Financial Market Uncovered – Article 11

Demystifying Interest Rate Models: From Short Rates to Market Curves

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Summary

1 Introduction

In the architecture of financial markets, interest rates play the role of gravitational force — invisible yet omnipresent, influencing the pricing, valuation, and behaviour of virtually every financial instrument. Whether you're valuing a government bond, pricing a swaption, or computing the cost of carry for an FX forward, some form of interest rate modelling is at work behind the scenes.

Despite this centrality, interest rate modelling is often misunderstood. It is not a monolithic practice, but a layered discipline with multiple paradigms. Some models focus on the *short rate* — the instantaneous interest rate at a given time — while others aim to describe the *entire yield curve* or even the behaviour of traded *market instruments* like forward rate agreements and swaptions. Each approach brings different strengths, assumptions, and degrees of realism, and each responds to a distinct practical need: pricing, risk management, calibration, or arbitrage-free simulation.

Over the past two decades, the field has evolved rapidly. The simple elegance of early models like Vasicek or CIR gave way to more flexible but computationally demanding frameworks like Heath–Jarrow–Morton (HJM) and the LIBOR Market Model (LMM). Meanwhile, real-world developments — such as the 2008 financial crisis and the transition away from LIBOR — have exposed the limitations of traditional models, spurring innovation in multi-curve frameworks and stochastic basis modelling.

Understanding interest rate models means understanding three things:

- How interest rates behave dynamically over time
- How instruments are priced under no-arbitrage conditions
- And how to reconcile theoretical constructs with actual market data

2 Short-Rate Models: The Starting Point

The earliest and most intuitive class of interest rate models begins with a single quantity: the short rate. Instead of modelling the entire yield curve, these models track the evolution of the instantaneous interest rate over time. Their appeal lies in mathematical tractability and analytical clarity — many allow closed-form solutions for bond prices and derivatives. But simplicity comes with trade-offs: they often struggle to match observed market prices or replicate complex curve dynamics.

2.1 What is the Short Rate?

At the core of many interest rate models lies a deceptively simple concept: the **short rate**. Often denoted by $r(t)$, the short rate is the instantaneous interest rate at time t — the rate at which an investor can borrow or lend money for an infinitesimally small period starting at t. It is the theoretical equivalent of a one-day overnight rate taken to the limit of zero time.

Formally, if you invest $1 at time t at the short rate $r(t)$, its value at time $t + dt$ is given by:

$$1 + r(t)dt$$

This rate is continuously compounded and underpins the pricing of fixed-income instruments in continuous-time models. In particular, the price of a **zero-coupon bond** maturing at time T, denoted $P(t,T)$, is expressed in terms of the expected path of the short rate between t and T, under a risk-neutral measure.

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\middle|\mathcal{F}_t\right]$$

This expression reveals two important features of the short rate:

- **It determines the entire term structure of interest rates** — once the dynamics of $r(t)$ are specified, one can theoretically derive the price of any bond and, by extension, the yield curve.
- **It forms the basis for arbitrage-free pricing** — since the price of a bond must reflect the time value of money under the appropriate probability measure, the short rate naturally anchors all no-arbitrage pricing frameworks.

However, the short rate is not directly observable in the market. Traders typically quote and observe longer-dated rates (e.g., 3-month LIBOR, 5-year swap rates, 10-year bond yields), not the instantaneous rate.

Nonetheless, the short rate remains conceptually elegant and mathematically tractable. By specifying how $r(t)$ evolves stochastically — whether through mean reversion, volatility, or external factors — we can build models that attempt to capture the behaviour of interest rates over time.

2.2 Vasicek Model

The **Vasicek model**, introduced in 1977, is one of the earliest and most influential models in interest rate theory. It proposes a simple stochastic process for the short rate, built around two core economic intuitions: **mean reversion** and **randomness**. In Vasicek's framework, interest rates tend to revert to a long-term equilibrium, but do so with some randomness over time.

The model defines the short rate dynamics under the risk-neutral measure $\mathbb{Q}$ as:

$$dr(t) = \kappa\big(\theta - r(t)\big)dt + \sigma W_t$$

where:

- $\kappa > 0$ is the **mean reversion speed**
- θ is the **long-term mean level** of the short rate
- σ is the **volatility** of the short rate
- W_t is a standard Brownian motion under $\mathbb{Q}$

This stochastic differential equation is a special case of the **Ornstein–Uhlenbeck process**, which has several desirable analytical properties.

Key Features

- *Mean-Reverting Behaviour:* The term $\kappa(\theta - r(t))$ ensures that the rate is pulled toward the long-term level θ. The higher κ, the faster the reversion.
- *Normally Distributed Rates:* The Vasicek model implies that $r(t)$ is normally distributed for any finite t. This allows closed-form solutions for zero-coupon bond prices and other derivatives.
- *Analytical Tractability:* The bond price under the Vasicek model can be written in exponential affine form:

$$P(t,T) = A(t,T) * exp(-B(t,T)r(t))$$

with:

$$B(t,T) = \frac{\left(1 - e^{-\kappa(T-t)}\right)}{\kappa}, A(t,T) = \exp\left(\left(\theta - \frac{\sigma^2}{2\kappa^2}\right)\big(B(t,T) - (T-t)\big) - \frac{\sigma^2}{4\kappa}B(t,T)^2\right)$$

This closed-form pricing formula is one of the main reasons why the Vasicek model is so widely studied.

The exponential term tells us: *"the higher the current short rate r(t), the lower the bond's price."* That's intuitive: higher rates mean future cash flows are discounted more.

The functions $A(t,T)$ and $B(t,T)$ adjust the price based on how quickly rates mean-revert, how volatile they are, and how far we are from maturity.

Limitations

The Vasicek model suffers from one major flaw: *It allows negative interest rates, since the short rate is normally distributed and unbounded below.*

This was long seen as a theoretical defect. However, the global shift to negative rates in Europe and Japan has made this feature more relevant in practice — though still inconsistent with the non-negative nature of many real-world rate processes (especially in emerging markets).

Moreover:

- The volatility is constant, regardless of the level of the rate, which is empirically unrealistic.
- The model is not market-consistent — it does not fit today's yield curve or volatility structure without recalibration.

Interpretation and Use

The Vasicek model remains a cornerstone of interest rate theory because:

- It illustrates the principle of mean reversion clearly.
- It provides a clean laboratory for analytical derivations (bond pricing, yield curves, duration).
- It serves as a foundation for more complex models (e.g., Hull-White extends Vasicek with time-dependent parameters).

2.3 CIR Model

The Cox–Ingersoll–Ross (CIR) model, introduced in 1985, builds upon the Vasicek model's core insight — that interest rates are mean-reverting — while addressing one of its key flaws: the possibility of negative interest rates.

The CIR model modifies the short-rate dynamics by making the volatility of the rate depend on its level. Specifically, the short rate evolves according to the following stochastic differential equation:

$$dr(t) = \kappa\big(\theta - r(t)\big)dt + \sigma\sqrt{r(t)}dW_t$$

where, as before:

- $\kappa > 0$ is the mean reversion speed
- θ is the long-term mean level of the short rate
- σ is the volatility of the short rate
- W_t is a standard Brownian motion under $\mathbb{Q}$

What's Different?

The key innovation lies in the volatility term:

$$\sigma\sqrt{r(t)}$$

This means:

- When $r(t)$ is high, the short rate becomes more volatile.
- When $r(t)$ is near zero, the volatility shrinks — helping keep the short rate positive.

This square-root diffusion process is called the Feller process, and under specific parameter conditions (notably $2\kappa\theta > \sigma^2$), the model guarantees that $r(t)$ stays strictly non-negative — a critical property in many fixed income markets.

Key Features

- Non-Negativity: Unlike Vasicek, the CIR model keeps rates positive without artificial constraints.
- Level-Dependent Volatility: More realistic rate behaviour — rates are more volatile when high, and more stable when low.
- Tractability: The model retains an exponential-affine bond pricing formula similar in structure to Vasicek:

$$P(t,T) = A(t,T) * exp(-B(t,T)r(t))$$

though the exact formulas for $A(t,T)$ and $B(t,T)$ are more complex and involve modified Bessel functions (often handled numerically).

Limitations

While the CIR model improves realism by enforcing non-negative rates and making volatility state-dependent, it has limitations:

- Limited Flexibility: Like Vasicek, CIR has constant parameters and cannot fit today's term structure exactly without time-dependent extensions.
- Calibration Challenges: The square-root volatility structure complicates numerical calibration, especially when market rates are close to zero (as they have been for much of the 2010s and 2020s).
- Still Not Market-Consistent: It is a model of rates, not of tradable instruments — meaning it often requires adjustments to match observed derivative prices.

When Is CIR Useful?

