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A TABLE OF SOLUTIONS OF THE ONE-DIMENSIONAL BURGERS EQUATION*

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Abstract. The literature relating to the one-dimensional Burgers equation is surveyed. About thirty-five distinct solutions of this equation are classified in tabular form. The physically interesting cases are illustrated by means of isochronal graphs.

Introduction and survey of literature. The quasilinear parabolic equation now known as the "one-dimensional Burgers equation,"

$$(\partial u/\partial t) + u(\partial u/\partial x) = \nu(\partial^2 u/\partial x^2), \tag{1}$$

first appeared in a paper by Bateman [4], who derived two of the essentially steady solutions (1.3 and 1.5 of our Table). It is a special case of some mathematical models of turbulence introduced about thirty years ago by J. M. Burgers [10], [11].

The distinctive feature of (1) is that it is the simplest mathematical formulation of the competition between convection and diffusion. It thus offers a relatively convenient means of studying not only turbulence but also the distortion caused by laminar transport of momentum in an otherwise symmetric disturbance and the decay of dissipation layers formed thereby. Moreover, the transformation

$$u = -(2\nu/\theta)(\partial\theta/\partial x) \tag{2}$$

relates $u(x, t)$ and $\theta(x, t)$ so that if θ is a solution of the linear diffusion equation

$$(\partial\theta/\partial t) = \nu(\partial^2\theta/\partial x^2), \tag{3}$$

then u is a solution of the quasilinear Burgers equation (1). Conversely, if u is a solution of (1) then θ from (2) is a solution of (3), apart from an arbitrary time-dependent multiplicative factor which is irrelevant in (2).

In connection with the Burgers equation, transformation (2) appears first in a technical report by Lagerstrom, Cole, and Trilling [38, especially Appendix B], and was published by Cole [21]. At about the same time it was discovered independently by Hopf [30] and also—in the context of the similarity solution $u = t^{-1/2}S(z)$, $z = (4\nu t)^{-1/2}x$—by Burgers [14, p. 250]. The similarity form of the Burgers equation—the quasilinear ordinary differential equation for $S(z)$—is a Riccati equation [51], and can thus be regarded as a basis for motivating transformation (2) inasmuch as (2) is a standard means of linearizing the Riccati equation. More general hydrodynamical applications of this transformation have been discussed by Ames [1, chapter 2], Chu [20], and Shvets and Meleshko [55].

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Another virtue of (1) is that although it appears to lack a pressure gradient term, it is in fact a deductive approximation for propagation of one-dimensional disturbances of moderate amplitude in a uniform diffusive compressible medium, if the variables are interpreted in a particular way. In this "acoustic" analogy for an ideal gas, the u, ν and x, t of (1) are, respectively,

$$\begin{aligned} &\tfrac{1}{2}(\gamma + 1)u', \qquad \tfrac{1}{2}[\tfrac{4}{3}\nu' + \nu_v' + (\gamma - 1)\kappa'], \\ &x' - a_0t', \qquad t', \end{aligned} \tag{4}$$

where u' is the fluid velocity, ν' and ν_v' are respectively the kinematic shear and bulk viscosities, κ' is the thermal diffusivity, γ the specific-heat ratio (assumed constant), x' is an axis fixed in the undisturbed medium, t' is the actual time, and a_0 is the speed of a linear sound wave. According to this analogy, t in (1) is the actual time and x is position in a coordinate frame moving with speed a_0 relative to the undisturbed medium; thus $\partial u/\partial t$ in (1) expresses the relatively slow wave-form distortion caused by convection and diffusion, rather than the comparatively fast local changes associated with ordinary propagation. The restriction to moderate amplitude is necessary because the analogy depends upon approximations valid only for small Mach number of the disturbance. Moreover, the analogy is derivable only when the propagation is unidirectional, as typically for a "simple" wave or disturbance that moves into an infinite resting medium.

Subject to the same restriction, an alternative statement of the acoustic analogy is to take u, ν and x, t in (1) as [43],

$$\begin{aligned} &-\tfrac{1}{2}(\gamma + 1)u', \qquad \tfrac{1}{2}[\tfrac{4}{3}\nu' + \nu_v' + (\gamma - 1)\kappa'], \\ &a_0t' - x', \qquad x'/a_0. \end{aligned} \tag{5}$$

The first term of (1) then expresses spatial variation in the translating frame, variation which is equivalent to the temporal changes of the first analogy and corresponds to the same effects of convection and diffusion. Analogy (4) is convenient for initial-value problems, since when $t' = 0$ we have $(x, t) = (x', 0)$ in this analogy; whereas (5) is convenient for boundary-value problems, since then $(x, t) = (a_0t', 0)$ when $x' = 0$.

With varying degrees of rigor the derivation of the one-dimensional Burgers equation from the fundamental gas-dynamic equations under the restrictions inherent in the acoustic analogy has been accomplished for viscous, non-conducting ideal gases by Lagerstrom, Cole, and Trilling [38, Appendix B], and by Su and Gardner [59]; for viscous, nonconducting fluids with quadratic dependence of pressure on rate of expansion by Mendousse [43]; for viscous conducting ideal gases by Lighthill [40] and by Soluyan and Khokhlov [58]; and for viscous, conducting fluids of general equation of state by Hayes [29]. The equation also describes finite-amplitude transverse hydromagnetic waves [27], longitudinal elastic waves in an isotropic solid [48], and disturbances on glaciers [39]. An equation related very simply to the Burgers equation arises in a problem of number theory [62].

In a remarkable series of papers extending over many years, Burgers (see references) studied statistical and spectral aspects of the equation (and related systems of equations) when initial conditions are given stochastically. Various aspects of the energy spectrum have also been investigated by Reid [49], Ogura [45] and Tatsumi [60]. More recently, the new deductive theories of turbulence have been "tested" on the Burgers model (see [23], [32], [33], [36], [41], [42], [46], [56], [57]) and various numerical experiments have been made on the Burgers equation (for example, see [5], [26], [31]). Saffman [54] has questioned the basis of the Kolmogorov law by using results derived from the Burgers model.

The Burgers equation gives an analytic framework for a second-order theory of finite-amplitude dissipative sound propagation [8], [9], [34], [35], [43], [58]. It has been used in discussions of shock structure in a Navier-Stokes fluid principally by Lagerstrom, Cole, and Trilling [38], by Lighthill [40] and by Hayes [29]. In the notable work of Lighthill, the conflict between the steepening effect of nonlinear convection and the broadening trend of dissipation is made especially clear; this dual process is the essence of the Burgers equation.

