Explaining Early Bidding in Informationally-Restricted, Ascending-Bid Auctions

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Abstract

We introduce a dynamic model of rationally-inattentive bidding to explain early bidding in online auctions we call *Korean auctions* invented in 2002 by an executive of a rental-car company in Korea to thwart suspected collusion. Bidders do not see each others' bids or identities or even the number of other bidders. The only information bidders receive is whether their high bid is is the highest bid among all bidders at each instant in the auction. We argue that perfect Bayesian equilibrium models cannot explain the early bidding behavior we observe at these auctions. We introduce a new dynamic model of rationally-inattentive bidding subject to bidding frictions under the concept of *anonymous equilibrium*. This model can predict early bidding and final high bids, but underpredicts the magnitude of first bids—a phenomenon we refer to as *early overbidding*. This is inconsistent with rational competitive bidding behavior as well as the hypothesis of collusion because auction prices would be lower and bidder profits would be significantly higher if they used algorithmic bidding strategies based on our structural model as their "agents". However if all bidders in the auction used frictionless versions of our optimal bidding algorithm as their agents, there would be no early bidding in anonymous equilibrium, efficiency would be 100%, and auction prices would be 3% higher that what we observe in actual auctions. We predict auction revenue would have been even higher, by 12%, had the executive used static second-price sealed-bid auctions to sell cars.

Keywords: dynamic auctions; behavioral economics; used cars; wholesale auctions; ascending-bid, sealed-bid, second-bid, English, Japanese, open-outcry, and Korean auctions; bid sniping; jump-bidding and bid creeping; collusion; algorithmic collusion; informational restrictions; perfect Bayesian equilibrium; anonymous equilibrium; rational inattention; bidding frictions; early bidding and overbidding; bounded rationality; dynamic programming; structural estimation; quasi-maximum likelihood estimation; empirical mechanism design; bidding paradox.

JEL Classification Numbers: C57, C61, C72.

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1 Introduction

We structurally estimate a dynamic model of bidding that we developed to explain early bidding behavior in online auctions we refer to as *Korean auctions*, a new type of auction invented in 2002 by an executive of a rental-car company in Korea.[1] A Korean auction is similar to a traditional oral, ascending-bid auction (often referred to as an *English auction*), except that it is conducted online, lasts for two minutes, and is informationally restricted: bidders are unable to observe each other's bids, the number of other bidders, or their identities. At each instant, bidders only know whether they hold the current highest bid. At the end of the auction, the bidder who submitted the highest bid wins and pays the highest bid they submitted at the auction. Time priority is enforced: a bidder who matches the current high bid does not become the new current high bidder.

The rental-car company sells hundreds of it vehicles in wholesale auctions held each month. Before the invention of the Korean auction, the company sold its vehicles at different rental-car locations using oral, ascending-bid (English) auctions, where bidders verbally cried out bids under the oversight of an auctioneer.[2] Over sixty professional bidders were registered to participate in these auctions: most are used-car dealers seeking to buy vehicles at wholesale prices for resale to their customers at a markup. Around 2000, an executive suspected collusion among some of the bidders at these auctions. In response, in 2002 he invented the informationally-restricted Korean auction that he claimed was successful in defeating this collusion. The new online auctions began in January 2003 and continued until 2007, when the executive decided that the fixed costs of running them were too high, so he reverted to selling cars at oral, ascending-bid auctions again—this time through an independent auction house in Seoul.

We refer to the oral, ascending-bid auctions held at the rental-car company as *Regime 0*, the on-

[1] Because of a confidentiality agreement we signed in order to obtain the data, we are prohibited from revealing the company's name or the identities of the bidders participating in these auctions.

[2] The oral, ascending-bid (English) auctions that have been studied theoretically are actually *Japanese auctions* (also referred to as as *thermometer* or *button* auctions by Milgrom and Weber [1982]), where bidders observe an exogenous, continuously-rising price and press buttons (or keep hands raised) to indicate their willingness to pay that price, with irreversible exit once they release the button or lower their hands. The winner is the last remaining bidder and the winning price is the value at which the penultimate bidder dropped out—within the independent private-values environment, an outcome that is strategically equivalent to the equilibrium of a second-price, sealed-bid auction, often referred to by economists as the *Vickrey auction* in honor of Vickrey [1961]. English auctions differ from Japanese auctions in that bid submission is endogenous and at the discretion of bidders. English auctions may have very different outcomes from Japanese ones—including complex patterns of jump bidding and sniping; see, for example, Avery [1998] as well as Isaac et al. [2007] who numerically computed equilibria in open-ended auctions with two bidders and showed that "jump bidding occurs due to strategic concerns and impatience." Little is known theoretically concerning bidding at English auctions with more than two bidders under alternative formats (for example, a two-minute time limit) or with informational restrictions. Cui and Lai [2013] analyzed heterogeneous bidding behavior in single-unit online ascending bid auctions that revealed similar types of bidding behavior that we find in our reduced-form analysis of Korean auctions in section 2.

line informationally-restricted Korean auctions as *Regime 1*, and the oral, ascending-bid auctions conducted by the auction house as *Regime 2*. Cho et al. [2014] analyzed data from over $30,000$ auctions conducted under Regimes 1 and 2: after controlling for car make and model as well as odometer reading, they concluded that prices under Regime 2 were nearly 10% higher than under Regime 1. Thus, net of the auction house's ten percent commission, the rental-car company received the same revenue per car sold, but saved the fixed costs of running its own online auctions. If collusion had resumed under Regime 2, then Cho et al. [2014] should have observed lower prices compared to Regime 1. Bidders can observe each other as well as each others' bids at oral, ascending-bid auctions. In the absence of collusion and assuming car valuations are affiliated, the linkage principle of Milgrom and Weber [1982] predicts that prices in Regime 2 should be higher than in the informationally-restricted auctions in Regime 1. Cho et al. [2014] noted, however, that the higher prices could be explained by a larger number of bidders participating in auctions conducted under Regime 2 than under Regime 1.

In this paper, we focus on understanding bidding at the Korean auctions (Regime 1) under the assumption that the informational restrictions were successful in defeating collusion, which the rental-car executive strongly believes was the case. In section 2, we compare limited data concerning auctions at the end of Regime 0 with auction prices of specific makes and models of cars just after the transition to Regime 1. We find mixed evidence of higher prices under Regime 1—in contrast to the executive's claims. Previous researchers have found that restricting information in dynamic auctions can hinder collusion; Cramton and Schwartz [2000] recommended informational restrictions in FCC spectrum auctions to reduce suspected collusion. These include anonymizing bidder identities to reduce the possibility of retaliation for deviating from collusive agreements and coarsening bids to three significant digits because bids in the billions "allowed for all kinds of signaling" in the less significant digits of the bids. Marshall and Marx [2009] also showed that suppressing bidder identities during and after an auction may successfully inhibit certain types of collusion. Bajari and Yeo [2009] studied FCC auction-design changes in response to these recommendations—including "click box bidding" that reduces jump bidding and "code bidding" and anonymization of bidder identities. They concluded that "these rule changes have limited firms' ability to tacitly collude" (p. 90) although "detecting collusion based solely on auction data can be difficult" (p. 100).

Few previous structural empirical analyses of bidding at dynamic, ascending-bid auctions exist—with or without the sort of informational restrictions employed at the Korean auctions.

Also, as we noted, very little theory exists characterizing equilibrium bidding behavior in online or oral ascending-bid auctions. The closest previous analysis to our's is by Barkley et al. [2021], who analyzed data from Texas auctions of certificates of deposit (CD) at thirty-minute online auctions that are informationally restricted in the same way as the Korean rental-car auctions: bidders cannot see each others' bids or identities, but each bidder is continually informed whether his bid is the highest ("in the money"). Barkley et al. [2021] found that the informational restrictions as well as bidding frictions resulted in significant inefficiencies and money left on the table due to "submitting winning bids at rates well above the lowest bid needed to win" (p. 380) that "are costly both for revenue and allocative efficiency" (p. 376). They concluded:

> The choice of auction mechanism is puzzling given the estimated losses due to bidding frictions. Why should the auctioneer not run a sealed-bid uniform price auction to allocate funds? We suggest two reasons why the current mechanism may be preferred to such an alternative mechanism despite the losses due to frictions: collusion and corruption.

Static first-price, sealed-bid or second-price, sealed-bid auctions reveal even less information than the informationally-restricted, online, ascending-bid auctions so the question remains: why would a dynamic auction be preferred to a static one in terms of its ability to defeat collusion? It is well known that repeated static auctions can also support collusion via bidding rings that employ self-enforcing, side-payment mechanisms to reduce the bids at auctions and lower winning prices; see, for example, the research of Asker [2010]; Graham and Marshall [1987]; Mailath and Zemsky [1991]; and McAfee and McMillan [1992]. In any event, we agree that it is difficult to detect collusion from bid data alone: if there were collusion that lowers bids, then our structural analysis would reflect this via estimated valuations that are lower than they would be absent collusion, but the auctions might otherwise appear competitive at the lower inferred valuations.

Our analysis proceeds under the assumption that no collusion exists. However our empirical analysis does not uncover any obvious evidence of collusion—such as artificially-low fake bids designed to create the impression of active, competitive bidding by members of a bidding ring. Instead, we find that bidders are *overbidding* at these auctions, or at least they are bidding more than our model of rationally inattentive bidding with bidding frictions predicts they should be bidding. The fact that auction prices increased by 10% when the company abandoned its Korean auctions (Regime 1) and moved to the auction house (Regime 2) is not evidence of collusion in the Korean auction. Instead, we believe the lower prices were caused by the *inefficiency* of the Korean auction, a conclusion that is consistent with the findings of Barkley et al. [2021]. Auction outcomes might have been more efficient under Regime 2. In fact, our results predict that the rental car company could raise revenues even more, by 12%, by adopting a static second-price sealed-bid auction format which results fully *ex post* efficient outcomes.

Our main goal, and the key contribution of this paper, is to improve our understanding of bidding behavior at dynamic auctions, and to conduct a limited type of *empirical mechanism design* designed to assess whether, in the absence of collusion, the rental-car company could have increased revenues had it removed the informational restrictions or adopted simpler static auction formats—such as first-price, sealed-bid or Vickrey auctions.

A natural starting point for analyzing bidding behavior in Korean auctions is to posit that in the absence of bidding frictions, the informational restrictions at the Korean auctions do not constitute a binding restriction on bidders compared to traditional oral, ascending-bid auctions. By a process we refer to as *bid creeping*, which we illustrate in section 2, bidders can learn the high bid at any moment of the auction and, thus, avoid the early overbidding that Barkley et al. [2021] found to be so prevalent at the Texas CD auctions. This would imply that bidding in Korean auctions should be strategically equivalent to the Japanese auction. In other words, bidders should keep bidding until the high bid exceeds their valuation, and then drop out, a strategy known as *straightforward bidding,* see Milgrom and Weber [1982]. Thus the bidder with the highest valuation wins with a winning bid equal to the valuation of the second-highest bidder, an outcome that is strategically equivalent to the outcome of a static second-price sealed-bid auction.

However as we noted above, the mechanics of how straigthforward bidding is actually implemented have never been described in detail, such as how often bidders should bid and how much to raise each successive bid. Well known lags in human perception and the speed at which they can call out or type in new bids limits how frequently bidders can revise their bids during the auction, and this puts inherent limits on the ability to carry out a straightforward bidding strategy. We reflect these limits by adopting a discrete time model of bidding, where bidders can only react and update their bids at one second intervals in the two minute auction. We prove that with informational restrictions on the identities of bidders but no restrictions on the values of their bids, a rational and frictionless bidder (which we will define in section 3) will find it optimal engage in *informational free-riding* and not bid until the last possible second in the auction. That is, all such bidders will *snipe* and there will be no early bidding as imagined under the usual story of straightforward bidding. Further bidders will shade their bids (i.e. bid less than their valuations) and not be willing to continue bidding up to their valuation for the object. This implies that anonymized

English auctions conducted in discrete time reduce to anonymized static sealed-bid auctions that are *not* strategically equivalent to second-price auctions.

However the informational restriction in Korean auctions does create an inherent incentive for early bidding, namely, to acquire valuable information about what the high bid currently is to avoid overpaying to win the auction. We show this incentive exists even for frictionless, perfectly attentive bidders *under certain conditions.* Thus, an optimal dynamic bidding strategy will typically entail early bidding, although in Korean auction with or without bidding frictions, straightforward bidding is generally not an equilibrium strategy. However if all bidders are perfectly attentive and use rational, frictionless bidding strategies, we show there is a *bidding paradox* — there will be no early bidding in equilibrium. Thus, early bidding in Korean auctions can only happen if some bidders are irrational, or are subject to various bidding frictions such as restrictions on how fast or often they can bid or the amount of attention they can devote to the auction.

The "gold standard" approach to explaining bidding behavior in dynamic auctions with incomplete information is *perfect Bayesian equilibrium* (PBE), where all bidders employ dynamic Nash-equilibrium strategies and Bayesian updating to determine when and how much to bid during the auction. We demonstrate that when bidding occurs in discrete time and the auction has a hard close (so all bidders can submit bids in the last instant of the auction and be guaranteed their bids will be accepted), an *uninformative PBE* exists that is strategically equivalent to the outcome of a first-price, sealed-bid auction (although it is a Bayesian version of this equilibrium since bidders only have a prior distribution concerning the number of bidders who might participate in any given auction; see McAfee and McMillan [1987a]). In other words, in the uninformative PBE, all bidders snipe: they make no early bids and only bid at the last possible instant.

Our auction data strongly reject the hypothesis that bidding behavior is consistent with an uninformative PBE of the Korean auction game, and so is the bidding behavior at the Texas CD auctions reported by Barkley et al. [2021]. Could other *informative* PBEs exist that are consistent with the early bidding we observe? In section 3, we construct a simple two-bidder, two-period example of the Korean auction, in which the only PBE is the uninformative PBE that is strategically equivalent to a first-price, sealed-bid auction. Whether informative PBEs exist in richer environments than our example remains an open question, but we agree with Barkley et al. [2021] that it is currently infeasible to compute or even to characterize such equilibria.

In section 4, we introduce our new, computationally-feasible approach to modeling bidding behavior at the Korean auction. We assume that bidders have rational, but non-Bayesian beliefs and face bidding frictions that include a form of rational inattention inspired by the research of Sims [2003] and Matějka and McKay [2015].[3] Bidders rationally account for their periodic lack of attention to the auction and, thus, bid earlier and higher than they otherwise would if they were able to pay attention at every instant and type in updated bids in under 1 second. Furthermore, their beliefs concerning the conditional probability distribution of the high bid at any point during the auction constitutes a "sufficient statistic" for calculating their optimal bidding strategy. We assume that rational, experienced bidders know this family of conditional probability distributions and, thus, bypass the intractable and extremely high-dimensional Bayesian updating problem involved in calculating a PBE. This approach allows us to recast the problem from one of computing a PBE into the much simpler problem of computing a Nash equilibrium to an *anonymous game*, where we can solve for each bidder's equilibrium strategy as a single-agent, dynamic-programming (DP) problem. An *anonymous equilibrium* of this game is a set of DP bidding strategies that satisfy the constraint that all bidders have rational beliefs concerning the stochastic process governing the high bid at the auction, thereby making their strategies mutual best-responses.[4]

We solve for optimal bidding strategies by discretizing the two-minute auction into 121 one second bidding intervals t during the auction running from $t = 0$ to the final possible bid at $t = 120$. We apply numerical DP to compute bidding strategies that maximize bidders' expected payoffs from participating in the auction. Our model involves a vector of unknown parameters (v, c, p, σ) where v is the bidder's valuation of the car being auctioned, c is the psychological cost (or benefit if negative) of submitting a bid at any instant, p is the probability that a bidder is distracted and unable to bid at any instant t, and σ is a scale parameter of an extreme-value distribution representing idiosyncratic costs/benefits of bidding at any instant t.

We used a fixed-effects, quasi-maximum likelihood (QML) approach to estimate these four parameters for $4,029$ auction-bidder pairs who participated in 533 auctions for a specific make/model of passenger car, the Hyundai Avante XD. [5] Our approach differs from that of Barkley et al. [2021] in that we employ numerical DP to solve for the the bidding strategies of each bidder at each auction under the assumption that each bidder has rational beliefs, so the outcomes are realizations of anonymous equilibria of the Korean auction. The QML estimator of the four parameters for each

[3]See also Bhattacharya and Howard [2022] who used rational inattention to explain violations of mixed strategy Nash behavior for pitchers in professional baseball.

[4]Computing the Nash equilibria of anonymous games is tractable and, in some cases, can be completed in polynomial time; see the research of Daskalakis and Papadimitriou [2015] as well as Cheng et al. [2017].

[5]The Hyundai Avante (Korean: 현대 아반떼) is a compact car produced by the South Korean manufacturer Hyundai since 1990, but is marketed as the Hyundai Elantra outside of South Korea and Singapore.

auction (so some $16,000$ parameters in total for the $4,029$ auction-bidder pairs) is chosen to fit the observed sequences of bids for all bidders at each auction. Our approach allows us to conduct detailed simulations of bidding behavior under various counterfactuals—including predicting the effect of eliminating the informational restriction on bidding behavior.

Our key empirical findings are summarized in section 5. Specifically, our model can explain the early bidding behavior observed at these auctions, but the model substantially underpredicts the initial bids tendered at these auctions—even though it does a better job of predicting final high bids. In section 6 we present counterfactual simulations involving frictionless bidders (i.e. bidders whose type is of the form $\tau = (v, 0, 0, 0)$ so they have perfect attention and no costs associated with submitting/updating bids). We show that the frictionless bidders constitute "bidding algorithms" that earn higher expected profits than their human counterparts by submitting lower bids than human bidders typically submit. This finding is inconsistent with the hypothesis of sophisticated rational bidders who are colluding to lower bids in these auctions. We refer to the proclivity of the human bidders to submit first bids that are systematically higher than the model predicts is optimal as *early overbidding*. We interpret this behavior as a rejection of the assumption of bidder rationality, and hypothesize some sort of bounded rationality or animal spirits among the bidders causes them to bid up prices faster and earlier at the auction compared to what rational frictionless bidders would do.

We compute the anonymous equilibrium outcome in markets with only frictionless bidders and show that it involves no early bidding, and hence is equivalent to the anonymous equilibrium of a static first-price sealed-bid auction. We show the latter equilibrium is 100% *ex post* efficient, and as a result, rental car revenues increase by 3% and bidder profits increase by 283% relative to the actual Korean auction outcomes, which we show is only 84% efficient. We conduct a further counterfactual experiment to predict the impact of switching to a second-price sealed-bid auction format where truthful bidding is a dominant strategy. We show that the first and second-price auctions are not revenue equivalent, and switching to a second-price format increases auction revenues by 12% and average bidder profits by 283%.

The non-Bayesian learning by the bidders during the dynamic online auction enables them to acquire more information, which allows the winner to pay less for the item than in a comparable first-price or second-price, sealed-bid auction. However this logic depends on the existence of other "irrational" bidders who engage in early overbidding. If all bidders are rational and frictionless, the auction converges to an anonymous equilibrium with no early bidding since all bidders learn that there is nothing to be gained by bidding early.

We also find that when frictionless bidders believe that there are other irrational early bidders and hence can profit from bidding early, their optimal bidding strategies are not straightforward. That is, these bidders do not continue to bid up to their valuation before stopping. Instead the optimal bidding strategy involves final high bids that involve shading, i.e. bidding less than their valuation for the car.

Finally, we predict the effect of dropping the informational restriction in the Korean auction by allowing all participants to see the current high bid even if they haven't bid in the auction. This modification makes this an electronic version of an oral, ascending-bid auction. Doing so eliminates the incentive of bidders to bid early at the auction to learn what the current high bid is, and leads to informational free-riding. We prove that in this anonymized version of an English auction, if there are bidding frictions and bidders are attentive, it is never optimal to bid until the last possible instant of the auction. Thus, explaining early bidding at these auctions requires at least a) informational restrictions and b) bidding frictions and/or bidder inattention, and c) some amount of irrationality or bounded rationality on the part of the bidders. Otherwise the early overbidding we observe could be an indication of risk aversion or a sign that bidders are maximizing something other than expected profits.

2 Auction Data

A large rental-car company in Korea provided us with detailed data from 11,259 auctions of all vehicles sold under its new informationally-restricted online auction system between 2003 and 2007 (Regime 1), before it switched back to oral, ascending-bid auctions that were conducted through an auction house in Seoul (Regime 2). Bidders were given notice in advance of auctions, so they could physically inspect the cars prior to the auction. Typically, however, bidders did not undertake detailed mechanical inspections, but rather just brief walk-arounds to inspect the exterior and interior condition of the vehicle. Bidders could request copies of a vehicle's maintenance history—including the total amount spent on maintenance, dates of maintenance, records of accidents, and so forth. We have the same accident and maintenance records that were available to the bidders, but we do not have the information gained from the physical inspection of the vehicles.

Our data include time stamps and each bid as well as the identities of all bidders at each auction. Some time stamps are a few milliseconds past the two-minute closing time; we excluded auctions that took longer than 121 seconds because many of these auctions reflected special circumstances (such as communication delays with the auction server) that required extra time to complete the auction.[6] In all instances, these slightly-late bids were admitted as valid bids at the auction. No reserve price existed at any of the auctions, so at virtually every auction, the rental-car company sold the vehicle to the highest bidder regardless of the value of the winning bid. There were a few auctions that were redone, such as where a bidder made a data-entry error that resulted in bid far in excess of any reasonable value for the car. In such exceptions, the company invalidated the auction outcome and re-auctioned the car at a later date.

2.1 Effect of informational restrictions on potential collusion

We compare auction prices for specific makes and models under Regime 0 (the oral, ascending-bid auctions held onsite that the rental-car executive suspected were affected by collusion) and Regime 1 (the informationally-restricted online auctions that the executive is convinced defeated collusion). Our data concern only 568 auctions under Regime 0 in the last three months of 2002. In Regime 1, we have 580 auctions conducted using the company's new informationally-restricted online platform from January through March 2003, and an additional 633 auctions run from April through June, 2003. The rental-car company owns a diverse inventory of different makes and models of vehicles; to assess the causal effect of the auction format on possible collusion, we use a matching estimator that conditions on individual makes and models of vehicles. Unfortunately, not enough observations exist to control for such other differences in age or odometer reading; in most cases, even limiting the comparison to specific makes and models yields too few observations to conduct a two-sample t-test of differences in mean sale prices before and after the regime change that could corroborate the manager's claim that the Korean auction defeated collusion. In Table 1, we report the means, standard deviations, and numbers of observations N for the most popular makes and models auctioned by the rental-car company, for three periods: 1) Regime 0 from October through December, 2002; 2) Regime 1 from January through March 2003 (period 1); and 3) Regime 1, from April though June 2003 (period 2). We divided the first six months of auctions under Regime 1 to assess the natural variability in auction prices over time under Regime 1 and to assess any potential transition effects after the new auction format was adopted.

The final column reports the p-values of a two-sample t-test for equality of the means, against

[6] We excluded 198 auctions whose durations exceeded 121 seconds; most of the excluded auctions lasted ten minutes or longer.

Table 1: Two-sample t-tests for the effect of auction format on collusion

Model	**Regime 0** 2002-10-01 to 2002-12-31	**Regime 1** 2003-01-01 to 2003-03-31	**Regime 1** 2003-04-01 to 2003-06-30	***p*-values for two-sample t-tests for equal means**
EF Sonata 1.8	5279 (1048),N=81	5148 (746),N=72	4919 (700),N=89	0.815, 0.995 0.975, 0.976
EF Sonata 2.0	5867 (1594),N=137	6043 (1359),N=46	7161 (1432),N=17	0.235, 0.001 0.019, 0.005
Dynasty 3.0	11633 (2496),N=25	13043 (2458),N=23	12934 (1757),N=13	0.027, 0.035 0.016, 0.560
Grandeur XG 2.0	11295 (1399),N=18	11081 (1055),N=14	11123 (978),N=15	0.687, 0.659 0.692, 0.456
Grandeur XG 2.5	12626 (2150),N=67	11504 (1974),N=50	11827 (1356),N=78	0.998, 0.995 0.999, 0.157
Galloper 7	7109 (1480),N=45	7477 (1473),N=61	7776 (1263),N=53	0.103, 0.010 .025, 0.123
Magnus 2.0	7614 (1170),N=11	6665 (1576),N=16	6503 (506),N=6	0.957, 0.992 0.980, 0.640

the alternative hypothesis that mean prices under Regime 1 are higher than under Regime 0. The final column reports p-values for the hypothesis of equal means: 1) Regime 0 versus Regime 1 (period 1), 2) Regime 0 versus Regime 1 (period 2), 3) Regime 0 versus pooled data for Regime 1 periods 1 and 2, and 4) Regime 1 period 1 versus Regime 1 period 2. The p-values for tests 1 and 2 are on the top line and those for tests 3 and 4 are on the bottom line of the last column of Table 1.

Because of limited numbers of observations and large standard deviations of the auction prices, we do not find strong evidence supporting the executive's claim that the auction he invented had successfully defeated collusion. Prices are not typically statistically significantly higher in either period of Regime 1 when compared to Regime 0. Only for three particular models—highlighted in the gray rows of the table, so EF Sonata 2.0, Dynasty 3.0, and Galloper 7—is there even moderate evidence of significantly higher prices under Regime 1. Mean prices for Magnus 2.0 and Grandeur XG 2.5 and EF. Sonata 1.8 models are actually lower under Regime 1, but the difference is not statistically significant.

In the remainder of this paper we analyze the bidding behavior under the maintained hypothesis that bidders did not collude in the Korean auction. Our structural estimation results in section 5 provide independent evidence supporting the executive's strong belief that this auction format successfully defeated collusion, though we suggest that switching to a static second-price sealed-bid auction format would have increased revenues by 12%

2.2 Detailed analysis of bidding strategies in individual auctions

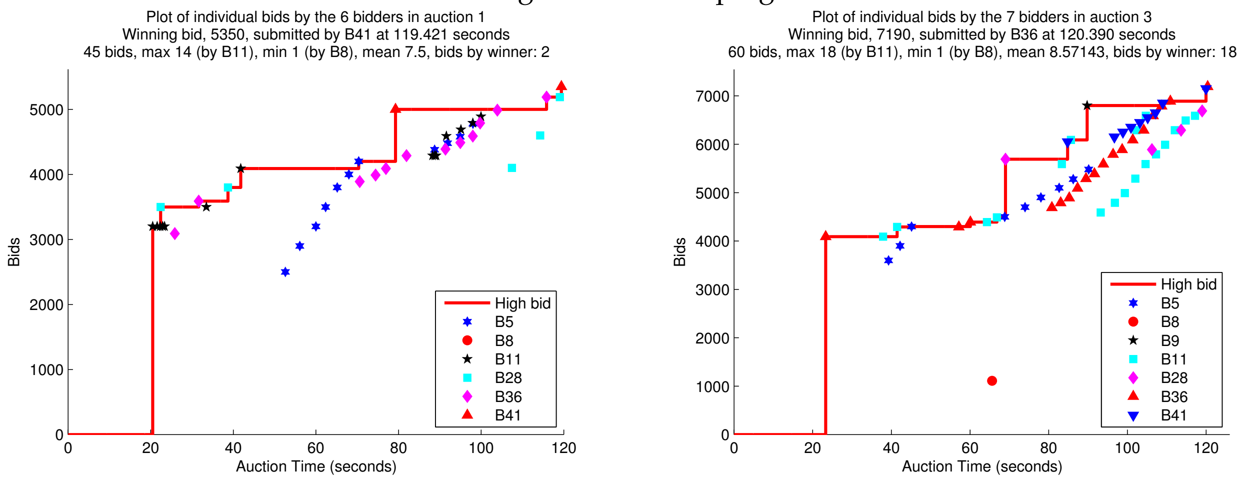
The left-hand panel of Figure 1 depicts bids at an auction held on January 26, 2005, which we refer to as Auction 1 because it was the first auction conducted on that day: a four-door, mid-sized sedan was sold at this auction. We have more precise data concerning the exact make and model, but for purposes of explaining how the auction works it suffices here to mention that the car was about two years old with approximately $40,000$ miles on its odometer at the time of the sale. As we can see from the figure, six bidders participated in this auction. We also see that the bids are generally monotonically increasing, although not all bidders were active at every possible instant. The winning bidder at this auction, B41, delayed submitting their bid until approximately fifty seconds remained in this two-minute auction; they made only one further revision to their first bid, raising it from $5,000 to approximately $5,400 at the very last instant of the auction.

