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On Some Approximate and Exact Solutions of Boundary Value Problems for Burgers' Equation

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1. INTRODUCTION

In the last few years, the nonlinear partial differential equation,

$$v_t + vv_x = \delta v_{xx}, \tag{1}$$

known as Burgers' equation, received considerable attention ([1], [2], [3], [4], [5], [6]), both because of its statistical properties ([1], [5], [6]) and because of the role that it occupies in the hierarchy of approximations emanating from the Navier-Stokes equations [4]. In particular, as a descriptor of appropriately restricted one-dimensional gas dynamic [4] and acoustical ([7], [8]) phenomena, it was shown on several occasions to be a very successful model equation. Its main "claim to fame," however, is that it belongs to that rather small [19] class of physically significant nonlinear partial differential equations, for which exact and complete solutions are known, in terms of the *initial values* ([2], [3]).

More precisely, the situation is the following. The Hopf transformation [2] carries Eq. (1) into the linear heat equation. Furthermore, as a *completely accidental feature* of this transformation, initial values prescribed for (1) are transformed in a very simple manner to initial values for the heat equation. The same is not true, however, concerning boundary values. As a result, problems yielding unique solutions under initial conditions alone (e.g., for $-\infty < x < \infty$) have been analyzed and utilized extensively; while others (e.g., $0 < x < \infty$, $a < x < b$) have been neglected almost completely. A notable exception is the paper by Kochina [18]: she discusses Eq. (1) for the ray $0 \leqslant x < \infty$.

From a physical point of view, problems for the halfplane $0 < x < \infty$

or the strip $a < x < b$ are rather important. In order to solve them, however, one has to resort to numerical techniques in almost every case, and lose thereby the qualitatively very important information that is available from a closed form solution.

Our aim here, therefore, is to present a technique for the closed form solution of (1) in the halfplane $0 < x < \infty$, by "sacrificing" a part of the information concerning the initial distribution of values, but retaining the exact boundary condition. We shall also discuss the connection of our results with some others, and obtain a solution for (1) under periodic boundary conditions.

2. Physical Interpretation

In order to keep our presentation as general as possible, while not quite discussing matters *in abstracto*, let us define the symbols in (1) in a somewhat general manner. Thus, we shall take the gas dynamic point of view and let v be the constant multiple of one of the significant flow quantities; t is a time-like, while x a space-like variable. δ, a positive constant, is the diffusion number. A particular example would be the treatment by Lighthill [4] of one-dimensional waves of finite amplitude. He interprets these quantities in the following way:

v = excess wavelet velocity;

t = time;

$x = X - ct$ (moving frame of reference; c = ambient sound speed);

$$\delta = \frac{\nu}{2}\left[2 + \frac{\eta'}{\eta} + \frac{\gamma - 1}{\mathrm{Pr}}\right];$$

ν = kinematic viscosity;

η, η' = shear and dilational viscosity numbers;

Pr = Prandtl number;

γ = ratio of specific heats.

An easy way to see the physical balance, as described by (1), is given by Cole [3]. He shows that multiplication of (1) by v and a subsequent integration with respect to x yields the following energy relation: the total rate of change of kinetic energy in the system, together with the net flow of this energy out across the boundaries and with the total dissipation present, is exactly balanced by the rate at which work is done at the boundaries.

3. General Solution; Infinite Line

As Hopf [2] and Cole [3] have shown, the transformation

$$v = -2\delta \frac{\partial}{\partial x} \ln \Theta(x, t) \tag{2}$$

changes (1) into the linear heat equation:

$$\Theta_t = \delta \Theta_{xx}; \tag{3}$$

and, therefore, any boundary and/or initial conditions prescribed for (1) must be appropriately "translated" to the framework of reference of (3). This is possible in the important case of the pure initial value problem, where

$$v(x, 0^+) = f(x) \qquad -\infty < x < \infty, \tag{4}$$

is prescribed. For then, by writing the transformation (2) in its inverted form,

$$\Theta(x, t) = \exp\left(-\frac{1}{2\delta}\int^x v(y, t)\, dy\right), \tag{2'}$$

one obtains the appropriate initial condition for (3):

$$\Theta(x, 0^+) = \exp\left[-\frac{1}{2\delta}\int^x v(y, 0^+)\, dy\right] = \exp\left[-\frac{1}{2\delta}\int^x f(y)\, dy\right],$$
$$-\infty < x < \infty. \tag{4'}$$

Now (4′) will give a solution, unique up to a multiplicative constant, for (3); and then, formula (2) yields the unique solution from this for (1).

