IO Class Notes: A Framework for Dynamic Analysis.

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January 26, 2024

Review: Static analysis in applied I.O.

Recall that static analysis conditions on

- the goods marketed (or their characteristics) and their cost functions,
- preferences over the goods (or over characteristics tuples).
- “institutional” features, like the type of equilibrium, structure of ownership, e.g. regulatory rules

and then analyzes how prices, quantities, and the distribution of profits and consumer surplus, are determinned (or how they might change if we changed one or the other element of the conditioning set).

To do the static analysis we need the following primitives.

1. the demand system [can estimate it]

2. the cost system [can estimate it]
3. the equilibrium assumption [usually taken as given from knowledge of institutions, but could "test"]

Simplifying Assumptions for Static Analysis.

Assuming, then, that there is a unique equilibrium and that we limit strategies to depend on "payoff relevant" random variables (variables which effect either demand or costs), our three primitives will enable us to

- solve for each firms prices (quantities) as a function of
  - that firm's own state variables. These typically include the characteristics of its cost function and the characteristics of its products.
  - the state variables of the other firms active in the industry,
  - and "exogenous" state variables. These typically include factor prices (or factor supply functions if these have some elasticity), exogenous factors which determine demand conditions, institutional detail like tariffs and taxes, ...
- Substitute these price and quantities into the profit functions and solve for the profits of each firm as a function of all state variables (those of the firm in question, its competitors, and exogenous state variables)

- Similarly we can solve for (the distribution of) consumer surplus as a function of the prices (i.e. the state variables) and the distribution of preferences.

What static analysis allows us to do.

- Numerical analysis of price, quantity, profitability, and consumer surplus responses to policy & or environmental change, conditional on the state vectors of the firms.
- Allows us to feed the profit function into a dynamic problem that allows us to analyze, entry, exit, investment, and (at least potentially) ownership structure decisions.

Analysis of responses over time.

Of course once there is, say, a policy or environmental change, it will generally also produce an incentive to change the "state variables" of the system (the goods produced, their cost functions, perhaps even ownership and "preferences").

- E.g. merger activity. Typical analysis focuses on the impact of mergers on prices *conditional* on both the products marketed and their production costs. Ownership changes cause changes in investment (as well as in prices)..., and these can offset the static effects. Similar for collusion.
- Environmental change. Realistic analysis of even intermediate run responses requires some sort of dynamic analysis. Empirical e.g.: Gas prices and average mpg of new car fleet from PLB.

So we want also a model to analyze the responses of the state variables, or a "dynamic" model.

- To be realistic a dynamic model of even the simplest of markets has to be quite complex; too complex to admit analytic results with much applied content.

- This the reason that the theory papers on dynamics work in a highly "stylized" environment; i.e. an environment that is designed more for the clarity it allows in developing intuitions than for its realism.

- We give up on the goal of analytic elegance.

- Instead they provide a framework for obtaining *quantitative* responses for dynamic policy or descriptive analysis in a more realistic, user specified, environment. The goal is to enable us to analyze the equilibrium implications of the more complex institutional structures that we actually observe.

A "Framework" for Analysis.

As noted, a lot can happen in dynamic analysis. What does happen depends on the primitives of the problem, and the nature of the equilibrium that is established.

• We start with Markov Perfect equilibrium where strategies are a function of payoff random variables. So we will not allow strategies to depend on past play, which essentially rules out collusion, and we will not allow for serially correlated asymmetric information (which will be covered next semester). This does limit what you can use the framework for, some of which are easy to extend the framework to, and some of them require new concepts of equilibrium. We come back to this periodically.

• The real sense in which this makes the framework incomplete, is that one might also want to relate the choice of equilibrium back to primitives (just when might we expect collusion or asymmetric information to play a big role in explaining the evolution of an industry). We simply do not have a useful "megamodel" that choses among alternatives, and we proceed assuming the researcher can make many choices based on common sense and knowledge of institutional detail.

• What the framework does is

- Take realistic
    - primitives and
    - and a particular notion of equilibrium

  (and here realism takes precedence over simplicity),

- Compute equilibrium policies,
- Use these policies to generate quantitative responses to environmental or policy changes from a pre-specified initial condition.

• The primitives that one has to know before the framework can be used include the primitives of the static analysis

- distribution of cost functions (estimated)
- distribution of preferences (estimated)
- equilibrium assumption for any "static controls"(assumed from knowledge of industry, perhaps tested).

and, in addition, at least some notion of

- sunk costs (entry, exit).
- the cost and impacts of investments.
- the discount rate.

Characteristics of Framework.

Framework is a dynamic game that allows for

- heterogeneity among firms,
- both firm and industry sources of uncertainty (so that firms that are relatively "poor performers" in one period can become relative "good performers" in another)
- investment, and entry and exit.

Core Version: Public Program (available on my web site.

Consists of dynamic analogues of standard classroom static models

- differentiated products,
- homogeneous products - capacity constraints, and
- homogeneous products with different marginal costs.

Simplifications These versions stick to the assumptions that the current "q" or "p" choice does not have an independent effect on;

- future costs [l.b.d., adjustment costs, networks,...]
- future demand [durable or experience goods, networks,...]
- future equilibrium choices [collusion]

Examples of papers that weaken these assumptions and extend the framework in conceptually straightforward ways

- "Dynamic" supply. We do Benkard (2000 *AER*, 2004 *Restud*) on learning by doing. However one could also have adjustment costs, or the economics of networks.
- "Dynamic" demand (experience goods, durable goods, networks). Example, Robin Lee "Vertical Integration and Exclusivity in Platform and Two-Sided Markets," December 2013, *American Economic Review*, 103(7): 2960-3000.
- Predation Besanko, Doraszelski, and Krylov, AER,2011,
- Collusion (Fershtman Pakes,2000 *Rand*); policies are a function of past states

Lectures.

- Describe “core version”.
- Describe “backward solution” technique (publicly available program).
- A numerical example.
- Extensions.

Depending on time constraints we might consider computational burden and more powerful computational techniques

- Deterministic Approximations(Judd)
- Reinforcement Learning, Artificial Intelligence, Stochastic Algorithms (Pakes and McGuire),
- Continuous Time (Doraszelski and Judd, 2007).

Core Version (Ericson and Pakes, 1995, *Restud*).

Note. A more formal treatment of what follows is given in Doraszelski and Pakes (2007 *Handbook of Industrial Organation*). From an assumption point of view, the big difference between the material presented here and that in their article is that I will ignore random entry and exit fees. Proofs of existence require those sources of randomness. However most of the examples that I will go over do not have it, and still compute equilibria (or at least “ϵ-equilibria”). I ignore them here largely for pedagogical reasons; they make the notation much more complex.

States.

- $i \in \mathcal{Z}^+$.
- s_i will be the number of firms with efficiency level i,
- $s = [s_i;\ i \in \mathcal{Z}^+]$ is the "industry structure" (the number of firms at each different efficiency level).

Assumption. Given investment decisions, the distribution of future values of the state of the system (of (i, s)) are independent of the choice of prices or quantities.
$\Rightarrow$ changes in price (quantity) affect only current profits.
$\Rightarrow \pi(i, s)$ can be calculated "off-line" (without needing to compute the value function).

Differentiated Product Example

The publically available program gives three examples of markets, and then lets the user chose the one to analyze. Each example contains a demand system, a cost function, and an equilibrium assumption. The program then calculates the "reduced form", $\pi(i, s)$, which can be printed out and examined.

We will focus on one of these three. It is a differentiated product model with a "logit" type demand system, spot market equilibrium which is Nash in prices, and investment in the "quality" of the product marketed (the same example analyzed in Pakes and McGuire, 1994). Later we will describe the other examples on the publically available program.

Static Profit Function.

$$U_{i,j} = v_j - p_j^* + \epsilon(i,j)$$

where v_j is the "quality" of the good and p_j^* is the price (for $j = 1, \ldots, J$.

Consumer i chooses good j if and only if

$$\epsilon(i,j) - \epsilon(i,q) > [v_q - v_j] - [p_q^* - p_j^*]$$
$$\equiv [v_q - \zeta] - [p_q^* - p_0^*] - \{[v_j - \zeta] - [p_j^* - p_0^*]\}$$

where ζ is the quality of the outside alternative,

$$\equiv i_q - p_q - [i_j - p_j]$$

which implicitly defines "real prices" and "real quality" (i.e. both relative to the outside alternative).

Define $C(i_j; s, p) = \{\epsilon$: the consumer chooses good $j\}$ Assuming the distribution of ϵ is i.i.d. extreme value, then the market shares have the "logit form"

$$\sigma(i_j; s, p) = \frac{exp[i_j - p_j]}{[1 + \sum exp[i_q - p_q]}.$$

Profits, assuming a constant marginal cost of mc are then

$$[p_j - mc]M\sigma(i_j; s, p).$$

This implies that the Nash pricing equilibrium pricing vector satisfies

$$-[p_j - mc]\sigma_j[1 - \sigma_j] + \sigma_j = 0.$$

There is a unique solution to this system, say $p(i_j; s)$, (see Caplin and Nalebuff,1991) and profits become

$$\pi(i_j; s) = [p(i_j; s) - mc]M\sigma[i_j; s, p(s)].$$

Bellman Equation for Incumbent Behavior.

$$V(i, s) = \max\{\phi, \pi(i, s)+$$
$$\sup_{(x \geq 0)} [-cx + \beta \sum V(i', s')pr(i', s'|x, i, s)]\}.$$

- $pr(i', s'|x, i, s) = pr(i'|i, x)pr(s'|i, x, s)$. Games where my own investment only affects my own state variables are often called "capital accumulation games".