We observe a variety of bidding behaviors by the other bidders: some posted bids much earlier in the auction and made frequent changes to their bids. These bidders appeared to be attempting to probe or to test the market to find the smallest bid they could submit that would make them the highest bidder. They did this by making small and frequent increases in their bids as in the case of B5. We refer to such bidding behavior as *bid creeping*. B5 never succeeded in placing a highest bid, and only learned at most that the high bid was higher than each of their sucessive bids, with the last bid reaching just over $4,500 with less than thirty seconds remaining at the auction, after which this bidder appeared to have given up and declined to submit any further bids. In all likelihood, this bidder had a reservation price for the vehicle and was unwilling to bid above this reservation price, so their final bid could reveal their reservation value.[7]

The right-hand panel of Figure 1 depicts bids at another auction, where seven bidders participated and a different bidder won the auction, B36. B36 behaved differently from the winning bidder of auction 1, B41: first, by virtue of being the first bidder to place a bid at the auction—with a bid of $4,000 just seconds after the start of the auction—and then by consistently increasing their bid in a series of small steps over nearly the entire duration of the auction until B36 tendered the winning bid of approximately $7,100 in the final second of the auction. B36 and B41 appeared to be dueling with one another to maintain the highest bid, even though neither used the strategy of jump bidding to try to win, nor was either aware of the other's bids. B41 delayed their first bid

[7]Note that the reservation price is not the same as the bidder's *valuation* of the car. As we will show, bid creeping is not equivalent to straightforward bidding, and the reservation price at which a bidder stops bidding is lower than the bidder's valuation of the car.

Figure 1: Bid creeping

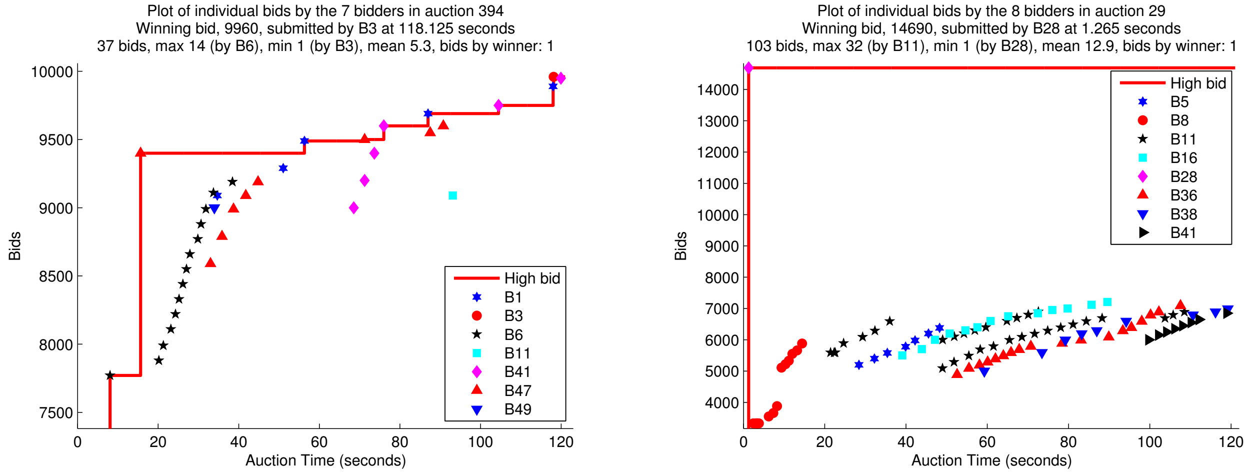


until approximately the last thirty seconds of the auction, and their first bid was higher, \$6,000. But at this auction, unlike at auction 1, B41 did increase their subsequent bids in small increments and appeared to be dueling with B36 to maintain the high bid. Both B41 and B36 placed bids in the very last instant of the auction, even though B36 succeeded in bidding just slightly higher—thus winning the auction.

In addition to bid creeping, we also see bidders who engaged in jump bidding and bid sniping strategies. The latter are bidders who place a single large bid at the very end of the auction. The left-hand panel of Figure 2 depicts the bids placed at auction 394. This auction was won by B3—by just a hair—with a bid of \$9,960 at 118.125 seconds, which exceeded the final bid by B41 of \$9,950 at 119.953 seconds. B3 won by placing a single bid 1.875 seconds before the end of the auction, while B41 opened their bidding with an initial bid of \$9,000 at 68 seconds into the auction and steadily increased their bid in five subsequent revisions until placing its final bid of \$9,950 less than one tenth of a second before the end of the auction. As we show shortly, bid sniping is a relatively infrequently-used strategy at these auctions.

The right-hand panel of Figure 2 illustrates an extreme form of early overbidding at auction 29, where B28 placed their first and only bid of \$14,690 at the 1.265-second mark of the auction. This bid is far higher than all of the other bids. Consequently, we see all of the other bidders fruitlessly trying to increase their bids to become the highest bidder. The highest of all of these other bids was \$7,210—tendered by B16—is less than half of B28's bid. Thus, it appears, at least from a simple *ex post* analysis of this auction, that B28 could have purchased this vehicle much more cheaply by starting with a lower bid and gradually increasing it over the course of the auction, instead of

Figure 2: Bid sniping and high early bidding

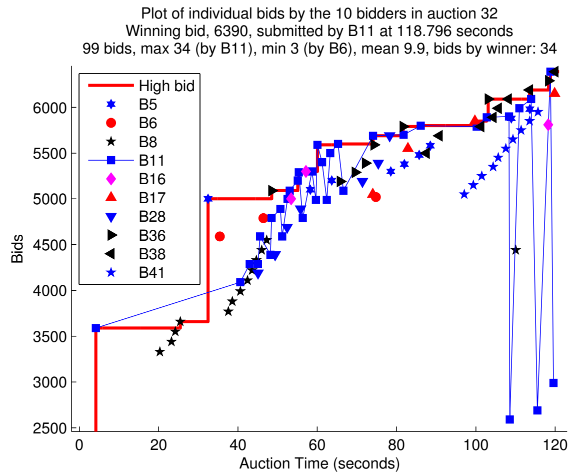


pre-committing to a very high bid at the beginning of the auction. Even though it is easy to make these judgements in hindsight with access to data that no single bidder possesses individually, it appears difficult to rationalize B28's high initial bid on purely *a priori* grounds. Bidding a high amount early in these auctions comes with the risk of overpaying, whereas making a low bid early in the auction, entails a risk of being outbid later in the auction and of revealing potentially valuable information to other bidders. But this risk can be controlled by delaying the bid.

Data from auction 32, depicted in Figure 3 below, illustrate non-monotonic bidding behavior that we observed by some bidders at some auctions. We see that the winning bidder at this auction B11 frequently reduced their bid, including reducing their bid to very low values, \$3,000 and below, lower than any other bids, including their own bids, earlier in the auction, but then dramatically increased their bid to win in the final milliseconds of the auction. It does not seem reasonable to attribute the frequent reductions in bids to keyboard errors or "trembles" on the part of B11. Instead, the bids seem to be intentional—perhaps out of boredom or to learn about some feature of the auction software, in an effort to exploit some unintended bug. The auction rules guarantee that other bidders did not observe this behavior and, thus, were completely unaware of the bid reductions by B11. In short, reducing a bid serves no informational or strategic purpose, so it is difficult to rationalize such behavior. Most of the other bidders placed monotonically increasing bids, although one other bidder B8 also reduced their bid to just below \$4,500 from a previous bid of \$6,000 before raising their bid again to about \$6,200 in the final seconds of the auction.

What can we conclude from our initial, simplistic, descriptive view of the data? First, we are interested in understanding the learning that derives from early bidding—how bidders use the

Figure 3: Non-monotonic bidding



option to place early bids affects their subsequent bidding decisions. From a single bidder's standpoint, it appears that having the option to bid early has value because it enables a bidder to test the waters safely: placing low bids and gradually increasing them sometimes yields good deals. On the other hand, given the coarse nature of information revealed over the course of the auction (since each bidder cannot see competing bids) it is not immediately clear that bidders will do any better under this auction format than they would in a standard one-shot, first-price, sealed-bid auction—a selling method that precludes any early bidding and within-auction learning. Given the large number of auctions we observe and the large numbers of bids tendered by the bidders at each auction, we believe these data present an interesting challenge both theoretically and empirically. We observe a variety of bidding behaviors in these auctions, with a combination of early bidders and late bidders, bid creepers, jump bidders, and in the extreme, the snipers who come in with high bids in the very last instants of auctions. The number of bids submitted per second increases dramatically in the final second of the auction as bidders jockey frenetically to submit the winning bid.

Overall, even though we do observe pre-emptive early bidding as well as bid sniping, and a high incidence of uninformative bidding in these auctions, by far the most commonly-observed bidding behavior is bid creeping, where a bidder makes a succession of increasing bids closely spaced in time, in an attempt to find out what the current high bid is. Examples of bid creeping are the bids by B5, B11, B28 and B36 in auction 1 depicted in Figure 1. Bid creeping seems to be a reasonable strategy for learning what the current high bid at the auction is because it avoids the risk of overbidding that might be implied by a bid-jumping strategy, which is similar to bid creeping, but involves bids that are spaced farther apart and jumps of higher increments compared to bid creeping. Examples of bid-jumping strategies include the sequences of bids by B41 in auction 1 depicted in Figure 1, and B1 in auction 394 depicted in Figure 2. Of course, the dividing line between bid creeping and bid jumping is a fuzzy one: the two behaviors are both consistent with a desire to learn what the current high bid in the auction is, but in bid creeping the bidder is willing to make a much larger number of bids in rapid succession, each one only slightly higher than the previous one, whereas in bid jumping the bidder seems to have a higher psychic cost of placing bids and tends to make fewer bids at more widely spaced intervals of time in the auction, and the increments over the previous bids are larger. Thus, bid jumpers seem to behave as if they have a higher cost of submitting bids and/or are more willing to take the risk of overbidding to become the current higher bidder relative to what we observe for bid creepers.

In summary, we have identified a number of different bidding behaviors at these auctions: 1) pre-emptive early bidding; 2) bid sniping; 3) non-monotonic bidding; 4) bid jumping; and 5) bid creeping. We have analyzed the $11,259$ auctions in our database with regard to the type of strategies employed by the winning bidder and found that bid creeping was the predominant strategy employed by the winning bidders—in over one-half of all auctions. We found that bid jumping and behaviors that involve a mix of creeping and sniping were the next most common behavior, used by the winning bidder in twenty percent of the auctions. We observed bid sniping in nearly five percent of all auctions, where the winner submitted a single bid in the final two seconds of the auction, and pre-emptive early bidding in nearly three percent of all auctions, where the winner submitted a single bid in the first two seconds of the auction.

When we analyzed the types of bidding behaviors on a bidder-by-bidder basis, we found a distribution of behaviors for each of the bidders; that is, no bidder exclusively followed one type of strategy (for instance, bid sniping) at all of the auctions in which they participated. We tabulated the distribution of various types of bidding behaviors for the 67 bidders at the auctions: the most common behavior for virtually all of the bidders was bid creeping, while the next most common behavior was bid jumping—or a mix of creeping and jumping behaviors.

3 Can Game-Theoretic Models Explain Early Bidding?

In this section, we investigate models of dynamic, equilibrium bidding at the rental-car auctions—specifically, whether the behavior we observe is consistent with a perfect Bayesian equilibrium

(PBE) of the Korean auction formulated as a dynamic game of incomplete information. Due to the severe restrictions concerning information provided to bidders, the amount a bidder can learn about his opponents during the course of the auction is limited. Indeed, at 22 auctions only a single bidder participated. Yet the average number of bids submitted at these 22 auctions was essentially the same as at auctions when several bidders participated. Apparently, the informational restrictions make it difficult for bidders to learn even the most basic fact: whether opponents are present!

In the introduction, we considered an alternative model of bidding in the Korean auction, namely *straightforward bidding* where all bidders bid frequently enough to learn the high bid at each point during the auction. If bidders could bid fast enough and in small enough increments, this reasoning suggests that the outcome of straightforward bidding should approximate the outcome at a Japanese auction: namely, using bid-creeping strategies, a bidder can learn the current high bid and, thus, remain in the auction until he exceeds his valuation. Except in the case of only one bidder, it implies that a Korean auction should be strategically equivalent to a Japanese auction, which in turn is strategically equivalent to a static second-price, sealed-bid auction, at least within the independent private-values model.

This casual intuition is not a substitute for developing a rigorous game-theoretic model of bidding—as has been recognized by Isaac et al. [2007, p. 145], who showed that at oral, ascending-bid auctions "straightforward bidding is not even typically part of a Nash equilibrium in the non-clock ascending auction, much less a dominant strategy." Things are even more complicated at a Korean auction, where the informational restrictions suggest that the natural equilibrium concept is the perfect Bayesian equilibrium (PBE). Further, in the previous section we presented empirical evidence inconsistent with straightforward bidding.

In this section, we sketch the elements of a PBE model of bidding at the Korean auction. We show that when the auction has a hard close — where all bidders who wait to submit their bids in the last second ($T = 120$) of the two-minute auction are guaranteed that their bids will be recorded — there is always an uninformative PBE that is strategically equivalent to a first-price, sealed-bid auction, but modified as in McAfee and McMillan [1987a] and Harstad et al. [1990] to account for a common knowledge prior over the distribution of unknown numbers of bidders participating in the auction. As we noted in the introduction, this equilibrium implies that all bidders use bid sniping strategies, which is manifestly inconsistent with the actual bidding behavior we observe at these auctions as we showed in the previous section. We summarize this more formally as

Lemma 1: *Suppose the Korean auction has a hard close: that is, any bidder can wait until the final instant T and be guaranteed that the bid he submits at that last instant will be recorded. Let $b_T = \beta(v)$ be the symmetric Bayesian equilibrium bidding strategy to a first-price, sealed-bid auction when the number of bidders is unknown but all bidders have common knowledge of the distribution of the number of bidders participating in the auction and of the density for valuations of other competing bidders that is described as the "contingent bid strategy" in equation (9) of Theorem 1 of Harstad et al. [1990]. Then β constitutes a symmetric, uninformative PBE of the Korean auction in which all bidders wait until the final instant T and submit their bids $b_T = \beta(v)$.*

The proof of Lemma 1 is quite elementary. First, by construction, there is no deviation bid by any of the bidders at time T that can improve his expected payoff under this candidate uninformative equilibrium. So we only need to check if there is any profitable unilateral deviation by any of the bidders prior to T. It is easy, however, to see that there isn't because if it is common knowledge that all competing bidders will submit their bids at the last instant T, then there is nothing a deviating bidder can learn by submitting his bid prior to T. This early bid also cannot have any impact on the bids that will be submitted by opponents, so we conclude there is no profitable deviation from this candidate PBE. Given that no bids are submitted prior to T, the only relevant information for submitting a bid at time T is just the bidder's valuation v, so each bidder uses the symmetric Bayes-Nash equilibrium bidding strategy $b_T = \beta(v)$ in equation (9) of Theorem1 of Harstad et al. [1990] and all the criteria for a PBE given in our definition are satisfied.

Do other informative PBEs exist—ones involving the type of early bidding consistent with what we observed in our data, as well as the other features, such as jump bidding, bid creeping, and the occasional bid sniping that we documented in the previous section? Using a simple two-period, two-bidder example, we show that informative PBE do not generally exist. Furthermore, as we noted in the introduction, even if an informative PBE do exist in more complex settings than the simple two-by-two case we have examined in this section, computing that informative PBE is an intractable computational problem. In short, the main message of this section is that we need to consider an alternative theoretical approach to explain the early bidding observed at the Korean auctions. Because of the computational and theoretical difficulties in characterizing (let alone computing) PBEs, below we only sketch the elements of a symmetric PBE bidding model—without attempting to formulate a version with maximum generality.

We consider the simplest possible case, where time is discrete and only two bidding opportunities exist: at $t = 1$ and at $t = 2$. We also assume that only two bidders participate in the auction, and this is common knowledge. Consider now a candidate informative symmetric equilibrium: This means that in period $t = 1$ the two bidders use the same bidding strategy $\beta_1(v)$ to place bids given their valuation v at time $t = 1$, and then at time $t = 2$ they place updated bids from two bid functions $\{\beta_2(v, b, \rho_0, 0), \beta_2(v, b, \rho_1, 1)\}$ that condition on their bid b at $t = 1$ and whether they were revealed as the high bidder or not at $t = 1$, including their posterior belief ρ_0, and ρ_1, respectively. If $h_1 = 1$, then the bidder is informed he had the high bid in period $t = 1$, so he updates his posterior belief concerning his opponent's valuation from $F(v|\mu)$ to the conditional probability distribution ρ_1 given below, and he uses the bid function $b_2 = \beta_2(v, b, \rho_1, 1)$ to compute his bid in period 2. On the other hand, if he is not revealed to be the high bidder at $t = 1$, then his bid function is given by $\beta_2(v, b, \rho_0, 0)$ where ρ_0 is the bidder's posterior belief about his opponent's valuation if he learns that he is not the high bidder at $t = 1$, which we also derive below.

Since both bidders are using the same bid function in period $t = 1$, if this bid function is strictly monotonic (as seems natural given we are considering an informative PBE), then the information concerning whether he submitted the high bid is equivalent to learning whether his opponent's valuation of the car being sold is higher or lower than his own known valuation. Applying Bayes' rule to update beliefs based on the information learned at $t = 1$, if bidder 1 has valuation v_1 and learns he submitted the high bid at $t = 1$, his posterior belief concerning the valuation of his opponent, bidder 2, at the start of period $t = 2$ is given by the conditional distribution (CDF)

$$\rho_1(v|\mu, v_1) \equiv F(v|\mu, v \leq v_1) = \frac{F(v|\mu)}{F(v_1|\mu)}, \quad v \leq v_1, \text{ 0 otherwise} \tag{1}$$

where $F(v|\mu)$ is his prior belief of the CDF of his opponents' valuation which can potentially depend on the public signal μ both bidders received concerning the quality of the car being sold prior to the start of the auction. Similarly, if bidder 2 learns he is the low bidder in stage $t = 1$ his posterior belief concerning his opponent's valuation is given by

$$\rho_0(v|\mu, v_2) = F(v|\mu, v \geq v_2) = \frac{F(v|\mu) - F(v_2|\mu)}{1 - F(v_2|\mu)} \quad v \geq v_2, \text{ 0 otherwise.} \tag{2}$$

Consider now the equations determining the equilibrium bidding strategies, where we assume a symmetric Bayes–Nash equilibrium of the two-period game. The strategies are *ex ante* symmetric in the sense that both bidders use the same bidding strategy $\{\beta_1(v), \beta_2(v, b, \rho_1, 1), \beta_2(v, b, \rho_0, 0)\}$ to determine their bids in both periods of the auction, even though this symmetric equilibrium does reflect an *endogenous information asymmetry* that arises in period $t = 2$ when the bidders submit their stage $t = 2$ bids. Of course, this asymmetry results from the information the bidders receive concerning whether they submitted the high bid in period $t = 1$.

We solve the bidding game by backward induction starting in stage $t = 2$. The bidding strategies $\beta_2(v, b, \rho_1, 1)$ and $\beta_2(v, b, \rho_0, 0)$ must be mutual best responses, so they must satisfy the following equations:

$$\begin{aligned}
\beta_2(v,b,\rho_1,1) &= \underset{b'\geq b}{\operatorname{argmax}}(v-b')\int_0^v \mathbf{1}[\beta_2(v',\beta_1(v'),\rho_0,0)\leq b']f(v'|\mu)\,\mathrm{d}v'/F(v|\mu)\\
\beta_2(v,b,\rho_0,0) &= \underset{b'\geq b}{\operatorname{argmax}}(v-b')\int_v^\infty \mathbf{1}[\beta_2(v',\beta_1(v'),\rho_1,1)\leq b']f(v'|\mu)\,\mathrm{d}v'/[1-F(v|\mu)],
\end{aligned}$$

where

$$\beta_1(v) = \underset{b\geq 0}{\operatorname{argmax}}\, W_1(v,b), \tag{3}$$

Here, $W_1(v,b)$ is the bidder's expected payoff from bidding b at period $t=1$ given by

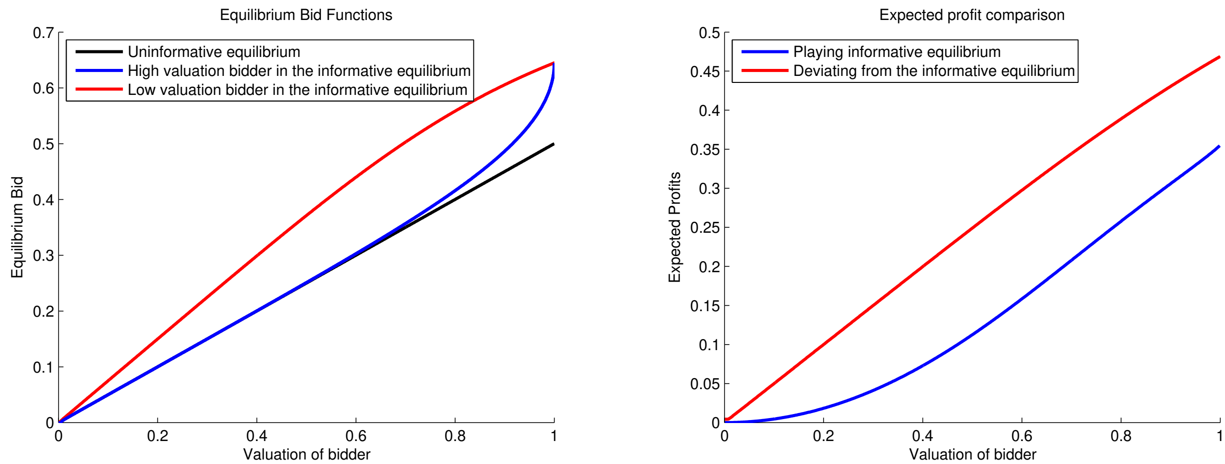
$$\begin{aligned}
W_1(v,b) &= [v-\beta_2(v,b,\rho_1,1)]\left(\int_0^v \mathbf{1}[\beta_2(v',\beta_1(v'),\rho_0,0)\leq \beta_2(v,b,\rho_1,1)]f(v'|\mu)\,\mathrm{d}v'\right)+\\
&\quad [v-\beta_2(v,b,\rho_0,0)]\left(\int_v^\infty \mathbf{1}[\beta_2(v',\beta_1(v'),\rho_1,1)\leq \beta_2(v,b,\rho_0,0)]f(v'|\mu)\,\mathrm{d}v'\right).
\end{aligned}$$

This is a system of functional equations whose solution gives the symmetric PBE of the Korean auction, assuming a solution exists. Notice that the period $t=1$ bid function $\beta_1(v)$ affects the period $t=2$ bid functions $\{\beta_2(v,b,\rho_1,1),\beta_2(v,b,\rho_0,0)\}$, and conversely the period $t=2$ bid functions determine the period $t=1$ bid function β_1. Note, too from equation (4) that β_1 must also be a best response to itself.

Assume that $f(v|\mu)$ is a uniform distribution, so that both of the bidders' valuations are independent and identically-distributed (IID) uniform random variables on the unit interval. $U(0,1)$. We conjecture further, and subsequently verify, that the period $t=2$ bid functions do not depend on the bids submitted in period $t=1$, so we can write them as $\beta_2(v,\rho_0,0)$ and $\beta_2(v,\rho_1,1)$, respectively. The only additional restriction we need to verify is that the stage $t=1$ equilibrium bid function $\beta_1(v)$ is strictly monontonic and positive for $v>0$.

Now, assume there is a unique asymmetric Bayesian equilibrium to the period $t=2$ bidding game defined by the solution to the system of ordinary differential equations (ODEs) for the inverse bid functions $\{\beta_2^{-1}(b,\rho_1,1),\beta_2^{-1}(b,\rho_0,0)\}$ (where the inverse is in the first argument; that is, $\beta_2^{-1}(b,\rho_i,i)$ is the valuation that results in a bid equal to b for $i\in\{0,1\}$), with the boundary conditions $\beta_2^{-1}(0,\rho_1,1)=0$ and $\beta_2^{-1}(0,\rho_0,0)=0$ and $\beta_2^{-1}(\overline{b},\rho_1,1)=\beta_2^{-1}(\overline{b},\rho_0,0)$ where $\overline{b}=\beta_2(1,\rho_1,1)=\beta_2(1,\rho_0,0)$ is the maximum bid that either bidder would submit in the second stage of the auction for any possible bid b in the first stage of the auction. The system of ODEs for $\{\beta_2^{-1}(b,\rho_0,0),\beta_2^{-1}(b,\rho_1,1)\}$ can be derived from first-order conditions to each of the bidder's optimal bidding strategies in the second stage of the game, from equation (3) above. This system

Figure 4: Equilibrium bids and deviation payoffs in a candidate informative PBE



is given by

$$\begin{aligned} \frac{\partial \beta_2^{-1}}{\partial b}(b, \rho_0, 0) &= \frac{\beta_2^{-1}(b, \rho_0, 0)}{\beta_2^{-1}(b, \rho_1, 1) - b} \\ \frac{\partial \beta_2^{-1}}{\partial b}(b, \rho_1, 1) &= \frac{\beta_2^{-1}(b, \rho_1, 1) - \beta_2^{-1}(b, \rho_0, 0)}{\beta_2^{-1}(b, \rho_0, 0) - b}. \end{aligned} \tag{4}$$

We solved system (4) as a free boundary-value problem because the end-point boundary condition $\beta_2^{-1}(\overline{b}, \rho_0, 0) = \beta_2^{-1}(\overline{b}, \rho_1, 1) = 1$ involves the unknown maximum bid $\overline{b}$.

For comparison, the left-hand panel of Figure 4 depicts the equilibrium bid functions for the uninformative PBE in period $t = 2$ where $\beta_1(v) = 0$ and the period $t = 2$ bid function is the unique symmetric equilibrium to a first-price, sealed-bid auction, $\beta_2(v) = v/2$. There is no endogenous asymmetry in the stage-two bid functions in this case, of course, because there are no bids placed in period $t = 1$ by either bidder and, thus, neither bidder learns anything from period $t = 1$ of the game. It follows that stage $t = 2$ is equivalent to the BNE of a single stage first-price, sealed-bid auction with uniform valuations. By Lemma 1, this is also a PBE of the overall second-period Korean auction.

Now suppose an informative PBE exists. Then, β_1 is strictly monotonic and strictly positive for $v > 0$ and the players observe $I\{\beta_1(v_1) > \beta_1(v_2)\}$, and this allows each of them to deduce whether they have the high valuation for the item, which creates the endogenous asymmetry in period $t = 2$. The red and blue bid functions in the left-hand panel of Figure 4 depict the unique equilibrium for the period $t = 2$ bid functions $\beta_2(v, \rho_0, 0)$ and $\beta_2(v, \rho_1, 1)$ and several things are immediately apparent: First, we see that $\beta_2(v, \rho_0, 0) \geq \beta_2(v, \rho_1, 1) \geq v/2$, with strict inequality for sufficiently large values of $v \in (0, 1]$. This implies that the bidder who learns he is the low-valuation bidder will bid more aggressively to win the auction in stage $t = 2$ than the bidder who learns he has the high valuation for the item. In fact, both types of bidders bid strictly more than the bidders would bid in period $t = 2$ under the uninformative equilibrium. Hence, it is not in the interest of either bidder to reveal his hand by submitting an informative bids in period $t = 1$ of the auction. The right-hand panel of Figure 4 verifies this to be the case—demonstrating that a symmetric informative PBE cannot exist in this case.

The blue line in the right-hand panel of Figure 4 depicts the conditional expected payoff to a bidder at the start of the game, as a function of his valuation v. This bidder reasons that if he placed an informative bid at this stage, with probability v he will turn out to be the high valuation bidder and so in stage $t = 2$ he will receive an expected payoff of $(v - \beta_2(v, \rho_1, 1))\beta_2^{-1}(\beta_2(v, \rho_1, 1), \rho_1, 1)/v$. With probability $(1 - v)$, he will learn he has the low valuation and will receive an expected payoff of $(v - \beta_2(v, \rho_0, 0))(\beta_2^{-1}(v, \rho_0, 0) - v)/(1 - v)$. The red line in the right-hand panel of Figure 4 is simply the weighted average of these two expected payoffs at period $t = 2$ using weights v and $(1 - v)$, respectively.

Consider next the red line in the right-hand panel of Figure 4, which depicts the deviation payoff as a function of the bidder's valuation v from submitting a stage $t = 1$ bid of 0 rather than the equilibrium bid $\beta_1(v)$. Assume that the other bidder is playing the informative PBE, then if the conjectured informative PBE is indeed an equilibrium, it should not pay for the bidder to deviate and submit a bid of 0 in the first stage. In fact, we see that it does pay to deviate. The deviation payoff is given by $\max_b (v - b)\beta_2^{-1}(b, \rho_1, 1)/v$, which is the payoff a bidder expects from submitting a bid of 0, which leads the other bidder to conclude with probability one that he is the high-valuation bidder and, thus, uses the less aggressive bidding stategy $\beta_2(v, \rho_1, 1)$ in stage $t = 2$. It is better for a bidder to be certain of bidding against the less aggressive bidder than have some probability of facing a more aggressive bidder in period $t = 2$ and, thus, the bidder concludes that there is no advantage to him to submitting a serious bid in stage $t = 1$ and, thereby, revealing information concerning his valuation.

