Financial Market Uncovered – Article 4

Decoding Volatility: How Market Uncertainty Shapes Asset Prices and Derivatives

Kilian Voillaume

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Summary

1 Introduction: Understanding volatility

Volatility is one of the most significant concepts in financial markets. It plays a central role in risk estimation, options pricing, and trading strategies. Volatility quantifies the extent to which the price has moved over time, indicating both market risk and investor sentiment.

Volatility is an estimate of statistical dispersion. Volatile assets exhibit large price movements, and low-volatility assets exhibit more stable behaviour. Volatility thus becomes a fundamental component of options pricing models, with a direct effect on option premiums.

In this article, we will explore volatility from multiple perspectives. We will begin with *historical volatility (HV)*, *realized volatility*, and *implied volatility (IV)*, which represents market expectations of future fluctuations. We will also talk about the *volatility surface*, *volatility skew*, and *term structure*, which show how volatility behaves across different strike prices and maturities. We will then examine *volatility surface (3D)* and volatility modelling approaches, from Black-Scholes to stochastic models like Heston and SABR, and their practical implications for trading strategies.

1.1 Volatility: A measure of market uncertainty

Volatility is commonly defined as the standard deviation of asset returns over a given period. Mathematically, it is expressed as:

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2}$$

where r_i represents individual returns, $\bar{r}$ is the mean return, and N is the number of observations. This formula provides a backward-looking measure of past price dispersion. Volatility also reflects the psychology of the market, capturing fear, uncertainty, and investor sentiment.

Volatility is often mean-reverting, meaning that it fluctuates over time but tends to return to a long-term average. During crises, volatility spikes dramatically, reflecting heightened uncertainty. On the other hand, volatility stays low during times of economic stability, which results in smaller option premiums and fewer opportunities for trading volatility-based methods.

1.2 Historical vs Implied volatility

One of the fundamental distinctions in volatility analysis is between *historical volatility (HV)* and *implied volatility (IV)*.

- *Historical Volatility (HV)* measures past price fluctuations. It is computed using realized returns over a set period (e.g., 30-day rolling volatility). It is mostly used for assessing past market turbulence. However, it does not predict future movements.
- *Implied Volatility (IV)* is forward-looking. It is computed from option prices and reflects the market's collective expectation of future price fluctuations. Unlike *HV*, which is

based on actual price movements, *IV* is driven by supply and demand in the options market.

One important lesson is that because traders expect big price movements, IV frequently surges before important events like earnings releases, economic statements, or geopolitical uncertainty. In equities like Tesla (TSLA), this behaviour is especially noticeable. Prior to earnings announcements, IV often rises, but then falls subsequently in a phenomenon known as a volatility crush.

1.3 Why volatility is a key factor in options pricing

Volatility, along with the Black-Scholes model, is a key input for models pricing options. It has a direct impact on the value of both *call options* and *put options*. If everything else is equal:

- Higher volatility increases the probability of large price movements, which leads to higher option premiums.
- Lower volatility reduces option prices, as the likelihood of significant price swings decreases.

Traders keep an eye on implied volatility rankings to determine whether options are overpriced or underpriced relative to historical norms. Market makers use volatility models to hedge their positions dynamically.

1.4 Role of volatility in market behaviour and trading strategies

Volatility not only affects option pricing but also influences market structure, liquidity, and trading strategies.

- Institutional investors: Hedge risk and optimize portfolio allocations based on volatility forecasts.
- Market makers: Adjust bid-ask spreads based on implied volatility fluctuations.
- Retail traders: Implement strategies such as straddles, strangles, and iron condors to capitalize on volatility shifts.

The Volatility Index (VIX) is one of the most widely watched volatility indicators. It is also referred to as the “Fear Index”. It measures the market’s expectation of volatility over a certain window of time for S&P 500 options and serves as a real-time barometer of market sentiment.

A rising VIX typically signals investor uncertainty, while a declining VIX reflects market confidence.

2 Historical volatility

Historical volatility (HV) measures how much an asset price has moved in the past, serving as a retrospective indicator of risk. It is a measure of market variability. Despite its simplicity, *HV* remains a cornerstone of risk management, portfolio construction, and preliminary options pricing models.

2.1 Definition and motivation

HV is computed as the standard deviation of log returns over a given time horizon. Let P_t be the closing price of an asset on day t. The log return at time t is defined as:

$$x_t = \ln\left(\frac{P_t}{P_{t-1}}\right)$$

Given a series of n such returns, the daily historical volatility is computed as:

$$HV_{daily} = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(x_t - \bar{x})^2} \; and \; \bar{x} = \sum_{t=1}^{n} x_t$$

Where $\bar{x}$ is the average log return over the period. This standard deviation gives a sense of how spread out the returns are from their mean—i.e., how volatile the asset has been.

To make this measure comparable across assets and relevant for annualized trading decisions, we scale it by the square root of time:

$$HV_{annual} = HV_{daily} * \sqrt{252}$$

Here, 252 represents the typical number of trading days in a year.

This simple formula provides a robust statistical proxy for past risk. For example, a stock with a historical volatility of 15% is considered less volatile than one with 60%, indicating lower average price fluctuations over the observed period.

2.2 Interpretation and use in practice

Historical volatility provides information about how volatile or stable an asset has been over a time period. *HV* is used by portfolio managers to assess the effectiveness of hedging strategies, quantify downside risk, and calibrate risk parity allocations. However, *HV* has a key limitation: it only reflects the past.

A period of low historical volatility does not guarantee future stability, especially in financial markets where shocks and regime changes are frequent occurrences. Because of this, traders and quants use forward-looking metrics like implied volatility or model-based volatility predictions to supplement *HV*.

Example:

Consider two assets over the last 30 trading days:

- Asset A: HV ≈ 12%.
- Asset B: HV ≈ 70%.

Even if both have similar prices and returns, the risk-adjusted performance and capital requirements to trade them would differ drastically. High *HV* implies larger potential price swings and, consequently, higher margin and risk exposure.

