Disequilibrium Play in Tennis

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Do the world's best tennis pros play Nash equilibrium mixed strategies? We answer this question using data on serve-direction choices (to the receiver's left, right, or body) from the Match Charting Project. Using a new approach, we test and reject a key implication of a mixed-strategy Nash equilibrium: that the probability of winning the service game is identical for all possible serve strategies. We calculate best-response serve strategies by dynamic programming (DP) and show that for most elite pro servers, the DP strategy significantly increases their win probability relative to the mixed strategies they actually use.

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I. Introduction

Walker and Wooders (2001) analyzed 40 tennis "point games" from Grand Slam tournaments, focusing on the server's choice of first-serve direction: to the receiver's left or right. They analyzed first serves to the ad (left) and deuce (right) courts separately, with each treated as repeated IID (independent and identically distributed) one-shot simultaneous-move games between the server and receiver. They then concluded that serve-location choices are consistent with a mixed-strategy Nash equilibrium in their hypothesized static game. In particular, the server's chance of winning a point is the same whether the serve is to the left or to the right. Equality of win rates across serve directions has been confirmed in several follow-up studies using additional data. In contrast, our tests typically reject the hypothesis of equal win probabilities across serve directions. We find that most elite professional players—such as Roger Federer, Rafael Nadal, and Novak Djokovic—could significantly increase their chances of winning if they were to systematically exploit these differences.

Our analysis differs from that of Walker and Wooders (2001) by considering three serve directions (left, right, and body) and modeling tennis as a dynamic game. We allow for body serves because tennis pros believe that they are important; see, for example, Rive and Williams (2011). Dynamics are relevant because the server's strategy and probability of winning the service game depend on the score state as well as muscle-memory effects. For instance, the server may be more likely to be successful when serving to the same location as the previous serve. Alternatively, the receiver may have more success receiving a serve hit to the same location as the previous serve. We capture muscle memory via the directions of the two previous first serves and show that it can explain serial correlation in serves even when play is in Nash equilibrium. Previous studies, including Walker and Wooders's, have found serial correlation and interpreted it as evidence against Nash equilibrium.

Accounting for dynamics, a third serve direction, and state-dependent serve-direction probabilities leads to more powerful tests of mixed-strategy play. Our analysis is based on an online database called the Match Charting Project (MCP), run by Sackmann (2013), which crowdsources play-by-play data from professional tennis matches and records all three serve directions used in our analysis. Even after restricting our sample to matches played on hard courts,[1] we end up with roughly 10 times as many serves per server-receiver pair as Walker and Wooders (2001).

However, the main reason why we reject the hypothesis of Nash equilibrium play is a new methodology that models tennis as a dynamic game, in

[1] We do this to eliminate a potential source of heterogeneity that could confound our results, since playing characteristics differ across surfaces. We extend our analysis to grass and clay courts in sec. V.C.

contrast to previous work that treated serves as choices in repeated static games. We model serves as decisions at each subgame of the overall service game between a server and receiver, which ends when one of the players has won at least four points and at least two more points than their opponent. The server chooses the location, speed, and spin of each serve, while the receiver allocates a fixed attention budget to the three serve locations. We test the null hypothesis that observed play in a tennis service game is realization of a Markov perfect equilibrium (MPE), in which the server and receiver's strategies depend only on the muscle memory and score state.[2]

We prove that an MPE exists and is unique in the sense that all subgame perfect equilibria result in the same win probability for the server. Serve strategies are also *completely mixed*; that is, at every state of the service game, the server has a positive probability of choosing any of the three possible serve directions. We define *point outcome probabilities* (POPs) as the equilibrium probabilities that a serve to a given direction is in, as well as the probability that the server wins the rally, given that the serve is in. Both are conditional on muscle memory, score state, and serve direction. The POPs are endogenous objects, since they depend on unobserved choices by the server and receiver. However, in a completely mixed MPE, the POPs can be treated as fixed and invariant to temporary changes in serve strategy. The reason is that a receiver would not be able to detect any deviation in serve strategy from a small number of observations if serve directions are chosen randomly at every stage of the service game.

This fact allows us to estimate the POPs as "reduced-form objects" or "projections" that reflect the unobserved strategic decisions of both players in terms of observable outcomes (i.e., faults and points following rallies), conditional on the choice of serve direction, which we do observe. This converts the dynamic game to a single-agent dynamic programming (DP) problem, since the POPs constitute the payoff-relevant beliefs that the server needs to evaluate different serve strategies. According to the *one-shot deviation principle* of game theory, a necessary condition for a serve strategy to be an MPE strategy is that there is no deviation at any stage of the dynamic game that strictly increases the server's expected win probability. In most games, this means that any deviation in serve strategy strictly reduces the probability of winning. However, in a completely mixed MPE, a much stronger restriction holds: all temporary deviations in serve strategy have the same win probability. This is an extremely strong implication of a completely mixed MPE that results in infinitely many testable restrictions, which we exploit to develop powerful new tests of equilibrium play.

[2] While the underlying characteristics of the game do not directly depend on the current score, it is the case that with muscle-memory effects, strategies generally depend on the score state.

In particular, the service-game win probability must be the same for all serve directions in all states of the service game. We test these strong implications of a mixed-strategy equilibrium by estimating the POPs and the actual serve strategy used in the service game. Since our model has 324 muscle-memory/score states and three serve directions, a fully unrestricted estimator of the serve strategy and the POPs would require 4,536 parameters for each server-receiver pair—far too many to estimate, given the size of our dataset. In section IV, under a testable assumption that actual serve strategies and POPs are stationary and Markovian (but not necessarily MPE strategies), we estimate flexible reduced-form parametric models of serve strategies and the POPs that include the unrestricted specification as a special case. We use the Akaike information criterion (AIC) to select a preferred specification with 44 parameters (12 for the server's strategy and 32 for the POPs) that balances the desire for flexibility against the danger of overfitting.

Rather than separately testing for equal win probabilities across serve locations by aggregating data across individual points (treating first serves as independent, static games, as Walker and Wooders 2001 did), we introduce a new, more powerful "omnibus Wald test" of the hypothesis of equal win probabilities that must hold across all possible states of the service game simultaneously. We also derive Wald tests of the other key restriction of a completely mixed MPE: that win probabilities are the same for all possible deviation strategies. These tests strongly reject the equal win probabilities for all serve directions implied by completely mixed MPE play for the majority of the elite pros we analyzed, including the very top players, such as Federer, Nadal, and Djokovic. The tests are based on recursive calculations of the conditional probability of winning the entire service game for any given serve strategy in all game states and serve directions. Our tests allow for serial correlation in serve directions and the POPs due to muscle-memory effects, thus allowing for state dependence in serve behavior and outcomes that cannot be captured in static approaches to testing for equal win probabilities. Not only does muscle memory explain serial correlation in serve directions: accounting for it is the key to the strong rejections of equal win probabilities, even under tests similar to the static testing methodology that Walker and Wooders employed that assume that servers maximize the probability of winning each point rather than the overall service game.

To quantify the potential deviation gains from systematically exploiting the unequal win probabilities that our tests reveal, we use DP to calculate best-response serve strategies for individual server-receiver pairs, using the estimated POPs to provide outcome probabilities for each point, given the choice of serve direction. For all the elite pros we analyze, the DP strategy significantly increases win probabilities relative to the mixed serve strategies implied by our reduced-form estimates of their serve behavior. Adopting the DP serve strategy would improve Nadal's probability of winning a service game against Djokovic from 71% (his current win rate) to 91.5% and Djokovic's chance of winning against Nadal from 83% to 93.7%.[3]

Thus, the play of elite tennis pros does not constitute an MPE: our empirical analysis reveals many small advantageous one-shot deviations (i.e., changes in serve direction at individual points), and the DP strategy takes maximal advantage of all of them, resulting in much more significant deviation gains at the level of the entire service game.[4] The reason why we find much larger deviation gains by modeling tennis as a dynamic game rather than as a sequence of repeated static games is an implication of the tennis scoring system we call the "magnification effect."[5] For example, if we assume that each point is an IID Bernoulli draw with a 50% chance of a win for the server, the server will also win the service game with 50% probability, since the rules of tennis imply that points evolve as a random walk with absorbing states of win and loss. However, if a change in serve strategy results in a small increase in winning each point, say an increase to 55% (a 10% increase), then tennis scores evolve as a random walk with drift. This causes the probability of winning the service game to increase to 62.3%, a nearly 25% increase.

Though the majority of our analysis focuses on elite male pros playing on hard courts, we show that our findings extend to elite women and other less elite pros, as well as to play on clay and grass courts. In general, we find that the magnitude of deviation gains from adopting the DP best-response serve strategy is a declining function of "relative ability," as proxied by the server's probability of winning the service game against specific opponents. We do not advise tennis pros to adopt our best-response serve strategies, since they are pure strategies that the receiver would eventually learn and adapt to. However, we also calculate "robust" mixed serve strategies that account for estimation error and uncertainty about the POPs and the strategy of the receiver. The robust strategies also significantly increase servers' win probabilities but are more difficult for a receiver to detect and adapt to.

[3] Traditional game theory has little to say about "mental ability," since all players are equally rational and intelligent. In the context of our model, these increases in win rates result from a better mental approach to the game. This is because the estimates assume that the receiver's strategy and other aspects of the server's play are unchanged under the DP serve strategy. Therefore, relative physical ability is held constant.

[4] We find significant improvements from the DP best-response serve strategies for all 94 server-receiver-surface combinations for which we have sufficient data to precisely estimate our model. See sec. V.C for details.

[5] See sec. 4 of the appendix (the appendix is available online) for further discussion of this "magnification effect" that causes our omnibus Wald test of equal win probabilities over the different possible serve directions to have far greater power than Walker and Wooders's (2001) tests.

To gain insight into the reasons for suboptimal serve choices, we estimate three structural models of the serve-direction choices involving increasing degrees of farsightedness. These models allow for persistent shocks to server performance (muscle memory) as well as IID shocks that reflect unobserved transitory factors that affect servers' choices. In the *fully dynamic* model, the server uses backward induction to maximize the probability of winning the entire service game, which is effectively an infinite-horizon problem because service games must be won by at least two points. In the *point-myopic* model, the server solves a two-period DP problem to maximize the probability of winning the current point. Here, the server accounts for the option value of a second serve but ignores the effect of current decisions on the future state of the service game. Finally, in the *serve-myopic* model, the server maximizes the probability of winning on each serve, a completely static problem that ignores even the option value of the second serve.

The serve-myopic model is typically rejected because of significant differences in observed serve directions between first and second serves that result from the option value of the second serve. The fully dynamic model is also nearly always rejected because it implies subjective POPs that are too "pessimistic," compared to our unrestricted estimate of the actual objective POPs. In most cases, the best-fitting model is the point-myopic model. It implies mixed serve strategies that are close to the ones players actually use while constituting a nearly optimal response to the "subjective POPs," in the sense that additional increases in win probability from adopting a full DP serve strategy are negligible. The suboptimality in serve behavior we identify is primarily driven by incorrect server beliefs, that is, a lack of rational expectations of the server and receiver's strengths and weaknesses as captured by the POPs, rather than players' inability to optimize.

We address concerns that we have, and therefore use, only estimates of the POPs, rather than the true POPs, which can result in spurious, upward-biased estimates of the deviation gains. To account for this, we derive an approximate probability distribution for the true POPs based on the observed data. We calculate win probabilities for the fully dynamic, point-myopic, and serve-myopic serve strategies, using a random sample of POPs drawn from the asymptotic distribution centered on the point estimates of the POPs. This robustness exercise confirms our core finding: the fully dynamic and point-myopic strategies based on "rational POPs" have significantly higher win probabilities (in the sense of first-order stochastic dominance) than those implied by the mixed serve strategies the elite pros actually use.

The paper is organized as follows. In section II, we briefly review previous work on testing for minimax play in tennis. Section III introduces our dynamic models of tennis serve behavior and the relevant implications from game theory for equilibrium play that we test empirically in this paper. In section IV, we summarize the key findings from our reduced-form empirical analysis of the MCP database, including our key finding: the frequent rejection of the hypothesis of equal win probabilities for all serve directions. In section V, we present estimation results for the three structural models of tennis serve behavior discussed above, and we calculate the deviation gains from using unrestricted estimates of the "objective POPs" to compute optimal serve strategies. Section VI concludes with further discussion/speculation as to why many elite tennis pros appear to fail to adopt optimal serve strategies, given the strong incentives to do so.

II. Previous Literature

The first empirical analysis of tennis using statistical methods that we are aware of is by George (1973), who analyzed the decision of whether the serve should be strong (i.e., fast and more difficult to return, but with a higher probability of faulting) or weak (i.e., slow and easier to return, but with a lower probability of faulting). The first analysis of the tennis service game using DP that we are aware of is by Norman (1985), who used it to determine "whether to serve fast or slow on either or both serves at each stage in a game, and a simple policy is found" (75).

We already noted the seminal work of Walker and Wooders (2001), who focused on first serves modeled as independent static games and were unable to reject the hypothesis of equal win probabilities for serving left or right. They also found negative serial correlation in serve directions across individual points in tennis, which they interpreted as evidence against equilibrium play. With a larger dataset, Hsu, Huang, and Tang (2007) confirmed Walker and Wooders's conclusions regarding equal win probabilities across observed serve directions, but they did not find serial correlation. Wiles (2006) showed that serial correlation may not necessarily be evidence of disequilibrium play because of the presence of a "timing variable," which is analogous to our muscle-memory effects.

In addition, Walker, Wooders, and Amir (2011) showed that if a monotonicity condition holds—namely, if it is always better to win the current point than lose it—then the strategy that maximizes the probability of winning each point also maximizes the probability of winning the service game. This could explain why the "point-myopic" serve behavior we find is not necessarily suboptimal, as we discuss in section III.E. Also, Lasso de la Vega and Volij (2020) proved that MPEs exist in a class of recursive zero-sum games that include our model without muscle-memory effects. Most recently, Gauriot, Page, Wooders (2023), using data from 3,000 matches and nearly 500,000 serves, confirmed Walker and Wooders's conclusions and noted that "the behavior in the field of more highly ranked (i.e., better) players conforms more closely to theory" (981). But unlike Hsu, Huang, and Tang (2007), they find evidence against serial independence of serve directions as a result of having a large dataset that includes non-elite pros.

Other related papers include Klaassen and Magnus (2001, 2009). Klaassen and Magnus (2001) tested whether successive points in tennis are IID binary random variables, using 481 Wimbledon matches containing nearly 90,000 points. They rejected the IID hypothesis, but they found that "deviations from iid are small, however, and hence the iid hypothesis will still provide a good approximation in many cases" (Klaassen and Magnus 2001, 500). Klaassen and Magnus (2009) abstracted from serve direction and focused on the trade-off between making a serve hard to return and faulting on the serve, considering both the first and second serves of a point. They rejected the hypothesis that servers optimally solve this trade-off, but found that "the estimated inefficiencies are not large" (72). In section VI, we also discuss empirical evidence for disequilibrium play in other sports, including soccer, football, and baseball. The findings are mixed: there is strong support for minimax play in penalty kicks in soccer but strong evidence against equilibrium play in first-down decisions in football and pitch locations in baseball.

III. Modeling Tennis as a Dynamic Game

Tennis is two-player game between a server and a receiver played in tournaments composed of matches. A match consists of a sequence of sets.[6] A set, in turn, is a sequence of service games in which one of the two players is the server. The server of the first game is chosen by a flip of a coin, and the identity of the server alternates in each game thereafter. Winning a set typically requires winning six games with a lead of at least two games.[7] Each service game consists of a sequence of subgames that are called points. A point consists of a first serve, plus an option for a second serve after a "faulted," or missed, first serve.[8] First serves alternate between the right (deuce) and left (ad) sides of the court. The service game ends when one of the players wins at least four points in total and at least two more points than their opponent.

A. *Dynamic Theory of the Service Game*

We use the scalar x to track both the cumulative points scored by each player in the current service game and whether or not the server is attempting a

[6] Depending on the tournament, a player must win two of three sets or three of five sets to win the match.

[7] Alternatively, if the score is tied at 6-all, the set is decided by a "tiebreak," in which the winner is the first to score seven points and be ahead by at least two points.

[8] If a serve touches the net and lands in the field of play (a "let"), then the serve (first or second) is redone. Since our data do not record lets, we do not include them in our model.

first or second serve. Figure 1 is a directed graph of all the transitions for the point-state variable x within a service game. The circular nodes indicate first serves, whereas the square nodes indicate second serves. The game starts in state $x = 1$, which corresponds to a first serve at the tennis

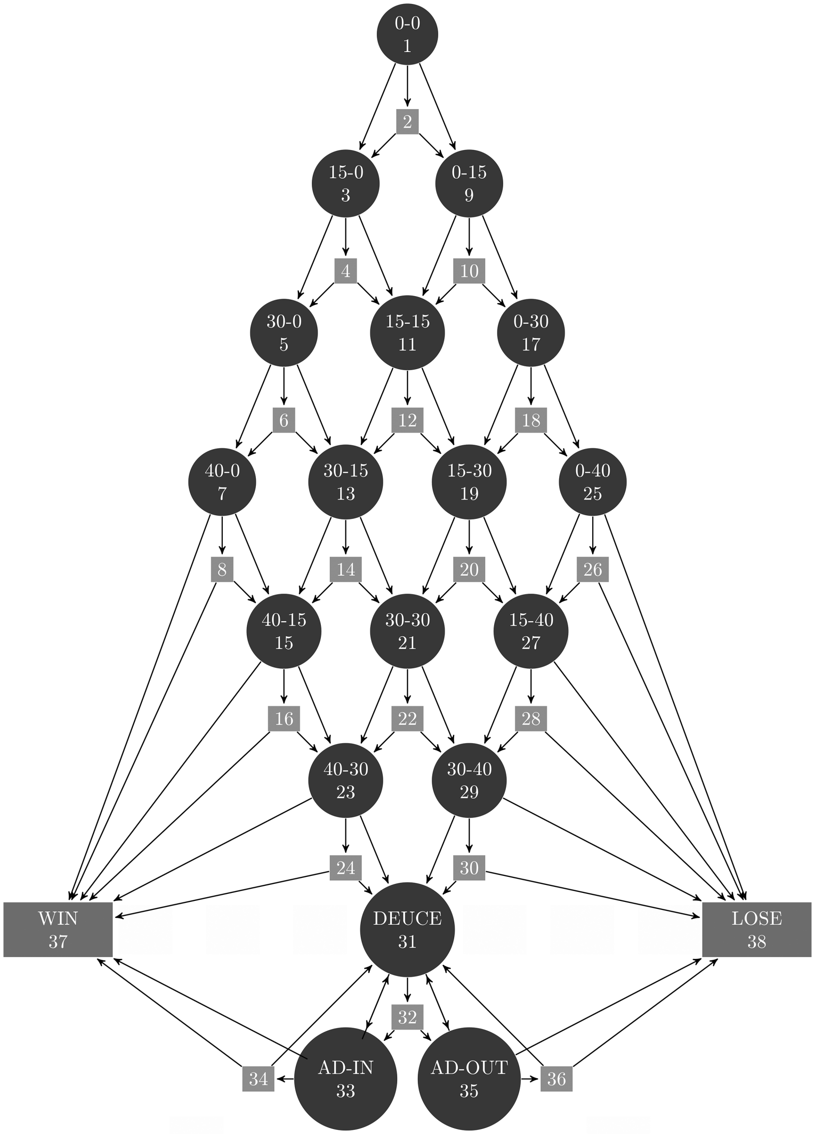

Fig. 1.—Score states and transitions in the service game.

score 0–0. If the server wins the point on that first serve, the point state transits to $x = 3$, corresponding to a first serve at the score 15–0. If the server faults the first serve, the state transits to $x = 2$, which is a second serve at 0–0, and so forth. There are three possible transitions at every first-serve node, two possible transitions at every second-serve node, and two absorbing states (i.e., terminal nodes whose arrows point only in): the server wins ($x = 37$) or loses ($x = 38$) the game.