Thus, while (2) is going to have the form of a quotient of two infinite series or integrals in general, its explicit form facilitates the analysis of the solution to a very great degree.

4. General Solution; Half-Infinite Line

The relative simplicity of the solution of the pure initial value problem disappears when we consider a mixed initial-boundary value problem. One of some importance is described by the conditions

$$\begin{aligned} v(x, 0^+) &= f(x) \qquad 0 \leqslant x < \infty, \\ v(0, t) &= g(t) \qquad t > 0. \end{aligned} \tag{5}$$

It can be easily established that, in order to obtain the solution for (1) under the conditions (5), one must first solve the integral equation

$$g(t)\, h(t) = 2\delta \int_0^t \frac{h'(\tau)}{\sqrt{\pi(t-\tau)}}\, d\tau + P(t). \tag{6}$$

This is a singular linear Volterra equation of the second kind for a new unknown function $h(t)$. The forcing function $P(t)$ is an explicit, but rather complicated expression which is calculable from the initial function $f(x)$ in (5); $g(t)$ of (6) is also from (5). The theory of these equations is well developed [20] and it is easy to find conditions under which solutions to (6) exist [21].

However, because of the variable multiplier on the left side of (6), this solution is generally available only as the limit of a sequence of approximations ([20], [21]). Thus, in order to solve (1) under conditions (5) explicitly, one first has to solve (6); then, using $h(t)$ from there, solve the system

$$\begin{aligned} &\Theta_t = \delta\Theta_{xx} && 0 < t, \quad 0 < x < \infty, \\ &\Theta(x, 0) = \exp\left[-\frac{1}{2\delta}\int^x f(y)\, dy\right] && 0 < x < \infty, \\ &\Theta(0, t) = h(t) && 0 < t, \end{aligned} \tag{7}$$

and then substitute the result in (2). Since, however, the third line of (7) is available only approximately, because of the difficulty connected with (6), the solution to (1) will also be approximate (in the sense that the initial condition will be satisfied exactly, but the boundary condition only approximately).

5. Approximate Methods

Because of the theoretical and practical importance of Eq. (1), several techniques have been used to solve the boundary-initial value problem (5). Perhaps the three most interesting ones are those of Kochina [18], Bass [17], and Blackstock [8].

The method of Kochina consists of changing condition (5) to the following:

$$v(0, t) = g(t), \qquad t > 0,$$
$$\lim_{x\to\infty} v(x, t) = v_\infty \leqslant 0.$$

Under these two conditions, and by taking a periodic $g(t)$, she obtains periodic solutions to (1). Nevertheless, her method still requires the solution of an infinite system of coupled algebraic equations. Thus, the solution, for practical purposes, is still approximate.

Bass approaches the problem in a somewhat different way. He too is interested in finding solutions to (1) which satisfy certain periodicity requirements; thus, he writes the solution as

$$v(x, t) = \frac{\Theta_x(x, t)}{1 + \Theta(x, t)};$$

and by long division of the quotient of the right side, he obtains a two term approximation and, thus, his solution.