- $i_{t+1} - i_t \equiv \tau_{t+1}$

- $\tau_t \equiv \nu_t - \zeta_t$

- $\nu =$ firm's investment outcome. $\mathcal{P} = \{p_\nu(\cdot|x),$ $x \in \mathcal{R}^+\}$, stochastically increasing in x. Further we assume that $Pr\{\nu > 0|x = 0\} = 0$. In the example we use $\nu = 1$ with probability $p(x) = ax/[1 + ax]$ and zero otherwise.

- $\zeta =$ common industry shock. Density $\mu(\zeta)$. In the example we assume $\zeta = 1$ with probability δ and zero otherwise.

Let $\hat{s}_i = s - e_i$ provide the states of the competitors of a firm at state i for a particular s, and $q[\hat{s}_i{}'|i, s, \zeta]$ provide the firm's perceived probability of its competitors future states conditional on a particular value of ζ. Then given the above

$$pr(i' = i^*, s' = s^*|x, i, s) =$$

$$\Sigma_\zeta p(\nu = i^* - i - \zeta|x)q[\hat{s}_i{}' = s^* - e(i^*)|i, s, \zeta]\mu(\zeta).$$

where $e(i)$ has a one in the i^{th} slot and zero elsewhere.

Note that $q[\cdot|i, s, \zeta]$ embodies the incumbent's beliefs about entry and exit.

Entry model.

- Must pay an amount x_e ($> \beta\phi$) to enter,
- Enters one period later at state $i^e \in \Omega^e \subset \mathcal{Z}^+$ with probability $p^e(\cdot)$.
- Only enters if the expected discounted value of future net cash flows from entering is greater than the cost of entry.
- The cost of entry can be specified as either x_e, or as a random variable which distributes uniformly on $[x_{e,l}, x_{e,u}]$. When random entry costs are used only the potential entrant knows the realization of the entry costs, the other incumbents know only that entry costs will be a random draw from this uniform distribution.

Dynamic Equilibrium:Characterization Results.

Explain "equilibrium".

- Every agent chooses optimal policies given its perceptions on likely future industry structures
- Those perceptions are consistent with the behavior of the agent's competitors.

This framework is due to E-P(1995). Doraszelski and Satterwaite (2003) prove that a Markov Perfect equilibrium exists for this model (at least if we have random entry fees and exit costs), and E-P show that any such equilibria has the following characteristics

1. It is "computable", i.e.. Never more than $\overline{n}$ firms active & $\#\Omega isfinite \Rightarrow$ need only compute equilibria for $(i, s) \in \Omega \times S$, where

$$S \equiv \{s = [s_1, ..., s_k] : \sum s_j \leq \overline{n} < \infty\}$$

2. It is Markov. Indeed equilibrium policies generate a homogeneous Markov chain for industry structures [for $\{s_t\}$], i.e.

$$Pr[s_{t+1} = s'|s^t] = Pr[s_{t+1} = s'|s_t] \equiv Q[s'|s_t].$$

   with the Markov transition "kernel" $Q(\cdot, \cdot)$ on $S \times S$.

3. They provide conditions on the primitives such that insure that any equilibrium $Q[\cdot|\cdot]$ is *ergodic*. [Picture].

   - Note that the nature of states in R, how states cycle in R, and the transitions to R depend on primitives.
   - R is frequently much smaller than S (and the divergence is greatest for large markets with many state variables).
   - In the limit the probabilities of being at the various points in R converges to an invariant measure (invariant to how the state enters R). This invariant measure is often referred to as a "steady state" of the system, though "steady" seems to be a misnomer (as the state is not constant).

Computation 1 (Pakes and McGuire,1994, *RAND*).

This is a "brute force" algorithm, but it is instructive, and surprisingly often still used.

Assume temporarily that Ω and $\overline{n}$ are known. The first algorithm we consider is a "backward solution" algorithm that computes the value and policy functions pointwise. It is the multiple agent analogue of what we did to compute single agent dynamic problems. As a result, it is easy to follow.

- In memory. Estimates of the value function and policies associated with each $(i,s) \in \Omega \times S$.
- Updating. *Synchronous*; i.e. it circles through the points in S in some fixed order and updates all estimates associated with *every* $s \in S$ at each iteration (here updating estimates at s involves updating estimates at each (i,s) that has $s_i > 0$).
- Convergence. The values and policies from successive iterations are the same. Converged policies and values satisfy all the properties of equilibrium values and policies (see below).

Updating 1: Rewrite Bellman Equation.

$$V(i,s) = \max_{\chi \in \{0,1\}} \{[1-\chi]\phi + \tag{1}$$

$$\chi\{\pi(i,s) - \sup_{x \geq 0}[-cx + \beta \sum_{\nu} w(\nu; i,s) p(\nu|x_1)]\},$$

where

$$w(\nu;\ i,s) \equiv$$

$$\sum_{(\hat{s}_i', \zeta)} V(i+\nu-\zeta, \hat{s}_i' + e(i+\nu-\zeta)|w) q[\hat{s}_i'|i,s,\zeta]\mu(\zeta), \tag{1a}$$

and

$$q[\hat{s}_i^{'} = s_i^{*} | i, s, \zeta] \equiv$$
$$Pr\{\hat{s}_i^{'} = \hat{s}_i^{*} | i, s, \zeta, \text{equilibrium policies }\} \quad (1b).$$

Here $w(\nu; i, s)$ is the expected discounted value of future net cash flow conditional on the current year's investment resulting in a particular value of ν, and the current state being (i, s) (it integrates out over the possible outcomes of both the investment strategies of competitors (the $\hat{s}_i^{'}$), and over the outside alternative (the ζ)).

Note: Just as in the single agent problem $w(\nu; i, s)$ is all the firm needs to know in order to make decisions. It is thus a sufficient statistic for decision making purposes (note that it is sufficient for a very complicated object, the expected discounted value of future net cash flows given a realization for the investment process).

Updating Rules.

- calculates $w^{k-1}(\cdot|i, s)$ from the information in memory, i.e. from (x^{k-1}, V^{k-1}) (as in 1a),
- substitutes $w^{k-1}(\cdot)$ for $w(\cdot)$ in (1) and then solve the resultant *single agent* optimization problem for the j^{th} iteration's entry, exit and investment polices at (i, s). That is

– Incumbents solve for (χ^k, x^k) that

$$\max_{\chi\in\{0,1\}}\{[1-\chi]\phi+$$
$$\chi \sup_{x\geq 0}[\pi(i,s)-cx+\beta\sum_{\nu} w^{k-1}(\nu;i,s)p(\nu|x)]$$

I.e. we solve the Kuhn-Tucker problem for investment conditional on continuing which in the example works out to be

$$\frac{\partial p(x)}{\partial x}[w(1;i,s)-w(0;i,s)]-c\leq 0,$$

with stict inequality if and only if $x=0$. We then substitute the solution for x in the continuation value above and determine whether it is greater than ϕ.

– Potential entrants compute

$$V_e^k(s)=\beta\sum_{\zeta} w^{k-1}(\zeta;i_e,s+e(i_e))\mu(\zeta).$$

and set $\chi_e^k=1 \Leftrightarrow V_e^k(s)>x_e$,

- substitutes these policies and the w^{k-1} for the w, x and the max operator in (1), and label the result $V^k(\cdot)$,
- calculates $V^k(\cdot)-V^{k-1}(\cdot)$ and then substitutes $V^k(\cdot)$ and the policies from iteration k, for the iteration $k-1$ values that were in memory.

Convergence.

At the end of the iteration calculate $||V^{k-1}(\cdot)-V^k(\cdot)||$ and $||x^{k-1}(\cdot)-x^k(\cdot)||$. If both are sufficiently small, stop. Else continue. The program prints out both the L^2 norm and the sup norm.

At fixed point each incumbent and potential entrant

- uses, as its perceived distribution of the future states of its competitors the actual distribution of future states of those competitors[1]
- chooses its policy to maximize its expected discounted value of future net cash flow given this distribution of the future of its competitors.

Star and Ho 1969, provide a proof that this is all that is needed for a MPE.

Setting K and $\overline{n}$

K.
Start with the monopoly problem ($\overline{n} = 1$) and an oversized K; $\rightarrow$ a lowest i at which the monopolist remains active and a highest i at which the monopolist invests. $\rightarrow$ 1 and K in Ω.

$\overline{n}$.
Set $\overline{n} = 2$ and do the iterative calculations again starting at $V^0(i_1, i_2) = V^*(i_1)$. Then set $\overline{n} = 3$ and set $V^0(i_1, i_2, i_3) = V^*(i_1, max(i_2, i_3))$. Continue until we reach an $\overline{n}$ so high that whenever there are $\overline{n} - 1$ firm's active there is no possible structure at which an entrant would want to enter. This is $\overline{n}$.

[1] That is how it is usually stated, but actually all we insure is that at equilibrium each firm's perceptions of the EDV of future net cash flows, is equal to the actual distribution of the EDV of future net cash flows (i.e. two distributions could yield the same expectation and the equilibrium would not distinguish between them).

For Student Use.

Notes on the (Updated) Publicly Available Algorithm.

- Choose to compute:
  - Differentiated product model,
  - homogeneous products with capacity constraint,
  - homogeneous products, differences in marginal cost.
- Choose to compute:
  - MPN equilibrium,
  - Social Planner Problem (ignoring information and incentive problems). This is a single agent dynamic programming problem. It sets price=mc, and choose all entry, exit, and investment policies to maximize the discounted value of consumer surplus minus the costs of entry and exit plus the selloff value. Designed to give you some idea if there is "room" for policy intervention to improve social surplus (of course if there is room there is still a question of whether we can find a mechansim that can lead us to an improvement without counterveiling costs).
  - Perfect collusion (ignoring the problem of supporting collusion). Also a single agent problem. Multi-product monopoly with no one else able to enter. Single agent

chooses all prices, entry, exit and investment to maximize the expected discounted value of net cash flow. Designed to give you some idea of how profitable collusion could possibly be (of course the fact that collusion would be profitable were we not required to support it, does not mean that collusion would be profitable).