- The CIR model is popular in credit risk modelling, particularly in the intensity-based models of default (e.g., the CIR++ framework).

- It’s also well-suited to emerging market rates, where keeping rates strictly positive is important and volatility often scales with level.
- In interest rate modelling, CIR often serves as a benchmark model, especially when analytical tractability and positivity are both desired.

2.4 Hull–White Model

The Hull–White model, introduced in the early 1990s, is arguably the most widely used short-rate model in practice — particularly in quantitative finance desks and risk systems. It extends the Vasicek framework by allowing the mean reversion level to vary over time, which makes it flexible enough to match today’s observed term structure exactly.

The short rate $r(t)$ evolves under the risk-neutral measure according to:

$$dr(t) = [\theta(t) - \kappa r(t)]dt + \sigma dW_t$$

where:

- κ is the speed of mean reversion (constant)
- $\theta(t)$ is a time-dependent function calibrated to fit the current yield curve
- σ is the volatility of the short rate
- W_t is a Brownian motion under $\mathbb{Q}$

This is still an Ornstein–Uhlenbeck process, like Vasicek — but much more adaptable.

Why the Time-Dependent Drift?

In the Vasicek model, the long-term rate θ is fixed. That means the model cannot perfectly reproduce today’s yield curve unless you're lucky. The Hull–White model solves this by making $\theta(t)$ a deterministic function that changes over time and is **calibrated** so that the model fits the current term structure *by construction*.

This ability to fit the initial curve exactly is critical when pricing interest rate derivatives like:

- Caps/floors
- Swaptions
- Callable bonds

In practice, $\theta(t)$ is often derived using analytical expressions that reverse-engineer the yield curve into the model — a process that remains tractable thanks to the model's linearity.

Key Features

- **Analytical Tractability**: The Hull–White model retains a closed-form solution for zero-coupon bond prices:

$$P(t,T) = A(t,T) * exp(-B(t,T)r(t))$$

with known expressions for $A(t,T)$ and $B(t,T)$, making it efficient for valuation and Greeks.

- Calibrated Realism: Thanks to $\theta(t)$, the model aligns with observed yield curves and can be recalibrated daily to changing market conditions.
- Gaussian Distribution: Like Vasicek, the Hull–White model allows negative rates — a drawback in some cases, but less controversial in a post-2010 world where real negative rates are observable.

Limitations

- *Negative Rates:* Although more acceptable today, the possibility of deeply negative rates can still be problematic for long-dated risk assessments or regulatory models.
- *Constant Volatility:* While $\theta(t)$ is flexible, the volatility σ remains constant — making it harder to capture term structure dynamics in volatility.
- *Calibration Complexity:* While tractable, fitting the model to market-observed cap/floor or swaption volatilities requires bootstrapping and interpolation, especially in multi-factor extensions.

Practical Applications

The Hull–White model is widely used in:

- Interest rate derivatives pricing (e.g., Bermudan swaptions, callable bonds)
- Risk-neutral valuation in fixed income platforms and XVA engines
- Yield curve simulation for stress testing and scenario generation

2.5 Model Comparisons

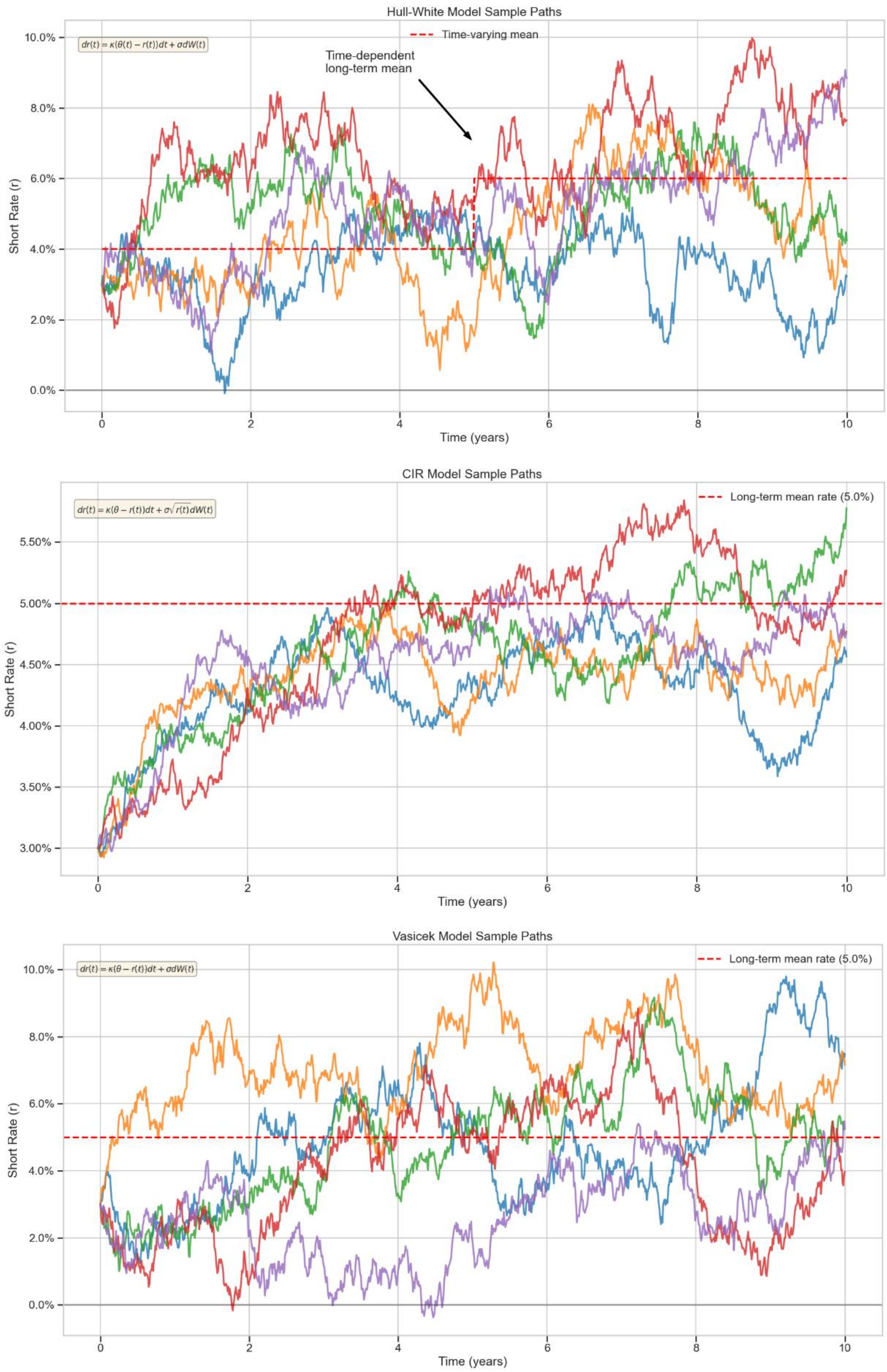

Figure 1: Hull-White, CIR, and Vasicek models sample paths

The sample paths of the Vasicek, CIR, and Hull–White models clearly illustrate the behavioural differences between the three short-rate frameworks.

- The **Hull–White model** displays greater flexibility with its time-dependent mean, allowing the rate to revert to different levels over time — well suited for environments where central bank guidance or macro regimes shift gradually.
- The **CIR model**, due to its square-root volatility term, strictly enforces positivity of the short rate — a critical feature for credit-sensitive applications.
- The **Vasicek model** shows smooth mean-reverting dynamics, but allows for negative rates, which may be unrealistic in some market regimes.

Together, these plots highlight how small structural changes in model assumptions lead to significantly different rate dynamics — especially in long-term forecasting or stress testing.