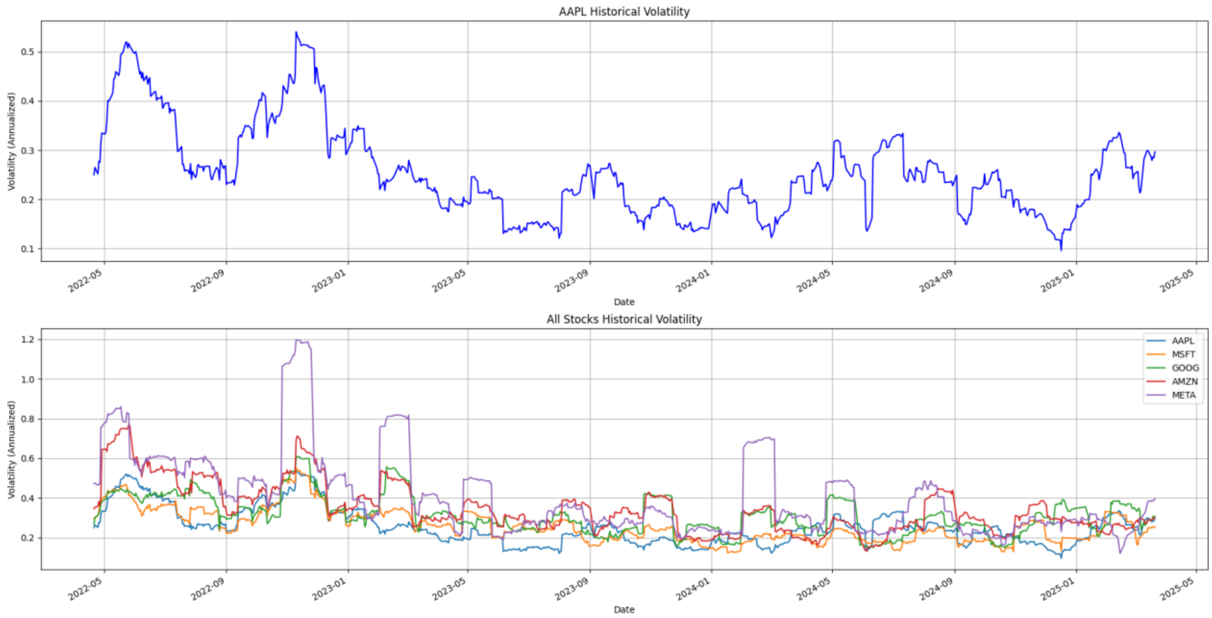


Figure 1: Historical volatility of AAPL compared to other US companies over the past 3 years

2.3 Limitations of historical volatility

While *historical volatility* is easy to compute and understand, it carries several caveats:

- Backward-looking: *HV* assumes that past price behaviour is indicative of future uncertainty, which may not hold true in the presence of regime shifts or macroeconomic shocks.
- Sensitive to window size: The choice of rolling period (e.g., 10, 30, or 60 days) can significantly alter the result.
- Ignores intraday dynamics: Daily closing prices may miss important high-frequency movements.
- Does not capture asymmetries: *HV* treats upward and downward moves symmetrically, while markets often respond asymmetrically to downside risk.

These limitations motivated the development of more sophisticated metrics like *implied volatility*, which captures the expectations of the market, and realized volatility, which takes intraday data into account.

3 Realized volatility

Historical volatility provides a basic retrospective assessment of price variations using daily closing prices. Realized volatility improves this concept by incorporating higher-frequency data, such as intraday returns. It offers a more accurate and detailed estimation of the true volatility encountered over a given time period. Realized volatility is particularly important in contemporary quantitative finance for model calibration, volatility forecasting, and pricing volatility derivatives like volatility swaps and variance swaps.

3.1 Definition and motivation

The ex-post volatility of an asset is referred to as realized volatility. It is determined by intra-period returns, usually across time intervals of one, five, or thirty minutes. In contrast, historical volatility, which may overlook the complex dynamics that takes place during a trading day because it only analyses daily data.

For a given trading day divided into M intervals (e.g., 5-minute bins), and over a window of n days, realized volatility is computed as:

$$RV_t = \sqrt{\sum_{i=1}^{M*n} r_i^2}$$

where r_i are the intra-period log returns. The result is typically annualized using a scaling factor that reflects the frequency of the observations (e.g., $\sqrt{252}$ for daily aggregation).

This measure is widely used in volatility modelling frameworks such as GARCH-type extensions and HAR-RV (Heterogeneous Autoregressive model of Realized Volatility), and it is now considered standard in high-frequency econometrics.

3.2 Realized vs historical volatility

Though conceptually similar, realized volatility has several advantages over its historical counterpart:

- Higher precision: By including intraday data, RV captures volatility clusters and sudden spikes more accurately.
- Lower estimation bias: Daily returns often underestimate actual price movement due to overnight gaps or intraday reversals.
- Better for short horizons: In options markets, especially for short-dated contracts, realized volatility is a more accurate input for hedging and risk metrics.
- Essential for variance swap pricing: Because these products settle based on actual variance observed over the life of the contract, realized volatility is the benchmark used for valuation and settlement.

Example:

Suppose we compute realized volatility over 10 trading days using 5-minute returns of the S&P 500.There are 78 five-minute intervals per day, so we will use $10 * 78 = 780$ squared returns. This produces a much richer estimate than using only 10 daily closing prices.

3.3 Applications in financial market

In many aspects of quantitative finance, realized volatility is crucial.

- Variance and volatility swaps: These instruments' payouts are correlated with actual variance or volatility over a predetermined time period.
- Model calibration: Realized volatility is frequently used as an input in risk models like GARCH or HAR or to calibrate stochastic volatility models (like Heston).
- Forecasting: When properly aggregated, realized volatility can be used as a predictor of future volatility as well as a target variable (e.g., weekly RV to forecast monthly RV).

3.4 Limitations of realized volatility

Despite its advantages, realized volatility is not without caveats:

- Market microstructure noise: High-frequency returns are affected by bid-ask bounce, price discreteness, and latency effects, which can bias volatility estimates.

- Data requirements: Calculating RV accurately requires clean, high-quality intraday data, often unavailable or expensive for some assets.

- Over-sensitivity: RV may react excessively to temporary shocks (e.g., news events) that do not reflect longer-term volatility regimes.

To mitigate some of these issues, techniques such as subsampling, kernel-based estimators, and noise-robust RV measures have been proposed in the econometrics literature.

