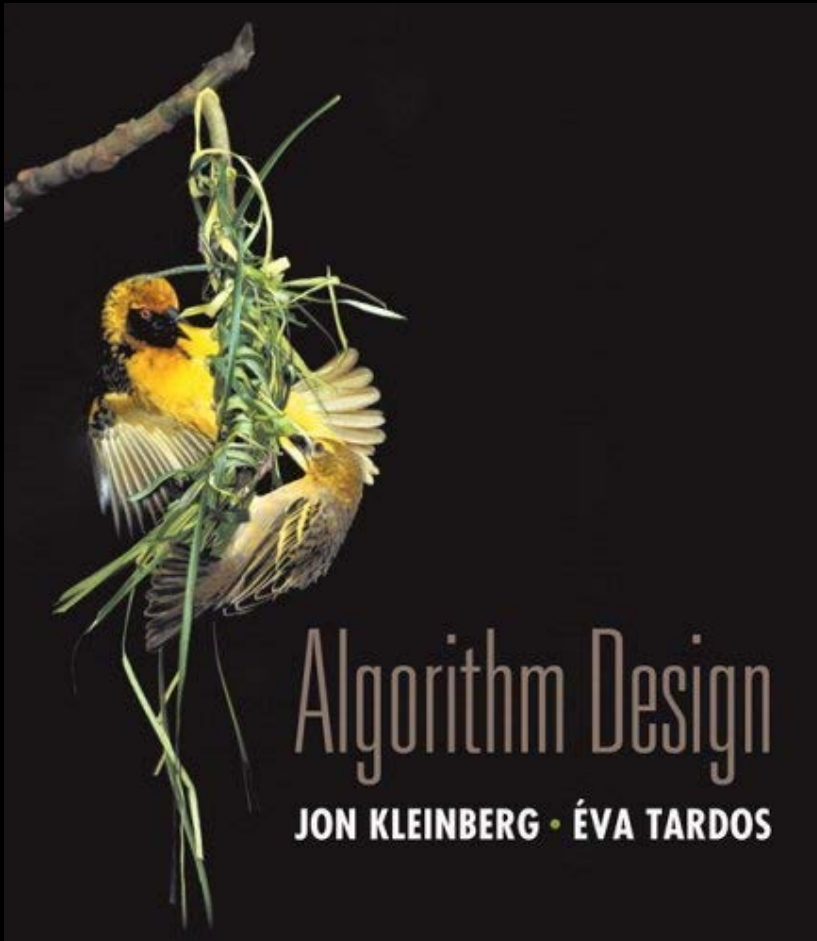


Chapter 6

Dynamic Programming



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Algorithmic Paradigms

Greed. Build up a solution incrementally, myopically optimizing some local criterion.

Divide-and-conquer. Break up a problem into two sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.

Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.

Dynamic Programming Applications

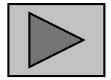
Areas.

- Bioinformatics.
- Control theory.
- Information theory.
- Operations research.
- Computer science: theory, graphics, AI, systems,

Some famous dynamic programming algorithms.

- Viterbi for hidden Markov models.
- Unix diff for comparing two files.
- Smith-Waterman for sequence alignment.
- Bellman-Ford for shortest path routing in networks.
- Cocke-Kasami-Younger for parsing context free grammars.

Fibonacci Sequence



Fibonacci Sequence

- Sequence defined by

- $a_1 = 1$

- $a_2 = 1$

- $a_n = a_{n-1} + a_{n-2}$

1, 1, 3, 5, 8, 13, 21, 34, ...

- How should you compute the Fibonacci sequence?
- Recursive algorithm:

```
Fib(n)
1.  If n =1 or n=2, then
2.      return 1
3.  Else
4.      a = Fib(n-1)
5.      b = Fib(n-2)
6.      return a+b
```

- Running Time?

Review Question

- Prove that the solution to the recurrence $T(n) = T(n-1) + T(n-2) + \Theta(1)$ is exponential in n .

Review Question

- Prove that the solution to the recurrence $T(n) = T(n-1) + T(n-2) + \Theta(1)$ is exponential in n .

Easy to show: $\Omega\left((\sqrt{2})^n\right)$ by inspecting the recursion tree

Later in the course (if time permits):

$\Theta(\phi^n)$ where $\phi \approx 1.618$ is the golden ratio

Computing Fibonacci Sequence Faster

- **Observation:** Lots of redundancy! The recursive algorithm only solves $n-1$ different subproblems
- “**Memoization**”: Store the values returned by recursive calls in a sub-table
- Resulting Algorithm:

```
Fib(n)
1.  If n = 1 or n = 2, then
2.      return 1
3.  Else
4.      f[1] = 1; f[2] = 1
5.      For i = 3 to n
6.          f[i] ← f[i-1] + f[i-2]
7.      return f[n]
```

- Running Time?

$O(n)$ if integer operations take constant time.

Computing Fibonacci Sequence Faster

- Observation: Fibonacci recurrence is linear

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} = \cdots = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$$