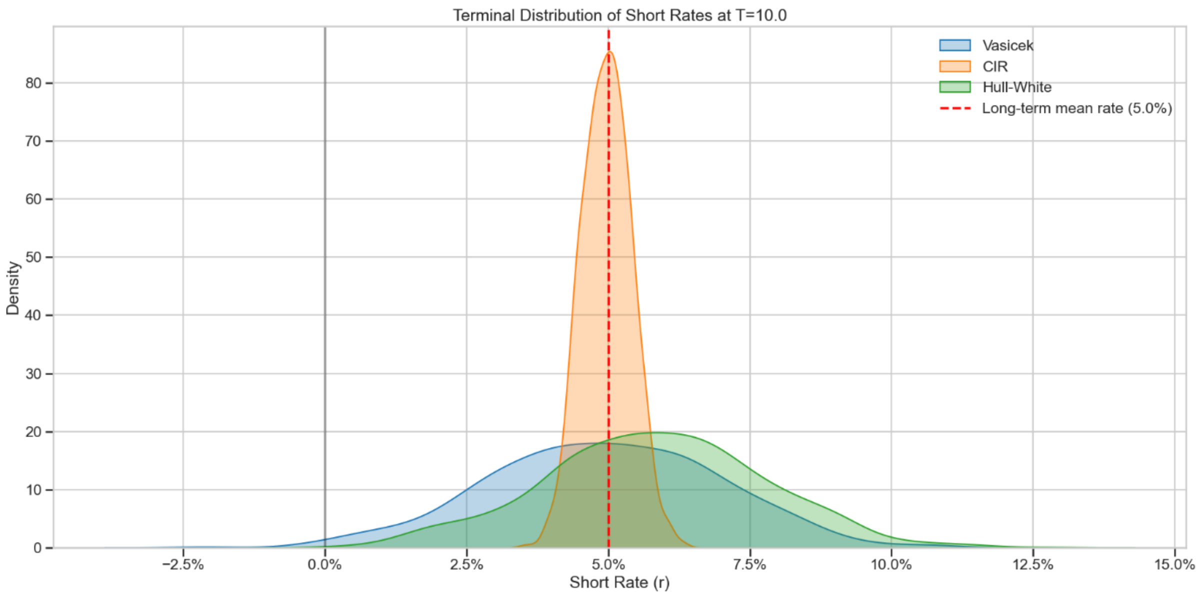

Figure 2: Terminal distribution of short rates

At a 10-year horizon, the terminal distributions of short rates differ markedly across the three models:

- **CIR** produces a sharply peaked distribution centred around the long-term mean, reflecting strong mean reversion and low dispersion.
- **Hull–White** offers a wider spread with more flexible tails, accommodating scenarios where macroeconomic regimes or policy anchors shift.
- **Vasicek** shows the broadest distribution — including a non-negligible probability of negative rates — due to its constant volatility and Gaussian assumptions.

This comparison reinforces that model selection isn't just a matter of pricing instruments — it's about understanding how rates behave under uncertainty, especially over long horizons.

3 Bond Pricing and Term Structure from Short-Rate Models

Once the dynamics of the short rate are specified, the next step is to derive asset prices — and the most fundamental asset in this context is the zero-coupon bond. These instruments, which pay a single cash flow at maturity, are the cornerstone of the fixed-income world. Their prices not only reflect interest rate expectations but also form the basis of the yield curve and the valuation of more complex derivatives.

3.1 Pricing Zero-Coupon Bonds: PDEs and Risk-Neutral Expectations

At the heart of every short-rate model lies a fundamental objective: pricing zero-coupon bonds. These bonds are the building blocks of all fixed-income instruments — their prices define the yield curve, discount factors, and ultimately the valuation of swaps, options, and structured products.

The key idea is this:

The value of a zero-coupon bond maturing at time T, denoted $P(t,T)$, is equal to the expected value — under a risk-neutral measure — of receiving one unit of currency at time T, discounted continuously at the short rate $r(t)$ over time.

This gives rise to the pricing formula:

$$P(t,T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\middle|\mathcal{F}_t\right]$$

Here:

- $\mathbb{E}^{\mathbb{Q}}$ is the expectation under the risk-neutral measure
- $\int_t^T r(s)ds$ is the cumulative discount factor from now until maturity

This expression defines the bond price as the expected value of a discounted cash flow, where the discounting is stochastic — it depends on how the short rate evolves over time.

3.1.1 Risk-Neutral Expectation (Simulation/Analytical)

If the model allows, we can directly evaluate the expectation above. This is possible in models like Vasicek and Hull–White, where the distribution of $r(t)$ is known and tractable. The result is often an exponential-affine formula:

$$P(t,T) = A(t,T) * exp(-B(t,T)r(t))$$

This form is widely used in pricing systems because it's fast, intuitive, and analytically sound.

3.1.2 Partial Differential Equations (PDE Method)

Alternatively, we can derive a **PDE** that the bond price must satisfy. From no-arbitrage arguments, it can be shown that $P(t,T)$ must satisfy:

$$\frac{\partial P}{\partial t} + \mu(t,r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma(t,r)^2\frac{\partial^2 P}{\partial r^2} - rP = 0$$

with the boundary condition $P(T,T) = 1$. Here, $\mu(t,r)$ and $\sigma(t,r)$ are the drift and volatility of the short-rate process under the risk-neutral measure.

This approach is useful when closed-form solutions are unavailable — for example, in more complex models like CIR or time-dependent stochastic volatility models.

From Bonds to Yield Curves

Once we have the bond price $P(t,T)$, we can define the continuously compounded yield $y(t,T)$ as:

$$y(t,T) = \frac{-\ln P(t,T)}{T-t}$$

The function $T \mapsto y(t,T)$ is the zero-coupon yield curve at time t. All other fixed-income products — coupon bonds, swaps, forwards — are ultimately constructed from this curve.

3.2 Closed-Form vs Numerical Integration Techniques

When working with short-rate models, computing the price of a zero-coupon bond often comes down to one question:

Can the expectation be evaluated analytically, or must we rely on numerical methods?

This distinction is not just theoretical — it affects speed, accuracy, and even model selection in trading systems and risk engines.

Closed-Form Solutions: Elegant and Efficient

Some short-rate models — particularly **Vasicek** and **Hull–White** — are structured in a way that allows for **closed-form bond pricing**. In these models, the stochastic process for the short rate is linear and Gaussian, which enables the use of tools from stochastic calculus (e.g., the Feynman–Kac theorem) to derive exact expressions.

These solutions take the **exponential-affine form**:

$$P(t,T) = A(t,T) * exp(-B(t,T)r(t))$$

where $A(t,T)$ and $B(t,T)$ are deterministic functions that depend on the model parameters (mean reversion, volatility, time horizon). These expressions can be computed very efficiently, making them ideal for:

- Real-time pricing
- Risk calculations (Greeks, sensitivities)
- Monte Carlo simulation of path-dependent products

They are also widely used in analytic approximations for more complex instruments like swaptions and callable bonds.

Numerical Methods: General but Costly

In more complex models — such as the CIR model or multi-factor frameworks — the expectation

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(-\int_t^T r(s)ds\right)\right]$$

cannot be simplified to a closed-form expression. In these cases, we must rely on numerical techniques, such as:

- Finite Difference Methods: Solve the pricing PDE on a grid of rate and time points. Requires discretisation, boundary conditions, and stability checks.
- Monte Carlo Simulation: Generate thousands of short-rate paths and estimate the expectation empirically. Flexible but computationally expensive.
- Quadrature or Tree Methods: Use lattice approximations or numerical integration over possible future rates (e.g., binomial trees or Gauss–Hermite quadrature).

Each method comes with trade-offs in speed, accuracy, and dimensionality. For instance, Monte Carlo scales well with complexity but is slow to converge; PDEs are faster but limited to low-dimensional models.

Why This Matters

The choice between closed-form and numerical methods is not purely academic:

- In trading systems, speed is essential — closed-form models dominate for this reason.
- In risk and regulatory contexts, accuracy and robustness may take priority — favouring more flexible but intensive numerical approaches.
- For exotic products or stress-testing, numerical techniques become indispensable.

3.3 Yield Curves and the Link to Observed Market Rates

While short-rate models provide a theoretical framework for pricing bonds, traders and investors operate in a world dominated not by abstract models, but by **observed yield curves**. These curves — representing interest rates across different maturities — are the practical interface between mathematical modelling and real-world pricing.

So how do we connect the **model-driven world** of short-rate dynamics with the **market-driven world** of yield curves?

From Bond Prices to Yields

The first step is to convert the model-implied bond prices $P(t,T)$ into continuously compounded zero-coupon yields:

$$y(t,T) = \frac{-\ln P(t,T)}{T-t}$$

This equation tells us:

The yield $y(t,T)$ is the average rate that equates the present value of \$1 received at time T to the observed (or modelled) bond price.

Plotting $y(t,T)$ against maturity T gives us the zero-coupon yield curve implied by the model.

Observed Market Curves

In practice, yield curves are not derived from theoretical models — they are constructed from a set of liquid market instruments, such as:

- Government bonds (e.g., OATs, Treasuries, Bunds)
- Interest rate swaps
- Deposit rates and futures contracts
- Forward rate agreements (FRAs)

Each of these instruments reflects market expectations of future interest rates. Financial institutions use bootstrapping techniques to infer the yield curve that is consistent with today's prices — this becomes the input for pricing and discounting across asset classes.

The Calibration Gap

Here lies the core challenge of short-rate models:

Most short-rate models cannot reproduce the entire observed yield curve *unless* they include time-dependent parameters (like $\theta(t)$ in Hull–White).

In fixed-parameter models (e.g. Vasicek, CIR), the implied yield curve is typically too smooth or too rigid to match market data. This creates a calibration gap — the model is internally consistent, but externally inaccurate.