Lemma 2: *In the two-period, two-bidder example of the Korean auction, if bidders have independent uniformly-distributed valuations, then no symmetric informative PBE exists.*

Proof: Even though the solutions above were calculated numerically, we can solve for the bidding strategies analytically using the results of Kaplan and Zamir [2012], so the conclusion of Lemma 2 does not rest on numerical calculations.

We do not know whether informative PBEs exist in other examples of the Korean auction with more time periods and bidders, or whether informative asymmetric PBEs may exist—even in the 2×2 case above. We do, however, believe it is quite challenging to demonstrate the existence of a non-trivial informative PBE to this bidding game. To the extent that our result on the non-existence of informative PBE extends to other versions of the Korean auction, it casts doubt on the relevance of the PBE solution concept—if we think that human bidders are incapable of behaving according to the extremely demanding standard of rationality implicit in this definition of equilibrium.

4 A Dynamic Model of Rationally-Inattentive Bidding

In this section, we develop an alternative model of dynamic bidding behavior at the Korean auction—one that is computationally tractable and capable of explaining the early bidding we observe, including the heterogeneous bidding strategies we documented in section 2. In many respects, our new approach is similar to the game-theoretic approach underlying the definition of a PBE presented in the previous section. Specifically, we assume bidders are rational optimizers who adopt bidding strategies that maximize their expected payoffs from bidding at the auction. Instead of the PBE concept, however, we use an alternative equilibrium concept of Nash equilibrium that has been recently developed for anonymous games—one similar to the notion of a rational expectations equilibrium (or self-confirming equilibrium) in market games, but extended to environments where agents playing these games are nonatomic and can, therefore, have measurable influence on the outcome; see Cerreia-Vioglio et al. [2022]. Two key differences exist between our notion of anonymous equilibrium applied to the Korean auction model and a PBE:

1. Instead of continually updating beliefs concerning the number of other bidders and their valuations using Bayes' rule, as in a PBE, our equilibrium only requires bidders to have beliefs concerning the stochastic process of the high bid at the auction and these beliefs are fixed and, thus, not carried as state variables in bidders' DP problems.
2. As in Barkley et al. [2021], we admit several bidding frictions, including time-varying psychological costs/benefits to submitting and updating bids, as well allowing for rationally inattentive bidding.

The anonymous equilibrium concept requires bidders to use dynamically optimal strategies, but maintains that after their experience in bidding in hundreds or thousands of individual rental-car auctions, learning by bidders leads them to converge to fixed, rational beliefs concerning the stochastic process of the high bid at the auction, which we shall demonstrate constitutes the relevant "sufficient statistic" for successful bidding. Unlike in a PBE, where Bayes' rule provides an operational procedure for belief formation and updating, our equilibrium concept is agnostic concerning how bidders learn and converge to rational beliefs concerning the stochastic process of the high bid in the auction. Indeed, it may seem to be an unrealistic requirement given that the informational restrictions at these auction only admit *endogenous sampling* of the high bid over the course of each auction. Research by George A. Hall and Rust [2021] has demonstrated that the endogenous sampling problem can be overcome, so it may not be unrealistic to assume that experienced bidders converge to accurate beliefs concerning the stochastic process of the high bid at these auctions. Furthermore, the computational savings from this assumption are enormous because it implies that we no longer have to carry around high-dimensional posterior beliefs ρ_t as a state variable and perform the subtle updating of beliefs using knowledge of the equilibrium bidding strategies $\{\beta_t\}$ in order to solve a bidder's DP problem.[8]

We are able to generate early informative bidding in an anonymous equilibrium because of the exogeneity of bidders' beliefs. Even though bidders realize that their own bidding behavior enables them to affect the winning price when they are the high bidder, in an anonymous equilibrium their beliefs concerning the stochastic process of the high bid is fixed and, therefore, unaffected by their bidding strategies. This structure allows us to convert the problem of finding an equilibrium bidding strategy into a single-agent DP problem, where bidders' beliefs concerning the law of motion for the high bid constitutes the law of motion of "nature." In contrast, the PBE solution concept involves much more complicated reasoning on the part of bidders—one that causes them to realize that their strategies affect their beliefs concerning the stochastic process for the high bid at the auction. The example we provided in the previous section illustrates how this more subtle reasoning leads bidders to conclude that their attempts to gain information early in the auction work to their collective disadvantage later, ruling out existence of an informative PBE. In contrast, we demonstrate that in an anonymous equilibrium, early bidding helps bidders to learn what the high bid at the auction is—thus enabling them to win the auction by paying less, on average, than they would pay compared to adopting a bid sniping strategy and submitting their bid at the last instant of the auction.

[8]Our equilibrium concept can extended to allow for ϵ-equilibrium versions of anonymous equilibrium, including the ϵ-estimated equilibrium concept of Cerreia-Vioglio et al. [2022] that involves two key requirements: "1) Every player best-responds to their beliefs (optimality). 2) The belief of every player is consistent with what they can observe (ϵ-discrepancy)" (p. 111). Daskalakis and Papadimitriou [2015] and Cheng et al. [2017] show that anonymous equilibria are easier to compute compared than standard (nonanonymous) Nash equilibrium or PBE.

4.1 Definition of anonymous equilibrium in the Korean auction

Before going into the details of our model of rationally-inattentive bidding, we follow Cerreia-Vioglio et al. [2022] and provide a high-level description of a successive-approximations algorithm we use to compute an ϵ-anonymous equilibrium in the Korean auction. Here, ϵ is a tolerance defining how closely we require bidders' beliefs to be in order to be correct—self-confirming. Let $\mathcal{B}_0$ be any initial guess for bidders' beliefs concerning the stochastic process governing the high bid in a set of Korean auctions of a homogenous type of used rental car. We shall describe precisely what the notation $\mathcal{B}_0$ means in more detail below, but at this point assume that given $\mathcal{B}_0$ we can solve bidders' DP problems to determine their optimal bidding strategies which we can write as functions $\beta(\mathcal{B}_0, \boldsymbol{\tau})$ that depend on the bidder's type $\boldsymbol{\tau}$ and their (common) belief $\mathcal{B}_0$ concerning the stochastic process of the high bid at the auction. Via repeated IID stochastic simulations of a given set of n bidders with types $[\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n]$ using the calculated strategies $[\beta(\mathcal{B}_0, \boldsymbol{\tau}_1), \ldots, \beta(\mathcal{B}_0, \boldsymbol{\tau}_n)]$, we can construct updated beliefs $\mathcal{B}_1$ that constitute the stochastic process for the high bid track implied by the initial guess for bidders' beliefs, $\mathcal{B}_0$. We can write the composition of these two operations (for example, solving for the optimal strategies followed by simulating auctions using them to generate and realized high bid tracks), as the mapping $\mathcal{B}_1 = \Lambda(\mathcal{B}_0)$, which can be viewed as the updated beliefs implied from the initial guess $\mathcal{B}_0$. We can repeat this sequence of solution and simulations repeatedly, so that at the generic iteration t of this updating process we have the following successive-approximations iteration or best-response belief mapping:

$$\mathcal{B}_{t+1} = \Lambda(\mathcal{B}_t). \tag{5}$$

Definition 1 (Anonymous Equilibrium): *If the best-response belief mapping (5) has a fixed point $\mathcal{B}_\infty$, then the corresponding collection of optimal bidding strategies $\{\beta(\mathcal{B}_\infty, \boldsymbol{\tau})\}$ for an appropriately-defined collection of bidder types $\boldsymbol{\tau}$ constitutes an anonymous equilibrium of the Korean auction.*

In practice, we do not find an exact fixed point via the successive-approximations procedure, but only an approximate fixed point after t iterations where a convergence critierion $\|\mathcal{B}_{t+1} - \Lambda(\mathcal{B}_t)\| \leq \epsilon$ is satisfied, where $\|\cdot\|$ is an appropriate distance function over beliefs that we define below.

Definition 2 (ϵ-Anonymous Equilibrium): *An ϵ-anonymous equilibrium is any belief $\mathcal{B}$ for the stochastic process of the high bid at the Korean auction and the corresponding collection of optimal bidding strategies $\{\beta(\mathcal{B}, \boldsymbol{\tau})\}$ for an appropriately-defined collection of bidder types $\boldsymbol{\tau}$ satisfying $\|\mathcal{B} - \Lambda(\mathcal{B})\| \leq \epsilon$.*

We note that in a static first price sealed bid auction, an Anonymous Equilibrium equilibrium will generally *not* be the same as the Bayes-Nash equilibrium with an unknown number of bidders, as given by the contingent bidding strategy in equation (9) of Harstad et al. [1990]. The reason is that Anonymous Equilibrium is a weaker concept of equilibrium that does not require the strong common knowledge assumptions of a Bayes-Nash equilibrium, such as that all bidders have common knowledge of each others' distributions of valuations for the item, as well as a common prior distribution for the number of bidders participating in any given auction. Anonymous equilibrium can be defined for asymmetric auctions where different types of bidders have different valuations for the item being sold, but bidders do not need to have common knowledge of the types of their opponents. Anonymous equilibrium only requires that all bidders have a common, rational belief about the distribution of the high bid in the auction. Since the number of bidders is unknown, bidders do not attempt to adjust the distribution of the high bid in the auction for the number of competing bidders except themselves. For this reason, the (common) bidding strategy used by bidders in an Anonymous Equilibrium will generally differ from the Bayes-Nash equilibrium bidding strategy in an auction model where bidders do have common knowledge over their opponents' distribution of valuations and share a common prior distribution over the number of entrants.

In Appendix A we provide a simple example to show how the equilibrium bidding strategy in an Anonymous Equilibrium differs from a Bayesian-Nash equilibrium, and how the Revenue Equivalence Theorem fails as a result. In the example in Appendix A, the expected revenue from a first price sealed bid auction exceeds the expected revenue in a second price auction. However this is not a general result: we will show in section 6.1 that in the case of the Korean auction this inequality is reversed: expected revenues from a second price auction format exceed the expected revenues from a static first price sealed bid auction by 8.5%.

With this overview of our concept of equilibrium, we now provide additional details concerning the underlying assumptions on which these definitions are based; we then introduce the specifics of our model of rationally-inattentive bidding at Korean auctions.

First, our analysis conditions on auctions of a specific make and model of commonly-auctioned rental cars: the Avante XD. This is a fairly generic passenger car in Korea and one of the most frequently-purchased car used by rental companies. We assume that not only are these cars generic, but the stochastic properties of the auctions in which they are sold are also generic as well and, hence, the data from these auctions can be aggregated for purposes of statistical analysis and estimating bidders' beliefs. In particular, we rule out the possibility of unique items being auctioned that would invalidate our assumption that the stochastic process of the high bid is made up of IID realizations from an underlying anonymous equilibrium for Korean auctions of generic Avante XD rental cars. We make this operational via the following assumptions:

Assumption 1 (Conditionally-independent private values): *If there are n bidders at an auction, their valuations of the rental car on sale are IID draws from a conditional density $f(v|\mu)$ where we refer to μ as public value of the car. It is a random variable that is a function of a vector of characteristics $\mathbf{x}$ of a specific car being auctioned and additional information ϵ that bidders can observe from a physical inspection of the car prior to the auction but which we as the econometrician cannot observe. The public variable μ is common knowledge to the bidders, as is the distribution $f(v|\mu)$. If there are K cars being auctioned, we also assume that their public values $\{\mu_1, \ldots, \mu_K\}$ are IID draws from some distribution $H(\mu)$, which is also common knowledge among the bidders.*

Assumption 2 (Random arrival of bidders): *The number of bidders n arriving to participate in a auction of a car with public value μ is a realization from a discrete probability distribution $g(n|\mu)$ which is common knowledge among the bidders.*

Assumption 3 (Time discretization): *Bidding at the Korean auction occurs during $T = 121$ discrete bidding seconds $t = 0, 1, \ldots, 120$ during the auction. Any bid that is recorded in the continuous-time interval $[t, t+1)$ is treated as having been submitted at bidding instant t. All bidders are informed whether any bid they submitted at second t (or high bid submitted at some previous bidding second) which we denote by b_t is the high bid at the start of second $t+1$. The final auction outcome (the winning bidder and bid) are determined after the last bidding second $T - 1 = 120$ so each bidder is informed if they won or not at second $T = 121$.*

Assumption 4 (Bidder types): *Consider an auction of car with public value μ. The type of a bidder is a vector $\boldsymbol{\tau} = (v, c, p, \sigma)$ where v is the bidder's valuation for the car being auctioned, c and σ are location/scale parameters of an extreme-value distribution governing the bidder's psychological cost of updating bids during the auction, and p is the probability that the bidder is distracted and unable to bid at a given bidding instant during the auction. he psychological bidding costs are IID extreme-value draws, and bidder distraction are IID Bernoulli draws at each of the $T = 121$ bidding seconds during the auction. Conditional on a bidder's valuation v, the remaining components of the bidder's type (c, p, σ) IID draws from a conditional distribution $Q(c, p, \sigma|v)$ which is common knowledge among the bidders.*

Assumption 5 (Bidder beliefs): *Bidders individually regard auctions as having generic IID outcomes and share a common belief $\mathcal{B}$ about the probability measure governing the stochastic process for the high bid in each auction, which we denote by $\{\overline{b}_t\}$, where $\overline{b}_t$ is the highest submitted bid up to bidding instant t in the auction.*

Assumption 6 (Bidder optimality): *Bidders adopt bidding strategies that maximize their expected payoff from participating in the auction, formulated as a single agent game against nature where "nature" is the combination of the rules of the Korean auction and the bidder's beliefs $\mathcal{B}$ about the probability law of $\{\overline{b}_t\}$, the stochastic process for the high bid in the auction. Bidders are rationally-inattentive: they take into account the possibility that they will be inattentive in later seconds of the auction when calculating their optimal bids in earlier seconds, and the also account for the impact of their psychological bidding costs that constitute additional bidding frictions' when determining their optimal bidding strategies.*

Assumptions 1 to 6 imply that individual auctions are "generic" and have generic realized "high bid tracks" $\{\overline{b}_t\}$ where $\overline{b}_t$ denotes the high bid at time t in the auction. If there are n bidders partcipating in the auction with types $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$, then the assumptions imply that the bidders use optimal bidding strategies in response to a common belief $\mathcal{B}$ concerning the stochastic process of $\{\overline{b}_t\}$ that take the form $[\beta(\mathcal{B}, \boldsymbol{\tau}_1), \ldots, \beta(\mathcal{B}, \boldsymbol{\tau}_n)]$. By Assumption 3 the IID structure of psychological bidding costs and bidder attention, implies these strategies will appear stochastic from the standpoint of outsiders who do not observe these costs or whether the bidder is inattentive at any given instant. Together with the rules of the Korean auction, this implies that the high bid track $\{\overline{b}_t\}$ is a well defined stochastic process, and the high bid tracks for different auctions will IID stochastic processes.

Lemma 3: *Assumptions 1 to 6 imply that the high bids at each instant of the Korean auctions $\{\overline{b}_t\}$ are well defined stochastic processes that are IID across different auctions.*

4.2 DP solution for the optimal bidding strategy

In this section, we demonstrate that a bidder's belief $\mathcal{B}$ concerning the stochastic process of the high bid at the Korean auction constitutes a "sufficient statistic" that enables the bidder to calculate their optimal bidding strategy $\beta(\mathcal{B}, \boldsymbol{\tau})$ using DP. As we noted in the introduction, we discretize time into 121 bidding seconds, $t = 0, 1, \ldots, 120$ and assume that bids are only submitted at those seconds. The auction software informs each bidder at $t = 1, \ldots, 121$ whether they have the high bid based on the history of all bids submitted in the auction through second $t - 1$. When considering whether to bid at second t each bidder has the information (b_t, h_t) where b_t is the high bid submitted by the bidder up to and including second $t - 1$ and h_t is the high bid indicator: $h_t = 1$ if the bidder's high bid b_t is also the high bid in the auction at the start of second t, and 0 otherwise.

The bidder's beliefs about the stochastic process for the high bid at the auction $\{\tilde{b}_t\}$ can be encoded by a family of conditional CDFS $\{\lambda_t\}$, where $\lambda_{t+1}(b|b_t, h_t)$ denotes the conditional probability that the high bid in the auction at second $t + 1$ is less than or equal to b given the bidder's information at time t, (b_t, h_t). At $t = 0$, none of the bidders have submitted a bid. Therefore, by definition, $b_0 = 0$ and $h_0 = 0$ and $\lambda_1(b|b_0, h_0)$ is the CDF of the high bid submitted at $t = 0$. If the bidder did not bid at $t = 0$, then we set $b_1 = 0$. In this case, $h_1 = 0$, even if no other bidders submitted a bid at $t = 0$. So h_t will only equal to one in the first period t when a positive bid has been tendered and the bidder in question has submitted the highest bid thus far at the auction.

Assumption 7: *Bidders have rational beliefs about the stochastic process* $\{\tilde{b}_t\}$ *for the high bid at the auction, and they are encoded by the family of conditional CDFs given by*

$$\mathcal{B} \equiv \{\lambda_{t+1}(b|b_t, h_t)|t = 0, 1, \ldots, 120\}. \quad (6)$$

We can now to write the Bellman equation recursions needed when solving for the bidder's optimal bidding strategy at the Korean auction. The solution will depend on the bidder's type $\boldsymbol{\tau} = (v, c, p, \sigma)$, his beliefs $\mathcal{B}$, and their state (b_t, h_t), for $t = 0, \ldots, 121$. For notational, we shal drop the non-time-varying variables $\boldsymbol{\tau}$ and $\mathcal{B}$ from expressions for the bidder's value functions, which we denote by W_t and their optimal bidding decision rule, which we denote by β_t.

Backward induction begins at the termination of the auction at $T + 1 = 121$ seconds, where the bidder has potentially submitted a final bid at second $T = 120$, which is denoted by b_{121} in our notation. The auction software then transmits the information h_{121}, which equals one if the bidder had submitted the highest bid at the last period (and, thus, won the auction), or zero otherwise. The terminal value function is $W_{T+1}(b_{T+1}, h_{T+1})$ given by

$$W_{T+1}(b_{T+1}, h_{T+1}) = (v - b_{T+1})I\{1 = h_{T+1}\}. \quad (7)$$

W_{T+1} is a post-decision value function, since it specifies the *ex post* payoff at the end of the auction. Let $w_T(b, b_T, h_T)$ denote the bid-specific value function. This is the *ex ante* expected value to the bidder at $T = 120$ from submitting a bid of b given their state at T is (b_T, h_T). The bid-specific value function is given by

$$w_T(b, b_T, h_T) = \mathbb{E}[W_T(b_{T+1}, h_{T+1})|b, b_T, h_T] = (v - b)\lambda_{T+1}(b|b_T, h_T). \quad (8)$$

Note that this is the same expected payoff function as a first-price, seald-bid auction when all bidders choose to snipe In this sense, our model includes a theory of bidding at a static first-price, sealed-bid auction as a special case.[9] Assuming the bidder is not distracted and decides to submits a bid at time T, the optimal bid is given by

$$\beta_T(b_T, h_T) = \underset{b \geq b_T}{\operatorname{argmax}} \left[w_T(b, b_T, h_T) \right]. \quad (9)$$

Generally, it is optimal to improve the current bid, so $\beta_T(b_T, h_T) > b_T$. As we saw in section 2, however, bidders frequently choose not to improve their bids, perhaps because they are distracted at

[9] It will, however, only be an anonymous equilibrium of the larger dynamic Korean auction if all bidders' beliefs assign zero probability to early bidding in the auction, that is, only if $\lambda_{t+1}(0|0, 0) = 1$ for $t < T = 120$.

certain seconds in the auction, but we also want to allow the possibility that a bidding friction—a psychological cost a bidder incurs from calculating an improved bid—might deter the bidder from submitting a strictly improved bid even if the bidder was not distracted. We admit this using another state variable $\epsilon_T = [\epsilon_T(0), \epsilon_T(1)]$, where $\epsilon_T(0)$ is the monetary equivalent of a psychological cost (if negative) or benefit (if positive) associated with not improving the current bid b_T and $\epsilon_T(1)$ is the corresponding benefit or cost associated with submitting the optimal bid $\beta_T(b_T, h_T)$. Thus, we define the value function $W_T(b_T, h_T, \epsilon_T)$ by

$$W_T(b_T, h_T, \epsilon_T) = \max\left[w_T(b_T, b_T, h_T) + \epsilon_T(0), \max_{b \geq b_T}[-c + w_T(b, b_T, h_T) + \epsilon_T(1)]\right]. \tag{10}$$

By Assumption 4, the shocks follow a Type-1 extreme-value distribution, having mean zero and scale parameter σ. This implies that the probability of submitting a bid $\beta_T(b_T, h_T)$ at the last instant $T = 120$ is $P_T(\beta_T(b_T, h_T)|b_T, h_T)$ given by a bivariate logit formula

$$P_T(\beta_T(b_T, h_T)|b_T, h_T) = \frac{\exp\{[w_T(\beta_T(b_T, h_T), b_T, h_T) - c]/\sigma\}}{\exp\{w_T(b_T, b_T, h_T)/\sigma\} + \exp\{[w_T(\beta_T(b_T, h_T), b_T, h_T) - c]/\sigma\}}. \tag{11}$$

Define the expected value function $EW_T(b_T, h_T)$ as the expectation of $W_T(b_T, h_T, \epsilon_T)$ with respect to ϵ_T but conditioning on (b_T, h_T). By the well-known property of Type-1 extreme-value distributions, we have the following closed-form solution for this expectation:

$$\begin{aligned} EW_T(b_T, h_T) &= \int_{\epsilon_T} W_T(b_T, h_T) q(\epsilon_T) d\epsilon_T \\ &= \sigma \log\Big(\exp\{w_T(b_T, b_T, h_T)/\sigma\} + \exp\{w_T(\beta_T(b_T, h_T), b_T, h_T)/\sigma\}\Big), \end{aligned} \tag{12}$$

where $q(\epsilon_T)$ is the probabilty density function of a bivariate Type-1 extreme-value random variable having mean zero and scale parameter σ. $EW_T(b_T, h_T)$ is relevant only if the bidder was not distracted at instant T and so observed ϵ_T and made a choice of whether to improve their bid. If the bidder was distracted, then no improved bid would be submitted and the value in this case is just $w_T(b_T, b_T, h_T)$, the same value as a conscious decision not to bid. Therefore, the bid-specific value function at bidding instant $T-1$ is $w_{T-1}(b, b_{T-1}, h_{T-1})$ given by

$$\begin{aligned} w_{T-1}(b, b_{T-1}, h_{T-1}) = {} & [p w_T(b, b, 1) + (1-p) EW_T(b, 1)]\, \lambda_T(b|b_{T-1}, h_{T-1}) + \\ & [p w_T(b, b, 0) + (1-p) EW_T(b, 0)]\, [1 - \lambda_T(b|b_{T-1}, h_{T-1})]. \end{aligned} \tag{13}$$

Equation (13) shows the rational-inattention aspect of our model: at second $T-1$ the bidder is not distracted and is considering the value of making alternative bids $b \geq b_{T-1}$. Yet the bidder is self-aware that at second T there is a probability p that they will be distracted and, thus, unable to make any further adjustments to their bid in that event. If the bidder tenders b at $T-1$, then they will be the high bidder at instant T with probability $\lambda_T(b|b_{T-1}, h_{T-1})$ so $h_T = 1$. In equation (13), the first line shows the expected payoff of this outcome. The second line covers the case when b is not the high bid, so $h_T = 0$. In both cases, if the bidder is distracted, then no further bid is made and this has value $w_T(b, b, 1)$ or $w_T(b, b, 0)$ depending on whether b is the high bid. If the bidder is not distracted, their expected values are $EW_T(b, 1)$ and $EW_T(b, 0)$, respectively.

Using $w_{T-1}(b, b_{T-1}, h_{T-1})$ we can define the optimal bid function $\beta_{T-1}(b_{T-1}, h_{T-1})$ in the same way as we did for period T in equation (9). The probability of bidding and the expected value functions are defined defined with equations similar to their time T counterparts in equations (11) and (12), respectively. Backward induction then proceeds using the same formulae for all other bidding seconds for $t = T-2, \ldots, 1, 0$. We shall denote by $\beta(\mathcal{B}, \boldsymbol{\tau})$ the full sequence of optimal bid functions

$$\beta(\mathcal{B}, \boldsymbol{\tau}) = (\beta_0, \beta_1, \ldots, \beta_T), \tag{14}$$

where we have suppressed the dependence on the arguments $\mathcal{B} = \{\lambda_{t+1}(b|b_t, h_t)|t = 0, \ldots, T\}$ and $\boldsymbol{\tau} = (v, c, p, \sigma)$ to simplify notation.

Note that the actual bidding strategy $\beta(\mathcal{B}, \boldsymbol{\tau})$ is stochastic due the effect of the IID bidding cost state variables $\{\epsilon_t\}$ and the Bernoulli process for bidder attention. Let a_t be a Bernoulli random variable that equals one if the bidder is paying attention at bidding instant t and zero otherwise. Also, let $\delta_t(b_t, h_t, \epsilon_t)$ be another Bernoulli random variable that equals one if it is optimal to bid in state (b_t, h_t, ϵ_t) when $a_t = 1$, that is,

$$\delta_t(b_t, h_t, \epsilon_t) = \begin{cases} 1 & \text{if } w_t(b_t, b_t, h_t) + \epsilon_t(0) \geq w_t(\beta_t(b_t, h_t), b_t, h_t) - c + \epsilon_t(1) \\ 0 & \text{if } w_t(b_t, b_t, h_t) + \epsilon_t(0) < w_t(\beta_t(b_t, h_t), b_t, h_t) - c + \epsilon_t(1). \end{cases} \tag{15}$$

We can then write the full bidding strategy as the function $\beta_t(b_t, h_t, \epsilon_t, a_t)$ given by the following:

$$\beta_t(b_t, h_t, \epsilon_t, a_t) = \begin{cases} b_t & \text{if } a_t = 0 \\ b_t & \text{if } a_t = 1 \text{ and } \delta_t(b_t, h_t, \epsilon_t) = 0 \\ \beta_t(b_t, h_t) & \text{if } a_t = 1 \text{ and } \delta_t(b_t, h_t, \epsilon_t) = 1. \end{cases} \tag{16}$$

The extended bidding strategy given in equation (16) above is what we will use both to simulate outcomes in the Korean auction and as a basis for inference. Intuitively, the bidder will not improve their current bid b_t at bidding instant $t+1$ if either 1) they are inattentive $a_t = 0$ or 2) they are attentive, but due to the realized psychological bidding shocks $[\epsilon_t(0), \epsilon_t(1)]$ they conclude it is suboptimal for him to improve their bid. A key restriction of this model is that in the third case where the bidder is paying attention and the shocks $[\epsilon_t(0), \epsilon_t(1)]$ are such that it is optimal for him to place a bid, where the amount of the bid, $\beta_t(b_t, h_t)$, is a deterministic function of (b_t, h_t). This implies that our model of bidding in the Korean auction is *statistically degenerate.* That is, the probability of observing any other bid $b \neq \beta_t(b_t, h_t)$ is zero under our rationally-inattentive model of bidding behavior and this rules out the direct use of maximum likelihood for inference. For this reason, the next section introduces a quasi-maximum likelihood estimator (QMLE) estimation strategy for conducting inference in this model that is tolerant of observed bids b that are not equal to the predicted optimal bid $\beta_t(b_t, h_t)$ and finds values for the bidder's type $\boldsymbol{\tau} = (v, c, p, \sigma)$ that best fits their behavior at individual auctions. Using the QMLE estimates of bidder types, we will show in section 5 that the estimated model is capable of explaining the observed early bidding behavior and heterogeneous bidding strategies that we documented in section 2.

4.3 Properties of the optimal bidding strategy and the value of learning the high bid

The DP solution for the optimal dynamic bidding strategy is rather complex and not easy to summarize concisely, but in this section we attempt to provide some insight by focusing on the case of a frictionless bidder, $\tau = (v, p, c, \sigma) = (v, 0, 0, 0)$, and illustrating the 'value of learning" via early bidding in the auction.