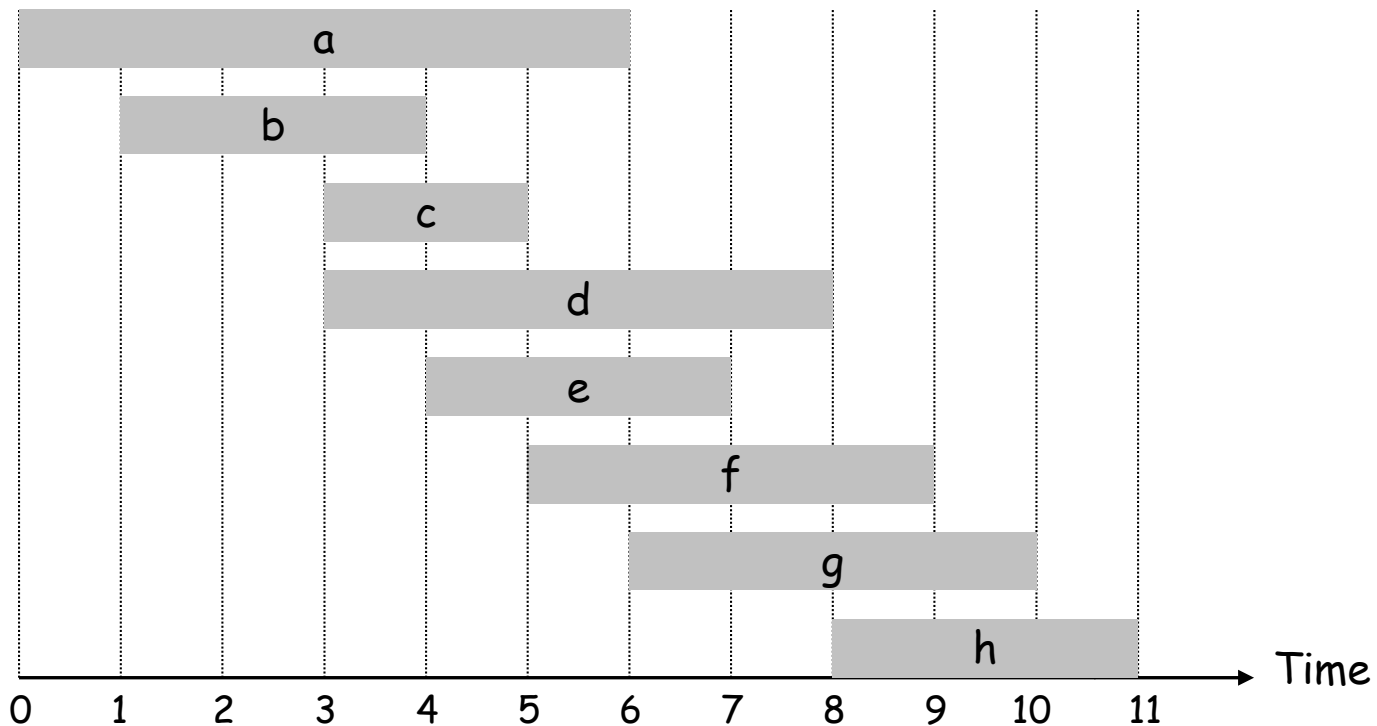
- Can compute A^n using only $O(\log n)$ matrix multiplications; each one takes $O(1)$ integer multiplications and additions.
- Total running time?
 $O(\log n)$ integer operations. Exponential improvement!
- **Exercise:** how big an improvement if we count bit operations?
 - Multiplying k -bit numbers takes $O(k \log k)$ time.
- How many bits needed to write down a_n ?

6.1 Weighted Interval Scheduling

Weighted Interval Scheduling

Weighted interval scheduling problem.

- Job j starts at s_j , finishes at f_j , and has weight or value v_j .
- Two jobs **compatible** if they don't overlap.
- Goal: find maximum **weight** subset of mutually compatible jobs.

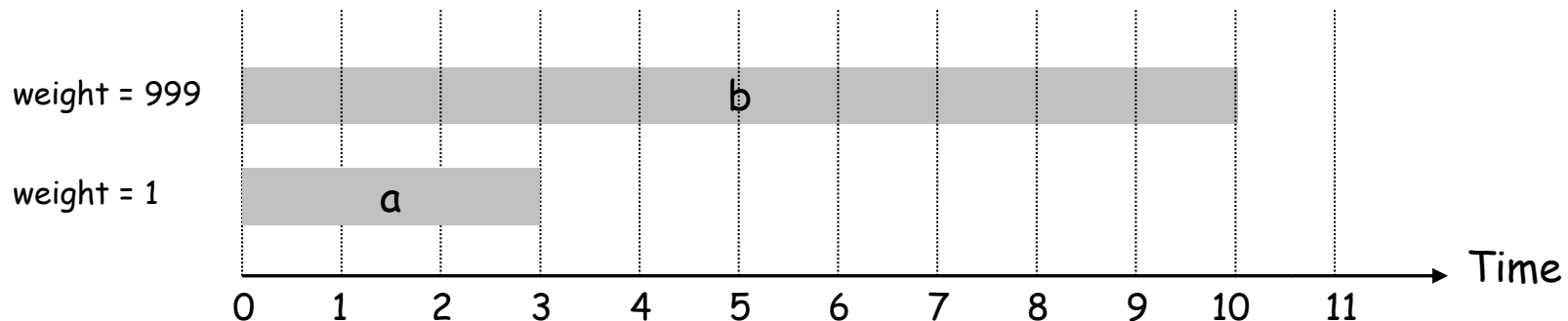


Unweighted Interval Scheduling Review

Recall. Greedy algorithm works if all weights are 1.

- Consider jobs in ascending order of finish time.
- Add job to subset if it is compatible with previously chosen jobs.

Observation. Greedy algorithm can fail spectacularly if arbitrary weights are allowed.