To bridge this gap:

- Models like Hull–White allow θ(t)\theta(t)θ(t) to be calibrated to fit today's yield curve exactly.
- More advanced frameworks (e.g. HJM, LMM) directly model instruments or forward rates observed in the market, which improves calibration but increases complexity.

3.4 Interest Rate Derivatives Under Short-Rate Dynamics

Short-rate models were originally developed to price **zero-coupon bonds**, but their real value lies in how they extend to the pricing of **interest rate derivatives** — the instruments that dominate modern fixed-income trading: caps, floors, swaptions, callable bonds, and more.

These derivatives are sensitive not only to the level of rates but to how rates **move and evolve** over time. Short-rate models offer a coherent, arbitrage-free framework to simulate that evolution and derive fair prices under the **risk-neutral measure**.

Core Principle: Expectation Under the Risk-Neutral Measure

For any interest rate derivative with payoff Φ at future time T, its price at time t is given by:

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r(s)ds}\Phi \middle| \mathcal{F}_t\right]$$

This is the natural extension of the zero-coupon bond pricing logic: the derivative's payoff is discounted stochastically using the path of the short rate, and the expected value is taken under the risk-neutral measure.

Common Derivatives and How Short-Rate Models Handle Them

1. Caps and Floors

- Caps are options on floating interest rates (e.g. 3M LIBOR).
- In a short-rate model, we compute the expected payoff of each caplet by integrating over the stochastic discount factor and the projected rate.
- Models like Hull–White and CIR allow for semi-analytical pricing using Jamshidian's decomposition.

2. Swaptions

- A swaption is an option on a future interest rate swap.
- In one-factor short-rate models, the underlying swap is priced as a portfolio of zero-coupon bonds.
- Swaptions can be priced using analytic approximations, closed-form solutions (in specific cases), or numerical methods (Monte Carlo, PDE).

3. Callable Bonds and Embedded Options

- Callable and puttable bonds have embedded options depending on rate paths.
- These require backward induction or tree methods under short-rate dynamics to capture the optimal exercise strategy.

Advantages of Using Short-Rate Models

- No-Arbitrage Structure: All cash flows are discounted consistently with the model.

- Analytical Tools: Many products (e.g., European options) admit semi-closed forms.
- Simulation Capability: The entire term structure can be simulated across future scenarios, making short-rate models powerful tools for risk management and ALM (asset-liability management).

Limitations and Market Gaps

Despite their consistency, short-rate models face practical limitations:

- Market Calibration: These models do not directly price market instruments (e.g. caps, swaptions) and often require complex adjustments to match observed implied volatilities.
- Smile and Skew: They typically assume lognormal or normal distributions of rates, failing to capture volatility smiles or skews seen in real-world option markets.
- Limited Flexibility: One-factor models cannot capture the richness of real curve dynamics (e.g. twists, butterflies).

This is one of the key reasons why market models (such as the LIBOR Market Model) were developed — to directly model the instruments that are traded, not just the rates that underlie them.

4 The HJM Framework

While short-rate models describe how the spot interest rate evolves over time, they do not directly control the shape or dynamics of the entire yield curve. In practice, however, it is the whole term structure — not just the short rate — that matters for pricing, hedging, and managing interest rate risk. The Heath–Jarrow–Morton (HJM) framework addresses this by modelling forward rates at every maturity directly. This approach provides greater flexibility and allows the modeller to ensure that the entire yield curve evolves in a way that remains arbitrage-free.

4.1 Motivation: Why Forward-Rate Models Emerged

Short-rate models offer mathematical clarity and a structured approach to pricing bonds and derivatives. However, they model only a single point on the yield curve — the instantaneous short rate — and derive the rest of the curve indirectly. This becomes a major limitation when practitioners want to model or hedge movements across the entire term structure, especially for instruments sensitive to specific maturities.

This is where forward-rate models come in.

Instead of modelling just the short rate, forward-rate frameworks — like the Heath–Jarrow–Morton (HJM) model — describe how the entire yield curve evolves over time. This shift allows the modeller to directly capture the dynamics of forward rates and ensure consistency with any observed initial yield curve and arbitrage-free conditions.

In other words:

Forward-rate models were designed to let us model *what we actually see and trade* — the full structure of future interest rates, not just one point at a time.

4.2 The Structure of Instantaneous Forward Rates

In the HJM framework, the object of interest is not the short rate, but the instantaneous forward rate curve. For each time t, we define a forward rate $f(t,T)$, which represents the interest rate agreed at time t for borrowing over an infinitesimally short period at future time T. In contrast to the short rate $r(t)$, which is a single number at time t, the function $T \mapsto f(t,T)$ is a curve — a function of maturity.

Formally, the instantaneous forward rate is defined as:

$$f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T)$$

where $P(t,T)$ is the price at time t of a zero-coupon bond maturing at T.

Interpreting $\boldsymbol{f(t,T)}$

- If you know all bond prices $P(t,T)$, you can construct the entire forward rate curve $f(t,T)$.
- The short rate is simply the forward rate evaluated at $T = t$:

$$r(t) = f(t,t)$$

In this sense, the forward rate curve contains all information about current and future expected interest rates and underlies all pricing in the HJM model.

Why This Matters

The key advantage of modelling $f(t,T)$ directly is that it gives us explicit control over the shape and dynamics of the yield curve. This is particularly useful when:

- Managing portfolios exposed to different curve points (e.g. 2s10s flatteners)
- Pricing exotic products linked to multiple maturities
- Simulating the term structure under different scenarios

In the HJM framework, the forward rate curve becomes the dynamic object that evolves through time — not just a derived result, but a primary modelling target.

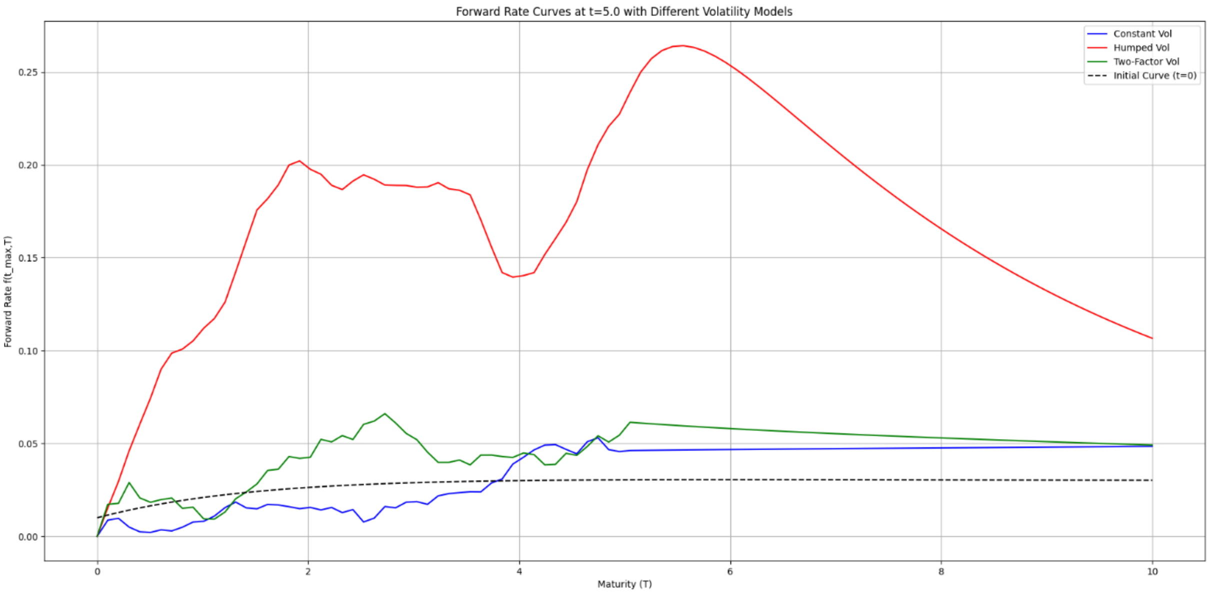

Figure 3: Forward rate curves with different volatility

This forward rate snapshot shows how the term structure evolves under different volatility assumptions in the HJM framework.

- The **constant volatility** model results in a smooth, gently upward-sloping curve, consistent with the initial structure.
- The **humped volatility** model generates a more realistic forward curve, capturing mid-curve steepening and long-end flattening — common features in empirical yield curves.

- The **two-factor volatility** setup adds another layer of realism, introducing twists and local bumps that cannot be captured by single-factor models.

This demonstrates the flexibility of HJM in capturing complex curve behaviours. In practice, traders and risk managers often prefer these richer models to match swaption surfaces and simulate curve evolution under economic scenarios.