Figure 5: Value functions for a frictionless bidder with $v = 5562$ at second $t = 120$

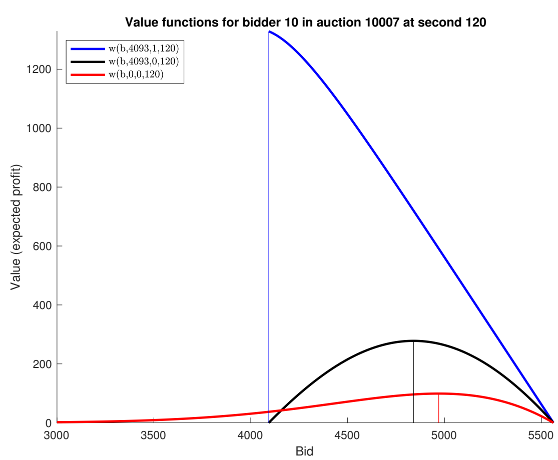


Figure 5 plots the bid-specific value functions, $w_{120}(b, b_{120}, h_{120})$, over possible bids b at $t = 120$ for a frictionless bidder with valuation $v = 5562$ with beliefs about the high bid estimated from data on the human bidders in the Avante auctions (see section 5.1). There are three different value functions for three different states (b_{120}, h_{120}). The red curve plots $w_{120}(b, b_{120}, h_{120})$ for an "uninformed sniper" who has not yet bid in the auction, $(b_{120}, h_{120}) = (0,0)$. The black line plots $w_{120}(b, b_{120}, h_{120})$ for a bidder whose highest bid so far, $b_{120} = 4093$, is not the high bid in the auction, so $(b_{120}, h_{120}) = (4093, 0)$. The blue line plots $w_{120}(b, b_{120}, h_{120})$ for a bidder whose highest bid so far is also 4093, but now we assume it is the high bid in the auction at $t = 120$, so $(b_{120}, h_{120}) = (4093, 1)$. Note that the latter two value functions are only defined over a restricted domain, $[b_{120}, v] = [4093, 5562]$, whereas the set of possible bids b for a bidder who has not yet bid is the full interval $[0, v] = [0, 5562]$.

It is clear that the bidder obtains valuable information from having placed a previous bid of $b_{120} = 4093$ compared to not bidding and remaining uninformed, $b_{120} = 0$, and this is true even if the bidder is not the high bidder at $t = 120$, so $h_{120} = 0$. But clearly, it is of highest value for the bidder to know their bid is the highest at $t = 120$.

Table 2: Expected high bids, optimal bids and expected profits for a frictionless bidder at $t = 120$

State ($v = 5562$)	$E\{\overline{b}_{121} \mid b_{120}, h_{120}\}$	$\beta_{120}(b_{120}, h_{120})$	$W_{120}(b_{120}, h_{120})$
$b_{120} = 0$, $h_{120} = 0$	5825	4970	99
$b_{120} = 4093$, $h_{120} = 0$	5173	4839	278
$b_{120} = 4093$, $h_{120} = 1$	4113	4093	1329

Table 2 plots the expectations of the bidder in each of the three cases, along with their optimal bids and expected profits. The uninformed sniper expects the winning price in the auction to be $E\{\overline{b}_{121} | 0, 0\} = 5825$ but bids less than this amount, $\beta_{120}(0, 0) = 4970$, resulting in an expected profit of $W_{120}(0, 0) = 99$. However if the bidder had bid earlier in the auction but was not the highest bidder, the knowledge that the winning bid in the auction will have to exceed their bid of $b_{120} = 4093$ actually *reduces* their expectation of the winning price in the auction to $E\{\overline{b}_{121} | 4093, 0\} = 5173$. The optimal bid is also lowered relative to what a sniper would optimally bid to $\beta_{120}(4093, 0) = 4839$, which nearly triples their expected profit, $W_{120}(4093, 0) = 278$, relative to the uninformed sniper. However, the best case is where the bidder knows they are the high bidder so far when they make their last bid. This knowledge reduces their expectation of being significantly outbid in the last second, and their expectation of the winning price is $E\{\overline{b}_{121} | 4093, 1\} = 4113$, only slightly higher than their current bid $b_{120} = 4093$. As a result, this bidder decides to "hold firm" and not raise their bid in the last second, so $\beta_{120}(4093, 1) = 4093$, and their expected profit is more than 13 times higher compared to the case of the informed sniper.

4.4 Effect of informational restrictions on early bidding

Before we describe the QMLE estimator, we consider how the bidder's DP problem can be modified to solve for the optimal bidding strategy when a key informational restriction of the Korean auction is dropped. Specifically, suppose that the auction software is modified to display the highest submitted bid at each instant during the auction, although the identity of the bidder holding the highest bid is still suppressed. The high bid indicator h_t still shows each bidder whether they are currently the high bidder, but now every bidder observes the high bid at the auction even if $h_t = 0$. We refer to this auction as an *anonymous open-outcry auction*. This change in rules makes the anonymous open-outcry auction a slightly informationally-restricted version of a standard open-outcry auction—the difference being that at most open-outcry auctions all bidders can see the identities of competing bidders, not just the high bid at any given moment. As the name suggests, the anonymous open-outcry auction can be formulated as an anonymous game, and an anonymous equilibrium can be computed using the same approach we outlined above for the Korean auction.

How are bidding strategies affected by dropping the informational restriction that each bidder only knows the high bid when they are the high bidder? It is easy to see that their beliefs about the CDF of the high bid at the auction no longer depends on the binary indicator h_t. Instead, we reinterpret the state variable b_t in the DP problem to be the current high bid at the auction, as opposed to the bidder's own high bid, our intrerpretation of b_t under the Korean auction's informational restrictions. Without this restriction, bidders' beliefs can be fully described by a family of conditional CDFs $\{\lambda_{t+1}(b|b_t)|t = 0, 1, \ldots, T\}$, where $\lambda_{t+1}(b|b_t)$ is the conditional probability that the high bid submitted at bidding instant t is less than or equal to b given that the high bid submitted up through time $t-1$ is b_t. The Bellman equations and optimal bidding strategies do not depend on the high bid indicator h_t. The intuitive implication of relaxing the informational restriction at the Korean auction is that bidders in an anonymous open-outcry auction have no incentive to bid early in the auction to learn the high bid since they are provided this information for free. This implies that relaxing the informational resriction leads to informational free-riding.

We nnow show that frictionless bidders will prefer to remain in the background and only submit a single bid at the last instant, T. That is, bid-sniping is the only anonymous equilibrium of this game, so it is strategically equivalent to the equilibrium of an anonymous, static, first-price, sealed-bid auction. In order to establish this result, we need to introduce a mild assumption about bidders' beliefs concerning the distribution of the high bid—namely, that $\lambda_{t+1}(b|b_t)$ is stochastically increasing in the current high bid b_t, an assumption that seems reasonable in an ascending bid auction. When this holds, in the absence of bidding frictions, we can show that no strategic reason exists for early bidding. However in the presence of rational-inattention there can be a motive for early bidding since it plays a role similar to a "soft close" in online auctions on Amazon discussed by Alvin E. Roth and Axel Ockenfels [2002].

Definition 3: We say the family $\{\lambda_{t+1}(b|b_t)|t = 0, 1, \ldots, T\}$ is *stochastically increasing in* b_t if for any $b'_t \geq b_t$ we have:

$$\lambda_{t+1}(b|b'_t) \leq \lambda_{t+1}(b|b_t), \tag{17}$$

for all $b \geq b_t$ and all $t \in \{0, 1, \ldots, T\}$.

Assumption 8: *The family of beliefs concerning the high bid* $\{\lambda_{t+1}(b|b_t)|t = 0, 1, \ldots, T\}$ *is stochastically increasing in* b_t *and satisfies the derivative condition*

$$\nabla_{b_t}\lambda_{t+1}(b|b_t) \leq 0, \tag{18}$$

where ∇_{b_t} *denotes the derivative of* $\lambda_{t+1}(b|b_t)$ *with respect to its conditioning argument,* b_t*, for all* $b \geq b_t$ *and* $t \in \{0, 1, \ldots, T\}$*. Furthermore,* $\lambda'_{t+1}(b|b_t) > 0$ *for* $b > b_t$ *where* $\lambda'_{t+1}(b|b_t)$ *is the derivative of the CDF* $\lambda_{t+1}(b|b_t)$ *with respect to b, which we assume is a continuous function of b for any* $b > b_t$ *and all* $t \in \{0, 1, \ldots, T\}$*.*

Theorem 1: *Suppose the auction rules are changed to be an anonymous version of a Japanese auction, i.e. the current high bid is made public to all bidders at every instant t during the auction, but bidders do not observe the identities of other bidders or even how many other bidders are present. Then anonymous equilibrium bidding strategies in the anonymous Japanese auction do not depend on* h_t*, the indicator for whether a bidder holds the high bid at each instant t. Furthermore, under Assumptions 1 to 8, if* $p = c = \sigma = 0$ *(that is, no bidding frictions), the anonymous equilibrium bidding strategies entail bidding zero until the final instant T of the auction. Thus all bidders adopt optimal bid-sniping strategies and the anonymous equilibrium of the anonymous open outcry auction is strategically equivalent to the anonymous equilibrium of an anonymous static first-price, sealed-bid auction.*

The proof of Theorem 1 is by induction, see Appendix B. The theorem encapsulates the idea of informational free-riding that rules out early bidding in ascending bid auctions without informational restrictions preventing bidders from seeing the amount of the high bid at each second in the auction, at least for rational, frictionless bidders.

We now consider the question of whether the key informational restriction at the Korean auction — only revealing the current high bid if a bidder submits a bid and their bid is the highest bid so far — is sufficient to generate early bidding. In fact, the example we discussed in section 4.3 shows that it is: in that case we showed that frictionless bidders will engage in early bidding in the auction due to the substantial "value of information" from learning what the high bid in the auction is. By bidding early, bidders can learn the learn the high bid and increase their chance of winning the auction without "overpaying".

However this intuition depends on the beliefs of the bidders: they need to believe there is valuable information to be obtained from bidding early in the auction. In the example in section 4.3 we endowed the frictionless bidder with beliefs based on the actual bidding behavior of human bidders in the Avante auctions, and for these beliefs we showed there can be substantial gains from early bidding. We now prove a result similar to Lemma 1 in section 3, namely that there always exists an anonymous equilibrium with no early bidding in auctions with frictionless bidders.

Theorem 2: *In the Korean auction with frictionless bidders, there is an anonymous equilibrium with no early bidding. This equilibrium is equivalent to the anonymous equilibrium of a static, sealed-bid first-price auction.*

Theorem 2 can be proved by induction using a similar argument as the proof of Lemma 1 in section 3. Essentially, if a bidder believes there will be no early bidding in the auction by other bidders, then there is nothing to be learned and hence no incentive for this bidder to bid early in the auction as well. As a result, no early bidding is a best response for any frictionless bidder in the auction when each bidder also believes none of the other bidders is engaging in early bidding.

This paradoxical outcome shows that the informational restrictions underlying the Korean auction are not sufficient in themselves to generate the early bidding behavior we observe. We need bidding frictions or beliefs that other bidders engage in early bidding to generate this behavior. Rational-inattention can generate early bidding in the Korean auction, though it also produces early bidding in Japanese auctions that do not involve the informational restriction of the Korean auctions due to the"soft-close" uncertainty that rational inattention creates among bidders about whether they can submit a bid in the last second of the auction.

Theorem 1 implies that early bidding will not occur in an anonymous Japanese auction with frictionless bidders (and no rational-inattention) yet the example of section 4.3 shows that we can obtain early bidding in the Korean auctions with frictionless, perfectly attentive bidders. However this early bidding depends on beliefs that they are other bidders who, for whatever reason, engage in early bidding. We do not know if there exist anonymous equilibria involving early bidding in the Korean auction where all bidders are perfectly attentive and frictionless, but we do show in our counterfactual analysis of the Korean auction in section 6 that a successive approximations algorithm for computing an anonymous equilibrium to the Korean auction with frictionless, perfectly attentive bidders converges to an equilibrium involving no early bidding.

4.5 Two-step, quasi maximum-likelihood, fixed-effects estimator

We employ a two-step estimation approach, where the first step involves estimating bidders' beliefs $\mathcal{B} = \{\lambda_{t+1}(b|b_t, h_t)|t = 0, \ldots, 120\}$, and the second step involves estimating bidder/auction-specific types $\boldsymbol{\tau} = (v, p, c, \sigma)$ for each bidder at each auction in our auction dataset. The latter is done using a structural nested DP quasi-maximum likelihood estimator (QMLE) where we repeatedly solve for a bidder's optimal bidding strategy for different candidate values of $\boldsymbol{\tau}$ using the first stage estimates of beliefs about the high bid, $\hat{\mathcal{B}}$. Since we estimate the conditional distribution of the high bid $\lambda_{t+1}(b|b_t, h_t)$ in the first stage using actual bidding data for the Avante car auctions, our estimation approach imposes rational expectations on the part of all bidders.[10] We recognize, however, that estimation noise exists in our first-stage estimates of beliefs, so we subsequently check whether the weaker condition of ϵ-anonymous equilibrium holds by resimulating data using the estimated bidding strategies for all bidders at all auctions and calculating the difference $\epsilon \equiv \|\hat{\mathcal{B}} - \Lambda(\hat{\mathcal{B}})\|$. If this difference is sufficiently small, then we can conclude that our estimated structural model of rationally-inattentive bidding constitutes an ϵ-anonymous equilibrium of the Korean auction.

We adopt a fixed-effects approach to estimation of the unknown parameters of our model of rationally-inattentive bidding in Korean auctions. The values v are clearly both bidder and auction-specific by Assumption 1. To allow for maximum heterogeneity, we estimate the other three parameters (p, c, σ) separately for each bidder/auction pair in our dataset as well. It is easiest to begin by explaining how to estimate $\boldsymbol{\tau}$ via full maximum-likelihood, and after deriving the full likelihood it will help motivate why we chose to estimate $\boldsymbol{\tau}$ using a QMLE instead.

To derive the likelihood, we consider an extension of the model developed in the previous section that is statistically nondegenerate; that is, it assigns positive probability to any possible observed sequence of bids by a given bidder at the auction. The data we have for an individual bidder at a specific auction are $\{(b_t, h_t)|t = 1, \ldots, T\}$ where $T = 121$, where we cleaned the data to remove non-monotonic bids submitted by any bidder. Recall our timing convention: b_1 denotes the bid the bidder submitted at $t = 0$ (or zero if no bid was submitted) and $h_1 = 1$ if that was the highest bid among all bids submitted in the interval $[0, 1)$. Continuing, b_t is the bid submitted at second $t - 1$; that is, in the time interval $[t - 1, t)$ and $h_t = 1$, if that bid were the highest bid submitted at the auction up to time t, and 0 otherwise. Finally, b_{121} denotes the final bid submitted

[10] We group all bids arriving in the time interval $[t, t+1)$ as having been submitted at bidding instant t to match our continuous time data to our discrete time DP model of the Korean auction.

at the last second of the auction $T = 120$; that is, in the time interval $[120, 121)$ where as noted in section 2 the auction software accepts bids that arrive slightly after the two-minute mark.

Suppose our extended, nonstatistically degenerate model results in a bid-transition probability of the form $P(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$. Then the likelihood for a bidder in a particular auction is given by

$$L(\boldsymbol{\tau}) = \prod_{t=0}^{120} f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) \tag{19}$$

where f is a conditional probability given by

$$f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) = \begin{cases} p + (1-p)[1 - P(b_{t+1}|b_t, h_t, \boldsymbol{\tau})] & \text{if } b_{t+1} = b_t \\ (1-p)P(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) & \text{if } b_{t+1} > b_t \end{cases} \tag{20}$$

and subject to the initial condition $(b_0, h_0) = (0, 0)$. The conditional probability $f(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ in equation (20) reflects two possibilities: 1) a bidder may fail to improve their bid at instant t, ($b_{t+1} = b_t$), or 2) the bidder can submit an improved bid, $b_{t+1} > b_t$. The probability that there is no improvement in the bid equals the sum of the probability p that the bidder was not paying attention at t plus the probability the bidder was paying attention but chose not to improve the bid. This is reflected by the second term in the first line of equation (20) where $P(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ is a multinomial logit formula for the probability of submitting a bid of b_{t+1} at instant t that we derive below. If the bidder does improve their bid ($b_{t+1} > b_t$), then the probability of this occurring is in the second line of equation (20) and it requires that the bidder not be inattentive and also actively chooses a bid higher than their best previous bid b_t.

The MLE uses a nested solution approach where an outer hill climbing algorithm searches for a 1×4 vector $\hat{\boldsymbol{\tau}}$ that maximizes $L(\boldsymbol{\tau})$, and an inner DP algorithm is called to solve the bidder's DP problem each time the likelihood is evaluated for a given value of $\boldsymbol{\tau}$. Note that bidder beliefs are fixed at the estimates $\hat{\mathcal{B}}$ from the first step throughout all second step structural estimations for all bidder/auction pairs.

We can modify the DP model in the previous section to be statistically nondegenerate by 1) assuming bids are discrete, say, restricted to the positive integer values (which is the case in our data), and 2) introducing a separate Type-1 extreme *bidding shock* for every possible integer bid b. When we do the latter, we need to modify the formula for the value function in equation (10), and then generic time t step of the backward induction the value function is $W_t(b_t, h_t, \epsilon_t)$ given by

$$W_t(b_t, h_t, \epsilon_t) = \max\left[w_t(b_t, b_t, h_t) + \epsilon_t(0), \max_{b > b_t}[-c + w_t(b, b_t, h_t) + \epsilon_t(b)]\right]. \tag{21}$$

This version of the model has enough bid-specific shocks to generate a positive probability of observing any possible bid b, since it implies the following multinomial logit conditional choice probability for $P(b|b_t, h_t)$

$$P_t(b|b_t, h_t) = \begin{cases} \frac{\exp\{[w_t(b,b_t,h_t)-c]/\sigma\}}{\exp\{w_t(b_T,b_T,h_T)/\sigma\}+\sum_{b>b_t}\exp\{[w_T(b,b_t,h_t)-c]/\sigma\}} & \text{if } b > b_t \\ \frac{\exp\{w_t(b,b_t,h_t)/\sigma\}}{\exp\{w_t(b_T,b_T,h_T)/\sigma\}+\sum_{b>b_t}\exp\{[w_T(b,b_t,h_t)-c]/\sigma\}} & \text{if } b = b_t \end{cases} \tag{22}$$

Unfortunately, the full likelihood approach involves a substantial computational burden because it requires exhaustive evaluation of the value functions at all possible integer bids b greater than or equal to b_t. The model we presented in the previous section only involves two shocks, $[\epsilon_t(0), \epsilon_t(1)]$, where $\epsilon_t(0)$ represents unobserved costs/benefits of not improving the current bid, and $\epsilon_t(1)$ are the unobserved costs/benefits corresponding to submitting the optimal bid $\beta_t(b_t, h_t)$. We use a more efficient continuous maximization algorithm to compute $\beta_t(b_t, h_t)$ that requires relatively few evaluations of an interpolated version of the bid-specific value functions $w_t(b, b_t, h_t)$. This is far faster than the brute-force exhaustive evaluation required by the full-likelihood approach.

Another attractive feature of the model we presented in the previous section is that its tight prediction of the optimal bid $\beta_t(b_t, h_t)$ is useful in assessing how well our model actually fits the data. We can think of $\beta_t(b_t, h_t)$ as akin to a nonlinear regression function that constitutes the model's predicted optimal bid, so we can directly evaluate the "residuals" $e_t(\hat{\boldsymbol{\tau}}) = b_{t+1} - \beta_t(b_t, h_t, \hat{\boldsymbol{\tau}})$ at the estimated value of each bidder's type $\hat{\boldsymbol{\tau}}$. In the next section, we compare actual bids to those predicted by our model. This is not only extremely informative not only about model fit, but also about the behavior of the bidders. The logit model in equation (22) has such a rich specification of unobservables that it can "rationalize" any observed bidding behavior, even though the fully saturated specification still imposes testable restrictions that can be assessed by a variety of specification tests. That said, we chose to begin with the QMLE estimator.

To implement the QMLE estimator, we need to define a "quasi-likelihood" of observing a bid of b that differs from the optimal bid $\beta_t(b_t, h_t, \boldsymbol{\tau})$ predicted by the model when the current parameter is $\boldsymbol{\tau}$. For this purpose, we use the following binary logit probability:

$$\Pi(b|b_t, h_t, \boldsymbol{\tau}) = \frac{\exp\{w_t(b, b_t, h_t)/\omega\}}{\exp\{w_t(b, b_t, h_t)/\omega\} + \exp\{[w_t(\beta_t(b_t, h_t), b_t, h_t) - c]/\omega\}}, \tag{23}$$

where $\omega \geq 0$ is a penalty parameter that controls how hard the QMLE tries to fit observed bids b_{t+1} via the predicted optimal bids from the model $\beta_t(b_t, h_t, \boldsymbol{\tau})$. Note that for any bid b, we have $w_t(b, b_t, h_t) \leq w_t(\beta_t(b_t, h_t), b_t, h_t)$, so it follows that $\Pi(b|b_t, h_t, \boldsymbol{\tau}) \leq 1/(1 + \exp\{-c/\omega\})$

and $\Pi(b|b_t, h_t, \boldsymbol{\tau})$ is maximized at $b = \beta_t(b_t, h_t)$. When ω is small, there is high penalization and $\Pi(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ will be close to zero for observed bids that are far fom the optimal bid predicted by the model.

The QMLE is defined in a similar way to the MLE using the following formula for the quasi-likelihood function of the observed bidding data:

$$QL(\boldsymbol{\tau}) = \prod_{t=0}^{120} f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}), \tag{24}$$

where f is a conditional probability given by

$$f(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) = \begin{cases} p + (1-p)[1 - P(\beta_t(b_t, h_t)|b_t, h_t, \boldsymbol{\tau})] & \text{if } b_{t+1} = b_t \\ (1-p)P(\beta_t(b_t, h_t)|b_t, h_t, \boldsymbol{\tau})\Pi(b_{t+1}|b_t, h_t, \boldsymbol{\tau}) & \text{if } b_{t+1} > b_t \end{cases} \tag{25}$$

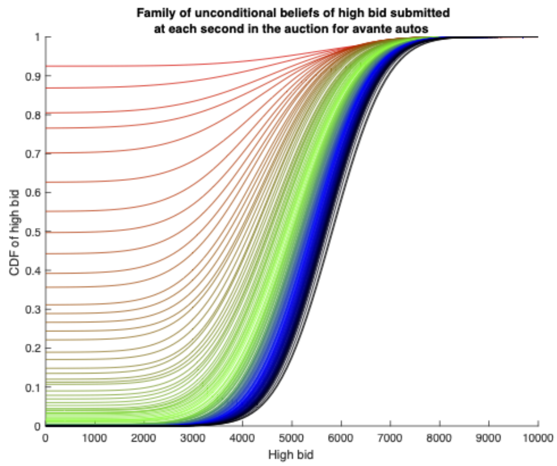
Comparing equations (20) and (25), we see that the key difference between the likelihood function $L(\boldsymbol{\tau})$ and the quasi-likelihood function $QL(\boldsymbol{\tau})$ is the presence of the binary logit probability $\Pi(b_{t+1}|b_t, h_t, \boldsymbol{\tau})$ that constitutes a penalty term for observed bids b_{t+1} that differ too much from the predicted optimal bid $\beta_t(b_t, h_t, \boldsymbol{\tau})$.[11]

5 Results

In this section, we report the results of our analysis of bidding data from 533 auctions of Hyundai Avante Elanta XD vehicles with 1.6L engines. These are generic passenger sedans without any unique features that would suggest we should treat different units as unique items—in violation of Assumption 1 of section 4. Our dataset contains a total of 4,029 bidder-auction histories from the 533 auctions, which we used to estimate an equivalent number of (4×1) type vectors $\boldsymbol{\tau}$ using the QMLE estimator defined in section 5. The data are $\{b_{tba}, h_{tba}|t = 1, \ldots, T\}$, the history of bids and signals for a given bidder b in auction a. The first step of our two-step estimation procedure involved pooling the bid data from all 4,029 bidding histories to estimate the beliefs of bidders concerning the high bid $\hat{\mathcal{B}}$ using a truncated normal specification. We found these parametric estimates matched closely the nonparametric estimates, but are substantially smoother and less noisy. Therefore, in our actual estimation results, we opted to use a parametric truncated normal family of beliefs.

[11] The likelihood $L(\boldsymbol{\tau})$ and quasi-likelihood $QL(\boldsymbol{\tau})$ are smooth functions of $\boldsymbol{\tau}$, but calculating their gradients requires recursive evaluation of the gradients of the bid-specific value functions $\{\nabla_{\boldsymbol{\tau}} w_t(b, b_t, h_t, \boldsymbol{\tau})\}$. This is done in tandem with the recursive calculation of $\{w_t(b, b_t, h_t, \boldsymbol{\tau})\}$ itself, using piecewise polynomial interpolation over a grid of bid values. MATLAB code to solve for optimal bidding strategies and estimate this model via the structural two-step QMLE is available on request.

Figure 6: Unconditional CDFs for high bids for Avante cars by elapsed time at auction



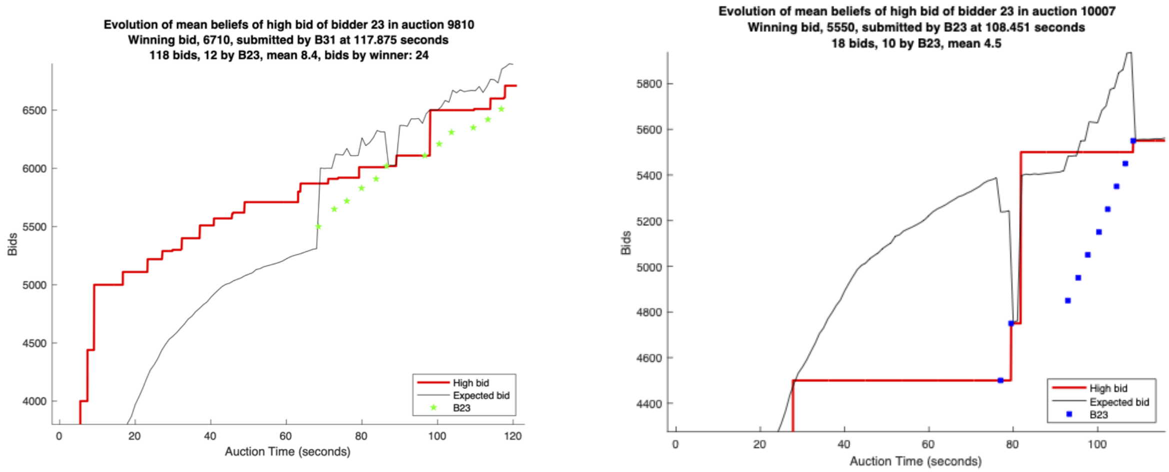
5.1 Estimated beliefs

The bid data from the $4,029$ bidder-auction observations provide 121 individual bid-second level observations of the high bid at each second of the auction. Figure 6 displays our estimates of the family of unconditional beliefs $\{\hat{\lambda}_{t+1}(b_{t+1}|0,0)\}$ at bid instants $t = 0, 1, \ldots, 120$. This family constitutes the *ex ante* beliefs concerning the distribution of high bids at each second of the auction for bidders who have not yet tendered bids at the auction. As would be expected at these ascending-bid auctions, strict stochastic dominance exists in the CDFs at successive seconds of the auction. The probability that the high bid equals zero (that is, no bids has been submitted) starts at over ninety percent in the first second of these auctions and rapidly drops to zero as t approaches $T = 120$.

We also estimated separate families of conditional CDFs $\{\hat{\lambda}_{t+1}(b_{t+1}|b_t, 1)\}$ that condition on the previous high bid b_t in the preceding second of the auction (and, hence, the second conditioning argument is $h_t = 1$), and a third family of CDFs $\{\hat{\lambda}_{t+1}(b_{t+1}|b_t, 0)\}$ that conditions on bids b_t that are lower than the current high bid $\overline{b}_t$ in the preceding second of the auction. We estimated these families via truncated regression methods—essentially Tobit models.

Figure 7 illustrates how bidder beliefs evolved during an auction by plotting two example paths for expected high bids, $\mathbb{E}(\overline{b}_{t+1}|b_t, h_t) = \int_{b_t}^{\infty} b\lambda_{t+1}(db|b_t, h_t)$ of bidder ID 23 (B23) at two different auctions. The left-hand panel of the figure illustrates the evolution of the expected high bid at auction 9810: B23 did not submit a bid until approximately 70 seconds into the auction, and the first bid submitted did not turn out to be the high bid. This is reflected in the concave

Figure 7: Expected high bids for specific examples



shaped black line in the figure that roughly parallels the red high bid track until the instant B23 submits their bid, at which point B23's expectation of the high bid jumps to a value above the high bid track. B23 continued to bid-creep, but did not succeed in capturing the high bid until around $t = 90$, when B23's expectation of the high bid decreases to a value virtually equal to b_{90}. Thus, when a bidder becomes the high bidder at some point in the auction, they have high confidence that they will remain the high bider for the next few seconds. A few seconds later, however, B23's high bid is eclipsed by a higher bid and then B23's belief concerning the expected high bid jumps back up above the high-bid track and remains there until the end of the auction because B23's subsequent bid creeping did not succeed in enabling B23 to regain the high bid and win the auction.