- Set parameter values.
  - maximum number of firms active ($\overline{n}$).
  - Highest efficiency level available (K).
  - Static profit function parameters (differ somewhat with problem)
    For differentiated products; marginal cost (mc), market size (M), inflection point in utility.
    For homogenous products, investment in marginal cost; demand intercept,fixed cost, minimum marginal cost.
    For homogenous products, investment in capacity; demand intercept, marginal cost, maximum capacity.

  - Dynamic parameters; discount rate, scrap value, entry sunk costs (deterministic or stochastic), sunk costs, efficiency level at which entrant enters, "a" parameter setting $p(x)$, δ parameter setting how the outside alternative moves,
- Compute $\pi(\cdot)$. This has to be done before computing the dynamic equilibrium. You can now go to the descriptive statistics part of the problem and just look at $\pi(\cdot)$.
- Compute entry, exit, and investment. Can now look at part of descriptive statistics which has to do with value

function (which gives you value function, investment, $p(x)$ entry, exit)

- Simulate industry structures and provide descriptive statistics from user specified initial conditions. Currently working part gives you
  - number of periods with; n firms active, entry, exit, entry and exit
  - mean and standard deviation of; investment, price-cost margin, one-firm concentration ratio, lifespan
- Simulate industry structures and compute mean and standard deviations of consumer, producer, and total surplus for runs of user specified length from user specified initial conditions. specified initial conditions.

Numerical Example.

Table 1. Generate 10,000 periods of industry evolution, and describe results. Note: The total current cost of producing the typical output of this industry is about 25. So setting the entry cost at .2 means it is about 1/125 of the cost of production in a given period. It is not very costly to enter in our base case.

Table Cases:

- MP is base case.
- $\sigma* = .65$. This is an institutionally created upper bound to the max market share of the largest firm.
- $x^e = 2$. Increase sunk entry costs by a factor of 10.
- PP is planners market.
- Coll. Is colluders market.

Base Case. Mostly 3 or 4 firms active. Not same firms and entry and exit positively correlated (as in cross-sectional data, but this is only sometimes true in time series on industries, as exit leaves an opportunity for entry but also sometimes occurs because of common shocks to the industry).

Descriptive Results.

Table 1

Characteristics of Ergodic Distribution[1]

$\delta = .7 \quad \alpha = 3 \quad \beta = .925 \quad xe = .2 \quad \phi = .1$
$m = 5 \quad c = 5$

No. of Time Periods	10,000				
	MP	$\sigma* = .65$	$x^e = 2$	PP	Coll.
% with 6 firms active	.1	.2	.0	0	0
% with 5 firms active	1.6	3.1	.3	.1	.1
% with 4 firms active	35.3	33.3	1.2	5.8	1.1
% with 3 firms active	63.0	63.3	17.1	44.5	23.1
% with 2 firms active	.0	0.0	81.4	49.6	75.6
% with entry and exit	13.1	11.5	.3	10.1	10.7
% with entry only	4.8	4.8	.7	3.0	1.9
% with exit only	2.0	2.5	.6	2.3	1.7
% with entry or exit	20.1	18.7	1.6	15.4	14.2
Gross job creation	.086	.086	.031	.033	.027
Gross job destruction	.087	.088	.032	.033	.027

1. Legend

MP	=	Markov Perfect Nash Equilibruim
δ*	=	.65, MP with market share constrained to be below .65.
x^e	=	2, MP with sunk entry costs increased to 2.
PP	=	planner's problem (see the text below).
Coll.	=	perfect cartel (see the text below).

Table 2:

- Lifetime Distribution very skewed.
- 1,800 firms in 10,000 periods
- Mode of lifespan distribution is 1, mean is 18.4, and standard deviation is 77.5. This is a very skewed distribution. Most entrants try something and fail. Those who do survive, persist for a very long time. They eventually exit, but they may have made a lot of money in the interim.
- Analytic reason is approximate "S" shape of sections of the value function. Initially the most likely direction is downward movement, but if we get some success we increase investment until we get over the point of inflection. Once we get over the point of inflection we tend to stay over it even though investment is decreasing. This because if we start dropping down we pick up investment.

Table 2

Lifetime Distribution

(Based on 10,000 time periods)

$\delta = .7 \quad \alpha = 3 \quad \beta = .925 \quad \text{xe} = .2 \quad \phi = .1$
$\text{m} = 5 \quad \text{c} = 5$

Life time	Percent			Implied Hazard			Cumulative Percent		
	MP	$x^e = 2$	PP	MP	$x^e = 2$	PP	MP	$x^e = 2$	PP
1	48.3	17.5	25	48.3	17.5	12.2	48.3	17.5	25.0
2	24.7	14.6	44.0	47.7	19.6	24.0	73.0	35.0	68.9
3	6.1	4.7	9.0	22.6	7.3	17.9	79.1	40.7	77.9
4	3.6	8.5	4.4	17.2	14.3	17.9	82.7	49.1	82.3
5	2.5	3.8	1.9	14.4	7.5	10.9	85.2	52.1	84.2
6	1.3	2.8	2.2	8.78	5.9	8.9	86.5	55.7	86.5
7	.8	1.9	1.4	5.9	4.5	4.1	87.3	57.2	87.9
8	.7	.9	.8	5.5	2	4.8	88.1	58.5	88.6
9	.7	.9	.5	5.9	2.9	5.7	88.8	59.5	89.2
10	.3	1.9	.7	2.7	4.6	5.7	89.1	61.3	89.9

	Number Active	Mean	Median	Standard Deviation
MP	1,800	18.4	2	73.2
S	103	191.5	6	428.0
PP	1,301	19.5	2	93.9

1. Legend

MP = Markov Perfect Nash Equilibruim
δ^* = .65, MP with market share constrained to be below .65.
x^e = 2, MP with sunk entry costs increased to 2.
PP = planner's problem (see the text below).
Coll. = perfect cartel (see the text below).

Table 3:

- Value Distribution very skewed.
- Only 10% make positive returns, but those who make it over the initial high mortality period do very well.
- Note the values and rates of returns of firms operating at one time will tend to be “supernormal”, but this is no indication of there being something “wrong” (some barrier to entry); it is just that those who are in tend to be those whose investments were profitable; the expected value of an entrant is just slightly more than the sunk cost of entry.

Table 3

Realized Value Distribution

$\delta = .7$ $\alpha = 3$ $\beta = .925$ xe = .2 $\phi = .1$
m = 5 c = 5

Obs/Num	Realized Values			Life Time	Sum of Realized Val	
	MP	$\sigma^* = .65$	$x^e = 2$	MP	MP	$\sigma^* = .65$
1	72.8	45.1	83.0	79	72.8	45.0
2	52.6	38.9	34.0	247	125.4	84.0
3	33.1	36.3	32.4	718	158.57	120.3
4	32.7	35.7	28.4	118	191.3	156.0
5	29.3	30.2	20.2	102	220.6	186.3
10	22.8	21.2	15.74	5	343.8	301.3
100	7.11	6.8	-4.43	215	1462.6	1313.9
150	1.81	1.1	—	37	1700.8	1508.5
170	.1	-.06	—	3	1717.6	1514.0
171	-.05	-.07	—	2	1717.5	1514.0
1491	-.10	-.7	—	4	1586.5	1336.1
1800	-4.08	—	—	15	1282.05	—
	Mean	Median	Std. Dev.	# Positive	Mean of Positive	# Negative
MP	.71	-.1	4.09	170	10.0	1630
$\sigma^* = .65$	.68	-.1	3.74	150	9.5	1465
$x^e = 2$	2.37	-1.89	11.28	329	14.2	74

1. Legend

MP = Markov Perfect Nash Equilibruim
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PP = planner's problem (see the text below).
Coll. = perfect cartel (see the text below).

Table 4 and figures.

- In the base case this is often a reasonably fracturred industry (concentration ratio averages .39 when usually 3 or 4 firms active
- but it goes through cycles of concentration that have simply to do with the logic of the technology; that is periodically a firm rises to dominate the industry and stays their for a long time. At some point either there is a series of negative shocks to it, or a competitor receives a series of positive shocks, and a more competitive structure is generated. This is all internal dynamics; there is no change in institutional environment.
- Markups 27 to 44%. These are considered reasonable in this kind of environment. Note that they are what generates the incentives for the entrants, but the expected value of the entrants is just equal to their sunk costs.

Table $4A$[1]

One-Firm Concentration Ratios

	MP	$\sigma^* = .65$	$x^e = 2$	PP	Coll.
.95 quantile	.62	.61	.63	1.0	1.0
.90 quantile	.5	.5	.63	.84	1.0
.75 quantile	.43	.43	.51	.52	.72
.50 quantile	.34	.34	.5	.5	.5
.25 quantile	.33	.33	.5	.46	.5
.10 quantile	.26	.3	.34	.33	.33
.05 quantile	.25	.25	.53	.33	.33
mean	.39	.38	.51	.54	.60
standard deviation	.11	.097	.09	.20	.20

1. Legend

MP = Markov Perfect Nash Equilibruim

σ^* = .65, MP with market share constrained to be below .65.

x^e = 2, MP with sunk entry costs increased to 2.

PP = planner's problem (see the text below).

Coll. = perfect cartel (see the text below).

