# *Financial Market Uncovered – Article 2*

## *Black-Scholes Explained: The Formula That Changed Finance*

## *“The Trillion Dollar Equation”*

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# Summary

# 1 The Black-Scholes model, the formula that changed options trading forever?

## 1.1 *What is the Black-Scholes model?*

Options pricing has been largely a guessing game. Traders mainly relied on intuition, experience, and gut feelings to decide how much an option was worth. We didn’t have a precise way to determine whether an option was fairly priced or not. But in 1973, Fischer Black and Myron Scholes, along with Robert Merton, introduced a model that completely transformed the world of options trading.

The Black-Scholes model gave traders a mathematical formula to price European options. The main goal of this model was to provide a systematic way to price an option. Black and Scholes developed this model to address a simple question:

*"If an option gives the right to buy or sell a stock at a fixed price in the future, how much should that option be worth today?"*

This model laid the foundation for modern derivatives trading, making options more accessible and predictable. The formula considers several key factors:

1. The current stock price (because an option’s value depends on the underlying asset).
2. The strike price (the agreed price at which the option can be exercised).
3. Time to expiration (since options lose value as they approach expiry).
4. The risk-free interest rate (because money today is worth more than money in the future).
5. Volatility (how much the stock price is expected to fluctuate).

The final output of the Black-Scholes formula is the option’s theoretical fair value. The actual price in the market may be different, depending on supply and demand, but the formula provides a benchmark for traders to compare to.

## 1.2 *How Black-Scholes’s model revolutionized options trading*

### *1.2.1 Standardizing options pricing*

Before Black-Scholes, options pricing was highly subjective. The model gave traders a common reference point by creating a consistent, mathematical way to determine a fair price.

### *1.2.2 Modern options market*

In 1973 the Chicago Board Options Exchange (CBOE) opened and became the first regulated exchange for listed options. The introduction of the Black-Scholes model made it easier for institutional investors to trade options, which led to incredible growth in the options market.

### *1.2.3 Rise of market makers*

With a standardized pricing formula, financial firms could act as market makers. They continuously bought and sold options at fair prices. This resulted in increased liquidity, making it easier for traders to enter and exit positions without huge price swings.

### *1.2.4 Hedging and risk management*

The Black-Scholes model also introduced the concept of delta hedging, which allows traders to adjust their positions based on how much the option price moves relative to the stock price. This innovation paved the way for more sophisticated risk management strategies.

# 2 Mathematical foundations

## 2.1 *How do we model stock prices?*

Stock prices don't move in a predictable straight line, unlike fixed-income investments or real estate. They fluctuate randomly based on news, market sentiment, and economic events. However, these fluctuations aren't completely chaotic—markets tend to have long-term trends, and the movements of stock prices can be modelled mathematically.

The Black-Scholes model assumes that stock prices follow a specific type of random motion called *Geometric Brownian Motion (GBM)*. But before that, we need to understand what a Wiener process is.

## 2.2 *The Wiener process*

A *Wiener process* (also known as *Brownian motion*) is a mathematical way to describe a random walk, a path where each step is completely unpredictable.

Mathematically, a *Wiener process* $W_t$ has two key properties:

1. **It has independent and normally distributed increments**. This means that the change in $W_t$ over any time period follows a normal distribution:

$$dW_t \sim N(0, dt)$$

This means that the expected change in $W_t$ is zero, but it has a variance, a measure of how spread out its values can be, that grows over time.

2. **It has continuous paths:** Unlike a jump process (where prices can suddenly leap from one level to another), a *Wiener process* moves in a smooth but random way.

In simple terms, a *Wiener process* represents pure randomness—it's the core ingredient for modelling uncertainty in stock prices.