The arrows connecting most nodes are unidirectional, leading to higher states x. But state $x = 31$ (deuce) is connected by a bidirectional arrow to both $x = 33$ and $x = 35$. This follows from the fact that when the players are tied at 40–40 (i.e., deuce), one of the players must win by two points to win the game. Collectively, we refer to states $31 - 38$ as the *deuce endgame*.[9]

Given these scoring rules, the probability of winning an individual point is generally not the same as the probability of winning the service game, as illustrated in figure 2. It plots the service-game win probability $g(p)$ as a function of the point win probability p under the assumption that each point of tennis is an IID Bernoulli draw with probability p of success. Although we relax the assumption that plays at different points are independent draws in our model below, the IID Bernoulli assumption implies that the point state in tennis evolves as a random walk with drift, with absorbing states $x = 37$ (win for the server) and $x = 38$ (loss for the server). The game win probability $g(p)$ equals p at $p = 0.5$, as we noted in the introduction. However, any changes in serve strategy that increase the probability of winning each point have a magnified effect on the probability of winning the game. Near $p = 0.5$, the slope of the game win probability $g(p)$ is approximately 2.5, so each 10% increase in the point win probability increases the game win probability by 25%. The magnification effect shows how small, hard-to-detect deviation gains at each point of tennis cumulate into much bigger and easier-to-detect deviation gains in the overall service game, a feature we exploit to derive more powerful tests of Nash play.

At each tennis serve, the server chooses the serve type $t = (s, d)$, where $d \in \{\mathrm{l}, \mathrm{r}, \mathrm{b}\}$ indicates the direction: to the receiver's left (l) or right (r) or directly into the receiver's body (b).[10] Moreover, $s \in \mathcal{S} \subset \mathbb{R}^2$ indicates the speed and spin of the serve ($\mathcal{S}$ is nonempty, closed, and bounded). The receiver anticipates the direction choice of the server. Anticipation

[9] Note that states 23 and 24 (29 and 30) are strategically equivalent to 33 and 34 (35 and 36): the transitions to future states in the game including winning or losing are identical.

[10] We follow the literature in assuming that servers "choose" a location, and we believe that it is a reasonable fit for the players we analyze. After all, these location categories are broad, and our servers are the "best of the best." Tea and Swartz (2023) group serves into our three categories on the basis of "heat maps" of the directions of tens of thousands of men and women's serves at the Grand Slam tournament Roland-Garros in 2019 and 2020.

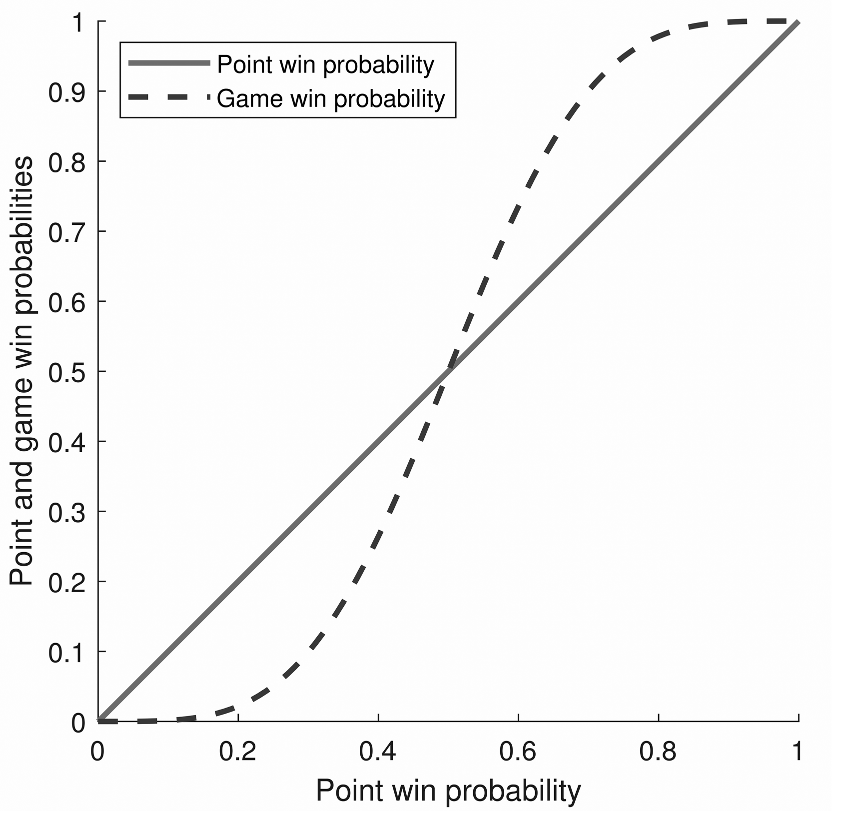

FIG. 2.—Probabilities of winning the point versus the service game.

includes observable (e.g., where to stand) and unobservable choices. We model anticipation with an attention vector $(a^{\mathrm{l}}, a^{\mathrm{r}}, a^{\mathrm{b}}) \geq 0$, where a^{d} denotes the attention the receiver devotes to serve location d. We normalize the attention budget $a^{\mathrm{l}} + a^{\mathrm{r}} + a^{\mathrm{b}} = 1$. We assume throughout that the serve-direction choice weakly follows the choice of a. This captures both the case in which a is a pure location choice, chosen strictly before the server chooses a direction, and the case in which a represents a simultaneous pure mental choice of anticipation.[11]

The probability ℓ that a serve "lands in" (i.e., is not a fault) depends on the court $(c \in \{0, 1\})$[12] and serve type t, while the probability ω that the

[11] All results extend to a model in which the receiver first chooses a subset of the unit triangle and then chooses a specific element of this subset, simultaneous to the server choosing t. This allows for the realistic case in which the physical location of the receiver on the court constrains, but does not fully determine, the attention vector.

[12] Recall that first serves alternate between the deuce (0) and ad (1) courts.

server wins the subsequent rally (conditional on serving in) depends on the serve type t, court, and attention vector a. We also assume that these probabilities can be affected by "muscle memory" m, which we encode as the directions of the previous two first serves. Thus, $m = (d_1, d_2)$, where d_1 is the direction of the previous first serve and d_2 is the direction chosen two first serves ago. We track the previous two first serves because of the alternation of serves between ad and deuce courts and the possibility that muscle memory may be more affected by the last serve to the same court rather than by the last serve, which is to a different court. We initialize muscle memory to null $m = (\varnothing, \varnothing)$ at the start of the service game, and after any first serve, we update muscle memory from $m = (d_1, d_2)$ to $m' = f(m, d) = (d, d_1)$, reflecting the direction of the current first serve. We assume that muscle memory is updated only after first-serve states. This still allows m to capture muscle-memory effects of the faulted first serve on the subsequent second serve, as well as allowing first-serve directions to depend on the direction of the previous first serve to the same court.

We assume that the probabilities $\ell(m, d, c, s)$ and $\omega(m, d, c, s, a)$ are continuous in (s, a), satisfy $\ell\omega \in [\underline{w}, \bar{w}]$ for some $0 < \underline{w} < \bar{w} < 1$), and are stationary; namely:

Assumption 1 (Stationarity I). The functions ℓ and ω may vary across server-receiver pairs but do not vary over time (independent of m and x) or across service games.

We assume that each player's objective is to win the service game, and so we normalize the winning payoff to 1 and the losing payoff to 0. Since $\ell\omega$ is strictly interior, the game will almost surely end in a finite number of serves. But for completeness' sake, we assume that each player earns payoff 1/2 if the game never ends.

Let (σ_S, σ_R) denote the server and receiver's strategies (perhaps mixed and arbitrarily history dependent) in the service game. And let $\mathcal{W}(x, m)$ be the set of probabilities that the server wins the game starting in state (x, m) induced by some pair of (not necessarily Markovian) subgame perfect equilibrium (SPE) strategies (σ_S^*, σ_R^*) for the server and receiver. Section 1 of the appendix proves theorem 1:

Theorem 1. All subgames have a unique value (i.e., $\mathcal{W}(x, m)$ is a singleton), and there exists an MPE in which strategies depend only on the current state (x, m).

Our empirical approach is valid in any MPE in which all strategies depend only on (x, m), and theorem 1 guarantees that there exists an equilibrium with this property. But if there are multiple MPEs, then one can construct nonstationary equilibria by making history-dependent selections from the set of MPEs. In fact, our empirical approach does not rely on Markovian choices of serve direction, meaning that it is sufficient for the server's (speed, spin) strategy and the receiver's attention strategy to be Markovian in (x, m). The next assumption guarantees that this is true in any SPE; the proof of theorem 2 is also in section 1 of the appendix.

ASSUMPTION 2. The win chance ω and the chance of not faulting ℓ obey three conditions: (i) ω is strictly convex in attention a, (ii) $\ell\omega$ is strictly concave in (speed, spin) s, and (iii) $\ell(\omega - 1)$ is concave in s.

THEOREM 2. If assumption 2 holds, then every SPE has the same attention strategy and the same (speed, spin) strategy, and each of these strategies is Markovian in the current state (x, m).

B. *Serve-Direction Strategies from the Induced Dynamic Program*

Our empirical analysis uses MCP data, which do not record the serve speed or spin or the location of the receiver. To overcome this shortcoming, we use theorem 1 to project any MPE into the induced DP problem that the server faces when choosing serve directions to maximize the chances of winning the service game. To do so, let $\rho(s|x, m)$ denote a Markov mixed strategy over the speed-and-spin vector $s \in \mathcal{S}$ for the server, and let $\alpha(a|x, m)$ denote a Markov mixed strategy over attention for the receiver.

DEFINITION 1. Given any MPE (ρ^*, α^*), the POPs Π are

$$\pi(\text{in}|x, m, d) \equiv \int \ell(m, d, c(x), s)\, d\rho^*(s|x, m),$$

$$\pi(\text{win}|x, m, d) \equiv \int\int \omega(m, d, c(x), s, a)\, d\rho^*(s|x, m)\, d\alpha^*(a|x, m).$$

Note that the mixing probabilities (ρ^*, α^*) will generally depend on the state of the game (x, m), so the POPs will depend on (x, m) even if the underlying conditional probabilities ℓ and ω do not. Given any MPE strategies (ρ^*, α^*), the probabilities π define a single agent "game against nature," that is, a dynamic optimization problem in which the server chooses a serve direction at each node in figure 1 in order to maximize the probability of winning the service game. Figure 3 illustrates the extensive form of the point subgame, namely, the subset of the larger directed graph starting at every odd point state x. In the point subgame, the server chooses a serve direction for the first serve d_1, and in the event of a fault, the direction of a second serve d_2. The point subgame ends with the server winning or losing a point at each square node.

Building on Norman (1985), we describe the server's DP problem, given π. Let $W(x, m)$ denote the server's maximal conditional win probability in state (x, m). Let $W(x, m, d)$ be the conditional win probability for the server

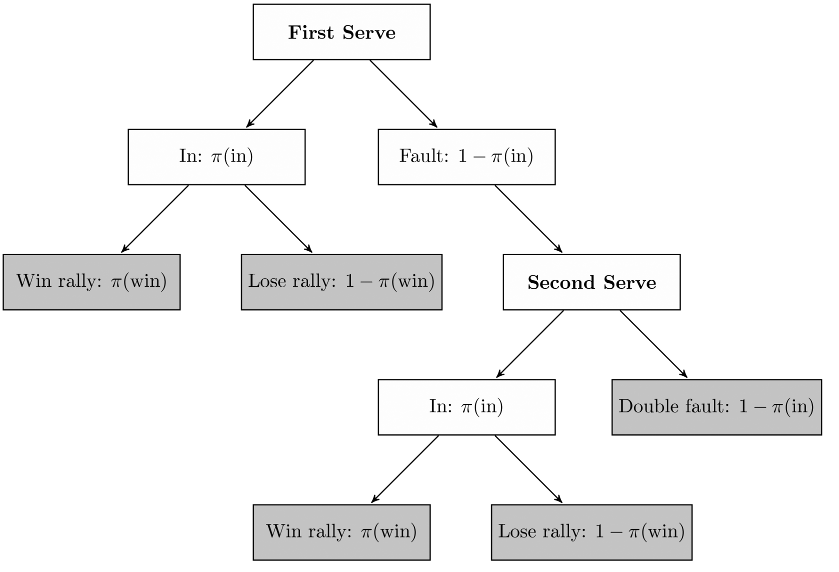

FIG. 3.—Details of the point subgame of tennis.

under the assumption that he serves to direction d on the current serve and behaves optimally on all following serves. Finally, let $x^+(x)$ and $x^-(x)$ denote the successor state in the event that the server wins or loses the point on the current serve, respectively.

The optimal serve strategy can be calculated recursively with the Bellman equation given by

$$W(x, m) = \max_{d\in\{l,b,r\}} W(x, m, d). \tag{1}$$

$$\begin{aligned} W(x, m, d) = {} & \pi(\text{in}|x, m, d)[\pi(\text{win}|x, m, d)W(x^+(x), m') \\ & + (1 - \pi(\text{win}|x, m, d))W(x^-(x), m')] \\ & + (1 - \pi(\text{in}|x, m, d))W(x + 1, m') \end{aligned} \tag{2}$$

when x is a first-serve state (i.e., x is one of the odd-numbered circular nodes in fig. 1), and

$$\begin{aligned} W(x, m, d) = {} & \pi(\text{in}|x, m, d)[\pi(\text{win}|x, m, d)W(x^+(x), m) \\ & + (1 - \pi(\text{win}|x, m, d))W(x^-(x), m)] \\ & + [1 - \pi(\text{in}|x, m, d)]W(x^-(x), m) \end{aligned} \tag{3}$$

when x is a second-serve state (i.e., x is one of the even-numbered square nodes in fig. 1). The optimal serve strategy $D^*(x, m)$ is the set of serve directions that maximize the win probability

$$D^*(x, m) = \underset{d \in \{\mathrm{l,b,r}\}}{\operatorname{argmax}} W(x, m, d). \tag{4}$$

A necessary condition for an MPE serve strategy to be a mixed strategy is that $D^*(x, m)$ contains more than one serve direction. A completely mixed MPE serve strategy requires equality of the three win probabilities $\{W(x, m, \mathrm{l}), W(x, m, \mathrm{b}), W(x, m, \mathrm{r})\}$ in all states (x, m). Since the win probability does not depend on the serve direction in any state (x, m) in a completely mixed MPE, this immediately implies the very strong *strategy-independence result*, which is that in equilibrium, any deviation in serve strategy implies the same win probability $\mathcal{W}(x, m)$ in all states (x, m). In section IV, we use this strong implication of completely mixed MPE play to construct powerful tests of equilibrium play.

Tennis can be viewed as an example of a *directional dynamic game*, defined by Iskhakov, Rust, and Schjerning (2016), with the exception of the deuce endgame, where directionality is not present. While most service games are reasonably short in practice (fewer than 10 points), there is no fixed upper bound on the duration of the deuce endgame, the subgame starting at $x = 31$.[13] As a result, tennis must be analyzed as an infinite-horizon dynamic game, starting with the deuce endgame, which is a fully recursive subgame where win probabilities are determined by solving the Bellman equation simultaneously as the unique fixed point $W = \Gamma(W)$. After solving the deuce endgame, we use state recursion to solve the rest of the game by backward induction across the remaining directionally ordered states $x < 31$.[14]

C. *Calculating Win Probabilities for Stationary Serve Strategies*

It is sufficient for our empirical analysis that the unobserved elements of choice (speed, spin, and receiver attention) are Markovian in (x, m), but this condition is not necessary. Instead, we can make the following assumption directly on the induced probabilities:

[13] The longest deuce endgame that we are aware of was between Anthony Fawcett and Keith Glass in 1975. The score reverted back to deuce 37 times before Glass won the game. Fawcett, however, won the match.

[14] Norman (1985) recognized the directionality of tennis and grasped the essence of state recursion when he described how the optimal tennis serve strategy and corresponding win probabilities could be calculated by DP.

ASSUMPTION 3 (Stationarity II). The actual POPs (those implied even if players are not using MPE strategies) are given by families of conditional probabilities $\{\pi(\text{in}|x, m, d), \pi(\text{win}|x, m, d)\}$ that do not vary over time (independent of (x, m)) or across service games.

Assumption 1 and MPE (serve, speed, and attention) strategies are jointly sufficient, but not necessary, for assumption 3. While assumption 3 does not impose equilibrium, it does implicitly assume that the players are unaware of whether they are failing to play mutual best responses. Otherwise, they would have an incentive to alter their strategies, perhaps touching off a learning and adaptation process that would violate stationarity.