One of the most interesting solutions of the Burgers equation, the only known exact time-dependent spectral solution (2.6 in the Table and Figure 9), appears first in a paper by Fay [24] where it was derived in the acoustic framework but without the aid of the Burgers equation and with the role of space and time inverted as in (5) (as pointed out by Rudnick [53], there is a minor error in Fay's Eq. (14): the correct numerical factor is 2, not 8). The Fay series was re-discovered by Cole [21] as an approximate solution of the Burgers equation for a sinusoidal initial condition, and by Benton [6], [7] as an exact solution. The relation of Fay's solution to the corresponding inviscid spectral solution of (1),

$$u(x, t) = -2 \sum_{n=1}^{\infty} (nt)^{-1} J_n(nt) \sin nx,$$

is thoroughly discussed by Blackstock [9] in connection with the sound field generated by sinusoidal motion of a one-dimensional piston (see also [2], [34], [52], [65]). This inviscid solution is known as the Fubini solution in the acoustics literature because of the work of Fubini-Ghiron [25]. It has been rediscovered by workers in several different fields (see [22], [28], [37], [44], [47], [66]).

Description of table. The correspondence between (1) and (3) through (2) makes it easy to construct exact solutions of (1) by starting from solutions of (3). Although the general solution of (3) is known for arbitrary initial conditions, and the transformation (2) is trivial to perform, not all special solutions are physically "interesting." It seems worthwhile, therefore, to call attention to those that are. The purpose of the following Table is to present a list of such solutions, arranged in a somewhat systematic way, as a possible aid in further investigations of the Burgers equation. Some new solutions are included, but we have primarily aimed at collecting and organizing numerous results scattered through a somewhat diffuse literature.

Eqs. (1, 2, 3) are invariant to a shift of origin

$$x - x_0 \rightarrow x, \quad t - t_0 \rightarrow t; \qquad u \rightarrow u, \quad \theta \rightarrow \theta, \tag{6}$$

where x_0 and t_0 are arbitrary, independent constants. They are also invariant under a change of scale:

$$x/\alpha \rightarrow x, \quad t/\alpha^2 \rightarrow t; \quad \alpha u \rightarrow u, \quad \beta\theta \rightarrow \theta, \tag{7}$$

where α and β are arbitrary, independent scale factors. Special cases of (7) that are useful in constructing the Table are

$$-x \rightarrow x, \quad t \rightarrow t; \qquad -u \rightarrow u \tag{7a}$$

which reverses the direction of the x-axis, and

$$ix \to x, \ -t \to t; \ -iu \to u \tag{7b}$$

which rotates the x-axis 90 degrees in the complex x-plane. The third important invariance transformation,

$$x - Ut \to x, \quad t \to t; \qquad u - U \to u, \quad \theta \exp\left(U(x - Ut)/2\nu\right) \to \theta, \tag{8}$$

represents translation of the reference frame at the constant speed U (Galilean invariance). In the Table we use nondimensional variables obtained by making the substitutions

$$x/L \to x, \ \nu t/L^2 \to t; \ uL/\nu \to u. \tag{9}$$

The insertion of ν here formally has the effect $\nu \to 1$ in (1, 2, 3), whereas the length scale enters as in (7) and thus does not alter the equations. It should be noted that in many cases L is not an external parameter (for example, see Note 1.3).

If two solutions are related through one or more invariance transformations, we say they are "equivalent" (for example, see Note 1.0 in the Table). If all the transformations in question are real, they do not alter the shape of the function on which they operate; we therefore say that such solutions are "isomorphic" (for example, Note 1.0). In the Table we list only real solutions (some of which are equivalent through complex transformations such as (7b)), and in the Notes we call attention to specific isomorphisms. In the diagrams, only non-isomorphic solutions are illustrated.

The presence of θ in the denominator of (2) calls for comment about the effect of a zero in θ; in general, a zero of any order produces a simple pole in u. The simplest example is $\theta = x$ which by (2) corresponds to the steady solution $u = -2/x$ of the Burgers equation, in nondimensional notation. The "energy" equation associated with (1) shows that this solution is steady only because the (infinitely large) dissipation in $x > 0$ and $x < 0$ is balanced by (infinite) flux of energy into the region from a maintained source at $x = 0$, a situation without much physical interest. On the other hand, if $-2/x$ is regarded as an *initial* condition $u(x, 0)$, we can solve the Burgers equation by calculating the corresponding initial $\theta(x, 0)$ from (2) and then integrating (3). In this process we meet the fact that (2) does not determine the sign of θ. Indeed, if the inversion of (2) is written

$$\theta(x, t) = \theta(a, t) \exp\left(-\frac{1}{2}\int_a^x u(\xi, t)\, d\xi\right), \tag{10}$$

then the result of taking the principal value of the integral when $u(x, 0) = -2/x$ is $\theta(x, 0) = \theta(a, 0)\cdot|x/a|$, and since the factor $\theta(a, 0)/|a|$ is irrelevant in (2), this is equivalent to $\theta(x, 0) = |x|$ for the purpose of solving the Burgers equation. The function $\theta(x, t)$ that comes from this initial condition is of course unsteady; the corresponding $u(x, t)$ has no energy source and decays to zero (solution (4.5)).

If all zeros of the general initial θ are treated in the same way, the result is a "positive" solution of (2), and the corresponding u of (1) will be free of singularities for $t > 0$ (except possibly at $|x| = \infty$). This process can be formalized by starting from the source representation of the solution of (2) with $f(x)$ as initial condition:

$$\theta(x, t) = \int_{-\infty}^{\infty} f(\xi)\phi(x - \xi, t)\, d\xi \tag{11}$$

where $\phi(x, t) \equiv (4\pi t)^{-1/2} \exp(-x^2/4t)$. If $f(x)$ has one or more zeros, this solution is not positive for all $t > 0$ (unless all the zeros are of even order), although it may become so for $t > T > 0$. However, the related solution

$$\int_{-\infty}^{\infty} |f(\xi)|\phi(x - \xi, t)\, d\xi \tag{12}$$

is strictly positive for $t > 0$. A typical example of (12) is solution (4.5) of the Table, for which $\theta(x, t) = x$ (solution (1.7)). On the other hand, if $f(x)$ is bounded, a linear transformation can always be found so that $\alpha + \beta f(x) > 0$, in which case

$$\theta'(x, t) \equiv \int_{-\infty}^{\infty} [\alpha + \beta f(\xi)]\phi(x - \xi, t)\, d\xi = \alpha + \beta\theta(x, t) \tag{13}$$

is positive for $t > 0$. The corresponding $u'(x, t)$ is therefore free of singularities; in the Table it is referred to as "allied" to $u(x, t)$. A typical example of (13) is solution (1.5′ (−)) in the Table.