## 2.3 *Geometric Brownian Motion (GBM)*

Let's apply the *Wiener process* to stock prices. The Black-Scholes model assumes that stock prices follow *Geometric Brownian Motion (GBM)*, which is defined as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Where:

- $S_t$ is the stock price at time t.
- $\mu$ is the **expected rate of return** of the stock.
- $\sigma$ is the **volatility** (a measure of how much the stock fluctuates).
- $dW_t$ is the **Wiener process**, representing the random component of price movement.

This equation says that the change in stock price $dS_t$ over a small time period dt has two components:

1. **A predictable drift term ($\mu S_t dt$)** – This is the expected return, which makes stock prices tend to increase over time (on average).
2. **A random term ($\sigma S_t dW_t$)** – This accounts for the unpredictable nature of price movements. The larger the volatility $\sigma$, the wilder the stock's fluctuations.

## *2.4 Ito's Lemma*

We now have a model for stock price, but we still need to understand how option's value changes over time, as the stock price moves. This is where we'll talk about **Ito's Lemma**—it allows us to determine how a function of a stochastic process behaves.

If we have a function $f(S,t)$ that depends on both stock price $S_t$ and time t, and if $S_t$ follows the *GBM* equation above, then Ito's Lemma tells us how $f$ changes:

$$df(S,t) = \left(\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S\frac{\partial f}{\partial S}dW_t$$

This equation shows that the total change in $f(S,t)$ has two terms:

1. A **deterministic part** (terms involving $dt$) that represents expected change over time.
2. A **random part** (term involving $dW_t$) that captures uncertainty.

# 3 The Black-Scholes formula

## *3.1 Applying Ito's Lemma to options pricing*

We apply Ito's Lemma to an option price, $C(S,t)$:

$$dC(S,t) = \left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW_t$$

Using this result, Black and Scholes built a risk-free hedging portfolio, removing the uncertainty term ($dW_t$) which led to the famous **Black-Scholes partial differential equation (PDE)**:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

This partial differential equation (PDE) describes how an option price $C(S,t)$ evolves over time, considering:

- The rate of change of the option price with respect to time ($\frac{\partial C}{\partial t}$).
- The sensitivity of the option price to changes in the stock price ($\frac{\partial C}{\partial S}$).
- The convexity of the option price relative to stock price movements ($\frac{\partial^2 C}{\partial S^2}$).

**I invite you to read Professor Martin Haugh's "The Black-Scholes Model" where he derives the Black-Scholes formula in detail using Ito's Lemma.**

## *3.2 Black-Scholes formula applied to call and put*

For a European *call* option, the Black-Scholes formula is:

$$C(S,t) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2)$$

For a European put option, the formula is:

$$P(S,t) = e^{-r(T-t)}K\Phi(-d_2) - S_t\Phi(-d_1)$$

where:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) * (T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

where:

$S_t$ = current stock price

$K$ = strike price

$T - t$ = time to maturity

$r$ = risk-free interest rate

$\sigma$ = volatility of the stock

$\Phi(d)$ = cumulative standard normal distribution function

**Understanding the formula**

- $\Phi(d_1)$ is the probability, under the risk-neutral measure, that the option will end *in-the-money (ITM)* at expiration. It adjusts for the fact that investors hedge dynamically. It represents the *delta* of a *European call* option, meaning it also measures the sensitivity of the option price to changes in the underlying asset's price.
- $\Phi(d_2)$ is the probability, under the risk-neutral measure, that the stock price will be above the strike price at expiration. This represents the actual probability of the option finishing *in-the-money (ITM)*, being exercised.
- $e^{-r(T-t)}K\Phi(d_2)$ discount the strike price to its present value.
- $S_t\Phi(d_1)$, the stock price term, adjust for the probability of exercising the option.

This formula provides a valuation of the option, based on the risk-neutral measure, which means that we assume all assets grow at the risk-free rate, and therefore discount future payoffs accordingly.

