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Question 1(a)

We execute this procedure with array A and x such that no element of A equals x. Therefore, for every iteration of the while loop, $A[i] \neq x$ is always true, and the algorithm proceeds to the else branch. If the coin flip results in "Head", the algorithm stops. The probability of $k-1$ iterations of "not Head" followed by a "Head" is:

$$(1-p)^{k-1} \cdot p$$

Let X be the number of iterations. We know from above that:

$$P(X=k) = (1-p)^{k-1} \cdot p$$

The expected number of iterations is:

$$E[X] = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = p \cdot \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}$$

Using the sum identity:

$$\sum_{k=1}^{\infty} k \cdot r^{k-1} = \frac{1}{(1-r)^2}, \quad \text{for } |r| < 1$$

Substitute $r = 1-p$:

$$E[X] = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

Specifically if p is zero, the code will never return. In that case, E[X] is infinity.

Question 1(b)

Now, suppose exactly one element in A equals x. The probability of selecting $A[i] = x$ is:

$$P(A[i] = x) = \frac{1}{n}, \quad P(A[i] \neq x) = \frac{n-1}{n}$$

We consider the probability of return or not return at each iteration:
1. If $A[i] = x$, the algorithm returns with probability $\frac{1}{n}$.
2. If $A[i] \neq x$, it may return if the coin flip is "Head", with probability $\frac{n-1}{n} \cdot p$.
Thus, the total probability of return at each iteration is:

$$P(\text{return}) = \frac{1}{n} + \frac{n-1}{n} \cdot p$$

The probability of not returning is:

$$P(\text{not return}) = \frac{n-1}{n} \cdot (1-p)$$

The probability of exactly k iterations is:

$$P(\text{k iterations}) = \left(\frac{n-1}{n} \cdot (1-p)\right)^{k-1} \cdot \left(\frac{1}{n} + \frac{n-1}{n} \cdot p\right)$$