When stationarity holds and we have enough data, we can consistently estimate Π and use DP to calculate optimal serve strategies numerically. We then compare optimal win probabilities to win probabilities, given actual serve strategies (which can also be consistently estimated, given sufficient observations on serve directions). Specifically, let $P(d|x, m)$ be an arbitrary (potentially suboptimal) Markovian serve strategy, that is, the probability that the server chooses direction d in state (x, m). Let $W_P(x, m)$ be the server's win probability starting in state (x, m) and using strategy P for all future serves, and let $W_P(x, m, d)$ be their win probability, given choice of serve direction d in state (x, m) and using strategy P for all future serves. We then have the analogue to equation (1):

$$W_P(x, m) = \sum_{d \in \{\text{l,b,r}\}} W_P(x, m, d) P(d|x, m), \tag{5}$$

where $W_P(x, m, d)$ is given by equations (2) and (3), with W_P in place of W. These equations make it clear that W_P is an implicit function of the POPs Π and the serve strategy P.

In fact, we can write an expression for W_P as the solution to a system of linear equations, as is well known in the DP literature on policy evaluation. Since there are 298 distinct states (x, m):[15]

$$W_P = w_P(P, \Pi) + M_P(P, \Pi) W_P. \tag{6}$$

Here, $w_P(P, \Pi)$ is a 298×1 vector providing the probability of directly winning the service game in each state (in most states this is zero), and $M_P(P, \Pi)$ is a 298×298 Markov subtransition matrix (i.e., not all of its rows sum to 1)[16] representing the probability of transitioning between any two states induced by the serve strategy P and POPs Π. Since $M_P(P, \Pi)$ is a Markov

[15] There is only one possible muscle-memory state at the start of the service game $x = 1$, three possible muscle-memory states for $x = 2$, 3, and 9, and nine possible muscle-memory states for the remaining 32 score states. Thus, $1 \times 1 + 3 \times 3 + 32 \times 9 = 298$ states (x, m).

[16] The rows of M_P do not all sum to 1; $\ell\omega \in [\underline{w}, \bar{w}]$ implies $\pi(\text{in})\pi(\text{win}) \in [\underline{w}, \bar{w}]$ for any strategy choices.

subtransition matrix, the linear system (6) has a unique solution W_P.[17] We can see from equation (6) that W_P is an implicit function of both P and Π. We use this result later in the paper to rapidly calculate win probabilities, and via the implicit-function theorem, the gradients of the win and conditional win probabilities with respect to model parameters. This enables us to compute standard errors for win probabilities and conduct efficient Wald tests of the hypothesis of equal win probabilities.

With enough observations of service games between a given server and receiver, $W_P(x, m, d)$ can be consistently estimated as the fraction of service games won when the server chose direction d in state (x, m). If the POPs and serve strategies are stationary and Markovian, any win probability W_P must obey identity (6). With 298 states (x, m), nonparametric estimation of W_P involves $298 \times 3 = 894$ individual probabilities $W_P(x, m, d)$. Our analysis also requires an estimate of the actual Markovian serve strategy P and the observed POPs Π. Nonparametric estimation of P requires 298 probabilities and Π a total of $894 \times 2 = 1{,}788$ probabilities. Together, nonparametric estimation of W_P, P, and Π involves a total of $894 + 298 + 1{,}788 = 2{,}980$ probabilities, which would require tens of thousands of service games to estimate with any accuracy.

However, in our dataset, we typically have only 100–200 service games per server-receiver pair. To overcome this data limitation, we introduce parametric reduced-form models for serve probabilities and the POPs in section IV. We still refer to these models as "unrestricted" models of serve behavior because, unlike the dynamic structural models we introduce next, we do not require serve-direction choices to be best responses to the server's beliefs about the POPs.

D. *Dynamic Discrete-Choice Models of Serve Behavior*

To get deeper insight into the behavior of elite servers, we introduce three different structural models of serve behavior that we use in our empirical analysis: (1) a fully dynamic model that assumes that the server chooses a strategy that maximizes the probability of winning the entire service game; (2) a point-myopic model that assumes that the server chooses serve directions to maximize the probability of winning each point; and (3) a serve-myopic model that assumes that the server chooses serve directions to maximize the probability of winning each serve, ignoring the option value of the second serve. For each of these models, we estimate the server's *subjective POPs*, which rationalize observed serve behavior as a best response to the server's potentially subjective beliefs about their own performance and the performance of the receiver.

[17] We prove this in a corollary of lemma 1 in step 3 of the proof of theorem 1 in sec. 1 of the appendix.

The other important aspect of these dynamic discrete-choice models is the introduction of unobserved shocks affecting a server's choice of serve direction. These shocks can be interpreted as idiosyncratic factors that affect the server's choice, which, unlike muscle memory, are not persistent over states of the game. Technically, the introduction of these shocks implies that the server is using a pure strategy that appears to be a mixed strategy only because of the effect of the unobserved "serve shocks," though it is tempting to interpret the conditional choice probabilities (CCPs) $P(d|x, m)$ implied by these models as mixed strategies.[18]

We assume that these trembles or preference shocks are IID across successive serves and are observed only by the server but not by the receiver or the econometrician. Let $\varepsilon(d)$ be the tremble associated with serving to direction d. We further assume that $\{\varepsilon(\mathrm{l}), \varepsilon(\mathrm{b}), \varepsilon(\mathrm{r})\}$ has a standardized Type 1 extreme-value distribution with location parameter normalized so that $E\{\max_d \varepsilon(d)\} = 0$ scaled by $\lambda \geq 0$. If λ is large enough, the server's behavior can mimic a mixed serve strategy even when the win probabilities for different serve directions are unequal. However, as $\lambda \downarrow 0$, the CCPs converge to a mixed strategy only if the subjective POPs satisfy the equal-win-probability restriction. Thus, dynamic discrete-choice models are a natural way to model server behavior while converting the test for equal win probabilities into a simpler test of whether the estimated value of λ equals 0.

Let $\sigma_{\mathrm{FD}}(x, m, \varepsilon)$ be the server's strategy under the fully dynamic structural model as a function of the observed state (x, m) and the unobserved trembles $\varepsilon = (\varepsilon(\mathrm{l}), \varepsilon(\mathrm{b}), \varepsilon(\mathrm{r}))$. The fully dynamic model presumes that for each (x, m, ε), the server chooses the serve direction that maximizes the probability of winning the service game, given by

$$\sigma_{\mathrm{FD}}(x, m, \varepsilon) = \underset{d \in \{\mathrm{l},\mathrm{b},\mathrm{r}\}}{\operatorname{argmax}}(\lambda \varepsilon(d) + V_\lambda(x, m, d)), \tag{7}$$

where $V_\lambda(x, m, d)$ is a conditional value function, the analogue of the conditional win probability $W(x, m, d)$ defined in equations (2) and (3) of section III.[19] Here, the analogue of the function $W(x, m)$ given by the Bellman

[18] An alternative, game-theoretic interpretation is that these shocks represent *trembles*, or incomplete information on players' preferences that imply a Bayesian Nash equilibrium. McKelvey and Palfrey (1995, 1998) studied games of this type and referred to them as "quantal response equilibria." However, the perspective we take is to model the server's direction choice as a single-agent dynamic discrete-choice problem, taking the receiver's behavior as given and embodied in the POPs. Under this interpretation, following Rust (1987), the ε shocks are unobserved state variables, i.e., idiosyncratic IID private information or preference shocks that are known by the server (though not the receiver or econometrician) but make the server's behavior appear random even if the actual serve strategy is a pure strategy (i.e., a deterministic function of the server's information).

[19] Generically, the optimal strategy $\sigma_{\mathrm{FD}}(x, m, \varepsilon)$ will be a pure strategy, since the probability of more than one serve direction resulting in the same expected reward (including the shock $\varepsilon(d)$) is zero. Below, we characterize a necessary condition for σ_{FD} to converge to a

equation (1) is replaced by $V_\lambda(m, d)$, which is given by

$$V_\lambda(x, m) = \lambda \log\left(\sum_{d\in\{\mathrm{l,b,r}\}} \exp\{V_\lambda(x, m, d)/\lambda\}\right). \tag{8}$$

The serve-direction probability implied by the fully dynamic model is denoted $P_{\mathrm{FD}}(d|x, m)$:

$$\begin{aligned} P_{\mathrm{FD}}(d|x, m) &= \Pr\{d = \sigma_{\mathrm{FD}}(x, m, \varepsilon)|x, m\} \\ &= \frac{\exp\{V_\lambda(x, m, d)/\lambda\}}{\sum_{d'\in\{\mathrm{l,b,r}\}} \exp\{V_\lambda(x, m, d')/\lambda\}}. \end{aligned} \tag{9}$$

Equation (9) gives the probability of choosing to serve to direction d in observed state (x, m) while accounting for the randomness of the unobserved trembles ε. Since the trembles are IID across serves, it would appear that this model should also imply conditional independence of serve directions across successive first and second serves. However, that will actually be true only if there is no muscle memory; that is, the variable m does not enter $V_\lambda(x, m, d)$ (recall that m is a vector that stores the directions of the two most recent first serves). With muscle memory present, we can still have serial correlation of serves even though the trembles are IID.

By theorem 3 of Iskhakov et al. (2017), we have

$$W(x, m, d) = \lim_{\lambda\downarrow 0} V_\lambda(x, m, d), \tag{10}$$

uniformly for all (x, m, d). This implies that the only way for $P_{\mathrm{FD}}(d|x, m)$ to converge to a completely mixed serve strategy as $\lambda \downarrow 0$ is if the limiting conditional win probabilities $W(x, m, d)$ obey the equal-win-probability constraints, $W(x, m, \mathrm{l}) = W(x, m, \mathrm{b}) = W(x, m, \mathrm{r})$ for all (x, m).

The point-myopic and serve-myopic models have the same general structure as the fully dynamic model, so the serve strategies, value functions, and choice (mixing) probabilities are given by the same equations: (7), (8), and (9), respectively. The difference is in the equations defining V_λ. In the serve-myopic model, we have

$$V_\lambda(x, m, d) = \pi(\mathrm{in}|d, x, m)\pi(\mathrm{win}|d, x, m); \tag{11}$$

that is, $V_\lambda(x, m, d)$ is the probability of winning the serve. A serve-myopic server maximizes the probability of winning each serve, while the point-myopic server's objective is to win each point. Thus, a point-myopic server

mixed strategy as $\lambda \downarrow 0$: this requires the limiting values of V_λ to be equal for all d. As a reviewer noted, using private-information shocks to provide an alternative interpretation of what might otherwise appear to be mixed strategies dates back to Harsanyi (1973) and can be used "as a way of justifying the paper's structural model." Indeed, we show in sec. 5.2 of the appendix that eqq. (7)–(9) hold in any Bayesian Nash equilibrium in which the server, but not the receiver, learns ε before choosing his serve strategy.

performs a two-period backward-induction calculation. In any second-serve state, the value of the point-myopic server $V_\lambda(x, m, d)$ coincides with the serve-myopic formula given in equation (11) above. But in any nonterminal first-serve state, V_λ is given by

$$\begin{aligned} V_\lambda(x, m, d) = {} & \pi(\text{in}|d, x, m)\pi(\text{win}|d, x, m) \\ & + (1 - \pi(\text{in}|d, x, m))V_\lambda(x + 1, m'), \end{aligned} \tag{12}$$

where $m' = (d^{-1}, d)$, and $V_\lambda(x + 1, m')$ is the maximum win probability over all the second-serve directions given in equation (11).

Note that all three structural models imply probabilistic serve strategies that are entirely determined by the POPs and the scale parameter λ for the trembles. In contrast, the reduced-form model of serve directions does not depend on the POPs, since it is estimated separately with a flexible parameterization of serve directions. The structural models can be viewed as restricted special cases of the most flexible specification of the reduced-form model. This enables us to conduct likelihood-ratio (LR) specification tests for the three structural models relative to the unrestricted reduced-form specification.[20]

E. *The Monotonicity Condition and Myopic Optimality*

Unlike chess, where a player's ability to look ahead and consider the consequences of different moves is critical to success, planning ahead may not be as critical to success in tennis. However, the ability to solve at least a two-period DP problem is important, and in section IV.A, we provide clear evidence that the option of a second serve affects the first-serve strategy. In particular, first serves are significantly faster but have a higher chance of faulting than second serves. Second serves are also more likely to be body serves, which are less likely to miss wide in either direction.

But it is less clear whether there is a payoff to solving an infinite-horizon DP problem to determine optimal serve strategies as we did in section III.B. Indeed, absent muscle-memory effects, point-myopic play is optimal. In particular, Walker, Wooders, and Amir (2011) show that tennis is a *binary Markov game*, which is a two-player constant-sum game with only two possible outcomes, for both the overall game and all component subgames.

[20] A valid LR specification test would be based on a fully unrestricted version of the reduced-form model with a total of 624 parameters, so that it has the flexibility to replicate any conditional probability $P(d|x, m)$. Given the limited number of observations for specific server-receiver pairs, our specification for $P(d|x, m)$ depends on only 12 parameters, though it produces estimates that fit the data well. While our reduced-form specification does not strictly nest the structural models, it has sufficient flexibility to closely approximate the structural serve probabilities. We can also perform tests using the nonnested specification test of Vuong (1989). However, we prefer the LR tests and also rely on the AIC model-selection criterion to select our preferred structural specification, similar to the way we used it to select our preferred specification for the reduced-form model.

They assume that the probability of winning a point is independent of the current score and all prior choices. They define a *minimax-stationary strategy* for the overall game as one where each player focuses only on winning the current point. They show that the minimax-stationary strategy coincides with the MPE of the overall game, provided that a *monotonicity condition* (MC) holds, namely that the probability of winning the service game is always higher after winning any point than after losing it. Thus, point-myopic play is optimal without muscle-memory effects, since the MC holds in this special case of our model.

However, the MC is not sufficient for Walker, Wooders, and Amir's (2011) decomposition result to hold in our model with muscle-memory effects.[21] Of course, their decomposition result is sufficient, but not necessary, for a point-myopic serve strategy to be optimal. For example, a point-myopic serve strategy will be optimal whenever points are IID Bernoulli outcomes, since we showed in section III.A that this implies that the tennis score state is a random walk with drift. Therefore, a strategy that increases the probability of winning any point also increases the probability of winning the service game.

In fact, a point-myopic serve strategy will be trivially optimal in our data as long as we are observing a Nash equilibrium in which the server is using a completely mixed strategy, since any deviation serve strategy is optimal in that case. So in this section, we assume that we are not observing a complete mixed Nash equilibrium, allowing for pure strategies in some states and/or disequilibrium play. The following testable condition implies that point-myopic play is optimal in the presence of muscle memory:

Definition 2 (Generalized MC [GMC]). The probability of winning the game is always higher after winning a point than after losing it:

$$W(x^{+}(x), m) > W(x^{-}(x), m) \quad \text{for all } (x, m). \tag{13}$$

Also, at any first-serve state x and for any direction d_1 of the previous first serve, if the probability of winning the point is higher for serving to direction d than to d', then

$$\begin{aligned} W(x^{+}(x), (d, d_1)) &\geq W(x^{+}(x), (d', d_1)), \\ W(x^{-}(x), (d, d_1)) &\geq W(x^{-}(x), (d', d_1)). \end{aligned} \tag{14}$$

[21] In the presence of dynamic effects such as muscle memory, we can imagine that there might be trade-offs, such as serve directions that increase the chance of winning the current point but compromise the ability to win subsequent points. For example, serving to the same direction as the previous serve may reduce the chance of faulting as a result of muscle-memory effects, but doing so might improve the receiver's ability to return future serves hit to that direction, reducing the server's effectiveness in subsequent states of the game.

The first inequality in (13) is the same monotonicity condition that Walker, Wooders, and Amir (2011) showed is sufficient to establish that an optimal point-myopic strategy also maximizes the probability of winning the service game when there is no "state dependence" other than through the score state x. When there are dynamic effects such as muscle memory, the MC alone will no longer be sufficient to establish this result. The new condition (14) imposes the additional restriction that if a particular serve direction d results in a higher probability of winning the current point than some other serve direction d', the choice of d will not lower the server's probability of winning the service game relative to d' in the subsequent states $x^{+}(x)$ and $x^{-}(x)$.

A stronger sufficient condition that implies condition (14) is to require that at any first-serve state x, the service-game win probability does not depend on the direction of the previous first serve (though it can depend on the serve direction d_2 two first serves ago when the server was serving to the same court, unless $d_2 = \varnothing$ for the first serves in the game to the deuce and ad courts). When there is muscle memory, the choice of first-serve direction has future consequences because it affects the evolution of muscle memory, which in turn affects the server effectiveness in future states of the game. However, if muscle memory operates only across successive serves to the same court, then condition (14) will hold, and the server will not have to consider the current serve direction's effects on the probabilities of winning in subsequent game states.

THEOREM 3. If the GMC holds, then the optimal point-myopic and fully dynamic serve strategies coincide and result in the same service-game win probabilities for the server.

The proof of theorem 3 is in section 1.4 of the appendix. In section IV, we show that the GMC is testable, and there are server-receiver pairs for which the GMC fails for our empirically estimated W. In these cases, point-myopic strategies are suboptimal, although we show that typically the cost of suboptimality in terms of reduced service-game win probability is small.

Consider the implications for serial independence of serve directions. If there are no muscle-memory effects, then Walker, Wooders, and Amir's (2011) MC is equivalent to inequality (13) and implies that any strategy that maximizes the probability of winning each point also maximizes the probability of winning the service game. This leads Walker and Wooders (2001, 1522) to the wrong conclusion that "in addition to equality of players' winning probabilities, equilibrium play also requires that each player's choices be independent draws from a random process." Independence in serve directions is a consequence of their assumption that serve strategies do not depend on previous choices and outcomes. When there is history dependence such as muscle memory, equilibrium strategies will generally depend on both the score state x and muscle memory m. This history dependence implies that a point-myopic serve strategy will generally be suboptimal in terms of maximizing the probability of winning the service game. However, even when the stronger form of the MC—our GMC assumption (14)—holds and a point-myopic serve strategy is optimal, serial correlation in serve directions will still be a general property of an MPE, as we show in section 1.4 of the appendix. Thus, serial independence of serve directions is generally not an implication (i.e., necessary condition) of a mixed-strategy MPE in the presence of muscle memory, though independence does hold in the absence of dynamic effects such as muscle memory.

We conclude by summarizing the testable implications of the theory we have presented:

1. Nash equilibrium: there should not be any alternative serve strategy that increases the server's probability of winning.
2. Mixed-strategy equilibrium: the probability of winning in state (x, m) should be equal for all serve directions chosen with positive probability in state (x, m).

We also test the following behavioral implications of the GMC:

3. Optimality of point-myopic serve strategies: when the GMC holds, it is optimal for the server to adopt a point-myopic strategy that focuses only on the goal of maximizing the probability of winning each point.
4. Serial independence: if GMC holds and there are no muscle-memory effects, the direction of a first serve should not depend on the direction of any previous first serve.