## *3.3 Adjusting the formula for dividends*

For a stock that pays a continuous dividend yield q, the formulas become:

$$C(S,t) = S_t e^{-q(T-t)}\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2)$$

$$P(S,t) = e^{-r(T-t)}K\Phi(-d_2) - S_t e^{-q(T-t)}\Phi(-d_1)$$

where $d_1$, and $d_2$ are now:

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - q + \frac{\sigma^2}{2}\right) * (T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- Here, the term $S_t e^{-q(T-t)}$ discounts the stock price to reflect the expected value of dividends, paid over the life of the option.

## *3.4 How dividends influence option prices*

For a *call option*, since the underlying stock loses value when dividends are paid, *call* options become less valuable.

For a *put option*, it becomes more valuable when dividends are expected, since the stock price drops by the dividend amount. So, the probability of the stock price being below the strike price at maturity increases.

## 3.5 Options' price visualization based on parameters' influence

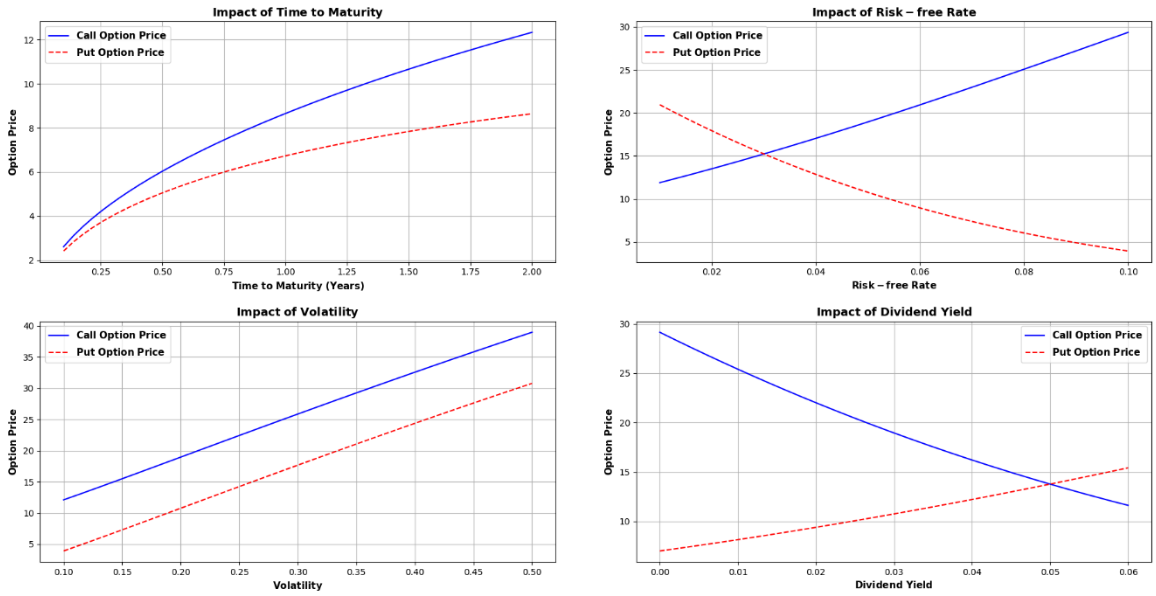


*Figure 1: Parameters' influence on options' price*

***The code can be found in the appendix.***

### 3.5.1 Impact of Time to maturity

An option becomes more valuable the longer the time remaining before expiration. This is because there is more time for the stock to move in a way that benefits the option holder. However, the option starts to lose value as the expiration date approaches. This is known as *time decay*. If the option is *out-of-the-money (OTM)*, it will lose value quickly as time passes. This is why short-term options lose value much faster than long-term options.

### 3.5.2 Impact of Interest rates

As we can see in the graphs, a *call* becomes more expensive as interest rates go up, while a p*ut* becomes cheaper. The reason is that, when interest rates are high, investing in risk-free assets, such as bonds, becomes more attractive for holding money. Buying a *call* is similar to delaying the purchase of an entire stock; therefore, *calls* are worth more when interest rates are higher.

