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On the Burgers equation with moving boundary

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Abstract

A general method for solving the Dirichlet problem for the Burgers equation with a moving boundary is introduced. The method reduces the initial value problem to a linear integral equation of Volterra type with mildly singular kernel, which admits a unique solution under rather general assumptions. Two explicit cases are considered: a boundary moving with constant velocity and a rapidly oscillating boundary. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

The Burgers equation

$$u_t = \left(\nu u_x - \sigma u^2\right)_x, \quad u \equiv u(x,t), \tag{1.1}$$

is well-known for its theoretical and practical interest (see, e.g., Refs. [1,2]). Boundary value problems for Eq. (1.1) also have great relevance in physical applications, and they have motivated several studies in the last few years (see, e.g., Refs. [3–9]). In particular, semiline solutions of Eq. (1.1) were obtained in Ref. [3] for the Dirichlet problem with a time dependent boundary condition (for the case of a constant boundary condition, see also Ref. [4]). The equivalence of such solutions to solutions of the forced Burgers equation was established in Refs. [5,6] in the case of an additive forcing term of distribution type. Moreover, the existence and uniqueness of semiline solutions of Eq. (1.1) was proven in Ref. [7] for a general class of boundary conditions at the origin, and explicit solutions were recently obtained in Refs. [8,9] in the case of flux-type boundary conditions.

All the above mentioned studies refer to the case of a static boundary. It is also the case that moving boundary problems have great interest both in mathematical and physical contexts (see, e.g., Refs. [10–13]). It is therefore natural to consider this kind of problems for the Burgers equation. In this Letter we introduce a general procedure to solve the Dirichlet problem for the Burgers equation in the case of a prescribed motion of the boundary. Namely, we consider Eq. (1.1) with initial–boundary conditions

$$u(x,0) = f(x), \tag{1.2a}$$

$$u\big(s(t),t\big) = g(t), \tag{1.2b}$$

where $s(t) \leqslant x < \infty$, $t \geqslant 0$ and $f(x)$, $s(t)$ and $g(t)$ are given functions of their arguments; in particular, $s(t)$ describes the motion of the boundary. Initial–boundary conditions such as (1.2) arise naturally in physical applications, and assume a special importance when considering the problem of a moving boundary be-

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tween two phases (see, e.g., Refs. [10,12]). It is important to note that, if the motion of the boundary is not known, one obtains a Stefan problem [14], whose solution demands the presence of an additional constraint, and requires the solution of a nonlinear integral equation [15,16].

The outline of this Letter is the following: In Section 2 we introduce an appropriate transformation of variables to linearize the problem given by (1.1) with (1.2). We reduce the problem to a linear integral equation of Volterra type in t, which admits a unique solution under the assumption that $s(t)$ is a smooth function of time and $g(t)$ is continuous and bounded. In Section 3 we consider the special case of a linear motion of the boundary, $s(t) = vt$, with $g(t) = c$ (v, c constants), and a step-like initial datum $f(x) = c\,\theta(x_0 - x)$, where $\theta(x)$ is the Heaviside step function. For moderate boundary velocities, the explicit solution behaves like a shock traveling to the right. This result is in agreement with the asymptotic behavior obtained when a Galileian transformation is made on the solution of the static boundary problem of Ref. [3]. In Section 4 we consider the case when $s(t)$ is a rapidly oscillating function of small amplitude, $s(t) = \varepsilon \sin(t/\varepsilon)$, with ε a small parameter. In this case we obtain the solution with a combination of analytical and numerical methods, and we compare the results to the unperturbed case $\varepsilon = 0$. Some technical details of the solution method are confined in the appendices. Finally, we mention that the same problem has been recently solved using a different but equivalent approach [17].

Since the values of the two constants ν and σ can be modified by trivial rescalings and in this Letter we are not interested in the limiting cases in which they vanish or diverge, we hereafter set $\nu = 1$ and $\sigma = \pm 1$, without loss of generality.

2. The linearizing transformation

In order to linearize (1.1) with initial–boundary conditions specified by (1.2) it proves convenient to perform the change of variables

$$u(x,t) = q(y,t), \tag{2.1a}$$

$$x = y + s(t). \tag{2.1b}$$

This transformation maps Eq. (1.1) into the following forced Burgers equation:

$$q_t - (q_{yy} + 2\sigma q q_y) = \dot{s}(t) q_y, \tag{2.2}$$

with initial and boundary conditions

$$q(y,0) = f(y), \tag{2.3a}$$

$$q(0,t) = g(t). \tag{2.3b}$$

The moving boundary problem (1.1)–(1.2) for the Burgers equation is then transformed into a fixed boundary problem for the parametrically forced nonlinear equation (2.2) in the quarter plane $y \geqslant 0$, $t \geqslant 0$ with boundary conditions (2.3).

We now introduce the generalized Cole–Hopf transformation [3]

$$q(y,t) = \frac{\phi(y,t)}{C(t) + \sigma \int_0^y dy'\, \phi(y',t)}, \tag{2.4a}$$

$$\phi(y,t) = C(t) q(y,t) \exp\left[\sigma \int_0^y dy'\, q(y',t)\right], \tag{2.4b}$$

with $C(0) = 1$. It is easily seen that this transformation implies that to the nonlinear (2.2) there corresponds the linear equation

$$\phi_t - \phi_{yy} = \dot{s}(t)\phi_y, \tag{2.5}$$

with the compatibility condition

$$\dot{C}(t) = \sigma \phi_y(0,t) + \sigma \dot{s}(t) \phi(0,t). \tag{2.6}$$

It is moreover clear that Eqs. (2.4) imply for the forced heat equation (2.5) the initial and boundary conditions

$$\phi(y,0) = C(0) f(y) \exp\left[\sigma \int_0^y dy'\, f(y')\right], \tag{2.7a}$$

$$\phi(0,t) = C(t) q(0,t) = C(t) g(t). \tag{2.7b}$$

In order to solve the linear equation (2.5) with the initial and boundary data (2.7) and the compatibility condition (2.6) we introduce the following generalized sine–Fourier transform:

$$\hat{\phi}(k,t,\gamma) = \int_0^\infty dy\, \sin\big[k\big(y + s(t) + \gamma\big)\big] \phi(y,t), \tag{2.8a}$$

which is inverted as

$$\phi(y,t)=\frac{2}{\pi}\int_0^\infty dk\,\sin\big[k\big(y+s(t)+\gamma\big)\big]\hat\phi(k,t,\gamma), \tag{2.8b}$$

provided that

$$s(t)+\gamma>0 \tag{2.9}$$

(see Appendix A), where explicit conditions on $f(x)$ for the existence of the above transform pair are also given). The parameter γ in the above formulae is arbitrary except for restriction (2.9) which must hold for all values of t. The solution of the linear problem $\phi(y,t)$ is, of course, independent of the value of γ (see Appendix A).