4.3 The No-Arbitrage Condition: Volatility Determines Drift

One of the most elegant and defining features of the Heath–Jarrow–Morton (HJM) framework is that it automatically enforces the absence of arbitrage, provided the volatility structure is specified appropriately. In contrast to short-rate models, where arbitrage-free pricing must be carefully constructed, the HJM model builds it in by design.

The key insight is this:

In the HJM framework, once you specify how the volatility of forward rates behaves, the model uniquely determines the corresponding drift needed to eliminate arbitrage.

Mathematical Formulation

The forward rate $f(t,T)$ under the risk-neutral measure evolves as:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

where:

- $\sigma(t,T)$ is the volatility of the forward rate at time t for maturity T
- $\alpha(t,T)$ is the drift term — which is not arbitrary

To prevent arbitrage, HJM imposes the following condition on the drift:

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)du$$

This remarkable result means:

- You only need to choose the volatility function $\sigma(t,T)$
- The drift $\alpha(t,T)$ is then fully determined to ensure no-arbitrage

This is known as the HJM drift condition, and it ensures that the bond price dynamics derived from $f(t,T)$ are consistent with risk-neutral valuation and arbitrage-free pricing.

Implication: Volatility is Everything

In traditional models, volatility is often seen as an adjustment — a factor that adds realism. In HJM, volatility is the driving force of the entire term structure. Once you define how uncertainty impacts different maturities, the model shapes the expected path of rates to ensure consistency.

This has profound consequences:

- Different volatility choices (e.g. constant, maturity-dependent, time-varying) lead to very different yield curve dynamics
- Calibration becomes focused on fitting volatility surfaces, not just interest rate levels

In short, the HJM framework turns modelling on its head: instead of choosing a process and checking for arbitrage, you start with a volatility structure and let the mathematics guarantee arbitrage-free dynamics.

Figure 4: HJM Forward Rate Evolution with Constant Volatility

This 3D plot illustrates the evolution of the forward rate surface $f(t,T)$ over time with a constant volatility specification. The smooth, monotonic growth highlights the direct relationship between volatility and forward drift:

In HJM, volatility drives the drift, enforcing arbitrage-free dynamics.

Here, the surface remains stable but lacks curvature — underscoring the limitations of constant-vol models in capturing observed market irregularities.

4.4 *Advantages of HJM*

The HJM framework stands out for one core reason: it models exactly what matters — the forward curve itself.

Because it directly captures the dynamics of the entire term structure, HJM offers several key advantages over short-rate models:

- *Perfect Fit to the Initial Yield Curve:* No need for ad hoc adjustments — the model starts from observed bond prices and builds from there.
- *Flexible Volatility Structures:* You can specify how different maturities behave under volatility, which is critical for pricing and hedging instruments across the curve.
- *Naturally Arbitrage-Free:* As seen in the previous section, the no-arbitrage condition is automatically enforced once the volatility is set.
- *Customisable Dynamics:* You can tailor the model to reflect market realities — steepening, flattening, twists, and more — by adjusting the volatility function.

These features make HJM particularly appealing for long-dated instruments, structured products, and curve-sensitive derivatives where full control over the term structure is essential.

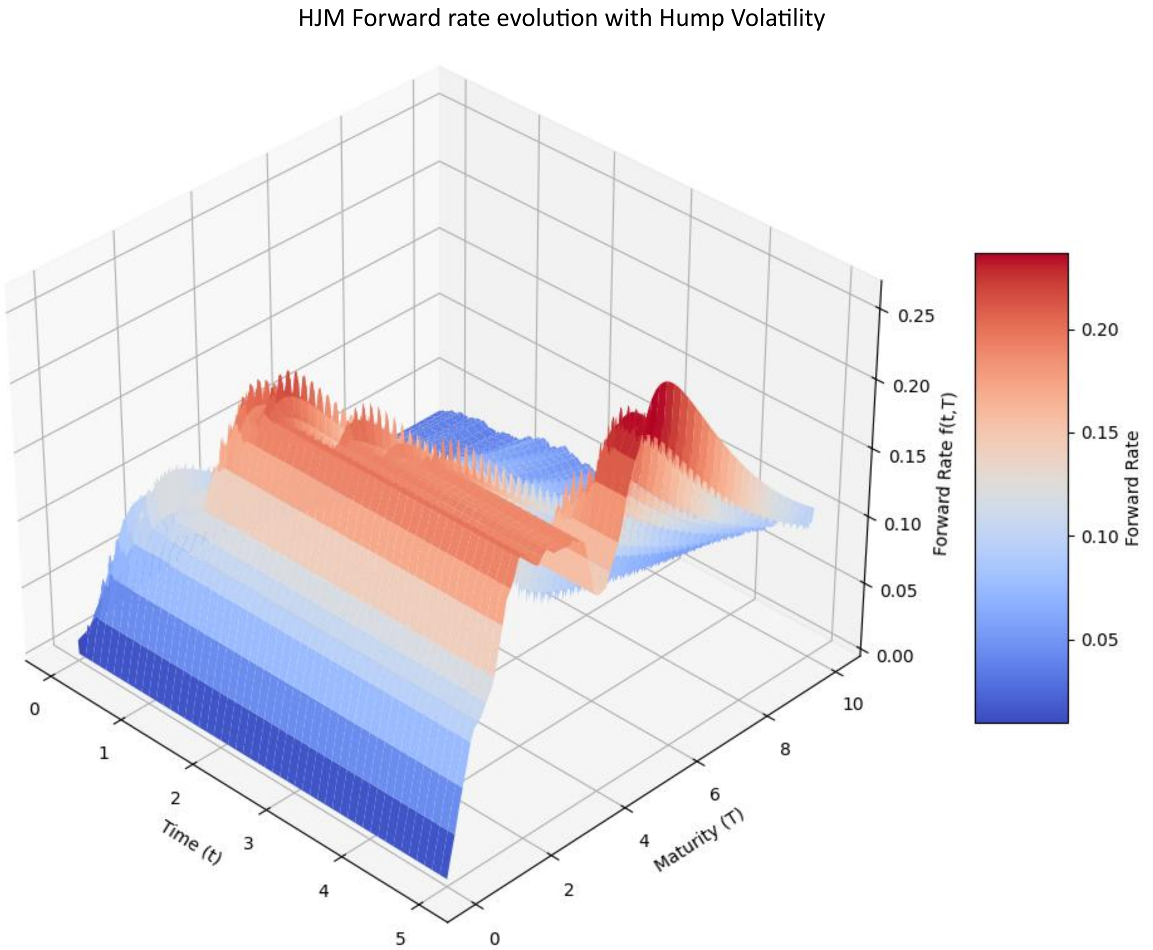

Figure 5: HJM Forward rate evolution with Hump Volatility

In contrast, the hump volatility case leads to a forward surface that develops rich features:

- Pronounced mid-curve bulges
- Dynamic evolution over time
- Enhanced curvature across both time and maturity axes

This is far more reflective of real-world forward curves and shows how modifying volatility functions in HJM can lead to substantially different rate paths — a powerful feature for pricing and risk analysis.

4.5 Drawbacks of HJM

While the HJM framework offers exceptional flexibility, that power comes with trade-offs — particularly when it comes to implementation:

- High Dimensionality: Because HJM models an entire curve of forward rates $f(t, T)$ for every maturity T, the model is inherently infinite-dimensional. In practice, this means numerical methods can become heavy and difficult to stabilise.
- Calibration Challenges: Fitting the model to both the yield curve *and* implied volatility surfaces requires sophisticated techniques, especially when the volatility function $\sigma(t, T)$ is rich or time-varying.
- Simulation Complexity: Forward rates are strongly correlated across maturities, and simulating their joint evolution can lead to instability unless approximations or dimensionality reductions (e.g., principal component analysis) are used.
- Limited Analytic Tractability: Unlike short-rate models, most pricing must be done via Monte Carlo methods, which increases computational cost — particularly for path-dependent instruments.

In short, HJM is powerful, but not lightweight — its realism makes it better suited to strategic pricing engines than to high-frequency trading applications.

5 Market Models: LIBOR, Swaptions and the Instruments That Trade

While short-rate and forward-rate models provide powerful tools for modelling interest rate dynamics, they are often disconnected from the actual instruments traded in financial markets. Traders quote caplets, swaptions, and forward rate agreements — not instantaneous forward curves or abstract bond price dynamics. Market models were developed to fill this gap by directly modelling the evolution of market-observed forward rates, ensuring consistency with both pricing conventions and the volatility surfaces seen in practice.

5.1 Forward Rates vs. Forward Prices: A Subtle but Critical Shift

The HJM framework models instantaneous forward rates — mathematical objects that are extremely useful for term structure modelling but not directly observable or traded. In contrast, financial markets operate on discrete instruments: forward rate agreements (FRAs), swaps, caplets, and swaptions, which are priced and quoted based on market-observed forward rates.