On the contrary, *put* options become less valuable because put buyers are giving up the opportunity to invest in interest-bearing assets. As interest rates have a bigger impact over longer timeframes, this effect is greater for a longer period.

### 3.5.3 Impact of Volatility

Volatility is one of the biggest factors affecting option prices. Both *call* and *put* options increase as volatility rises.

For a highly volatile stock, there is a greater probability that its price will move far enough for the option to be *in-the-money (ITM)* before expiration. Since options give the right, but not the obligation, to buy or sell at a fixed price, greater uncertainty (volatility) is better. This is why options with high volatility, like those on tech stocks, are more expensive than those of stable companies like utility providers.

Here is a heatmap of how both *call* and *put* options' price change with regard to volatility:

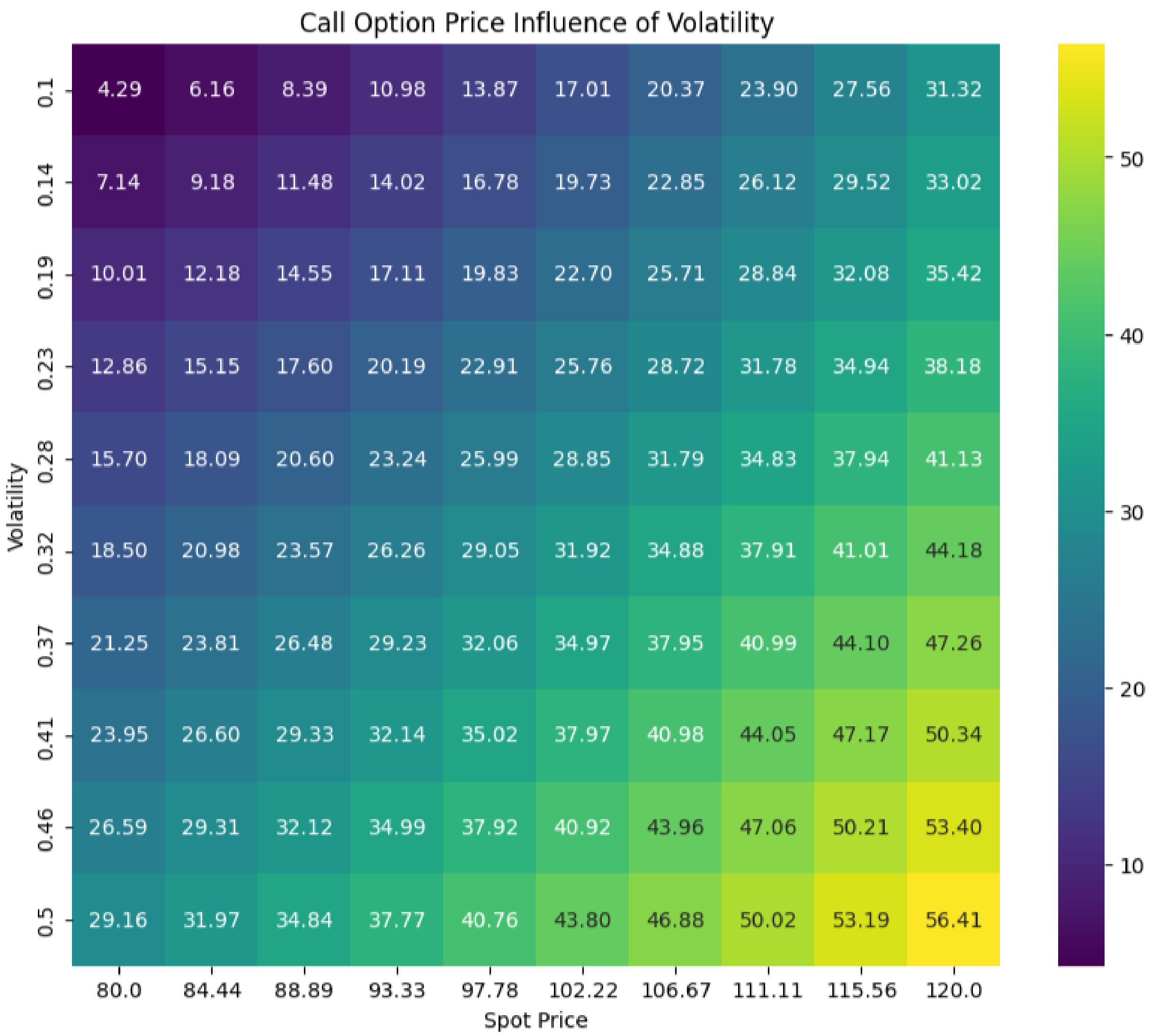


*Figure 2: Heatmap of a call price changes in regard to volatility*

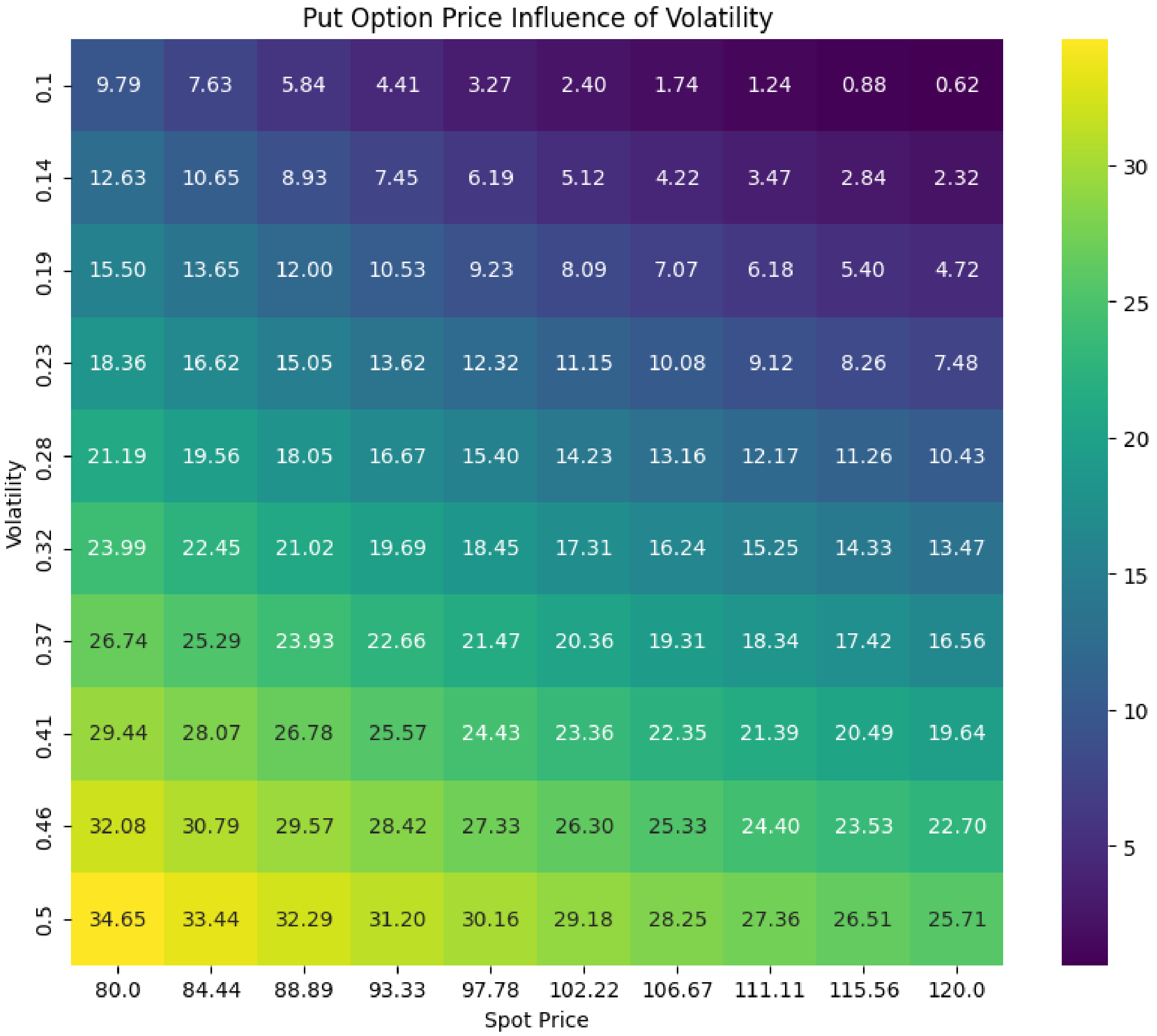


*Figure 3: Heatmap of a put price changes in regard to volatility*

### 3.5.4 *Impact of Dividends*

As we said in part ***3.4***, *call* options become cheaper when dividend yields increase, while *put* options become more expensive.

# 4 Assumptions and limitations of the Black-Scholes model

The Black-Scholes model relies on several assumptions that don't always hold in real markets.

## *4.1 Assumptions*

**Stock prices follow a Geometric Brownian Motion**

The model assumes that stock prices move continuously in a random, predictable way, following a *GBM*. This means that prices:

- Change gradually without sudden jumps.
- Follow a lognormal distribution.
- Cannot go negative.

**Volatility remains constant**

The model implies that a stock's volatility remains constant throughout the option's life. It simplifies calculations and provides a closed-form pricing solution. It suggests that volatility smiles do not exist.

**No transaction costs or market frictions**

Trading is considered to be frictionless, with no bid-ask spreads, commissions, or liquidity concerns. Traders can buy and sell as much as they want at the exact theoretical price.

**Options are European-Style**

The model is intended for *European options* that can only be exercised at expiration. It excludes *American options*, which can be exercised at any moment.

**Risk-free interest rate is constant**

The model assumes that the risk-free interest rates remain constant throughout the option's life. This makes it simpler to discount future cash flows.

## *4.2 Limitations*

**Stock prices can jump suddenly**

In reality, stock values do not always fluctuate smoothly. Market collapses, earnings surprises, and unexpected news events can all result in huge, rapid price spikes. These jumps are too large for the Black-Scholes model to handle, which might result in option mispricing.

**Volatility changes over time**

Volatility is not constant; it varies depending on market conditions, investor sentiment, and economic data. This results in the volatility smile, where options with different strike prices have varying implied volatilities.

**Trading costs and liquidity**

In the real market, traders suffer transaction fees, *bid-ask* spreads, and liquidity limits. These considerations make it hard to buy and sell at the exact theoretical price determined by the model.

**Interest rates fluctuate**

Interest rates fluctuate according to central bank policies, inflation, and economic factors. The Black-Scholes model assumes a fixed risk-free rate, which does not reflect reality.

## *4.3 Alternative models*

**The Binomial model**

One important restriction of the Black-Scholes model is that it only works with European-style options. Cox, Ross, and Rubinstein (CRR) developed the binomial model, which breaks down time into discrete steps to provide a more flexible framework for pricing choices. It simulates the stock price as rising or falling with each stage, allowing for a step-by-step approach.

It allows traders to price American options more accurately than Black-Scholes by considering early exercise opportunities.

**The Heston model**

Because the Black-Scholes model assumes continuous volatility, the Heston model, developed in 1993, treats volatility as a variable with its own stochastic process.

It made it possible to capture volatility smiles, and skews, making it more applicable for real-world option pricing.

**Jump-diffusion model**

Black-Scholes assumes that stock prices follow a smooth, continuous trend, which is not the case in reality. Jump-diffusion models, such as the Merton Jump-Diffusion model, improve on Black-Scholes by introducing random leaps into the stock price process.