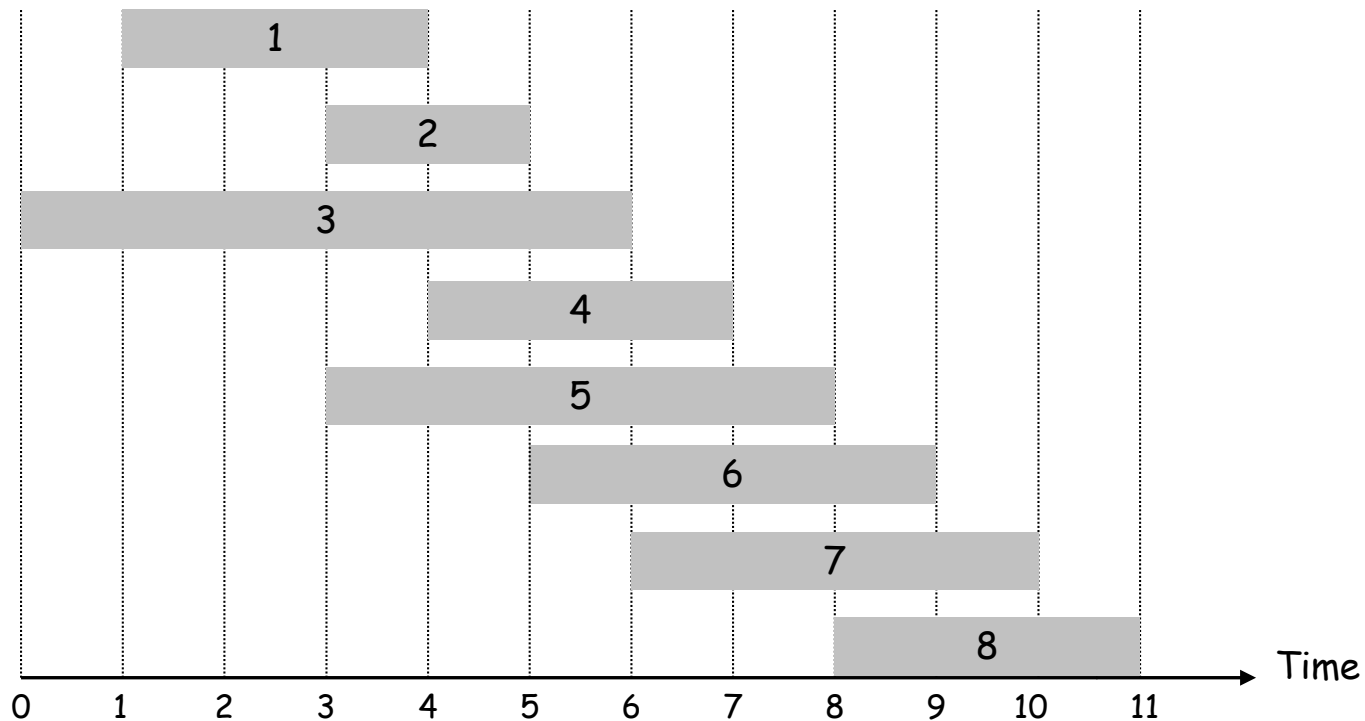


Weighted Interval Scheduling

Notation. Label jobs by finishing time: $f_1 \leq f_2 \leq \dots \leq f_n$.

Def. $p(j)$ = largest index $i < j$ such that job i is compatible with j .

Ex: $p(8) = 5$, $p(7) = 3$, $p(2) = 0$.



Dynamic Programming: Binary Choice

Notation. $OPT(j)$ = value of optimal solution to the problem consisting of job requests 1, 2, ..., j.

• Case 1: OPT selects job j.

- can't use incompatible jobs $\{ p(j) + 1, p(j) + 2, \dots, j - 1 \}$
- must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., $p(j)$

↖
↙
optimal substructure

• Case 2: OPT does not select job j.

- must include optimal solution to problem consisting of remaining compatible jobs 1, 2, ..., j-1

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \max \{ v_j + OPT(p(j)), OPT(j-1) \} & \text{otherwise} \end{cases}$$

Weighted Interval Scheduling: Brute Force

Brute force algorithm.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

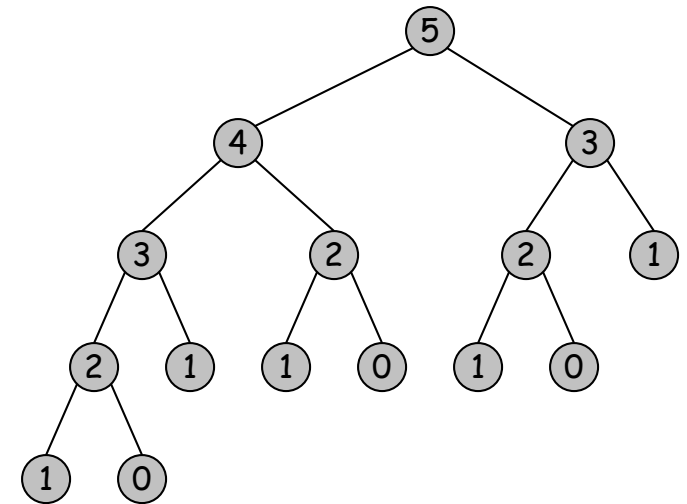
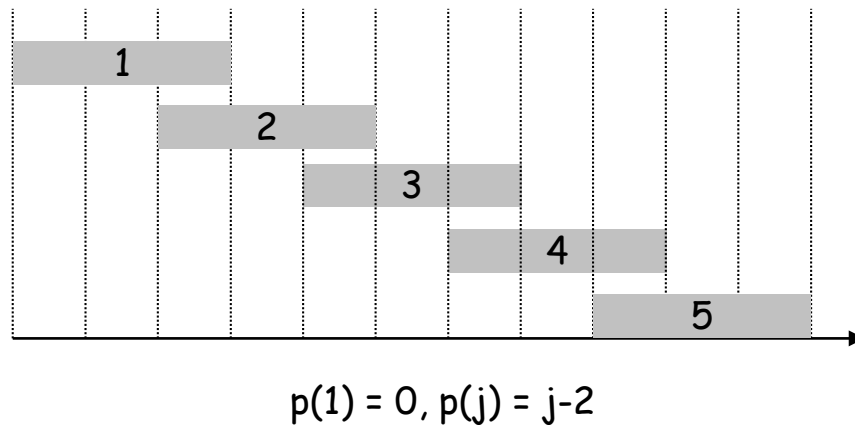
Compute $p(1), p(2), \dots, p(n)$

```
Compute-Opt(j) {  
    if (j = 0)  
        return 0  
    else  
        return max( $v_j + \text{Compute-Opt}(p(j))$ ,  $\text{Compute-Opt}(j-1)$ )  
}
```

Weighted Interval Scheduling: Brute Force

Observation. Recursive algorithm fails spectacularly because of redundant sub-problems \Rightarrow exponential algorithms.

Ex. Number of recursive calls for family of "layered" instances grows like Fibonacci sequence.



Weighted Interval Scheduling: Memoization

Memoization. Store results of each sub-problem in a cache; lookup as needed.

Input: $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$

Sort jobs by finish times so that $f_1 \leq f_2 \leq \dots \leq f_n$.

Compute $p(1), p(2), \dots, p(n)$

for $j = 1$ to n

$M[j] = \text{empty}$ \leftarrow global array

$M[j] = 0$

M-Compute-Opt(j) {

if ($M[j]$ is empty)

$M[j] = \max(w_j + \text{M-Compute-Opt}(p(j)), \text{M-Compute-Opt}(j-1))$

return $M[j]$

}

Weighted Interval Scheduling: Running Time

Claim. Memoized version of algorithm takes $O(n \log n)$ time.