4 Implied volatility

Implied volatility (IV) represents the market's estimate of future volatility, whereas historical and realized volatility gauge the asset's past behaviour. By reversing an option pricing model like Black-Scholes, it is inferred from option prices rather than directly observed from asset prices. Therefore, *implied volatility* is a predictive metric that incorporates supply-demand dynamics, market emotion, and expectations.

4.1 Definition and motivation

In theory, the price of a *vanilla* (standard) European option can be computed using a model like Black-Scholes, assuming we know all the inputs: current asset price S, strike price K, time to maturity T, risk-free interest rate r, and volatility σ. In practice, we observe the option's market price, and all parameters except volatility are known. The idea is to invert the pricing formula to solve for the value of σ that makes the model price equal to the observed market price.

This value of σ is the implied volatility, denoted σ_{impl}, and it represents the market's consensus view of the expected volatility of the underlying asset over the option's lifetime.

Because option prices are impacted by supply and demand, *IV* incorporates more than just statistical forecasts—it captures risk aversion, uncertainty, and event risk.

4.2 Calculation via the Black-Scholes Model

Let C_{market} be the observed market price of a European call option. Using the Black-Scholes formula, the model price is given by:

$$C_{model}(\sigma) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

with

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The implied volatility is then defined as the solution σ^* to the following equation:

$$C_{model}(\sigma^*) = C_{market}$$

The equation has to be solved with numerical methods.

4.3 Numerical methods for implied volatility

The most widely used approach for solving the implied volatility equation is the Newton-Raphson method, an iterative root-finding algorithm. Starting from an initial guess σ_0 , the algorithm updates volatility as follows:

$$\sigma_{n+1} = \sigma_n - \frac{f(\sigma_n)}{f'(\sigma_n)}$$

where:

$$f(\sigma) = C_{model}(\sigma) - C_{market}$$

$$f'(\sigma) = Vega = \frac{\partial C}{\partial \sigma} = S\Phi'(d_1)\sqrt{T-t}$$

with $\Phi'(d_1)$ the standard normal probability density function. *Vega* captures the sensitivity of the option price to changes in volatility, and it is essential to the convergence of Newton-Raphson.

If *Vega* is too small (e.g., deep *ITM* or *OTM options*), the Newton-Raphson method may become unstable. In those cases, alternative methods such as the bisection method or Brent's method are preferred for their robustness.

4.4 Why implied volatility matters

Implied volatility plays a central role in both pricing and trading:

- Option pricing: *IV* is the volatility input that reconciles theoretical price with the observed market price.
- Market expectations: It reflects anticipated risk—higher *IV* implies more uncertainty or expected price movement.
- Event anticipation: *IV* often spikes before major events (earnings, Fed decisions), pricing in potential moves.
- Volatility trading: Traders express directional views on volatility via long straddles, strangles, or *VIX* derivatives.
- Relative value: Comparing implied volatility to historical/realized volatility helps assess whether options are "cheap" or "expensive."

Example:

Suppose the market price of an at-the-money 30-day call on a stock is \$5. Using Black-Scholes, you find that for $\sigma = 25\%$, the model price is \$4.75. For $\sigma = 30\%$ the model price is \$5.25. Through iteration, you find that $\sigma_{impl} = 28.2\%$ makes the model price match the market price exactly. This is the implied volatility of that option.

4.5 The role of vega and time sensitivity

Vega is highest *at-the-money* because small changes in volatility most affect option prices when the probability of finishing *in-the-money* is balanced. It also declines as time to maturity decreases.

This behaviour explains why short-dated *OTM* options are less sensitive to volatility changes, and why *IV* estimates in those regions are more erratic. Understanding *Vega* is crucial not only for *IV* estimation, but also for risk management and option Greeks-based hedging.

4.6 *Python output for IV estimation*

```
Theoretical market price: 43.028

Iteration 1: Sigma = 1.80000000, Loss = 49.73102715
Iteration 2: Sigma = -0.55160635, Loss = -53.26443727
Iteration 3: Sigma = 0.89968050, Loss = 23.20864766
Iteration 4: Sigma = 0.26701000, Loss = 1.36231496
Iteration 5: Sigma = 0.19934404, Loss = 0.32204982
Iteration 6: Sigma = 0.16791871, Loss = 0.07650662
Iteration 7: Sigma = 0.15391739, Loss = 0.01325475
Iteration 8: Sigma = 0.15025271, Loss = 0.00080091
Iteration 9: Sigma = 0.15000115, Loss = 0.00000363
Iteration 10: Sigma = 0.15000000, Loss = 0.00000000

Estimated Implied Volatility: 0.15000
```

Figure 2: Python output to a code estimating implied volatility

The code in the appendix estimates the implied volatility of a European call option using the Newton-Raphson's method. Given a market price, it iteratively adjusts the volatility input in the Black-Scholes formula until the model price matches the observed price. The gradient of the pricing error with respect to volatility is computed using automatic differentiation via the *autograd* library, making the method both efficient and precise. This mirrors how implied volatility is typically derived in real financial markets.

5 Volatility smile

The volatility smile is one of the most well-known empirical deviations from the Black-Scholes model. Plotting *implied volatility (IV)* versus strike prices for options with the same maturity reveals a U-shaped pattern: *out-of-the-money (OTM)* and *in-the-money (ITM)* options tend to have higher IV than *at-the-money (ATM)* options.

This contradicts the Black-Scholes assumption of constant volatility across strikes and underlines the limits of the model in reflecting the underlying dynamics of option markets. The smile represents how market players view and value risk —especially in the tails of the return distribution —and is not merely a pricing anomaly.

5.1 Empirical observation of the smile

In an idealized Black-Scholes world, *IV* would be constant for all strike prices and depend only on time to maturity. However, in practice:

- *OTM put options* tend to have significantly higher *implied volatility* than *ATM* options.
- *OTM call options* may also exhibit elevated *IV*, though less consistently, depending on the asset class.
- The result is a U-shaped curve—the *volatility smile*.

This shape has been especially pronounced since the 1987 market crash, where the market's demand for downside protection caused *OTM put option* prices—and thus their *implied volatilities*—to rise sharply. That event permanently embedded a crash risk premium into the pricing of equity options, and volatility smiles have persisted ever since.

