

2024 Fall Discrete Mathematics
Final exam
2024/1/9

You may answer in Chinese or English. Please use a large and clear font. Please do not write in strange colors. Please indicate your **department**, **student ID**, and **name**. Please be sure to write according to the question number! No make-up submissions are allowed.

1. (5%) How many positive divisors do 378000 have?
Ans: $(4+1)(3+1)(3+1)(1+1)=160$
2. (5%) How many of the positive divisors of 378000 are multiples of 147?
Ans: 0. Because 378000 does not have enough powers of 7 and 3 in its factorization to satisfy $147=3*7^2$.
3. (5%) Let $m = 720720$ and $n = 104720$. Find $\gcd(m, n)$ and $\text{lcm}(m, n)$.
Ans: 6160, 12252240
4. (5%) Use the generating function $[x/(1-x)]$ to calculate the total number of compositions for $n = 6$.
Ans: $1+5+10+10+5+1=32$
5. (5%) Let a_n denote the number of bit strings of length n that do not have two consecutive 0's. Let A denote the set of bit strings without two consecutive 0's. Find a recurrence relation for a_n .
Ans: $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$
6. There are Fibonacci numbers $F_0, F_1 \dots F_n$. ($F_0 = 0, F_1 = 1$)
 - (a) (2%) For $n \in \mathbb{Z}^+$ with $n \geq 2$, what is $F_n = ?$
 - (b) (2%) What is $F_0^2 + F_1^2 + F_2^2 \dots + F_n^2 = ?$
 - (c) (1%) What is $F_0^2 + F_1^2 + F_2^2 \dots + F_{12}^2 = ?$

Ans:

(a) For $n \in \mathbb{Z}^+$ with $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$

(b)

$$\sum_{i=0}^n F_i^2 = F_n \times F_{n+1}$$

(c)

$$\sum_{i=0}^{12} F_i^2 = F_{12} \times F_{13} = 144 \times 233 = 33552$$
7. (5%) Given binary operations $f : A \times A \rightarrow A$, $A \subseteq \mathbb{Z}^+$, $x, y \in A$. Determine in each case whether f is commutative and/or associative, and/or closed. **Please fill the answers(V or X) in the table below.**

Functions	Commutative	Associative	Closed
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(a) $f(x, y) = x - y + xy$			
(b) $f(x, y) = x + y - 10$			

Ans:

Functions	Commutative	Associative	Closed
(a) $f(x, y) = x - y + xy$	X	X	V
(b) $f(x, y) = x + y - 10$	V	V	X

8. (5%) Let $S = \{0, 1, 2, 3, 4, \dots, 9\}$. How many elements must we select from S to insure that there will be at least two whose sum is 10?

Ans:

$$10=1+9=2+8=3+7=4+6$$

Pigeonholes are the subsets $\{0\}, \{1,9\}, \{2,8\}, \{3,7\}, \{4,6\}, \{5\}$

$$6+1=7$$

9. (5%) Find a_{10} if $a_{n+1}^2 = 3a_n^2$, where $a_n > 0$ for $n \geq 0$, and $a_0 = 2$. (Do not need to expand the multiplication.)

Ans:

$$\text{let } b_n = a_n^2$$

$$b_{n+1} = 3b_n, b_0 = 2^2 = 4$$

$$b_n = 4 * 3^n$$

$$\text{Therefore, } a_n = 2(\sqrt{3})^n \text{ for } n \geq 0, \text{ and } a_{10} = 2(\sqrt{3})^{10} = 486$$

10. (5%) Solve the recurrence relations:

$$2a_{n+2} - 11a_{n+1} + 5a_n = 0, n \geq 0, a_0 = 2, a_1 = -8$$

Ans:

$$\text{Let } a_n = cr^n$$

$$2r^2 - 11r + 5 = 0 \Rightarrow r = \frac{1}{2}, 5$$

$$a_n = c_1\left(\frac{1}{2}\right)^n + c_2(5^n)$$

$$2 = c_1 + c_2$$

$$-8 = c_1\left(\frac{1}{2}\right) + c_2(5)$$

$$\Rightarrow c_1 = 4, c_2 = -2$$

$$\Rightarrow a_n = 4\left(\frac{1}{2}\right)^n - 2(5^n), n \geq 0$$

11. (5%) Given closed binary operations $f : A \times A \rightarrow A$, determine whether the operations have an **identity element**. If an identity element exists, write its value; otherwise, answer "X".

