

Mathematics-1 (FCMT001)

Srijan Mahajan (2023UCM2326)

Prof. Sarika

Contents

1	Functions of Single Variable	4
1.1	Ranges	4
1.1.1	Maximum and Minimum	4
1.1.2	Infimum and Supremum	4
1.2	Bounded Set	5
1.3	Limits	5
1.3.1	Sequence Definition of a Limit	6
1.4	Differentiability	7
1.5	Tutorial-1 (19.08.2023)	8
1.5.1	Examine the limit and continuity of the following functions	8
1.5.2	Are the following functions continuous and differentiable	9
1.5.3	Question 3	11
1.5.4	Question 4	11
1.5.5	Applications of IVT, Rolle's Theorem and MVT	11
1.5.6	Problems on Hyperbolic Functions	13
1.5.7	Find the real and Imaginary parts of the following	13
1.5.8	Problems on Inverse Hyperbolic Functions	14
1.5.9	Separate the real and imaginary parts of the following	15
1.6	Successive Differentiation	18
1.6.1	n^{th} Order Derivatives of Common Functions	18
1.7	Tutorial-2	19
1.7.1	Solve the following	19
1.7.2	Here $y_n = \frac{d^n y}{dx^n}$ in the following parts:	20
1.7.3	Find the n^{th} derivative of the following functions	21
1.7.4	Question 4	22
1.7.5	Use Leibniz Theorem to solve the n^{th} order derivative of the following	23
1.7.6	If $y = (\sin^{-1} x)^2$, prove that,	25
1.7.7	Expand the following functions in power of x by Maclaurin's Theorem OR write the series expansion of function about the point $x = 0$	26
1.7.8	Using Taylor's Theorem	26
1.7.9	Question 9	27
1.8	Taylor's Theorem	29
1.8.1	Maclaurin Series	29
1.9	Hyperbolic Functions	29
1.9.1	Graphs	30
1.9.2	Identities of Conversion	31
1.9.3	Period	31
1.9.4	Inverse Hyperbolic Functions	32
1.9.5	More Identities of Conversion	32
2	Functions of Several Variables	33
2.1	Limit in Higher Dimensions	33
2.2	Continuity	34
2.3	Partial Derivatives	34
2.4	Composition of Functions	35
2.5	Jacobian	36
2.6	Explicit and Implicit Functions	37
2.7	Maxima and Minima of Two Variables	37
2.7.1	Critical Point	38
2.8	Tutorial-3	39
2.8.1	Determine the domain and range of the following functions	39
2.8.2	Examine if the limits of the following functions $f(x, y)$ as $(x, y) \rightarrow (0, 0)$ exist	39

2.8.3	Examine the continuity of the following functions at $(0, 0)$	40
2.8.4	Find the first order partial derivatives of the following functions at the given points	41
2.8.5	Calculate the partial derivatives of the following functions at $(0, 0)$	42
2.8.6	Find the Jacobian of the following functions	46
2.8.7	Find f_u , f_v and total derivative for the functions	47
2.8.8	Using change of variable, prove the following	47
2.8.9	Find $\frac{dy}{dx}$ for the following implicit functions	49
3	Multiple Integrals	51
3.1	Integration	51
3.2	Double Integrals	52
3.2.1	X-Regular and Y-Regular	52
3.3	Change of Variables	54
3.3.1	Integrals using Polar Coordinate Systems	55
3.4	Triple Integrals	56
3.5	Beta Function	58
3.6	Gamma Function	58
3.7	Tutorial-6	59
3.7.1	Evaluate the double integrals	59
3.7.2	Evaluate the following integrals by changing the order of integration if needed	60
3.7.3	Find the volume of the following	63
3.7.4	Use the given transformations to transform the integrals and evaluate them	63
3.7.5	Evaluate the following volume integrals	65
3.7.6	Using Beta and Gamma functions, evaluate the following	66

Chapter 1

Functions of Single Variable

Some common sets are,

- \mathbb{R} It is the set of all real numbers
- \mathbb{N} It is the set of all natural numbers
- \mathbb{Z} It is the set of all integers
- \mathbb{Q} It is the set of all rationals
- \mathbb{Q}^c It is the set of all irrationals

Also, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$.

Theorem 1 (Uncountable \mathbb{Q}^c). The set of all irrationals is uncountable.

Proof. We know, \mathbb{R} is uncountable and \mathbb{Q} is countable, since,

$$\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$$

\mathbb{Q}^c must be uncountable. □

1.1 Ranges

1.1.1 Maximum and Minimum

The maxima is the highest element in the set and similarly, minima is the lowest element in the set.¹

Example (Max and min not enough). Find the maximum and minimum of

$$S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

Solution. Here, the set can be represented as,

$$S \in (0, 1]$$

Here, the maximum value is clearly 1, however, the minimum value of the expression can't be defined^a.

^aSince 0 is not in the set and also, 0 is never actually obtained but only close to.

1.1.2 Infimum and Supremum

The infimum of a set is its greatest lower bound and supremum is the lowest upper bound. These are generalizations of maximum and minimum.

Example. Find the infimum and supremum of

$$S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

¹Maximum and minimum are unique for a set, i.e. there exists only one for a given set.

Solution. Similarly,

$$S = (0, 1]$$

Thus, the infimum is 0 and supremum is 1.

Example (Not always in set). Find the infimum and supremum of

$$S = \{x^2 \leq 2 | x \in \mathbb{Q}^+\}$$

Solution.

$$x \in (0, \sqrt{2}]$$

Here, x is defined in the natural numbers but the supremum is not in \mathbb{Q}^+ . Thus, infimum and supremum are not always a part of the set.

Example. Find the infimum and supremum of

$$S = \{1 + \frac{(-1)^n}{n} | n \in \mathbb{N}\}$$

Solution. The next few digits are, $0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \dots$. Hence, infimum is 0 and supremum is $\frac{3}{2}$.

1.2 Bounded Set

A set is said to be bounded if it has both upper and lower bounds. E.g. $(0, \infty)$ has a lower bound but no upper bound.

Example (Finite). Is the given set finite/infinite, bounded/unbounded and find infimum and supremum (if exists),

$$S = \{\sin\left(\frac{n\pi}{2}\right) | n \in \mathbb{Z}\}$$

Solution. The range of the set is,

$$\text{ran}(S) = \{-1, 0, 1\}$$

Since the cardinality is finite, the set S is said to be finite, and bounded. Clearly, the infimum is -1 and the supremum is 1 .