IV. Reduced-Form Analysis of Serve Strategies

In this section, we start with a model-free descriptive analysis of our data. Then we introduce a flexible reduced-form model of tennis that we use to test several of the key implications of game theory summarized in section III, particularly the implication that conditional win probabilities are the same for all serve directions.[22] Most of our analysis focuses on a set of elite professional tennis players, who have all been ranked number 1 in the world and won multiple Grand Slam tournaments. These players are Roger Federer, Rafael Nadal, Novak Djokovic, Andy Murray, Pete Sampras,

[22] Our analysis is not assumption-free, as we maintain assumption 2 for validity of our statistical tests.

and Andre Agassi.[23] We focus on these players for two reasons: first, we have the most observations for them, and second, if we can show that they serve suboptimally, that means that even the best of the best are susceptible to strategic errors.

A. Analysis of Play of Specific Server-Receiver Pairs

We have sufficient observations to analyze serve decisions of specific server-receiver pairs. Table 1 summarizes some of the key statistics for 10 selected elite server-receiver pairs, revealing a great deal of player-specific heterogeneity that would be masked in pooled statistics. The table presents the total number of service games and serves we observe for each pair. A typical service game ends after seven to nine serves. The third column breaks down the total number of serves we observe into first and second serves. We can see that the "crude fault rate" (fraction of total serves that are second serves) differs across servers, ranging from a low of 21% for Nadal serving to Federer to a high of 30% for Sampras serving to Agassi.

The three columns under "Serve Direction" list the fractions of first and second serves to the receiver's left, body, and right for each server-receiver pair. We see that in general, servers use mixed strategies, but the mixing probabilities for second serves differ significantly from those for first serves. The last column of the table includes the p-value of an LR test of the null hypothesis that the mixing probabilities for the first and second serves are equal. We see that for all servers, we can decisively reject this hypothesis. In general, the fraction of second serves to the body is about twice as large as that for first serves.

We also see that servers adjust their serve strategies for different receivers. For example, from table 1, we can see that Nadal uses a different serve strategy when serving to Federer than when serving to Djokovic. The penultimate column of table 1 shows the empirical service-game win probability for the server and its estimated standard error (i.e., the fraction of games the server won). We see quite a bit of variation in service-game win probabilities across different server-receiver pairs, ranging from a low of 72% for Nadal serving to Djokovic, to a high of 90% for Sampras serving to Agassi. Even controlling for the same server, we see a fairly big variation in win probabilities, depending on the receiver: for example, Nadal has an 81% service-game win probability when serving to Federer, as Federer is a weaker receiver than Djokovic. Given the relatively small standard deviations in estimated win probabilities, we can strongly reject the null hypothesis that the variation in estimated win probabilities is due to sampling error.

[23] We provide results for women and additional men in sec. V.C.

TABLE 1
WIN PROBABILITIES AND MIXED SERVE STRATEGIES FOR SELECTED ELITE SERVER-RECEIVER PAIRS

				Serve Direction				
Server → Receiver	Games; Serves	Serves/Game	Serves	Left	Body	Right	Win Probability (SE)	p-Value: $P_1 = P_2$
Federer → Nadal	523; 4,732	8.36					.7686 (.0184)	5.1×10^{-60}
First serves			3,208	.4402	.1007	.4592		
Second serves			1,164	.2174	.2698	.5129		
Nadal → Federer	519; 4,081	7.86					.8092 (.0172)	6.3×10^{-12}
First serves			3,227	.6616	.2048	.1336		
Second serves			854	.5937	.3208	.0855		
Federer → Djokovic	411; 3,501	8.52					.8200 (.0190)	6.7×10^{-68}
First serves			2,524	.4521	.0939	.4540		
Second serves			977	.4084	.3408	.2508		
Djokovic → Federer	407; 3,653	8.98					.8010 (.0198)	1.0×10^{-33}
First serves			2,696	.4640	.1565	.3795		
Second serves			957	.4389	.3365	.2247		
Nadal → Djokovic	346; 2,937	8.49					.7197 (.0241)	2.4×10^{-64}
First serves			2,230	.3964	.2825	.3211		
Second serves			707	.4073	.5403	.0523		
Djokovic → Nadal	356; 2,877	8.08					.7528 (.0222)	1.2×10^{-40}
First serves			2,149	.4067	.1619	.4314		
Second serves			728	.1484	.2940	.5577		
Djokovic → Murray	230; 1,958	8.51					.7696 (.0278)	3.0×10^{-53}
First serves			1,447	.4651	.1244	.4105		
Second serves			511	.2192	.4618	.3190		
Murray → Djokovic	230; 2,141	9.31					.7435 (.0288)	5.8×10^{-122}
First serves			1,522	.3863	.0841	.5296		
Second serves			619	.4233	.4782	.0985		
Sampras → Agassi	140; 1,275	9.11					.9000 (.0254)	5.3×10^{-8}
First serves			884	.4434	.0724	.4842		
Second serves			391	.4680	.1765	.3555		
Agassi → Sampras	135; 1,125	8.33					.8666 (.0293)	7.2×10^{-16}
First serves			825	.5127	.1115	.3758		
Second serves			300	.5766	.2700	.1533		

B. A Flexible, Agnostic Reduced-Form Probability Model of Tennis

In order to test the key necessary condition for a mixed-strategy equilibrium—equality of win probabilities for all serve directions—a deeper econometric analysis is required. As discussed in section III, we do not have enough data to estimate a nonparametric model. Instead, we estimate a flexibly parameterized reduced-form specification for serve strategies $P(d|x, m)$ and POPs $(\pi(\text{in}|x, m, d), \pi(\text{win}|x, m, d))$. Following standard terminology in the dynamic discrete-choice literature, we refer to the serve probabilities below as CCPs. Let $f(x, m, d)$ be a $1 \times K_P$ vector of indicators for various subsets of the state/action space. We describe specific choices for f below. In general, f partitions the state space into subsets where serve-direction probabilities are similar. Let θ_P be a conformable $K_P \times 1$ vector of coefficients to be estimated. We use the following flexible logit model for the CCPs:

$$P(d|x, m, \theta_P) = \frac{\exp\{f(x, m, d)'\theta_P\}}{\sum_{\delta\in\{\text{l,b,r}\}} \exp\{f(x, m, \delta)'\theta_P\}}. \tag{15}$$

Similarly, let $g_{\text{in}}(x, m, d)$ and $g_{\text{win}}(x, m, d)$ be $1 \times K_{\text{in}}$ and $1 \times K_{\text{win}}$ vectors of indicators used to define the following binary logit models for $\pi(\text{in}|x, m, \theta_{\text{in}})$ and $\pi(\text{win}|x, m, \theta_{\text{win}})$ that depend on parameter vectors $(\theta_{\text{in}}, \theta_{\text{win}})$:

$$\pi(\text{in}|x, m, d, \theta_{\text{in}}) = \frac{\exp\{g_{\text{in}}(x, m, d)'\theta_{\text{in}}\}}{1 + \exp\{g_{\text{in}}(x, m, d)'\theta_{\text{in}}\}}, \tag{16}$$

$$\pi(\text{win}|x, m, d, \theta_{\text{win}}) = \frac{\exp\{g_{\text{win}}(x, m, d)'\theta_{\text{win}}\}}{1 + \exp\{g_{\text{win}}(x, m, d)'\theta_{\text{win}}\}}. \tag{17}$$

We estimate the parameter vector $\theta = (\theta_P, \theta_{\text{in}}, \theta_{\text{win}})$ by maximum likelihood, using the log-likelihood function $L(\theta)$ given by

$$L(\theta) = \sum_{n=1}^{N}\sum_{s=1}^{S_n}(\log(P(d_{s,n}|x_{s,n}, m_{s,n}, \theta_P)) + \log(h(o_{s,n}|x_{s,n}, m_{s,n}, d_{s,n}, \theta_{\text{in}}, \theta_{\text{win}}))), \tag{18}$$

where N is the total number of service games observed for a particular server-receiver pair, S_n is the number of serves in game n, and $(d_{s,n}, x_{s,n}, m_{s,n})$ are the observed serve direction, game state, and muscle-memory state, respectively, at serve s in game n. The variable $o_{s,n}$ is the outcome of serve s of game n and takes one of three possible values: $o_{s,n} = 1$ if the serve is in (i.e., not faulted) and the server wins the subsequent rally, $o_{s,n} = 2$ if the serve is in and the server loses the subsequent rally, or $o_{s,n} = 3$ if the serve is faulted. In all first-serve states (i.e., odd values of x), the service-game state transits to a second serve in the event that $o = 3$, but in any second-serve state the server loses the point when $o = 3$ (i.e., the server "double-faults"). The conditional probability $h(o|x, m, d, \theta_{\text{in}}, \theta_{\text{win}})$ is defined in terms of the POPs as follows:

$$h(o|x, m, d, \theta_{\text{in}}, \theta_{\text{win}}) = \begin{cases} \pi(\text{in}|x, m, d, \theta_{\text{in}})\pi(\text{win}|x, m, d, \theta_{\text{win}}) & \text{if } o = 1, \\ \pi(\text{in}|x, m, d, \theta_{\text{in}})(1 - \pi(\text{win}|x, m, d, \theta_{\text{in}})) & \text{if } o = 2, \\ 1 - \pi(\text{in}|x, m, d, \theta_{\text{in}}) & \text{if } o = 3. \end{cases} \tag{19}$$

Different specifications correspond to different partitions of the state/action space. The finest partition, in which every pair (x, m) is a partition element, yields the full nonparametric specification for (P, Π). Since we do not have sufficient observations to reliably estimate a fully nonparametric model, we face a classic bias/variance trade-off between estimating a flexible model with many parameters and estimating a more parsimonious model with sufficiently many observations per parameter to guard against overfitting plus outliers that could distort our estimates of the POPs.

We manage this trade-off using model-selection techniques, particularly the AIC, which penalizes model complexity. Specifically, $\text{AIC} = 2(K - L(\hat{\theta}))$, where K is the total number of parameters estimated in a given model, $L(\hat{\theta})$ is the maximized value of the log-likelihood function, and $\hat{\theta}$ is the maximum-likelihood estimate of the parameters of the particular model. We evaluated several different specifications (i.e., choices for f, g_{in}, and g_{win} with different numbers of parameters and different partitions of the state space) and chose as our preferred specification (see sec. 2 of the appendix) the model with the lowest AIC.[24]

Our preferred specification still involves a large number of parameters per server-receiver pair. In particular, the serve-direction probabilities $P(d|\cdot)$ are governed by 12 parameters. Eight are for the full set of interactions between direction $d \in \{\text{l}, \text{r}\}$[25] and the court (deuce vs. ad) and serve (first vs. second) dummy variables. The other four are muscle-memory parameters that interact the court and serve dummies with a dummy indicating whether the direction of the current serve equals the direction of the previous first serve to the same court. Each of the POPs $(\theta_{\text{in}}, \theta_{\text{win}})$ is determined by 16 parameters, which correspond to the indicators just described for serve probabilities, except that the current serve

[24] We also used the Bayesian information criterion (BIC) $\text{BIC} = K\log(N) - 2L(\hat{\theta})$, which has a stronger penalty for model complexity. But we found that the higher complexity penalty caused the BIC to select models with fewer parameters. In cases where one model specification was nested within another encompassing specification, the BIC would choose the more parsimonious restricted specification even though LR tests would lead us to reject the parsimonious restricted specification relative to the less restricted encompassing model.

[25] Since $P(d|x, m, \theta_P)$ sums to 1 across directions, we need include only two dummies for current serve direction.

direction must include all three directions, since the probabilities $\pi(\text{in})$ and $\pi(\text{win})$ need not sum to 1 across serve directions.[26]

We do not have the space to present all these parameter estimates and the associated standard errors for each of the server-receiver pairs we analyzed, though we provide them for Federer versus Djokovic in section 2 of the appendix and can provide the rest on request. As we will describe further in the next sections, our preferred specification balances the tradeoff described above: it provides an accurate probability model of the entire service game for individual server-receiver pairs while avoiding the dangers of overfitting. In the remainder of this section, we will use this model to test several of our key assumptions, including the key hypothesis of Nash equilibrium play in tennis.

C. *Testing for Stationarity across Matches*

We now test assumption 3, that is, stationarity of the POPs $(\pi(\text{in}|x, m, d, \theta_{\text{in}}), \pi(\text{win}|x, m, d, \theta_{\text{win}}))$ over time and across service games. Suppose that the CCPs are also stationary in this same sense. Then the stochastic processes of serves and serve outcomes in any given service game of a given server-receiver pair on a given type of court are Markovian, and the realizations of these Markov processes are IID across successive service games. While the presence of muscle memory and the scoring rules of tennis imply that the sequences of serve directions and serve outcomes will be serially correlated within a service game, there will be no dependence across successive games, because we assume that muscle memory is reset at the start of each service game and there are no other effects that lead to dependence across successive games.

It is easy to think of reasons why assumption 3 may not hold. For example, if a server injures his shoulder, this can adversely affect the POPs. Or there might be psychological effects, such as confidence or a “hot hand,” that could lead to serial correlation across successive service games served by the same player. Finally, if a player is learning and adapting, his strategy may slowly evolve as he learns more about his opponent’s weaknesses and adjusts to exploit them.

On the other hand, we need to pool across service games to have any hope of efficiently estimating the parameters determining the CCPs and POPs. From the previous section, our preferred reduced-form model has a total of 32 parameters (24 if we exclude muscle-memory effects). Given that a typical service game lasts for about seven to nine serves, we need at least

[26] We also estimated our models with a reduced-form specification that adds a binary partition of the score state capturing how far ahead (or behind) the server is in the current service game to the reduced-form specification of the POPs and CCPs. All of our qualitative results are robust to this alternative specification.

100 games of data to estimate these 32 (or even 24 parameters) with sufficient accuracy. We are particularly concerned about overfitting, along with the possibility that the model's predictions of conditional win probabilities will be incredibly high or low because of the lack of sufficient observations.

The stationarity assumption is testable, and we present results from a simple way of testing for stationarity in tables 2 and 3 below. For the same set of 10 server-receiver pairs in table 1, we estimate separate CCPs and POPs for different subsets of service games on the basis of year groupings of our data.[27] For example, for Agassi and Sampras, we divide the data into two subperiods, one from 1995 to 1999, where we have 67 service games, and another from 2000 to 2002, where we have 60 games. For Federer and Nadal, we have sufficient data to create three subperiods: 2004–7, 2008–12, and 2013–17, with 67, 81, and 91 service games, respectively. We estimate a pooled, or "restricted," model using all games in all years and imposing stationarity. Next, we estimate an "unrestricted" model that allows the CCPs and POPs to be different in each subperiod.

We calculate an LR test statistic of the stationarity hypothesis by taking two times the difference between the log likelihood for the unrestricted model (i.e., summing the individual subperiod log likelihoods) and the log likelihood for the restricted model. The unrestricted model with two subperiods has a total of $2 \times 32 = 64$ (with muscle memory) or $2 \times 24 = 48$ (without muscle memory) parameters, which are estimated separately without placing any equality restrictions across the two sample subsets. Thus, the LR test has 32 degrees of freedom for the specification with muscle memory and 24 degrees of freedom for the specification without muscle memory. For the player pairs where we have enough data to divide the sample into three subperiods, the test has 64 and 48 degrees of freedom, respectively.

Table 2 shows that we are unable to reject our stationarity assumption 3 at the 5% level for any of the 10 player pairs we analyzed under the muscle-memory specification. For the specification without muscle memory (which is the preferred one for all 10 pairs under the AIC criterion), we reject stationarity only for Agassi serving to Sampras. We conclude that assumption 3 is a reasonable approximation to the data, which justifies pooling across service games to get the most reliable possible estimates of the CCPs and POPs.

Table 3 displays the results of LR tests of stationarity of the CCPs. The AIC is lowest for the muscle-memory specification for seven of the 10 player pairs. We reject stationarity of the CCPs for eight of the 10 and nine of the 10 pairs under the muscle-memory and no-muscle-memory specifications, respectively. Under the assumptions in section III, if the POPs are stationary, then players use MPE strategies, and if the MPE is unique, then

[27] We provide test results for stationarity with an alternative partition of the data in sec. 3 of the appendix.

TABLE 2

TESTS FOR STATIONARITY OF POPS: $\{\pi(\text{IN}|x, m, d, \theta_{\text{in}}), \pi(\text{win}|x, m, d, \theta_{\text{win}})\}$

	Muscle Memory				No Muscle Memory			
Server → Receiver	Restricted	Unrestricted	LR Test (df)	*p*-Value of LR Test	Restricted	Unrestricted	LR Test (df)	*p*-Value of LR Test
Federer → Nadal			64.9 (64)	.447			46.5 (48)	.533
LL	−1,934.3	−1,901.9			−1,940.1	−1,916.9		
AIC	**3,932.6**	3,995.7			**3,928.2**	3,977.7		
Nadal → Federer			75.7 (64)	.150			58.6 (48)	.140
LL	−1,880.9	−1,843.1			−1,883.2	−1,853.9		
AIC	**3,825.9**	3,878.2			**3,814.5**	3,851.9		
Federer → Djokovic			77.4 (64)	.122			55.8 (48)	.206
LL	−2,280.7	−2,242.0			−2,284.7	−2,256.9		
AIC	**4,625.4**	4676.0			**4,617.5**	4,657.7		
Djokovic → Federer			80.1 (64)	.084			56.7 (48)	.183
LL	−2,403.9	−2,363.9			−2,411.7	−2,383.3		
AIC	**4,871.9**	4,919.7			**4,871.3**	4,910.7		
Nadal → Djokovic			24.3 (32)	.832			14.6 (24)	.932
LL	−1,414.2	−1,402.0			−1,415.8	−1,408.5		
AIC	**2,892.4**	2,932.1			**2,879.6**	2,913.0		
Djokovic → Nadal			42.7 (32)	.098			37.1* (24)	.043
LL	−1,302.1	−1,280.7			−1,304.5	−1,285.9		
AIC	**2,668.1**	2,689.4			**2,656.9**	2,667.9		
Djokovic → Murray			35.0 (32)	.326			26.7 (24)	.317
LL	−1,183.2	−1,165.6			−1,188.7	−1,175.4		
AIC	**2,430.3**	2,459.3			**2,425.5**	2,446.8		
Murray → Djokovic			43.0 (32)	.092			30.0 (24)	.186
LL	−1,280.1	−1,258.5			−1,287.9	−1,273.0		
AIC	**2,624.1**	2,645.1			**2,623.9**	2,641.9		
Sampras → Agassi			40.9 (32)	.135			33.7 (24)	.091
LL	−1,117.9	−1,097.4			−1,124.1	−1,107.3		
AIC	**2,299.7**	2,322.9			**2,296.2**	2,310.5		
Agassi → Sampras			44.0 (32)	.077			40.6* (24)	.019
LL	−1,031.1	−1,009.1			−1,032.6	−1,012.3		
AIC	**2,126.2**	2,146.2			**2,113.2**	2,120.6		

NOTE.—LL = log likelihood. Boldface indicates the preferred model (lower AIC value).
* Statistically significant at the 5% level or higher.