Table $4B^2$

Price / Cost Ratios

	MP	$\sigma^* = .65$	$x^e = 2$	Coll.
MAX	2.11	2.21	2.18	2.45
.95	1.44	1.48	1.46	2.34
.90	1.40	1.39	1.46	2.33
.75	1.33	1.33	1.40	2.33
.50	1.30	1.30	1.40	2.33
.25	1.30	1.30	1.40	2.24
.10	1.27	1.27	1.30	2.22
.05	1.27	1.27	1.30	2.20
MIN	1.24	1.23	1.27	1.29
Mean	1.33	1.32	1.41	2.30
Standard Deviation	.085	.073	.099	(.089)

1. Legend

MP = Markov Perfect Nash Equilibruim

$\sigma^* = .65$, MP with market share constrained to be below .65.

x^e = 2, MP with sunk entry costs increased to 2.

Coll. = perfect cartel (see the text below).

Policy Analysis.

Consumer surplus given by

$$\int max(\omega_j - p_j + \epsilon_j) dG(\epsilon_1, \dots \epsilon_n) \equiv log \sum_j exp[\omega_j - p_j]$$

Note that there are a distribution of welfare results associated with any given set of institutions; it depends on the realizations of the random terms. We run 100 runs of 100 periods and give mean and variance.

Table 5.

- Initial condition always $s_0 = e(\omega_0)$.
- Temporary monopolies correspond to patents. Difference between monopoly and MPN is seen by noting that consumer benefits are large in MPN and producer benefits are large in Monopoly. Monopoly also decreases total surplus by 20%.
- Multiproduct monopolist (same as colluder without incentive compatibility constraints). Now 10% less than MPN, and both producer and consumer surplus increase.
- Note that the standard error of welfare gains is $\approx$ 20%. That is with the same institutions just differences in random outcomes generate, on average, 20% standard error of outcomes. Problem for case studies as here we have set everything exactly the same, and still there are large differences between cases due to randomness in outcomes of innovative process.
- Compare MPN to the planner's problem. Not much difference in welfare but huge difference in market structure.

Planner is much more similar to colluder in observables (in entry exit rates, number of firms....). The colluder however charges much higher prices. Its P/C ratio is 2.3 whereas the MPN average is 1.3 and the planner's is one.

- Look at number of firms and investment. MPN has most firms and investment. Mankiw and Whinston story; that is incumbents and potential entrants do not internalize the buisness steeling effects and hence "overinvest". Then comes the planner; it internalizes the buisness steeling which causes lower investment, but it also internalizes the consumer surplus which causes higher. Then the colluder; it internalizes buisness steeling but not consumer surplus.

- $\sigma* = .65$. Here when an unconstrained firm would have gotten a market share greater thean .65, that firm increases its price until the market share goes back to this number. This has an effect on investment (dampening it) as well as on pricing. Market structures look much the same as unconstrained MPE but welfare falls 5%.

- Increase sunk costs. Market structures change dramatically, look like colluder, but welfare does not change much (and if we added back in the sunk costs it would be even closer). Actually it is likely that there is a sunk cost inbetween which maximizes welfare from MPE (since when the entry costs are low there is too much entry and investment). Still there is not too much fine tuning possible here, as the MPN is pretty close to the social planner.

- The relationship between market structure and welfare is quite complex.

Table 5

Social Welfare From
Alternative Market Structures*

Benefits/ Market Structure	Total Firm Cash Flows		Consumer Benefits		Total Benefits**	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
Monopoly	207	66	96	17	303	83
20 Year Monopoly then free entry	180	57	140	21	320	74
10 Year Monopoly then free entry	146	54	186	31	331	71
Markov-Perfect Nash	70	26	301	65	369	68
Perfect Collusion	218	55	115	19	332	74
Social Planner	—	—	—	—	377	—
$\sigma^* = .65$	61.7	15.1	289.7	64.4	349.6	67.4
$\sigma^* = .55$	54.4	12.0	284.8	66.1	337.5	73.1
Sunk Costs = 2	76.5	26.3	293.4	55	361.5	69.8

Table 6

Average Investment and Number of Active Firms Under Alternative Institutional Arrangements

	Investment	Active Firms
Markov Perfect Nash	2.57	3.4
$\sigma^* = .65$	2.56	3.4
$x^e = 2$	1.97	2.2
Planner	1.95	2.6
Perfect Collusion	1.75	2.3

The Core Version and Extensions.

Core version used primarily in teaching. Allows us to illustrate dynamic responses to say a tax (or tariff), entry barrier, differences in monopoly policy .... Also allows us to teach estimation; especially in relatioship to estimating sunk costs.

There are a large number of potentially interesting extensions. Suffice with a review of those that have been computed, and some straightforward extensions of obvious applied importance. The extensions I will outline are interrelated but I will discuss them one at a time, allowing, in turn, for

- Change the static profit function.
  - Shihua Liu (1998) modifies the differentiated products example used here to allow for different substitution patterns between domestic goods then between domestic and imported goods (using a nested logit assumption). Liu also shuts down the investment process, using instead an exogenously evolving "quality" parameter. He however does fit the model to data on the evolution of the Pulp and Paper Industry in Colombia. His goal is to investigate the effects of changing trade regimes on welfare (including consumer surplus, but also the costs of labor turnover), productivity (focussing on reallocation of output to more productive firms), and profits, in a world with heterogeneous producers and imperfect competition. He is particularly concerned with the effects of trad liberalization on developing countries.
  - Allison Oldale (1998) ($\cdot$) computes an infinite period generalization to Sutton's two period endogogenous sunk cost and exogensous sunk cost models, and gets results somewhat similar to those presented in his two books.
  - An important example that has not been explicitly analyzed is computing expected profit from an assymetric auctions conditional on state variables that determines the asymmetry. This would allow us to analyze sequential auctions when we can invest in those state variables (for a related two period model see Cantellon and ?).
- multiple states per firm (e.g. Gowrisankaran and Town, 1998, allow for for-profit and not for-profit hospitals in their empirical analysis of

the impact of new demand management policies on the structure of the hospital market).

- Evolution of state variables not independent of price or quantity choices
  - collusive price (or quantity) setting (allow current price or quantity choice to depend on past price or quantity choices) (Fershtman and Pakes, 2000, provide a theoretical analysis of a dynamic game with collusion, and De Roos, 2001, uses a related model to analyze collusion in agricultural chemical markets),
  - "dynamic" cost functions (Benkard, forthcoming, analyzes learning by doing in Aircraft production, adjustment costs and capacity constraints are similar)
  - "dynamic" demand functions (Marcovitch 1998, provides a theoretical analysis of a market with hardware-software connections, Ching, 1999, considers expeience goods),
- nonpecuniary externalities to investment (Tien, 2001, provides a theoretical analysis of "learning from others" in the context of internet useage),
- merger activity (Gowrisankaran, 1999, provides a theoretical analysis of merger activity)

The stress here will be on the conceptual problems that arise when we extend the model in these ways. The extensions can also have important impacts on the computational properties of the algorithm, and we discuss these in the next section.

Multiple States Per Firm.

Incorporating additional state variables does not, per se, generate any additional conceptual problems and, by enabling a better approximation to real world institutions, can produce a much richer analysis of the problems at hand.

Exogenously Evolving States.

This includes

- states that evolve as exogenous Markov processes
- states that differentiate between different (time invariant) types of firms.

In applied work movements in industry wide demand, technology, or factor prices, are often incorporated by adding exogenous Markov processes whose realizations take on the same value for all firms. This just mimics what we do when we add exogenously differing state variables to single agent problems.

Gowrisankaran and Town (1997) extend the core version to allow for two different types of firms, for profit and not for profit hospitals, and then investigate the likely impacts of health policy changes on the evolution of the hospital industry. Not for profit hospitals care about quality, as well as profits, and have certain tax advantages. Using (largely) estimated parameters, they investigate the impact of policy changes that were designed primarily to change the conditions determining demand for hospital services (changes in medicare reimbursement rates, insuring uninsured patients, and taxing not for profit hospitals) on supply conditions in that industry. That is, by endogenizing the impact of the demand side changes on market structure (through induced changes in investment, entry, and exit), they are able to analyze the *equilibrium* impacts of the policy changes on patient welfare. They find that the induced changes in market structure can have large (and largely unintended) impacts on patients. Though one has to make a lot of assumptions to get the exact results, there is no question that they document the likely importance of endogenizing the supply side of the market when considerring the impacts of demand.

States that Evolve Endogenously.

The conceptual problems that arise in incorporating multiple endogneously evolving states per firm differ with the details needed for the application at hand. Two extensions which are important for applied work are

- allowing for products that are differentiated by more than one characteristic, and
- allowing for firms that market more than one product.

To dynamize these models we need to formulate investment and entry processes that operate on a multidimensional space. I have computed several such models for the new car market, focussing on the trade off between miles per gallon and car size, and on how various policies can effect it. Probably the most important problem that arose in this context was the problem of determining families of transition functions that reflect the technological and institutional conditions in the industry of interest (transitions for both the couple of states of the active products, and for the locations of entrants). For example, it is undoubtedly much harder to increase the mpg of a large car to 35 than to do so to a small car and that has to be incorporated into the analysis in a realistic way. Modulus these problems, the rest of the analysis was a relatively straightforward extension of the techniques we used in the core version.

To my knowledge the framework has not been used to study markets with multiproduct firms; despite its potential importance to merger analysis. Once we allow for multiproduct firms it is natural to allow incumbents (as well as new entrants) to introduce new products. Even in the simplest such case, say when a single quality dimension is the sole source of differentiation among products and entry occurs as a random draw from an exogenously specified distribution of these qualities, when incumbents (as well as the potential entrant) can introduce new products, there will be many possible equilibria.