To preserve the organization of the Table, some solutions are included that have infinite energy sources (non-positive θ) and thus are not physically interesting, but these solutions are not displayed graphically, and little or no comment about them is made in the Notes. Verbal description of many of the figures is awkward and largely superfluous, so we have avoided it and let the diagrams speak for themselves. Where descriptions are given, for historical or other reasons, we have chosen the acoustic analogy of the Burgers equation, rather than the turbulence analogy used originally by Burgers. In the acoustic analogy the "condensation" (density excess) is proportional to u, so there is compression or expansion according to whether u is positive or negative.

Finally, we emphasize that the Table includes explicitly only solutions of the initial-value problem for (1) and (3) on the infinite interval $-\infty < x < \infty$. Some solutions, such as (2.6) and (5.2), are spatially periodic and keep $u = 0$ at $x = \pm\pi$. These may be thought of as solutions of the combined initial- and boundary-value problem in which initial values of u (or θ) are assigned on $-\pi < x < \pi$ and boundary values $u = 0$ (or $\partial\theta/\partial x = 0$) at $x = \pm\pi$. The general problem with arbitrary, time-dependent boundary values of u is more difficult (see [2], [35], [50], [52], and [63] where further references are given).

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A Table of Solutions of the One-Dimensional Burgers Equation
(Each group of solutions is followed by explanatory notes.)

1. Solutions in which $\partial u/\partial t = 0$

	$\theta(x,t)$	$u(x,t) = -2\ \partial \ln\theta/\partial x$
1.1	e^{t+x}	-2
1.2	e^{t-x}	2
1.3	$e^{t} \cosh x$	$-2 \tanh x$
1.4	$e^{t} \sinh x$	$-2 \coth x$
1.5	$e^{-t} \cos x$	$2 \tan x$
1.6	$e^{-t} \sin x$	$-2 \cot x$
1.7	x	$-2/x$

(1.0). *Isomorphisms*: (1.2) and (1.1), (1.6) and (1.5). When $\partial u/\partial t = 0$ then $\partial^2 \ln \theta/\partial t\partial x = 0$ so $\theta = A(t)\cdot B(x)$. The only such solutions of the diffusion equation are those given above. Apart from (1.1) and (1.2), which are trivial, there are four and only four distinct (non-isomorphic) steady solutions of the Burgers equation, namely (1.3), (1.4), (1.5) or (1.6) and (1.7). Of these, only (1.3) is free of singularities. (The steady solutions are "equivalent," since each can be obtained from (1.3) by a scale transformation and shift of origin: starting from $-2a \tanh a(x - x_0)$ we have (1.4) with $a = 1$, $x_0 = \frac{1}{2}i\pi$; (1.5) with $a = i$, $x_0 = 0$; (1.6) with $a = i$, $x_0 = \frac{1}{2}\pi$; (1.7) with $a \to 0$, $ax_0 \to \frac{1}{2}i\pi$.) *References* (apart from (1.3)): Rodin [50], Benton [7].

(1.1) and (1.2). Although trivial, these solutions correspond to regions of uniform convection of energy, as in the steady shock (1.3).

(1.3). Figure 1: the steady shock. If $2U$ is the dimensional value of $|u|$ at $x = \pm\infty$, then in view of (9) the length scale is the intrinsic shock thickness ν/U. *References*: Taylor [61], Bateman [4], Burgers [10].

(1.7.) Behavior of u at $\theta = 0$ (compare with (4.5)).

1'. Solutions allied to those of Group 1

	$\theta'(x,t) = \alpha + \theta(x,t)$	$u'(x,t) = -2\,\partial \ln\theta'/\partial x$
1.1'(±)	$\pm 1 + e^{t+x}$	$-\dfrac{2}{1 \pm e^{-t-x}}$
1.2'(±)	$\pm 1 + e^{t-x}$	$\dfrac{2}{1 \pm e^{-t+x}}$
1.3'(±)	$\pm 1 + e^{t} \cosh x$	$-\dfrac{2 \sinh x}{\cosh x \pm e^{-t}}$
1.4'(±)	$\pm 1 + e^{t} \sinh x$	$-\dfrac{2 \cosh x}{\sinh x \pm e^{-t}}$
1.5'(±)	$\pm 1 + e^{-t} \cos x$	$\dfrac{2 \sin x}{\cos x \pm e^{t}}$
1.6'(±)	$\pm 1 + e^{-t} \sin x$	$-\dfrac{2 \cos x}{\sin x \pm e^{t}}$
1.7'	$\alpha + x$	$-\dfrac{2}{\alpha + x}$

(1.0′). *Isomorphisms*: (1.1′ (+)) and (1.3), (1.1′ (−)) and (1.4), (1.2′ (±)) and (1.1′ (±)), (1.4′ (−)) and (1.4′ (+)), (1.5′ (−)) and (1.5′ (+)), (1.6′ (±)) and (1.5′ (±)), (1.7′) and (1.7). *Persistent singularities*: (1.1′ (−)), (1.2′ (−)), (1.4′ (±), (1.7′). Except in (1.7′) the function θ' has been divided by $|\alpha|$ and the factor $|\alpha|$ absorbed in θ by shifting the origin of t. Owing to isomorphisms, only four of the solutions in Group 1′ are distinct and differ from those of Group 1, namely (1.3′ (±)), (1.4′ (+) or (−)), and (1.5′ (+) or (−)).

(1.3′ (+)). Figure 2: coalescence of two equal shocks. The inflection points of the coalescing shocks disappear at $t = -\ln 2$. The configuration at $t = \infty$ is the steady shock of (1.3). *References*: Lighthill [40]; attributed to Howard by Hayes [29]. These authors give the general solution for two unequal shocks.

(1.3′ (−)). Figure 3. Singular for $t \leq 0$. The configuration at $t = \infty$ is the steady shock of (1.3).

(1.5′ (−)). Figure 4. Singular for $t \leq 0$.

2. Discrete distributions of instantaneous point sources in θ

	$\theta(x,t)$	$u(x,t) = -2\,\partial\ln\theta/\partial x$
2.1	$(4\pi t)^{-1/2}\, e^{-x^2/4t} \equiv \phi(x,t)$	x/t
2.1'(±)	$\pm 1 + t^{-1/2}\, e^{-x^2/4t}$	$\dfrac{x/t}{1 \pm t^{1/2}\, e^{x^2/4t}}$
2.2	$\phi(x+2,t) + \phi(x-2,t) =$ $= (\pi t)^{-1/2}\, e^{-(x^2+4)/4t} \cosh(x/t)$	$\frac{x}{t} - \frac{2}{t} \tanh \frac{x}{t}$
2.3	$\phi(x-2,t) - \phi(x+2,t) =$ $= (\pi t)^{-1/2}\, e^{-(x^2+4)/4t} \sinh(x/t)$	$\frac{x}{t} - \frac{2}{t} \coth \frac{x}{t}$
2.4	$\phi(x-2i,t) + \phi(x+2i,t) =$ $= (\pi t)^{-1/2}\, e^{-(x^2-4)/4t} \cos(x/t)$	$\frac{x}{t} + \frac{2}{t} \tan \frac{x}{t}$
2.5	$\phi(x-2i,t) - \phi(x+2i,t) =$ $= i(\pi t)^{-1/2}\, e^{-(x^2-4)/4t} \sin(x/t)$	$\frac{x}{t} - \frac{2}{t} \cot \frac{x}{t}$
2.6	$\sum_{n=-\infty}^{\infty} \phi(x+2n\pi,t) \equiv \psi(x,t)$	$-2 \sum_{n=1}^{\infty} (-)^n \frac{\sin nx}{\sinh nt}$
2.7	$\sum_{n=-\infty}^{\infty} (-)^n\, \phi(x+2n\pi,t)$	$\tan \frac{1}{2} x - 2 \sum_{n=1}^{\infty} (-)^n\, e^{-nt} \frac{\sin nx}{\sinh nt}$