The right-hand panel of Figure 7 illustrates the evolution of B23's mean beliefs concerning the high bid at auction 10007. As in the previous case, the *ex ante* expectation of the high bid (*ex ante* in the sense that B23 has not submitted any bids yet) follows a concave shape, but in this case it lies substantially above the red realized high-bid track. At $t = 78$, B23 submits a bid equal to the current high bid $\overline{b}_{78}$, but because of the time-priority rules of the Korean auction, B23 is not informed that their bid equals the high bid, so B23's information is $h_{78} = 0$; that is, they are not told that they are the high bidder at that point. Nevertheless, B23's expectation of the high bid falls in response to this information, so $\mathbb{E}(\overline{b}_{79}|b_{28}, 0) < \mathbb{E}(\overline{b}_{79}|0, 0)$. At $t = 80$, B23 submits a higher bid that does become the high bid at that point, so B23 is informed that $h_{80} = 1$. This information decreases B23's expectation of the high bid to a value only slightly higher than b_{80}. A few seconds later, some other bidder tenders a substantially higher bid that increases the high bid to around

\$5,500, which causes B23's beliefs concerning the expected high bid to jump up to approximately \$5,400 in value.

In the remaining thirty seconds of auction 10007, B23 executed a sequence of bid creeps that are initially unsuccessful in capturing the high bid status until $t = 108$, when B23 becomes the high bidder with a bid of $b_{108} = 5,550$. B23 opted to make no further improvements in bid for the remainder of the auction, retaining their high bid status for the rest of the auction and winning it with a bid of $b_{108} = 5,550$. Thus, we can see how even with fixed beliefs the model captures learning in these auctions.

Note that bidders are not learning in a Bayesian sense: instead, they are learning about the high bid at the auction through the process of bidding during the auction. We now show how bidders' rational beliefs concerning the evolution of the high bid at the auction is reflected in the magnitudes of the bids; that is, we try to assess how close their actual bids $\{b_t\}$ are to the optimal bids $\{\beta_t(b_t|b_{t-1}, h_{t-1})\}$ predicted by the structural estimates of our model of rationally-inattentive bidding.

5.2 Estimated types

We estimated $4,029$ (4×1) type vectors $\hat{\boldsymbol{\tau}}$ using our structural nested DP QMLE estimator by repeatedly solving for optimal bidding strategies at the Korean auction to find ones that best fit the actual bidding behavior of the $4,029$ bidder-auction observations in our dataset. In all of these solutions, we keep the beliefs of all bidders fixed at the values we estimated in the first stage, $\hat{\mathcal{B}}$ described above. Even though at most 121 bidding instants exist at a single auction, we have sufficient information to point-identify the $4,029$ type vectors for all bidder-auction pairs in our dataset. We acknowledge potential econometric issues with our QMLE estimation approach, including the standard incidental parameters problem that can affect the consistency of fixed-effect estimators, as well as the issue of how to think about asymptotics when the data we use to estimate the type of each bidder are limited to the observed bids submitted during each 120-second auction. For now, we do not worry about these econometric issues and treat our QMLE estimation approach as a calibration exercise that provides substantial flexibility to fit observed bidding behavior best, as well as to capture the heterogeneity in bidding strategies that we documented in section 2.

With estimates of over $16,000$ parameters, it is obviously impractical to display them in a table (although we can provide a dataset containing the estimated parameters on request). Instead, we display the results graphically via univariate kernel-density plots of each of the four components of bidders' types, $\boldsymbol{\tau} = (v, c, p, \sigma)$ in Figures 8 and 9. The left-hand panel of Figure 8 shows the

Figure 8: Estimated valuations v and costs of bidding c

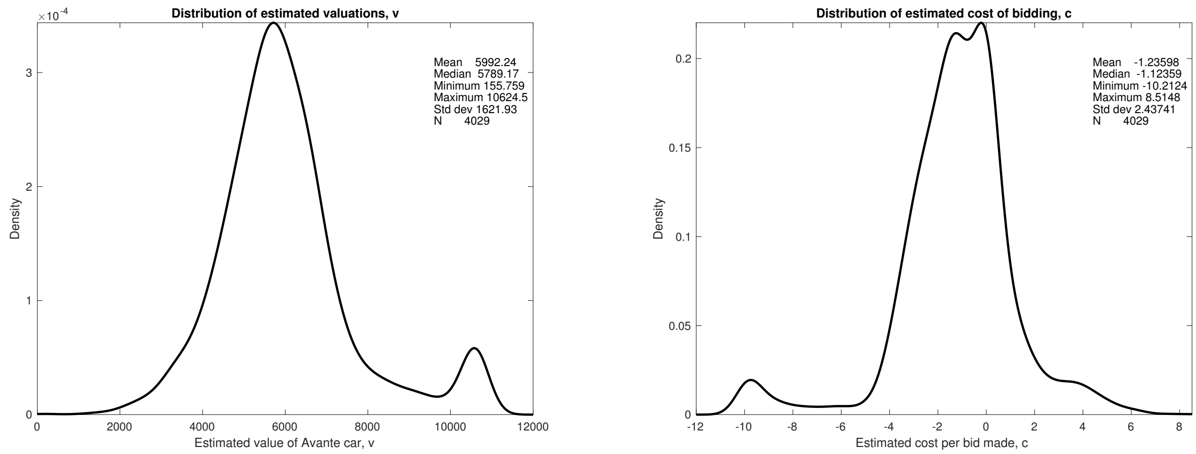
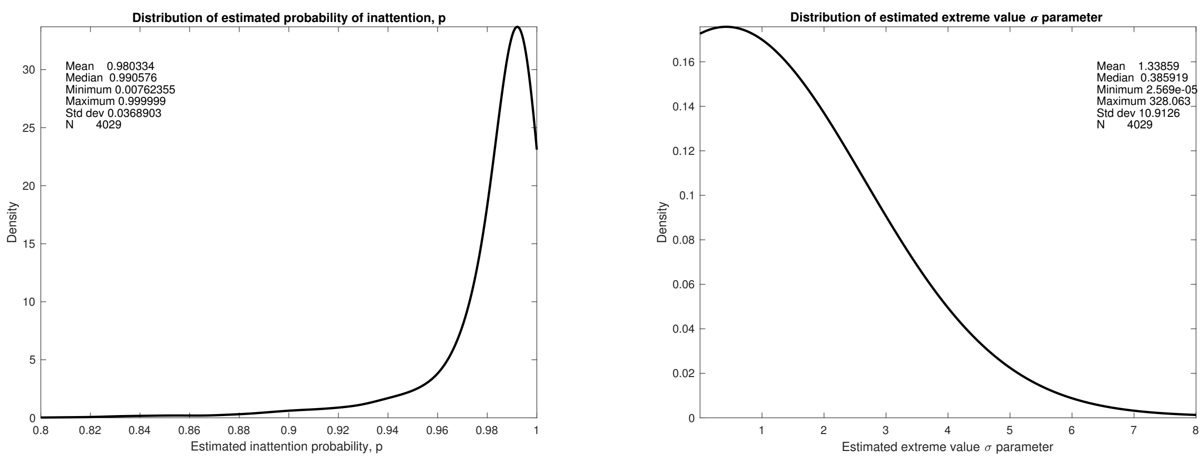


Figure 9: Estimated inattention probabilities p and scale parameters σ



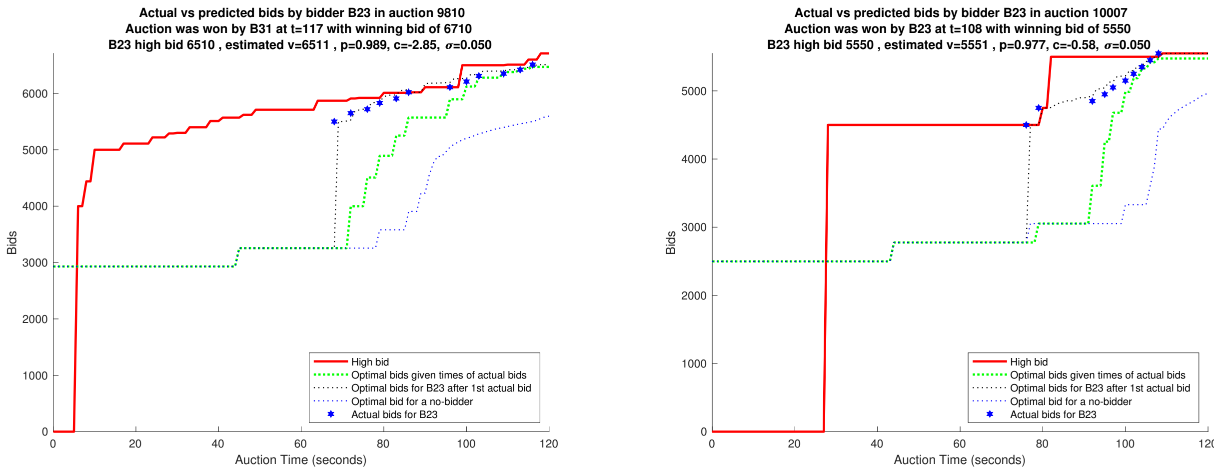
distribution of estimated valuations, v. The mean valuation for the 4,029 bidders participating in the 533 Avante XD 1.6L auctions was \$5,992, not much above the mean winning bid of \$5,824. The distribution seems to be approximately normally distributed, although an interesting hump exists in the upper tail of valuations. Further analysis is required to see whether something unique exists concerning these valuations—such as the potential that we mis-identified a subset of Avante XD 1.6L cars that have additional features (for example, luxury interiors) that make them more valuable to bidders than the generic Avante XD 1.6L cars that we identified. Since unique features for a subset of vehicles would violate our Assumption 1, it would be appropriate to remove these cases from our estimation results. The right-hand panel of Figure 8 shows the distribution of the location term c for the distribution of idiosyncratic psychological costs or benefits a bidder perceives from updating the current bid. The mean value of c is -1.2, indicating that the average bidder perceives a small "curiousity benefit" to updating their bid, above and beyond the benefit inherent from learning more about the current high bid. The mode of the distribution is, however, positive, so about 40 percent of all bidders perceive a psychological cost that constitutes a slight deterrent to frequently updating their bid—to wit, a bidding friction.

The left-hand panel of Figure 9 plots the distribution of inattention probabilities, p. Surprisingly, the mean probability of being inattentive at any point during the auction is nearly 98 percent. The reader may reasonably wonder: if bidders are inattentive 98 percent of the time, then how can they ever find an opportunity to place even a single bid at the auction? Recall that our implicit assumption is that inattention is an IID Bernoulli process with parameter p and there are 121 bidding instants during the auction. The probability is $1-(0.98)^{121}=0.91$ that the bidder will be paying attention at one or more instants and, therefore, able to submit one or more bids during an auction. Nevertheless, this does highlight a shortcoming of our specification: with such a large value of p, it is difficult for the model to replicate bid creeping when inattention is an IID Bernoulli process: this makes it unlikely to observe a sequence of several bids occuring only seconds apart from each other.[12]

The final, right-hand panel of Figure 9 depict the distribution of σ scale parameters for the extreme-value distributed idiosyncratic component of psychological bidding costs. On average, the values of σ are large— indicative of a fair degree of idiosyncratic variation in the costs that determine whether a bidder will update their bid, conditional on not being inattentive at a given instant in the auction. The overall message is: the estimated model predicts a fair degree of randomness in the times at which different bidders submit their bids at Korean auctions. We shall explore this further below, but to foreshadow, our model of rationally-inattentive bidding is much better at predicting how much bidders bid, but it is much harder to predict the total number of bids and the specific times at which bidders submit their bids.

[12]Of course, bidders who bid creep have smaller estimated values of p, since the QMLE estimator reduces p to increase the likelihood of observing sequences of such sequences of bids. We can also, however, estimate a two-state Markov model of rational inattention in future work. In a Markovian model, if a bidder is attentive at instant t they may be more likely to be attentive at $t+1$, and conversely for bidders who are inattentive. A Markov model requires estimation of two probabilities for bidders who are currently attentive and inattentive, respectively.

Figure 10: Actual versus predicted optimal bids by B23 at auctions 9810 and 10007



5.3 Evaluation of model fit at level of individual bidders

We now present evidence on the ability of the model to fit bidder behavior, particularly to assess how closely the model predicts a bidder's actual sequence of bids at an auction, conditioning on the times when the bidder submitted those bids. Figure 10 compares the actual bids of B23 with the bids that the estimated model of rationally-inattentive bidding predicts are optimal in the two auctions we used previously to illustrate the evolution of B23's beliefs about the high bid; see Figure 7. In each panel, the blue stars depict the actual bids submitted by B23 at auctions 9810 and 10007, respectively. The red line is the high bid track and the dotted lines are counterfactual optimal bids predicted by the estimated model under three scenarios:

1. The blue dotted line is the optimal bid for a bidder who has not submitted a bid up to this instant at the auction; that is, it graphs $\{\beta_t(0,0)|t = 0,\ldots,120\}$.

2. The black dotted line is the optimal bid for a bidder conditioning on the times and bids the bidder actually submitted at the auction. This line is the path of bids $\{\beta_t(b_t, h_t^c)|t = 0,\ldots,120\}$ where $\{(b_t, h_t^c)\}$ is the actual sequence of realized bids and h_t^c is a counterfactual high bid indicator that may differ from the actual high bid indicator h_t to the extent that the predicted optimal bid from the model may be above or below the high bid track (which is treated as fixed except in cases where the counterfactual optimal bid becomes the high bid in the auction).

3. The green dotted line is the optimal bid for a bidder that condition on the times at the auction that the bidder submitted their bids but not the values of their actual bids. This line is the recursively calculated path of bids $\{\beta_t(b_t^*, h_t^*)|t = 0,\ldots,120\}$ where (b_t^*, h_t^*) is the model's counterfactual prediction of the bidder's highest optimal bid up to second t and high bid indicator treating the bids of all other bidders as fixed.

To understand the difference in these three counterfactual optimal bid paths, consider the left-hand panel of figure 10. B23 did not submit a first bid at auction 9810 until $t = 68$. The model's prediction of the first bid is about \$3,256, but the actual first bid submitted by B23 was \$5,500, an overbid of nearly \$2300. If we constrain the model to bid only at the times B23 actually bid at this auction, the predicted counterfactual bid path for B23 is the green dotted line. The black dotted line can be viewed as an off-the-equilibrium bid path. That is, the model takes the initial overbid as a given, and then calculates what bids would be subsequently optimal given that initial deviation. The model's prediction of the optimal second bid taking the initial overbid as given is actually slightly higher than the actual second bid by B23. B23's second bid is another deviation off the equilibrium path, and the optimal third bid is also higher than B23's actual third bid.

Thus, except for the initial overbid at $t = 68$, the model predicts the subsequent path of actual bids well. Note, too, how the unrestricted optimal bid path (green dotted line) rapidly increases and catches up to nearly equal the actual bids after the first bid $t = 68$. Consequently, the model almost perfectly predicts the actual final bid of \$6,510 by B23 at $t = 116$. The blue dotted line can be used to see what bid B23 should have optimally submitted had B23 followed a bid sniping strategy and waited until $T = 120$ to submit a single bid. The model predicts that the optimal snipe bid is the last point on the blue dotted line at $T = 120$, a bid of \$5,596. The bidder would have lost if they had followed the model's advice, but B23's actual bids lost this auction, so we might wonder if the initial "overbid" has any real consequence. Notice that B23's final bid of \$6510 is just \$1 short of B23's estimated valuation of the car of \$6,511. Thus, it appears as if B23 is using a straightforward bidding strategy — bidding up to \$1 below their valuation before dropping out.

Consider now the right-hand panel of Figure 10, which compares actual and counterfactual bids for auction 10007. Once again we observe a substantial overbid in the first bid B23 submits $t = 76$, but the optimal bid path (green dotted line) rapidly catches up to the actual bid path, so the model perfectly predicts B23's final bid of \$5,550 at $t = 108$. In this case, B23 won the auction, but B23 would have lost this auction had they followed the predicted optimal bidding strategy because B23's final predicted optimal final bid of \$5,475 is less than a bid of \$5,500 by B16 at $t = 81$. One other feature of Figure 10 is worth noting: B23's final bid in this auction was again just \$1 less than their estiated valuation for this car, consistent with straightforward bidding.

B23's actual bidding behavior (involving both frequent early bidding and initial overbidding) led to B23's leaving these auctions with very little surplus. B23 lost auction 9810, so they received zero profit. B23 won auction 10007, but thier final bid of \$5,550 was only \$1 less than B23's estimated valuation of $v = 5551$. This raises the question: can it be optimal for the winner of an auction to bid so aggressively that they end up earning zero profits from winning?

Figure 11: Actual versus predicted optimal bids by B10 and B14 at auction 9248

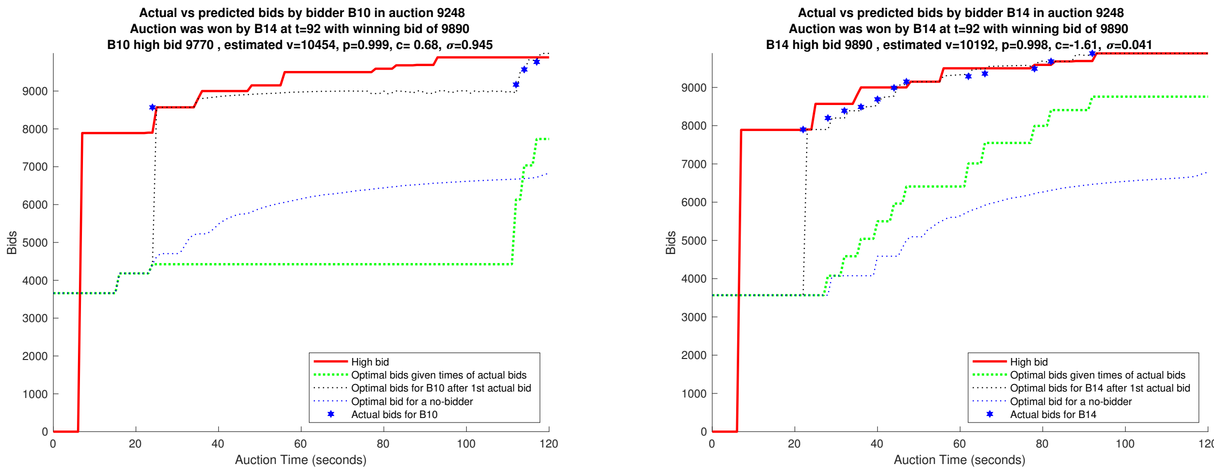


Figure 11 shows that not all bidders appear to be using straightforward bidding strategies and continuing to raise bids up to $1 below their valuation. The figure compares the actual and predicted optimal bids for B10 (a losing bidder) and B14 (the winner) of auction 9248. In both cases, the final bids of both bidders are well below their estimated valuations. For this auction, we estimated the valuation of B10 to be $v = 10454$, while the valuation of the winner B14 is estimated to be $v = 10192$. Another bidder (not shown) is B1, whose estimated valuation is $v = 10624$. If the Korean auction were strategically equivalent to a Japanese auction, then B1 should have won the auction and paid a price of $1 more than B10's valuation, namely, $10,455. However B10's and B14's actual final bids were $9,770 and $9,890, respectively, well below their estimated valuations.

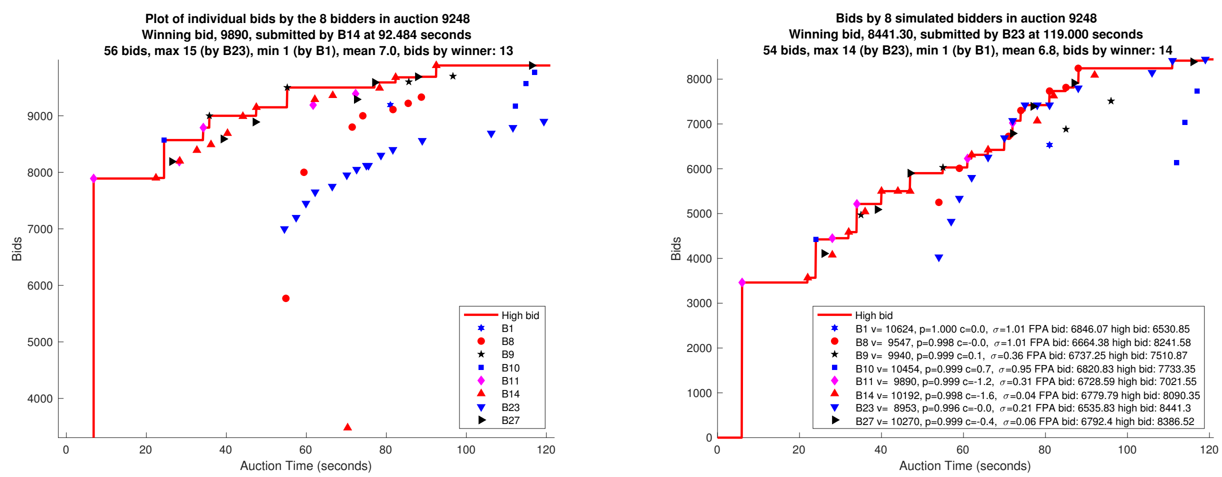
Comparing Figures 10 and 11, we see that the final optimal bids predicted by the DP model conditioning on the times the bidders actual bid in these auctions (green dotted line) are closer to the actual high bids of the lower valuation bidders in figure 10 than they are for the high valuation bidders in figure 11. The DP model predicts that bidders with low valuations are weak bidders, who tend to bid more aggressively and in some cases nearly up to their valuations for the car before stopping (as we illustrated for B23 above), but bidders with relatively high valuations take advantage of being strong bidders and bid less aggressively.

Another key point to note from Figure 11 is that we continue to find a pattern of early overbidding on the part of both B10 and B14: in both cases, their first bids are significantly higher than what the DP model predicts is optimal. But in these cases, the early overbidding is not innocuous: it pushes up the path of subsequent bids so that the final high bids are significantly higher than the optimal bids predicted by the DP model. We will show below that this is a general pattern for most bidders, and results in collective overbidding in these auctions to their detriment, but of course to the benefit of the rental company.

5.4 Evaluation of model fit at the level of individual auctions

Now we compare actual bidding behavior to simulated bidding in individual auctions, where we allow all bids to be the optimal ones computed by the DP model to demonstrate that the early overbidding at the individual bidder level does indeed push up the final winning bid to a higher value than would occur in our model of rationally inattentive bidders. To see this, consider Figure 12 that compares the actual outcome of auction 9248 (left panel) with the predicted outcome if *all bidders* would have bid at exactly the same times that they actually did, but at the values that are optimal according to the estimated models for each of the eight participating bidders.[13]

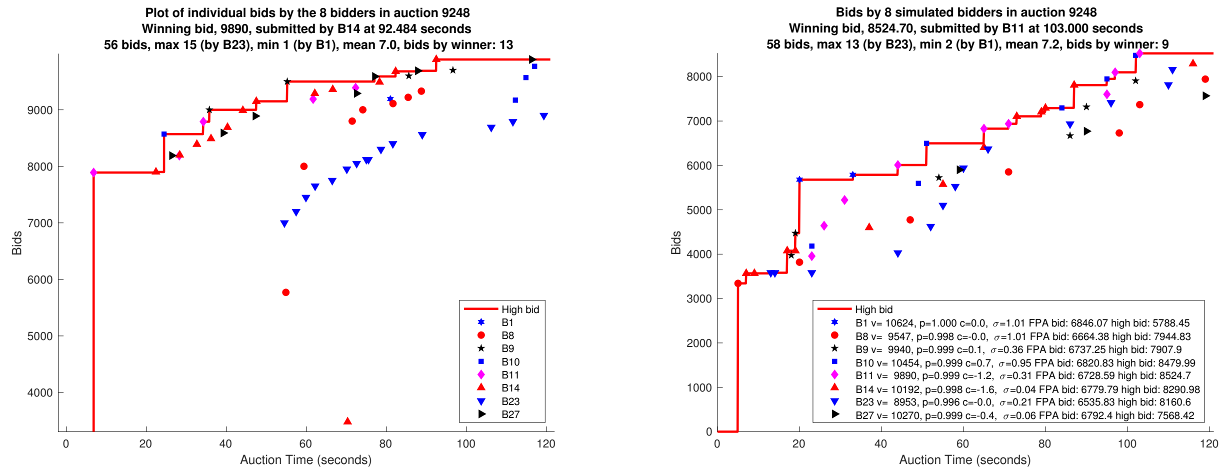
Figure 12: Actual versus simulated bids at auction 9248 (conditioning on actual bid times)



In particular, even though the actual winning bid at auction 9248 was \$9,890 by B14 at $t = 92$, in the counterfactual simulation of auction 9248 the winning bid is \$800 lower: \$8,441 by B23 at $t = 119$. From the left panel of figure 12 we see the high actual initial bid of B11 did push up the high bid track and all 14 actual bids of B23 were below the high bid. However in the simulated auction, the first bid of B11 is much lower, \$3,500, and along with lower optimal bids of the other simulated bidders, the simulated high bid track (red line) follows a lower trajectory compared to the actual one. In particular, B23's lower optimal bids eventually become the high bid and B23 is

[13]The right-hand panel of Figure 12 illustrates the 54 bids placed in the conditional simulation of action 9248, whereas the actual number of bids submitted in this auction was 56. This is because in two instances the optimal bidding strategy chose not to improve the bid whereas the actual bidders did improve their bids at the two corresponding bid times.

Figure 13: Actual versus simulated bids at auction 9248 (simulated bid times)



able to win the simulated auction at a lower bid of $8,441 than B23's actual high but losing bid of $8,900 in the actual auction.

Our comparison of the high bid tracks in the two panels of Figure 12 suggests that the early overbidding we observe in the actual auctions pushes up bids over the entire course of the auction and causes the winning bidder to end up paying significantly more than the counterfactual winning bid predicted by the model. Note also that in this case the simulated winner, B23, is the lowest valuation bidder, so the simulated outcome is highly inefficient. This unfortunate outcome may be an artifact of conditioning on the actual bid times of the human bidders, which can artificially reward more aggressive and frequent human bidders. In the actual auction, B23's 14 bids were a result of unsuccessful attempt at bid creeping to learn the high bid in the auction.

Figure 13 shows a full stochastic simulation were we allow all simulated bidders to choose both the times and values of their bids according to the estimated DP bidding strategy. We find that when we use the estimated inattention probabilities to simulate bids for each bidder, there is too little bidding during the auction and simulated winning prices are significantly lower than actual prices. Therefore in the simulation shown in the right hand panel of figure 13 we used *empirical inattention probabilities* to simulate each bidder. These probabilities are defined as the fraction of discretized bidding seconds $t \in \{0, 1, \ldots, 120\}$ that each bidder does not bid during the auction. The empirical inattention probabilities are lower than the structural estimates of p. For example for B23 in auction 9248, the structural estimate of p is .996 whereas the empirical probability that B23 does not bid in any given second during the auction is .884. The reason why the structural model over-estimates p is that inattention is a powerful force motivating earlier higher bidding in the auction. To maximize the quasi-likelihood function, there is a tradeoff between choosing a lower value of p to match the empirical probabilities of bidding in any given second versus choosing a higher value of p so the model can better match the high early bids we observe.