The researcher will then have to formulate mechanisms that direct the algorithm to computing one of them. These can be as simple as allowing random draws to determine the order of moves. Alternatively we could let the firm for whom entry generates the highest increment to its value get the opportunity to move first (or a higher probability of the opportunity to move first). More interestingly (and probably more realistically), we could worry explicitly about the investment required before launching a new product (either through in-house development or through buying out potential new entrants), and let the outcomes of such a process take away much (though possibly not all) of the burden of choosing between alternative equilibria. More detailed empirical knowledge of how entry actually occurs in particular industries would be helpful in this context. Note that entry issues get both more complex, and more interesting, when products are differentiated in richer ways and firms have some choice of where to enter (see, for e.g. Schmalensee's (1978) paper on cereal's and Judd's (1985) comment). Much potentially important resarch remains to be done here.

Dynamic Costs and/or Demand.

The core version of our model assumes that the distribution of future states, conditional on current states and all investments, is independent of the price or quantity choices of the agents. This assumption is inappropriate whenever either the current cost or the current demand function depends on the quantities sold (or the prices set) in previous periods. Quantities have an independent effect on future costs when learning by doing is important or when there are adjustment costs, and they have an independent effect on future demand when the goods being marketed are either durable, evaluated through personal experience, or addictive. Network effects can cause either future demand or future costs to depend on current price and/or quantity choices.

Recall that in the core version of the algorithm we can compute the current profit function "off line" and then simply import a "table" of the needed values for the profit function into the algorithm designed to compute equilibrium entry, exit and investment policies. Once current demand has an independent impact on either future costs, or future demand, this is no longer true. In these cases the Nash first order condition for equilibrium quantities (prices) has a term for the impact of current quantities (or prices) on future net cash flow as well as a term for their impact on current profits. As a result we cannot obtain either the "static" control or profits without computing the entire value function, and this makes for a more difficult computational problem.

The modification to the first order condition needed to accommodate these cases differs depending on whether the choice of the control for one firm effects the transition probabilities for the states of *all* firms, or just the transition probabilities for the states of the given firm. In games of quantity competition in which the transition probability for a firm's own states are determined only by own quantities, the simpler case where a firm's choice of controls only effects its own state prevails. Benkard (2000) analyzes a model of this sort and since his work has the added realism of a model build up from estimated parameters, we begin with an outline of it. When the control of a given firm is a determinant of the distribution of quantity outcomes for all competitors, as is usually the case when the control is price (see Berry and Pakes, in process), or when the control is a bid in a repeated auction with capacity constraints (see Joffre Bonet and Pessendorfer, 2000), then the first order condition must account for the fact that a given firm's choice of control affects the evolution of the states of all competitors. Later we illustrate what happens in these cases by modifying Benkard's model to allow for price, rather than quantity, competition.

Benkard and Learning by Doing in Aircraft.

Benkard (2000)'s goal is to analyze competition in the market for wide bodied commercial aircraft. Starts with the Boeing 747 in 1969. By 1997 aircraft production is a $60B\$$ industry, $40B\$$ in the U.S. (and often the biggest export industry of the U.S.). There are two technological features of the industry that have a big impact on the nature of competition.

- Huge sunk costs of development (over $2-3B\$$)
- Importance of learning by doing; frequently initial mc is five to six times higher than eventual mc.

Both oligopoly theory (Fudenberg and Tirole $\cdot$, Cabral and Riordan, $\cdot$) and empirical work indicate that the learning by doing will lead to costs much greater than price in early years (at least if there is more than one product available). This leads to what is usually called "intense competition", with alot at stake (the sunk cots and the early losses), so it is not infrequent to see various forms of government support for this industry, international complaints about that support, a suspension of DOJ rules, ...

Benkard ignores traditional investment in increasing the quality of the product (i.e. investment of the type we focused on in the early section), but allows current quantity choices to effect an experience variable, which in turn affects cost of production in future yearss (this was for tractability). Using estimated parameters for both the demand and cost functions, he then computes and analyzes a model of dynamic quantity competition among producers which is a reasonably realistic approximation to the competition that occurred in the commercial aircraft market at the time the Lockheed Tristar was being introduced.

Letting i index experience levels (or cost functions), and taking some liberties to place his model in the confines of our notation, he has $i_{t+1} = i_t + \tau_{t+1}$, where if Q is quantity produced, the distribution of τ_{t+1} is determined by the family $\{\lambda(\cdot|Q) \mid Q \in R^+\}$ which is stochastically increasing in Q. Quantities are a choice variable and the vector of quantities determine all prices. As a result the profits of the j^{th} firm can be written as

$$\pi(i_j, Q_j, Q_{-j}, M) = Q_j \left(p(Q_j, Q_{-j}, M) - mc(i_j)\right),$$

where $p(\cdot)$ provides the pricing function, and M represents a set of state variables that evolve exogenously (there are cyclical movements in the demand for aircraft which he pick up in M). Letting $\underline{i}$ be the vector of states of the active firms (ordered so $i_r > i_{r-1}$), the Bellman equation in (5) becomes

$$V(i_j, \underline{i}, M) = max_{\chi\in\{0,1\}}\{[1-\chi]\phi + \chi sup_{Q\geq 0}[\pi(i_j, Q_j, Q_{-j})$$

$$+\beta \sum V(i_j+\nu-\xi, \hat{\underline{i}_j}'(\xi)+e(i_j+\nu-\xi))q(\hat{\underline{i}_j}'(\xi)|i, \underline{i}, \xi, M)\mu(M', \xi|M)\lambda(\nu|Q_j)]\}$$

where $\hat{\underline{i}_j}'(\xi)$ is notation for the locations of firm $j's$ competitors in the next period for a given realization of ξ.

Note that there is no independent effect of the firm's own quantity choice on the distribution of its competitor's future states.

Summing out over the $(\hat{\underline{i}_j}'(\xi), \xi, M')$ we get

$$max_{\chi\in\{0,1\}}\{[1-\chi]\phi + \chi sup_{Q\geq 0}[\pi(i, Q, Q_-) + \beta \sum w(\nu; i, \underline{i}, M)\lambda(\nu|Q)]\}$$

with appropriate defintion of $w(\cdot)$.

Now assuming that $\chi = 1$ (the firm continues in operation) and that $Q_j > 0$, the first order condition which sets Q_j is modified to read

$$\frac{\partial\pi(i, Q, Q_-)}{\partial Q} + \beta \sum w(\nu; i, \underline{i}, M)\frac{\partial\lambda(\nu|Q)}{\partial Q} = 0. \qquad (2)$$

Here are some other points on this f.o.c.

- The standard Nash equilibrium in quantities, when quantites do not impact on future costs is obtained from this by setting $\beta = 0$.

- The standard investment equation is obtained from this by letting Q be investment and $\partial\pi(\cdot)/\partial Q$ be the cost of investment. I.e. now the control affects both current and future profits.
- Since the value function is increasing in experience (in i) and the distribution of future experience is stochastically increasing in quantity, this model implies that *conditional* on the other firm's outputs, this firm will put more output will be put on the market than standard static Nash in quantities model would predict. Note further that
  - The discrepancy between the output's generated by this and the standard model will be greatest when the "derivative" of the value function with respect to experience is steep. For a given distribution of competitors, this will tend to occur for values of the state vector at which the derivative of the learning curve is steep. We would expect the increase to be even larger, and the price to be lower, if there is a competitor active in the early period.
  - The partial equlibrium analysis (holding other firm's output's constant) is inappropriate. That is if this firm'squantities go up is no longer a Nash equilibrium for other firms to hold their quantities constant. Whether other firm's quantities go up or down depends on whether they are strategic complements or strategic substitutes, but whatever they are once they move, it is optimal for me to move again. In the end equilibrium outputs with learning need not be higher than those without (see Cabrel and Villa-Boas,2001, for the static analysis of related problems). On the other hand, the model without learning cannot generate price less than marginal cost, so at least in this region the model with learning has more output than does the model without.
- With this change in the first order condition, the equilibrium for Benkard's model can be computed iteratively as before.

Some details of his specification.

- Three state variables for each firm; i.e.

$$i = \text{level of experience, size of airplane, quality of airplane.}$$

Only experience is subject to control; the others evolve exogenously. Size of airplane is constant over its lifespan but differentiates planes. For quality see below.

  - The fact that there is only one endogenous state implies that there are only two $w(\cdot)'s$ that need to be calculated for each state vector,
  - However there are many possible states (if each element of the triple can take on K values there are now K^3 possible states for each firm, and remember the state is not only where the firm is, but also where its competitors are).

- Costs (see his Table 1). Include a fixed cost, and a variable cost which depended on size of aircraft, and experience. He used random entry costs as in the table gotten from industry publications.

- Demand. Assumes a perfect second hand market; i.e. each airline optimally reallocates its entire fleet each year choosing from all avilable new and used planes. Nested logit. One nest is new wide-bodied planes;the other is used. There is an unobserved product characteristic, which becomes product quality. Quality is then estimated pointwise, and its distribution over time is estimated.

- Demand and Dynamic Parameters. Table 3. pp 24. Note takes out trend from demand before estimating its Markov process.

Uses $\overline{n} = 4$. This gives us 13 state variables and on the order of 7 million states (uses both parallel processesing and stochastic algorithm). Here are some of the results of the excercise;

- Figure 3 p.27 p/mc ratios. As competitor's quality and experience go up you cut your price further

- Figure 4. Model's prediction for Markup at actual states for years 1972 to 1985 vs. actual markups in those periods. He predicts too high a markup in 73-75 an claims this is because the demand system underestimates the impacts of the similarity between the L-1011 and the DC-10. Also predicts the variance in prices much less than costs and also gets that.