1.3 Limits

Theorem 2 (Continuity). A function $f : S \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is continuous at $x = a$ if,

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a} f(x) = f(a)$$

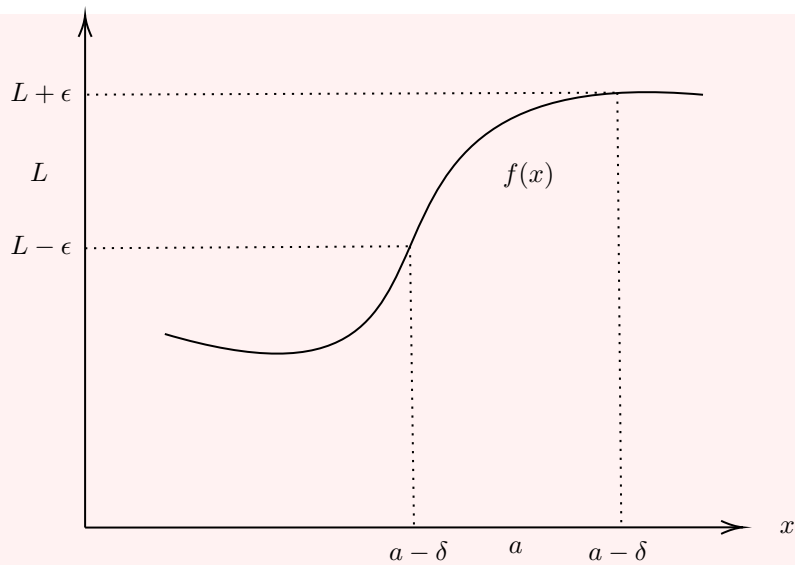
Theorem 3 (Limit Exists). Let $S \subset \mathbb{R}$ and $f : S \rightarrow \mathbb{R}$, we can say $\lim_{x \rightarrow a} f(x) = L$ if,

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

We can say that the limit exists at some point when the function approaches the value from both sides (left and right hand limit), however, this works only in the case of \mathbb{R} . It fails in \mathbb{R}^n , as the point can be achieved from infinitely many directions (disc/sphere).

Theorem 4 ($\epsilon - \delta$ definition of a limit). Let $f(x)$ be a real valued function defined on an open interval around a .

$$\lim_{x \rightarrow a} f(x) = L, \text{ if } \forall \epsilon > 0, \exists \delta > 0 (\forall x > 0, 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon)$$



Clearly, if $f(a) = L$, then f is continuous.

1.3.1 Sequence Definition of a Limit

A sequence $\{x_n\}$ is a $f : \mathbb{N} \rightarrow \mathbb{R}$ such that $f(n) = x_n$.

If $x_n \rightarrow a$ ($a \in \mathbb{R}$), then the sequence is said to be convergent, else convergent/oscillating.

Theorem 5 (Non-existing Limit). $\lim_{x \rightarrow a} f(x)$ doesn't exist iff

$$\begin{aligned}\lim_{n \rightarrow \infty} f(x_n) &= L_1 \\ \lim_{n \rightarrow \infty} f(y_n) &= L_2\end{aligned}$$

where $L_1 \neq L_2$ and $x_n \rightarrow a \wedge y_n \rightarrow a$ as $n \rightarrow \infty$.

To prove that a limit exists, it must satisfy $\lim_{n \rightarrow \infty} f(x_n) = L$ for all sequences x_n .

Example. Find the limit

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$$

For these, we need to work backwards, first we take two values that the function can take (here 0 and 1), then find two sequences that converge to the same but give the respective values.

Solution. Consider,

$$x_n = \frac{1}{2n\pi} \wedge y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

Clearly,

$$f(x_n) = 0 \wedge f(y_n) = 1$$

But $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 0$. Thus, the limit doesn't exist.

Example (Someone). Show that the limit $\lim_{x \rightarrow 0} \frac{|x|}{x}$ doesn't exist.

Solution. Consider the sequences,

$$x_n = \frac{1}{n} \wedge y_n = -\frac{1}{n}$$

Clearly, $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = 0$ Also,

$$\lim_{n \rightarrow \infty} \frac{|\frac{1}{n}|}{\frac{1}{n}} = 1 \wedge \lim_{n \rightarrow \infty} \frac{|-\frac{1}{n}|}{-\frac{1}{n}} = -1$$

Thus, the limit doesn't exist.

Example. Show that f is continuous at $x = \frac{1}{2}$ and discontinuous everywhere else, where f is,

$$f(x) = \begin{cases} x & , x \in \mathbb{Q} \\ \frac{1}{2} & , x \notin \mathbb{Q} \end{cases}$$

Solution. Consider the Left Hand Limit,

$$\lim_{x \rightarrow \frac{1}{2}^-} f(x) = \lim_{h \rightarrow 0} f\left(\frac{1}{2} - h\right) = \begin{cases} \lim_{h \rightarrow 0} \frac{1}{2} - h & , x \in \mathbb{Q} \\ \lim_{h \rightarrow 0} \frac{1}{2} & , x \notin \mathbb{Q} \end{cases} = \frac{1}{2}$$

Similarly, Right Hand Limit is $\frac{1}{2}$, thus the function is continuous at $x = \frac{1}{2}$.
Consider any point $x = a \in \mathbb{R} - \{\frac{1}{2}\}$,

$$\lim_{x \rightarrow a^-} f(x) = \begin{cases} \lim_{h \rightarrow 0} a - h & , a - h \in \mathbb{Q} \\ \lim_{h \rightarrow 0} \frac{1}{2} & , a - h \notin \mathbb{Q} \end{cases}$$

Thus the limit doesn't exist.

Theorem 6 (Intermediate Value Theorem: A Special Case). Consider a continuous function f on $[a, b]$ such that $f(a)f(b) < 0$, then, $\exists c \in (a, b)$ such that $f(c) = 0$.

1.4 Differentiability

Theorem 7 (A differentiable function). A function $f : S \neq \emptyset \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is called differentiable at $x = a$ iff the following limit exists,

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

If the limit does exist,

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Example. Show f is continuous at $x = 0$,

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Solution. Suppose $\epsilon > 0$. Now choose $\delta = \sqrt{\epsilon}$,
Let $|x - 0| < \delta$. Check,

$$\begin{aligned} & |x^2 \sin\left(\frac{1}{x}\right)| \\ & \leq |x|^2 \\ & \leq \delta^2 \\ & = \epsilon \end{aligned}$$

Thus, $|x^2 \sin\left(\frac{1}{x}\right)| < \epsilon$, so, f is continuous at $x = 0$.

Theorem 8 (Rolle's Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on its domain and be differentiable on (a, b) , such that $f(a) = f(b)$, then $\exists c \in (a, b)$ such that, $f'(c) = 0$.

