3. (**Flow decomposition**) A flow f is acyclic if the subgraph of directed edges with positive flow contains no directed cycles.

(a) Prove that every flow f has at least one corresponding acyclic flow that has the same value. (In other words, for every graph, at least one maximum flow is acyclic.)

Let f be a flow with at least one cycle. We generate another flow f' with the same value as f but with less cycles. An acyclic flow with the same value is obtained by repeating this procedure until all the cycles are eliminated.

Consider any cycle in the graph of edges carrying positive flow, and let b be the smallest (bottleneck) flow on an edge in the cycle. Flow f' is obtained from f by decreasing the flow by b on every edge in the cycle.

Observe that f' is still a valid flow. The capacity constraint still holds since we only decrease the flow. For the flow conservation, consider a node in the graph. This node is either in the cycle or not. If it is in the cycle, the flow into it decreases by b but the flow out of it also decreases by b. If it is not in the cycle, the total flow into and out of it are not affected. Hence, the flow conservation also holds.

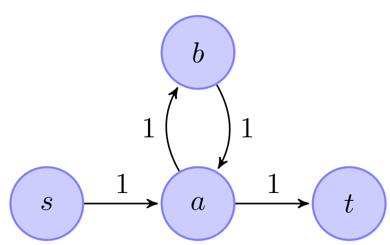
Finally, we show that f' has the same value as f. Notice that the source cannot be in a cycle since it does not have any incoming edges. When a flow f' is generated by decreasing the flow in a cycle, the flow out of the source remains the same as in the original flow f. As f' is a valid flow, f and f' have the same value.

(b) A **path flow** is a flow that gives positive values to a simple, directed path from source to sink. Prove that every acyclic flow is a finite combination of path flows.

To prove by contradiction, assume that an acyclic flow cannot be written as the sum of a finite number of path flows. It means either (1) some part the flow is not represented as a path flow, or (2) an acyclic flow is written as the sum of infinitely many path flows. The first case does not happen since it means the flow has a cycle. The second case does not happen either because there is only a finite number of s-t paths in a graph and each edge has a finite amount of capacity. This completes the proof.

(c) Some flows for a directed graph are *not* a combination of path flows. Give an example of one.

The contrapositive of part (b) is: if a flow cannot be written as a (finite) sum of path flows, it must contain a cycle. A simple example is shown below.



The flow takes the path $s \to a \to b \to a \to t$ but it cannot be represented as a sum of path flows since the path flow $s \to a \to t$ is the only simple s-t path in the graph.