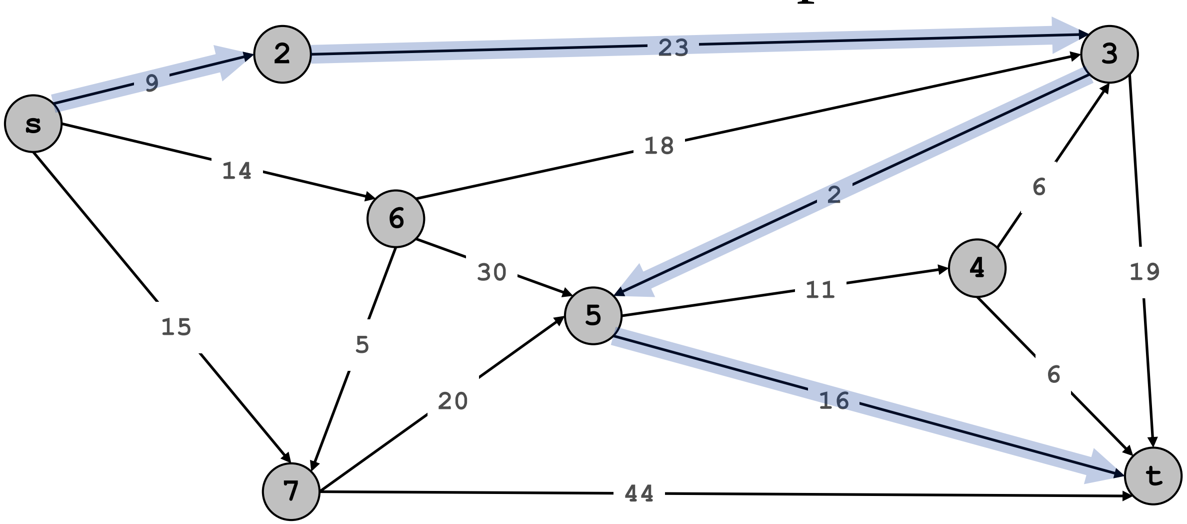
Shortest Paths

Based on slides by S. Raskhodnikova and A. Smith

Shortest Path Problem

- **Input:**
  - Directed graph G = (V, E).
  - Source node s, destination node t.
  - for each edge e, length $\ell(e)$ = length of e.
  - length of a path = sum of lengths of edges on the path
- **Find:** shortest directed path from s to t.



Length of path (s,2,3,5,t)
is 9 + 23 + 2 + 16 = 50.

Based on slides by S. Raskhodnikova and A. Smith

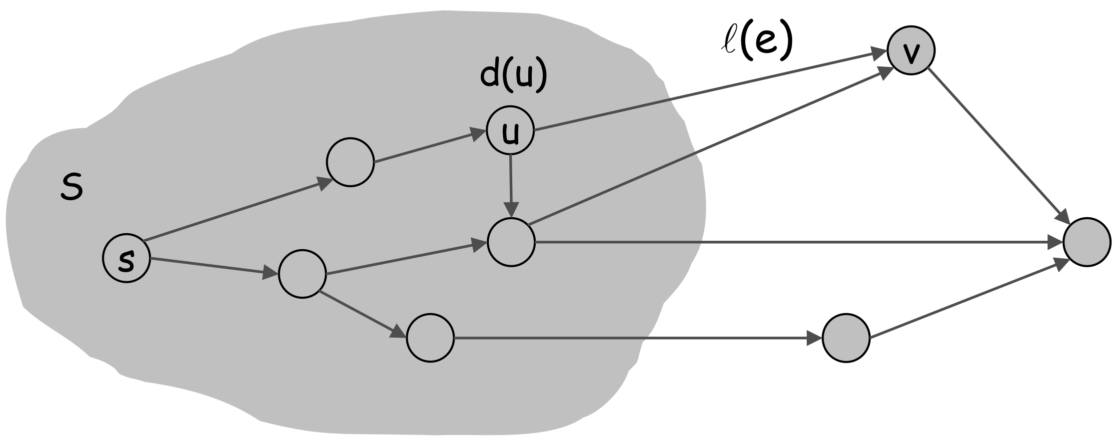
Dijksta's Algorithm: Overview

- Maintain a set of **explored nodes** S whose shortest path distance d(u) from s to u is known.
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e=(u,v):u\in S} \big(d(u) + \ell(e)\big)$$

- add v to S, and set $d(v) = \pi(v)$.

shortest path to some u in explored part, followed by a single edge (u, v)



Dijksta's Algorithm: Overview

- Maintain a set of **explored nodes** S whose shortest path distance d(u) from s to u is known.

Invariant: d(u) is known for all vertices in S

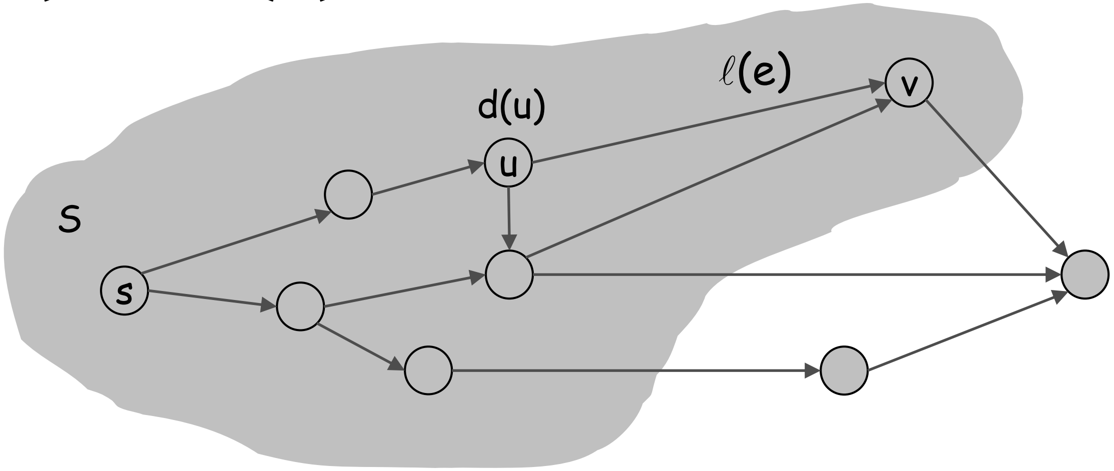
- Initialize S = { s }, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes

$$\pi(v) = \min_{e=(u,v):u\in S} \left(d(u) + \ell(e)\right)$$

shortest path to some u in explored part, followed by a single edge (u, v)

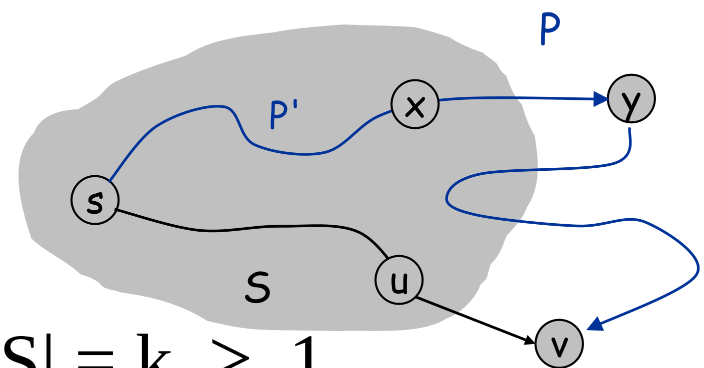
- add v to S, and set $d(v) = \pi(v)$.

Intuition: like BFS, but with weighted edges



Correctness Proof of Dijkstra's
(Greedy Stays Ahead)

Invariant. For each node $u \in S$, $d(u)$ is the length of the shortest path from s to u.