Theorem 9 (Lagrange Mean Value Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function on its domain and be differentiable on (a, b) , then $\exists c \in (a, b)$ such that, $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof. We can prove this using Rolle's Theorem. It is an extension of Rolle's Theorem. □

1.5 Tutorial-1 (19.08.2023)

1.5.1 Examine the limit and continuity of the following functions

Question.

$$f(x) = \lim_{n \rightarrow \infty} \frac{e^x - x^n \sin x}{1 + x^n}, 0 \leq x \leq \frac{\pi}{2} \text{ at } x = 1$$

Answer. We can split the function at $x = 1$,

$$\begin{aligned} f(x) &= \begin{cases} \lim_{n \rightarrow \infty} \frac{e^x - x^n \sin x}{1 + x^n} & , 0 \leq x < 1 \\ \lim_{n \rightarrow \infty} \frac{e^x - x^n \sin x}{1 + x^n} & , x = 1 \\ \lim_{n \rightarrow \infty} \frac{e^x - x^n \sin x}{1 + x^n} & , 1 < x \leq \frac{\pi}{2} \end{cases} \\ &= \begin{cases} e^x & , 0 \leq x < 1 \\ \frac{e - \sin 1}{2} & , x = 1 \\ -\sin x & , 1 < x \leq \frac{\pi}{2} \end{cases} \end{aligned}$$

Thus, clearly, f is discontinuous at $x = 1$.

Question.

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases} \text{ at } x = 0$$

Answer. First we calculate the limit

$$\begin{aligned} L &= \lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) \\ &= 0 \end{aligned}$$

Here, $\lim_{x \rightarrow 0} f(x) = f(0)$, thus the function is continuous.

Question.

$$f(x) = \begin{cases} \sin\left(\frac{1}{x-a}\right) & , x \neq a \\ 0 & , x = a \end{cases} \text{ at } x = a$$

Answer. Consider,

$$\begin{aligned} x_n &= a + \frac{1}{2n\pi} & y_n &= a + \frac{1}{2n\pi + \frac{\pi}{2}} \\ \lim_{n \rightarrow \infty} x_n &= a & \lim_{n \rightarrow \infty} y_n &= a \end{aligned}$$

Also,

$$f(x_n) = 0 \wedge f(y_n) = 1$$

Thus, by Sequence Criteria, f is discontinuous at $x = a$.

Question.

$$f(x) = \begin{cases} \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \text{ at } x = 0$$

Answer. Consider the Left Hand Limit,

$$\lim_{x \rightarrow 0^-} \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} = \lim_{x \rightarrow 0^-} \frac{e^{\frac{2}{x}} - 1}{e^{\frac{2}{x}} + 1} = -1$$

As $\lim_{x \rightarrow 0^-} f(x) \neq f(0)$ the function is discontinuous at $x = 0$.

Question.

$$f(x) = \begin{cases} \frac{\sin^2(ax)}{x^2} & , x \neq 0 \\ 0, & , x = 0 \end{cases} \quad a \in \mathbb{R}$$

Answer.

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin^2(ax)}{(ax)^2} \cdot a^2 = a^2$$

Thus, the function is discontinuous for all $a \in \mathbb{R} - \{0\}$, i.e. it is continuous at $a = 0$.

Question.

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2} & , x \neq 0 \\ \frac{1}{2} & , x = 0 \end{cases} \quad \text{at } x = 0$$

Answer.

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2} = \frac{1}{2}$$

Since $\lim_{x \rightarrow 0} f(x) = f(0)$, the function is continuous at $x = 0$.

Question.

$$f(x) = \begin{cases} \frac{(x-1)-|x-1|}{x-1} & , x \neq 1 \\ 2 & , x = 1 \end{cases} \quad \text{at } x = 1$$

Answer. Consider the Right Hand Limit,

$$\lim_{x \rightarrow 1^+} \frac{(x-1)-|x-1|}{x-1} = \lim_{x \rightarrow 1^+} \frac{(x-1)-(x-1)}{x-1} = 0$$

As $\lim_{x \rightarrow 1^+} f(x) \neq f(1)$, the function is discontinuous at $x = 1$.

1.5.2 Are the following functions continuous and differentiable

Question.

$$f(x) = \begin{cases} 2+x & , x \geq 0 \\ 2-x & , x \leq 0 \end{cases} \quad \text{at } x = 0$$

Answer.

$$\lim_{x \rightarrow 0^-} f(x) = 2 \wedge \lim_{x \rightarrow 0^+} f(x) = 2 \wedge f(0) = 2$$

Thus, the function is continuous at $x = 2$. For differentiability,

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} \frac{2+x-2}{x} = 1 \wedge \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^+} \frac{2-x-2}{x} = -1$$

Thus, the function is not differentiable at $x = 0$.

Question.

$$f(x) = \begin{cases} x^2 - 1 & , x \geq 1 \\ 1 - x & , x \leq 1 \end{cases} \quad \text{at } x = 1$$

Answer.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = f(1) = 0$$

Thus, the function is continuous at $x = 1$. For differentiability,

$$\lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^-} \frac{x^2 - 1 - 0}{x - 1} = 2 \wedge \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1^+} \frac{1 - x - 0}{x - 1} = -1$$

Thus, the function is not differentiable at $x = 1$.

Question.

$$f(x) = \begin{cases} x \frac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \text{ at } x = 0$$

Answer. Clearly, due to multiplication of x , the function is continuous at $x = 0$. For differentiability,

$$\lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = -1 \wedge \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = 1$$

Thus, the function is not differentiable at $x = 0$.

Question.

$$f(x) = \begin{cases} x \tan^{-1}\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases} \text{ at } x = 0$$

Answer. We know,

$$\begin{aligned} -\frac{\pi}{2} &< \tan^{-1}\left(\frac{1}{x}\right) < \frac{\pi}{2} \\ \Rightarrow -\frac{\pi}{2}x &< x \tan^{-1}\left(\frac{1}{x}\right) < \frac{\pi}{2}x \\ \therefore 0 &< \lim_{x \rightarrow 0} x \tan^{-1}\left(\frac{1}{x}\right) < 0 \end{aligned}$$

Thus, $\lim_{x \rightarrow 0} x \tan^{-1}\left(\frac{1}{x}\right) = 0$. Hence, the function is continuous at $x = 0$. For differentiability,

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \tan^{-1}\left(\frac{1}{x}\right) \text{ Does not exist}$$

Thus, the function is not differentiable at $x = 0$.

Question.

$$f(x) = \begin{cases} 2x & , x \in \mathbb{Q} \\ -2x & , x \notin \mathbb{Q} \end{cases} \text{ at } x = 0$$

Answer.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0^-} f(h) = \begin{cases} 2h & , h \in \mathbb{Q} \\ -2h & , h \notin \mathbb{Q} \end{cases} = 0$$

Similarly, for the Right Hand Limit. Thus, the function is continuous at $x = 0$. For differentiability,

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \begin{cases} 2 & , x \in \mathbb{Q} \\ -2 & , x \notin \mathbb{Q} \end{cases}$$

Thus, the function is not differentiable at $x = 0$.

Question.

$$f(x) = \begin{cases} x^2 & , x \in \mathbb{Q} \\ 2x & , x \notin \mathbb{Q} \end{cases} \text{ at } x = 0$$

Answer.