It is easily seen that (2.8a), (2.6) and (2.7) imply

$$\begin{aligned}\hat\phi(k,t,\gamma)\\ &=\hat\phi_0(k,\gamma)e^{-k^2t}\\ &\quad+k\int_0^t dt'\,\cos\big[k\big(s(t')+\gamma\big)\big]C(t')g(t')e^{-k^2(t-t')}\\ &\quad+\int_0^t dt'\,\sin\big[k\big(s(t')+\gamma\big)\big]\dot C(t')e^{-k^2(t-t')},\end{aligned} \tag{2.10}$$

where, of course,

$$\begin{aligned}\hat\phi_0(k,\gamma)&=\hat\phi(k,0,\gamma)\\ &=\int_0^\infty dy\,\sin\big[k\big(y+s(0)+\gamma\big)\big]\phi(y,0).\end{aligned} \tag{2.11}$$

As in the case of a static boundary, the strategy of solution is to assume temporarily that the function $C(t)$ is known (cf. Ref. [3]). The evaluation of $q(y,t)$ is then explicitly performed through the following three steps:

1. Given $f(y)$, compute $\phi(y,0)$ via (2.7a).
2. Compute $\phi(y,t)$ via (2.8b) and (2.10).
3. Recover $q(y,t)$ from $\phi(y,t)$ via (2.4a).

The explicit solution for $\phi(y,t)$, which is found by substituting (2.10) into (2.8b), reads

$$\begin{aligned}\phi(y,t)&=\int_0^\infty dy'\,\phi(y',0)\,I_1(y,y',t,0)\\ &\quad-\sigma\int_0^t dt'\,\dot C(t')\,I_1(y,0,t,t')\\ &\quad+\int_0^t dt'\,C(t')g(t')\,I_2(y,0,t,t'),\end{aligned} \tag{2.12}$$

with kernels $I_1(y,y',t,t')$, $I_2(y,y',t,t')$ given by

$$\begin{aligned}I_1(y,y',t,t')\\ &=\frac{1}{2\sqrt{\pi}(t-t')^{1/2}}\\ &\quad\times\big[e^{-(y-y'+s(t)-s(t'))^2/4(t-t')}\\ &\qquad-e^{-(y+y'+s(t)+s(t')+2\gamma)^2/4(t-t')}\big],\end{aligned} \tag{2.13a}$$

$$\begin{aligned}I_2(y,y',t,t')\\ &=\frac{1}{4\sqrt{\pi}(t-t')^{3/2}}\\ &\quad\times\big[\big(y-y'+s(t)-s(t')\big)\\ &\qquad\times e^{-(y-y'+s(t)-s(t'))^2/4(t-t')}\\ &\qquad+\big(y+y'+s(t)+s(t')+2\gamma\big)\\ &\qquad\times e^{-(y+y'+s(t)+s(t')+2\gamma)^2/4(t-t')}\big].\end{aligned} \tag{2.13b}$$

It is easily seen that, in the special case $s(t)=\gamma=0$, Eq. (2.12) reduces to the formulae for the semiline Burgers equation found in Ref. [3].

Let us emphasize that the final result for $q(y,t)$ is independent of the value of the parameter γ, which nevertheless enters in a non-trivial manner into the above equations. In fact, for every value of γ, Eqs. (2.12), (2.13) yield a different but equivalent representation of the solution $\phi(y,t)$ of the associated linear problem. In the following we choose γ to be positive and very large. This choice enables us to consider the asymptotic, large γ, expansion of $\phi(y,t)$ (obtained via (2.12) with (2.13)) and neglect all the exponentially small γ-dependent contributions. The functions $I_{1,2}(y,y',t,t')$ in this case reduce to

$$\begin{aligned}I_1(y,y',t,t')&=\frac{1}{2\sqrt{\pi}(t-t')^{1/2}}\\ &\quad\times e^{-(y-y'+s(t)-s(t'))^2/4(t-t')},\end{aligned} \tag{2.14a}$$

$$I_2(y,y',t,t') = \frac{y-y'+s(t)-s(t')}{4\sqrt{\pi}(t-t')^{3/2}} \times e^{-(y-y'+s(t)-s(t'))^2/4(t-t')}, \tag{2.14b}$$

with a great simplification of all subsequent calculations. However, it is essential to note that, since the value of $\phi(y,t)$ (as computed by (2.12)) is independent from the explicit value of γ, the first term (i.e., the γ-independent) in the large-γ asymptotic expansion yields an exact analytical result. Or, in other words, the use of Eqs. (2.14) instead of (2.13) does not constitute an approximation for $\phi(y,t)$.

Our last task is to compute $C(t)$. To this aim we evaluate (2.12) at $y=0$, $t=\tau$ and integrate in $d\tau/(t-\tau)^{1/2}$ from 0 to t. In this way, using (2.7b) together with (2.14), and after some appropriate integration by parts, we get the following linear integral equation for $C(t)$:

$$C(t) = 1 + H(t) + \int_0^t dt'\, K(t,t')C(t'), \tag{2.15}$$

where

$$H(t) = \frac{2}{\sqrt{\pi}}\left[A_1(t,0) + \sigma\int_0^\infty dy\,\phi(y,0)A_0(y,t)\right] - 1, \tag{2.16a}$$

$$K(t,t') = \frac{2}{\sqrt{\pi}}\left[\frac{\partial A_1(t,t')}{\partial t'} + \sigma A_2(t,t')g(t') - \sigma\frac{g(t')}{2(t-t')^{1/2}}\right], \tag{2.16b}$$

$$A_0(y,t) = \frac{1}{2\sqrt{\pi}}\int_0^t d\tau\,\frac{1}{(t-\tau)^{1/2}\tau^{1/2}} \times e^{-(y-s(\tau))^2/4\tau}, \tag{2.16c}$$

$$A_1(t,t') = \frac{1}{2\sqrt{\pi}}\int_{t'}^t d\tau\,\frac{1}{(t-\tau)^{1/2}(\tau-t')^{1/2}} \times e^{-(s(\tau)-s(t'))^2/4(\tau-t')}, \tag{2.16d}$$

$$A_2(t,t') = \frac{1}{4\sqrt{\pi}}\int_{t'}^t d\tau\,\frac{s(\tau)-s(t')}{(t-\tau)^{1/2}(\tau-t')^{3/2}} \times e^{-(s(\tau)-s(t'))^2/4(\tau-t')}. \tag{2.16e}$$

Eq. (2.15) is a linear integral equation of Volterra type with mildly singular kernel; it admits a unique continuous solution under the assumptions that $s(t)$ is a smooth function and $g(t)$ is continuous and bounded [18]. Formally differentiating Eq. (2.15), an integral equation for $\dot{C}(t)$ can be obtained, and, once the existence of $C(t)$ is established, it is straightforward to show that a unique continuous solution for $\dot{C}(t)$ (which is required to ensure the validity of the compatibility condition (2.6)) exists under the same hypotheses. The problem is thereby solved.

