Lecture 19: Solving directional dynamic games for all Markov perfect equilibria

Econometric Society Summer Schools in Dynamic Structural Econometrics

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ROAD MAP

1. Collusion of Australian corrugated fibre packaging (CFP) producers
   - Collusion between Amcor and Visy
   - Bertrand pricing and investment game
   - Solution concept: Markov perfect equilibrium (MPE)
2. Experiment with the model
3. State recursion algorithm
   - Theory of directional dynamic games (DDGs)
4. Recursive lexicographical search (RLS) algorithm
5. Full solution for the leapfrogging game
6. Structural estimation of directional dynamic games with Nested RLS method

Estimation of directional dynamic games:
Full solution nested MLE estimation

Nested Recursive Lexicographic Search algorithm

Markov Perfect Equilibria

- MPE is a pair of strategy profile and value functions
- In compact notation

$$\begin{aligned} \mathsf{V} &= \Psi^{\mathsf{V}}(\mathsf{V}, \mathsf{P}, \theta) \\ \mathsf{P} &= \Psi^{\mathsf{P}}(\mathsf{V}, \mathsf{P}, \theta) \end{aligned}$$

- Set of all Markov Perfect Equilibria

$$SOL(\Psi, \theta) = \left\{ (\mathsf{P}, \mathsf{V}) \;\middle|\; \begin{array}{l} \mathsf{V} = \Psi^{\mathsf{V}}(\mathsf{V}, \mathsf{P}, \theta) \\ \mathsf{P} = \Psi^{\mathsf{P}}(\mathsf{V}, \mathsf{P}, \theta) \end{array} \right\}$$

- $\Psi^{\mathsf{V}}: \ \mathsf{V}, \mathsf{P} \longrightarrow \mathsf{V}$ Bellman operator
- $\Psi^{\mathsf{P}}: \ \mathsf{V}, \mathsf{P} \longrightarrow \mathsf{P}$ Choice probability formulas (logit)
- $\Gamma: \ \mathsf{P} \longrightarrow \mathsf{V}$ Hotz-Miller inversion

Estimation methods for *dynamic* stochastic games

- Two step (CCP) estimators
  - Fast, potentially large finite sample biases

Hotz, Miller (1993); Altug, Miller (1998); Pakes, Ostrovsky, and Berry (2007); Pesendorfer, Schmidt-Dengler (2008)

1. Estimate CCP $\rightarrow \hat{\mathsf{P}}$
2. Method of moments • Minimal distance • Pseudo likelihood

$$\min_{\theta} \left[\hat{\mathsf{P}} - \Psi^{\mathsf{P}}(\Gamma(\theta, \hat{\mathsf{P}}), \hat{\mathsf{P}}, \theta)\right]' W \left[\hat{\mathsf{P}} - \Psi^{\mathsf{P}}(\Gamma(\theta, \hat{\mathsf{P}}), \hat{\mathsf{P}}, \theta)\right]$$

$$\max_{\theta} \mathcal{L}(\mathsf{Z}, \Psi^{\mathsf{P}}(\Gamma(\theta, \hat{\mathsf{P}}), \hat{\mathsf{P}}, \theta))$$

- Nested pseudo-likelihood (NPL)
  - Recursive two step pseudo-likelihood
  - Bridges the gap between efficiency and tractability
  - Unstable under multiplicity

Aguirregabiria, Mira (2007); Pesendorfer, Schmidt-Dengler (2010); Kasahara and Shimotsu (2012); Aguirregabiria, Marcoux (2021)

Estimation methods for *dynamic* stochastic games

- Equilibrium inequalities (BBL)
  - Minimize the one-sided discrepancies
  - Computationally feasible in large models

  Bajari, Benkard, Levin (2007)

- Math programming with equilibrium constraints (MPEC)
  - MLE as constrained optimization
  - Does not rely on the structure of the problem
  - Much bigger computational problem

  Su (2013); Egesdal, Lai and Su (2015)

$$\max_{(\theta,\mathsf{P},\mathsf{V})} \mathcal{L}(\mathsf{Z},\mathsf{P}) \text{ subject to } \mathsf{V} = \Psi^{\mathsf{V}}(\mathsf{V},\mathsf{P},\theta), \mathsf{P} = \Psi^{\mathsf{P}}(\mathsf{V},\mathsf{P},\theta)$$

- All solution homotopy MLE

  Borkovsky, Doraszelsky and Kryukov (2010)

Overview of NRLS

- Robust and *computationally feasible*$^{(?)}$ MLE estimator for directional dynamic games (DDG)
- Rely of full solution algorithm that provably computes all MPE under certain regularity conditions
- Employ smart discrete programming method to maximize likelihood function over the finite set of equilibria
- Fully robust to multiplicity of MPE
- Relax single-equilibrium-in-data assumption
- Path to estimation of equilibrium selection rules

Nested Recursive Lexicographical Search (NRLS)

- Data from M independent markets from T periods
  $Z = \left\{a^{jt}, x^{jt}\right\}_{j \in \{1,\dots,N\}, t \in \{1,\dots,T\}}$
- Let the set of all MPE equilibria be $\mathcal{E} = \{1, \dots, K(\theta)\}$

1. Outer loop
   Maximization of the likelihood function w.r.t. to structural parameters θ
   $$\theta^{ML} = \arg\max_{\theta \in \Theta} \mathcal{L}(Z, \theta)$$
2. Inner loop
   Maximization of the likelihood function w.r.t. equilibrium selection
   $$\mathcal{L}(Z, \theta) = \arg\max_{k \in \{1,\dots,K(\theta)\}} \mathcal{L}(Z, \theta, V_\theta^k)$$

Max of a function on a discrete set organized into RLS tree

Likelihood over the state space

- Given equilibrium k choice probabilities $P_i^k(a|x)$, likelihood is

$$\mathcal{L}(Z, \theta, V_\theta^k) = \sum_{j=1}^{N} \sum_{t=1}^{T} \sum_{i=1}^{J} \log P_i^k(a_i^{jt}|x^{jt}; \theta)$$

- Let ι index points in the state space
  $\iota = 1$ initial point, $\iota = S$ the terminal state
- Denote n_ι the number of observations in state x_ι and $n_\iota^{a_i}$ the number of observations of player i taking action a_i at x_ι