- Sort by finish time: $O(n \log n)$.
- Computing $p(\cdot)$: $O(n)$ after sorting by start time.
- $M\text{-Compute-Opt}(j)$: each invocation takes $O(1)$ time and either
 - (i) returns an existing value $M[j]$
 - (ii) fills in one new entry $M[j]$ and makes two recursive calls
- Progress measure $\Phi = \#$ nonempty entries of $M[\]$.
 - initially $\Phi = 0$, throughout $\Phi \leq n$.
 - (ii) increases Φ by 1 \Rightarrow at most $2n$ recursive calls.
- Overall running time of $M\text{-Compute-Opt}(n)$ is $O(n)$. ▪

Remark. $O(n)$ if jobs are pre-sorted by start and finish times.

Weighted Interval Scheduling: Finding a Solution

Q. Dynamic programming algorithms computes optimal value. What if we want the solution itself?

A. Do some post-processing.

```
Run M-Compute-Opt(n)
Run Find-Solution(n)

Find-Solution(j) {
    if (j = 0)
        output nothing
    else if ( $v_j + M[p(j)] > M[j-1]$ )
        print j
        Find-Solution(p(j))
    else
        Find-Solution(j-1)
}
```

$\#$ of recursive calls $\leq n \Rightarrow O(n)$.

Weighted Interval Scheduling: Bottom-Up

Bottom-up dynamic programming. Unwind recursion.

```
Input:  $n, s_1, \dots, s_n, f_1, \dots, f_n, v_1, \dots, v_n$ 
```

```
Sort jobs by finish times so that  $f_1 \leq f_2 \leq \dots \leq f_n$ .
```

```
Compute  $p(1), p(2), \dots, p(n)$ 
```

```
Iterative-Compute-Opt {  
     $M[0] = 0$   
    for  $j = 1$  to  $n$   
         $M[j] = \max(v_j + M[p(j)], M[j-1])$   
}
```

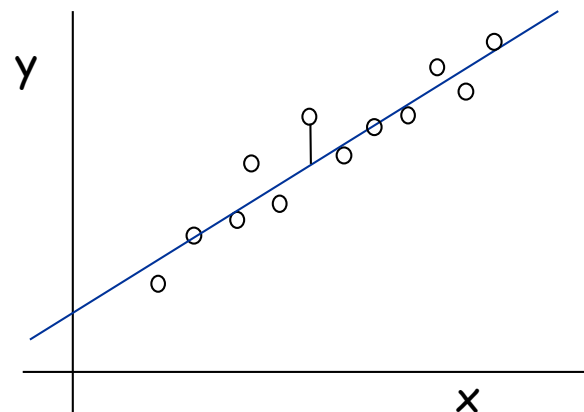
6.3 Segmented Least Squares

Segmented Least Squares

Least squares.

- Foundational problem in statistic and numerical analysis.
- Given n points in the plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$.
- Find a line $y = ax + b$ that minimizes the sum of the squared error:

$$SSE = \sum_{i=1}^n (y_i - ax_i - b)^2$$



Solution. Calculus \Rightarrow min error is achieved when

$$a = \frac{n \sum_i x_i y_i - (\sum_i x_i) (\sum_i y_i)}{n \sum_i x_i^2 - (\sum_i x_i)^2}, \quad b = \frac{\sum_i y_i - a \sum_i x_i}{n}$$

Segmented Least Squares

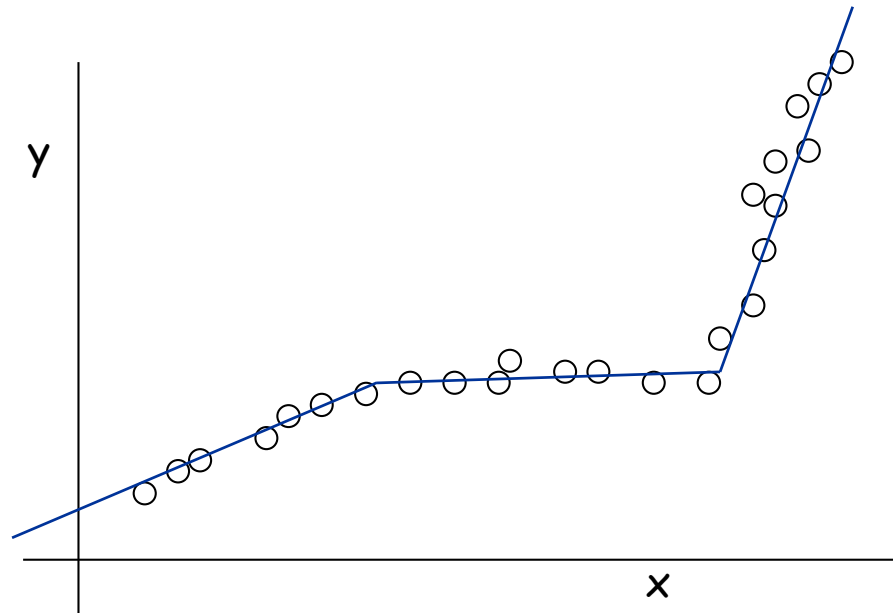
Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with
- $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes $f(x)$.

Q. What's a reasonable choice for $f(x)$ to balance accuracy and parsimony?

↑
number of lines

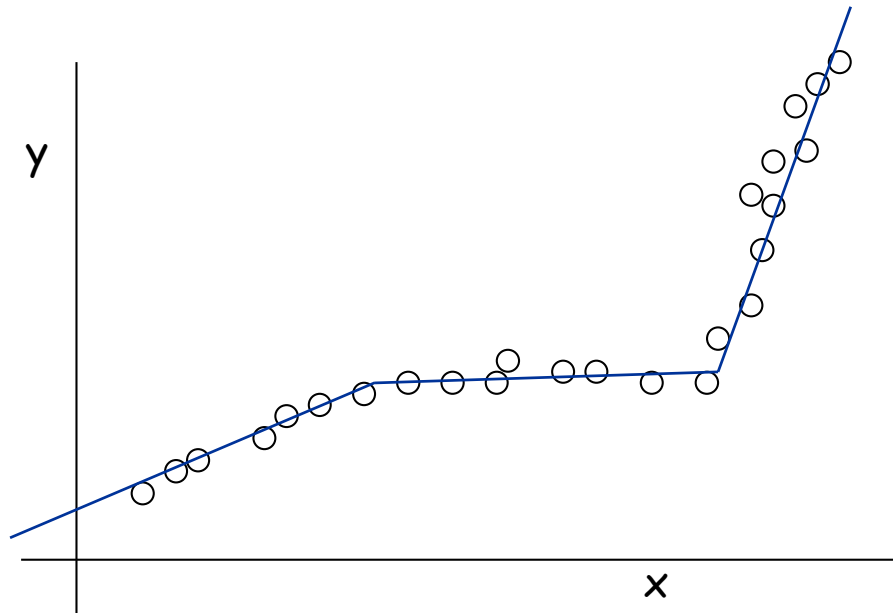
↑
goodness of fit



Segmented Least Squares

Segmented least squares.