This brings us to a subtle but fundamental distinction:

HJM models forward rates continuously across all maturities; market models focus on the finite set of forward prices that are actually traded.

What Is a Market Forward Rate?

In practice, when traders refer to a "forward rate," they usually mean a forward LIBOR or EURIBOR rate over a future period $[Ti, Ti + 1]$, defined by:

$$L(t; T_i T_{i+1}) = \frac{1}{\delta}\left(\frac{P(t, T_i)}{P(t, T_{i+1})} - 1\right)$$

where $\delta = Ti + 1 - Ti$ is the year fraction, and $P(t, T)$ is the zero-coupon bond price.

This quantity reflects the actual forward lending rate over that period — and it’s what FRAs, interest rate swaps, and caplets are priced from.

Why the Shift Matters

The HJM model can describe the dynamics of the entire yield curve, but it does so in terms of abstract objects (instantaneous forwards) that aren't quoted in practice. Traders, on the other hand, need models that:

- Evolve discrete forward rates, not continuous ones
- Are consistent with market-quoted volatility surfaces
- Allow direct pricing of caplets, swaptions, and other real-world instruments

This leads to the development of market models — particularly the LIBOR Market Model (LMM) — which describe the stochastic evolution of forward LIBOR rates themselves, under their associated forward measures.

In short:

Market models bridge the gap between theoretical consistency and traded reality.

5.2 *LIBOR Market Model: Structure and Logic*

The LIBOR Market Model (LMM) — also known as the Brace–Gatarek–Musiela (BGM) model — is the cornerstone of modern interest rate derivatives pricing. It was specifically designed to model the dynamics of forward LIBOR rates in a way that is both arbitrage-free and market-consistent.

Unlike short-rate or HJM models, which model rates that are either unobservable or continuous across maturities, the LMM focuses on a discrete set of forward rates that match the maturities of real-world traded instruments such as:

- Forward Rate Agreements (FRAs)
- Interest rate caps and floors
- Swaptions

Core Idea

The LMM models each forward LIBOR rate $L_i(t)$— the rate fixed at time t for borrowing over the interval $[Ti, Ti + 1]$ — as a **lognormal process** under its own **forward measure** $\mathbb{Q}^{T_{i+1}}$:

$$dL_i(t) = \mu_i(t)dt + \sigma_i(t)L_i(t)dW_i^{\mathbb{Q}^{T_{i+1}}}(t)$$

Here:

- $\sigma_i(t)$ is the volatility of the forward rate, which can depend on time and maturity
- $\mu_i(t)$ is the drift, derived to ensure no-arbitrage across the full curve
- Each $L_i(t)$ has evolved under its own forward measure, which simplifies pricing of caplets and other derivatives linked to that specific forward

Key Properties

- No-Arbitrage Structure: Just like HJM, the LMM enforces arbitrage-free dynamics by linking volatilities to drifts — but it does so on discrete, tradable forward rates.
- Market Consistency: The model can be calibrated directly to the implied volatility surfaces of caplets and swaptions, making it highly practical.
- Lognormal Distribution: Under each forward measure, forward rates follow a lognormal process — enabling the use of Black's formula for derivative pricing.

Why It Matters

The LIBOR Market Model represents a paradigm shift: instead of deriving market prices from an abstract rate process, it starts with market observables and evolves them consistently. It respects market quoting conventions, handles volatility smiles (with extensions), and serves as the backbone of curve-consistent derivative pricing engines in banks and asset managers worldwide.

5.3 Caplets and Swaptions

One of the key strengths of the LIBOR Market Model (LMM) is that it directly models the forward rates used to price the instruments that actually trade — namely caplets and swaptions. This makes it naturally consistent with market quotes, something that traditional short-rate models often struggle to achieve.

Caplets: Options on Forward Rates

A caplet is an option that pays off when a specific forward LIBOR rate exceeds a strike. Under the LMM, since each forward rate $L_i(t)$ follows a lognormal process under its own forward measure $\mathbb{Q}^{T_{i+1}}$, the price of a caplet maturing at T_i can be calculated using Black's formula — the market standard for quoting cap volatilities.

This means:

The LMM reproduces caplet prices exactly if calibrated to the observed cap volatility surface.

Swaptions: Options on Swaps

A swaption is an option to enter into a fixed-for-floating interest rate swap. In market practice, swaptions are quoted in terms of Black volatilities on the swap rate — which is a function of several forward LIBOR rates.

In the LMM:

- The swap rate becomes a function of the forward rates $L_i(t)$.
- Though the swap rate is not lognormal, approximations (e.g. Rebonato's formula) or dedicated Monte Carlo methods allow the model to match swaption prices and volatilities observed in the market.

Why This Matters

In contrast to short-rate models:

- Caplet and swaption prices are inputs, not outputs.
- Calibration is direct: you fit the model to market prices, ensuring alignment with traded instruments.

- Risk management (e.g. delta, vega) is more reliable because sensitivities reflect market-quoted instruments.

This calibration-first approach makes the LMM the preferred model for:

- Pricing vanilla IR options
- Hedging trading books
- Building volatility cubes for structured products

5.4 Shifted and Displaced Diffusion Models

The original LIBOR Market Model (LMM) assumes that forward rates follow a lognormal distribution, which implies they remain strictly positive. This was once a safe assumption — until global interest rates approached, and even crossed, zero in the aftermath of the 2008 crisis and the subsequent monetary easing of the 2010s and 2020s.

To adapt, practitioners extended the LMM with simple but powerful adjustments that allow it to handle low or negative rates without abandoning its core structure.

1. Shifted Lognormal Models

In a shifted diffusion framework, each forward rate $L_i(t)$ is modelled not directly, but as:

$$d(L_i(t) + \phi = \mu_i(t)dt + \sigma_i(t)(L_i(t) + \phi)dW_t$$

where:

- $\phi > 0$ is a positive shift applied to each forward rate
- This allows $L_i(t)$ to go negative, while ensuring the shifted process $L_i(t) + \phi$ remains positive and lognormal

This adjustment preserves analytical tractability (e.g. modified Black's formula) while making the model compatible with negative-rate environments.

2. Displaced Diffusion Models

An alternative approach models the forward rate as a linear combination of a constant and a stochastic term:

$$dL_i(t) = \mu_i(t)dt + \sigma_i(t)(\alpha + (1 - \alpha)L_i(t))dW_t$$

where $\alpha \in [0,1]$ is the displacement coefficient:

- $\alpha = 0$ → standard lognormal model
- $\alpha = 1$ → normal (Bachelier) model
- $0 < \alpha < 1$ → interpolation between normal and lognormal

This gives traders more control over how the distribution of rates behaves near zero, and allows smoother calibration across positive and negative domains.

Why These Extensions Matter

- They preserve the structure and intuition of the standard LMM
- They enable pricing using adjusted versions of Black's formula
- They allow consistent calibration to market-implied volatilities, even when rates are negative or close to zero

In today's yield environment, where negative rates are no longer theoretical, these models have become the market standard for derivative desks needing both realism and analytical convenience.

6 Model Choice in Practice

Having explored the major interest rate modelling frameworks — short-rate, forward-rate, and market models — the next natural step is to ask: *Which model should you actually use?* In real-world applications, the answer depends on your objectives: pricing accuracy, calibration speed, hedging precision, or curve simulation. This section provides a practical lens through which to compare the models — highlighting the trade-offs and best-fit use cases that should guide your selection.

6.1 Short-Rate vs. Forward-Rate vs. Market Models

By this point, we've explored the three major families of interest rate models — each with its own philosophy, mechanics, and strengths. But in practice, choosing the right model isn't about elegance alone. It's about fit-for-purpose decision-making.

Here's a side-by-side comparison to clarify the **core differences**:

Feature	Short-Rate Models	Forward-Rate Models (HJM)	Market Models (LMM)
Core Object	Short rate r(t)	Instantaneous forward rate f(t,T)	Market forward rate L(t;Ti,Ti+1)
Curve Coverage	Implied from r(t)	Models entire yield curve directly	Discrete forward curve (observed tenors)
Arbitrage-Free	Yes (if correctly specified)	Enforced via drift condition	Enforced via measure changes
Market Fit (Yield Curve)	Poor to Moderate	Excellent	Excellent
Market Fit (Vol Surfaces)	Weak	Moderate (with volatility functions)	Excellent (direct calibration)
Numerical Complexity	Low to Moderate	High (infinite-dimensional)	Moderate to High (many forwards)
Analytical Tractability	Often closed-form	Rare	Partial (caplets via Black, others need MC)
Best Use Case	Theoretical insight, simple bonds	Term-structure simulation, ALM	Caplets, swaptions, trading desks

How to Think About It

- Use short-rate models when you need closed-form solutions, fast prototyping, or a didactic framework for fixed income pricing.
- Choose forward-rate models when modelling the dynamics of the yield curve itself is essential (e.g. for ALM or risk simulation).
- Apply market models when pricing or hedging real-world instruments that are quoted in the market — especially caps, floors, and swaptions.