Proof: (by induction on $|S|$)

- **Base case:** $|S| = 1$; $d(s)=0$.
- **Inductive hypothesis:** Assume for $|S| = k \geq 1$.
  - Let v be next node added to S, and let (u,v) be the chosen edge.
  - The shortest s-u path plus (u,v) is an s-v path of length $\pi(v)$.
  - Consider any s-v path P. We'll see that it's no shorter than $\pi(v)$.
  - Let (x,y) be the first edge in P that leaves S, and let P' be the subpath to x.
  - P' + (x,y) has length $\underset{\text{inductive hypothesis}}{\geq} d(x) + \ell(x,y) \underset{\text{defn of } \pi(y)}{\geq} \pi(y) \underset{\text{Dijkstra's chose v instead of y}}{\geq} \pi(v)$

Based on slides by S. Raskhodnikova and A. Smith

Implementation

- For unexplored nodes, maintain

$$\pi(v) = \min_{e=(u,v):u\in S} \left(d(u) + \ell(e)\right)$$

  – Next node to explore = node with minimum $\pi(v)$.

  – When exploring v, for each edge e = (v,w), update

$$\pi(w) = \min\{\pi(w), \pi(v) + \ell(e)\}.$$

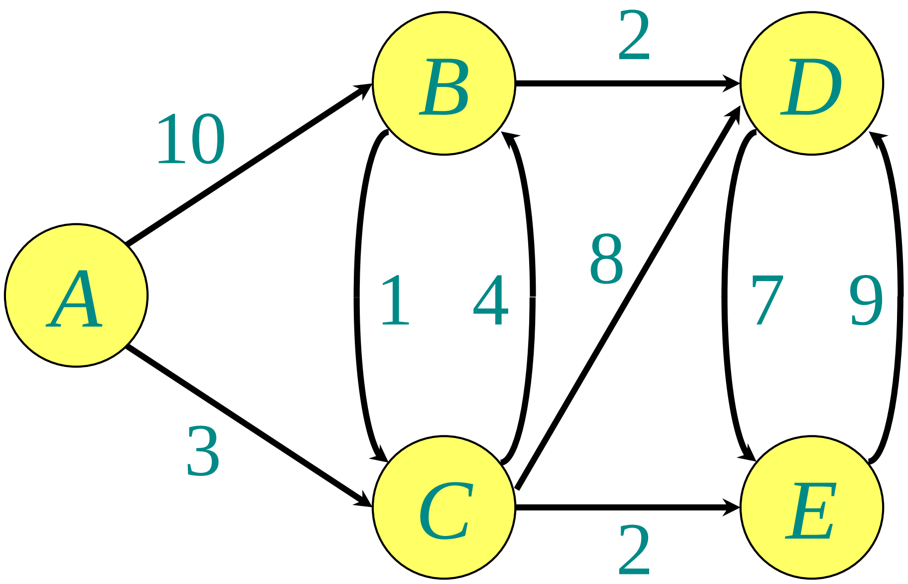
- **Efficient implementation:** Maintain a **priority queue** Q of unexplored nodes, prioritized by $\pi(v)$.

Implementation: priority queues

- Maintain a set of items with priorities (= “keys”)
  - Example: jobs to be performed
- Operations:
  - INSERT
  - DECREASE-KEY
  - EXTRACT-MIN: find and remove item with least key
- Common data structure: heap
  - Time: O(log n) per operation

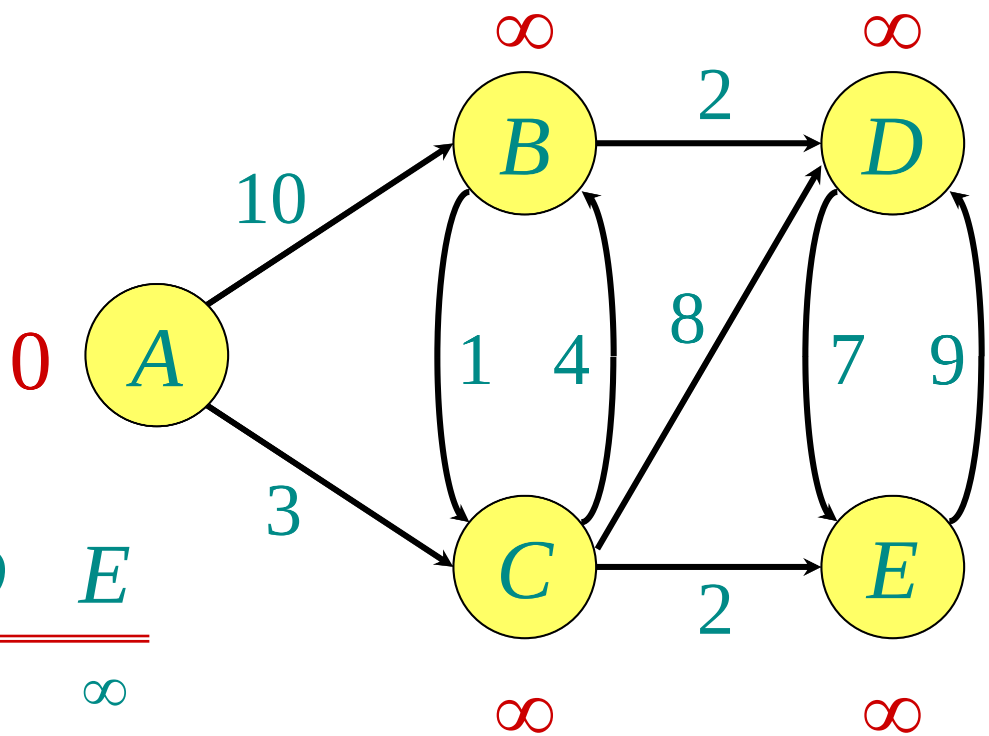
Demo of Dijkstra's Algorithm

Graph with nonnegative edge lengths:



Demo of Dijkstra's Algorithm

Initialize:

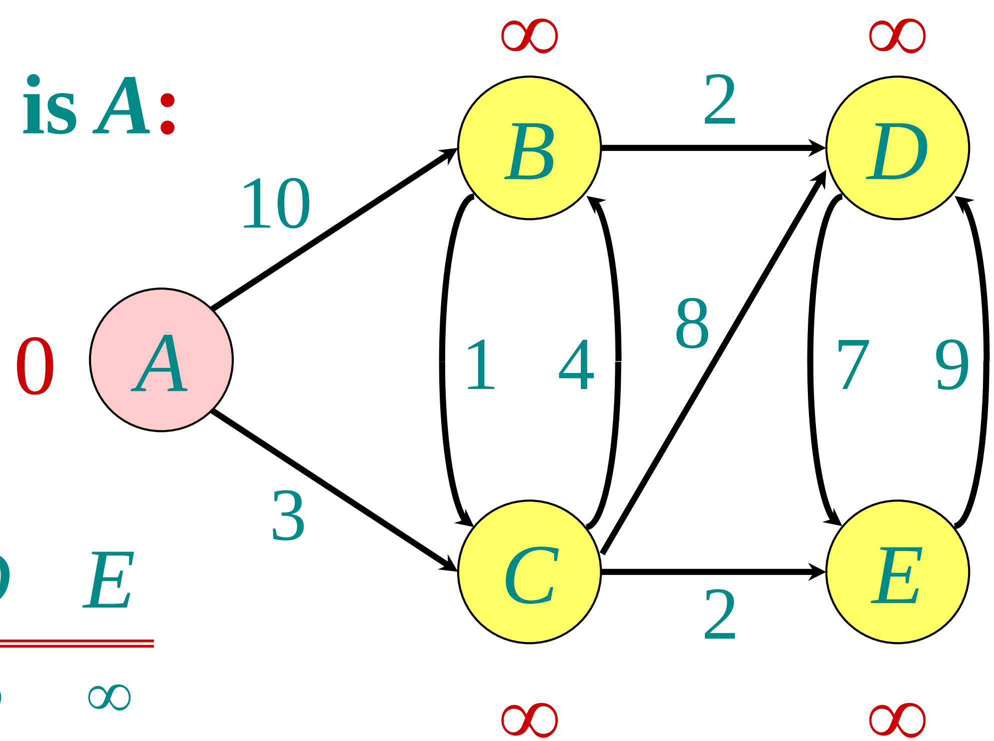


Q:	A	B	C	D	E
$\pi(v)$:	0	∞	∞	∞	∞

S: { }

Demo of Dijkstra's Algorithm

EXTRACT-MIN(*Q*) is *A*:

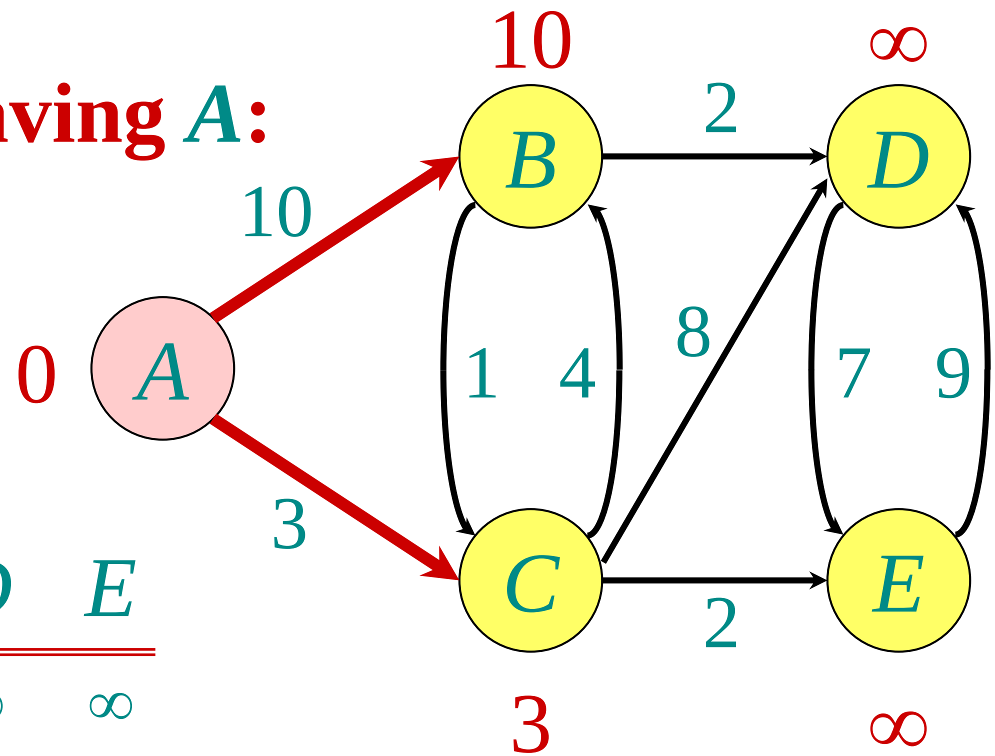


Q:	A	B	C	D	E
$\pi(v)$:	0	∞	∞	∞	∞

S: { A }

Demo of Dijkstra's Algorithm

Explore edges leaving *A*:

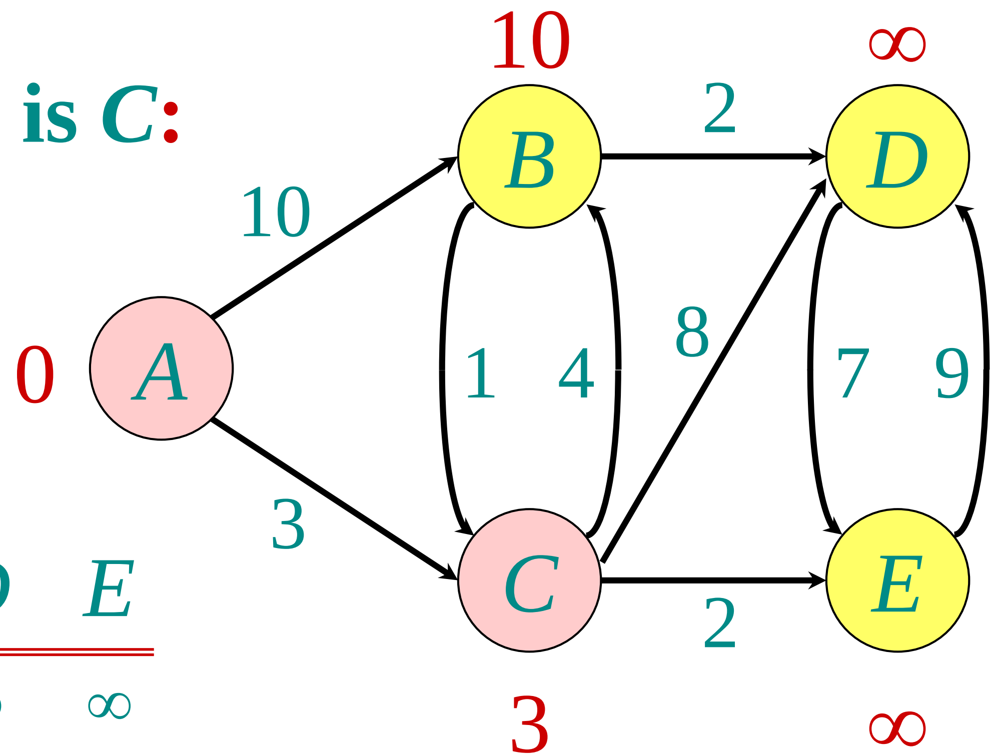


Q:	A	B	C	D	E
π(v):	0	∞	∞	∞	∞
		10	3	∞	∞

S: { A }

Demo of Dijkstra's Algorithm

EXTRACT-MIN(*Q*) is *C*:

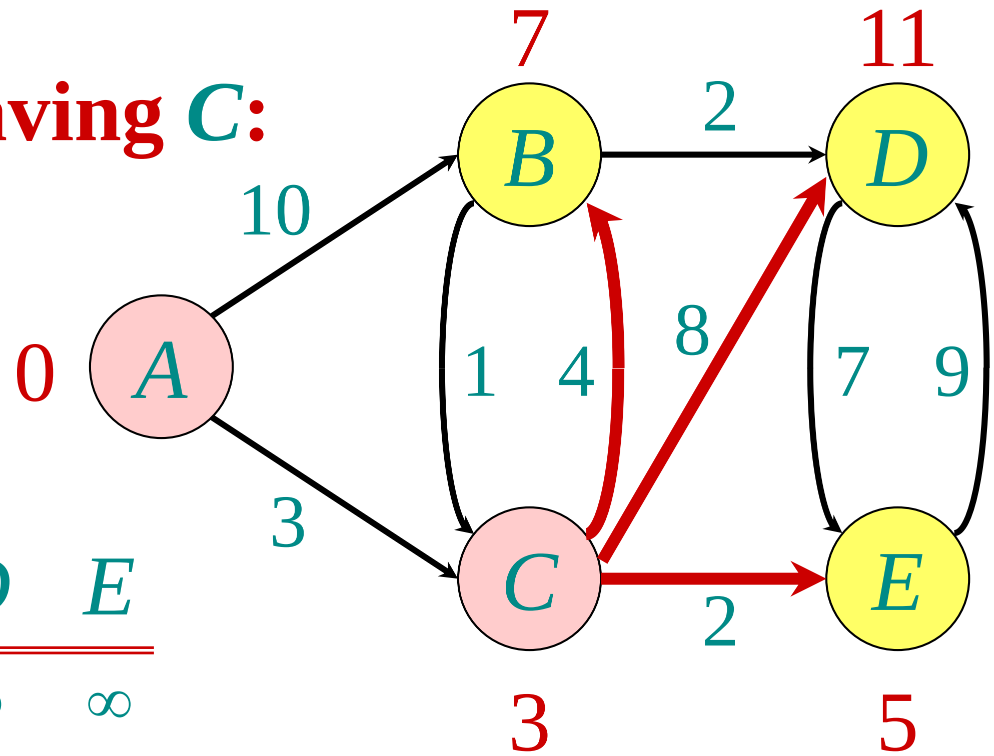


Q:	*A*	*B*	*C*	*D*	*E*
π(v):	0	∞	∞	∞	∞
		10	3	∞	∞

S: { *A*, *C* }

Demo of Dijkstra's Algorithm

Explore edges leaving *C*:

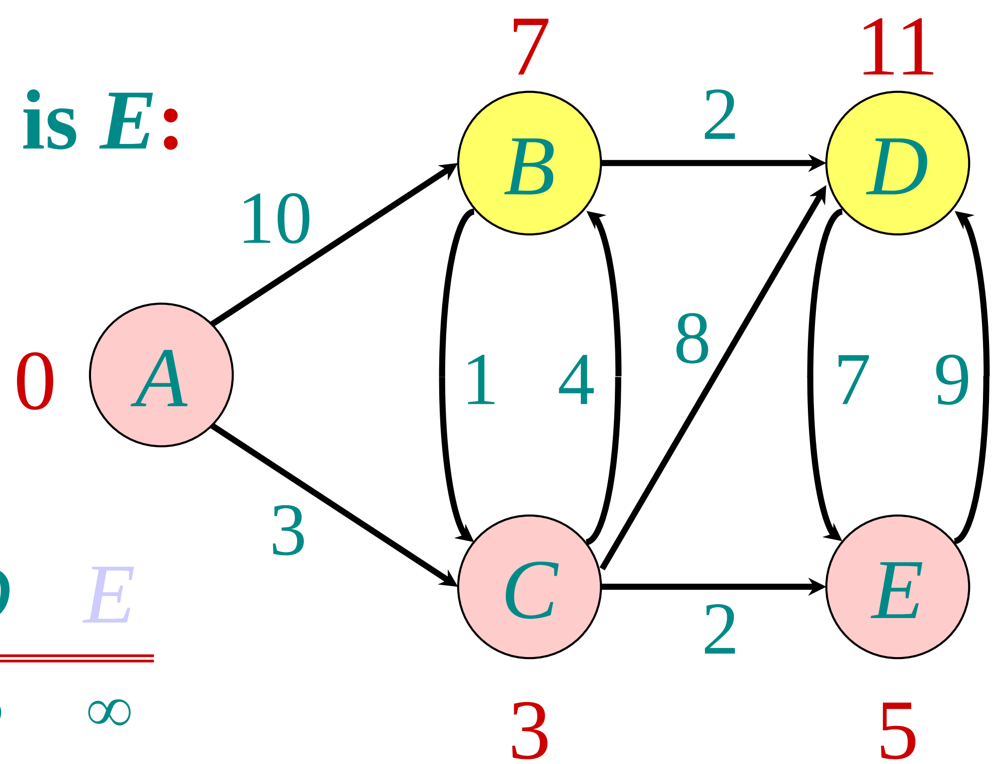


Q:	*A*	*B*	*C*	*D*	*E*
π(v):	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { *A*, *C* }

Demo of Dijkstra's Algorithm

EXTRACT-MIN(Q) is E:

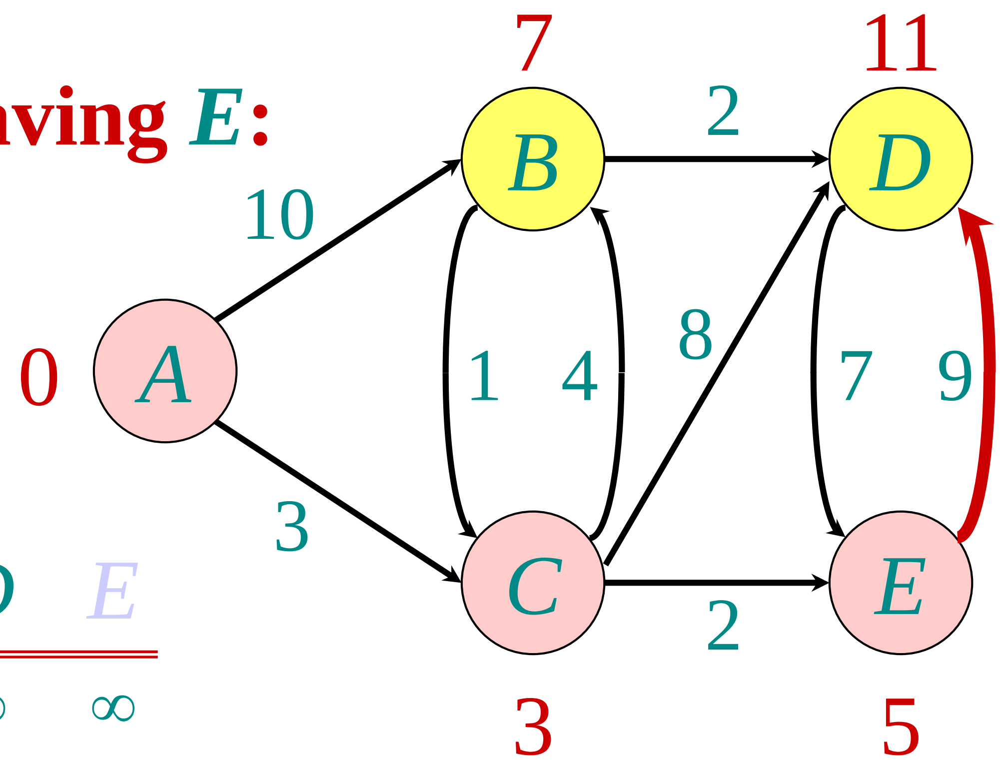


Q:	A	B	C	D	E
π(v):	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5

S: { A, C, E }

Demo of Dijkstra's Algorithm

Explore edges leaving *E*:



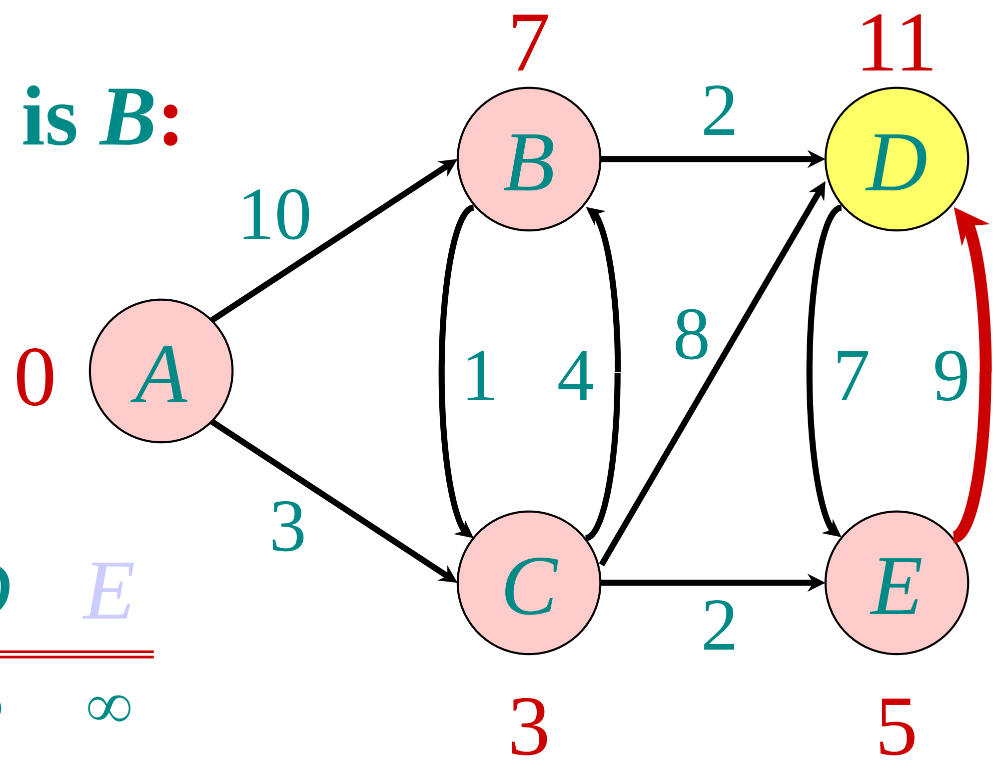
Q:	*A*	*B*	*C*	*D*	*E*
$\pi(v)$:	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { *A*, *C*, *E* }