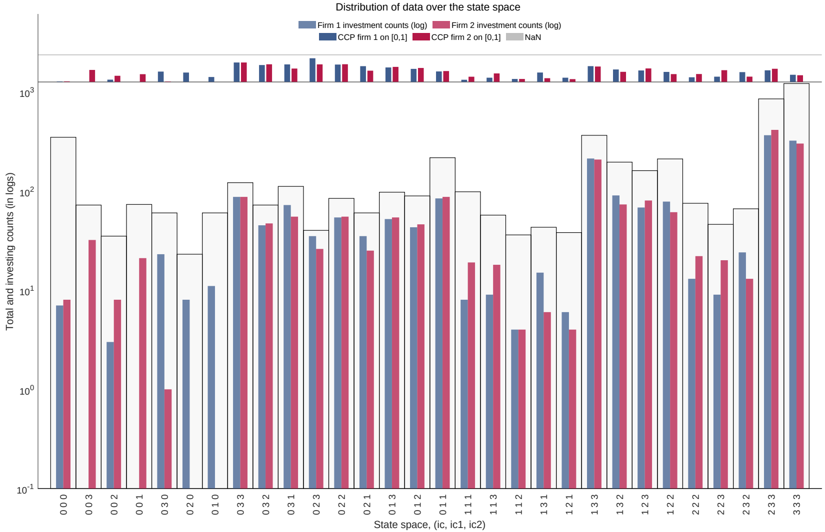
$$n_\iota = \sum_{j=1}^{N} \sum_{t=1}^{T} \mathbb{1}\{x^{jt} = x_\iota\} \qquad n_\iota^{a_i} = \sum_{j=1}^{N} \sum_{t=1}^{T} \mathbb{1}\{a_i^{jt} = a_i, x^{jt} = x_\iota\}$$

- Then equilibrium-specific likelihood can be computed as

$$\mathcal{L}(Z, \theta, V_\theta^k) = \sum_{\iota=1}^{S} \sum_{i=1}^{J} \sum_{a} n_\iota^{a_i} \log P_i^k(a|x_\iota; \theta)$$

Data distribution over the state space

1000 markets, 5 time periods, init at apex of the pyramid



Branch and bound (BnB) method

Land and Doig, 1960 *Econometrica*

- Old method for solving discrete programming problems
- Maximizing/minimizing a function over a discrete set

1. Form a tree of subdivisions of the set of admissible plans
2. Specify a bounding function representing the best attainable objective on a given subset
   - Monotonicity: the bounding function has to be weakly decreasing in the cardinality of the set argument (for max problem)
   - Has to equal the criterion function when computed at singletons
3. Dismiss the subsets of the plans where the bound is below the current best attained value of the objective

- There are several flavors of BnB method, differences in implementation
- There are several extensions to the BnB method

Theory of BnB: branching

$$\max f(x) \text{ s.t. } x \in \Omega$$

$f(x) : \mathbb{R}^n \to \mathbb{R}$ objective function

Ω set of feasible x

$\mathcal{P}_j(\Omega)$ partition of Ω into $k_j + 1$ subsets, $k_0 = 0$, $\mathcal{P}_0(\Omega) = \Omega$

$$\mathcal{P}_j(\Omega) = \{\Omega_{j1}, \ldots, \Omega_{jk_j} : \Omega_{ji} \cap \Omega_{ji'} = \varnothing, i \neq i', \cup_{i=1}^{k_j} \Omega_{ji} = \Omega\}$$

$\{\mathcal{P}_j(\Omega)\}_{j=1,\ldots,J}$ a sequence of J gradually refined partitions

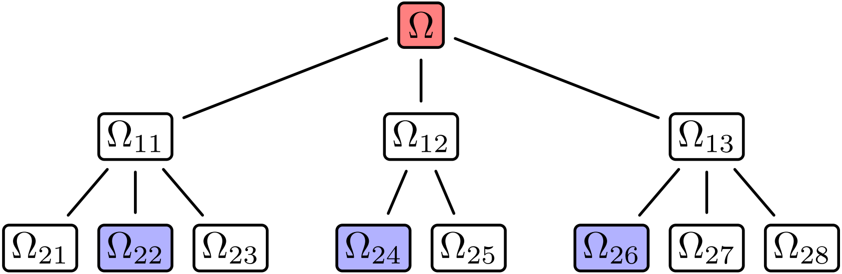
$$0 = k_0 \leq k_1 \leq \cdots \leq k_j \leq \cdots \leq k_J$$

$$|\Omega| \geq \max_i |\Omega_{k_1 i}| \geq \cdots \geq \max_i |\Omega_{k_j i}| \geq \cdots \geq \max_i |\Omega_{k_J i}|$$

$$\forall j = 1, \ldots, J, \forall i = 1, \ldots, k_j, \forall j' < j : \exists i' \in \{1, \ldots, k_{j'}\} \text{ such that } \Omega_{ji} \subset \Omega_{j'i'}$$

Theory of BnB: branching

$$\max f(x) \text{ s.t. } x \in \Omega$$



Theory of BnB: bounding

$$\max f(x) \text{ s.t. } x \in \Omega$$

$g(\Omega_{ij}) : 2^{\Omega} \to \mathbb{R}$ bounding function: from subsets of Ω to real line
$g(\{x\}) = f(x)$ for singletons, i.e. when $\Omega_{ij} = \{x\}$

Monotonicity of bounding function

$$\forall \Omega_{j_1,i_1} \supset \Omega_{j_2,i_2} \supset \cdots \supset \Omega_{j_k,i_k}$$

$$g(\Omega_{j_1,i_1}) \geq g(\Omega_{j_2,i_2}) \geq \cdots \geq g(\Omega_{j_k,i_k})$$

- Inequalities should be reversed for the minimization problem

BnB with NRLS

- **Branching**: RLS tree
- **Bounding**: The bound function is partial likelihood calculated on the subset of states that

$$\mathcal{L}^{\text{Part}}(Z, \theta, \mathcal{S}) = \frac{1}{M} \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{t=1}^{T} \log P_i^{\ell}(a_i^{jt} | x^{jt}; \theta)$$

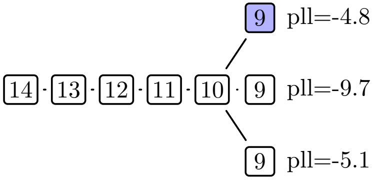
$$\text{s.t. } (x^{jt}, a_i^{jt}) \in \mathcal{S}$$

- Monotonically declines as more data is added
- Equals to the full log-likelihood at the leafs of RLS tree

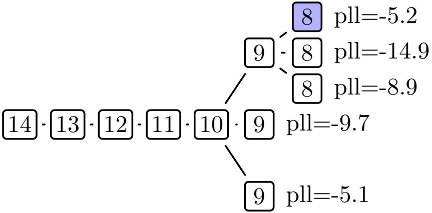
BnB on RLS tree, step 1

$\boxed{14} \cdot \boxed{13} \cdot \boxed{12} \cdot \boxed{11} \cdot \boxed{10}$ Partial loglikelihood = -3.2