Despite the fact that it requires calculating jump frequency and magnitude, it is an effective method for pricing options prior to events involving abrupt price fluctuations.

# 5 How the model is used in practice

## *5.1 Pricing options and determining fair value*

The Black-Scholes model is primarily used to compute an option's theoretical price. This theoretical pricing serves as a standard for determining whether an option is overpriced or underpriced.

- If an option is trading at a price higher than the Black-Scholes price, it may be overvalued, so traders would consider selling it.
- If an option is trading at a lower price than the Black-Scholes price, it may be undervalued, so trader would consider buying it.

Here is a streamlit app application that calculates an option's price based on several input parameters:

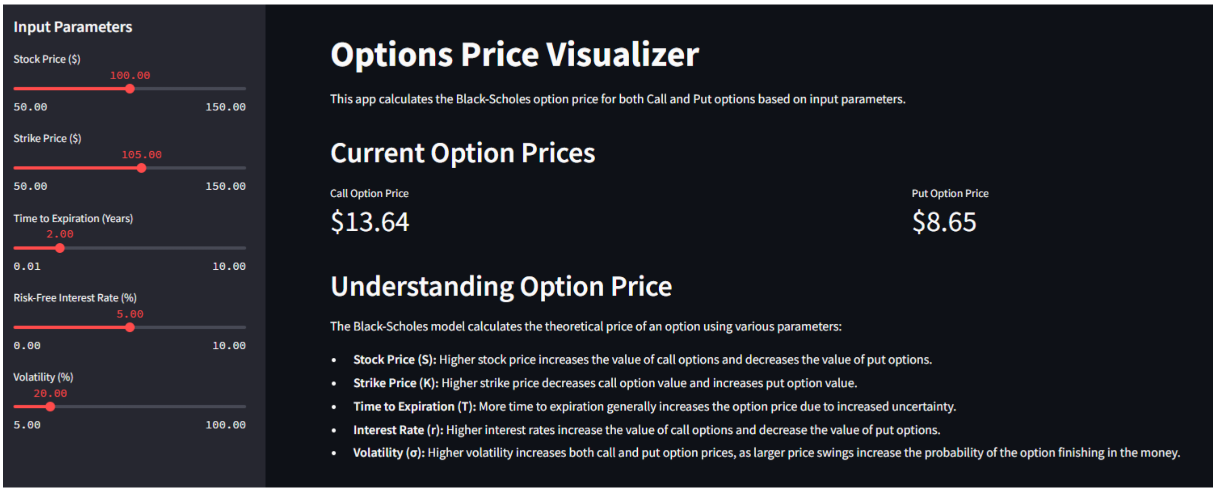


*Figure 4: Streamlit application to calculate call and put price*

## *5.2 Implied volatility*

Instead of using historical volatility as an input, traders reverse-engineer the Black-Scholes model to derive implied volatility which is the volatility that the market assigns to an option. It can then be compared to historical volatility to determine whether the options are expensive or cheap.

- If implied volatility is high, options are expensive, and traders might prefer selling to collect option premiums
- If implied volatility is low, options are cheaper, making it a better time to buy options.

Most of the time, implied volatility is monitored to spot market trends, potential uncertainty, and major events risks.

## 5.3 Hedging and Risk management

Options market makers, institutional investors, and hedge funds use the model to manage risk by hedging their positions, usually using what is called *delta hedging*.

Since options do not move in a 1:1 ratio with the stock price, traders use *delta hedging* to adjust their positions dynamically.

- *Delta* ($\Delta$) measures how much an option's price changes for every $1 movement in the underlying stock.
- A trader who is long a call option can hedge by short-selling the underlying stock in proportion to the option's delta.