TABLE 3

Tests for Stationarity of CCPs: $\{P(d|x, m)\}$

	Muscle Memory				No Muscle Memory			
Server → Receiver	Restricted	Unrestricted	LR Test (df)	p-Value of LR Test	Restricted	Unrestricted	LR Test (df)	p-Value of LR Test
Federer → Nadal			65.6* (24)	.000			60.6* (16)	.000
LL	−1,844.8	−1,812.0			−1,874.4	−1,844.0		
AIC	3,713.5	**3,696.0**			3,764.7	**3,736.1**		
Nadal → Federer			104.2* (24)	.000			95.2* (16)	.000
LL	−1,688.2	−1,636.1			−1,690.9	−1,643.3		
AIC	3,400.3	**3,344.1**			3,397.8	**3,334.5**		
Federer → Djokovic			64.1* (24)	.000			58.4* (16)	.000
LL	−2,265.1	−2,233.0			−2,293.9	−2,264.6		
AIC	4,554.1	**4,538.1**			4,603.7	**4,577.3**		
Djokovic → Federer			62.2* (24)	.000			58.1* (16)	.000
LL	−2,423.8	−2,392.7			−2,454.8	−2,425.7		
AIC	4,871.6	**4,857.4**			4,925.5	**4,899.4**		
Nadal → Djokovic			26.2* (12)	.010			21.4* (8)	.006
LL	−1,432.6	−1,419.5			−1,437.5	−1,426.8		
AIC	2,889.3	**2,887.0**			2,891.0	**2,885.6**		
Djokovic → Nadal			8.6 (12)	.740			7.8 (8)	.450
LL	−1,347.4	−1,343.2			−1,364.2	−1,360.3		
AIC	**2,718.9**	2,734.3			**2,744.4**	2,752.6		
Djokovic → Murray			57.2* (12)	.000			58.6* (8)	.000
LL	−1,201.6	−1,173.0			−1,221.0	−1,191.7		
AIC	2,427.1	**2,393.9**			2,458.0	**2,415.4**		
Murray → Djokovic			134.6* (12)	.000			130.9* (8)	.000
LL	−1,250.0	−1,182.7			−1,254.9	−1,189.5		
AIC	2,524.0	**2,413.4**			2,525.9	**2,411.0**		
Sampras → Agassi			30.3* (12)	.003			25.0* (8)	.002
LL	−1,085.4	−1,070.3			−1,096.4	−1,083.9		
AIC	2,194.9	**2,188.6**			2,208.8	**2,199.8**		
Agassi → Sampras			24.2* (12)	.019			22.1* (8)	.005
LL	−931.8	−919.7			−945.2	−934.2		
AIC	1,887.6	**1,887.4**			1906.5	**1900.4**		

Note.—LL = log likelihood. Boldface indicates the preferred model (lower AIC value).

* Statistically significant.

the CCPs must be stationary as well. Thus, we conclude that the rejections in table 3 indicate that either (*a*) there are multiple MPEs, and the players "select" different MPEs in different time periods, or (*b*) players are not playing MPE strategies, and the variation in CCPs reflects the effect of some sort of learning or experimentation with different serve strategies over time.

D. *Testing Equality of Win Probabilities over Directions and Strategies*

We now present tests of the key implication of a completely mixed MPE that point and service-game win probabilities are independent of serve direction, plus the stronger implication that all deviation serve strategies imply the same win probability. We strongly reject these implications in models with muscle memory. As we show below, the data support the presence of muscle memory for almost all player pairs because of strong evidence of serial dependence in serve directions. Accounting for this dependence is key to our ability to detect violations of equal win probabilities.

Table 4 compares the recursively calculated game win probabilities from equation (6) to nonparametric estimates (i.e., simply the fraction of games won) of these probabilities at the first serve of each service game. We restrict attention to the first serve of the game because it provides the most observations to reliably estimate the game win probability nonparametrically. The final column shows the p-value of a Durbin-Hausman-Wu (DHW) test of our preferred reduced-form specification. Recall that the DHW test compares a consistent but inefficient nonparametric estimator of the game win probability to a relatively efficient estimate of it from equation (6).[28]

We see that the calculated win probabilities are close to the nonparametric estimates and are almost always within a standard deviation of each other. The high p-values of the DHW specification tests in the final column of the table show that for all servers except Federer serving to Nadal, we are unable to reject the reduced-form specification and its implied win probability. In the case of Federer serving to Nadal, the reduced-form estimate of the win probability is 0.796, slightly more than 1 standard

[28] The DHW specification test compares two estimators of a given quantity or parameter: an inefficient but $\sqrt{N}$-consistent estimator that is consistent under both the null and alternative hypotheses, and an efficient estimator that is also $\sqrt{N}$-consistent for the true parameter under the null hypothesis but may be inconsistent under the alternative hypothesis (N denotes the sample size). In our case, the relevant null hypothesis is that our reduced-form specification for (P, Π) is correct, and the nonparametric estimates of the win probabilities in table 4 are inefficient but consistent even if the null hypothesis is false (i.e., our reduced-form model is misspecified). Under the null, the DHW test statistic is equal to the square of the two estimates of the win probability divided by the differences in the asymptotic variances, and it converges to a χ^2 random variable with 1 degree of freedom.

TABLE 4
Estimated Win and Conditional Win Probabilities at First Serve of Service Game

Server → Receiver	Win Probability, First Serve	Conditional Win Probability, First Serve: Left	Body	Right	DHW Test p-Value
Federer →					
Nadal					.004
NP estimate	.796 (.026)	.816 (.025)	.650 (.030)	.803 (.025)	
RF estimate	.829 (.023)	.828 (.024)	.819 (.027)	.833 (.022)	
Nadal →					
Federer					.107
NP estimate	.786 (.026)	.748 (.028)	.896 (.020)	.762 (.028)	
RF estimate	.807 (.023)	.808 (.023)	.807 (.025)	.803 (.025)	
Federer →					
Djokovic					.504
NP estimate	.810 (.024)	.844 (.022)	.867 (.021)	.767 (.025)	
RF estimate	.818 (.020)	.826 (.020)	.812 (.023)	.813 (.021)	
Djokovic →					
Federer					.910
NP estimate	.782 (.025)	.769 (.026)	.710 (.028)	.815 (.024)	
RF estimate	.781 (.022)	.792 (.022)	.769 (.026)	.774 (.024)	
Nadal →					
Djokovic					.992
NP estimate	.712 (.035)	.685 (.036)	.726 (.035)	.750 (.034)	
RF estimate	.712 (.034)	.712 (.034)	.701 (.035)	.718 (.034)	
Djokovic →					
Nadal					.278
NP estimate	.829 (.029)	.868 (.026)	.735 (.034)	.833 (.029)	
RF estimate	.848 (.023)	.854 (.023)	.830 (.027)	.849 (.023)	
Djokovic →					
Murray					.871
NP estimate	.794 (.034)	.759 (.036)	.750 (.036)	.841 (.031)	
RF estimate	.791 (.029)	.796 (.031)	.758 (.034)	.799 (.029)	
Murray →					
Djokovic					.675
NP estimate	.721 (.038)	.816 (.033)	.500 (.042)	.701 (.038)	
RF estimate	.717 (.036)	.735 (.036)	.712 (.039)	.703 (.037)	
Sampras →					
Agassi					.150
NP estimate	.885 (.028)	.894 (.027)	1.00 (.000)	.859 (.030)	
RF estimate	.866 (.024)	.866 (.025)	.839 (.029)	.872 (.024)	
Agassi →					
Sampras					.362
NP estimate	.874 (.029)	.907 (.026)	.867 (.030)	.852 (.032)	
RF estimate	.859 (.024)	.861 (.026)	.853 (.026)	.859 (.024)	

Note.—NP estimate = nonparametric estimate; RF estimate = reduced-form estimate. Standard errors are in parentheses.

deviation away from the nonparametric estimate of the win probability, 0.829. The middle columns compare the nonparametric estimates of the conditional win probabilities with the corresponding estimates implied by the reduced-form model, $W_P(1, 1, d)$ for $d \in \{l, b, r\}$. The estimates are generally close to each other, though there are some cases where there are large differences due to small numbers of observations, resulting in noisy nonparametric estimates.[29]

In contrast, the middle columns of table 4 reveal big differences in game win probabilities for different serve directions. The largest gap is a 31 percentage point difference between the win probability for left (81.6%) versus body (50%) for Murray serving to Djokovic, roughly 10 times their estimated standard errors. The average value of the maximum deviation in game win probabilities over all states and serve directions in table 4 is 19 percentage points, nearly four times as large as the estimated standard errors of these maximum deviations.

Although table 4 reassures us that our recursive calculation of game win probabilities results in accurate and efficient estimates, some readers may be skeptical that the evidence against equal game win probabilities is as convincing as the tests of equal point win probabilities that Walker and Wooders (2001) and most of the subsequent literature have focused on. In table 5, we present omnibus Wald tests of equality of the point win probabilities at all states (x, m) of tennis simultaneously. Recall that under the point-myopic theory of play, the server does not consider the future consequences of different serve directions and instead maximizes the probability of winning each point, which is a two-period DP problem. Starting at the second serve, the restriction that point win probabilities are the same for all serve directions holds if the serve win probability $V(x, m, d)$ given by

$$V(x, m, d) = \pi(\text{in}|x, m, d)\pi(\text{win}|x, m, d) \tag{20}$$

is the same for all three serve directions in all second-serve states (x, m). In any first-serve state, the point win probability $V(x, m, d)$ is given by

$$\begin{aligned} V(x, m, d) &= \pi(\text{in}|x, m, d)\pi(\text{win}|x, m, d) \\ &\quad + (1 - \pi(\text{in}|x, m, d))\left(\sum_{d' \in \{\text{l,b,r}\}} P(d'|x, m')\pi(\text{in}|x+1, m', d')\pi(\text{win}|x+1, m', d')\right), \end{aligned} \tag{21}$$

and it should also be the same for all d, where $m' = f(m, d')$ is the new muscle-memory state implied by serve direction d', which is updated only in first-serve states.

Table 5 provides the test statistics, p-values, and degrees of freedom for the omnibus Wald test of equality of conditional win probabilities for

[29] For example, in the case of Sampras serving to Agassi, because of the low probability that Sampras serves to the body (approximately 7%; see table 1) and the relatively low number of games in which we observe him serving (140), the nonparametric estimate of the conditional win probability of serving to the body equals 1. Of course, this nonparametric estimate is probably not a reasonable estimate: instead it is likely to be a lucky outcome for Sampras, who happened to win every one of the eight games where he served to Agassi's body on the very first serve of the game.

TABLE 5
Wald Tests of Equal Point Win Probabilities,
Muscle-Memory Specification

Server → Receiver	Wald Statistic	Degrees of Freedom	p-Value
Federer → Nadal	405.4	29	5.9×10^{-68}
Nadal → Federer	243.2	30	2.9×10^{-35}
Federer → Djokovic	23.6	30	.75
Djokovic → Federer	274.5	27	8.9×10^{-43}
Nadal → Djokovic	83.5	29	3.5×10^{-7}
Djokovic → Nadal	69.6	28	2.1×10^{-5}
Djokovic → Murray	52.3	30	.007
Murray → Djokovic	212.0	30	2.7×10^{-29}
Sampras → Agassi	146.4	30	2.9×10^{-17}
Agassi → Sampras	198.6	30	8.9×10^{-27}

all serve directions, that is, the restrictions that $V(x, m, \mathrm{l}) = V(x, m, \mathrm{b}) = V(x, m, \mathrm{r})$ for all 298 states (x, m), where $V(x, m, d)$ is given in equations (20) and (21) above. We see that there are strong rejections of the hypothesis of equal win probabilities for all player pairs except for Federer serving to Djokovic. Overall, we also see big differences in point win probabilities across different serve directions: the average maximum deviation over all 10 player pairs is .275, with a standard deviation of .083.

Why are we able to reject the hypothesis of equal point win probabilities so strongly when previous studies were unable to do so? We believe that accounting for muscle memory is a large part of the story. If we repeat the Wald tests in table 5 under the no-muscle-memory specification, we find smaller maximum deviations in win probabilities over serve directions over the reduced state space, and we reject the equal-win-probability hypothesis for only two of the 10 player pairs above.[30] Previous tests, such as those by Walker and Wooders (2001), focused only on first serves and pooled all first-serve observations into just two groups: the deuce and ad courts. Pooling the data in this way masks big differences in win probabilities for different serve directions that appear once we control for serial correlation in serves by conditioning on previous serve history via the muscle-memory state. The importance of controlling for muscle memory is confirmed in section 4 of the appendix, where we present the results of Wald tests of equal win probabilities under the no-muscle-memory specification, essentially replicating Walker and Wooders's approach but using our data and including second serves. Like Walker and Wooders's, these tests usually fail to reject equal win probabilities.

[30] These pairs are Djokovic serving to Murray and Djokovic serving to Nadal (the Wald statistics are 18.4 and 23.0, with p-values of .018 and .003, respectively, under 8 degrees of freedom). Also, the average value of the maximum difference in point win probabilities over all directions and states for these 10 pairs is .20 (standard deviation = .08).

We now turn to testing for equal service-game win probabilities using the fully dynamic version of the model with recursively calculated game win probabilities W_P from equation (5). We also use equations (2), (3), and (5) to calculate the direction-specific win probabilities $W_P(x, m, d)$ entering the recursive formula for $W_P(x, m)$. The omnibus Wald test of equal win probabilities for all serve directions involves testing in all 298 states (x, m) the equality restrictions:

$$W_P(x, m, \mathrm{l}) = W_P(x, m, \mathrm{b}) = W_P(x, m, \mathrm{r}). \tag{22}$$

In our preferred specification with muscle memory, this test amounts to a test of 596 equality restrictions of the form given in equation (22).[31]

Since the conditional win probabilities are implicit functions of (P, Π) and (P, Π) are functions of the 44-dimensional parameter vector $\hat{\theta} = (\hat{\theta}_P, \hat{\theta}_{\text{in}}, \hat{\theta}_{\text{win}})$, we use the delta method to construct the omnibus Wald test statistic. This is a quadratic form in the 596×1 vector of differences in conditional win probabilities between serve directions over all states, using the Moore-Penrose inverse of the 596×596 covariance matrix of win probability differences, expressed as a sandwich formula in terms of the 44×44 variance-covariance matrix for the reduced-form parameter vector $\hat{\theta}$. We need to use the Moore-Penrose inverse rather than the standard matrix inverse because the rank of the covariance matrix (which equals the degrees of freedom of the χ^2 distribution of the omnibus test statistic under the null hypothesis) is at most 44.[32]

We find that the omnibus Wald test results in rejections of equal service-game win probabilities even stronger than those we obtained when testing for the equality of point win probabilities in table 5, with p-values of nearly 0 for all player pairs (see table 6). However, there are reasons to distrust such strong rejections due to small-sample numerical issues with the Moore-Penrose inverse, which is not a continuous function of its matrix argument. The discontinuity can invalidate the standard χ^2 asymptotic distribution of the Wald test statistic under the null hypothesis. Andrews (1987) provides a sufficient condition for "generalized Wald tests" (which rely on the Moore-Penrose inverse) to have the usual asymptotic χ^2 distribution: namely, the rank of the finite sample covariance matrix of the restrictions must converge with probability 1 to the rank of the limiting covariance matrix. Matrix rank is not a continuous function either, but it is

[31] The specification with muscle memory is our preferred specification, since the no-muscle-memory specification is strongly rejected for all but one of the player pairs (see table 7 in sec. IV.E below).

[32] The rank of the covariance matrix is generally even lower than 44 (the number of parameters) because the rank of the 596×44 gradient matrix of the win probability differences is often less than 44.

TABLE 6
Wald Tests of Equal Service-Game Win Probabilities, Muscle-Memory Specification

Server → Receiver	Four Fixed Serve Strategies at Three States (9 df)	Reduced-Form Serve Strategy at Four States (12 df)
Federer → Nadal	1.4×10^{-11}	.605
Nadal → Federer	.873	.018
Federer → Djokovic	6.5×10^{-30}	1.6×10^{-68}
Djokovic → Federer	.0009	.220
Nadal → Djokovic	2.2×10^{-254}	.526
Djokovic → Nadal	4.0×10^{-91}	4.5×10^{-44}
Djokovic → Murray	1.4×10^{-69}	.018
Murray → Djokovic	9.1×10^{-91}	.00001
Sampras → Agassi	.787	.003
Agassi → Sampras	.764	.667

semicontinuous, so the rank condition of Andrews (1987) should hold generically.[33]

Nevertheless, we have observed a tendency for Wald test statistics to grow rapidly with the total number of restrictions being tested, so we have opted to adopt a more conservative approach to testing for equal win probabilities using small subsets of the total number of restrictions. Since matrix inversion is continuous, our conservative approach reduces the problem of spurious rejections, though it does lead to power/size trade-offs in the choice of how many restrictions to test. It also requires additional choices over which subset of restrictions we choose to test.

The last column of table 6 presents the p-values for our more conservative test of equal win probabilities at a subset of six points in the state space: (1) 0–0, (2) 15–0, (3) 0–15, (4) 40–15, (5) 15–40, and (6) deuce. This test has 12 restrictions, and since the covariance matrix for these restrictions is invertible, the test has 12 degrees of freedom. It rejects the hypothesis of equal game win probabilities at the 5% level for six of the 10 player pairs in the table.[34]

The second column of table 6 reports p-values for a Wald test of the invariance of win probabilities with respect to strategy deviations that must hold when MPE serve strategies are completely mixed. We compute the win probabilities of four different fixed strategies at three points in the state space, resulting in a test with nine restrictions and degrees of freedom (since the covariance matrix for this reduced set of restrictions is invertible). The four fixed serve strategies are (1) always serve left, (2) always

[33] A sufficient condition for the rank condition in Andrews (1987) to hold is that the limit covariance matrix of the restrictions must be *regular*: i.e., the rank must be the same for all covariance matrices in a neighborhood of the limiting value. In addition, proposition 4 of Lewis (2009) establishes that the set of regular matrices is an open and dense set of the space of all matrices.