We should point out that in Ref. [17] the moving boundary problem for the Burgers equation was solved by transforming it into a *moving boundary problem for the unforced heat equation* through a different generalization of the Cole–Hopf transformation, and the solution of the linear problem was then obtained by employing Green's functions. Here, we transform the moving boundary problem for the Burgers equation into a forced problem for the Burgers equation on the half-line, and we then use a "standard" generalization of the Cole–Hopf transformation (as employed in Refs. [3,5–7,9]) to reduce the latter to a *forced problem for the heat equation on the half-line*. The heat equation is then solved by introducing a generalized sine transform. While the two solution methods obviously have a certain similarity, they are nonetheless clearly different, and they represent two different approaches to the problem. It is also immediate to see that in the case $s(t)=0$ Eq. (2.15) reproduces the static boundary result for $C(t)$ obtained in Ref. [3].

In the remaining part of this Letter we apply the formulae derived above in a few explicit examples. We will consider the case of a linear motion and of a rapid vibration of the boundary.

3. Linearly moving boundary

As a special case of the general formulae derived in the previous section we consider the situation of a linearly moving boundary, i.e., we take

$$s(t) = \nu t, \tag{3.1}$$

with $\sigma=-1$ and initial–boundary conditions given by

$$g(t) = c, \tag{3.2a}$$

$$f(x) = c\,\theta(x_0 - x), \tag{3.2b}$$

where c and x_0 are constant and $\theta(x)$ is the Heaviside step function.

The solution of the linear problem $\phi(y,t)$ can be decomposed as

$$\phi(y,t) = \phi_1(y,t) + \phi_2(y,t), \tag{3.3}$$

where $\phi_1(y,t)$ and $\phi_2(y,t)$ represent the parts of $\phi(y,t)$ which depend on the initial datum and on the boundary condition, respectively (cf. Eq. (2.12)). Explicitly, when Eqs. (3.1), (3.2) are used, Eqs. (2.12)–(2.14) yield

$$\phi_1(y,t) = \frac{c}{2(\pi t)^{1/2}} \int_0^{x_0} dy'\, e^{-cy'-(y-y'+vt)^2/4t}, \tag{3.4a}$$

$$\phi_2(y,t) = \frac{1}{4\pi^{1/2}} \int_0^t dt'\, e^{-(y+v(t-t'))^2/4(t-t')} \times \frac{1}{(t-t')^{1/2}} \left[2\dot{C}(t') + \left(cv + \frac{y}{t-t'} \right) \times C(t') \right]. \tag{3.4b}$$

Calculation of $\phi_1(y,t)$ is straightforward and yields

$$\phi_1(y,t) = \frac{1}{2} c\, e^{(c^2-vc)t-cy} \times \left\{ \operatorname{erfc}\left[\frac{y-x_0}{2t^{1/2}} - \left(c - \frac{1}{2}v \right) t^{1/2} \right] - \operatorname{erfc}\left[\frac{y}{2t^{1/2}} - \left(c - \frac{1}{2}v \right) t^{1/2} \right] \right\}. \tag{3.5a}$$

However, $\phi_2(y,t)$ depends on $C(t)$ and requires some attention. In Appendix B we solve the corresponding integral equation for $C(t)$. We will deal explicitly with two limiting cases: $v \ll c$ and $v \gg c$.

If $v \ll c$ the behavior of $C(t)$ for large t can be approximated by $C(t) \sim e^{(c^2-vc)t}$ (cf. Eq. (B.17a)). We substitute this asymptotic behavior in $\phi(y,t)$ (Eq. (3.4b)) and compute the resulting integrals using Laplace transforms and the same procedure used in Appendix B to invert the Laplace transform of $C(t)$. In this way we find

$$\phi_2(y,t) \underset{t\to\infty}{\sim} \frac{1}{2} c\, e^{(c^2-vc)t-cy} \times \operatorname{erfc}\left[\frac{y}{2t^{1/2}} - \left(c - \frac{1}{2}v \right) t^{1/2} \right]. \tag{3.5b}$$

Note that $\phi_2(y,t)$ exactly cancels the second part of $\phi_1(y,t)$. Calculating the integral of $\phi(y,t)$ is immediate, and allows us to obtain, via Eq. (2.4a), an asymptotic expansion for the solution of the forced Burgers equation, $q(y,t)$, in the comoving frame of reference. From $q(y,t)$ we go back to $u(y,t)$ via Eqs. (2.1). The corresponding expression is

$$u(x,t) \underset{t\to\infty}{\sim} c\, e^{c^2t-c(x-x_0)} \operatorname{erfc}\left[\frac{x-x_0}{2t^{1/2}} - ct^{1/2} \right] \Big/ \left\{ e^{c^2t-c(x-x_0)} \operatorname{erfc}\left[\frac{x-x_0}{2t^{1/2}} - ct^{1/2} \right] + \operatorname{erf}\frac{x-x_0}{2t^{1/2}} - \operatorname{erf}\frac{vt-x_0}{2t^{1/2}} \right\}. \tag{3.7}$$

As the corresponding formula in the static case (to which it reduces if $v \to 0$), this solution represents a shock of amplitude c that moves to the right with velocity c. More precisely, in the region $x < x_0 + ct$ the argument of the exponentials is positive, and the first term on the denominator dominates, so that the value of $u(x,t)$ is approximately equal to c. Vice versa, if $x < x_0 + ct$ the exponential term in the denominator is negligible, and $u(x,t)$ becomes exponentially small. The transition point between the two regions of the solution is found by setting to zero the argument of the exponentials, and is $x = x_0 + ct$. We observe that, for the specific case considered here, the propagation of the shock is unaffected by the presence of the moving boundary in the fixed frame of reference. That is, if $v < c$ the solution for $u(x,t)$ coincides asymptotically with that of the Burgers equation with static boundary.