*Delta hedging will be explained in more detail in the next article, dedicated to Greeks.*

# 6 Conclusion

The Black-Scholes model permanently altered the way options trading was conducted. Previously, options were priced according to intuition and experience. The introduction of a structured formula gave traders, institutions, and investors a mathematical framework for calculating fair value, controlling risk, and devising more sophisticated strategies.

Despite its faults and limitations, the Black-Scholes model remains the foundation for modern options pricing. It resulted in the phenomenal growth of the options markets, allowing market makers to quote prices more effectively and introducing concepts such as implied volatility, which traders now use daily.

However, the market is not perfect. Markets do not always run smoothly and predictably; volatility fluctuates, prices may rise unexpectedly, and options traders face transaction fees and liquidity constraints. These real-world factors drove the development of more sophisticated pricing models.

Black-Scholes remains relevant. Many of the more complex models build upon its foundation rather than replacing it entirely. Even as quantitative finance, machine learning, and AI-powered models develop, Black-Scholes continues to serve as a baseline for understanding and pricing options.

Finally, the Black-Scholes model is more than just a mathematical formula; it marks a turning point in financial history. It provided the framework for the modern options market and remains today an indispensable instrument for traders and analysts.

# 7 Appendix

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

#++++++++++++++++++++
#++++ FUNCTIONS ++++
#++++++++++++++++++++
def black_scholes_call(S, K, T, r, sigma, q=0):
    d1 = (np.log(S / K) + (r - q + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    call_price = S * np.exp(-q * T) * norm.cdf(d1) - K * np.exp(-r * T) * norm.cdf(d2)
    return call_price

def black_scholes_put(S, K, T, r, sigma, q=0):
    d1 = (np.log(S / K) + (r - q + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)
    put_price = K * np.exp(-r * T) * norm.cdf(-d2) - S * np.exp(-q * T) * norm.cdf(-d1)
    return put_price

import numpy as np
import matplotlib.pyplot as plt

def plot_sensitivity(S, K, T, r, sigma, q=0):
    fig, axes = plt.subplots(2, 2, figsize=(10, 9))  # 2x2 grid of subplots

    # Time to Maturity Sensitivity
    T_range = np.linspace(0.1, 2, 50)
    prices_T_call = [black_scholes_call(S, K, t, r, sigma, q) for t in T_range]
    prices_T_put = [black_scholes_put(S, K, t, r, sigma, q) for t in T_range]
    axes[0, 0].plot(T_range, prices_T_call, label=r'$\bf{Call\ Option\ Price}$', color='b')
    axes[0, 0].plot(T_range, prices_T_put, label=r'$\bf{Put\ Option\ Price}$', linestyle='dashed', color='r')
    axes[0, 0].set_xlabel(r'$\bf{Time\ to\ Maturity\ (Years)}$', fontsize=11)
    axes[0, 0].set_ylabel(r'$\bf{Option\ Price}$', fontsize=11)
    axes[0, 0].set_title(r'$\bf{Impact\ of\ Time\ to\ Maturity}$', fontsize=13)
    axes[0, 0].legend(fontsize=11, loc='best', frameon=True)
    axes[0, 0].grid()

    # Same logic for the rest of the plots...

    plt.tight_layout()  # Adjusts spacing to prevent overlap
    plt.show()

#++++++++++++++++++++++++++
#++++ INPUT PARAMETERS ++++
#++++++++++++++++++++++++++
S = 100         # Stock price
K = 110         # Strike price
T = 2           # Time to maturity in years
r = 0.05        # Risk-free interest rate
sigma = 0.2     # Volatility
q = 0.03        # Dividend yield

plot_sensitivity(S, K, T, r, sigma, q)
```

*Figure 5: Code for parameters' influence on options' price*

# 8 References

[1] Hull, J. C. (2018). Options, Futures, and Other Derivatives (10th ed.). Pearson.

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[6] Ramirez, J. (2011). Handbook of corporate equity derivatives and equity capital markets. Wiley.