# A4 Q2 Wrote Jingbo Yang Read Muzi Zhao

## (a)

We do not care about the outcomes of the coins before $c_i$, as $x$ will record the last coin that gives "Head." We only care about what happens after $c_i$, as we need no further heads to update $x$. Therefore, there are $n-i$ coins after $c_i$, and the probability $\Pr[x=i]$ is:

$$\Pr[x=i] = p_i \cdot \prod_{j=i+1}^{n} (1-p_j)$$

When $p_2 = p_3 = \cdots = p_n = \frac{1}{2}$, the probability becomes:

$$\Pr[x=i] = \begin{cases} \left(\frac{1}{2}\right)^{n-1}, & \text{if } i=1 \\ \left(\frac{1}{2}\right)^{n-i+1}, & \text{if } i>1 \end{cases}$$

## (b)

From $Pr[x=n] = p_n = \frac{1}{n}$, and solving $Pr[x=n-1] = p_{n-1}(1-p_n) = \frac{1}{n}$, we find $p_{n-1} = \frac{1}{n-1}$. Continuing similarly:

$$p_{n-2}(1-p_{n-1})(1-p_n) = \frac{1}{n} \implies p_{n-2} = \frac{1}{n-2}$$

Thus, we conclude $p_i = \frac{1}{i}$.
**Induction Proof:** We prove the general case by induction.
We want to show that for $p_i = \frac{1}{i}$, the probability $\Pr[x=i]$ is $\frac{1}{n}$ for all $1 \le i \le n$.
**Base case:**
For i = 1 ,

$$\Pr[x=1] = p_1(1-p_2)\cdots(1-p_n) = 1 \cdot \frac{1}{2} \cdots \frac{n-1}{n} = \frac{1}{n}$$

For i = 2 ,

$$\Pr[x=2] = p_2(1-p_3)\cdots(1-p_n) = \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n-1}{n} = \frac{1}{n}$$

Thus, the base cases hold.
**Induction step:** Assume the statement holds for $i=k$, $2 <= k < n$

$$\Pr[x=k] = p_k \prod_{j=k+1}^{n} (1-p_j) = \frac{1}{n}$$

We need to show that the statement holds for $i=k+1$:

$$\Pr[x=k+1] = p_{k+1} \prod_{j=k+2}^{n} (1-p_j)$$

By assumption:

$$\Pr[x=k] = \frac{1}{k}(1-\frac{1}{k+1}) \prod_{j=k+2}^{n} (1-p_j) = \frac{1}{n}$$

Thus:

$$\Pr[x=k+1] = \frac{1}{k+1} \cdot \prod_{j=k+2}^{n} (1-p_j) = \frac{1}{k+1} \cdot \frac{1}{n} \cdot (k+1) = \frac{1}{n}$$

Therefore, by induction, $\Pr[x=i] = \frac{1}{n}$ for all $i$.