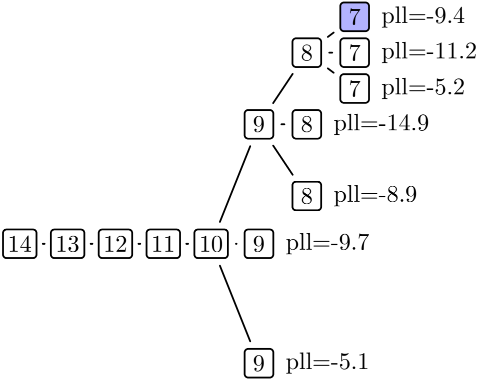
BnB on RLS tree, step 2



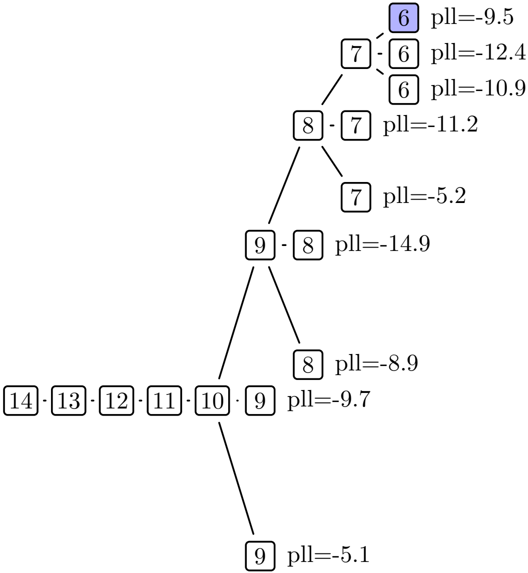
BnB on RLS tree, step 3



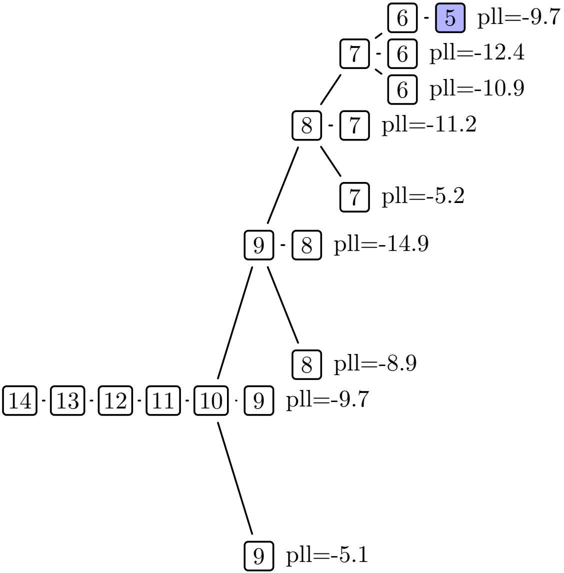
BnB on RLS tree, step 4



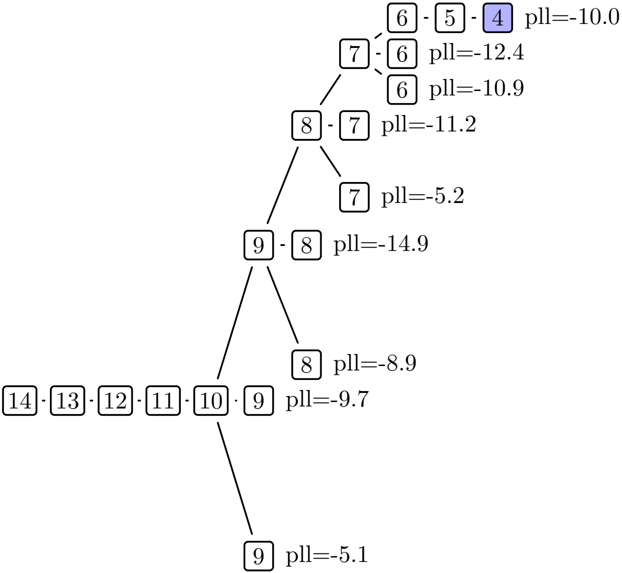
BnB on RLS tree, step 5



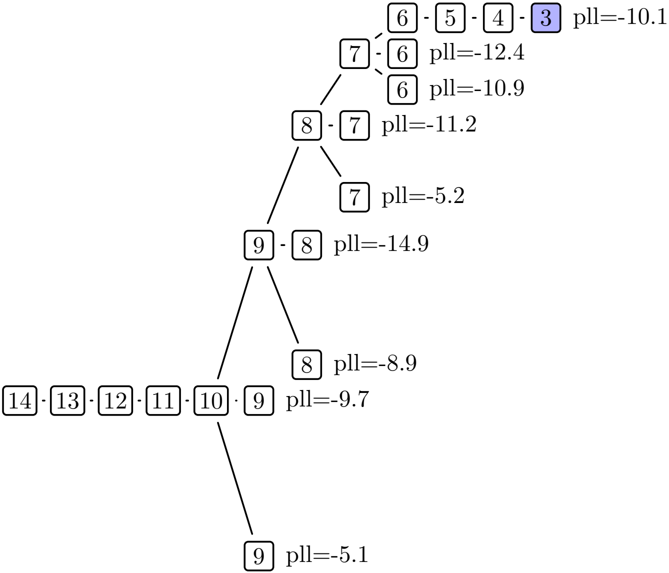
BnB on RLS tree, step 6



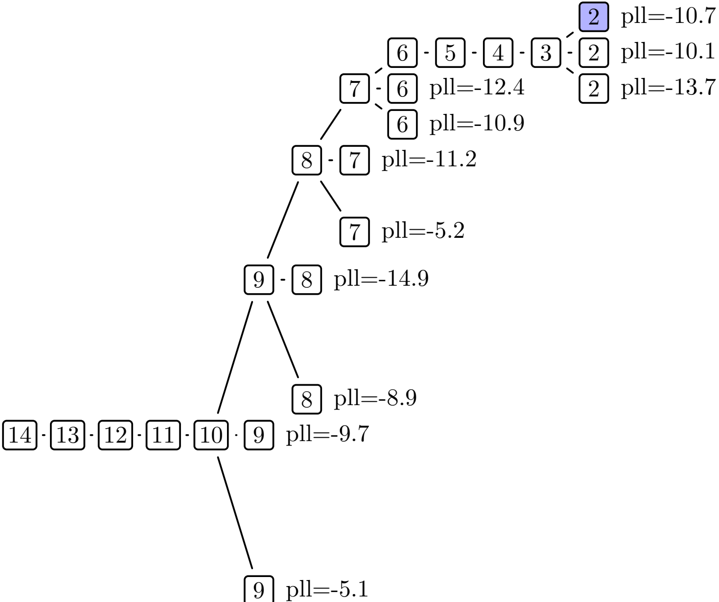
BnB on RLS tree, step 7



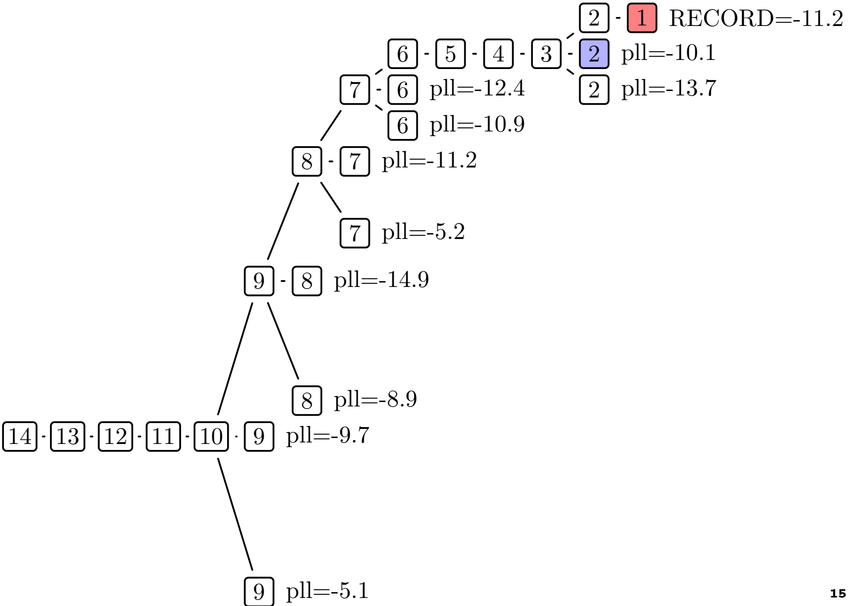
BnB on RLS tree, step 8



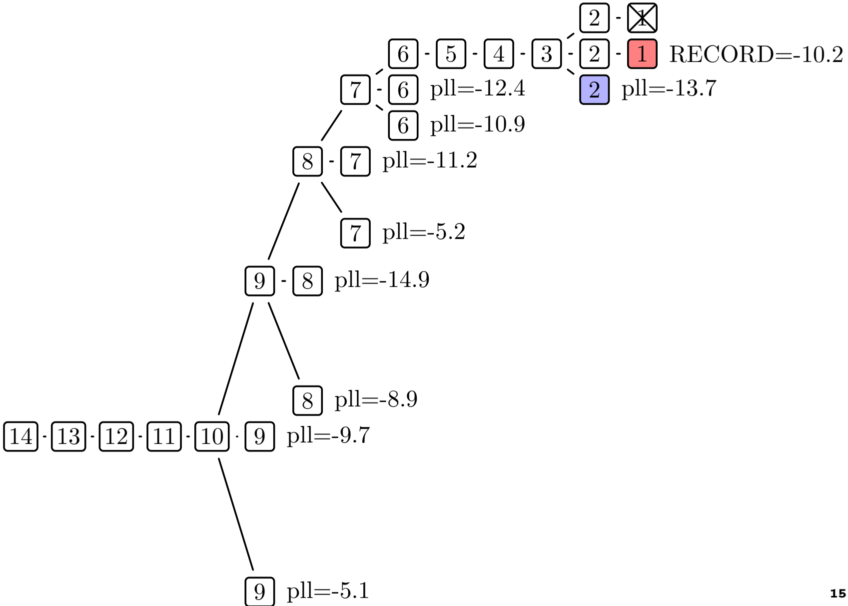
BnB on RLS tree, step 9



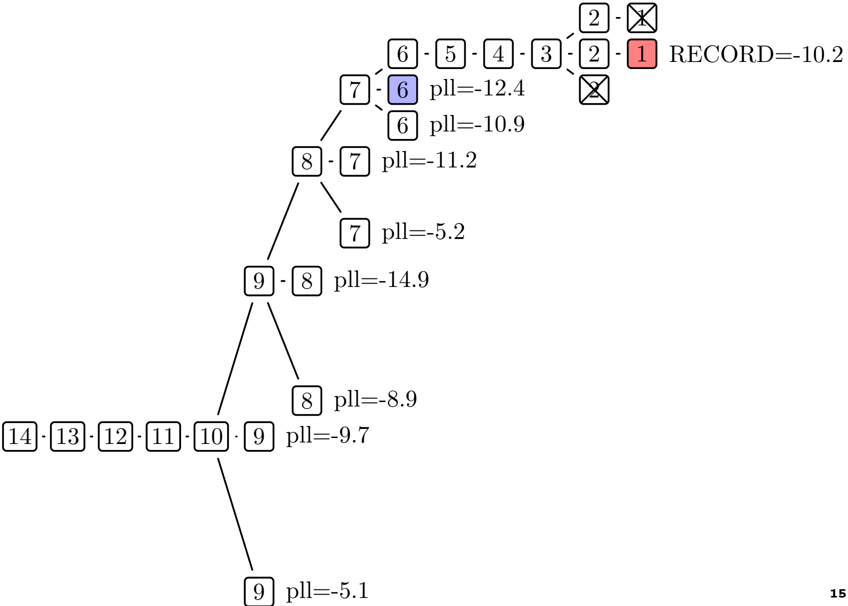
BnB on RLS tree, step 10



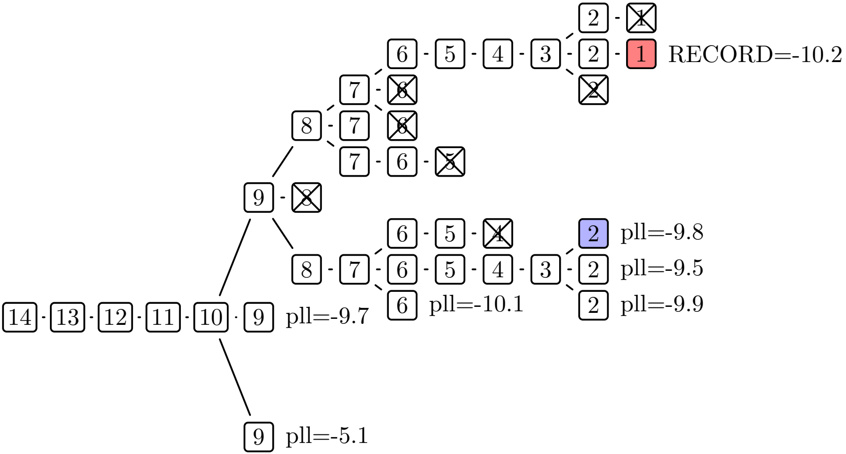