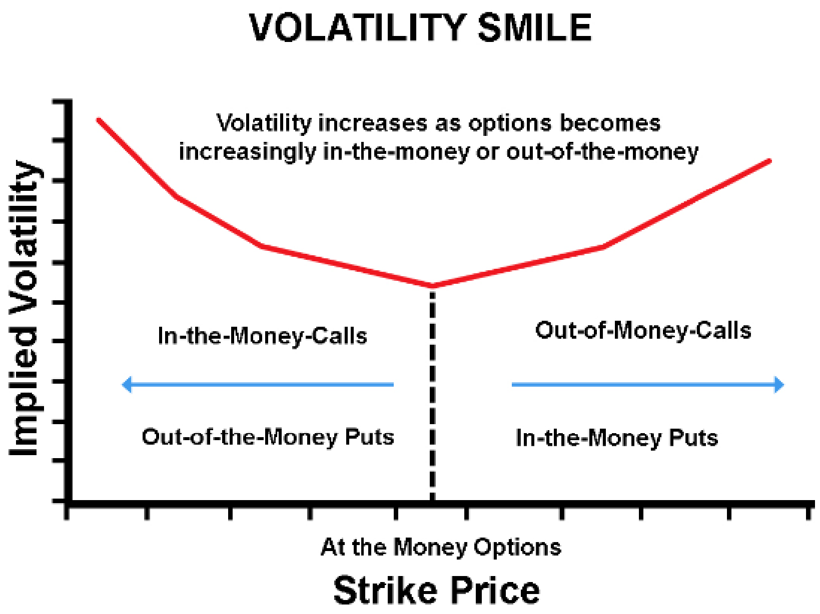


Figure 3: Volatility smile explanation for both call and put options

5.2 *Interpreting the smile*

The volatility smile is not just a mathematical curiosity. It provides key insights into:

- Tail risk pricing: Higher *IV* for *OTM* options reflects a greater perceived probability of extreme events, especially on the downside.
- Skewness in return distributions: The smile implies that asset returns are not normally distributed, but instead exhibit fat tails and asymmetry.
- Market sentiment: The steepness or flatness of the smile reflects investor fear, hedging demand, and speculative positioning.

In equity markets, the left side of the smile (*OTM put options*) is usually more elevated than the right (*OTM call options*), leading to a *volatility skew*. In FX and commodities, more symmetrical smiles are common, reflecting more balanced tail risk.

5.3 *Modelling the smile: limitations of Black-Scholes*

The classical Black-Scholes model, by assuming constant volatility, cannot generate a volatility smile. It relies on several assumptions that break down in real markets:

- Constant volatility across time and strike.
- Lognormal asset return distribution.
- No jumps or stochasticity in volatility.

To capture smiles, alternative models have been developed. One class of such models includes stochastic volatility models, where volatility is itself a random process. Among them, the *SABR model* is particularly suited to generating smiles in interest rate and FX derivatives.

5.4 *The SABR model*

The *SABR model (Stochastic Alpha, Beta, Rho)*, introduced by Hagan et al. (2002), was designed specifically to produce implied volatility smiles that align with observed market data. *Alpha* represents the volatility level, *Beta* is the forward elasticity, and *Rho* is the correlation between price and volatility.

Let f_t be the forward price (e.g., forward rate, swap rate). The SABR model assumes:

$$df_t = \sigma_t f_t^{\beta} dW_t^1 \qquad (1)$$

$$d\sigma_t = \nu \sigma_t dW_t^2 \qquad (2)$$

$$dW_t^1 dW_t^2 = \rho dt \qquad (3)$$

(1) is the forward rate dynamics. It says that the change in the forward price f_t depends on its current level (via f_t^{β}) and on a stochastic volatility σ_t, plus random shocks from dW_t^1.

(2) describes a geometric Brownian motion for volatility, σ_t can vary over time and itself has a volatility. So, volatility is not constant but follows its own stochastic path, enabling the model to generate volatility smiles and skew.

(3) describes the correlation between shocks. It introduces asymmetry in the distribution of returns; If the forward drops, volatility tends to rise.

Where:

- σ_t is the *stochastic volatility* of the forward.
- ν is the *volatility of volatility*.
- dW_t^1 is a *Brownian motion*, modelling randomness in the forward price.
- dW_t^2 is another *Brownian motion for volatility*, driving the random shocks in volatility.
- β determines the *forward dynamics*:
  - $\beta = 0$: normal model with additive moves, constant absolute volatility.
  - $\beta = 1$: lognormal model with multiplicative moves, constant relative volatility (like Black-Scholes).
  - $\beta = 0.5$ which represents a square-root dynamics.
- ρ is the *correlation* between the asset and its volatility:
  - $\rho < 0$: typical in equity/interest rate markets, leads to negative skew (higher *IV* for OTM puts).
  - $\rho > 0$: might occur in commodities (positive skew).

The model does not directly price options. Instead, it produces implied volatilities as a function of strike, which are then used in Black's model to price European options.

5.5 SABR implied volatility formula (Hagan et al.)

The SABR approximation for implied volatility σ_{BS} as a function of strike K and forward price f is:

$$\sigma_{BS}(K) = \frac{\alpha}{(fk)^{\frac{1-\beta}{2}}} * \frac{z}{x(z)} * \left[1 + T * \left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\nu\beta\alpha}{(fK)^{\frac{1-\beta}{2}}} + \frac{2-3\rho^2}{24}\nu^2\right)\right]$$

where:

$$z = \frac{\nu}{\alpha}(fK)^{\frac{1-\beta}{2}} \ln\left(\frac{f}{k}\right)$$

$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right)$$

$\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}$ accounts for how steeply the smile changes with strike

$\frac{1}{4}\frac{\rho\nu\beta\alpha}{(fK)^{\frac{1-\beta}{2}}}$ introduces skew via correlation

$\frac{2-3\rho^2}{24}\nu^2$ captures convexity due to vol-of-vol

5.6 What makes SABR powerful and how it is used in practice

The *SABR model* is widely used in trading as:

It captures market smiles & skews

- Equity & rates markets tend to have left skew: *OTM put options* are expensive.
- Commodities may show right skew.
- FX options often show symmetric smiles.
- *SABR* is flexible enough to model these different shapes just by adjusting β, ρ, ν

Traders calibrate *SABR* to market quotes of *ATM* volatility and volatility at different deltas/strikes, to build a consistent implied volatility curve.

It is used in quoting, risk, and hedging

- Volatility traders use *SABR* curves to interpolate or extrapolate volatilities.
- Risk managers use it to compute greeks consistently across the smile.
- Exotics desks use it as a benchmark to price more complex options.