A different result is found if $v \geqslant c$. In fact, in the appendix we show that, if $v \gg c$, $C(t) \simeq 1$ asymptotically in time (cf. Eq. (B.17b)). Then, by computing $\phi_2(y,t)$ with the same procedure used above, we find

$$\phi_2(y,t) \underset{t\to\infty}{\sim} \frac{1}{2} c\, e^{-vy} \operatorname{erfc}\left(\frac{y}{2t^{1/2}} - \frac{1}{2} vt^{1/2} \right), \tag{3.8}$$

while $\phi_1(y,t)$ is still given by (3.4a). Since $c - v \ll 0$, now $\phi_1(y,t)$ is exponentially smaller than $\phi_2(y,t)$. Hence, neglecting small terms, we can express the behavior of $q(x,y)$ as

$$q(y,t) \underset{t\to\infty}{\sim} c\, e^{-vy}. \tag{3.9}$$

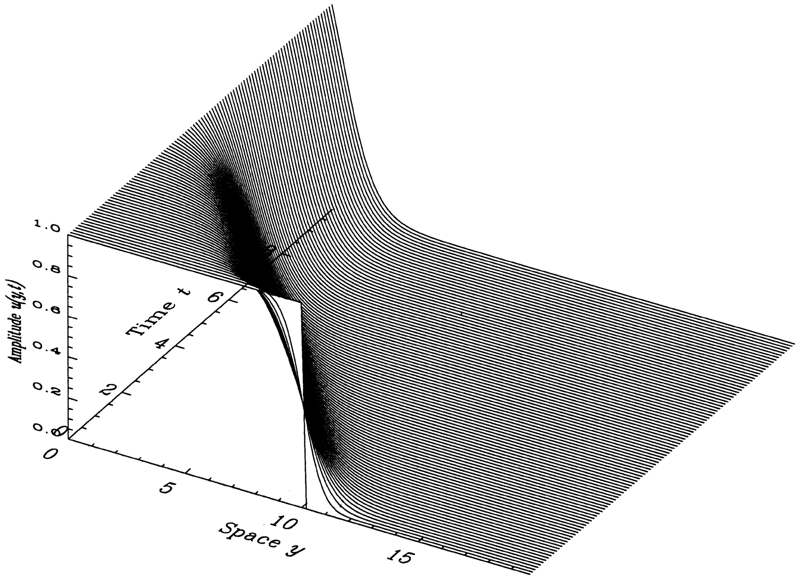

Fig. 1. The solution $q(y,t)$ of the forced Burgers equation in the moving frame of reference for a linearly moving boundary $s(t) = vt$, with $c = 1$ and $v = 1.5$. Since the velocity of the boundary is greater that that of the shock, in the moving frame the solution is seen to move backwards, towards the boundary $y = 0$.

Since $v \gg c$, the boundary moves much faster than the shock. Thus, after a sufficiently large time, it will eventually "hit" the shock, producing a boundary layer — in our particular case, a solution which decays exponentially fast away from the boundary, as shown by Eq. (3.9). To check this prediction, we integrated numerically the forced Burgers equation in the moving frame of reference (2.2), using a finite difference scheme in space and an adaptive variable-order backward differentiation time integrator. The resulting solution $q(y,t)$ in the moving frame of reference is shown in Fig. 1 for $c = 1$ and $v = 1.5$.

It is worth noticing that, in the special case of a linearly moving boundary, the solution of the Burgers equation can be found directly from Eq. (2.2), via a change of dependent variable. In fact, if $s(t) = vt$, Eq. (2.2) becomes

$$q_t = q_{yy} - 2\left(q - \tfrac{1}{2}v\right)q_y, \tag{3.10}$$

which, with the change of variable

$$w(y,t) = q(y,t) - \tfrac{1}{2}v, \tag{3.11}$$

is reduced to the standard semiline Burgers equation for w:

$$w_t = w_{yy} - 2ww_y, \tag{3.12}$$

together with

$$w(y,0) = f(y) - \tfrac{1}{2}v, \tag{3.13a}$$

$$w(0,t) = g(t) - \tfrac{1}{2}v. \tag{3.13b}$$

The problem specified by Eqs. (3.12), (3.12) can now be solved using the technique developed in Ref. [3]. If $g(t) = c$ and $f(y) = c\,\theta(x_0 - y)$ the solution for $q(y,t)$ represents a shock propagating to the right with velocity $c - v$. The corresponding expressions, when translated back into the original variables $q(y,t)$ and $u(x,t)$, are in perfect agreement with the results obtained with the general method outlined before. This explicit test provides a further check about the validity of the solution technique presented in the previous section.

4. Rapidly oscillating boundary

As a second example we discuss the case of a periodic, high-frequency and small-amplitude motion of the boundary. That is, we let

$$s(t) = \varepsilon \sin(t/\varepsilon), \tag{4.1}$$

with $\sigma = -1$ and $\varepsilon \ll 1$. Also, we take a constant boundary condition and a quasi-null initial datum:

$$f(x) = c\,\theta_-(x), \tag{4.2a}$$

$$g(t) = c, \tag{4.2b}$$

where $\theta_-(x)$ is defined as

$$\theta_-(x) := \theta(-x+0), \tag{4.3}$$

so that $\theta_-(0) = c$ and $\theta_-(x) = 0$ for all $x > 0$. This particular choice of initial condition is required in order to satisfy the compatibility condition $f(0) = g(0)$. It should be emphasized that the singularity at the origin at $t = 0$ does not represent a problem, since the parabolic nature of the Burgers equation implies that the discontinuity is smoothed out at any $t \neq 0$, just like in the case of a linearly moving boundary (see also Appendix B).

We observe that, unlike the case of a linear motion of the boundary, this problem cannot be solved with any simple change of variable like (3.11). At the same time, the motion of the boundary is confined to a region which is very close to the origin. Therefore we treat $s(t)$ as a perturbation to the static problem. That is, we expand

$$u(x,t) = \varepsilon^{n_0} u^{(n_0)}(x,t) + \varepsilon^{n_1} u^{(n_1)}(x,t) + \varepsilon^{n_2} u^{(n_2)}(x,t) + \cdots, \tag{4.4}$$

where the exponents n_j and the corresponding terms $u^{(n_j)}(x,t)$ are to be determined in the expansion.