Each model has its place. The real skill lies in matching the tool to the task — and recognising when the elegance of theory must yield to the rough edges of market practice.

6.2 Analytical vs. Numerical Trade-Offs

One of the most practical considerations when selecting an interest rate model is the balance between analytical tractability and numerical flexibility. Every model sits somewhere on this spectrum — and understanding where is crucial when implementing, calibrating, or using it in production environments.

Analytical Models: Fast, Transparent, but Rigid

Models like Vasicek and Hull–White offer closed-form solutions for zero-coupon bonds, options, and basic interest rate derivatives. Their advantages are clear:

- Speed: Pricing is instantaneous and easy to differentiate for risk calculations.
- Transparency: Behaviour is well understood, which aids communication and model validation.
- Ease of Calibration (to curves, not volatilities): Fitting yield curves is straightforward.

However, their main limitation is inflexibility. They cannot easily accommodate:

- Complex payoffs (e.g. callable exotics)
- Market volatility smiles or skews
- Arbitrary changes in the curve shape

Numerical Models: Flexible, Realistic, but Heavy

Models like CIR, HJM, and LMM typically require:

- Monte Carlo simulation for derivative pricing
- PDE solvers for early-exercise features
- Root-finding and interpolation for calibration to vol surfaces

These models offer the ability to:

- Match observed market prices more precisely
- Handle exotic and path-dependent products
- Simulate realistic curve dynamics across many scenarios

But they also come at a cost:

- Higher computational load
- Longer calibration time
- Greater complexity in implementation and validation

Choosing Your Trade-Off

In practice, the choice depends on the context:

- Trading desks prioritise speed and accuracy for market instruments — hybrid approaches (e.g. shifted LMM with semi-analytical pricing) often win.
- Risk engines and ALM teams value long-term realism — favouring multi-curve HJM or Monte Carlo-based LMMs.

- Academics and quants building intuition may start with Vasicek or Hull–White for clarity and pedagogical value.

In short:

Analytical models are efficient. Numerical models are expressive. Most real-world systems blend both.

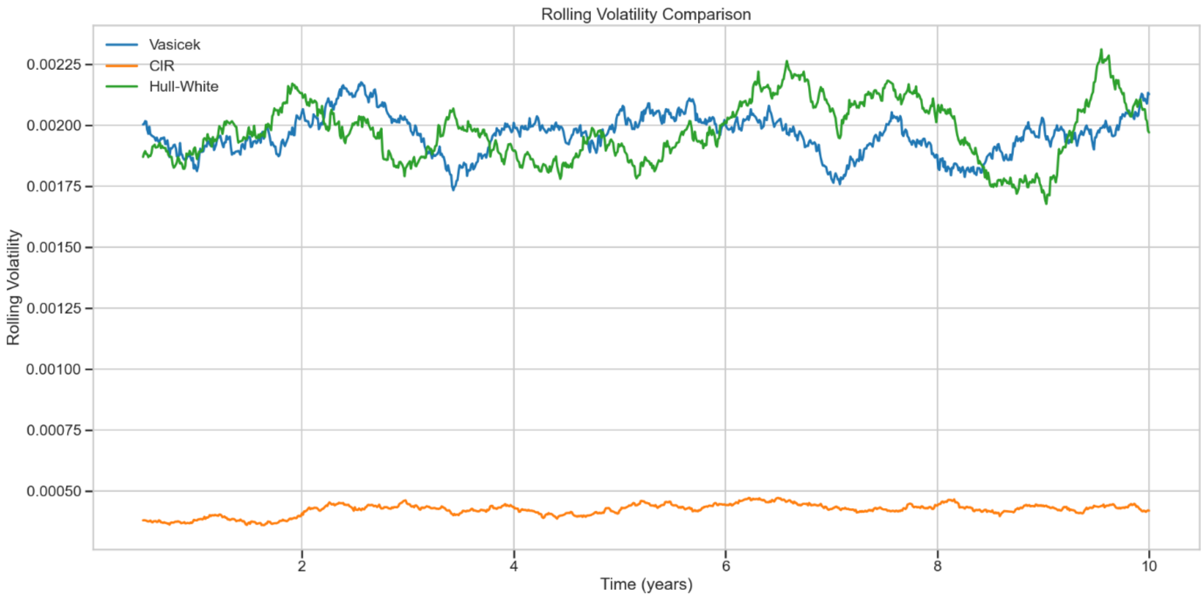

Figure 6: Vasicek, CIR, and hull-White rolling volatility comparison

When comparing rolling volatilities, we see that:

- The **CIR** model has consistently lower volatility, due to its stabilising effect near zero — volatility in CIR scales with the level of the rate.
- The **Vasicek and Hull–White** models exhibit more volatility, especially as the rate moves around the long-term mean, with Hull–White showing the most flexibility due to its evolving drift.

This difference has major implications for hedging and risk attribution: strategies built using Vasicek might underestimate tail risks when interest rates approach zero, while CIR may underreact to high-volatility regimes. Hull–White, on the other hand, provides a more adaptable structure — at the cost of greater calibration complexity.

6.3 Risk Management Implications of Model Selection

Model choice isn't just a pricing concern — it has direct consequences for risk measurement, hedging strategies, and portfolio sensitivity analysis. Whether you're managing a trading book, an asset-liability portfolio, or a derivatives exposure, the model you choose shapes how you see and respond to risk.

Short-Rate Models: Simplicity, but Coarse Sensitivities

Short-rate models like Vasicek and Hull–White provide:

- Clean delta and duration estimates
- Fast scenario analysis (e.g. parallel shifts, twists)
- Useful intuition for rate-level sensitivity

But they lack granularity. Since they only model one factor (the short rate), they fail to capture risk across different curve segments. A hedge calibrated using a one-factor model may miss exposure to curve shape changes — like steepening or butterfly shifts — that matter in real portfolios.

Forward-Rate and Market Models: Granular but Complex

In contrast, HJM and LMM frameworks:

- Decompose curve risk across maturities
- Enable bucketed sensitivity (key rate durations, PV01s per tenor)
- Capture volatility risk (vega) aligned with market surfaces

This makes them much better suited for:

- Dynamic hedging with instruments of various maturities
- Stress testing under non-parallel curve moves
- VaR and CVaR simulation where curve behaviour matters

However, these benefits come at the cost of:

- Higher computational burden
- More complex hedge construction
- Greater sensitivity to model assumptions and calibration stability

Bottom Line

- Use short-rate models for broad rate-level risk
- Use forward-rate or market models for curve-sensitive, option-sensitive, or volatility-sensitive positions
- Ensure that your risk view matches your pricing model — or inconsistencies will emerge in hedges, P&L attribution, and regulatory reporting

In risk management, the right model is the one that captures *your portfolio's true exposures* — not just the one that's easiest to compute.

6.4 Where Each Model Fits

No single interest rate model is best for all tasks. Each has its own domain of relevance, depending on whether you're trying to price a vanilla product, hedge complex exposures, build structured instruments, or simulate future scenarios. Here's how they map out:

Pricing

- *Short-rate models:* Suitable for standard fixed-income products and theoretical pricing. Quick, clean, but limited for derivatives.
- *Market models (LMM):* Industry standard for pricing caps, floors, and swaptions, thanks to alignment with market volatility quotes.
- *Forward-rate models (HJM):* Less common in direct pricing due to numerical complexity, but valuable for structured products needing full curve dynamics.

Hedging

- *Short-rate models:* Useful for broad directional hedges (duration-based).
- *Market models:* Superior for constructing delta-vega hedges that align with actual traded instruments.
- *Forward-rate models:* Ideal when hedging across multiple curve segments or under regulatory stress-testing environments.

Structuring

- *Market models:* Best for bespoke instruments tied to market-quoted forward rates (e.g. callable structured notes, CMS spreads).
- *HJM:* Strong when structuring long-dated or curve path-dependent instruments that require full-term structure modelling.

Forecasting and Scenario Analysis

- *Short-rate models:* Easy to simulate but overly simplistic. Not well suited for realistic stress scenarios or forecasting curve shape.
- *HJM:* Best choice for yield curve simulation in ALM, macro stress testing, or Monte Carlo engines.
- *Market models:* Used less for long-term forecasting; more common in forward-looking pricing with volatility views.