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0^-} f(h) = \begin{cases} h^2 & , h \in \mathbb{Q} \\ 2h & , h \notin \mathbb{Q} \end{cases} = 0$$

Similarly, for the Right Hand Limit. Thus, the function is continuous at $x = 0$. For differentiability,

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \begin{cases} x & , x \in \mathbb{Q} \\ 2 & , x \notin \mathbb{Q} \end{cases}$$

Thus, the function is not differentiable at $x = 0$.

1.5.3 Question 3

Question. Show that the function f defined by $f(x) = \begin{cases} x^2 \cos(\frac{1}{x}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ is continuous and differentiable at $x = 0$ but its derivative is not.

Answer. We know,

$$\begin{aligned} -1 &< \cos\left(\frac{1}{x}\right) < 1 \\ \implies -x^2 &< x^2 \cos\left(\frac{1}{x}\right) < x^2 \\ \therefore 0 &< \lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) < 0 \end{aligned}$$

Thus, $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{1}{x}\right) = 0$. Thus, f is continuous at $x = 0$. For differentiability,

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} x \cos\left(\frac{1}{x}\right) = 0$$

Thus, the function is differentiable at $x = 0$. The derivative is,

$$f'(x) = \begin{cases} 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Clearly, $\sin\left(\frac{1}{x}\right)$ is discontinuous at $x = 0$, thus, $f'(x)$ is discontinuous at $x = 0$.

1.5.4 Question 4

Question. Show that the function f defined by $f(x) = \begin{cases} x^p \cos(\frac{1}{x}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ is continuous and differentiable at $x = 0$ but its derivative is not.

Answer. We know,

$$-x^p < x^p \cos\left(\frac{1}{x}\right) < x^p$$

Thus, the limit exists only if $p > 0$. Thus, f is continuous for $p > 0$. For differentiability,

$$\lim_{x \rightarrow 0} x^{p-1} \cos\left(\frac{1}{x}\right)$$

This limit exists only when $p > 1$. Thus, f is differentiable for $p > 1$.

$$f'(x) = \begin{cases} px^{p-1} \cos\left(\frac{1}{x}\right) + x^{p-2} \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

This is continuous for $p > 2$. Thus $f'(x)$ is continuous for $p > 2$.

1.5.5 Applications of IVT, Rolle's Theorem and MVT

Determine if the following equations admits solution in the interval mentioned

Question.

$$x^5 - 3x^2 = -1$$

in $[0, 1]$

Answer. Let,

$$f(x) = x^5 - 3x^2 + 1 \implies f(0) = 1 \wedge f(1) = -1$$

Since $f(0)f(1) < 0$, by Intermediate Value Theorem, there is a solution in the interval.

Question.

$$\sin^2 x - 2 \cos x = -1$$

in $[0, \frac{\pi}{2}]$

Answer. Let,

$$f(x) = \sin^2 x - 2 \cos x + 1 \implies f(0) = -1 \wedge f(\frac{\pi}{2}) = 2$$

Since $f(0)f(\frac{\pi}{2}) < 0$, by Intermediate Value Theorem, there is a solution in the interval.

Question.

$$(1 - x) \cos x = \sin$$

in $(0, 1)$

Answer. Let,

$$f(x) = (1 - x) \cos x - \sin x \implies f(0) = 1 \wedge f(1) = -\sin 1$$

Since $f(0)f(1) < 0$, by Intermediate Value Theorem, there is a solution in the interval.

Question. Show that a polynomial of odd degree has at-least one real root.

Answer. Consider a polynomial $f : \mathbb{R} \rightarrow \mathbb{R}$, since it is a polynomial of odd degree,

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \wedge \lim_{x \rightarrow \infty} f(x) = \infty$$

Thus, by Intermediate Value Theorem, $\exists c \in \mathbb{R}$ such that $f(c) = 0$.

Question. Does there exist a $c \in (1, 3)$ for the function $f(x) = x^3$ such that $f'(c) = 13$.

Answer. Using Lagrange Mean Value Theorem for f between the points 1 and 3,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = 13$$

Question. Show that $x^5 + 4x = 1$ has exactly one solution.

Answer. Let,

$$f(x) = x^5 + 4x - 1$$

Clearly, $f(0) = -1 \wedge f(1) = 4$, thus, there is one root in $(0, 1)$ (By Intermediate Value Theorem).

By way of contradiction, assume there to be two roots, $x = a, b$, i.e. $f(a) = f(b) = 0$, then by, Rolle's Theorem, $\exists c \in (a, b)$ such that,

$$f'(c) = 0$$

, but,

$$f'(x) = 4x^4 + 4$$

This has no real solutions. Thus, f can't have two roots and the root it has lies between 0 and 1.

Question. Show that $||\sin x| - |\sin y|| < |x - y|$.

Answer. Let,

$$f(x) = \sin x$$

By Lagrange Mean Value Theorem,

$$|\frac{f(x) - f(y)}{x - y}| = |f'(c)| = |\cos c| \leq 1$$

Thus, we can say that,

$$|\sin x - \sin y| \leq |x - y|$$

Using the triangle inequality,

$$||\sin x| - |\sin y|| \leq |\sin x - \sin y| \leq |x - y|$$

1.5.6 Problems on Hyperbolic Functions

Question. If $\cosh \alpha = \sec \theta$, then prove that $\tanh^2(\frac{\alpha}{2}) = \tan^2(\frac{\theta}{2})$.

Answer. Clearly,

$$\begin{aligned}\tanh^2\left(\frac{\alpha}{2}\right) &= \left(\frac{e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}} + e^{-\frac{\alpha}{2}}}\right)^2 \\ &= \frac{e^{\alpha} + e^{-\alpha} - 2}{e^{\alpha} + e^{-\alpha} + 2} \\ &= \frac{\sec \theta - 1}{\sec \theta + 1} \\ &= \frac{1 - \cos \theta}{1 + \cos \theta} \\ \therefore \tanh^2\left(\frac{\alpha}{2}\right) &= \tan^2\left(\frac{\theta}{2}\right)\end{aligned}$$

Question. If $\tan \theta = \tanh x \cot y$ and $\tan \phi = \tanh x \tan y$ then show that,

$$\frac{\sin 2\theta}{\sin 2\phi} = \frac{\cosh 2x + \cos 2y}{\cosh 2x - \cos 2y}$$

Question. If $\tan(\theta + i\phi) = \tan \alpha + i \sec \alpha$, then prove that $e^{2\phi} = \pm \cot \frac{\alpha}{2}$ and $2\theta = n\pi + \frac{\pi}{2} + \alpha$.