The first contribution to (4.4) is obtained when $n_0 = 0$, and it yields the unperturbed solution, $u^{(0)}(x,t)$. It represents a shock moving to the right with velocity c, and is obtained by Eq. (3.7) taking $v = x_0 = 0$. To compute the corrections to $u^{(0)}(x,t)$ we substitute (4.2) into (2.15), and write a corresponding small-ε asymptotic expansion for $C(t)$. Again, the unperturbed solution is given by the case of a static boundary: $C^{(0)}(t) = e^{c^2 t}(1 + \operatorname{erf}(ct^{1/2}))$ (cf. Eq. (B.10)). In Appendix B we study the integral equation for $C(t)$ and we show that: (i) the asymptotic expansion contains semi-integer powers of ε; (ii) the first correction to the unperturbed solution is $O(\varepsilon^{3/2})$. As a result, the expansion of $u(x,t)$ will also contain semi-integer powers of ε. Moreover, to compute corrections to $u(x,t)$ up to $O(\varepsilon)$ we can use the unperturbed solution of $C(t)$. Therefore we substitute the asymptotic behavior of the static case, $C(t) \sim 2\,e^{c^2 t}$, in Eq. (2.12). This allows us to write $\phi(y,t)$ as

$$\phi(y,t) \underset{t\to\infty}{\sim} J\big(y + \varepsilon \sin(t/\epsilon), t\big) + O\big(\varepsilon^{3/2}\big), \tag{4.5}$$

where we have defined

$$J(\zeta,t) := \frac{c}{\sqrt{\pi}}\left(c - \frac{\partial}{\partial \zeta}\right) \int_0^t dt'\, \frac{1}{(t-t')^{1/2}} \times e^{c^2 t' - (\zeta - \varepsilon \sin(t'/\varepsilon))^2/4(t-t')}. \tag{4.6}$$

If we let $\varepsilon = 0$ in (4.6) we can compute $J(\zeta,t)$ exactly, and we recover the static result for $u(x,t)$, which is a special case of the shock solution found in previous section:

$$u^{(0)}(x,t) \underset{t\to\infty}{\sim} \frac{c\, e^{c^2 t - cx} \operatorname{erfc}[(x-2ct)/2t^{1/2}]}{e^{c^2 t - cx} \operatorname{erfc}[(x-2ct)/2t^{1/2}] - \operatorname{erf}[x/2t^{1/2}]}. \tag{4.7}$$

To obtain the correction to this value we need to compute the integral in Eq. (4.6) for values of $\varepsilon \neq 0$. We do this numerically, using an adaptive numerical quadrature routine from the NAG library (code D01APF). To check our model, we integrate numerically the Burgers equation with periodic boundary (Eq. (2.2)), using a finite difference scheme in space and an adaptive variable-order backward differentiation time integrator (code D01EJF from the NAG library). The results of these simulations confirm the theoretical predictions based on the asymptotic expansion of the solution (Eq. (4.5)). The resulting solution $q(y,t)$ in the moving frame of reference is shown in Fig. 2 for $\varepsilon = 1/\pi$.

The solution $q(y,t)$ is now represented by a shock whose spatial location oscillates in time around the position of the unperturbed shock. That is, the correction to the unperturbed solution appears as a modification of the values in the transition region of the shock. This result can be explained by noting that, in Eq. (2.2), the only region in which the parametric forcing term $s(t)q_y$ can have an effect on the solution is where the spatial derivative of u is different from zero. In our case, this condition is verified only in the transition region of the shock solution. However, it is interesting to note that, when the solution is translated into the fixed frame of reference, the oscillations tend to disappear, and the solution $u(x,t)$ is found to be extremely simi-

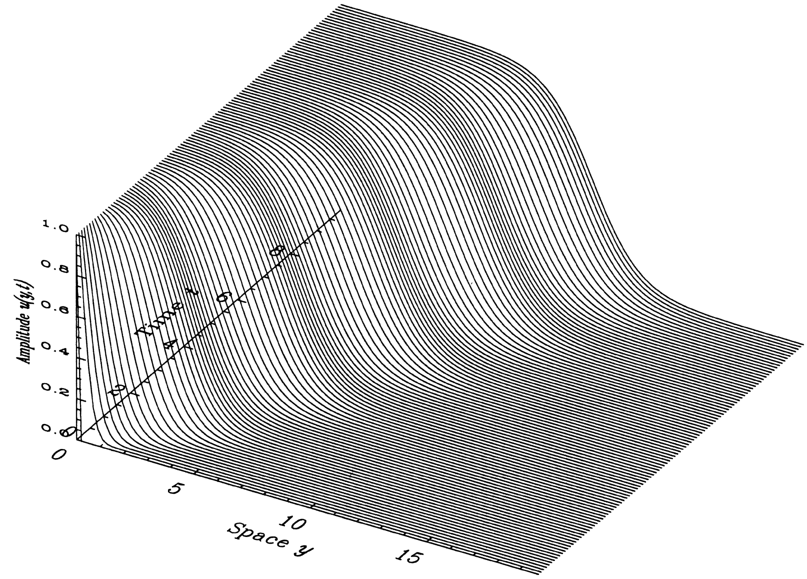

Fig. 2. The solution $q(y,t)$ of the forced Burgers equation in the moving frame of reference for a periodic boundary $s(t) = \varepsilon \sin(t/\varepsilon)$, with $c = 1$ and $\varepsilon = 1/\pi$ as computed numerically (see Section 4).

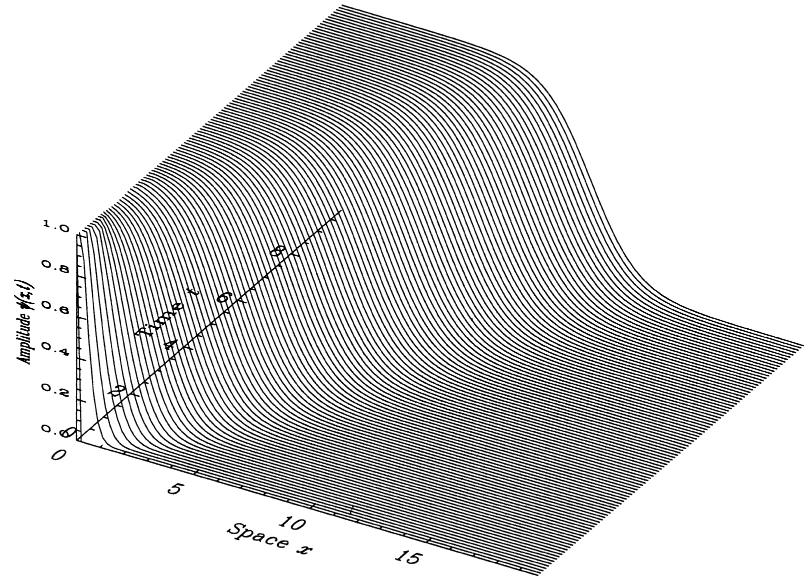

Fig. 3. The solution $u(x,t)$ of the Burgers equation in the fixed frame of reference corresponding to $q(y,t)$, for the case shown in Fig. 2, i.e., a periodic boundary $s(t) = \varepsilon \sin(t/\varepsilon)$, with $c = 1$ and $\varepsilon = 1/\pi$.

lar to the solution in the unperturbed case, as shown in Fig. 3. In fact, the only visible effect of the oscillating boundary is a constant time shift in the position of the shock, as shown in Fig. 4.