Finally, Blackstock, in his study, observed that Eq. (1) had been derived from a more general equation by Mendousse [16]; and that by another method of approximating the magnitudes of the terms in that original equation, Mendousse also obtained a companion equation:

$$u_x + uu_t = \delta u_{tt}.$$

In this last equation, the derivatives taken with respect to x and t in (1) are interchanged. Thus, it becomes possible to solve this new equation for the single condition

$$u(0^+, t) = p(t), \qquad -\infty < t < \infty.$$

6. Solutions Satisfying the Boundary Conditions

In order to construct solutions to (1), which satisfy the boundary conditions exactly, we start with a formal expression for the solutions of the heat equation (3):

$$\Theta(x, t) = \sum_{n=0}^{\infty} F^{(n)}(t) \frac{x^{2n}}{\delta^n (2n)!} + \sum_{n=0}^{\infty} G^{(n)}(t) \frac{x^{2n+1}}{\delta^n (2n+1)!}. \tag{8}$$

Solutions of the form (8) have the property that

$$\Theta(0, t) = F(t), \qquad \Theta_x(0, t) = G(t); \tag{9}$$

and, for F, G, analytic, they are unique on $-\infty < x < \infty$ [11]. Therefore, the solutions of (1) constructed from (8) will have the form

$$v(x, t) = -2\delta \frac{\sum_{n=0}^{\infty} F^{(n+1)}(t) \frac{x^{2n+1}}{\delta^{n+1}(2n+1)!} + \sum_{n=0}^{\infty} G^{(n)}(t) \frac{x^{2n}}{\delta^n (2n)!}}{\sum_{n=0}^{\infty} F^{(n)}(t) \frac{x^{2n}}{\delta^n (2n)!} + \sum_{n=0}^{\infty} G^{(n)}(t) \frac{x^{2n+1}}{\delta^n (2n+1)!}} \tag{10}$$

Thus, from (10),

$$v(0, t) = -2\delta \frac{G(t)}{F(t)} \tag{11a}$$

and

$$v_x(0, t) = 2\delta \left[\frac{G^2(t)}{F^2(t)} - \frac{F'(t)}{\delta F(t)}\right]. \tag{11b}$$

These simple results can be used to solve a number of problems. We take first

$$v_t + vv_x = \delta v_{xx}, \qquad v(0, t) = P(t); \tag{12}$$

and disregard any initial conditions. Then, by choosing $F(t) = -2\delta$ and $G(t) = P(t)$ in (11a), (10) gives the solution

$$v(x, t) = \frac{-2\delta \sum_{n=0}^{\infty} G^{(n)}(t) \frac{x^{2n}}{\delta^n (2n)!}}{-2\delta + \sum_{n=0}^{\infty} G^{(n)}(t) \frac{x^{2n+1}}{(2n+1)!}}. \tag{13}$$

Another type of problem arises when one considers the situation produced by a moving piston, with displacement $H(t)$ and velocity $H'(t)$, at, say, $x = 0$. The boundary condition for such a situation, and for subsonic piston movement, is [22]

$$v[H(t), t] = H'(t). \tag{14}$$

Condition (14), if expanded in a Taylor series, where all but the first two terms are neglected, becomes

$$v(0, t) + H(t)\, v_x(0, t) = H'(t). \tag{14'}$$

When the displacement and the velocity of the piston are large (a rocket, for instance, can, in this context, be considered as a forward moving piston), then the first term in (14') can be neglected. This, together with (1), defines the problem

$$\begin{aligned} v_t + vv_x &= \delta v_{xx}, \\ v_x(0, t) &= \frac{H'(t)}{H(t)}, \\ v(0, t) &= 0. \end{aligned} \tag{15}$$

Observe that while problem (12) is an incompletely defined one, (15)—if regarded as a problem for $-\infty < x < \infty$—is well defined. In fact, it is equivalent to an initial value problem [11].

Now (15) and (11) together imply that we must take

$$G(t) = 0 \qquad \text{and} \qquad \frac{H'(t)}{H(t)} = -2\frac{F'(t)}{F(t)};$$

so that

$$F(t) = [H(t)]^{-1/2} \qquad \text{and} \qquad G(t) = 0. \tag{16}$$

Thus, from (10) we obtain the solution as

$$v(x, t) = -2\delta \frac{\sum_{n=0}^{\infty} [(H(t))^{-1/2}]^{(n+1)} \frac{x^{2n+1}}{\delta^{n+1}(2n+1)!}}{\sum_{n=0}^{\infty} [(H(t))^{-1/2}]^{(n)} \frac{x^{2n}}{\delta^{n}(2n)!}}. \tag{17}$$

7. Some Observations

The physical motivation for the entire preceding discussion was the study of propagation of waves of finite amplitude in thermoviscous media. This is the reason why we chose, for instance, the series form for solutions of the heat equation, instead of the more easily analyzable integral formulation: for almost all the known results in this area are available in terms of the interaction of harmonics [23].