7 Post-LIBOR and Modern Extensions

The evolution of interest rate modelling has been shaped not only by theoretical progress, but by structural changes in the financial system. The global transition away from LIBOR, the rise of overnight risk-free rates like SOFR and €STR, and the adoption of multi-curve frameworks have redefined how interest rates are quoted, traded, and modelled. This section explores how modern models have adapted to this new reality — one where legacy assumptions no longer hold, and flexibility across curves and instruments is essential.

7.1 The Rise of OIS and the Need for Multi-Curve Frameworks

For decades, most interest rate models operated under a single-curve paradigm: the same curve was used for both projecting forward rates and discounting cash flows. That world changed dramatically after the 2008 financial crisis.

As counterparty risk and liquidity premiums became visible in interbank lending markets, it became clear that forward rates (e.g. LIBOR, EURIBOR) and discounting curves (e.g. OIS rates) could no longer be treated as equivalent. This gave rise to a new standard: the multi-curve framework.

What Is OIS?

An Overnight Indexed Swap (OIS) is a swap where the floating leg is based on compounded overnight rates (e.g. Fed Funds, EONIA, €STR). OIS rates:

- Reflect minimal credit risk
- Are highly liquid and closely aligned with risk-free rates

After 2008, OIS curves became the market standard for discounting collateralised cash flows, especially under CSAs (Credit Support Annexes).

Why Multi-Curve Modelling Became Essential

In the multi-curve world:

- Forward curves (e.g. 3M, 6M, 12M LIBOR) are used to project future cash flows
- Discount curves (typically OIS) are used to compute present values

This distinction means:

- The pricing of even vanilla instruments requires at least two curves
- Models must now handle multiple forward curves and one or more discount curves
- Hedging and risk management must account for basis risk — the spread between different curves

Impact on Modelling

Traditional models like Vasicek or even LMM were not designed to accommodate this complexity. In response, new approaches were developed:

- Multi-curve extensions of the LMM, with separate stochastic processes per tenor
- Curve-specific volatility and basis dynamics, often calibrated jointly

In short:

The rise of OIS discounting and multi-curve pricing was not a theoretical evolution — it was a market-driven necessity, and one that reshaped how interest rate models are built, calibrated, and applied.

7.2 Modelling Overnight Rates (SOFR, €STR, SONIA)

The shift away from LIBOR has brought overnight risk-free rates (RFRs) to the forefront of interest rate markets. Benchmarks like SOFR (USD), €STR (EUR), and SONIA (GBP) have become the new standards for derivatives and secured funding, replacing term-based interbank rates with compounded overnight rates.

This transformation poses a fresh modelling challenge:

How do we represent the behaviour of rates that are overnight, compounded, and backward-looking, rather than fixed in advance like LIBOR?

Core Features of Overnight Rates

- **Nearly risk-free: Reflect central bank policy and secured lending**
- **Daily compounding: Used in OIS contracts and fallback replacement structures**
- **Lack of term structure: Rates are not quoted for future tenors; they are built from historical compounding**

Modelling Approaches

1. **Simple Forward Modelling**

For short-term forecasting and pricing of OIS, models treat the overnight rate as analogous to the short rate r(t)r(t)r(t), often using a modified Hull–White or CIR specification.

2. **Term Structure Construction via Compounding**

The term structure is not directly observable. It is built by compounding daily overnight rates, requiring integration over future paths — often using Monte Carlo simulation.

3. **Hybrid Multi-Curve Models**

In the post-LIBOR world, some institutions model:

- The overnight rate as a base curve (e.g. for discounting and collateralised trades)

- Add basis spreads to approximate forward-looking tenors for derivatives referencing term structures (e.g. legacy LIBOR deals)

Key Implications

- The stochastic nature of compounding introduces path dependency, complicating pricing of instruments like compounded coupons with observation lags.
- Traditional lognormal models are often unsuitable — instead, modellers may adopt normal or shifted normal models to reflect the lower and sometimes negative rate regime.
- Since RFRs are central to discounting, their modelling affects almost every valuation, even for instruments that reference other curves.

7.3 Term Structure Volatility, Stochastic Spreads, and Hybrid Models

As interest rate modelling adapts to a multi-curve and post-LIBOR world, one challenge becomes increasingly important: how to capture the volatility and dynamics across multiple curves, not just a single risk-free rate. This requires more than just extending old models — it demands hybrid frameworks that can represent basis spreads, curve-dependent volatilities, and their interactions.

Term Structure Volatility: Beyond Level Shifts

In single-curve models, volatility is often tied to the level of the short rate or forward rate. But real-world movements involve:

- Twists: front-end vs. back-end curve changes
- Butterflies: shifts in the middle of the curve
- Non-parallel shifts: common in stress and central bank scenarios

To simulate these, modern models must include:

- Time- and maturity-dependent volatilities
- Multiple stochastic factors, each affecting different segments of the curve

This leads to multi-factor HJM or multi-factor LMM frameworks calibrated to the principal components of historical curve changes.

Stochastic Basis Spreads

In a multi-curve environment, the spread between two tenors (e.g. 3M vs. 6M) or between a term rate and OIS is itself a stochastic process. Ignoring this dynamic can result in:

- Mispricing of basis swaps
- Poor hedge performance for cross-curve exposures

To address this, modern models treat the basis as a mean-reverting stochastic spread, modelled either:

- Directly as a diffusion process, or
- Implicitly, by specifying separate stochastic processes for each curve

Hybrid Models: Flexibility Meets Complexity

Hybrid models combine elements of different modelling approaches:

- A short-rate or HJM process for the overnight rate (e.g. SOFR or €STR)
- Additional processes for LIBOR-like forward curves
- Stochastic spreads or correlated noise terms to link curves consistently

These models are increasingly used in:

- Structured product pricing (e.g. CMS spread notes)
- XVA and risk engines
- Cross-currency and curve-arbitrage strategies

8 Conclusion

Interest rate modelling is both foundational and perpetually evolving. From the elegance of short-rate models to the flexibility of forward-rate frameworks and the market precision of LMMs, each generation of models has responded to the needs of its time — whether driven by mathematical rigour, product innovation, or structural market change.

In today's world, modelling interest rates requires more than a single curve or process. It demands tools that can handle:

- Multi-curve dynamics and stochastic spreads
- Realistic volatility structures across tenors
- Regulatory consistency and alignment with market conventions
- Negative rates, basis risk, and overnight benchmarks like SOFR or €STR

There is no one-size-fits-all approach. The "right" model depends on your objective: analytical speed, market calibration, risk simulation, or pricing exotics. Most institutions, in practice, blend models — using Hull–White for fast bond pricing, LMM for caps and swaptions, and hybrid frameworks for structured or collateralised instruments.

What matters most is not just choosing the right model, but understanding what it can and cannot do, and aligning it with the reality of traded markets.

As we move into a post-LIBOR landscape defined by overnight rates, regulatory stress testing, and volatility-aware products, interest rate modelling remains a living discipline — still deeply mathematical, but now more than ever, shaped by the markets it seeks to represent.

9 Python code

```python
# Sample paths for the Vasicek model
def vasicek_model(r0, kappa, theta, sigma, T, N, paths):
    dt = T / N
    r = np.zeros((paths, N+1))
    r[:, 0] = r0
    for i in range(paths):
        dW = np.random.normal(0, np.sqrt(dt), N)
        for t in range(N):
            dr = kappa * (theta - r[i, t]) * dt + sigma * dW[t]
            r[i, t+1] = r[i, t] + dr
    return r

# Sample paths for the CIR model
def cir_model(r0, kappa, theta, sigma, T, N, paths):
    dt = T / N
    r = np.zeros((paths, N+1))
    r[:, 0] = r0
    for i in range(paths):
        dW = np.random.normal(0, np.sqrt(dt), N)
        for t in range(N):
            dr = kappa * (theta - r[i, t]) * dt + sigma * np.sqrt(max(r[i, t], 0)) * dW[t]
            r[i, t+1] = max(r[i, t] + dr, 0)  # Ensure rates don't go negative
    return r

# Function for Hull-White with time-dependent drift
def hull_white_model(r0, kappa, sigma, T, N, paths):
    dt = T / N
    r = np.zeros((paths, N+1))
    r[:, 0] = r0

    def theta_t(t):
        if t < T/2:
            return 0.04  # 4% initially
        else:
            return 0.06  # 6% later

    theta_values = np.array([theta_t(t) for t in time_grid])
    for i in range(paths):
        dW = np.random.normal(0, np.sqrt(dt), N)
        for t in range(N):
            theta_current = theta_values[t]
            dr = kappa * (theta_current - r[i, t]) * dt + sigma * dW[t]
            r[i, t+1] = r[i, t] + dr
    return r, theta_values
```

Figure 7: Short rate model comparison Python code

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