Traders use the *SABR model* using observed *IV* at various strike as an input parameter for the model. They then optimize α, β, ν, ρ to best match the market volatilities. They build a volatility smile/surface using the fitted formula to generate *IV* for any strike and maturity. Finally, they use the *SABR IV* as an input for the Black-Scholes model to price plain vanilla options.

6 Volatility skew

Volatility skew refers to the asymmetry in *implied volatility* across different strike prices. Market data continuously defies the Black-Scholes model's assumption that volatility is constant independent of strike or moneyness. In reality, *IV* frequently varies asymmetrically among strikes, especially in commodity and equities markets. This *skew* is not a technical anomaly; it reflects underlying structural aspects of market expectations, including pricing of tail events, hedging flows, and unequal risk perception.

In equity options markets, *IV* typically declines as strike increases, meaning that *out-of-the-money (OTM) put options* have higher *IV* than *OTM call options*. This leads to what is called a *left skew* or *smirk*, a downward-sloping *implied volatility* curve. This pattern suggests that investors are more concerned about downside risk than upside potential—a phenomenon aligned with crash aversion and the demand for portfolio protection.

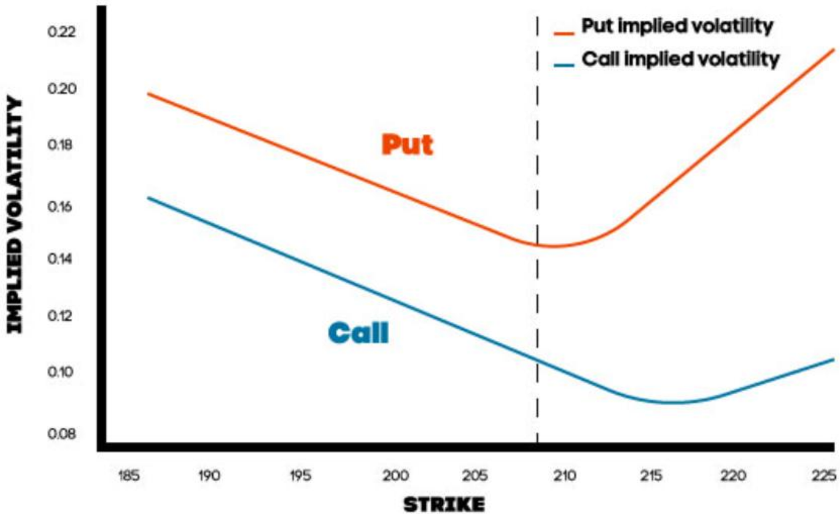


Figure 4: Volatility skew for both call and put options

6.1 Quantifying the skew

There are several ways to formally define and compute skew, depending on the precision required and the market convention. At the most basic level, skew can be approximated using the implied volatilities of two symmetric options—typically an OTM put and an OTM call. A discrete approximation is given by:

$$Skew = \frac{IV_{put} - IV_{call}}{K_{put} - K_{call}}$$

This is widely used in practice, especially when computing risk reversals—a standard measure of skew in FX and commodities. For more refined modelling, skew can be defined as the first derivative of implied volatility with respect to strike (or log-moneyness). This continuous definition captures the local slope of the volatility smile at any given strike:

$$Skew = \frac{\partial \sigma impl(K)}{\partial K} \; or \; \frac{\partial \sigma impl(K)}{\partial K}, x = \ln\left(\frac{K}{F}\right)$$

These slope-based formulations are crucial in quantitative modelling, particularly for calibrating local volatility or stochastic local volatility models.

6.2 Interpreting skew and its variants

The market's overall perception of asymmetric risks is reflected in the volatility skew's shape, which differs across asset classes.

The skew in equity markets is usually negative, meaning that as strike rises, *implied volatility* falls. Crash risk aversion causes this left skew because investors are more prepared to pay more for *put options*, which provide downside protection, thereby raising *implied volatility*. This situation is largely caused by institutional actors' (such as insurers and pension funds') demand for *tail risk* hedging. *Tail risk* refers to the probability that an investment will fall by more than three standard deviations.

On the other hand, we frequently see positive skew in commodity markets like those for oil or agricultural items, which indicates that higher strike *(OTM call)* options have more *implied volatility*. This is often linked to short squeezes or supply shocks, which can cause prices to spike upward. Investors in these markets might be more worried about upside leaps brought on by inventory shortages or geopolitical disruptions.

Foreign exchange (FX) markets often exhibit more symmetrical smiles or skewed smiles depending on the currency pair and macroeconomic backdrop. In major currency pairs, the symmetry in monetary policy expectations can result in a smile shape, where both wings *(OTM call and put options)* have higher *IV* than *ATM* strikes.

The forward skew is another variant of interest. It refers to the skew of options that start at a future date (e.g., forward-start options). It can differ significantly from spot skew and plays a key role in pricing exotic instruments. Understanding the time evolution of skew—how it shifts and rotates with maturity—is critical in structured products and volatility modelling.

6.3 Skew and risk reversals

A practical way to measure and trade skew is through *risk reversals (RR)*. In the FX world, a 25-delta risk reversal is defined as the difference in implied volatility between a 25-delta call and a 25-delta put:

$$RR_{25} = IV_{25\Delta\text{−call}} - IV_{25\Delta\text{−put}}$$

A *negative risk reversal* indicates a *left skew* (common in equities and some emerging market currencies), while a *positive RR* signals *right skew*. Traders often use this metric as a gauge of market sentiment and tail-risk premium.

For example, if the *RR* steepens negatively, it may indicate rising fear of a downside move and growing demand for puts.

6.4 Origins of skew in volatility models

From a modelling perspective, the presence of skew arises naturally in stochastic volatility models. For example, in the *SABR model*, skew is controlled by the correlation ρ between the asset's returns and its volatility.

When $\rho < 0$, a drop in the asset price leads to an increase in volatility—resembling what happens during equity market crashes. This produces a left-skewed IV curve.

Conversely, $\rho > 0$ creates right skew, consistent with commodity markets where price spikes lead to rising volatility.

The Heston model and other stochastic volatility frameworks similarly allow for skew generation via negative correlation and volatility-of-volatility parameters. In local volatility models like Dupire's, skew is embedded directly into the volatility surface by making volatility a function of both time and the underlying asset price.

