

# 2024 Fall Discrete Mathematics

## Final exam

### 2024/1/9

You may answer in Chinese or English. Please use a large and clear font. Please do not write in strange colors. Please indicate your **department**, **student ID**, and **name**. Please be sure to write according to the question number! No make-up submissions are allowed.

1. (5%) How many positive divisors do 378000 have?  
**Ans:  $(4+1)(3+1)(3+1)(1+1)=160$**
2. (5%) How many of the positive divisors of 378000 are multiples of 147?  
**Ans: 0. Because 378000 does not have enough powers of 7 and 3 in its factorization to satisfy  $147=3*7^2$ .**
3. (5%) Let  $m = 720720$  and  $n = 104720$ . Find  $\gcd(m, n)$  and  $\text{lcm}(m, n)$ .  
**Ans: 6160, 12252240**
4. (5%) Use the generating function  $[x/(1-x)]$  to calculate the total number of compositions for  $n = 6$ .  
**Ans:  $1+5+10+10+5+1=32$**
5. (5%) Let  $a_n$  denote the number of bit strings of length  $n$  that do not have two consecutive 0's. Let  $A$  denote the set of bit strings without two consecutive 0's. Find a recurrence relation for  $a_n$ .  
**Ans:  $a_n = a_{n-1} + a_{n-2}, n \geq 3$**
6. There are Fibonacci numbers  $F_0, F_1 \dots F_n$ . ( $F_0 = 0, F_1 = 1$ )
  - (a)(2%) For  $n \in \mathbb{Z}^+$  with  $n \geq 2$ , what is  $F_n = ?$
  - (b)(2%) What is  $F_0^2 + F_1^2 + F_2^2 \dots + F_n^2 = ?$
  - (c)(1%) What is  $F_0^2 + F_1^2 + F_2^2 \dots + F_{12}^2 = ?$**Ans:**  
**(a) For  $n \in \mathbb{Z}^+$  with  $n \geq 2, F_n = F_{n-1} + F_{n-2}$**   
**(b)**  

$$\sum_{i=0}^n F_i^2 = F_n \times F_{n+1}$$
  
**(c)**  

$$\sum_{i=0}^{12} F_i^2 = F_{12} \times F_{13} = 144 \times 233 = 33552$$
7. (5%) Given binary operations  $f : A \times A \rightarrow A, A \subseteq \mathbb{Z}^+, x, y \in A$ . Determine in each case whether  $f$  is commutative and/or associative, and/or closed. **Please fill the answers( V or X ) in the table below.**

| Functions | Commutative | Associative | Closed |
|-----------|-------------|-------------|--------|
|-----------|-------------|-------------|--------|

|                            |  |  |  |
|----------------------------|--|--|--|
| (a) $f(x, y) = x - y + xy$ |  |  |  |
| (b) $f(x, y) = x + y - 10$ |  |  |  |

Ans:

| Functions                  | Commutative | Associative | Closed |
|----------------------------|-------------|-------------|--------|
| (a) $f(x, y) = x - y + xy$ | X           | X           | V      |
| (b) $f(x, y) = x + y - 10$ | V           | V           | X      |

8. (5%) Let  $S = \{0, 1, 2, 3, 4, \dots, 9\}$ . How many elements must we select from  $S$  to insure that there will be at least two whose sum is 10?

Ans:

$$10 = 1 + 9 = 2 + 8 = 3 + 7 = 4 + 6$$

Pigeonholes are the subsets  $\{0\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$

$$6 + 1 = 7$$

9. (5%) Find  $a_{10}$  if  $a_{n+1}^2 = 3a_n^2$ , where  $a_n > 0$  for  $n \geq 0$ , and  $a_0 = 2$ . (Do not need to expand the multiplication.)

Ans:

$$\text{let } b_n = a_n^2$$

$$b_{n+1} = 3b_n, \quad b_0 = 2^2 = 4$$

$$b_n = 4 \cdot 3^n$$

$$\text{Therefore, } a_n = 2(\sqrt{3})^n \text{ for } n \geq 0, \text{ and } a_{10} = 2(\sqrt{3})^{10} = 486$$

10. (5%) Solve the recurrence relations:

$$2a_{n+2} - 11a_{n+1} + 5a_n = 0, \quad n \geq 0, \quad a_0 = 2, \quad a_1 = -8$$

Ans:

$$\text{Let } a_n = cr^n$$

$$2r^2 - 11r + 5 = 0 \Rightarrow r = \frac{1}{2}, 5$$

$$a_n = c_1\left(\frac{1}{2}\right)^n + c_2(5^n)$$

$$2 = c_1 + c_2$$

$$-8 = c_1\left(\frac{1}{2}\right) + c_2(5)$$

$$\Rightarrow c_1 = 4, \quad c_2 = -2$$

$$\Rightarrow a_n = 4\left(\frac{1}{2}\right)^n - 2(5)^n, \quad n \geq 0$$

11. (5%) Given closed binary operations  $f : A \times A \rightarrow A$ , determine whether the operations have an **identity element**. If an identity element exists, write its value; otherwise, answer "X".