(2.0). *Isomorphisms*: (2.5) and (2.4). *Persistent singularities*: (2.2), (2.3), (2.4) (equivalent to (2.2) by (7b)), (2.5) (equivalent to (2.3) by (7b)), (2.7).

(2.1). Figure 5 (after Burgers). This solution satisfies the inviscid equation $\partial u/\partial t + u\partial u/\partial x = 0$ exactly, and gives the "saw-tooth" limit which approximates solutions of the Burgers equation for large Reynolds number in regions between dissipation layers. In Figure 5 we have joined segments of $(x + 2)/(t - 2)$, x/t, $(x - 3)/(t - 1)$, and $x/(t + 2)$ to form the sawtooth. *References*: Burgers [11], [14].

(2.1′ (+)). Figure 6: decay of a solitary pair of equal compression and expansion pulses. *Reference*: Lighthill [40].

(2.1′ (−)). Figure 7. Singular for $t \leq 1$.

(2.2) and (2.4). *References*: Cole [21], Benton [6].

(2.6). Figure 8: decay of a spatially periodic wave initially in a saw-tooth configuration with infinite intensity. The spatially periodic function $\psi(x, t)$ represents an infinite row of sources in θ and has the Fourier series

$$(2\pi)^{-1} \sum_{n=-\infty}^{\infty} \exp(-n^2 t) \cos nx = (2\pi)^{-1} \vartheta_3(\tfrac{1}{2}x, e^{-t})$$

where ϑ_3 is a theta function in the notation of Whittaker and Watson. The expression given for u follows from standard formulas for the logarithmic derivative of theta functions. It is the only known spatially periodic solution of the Burgers equation whose Fourier coefficients can be stated explicitly as functions of t. *References*: Cole [21], Lighthill [40], Benton [6]; see also Fay [24].

(2.7). The Fourier series for θ is

$$(2\pi)^{-1} \sum_{n=-\infty}^{\infty} \exp\left[-(n+\tfrac{1}{2})^2 t\right] \cos(n+\tfrac{1}{2})x = (2\pi)^{-1}\vartheta_2(\tfrac{1}{2}x, e^{-t})$$

Reference: Benton [6].

3. Similarity solutions

	$\theta(x,t)$	$u(x,t) = -2\,\partial \ln\theta/\partial x$
3.1	$2\int_x^\infty \phi(\xi,t)\,d\xi \equiv \text{erfc } z.$	$\dfrac{2}{\sqrt{\pi t}}\dfrac{e^{-z^2}}{\text{erfc } z}$
3.2	$2\left(\int_x^\infty d\xi\right)^2 \phi(\xi,t) = 2\sqrt{t}\int_z^\infty \text{erfc }\zeta\,d\zeta$	$\dfrac{1}{\sqrt{t}}\dfrac{\text{erfc } z}{i^1\,\text{erfc } z}$
3.3	$2\left(\int_x^\infty d\xi\right)^{n+1} \phi(\xi,t) = (4t)^{n/2}\, i^n\,\text{erfc } z$	$\dfrac{1}{\sqrt{t}}\dfrac{i^{n-1}\,\text{erfc } z}{i^n\,\text{erfc } z}$ $(n = 0,1,2,\ldots)$
3.4	$\sqrt{\pi}\left(-\dfrac{\partial}{\partial x}\right)^{n-1} \phi(x,t) = (4t)^{-n/2}\, e^{-z^2}\, H_{n-1}(z)$	$\dfrac{1}{\sqrt{t}}\dfrac{H_n(z)}{H_{n-1}(z)}$ $(n = 1,2,3,\ldots)$
3.5	$\frac{1}{2}\, t^{-1}\, z\, e^{-z^2}$	$\dfrac{1}{\sqrt{t}}\left(2z - \dfrac{1}{z}\right) = \dfrac{x}{t} - \dfrac{2}{x}$

(3.0). *Notation*: ϕ is defined in (2.1) and $z \equiv x/2\sqrt{t}$. The notation for erfc z and the nth integral i^n erfc z is standard, and $H_n(z)$ is the Hermite polynomial. *Persistent singularities*: all solutions in this group.

(3.3). For $n = 0$ see (3.1); for $n = 1$ see (3.2).

(3.4). For $n = 1$ see (2.1); for $n = 2$ see (3.5).

(3.5). This solution is the sum of two others: (2.1) and (1.7). Superposition is possible because the product is a function only of t [7].

3'. Solutions allied to those of Group 3

	$\theta'(x,t) = \alpha + \theta(x,t)$	$u'(x,t) = -2\,\partial \ln\theta'/\partial x$
3.1'	$\alpha + \operatorname{erfc} z$	$\frac{2}{\sqrt{\pi t}} \frac{e^{-z^2}}{\alpha + \operatorname{erfc} z}$
3.2'(±)	$\pm 1 + \sqrt{t}\, i^1 \operatorname{erfc} z$	$\frac{\operatorname{erfc} z}{\pm 1 + \sqrt{t}\, i^1 \operatorname{erfc} z}$
3.3'(±)	$\pm 1 + t^{n/2}\, i^n \operatorname{erfc} z$ (n = 1,2,3,...)	$\frac{t^{(n-1)/2}\, i^{n-1} \operatorname{erfc} z}{\pm 1 + t^{n/2}\, i^n \operatorname{erfc} z}$
3.4'(±)	$\pm 1 + t^{-n/2} e^{-z^2} H_{n-1}(z)$ (n = 1,2,3,...)	$\frac{1}{\sqrt{t}} \frac{H_n(z)}{H_{n-1}(z) \pm t^{n/2} e^{z^2}}$
3.5'(±)	$\pm 1 + t^{-1} e^{-z^2} \cdot 2z$	$\frac{1}{\sqrt{t}} \frac{4z^2 - 2}{2z \pm t\, e^{z^2}}$

(3.0′). *Notation*: see Note 3.0. *Isomorphisms*: for n even, (3.4′ (−)) and (3.4′ (+)); (3.5′ (−)) and (3.5′ (+)). *Persistent singularities*: (3.2′ (−)). Except in (3.1′) we take $|\alpha| = 1$, because two different values of α ($\neq 0$) with the same sign correspond to isomorphic solutions.