6.5 Implications for trading and hedging

Understanding and modelling skew is essential for both *vanilla and exotic option* pricing, as well as for hedging strategies. Many exotic products are highly sensitive to the shape of the skew. Misestimating skew can lead to significant hedging errors and P&L drift, even if volatility levels are correctly predicted.

Traders may also take direct positions on *skew* itself. For example, by using *risk reversals*, they can express a view on the steepness of the *IV* curve. Others trade skew spread strategies particularly around macro events or earnings announcements. In institutional contexts, *skew* serves as a critical stress-testing variable, informing how portfolios would react to non-linear changes in volatility across strikes.

7 Volatility surface (3D)

The *implied volatility* as a function of strike price and time to maturity is represented in three dimensions by the volatility surface. It provides a comprehensive view of how options are priced throughout the market by fusing the horizontal term structure with the vertical volatility skew structure.

The volatility surface encompasses both implied volatility variation with strike (for a constant maturity) and changes in *IV* over time (for a fixed strike), while the volatility smile incorporates *implied volatility* variation with strike only. This makes it a vital tool for pricing derivatives, calibrating models, anticipating volatility, and structuring exotic options.

7.1 Constructing the surface

To build the volatility surface, traders or quants collect implied volatilities across a range of strikes and maturities. Mathematically, the surface is defined by:

$$\sigma_{impl}(K, T)$$

Each point on the surface corresponds to the implied volatility of an option with a particular strike and expiry. In practice, data is obtained from market quotes of *vanilla options*, and interpolation methods are used to construct a smooth surface from discrete data.

7.2 Interpreting the shape of the surface

In equity markets, the *volatility surface* typically displays a left skew and a term structure in contango. This means:

- Across strikes: *OTM puts* (low strikes) have higher *IV* than *ATM* or *OTM calls* → negative skew.
- Across maturities: *IV* tends to increase with maturity during calm markets (i.e., long-dated options are more volatile), though this relationship may invert in times of crisis.

This surface is shaped by a combination of market microstructure, risk aversion, event risk, and liquidity imbalances. The precise form of the volatility surface can also vary considerably between asset classes.

For example:

- FX markets often exhibit symmetrical smiles across strikes, with surface curvature that reflects macroeconomic regime shifts.
- Commodities may show a steep slope at short maturities due to seasonal volatility or storage constraints.
- Interest rate derivatives, especially swaptions, require complex models like SABR to capture the evolving dynamics of their surfaces.

7.3 *Why the surface matters*

Under the Black-Scholes assumptions of constant volatility, the entire volatility surface would be a horizontal plane—flat across all strikes and maturities. However, the actual observed surfaces contradict this. The existence of curvature, slope, and time dependency in the surface demonstrates that volatility is neither constant nor independent of strike or maturity.

These deviations are not just theoretical observations—they have profound practical implications. Traders who hedge using models that assume flat volatility may experience significant hedging error when skew or term effects are pronounced. This is especially true for exotic options and structured products, where sensitivity to both skew and term structure is heightened.

7.4 *Applications of the volatility surface*

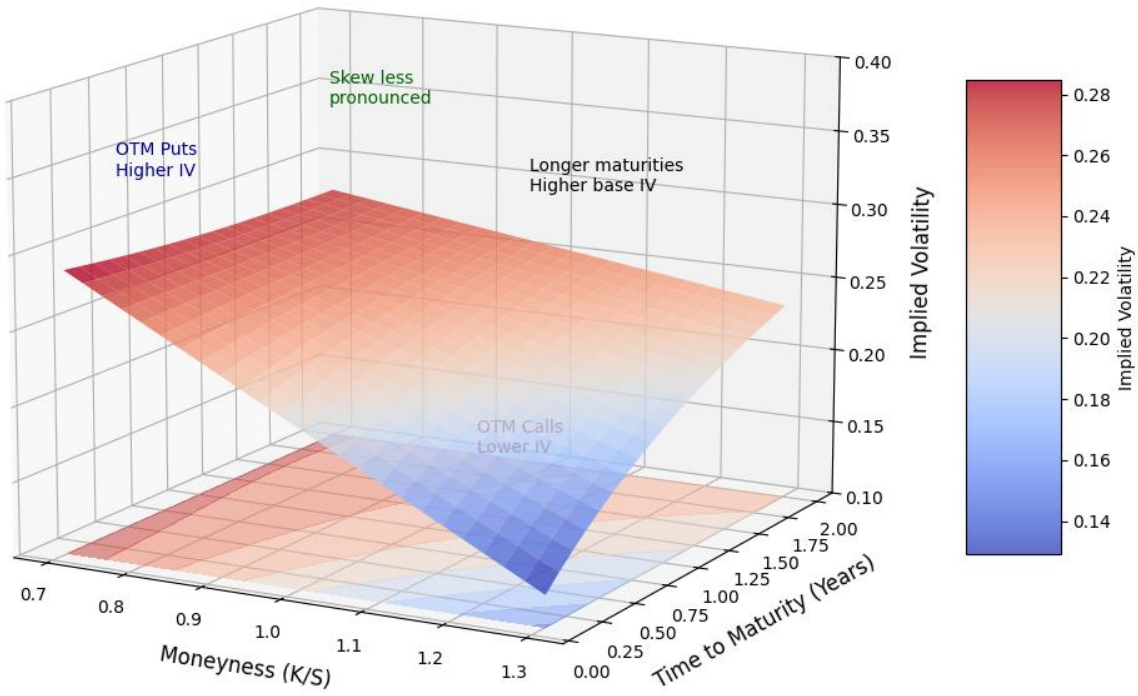
The volatility surface is used extensively in both quantitative modelling and day-to-day trading operations. Its applications include:

- Model Calibration: Quants use the surface to calibrate local volatility models, stochastic volatility models, and hybrid models.
- Volatility Trading: Traders express views on relative *IV* levels across the surface by constructing strategies.
- Exotic Option Pricing: Pricing and hedging path-dependent or barrier options requires full knowledge of how volatility behaves across both dimensions.
- Scenario Analysis and Risk Management: Stress-testing under various volatility surface shifts is a key element of managing portfolio risk.