(a) Let  $A \in \mathbb{Z}$ , and  $f(a, b) = a + b$

(b) Let  $A \in \mathbb{R}$ ,  $a \neq 0, b \neq 0$ , and  $f(a, b) = a/b$

Ans: a) 0 b) X

12. (5%) Function  $f : \mathbb{R} \rightarrow \mathbb{R}^+$  defined by  $f(x) = 2^x$ . If  $f$  is invertible, write  $f^{-1} = ?$ . If not, write the reasons.

Ans:  $f^{-1}(x) = \log_2(x)$

13. (5%) Let  $\lambda$  be empty string.  $\{\lambda\} = \emptyset$  is true or false? Write your reasons.

Ans: False, since  $|\{\lambda\}| = 1 \neq 0 = |\emptyset|$ .

14. (5%) Find the coefficient of  $x^7$  in  $\frac{1}{1-3x}$ .

Ans:  $\frac{1}{1-3x} = \sum_{i=0}^{\infty} (3x)^i$

The coefficient of  $x^7 : (3)^7$

15. (5%) Find the coefficient of  $x^4$  in  $(1 + 2x)^{-8}$ .

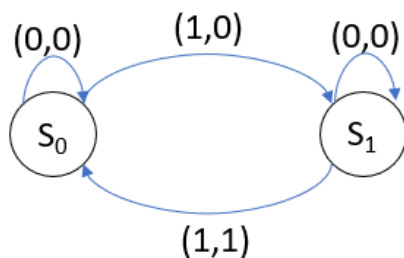
Ans:  $(1 + 2x)^{-8} = \sum_{r=0}^{\infty} \binom{-8}{r} (2x)^r$

The coefficient of  $x^4 : \binom{-8}{4} (2)^4 = (-1)^4 \binom{8+4-1}{4} (16) = (16) \binom{11}{4}$

16. (5%) Let  $A, B$  be the finite languages, prove or disprove that " $|AB|=|BA|$ ", and if disprove, just give a counterexample simply.

Ans: ppt.27.

17. (5%) Given the following FSM, find the language that the FSM recognized.



Ans: recognize the string with the even number of 1s and the last bit is 1.

18. (5%) The partition function  $P(n)$  is the number of partitioning a positive integer  $n$ , please write its generating function.

Ans:  $P(x) = \prod_{i=1}^{\infty} \frac{1}{(1-x^i)}$

19. (5%) Take over the previous question, find  $P(7)$ .

Ans: 15

20. (5%) Solve the relation  $a_n = n \cdot a_{n-1}$ , where  $n \geq 1$  and  $a_0 = 1$

Ans:

$$\begin{aligned}
 a_0 &= 1 \downarrow \\
 a_1 &= a_0 \times 1 \downarrow \\
 a_2 &= a_1 \times 2 \downarrow \\
 &\dots \downarrow \\
 a_n &= a_{n-1} \times n \downarrow \\
 a_0 \times a_1 \times a_2 \times \dots \times a_n &= a_0 \times a_1 \times a_2 \times \dots \times a_{n-1} \times n! \downarrow \\
 a_n &= n! \leftarrow
 \end{aligned}$$

**Table 10.2**

|                           | $a_n^{(p)}$   |
|---------------------------|---|
| $c$ , a constant          | $A$ , a constant  |
| $n$                       | $A_1 n + A_0$   |
| $n^2$                     | $A_2 n^2 + A_1 n + A_0$                                 |
| $n^t, t \in \mathbf{Z}^+$ | $A_t n^t + A_{t-1} n^{t-1} + \dots + A_1 n + A_0$       |
| $r^n, r \in \mathbf{R}$   | $Ar^n$  |
| $\sin \theta n$           | $A \sin \theta n + B \cos \theta n$                     |
| $\cos \theta n$           | $A \sin \theta n + B \cos \theta n$                     |
| $n^t r^n$                 | $r^n (A_t n^t + A_{t-1} n^{t-1} + \dots + A_1 n + A_0)$ |
| $r^n \sin \theta n$       | $Ar^n \sin \theta n + Br^n \cos \theta n$               |
| $r^n \cos \theta n$       | $Ar^n \sin \theta n + Br^n \cos \theta n$               |