[34] A test of equal point win probabilities over the same score states rejects for three of the 10 pairs at the 5% level.

serve to the body, (3) always serve right, and (4) serve to each direction with probability 1/3 (i.e., a uniform distribution across the serve directions). The three particular score states used in these tests are (1) 40–15, (2) 15–40, and (3) deuce. This test strongly rejects the hypothesis of equal win probabilities for seven of the 10 player pairs. Thus, our new approach to testing for equal win probabilities, allowing for serial correlation in serve directions via muscle-memory effects, and our inclusion of body serves and more observations explain why we reject mixed-strategy Nash play for the majority of the elite player pairs in our data set.

E. Testing for "Muscle-Memory" Effects

We conclude this section by presenting evidence of serial dependence in the CCPs and, to a lesser extent, the POPs. We have already shown in section IV.A that there are significant differences between the mixture probabilities for first and second serves, so it should not be surprising that we also find significant serial dependence between first and second serves. However, this serial dependence is not necessarily inconsistent with equilibrium play, since the server considers the option value of the second serve when choosing the speed, spin, and direction of the first serve.

The more important question is whether there is serial correlation across successive first serves hit to the same court. Our preferred specification for our reduced-form model of serve directions conditions on the deuce versus the ad court, so the server's strategy can alternate across courts. However, this effect does not induce serial correlation in serve directions across successive first serves to the same court. We capture serial correlation in such serve directions via muscle memory. The muscle-memory specification also induces serial correlation between first and second serves, since the CCPs for second serves depend on the direction of the faulted first serve. The specification without muscle memory allows for serial dependence as play alternates between courts, but it implies zero serial correlation across successive first serves to the same court.

We use LR tests for serial correlation by comparing the likelihood that includes muscle memory with the restricted likelihood that excludes the muscle-memory variable m, since, as we showed in section III, serve directions become serially independent under this specification. Table 7 presents the results of LR tests of the hypothesis of "no muscle-memory effects." The last column of this table shows that except for Nadal serving to Federer, we can reject the hypothesis of no muscle memory in the CCPs at the 5% level. However, when it comes to the POPs, we have far weaker evidence of serial correlation. For most of the server-receiver pairs in table 7, we are unable to reject the hypothesis of no muscle-memory effects in the POPs.

TABLE 7

Tests for Muscle-Memory Effects in (P, Π)

	No Muscle Memory		Muscle Memory			
Server → Receiver	AIC	LL	AIC	LL	LR Test	p-Value of LR Test
Federer → Nadal					4.3×10^{-12}	.170
Serves model	3,764.7	−1,874.4	3,713.5*	−1,844.8		
POPs model	3,928.3*	−1,940.1	3,932.6	−1,934.3		
Nadal → Federer					.249	.779
Serves model	3,397.8*	−1,690.9	3,400.3	−1,688.2		
POPs model	3,814.5*	−1,883.3	3,824.9	−1,880.9		
Federer → Djokovic					9.3×10^{-12}	.414
Serves model	4,603.7	−2,293.9	4,554.1*	−2,265.1		
POPs model	4,617.4*	−2,284.8	4,625.4	−2,280.7		
Djokovic → Federer					1.1×10^{-12}	.048
Serves model	4,925.5	−2,454.8	4,871.6*	−2,423.8		
POPs model	4,871.3*	−2,411.7	4,871.9	−2,403.9		
Nadal → Djokovic					.044	.921
Serves model	2,891.0	−1,437.5	2,889.3*	−1,432.6		
POPs model	2,879.6*	−1,415.8	2,892.4	−1,414.2		
Djokovic → Nadal					9.0×10^{-7}	.779
Serves model	2,744.8	−1,364.2	2,718.9*	−1,347.4		
POPs model	2,656.9*	−1,304.5	2,668.1	−1,302.1		
Djokovic → Murray					7.7×10^{-8}	.202
Serves model	2,458.0	−1,221.0	2,427.1*	−1,201.6		
POPs model	2,425.5*	−1,188.7	2,430.3	−1,183.2		
Murray → Djokovic					.044	.049
Serves model	2,525.9	−1,254.9	2,524.0*	−1,250.0		
POPs model	2,623.9	−1,287.9	2,524.1*	−1,280.1		
Sampras → Agassi					2×10^{-4}	.134
Serves model	2,208.8	−1,096.4	2,194.9*	−1,085.4		
POPs model	2,296.2*	−1,124.1	2,299.7	−1,117.9		
Agassi → Sampras					2×10^{-5}	.934
Serves model	1,906.5	−945.3	1,887.6*	−931.8		
POPs model	2,113.2*	−1,032.6	2,126.2	−1,031.1		

Note.—LL = log likelihood.

* Statistically significant.

Why is this the case? We think that it may have to do with the receiver's behavior. Specifically, if muscle-memory effects are real and the receiver shifts his position accordingly, then the receiver can effectively cancel out any effect that muscle memory would impart on the POPs. As a result, we observe serial correlation in the server's directional choices but not in the POPs.[35] This results can be consistent with Nash equilibrium play, as we demonstrate in section 6 of the appendix.

V. Dynamic Structural Analysis of Serve Strategies

In the previous section, we estimated an unrestricted reduced-form model of serve directions and POPs and showed that this flexible, agnostic model of tennis rejects the key implication of a mixed-strategy Nash equilibrium: namely, that the POPs satisfy the restriction that the server's win probability is the same for all serve directions in every state of the service game (and thus that all possible serve strategies have equal win probabilities). These tests did not require us to make any assumptions about server behavior beyond assumption 3 (stationarity). This section provides more insight into server behavior by presenting estimation results for the three structural models we introduced in section III.D. We estimate their parameters by maximum likelihood, using the full-panel likelihood function (18) on data from hard courts for the 10 elite server-receiver pairs listed in table 8.[36] On the basis of our findings in section IV.E, which provide strong evidence of serial correlation in serve directions across successive points, we focus on the specification with muscle memory. For comparability, we use the same specification of the POPs as in our reduced-form model presented in section IV. Therefore, our structural models have a total of 33 parameters: the 32×1 vector of POP parameters (θ_{in}, θ_{win}) plus the extreme-value scaling parameter λ.

The structural estimates of the POPs can be regarded as estimates of the server's subjective beliefs that may or may not correspond to rational objective beliefs about the true POPs, which we estimate via our unrestricted POP estimates. As we discussed in section III.D, the structural model implies mixed-strategy Nash equilibrium play if two key restrictions are satisfied: (1) $\lambda = 0$ (i.e., players use mixed strategies, which can hold only if the POPs obey the equal-win-probability restrictions), and (2) the subjective POPs equal the objective POPs.

Unlike the reduced-form specification, the assumption of optimal play implicit in the structural models imposes "cross-equation restrictions" on

[35] We solved for the MPE in a two-direction version of our model and confirm that muscle-memory effects induce much larger changes in the server and receiver's equilibrium mixed strategies than in the POPs.

[36] We extend our analysis to grass and clay courts and a much larger set of server-receiver pairs in sec. V.C.

TABLE 8
Summary of Structural Estimation Results for Selected Elite Server-Receiver Pairs

Server → Receiver	Reduced-Form Model	Serve-Myopic Model	Point-Myopic Model	Fully Dynamic Model
Federer → Nadal:				
LL (observations)	−3,779.1 (2,011)	−3,788.2	−3,783.8	−3,817.3
$\hat{\lambda}$		6.1×10^{-4}	5.5×10^{-3}	2.9×10^{-5}
AIC	7,646.1	7,642.7	**7,633.7**	7,700.7
LR p-value		.074	**.571**	7.1×10^{-12}
Nadal → Federer:	−3,569.1 (1,882)	−3,571.3	−3,570.6	−3,632.1
$\hat{\lambda}$		6.1×10^{-3}	2.7×10^{-3}	8.4×10^{-5}
AIC	7,226.2	7,208.6	**7,207.3**	7,330.3
LR p-value		.957	**.990**	1.1×10^{-21}
Federer → Djokovic:	−4,545.8 (2,333)	−4,551.2	−4,552.2	−4,576.0
$\hat{\lambda}$		.010	4.4×10^{-3}	9.0×10^{-4}
AIC	9,179.5	**9,168.4**	9,170.4	9,128.0
LR p-value		**.457**	.300	7.5×10^{-9}
Djokovic → Federer:	−4,827.7 (2,372)	−4,840.0	−4,842.0	−4,844.8
$\hat{\lambda}$		.011	1.8×10^{-3}	2.4×10^{-4}
AIC	**9,743.5**	9,746.0	9,750.0	9,755.6
LR p-value		.010	2.7×10^{-3}	3.5×10^{-4}
Nadal → Djokovic:	−2,846.8 (1,405)	−2,853.8	−2,853.2	−2,864.5
$\hat{\lambda}$		5.8×10^{-6}	6.5×10^{-6}	6.4×10^{-7}
AIC	5,781.7	5,773.7	**5,772.4**	5,795.0
LR p-value		.232	**.310**	2.2×10^{-4}

Djokovic → Nadal:	−2,649.5 (1,344)	−2,659.9	−2,656.1	−2,654.7
$\hat{\lambda}$		.070	.097	5.8×10^{-6}
AIC	5,387.0	5,385.9	5,378.2	**5,375.3**
LR p-value		.035	.285	**.505**
Djokovic → Murray:	−2,384.7 (1,201)	−2,396.2	−2,396.9	−2,413.0
$\hat{\lambda}$		9.6×10^{-3}	.044	2.2×10^{-4}
AIC	4,857.5	**4,858.3**	4,859.8	4,892.0
LR p-value		**.018**	.011	4.1×10^{-8}
Murray → Djokovic:	−2,530.1 (1,328)	−2,536.4	−2,539.8	−2,556.3
$\hat{\lambda}$		.014	6.8×10^{-3}	1.1×1^{-5}
AIC	5,148.1	**5,138.9**	5,145.7	5,178.6
LR p-value		**.310**	.052	2.2×10^{-7}
Sampras → Agassi:	−2,203.3 (1,181)	−2,219.6	−2,217.7	−2,240.2
$\hat{\lambda}$		.031	.037	3.1×10^{-4}
AIC	**4,494.6**	4,505.3	4,501.4	4,546.3
LR p-value		5.9×10^{-4}	2.5×10^{-3}	2.4×10^{-11}
Agassi → Sampras:	−1,962.9 (1,050)	−1,973.0	−1,970.9	.8
$\hat{\lambda}$		6.7×10^{-4}	3.2×10^{-6}	5.1×10^{-6}
AIC	4,013.8	4,011.9	**4,007.8**	4,077.6
LR p-value		.043	**.140**	1.1×10^{-13}

NOTE.—LL = log likelihood. Boldface indicates the preferred model (lowest AIC value).

the serve probabilities: they are implicit functions of the POP parameters as well as the scale parameter λ for the extreme-value-distributed trembles. This implies that the likelihood function is no longer block-diagonal between the POP parameters $(\theta_{\text{in}}, \theta_{\text{win}})$ and λ, unlike the unrestricted reduced-form model, where we have block diagonality between the 12×1 parameter θ_P determining serve-direction probabilities and the POPs $(\theta_{\text{in}}, \theta_{\text{win}})$. Thus, in the structural model, there is a tension between maximizing the likelihood for the POPs and maximizing that for the serve directions. As we will see, maximum likelihood resolves this tension by distorting the estimates of the POPs while also driving the estimate of λ close to zero. As we noted in equation (10) of section III.D, the only way the model can explain mixed-strategy play as $\lambda \downarrow 0$ is to force the POPs to obey the equal-win-probability restrictions. Maximum likelihood results in distorted POPs that satisfy equal-win-probability restrictions because it enables the model to match observed serve-direction probabilities.

Table 8 summarizes the structural estimation results for the same 10 elite server-receiver pairs that we analyzed in section IV.[37] For comparison, we show the optimized log-likelihood function for the reduced-form model and the number of serve observations used to estimate the parameters, along with the point estimates of λ for each of the structural models. The third and fourth rows of numbers for each server-receiver pair report, respectively, the AIC value and the p-value of an "LR test" of each structural model relative to the reduced-form model. As per our discussion above, these models are not strictly nested within each other, though the reduced-form model is the more flexible specification, with a total of 44 parameters.

In light of this, we follow our approach in section IV and select our preferred model as the one with the smallest value of the AIC, which we present in bold font. Note that the best-fitting model per the AIC is also the model with the highest p-value for a quasi-LR test of each the structural models relative to the reduced-form model. Thus, the model with the lowest AIC is generally also the model for which there is the least evidence (from the quasi-LR test) against it relative to the reduced-form model. In two cases, Djokovic serving to Federer and Sampras serving to Agassi, the AIC selects the reduced-form model and the LR test strongly rejects all three structural models.

For the other eight servers, the AIC selects the fully dynamic model in only one case, Djokovic serving to Nadal. It selects the point-myopic model for four other servers and the serve-myopic model for the remaining three. We would expect the serve-myopic model to be resoundingly rejected because it does not allow the server to look even just one serve ahead to take

[37] Because of limited space, we do not provide the 32 parameter estimates of $(\theta_{\text{in}}, \theta_{\text{win}})$ and their standard errors for all 10 servers and all three structural models. We are happy to provide these results to interested readers on request.

advantage of the option value of a second serve when hitting a first serve. However, the serve-myopic model does implicitly reflect adjustments in serve strategy via the POPs that may reflect a server's ability to look ahead. For example, the estimated POPs for the second serve in the serve-myopic specification show a lower probability of faulting (presumably because the server reduces the speed of the second serve) but a lower probability of winning the rally, given that the second serve is in (presumably because of the receiver's improved ability to return a slower serve). Therefore, the serve-myopic model is able to reflect state dependence in tennis serves via its effect on the POPs, which is why it is not so surprising that this model performs as well as it does.

Note that the estimated scale parameters $\hat{\lambda}$ for all specifications are uniformly small, so we find a limited role for "trembles" to explain the observed mixed serve strategies of these players. Instead, the maximum-likelihood estimates of the POPs $(\hat{\theta}_{\text{in}}, \hat{\theta}_{\text{win}})$ are distorted in a manner that results in conditional win probabilities much closer to equal than the ones implied by the reduced-form estimates of the POPs. Note that the λ estimates decline for the structural models that require increasingly "farsighted" calculations by the server. When λ is sufficiently small, the conditional value functions $V_\lambda(x, m, d)$ are extremely close to the conditional win probabilities, per the limiting result in equation (10). But when λ is larger, the trembles play a more important role in the mixed serve strategies, allowing more freedom for the conditional value functions (and the conditional win probabilities) to differ across serve directions.

Table 9 provides the estimated service-game win probabilities and p-values of DHW tests of the different model specifications. Recall that this test is based on comparing the implied win probabilities calculated via equation (6) to the nonparametric estimate of those probabilities; the latter is simply the fraction of service games between a given server and receiver that the server won. The first column of table 9 presents the nonparametric estimates of the win probabilities and their standard errors, and the remaining columns present the estimated win probabilities implied by equation (6), with standard errors calculated via the delta method.[38]

We see that the specification tests strongly reject the fully dynamic model, with the exception of Djokovic serving to Nadal. Recall from table 8 that the AIC selects the fully dynamic model as the preferred specification for Djokovic serving to Nadal, so it is reassuring to know that it is

[38] Note that the model estimates are relatively efficient estimates of the win probabilities (as reflected by their smaller standard errors), but they are consistent only if the model specification is correct. The less efficient nonparametric estimator of the win probabilities is consistent regardless.

TABLE 9

Service-Game Win Probabilities and DHW Tests

Server → Receiver	Nonparametric Win Probability $\hat{W}(1,1)$	Reduced-Form Model $W(1,1)$	Serve-Myopic Model $W(1,1)$	Point-Myopic Model $W(1,1)$	Fully Dynamic Model $W(1,1)$
Federer → Nadal	.796 (.026)	.829 (.023)	.825 (.021)	.830 (.021)	.749 (.021)
P-value		.004	.050	.025	1.3×10^{-3}
Nadal → Federer	.786 (.026)	.807 (.023)	.806 (.022)	.807 (.022)	.641 (.022)
P-value		.107	.147	.132	1.0×10^{-22}
Federer → Djokovic	.810 (.024)	.818 (.020)	.818 (.020)	.818 (.020)	.759 (.020)
P-value		.504	.506	.509	1.0×10^{-4}
Djokovic → Federer	.781 (.025)	.781 (.023)	.779 (.022)	.778 (.022)	.746 (.021)
P-value		.910	.838	.756	.011
Nadal → Djokovic	.712 (.035)	.712 (.034)	.703 (.033)	.706 (.033)	.650 (.032)
P-value		.992	.485	.675	4.1×10^{-5}
Djokovic → Nadal	.829 (.029)	.848 (.023)	.846 (.023)	.850 (.023)	.797 (.024)
P-value		.278	.318	.231	.052
Djokovic → Murray	.794 (.034)	.792 (.029)	.791 (.030)	.791 (.030)	.750 (.029)
P-value		.871	.840	.844	.012
Murray → Djokovic	.721 (.038)	.717 (.036)	.717 (.034)	.718 (.035)	.584 (.032)
P-value		.675	.792	.825	1.6×10^{-11}
Sampras → Agassi	.885 (.028)	.866 (.024)	.863 (.024)	.866 (.024)	.757 (.028)
P-value		.150	.130	.187	0
Agassi → Sampras	.874 (.029)	.859 (.024)	.854 (.026)	.853 (.026)	.715 (.028)
P-value		.362	.183	.150	1.4×10^{-58}

Note.—Standard errors are in parentheses.

not rejected by the specification tests. But for the other servers, we note that the fully dynamic model typically significantly underestimates the true service-game win probability. This is caused by the need to distort the POPs to rationalize serve behavior as a best response to the estimated POPs in the fully dynamic model. As we will show in the next subsection, the serve strategy for the fully dynamic model is close to the "true" serve strategy captured by the reduced-form model, but the estimated POPs from the fully dynamic model imply far less favorable performance for the server than the POPs estimated from the reduced-form model. Indeed, the fully dynamic POPs generally imply both a higher probability of faults and a lower probability of winning the rally, given that a serve is in, compared to the reduced-form POPs. In contrast, the specification tests are generally unable to reject the point-myopic and serve-myopic models. This is consistent with the results we reported in table 8, where we showed that these models were the ones most frequently selected as having the lowest AIC values.