(3.1′). Figures 9, 10 and 11: a solitary compression pulse. This is the only similarity solution in Group 3′. Since $0 < \operatorname{erfc} z < 2$, the solutions for $-2 < \alpha < 0$ are singular, and since $\operatorname{erfc}(-z) = 2 - \operatorname{erfc} z$, the solutions corresponding to α and $-(\alpha + 2)$ are isomorphic. Hence it suffices to consider $\alpha > 0$. The intensity of the pulse, $R \equiv \int_{-\infty}^{\infty} u'\,dx = 2 \ln (1 + 2\alpha^{-1})$ is independent of t and may be regarded as a Reynolds number. The spatial shape of the pulse is prescribed by $e^{-z^2}/(\alpha + \operatorname{erfc} z)$. This is Gaussian on the far left ($z \ll -1$ so $\operatorname{erfc} z \approx 2$) and on the far right ($z \gg 1$ so $\operatorname{erfc} z \approx 0$) for any $\alpha > 0$. Its peak is located at z_0 determined by $\alpha = -\operatorname{erfc} z_0 + (\exp(-z_0^2))/z_0 \sqrt{\pi}$. For small R (specifically $R < 0.2$ so $\alpha > 20$) we have $\alpha \gg \operatorname{erfc} z$ (all z) so the pulse is dominated by diffusion and its shape is Gaussian throughout (Figure 9). (Note that $\operatorname{erfc} z$ in the shape function arises directly from the convection term of the Burgers equation.) For large R (specifically $R > 20$ so $\alpha < 10^{-4}$) we have $z_0 > 2$ at the peak of the pulse, so to the left of the peak there is a range where $\alpha \ll \operatorname{erfc} z \approx \exp(-z^2)/z \sqrt{\pi}$, which gives a convective "sawtooth" regime $u' \sim x/t$ (Figure 10. This case is equivalent to that plotted by Burgers [14] in his Figure 1, page 251.) To the right of the peak there is an abrupt transition (shock) to the Gaussian forerunner. The shape of the pulse for several values of R is shown in Figure 11. *References*: Burgers [14], Lighthill [40].

3.2′ (+). Figure 12: decay of a sharp compression front. Initially $u' = 2/(1 - x)$ for $x < 0$ and $u' = 0$ for $x > 0$.

3.5′ (−). Figure 13. Singular for $t \leq (2/e)^{1/2} \approx 0.86$.

4. Direct assignment of aperiodic initial conditions

	$\theta(x,t)$	$u(x,t) = -2\,\partial\ln\theta/\partial x$
4.1	$\int_{-\infty}^{\infty}\theta(\xi,0)\,\phi(x-\xi,t)\,d\xi = \int_{-\infty}^{\infty} a_n\, e^{inx-n^2t}\,dn$ $a_n \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty}\theta(\xi,0)e^{-in\xi}\,d\xi$	$\dfrac{\int_{-\infty}^{\infty}\theta(\xi,0)\,\frac{\partial}{\partial\xi}\,\phi(x-\xi,t)\,d\xi}{\frac{1}{2}\int_{-\infty}^{\infty}\theta(\xi,0)\,\phi(x-\xi,t)\,d\xi} = \dfrac{\int_{-\infty}^{\infty} na_n\, e^{inx-n^2t}\,dn}{\frac{1}{2}\,i\int_{-\infty}^{\infty} a_n\, e^{inx-n^2t}\,dn}$
4.2	$F(x,t) + F(-x,t)$ $F(x,t) \equiv \frac{1}{2}\,e^{t-x}\,\mathrm{erfc}\,\frac{x-2t}{2\sqrt{t}}$	$2\cdot\frac{F(x,t) - F(-x,t)}{F(x,t) + F(-x,t)}$
	$\theta(x,0) = e^{\lvert x\rvert}$	$u(x,0) = -2\ \mathrm{sgn}\ x$
4.3	$G(x,t) + G(-x,t)$ $G(x,t) \equiv \frac{1}{2}\,e^{t-x}\,\mathrm{erfc}\,\frac{2t-x}{2\sqrt{t}}$	$2\cdot\frac{G(x,t) - G(-x,t)}{G(x,t) + G(-x,t)}$
	$\theta(x,0) = e^{-\lvert x\rvert}$	$u(x,0) = 2\ \mathrm{sgn}\ x$
4.4	$I^- + I + I^+$ $I^\pm \equiv \frac{1}{2}\,e^{\pm\frac{1}{4}R}\,\mathrm{erfc}\,\frac{\frac{1}{2}\pm x}{2\sqrt{t}}$ $I \equiv \frac{1}{2}\,e^{-\frac{1}{2}R(x-\frac{1}{2}Rt)}\left(\mathrm{erfc}\,\frac{x-\frac{1}{2}-Rt}{2\sqrt{t}} - \mathrm{erfc}\,\frac{x+\frac{1}{2}-Rt}{2\sqrt{t}}\right)$	$R\cdot\frac{I}{I^- + I + I^+}$
	$\theta(x,0) = \begin{cases} e^{-\frac{1}{2}Rx} & \text{in } \lvert x\rvert \le \frac{1}{2} \\ e^{\pm\frac{1}{4}R} & \text{in } \lvert x\rvert \ge \frac{1}{2}\end{cases}$	$u(x,0) = \begin{cases} R & \text{in } \lvert x\rvert > \frac{1}{2} \\ 0 & \text{in } \lvert x\rvert > \frac{1}{2}\end{cases}$
4.5	$x + 2\sqrt{t}\ i^1\,\mathrm{erfc}\ z$	$-\frac{2\ \mathrm{erf}\ z}{x + 2\sqrt{t}\ i^1\,\mathrm{erfc}\ z}$
	$\theta(x,0) = \lvert x\rvert$	$u(x,0) = -2/x$
4.5'	$1 + x + 2\sqrt{t}\ i^1\,\mathrm{erfc}\ z$	$-\frac{2\ \mathrm{erf}\ z}{1 + x + 2\sqrt{t}\ i^1\,\mathrm{erfc}\ z}$
	$\theta(x,0) = 1 + \lvert x\rvert$	$u(x,0) = -\frac{2\ \mathrm{sgn}\ x}{1 + \lvert x\rvert}$

(4.1). The source representation and Fourier-integral representation of $\theta(x, t)$ in terms of initial conditions arbitrary on the infinite interval. See (2.1) for the definition of ϕ.