Based on slides by S. Raskhodnikova and A. Smith

Demo of Dijkstra's Algorithm

EXTRACT-MIN(*Q*) is *B*:



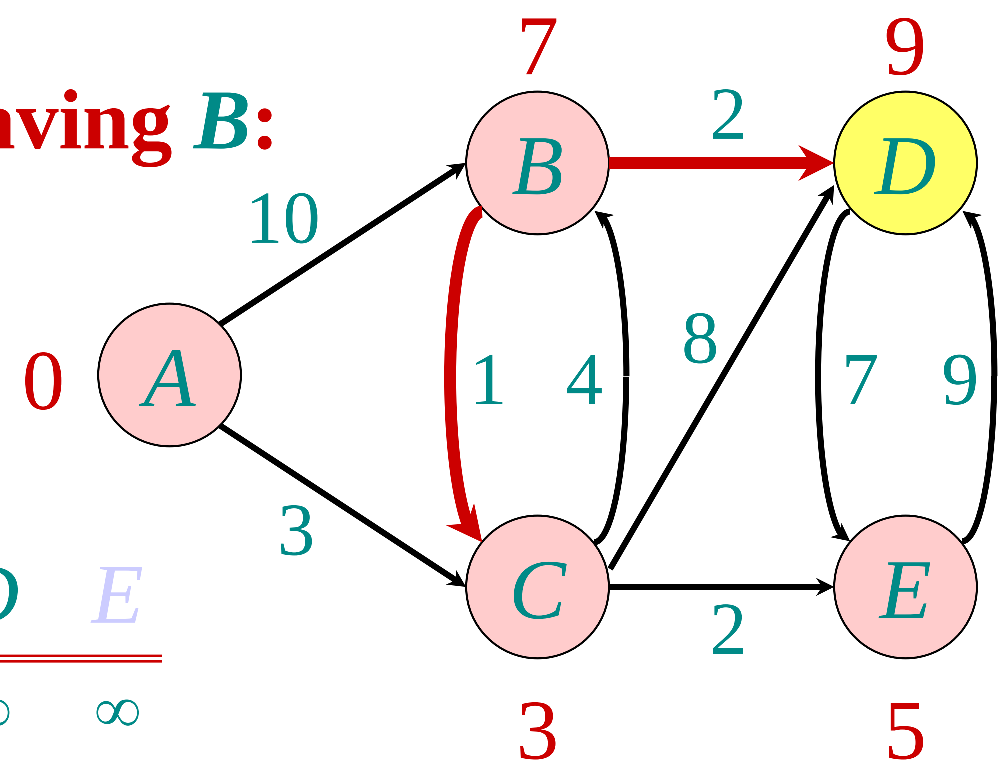
Q:	A	B	C	D	E
$\pi(v)$:	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	

S: { A, C, E, B }

Based on slides by S. Raskhodnikova and A. Smith

Demo of Dijkstra's Algorithm

Explore edges leaving *B*:

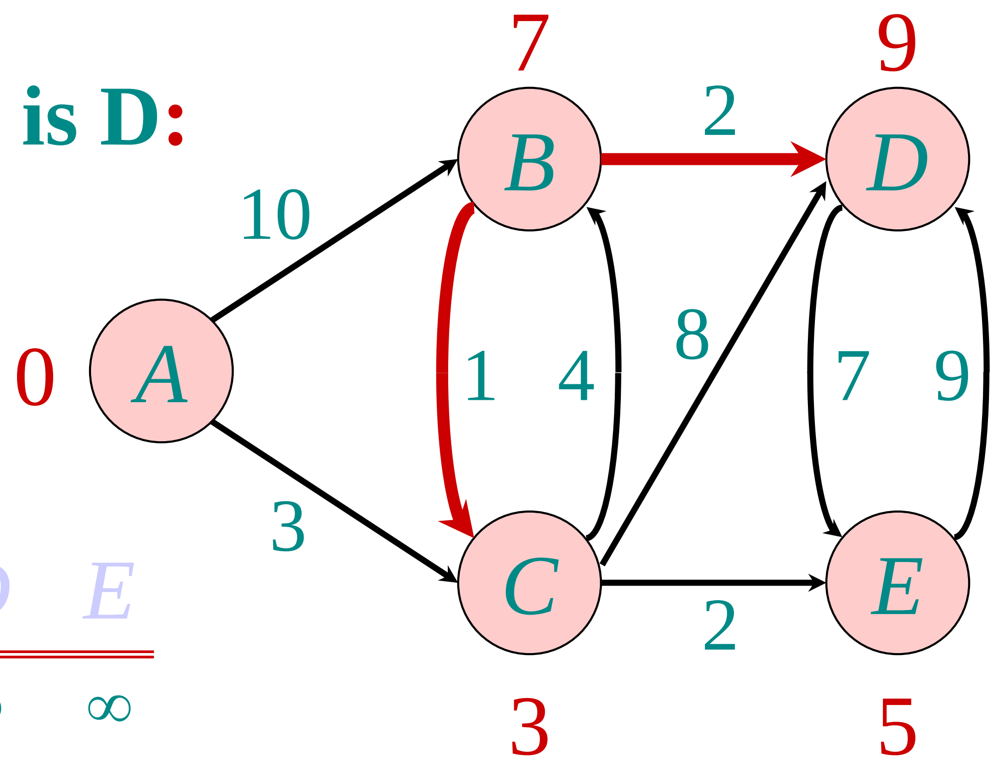


Q:	*A*	*B*	*C*	*D*	*E*
π(v):	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B* }

Demo of Dijkstra's Algorithm

EXTRACT-MIN(*Q*) is D:



Q:	*A*	*B*	*C*	*D*	*E*
π(v):	0	∞	∞	∞	∞
		10	3	∞	∞
		7		11	5
		7		11	
				9	

S: { *A*, *C*, *E*, *B*, *D* }

Pseudocode for Dijkstra(G, ℓ)

$d[s] \leftarrow 0$
for each $v \in V - \{s\}$
    do $d[v] \leftarrow \infty$; $\pi[v] \leftarrow \infty$
$S \leftarrow \varnothing$
$Q \leftarrow V$ ⊳ Q is a priority queue maintaining $V - S$, keyed on $\pi[v]$
while $Q \neq \varnothing$
    do $u \leftarrow$ EXTRACT-MIN(Q)
        $S \leftarrow S \cup \{u\}$; $d[u] \leftarrow \pi[u]$
        for each $v \in \mathit{Adjacency\text{-}list}[u]$
            do if $\pi[v] > \pi[u] + \ell(u, v)$
                then $\boldsymbol{\pi}[v] \leftarrow d[u] + \ell(u, v)$

explore edges leaving v

Implicit DECREASE-KEY

Analysis of Dijkstra

n times

while $Q \neq \emptyset$
  do $u \leftarrow$ EXTRACT-MIN(Q)
    $S \leftarrow S \cup \{u\}$
    for each $v \in Adj[u]$
explore an edge
      do if $d[v] > d[u] + \ell(u, v)$
        then $d[v] \leftarrow d[u] + \ell(u, v)$

\\\\ m implicit DECREASE-KEY's.