In addition to this, the ad-hoc nature of (13), and the approximations involved in arriving at (17) necessitate some scrutiny of these solutions. To mention but one important question: conditions for G in (13) and for H in (17) must be prescribed, such that the denominators there will be positive—i.e., that these solutions of the heat equation will be strictly positive ones.

While it is quite possible to utilize known results, such as some of Bernstein's theorems [24] on absolutely and completely monotonic functions, and Widder's results [11] on nonnegative solutions of the heat equation, in order to characterize classes of admissible functions G and H, and ranges of convergence for the series, it seems to be more desirable to take our solutions as formal ones and analyze their validity as the need arises. The principal reasons for this are that most such results are used qualitatively in any case, and that because of the approximations involved in obtaining Eq. (1) (see again Mendousse [16]) and the boundary conditions, the mathematical restrictions imposed on these conditions would be usually physically unrealistic. Thus, the presence of the next section is partly due to a desire to illustrate precisely this point. We are using (17) there; for applications of (13) are somewhat trivial.

8. Fay's Solution

Because of the physical background motivating this study, it seems desirable to compare expressions (13) and (17) to solutions derived from other principles. An instance of this is the work of Fay [9], now considered classical. An advantage of this particular comparison is that, except for minor modifications, Fay's results have not been generalized.

In his paper, Fay was not concerned with piston motion, or indeed with any particular boundary value problem. Rather, he sought to determine the periodic elements of the most stable wave form for propagation of the type that we are considering; i.e., waves of finite amplitude in thermoviscous media. The analytical vehicle that he used was an equation of the form

$$\alpha(\omega_x)^k \, \omega_{xx} = \omega_{tt} - c\omega_{xxt}; \tag{F}$$

which, while from certain points of view similar, is still quite different from Burgers' equation. Here ω is the variable flow quantity, while α, k, and c are constants. Equation (F) was one cast in a Lagrangian frame of reference. Because of that, his solution was found as an approximated expression for the pressure P:

$$P = a_0 \sum_{n=1}^{\infty} \frac{\sin nX}{\sinh n\tau}, \tag{18}$$

with a_0 a group of constants, X a space-like and τ a time-like quantity.

With the freedom of interpretation of the variables that we reserved for ourselves at the very outset, we can reduce (17) to (18), by assuming an appropriate $H(t)$. Since (18) represents the most stable periodic wave form, it is to be expected that a rather simple and "gentle" kind of piston motion should produce it; as indeed is the case.

We can obtain Fay's result from our solution (17) by letting

$$H(t) = \left[1 + 2 \sum_{k=1}^{\infty} e^{-k^2 t}\right]^{-2}. \tag{19}$$

We note that (19) is a strictly increasing function of t, such that $H(0) = 0$, $H(\infty) = 1$; so that H is a distribution function on $[0, \infty)$. This is particularly significant because of the statistical importance of Burgers' equation; see (5) and (6) for example.

Substitution of (19) into (17) yields the following sequence of equalities:

$$
\begin{aligned}
v(x, t) &= -2\delta \frac{\partial}{\partial x} \ln \left\{ \sum_{n=0}^{\infty} \left[\frac{1}{\delta^n} \left(1 + 2 \sum_{k=1}^{\infty} e^{-k^2 t}\right)^{(n)} \frac{x^{2n}}{(2n)!} \right] \right\} \\
&= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + \sum_{n=0}^{\infty} \left[\frac{2}{\delta^n} \left(\sum_{k=1}^{\infty} e^{-k^2 t} \right)^{(n)} \frac{x^{2n}}{(2n)!} \right] \right\} \\
&= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + \sum_{n=0}^{\infty} \left[\frac{1}{\delta^n} \left(\sum_{k=1}^{\infty} (-1)^n k^{2n} e^{-k^2 t} \right) \frac{x^{2n}}{(2n)!} \right] \right\} \\
&= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + 2 \sum_{k=1}^{\infty} \left[e^{-k^2 t} \left(\sum_{n=0}^{\infty} (-1) \frac{n(k\Delta x)^{2n}}{(2n)!} \right) \right] \right\} \qquad (20) \\
&= -2\delta \frac{\partial}{\partial x} \ln \left\{ 1 + 2 \sum_{k=1}^{\infty} [e^{-k^2 t} \cos(k\Delta x)] \right\} \\
&= -2\delta \frac{\partial}{\partial x} \ln \left\{ \Theta_3 \left(\frac{\Delta}{2} x, t \right) \right\} \\
&= -4\delta \sum_{n=1}^{\infty} (-1)^n \frac{\sin(n\Delta x)}{\sinh(nt)} .
\end{aligned}
$$

Here we are using the notation

$$\delta^{-1/2} = \Delta. \qquad (20')$$

The reader will note that the last formula in (20) is not quite identical with that of Fay, formula (18). We could have obtained (18) exactly, had we inserted a factor of $(-1)^k$ in the summation of (19); the reason that we chose not to do so is our intention to point out that (17) is indeed a generalization of (18), not only qualitatively, but also from a quantitative point of view.

The development of (20) can be justified by noting the following: the first step is a mere rearrangement of the series; which, incidentally, is convergent absolutely and uniformly, together with its derivatives, on $t > 0$. Because of this convergence, the term by term differentiation in the next step can seen to be justified, as can the interchange of summations in the following one. In the next to the last step, we are using the definition of a Jacobian Theta function [15]; ending up, finally, with an application of the simple series expression for the logarithmic derivative of the Θ function [15];

$$\frac{\partial}{\partial x} [\ln(\Theta_3(x, t))] = -8 \sum_{n=1}^{\infty} (-1)^n \frac{\sin(2nx)}{\sinh(n\pi t)},$$

where

$$\Theta_3(x, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2 \pi t} \cos 2nx.$$

Observe also, that for

$$\Theta_4(x, t) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-n^2 \pi t} \cos 2nx,$$

the logarithmic derivative is given by

$$\frac{\partial}{\partial x} [\ln(\Theta_4(x, t))] = -8 \sum_{n=1}^{\infty} \frac{\sin(2nx)}{\sinh(n\pi t)} .$$

9. Benton's Solution

It may serve a useful purpose to point out that (13) and (17) can be utilized to obtain "new" solutions for Burgers' equation. Thus, we shall now show how (17) yields the two expressions which Benton presented in his paper [10], without any indication as to how they were obtained. In our notation, Benton's solutions can be written in the form

$$v(x, t) = \frac{\delta}{\alpha + \delta t} \left\{ x + \beta \tan \left[\frac{\beta x}{2(\alpha + \delta t)} \right] \right\} \tag{21}$$

with α and β arbitrary constants. The case $\beta =$ (real number) yields Benton's first solution, and $\beta =$ (pure imaginary number) yields his second. Taking the inessential constant $\alpha = 0$, we observe that (21) is that solution of (1), in which

$$v(0, t) = a(t) = 0, \qquad v_x(0, t) = b(t) = \frac{1}{t} \left[1 + \frac{\beta^2}{2\delta t} \right] . \tag{22}$$

Therefore, the solution of (1) and (22) corresponds to solving the associated linear heat equation $\Theta_t = \delta \Theta_{xx}$ with the boundary conditions

$$\Theta(0, t) = t^{-1/2} \exp \frac{\beta^2}{4\delta t}; \qquad \Theta_x(0, t) = 0. \tag{22'}$$

It is now clear why $\beta^2 > 0$ yields the shock solutions, and $\beta^2 < 0$ the smooth solutions of (21); this is a result of the well-known behavior ([11], [12]) of the source solution present in (22′).