(4.2). Figure 14: an initial compression step diffuses to the steady shock of (1.3). *References*: Cole [21], Lighthill [40].

(4.3.) Figure 15: an initial expansion step decays to zero.

(4.4). An initially square compression pulse decays to zero, with constant intensity $R \equiv \int_{-\infty}^{\infty} u\, dx$. For small R the pulse retains its symmetry (as in Figure 9). For large R it is convectively distorted (as in Figure 10). This solution and (3.1′) (in which the initial u is a delta function) are isomorphic in the sense that (3.1′) is obtained in the limit $\epsilon \to 0$, if in (4.4) the variables are shifted to x/ϵ, t/ϵ^2, ϵu. *Reference*: Jeng et al. [32], where the more general solution is given for a set of square waves of equal width and arbitrary amplitude in a finite interval.

(4.5). Figure 16: This similarity solution has an initial condition for u identical to the steady solution (1.7). It corresponds to decay in the absence of energy sources, whereas (1.7) must be maintained by an infinite source (see Introduction). See (3.0) for notation.

(4.5′). Figure 17. (Compare with Figure 12.) See (3.0) for notation.

5. Direct assignment of periodic initial conditions

	$\theta(x,t)$	$u(x,t) = -2\, \partial \ln\theta/\partial x$
5.1	$\int_{-\pi}^{\pi} \theta(\xi,0)\, \psi(x-\xi,t)\, d\xi =$ $= \sum_{n=-\infty}^{\infty} a_n\, e^{inx-n^2t}$ $a_n \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(\xi,0)\, e^{-in\xi}\, d\xi$	$\dfrac{\int_{-\pi}^{\pi} \theta(\xi,0) \frac{\partial}{\partial\xi} \psi(x-\xi,t)\, d\xi}{\frac{1}{2}\int_{-\pi}^{\pi} \theta(\xi,0)\, \psi(x-\xi,t)\, d\xi} =$ $= \dfrac{\sum_{n=-\infty}^{\infty} n\, a_n\, e^{inx-n^2t}}{\frac{1}{2}i \sum_{n=-\infty}^{\infty} a_n\, e^{inx-n^2t}}$
5.2	$a_0 + 2 \sum_{n=1}^{\infty} a_n\, e^{-n^2t} \cos nx$ $a_n \equiv (-)^n I_n(\frac{1}{2} R)$	$\dfrac{4 \sum_{n=1}^{\infty} n\, a_n\, e^{-n^2t} \sin nx}{a_0 + 2 \sum_{n=1}^{\infty} a_n\, e^{-n^2t} \cos nx}$
	$\theta(x,0) = e^{-\frac{1}{2}R \cos x}$	$u(x,0) = -R \sin x$

(5.1). The source representation and Fourier-series representation of $\theta(x, t)$ in terms of periodic initial conditions arbitrary in $-\pi < x < \pi$. See (2.6) for the definition of ψ.

(5.2). Figures 18 and 19: An initially simple-harmonic wave form decays to zero. In this solution, obtained by using the Fourier series for ψ (see (2.6)), I_n is the exponentially increasing Bessel function of the second kind. The Reynolds number R specifies the initial intensity of the wave. If R is small, diffusion dominates and the decay proceeds with little harmonic distortion of the wave form (Figure 18). This is expressed formally by the approximation $a_n \approx (-\frac{1}{4}R)^n/n!$ for $R \ll 1$, which makes the series converge rapidly. For large Reynolds number, convective distortion dominates initially and produces a typical sawtooth wave (Figure 19). This is expressed by $a_n \sim (-)^n \exp(\frac{1}{2}R)/(\pi R)^{1/2}$ for $R \gg n^2$, which makes the series converge slowly if t is small. More precisely, the sawtooth phase dominates in the interval $R^{-1} \ll t \ll 1$. Then $\theta(x, t) \approx \psi(x + \pi, t)$; that is, solution (5.2) is approximately equal to solution (2.6), apart from a phase shift of π in x. When $t \sim R^{-1}$ or less, there has not been sufficient time for the effect of convection to appear; when $t \sim 1$ or greater, the wave has been restored to its initial form (rapid convergence of the series) and is in the final period of decay. *References*: Cole [21], Lighthill [40].

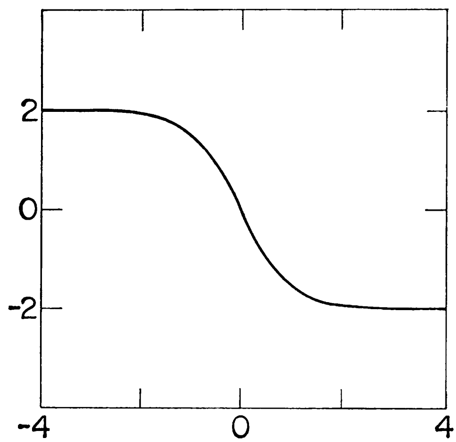

FIG. 1. Solution 1.3 (steady "shock").

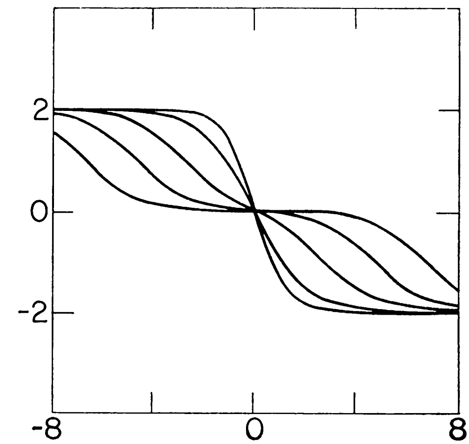

FIG. 2. Solution 1.3′ (+): $t = -6, -4, -2, 0, 2$.

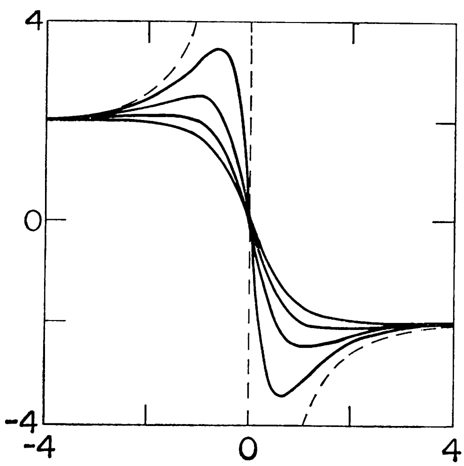

FIG. 3. Solution 1.3′ (−): $t = 0$ (broken), 0.2, 0.5, 1, 2.