Acknowledgements

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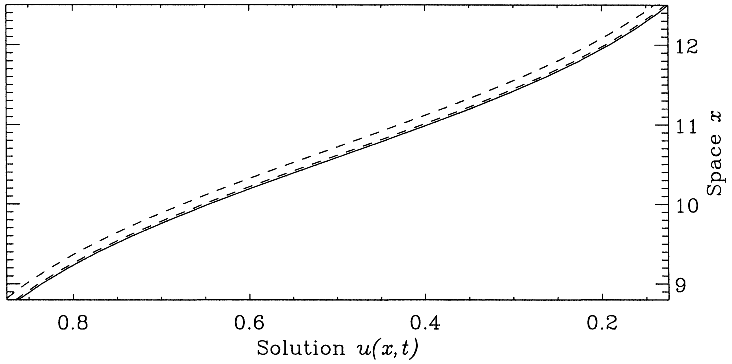

Fig. 4. A comparison between the solution of the Burgers equation for a periodic boundary in the fixed frame of reference, $u(x,t)$, and the corresponding solution in the unperturbed case, for $t=10$. Solid line: $\varepsilon = 1/\pi$, dashed line: $\varepsilon = 0$.

Appendix A. The generalized sine transform

In this section we consider the generalized sine–Fourier transform (2.8a) introduced in Section 2,

$$\tilde{\phi}(k,t,\gamma) = \int_0^\infty dx\, \sin\big[k\big(x+s(t)+\gamma\big)\big]\phi(x,t), \tag{A.1a}$$

and we show that, in the appropriate functional space, it is inverted through

$$\phi(x,t) = \frac{2}{\pi}\int_0^\infty dk\, \sin\big[k\big(x+s(t)+\gamma\big)\big]\tilde{\phi}(k,t,\gamma), \tag{A.1b}$$

provided that

$$s(t)+\gamma > 0. \tag{A.2}$$

We consider an initial datum $\phi(x,0) \in L^1 \cap L^2[0,\infty)$. Using (2.10), it is easy to see that, if $\phi(x,0)$ is in this class, $\phi(x,t)$ remains in the same class, provided that $C(t)$, $\dot{C}(t)$ and $g(t)$ are continuous functions of time. This condition also ensures the existence of the usual sine and cosine Fourier transform, defined as

$$\tilde{\phi}_s(k,t) = \int_0^\infty dx\, \sin kx\, \phi(x,t), \tag{A.3a}$$

$$\tilde{\phi}_c(k,t) = \int_0^\infty dx\, \cos kx\, \phi(x,t), \tag{A.3b}$$

together with the inversion formulae:

$$\phi(x,t) = \frac{2}{\pi}\int_0^\infty dk\, \sin kx\, \tilde{\phi}_s(k,t) = \frac{2}{\pi}\int_0^\infty dk\, \cos kx\, \tilde{\phi}_c(k,t). \tag{A.4}$$

Therefore we can rewrite (A.1a) as

$$\tilde{\phi}(k,t,\gamma) = \cos\big[k\big(s(t)+\gamma\big)\big]\tilde{\phi}_s(k,t,\gamma) + \sin\big[k\big(s(t)+\gamma\big)\big]\tilde{\phi}_c(k,t,\gamma), \tag{A.5}$$

which proves the existence of $\tilde{\phi}(k,t,\gamma)$. In the same way we can prove the existence of the RHS of (A.1b). Therefore all is left to prove is that, when Eq. (A.1a) is substituted back into the right-hand side of (A.1b), there results an identity. By virtue of Fubini's theorem we can interchange the order of integration. Then, making use of dominated convergence and the integral representation of the Dirac delta

$$\delta(z) = \frac{1}{2\pi}\int_{-\infty}^\infty dk\, e^{ikz} \tag{A.6}$$

(which is valid for the class of functions considered), we get

$$\phi(x,t) = \int_0^\infty dx'\, \phi(x',t)\delta(x-x')$$

$$- \int_0^\infty dx'\,\phi(x',t)\delta\bigl(x + x' + 2\bigl(s(t)+\gamma\bigr)\bigr). \tag{A.7}$$

Eq. (A.7) reduces to an identity provided that the second integral is zero for all $x' \geqslant 0$. In turn this requirement holds if $x + x' + \gamma + s(t) > 0$ for all $x, x' \geqslant 0$, which yields condition (A.2).

Appendix B. The integral equation

In this section we consider the integral equation (2.15) for the function $C(t)$ derived in Section 2, and we show how it can be solved in a number of cases. For convenience, we rewrite here the integral equation:

$$C(t) = 1 + H(t) + \int_0^t dt'\,K(t,t')C(t'), \tag{B.1}$$

where, taking $\sigma = -1$,

$$H(t) = -\frac{2}{\sqrt{\pi}} \int_0^\infty dy\,\phi(y,0)A_0(y,t) + \frac{2}{\sqrt{\pi}} A_1(t,0) - 1, \tag{B.2a}$$

$$K(t,t') = \frac{2}{\sqrt{\pi}} \frac{\partial A_1(t,t')}{\partial t'} - \frac{2}{\sqrt{\pi}} A_2(t,t')g(t') + \frac{g(t')}{\sqrt{\pi}(t-t')^{1/2}}, \tag{B.2b}$$

$$A_0(y,t) = \frac{1}{2\sqrt{\pi}} \int_0^t d\tau\, \frac{1}{(t-\tau)^{1/2}\tau^{1/2}} \times e^{-(y-s(\tau))^2/4\tau}, \tag{B.2c}$$

$$A_1(t,t') = \frac{1}{2\sqrt{\pi}} \int_{t'}^t d\tau\, \frac{1}{(t-\tau)^{1/2}(\tau-t')^{1/2}} \times e^{-(s(\tau)-s(t'))^2/4(\tau-t')}, \tag{B.2d}$$

$$A_2(t,t') = \frac{1}{4\sqrt{\pi}} \int_{t'}^t d\tau\, \frac{s(\tau)-s(t')}{(t-\tau)^{1/2}(\tau-t')^{3/2}} \times e^{-(s(\tau)-s(t'))^2/4(\tau-t')}, \tag{B.2e}$$

and

$$\phi(y,0) = f(y)\exp\left[-\int_0^y dy'\,f(y')\right]. \tag{B.3}$$

We will deal with three explicit cases. The first two are relative a linearly moving boundary, $s(t) = vt$, with $g(t) = c$. First we concentrate on the homogeneous case, i.e., $f(x) = 0$; then we deal with a step-like initial datum, $f(x) = c\,\theta(x_0 - x)$. In both cases we obtain the static result in the limit $v \to 0$. Finally, we estimate the first correction to the unperturbed solution for a rapidly oscillating boundary of the type $s(t) = \varepsilon \sin(t/\varepsilon)$, with $g(t) = c$ and $f(x) = 0$.