Answer.

$$\tan(\theta + i\phi) = \tan \alpha + i \sec \alpha \implies \tan(\theta - i\phi) = \tan \alpha - i \sec \alpha$$

Hence,

$$\begin{aligned}\tan((\theta + i\phi) + (\theta - i\phi)) &= \frac{\tan \alpha + i \sec \alpha + \tan \alpha - i \sec \alpha}{1 - (\tan \alpha + i \sec \alpha)(\tan \alpha - i \sec \alpha)} \\ \tan 2\theta &= \tan\left(\frac{\pi}{2} + \alpha\right) \\ \therefore 2\theta &= n\pi + \frac{\pi}{2} + \alpha \\ \tan((\theta + i\phi) - (\theta - i\phi)) &= \frac{(\tan \alpha + i \sec \alpha) - (\tan \alpha - i \sec \alpha)}{1 + (\tan \alpha + i \sec \alpha)(\tan \alpha - i \sec \alpha)} \\ \tan(2i\phi) &= i \cos \alpha \\ \frac{e^{2\phi} - e^{-2\phi}}{e^{2\phi} + e^{-2\phi}} &= \cos \alpha \\ \implies e^{4\phi} &= \cot^2 \frac{\alpha}{2} \\ \therefore e^{2\phi} &= \pm \cot \frac{\alpha}{2}\end{aligned}$$

1.5.7 Find the real and Imaginary parts of the following

Question.

$$\tan(\alpha + i\beta)$$

Answer.

$$\begin{aligned}\tan(\alpha + i\beta) &= \frac{\sin(\alpha + i\beta)}{\cos(\alpha + i\beta)} \\ &= \frac{\sin(\alpha + i\beta)}{\cos(\alpha + i\beta)} \times \frac{\cos(\alpha - i\beta)}{\cos(\alpha - i\beta)} \\ &= \frac{\sin 2\alpha + \sin 2i\beta}{\cos 2\alpha + \cos 2i\beta} \\ &= \frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta} + i \frac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}\end{aligned}$$

Question.

$$\frac{\cos(x+iy)}{(x+iy)+1}$$

Answer.

$$\begin{aligned}\frac{\cos(x+iy)}{(x+iy)+1} &= \frac{\cos x \cos iy - \sin x \sin iy}{(x+iy)+1} \times \frac{(x+iy)-1}{(x+iy)-1} \\ &= \frac{[(x+1)-iy](\cos x \cosh y - i \sin x \sinh y)}{(x+1)^2 + y^2}\end{aligned}$$

Question.

$$\sin^2(x+iy)$$

Answer.

$$\begin{aligned}\sin^2(x+iy) &= (\sin x \cos iy + \sin iy \cos x)^2 \\ &= (\sin x \cosh y + i \sinh y \cos x)^2 \\ &= (\sin^2 x \cosh^2 y - \sinh^2 y \cos^2 x) + i(2 \sin x \sinh y \cos x \cosh y)\end{aligned}$$

Question.

$$\frac{e^{i\theta}}{1 - ke^{i\theta}}$$

Answer.

$$\begin{aligned}\frac{e^{i\theta}}{1 - ke^{i\theta}} &= \frac{\cos \theta + i \sin \theta}{(1 - k \cos \theta) - i(k \sin \theta)} \\ &= \frac{\cos \theta - k + i \sin \theta}{k^2 - 2k \cos \theta + 1}\end{aligned}$$

Question.

$$e^{\sin(x+iy)}$$

Answer.

$$\begin{aligned}e^{\sin(x+iy)} &= e^{\sin x \cosh y} e^{i(\cos x \sinh y)} \\ &= e^{\sin x \cosh y} (\cos(\cosh x \sin y) + i \sin(\cosh x \sin y))\end{aligned}$$

1.5.8 Problems on Inverse Hyperbolic Functions

Question. Express $\cos^{-1}(x+iy)$ in the form of $A+iB$.

Answer.

$$\cos^{-1}(x+iy) = A+iB \implies \cos^{-1}(x-iy) = A-iB$$

Thus, we can say,

$$\begin{aligned}2A &= \cos^{-1}(x+iy) + \cos^{-1}(x-iy) \\ \therefore A &= \frac{1}{2} \cos^{-1} \left[x^2 + y^2 - \sqrt{(1-x^2+y^2)^2 + 4x^2y^2} \right] \\ 2iB &= \cos^{-1}(x+iy) - \cos^{-1}(x-iy) \\ \therefore B &= \frac{1}{2i} \cos^{-1} \left[x^2 + y^2 + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2} \right] = \cosh^{-1} \left[x^2 + y^2 + \sqrt{(1-x^2+y^2)^2 + 4x^2y^2} \right]\end{aligned}$$

Question. Express $\tanh^{-1}(x+iy)$ into the form $\alpha+i\beta$ and hence deduce the value of $\tanh^{-1}iy$.

Answer.

$$\tanh^{-1}(x+iy) \implies -y+ix = \tan(-\beta+i\alpha) \wedge -y-ix = \tan(-\beta-i\alpha)$$

Thus,

$$\begin{aligned}\tan 2\beta &= -\tan [(-\beta + i\alpha) + (-\beta - i\alpha)] \\ \therefore \beta &= \frac{1}{2} \tan^{-1} \frac{2y}{1 - x^2 - y^2} \\ \tan 2i\alpha &= \tan [(-\beta + i\alpha) - (-\beta - i\alpha)] \\ \Rightarrow i \tanh \alpha &= \frac{2ix}{1 + x^2 + y^2} \\ \therefore \alpha &= \frac{1}{2} \tanh^{-1} \frac{2x}{1 + x^2 + y^2}\end{aligned}$$

Thus, if $x = 0$, then

$$\tanh^{-1} iy = \frac{i}{2} \tan^{-1} \frac{2y}{1 - y^2} = i \tan^{-1} y$$

Question. Show that $\sinh^{-1}(\cot x) = \ln(\cot x + \csc x)$.

Answer.

$$\begin{aligned}\sinh^{-1}(\cot x) &= \ln\left(\cot x + \sqrt{\cot^2 x + 1}\right) \\ &= \ln(\cot x + \csc x)\end{aligned}$$

Question. Show that the general term of $\tan^{-1}(x + iy)$ is $n\pi + \tan^{-1}(x + iy) = n\pi + \frac{1}{2} \tan^{-1} \frac{2x}{1 - x^2 - y^2} + \frac{i}{2} \tanh^{-1} \frac{2y}{1 + x^2 + y^2}$.

Answer. Clearly,

$$\tan y = x \implies \tan(y + n\pi) = x \implies n\pi + y = \tan^{-1} x$$

Thus, it follows that (from last question),

$$\tan^{-1}(x + iy) = n\pi + \tan^{-1}(x + iy) = n\pi + \frac{1}{2} \tan^{-1} \frac{2x}{1 - x^2 - y^2} + \frac{i}{2} \tanh^{-1} \frac{2y}{1 + x^2 + y^2}$$

Question. If $\sin^{-1}(\theta + i\phi) = \alpha + i\beta$, then prove that $\sin^2 \alpha$ and $\cosh^2 \beta$ are the roots of the equation $x^2 - x(1 + \theta^2 + \phi^2) + \theta^2 = 0$.