7.5 *Example: Equity volatility surface*

To illustrate, consider a typical volatility surface for a major equity index like the S&P 500. The implied volatility is plotted on the vertical axis, strike on the X-axis (expressed in terms of moneyness), and time to maturity on the Y-axis.

The surface typically shows as a plane that slopes downward along the strike axis, which represents the skew, and somewhat upward along the maturity axis, which represents the term structure. The surface is most curved around *ATM* strikes and short maturities, indicating the areas with the highest trading activity and the most intricate hedging.



The higher implied volatility observed for *OTM puts* reflects the demand for crash protection—investors are willing to pay more to hedge against sharp downside moves, producing a left-skewed surface. Conversely, *OTM calls* exhibit lower *implied volatility*, as markets typically assign less probability and hedging demand to extreme upward moves. *Implied volatility* tends to increase with maturity, illustrating a normal term structure (*contango*), where long-dated options incorporate more uncertainty. Finally, the skew becomes less steep as maturity increases, suggesting that near-term options price in sharper tail risks.

8 Term structure of volatility

The way *implied volatility* for options on a particular underlying asset changes with time to maturity is referred to as the *term structure* of volatility. *Term structure* captures longitudinal fluctuations across time, whereas skew represents cross-sectional asymmetries over strikes. It reflects the market's expectations of future uncertainty, macro event scheduling, and underlying volatility regimes.

The *term structure* curve, which is obtained by plotting *implied volatility* for *at-the-money (ATM)* options across maturities, is not flat. Building volatility-based strategies and hedging portfolios requires a grasp of how its shape changes based on market conditions.

8.1 Contango and Backwardation

The two primary shapes of the volatility term structure are contango and backwardation.

In a contango regime, *implied volatility* rises with maturity. In other words, the *IV* of long-dated options is higher than that of short-dated options. In quiet or "normal" markets, where uncertainty is anticipated to build up over time, this is the typical shape. Intuitively, the risk premium embedded in longer-dated options increases as the number of unknowns (such as elections, economic cycles, and interest rate changes) increases.

In a backwardation regime, short-dated *implied volatility* exceeds long-dated *IV*. This shape is often observed during periods of market stress or anticipated high-impact events, such as a financial crisis, geopolitical tensions, etc. In these situations, traders anticipate large moves in the immediate future and are willing to pay a premium for short-dated protection—driving up *IV* at the front end of the curve.

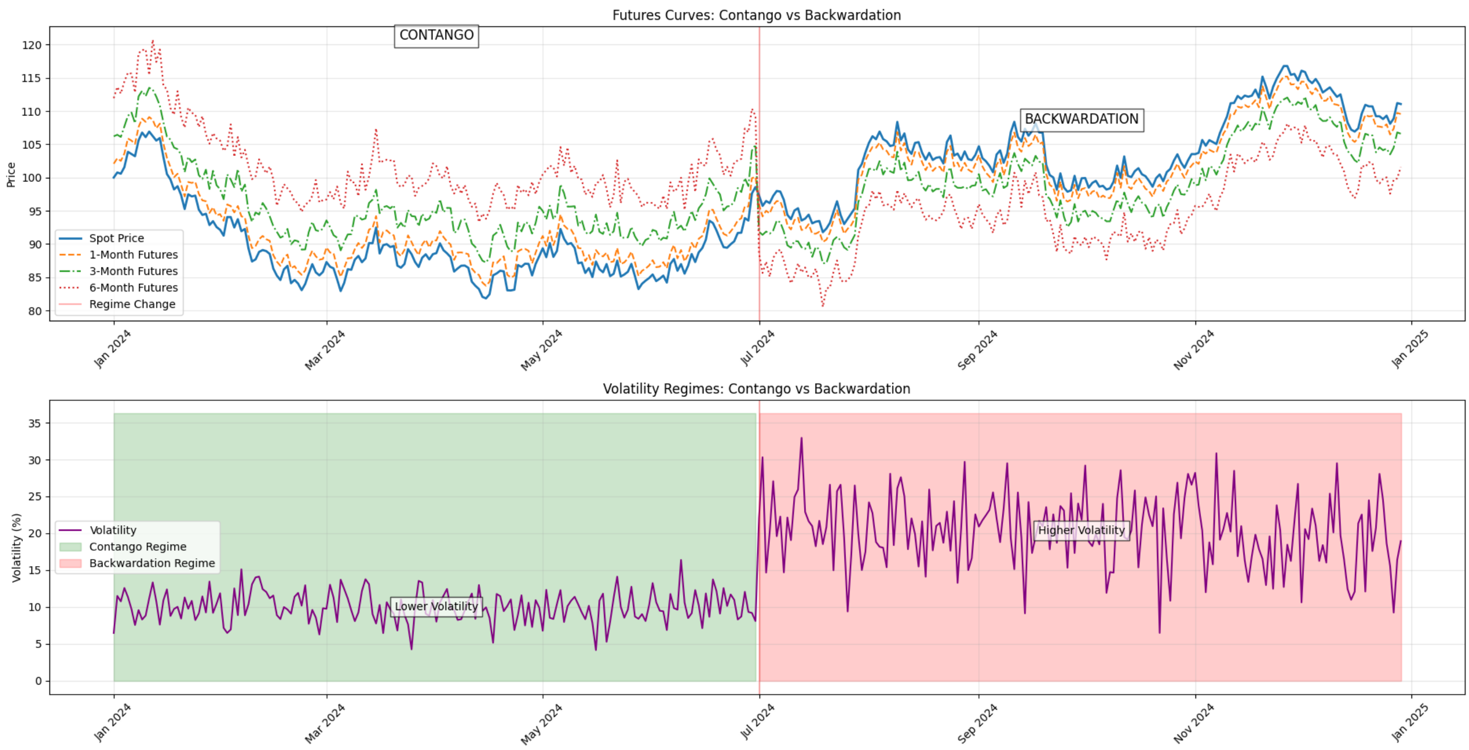


Figure 5: Contango and backwardation example for the term structure of volatility

8.2 Drivers of the term structure

Several factors influence the shape and dynamics of the volatility term structure:

- Macro events and news cycles: Implied volatility often spikes before known events (e.g., FOMC decisions, elections), creating temporary humps in the curve.
- Mean-reversion in volatility: Under stochastic volatility models (e.g., Heston), volatility tends to revert to a long-term average. This dampens long-dated IV and steepens backwardation in crises.
- Volatility risk premium: Investors demand compensation for bearing volatility risk, especially in long-dated instruments.
- Calendar effects: Known seasonal effects (e.g., end-of-quarter earnings cycles) can create periodic distortions in short-term IV.