We start with the case of a linearly moving boundary, constant boundary condition and quasi-zero initial datum, i.e., we take $s(t) = vt$ and $g(t) = c$ (with v, c constant). Note that the compatibility condition $f(0) = g(0)$ does not allow us to take $f(x) \equiv 0$, and requires us to use $f(x) = c\,\theta_-(x)$, where $\theta_-(x)$ is defined as $\theta_-(x) := \lim_{\varepsilon\to 0^+} \theta(-x+\varepsilon)$. Note also that, even if the initial datum is discontinuous, at any time $t \neq 0$ the discontinuity is smoothed out due to the parabolic nature of the partial differential equation, as in the case of a step-like initial condition for the Burgers equation on the infinite line (cf. Ref. [2]).

With the previous assumptions the integral Eq. (B.1) is

$$C(t) = 1 + H(t) + \int_0^t dt'\,K(t-t')C(t'), \tag{B.4}$$

where

$$H(t) = B(t) - 1, \tag{B.5a}$$

$$K(t) = \frac{c}{(\pi t)^{1/2}} - \dot{B}(t) - kcB(t), \tag{B.5b}$$

$$B(t) = e^{-(1/2)k^2 t} I_0\bigl(\tfrac{1}{2}k^2 t\bigr), \tag{B.5c}$$

the $I_\mu(x)$ being the Bessel functions of an imaginary argument [19], and $k \equiv \frac{1}{2}v$. We solve Eq. (B.4) using Laplace transforms: if

$$\tilde{C}(s) := \mathcal{L}_s\bigl[C(t)\bigr] = \int_0^\infty dt\,e^{-st}C(t), \tag{B.6}$$

Eqs. (B.4)–(B.5c) imply

$$\tilde{C}(s) = \bigl[s - c\bigl(s+k^2\bigr)^{1/2} + kc\bigr]^{-1}. \tag{B.7}$$

Thus the solution of the integral Eq. (B.4) is reduced to the problem of finding the inverse transform of $\tilde{C}(s)$. Explicitly,

$$C(t) = \mathcal{L}_t^{-1}\big[\tilde{C}(s)\big] = \frac{1}{2\pi i}\int_{\Gamma-i\infty}^{\Gamma+i\infty} ds\, \frac{e^{st}}{s - c(s+k^2)^{1/2} + kc} = \frac{1}{2\pi i}\, e^{-k^2 t}\int_{\Gamma-i\infty}^{\Gamma+i\infty} ds\, \frac{e^{st}}{\zeta(s)}, \tag{B.8}$$

where $\zeta(s) = s - cs^{1/2} + kc - k^2$ and Γ is any real number to the right of all the singularities of the integrand. We use the principal branch of the square root, with $-\pi \leqslant \arg(s) < \pi$. The function $\zeta(s)$ is analytic everywhere except on the negative real axis, where it has a branch cut. Also, $\zeta(s)$ has two simple zeros at $s = \kappa_\pm^2$, where $\kappa_\pm = \frac{1}{2}(c \pm |c-2k|)$. Hence we can use the contour of integration illustrated in Fig. 5 (see, e.g., Ref. [21]). It is easy to see that the integrals over $C_{\pm R}$ and C_ε go to zero as $R \to \infty$ and $\varepsilon \to 0$.

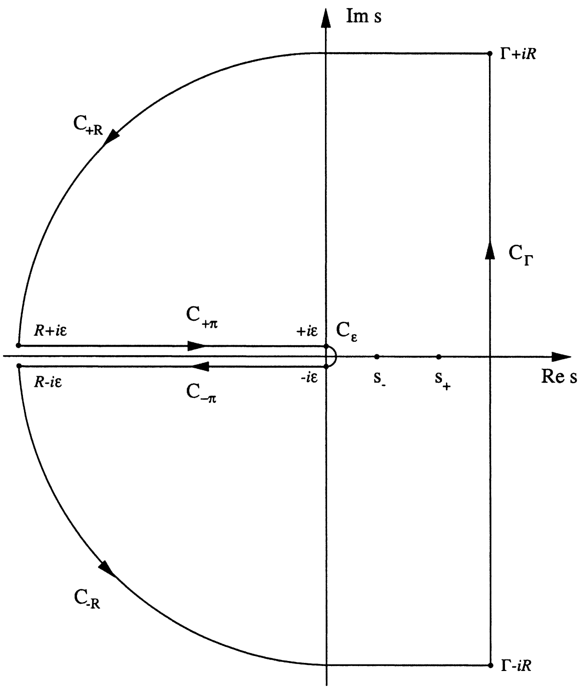

Fig. 5. The contour of integration for the inversion of the Laplace transform (B.7).

Then, by computing the integrals over $C_{\pm\pi}$ and the pole contributions, we find

$$C(t) = \frac{e^{-k^2 t}}{|c-2k|}\Big\{\kappa_+ e^{\kappa_+^2 t}\big[1 + \mathrm{erf}\big(\kappa_+ t^{1/2}\big)\big] - \kappa_- e^{\kappa_-^2 t}\big[1 + \mathrm{erf}\big(\kappa_- t^{1/2}\big)\big]\Big\}. \tag{B.9}$$

In particular, if $v = 0$, it is $\kappa_+ = c$, $\kappa_- = 0$, and we obtain the value of $C(t)$ for a static boundary:

$$C(t) = e^{c^2 t}\big[1 + \mathrm{erf}\big(ct^{1/2}\big)\big]. \tag{B.10}$$

Note that, asymptotically in time, $C(t) \sim 2\,e^{c^2 t}$.