BnB on RLS tree, step 11



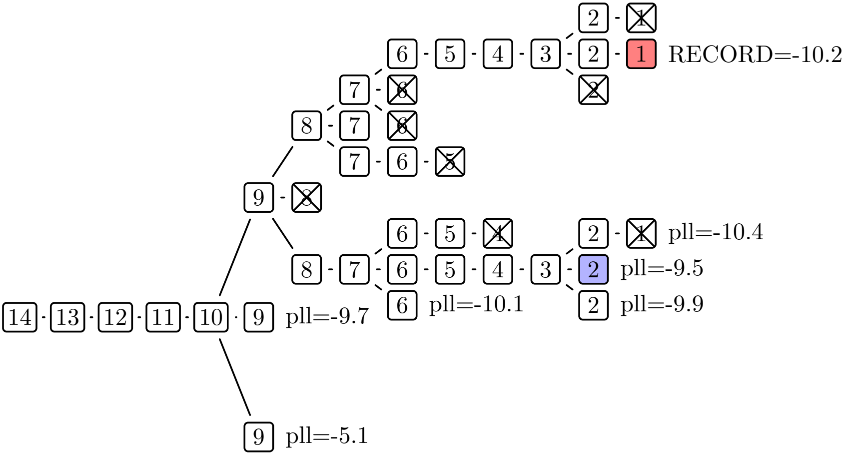
BnB on RLS tree, step 12



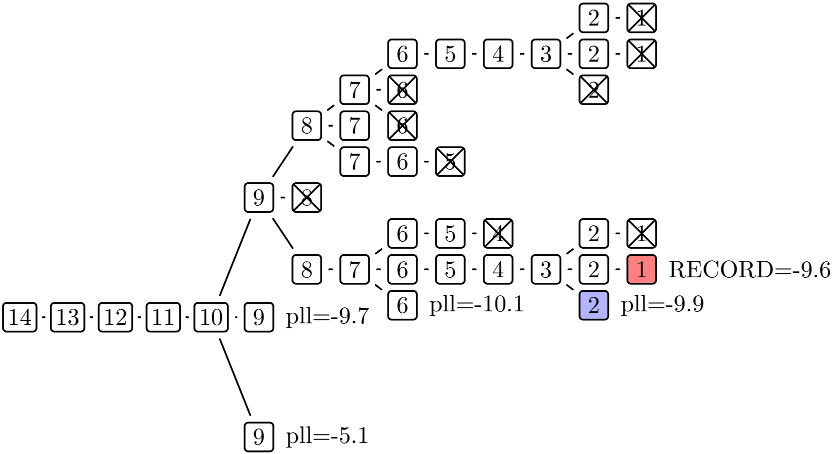
BnB on RLS tree, step 28



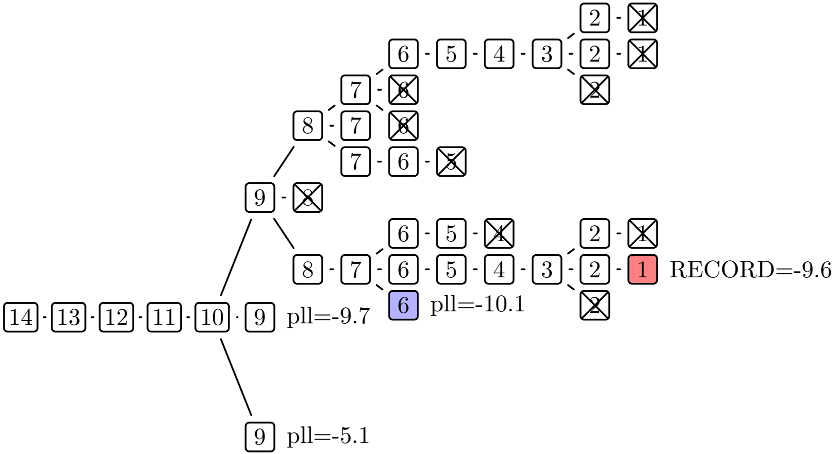
BnB on RLS tree, step 29



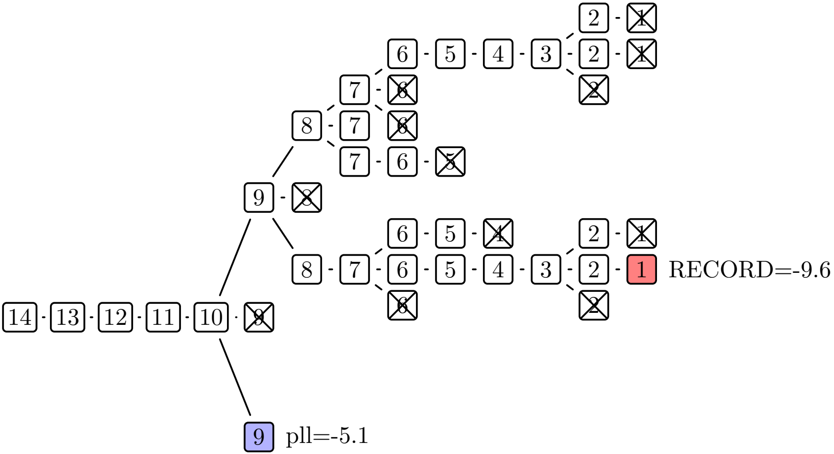
BnB on RLS tree, step 30



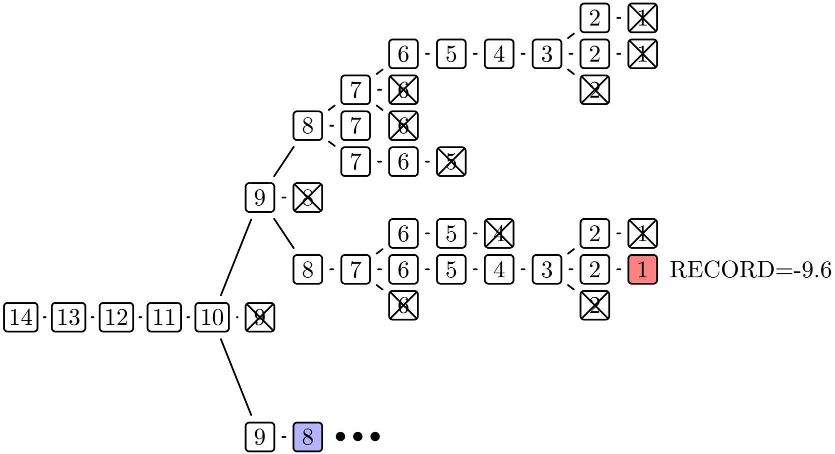
BnB on RLS tree, step 31



BnB on RLS tree, step 33



BnB on RLS tree, step 34



Non-parametric likelihood bounding

- Replace choice probabilities $P_i^k(a|x_\iota;\theta)$ with frequencies n_ι^a/n_ι

$$\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}}) = \sum_{\iota\in\mathcal{S}} \sum_{i=1}^{J} \sum_{a} n_\iota^{a_i} \log(n_\iota^a/n_\iota)$$