8.3 Measuring and using the term structure

Practitioners typically extract the term structure using implied volatilities from ATM options across various expirations (1 week, 1 month, 3 months, 6 months, 1 year, etc.). Plotting these points gives the curve, and its slope and curvature are actively monitored by option desks and volatility traders.

This structure is particularly relevant when constructing volatility trading strategies, such as:

- Calendar spreads: Buy a long-dated option and sell a short-dated option (or vice versa) to exploit differences in IV across maturities. This is a direct bet on the term spread.
- VIX futures curve: The term structure of VIX futures (which measure forward-looking S&P 500 volatility) is a key signal in equity volatility trading. Contango or backwardation in VIX futures directly maps to trader sentiment and hedging cost.

8.4 Volatility products

The shape of the term structure is especially important for pricing and managing volatility-linked instruments, such as:

- Volatility ETFs: These are particularly sensitive to roll yield driven by the VIX term structure and are connected to short-term VIX (volatility index) futures.
- Variance swaps: The term structure of implied variance determines the swap rate, and the payout is based on realized variance over a predetermined horizon.
- Forward-start options: The pricing of these contracts necessitates knowledge of forward IV levels, and they start at a future date.

8.5 Term structure in modelling

Volatility models must incorporate time dynamics to accurately price and hedge derivatives. Stochastic volatility models like Heston and SABR can replicate the term structure of IV via parameters such as:

- Volatility of volatility (ν) — governs curvature of term structure.
- Mean reversion (κ) — controls the speed at which volatility returns to long-run mean.
- Initial variance (v_0) and long-run variance (θ) — anchor the level of the curve.

Dupire's local volatility framework, on the other hand, captures term structure explicitly by making volatility a function of both time and underlying price:

$$\sigma_{loc}(S, t)$$

This is critical in exotic option pricing and for maintaining arbitrage-free volatility surfaces.

9 Conclusion

Volatility is far more than a mere statistical parameter — it is the language through which markets express uncertainty, fear, confidence, and risk appetite. Throughout this article, we have dissected its many dimensions: from backward-looking measures such as historical and realized volatility, to the forward-looking implied volatility that governs the pricing of derivative instruments.

We explored how implied volatility is not constant, but instead forms rich patterns across both strike and maturity. These patterns — the volatility smile, the skew, and the full volatility surface — reflect market participants' asymmetric beliefs about future price paths. The left skew in equities, the right skew in commodities, and the smile in FX are not just anomalies; they are structural fingerprints of the underlying market mechanics and investor behaviour.

The term structure of volatility, meanwhile, provides a temporal lens into market psychology. Whether upward-sloping in times of calm (contango) or inverted in times of panic (backwardation), it reflects the market's evolving perception of near- and long-term risk. Together, skew and term structure give rise to the volatility surface — a three-dimensional map of option-implied uncertainty across the full option chain.

These volatility structures are not just descriptive — they are actionable. They inform the pricing of complex derivatives, the construction of hedging strategies, and the calibration of sophisticated models such as Heston, SABR, and local volatility frameworks. For traders, they are opportunities; for risk managers, warnings; and for quantitative modelers, the object of continuous refinement.

Ultimately, to understand volatility is to understand the market's collective imagination: not just what investors believe will happen, but what they fear, hope, and are willing to pay to avoid or embrace. It is where finance meets psychology, probability, and strategy — and it is indispensable to anyone seeking to navigate the derivative landscape with precision.

10 Python code

```python
import autograd.numpy as np
from autograd.scipy.stats import norm
from autograd import grad

# ++++++++++++
# FUNCTIONS
# ++++++++++++
def N(x):
    return norm.cdf(x)

def black_scholes(S0, K, T, r, sigma, Q, option_type='call'):
    d1 = (np.log(S0 / K) + (r - Q + 0.5 * sigma**2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)

    if option_type == 'call':
        return S0 * np.exp(-Q * T) * N(d1) - K * np.exp(-r * T) * N(d2)
    elif option_type == 'put':
        return K * np.exp(-r * T) * N(-d2) - S0 * np.exp(-Q * T) * N(-d1)
    else:
        raise ValueError("Invalid option type. Choose 'call' or 'put'.")

def value_loss(S0, K, T, r, sigma_guess, Q, market_price, option_type='call'):
    return black_scholes(S0, K, T, r, sigma_guess, Q, option_type) - market_price

# Compute the gradient of value_loss with respect to sigma (4th argument)
loss_grad = grad(value_loss, argnum=4)

# ++++++++++++++++++++++++++++++++
# IMPLEMENTING NEWTON'S METHOD
# ++++++++++++++++++++++++++++++++
def implied_volatility(S0, K, T, r, market_price, Q, sigma_guess=1.2, tol=1e-6, max_iter=50, verbose=True):
    IV = sigma_guess  # Initial guess
    for i in range(max_iter):
        loss = value_loss(S0, K, T, r, IV, Q, market_price, option_type='call')

        if verbose:
            print(f"Iteration {i+1}: Sigma = {IV:.8f}, Loss = {loss:.8f}")
        if abs(loss) < tol:  # If the loss is small enough, we have converged
            break
        grad_loss = loss_grad(S0, K, T, r, IV, Q, market_price)  # Compute gradient
        if grad_loss == 0:  # Prevent division by zero
            raise ValueError("Gradient is zero, Newton's method failed to converge.")

        IV = IV - loss / grad_loss  # Newton's update step

    return IV

# ++++++++++++++++++
# TEST PARAMETERS
# ++++++++++++++++++
S0 = 110
K = 100
T = 2  # Time to maturity (years)
r = 0.2
sigma_guess = 0.15  # Initial guess for volatility
Q = 0

# Compute the market price using Black-Scholes
market_price = black_scholes(S0, K, T, r, sigma_guess, Q, option_type='call')
print(f"\nTheoretical market price: {market_price:.3f}\n")

# Compute implied volatility
calculated_IV = implied_volatility(S0, K, T, r, market_price, Q, sigma_guess=1.8)
print(f"\nEstimated Implied Volatility: {calculated_IV:.5f}\n")
```

11 References

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