- Points lie roughly on a sequence of several line segments.
- Given n points in the plane $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with
- $x_1 < x_2 < \dots < x_n$, find a sequence of lines that minimizes:
 - the sum of the sums of the squared errors E in each segment
 - the number of lines L
- Tradeoff function: $E + c L$, for some constant $c > 0$.



Dynamic Programming: Multiway Choice

Notation.

- $OPT(j)$ = minimum cost for points p_1, p_2, \dots, p_j .
- $e(i, j)$ = minimum sum of squares for points p_i, p_{i+1}, \dots, p_j .

To compute $OPT(j)$:


- Last segment uses points p_i, p_{i+1}, \dots, p_j for some i .
- Cost = $e(i, j) + c + OPT(i-1)$.

$$OPT(j) = \begin{cases} 0 & \text{if } j = 0 \\ \min_{1 \leq i \leq j} \{ e(i, j) + c + OPT(i-1) \} & \text{otherwise} \end{cases}$$

Segmented Least Squares: Algorithm

INPUT: n, p_1, \dots, p_N, c

```
Segmented-Least-Squares() {  
    M[0] = 0  
    for j = 1 to n  
        for i = 1 to j  
            compute the least square error  $e_{ij}$  for  
            the segment  $p_i, \dots, p_j$   
  
        for j = 1 to n  
            M[j] =  $\min_{1 \leq i \leq j} (e_{ij} + c + M[i-1])$   
  
    return M[n]  
}
```

Running time. $O(n^3)$.  can be improved to $O(n^2)$ by pre-computing various statistics

n Bottleneck = computing $e(i, j)$ for $O(n^2)$ pairs, $O(n)$ per pair using previous formula.

6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

$$W = 11$$

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Greedy: repeatedly add item with maximum ratio v_i / w_i .

Ex: { 5, 2, 1 } achieves only value = 35 \Rightarrow greedy not optimal.

Dynamic Programming: False Start

Def. $OPT(i)$ = max profit subset of items $1, \dots, i$.

- Case 1: OPT does not select item i .
 - OPT selects best of $\{ 1, 2, \dots, i-1 \}$
- Case 2: OPT selects item i .
 - accepting item i does not immediately imply that we will have to reject other items
 - without knowing what other items were selected before i , we don't even know if we have enough room for i

Conclusion. Need more sub-problems!

Dynamic Programming: Adding a New Variable

Def. $OPT(i, w)$ = max profit subset of items 1, ..., i with weight limit w.

- Case 1: OPT does not select item i.
 - OPT selects best of $\{ 1, 2, \dots, i-1 \}$ using weight limit w
- Case 2: OPT selects item i.
 - new weight limit = $w - w_i$
 - OPT selects best of $\{ 1, 2, \dots, i-1 \}$ using this new weight limit

$$OPT(i, w) = \begin{cases} 0 & \text{if } i = 0 \\ OPT(i-1, w) & \text{if } w_i > w \\ \max \{ OPT(i-1, w), v_i + OPT(i-1, w - w_i) \} & \text{otherwise} \end{cases}$$

Knapsack Problem: Bottom-Up

Knapsack. Fill up an n -by- W array.

```
Input:  $n, w_1, \dots, w_N, v_1, \dots, v_N$ 

for  $w = 0$  to  $W$ 
     $M[0, w] = 0$ 

for  $i = 1$  to  $n$ 
    for  $w = 1$  to  $W$ 
        if ( $w_i > w$ )
             $M[i, w] = M[i-1, w]$ 
        else
             $M[i, w] = \max \{M[i-1, w], v_i + M[i-1, w-w_i]\}$ 

return  $M[n, W]$ 
```

Knapsack Algorithm

		W + 1 →											
		0	1	2	3	4	5	6	7	8	9	10	11
n + 1 ↓	ϕ	0	0	0	0	0	0	0	0	0	0	0	0
	{ 1 }	0	1	1	1	1	1	1	1	1	1	1	1
	{ 1, 2 }	0	1	6	7	7	7	7	7	7	7	7	7
	{ 1, 2, 3 }	0	1	6	7	7	18	19	24	25	25	25	25
	{ 1, 2, 3, 4 }	0	1	6	7	7	18	22	24	28	29	29	40
	{ 1, 2, 3, 4, 5 }	0	1	6	7	7	18	22	28	29	34	34	40

OPT: { 4, 3 }
value = 22 + 18 = 40

W = 11

Item	Value	Weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

Knapsack Problem: Running Time

Running time. $\Theta(n W)$.

- Not polynomial in input size!
- "Pseudo-polynomial."
- Decision version of Knapsack is NP-complete. [Chapter 8]

Knapsack approximation algorithm. There exists a polynomial algorithm that produces a feasible solution that has value within 0.01% of optimum. [Section 11.8]

6.6 Sequence Alignment

String Similarity

How similar are two strings?

ⁿ **ocurrence**

ⁿ **occurrence**

o	c	u	r	r	a	n	c	e	-
---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	n	c	e
---	---	---	---	---	---	---	---	---	---

6 mismatches, 1 gap

o	c	-	u	r	r	a	n	c	e
---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	n	c	e
---	---	---	---	---	---	---	---	---	---

1 mismatch, 1 gap

o	c	-	u	r	r	-	a	n	c	e
---	---	---	---	---	---	---	---	---	---	---

o	c	c	u	r	r	e	-	n	c	e
---	---	---	---	---	---	---	---	---	---	---

0 mismatches, 3 gaps

Edit Distance

Applications.

- Basis for Unix diff.
- Speech recognition.
- Computational biology.

Edit distance. [Levenshtein 1966, Needleman-Wunsch 1970]

- Gap penalty δ ; mismatch penalty α_{pq} .
- Cost = sum of gap and mismatch penalties.