Note that when λ is sufficiently small, the structural models predict that the effect of trembles is negligible, and servers will choose to serve to the direction with the highest win probability. In this situation, in order to fit the observed mixed serve strategies, the model is forced to equate conditional win probabilities. We see this most clearly in the inability of the omnibus Wald test to reject the hypothesis of equal conditional win probabilities for the fully dynamic model (not reported, for space considerations). For the point-myopic and serve-myopic models, we showed that the estimated λ values are larger, so trembles play a greater role in explaining serve strategies. This allows more freedom for these models to rationalize the observed mixed strategies without having to equate conditional win probabilities, which is reflected in turn by more rejections of equal win probabilities for these models, especially the serve-myopic model. The reduced-form model places no constraint on the estimation of the POPs, since it estimates separate parameters and likelihoods for the CCPs and POPs. This flexibility results in nearly unbiased estimates of the POPs and their implied service-game win probabilities.

We also observe significant dynamic attenuation in the restricted structural estimates of the POPs. That is, as we noted in the previous section, the reduced-form estimation results reveal much stronger evidence of serial correlation in serve directions, compared to the POPs. In the fully dynamic model, the degree of serial correlation in both serve directions and the POPs is attenuated (i.e., closer to zero), and it is thus less likely to be statistically significant. In fact, for most servers, the fully dynamic model does not exhibit any statistically detectable serial correlation in the structural estimates of the POPs, though it does predict serial correlation in serve directions. What explains this paradox? The explanation is that when λ is close to zero, serve strategies are very sensitive to small changes in the POPs, since trembles play a negligible role, and the server chooses to serve to the direction with the highest win probability. Thus, it is possible to produce significant muscle-memory effects in serve strategies (i.e., the current serve direction, depending on the direction of the previous serve to the same court) via very tiny oscillations in the POPs that are hard to detect statistically.

Now we return to the key question of this paper: Do these distorted/attenuated estimates of the POPs enable the structural models to rationalize observed serve behavior as mixed strategies consistent with Nash equilibrium? We have shown that, at best, the structural models are able to rationalize observed serve behavior as a best response, but only relative to the server's subjective perceptions of their environment and the receiver, as captured by the structural estimates of the POPs. These subjective beliefs are distorted estimates of the true POPs, which are consistently estimated by the unrestricted reduced-form model. A Nash equilibrium entails a key assumption of rationality, that is, that the players' subjective beliefs about each other coincide with the truth. In the next section, we use DP to directly calculate best-response strategies for our estimates of the true POPs and compare how well these strategies perform relative to the mixed serve strategies the players actually use.

A. *Calculating Best-Response Serve Strategies*

We now provide a more powerful direct test of Nash equilibrium play in tennis: we construct alternative deviation serve strategies that significantly increase a server's chance of winning the service game, compared to the mixed strategy they are actually using. If the hypothesis of Nash equilibrium is correct, it should be impossible to construct any such deviation strategies. We construct optimal deviation strategies via DP, setting $\lambda = 0$ and using the reduced-form estimates of the POPs. The DP solution results in pure serve strategies that exploit the unequal win probabilities reflected in the reduced-form estimates of the POPs. At each stage of the game, the DP serve strategy chooses the serve direction that has the maximum conditional win probability (see eq. [4] of sec. III), where the optimal conditional win probability $W(x, m, d)$ is calculated via the Bellman equations given in equations (1)–(3) of section III.

Table 10 presents the optimal DP service-game win probability, as well as game win probabilities implied by three other potentially suboptimal serve strategies. For convenience, we repeat the first three columns of table 9, which show the 10 player pairs, the nonparametric win probabilities, and the reduced-form estimates of the win probabilities. The latter are calculated from the estimated POPs and mixed serve strategies of the reduced-form model using equation (5) in section III.C. As we noted, the reduced-form estimates generally closely match the nonparametric

TABLE 10
IMPROVEMENTS IN SERVICE-GAME WIN PROBABILITIES

Server → Receiver	Nonparametric Win Probability	Reduced-Form Model	Serve-Myopic Model	Point-Myopic Model	Fully Dynamic Model	Wald Test p-Value
Federer → Nadal	.796 (.026)	.829 (.023)	.856	.894	.894	.002
Nadal → Federer	.785 (.026)	.807 (.023)	.840	.884	.884	.014
Federer → Djokovic	.810 (.024)	.818 (.020)	.823	.870	.877	.024
Djokovic → Federer	.782 (.025)	.781 (.023)	.850	.863	.869	.002
Nadal → Djokovic	.712 (.035)	.712 (.034)	.855	.916	.916	.0004
Djokovic → Nadal	.829 (.029)	.848 (.023)	.937	.927	.937	.0001
Djokovic → Murray	.794 (.034)	.792 (.029)	.901	.905	.905	.0002
Murray → Djokovic	.721 (.038)	.717 (.036)	.845	.860	.869	.001
Sampras → Agassi	.885 (.028)	.866 (.024)	.942	.949	.949	.003
Agassi → Sampras	.874 (.029)	.859 (.024)	.912	.935	.936	.040

NOTE.—Standard errors are in parentheses.

estimates, and thus they constitute our best estimates of each server's win probability implied by the mixed serve strategy they actually use. The next three columns of table 10 show counterfactual game win probabilities for the serve-myopic, point-myopic, and fully dynamic serve strategies, also using equation (5). In all three cases, we calculate game win probabilities with the reduced-form estimates of the POPs, not the distorted structural estimates of the POPs in table 9. We also fix $\lambda = 0$, so we do not allow for any "trembles" in our calculated serve strategies.

By construction, the fully dynamic serve strategy maximizes the game win probability, which we see in table 10. However, if the GMC holds, the optimal point-myopic serve strategy coincides with the fully dynamic strategy and implies the same game win probability. Therefore, failures in the GMC are revealed by cases where the fully dynamic game win probability is strictly higher than the win probability implied by the optimal point-myopic serve strategy. We do observe some violations of the GMC in table 10, but in all cases, the incremental gain from using DP to compute an optimal dynamic serve strategy is small.

The last column of table 10 presents the p-value of a Wald test for Nash equilibrium. The test is constructed by appealing to the one-shot deviation principle, which states that there is no deviation at any stage of a dynamic game that can increase the server's chance of winning, given the strategy of the receiver and the service-game continuation values. We find that there are profitable one-shot deviations at many stages of the service game, and while each such deviation yields a modest improvement in the game win probability, the cumulative effect of all profitable deviations is often a large improvement in the game win probability. Of course, if a server were to switch to the DP best response, the receiver would eventually detect the change and adjust his own strategy, which would change the POPs and thus offset some of the gains we predict.

Recall that σ_S was used in section III to denote the optimal serve strategy, which is an implicit function of the POPs Π that we now make explicit by writing $\sigma_S(\Pi)$. Let P^* and Π^* denote the true equilibrium mixed serve strategy and POPs, respectively, in an MPE. By assumption, the players have common knowledge of these POPs. While we do not directly observe P^* and Π^*, we can consistently estimate them with sufficient data. In particular, the hypothesis of Nash equilibrium implies that for any alternative serve strategy σ, we have

$$W(P^*, \Pi^*) \geq W(\sigma, \Pi^*). \tag{23}$$

Let $\sigma_S(\Pi^*)$ be the optimal dynamic serve strategy (generally a pure strategy) calculated by DP for the true Nash equilibrium POPs Π^*. Then by definition of optimality, we have

$$W(\sigma_S(\Pi^*), \Pi^*) \geq W(P^*, \Pi^*) \geq W(\sigma, \Pi^*) \tag{24}$$

for all stationary Markovian serve strategies σ. Together, inequalities (23) and (24) imply the key equality:

$$W(P^*, \Pi^*) = W(\sigma_S(\Pi^*), \Pi^*), \tag{25}$$

which serves as the basis for our direct test of a mixed-strategy Nash equilibrium in tennis: the optimal DP serve strategy should not result in a higher win probability, compared to the mixed serve strategy P^* that the server actually uses.

Using consistent estimators of the game win probabilities on the left- and right-hand sides of equation (25), we can construct a test statistic based on the squared standardized difference of these win probabilities, which has a χ^2 distribution with 1 degree of freedom if the null hypothesis is true. The last column of table 10 presents the p-values for this test, and it shows that we strongly reject the best-response property for a mixed-strategy equilibrium (see eq. [25]) for all 10 player pairs.

B. Evaluating the Robustness of Deviation Gains

Our tests of the hypothesis of Nash equilibrium are of course based on estimates of the POPs rather than the true POPs. In small samples, estimation error could result in spurious, upward-biased estimates of service-game win probabilities from noisy estimates of the POPs (instead of the unobservable true POPs) in best-response serve strategies calculated by DP. To address this possibility, we use stochastic simulations to demonstrate the robustness of our conclusions by comparing game win probabilities of the estimated mixed strategies the players use and our calculated DP best-response strategies over a large number of randomly drawn POPs.

We draw the random POPs from the asymptotic distribution of the maximum-likelihood estimator centered on the point estimates of the reduced-form POP parameters $(\hat{\theta}_{\text{in}}, \hat{\theta}_{\text{win}})$. We then calculate POPs implied by these simulated parameter values to generate a set of POPs that are randomly distributed about the true POPs. For each realization of the POPs, we calculate the game win probability when fixing the mixed serve strategy at its estimated value $\hat{P}$ and fixing our estimated DP best-response serve strategy at the value calculated with the reduced-form point estimate of the POPs, $\sigma(\hat{\Pi})$. This results in a distribution of simulated win probabilities for the two fixed serve strategies, allowing us to determine whether the DP serve strategy outperforms the estimated mixed serve strategy in a range of environments near the true POPs. This eliminates any advantage the DP strategy obtains from assuming that the estimated POPs are the same as the true POPs. Thus, we force the DP strategy to confront POPs it is not "expecting."

We also calculate similar distributions of win probabilities, but using simulated draws from the structural estimates of the POPs. We call these random draws the "perturbed POPs," and figure 4*A* shows the cumulative distribution functions (CDFs) of simulated game win probabilities for three different cases of Federer serving to Djokovic: (1) the fully dynamic serve strategy $\sigma(\hat{\Pi})$ (calculated with $\lambda = 0$) against 500 random perturbations of the reduced-form estimates of the POPs $\hat{\Pi}$ (solid black line), (2) the fully dynamic strategy (calculated with $\hat{\lambda} > 0$) against 500 perturbations of the structural estimates of the POPs (solid gray line), and 3) the fully dynamic strategy (calculated with $\lambda = 0$) against 500 perturbations of the structural estimates of the POPs (dashed line). Note that the strategy used to construct CDF 2 is a mixed strategy, whereas CDFs 1 and 3 are calculated by DP with $\lambda = 0$ and thus are pure strategies.

We see that CDFs 2 and 3 (solid and dashed gray lines, respectively) are nearly identical, which is an illustration of the "no-deviation-gains" condition in equation (25) when we assume that the server is using an MPE strategy. Even though equal win probabilities do not hold for the 500 perturbations of the structural estimates of the POPs, they are close enough to holding that the win probabilities implied by the pure strategy (i.e., CDF 3) do not systematically outperform those implied by the mixed strategy (i.e., CDF 2). In contrast, CDF 1 (black line) is the win probability CDF implied by the DP serve strategy (with $\lambda = 0$) against perturbations of the

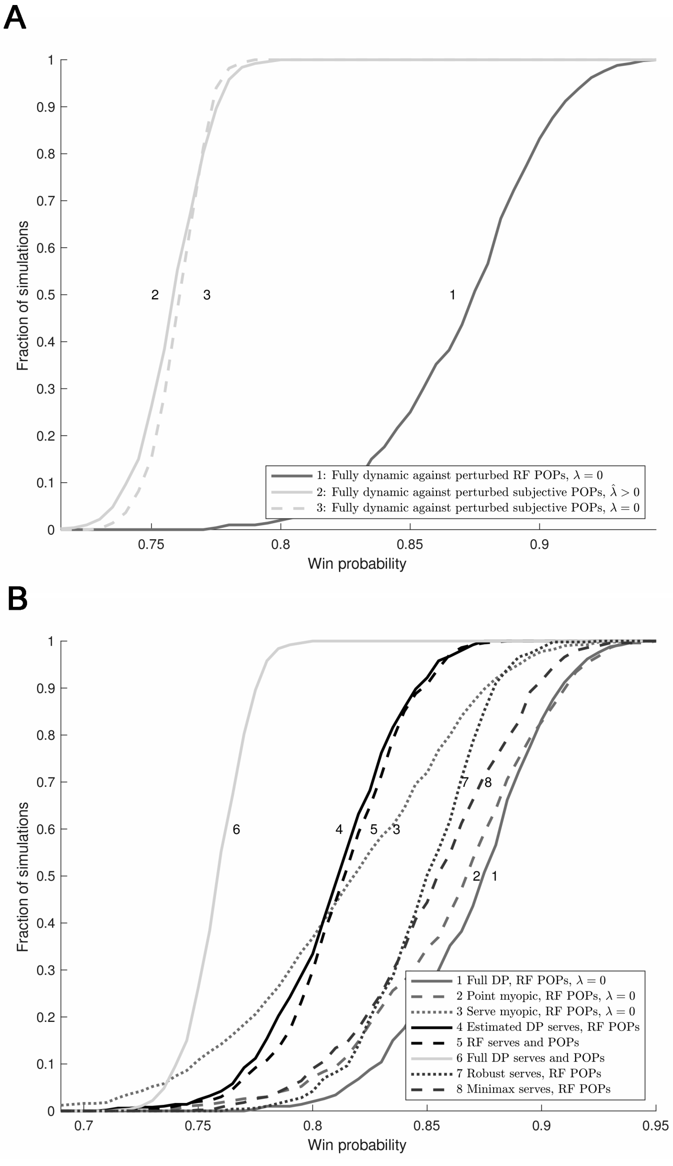

FIG. 4.—Distributions of win probabilities, Federer serving to Djokovic. RF = reduced-form.

reduced-form estimates of the POPs, and it clearly stochastically dominates the other two CDFs.

This result illustrates the large increase in game win probabilities resulting from an optimal serve strategy based on an unbiased estimate of the POPs. As we have already noted, the structural estimates of the POPs are distorted to rationalize the observed mixed serve strategy as a best response to the POPs. Even though the reduced-form estimates of the POPs may reflect some small-sample noise, they indicate sufficiently large departures from the equal-win-probabilities restriction to result in the large deviation gains illustrated by the solid black CDF 1 in figure 4. Essentially, while the structural model can "rationalize" mixed serve strategies, it can do so only via irrational, distorted estimates of the subjective POPs.

Figure 4*B* plots distributions of CDFs for other serve strategies. The line labeled 4 shows the CDF of game win probabilities implied by the estimated mixed strategy from the reduced-form model, whereas line 5 is the CDF of game win probabilities implied by the estimated strategy from the fully dynamic structural model. Both CDFs are calculated using random perturbations of the reduced-form estimates of the POPs. We see that CDF 4 lies nearly on top of CDF 5, indicating that the estimated strategy from the fully dynamic model is virtually the same as the mixed strategy estimated by the reduced-form model. This result illustrates how the fully dynamic structural model of serve behavior succeeds in "rationalizing" observed mixed serve strategies.

Note that CDF 6 and CDF 1 are the same as their counterparts in figure 4*A* and are included for reference (except that CDF 6 in *B* is CDF 2 in *A*). Recall that CDF 6 plots the game win probabilities implied by the estimated fully dynamic mixed strategy (i.e., with $\hat{\lambda} > 0$) against perturbations of the subjective POPs, whereas line 1 shows the CDF of game win probabilities implied by running the fully dynamic model (i.e., with $\lambda = 0$) on perturbations of the unrestricted reduced-form estimate of the POPs. Thus, the improvement from CDF 6 to CDF 5 can be thought of as the increase in win probabilities from replacing the distorted subjective POPs with the unrestricted reduced-form, or "rational," POPs (but in both cases fixing the estimated mixed strategy from the fully dynamic model). Meanwhile, the additional gain in win probabilities by moving from CDF 4 or 5 to CDF 1 comes from systematically exploiting unequal win probabilities at all points in the game tree. The fact that the CDF 1 clearly stochastically dominates CDFs 4 and 5 illustrates that Federer's statistically significant 5.9 percentage point increase in game win probabilities in table 10 not only holds at the point estimate of the POPs but also is robust to significant unexpected deviations of the POPs.

CDFs 2 and 3 in figure 4*B* are the distributions of win probabilities implied by the serve-myopic strategy and the point-myopic strategy, respectively, both computed with $\lambda = 0$ against 500 perturbations of the reduced-form estimates of the POPs. The fact that CDFs 1–3 are also ordered by stochastic dominance shows that the point-myopic strategy outperforms the serve-myopic strategy and that the fully dynamic strategy outperforms both. This figure illustrates a case where the GMC does not hold, though most of the gain comes from using a point-myopic strategy over a serve-myopic one. However, Federer would still benefit from the small but nonetheless significant additional gain from adopting the fully dynamic strategy.

We also use an informal "robust-control" approach to calculate two additional serve strategies illustrated by CDFs 7 and 8. CDF 7 is computed using a "robust strategy" that constitutes a simple average of the optimal fully dynamic strategies for each randomly drawn set of POPs. The robust strategy is a mixed strategy, which is a desirable property if receivers have more difficulty finding best responses to mixed than to pure strategies. In addition, CDF 8 shows the performance of a minimax strategy where we computed the fully dynamic strategy for the worst-case draw of POPs, that is, the fully dynamic best response for the set of POPs that results in the lowest game win probability over all the randomly drawn sets of POPs. Note that to construct these CDFs, we independently draw another 500 random perturbations of the POPs and run the robust and minimax strategies on them. Both these CDFs stochastically dominate CDF 4, where the latter is computed using our reduced-form estimate of Federer's actual mixed serve strategy. Neither of the robust strategies does as well as the fully dynamic strategy, CDF 1, however.

Overall, it appears that the optimal DP serve strategy, which is a pure strategy, performs surprisingly well in environments it is not "expecting." This could be because it is a pure strategy, and pure strategies may be fairly robust to perturbations in the POPs because they are frequently "corner solutions" that will not change in response to sufficiently small changes in the POPs. In any event, we leave further exploration of this topic, and a deeper assessment of the value of more sophisticated versions of robust control, to future work. Finally, we note that the optimal pure strategies that we calculate by DP are intuitive and relatively simple to describe verbally. For example, in the case of Djokovic serving to Nadal, the fully dynamic serve strategy generally entails serving to Nadal's right (i.e., backhand, since Nadal is a lefty) on first serves, whereas on second serves, the optimal direction depends on the whether Djokovic is serving to the deuce or ad court. To the deuce court, he should serve to Nadal's backhand, whereas to the ad court, he should serve to Nadal's forehand. That is, Djokovic should hit his second serve wide.