We use the empirical inattention probabilities to simulate times that bidders are paying attention and submit updated bids, resulting in more participation and active bidding and higher simulated prices. However even when we do this, the model typically underpredicts the winning bid in the auction even though it is able to qualitatively match several features of the actual auctions, including the total number of bids and the early bidding we observe in the actual auction. We can see this in figure 13, where there is early bidding but at significantly lower values: the first bid in the simulation is \$3,342 at $t = 5$ versus nearly \$7,890 at $t = 6$ in the actual auction. Even though bids steadily rise through the remainder of the simulation, the winning bid in the simulation is \$8,525, which is \$1,365 lower than the winning bid of \$9,890 in the actual auction.[14]

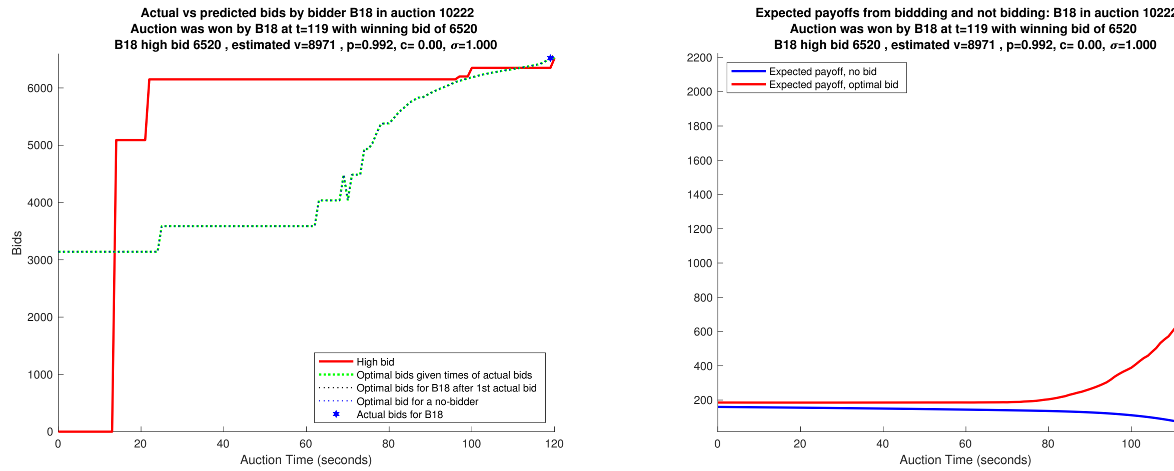
We conclude that our model of rationally inattentive bidding underpredicts bidding actvity in simulated auctions due to the high estimated probabilities of being inattentive and not bidding. Our solution is to use the empirical bid probabilities to simulate the times at which bids are made during the auctions. Our interpretation of this seemingly inconsistent approach to simulation is that it reflects bidders' over-estimation of the probability of being distracted at any given second during the auction when determining their bidding strategies. This causes them to bid higher and earlier out of an excessive concern that they will be distracted and unable to bid earlier in the auction. However their actual probability of inattention in the auction is lower than what they believe, so our model might be described as *irrational inattention.* In the next section we evaluate whether this model of irrationally inattentive bidding is capable of predicting the observed distribution of outcomes in all 533 Avante car auctions.

5.5 Comparing actual and simulated auctions of all Avante cars

Our model of (ir)rationally-inattentive bidding can qualitatively explain the early bidding behavior observed by individual bidders, but quantitatively it underpredicts bothh the number of bids in an auction and the magnitude of the first bids at these auctions — the phenomenon we have referred to as early overbidding. These high early bids affect subsequent bidding in the auction, so

[14] Note that the simulated final high bids of all 8 bidders are significantly lower than their estimated valuations. This is further evidence that the bidding behavior predicted by our anonymous equilibrium model of the Korean auction is not "straightforward bidding" so the auction is not strategically equivalent to bidding at a Japanese auction, where losing bidders exit the auction when the current high bid exceeds their valuations.

Figure 14: Actual versus simulated bids for B18 at auction 10222



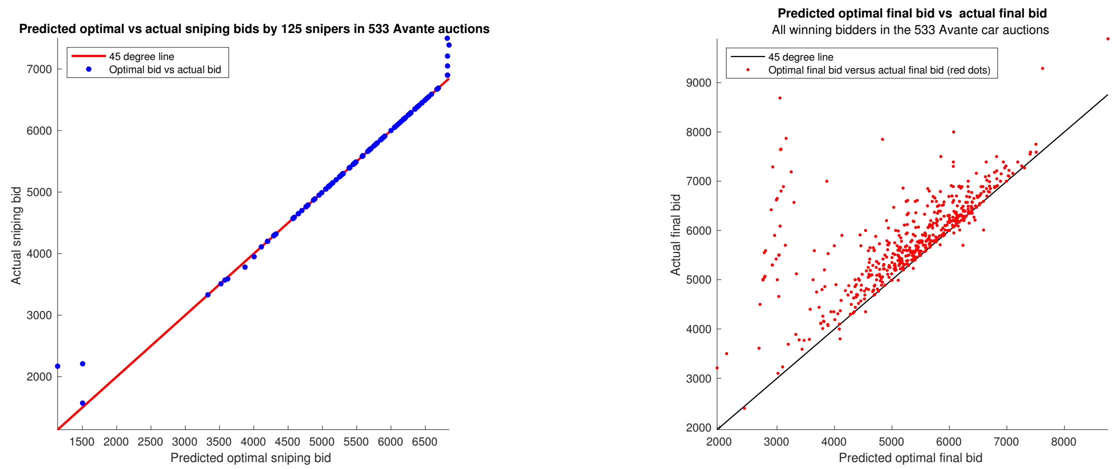
the predicted auction prices from the model tend to be significantly lower than what we observe in the actual auctions.

Of course, we can show many examples where the model does provide accurate predictions of all bids submitted during the auction. This is particularly true for bid snipers, since the model only needs to adjust its estimate of the valuation to match the single bid submitted by the bidder in the final few seconds of the auction. We show this in Figure 14: The left-hand panel plots the predicted optimal bid trajectory and illustrates that it correctly predicts the actual bid submitted by B18 at $t = 119$. The right-hand panel plots the expected value of submitting an optimal bid (red line) versus the value of not bidding (blue line). The two lines are close to each other for the first 80 seconds of the auction, indicating that no huge gain exist to B18's submitting a bid early in the auction. The value of submitting a bid starts to rise rapidly in the remaining forty seconds of the auction, whereas the value of not bidding converges to zero at t approaches $T = 120$. Note that, once B18 submits their optimal bid of 6520 at $t = 119$, the gain from further bidding falls to zero. In this case, B18 won the auction, and it is another example where the optimal bidding strategy implies a final high bid that is significantly below B18's estimated valuation of $v = 8971$. This case is not atypical: the model accurately predicts the bids by bid snipers, whereas it tends to underpredict the final high bids of non-snipers.

5.6 Evaluation of model over all **Avante** auctions

Thus far we have evaluated model fit at the level of individual bidders and auctions and showed that our estimated model of rationally-inattentive bidding captures the early bidding behavior

Figure 15: Actual versus predicted final bids 125 bid snipers and all 533 winning bidders



we observed in the data, as well as the heterogeneity in bidding strategies that we identified in section 2 including bid creeping and bid sniping. We have also shown examples of bids that the model cannot predict well, particularly the phenomenon of early overbidding—the tendency for the model to underpredict the magnitude of the first bid submitted during the auction.

However the model is able to predict the bids submitted by snipers very well. We illustrated that using a single example in Figure 14, but the left-hand panel of Figure 15 shows this is true in general by plotting predicted vs actual bids for 125 bid-snipers that we identified among the $4,029$ bidder-auction pairs in our dataset. We classified a bid to be a sniped bid if the bidder submitted only a single bid at the auction and this bid was submitted after second $t = 118$, that is, within the last two seconds of the auction. Except for a few cases, the model perfectly predicts their bids.

The right-hand panel of Figure 15 plots predicted versus actual final bids for all winning bidders in the 533 Avante auctions. These dots are also highly concentrated along the 45°-line, indicating that the model also does a very good job of predicting the final bids of non-bid-snipers. Notice that the red dots lie mostly above the 45°-line, indicating that the model generally underpredicts the final high bids submitted by winning bidders. The pattern of underprediction emerges even more clearly in Figure 16: The left-hand panel plots the predicted optimal final bid against the actual final bid for all $4,029$ bidder-auction pairs in our dataset, while the right-hand panel plots predicted versus optimal first bids for these same pairs. There is substantially greater unprediction of first bids than final bids, reflects the pattern we have already seen in our analysis of individual bidder data, thus documenting the phenomenon of early overbidding. We now consider the model's ability to fit other features of the data by comparing actual versus simulated distributions

Figure 16: Actual versus predicted final and first bids for all 4029 bidder-auction pairs

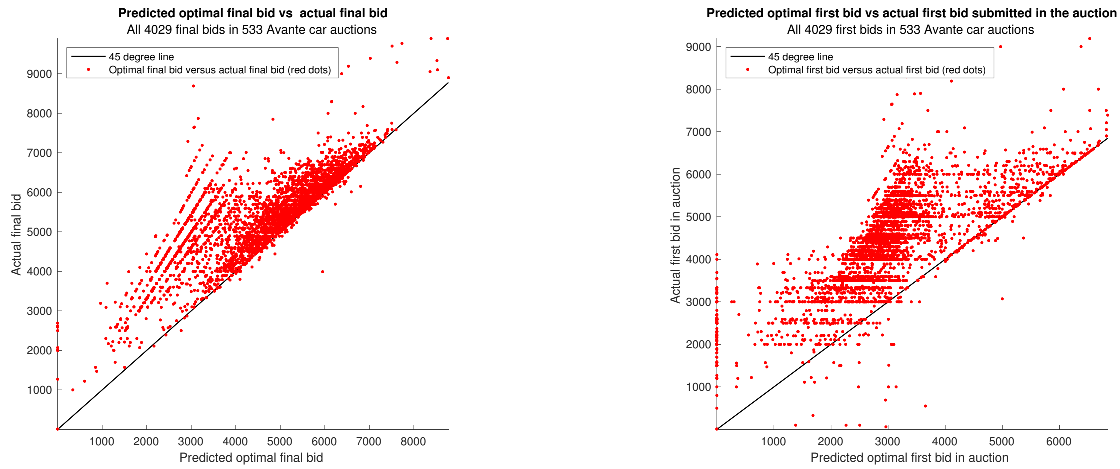
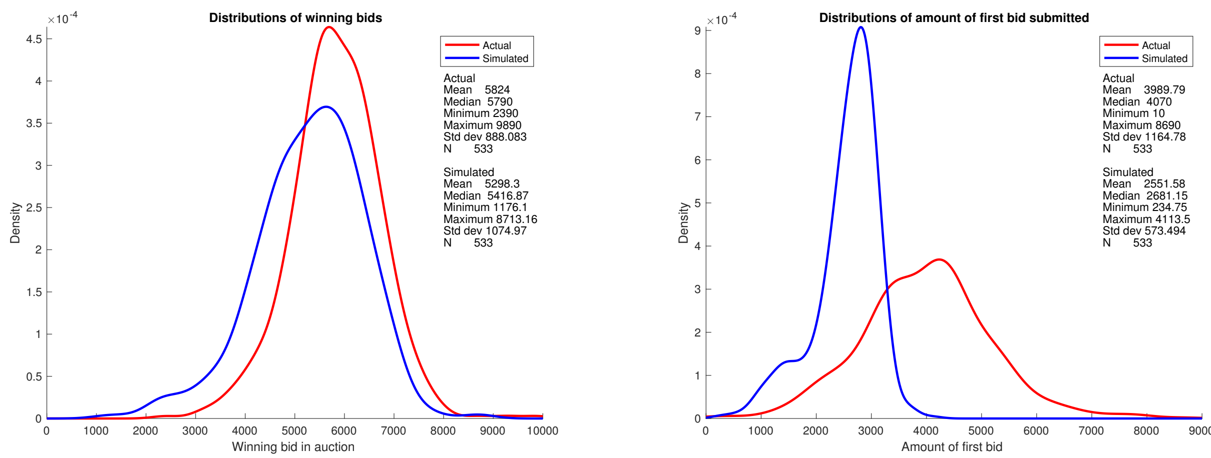


Figure 17: Actual versus simulated winning bids and first bids

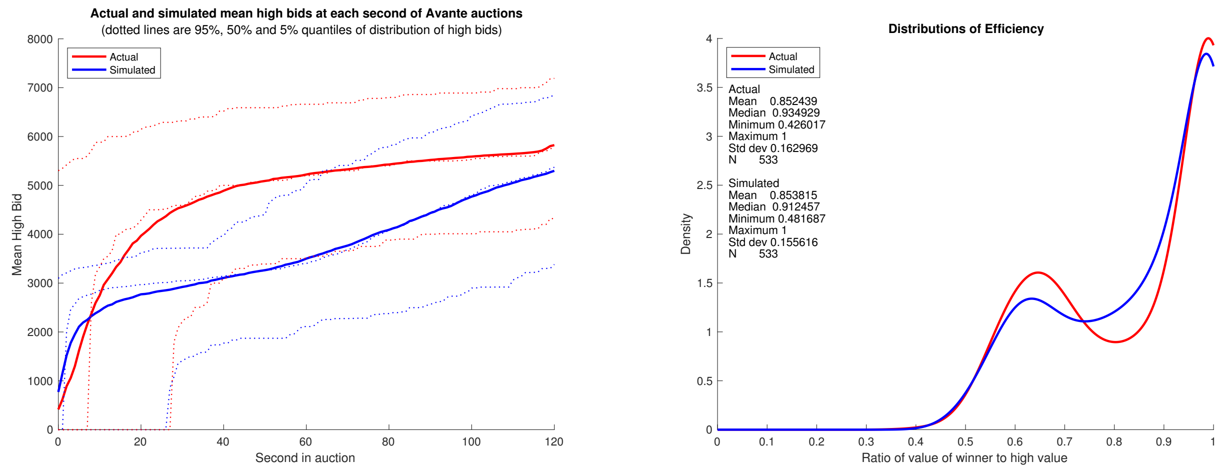


of other outcomes in the 533 Avante auctions.

By construction, our simulation strategy guarantees that the distribution of the number of bidders participating in individual auctions in our simulations is identical to the actual distribution. The model captures the general pattern that bidders wait until the last seconds of the auction to submit their final high bids, but the simulated bidders wait longer on average before submitting the winning bid, at $t = 111$ seconds into the auction compared to $t = 103$ in the data. But the right-hand panel of figure 16 shows that our model is generally unable to predict the high first bids in these aucctions.

We see this also in figure 17 which compares simulated versus actual distributions of first bids and final bids at the 533 Avante auctions. We see that our model underpredicts both quantities,

Figure 18: Actual versus simulated bid trajectories and distributions of efficiency losses



but the problem is much more severely for first bids than for final bids. The average underprediction in final bids is $\$5,550 - \$5,091 = \$459$, whereas the average underprediction in first bids is $\$3,990 - \$2,708 = \$1,282$, almost three times larger. This confirms the early overbidding phenomenon that we have already illustrated above is one of the problematic aspects of our model. The term overbidding suggests that the problem is with the bidders: but is this evidence of bidder irrationality or a symptom of a specification error in the model? We return to this question below.

We conclude our analysis of model fit with Figure 18 which compares the simulated versus the actual distributions of high bid trajectories and *ex post* efficiency losses at the 533 Avante auctions. The right-hand panel illustrates that the model captures the distribution of *ex post* efficiency at these auctions, where we define the *ex post* efficiency as the ratio of the valuation of the winning bidder to the highest valuation for all bidders in the auction. Auctions (such as the Japanese auction or the second-price, sealed-bid) are predicted to be fully efficient, so the distribution of efficiency for those auctions would be a unit mass at 100% efficiency. By contrast, at the Korean auctions, the average efficiency is 85%, which implies large *ex post* efficiency losses averaging over \$1,000 per auction. Considering that the average actual winning bid at these auctions is \$5,824, the predicted efficiency losses are indeed substantial.

The left-hand panel of figure 18 compares the mean value of the high bid trajectories as well as the median trajectory and the five percent and ninety-five percent quantiles of the coordinate distributions of high bid paths $\{\overline{b}_t\}$, that is, the marginal distributions of $\overline{b}_t$ for $t = 0, 1, \ldots, 120$). Recall our definition of ϵ-anonymous equilibrium, definition 4.1 in section 4. For the model to be in an ϵ-equilibrium, bidders' beliefs about the distribution of the high bid must be sufficiently close to the actual distribution of high bids resulting from those beliefs and the implied optimal bidding behavior. That is, beliefs should be approximately self-confirming in the sense that the difference $\|\mathcal{B} - \Lambda(\mathcal{B})\|$ should be small, where Λ is the operator that maps bidders' beliefs into actual auction outcomes, namely, the composition of the DP solution operator to compute optimal bidding strategies given beliefs, and a simulation operator that generates the implied distribution of high bid paths implied by bidders' optimal bidding strategies.

Looking at the left-hand panel of Figure 18, we conclude that regardless of how we might define the distance metric $\|\cdot\|$ to measure the difference between rational beliefs—namely, the ones implied by our data $\hat{\mathcal{B}}$ and $\Lambda(\hat{\mathcal{B}})$, the difference is too large to argue that our two-step estimation approach has indeed been able to explain observed bidding behavior as an ϵ-anonymous equilibrium of our model of rationally-inattentive bidding. We treat the discrepancy depicted in the left-hand panel of Figure 18 as further evidence that our model is unable to accurately predict observed bidding behavior at the Korean auctions.

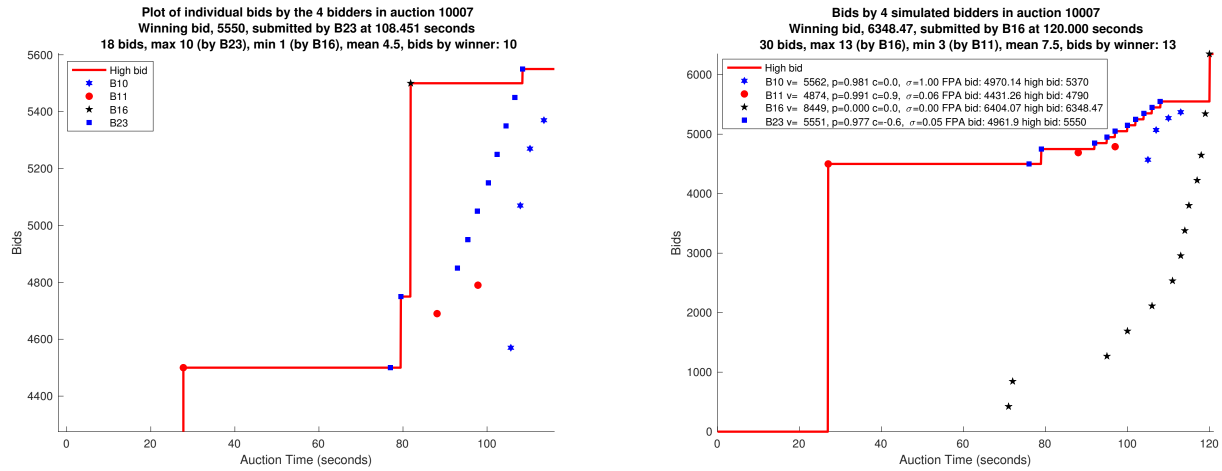
6 Counterfactual experiments

Thus far we have shown that our dynamic model of rationally-inattentive bidding is capable of providing a qualitative explanation of the early bidding behavior we observe in the Korean auctions, but it fails to provide a sufficiently accurate quantitative predictions of this behavior. In particular, the key feature of the data that our model fails to capture is early overbidding. In this section, we run several counterfactual simulations of our model to obtain insight into the nature of the nature of early overbidding and whether it is symptomatic of bidders' lack of rationality and inability to collude, causing them to bid more than necessary to win cars in the Korean auctions.

To convince the reader that early overbidding is problematic for bidders, we conducted a first counterfactual experiment where we calculate counterfactual profits for each of the bidders in the auction under the assumption that instead of using the bidding strategies they actually employed, their bids are those predicted by a frictionless robot "agent" using the estimated values for each car. In this counterfactual, we assume that all the bids of the other human bidders in the auction are unaffected by the bids of the counterfactual robot bidder, except to the extent that the counterfactual bids of the robot bidder changes the high bid track.

We illustrate this counterfactual in figure 19 where we replaced B16 in auction 10007 by a frictionless robot. The left panel of the figure shows the actual auction outcome, where B23 wins

Figure 19: Counterfactual effect of replacing B16 by a frictionless robot in auction 10007



the auction with a high bid of \$5,550 at second $t = 108$. Notice that the human bidder B16 placed a single bid of \$5,500 at $t = 81$. The right panel shows the counterfactual outcome when we replace B16 by a frictionless robot, which starts bidding earlier at $t = 71$ but with a much lower opening bid of \$423, then raising it in 12 subsequent bids toward the end of the auction and winning it with a final bid of \$6,348 in the final second, $t = 120$. Thus, B16's bidding robot earned a counterfactual profit of $v - b = 8449 - 6348 = 2101$, whereas the human bidder B16 earned \$0 in auction 10007, either due to inattention or complacency about remaining the high bidder from $t = 109$ to $t = 120$.

In the counterfactual simulation right panel, we kept all other human bidders' bids fixed at their actual values, but adjusted the high bid track (red line) to account for the counterfactual bids by the frictionless agent for B16. In particular, the agent did not place a high bid of \$5,500 at $t = 81$, so B23 becomes the new high bidder with their bid of \$4750 at $t = 79$. Note that B23 made a succession of "bid creeps" to learn the high bid of \$5500 in the actual auction, and we assume that B23 would have continued to do this in the counerfactual simulation even though B23 becomes the new high bidder at $t = 79$. One could argue that B23 might have stopped increasing their bid at $t = 79$ until $t = 97$ when B11 made a bid of \$4790. Note that B23 also submitted a bid of \$5,050 at $t = 97$ but B23 might have waited until $t = 98$ to learn that they had been outbid, and submitted the \$5,050 bid at $t = 98$ rather than $t = 97$. In the actual auction B23 submitted their winning bid of \$5,550 at $t = 108$, ending its series of bid creeps to become the new high bidder and B23 remained the high bidder for the remainder of the auction. In the counterfactual auction this is also the case, except that the frictionless agent for B16 sneaks in at the last second $t = 120$ and submits the winning bid of \$6,348. Thus, it seems plausible that the bids of the other human bidders would not have changed very much in this auction if B16's bids had been made by the frictionless robot bidder.

Tables 3 and 4 summarize the results of the first counterfactual, where we simulated all 4, 029 bidder-auction pairs and for each bidder calculated the profit that bidder would have earned had they used the frictionless bidding strategy as their agent, while all other bids in each auction are those given by the actual bids submitted by the human bidders, adjusting the high bid track appropriately as described above. We see that the adoption of bidding agents results in a win-win situation for both bidders and the car rental company: mean auction prices increase by $99 and mean profits earned by bidders more than doubles.

Table 3: Counterfactual simulation 1: mean values, profits, prices and efficiency

	Mean valuations	Mean profits	Mean winning bid	Efficiency
Actual	6843 (64)	659 (45)	5824 (38)	85.0%
Counterfactual	7423 (90)	1500 (67)	5923 (39)	85.8%

Table 4: Counterfactual simulation 1: outcomes

Counterfactual	Actual Bidder	
Bidder	Wins	Loses
Wins	128, 3.1%	265, 6.6%
Loses	405, 10.0%	3231, 80.2%

How is this possible? The first column of table 3 shows that the mean valuation of the winning bidder increases by $580. Notice also that the sum of winning price plus bidder profits equals the valuation of the winning bidder. So by bidding higher at the last second of the auction the frictionless bidding agents improved efficiency from 85% to 85.8% (see last column). The higher efficiency is reflected in the higher average valuation of the winning bidder in the auctions. The result is that the frictionless bidders improve both bidder profits and auction revenues by reducing the inefficiency of human bidders in the Korean auctions.

Table 4 provides additional evidence that the human bidders were bidding too high. Obviously, a human bidder was the winner in all actual Avante auctions. But the frictionless bidder won only 128+265=393 auctions. This is because the frictionless bidding agents were generally bidding less than their human counterparts, so this implies that the bidding agents won fewer auctions in total. This means that the bidding agents were earning sufficiently higher profits *con-*

ditional on winning to make up for the lower chance of winning any particular auction.

We illustrate this point in more detail in table 5 where we compare the actual and counterfactual outcomes for selected bidders. To aid comparison of statistically significant differences we report the standard errors of the mean value in each cell of the table. From the first column we see that not all human bidders participated in all 533 auctions, though some of the such as B8 and B9 bid in over 300 auctions.[15] There appears to be differential patterns of entry into bidding, since we can see from the third to last column, with the average valuations of all cars in the auctions each bidder participated in shows wide variation. For example the mean valuation of cars in the 315 auctions that B8 bid in was $5255 whereas for B9, the average valuation was $7043.[16]

The main take-aways from table 5 are listed below:

1. Bidding agents generally wins fewer auctions, as you can see by comparing actual and counterfactual win probabilities in column 2 of the table.
2. Bidding agents win less frequently due to bidding less on average in auctions compared to their human counterpart, as can be seen in the column High bids, all auctions.
3. However in the auctions the bidding agent wins, the mean winning bid is often as high or higher than their human counterparts.
4. Conditional on winning, the bidding agents earn far higher profit compared to their human counterparts as you can see from the column Mean profits, auctions won in the table.
5. Despite winning less frequently compared to their human counterparts, the bidding agents earn sufficiently more conditional on winning to raise their average (and thus total) profits over all auctions as shown in the columns average Profits, All auctions.
6. The bidding agents almost always submit their winning bid in the last second of the auction, whereas human bidders submit their final bid at approximately $t = 106$ well before $t = 120$.
7. Via the combination of bidding at the last second and bidding slightly higher at that time, the bidding agents are able to win without substantially overpaying in order to win.
8. The bidding agents do not engage in "straightforward bidding" by bidding up to their valuation for the car. Thus, bidding agents with lower valuations submit final bids at $t = 120$ well below their valuation v.
9. As a result of this, the bidding agent that does win an auction is more likely to have a higher valuation for the car, and this is reflected in the significantly higher mean valuation of cars conditional on winning compared to their human counterparts, as shown in the last two columns of the table.

[15] In a separate analysis not shown here in the interest of space, we do not find any relationship between bidder experience (as proxied by the number of auctions in which a bidder participated) and expected profits.

[16] Of course differences in valuation may also reflect differential opportunities to resell cars that vary over bidders and are reflected in their valuations in addition to differences in car values resulting from additional information bidders observe but we do not observe as the econometrician.

Table 5: Counterfactual 1: actual versus simulated outcomes for selected bidders

Bidder	**Win**	**Mean profits**				**High bids**				**Values**		
Auction	**Prob**	All		Won		All		Won		All	Won	
Count	Act/CF	Act	CF	Act	CF	Act	CF	Act	CF		Act	CF
1	20.2	237	470	1169	2128	5671	5440	5821	5841	6631	6990	7969
163	22.1	(58)	(87)	(223)	(237)	(77)	(66)	(164)	(146)	(132)	(306)	(341)
3	37.1	325	355	874	2483	6182	5705	6330	5988	7128	7204	8471
35	14.3	(118)	(167)	(257)	(583)	(125)	(125)	(181)	(500)	(291)	(393)	(1020)
5	5.5	16	144	298	2637	5146	4869	5955	6246	5571	6208	8883
146	5.5	(7)	(54)	(88)	(409)	(87)	(76)	(182)	(179)	(120)	(196)	(531)
6	13.5	33	22	244	3540	5708	5230	5925	6685	6049	6169	10225
163	0.6	(9)	(22)	(52)	(0)	(66)	(52)	(143)	(0)	(96)	(153)	(0)
8	10.8	35	138	324	1810	4935	4648	5168	5393	5255	5492	7204
315	7.6	(16)	(35)	(137)	(301)	(52)	(47)	(186)	(229)	(77)	(238)	(490)
9	9.9	159	1008	1609	2987	5448	5459	5527	6241	7043	7137	9228
323	33.7	(43)	(88)	(343)	(116)	(47)	(54)	(142)	(65)	(128)	(412)	(163)
10	12.3	84	259	682	2264	4879	4713	5566	5507	5459	6249	7771
227	11.5	(31)	(57)	(223)	(280)	(72)	(66)	(159)	(192)	(111)	(265)	(438)
11	9.7	29	125	303	1960	5344	5003	5707	5712	5778	6010	7672
361	6.3	(7)	(31)	(47)	(283)	(52)	(44)	(118)	(156)	(75)	(135)	(392)
14	15.6	54	107	349	1376	4690	4468	5270	5312	5004	5619	6687
167	7.8	(16)	(35)	(82)	(272)	(84)	(76)	(235)	(266)	(108)	(252)	(500)
15	6.8	5	37	79	1624	4664	4353	6370	4947	4784	6449	6561
44	2.3	(3)	(37)	(28)	(0)	(140)	(110)	(896)	(0)	(145)	(871)	(0)
16	15.2	95	175	629	1536	5428	5107	5368	5399	5910	5997	6935
158	11.4	(28)	(50)	(142)	(285)	(75)	(62)	(191)	(246)	(106)	(288)	(494)
17	14.9	61	174	406	2623	5528	5304	5781	5967	6316	6186	8591
181	6.6	(14)	(54)	(67)	(379)	(65)	(59)	(138)	(245)	(118)	(144)	(565)
23	21.6	109	86	505	1276	5383	5044	5504	5470	5711	6010	6745
148	6.8	(40)	(35)	(167)	(347)	(65)	(52)	(134)	(326)	(84)	(231)	(639)
28	15.6	129	257	830	2223	5641	5418	6158	6105	6478	6988	8328
225	11.6	(29)	(54)	(140)	(225)	(57)	(50)	(129)	(125)	(101)	(230)	(334)
32	33.7	61	23	181	1001	5828	5271	5982	5495	6023	6163	6496
86	2.3	(24)	(18)	(67)	(477)	(80)	(57)	(142)	(593)	(92)	(162)	(1070)
36	6.1	16	306	261	3112	5675	5368	6506	6477	6410	6767	9589
132	9.8	(7)	(86)	(85)	(86)	(73)	(63)	(393)	(97)	(138)	(350)	(354)
47	5.4	11	37	209	2512	5661	5175	5955	5754	5935	6164	8266
203	1.4	(5)	(24)	(79)	(931)	(54)	(43)	(315)	(932)	(81)	(265)	(1852)
49	2.9	5	130	190	3007	4916	4758	5352	6247	5495	5543	9254
139	4.3	(3)	(57)	(53)	(566)	(84)	(84)	(697)	(451)	(146)	(697)	(1012)

10. There are three human bidders, B6, B23 and B32, whose average profits exceed the average profits earned by their bidding agent. The bidding agents bid significantly less but as a result, they win far fewer auctions compared to these 3 bidders for the subset of auctions these bidders participated in.[17]

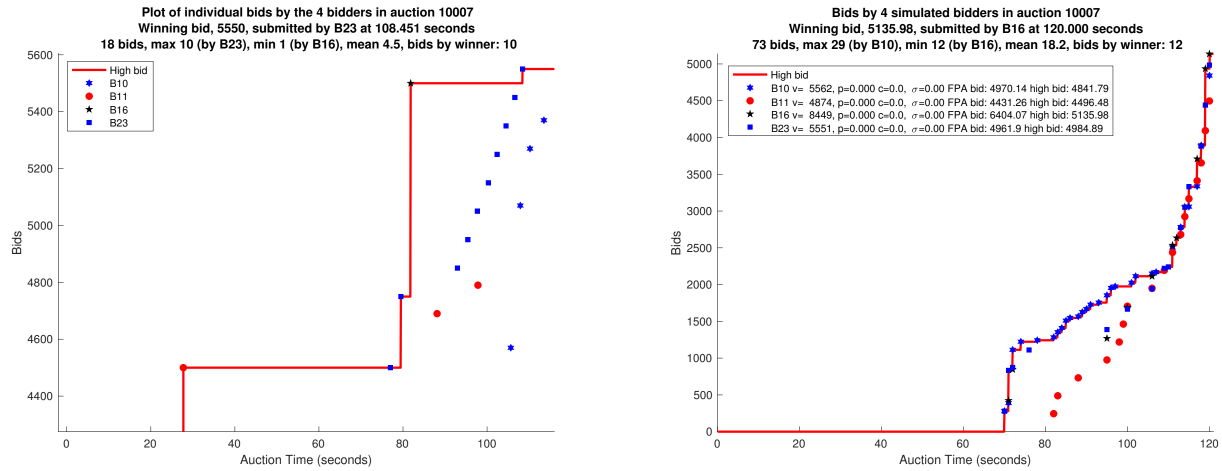
We take the results of Counterfactual 1 as strong evidence in favor of the hypothesis that the reason our model fails to explain the early overbidding by the bidders at these auctions is that most of the bidders are bidding suboptimally. It demonstrates that most bidders in our sample are not using best-response bidding strategies: we have constructed alternative bidding strategies that enables them to earn significantly higher expected profits. This should be impossible if the bidders were using equilibrium best-response bidding strategies, and seems inconsistent with the hypothesis of bidder collusion.