(a) Let $A \in \mathbb{Z}$, and $f(a, b) = a + b$

(b) Let $A \in \mathbb{R}, a \neq 0, b \neq 0$, and $f(a, b) = a/b$

Ans: a) 0 b) X

12. (5%) Function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ defined by $f(x) = 2^x$. If f is invertible, write $f^{-1} = ?$. If not, write the reasons.

Ans: $f^{-1}(x) = \log_2(x)$

13. (5%) Let λ be empty string. $\{\lambda\} = \emptyset$ is true or false? Write your reasons.

Ans: False, since $|\{\lambda\}| = 1 \neq 0 = |\emptyset|$.

14. (5%) Find the coefficient of x^7 in $\frac{1}{1-3x}$.

Ans: $\frac{1}{1-3x} = \sum_{i=0}^{\infty} (3x)^i$

The coefficient of x^7 : $(3)^7$

15. (5%) Find the coefficient of x^4 in $(1+2x)^{-8}$.

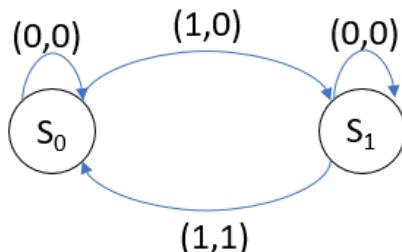
Ans: $(1+2x)^{-8} = \sum_{r=0}^{\infty} \binom{-8}{r} (2x)^r$

The coefficient of x^4 : $\binom{-8}{4} (2)^4 = (-1)^4 \binom{8+4-1}{4} (16) = (16) \binom{11}{4}$

16. (5%) Let A, B be the finite languages, prove or disprove that " $|AB|=|BA|$ ", and if disprove, just give a counterexample simply.

Ans: ppt.27.

17. (5%) Given the following FSM, find the language that the FSM recognized.



Ans: Ans: recognize the string with the even number of 1s and the last bit is 1.

18. (5%) The partition function $P(n)$ is the number of partitioning a positive integer n , please write its generating function.

Ans: $P(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)}$

19. (5%) Take over the previous question, find $P(7)$.

Ans: 15

20. (5%) Solve the relation $a_n = n \cdot a_{n-1}$, where $n \geq 1$ and $a_0 = 1$

Ans:

$$\begin{aligned}
a_0 &= 1 \downarrow \\
a_1 &= a_0 \times 1 \downarrow \\
a_2 &= a_1 \times 2 \downarrow \\
&\dots \downarrow \\
a_n &= a_{n-1} \times n \downarrow \\
a_0 \times a_1 \times a_2 \times \dots \times a_n &= a_0 \times a_1 \times a_2 \times \dots \times a_{n-1} \times n! \downarrow \\
&| a_n = n! \leftarrow
\end{aligned}$$

Table 10.2

	$a_n^{(p)}$
c , a constant	A , a constant
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
$n^t, t \in \mathbf{Z}^+$	$A_t n^t + A_{t-1} n^{t-1} + \dots + A_1 n + A_0$
$r^n, r \in \mathbf{R}$	Ar^n
$\sin \theta n$	$A \sin \theta n + B \cos \theta n$
$\cos \theta n$	$A \sin \theta n + B \cos \theta n$
$n^t r^n$	$r^n (A_t n^t + A_{t-1} n^{t-1} + \dots + A_1 n + A_0)$
$r^n \sin \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$
$r^n \cos \theta n$	$Ar^n \sin \theta n + Br^n \cos \theta n$