- $\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}})$ depends only on the counts from the data!
- Not hard to show algebraically that for any $Z^{\mathcal{S}}$ ($\approx$Gibbs inequality)

$$\mathcal{L}^{\text{non-par}}(Z^{\mathcal{S}}) > L^{\text{part}}(Z^{\mathcal{S}}, \theta, V_\theta^k)$$

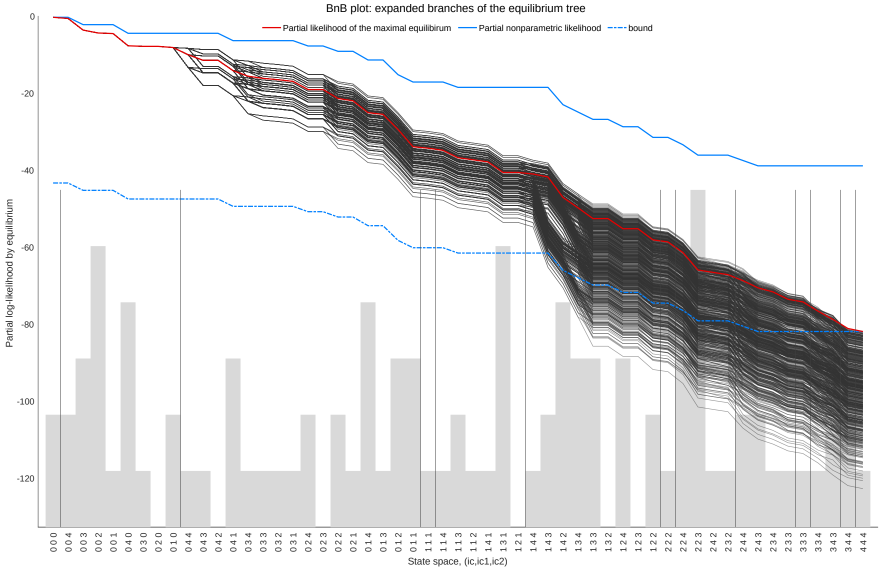
- Therefore partial likelihood can be optimistically extrapolated by empirical likelihood at any step ι of the RLS tree traversal

$$\mathcal{L}^{\text{part}}(Z^{\{S,S-1,\dots,\iota\}}, \theta, V_\theta^k) + \mathcal{L}^{\text{non-par}}(Z^{\{\iota-1,\dots,1\}})$$