Of course, the reader may be uneasy about the first counterfactual simulation because we have treated the bids of all other human bidders other than the deviation bidder as fixed (subject only to the minor caveats discussed above). To allay these concerns, we conducted a second counterfactual experiment in which *all human bidders* at each of the 533 auctions were replaced by frictionless bidding agents. In this counterfactual both the amount and the timing of the bids of all bidders are those chosen by the frictionless bidding agents so outcomes are fully endogenous.

Figure 20 compares the actual outcome of auction 10007 with the counterfactual outcome if all the bidders were frictionless agents using the estimated valuations of the cars from our analysis of human bidders. The left-hand panel repeats the actual data on bids at auction 10007 whereas the right panel depicts the counterfactual simulation outcome where all of four human bidders are replaced by their frictionless bidding agents. Paradoxically, there is no early bidding in the frictionless bidder simulation. We can also see that the bidder with the highest valuation, B16 with $v = 8449$, wins the auction when all bidders are frictionless bid, but loses the actual auction in favor of B23 who has the second lowest valuation. Notice also that while in this case, the counterfactual auction outcome is *ex post* efficient, the winning bid of \$5136 is lower than the second-highest valuation, \$5562. This is another reminder that frictionless bidding is not straightforward bidding, and the outcome of the Korean auction with frictionless bidders is not strategically equivalent to

[17] Even though the counterfactual prediction is supposed to be optimal, the optimality is relative to our Assumption 1 that all bidders have common beliefs $\mathcal{B}$ and participate in generic auctions for Avante. If, however, certain bidders self-select into participation for certain types of Avante cars (for example, screening on the μ parameter in our conditional independent private-values Assumption 1 where bidder valuations are drawn according to the conditional density $f(v|\mu)$), it is possible that such bidders have superior beliefs for the subset of auctions in which they actually participate, relative to the average beliefs $\hat{\mathcal{B}}$ for all 533 Avante auctions we have estimated. If so, these bidders would have an informational advantage that enables them to earn higher profits than optimal bidding strategies solved with average beliefs $\hat{\mathcal{B}}$.

Figure 20: Actual versus counterfactual outcomes with all frictionless bidders in auction 10007



a static second price sealed bid auction as it is in a Japanese or "thermometer" auction.

The right panel of figure 20 shows the bids and high-bid trajectory that emerges under frictionless bidding in the Korean auction. Paradoxically, we see no early bidding in the frictionless bidding scenario: first bids only arrive after second $t = 70$ and they start out much lower compared to human bidders. Although the frictionless bidders submit a much larger number of bids their bids are concentrated in the last half of the auction and rise most rapidly in the final five seconds of the auction, producing the convex shape of the mean high bid trajectory in the frictionless case, in contrast to the concave-shaped mean high-bid trajectory in the actual auctions.

Table 6: Counterfactual simulation 2: mean values, profits, prices and efficiency

	Mean valuations	Mean profits	Mean winning bid	Efficiency
Actual	6843 (64)	659 (45)	5824 (38)	85.0%
Counterfactual	7321 (89)	1895 (62)	5426 (33)	93.4%

Table 6 summarizes the results of counterfactual simulation 2 (or CF2 for short). The results are similar to those from CF1 in table 3 except: 1) efficiency is significantly higher, 93%, when all bidders are frictionless, 2) winning prices are significantly lower (\$5426 vs \$5923), 3) bidder profits are significantly higher (\$1895 vs \$1500), and 4) mean valuations of winners is slightly lower (\$7321 vs \$7423). Table 7 provides detailed results for the same set of bidders that we showed for the first counterfactual in table 5 with the following differences

1. Since all bidders are frictionless, the counterfactual win rates are not significantly lower com-

pared to actual win rates in CF2.

2. Winning bids are significantly lower: collectively the frictionless bidders are able to significantly reduce winning prices compared to both CF1 and human bidding outcomes.

3. Though counterfactual profits are now always higher than actual profits for all bidders in CF2 as you can see in table **??**, they are not always higher than the profits under CF1. This is due to the fact that in CF1 the frictionless bidders were able to exploit the human bidders and take some of their surplus, whereas in the all frictionless bidder CF2, all of the bidders are equally "able" bidders: the more successful bidders have higher valuations.

4. However due to the lower winning prices in CF2, bidders are better off in aggregate.

The results for CF2 shown above suggest the potential for *algorithmic collusion* using the frictionless bidding strategies we have developed. If all bidders were to switch to the frictionless bidding strategies acting as their agents, average winning bids in the auctions would be nearly \$400 lower, and average bidder profits would be over \$1200 higher. This is why we conclude that it seems unlikely that bidders are colluding in the Korean auctions. Instead, we think the more likely explanation is that the human bidders are engaging in early overbidding that is resulting in their paying higher prices and earning lower prices than we would expect to observe if all bidders were frictionless and rational.

However we note a flaw in CF2: we computed the frictionless bidding strategies using *human beliefs* — i.e. we used the data on human bidders to construct the beliefs about the stochastic process governing the high bid in the auction. This might be appropriate for predicting the immediate aftermath if all bidders switched to frictionless bidders as their agents. However since frictionless bidders are bidding later in the auction and more aggressively this alters the stochastic process for the high bid trajectory in the auction, implying different beliefs. Recall section 4.1 where we described a successive approximations algorithm for updating beliefs over time based on experience in the auction: $\mathcal{B}_{t+1} = \Lambda(\mathcal{B}_t)$ from equation (5).

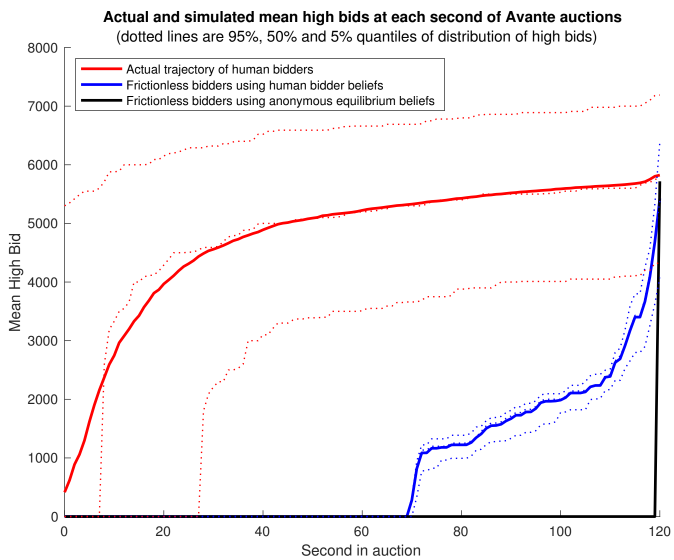
Figure 21 illustrates the result of this successive approximations algorithm for computing an anonymous equilibrium to the Korean auction with all frictionless bidders. The red line in the figure is the empirical mean high bid trajectory for *human bidders* from our data on the 533 Avante car auctions. Call this initial "guess" for the beliefs $\mathcal{B}_0$. We then solved the DP problems for 4029 bidder/auction pairs with these initial beliefs and then simulated 533 new auction outcomes under the initial beliefs $\mathcal{B}_0$ and the implied mean high bid trajectory is illustrated by the blue line in figure 21. We then re-estimated the beliefs using this simulated data resulting in updated beliefs, $\mathcal{B}_1 = \Lambda(\mathcal{B}_0)$ and resolved the 4029 DPs under these updated beliefs for all frictionless bidders.

Table 7: Counterfactual 2: actual versus simulated outcomes with all frictionless bidders

Bidder	**Win**	**Mean profits**				**High bids**				**Values**		
Auction	**Prob**	All		Won		All		Won		All	Won	
Count	Act/CF	Act	CF	Act	CF	Act	CF	Act	CF		Act	CF
1	20.2	237	393	1169	2066	5671	5196	5821	5418	6631	6990	7484
163	19.0	(58)	(83)	(223)	(285)	(77)	(60)	(164)	(145)	(132)	(306)	(375)
3	37.1	325	415	874	2076	6182	5405	6330	5692	7128	7204	7768
35	20.0	(118)	(176)	(257)	(555)	(125)	(112)	(181)	(293)	(291)	(393)	(808)
5	5.5	16	170	298	2702	5146	4783	5955	5682	5571	6208	7755
146	8.2	(7)	(57)	(88)	(396)	(87)	(72)	(182)	(161)	(120)	(196)	(549)
6	13.5	33	132	244	1651	5708	5113	5925	5543	6049	6169	7194
163	8.0	(9)	(45)	(52)	(378)	(66)	(48)	(143)	(222)	(96)	(153)	(568)
8	10.8	35	117	324	1186	4935	4546	5168	4774	5255	5492	5959
315	9.8	(16)	(28)	(137)	(210)	(52)	(42)	(186)	(147)	(77)	(238)	(335)
9	9.9	159	884	1609	2976	5448	5120	5527	5705	7043	7137	8679
323	29.7	(43)	(90)	(343)	(161)	(47)	(45)	(142)	(76)	(128)	(412)	(219)
10	12.3	84	159	682	1241	4879	4570	5566	5080	5459	6249	6322
227	12.7	(31)	(37)	(223)	(199)	(72)	(59)	(159)	(136)	(111)	(265)	(310)
11	9.7	29	196	303	1866	5344	4897	5707	5336	5778	6010	7202
361	10.5	(7)	(39)	(47)	(233)	(52)	(41)	(118)	(111)	(75)	(135)	(311)
14	15.6	54	187	349	1007	4690	4337	5270	4852	5004	5619	5859
167	18.6	(16)	(42)	(82)	(155)	(84)	(70)	(235)	(130)	(108)	(252)	(265)
15	6.8	5	57	79	1251	4664	4288	6370	5981	4784	6449	7232
44	4.5	(3)	(41)	(28)	(329)	(140)	(116)	(896)	(332)	(145)	(871)	(661)
16	15.2	95	154	629	1156	5428	4941	5368	4923	5910	5997	6079
158	13.3	(28)	(40)	(142)	(190)	(75)	(58)	(191)	(168)	(106)	(288)	(308)
17	14.9	61	244	406	1768	5528	5092	5781	5651	6316	6186	7418
181	13.8	(14)	(56)	(67)	(244)	(65)	(52)	(138)	(120)	(118)	(144)	(345)
23	21.6	109	129	505	1064	5383	4854	5504	5262	5711	6010	6326
148	12.2	(40)	(34)	(167)	(158)	(65)	(51)	(134)	(115)	(84)	(231)	(262)
28	15.6	129	305	830	1808	5641	5203	6158	5634	6478	6988	7442
225	16.9	(29)	(54)	(140)	(176)	(57)	(45)	(129)	(80)	(101)	(230)	(249)
32	33.7	61	103	181	986	5828	5182	5982	5316	6023	6163	6301
86	10.4	(24)	(37)	(67)	(180)	(80)	(59)	(142)	(155)	(92)	(162)	(326)
36	6.1	16	316	261	2610	5675	5226	6506	5842	6410	6767	8452
132	12.1	(7)	(87)	(85)	(386)	(73)	(53)	(393)	(143)	(138)	(350)	(490)
47	5.4	11	135	209	1707	5661	5085	5955	5490	5935	6164	7196
203	9.9	(5)	(43)	(79)	(375)	(54)	(41)	(315)	(179)	(81)	(265)	(541)
49	2.9	5	165	190	2292	4916	4679	5352	5827	5495	5543	8119
139	7.2	(3)	(60)	(53)	(473)	(84)	(76)	(697)	(251)	(146)	(697)	(685)

It turns out that after a single iteration of this successive approximations procedure, all frictionless bidders find it optimal to snipe, i.e. to wait until the final second of the auction $t = 120$ to place their bids. This implies a high bid trajectory given by the black link in figure 21. All subsequent iterations of the algorithm involve bidding behavior by the frictionless bidders that is strategically equivalent to bidding in a first price sealed-bid auction. The algorithm converged to an approximate fixed point that constitutes the anonymous equilibrium of a static first-price sealed-bid auction.

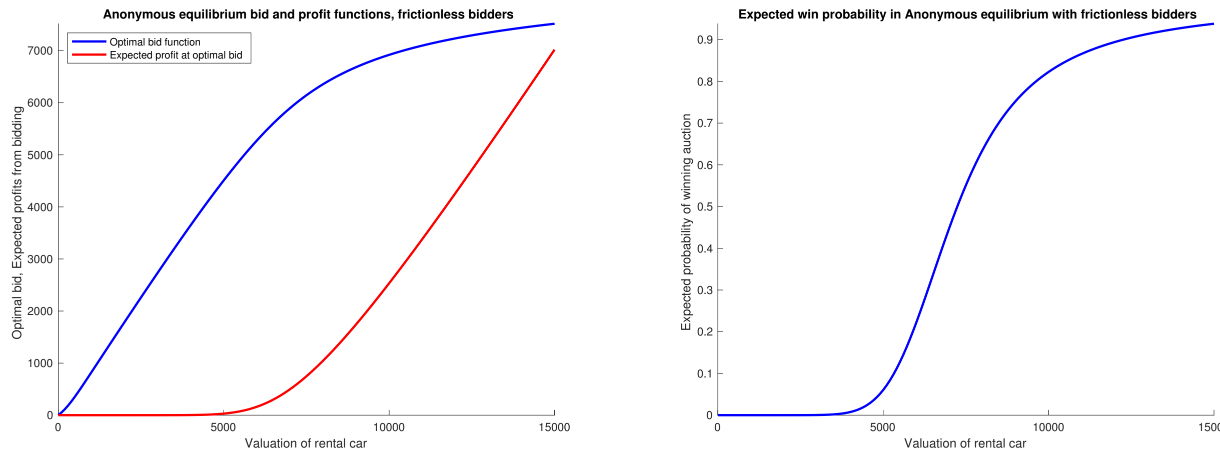
Figure 21: Convergence of beliefs to anonymous equilibrium with frictionless bidders



So our final counterfactual was computed using the optimal bid strategies for frictionless bidders from the anonymous equilibrium of the first-price sealed-bid auction, which, by Theorem 2 of section 4.4, is also an anonymous equilibrium of the Korean auction with frictionless bidders. We illustate the anonymous equilibrium bid function, expected profit function and win probability as a function of the bidder's valuation v in figure 22. Notice that win probabilities and expected profits are nearly 0 for valuations under \$5000. We also observe that since all frictionless bidders use the same monotonic bid function in an anonymous equilibrium, we conclude that this equilibium will be *ex post* efficient.

Summary results from CF3 are presented in table 8. We see that since auction outcomes are now 100% efficient (i.e. the highest valuation bidder always wins each auction), both the car rental company and the bidders can share in the "bounty" provided by this recaptured surplus. In particular, the mean winning bid in these auctions is \$6019, or approximately \$200 higher than the

Figure 22: Anonymous equilibrium bids, expected profits and win probabilities in Korean auction



winning bid in the Korean auctions with human bidders. Further, the mean profits per auction is nearly 3 times higher. So both the rental car company and the average bidder are substantially better off if bidding were conducted using a first-price sealed-bid auction format, which again is the anonynmous equilibrium to the dynamic Korean auction when all bidders are rational and frictionless.

More detailed bidder-specific results from CF3 are presented in Table 9 for the same selected set of bidders as we displayed in the corresponding tables for counterfactuals 1 and 2. In this case, expected profits (over all auctions each bidder participated in) are strictly higher for all bidders except B23 and B32 under a first-price sealed-bid auction format. The expected winning bids are uniformly higher too, as are expected profits conditional on winning the auction. Win probabilities are approximately the same as in the human Korean auctions, except for B23 and B32 where the CF3 win probabilities are less than half as large as the actual win probabilities, and this may explain why expected profits over all auctions are lower for these two bidders.

Table 8: Counterfactual simulation 3: mean values, profits, prices and efficiency

	Mean valuations	Mean profits	Mean winning bid	Efficiency
Actual	6843 (64)	659 (45)	5824 (38)	85.0%
Counterfactual	7888 (93)	1869 (53)	6019 (42)	100.0%

We conclude that the complex dynamics of bidding in the informationally restricted Korean auctions seems to create severe challenges for human bidders. The combination of inattention and

Table 9: Counterfactual 3: actual versus anonymous equilibrium of first-price auction

Bidder	**Win**	**Mean profits**				**High bids**				**Values**		
Auction	**Prob**	All		Won		All		Won		All	Won	
Count	Act/CF	Act	CF	Act	CF	Act	CF	Act	CF		Act	CF
1	20.2	237	373	1169	1962	5671	5489	5821	6017	6631	6990	7980
163	22.1	(58)	(75)	(223)	(236)	(77)	(74)	(164)	(194)	(132)	(306)	(420)
3	37.1	325	544	874	2115	6182	5769	6330	6189	7128	7204	8304
35	25.7	(118)	(197)	(257)	(473)	(125)	(136)	(181)	(304)	(291)	(393)	(769)
5	5.5	16	135	298	1790	5146	4836	5955	6106	5574	6253	7896
146	7.5	(7)	(46)	(88)	(336)	(87)	(82)	(182)	(246)	(120)	(185)	(570)
6	13.5	33	84	244	1522	5708	5206	5925	6037	6049	6169	7559
163	5.5	(9)	(31)	(52)	(300)	(66)	(57)	(143)	(205)	(96)	(153)	(499)
8	10.8	35	120	324	1220	4935	4603	5168	5190	5255	5492	6410
315	9.8	(16)	(29)	(137)	(205)	(52)	(51)	(186)	(222)	(77)	(238)	(417)
9	9.9	159	999	1609	2540	5448	5562	5527	6472	7043	7137	9013
323	39.3	(43)	(81)	(343)	(107)	(47)	(63)	(142)	(73)	(128)	(412)	(176)
10	12.3	84	210	682	1540	4879	4710	5566	5794	5459	6249	7334
227	13.7	(31)	(44)	(223)	(199)	(72)	(72)	(159)	(175)	(111)	(265)	(365)
11	9.7	29	125	303	1960	5344	5003	5707	5859	5778	6010	7479
361	9.1	(7)	(31)	(47)	(283)	(52)	(44)	(118)	(169)	(75)	(135)	(360)
14	15.6	54	136	349	1080	4690	4418	5270	5251	5004	5619	6859
167	12.6	(16)	(38)	(82)	(207)	(84)	(79)	(235)	(233)	(108)	(252)	(376)
15	6.8	5	57	79	1251	4664	4288	6370	5981	4784	6449	7232
44	4.5	(3)	(41)	(28)	(329)	(140)	(116)	(896)	(332)	(145)	(871)	(661)
16	15.2	95	193	629	1271	5428	5091	5368	5588	5910	5997	6859
158	15.2	(28)	(46)	(142)	(189)	(75)	(68)	(191)	(199)	(106)	(288)	(376)
17	14.9	61	277	406	1618	5528	5316	5781	5898	6316	6186	7517
181	17.1	(14)	(57)	(67)	(207)	(65)	(66)	(138)	(159)	(118)	(144)	(358)
23	21.6	109	107	505	992	5383	4998	5504	5342	5711	6010	6334
148	10.8	(40)	(33)	(167)	(200)	(65)	(56)	(134)	(195)	(84)	(231)	(384)
28	15.6	129	288	830	1659	5641	5432	6158	6030	6478	6988	7689
225	17.3	(29)	(50)	(140)	(162)	(57)	(56)	(129)	(126)	(101)	(230)	(281)
32	33.7	61	52	181	890	5828	5238	5982	5345	6023	6163	6235
86	5.8	(24)	(24)	(67)	(171)	(80)	(62)	(142)	(388)	(92)	(162)	(557)
36	6.1	16	284	261	2504	5675	5375	6506	6568	6410	6767	9072
132	11.4	(7)	(76)	(85)	(276)	(73)	(72)	(393)	(159)	(138)	(350)	(428)
47	5.4	11	86	209	1750	5661	5139	5955	5799	5935	6164	7550
203	4.9	(5)	(34)	(79)	(449)	(54)	(47)	(315)	(349)	(81)	(265)	(789)
49	2.9	5	94	190	2186	4916	4721	5352	6210	5495	5543	8396
139	4.3	(3)	(43)	(53)	(502)	(84)	(91)	(697)	(485)	(146)	(697)	(961)

other bidding frictions and perhaps additional elements of irrationality and "animal spirits" contribute to the early overbidding phenomenon that we have documented in the previous section. While early overbidding does contribute to higher prices for the car rental company, our counterfactuals suggest that the complexity of the Korean auction format and the presence of irrationality/bidding frictions contributes to the low efficiency of human bidders, 85%. By switching to a simpler static first-price sealed-bid auction format, which is arguably far less taxing and easier to bid in compared to the Korean auction, we have shown that a 100% efficient outcome can obtain and both the car rental company the the bidders in the auction are significantly better off than under the Korean auction. Thus it seems like a win-win situation for the car rental company to switch to an anonymized version of a static first-price sealed-bid auction for sales of its rental cars. We see no compelling *a priori* reason why the Korean auction format should be less susceptible to bidder collusion than a static first-price sealed-bid auction.

6.1 First-price vs Second-price auctions

But can the car rental company do even better? Our final counterfactual is to predict the effect of switching from a static first-price sealed-bid auction format to a second-price or Vickrey [1961] format. Recall that in a second-price auction the bidder who submits the highest bid is only required to pay the amount of the next highest bid. As is well known, it is a dominant strategy for rational bidders to bid truthfully, i.e. the optimal bid function is $\gamma(v) = v$. Further, under the assumption of independent private values, Myerson [1981] proved that the revenue-optimal auction design from the standpoint of the seller is second-price auction with an appropriately specified reservation price.

We assume that human bidders would bid truthfully under the second-price auction format, though we note that studies by Dyer et al. [1989] and Isaac et al. [2012] reported evidence from experimental studies of first-price and second-price static sealed-bid auctions where the number of bidders participating in the auctions were unknown to the bidders that are not always consistent with the predictions of these models. In particular, Kagel et al. [1987] found that subjects bid an average of 11% more than their dominant strategy bids in second-price auctions. Isaac et al. [2012] concluded: "We observe systematic deviations from risk neutral bidding in FP auctions and show theoretically that these deviations are consistent with risk averse preferences." (p. 516).[18]

[18]Isaac et al. [2012] argued that the predictions of the Nash equilibrium theory provide a better approximation to actual bidding behavior in their laboratory experiments if subjects are modeled as risk-averse, expected-utility maxi-

Our simulations reveal that while the first-price and second-price auctions are both 100% efficient, they are not revenue equivalent: the second-price auction results in average revenue of \$6532 per car auctioned compared to \$6019 for a first-price auction. Is this finding a violation of the Revenue Equivalence Principle of auction theory? We argue that Revenue Equivalence (i.e. that the expected revenue from first-price and second-price auction formats are the same) depends on a game-theoretic analysis that presumes common knowledge of the distribution of valuations and the distribution of the number of bidders participating as bidders in any particular auction. Our use of the anonymous equilibrium solution concept does not require common knowledge of those objects, and hence our finding that the revenue from a second-price auction significantly exceeds the revenue from a first-price auction does not contradict the Revenue Equivalence Principle.

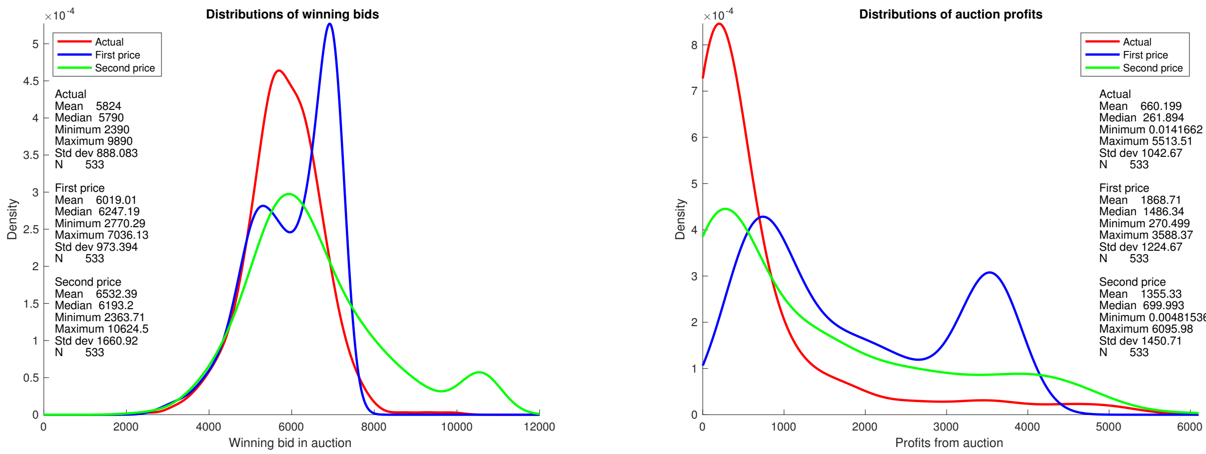
Figure 23 graphs the distribution of the winning bid and profits for the winning bidder in the 533 Avante auctions. Three curves are plotted: 1) the red line is the distribution of winning bids and profits under the actual Korean auction format, 2) the blue line plots the counterfactual predictions of these quantities under a static first-price auction format, and 3) the green line plots the distribution of these quantities under the second-price auction format. Due to the concavity of the optimal bid function in v (blue line in the left panel of figure 22), there are very few bids above \$7000 in the first-price auction, and this is reflected by a density of bids that is near zero in the blue line in the left panel of figure 23. However there is no "shading of bids" in a second-price auction, and we see this reflected in a significant density of bids in excess of \$7000 in the green line of the left panel of figure 23.

The fact that few bids are above \$7000 in the first-price auction means that the high valuation bidder in these auctions can gain a substantial profit when the bidder's valuation for the car is over \$10,000. We see this in the second mode in the distribution of profits between \$3000 and \$4000 (blue line in right panel of figure 23 which is not present in the distribution of profits under the second-price auction (green line). Thus, bidders are better off on average under the first-price auction than under the second-price auction, whereas the car rental company strictly prefers a second-price auction because it results in higher expected (and therefore total) revenues.