PQ Operation	Dijkstra	Array	Binary heap	d-way Heap	Fib heap †
ExtractMin	n	n	log n	HW	log n
DecreaseKey	m	1	log n	HW	1
Total		n^2	m log n	m $\log_{m/n}$ n	m + n log n

† Individual ops are amortized bounds

Physical intuition

- System of pipes filling with water
  - Vertices are intersections
  - Edge length = pipe length
  - $d(v)$ = time at which water reaches v
- Balls and strings
  - Vertices $\mapsto$ balls
  - Edge $e \mapsto$ string of length $\ell(e)$
  - Hold ball s up in the air
  - $d(v) =$ (height of s)-(height of v)
- Nature uses greedy algorithms

Review

- Is Dijsktra's algorithm correct with negative edge weights?
  Give either
  - a proof of correctness, or
  - an example of a graph where Dijkstra fails

Further reading

- Erickson's lecture notes:
  http://web.engr.illinois.edu/~jeffe/teaching/algorithms/notes/21-sssp.pdf

Minimum Spanning Tree

Based on slides by S. Raskhodnikova and A. Smith

Minimum spanning tree (MST)

Input: A connected undirected graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$.

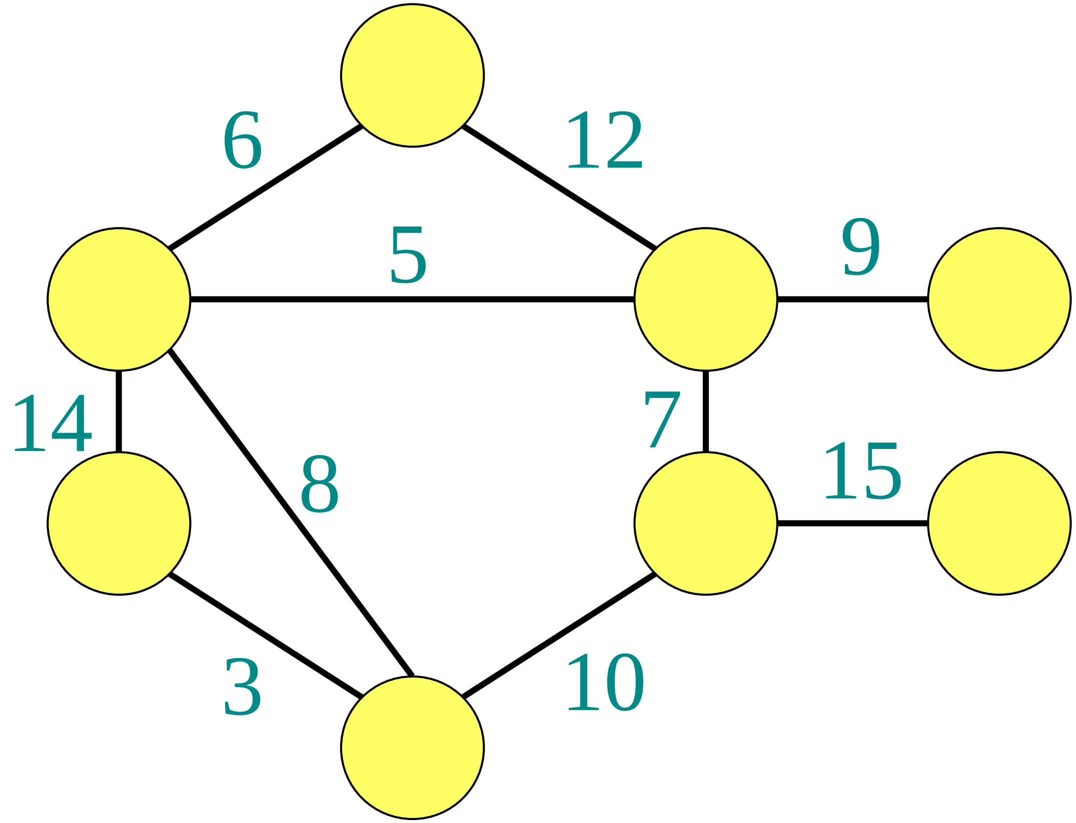
- For now, assume all edge weights are distinct.

Definition: A ***spanning tree*** is a tree that connects all vertices.

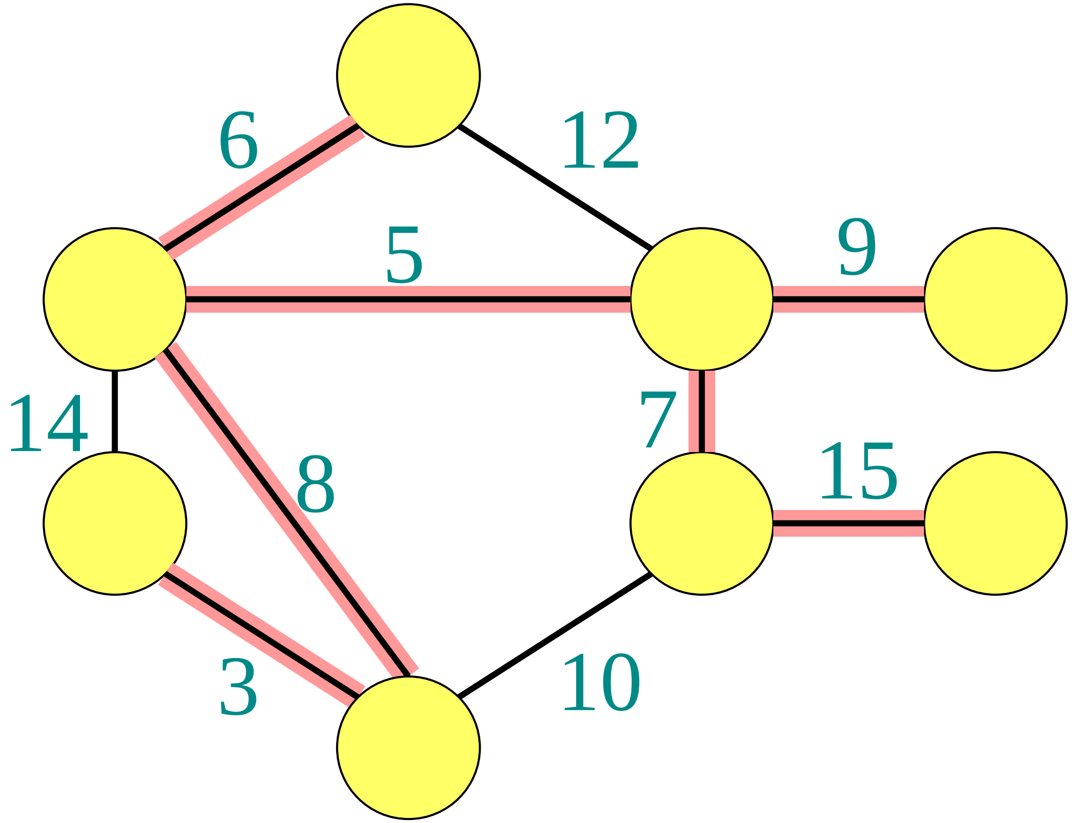
Output: A ***spanning tree*** T of minimum weight:

$$w(T) = \sum_{(u,v)\in T} w(u,v) .$$

Example of MST



Example of MST

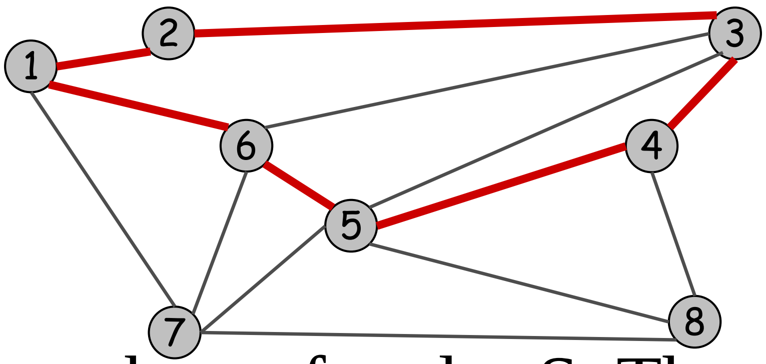


Greedy Algorithms for MST

- **Kruskal's:** Start with $T = \emptyset$. Consider edges in ascending order of weights. Insert edge e in T unless doing so would create a cycle.
- **Reverse-Delete:** Start with $T = E$. Consider edges in descending order of weights. Delete edge e from T unless doing so would disconnect T.
- **Prim's:** Start with some root node s. Grow a tree T from s outward. At each step, add to T the cheapest edge e with exactly one endpoint in 5.