- Augmented partial likelihood is much more powerful bound for BnB

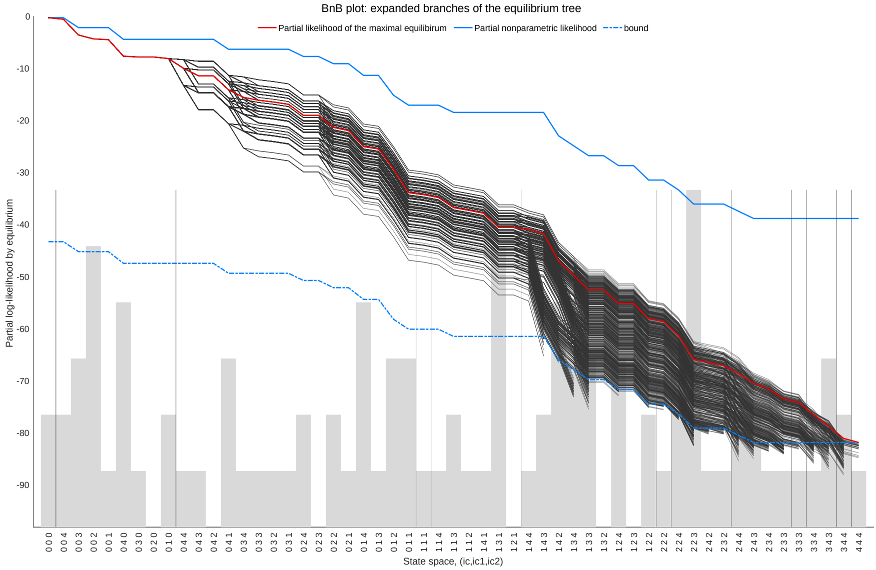
Non-parameteric likelihood bounding

$\iota = S = 14$ (terminal state) on the left, $\iota = 1$ (initial state) on the right



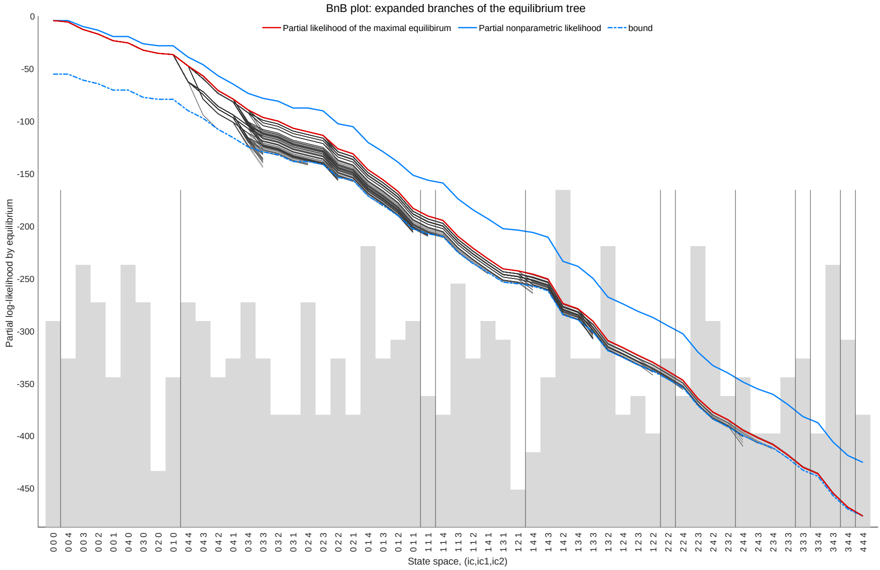
BnB with non-parameteric likelihood bound

Greedy traversal + non-parameteric likelihood bound



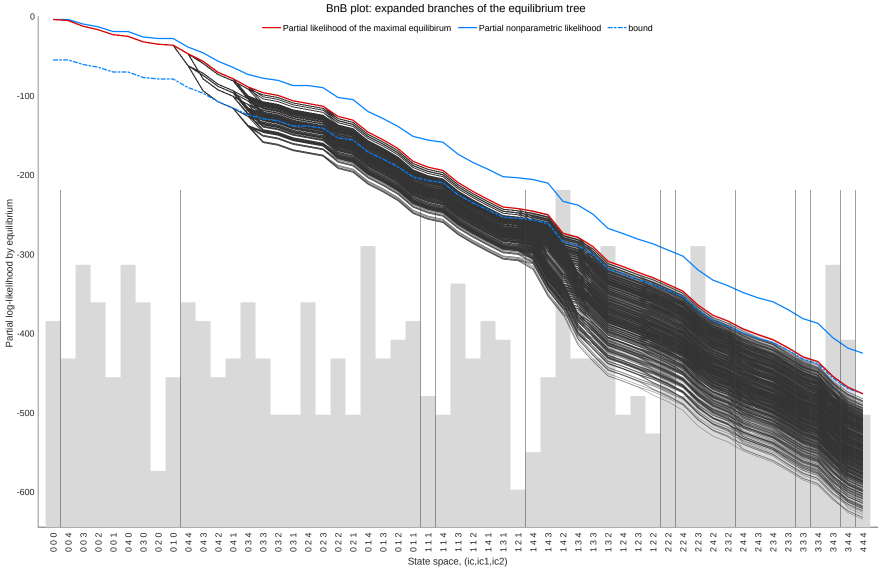
BnB with non-parameteric likelihood bound, larger sample

Non-parametric $\rightarrow$ parametric likelihood as $N \rightarrow \infty$ at true $\theta \Rightarrow$ even less computation



Full enumeration RLS in larger sample

Comparing with the previous slide most of the computation is avoided!



BnB refinement with non-parametric likelihood

- For any amount of data the non-parametric likelihood is greater or equal to the parametric likelihood *algebraically*
- BnB augmented with non-parameteric likelihood bound gives sharper Bounding Rules $\rightarrow$ less computation

- Wih more data as $M \rightarrow \infty$
- Non-parametric log-likelihood converge to the likelihood line
- The width of the band between the blue lines in the plots decreases

  $\rightarrow$ Even sharper Bounding Rules
  $\rightarrow$ Even less computation

MLE for any sample size, but easier to compute with more data!