We conclude this section with Table 10 which summarizes revenues and bidder profits under the first-price and second-price auction formats and compares them to actual revenues under the

mizers instead of risk-neutral expected-payoff maximizers. Other theories including loss aversion theories of decision making under ambiguity such as theories involving probability weighting have also been invoked to try to explain the discrepancy between the predictions of Nash equilibrium models of bidding in auctions and laboratory evidence.

Figure 23: Distribution of winning bids and profits in first-price and second-price auctions



dynamic, informationally restricted Korean auction. Mean auction revenues would be 3.3% higher under a static first-price format, and 12.2% under the second-price format. Average bidder profits are 205% and 283% higher, respectively, under the first-price and second-price auction formats compared to those earned under the dynamic Korean auction. As we noted above, the high gains to bidders is "financed" by the significant improvement in auction efficiency, which increases from 84% under the Korean auctions to 100% under the first-price and second-price auction formats.

Thus, our empirical findings are in line with Myerson's theoretical predictions, even though we have relaxed the notion of "equilibrium" in these auctions so that the Revenue Equivalence Principle does not necessarily apply in our case. It is beyond the scope of this paper to analyze other auction formats to see if they result in even higher expected revenues to the seller, and we have not attempted to calculate an optimal reserve price or predict how that would affect the expected winning bid.[19]

Table 10: Expected winning bids under different scenario and auction mechanisms

	Mean valuations	Mean profits	Mean winning bid	Efficiency
Actual	6843 (64)	659 (45)	5824 (38)	85.0%
First-price	7888 (93)	1869 (53)	6019 (42)	100.0%
Second-price	7888 (93)	1355 (53)	6532 (72)	100.0%

[19]Note that the optimal reserve price depends on the distribution of valuations $f(v|\mu)$, which in turn depends on the individual characteristics of each car being sold μ, which in turn depends on variables we do not observe.

Of course, our conclusions in Table 10 depend on the assumption of no collusion occuring. Were collusion present, it is unclear whether the information restrictions at the Korean auction are more effective in curbing it than the stronger informational restrictions inherent in anonymized versions of a static first-price or second-price sealed-bid auctions—especially with an appropriately calculated reserve price. Sealed-bid auctions (either first- or second-price) are much simpler to implement and to bid at, and convey even less information to bidders than the Korean auction does. The main argument for adopting the more complex Korean auction mechanism seems to be to exploit the irrational exuberance of the bidders. Extensive laboratory evidence exists documenting irrational bidding at second-price auctions, which leads to overbidding in that format as well.[20] Thus, it is unclear whether the Korean auction is the best choice of auction, even considering the possibility of collusion and irrationality on the part of bidders.

7 Conclusion

We have analyzed detailed bid-level data from a new type of auction we call the *Korean auction* — an informationally-restricted, online ascending-bid auction designed by an executive of a rental-car company in Korea. The executive suspected collusion by professional bidders participating in the company's previous selling method — English "open outcry" auctions — conducted on-site at each rental location. In the new online auction mechanism, bidders observed neither the values of other bids nor the identities of other bidders: the only information available to a bidder is whether they hold the current highest bid at any point during the 2 minute auction.

As we discussed in the introduction, informational restrictions such as anonymization of bidders and "coarsening" of bids have been used to deal with potential collusion in other contexts. However it is less clear why a complex dynamic ascending bid auction mechanism instead of a simpler static sealed bid format provides additional protection against collusion. The standard reason dynamic ascending bid auctions are preferred is the *linkage principle* — more information is released in dynamic auctions and this promotes bidder learning that can result in higher auction prices compared to static sealed-bid auctions that reveal far less information.

Dynamic ascending bid auctions such as FCC Spectrum auctions or Ebay auctions also invoke *activity rules* or *soft closures* that discourage bidders from "hiding in the background" and observ-

[20] See, for example, Kagel et al. [1987], who found that laboratory subjects overbid by eleven percent relative to their valuations at second-price, sealed-bid auctions. They concluded that "Although we observe persistent overbidding in second-price auctions, clear economic forces are at work limiting the size of the overbid." (p. 1302).

ing but not bidding in auctions until very near the end of the auction, resulting in too little early bidding and information disclosure. That is, dynamic auctions often have rules to discourage *informational free-riding*. The Korean auction has a "hard close" at 2 minutes but solves the problem of informational free-riding in an elegant and novel fashion. Namely, by further restricting information so that no bidder can see the bids of other bidders during the auction. The only information bidders receive is whether their bid is the highest at any instant during the auction. Intuitively, this latter restriction should create significant incentives for early bidding in the auctions in order to gather information to win the auction without substantially overpaying. However it is less clear whether this informational restriction benefits the seller unless the informational gains from the linkage principle outweigh the effect of learning what the high bid will be, since the latter type of learning should operate to reduce prices bidders pay.

Given the structure of the Korean auto auctions, where all bidders were allowed inspect cars before bidding, we would not expect a high degree of affiliation or "common values" in bidders' valuations for cars beyond the essentially public information obtained from inspections. For this reason we believe that a conditional independent private values model is more appropriate, i.e. one where bidders' values are independently distributed conditional on the public information on each car (which we as the econometrician do not observe). When values are conditionally independent, the linkage principle does not hold since even if bids during the auction are public, this information does not affect the bidders' values for the cars. In any event, other bids are not public in the Korean auction, so its design does not allow it to benefit from the linkage principle even if it were present.

In a previous study, Cho et al. [2014] found that average prices of cars auctioned increased by 10% when the executive abandoned his Korean auction mechanism in favor of English open-outcry auctions held at a large auction house in Seoul. What explains this increase? Is it evidence that collusion was still happening under the Korean auction, or was the linkage principle is operative, or was it due to the larger number of bidders participating in auctions held at the auction house? The reduced-form analysis of Cho et al. [2014] was inconclusive on this question.

Our more detailed structural analysis of the Korean auction data suggests a new explanation: the complexity of the dynamic Korean auction format interacts in a negative way with the bounded rationality of the bidders in these auctions, producing relatively inefficient outcomes. We have suggested that by switching to simpler static sealed-bid auction formats, this inefficiency can be eliminated resulting in both higher aggregate profits for bidders and higher average prices for the rental car company. Our counterfactual calculations suggest that if the executive had adopted a static second-price sealed-bid auction format, average auction prices would have increased by 12%, which is more than the 10% increase that the company obtained by switching to open-outcry auctions at the auction house in Seoul (which net of the auction house's 10% commission, involved no actual net improvement relative to the Korean auction).

This paper has focused on solving a key puzzle we call the *bidding paradox* — namely, we observe early high bidding by the professional human bidders in the Korean auctions but it has proved quite challenging to provide rational explanations of this behavior that are consistent with optimality and equilibrium. The standard way of studying dynamic auctions, formulated as dynamic games of incomplete information, is to view bidding behavior as perfect Bayesian equilibrium (PBE) outcomes. However we have argued that it is unclear whether early bidding can occur in a PBE. On the other hand we proved that there is always a trivial *uninformative* PBE involving no bidding until the last possible second in the auction. This uninformative PBE is strategically equivalent to the PBE of a static first-price sealed-bid auction where the number of bidders participating in the auction is unknown (though there is common knowledge of the distribution of bidders and the distribution of their valuations).

Instead of abbandoning a structural analysis of our rich dataset on bidding at these auctions due to our inability to find or solve for an informative PBE, we have introduced a computationally-tractable model of rationally-inattentive bidding under a relaxed definition of equilibrium known as an ϵ-anonymous equilibrium. We have shown that this model can explain the early bidding behavior we observe, at least in a qualitative sense. We assume that in any given instant during the auction there is a probability p that the bidder is distracted and not paying attention to the auction. Inattention is rational in the sense that bidders are aware of their inattention and compensate by bidding earlier and higher in the instants where they are not distracted. Thus, rational inattention has effects akin to introducing a soft close at the Korean auction, which is known to create an incentive for earlier bidding compared to bid sniping as Roth and Ockenfels [2002] have shown.

We introduced a new, fixed-effects, QMLE estimation algorithm to estimate the four type parameters $\boldsymbol{\tau} = (v, p, c, \sigma)$ of our structural model of bidding behavior using a nested numerical solution approach where we explicitly calculate optimal bidding strategies for each bidder using numerical DP. In all, we estimated $4,029$ bidder/auction-specific types for the set of all bidders who participated in 533 auctions of a generic type of rental car, Avante XD 1.6L. The key component of the type for each bidder is v, his private valuation of the car being auctioned. The other three components (c, p, σ) are parameters governing bidding frictions that include the probability of being inattentive p, the psychological cost or benefit of submitting a bid c, and a scale parameter σ of an extreme-value distribution governing idiosyncratic variations in bidding costs.

Even in the presence of frictions, our model is fundamentally a rational model of bidding behavior: we use rational inattention and modest bidding frictions to explain the bidding behavior we observe. We have found that introducing these modest frictions takes us a great distance toward explaining observed bidding behavior, but we conclude that not even our model is capable of providing a sufficiently accurate prediction of the magnitude of early overbidding prevalent in the data. That is, even though our econometric model does a reasonable job of predicting the final bids tendered at these auctions, it systematically underpredicts the size of the initial bid submitted by most bidders at these auctions.

Another aspect of the bidding paradox is that we would expect that frictionless bidders (i.e. those whose types are of the form $\boldsymbol{\tau} = (v, 0, 0, 0)$) should be the most likely to engage in early bidding in the Korean auction. However we have shown that there is plenty of time in the last half of the auction for the frictionless bidders to do all the bidding they need to do, so there is no bidding in the first half of the auction. In fact, if it is common knowledge that all bidders in the Korean auction are frictionless bidders, we have shown that similar to the case of an uninformative PBE, *there is no bidding prior to the last second of the auction.* That is, the anonymous equilibrium of the Korean auction collapses and is strategically equivalent to the anonymous equilibrium of static first-price sealed bid auction.

Via a series of counterfactual simulations, we have demonstrated that frictionless bidding algorithms that are endowed with the same estimated beliefs and valuations of their human counterparts end up bidding later, lower, and earn higher profits. This finding suggests that bidders in the Korean auction were not colluding, because our frictionless bidding algorithms constitute a set of feasible (though competitive) bidding strategies that significantly raise bidders' collective profits from participating in these auctions. We conclude that the early overbidding we observe is symptomatic of boundedly-rational behavior on the part of the professional bidders at this auction, reflecting bidding mistakes that result in higher winning prices and lower efficiency compared to bidding by frictionless bidding robots.

In the anonymous equilibrium of the Korean auction with only frictionless, rational bidders, outcomes are 100% *ex post* efficient, since the auction is then strategically equivalent to a static first-price sealed-bid auction and all bidders use a common optimal bid function. Paradoxically auction prices are *higher* in the anonymous equilibrium with all frictionless bidders than they are with human bidders, despite the prevalent early overbidding by human bidders.. We showed that both the bidders and the car rental company benefit from the increase in efficiency under the static first-price, sealed-bid format, raising auction revenues by 3% and bidder profits by 205%.

We are not the first to use rational inattention to explain discrepancies between the actual behavior of professionals in high stakes settings and the predictions of game theory. We already noted the work of Bhattacharya and Howard [2022] in the introduction, who found that major league baseball pitchers "are rationally inattentive to the state in a high-stakes setting" and as a result "Nash equilibrium play state by state does not explain the data" (p. 388).

However the reader may wonder if the inability of our model of bidding with rational inattention to fully explain early overbidding is evidence of a deeper form of irrationality on the part of bidders at these auctions or a symptom of some type of specification error in our model. We believe it may be possible to "rationalize" observed bidding behavior by allowing for some degree of "irrationality." One way to do this is via a bidding model that allows for *irrational subjective beliefs.* This is one of the avenues we plan to explore in future work following the approach of Anderson et al. [2024] who analyzed serves of elite tennis professionals and showed that irrational subjective beliefs can rationalize disequilibrium play while these players are otherwise behaving optimally.

However our results suggest that early overbidding may not be optimal, even with irrational subjective beliefs. There may be a deeper form of irrationality in dynamic bidding due to "animal spirits" or "auction fever". For example Offerman et al. [2022] conducted a laboratory study of the effect of different auction mechanisms in a common-values framework, including the "open outcry" ascending bid auction. "During an Oral Outcry auction, a standing bidder is identified, who is the highest bidder at that moment. The previous literature has established that this can induce a so-called auction fever (Heyman, Orhun, and Ariely (2004), Ehrhart, Ott, and Abele (2015)). A standing bidder may get used to the feeling of winning the good and become prepared to bid higher than she originally intended. If that happens, auction fever triggers a quasi-endowment effect." (p. 814). In their study, they find that due to the endogenous jump bidding in the oral outcry auction "the available information is least well processed, and the price paid by the winner is the worst approximation of the common value among all three formats." We think that similar issues may be causing the early overbidding and low efficiency we find in the Korean auctions.[21]

[21] We do not take a stand as to whether early overbidding is a result of an inability to calculate optimal bids perfectly consistent with the bounded-rationality notion of Herbert A. Simon [1957] or due to the effect of emotions or animal

Another interesting agenda for future research would be to explore the use of machine and reinforcement learning algorithms to see if these algorithms could be trained to engage in spontaneous collusion without explicit central oversight or control. One of the questions that we did not solve in this paper is whether the stochastic process for high bids is "learnable" given the limited information on bids that each individual bidder receives in each auction. Similar to other Nash equilibrium concepts, our paper assumed that somehow bidders can learn and bid according to an anonymous equilibrium bidding strategy, but we have not offered a constructive algorithm and demonstrated that bidders will eventually learn the correct stochastic process for the high bid in auctions via repeated play, in a way similar to the study of learning in games, i.e. whether boundedly rational agents can eventually learn Nash equilibrium strategies in other types of games.

Appendix

A: Revenue Equivalence Theorem Fails in Anonymous Equilbrium

This appendix provides a simple counterexample to show that the Revenue Equivalence Theorem fails in an Anonymous Equilibrium of a first price sealed bid auction. That is, we show that the expected revenue from a sealed bid first price auction will generally differ from the expected revenue from a second price auction, despite the fact there is a common (i.e. symmetric) bid bid function in Anonymous Equilibrium and hence the outcomes are *ex post* efficient. The example also establishes that an Anonymous Equilibrium of a static first price auction is generally not the same as the Bayes-Nash Equilibrium of the corresponding auction where we assume independent private values and symmetry (i.e. all bidders have the same distribution of valuations and a common prior over the number of bidders entering the auction).

Consider an example where all bidders have valuations that are independently and uniformly distributed, $U(0,1)$ but no bidder knows N the number of bidders participating in any auction. Bidders' beliefs $\mathcal{B}$ are given by a common CDF $F(b)$ representing the distribution of the high bid in any given first price auction. Given this belief, the bidders' common bid function $\beta(v)$ is given by

$$\beta(v) = \underset{b}{\operatorname{argmax}}\, F(b)(v-b). \tag{26}$$

spirits as emphasized by George A. Akerlof and Robert J. Shiller [2009] or other reasons (such as risk aversion or desire to win an auction for the sake of winning), even though exploring these different possible explanations is an interesting direction for future research.

While the bidders are unaware of the number of bidders participating in any given auction and make their bidding decision based on the common belief $F(b)$, suppose that in fact there are *always* N bidders in each auction. Then we claim that the following distribution will be an Anonymous Equilibrium for this first price auction

$$F(b) = \left(\frac{N+1}{N}\right)^N b^N. \tag{27}$$

Given these beliefs, the optimal bidding strategy for each bidder is $\beta(v) = Nv/(N+1)$ which differs from a Bayesian-Nash Equilibrium bid strategy $\beta(v) = (N-1)v/N$ when N is known by the bidders, and also differs from the symmetric contingent bidding strategy given in equation (9) of Harstad et al. [1990], i.e.

$$\beta(v) = \sum_N \frac{(N-1)v}{N} \pi(N|v), \tag{28}$$

where $\pi(N|v)$ is the bidders' posterior belief that there are N bidders in the auction given their own valuation v, where

$$\pi(N|v) = \frac{v^{N-1}\pi_N}{\sum_n v^{n-1}\pi_n}, \tag{29}$$

where π_n is a bidder's prior probability that n bidders (including themself) are bidding in the auction.

This example also illustrates why the Revenue Equivalence Theorem does not hold in Anonymous Equilibrium. First note that in a second price auction, bidders do not need to know the number of bidders since their dominant strategy is to bid truthfully, $\beta(v) = v$. This implies that the expected selling price of the item in an Anonymous Equilibrium of a second price auction is the same as in the Bayes-Nash equilibrium, i.e. the expectation of the second order statistic (i.e. 2nd highest valuation), $(N-1)/(N+1)$. When the number of bidders is known, the expected high bid in the first price auction is $(N-1)/N$ times the expectation of the first order statistic, $N/(N+1)$ which also equals $(N-1)/(N+1)$, verifying that the Revenue Equivalence Theorem holds when N is known, or even in the Bayes-Nash equilibrium when N is unknown as can be easily verified via the Law of Iterated Expectations using the contingent bidding strategy (28).

However in the Anonymous Equilibrium, the equilibrium bid function in a first price auction is $\beta(v) = Nv/(N+1)$ and each bidder bids more aggressively since they fail to account for the number of competing bidders in the auction, which are unknown to them. This implies that the expected selling price in a first price auction equals $N^2/(N+1)^2$ which exceeds the expected revenue in the Anonymous Equilibrium of a second price auction. It follows that the Revenue Equivalence Theorem does not hold in general in an Anonymous Equilibrium.

B: Proof of Theorem 1

In this appendix, we provide a proof of the Theorem in section 4.4. Our proof is by induction: starting in the final period T, we show inductively that a) the value functions and optimal bid functions are independent of the high bid indicator, h_t, $t = 0, 1, \ldots, T$; and b) for all $t < T$, the optimal bid function $\beta_t(b_t) = 0$, whereas in the last period, the optimal bid function $\beta_T(b_T)$ is the anonymous equilibrium bid function for an anonymous static first-price, sealed-bid auction, where beliefs are given by $\lambda_{T+1}(b|0)$. To wit, bidders' *ex ante* anonymous equilibrium beliefs concerning the high bid in an anonymous version of a static sealed-bid auction.

Consider the optimal bid function $\beta_T(b_T)$ in the last period T, where we assume the bidder is paying attention and observes the high bid so far in the auction, b_T, and must decide what final bid $\beta_T(b_T)$ to submit. If $b_T > v$, the optimal bid is zero, otherwise, it is given by

$$\beta_T(b_T) = \underset{b \geq b_T}{\text{argmax}}(v - b)\lambda_{T+1}(b|b_T). \tag{30}$$

In general, the optimal bid function will involve a strict improvement over the current high bid, that is, $\beta_T(b_T) > b_T$ and involves shading, namely, bidding below the bidder's true valuation v

$$\beta_T(b_T) = v - \frac{\lambda_{T+1}(\beta_T(b_T)|b_T)}{\lambda'_{T+1}(\beta_T(b_T)|b_T)}, \tag{31}$$

where $\lambda'_{T+1}(b|b_T)$ is the derivative of the CDF $\lambda_{T+1}(b|b_T)$ with respect to its first argument, b. It is possible, however, that the constraint $b \geq b_T$ is binding, so the optimal bid is equal to b_T. Since time priority is binding at the auction, matching the current high bid will result in zero probability of winning the auction. A necessary condition for $\beta_T(b_T) = b_T$ is that $w_T(b, b_T)$ is non-increasing in its first argument b, where $w_T(b, b_T) = (v - b)\lambda_{T+1}(b|b_T)$ is the terminal period bid-specific value function. Therefore, $\beta_T(b_T)$ is given by

$$\beta_T(b_T) = \begin{cases} 0 & \text{if } b_T > v \\ b_T & \text{if } b_T \leq v \text{ and } w_T(b, b_T) \text{ is non-increasing in } b \\ \beta_T(b_T) & \text{otherwise, for } \beta_T(b_T) \text{ given in (31).} \end{cases} \tag{32}$$

Substituting the optimal decision rule $\beta_T(b_T)$ into the decision-specific value function $w_T(b, b_T)$ we obtain the *ex ante* value function $W_T(b_T)$ given by

$$W_T(b_T) = \begin{cases} 0 & \text{if } v < b_T \\ 0 & \text{if } b_T \leq v \text{ and } w_T(b, b_T) \text{ is non-increasing in } b \\ w_T(\beta_T(b_T), b_T) & \text{otherwise, for } \beta_T(b_T) \text{ given in (31).} \end{cases} \tag{33}$$

Note that $w_T(b_T, b_T) = 0$ due to time priority rules: just matching an existing high bid of b_T will not succeed in winning the auction.

The next step is to show that $W'_T(b_T) \leq 0$ for any b_T, where $W'_T(b_T)$ is the derivative of W_T at b_T. Using formula (31), it is easy to see that the result holds when $v < b_T$ since $W_T(b_T) = 0$ in this region. If $b_T \leq v$ and $\beta_T(b_T) = b_T$ (the binding constraint case), then $W'_T(b_T) = 0$ for the same reason. Consider now the final case in the last equation of (33), where $\beta_T(b_T) > b_T$ and, hence, we have an interior optimum. In this region, we have

$$\begin{aligned} W'_T(b_T) &= \frac{\partial}{\partial b} w_T(\beta_T(b_T), b_T) + \frac{\partial}{\partial b_T} w_T(\beta_T(b_T), b_T) \\ &= 0 + (v - b_T)\nabla_{b_T}\lambda_{T+1}(b_T|b_T). \end{aligned} \tag{34}$$

In the last equation of (34), we appealed to the envelope theorem, so the first term equals zero when $b = \beta_T(b_T) > b_T$. By Assumption 8, $\lambda_{T+1}(b|b_T)$ is stochastically increasing in b_T, so we have $\nabla_{b_T}\lambda_{T+1}(b_T|b_T) \leq 0$. It follows that $W'_T(b_T) \leq 0$ in this region as well.

Now we define the bid-specific value function at period $T-1$, $w_{T-1}(b, b_{T-1})$, which is the expected payoff at time $t = T-1$ from submitting a bid of b given that the current high bid at the start of $T-1$ is b_{T-1}. Let $\beta_{T-1}(b_{T-1})$ be the optimal bid function at period $T-1$. We want to show that either $\beta_{T-1}(b_{T-1}) = 0$ or $\beta_{T-1}(b_{T-1}) = b_{T-1}$, so in either case either the bidder does not bid at $T-1$ or does not submit an improved bid at $T-1$, and with time priority for the high bid, either case can be viewed as equivalent to not bidding. Clearly, if $v < b_{T-1}$ we have $\beta_{T-1}(b_{T-1}) = 0$. Suppose now that $v \geq b_{T-1}$, so there is a potential for profiting by submitting an improved bid, $b > b_{T-1}$. When $v \geq b_{T-1}$, the bid-specific value function $w_{T-1}(b, b_{T-1})$ is

$$w_{T-1}(b, b_{T-1}) = W_T(b)\lambda_T(b|b_{T-1}) + \int_b^{\infty} W_T(b')\lambda'_T(b'|b_{T-1})db' \tag{35}$$

The term in the first line of equation (35) is the expected utility in the case where the bid the bidder submitted, b, is the high bid submitted at $T-1$. Taking the *partial derivative* of $w_{T-1}(b, b_{T-1})$ with respect to b, we obtain

$$\begin{aligned} \frac{\partial}{\partial b} w_{T-1}(b, b_{T-1}) &= W'_T(b)\lambda_T(b|b_{T-1}) + W_T(b)\lambda'_T(b|b_{T-1}) - W_T(b)\lambda'_T(b|b_{T-1}) \\ &= W'_T(b)\lambda_T(b|b_{T-1}). \end{aligned}$$

It is, however, clear that the last line of the expression for $\frac{\partial}{\partial b} w_{T-1}(b, b_{T-1})$ is non-positive since we have shown above that $W'_T(b) \leq 0$. It follows that $w_{T-1}(b, b_{T-1})$ is maximized at $b = b_{T-1}$, so

$\beta_{T-1}(b_{T-1}) = b_{T-1}$. We summarize this as

$$\beta_{T-1}(b_{T-1}) = \begin{cases} 0 & \text{if } v < b_{T-1} \\ b_{T-1} & \text{if } v \geq b_{T-1}, \end{cases} \tag{36}$$

and the corresponding value function $W_{T-1}(b_{T-1})$ is given by

$$W_{T-1}(b_{T-1}) \equiv w_{T-1}(\beta_{T-1}(b_{T-1}), b_{T-1}) = \begin{cases} 0 & \text{if } v < b_{T-1} \\ w_{T-1}(b_{T-1}, b_{T-1}) & \text{if } v \geq b_{T-1}. \end{cases} \tag{37}$$

We now calculate $W'_{T-1}(b_{T-1})$ and show that $W'_{T-1}(b_{T-1}) \leq 0$. This obviously holds when $v < b_{T-1}$, where $W_{T-1}(b_{T-1}) = 0$. Consider now the case where $v \geq v_{T-1}$. In this case, $W'_{T-1}(b_{T-1}) = \nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1})$, where $\nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1})$ is the *total derivative* of $w_{T-1}(b_{T-1}, b_{T-1})$ defined in equation (35). Calculating this, we have

$$\begin{aligned}
& \nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1}) \\
= \; & \nabla_{b_{T-1}} \left(W_T(b_{T-1})\lambda_T(b_{T-1}|b_{T-1}) + \int_{b_{T-1}}^{\infty} W_T(b')\lambda'_T(b'|b_{T-1})db' \right) \\
= \; & W'_T(b_{T-1})\lambda_T(b_{T-1}|b_{T-1}) + W_T(b_{T-1})\lambda'_T(b_{T-1}|b_{T-1}) \\
& + W_T(b_{T-1})\nabla_{b_{T-1}}\lambda_T(b_{T-1}|b_{T-1}) \\
& - W_T(b_{T-1})\lambda'_T(b_{T-1}|b_{T-1}) \\
& + \nabla_{b_{T-1}} \left[\int_{b_{T-1}}^{\infty} W_T(b')\lambda'_T(b'|b_{T-1})db' \right]. \\
= \; & W'_T(b_{T-1})\lambda_T(b_{T-1}|b_{T-1}) \\
& + W_T(b_{T-1})\nabla_{b_{T-1}}\lambda_T(b_{T-1}|b_{T-1}) \\
& + \nabla_{b_{T-1}} \int_{b_{T-1}}^{\infty} W_T(b')\lambda'_T(b'|b_{T-1})db' \\
\leq \; & 0.
\end{aligned} \tag{38}$$

The final inequality in (38) follows from the third equation for $\nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1})$; noting the first term of that third equation is non-positive since we have already shown that $W'_T(b_{T-1}) \leq 0$. The second term of the last equation in (38) follows from Assumption 8 and the property that $\lambda_{T-1}(b_{T-1}|b_{T-1})$ is stochastically monotone in the conditioning argument of the CDF $\lambda_{T-1}(b|b_{T-1})$ for any $b \geq b_{T-1}$. The final term in the third equation for $\nabla_{b_{T-1}} w_{T-1}(b_{T-1}, b_{T-1})$ in (38) also follows from Assumption 8, since the stochastic monotonicity of $\lambda_{T-1}(b|b_{T-1})$ in b_{T-1} implies that the expectation of any non-increasing function of b will be non-increasing.

We have now completed a full induction step. We proved that if $w_T(b, b)$ and $W_T(b)$ are non-increasing functions of b, then $\beta_{T-1}(b_{T-1})$ equals 0 or b_{T-1} depending on whether $v < b_{T-1}$ or

$b \geq b_{T-1}$. We then showed that $w_{T-1}(b, b)$ and $W_{T-1}(b)$ are also non-increasing functions of b. A similar argument as used above to derive $\beta_{T-1}(b_{T-1})$ implies that $\beta_{T-2}(b_{T-2})$ either equals 0 or b_{T-2}, depending on whether $v < b_{T-2}$ or $v \geq b_{T-2}$. By induction it follows that these properties hold for all time periods $t = 0, 1, \ldots, T-1$. At the start of the auction, however, the high bid is by definition equal to zero, that is, $b_0 = 0$. This implies that $\beta_t(b_t) = 0$ for $t = 0, 1, \ldots, T-1$. The only positive bid that is made is at the last period T where bidders, using the same inductive logic, will realize no other bidder will place a positive bid before T, so the distribution of the high bid will be $\lambda_{T+1}(b|0)$. Thus, all bidders will bid-snipe and the anonymous equilibrium of the anonymous dynamic ascending bid Japanese auction will be strategically equivalent to the anonymous equilibrium of an anonymous static first-price, sealed-bid auction as claimed in Theorem 1.

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