As a byproduct of this analysis, we also obtain another expression similar to the last one in (20), namely the equality

$$\frac{x}{t} + \frac{\beta}{t} \tan\left(\frac{\beta x}{2\delta t}\right) = -2\delta \frac{\partial}{\partial x} \ln\left[\sum_{n=0}^{\infty} \left[t^{-1/2} \exp\left(\frac{\beta^2}{4\delta t}\right)\right]^{(n)} \frac{x^{2n}}{\delta^n (2n)!}\right] \qquad (23)$$

valid for $t > 0$, $\delta > 0$, $\beta^2 < 0$, and for all x. Note that the right side could be continued analytically for values $\beta^2 > 0$, except on a null set; in which case we would have the convergent quotient of two unconditionally divergent series, representing the function on the left side.

One can, of course, obtain several other similar formulas and achieve the reduction of these results into simpler form. In particular, it can be shown that certain asymptotic estimations of the solutions of initial value problems (e.g., Cole [3] and Blackstock [7]) correspond to exact boundary value solutions.

10. Propagation of Periodic Piston Motion

We now investigate the velocity field created by sinusoidal piston motion. Since we are not interested in shocks, we shall take

$$H(t) = \exp[-\alpha \cos \omega t]. \qquad (24)$$

Then, substituting in (17),

$$\sum_{n=0}^{\infty} \frac{[H^{-1/2}(t)]^{(n)}}{\delta^n} \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{\left(\exp\left[\frac{\alpha}{2} \cos \omega t\right]\right)^{(n)}}{\delta^n} \frac{x^{2n}}{(2n)!}. \qquad (25)$$

In order to perform the differentiations in (25), we write $\exp[\alpha/2 \cos \omega t]$ as a series of modified Bessel functions, utilizing the formula [15]

$$\exp[z \cos \omega t] = I_0(z) + 2 \sum_{k=1}^{\infty} I_k(z) \cos(k\omega t). \qquad (26)$$

We shall also need the integral of this expression with respect to t:

$$\int_0^t \exp[z \cos \omega\tau]\, d\tau = tI_0(z) + 2 \sum_{k=1}^{\infty} \frac{I_k(z)}{k\omega} \sin(k\omega t), \qquad (26')$$

which is a well defined expression even for $\omega = 0$. We rewrite now (25) in terms of (26), to obtain

$$\sum_{n=0}^{\infty} \left\{ \left[I_0\left(\frac{\alpha}{2}\right) + 2 \sum_{k=1}^{\infty} I_k\left(\frac{\alpha}{2}\right) \cos k\omega t \right]^{(n)} \frac{x^{2n}}{\delta^n (2n)!} \right\}$$

$$= \exp\left(\frac{\alpha}{2} \cos \omega t\right)$$

$$+ 2 \sum_{n=1}^{\infty} \left\{ \sum_{k=1}^{\infty} (-1)^n I_k\left(\frac{\alpha}{2}\right) [(k\omega)^{2n-1} \sin k\omega t + (k\omega)^{2n} \cos k\omega t] \frac{x^{2n}}{\delta^n (2n)!} \right\}.$$

Because of the uniform convergence, we may interchange the order of the summations and rearrange the series:

$$= \exp\left(\frac{\alpha}{2} \cos \omega t\right) + 2 \sum_{k=1}^{\infty} \frac{I_k\left(\frac{\alpha}{2}\right) \sin(k\omega t)}{k\omega} \left\{ \sum_{n=1}^{\infty} (-1)^n \frac{(k\omega x)^{2n}}{\delta^n (2n)!} \right\}$$

$$+ 2 \sum_{k=1}^{\infty} I_k\left(\frac{\alpha}{2}\right) \cos(k\omega t) \left\{ \sum_{n=1}^{\infty} (-1)^n \frac{(k\omega x)^{2n}}{\delta^n (2n)!} \right\}.$$

Both inner series yield the same cosine function:

$$= \exp\left(\frac{\alpha}{2} \cos \omega t\right)$$
$$+ 2 \sum_{k=1}^{\infty} \left\{ I_k\left(\frac{\alpha}{2}\right) [\cos(k\omega\, \Delta x) - 1] \left[\frac{\sin(k\omega t)}{k\omega} + \cos(k\omega t)\right] \right\}. \tag{27}$$