- Tables 5 and 6; Market structure and plane type distribution. Simulate from actual initial condition (I.C. Simulation), and simulate invariant distribution on recurrent class. One and two plane concentration ratios are almost eactly right. Also gets the distribution between large, medium and small planes about right. Note that the invariant distribution has less large and more small planes, and this is what happened post 1985 (though for the first time just recently Airbus is thinking of another large plane; Boeing is presumably following with a fast plane.)
- Only half the planes make positive values, and the values are very variant. See also Figure 8 for profits. Firm 5 loses in every period; looks like Lockheed.

Also does some policy experiments.

- Table 7 considers the comparison to a social planner and to a multi-product monopolist. MPE has "too many" plane types, and too high a price.
- Very little difference in production cost between the MPE and Social Planner. Differences in welfare largely a result of; (i) too high a price (this is seen in the shift from consumer to producer benefits when we move to the social planner), and (ii) too much investment in MPE. The resultant fall in total surplus is about 10%.
- More problems would have arisen if there would have been a monopolist. There would have been much less variety in planes, much higher prices, and much less quantity. As a result total surplus would have fallen to under 70% of its potential value.
- Table 10 p.44 Also tries DOJ constraints on max 1-firm concentration ratio (once 60% and once 50%). For 50% the mean harm was about .8% of surplus which is about 1.4b$. Note that all the harm is on consumers, since now if a firm gets far ahead it also generates lower costs, and passes part of it on to consumers.

Benkard's Problem in a Price Rather than a Quantity Setting Game.

We now go back to Bellman equation for this problem and consider what would happen if the firms were playing a price, rather than a quantity setting game. We do this because this a form of the "dynamic" price or quantity game that shows up quite frequently when we are trying to model "dynamic" demand or cost.

Recall that the perception of the distribution of a firm's competitor's states in the future is given by

$$q(\underline{\hat{i}}'|i_j, \underline{i})$$

and in equilibrium must satisfy

$$q(\underline{\hat{i}}'|i_j, \underline{i}) = \Pi_{r \neq j} \lambda(i_r'|i_r, Q_r).$$

Now if we were in a price setting game the quantity of the r^{th} firm would be a function of the prices set by all firms, i.e.

$$Q_r = Q_r(p_r, p_{-r})$$

As a result the price the j^{th} firm sets not only effects its own quantity, and hence its own likely future experience levels, but it also effects its competitors quantities, and hence the competitors likely future experience levels.

As a result the first order condition that sets price (the analogue of (7) in the quantity setting model) is the more complicated expression

$$\frac{\partial \pi(i_j, p_j, p_{-j})}{\partial p_j}$$

$$+\beta \sum V(i_j', \underline{\hat{i}}') \left(\sum_r \frac{\partial \lambda(i_r'|i_r, Q_r)}{\partial Q_r} \frac{\partial Q_r}{\partial p_j} \Pi_{l \neq r} \lambda(i_l'|i_l, Q_l) \right) = 0.$$

Again, this is somewhat more complicated, but the iterative computational technique should still work; as Berry and Pakes $(\cdot)$ illustrate for a model of experience goods. There are also a number of other related papers in the area. Joffe and Pessendorfer (1999) consider repeated procurement auctions for California highway paving projects. Here the source of the dynamics is that firms that win a project today have less excess capacity and higher costs of production in the next period. The bids today, take into account the effect that winning today puts the winner in a disadvantaged spot in the future (and it is clear from a reduced form analysis of bids that this effect is empirically present). Here the bid of one firm can potentially effect the capacity of every firm, and we are back to the more complicated model immediately above. They do not actually compute the equilibria, but they do introduce a technique which allows you to estimate all the primitives of the problem and exhibit bid functions. We will come back to this paper if we do estimation.

Other related papers are;

- Judd's 1996 investigation of the robustness of Bertrand and Cournot competition to the ability to set both price and quantity, when there are costs of adjustment and no entry or exit. He uses a linear quadratic game and shows that outcomes mimic Cournot outcomes when adjustment costs are high but mimic Bertrand outcomes when they are low.

- Markovitch (1998) has modified the core version of our model to analyze numerical examples of dynamic demand emanating from the interactions between hardware and software choices (a "network" effect). Consumers make a hardware choice that lasts two periods. Software is designed to run on one, and only one, of the two types of hardware. Software firms must commit to one of the types of hardware when they enter, and then can invest to improve the quality of their product, or exit, just as in the core version of our model. The demand for a given software product depends not only on the vectors of qualities of software products available for each of the two hardware types, but also on the number of consumers who have purchased the different types hardware in the past. Thus consumer's demand for hardware products depends on their beliefs about the likelihood of future software products available for each hardware type, while the entry exit and investment decisions of software firm's depends on their beliefs on the future hardware purchases of consumers. She solves for a rational expectations Markov Perfect equilibrium and finds that if the industry's competitors (the outside alternative) are not progressing too quickly the equilibrium is one where both types of hardware are produced (the "variety" equilibrium), while if the competitors to the industry are growing quickly we see an equilibrium with only a single type of hardward produced (an equilibrium with "standardization".)

- Ching (1999) analyzes the market for pharmaceutical products, treating these products as an "experience" good. In his model the demand for a particular product depends on experience with that product to date, and pricing decisions take into account the fact that if more products are sold today there will be less variance in the perception of the quality of the product tomorrow. He also introduces a new estimation procedure using nonparametrics to overcome the problem of computing equilibrium prices at each value of the paramter vector in the estimation algorithm (in a similar way to the way in which Olley and Pakes use nonparametrics). The "experience" in this paper has the same value for each individual (thus different individuals do not react differently to the good) and there are some questionnable assumptions about price setting in this environment. Still there is little question that treating drugs as experience goods can explain many of the anomalies that have been problematic to researchers in this important industry, and the paper provides a start on how this can be done.

The fact that Markovich, Ching, and Berry Pakes, can incorporate dynamic demand in a tractable way gives us some hope that progress can be made on other markets where demand is by nature dynamic (this includes durable goods, as well as experience and network goods). The potential application here is enormous; much of manufacturring produces durable goods, and our marketing colleagues tell us that experience is an important determinant of demand for most consumer products (see Ackerberg,1997a and b, and the literature cited their).

Collusion.

Most of the theoretical work on collusion and price wars assumes identical firms and an unchanging environment (Green and Porter $(\cdot)$, Abreu Pearce and Stachetti $(\cdot)$, Rotemberg and Saloner $(\cdot)$). These assumptions help to clarify both the process by which collusion can be supported (continuation values which are dependent on the history of play) and its implicatons (e.g.,"pricewars" in both Green and Porter, and Rotenberg and Saloner) they often make the existing collusion models unappealing to applied researchers who are trying to understand the workings of particular industries because. I.e. in many applied problems we are simply not willing to assume pricing mechanisms that ignore differences in the policy cum profitability options of different firms. This, I should note, is despite the fact that applied people are often not satisfied with the simple static Nash pricing models that we generally bring to data. Further the assumption of an unchanging environment limits the investigation of the implications of collusion to its impact of prices; ignoring the (possibly equally important) effects of collusion on the costs, qualities, and varieties of the products marketed.

Fershtman and Pakes (2000) provide one (of many possible models), and it illustrates just how the analysis would proceed. This is a model with symmetric information in which it is hard to sustain collusion when either; one of the firms does not keep up with the advances of its competitors, or a "low quality" entrant enters. In either case there will be an active firm that is quite likely to exit in the near future. Not only is it hard to punish a firm who is likely to exit after it deviates, but if one of the competitors is near an exit state the other incumbent(s) has an incentive to price predatorily (that is to deviate themselves) in order to hasten that exit.

They assume that firm's either collude to set prices or set prices as in a static Nash pricing equilibrium. Collusive prices and profits are determinned by the outcomes of a Nash bargaining game in which the threat value is the profits from the static, Nash in prices, equilibrium. The difference between their model and our core model is that they introduce a second state variable for each firm and let pricing policy depend on it. The second state variable is an indicator function which is one if the given firm has ever deviated from the collusive agreement in the past. If any incumbent has deviated in the past, the static Nash in price solution is played whenever that incumbent is active. No incumbent ever deviates, but there are tuples of states for which the punishment of reverting to noncollusive prices is not sufficient to support collusion, and this generates price wars. To determine which states will support collusion we must calculate values for incumbents who have deviated; that is we need the value function for states which should never actually be observed (they are "off the equilibrium path").

Without going through the details it is instructive to look at their Bellman equation, and then explain it verbally. For each $(\omega, \alpha) \in S$, where α is a vector of indicator variables which tell us whether any firm has deviated in the past, the value of the j^{th} firm satisfies

$$V(\omega_j;\omega_{-j},\alpha) = max\{\phi, \ \ \pi(\omega_j;\omega_{-j},\alpha)$$

$$+max_{x\geq 0}[-x+\beta\Sigma_{\omega'}V(\omega_j';\omega_{-j}',\alpha')p(\omega_j'|\omega_j,x)p(\omega_{-j}'|\omega,\alpha)]\},$$

where

$$\pi(\omega_j;\omega_{-j},\alpha) \equiv I(\omega,\alpha)\pi^C(\omega_j;\omega_{-j}) + [1-I(\omega,\alpha)]\pi^N(\omega_j;\omega_{-j}),$$

and

$$I(\omega,\alpha) \in \{1,0\}$$

with

$$I(\omega,\alpha) = 1 \ \ \textit{if and only if}$$

- $\alpha = 0$, and
- for all j who continue

  $\pi_j^D + max_{x_j}[-x_j+\beta\Sigma_{\omega'}V(\omega_j';\omega_{-j}',\alpha')p(\omega_j'|\omega_j,x_j)p(\omega_{-j}'|\omega,x,\alpha_j=1,\alpha_{-j}=0)]$

  $< \ \pi_j^C + max_{x_j}[-x_j+\beta\Sigma_{\omega'}V(\omega_j',\omega_{-j}',\alpha')p(\omega_j'|\omega_j,x_j)p(\omega_{-j}'|\omega,x,\alpha=0)].$

The first max operator compares the exit value of the firm (ϕ) to its continuation value. If ϕ is larger the firm shuts down. We let $\chi(\omega_j;\omega_{-j},\alpha)$ be the indicator function which takes the value of one if the firm remains active and zero elsewhere.