C	T	G	A	C	C	T	A	C	C	T
---	---	---	---	---	---	---	---	---	---	---

-	C	T	G	A	C	C	T	A	C	C	T
---	---	---	---	---	---	---	---	---	---	---	---

C	C	T	G	A	C	T	A	C	A	T
---	---	---	---	---	---	---	---	---	---	---

C	C	T	G	A	C	-	T	A	C	A	T
---	---	---	---	---	---	---	---	---	---	---	---

$$\alpha_{TC} + \alpha_{GT} + \alpha_{AG} + 2\alpha_{CA}$$

$$2\delta + \alpha_{CA}$$

Sequence Alignment

Goal: Given two strings $X = x_1 x_2 \dots x_m$ and $Y = y_1 y_2 \dots y_n$ find alignment of minimum cost.

Def. An **alignment** M is a set of ordered pairs $x_i - y_j$ such that each item occurs in at most one pair and no crossings.

Def. The pair $x_i - y_j$ and $x_{i'} - y_{j'}$ **cross** if $i < i'$, but $j > j'$.

$$\text{cost}(M) = \underbrace{\sum_{(x_i, y_j) \in M} \alpha_{x_i y_j}}_{\text{mismatch}} + \underbrace{\sum_{i: x_i \text{ unmatched}} \delta + \sum_{j: y_j \text{ unmatched}} \delta}_{\text{gap}}$$

Ex: CTACCG **vs.** TACATG.

Sol: $M = x_2 - y_1, x_3 - y_2, x_4 - y_3, x_5 - y_4, x_6 - y_6$.

x_1	x_2	x_3	x_4	x_5		x_6
C	T	A	C	C	-	G

-	T	A	C	A	T	G
	y_1	y_2	y_3	y_4	y_5	y_6

Sequence Alignment: Problem Structure

Def. $OPT(i, j)$ = min cost of aligning strings $x_1 x_2 \dots x_i$ and $y_1 y_2 \dots y_j$.

\bullet Case 1: OPT matches x_i - y_j .

- pay cost of x_i - y_j + min cost of aligning two strings

$x_1 x_2 \dots x_{i-1}$ and $y_1 y_2 \dots y_{j-1}$

\bullet Case 2a: OPT leaves x_i unmatched.

- pay gap for x_i and min cost of aligning $x_1 x_2 \dots x_{i-1}$ and $y_1 y_2 \dots y_j$

\bullet Case 2b: OPT leaves y_j unmatched.

- pay gap for y_j and min cost of aligning $x_1 x_2 \dots x_i$ and $y_1 y_2 \dots y_{j-1}$

$$OPT(i, j) = \begin{cases} j\delta & \text{if } i = 0 \\ \min \begin{cases} \alpha_{x_i y_j} + OPT(i-1, j-1) \\ \delta + OPT(i-1, j) \\ \delta + OPT(i, j-1) \end{cases} & \text{otherwise} \\ i\delta & \text{if } j = 0 \end{cases}$$

Sequence Alignment: Algorithm

```
Sequence-Alignment( $m, n, x_1x_2\dots x_m, y_1y_2\dots y_n, \delta, \alpha$ ) {  
  for  $i = 0$  to  $m$   
     $M[0, i] = i\delta$   
  for  $j = 0$  to  $n$   
     $M[j, 0] = j\delta$   
  
  for  $i = 1$  to  $m$   
    for  $j = 1$  to  $n$   
       $M[i, j] = \min(\alpha[x_i, y_j] + M[i-1, j-1],$   
                     $\delta + M[i-1, j],$   
                     $\delta + M[i, j-1])$   
  
  return  $M[m, n]$   
}
```

Analysis. $\Theta(mn)$ time and space.

English words or sentences: $m, n \leq 10$.

Computational biology: $m = n = 100,000$. 10 billions ops OK, but 10GB array?

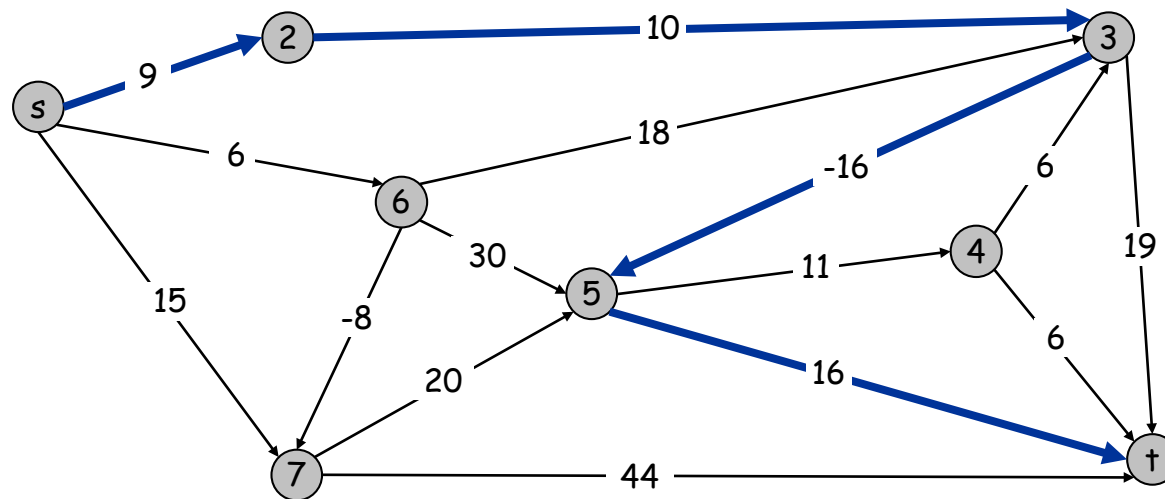
6.8 Shortest Paths

Shortest Paths

Shortest path problem. Given a directed graph $G = (V, E)$, with edge weights c_{vw} , find shortest path from node s to node t .