Answer.

$$\begin{aligned}\sin(\alpha + i\beta) &= \theta + i\phi \\ \implies \sin \alpha \cosh \beta + i \sinh \beta \cos \alpha &= \theta + i\phi\end{aligned}$$

Thus,

$$\theta^2 + \phi^2 + 1 = \sin^2 \alpha \cosh^2 \beta + \sinh^2 \beta \cos^2 \alpha + 1 = \sin^2 \alpha + \cosh^2 \beta$$

Moreover,

$$\theta^2 = \sin^2 \alpha \cosh^2 \beta$$

Consider a quadratic with roots $\sin^2 \alpha$ and $\cosh^2 \beta$,

$$x^2 - (\sin^2 \alpha + \cosh^2 \beta)x + \sin^2 \alpha \cosh^2 \beta = 0 \implies x^2 - x(1 + \theta^2 + \phi^2) + \theta^2 = 0$$

1.5.9 Separate the real and imaginary parts of the following

Question.

$$\cos^{-1}(\cos \beta + i \sin \beta)$$

Answer. Let,

$$\cos^{-1}(\cos \beta + i \sin \beta) = x + iy$$

Thus,

$$\begin{aligned}\cos x \cosh y &= \cos \beta \\ -\sin x \sinh y &= \sin \beta\end{aligned}$$

On solving we get,

$$\cos^{-1}(\cos \beta + i \sin \beta) = \sin^{-1} \sqrt{\sin \beta} + i \ln(\sqrt{1 + \sin \beta} - \sqrt{\sin \beta})$$

Question.

$$\tan^{-1}(\cos \beta + i \sin \beta)$$

Answer. Let,

$$\tan^{-1}(\cos \beta + i \sin \beta) = x + iy$$

Thus,

$$\cos \beta + i \sin \beta = \tan(x + iy) \implies \cos \beta - i \sin \beta = \tan(x - iy)$$

$$\tan 2x = \tan[(x + iy) + (x - iy)]$$

$$\implies x = \frac{\pi}{4}$$

$$\tan 2iy = \tan[(x + iy) - (x - iy)]$$

$$\implies 2y = \tanh^{-1} \sin \beta$$

$$\implies y = \frac{1}{2} \ln \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right) = \frac{1}{2} \ln \cot\left(\frac{\pi}{4} - \frac{\beta}{2}\right)$$

Hence^a,

$$\tan^{-1}(\cos \beta + i \sin \beta) = \frac{\pi}{4} + \frac{i}{2} \ln \cot\left(\frac{\pi}{4} - \frac{\beta}{2}\right)$$

^aCan't get the $\frac{n\pi}{2}$ term

Question.

$$\sin^{-1}(\cos \beta + i \sin \beta)$$

Answer. Let,

$$\sin^{-1}(\cos \beta + i \sin \beta) = x + iy$$

Thus,

$$\sin x \cosh y = \cos \beta$$

$$\cos x \sinh y = \sin \theta$$

On solving we get,

$$\sin^{-1}(\cos \beta + i \sin \beta) = \cos^{-1} \sqrt{\sin \beta} + i \ln(\sqrt{1 + \sin \beta} + \sqrt{\sin \beta})$$

Question. Show that $\sin^{-1} \csc \theta = [2n + (-1)^n] \frac{\pi}{2} + i(-1)^n \ln \cot \frac{\theta}{2}$

Question. If $\cosh^{-1}(x + iy) + \cosh^{-1}(x - iy) = \cosh^{-1} 1a$, show that $2(a - 1)x^2 + 2(a + 1)y^2 = a^2 - 1$

Answer. Let,

$$\cosh^{-1}(x + iy) = \alpha + i\beta \implies \cosh^{-1}(x - iy) = \alpha - i\beta$$

Thus,

$$2\alpha = \cosh^{-1} a \implies \cosh^2 \alpha = \frac{a+1}{2} \wedge \sinh^2 \alpha = \frac{a-1}{2}$$

Also,

$$x + iy = \cosh(\alpha + i\beta)$$

$$\implies x + iy = \cosh \alpha \cos \beta + i \sinh \alpha \sin \beta$$

Thus, solving,

$$\begin{aligned}\cos^2 \beta + \sin^2 \beta &= 1 \\ \Rightarrow \left(\frac{x}{\cosh \alpha} \right)^2 + \left(\frac{y}{\sinh \alpha} \right)^2 &= 1 \\ x^2 \sinh^2 \alpha + y^2 \cosh^2 \alpha &= \cosh^2 \alpha \sinh^2 \alpha \\ \therefore 2(a-1)x^2 + 2(a+1)y^2 &= a^2 - 1\end{aligned}$$

1.6 Successive Differentiation

Consider a function $y = f(x)$, then the notation for the n^{th} order derivatives is

$$\begin{aligned} y_1, y_2, \dots, y_n \\ y', y'', y''', y^{(iv)} \dots y^{(n)} \\ \frac{dy}{dx}, \frac{d^2y}{dx^2} \dots \frac{d^ny}{dx^n} \\ D_x, \dots D^n x \end{aligned}$$

1.6.1 n^{th} Order Derivatives of Common Functions

$$y = x^m \implies y^{(n)} = {}^mP_n x^{m-n} \quad (m > n)$$

$$y = \frac{1}{ax+b} \implies y^{(n)} = (-1)^n \frac{n!a^n}{(ax+b)^{n+1}}$$

$$y = a^{mx} \implies y^{(n)} = (m \ln a)^n a^{mx}$$

$$y = \sin(ax+b) \implies y^{(n)} = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)$$

$$y = \cos(ax+b) \implies y^{(n)} = a^n \cos\left(ax+b+n\frac{\pi}{2}\right)$$

$$y = e^{ax} \sin(bx+c) \implies y^{(n)} = (a^2+b^2)^{\frac{n}{2}} e^{ax} \sin\left(bx+c+n \tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$y = \ln(ax+b) \implies y^{(n)} = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

Example. Find the n^{th} derivative of $y = \tan^{-1}\left(\frac{x}{a}\right)$.

Solution.

$$\begin{aligned} y_1 &= \frac{a}{x^2+a^2} \\ &= \frac{1}{2i} \left[\frac{1}{x-ia} - \frac{1}{x+ia} \right] \\ \therefore y_n &= \frac{(-1)^{n-1}(n-1)!}{2i} \left[\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n} \right] \end{aligned}$$

Here, let $x = r \cos \theta$ and $a = r \sin \theta$,

$$\implies \frac{1}{(x-ia)^n} = r^{-n} e^{in\theta} \wedge \frac{1}{(x+ia)^n} = r^{-n} e^{-in\theta}$$

$$\begin{aligned} y_n &= (-1)^{n-1}(n-1)!r^{-n} \sin n\theta \\ &= (-1)^{n-1}(n-1)!a^{-n} \sin^n \theta \sin n\theta \\ &= (-1)^{n-1}(n-1)!(x^2+a^2)^{-\frac{n}{2}} \left(\frac{a}{\sqrt{x^2+a^2}} \right)^n \sin \left(n \sin^{-1} \left(\frac{a}{\sqrt{x^2+a^2}} \right) \right) \end{aligned}$$

Theorem 10 (Leibniz Theorem). If u and v are differentiable functions of the order n , then,

$$y = uv \implies \frac{d^ny}{dx^n} = \sum_{k=0}^n {}^nC_k u_{n-k} v_k$$

Proof. By Mathematical Induction □

1.7 Tutorial-2

1.7.1 Solve the following

Question. Differentiate $x^{\sin x}$ w.r.t. $(\sin x)^x$.