Monte Carlo simulations

A

Single equilibrium in the model
One equilibrium in the data

B

Multiple equilibria in the model
Same equilibrium played the data

C

Multiple equilibria in the model
Multiple equilibria in the data:

- Long panels, each market plays their own equilibrium
- Groups of markets play the same equilibrium

(*not today*)

Implementation details

- Two-step estimator, NPL and EPL
  - Matlab unconstrained optimizer (with numerical derivatives)
  - CCPs from frequency estimators
  - Max 120 iterations (for NPL and EPL)
- MPEC
  - Matlab constraint optimizer (interior-point) with analytic derivatives
  - MPEC-VP: Constraints on both values and choice probabilities (as in Egesdal, Lai and Su, 2015)
  - MPEC-P: Constraints in terms of choice probabilities + Hotz-Miller inversion (twice less variables)
  - Starting values from two-step estimator
- Estimated parameter k_1
- Sample size: 1000 markets in 5 time periods
- Parameters are chosen to ensure good coverage of the state space and non-degenerate CCPs in all states

Monte Carlo A, run 1: no multiplicity

Number of equilibria at true parameter: 1
Number of equilibria in the data: 1

	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True $k_1 = 3.5$	3.52786	3.49714	3.49488	3.49488	3.49486	3.49488
Bias	0.02786	-0.00286	-0.00512	-0.00512	-0.00514	-0.00512
MCSD	0.10037	0.06522	0.07042	0.07042	0.07078	0.07042
ave log-like	-1.16661	-1.16144	-1.16143	-1.16143	-1.16139	-1.16143
log-likelihood	-5833.07	-5807.21	-5807.16	-5807.16	-5806.95	-5807.16
log-like short	-	-0.050	-0.000	-0.000	-0.000	-0.000
KL divergence	0.03254	0.00021	0.00024	0.00024	0.00024	0.00024
$\|\|P - P0\|\|$	0.11270	0.00469	0.00495	0.00495	0.00500	0.00495
$\|\|\Psi(P) - P\|\|$	0.16185	0.0000	0.0000	0.0000	0.0000	0.0000
$\|\|\Gamma(v) - v\|\|$	0.87095	0.00000	0.00000	0.00000	0.00000	0.00000
Convrged of 100	-	100	100	100	99	100

- Equilibrium conditions satisfied (except 2step)
- Nearly all MLE estimators identical to the last digit
- NPL and EPL estimators approach MLE

Monte Carlo B, run 1: little multiplicity

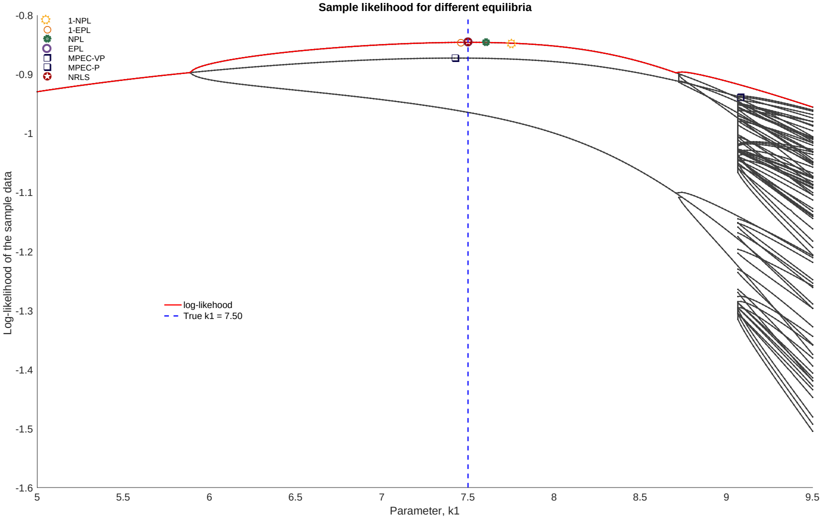
Number of equilibria at true parameter: 3
Number of equilibria in the data: 1
Data generating equilibrium: stable

	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True k1=7.5	7.55163	7.49844	7.49918	7.65318	7.35124	7.49919
Bias	0.05163	-0.00156	-0.00082	0.15318	-0.14876	-0.00081
MCSD	0.17875	0.06062	0.03413	0.99742	0.47136	0.03413
ave log-like	-0.84779	-0.84425	-0.84421	-0.88682	-0.87541	-0.84421
log-likelihood	-21194.86	-21106.33	-21105.13	-22170.40	-21885.37	-21105.13
log-like short	-	-1.206	-0.000	-1062.740	-776.809	-0.000
KL divergence	0.02557	0.00040	0.00013	0.23536	0.16051	0.00013
$\|\|P - P_0\|\|$	0.11085	0.00490	0.00280	0.17466	0.20957	0.00280
$\|\|\Psi(P) - P\|\|$	0.170940	0.000000	0.000000	0.000000	0.000000	0.000000
$\|\|\Gamma(v) - v\|\|$	1.189853	0.000000	0.000000	0.000000	0.000000	0.000001
N runs of 100	100	100	100	98	97	100

- MPEC convergence deteriorates
- Equilibrium conditions are satisfied, but estimators start to converge to *wrong* equilibria
  (as seen from KL divergence from the data generating equilibrium)

Likelihood correspondence

Lines are costructed using symmetric KL-divergence



Monte Carlo B, run 2: little multiplicity, unstable

Number of equilibria at true parameter: 3
Number of equilibria in the data: 1
Data generating equilibrium: unstable

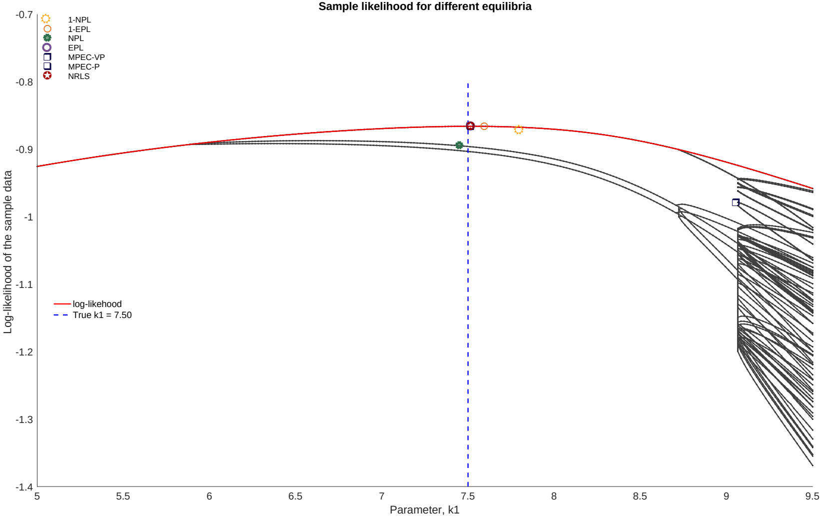
	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True k1=7.5	7.54238	7.39276	7.48044	7.73133	7.63100	7.50176
Bias	0.04238	-0.10724	-0.01956	0.23133	0.13100	0.00176
MCSD	0.17145	0.05608	0.15801	0.72988	0.89874	0.03820
ave log-like	-0.86834	-0.89374	-0.86550	-0.88512	-0.90196	-0.86504
log-likelihood	-21708.60	-22343.58	-21637.54	-22127.91	-22549.06	-21626.12
log-like short	-	-765.242	-11.413	-502.121	-920.643	-0.000
KL divergence	0.02271	0.15996	0.00257	0.11452	0.20182	0.00012
$\|\|P - P_0\|\|$	0.09757	0.20709	0.00619	0.03860	0.02504	0.00307
$\|\|\Psi(P) - P\|\|$	0.160102	0.000000	0.000000	0.000000	0.000000	0.000000
$\|\|\Gamma(v) - v\|\|$	1.126738	0.000000	0.000000	0.000000	0.000000	0.000001
N runs of 100	100	18	100	99	98	100