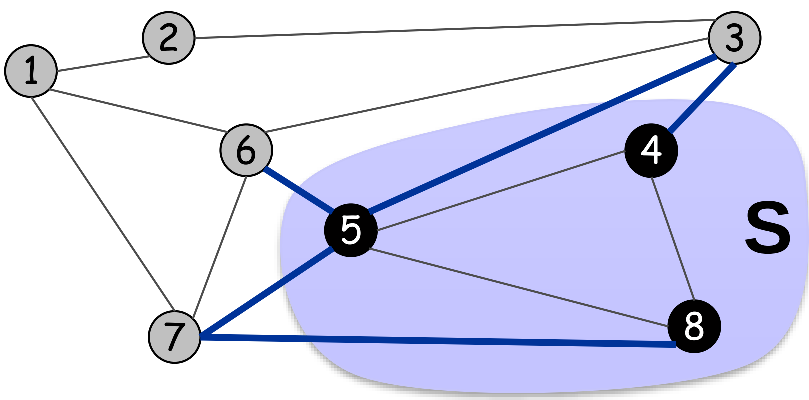
Cycles and Cuts

- **Cycle:** Set of edges of the form (a,b),(b,c),…,(y,z),(z,a).



Cycle C = (1,2),(2,3),(3,4),(4,5),(5,6),(6,1)

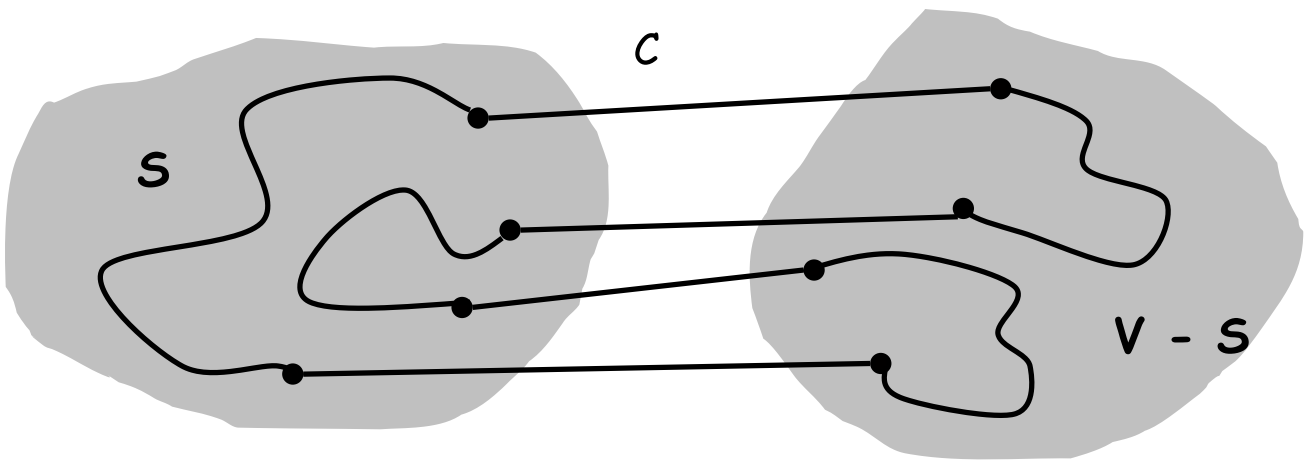
- **Cut:** a subset of nodes S. The corresponding **cutset** D is the subset of edges with exactly one endpoint in S.



Cut S = { 4, 5, 8 }
Cutset D = (5,6), (5,7), (3,4), (3,5), (7,8)

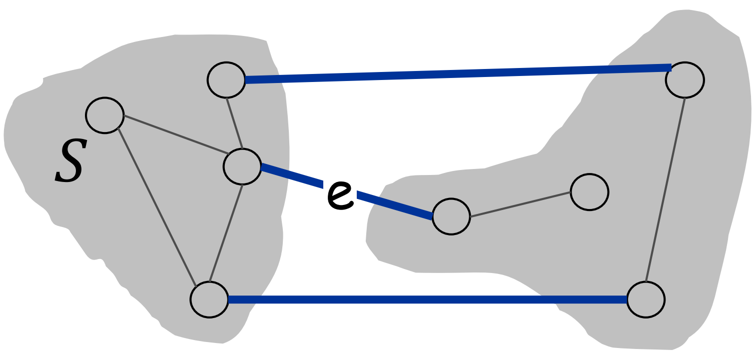
Cycle-Cut Intersection

- **Claim.** A cycle and a cutset intersect in an even number of edges.
- **Proof:** A cycle has to leave and enter the cut the same number of times.

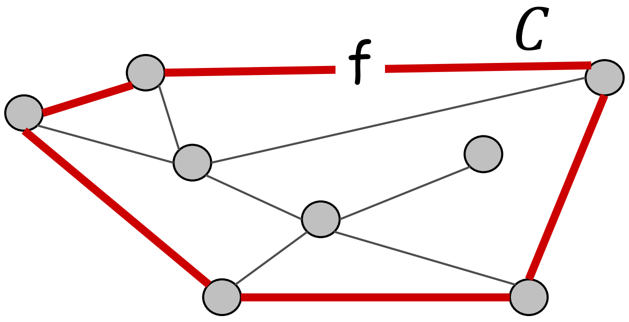


Cut and Cycle Properties

- **Cut property.** Let S be a subset of nodes. Let e be the min weight edge with exactly one endpoint in S. Then the MST contains e.
- **Cycle property.** Let C be a cycle, and let f be the max weight edge in C. Then the MST does not contain f.



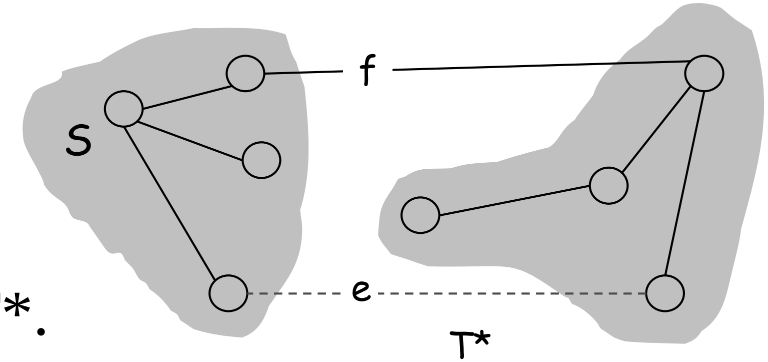
e is in the MST



f is not in the MST

Proof of Cut Property

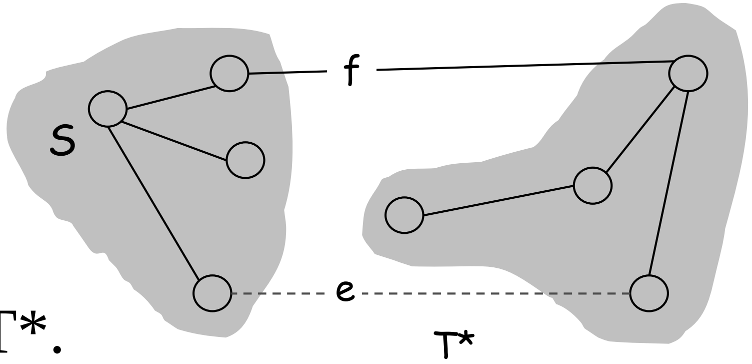
Cut property: Let S be a subset of nodes. Let e be the min weight edge with exactly one endpoint in S. Then the MST T* contains e.