Answer. Let,

$$\begin{array}{l} u = x^{\sin x} \\ \ln u = \sin x \ln x \\ \Rightarrow \frac{du}{dx} = x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right) \end{array} \quad \bigg| \quad \begin{array}{l} v = (\sin x)^x \\ \Rightarrow \frac{dv}{dx} = (\sin x)^x (\ln \sin x + x \cot x) \end{array}$$

Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{du}{dx}}{\frac{dv}{dx}} \\ \therefore \frac{dy}{dx} &= \frac{x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right)}{(\sin x)^x (\ln \sin x + x \cot x)} \end{aligned}$$

Question. Differentiate $\ln(x)^{\tan x}$ w.r.t. $\sin(m \cos^{-1} x)$.

Answer. Let,

$$\begin{array}{l} u = \ln(x)^{\tan x} \\ \Rightarrow \frac{du}{dx} = (\ln x)^{\tan x} \left[\frac{\tan x}{x \ln x} + \sec^2 x \ln(\ln x) \right] \end{array} \quad \bigg| \quad \begin{array}{l} v = \sin(m \cos^{-1} x) \\ \Rightarrow \frac{dv}{dx} = -\frac{m \cos(m \cos^{-1} x)}{\sqrt{1-x^2}} \end{array}$$

Thus,

$$\begin{aligned} \frac{dy}{dx} &= \frac{\frac{du}{dx}}{\frac{dv}{dx}} \\ \therefore \frac{dy}{dx} &= -\frac{\sqrt{1-x^2} (\ln x)^{\tan x} \left[\frac{\tan x}{x \ln x} + \sec^2 x \ln(\ln x) \right]}{m \cos(m \cos^{-1} x)} \end{aligned}$$

Question. If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx} = \frac{\ln x}{(1+\ln x)^2}$.

Answer.

$$\begin{aligned} x^y &= e^{x-y} \\ y &= \frac{x}{\ln x + 1} \\ \Rightarrow \frac{dy}{dx} &= \frac{\ln x}{(1 + \ln x)^2} \end{aligned}$$

Question. If $y = x^{x^{\dots \infty}}$, then show that $x \frac{dy}{dx} = \frac{y^2}{1-y \ln x}$.

Answer.

$$\begin{aligned} y &= x^y \\ \ln y &= y \ln x \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= \ln x \frac{dy}{dx} + \frac{y}{x} \\ \therefore x \frac{dy}{dx} &= \frac{y^2}{1 - y \ln x} \end{aligned}$$

Question. If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \dots \infty}}}$, then show that $(2y-1) \frac{dy}{dx} + \sin x = 0$.

Answer.

$$\begin{aligned}y &= \sqrt{\cos x + y} \\y^2 - y - \cos x &= 0 \\ \implies (2y - 1) \frac{dy}{dx} + \sin x &= 0\end{aligned}$$

Question. If $y = x^3 \sin ax$ then find $\frac{d^3y}{dx^3}$.

Answer. Using Leibniz Theorem,

$$y_3 = -a^3 x^3 \cos ax - 9a^2 x^2 \sin ax + 18ax \cos x + 6 \sin ax = (6 - 9a^2 x^2) \sin ax + (18ax - a^3 x^3) \cos ax$$

Question. If $x = a(\theta + \sin \theta)$, $y = a(1 + \cos \theta)$ then calculate $\frac{d^2y}{dx^2} \Big|_{\theta=\frac{\pi}{2}}$.

Answer.

$$\begin{aligned}x &= a(\theta + \sin \theta) & y &= a(1 + \cos \theta) \\ \implies \frac{dx}{d\theta} &= a(1 + \cos \theta) & \implies \frac{dy}{d\theta} &= -a \sin \theta\end{aligned}$$

Thus,

$$\frac{dy}{dx} = -\frac{\sin \theta}{1 + \cos \theta} = -\tan \frac{\theta}{2}$$

Now,

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d \frac{dy}{dx}}{d\theta} \frac{1}{\frac{dx}{d\theta}} \\ &= \frac{-1}{2a} \frac{\sec^2 \frac{\theta}{2}}{1 + \cos \theta} \\ \therefore \frac{d^2y}{dx^2} \Big|_{\theta=\frac{\pi}{2}} &= \frac{2 - \sqrt{2}}{a}\end{aligned}$$

1.7.2 Here $y_n = \frac{d^n y}{dx^n}$ in the following parts:

Question. If $y = (\sin^{-1} x)^2$, then show that $(1 - x^2)y_2 - xy_1 = 2$.

Answer.

$$\begin{aligned}y &= (\sin^{-1} x)^2 \\ \implies \frac{1}{2} \frac{1}{\sqrt{y}} y_1 &= \frac{1}{\sqrt{1 - x^2}} \\ (1 - x^2)(y_1)^2 &= 4y \\ \implies (-2x)(y_1)^2 + (1 - x^2)(2y_1)y_2 &= 4y_1 \\ \therefore (1 - x^2)y_2 - xy_1 &= 2\end{aligned}$$

Question. If $y = e^{ax} \sin(bx)$, then show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$.

Answer.

$$\begin{aligned}y_1 &= ae^{ax} \sin(bx) + be^{ax} \cos(bx) \\ &= ay + be^{ax} \cos(bx) \\ \implies y_2 &= ay_1 + b[ae^{ax} \cos(bx) - be^{ax} \sin(bx)] \\ &= 2ay_1 + (a^2 + b^2)y \\ \therefore y_2 - 2ay_1 + (a^2 + b^2)y &= 0\end{aligned}$$

Question. If $y = [\log(x + \sqrt{1+x^2})]^2$, prove that $(1-x^2)y_2 + xy_1 = 2$.

Answer.

$$\begin{aligned} y &= [\log(x + \sqrt{1+x^2})]^2 \\ \Rightarrow y_1 &= \frac{2y}{\sqrt{1+x^2}} \\ (1+x^2)(y_1)^2 &= 4y \\ \therefore (1-x^2)y_2 + xy_1 &= 2 \end{aligned}$$

Question. If $y^3 + x^3 - 3axy = 0$, show that $\frac{d^2y}{dx^2} = \frac{-2a^3xy}{(y^2-ax)^3}$.