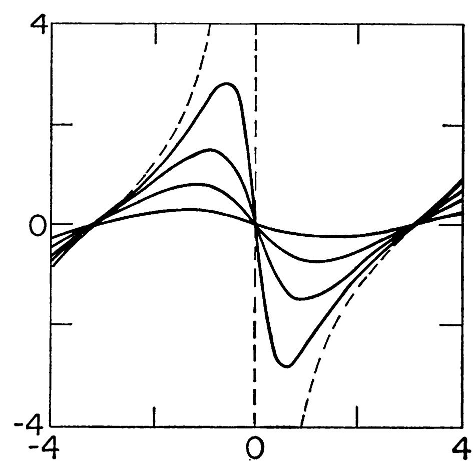

FIG. 4. Solution 1.5′ (−): $t = 0$ (broken), 0.2, 0.5, 1, 2.

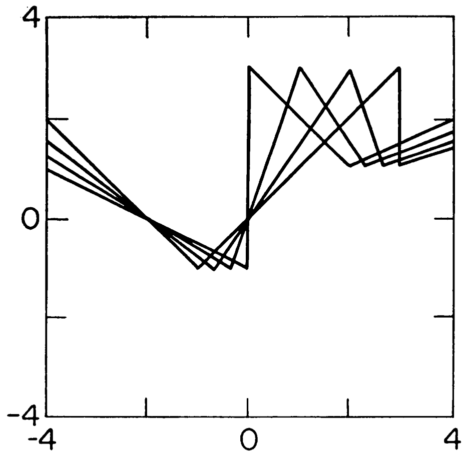

FIG. 5. Solution 2.1 (after Burgers [14]: $t = 0$, $\frac{1}{3}$, $\frac{2}{3}$, 1.

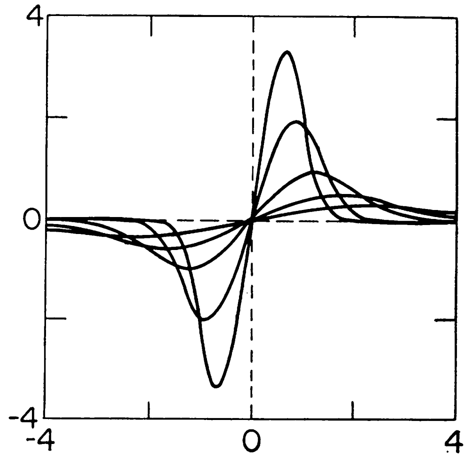

FIG. 6. Solution 2.1′ (+): $t = 0$ (broken), 0.1, 0.2, 0.5, 1, 2.

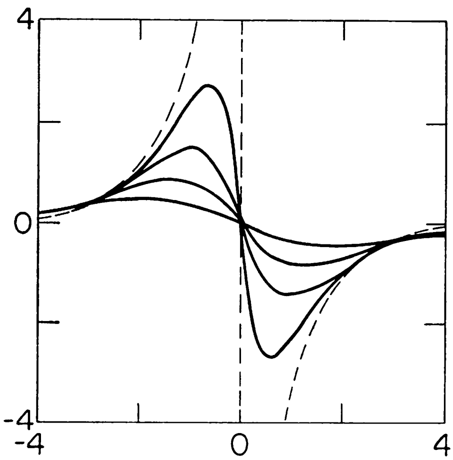

FIG. 7. Solution 2.1′ (−): $t = 1.0$ (broken), 1.2, 1.5, 2, 3.

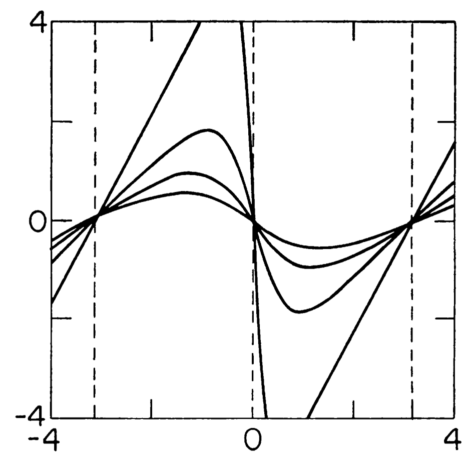

FIG. 8. Solution 2.6 (abscissa: $x - \pi$): $t = 0$ (broken), 0.5, 1.0, 1.5, 2.0.

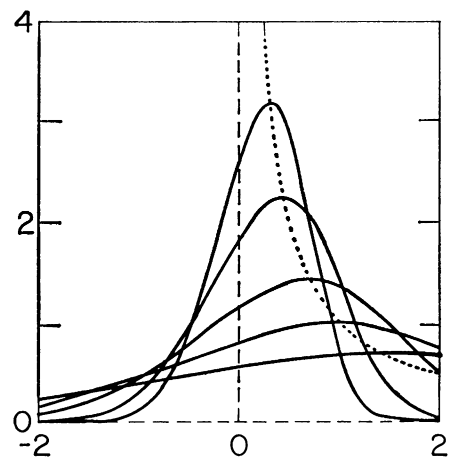

FIG. 9. Solution 3.1′, $R = 3.6$ ($\alpha = 0.40$): $t = 0$ (broken), 0.1, 0.2, 0.5, 1, 2 (dotted curve: locus of maxima).

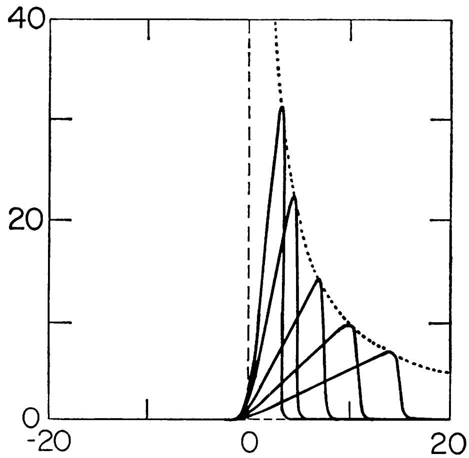

FIG. 10. Solution 3.1′, $R = 64$ ($\alpha = 3.0 \times 10^{-14}$): $t = 0$ (broken), 0.1, 0.2, 0.5, 1, 2 (dotted curve: locus of maxima).

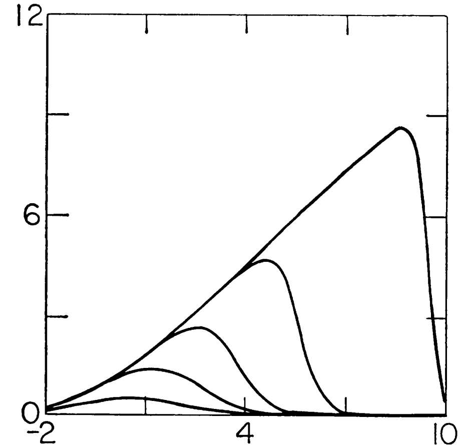

FIG. 11. Solution 3.1′: $t = 1$ for $R = 2, 5, 10, 20, 50$ (after Lighthill [40]).