- **Proof:** (exchange argument)
  - Suppose e does not belong to T*.
  - Adding e to T* creates a cycle C in T*.
  - Edge e is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say f, that is in both C and D.
  - T' = T* $\cup$ { e } - { f } is also a spanning tree.
  - Since $c_e < c_f$, cost(T') < cost(T*). Contradiction. ▪

Proof of Cycle Property

Cycle property: Let C be a cycle in G. Let f be the max weight edge in C. Then the MST T* does not contain f.



- **Proof:** (exchange argument)
  - Suppose f belongs to T*.
  - Deleting f from T* creates a cut S in T*.
  - Edge f is both in the cycle C and in the cutset D corresponding to S $\Rightarrow$ there exists another edge, say e, that is in both C and D.
  - T' = T* $\cup$ { e } - { f } is also a spanning tree.
  - Since $c_e < c_f$, cost(T') < cost(T*). Contradiction. ▪

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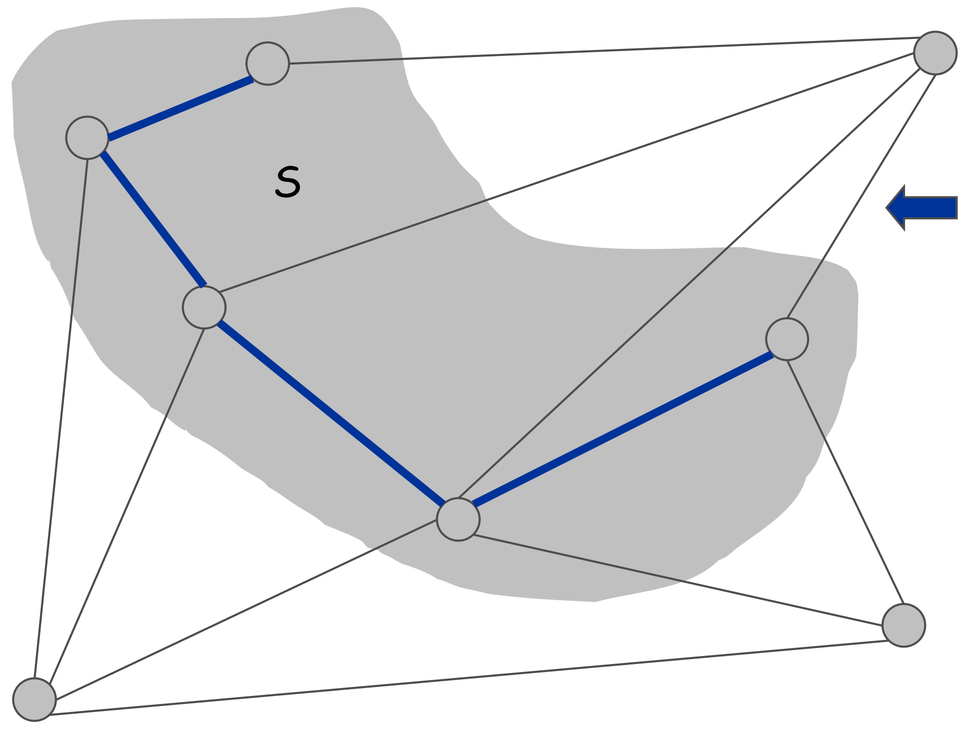
Based on slides by S. Raskhodnikova and A. Smith

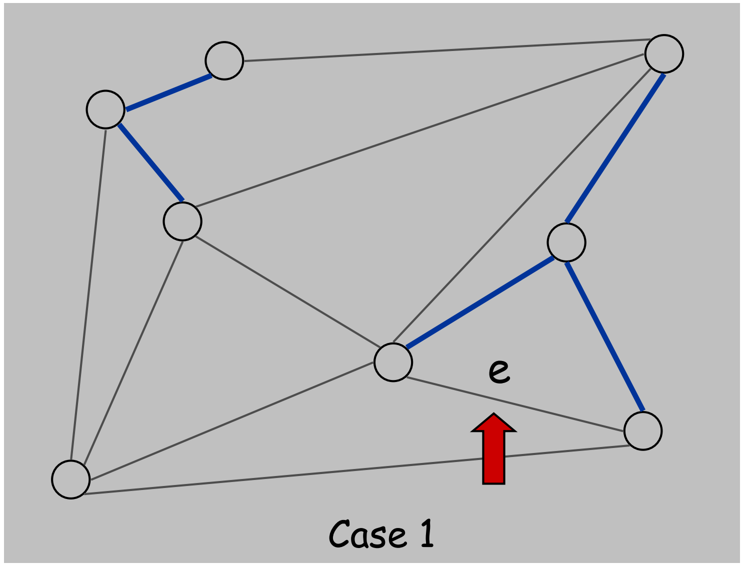
Prim's Algorithm: Correctness

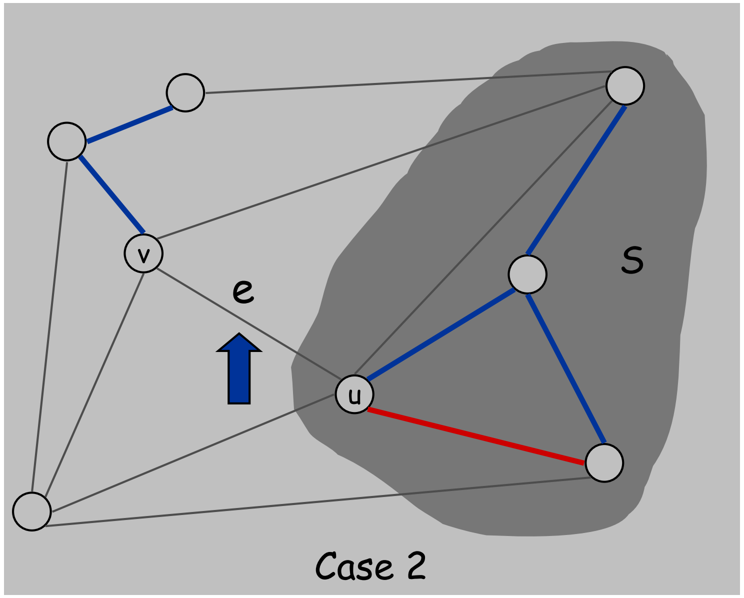
- Prim's algorithm. [Jarník 1930, Prim 1959]
  - Apply cut property to S.
  - When edge weights are distinct, every edge that is added must be in the MST
  - Thus, Prim's algorithm outputs the MST



Correctness of Kruskal

- [Kruskal, 1956]: Consider edges in ascending order of weight.
  - **Case 1:** If adding e to T creates a cycle, discard e according to cycle property.





  - **Case 2:** Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.

Review Questions

Let G be a connected undirected graph with distinct edge weights. Answer true or false:

- Let e be the cheapest edge in G. The MST of G contains e.

- Let e be the most expensive edge in G. The MST of G does not contains e.

Review Questions

Let G be a connected undirected graph with distinct edge weights. Answer true or false:

- Let e be the cheapest edge in G. The MST of G contains e.

(Answer: True, by the Cut Property)

- Let e be the most expensive edge in G. The MST of G does not contains e.

(Answer: False. Counterexample: if G is a tree, all its edges are in the MST.)

Exercise

- Given an undirected graph G, consider spanning trees produced by four algorithms
  - BFS tree
  - DFS tree
  - shortest paths tree (Dijsktra)
  - MST
- Find a graph where
  - all four trees are the same
  - all four trees must be different (note: DFS/BFS may depend on exploration order)