(We returned here to the notation of (20′): $\delta^{-1/2} \equiv \Delta$.) Multiplication of the two bracketed terms in this series allows us to write it as the sum of four series; in particular, the two arising from the products of (-1) with $[\sin(k\omega t)]/k\omega$ and with $\cos(k\omega t)$, yields from (26) and (26′),

$$t\left[I_0\left(\frac{\alpha}{2}\right) + 1\right] - \exp\left[\frac{\alpha}{2} \cos \omega t\right] - \int_0^t \exp\left[\frac{\alpha}{2} \cos \omega\tau\right] d\tau. \tag{28}$$

To reduce the trigonometric products, we use the identities:

$$\cos(k\omega\, \Delta x) \sin(k\omega t) = \tfrac{1}{2} [\sin k\omega(t + \Delta x) + \sin k\omega(t - \Delta x)],$$

$$\cos(k\omega\, \Delta x) \cos(k\omega t) = \tfrac{1}{2} [\cos k\omega(t + \Delta x) + \cos k\omega(t - \Delta x)].$$

Thus, we obtain for the sum of the other two series the expression

$$-I_0\left(\frac{\alpha}{2}\right)[t+1]+\frac{1}{2}\Big\{\exp\left[\frac{\alpha}{2}\cos\omega(t+\Delta x)\right]+\exp\left[\frac{\alpha}{2}\cos\omega(t-\Delta x)\right] \\ +\int_0^t\left[\exp\left[\frac{\alpha}{2}\cos\omega(\tau+\Delta x)\right]+\exp\left[\frac{\alpha}{2}\cos\omega(\tau-\Delta x)\right]\right]\Big\}\,d\tau. \tag{29}$$

Therefore, (27) can be written as the sum of the following: the first term in (27), together with (28) and (29). Some further simplifications yield, in this way, an expression which can be written in the following symmetric form:

$$t-I_0\left(\frac{\alpha}{2}\right)+\frac{1}{2}\{[F(t+\Delta x)-F(t)]+[F(t-\Delta x)-F(t)]\} \\ +\frac{1}{2}\int_0^t\{[F(\tau+\Delta x)-F(\tau)]+F(\tau-\Delta x)-F(\tau)\}\,d\tau, \tag{30}$$

where

$$F(s)=\exp\left[\frac{\alpha}{2}\cos\omega s\right]. \tag{31}$$

To recapitulate: expression (30) is the transformed form of (25), with $H(t)$ given by (24). Thus, the solution of (1), with the approximated piston condition (24), is given, according to (17), by the product of (-2δ) with the logarithmic x derivative of (30). This takes the form

$$v(x,t)=\frac{\frac{2}{\Delta}\{[F'(t-\Delta x)-F'(t+\Delta x)]+[F(t-\Delta x)-F(t+\Delta x)]\}}{\left[\begin{array}{l}M+\{[F(t+\Delta x)-F(t)]+[F(t-\Delta x)-F(t)]\} \\ \quad+\int_0^t\{[F(\tau+\Delta x)-F(\tau)]+[F(\tau-\Delta x)-F(\tau)]\}\,d\tau\end{array}\right]} \tag{32}$$

where

$$M=2\left[t-I_0\left(\frac{\alpha}{2}\right)\right].$$

The suggestiveness of (32) is obvious: the two types of waves, so well-known from linear theory, are both present. However, because of the viscous mechanism, they contain a "damping"—in fact, it is clear here how increased viscosity ($\delta\to\infty$ $\Delta\to 0$) reduces the propagation to zero. Moreover, the nonlinearity of the mechanism and/or of the medium are represented in a conceptually rather simple form. Note that shocks depend on the values of t and α, in particular. Finally, perhaps the most significant feature of (32) is that this expression is seen to depend on $(\delta)^{1/2}$ instead of δ: a circumstance which gives further credence to Burgers' equation as a model equation for phenomena of the type here discussed.

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