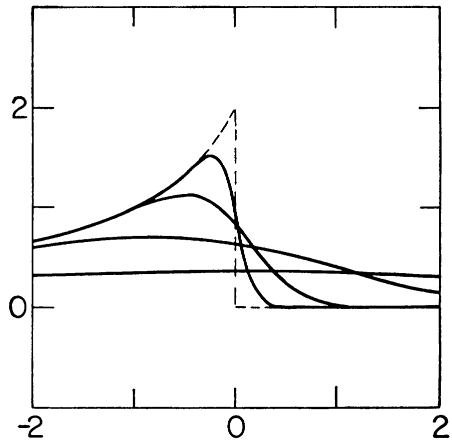

FIG. 12. Solution 3.2′ (+): $t = 0$ (broken), 0.01, 0.1, 1, 10 (Compare with Fig. 19.)

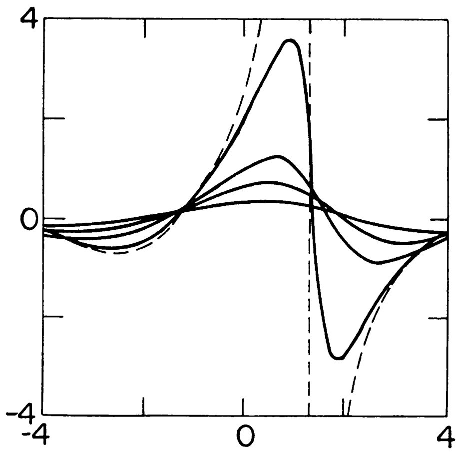

FIG. 13. Solution 3.5′ (−): $t = (2/e)^{1/2} = 0.86$ (broken), 1, 1.5, 2, 3.

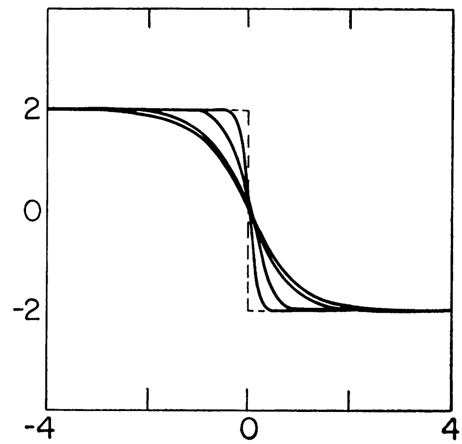

FIG. 14. Solution 4.2: $t = 0$ (broken), 0.01, 0.1, 1, 10.

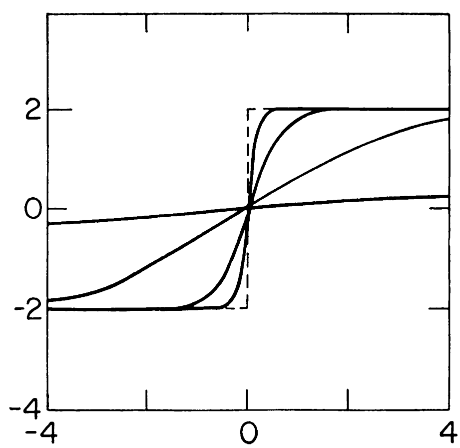

FIG. 15. Solution 4.3: $t = 0$ (broken), 0.01, 0.1, 1, 10.

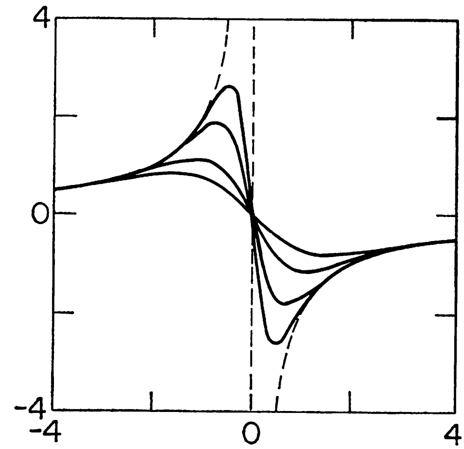

FIG. 16. Solution 4.5: $t = 0$ (broken), 0.1, 0.2, 0.5, 1.

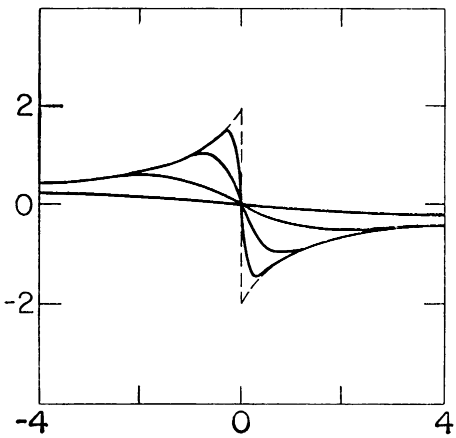

FIG. 17. Solution 4.5′: $t = 0$ (broken), 0.01, 0.1, 1, 10.

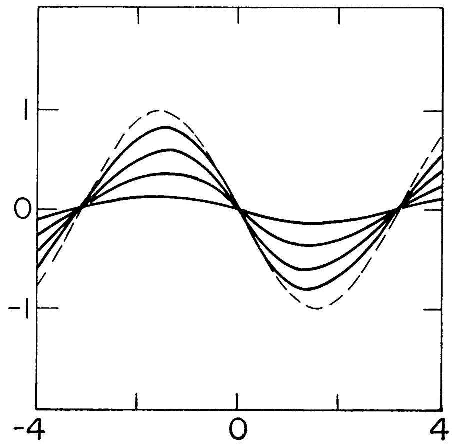

FIG. 18. Solution 5.2, $R = 1$: $t = 0$ (broken), 0.2, 0.5, 1, 2.

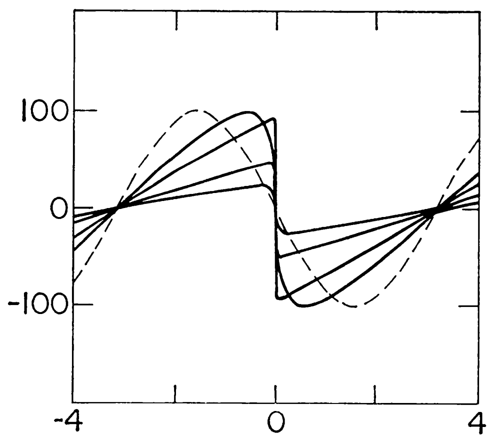

FIG. 19. Solution 5.2, $R = 100$: $t = 0$ (broken), 0.01, 0.02, 0.05, 0.1.