If the firm does continue it earns current profits plus the expected discounted value of future returns. Current profits are either Nash profits or collusive profits according as the indicator function, $I(\cdot)$, is zero or one. If $\alpha \neq 0$, that is if one of the current participants deviated in the past, then $I = 0$ and the firms earn the one shot Nash equilibrium profits. If $\alpha = 0$ then the firms only collude if collusion can be sustained; that is if, for every active firm, collusive profits plus the expected discounted value of future net cash flows conditional on the firm colluding are greater than defector's profits plus the expected discounted value of future net cash flows conditional on the firm defecting.

Their numerical results are instructive in that they illustrate the potential importance of dynamic considerations in evaluating the benefits and costs of collusion. Explain.

Allowing for Investment Externalities.

The framework has been used to study models with externalities by Tien, 2001. He considers externalities generated by people learning to use the internet. Externalities can be appended to our core models in several different ways. I will stick to a method slightly different than that in Tien, trying instead to keep to the spirit of the models used by Steve Klepper and his coauthors( ) in their study of industry evolution (once again, however, I will give myself considerable leeway in tranforming their ideas into my notation).

Those models have two types of investment. There is a "traditional" investment similar to our investment in that it increases the value of the firm's own state, and an R&D investment. The R&D investment, if successful, decreases the firm's marginal cost in the current period (marginal costs are constant over quantity levels) . However the firm can only appropriate the gains from the output of its R&D activities for a single period. Thus the marginal costs available to each firm at the beginning of each period is the minimum of the realized marginal costs of the preceeding period. So if $mc_{j,t}$ denotes marginal cost of the j^{th} firm in period t

$$mc_{j,t} = mc_{t-1} exp[-\epsilon_{j,t}]$$

where,

- $mc_{t-1} = min_j mc_{j,t-1}$, and
- there is a family of distributions for ϵ which is stochastically increasing in the amount or research investment (r); i.e. $\{P_\epsilon(\cdot|r), r \in R^+\}$.

When we append this research process to our framework, we will need to account for the fact that profits in the current period are not known at the beginning of the period (since they will depend on the marginal costs of all firms, which in turn depends on the realization of the $\epsilon's$). So instead of maximizing current profit plus the expected discounted value of future net cash flows we maximize expected profit plus this expectation. Expected profits will depend on the firm's own research investment, as well as the research investments of its competitors (since their marginal costs will help determine realized equilbrium quantities and prices). We let those expected profits be $E[\pi(i, s, \overline{mc})|i, s, \overline{r}]$, where $\overline{y}$ represents the vector of y variables of all active agents.

The Bellman equation for the value function is then

$$V(i,s,mc) = \max\{\phi, \sup_{r\geq 0, x\geq 0} [E[\pi(i,s,\overline{mc})|i,s,\overline{r}] - c(x+r)$$

$$+\beta \sum V(i+\nu-\zeta, \hat{s}_i + e(i+\nu-\zeta), mc') p_{\overline{\epsilon}}(mc'(\overline{\epsilon})|mc,\overline{r}) q[\hat{s}_i|i,s,\zeta] p(\nu|x)\mu(\zeta)]\}.$$

We can now proceed pretty much as before, realizing that we have two controls, and noting that the first order condition for one of them (r) will include a derivative with respect to current profits as well as a derivative of expected future net cash flow. A model in this spirit should not be too difficult to compute, and if modified to suit a particular industry and parameterized accordingly, could provide a basis for evaluating alternative R&D policies.

Horizontal Mergers.

Almost all of the formal models of merger activity *condition* on the cost, qualities, and variety of products sold in the market. These models hold the distribution of characteristics of the products being marketed (as well as the nature of competition) fixed, and analyze the impact of the ownership change on producer and consumer surplus. The producer surplus analysis provides a vehicle for analyzing the incentives to merge, while the addition of consumer surplus analysis is used to analyze whether the merger is beneficial to society.

As noted in Stigler's(1965) investigation of the US Steel merger, the results from such a "static" analysis of mergers can easily be overturned by simple dynamic considerations (his discussion allowed for adjustment costs in an analysis of mergers in a homgenous goods industry). The first attempts I know of to build a model to analyze the dynamic effects of mergers are in articles by Cheong and Judd (1992) , and Berry and Pakes (1993) . They both consider a "one-time" exogenously specified merger, and analyze its impacts in a dynamic model which allows for investment, but does not allow for any further mergers. These papers show that mergers can be beneficial to *both* the firms merging and to society, even if the profits of the merging firms *and* consumer surplus falls at the time of the merger.

To be realistic a model which investigated the dynamic impacts of mergers would want to allow mergers to arise endogenously, and not just investigate the impacts of a "one-time" exogenously specified merger. There are many unsolved problems here, not least among them being the diveristy of views on the factors motivating merger activity in different indsutries. In addition to specifying the possible sources of gains from mergers, a merger model must also specify a market mechanism for determinning which among the possible profitable mergers at any point of time are in fact consummated.

One such mechanism is provided in Gowrisankaran(1999). He takes the capacity constrained homogeneous goods version of our framework and adds to it a merger game. The merger game occurs at the beginning of each period and proceeds in the following sequential manner. The largest firm is allowed to choose a merger partner first. All other firms present the largest firm with a "take it or leave it" price at which they are willing to be bought. Information is symmetric except that the largest firm draws a "synergy" value for each merger which is known only to it; i.e. the synergy value for a given firm is not known to any of the firms that might be acquired. The largest firm chooses to merge with the firm which generates the highest net merger value provided that value is positive. The net merger value is the the expected discounted value of future net cash flow if a merger would take place net of the price of the acquisition and what the value of the firm would be if the merger did not take place. If a merger takes place the process restarts (there are new take it or leave if offers, and new synergy values), and the (new) largest firm can choose another merger partner. When the largest firm chooses not to merge further, the second largest firm gets to choose a merger partner in the same way. This process continues until the smallest active firm chooses not to merge further. At that point production, investment, and then exit followed by entry and investment decisions are made. All offers and actions are made to maximize the expected discounted value of future net cash flows given the agents' information sets, and the equilibrium is Markov Perfect.

Perhaps the most striking part of this analysis is that it in fact can be done. Given the quantitative magnitude of the merger phenomena in recent years and the extent that it can be impacted by policy, any step in developing a useable models of mergers is wellcome. Still, it is clear that we are only at the beginnings of developing an ability to provide realistic dynamic models that allow for mergers; alot of work remains to be done.

The Computational Burden of the Simple Algorithm.

The computational burden is (essentially) the product of three factors,

- the number of points evaluated at each iteration;
- the time per point evaluated;
- the number of iterations.

Number of Points.

Since each of the $\overline{n}$ active firms can only be at K distinct states, the number of points we need to evaluate at each iteration, or $\#S \leq K^{\overline{n}}$. Exchangeability, of the value and the policy functions in the state variables of a firm's competitors implies that we do not need to differentiate between two vectors of competitors that are permutations of one another notation does not. Pakes (1993) shows that an upper bound for $\#S$ is given by the combinatoric

$$\binom{K+\overline{n}-1}{\overline{n}} << K^{\overline{n}}.$$

but for $\overline{n}$ large enough this bound is tight.

Burden per Point.

Determined by the cost of calculating the expected value of future states (of obtaining the $w^j(\cdot; i, s)$ from the information in memory). The burden of obtaining the optimal polices and the new value function given $w^j(\cdot; i, s)$ does not depend on the other parameters of the problem.

Assume that there is positive probability on each of κ points for each of the $m-1$ active competitors of a given firm. Then we need to sum over κ^m possible future states and there are $\kappa \times m$ values of $w^j(\cdot)$ needed at that s. Average m should increase in $\overline{n}$, and κ should be determined by the nature of the state space per firm.

Little known about the relationship between the number of iterations and the number of state variables of the problem; but doesn't seem to be too bad.

Approximation Techniques.

- Pointwise algorithm has been used both as a tool for substantive problems and as a teaching device.
- Still many applied problems need more powerful computational tools.
- Two available. Each has their problems, but both compute equilibria with much less of a computational burden than the standard algorithm.
- I do not know of a publicly available versions of these algorithms, though the stochastic algorithm introduced below is much easier to program than the other algorithms discussed in this paper.

Deterministic Approximations or Curve Fitting.

These techniques begin by specifying a set of functions considered rich enough to contain an element which provides a good approximation to the value function. In the paper we consider polynomials of order d. They then use a small subset of the points in $\mathcal{S}$ to find an approximating member of the set of functions, and uses this approximating function to predict the value function at other points as needed.

The paper contains a fairly detailed description of how to construct an algorithm based on these techniques for our problem. This is new material and it shows that the tools that can be brought to this problem are quite powerful. However we were not terribly successful when we used them on problems the size of the problem computed in PM(1994). They are likely to be more successful when $\overline{n}$ is larger. Indeed Liu's thesis(1998) uses a combination of the polynomial approximation technique and simulation, and successfully computes equilibria for a 35 firm industry.

The Stochastic Algorithm; Pakes and McGuire,1997.