C. *Results for Additional Server-Receiver Pairs and Surfaces*

We summarize our core findings for every server-receiver-surface combination for which we have sufficient data to estimate our model. In total, we estimated the model for 99 distinct server-receiver-surface combinations. We decisively reject the hypothesis that the estimated mixed serve strategies are consistent with equilibrium play for all 99 cases, using the Wald test of the absence of deviation gains, that is, tests that equation (25) holds.

Figure 5 summarizes the deviation gains from switching to the best-response strategies we calculate by DP. This figure has four panels that plot the gain in win probability for different groups of players. In each panel, the vertical axis is the ratio of the mean win probabilities from the fully dynamic best-response serve strategy to the win probability implied by our reduced-form estimates of each server's actual serve strategy, and the horizontal axis is the mean win probability under the actual serve strategy. Thus, the ratio of mean win probabilities shows the relative improvement in the win probability from adopting the fully dynamic serve strategy.

We calculate the probabilities in figure 5 using the same procedure as in section V.B; namely, by calculating win probabilities that are robust to estimation error in the POPs. Specifically, we estimate our structural model for each server-receiver pair. We then calculate win probabilities for the observed server strategies and the fully dynamic best response for 500 different POPs that are IID draws from the estimated asymptotic distribution about the point estimates of the POPs for each server in each server-receiver pair.

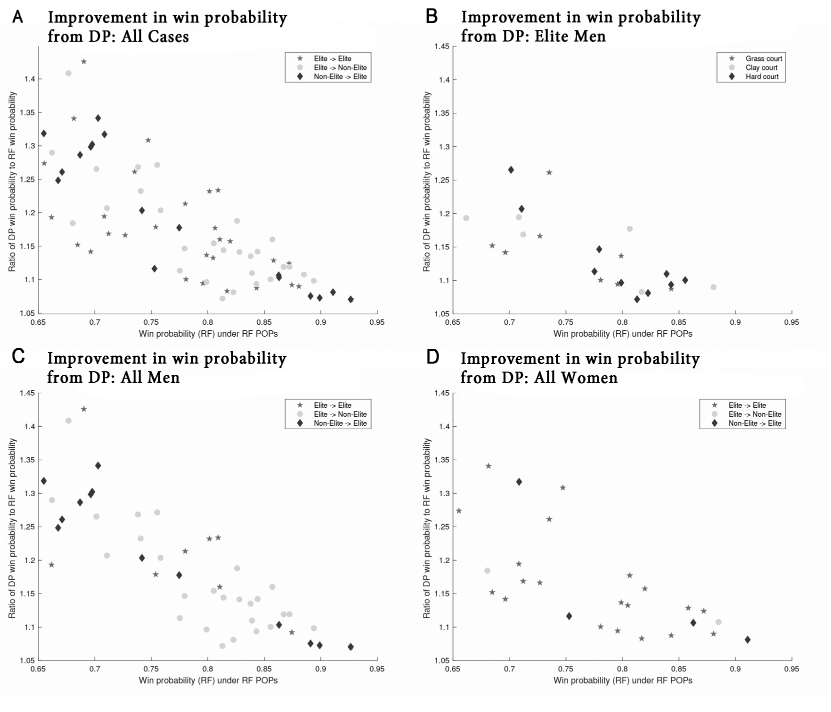

FIG. 5.—Relation between player ability and deviation gains for different player groups. RF = reduced-form.

Thus, each point plotted in the figures represents the average "deviation gains" from using the DP server strategy over the 500 randomly drawn POPs.

Figure 5*A* shows a scatter plot of the improvements for 99 top-ranked server-receiver pairs for which we had sufficient data from the MCP to reliably estimate the POPs. The points are coded with stars indicating our calculated mean deviation gains from using the optimal DP server strategy relative to the servers' existing mixed strategy for "elite" servers playing "elite" receivers (where we classify a player as "elite" if they were ever ranked first or second worldwide in their career). The circles plot results for elite servers serving to non-elite receivers, the diamonds are for non-elite servers serving to elite receivers (we do not show non-elite servers serving to non-elite receivers, since we had too few of these cases in our dataset).

The most striking finding in this graph is the obvious downward-sloping pattern in the scatter plot: we predict that servers with lower win probabilities experience the biggest relative deviation gains from switching to the DP serve strategy. Of course, since win probabilities cannot be higher than 1, the relative gain is constrained to decline as the win probability under the server's existing serve strategy approaches 1. Nevertheless, the results indicate a clear correlation between "ability," as measured by the server's existing win probability, and the extent of their suboptimality: we predict that the less effective servers have the most to gain from using DP to optimize their serve strategies.

Figure 5*B* plots the deviation gains across surfaces (stars for grass, circles for clay, and diamonds for hard courts) for the 10 most "elite" server-receiver pairs that we have focused our analysis on throughout this paper (i.e., the 10 pairs in table 10, resulting in 25 player pair/surface combinations). We see that the elite servers have somewhat higher service-game win probabilities (77%, vs. 73% for all 99 player pair/surface combinations) and also lower deviation gains compared to the set of all players plotted in the figure 5*A* (the mean deviation gain from adopting the DP strategy is 15% for the elite players vs. 26% for all 99 player pairs). The biggest relative gain from adopting the DP serve strategy is for the non-elite players serving to elite opponents: these players can expect a 30% improvement in their win probabilities from adopting the DP serve strategy.

Figures 5*C* and 5*D* plot the results for men and women, respectively. The same negative correlation between deviation gains and ability, as measured by win probability under their existing serve strategy, is apparent for both men and women servers. The relationship between average win probability and the calculated gain to switching to the fully dynamic serve strategy is also robust across the two sexes. In particular, the male servers in our analysis have a higher average probability of winning under their current serve strategy (74% for men vs. 70% for women) and a lower average deviation gain from switching to the DP serve strategy (23% for men vs. 30% for women).

VI. Conclusion

There is substantial evidence against Nash equilibrium and minimax play in laboratory experiments: see, for example, Brown and Rosenthal (1990) and Camerer (2003). However, a standard critique is that laboratory subjects are not sufficiently trained and incentivized to behave sufficiently closely to the predictions of game theory. The influential study by Walker and Wooders (2001, 1535) concluded that "the theory has performed far better in explaining the play of top professional tennis players in our data set." Similar results have been found in other sports, such as soccer (see, e.g., Chiappori, Levitt, and Groseclose 2002, who studied the direction of penalty kicks). The general conclusion is encapsulated in the title of the study by Palacios-Huerta (2003), "Professionals Play Minimax" (see also Palacios-Huerta 2014).

In contrast, we show that the serve strategies of elite tennis pros are inconsistent with the minimax prediction. Though they use mixed strategies, win probabilities are not the same for all serve directions at all stages of the game—the key restriction of the Nash equilibrium/minimax solution. There has also been considerable work on testing for serial independence in serve directions as an additional implication of mixed-strategy equilibrium. We argue in section 1 of the appendix that serial dependence, which has been found in many previous studies, including Walker and Wooders (2001), is not necessarily inconsistent with equilibrium play when we account for muscle-memory effects that reflect natural improvements from repeating recently performed actions. We also show that such muscle-memory effects can induce both positive and negative serial correlation in serve directions and that it is important to account for it to explain observed serve behavior.

Our empirical analysis exploits a new source of data, the Match Charting Project (MCP), that allows us to analyze a large number of professional tennis matches at the level of individual server-receiver pairs. We also include body serves—a feature of the MCP data—along with the left and right serves in the previous literature. Tennis players and coaches consider body serves to be an important component of an optimal serve strategy. Our analysis supports this view, since body serves are used frequently in the data and in the calculated optimal serve strategies.

However, the inclusion of body serves and access to more observations are not the main reason why we reject the hypothesis of Nash equilibrium play. Our main innovation is to provide new, more powerful tests of the dynamic implications of Nash equilibrium. Specifically, we introduce new tests of the one-shot deviation principle and an omnibus Wald test of the equal-win-probability restriction for all serve directions in all states that must hold in a completely mixed-strategy MPE. The latter test strongly rejects the hypothesis of equal win probabilities for the majority of the 10 elite professional server-receiver pairs we analyze, as well as the majority of an additional 66 male and female top-ranked professional pairs. We also introduce a new test of the one-shot deviation principle, that is, the restriction that in an MPE there is no deviation strategy that strictly improves the payoff of the players. Using numerical DP and our econometric estimates of the POPs that capture the probabilistic outcomes of serves to each possible direction, we reject the hypothesis that the observed mixed strategies of these elite pro servers constitute best responses.

Previous approaches to testing for equal win probabilities over serve directions focused on the probability of winning individual points, whereas we recursively calculate how the choice of serve direction affects the probability of winning the entire service game. Tests based on the former have low power to detect evidence of disequilibrium play because (as we show in sec. 4 of the appendix) deviation gains for individual points are smaller and statistically more difficult to detect. By focusing on the conditional win probabilities for the entire service game, we develop much more powerful tests of the key implications of Nash equilibrium play that exploit the magnification effect—that small deviation gains at each individual point cumulate into much more substantial and easier-to-detect deviation gains in the service game as a whole. Using DP to construct best-response serve strategies, we show that they significantly increase the probability of winning the overall service game. Then, using stochastic simulations, we show that our calculated deviation gains are robust, in the sense that they result in significantly higher win probabilities even when the true POPs differ from the estimated POPs that the strategies are "expecting."

Similar to that of Walker and Wooders (2001), our conclusion is based on a key stationarity assumption that all learning and strategy experimentation has already taken place and that strategies do not change across games. However, our stationarity assumption is substantially weaker than Walker and Wooders's: like them, we assume stationary play across service games, but unlike them, we relax the assumption of stationary play over different states within a service game. We show that serve strategies and win probabilities vary significantly across states within an individual service game in tennis. We also show that the stationarity assumption is testable and that we cannot reject stationarity of the POPs across service games, though we do reject stationarity of the serve strategies (CCPs) both within and across games. We interpret the latter rejection as further evidence against minimax play, since if the POPs are stationary across games and serve strategies correspond to a unique MPE of the service game, then serve strategies should be stationary across games as well.

A reviewer noted that our structural model fails to incorporate persistent private information of the players, such as their health or stamina during a match, and that this failure could result "in the econometrician observing POPs that are very different from the POPs observed by the player. Therefore, the econometrician will be using the wrong statistics to test the 'equal win probabilities' hypothesis, rendering the test invalid." We acknowledge this limitation, though we do account for persistent public information (e.g., muscle memory) and private information that is not persistent across serves. And although we do not account for persistent private information, we do not believe that this necessarily implies that our tests are invalid. Our estimates of the POPs, serve-direction strategies, and the implied conditional win probabilities can be viewed as projections onto information we do observe. Let $W(\xi)$ denote the expected win probability at the start of a tennis service game, where ξ denotes any persistent private information of the players at the start of the game. Though we do not observe ξ, using our data we can estimate $W(x) = E\{W(\xi)|x\}$, which can be treated as the "projection" of the random variable $W(\xi)$ onto the information x we do observe. Thus, our estimates of win probabilities can be viewed as "average win probabilities" that differ from the actual win probability $W(\xi)$ by a serially uncorrelated error term, $W(\xi) = E\{W(\xi)|x\} + \varepsilon$.

We have shown that there exist serve strategies that depend on the public information x and significantly increase the expected win probability $E\{W(\xi)|x\}$. That is, we construct alternative, feasible serve strategies P' that imply counterfactual win probabilities satisfying $W'(x) > E\{W(\xi)|x\}$. Even though these counterfactual serve strategies depend on less information than the players actually have, and even though there may be particular realizations of the unobserved private information ξ for which $W(\xi) > W'(x)$, our counterfactual strategies still improve win probabilities on average, relative to what we actual observe, and thus still constitute "one-shot deviations" that increase expected win probabilities, contradicting the implication that there are no deviation gains in the perfect Bayesian equilibrium (PBE) of the extended model of tennis that includes persistent private information. For this reason, we believe that our tests are still valid even though our model does not incorporate persistent private information.

It would be interesting to try to extend our model to incorporate persistent private information, but doing it involves significant computational and identification challenges. With persistent private information, the natural notion of equilibrium is a PBE, but calculating a PBE requires carrying a posterior belief for each player as a state variable, and these beliefs are potentially infinite-dimensional objects (i.e., conditional probability distributions that depend on the entire history of the game, and potentially previous games in a tennis match). Thus, it appears that computing a PBE in tennis is computationally infeasible, at least given our computational skill. Second, to form a likelihood for inference (or via other simulation-based approach), we would need to "integrate out" the unobserved states of the players, that is, their beliefs about the persistent private information of their opponent (which is also a function of their own persistent private information). It may be difficult or impossible to identify the parameters of such a model. Even in our framework, we do not directly observe the actions of the receiver. This already creates identification challenges, and it is why we limited our analysis to the behavior of the server, since we can reliably observe the serve direction, which is a key strategic decision of the server. However, we do not know how to test whether receivers are playing equilibrium strategies because of our inability to directly observe their choices, such as the anticipation vector, (a^l, a^r, a^b).

This raises the question whether trying to overcome the challenges involved in incorporating persistent private information into the model is justified by the likely promising new empirical insights that might emerge from this extension. Our opinion is that it is not, given the data we have to analyze. The main support for this opinion is the results of our stationarity tests. In particular, we conjecture that the POPs would be nonstationary in a model with persistent private information. For example, if the server has private information about some aspect of their current ability, then presumably the receiver will update their own beliefs about the server's ability throughout the match and adjust their strategy in turn, which, intuitively, will change the POPs. However, we cannot reject stationarity of the POPs across calendar years or between first sets and later sets in a match. We discuss private information in significantly more detail in section 5 of the appendix.

Our finding that many elite tennis pros fail to play serve strategies that are best responses to their opponents may also seem surprising, given the stakes involved in top-level tennis matches, and it is clearly contrary to the consensus in the literature noted above. We believe that we have convincing evidence of suboptimal serve strategies, but the ultimate test would be to run field experiments to verify whether our DP serve strategies really do deliver the increased win probabilities that we predict. Our predicted gains may dissipate rapidly in the field as the receiver recognizes and adapts to a change in the server's strategy. Ultimately, the issues raised by the possibility of learning and adaptation to changes in strategy are fascinating topics for further exploration but are beyond the scope of this analysis.

Though we introduce behavioral models that can explain disequilibrium play as a result of "distorted subjective beliefs," we have not explained why elite players seem to have less than fully rational expectations about their own strengths and weaknesses and those of their opponents. The monetary rewards to increasing the probability of winning by the magnitudes we estimate are very high. The usual presumption in economics and much of the previous literature on tennis is that when there are high rewards, we can expect that some incompletely defined learning process will lead to behavior consistent with Nash equilibrium. At the very least we should not see large gains left unexploited. Thus, our findings are only partially consistent with Simon's (1956) principle of "satisficing": "However adaptive the behavior of organisms in learning and choice situations, this adaptiveness falls far short of the ideal of 'maximizing' postulated in economic theory. Evidently, organisms adapt well enough to 'satisfice'; they do not, in general, 'optimize'" (129).

Our conclusion that even elite pro tennis players may have inadequate statistical knowledge or an inaccurate "mental model" of the POPs is corroborated by the nascent industry of sports analytics, which provides statistical analysis and advice to improve athletes' play in tennis and other sports. It is unlikely that the growth in tennis analytics would be as large as it is if most of the elite pros already have "rational beliefs" and are already playing best-response strategies on their own. Over time, if more elite pros use increasingly powerful analytics to help improve their play, the long-run outcome of this process of learning and experimentation could well be something that more closely approximates Nash equilibrium play.

Ours is not the first study to have provided evidence that suggests that highly compensated and motivated sports professionals may not be behaving optimally. There is the famous book *Moneyball*, by Michael Lewis (2003), that showed how analytics can improve the performance of entire baseball teams. Also, focusing on individual baseball players, Bhattacharya and Howard (2022, 350) found that while pitchers use mixed strategies over different pitches (fastball, curve, etc.), "payoffs differ significantly across pitch types." In football, Romer (2006) used DP to demonstrate that teams were making suboptimal decisions regarding when to go for it on fourth down, punt, or kick a field goal. Tennis may be another sport where econometrics, DP, and analytics can affect thinking, change behavior, and help guide players to play in a way that more closely corresponds to the predictions of Nash equilibrium.

Data Availability

Data for this study were downloaded from www.tennisabstract.com for a selected subset of elite professional men and women tennis players. This website is also known as the "Match Charting Project" (MCP; Sackmann 2013) and includes a huge amount of information on the outcomes of tennis matches including point-by-point descriptions of individual games in each set of each match of each tournament. An example is the point-by-point description of the match between Jannik Sinner and Daniil Medvedev at the 2024 Australian Open, which can be viewed at https://www

.tennisabstract.com/charting/ by appending the match identifier 20240 128-M-Australian_Open-F-Jannik_Sinner-Daniil_Medvedev.html.

We downloaded 3,587 matches from the www.tennisabstract.com website over the period 1970–2018, involving a total of 961 distinct professional tennis players and a total of 548,302 observations of individual serve directions. We focused on a smaller subset of the "elite pros," reducing our dataset to 46 server-receiver pairs that are listed in a text file all_players_list.txt in the Dataverse replication data (Anderson et al. 2024, in the Harvard Dataverse, https://doi.org/10.7910/DVN/RQ6JVL). Most of our analysis in the paper was confined to a subset these 46 pairs that we deemed as the "best of the best," and we further restricted our analysis to matches played on hard surfaces. The data used to estimate the various models in sections I–IV of the paper are contained in the ASCII text file lcrdata.txt. These numerical data were parsed and encoded from the original natural language play-by-play descriptions of tennis matches on https://www.tennisabstract.com/charting/. Our data also contain matches played on grass and clay surfaces. We extended our analysis to a larger subset of elite professional tennis players in section V.C, where we summarized results for three different court surfaces; grass, clay, and hard; for a total of 99 distinct server-receiver-surface combinations that were reported in figure 5 of section V.C. The names of these additional player pairs are in the file all_players_list_extra.txt and the serve and game outcomes are in the file lcrdata_extra.txt on the Dataverse website. Full documentation of the data can be found in the plain text file data_explanation.tex file in the data we uploaded to Anderson et al. (2024). Please contact the corresponding author (Rust) for further questions relating to the data, or for assistance in replicating any results in this paper.

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