Now we turn our attention to the inhomogeneous case. In particular, we consider $f(x) = c\,\theta(x_0 - x)$, where $\theta(x)$ is the Heaviside step function. The only change in Eq. (B.4) is

$$H(t) = B(t) - 1 - \frac{c}{\pi}\int_0^{x_0} dy\, e^{-cy} \times \int_0^t d\tau\, \frac{e^{-(y-vt)^2/4\tau}}{(t-\tau)^{1/2}\tau^{1/2}}, \tag{B.11}$$

which yields

$$\tilde{C}(s) = \frac{(s+k^2)^{1/2} + c\,e^{-[c+(s+k^2)^{1/2}]x_0}}{[c + (s+k^2)^{1/2}][s - c(s+k^2)^{1/2} + kc]}. \tag{B.12}$$

If $k = 0$ (i.e., for a static boundary) Eq. (B.12) reduces to

$$\tilde{C}(s) = \frac{1}{s - c^2}\left[1 - \frac{c}{s^{1/2}}\, e^{-(c+s^{1/2})x_0}\right]. \tag{B.13}$$

Then, using the same procedure adopted to invert (B.7) we find

$$C(t) = e^{c^2 t}\left[1 - \cosh cx_0 + e^{-2cx_0} + \frac{1}{2}e^{cx_0}\,\mathrm{erf}\left(ct^{1/2} + \frac{x}{2t^{1/2}}\right) - \frac{1}{2}e^{cx_0}\,\mathrm{erf}\left(ct^{1/2} - \frac{x}{2t^{1/2}}\right)\right]. \tag{B.14}$$

The asymptotic behavior of the solution is

$$C(t) \underset{t\to\infty}{\sim} 2\,e^{c^2 t - cx_0}\cosh cx_0. \tag{B.15}$$

In general, we see that the behavior of $C(t)$ depends on the value of x_0: if $cx_0 \gg 1$ we find $C(t) \sim e^{c^2 t}$, while if $x_0 \to 0$, we recover the result found in the homogeneous case: $C(t) \sim 2\,e^{c^2 t}$.

The same analysis can be performed when $k \neq 0$. However, since in this case we are only interested in the asymptotic behavior of $C(t)$, it is sufficient to look at the poles of $\tilde{C}(s)$ and compute the relevant residues. In this way we find

$$C(t) \underset{t\to\infty}{\sim} \frac{1}{|c-2k|}\, e^{-k^2 t} \times \left\{ \frac{2\kappa_+}{c+\kappa_+}\, e^{\kappa_+^2 t}\big[\kappa_+ + c\, e^{-(c+\kappa_+)x_0}\big] - \frac{2\kappa_-}{c+\kappa_-}\, e^{\kappa_-^2 t}\big[\kappa_- + c\, e^{-(c+\kappa_-)x_0}\big]\right\}. \tag{B.16}$$

In particular, we consider two limiting cases: $c \gg v$ and $c \ll v$ (with $cx_0 \gg 1$). In these two cases Eq. (B.16) becomes, respectively,

$$C(t) \underset{t\to\infty}{\sim} e^{c(c-v)t} \quad \text{if } c \gg k, \tag{B.17a}$$

$$C(t) \underset{t\to\infty}{\sim} 1 \quad \text{if } c \ll k. \tag{B.17b}$$

Our final task is the estimate of the correction to the unperturbed solution for a periodic boundary with large frequency and small amplitude. That is, we consider $s(t) = \varepsilon \sin(t/\varepsilon)$, $g(t) = c$ and $f(x) = \theta_-(x)$, with $\varepsilon \ll 1$. The unperturbed solution is obtained by taking $\varepsilon = 0$ and coincides with the solution of the static problem (Eq. (B.10)): $C(t) \sim 2\,e^{c^2 t}$. To compute the correction to this value we need to estimate the integrals $A_{1,2}(t,t')$ appearing in (B.1):

$$A_1(t,t') = \frac{1}{2\sqrt{\pi}} \int_{t'}^{t} d\tau\, \frac{1}{(t-\tau)^{1/2}(\tau-t')^{1/2}} \times e^{-\varepsilon^2(\sin(\tau/\varepsilon)-\sin(t'/\varepsilon))^2/4(\tau-t')}, \tag{B.18a}$$

$$A_2(t,t') = \frac{\varepsilon}{4\sqrt{\pi}} \int_{t'}^{t} d\tau\, \frac{\sin(\tau/\varepsilon)-\sin(t'/\varepsilon)}{(t-\tau)^{1/2}(\tau-t')^{3/2}} \times e^{-\varepsilon^2(\sin(\tau/\varepsilon)-\sin(t'/\varepsilon))^2/4(\tau-t')}. \tag{B.18b}$$

By expanding the exponentials in powers of ε it is immediate to see that the first correction to $A_1(t,t')$ is $O(\varepsilon^2)$ and that, to leading order,

$$A_2(t,t') = \frac{\varepsilon}{4\sqrt{\pi}(t-t')^{1/2}} \times \left[\cos(t'/\varepsilon)\int_0^1 du\, \frac{\sin[(t-t')u/\varepsilon]}{(1-u)^{1/2}u^{3/2}} + \sin(t'/\varepsilon)\int_0^1 du\, \frac{1-\cos[(t-t')u/\varepsilon]}{(1-u)^{1/2}u^{3/2}}\right]. \tag{B.19}$$

Both integrals in Eq. (B.19) are always finite for every value of t, t' and ε. However they have different behavior, since the second one contains a non-oscillatory part which yields the dominant contribution. The oscillatory parts of both integrals are easily expanded in asymptotic series in powers of ε (see, for example, Ref. [20]) and yield contributions which contain all half-integer powers of ε, thus producing corrections that are $O(\varepsilon^{3/2})$ and higher. The non-oscillatory part of the second integral is computed by the same procedure used to invert the Laplace transform of $C(t)$ in the case of a linearly moving boundary. In this way we find the following result:

$$A_2(t,t') = -\frac{1}{2}\varepsilon^{1/2} e^{-\varepsilon/4(t-t')} \sin\frac{t'}{\varepsilon} + O\big(\varepsilon^{3/2}\big). \tag{B.20}$$

When substituted back into $K(t,t')$, this result allows us to obtain the first correction to the unperturbed solution $C(t)$. Again, we expand the resulting integrals in asymptotic series in powers of ε, and find the first correction to $C(t)$ as

$$C^{(3/2)}(t) = \frac{2}{\sqrt{\pi}} c\,\varepsilon^{3/2} e^{-\varepsilon/4t}. \tag{B.21}$$

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