Two distinct ideas. First the algorithm is *asynchronous*, i.e. it only updates a single location at each iteration, and the selection process which chooses the transitions eventually confines its attention to the recurrent class of points, or R, a subset of S whose cardinality does not necessarily depend on the dimension of the state space. Note that

- If $s_t \in R$, then so will be all future s (with probability one),
- The algorithm is designed to tell the user whether the user-specified initial condition is in R.
- In many I.O. problems R is small relative to S as does not increase in size as the size of the state space does (consider increasing market size, or the multiple characteristic model)

Second, the algorithm estimates the integral of future values needed to determine current policies (i.e. the $w(\cdot)$) by an average of past monte carlo evaluations of those integrals. Tradeoff between speed and accuracy of the evaluation. Tradeoff tends to favor monte carlo when the state space is large.

Reinforcement Learning

- The stochastic algorithm is also iterative (though asynchronous). In memory is an estimate of w, say w^k, and by its location, an $s^k \in S$.
- Updating is done to mimics the rules used in the reinforcement learning literature (see Barto, ...).

Reinforcement learning is primarily concerned with formulating a set of rules that might actually be used by agents attempting to infer optimal behavior from past outcomes (indeed they are used in various "machine learning" situations). As a result it is possible to interpret our algorithm as a decision making process that agents in a particular market might actually use.

Updating Rules.

- Assume $s = s_t$ and that all agents believe that the expected discounted value of future net cash flows are given by $w(\cdot|\cdot, s_t) = w^+(\cdot|\cdot, s_t)$.

- They would then choose their policies to maximize the $V(\cdot, s_t|w^+)$ obtained by substituting w^+ for w in (1) above.

- These choices would generate a distribution of outcomes for each agent's competitors given by (i.e. a distribution for agent $i's$ competitors in the next period).

- Nature would then choose the market outcome as a random draw from this distribution.

- The current perception of the *value* of this outcome to firm i is obtained by substituting it into the evaluation function given by w^+ and (1).

- If these values are viewed as random realizations from the integral defining the appropriate components of the true w, denoted as $w*$, the agents might use them to update their estimates $w*$.

- Our algorithm for finding $w*$ mimics the learning algorithm just described.

We begin with the update for s. This requires policies for the incumbents and the potential entrant. The investment and exit policies policies for incumbents, say, $x(\cdot, s_k|w^k)$ and $\chi(\cdot, s_k|w^k)$ are obtained as the solution to

$$\max_{\chi \in \{0,1\}} \{[1-\chi]\phi \ + \ \chi sup_x[\pi(i, s^k) - cx + \beta \sum_{\nu} w^k(\nu; i, s^k)p(\nu|x_1)]\},$$

while the entry policy is given by

$$\chi_e(s_k|w^k) = 1 \Leftrightarrow \beta w^k(0; i_e, s^k + e(i_e)) > x_e,$$

where $e(i_e)$ is a K-vector which has one for its i_e element and zero elsewhere.

These policies determine a distribution for s^{k+1}. The actual s^{k+1} is obtained as a random draw from this distribution. To obtain the draw use $\chi(\cdot|w^k)$ to determine which of the firms in s^k remain active, and $\chi^e(\cdot|w^k)$ to determine whether or not their is an entrant. Now take random draws for the $\nu's$ of the incumbents that remain active and a random draw on ζ. This determines the $k+1^{th}$ iteration's state of each agent active in that period, and all that remains is to reorder those states into s^{k+1}. [2] Note that if $w^k = w*$ the process generating s^{k+1} would be an ergodic Markov process, and hence would wander into the recurrent class in a finite number of iterations and stay there.

We now update w^k. For each agent and each possible realization of ν, use $V(\cdot|w^k)$ [equation (7) with w^k substituted for w] to evaluate the state defined by the actual *simulated draws* for ζ and for the locations of the agent's competitors; i.e. use (7) and w^k to calculate

$$V(i+\nu-\zeta^{k+1}, \hat{s}_i^{k+1} + e(i+\nu-\zeta^{k+1})|w^k).$$

This expression is the k^{th} period evaluation of being in location $(i+\nu-\zeta^{k+1})$ when all the other competitors states are determined by their simulated draws. Its expectation conditional on information realized by iteration k is $\sum_{(\hat{s}_i',\zeta)} V(i+\nu-\zeta, \hat{s}_i' + e(i+\nu-\zeta^{k+1})|w^k) q^{w^k}[\hat{s}_i'|i, s^k, \zeta]\mu(\zeta)$. So if $w^k = w*$, then its expectation is $w*$.

Since $V(i+\nu-\zeta^{k+1}, \hat{s}_i^{k+1} + e(i+\zeta-\zeta^{k+1})|w^k)$ is the current period's perception of the value of a random draw from $w^*(\nu; i, s)$, it is used to update $w^k(\nu; i, s)$. In particular if $V(i+\nu-\zeta^{k+1}, \hat{s}_i^{k+1} + e(i+\zeta-\zeta^{k+1})|w^k)$ is different from $w^k(\nu; i, s)$, then set $w^{k+1} - w^k$ equal to a fraction of the difference. More formally if $\alpha(k, s^k) \in (0,1)$ set

$$w^{k+1}(\nu; i, s^k) - w^k(\nu; i, s^k) = \alpha(k, s^k) \times \tag{3}$$

$$\{V[i+\nu-\zeta^{k+1}, \hat{s}_i^{k+1} + e(i+\nu-\zeta^{k+1})|w^k] - w^k(\nu; i, s^k)\}.$$

Note that if we set $\alpha(k, s^k)$ equal to the inverse of the number of times the estimate of $w^*(\nu; i, s)$ has been updated in the past, then the $w^k(\cdot; i, s)$ are just the sample average of past draws on the expected discounted value of future net cash flows (i, s). Also if $V(i+\nu-\zeta^{k+1}, \hat{s}_i^{k+1} + e(i+\nu-\zeta^{k+1})|w^k) = w^k(\nu; i, s)$, then

[2] More formally let the r^{th} active agent's location be i_r^k and its investment be $x_r^k(\cdot|w^k)$. Then for each active agent draw a random variable from the distribution $p(\cdot|x_r^k)$, say ν_r^{k+1}. Also draw ζ^{k+1} from $\mu(\zeta)$. We obtain s^{k+1} as follows. Compute $i_r^k + \nu_{1r}^{k+1} - \zeta^{k+1}$ for each active agent. If $\chi_e(w^k) = 1$, also compute $i_e - \zeta^{k+1}$ for the potential entrant. Now count how many of these numbers equal i for each $i \in \Omega$. The vector of integers obtained by this procedure is s^{k+1}.

$w^{k+1}(\nu; i, s) = w^k(\nu; i, s)$. Consequently if $w^k = w*$ then the expectation of w^{k+1} is $w*$. I.e. if we are at $w*$ we will tend to stay their.

We have now shown how to update both w^k and s^k. This is the core of the algorithm. We keep updating until we can show that the values and policies in memory are equilibrium values and policies for a recurrent class of points. Pakes and McGuire (1997) provide a detailed definition of what we mean by this, and suggest a testing procedure. They also consider; alternative starting values, different sequences for the α in (9), procedures for estimating policies at an s not in R, and a host of other details.

0.0.1 Concluding Comments on the Stochastic Algorithm.

The economics of MP models typically indicate that a given set of primitives can only support certain configurations of firms in any lasting way. The lasting configurations become those in R. The reasons this occurs are typically different when S is large because market size (and hence $\overline{n}$) is large then when S is large because the number of state variables per firm is large.

Though as market size increases the model will support structures with more active firms, it will no longer support structures where there are a small number of active firms. So as market size increases we both add and subtract points from R. Thus the *net* effect of market size on $\#R$ is not obvious, and will depend on the primitives of the problem. We have typically found the cardinality of R to grow in $\overline{n}$, but only linearly at low $\overline{n}$, and if anything the rate *falls* as $\overline{n}$ increases. Recall that $\#S$ increases geometrically in $\overline{n}$.

As for the case with more than one state variables per firm, here we have found that the relationship between R and S can vary greatly with the economics underlying the transitions for the state variables. For example in differentiated product models where the state vector details different characteristics of the products (e.g.. the size, mpg, and hp of cars), the primitives often indicate that certain combination of characteristics are not demanded at a price greater than their marginal cost of production (e.g.. large cars with a high mpg, or small cars with a low mpg). Alternatively consider locational models where there is an initial locational choice and then a plant specific cost of production (or quality of product) which changes over time as the result of outcomes of an investment process. $\#R$ in these models tends to be linear in the number of locations at which their is entry.

All this was on the number of points visited in the stochastic algorithm. There is also the question of the computational burden per point at each iteration. There are two aspects to this. One is that we have to search our storage device for the $w(\cdot)'s$ associated with the point we visit at each iteration. We have been using a trinary tree to store the $w(\cdot)$ and the computational burden of finding a point in such a tree grows like the logarithm of the number of points. Second, the computation once we have found the $w(\cdot)$ grows linearly with the number of firms active at that point. Again this is to be compared to exponential and geometric growth in the computational burden per point in the other two algorithms considered in this paper.

As for number of iterations, though accuracy will require that the number of iterations be large when using the stochastic algorithm, it is not clear that it has to increase in any particular way with the dimension of the state space. Moreover, in distinct contrast to the other algorithms considered in this paper (particularly the algorithm which uses polynomial approximations), we have *never* encountered a convergence problem with the stochastic algorithm.

Indeed part of the reason we have not pushed forward on the algorithm based on polynomial approximations is that we have had so much success with the stochastic algorithm. Still as applied work moves to more detailed analysis and to larger problems we will encounter bigger recurrent classes, and a need for further computational simplifications. The artificial intelligence literature for large, single agent, dynamic programing problems has a set of papers which combine reinforcement learning techniques (similar to those used here) with approximation methods in the spirit of those described in the last section (see SSB,1993, sec. 9 for references in AI); a combination which is potentially quite powerful for the problems we are interested in.