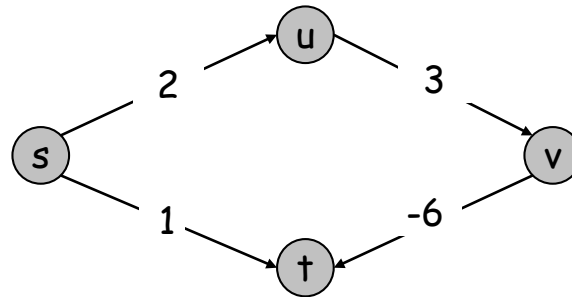
↖ allow negative weights

Ex. Nodes represent agents in a financial setting and c_{vw} is cost of transaction in which we buy from agent v and sell immediately to w .

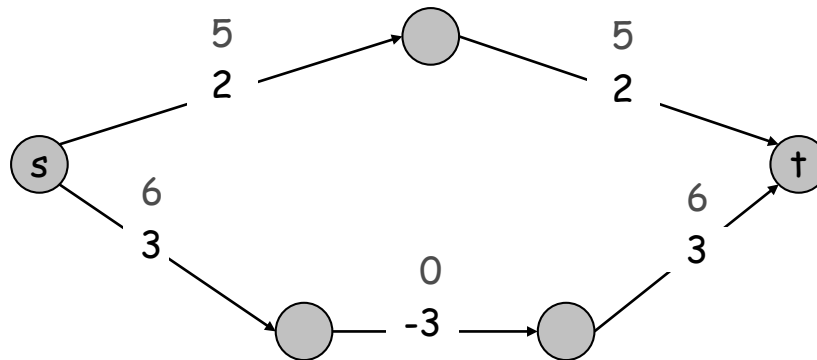


Shortest Paths: Failed Attempts

Dijkstra. Can fail if negative edge costs.

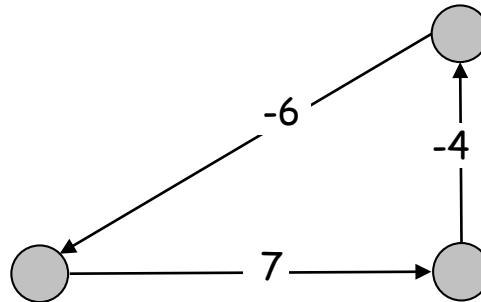


Re-weighting. Adding a constant to every edge weight can fail.

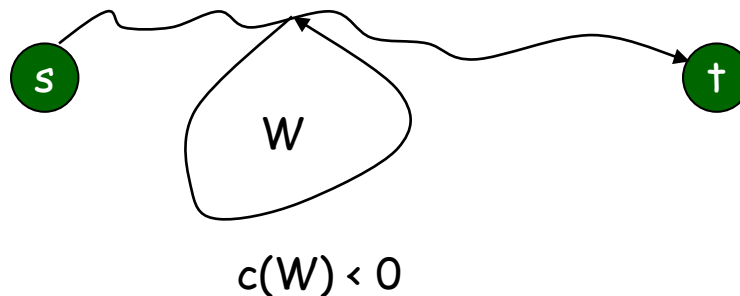


Shortest Paths: Negative Cost Cycles

Negative cost cycle.



Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s - t path; otherwise, there exists one that is simple.



Shortest Paths: Dynamic Programming

Def. $OPT(i, v)$ = length of shortest v - t path P using at most i edges.

- Case 1: P uses at most $i-1$ edges.
 - $OPT(i, v) = OPT(i-1, v)$
- Case 2: P uses exactly i edges.
 - if (v, w) is first edge, then OPT uses (v, w) , and then selects best w - t path using at most $i-1$ edges

$$OPT(i, v) = \begin{cases} \infty & \text{if } i = 0 \\ \min \left\{ OPT(i-1, v), \min_{(v, w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Remark. By previous observation, if no negative cycles, then $OPT(n-1, v)$ = length of shortest v - t path.

Shortest Paths: Implementation

```
Shortest-Path( $G, t$ ) {  
  foreach node  $v \in V$   
     $M[0, v] \leftarrow \infty$   
   $M[0, t] \leftarrow 0$   
  
  for  $i = 1$  to  $n-1$   
    foreach node  $v \in V$   
       $M[i, v] \leftarrow M[i-1, v]$   
      foreach edge  $(v, w) \in E$   
         $M[i, v] \leftarrow \min \{ M[i, v], M[i-1, w] + c_{vw} \}$   
}
```

Analysis. $\Theta(mn)$ time, $\Theta(n^2)$ space.

Finding the shortest paths. Maintain a "successor" for each table entry.

Shortest Paths: Practical Improvements

Practical improvements.

- Maintain only one array $M[v]$ = shortest v - t path that we have found so far.
- No need to check edges of the form (v, w) unless $M[w]$ changed in previous iteration.

Theorem. Throughout the algorithm, $M[v]$ is length of some v - t path, and after i rounds of updates, the value $M[v]$ is no larger than the length of shortest v - t path using $\leq i$ edges.

Overall impact.

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but substantially faster in practice.

Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path( $G, s, t$ ) {  
  foreach node  $v \in V$  {  
     $M[v] \leftarrow \infty$   
     $\text{successor}[v] \leftarrow \phi$   
  }  
  
