only one plane of symmetry $z = 0$.

We will again look at the intersection of the surfaces with the plane of symmetry, this is displayed on image 3.7.

For E_{7++} singularity, we will pick $(-1, 0, 0)$ and $(\frac{1}{2}r^{\frac{3}{2}}, -r, 0)$ as triangulation vectors. For E_{8++} singularity, we will pick $(-1, -1, 0)$ as a triangulation vector. These vectors are displayed on the image 3.7 as blue arrows.

Analytical calculation of local triangulation of some ADE singularities

For given edge size e, we want to calculate the local triangulation of ADE singularities, such that edges on the border of the local triangulation have length e.

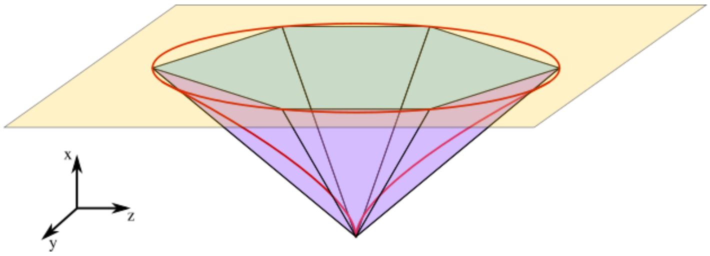

Figure 3.8: Triangulation of A_{n--} singularity.

A_{n--} singualrities

For A_{n--} singularities, we create a disc of 6 isosceles triangles with vertex in the singular point. The bases of these triangles create regular hexagon in the plane P parallel to the plane $x = 0$, as showed on the image 3.8 Given by equation $x^{n+1} - y^2 - z^2 = 0$, we will find the distance of the plane P from the plane $x = 0$ for the given length e of the sides of the hexagon.

Let e be the length of the side of the hexagon, then the circumscribed circle has radius e. This circle is identical with the intersection of the surface and the plane $x = h$. The equation of the intersecting circle is $y^2 + z^2 = h^{n+1}$ therefore, the radius can be also expressed as $r = h^{\frac{n+1}{2}}$, which emerges $h = e^{\frac{2}{n+1}}$. Knowing the distance of the plane, one can easily calculate the length of the arms of the triangles using Pythagorean theorem:

$$a^2 = h^2 + e^2 \implies a = \sqrt{e^{\frac{4}{n+1}} + e^2}$$

D_n singualrities

E_6, E_7 and E_8 singualrities

Numerical calculation of local triangulation of ADE singularities

3.3 Triangulation of non-isolated singularities of translation surfaces

Chapter 4

Results

4.1 Quality criteria

4.2 Comparison with TODO

Chapter 5

Future work

Conclusion

TODO Conclusion

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Appendix A

Appendix B