- NPL estimator fails to converge
- Similar convergence issues for MPEC
- EPL estimator performs well

Aguirregabiria, Marcoux (2021)

Likelihood correspondence

Lines are costructed using symmetric KL-divergence



Monte Carlo B, run 3: discontinuous likelihood

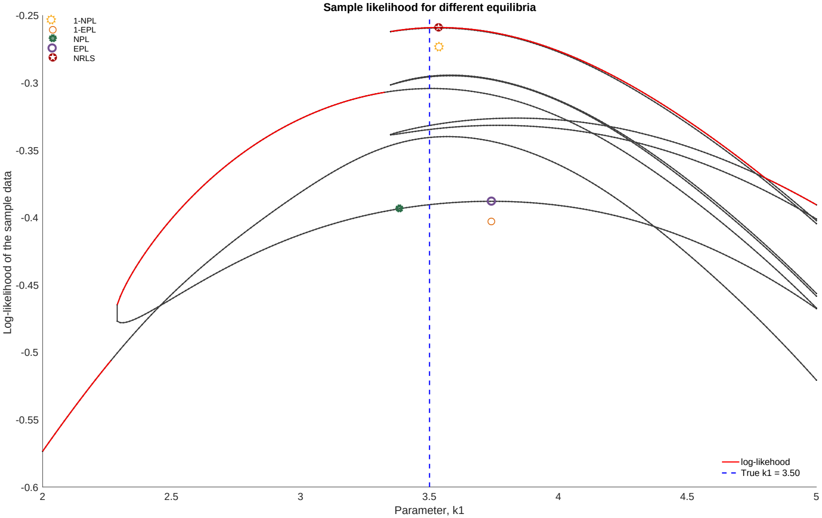
Number of equilibria at true parameter: 9
Number of equilibria in the data: 1
Data generating equilibrium: unstable, near "cliffs"

	2step	NPL	EPL	MPEC-VP	MPEC-P	NRLS
True k1=3.5	3.49739	3.55144	3.64772	3.65943	3.67027	3.50212
Bias	-0.00261	0.05144	0.14772	0.15943	0.17027	0.00212
MCSD	0.13999	0.07133	0.12900	0.12693	0.11583	0.03255
ave log-like	-0.27494	-0.29474	-0.29528	-0.30330	-0.30257	-0.25086
log-likelihood	-1374.721	-1473.695	-1476.425	-1516.503	-1512.847	-1254.320
log-like short	-	-219.375	-222.104	-270.999	-267.523	-0.000
KL divergence	0.01512	0.04889	0.04495	0.04102	0.04078	0.00016
$\|\|P - P_0\|\|$	0.62850	0.86124	0.83062	0.66562	0.65879	0.01610
$\|\|\Psi(P) - P\|\|$	0.763764	0.000000	0.000000	0.000000	0.000000	0.000002
$\|\|\Gamma(v) - v\|\|$	0.852850	0.000000	0.000000	0.000000	0.000000	0.000005
N runs of 100	100	100	100	28	27	100

- Similar convergence issues
- Poor estimates by EPL, NPL and MPEC
  (constraints are satisfied, yet low likelihood and high KL divergence)

Likelihood correspondence

Lines are costructed using symmetric KL-divergence



Monte Carlo B, run 4: massive multiplicity

Number of equilibria at true parameter: 2455
Number of equilibria in the data: 1
Time to enumerate all equilibria (RLS): 10m 39s

	1-NPL	NPL	EPL	NRLS
True k1=3.75	3.70959	3.71272	3.78905	3.74241
Bias	-0.04041	-0.03728	0.03905	-0.00759
MCSD	0.11089	0.06814	0.40716	0.03032
ave log-likelihood	-0.38681557	-0.37348793	-0.45256293	-0.35998461
log-likelihood	-1934.078	-1867.440	-2262.815	-1799.923
log-like shortfall	-	-66.529	-467.607	-0.000
KL divergence	Inf	14.07523	12231.59186	0.32429
$\|\|P - P_0\|\|$	0.82204	0.65580	0.79241	0.07454
$\|\|\Psi(P) - P\|\|$	0.963574	0.000000	0.000000	0.000006
$\|\|\Gamma(v) - v\|\|$	7.020899	0.000000	0.000000	0.000008
N runs of 100	100	18	68	100
CPU time	0.159s	11.262s	4.013s	4.731s

- Severe convergence problems for NPL and EPL
- Poor eqb identification (low likelihood and high KL divergence)
- NRLS has comparable CPU time (much faster than full enumeration)

Monte Carlo C, multiple equilibria in the data

The path forward:

- Assume that the same equilibrium is played in each market over time
- Grouped fixed-effects, groups defined by the equilibria played

1. Joint grouped fixed-effects estimation
   - Estimate the partition of the markets into groups playing different equilibria together with θ
   - For each market compute maximum likelihood over all equilibria and "assign" it to the relevant group (estimation+classification)
   - Computationally very demanding: BnB market-by-market, non-parametric refinement has no bite
2. Two-step grouped fixed-effects estimation
   - Step 1: partition the markets based on some observable characteristics (K-means clustering)
   - Step 2: estimate θ allowing different equilibria in different groups
   - Small additional computational cost!

Bonhomme, Manresa (2015); Bonhomme, Lamadon, Manresa (2022)

Conclusions: Bertrand investments model

- Many types of endogenous coordination is possible in equilibrium
  - Leapfrogging (alternating investments)
  - Preemption (investment by cost leader)
  - Duplicative (simultaneous investments)
- Full rent dissipation and monopoly outcomes are supported as MPE.
- Numerous MPE equilibria and "Folk theorem"-like result
- The equilibria are generally inefficient due to over-investment
  - Duplicative or excessively frequent investments

Conclusions: Solution of dynamic games

- When equilibrium is not unique the computation algorithm inadvertently acts as an equilibrium selection mechanism
- When directionality in the state space is present, state recursion algorithm is preferred to time iterations
- Plethora of Markov perfect equilibria poses new challenges:
  - How firms manage to coordinate on a particular equilibrium?
  - Increased difficulties for empirical applications.
  - Daunting perspectives for identification of equilibrium selection rule from the data.
- Estimation of dynamic games with multiple equilibria
  Nested Recursive Lexicographical Search (NRLS)

Conclusions: NRLS estimator

- Full solution MLE estimator for dynamic games of a particular type, namely directional dynamic games (DDGs)
- Nested loop: outer likelihood max + inner model solver
- Need to maximize over the set of all equilibria $\leftrightarrow$ daunting computational task
- Smart BnB algorithm not to waste time on unlikely MPE
- NRLS is MLE estimator for dynamic games of a particular type, directional dynamic games (DDGs)
  - Fully robust to multiplicity of equilibria
  - Able to identify multiple equilibria in the data