   $M[t] = 0$   
  for  $i = 1$  to  $n-1$  {  
    foreach node  $w \in V$  {  
      if ( $M[w]$  has been updated in previous iteration) {  
        foreach node  $v$  such that  $(v, w) \in E$  {  
          if ( $M[v] > M[w] + c_{vw}$ ) {  
             $M[v] \leftarrow M[w] + c_{vw}$   
             $\text{successor}[v] \leftarrow w$   
          }  
        }  
      }  
    }  
    If no  $M[w]$  value changed in iteration  $i$ , stop.  
  }  
}
```

6.10 Negative Cycles in a Graph

Detecting Negative Cycles

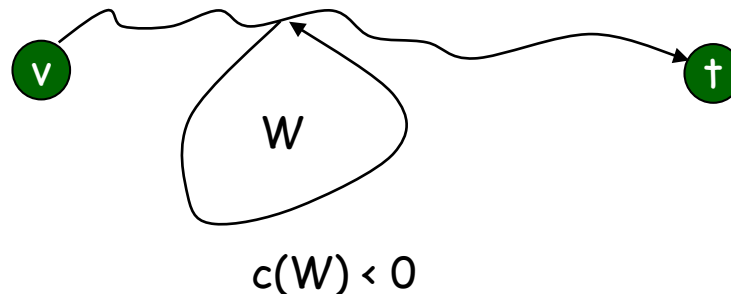
Lemma. If $\text{OPT}(n,v) = \text{OPT}(n-1,v)$ for all v , then no negative cycles.

Pf. Bellman-Ford algorithm.

Lemma. If $\text{OPT}(n,v) < \text{OPT}(n-1,v)$ for some node v , then (any) shortest path from v to t contains a cycle W . Moreover W has negative cost.

Pf. (by contradiction)

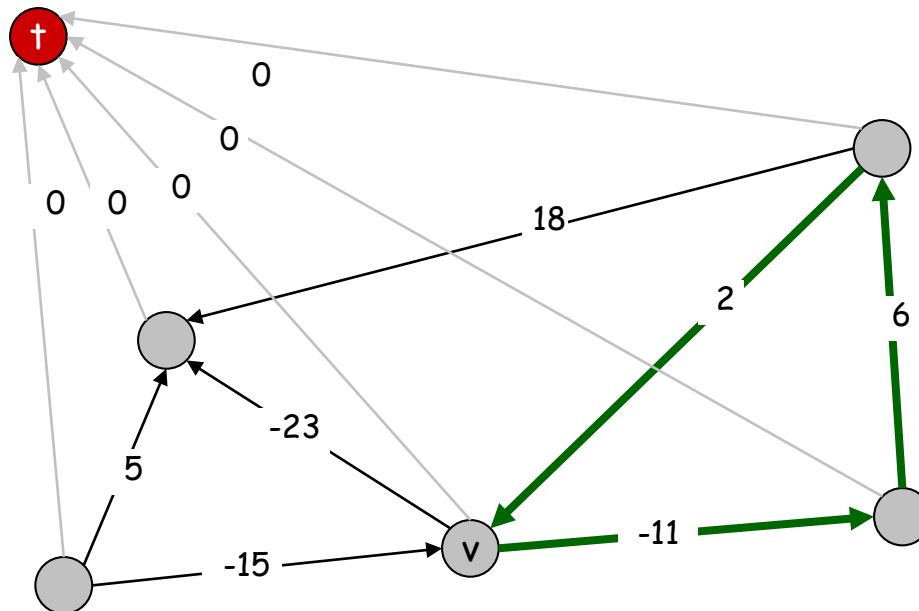
- Since $\text{OPT}(n,v) < \text{OPT}(n-1,v)$, we know P has exactly n edges.
- By pigeonhole principle, P must contain a directed cycle W .
- Deleting W yields a v - t path with $< n$ edges $\Rightarrow W$ has negative cost.



Detecting Negative Cycles

Theorem. Can detect negative cost cycle in $O(mn)$ time.

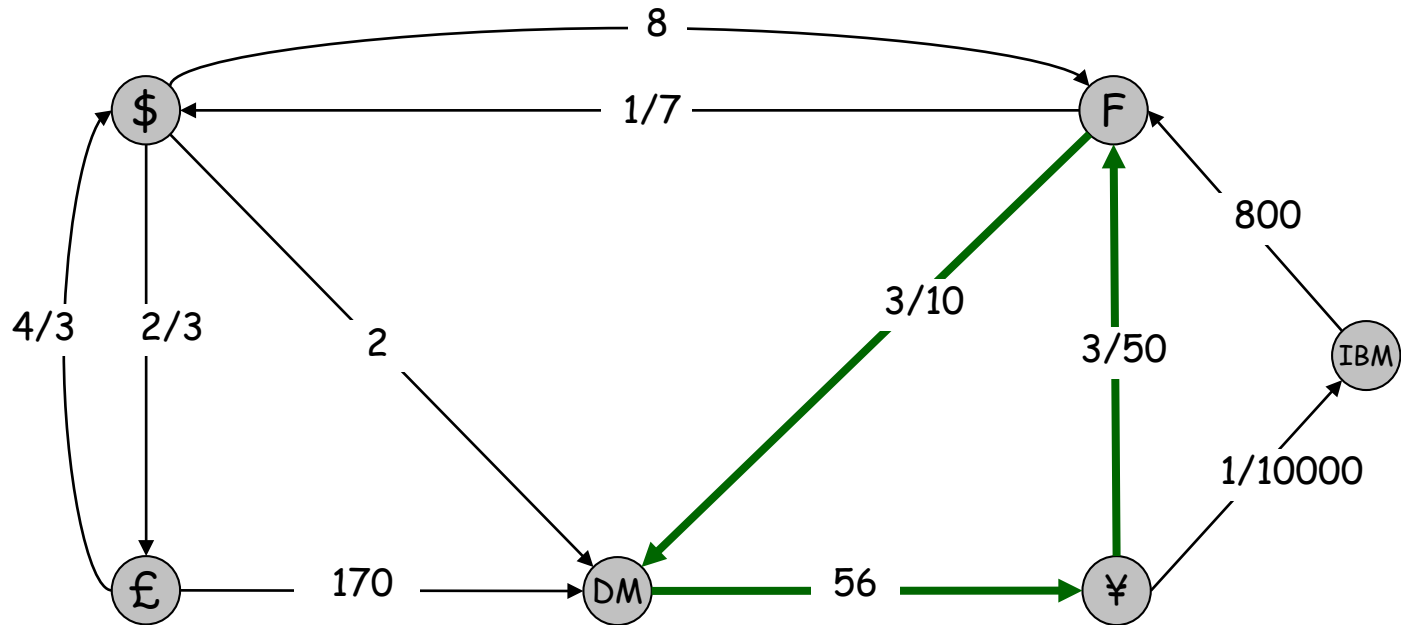
- Add new node t and connect all nodes to t with 0-cost edge.
- Check if $\text{OPT}(n, v) = \text{OPT}(n-1, v)$ for all nodes v .
 - if yes, then no negative cycles
 - if no, then extract cycle from shortest path from v to t



Detecting Negative Cycles: Application

Currency conversion. Given n currencies and exchange rates between pairs of currencies, is there an arbitrage opportunity?

Remark. Fastest algorithm very valuable!



Detecting Negative Cycles: Summary

Bellman-Ford. $O(mn)$ time, $O(m + n)$ space.

- Run Bellman-Ford for n iterations (instead of $n-1$).
- Upon termination, Bellman-Ford successor variables trace a negative cycle if one exists.
- See p. 288 for improved version and early termination rule.