Answer.

Question. If $x = \sin t$ $y = \sin pt$, prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$

Answer. Clearly, the function is,

$$\begin{aligned} y &= \sin(p \sin^{-1} x) \\ \sin^{-1} y &= p \sin^{-1} x \\ \Rightarrow \frac{y_1}{\sqrt{1-y^2}} &= \frac{p}{\sqrt{1-x^2}} \\ (1-x^2)(y_1)^2 &= p^2(1-y^2) \\ \Rightarrow 2(1-x^2)y_1y_2 + (-2x)(y_1)^2 &= p^2(-2y)y_1 \\ \therefore 2(1-x^2) - xy_1 + p^2y &= 0 \end{aligned}$$

Question. $y = e^{-x} \cos x$, then show that $y_4 + 4y = 0$.

Answer.

$$\begin{aligned} y &= e^{-x} \cos x \\ y_4 &= -e^{-x} \cos x \quad (\text{Leibniz Theorem}) \\ \therefore y_4 + 4y &= 0 \end{aligned}$$

1.7.3 Find the n^{th} derivative of the following functions

Question.

$$y = \cos^3 x$$

Answer.

$$\begin{aligned} y &= \cos^3 x \\ &= \frac{\cos 3x + 3 \cos x}{4} \\ \therefore \frac{d^n y}{dx^n} &= \frac{3^n \cos(3x + n\frac{\pi}{2}) + 3 \cos(x + n\frac{\pi}{2})}{4} \end{aligned}$$

Question.

$$y = e^{2x} \sin^3 x$$

Answer.

$$\frac{d^n y}{dx^n} = \sqrt{13}^n e^{2x} \sin\left(3x + n\frac{3}{2}\right)$$

Question.

$$y = \cos x \cos 2x \cos 3x$$

Answer.

$$y = \frac{1}{4}(\cos 6x + \cos 4x + \cos 2x + 1)$$
$$\therefore \frac{d^n y}{dx^n} = \frac{1}{4}(6^n \cos\left(6x + n\frac{\pi}{2}\right) + 4^n \cos\left(4x + n\frac{\pi}{2}\right) + 2^n \cos\left(2x + n\frac{\pi}{2}\right))$$

Question.

$$y = \frac{x^3}{x^2 - 1}$$

Answer.

$$y = x + \frac{1}{2}\left(\frac{1}{x+1} + \frac{1}{x-1}\right)$$
$$\therefore \frac{d^n y}{dx^n} = \begin{cases} 1 + \frac{(-1)^n}{2}\left(\frac{n!}{(x+1)^{n+1}} + \frac{n!}{(x-1)^{n+1}}\right) & , n = 1 \\ \frac{(-1)^n}{2}\left(\frac{n!}{(x+1)^{n+1}} + \frac{n!}{(x-1)^{n+1}}\right) & , n > 1 \end{cases}$$

Question.

$$y = \frac{x^2}{(x-1)^3(x-2)}$$

Answer.

$$y = \frac{-4}{x-1} + \frac{-3}{(x-1)^2} + \frac{-1}{(x-1)^3} + \frac{4}{x-2}$$
$$\therefore y_n = \frac{(-1)^n n!}{(x-1)^n} \left(\frac{-4}{x-1} + \frac{-3}{(x-1)^2} + \frac{-1}{(x-1)^3} \right) + 4 \frac{(-1)^n n!}{(x-2)^{n+1}}$$

Question.

$$y = a^x \sin x$$

Answer.

$$y = a^x \sin x$$
$$\Rightarrow y_n = a^x ((\ln x)^2 + 1)^{\frac{n}{2}} \sin\left(x + n \tan^{-1} \frac{1}{\ln x}\right)$$

Question.

$$y = \frac{1}{(x^2 - a^2)(x^2 - b^2)}$$

Answer.

$$y = \frac{1}{a^2 - b^2} \left(\frac{1}{x^2 - a^2} - \frac{1}{x^2 - b^2} \right)$$
$$= \frac{1}{a^2 - b^2} \left[\frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right) + \frac{1}{2b} \left(\frac{1}{x-b} - \frac{1}{x+b} \right) \right]$$
$$\therefore y_n = \frac{(-1)^n n!}{a^2 - b^2} \left[\frac{1}{2a} \left(\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}} \right) + \frac{1}{2b} \left(\frac{1}{(x-b)^{n+1}} - \frac{1}{(x+b)^{n+1}} \right) \right]$$

Question.

$$y = \ln(x^2 - a^2)$$

Answer.

$$y_n = (-1)^n (n-1)! \left[\frac{1}{(a+x)^n} + \frac{1}{(a-x)^n} \right]$$

1.7.4 Question 4

Question. If $A = \sin nx + \cos nx$, then prove that $A_p = n^p[1 + (-1)^n \sin 2nx]^{\frac{1}{2}}$, where A_p denotes the p^{th} derivative of A w.r.t x .

Answer.

$$\begin{aligned} A &= \sin nx + \cos nx \\ &= \sqrt{2} \sin\left(nx + \frac{\pi}{4}\right) \\ \Rightarrow A_p &= \sqrt{2} n^p \sin\left(nx + \frac{\pi}{4} + p\frac{\pi}{2}\right) \\ &= n^p[1 + (-1)^n \sin 2nx]^{\frac{1}{2}} \end{aligned}$$

1.7.5 Use Leibniz Theorem to solve the n^{th} order derivative of the following

Question.

$$y = e^x \ln x$$

Answer.

$$y_n = e^x \left[\ln x + \sum_{k=1}^n {}^nC_k \frac{(-1)^{k+1}}{x^k} \right]$$

Question.

$$y = x^2 \sin x$$

Answer. x^2 only has 2 derivatives before it is 0, thus,

$$\begin{aligned} y_n &= {}^nC_0 x^2 \sin\left(x + n\frac{\pi}{2}\right) + {}^nC_1 (2x) \sin\left(x + (n-1)\frac{\pi}{2}\right) + {}^nC_2 (2) \sin\left(x + (n-2)\frac{\pi}{2}\right) \quad (\text{Rest all terms are 0}) \\ &= x^2 \sin\left(x + n\frac{\pi}{2}\right) - 2nx \cos\left(x + n\frac{\pi}{2}\right) - n(n-1) \sin\left(x + n\frac{\pi}{2}\right) \end{aligned}$$

Question.

$$y = \frac{\ln x}{x}$$

Answer.

$$\begin{aligned} y_n &= {}^nC_0 \ln x \frac{(-1)^n n!}{x^{n+1}} + {}^nC_1 \frac{1}{x} \frac{(-1)^{n-1} (n-1)!}{x^n} + {}^nC_2 \frac{-1}{x^2} \frac{(-1)^{n-2} (n-2)!}{x^{n-1}} + \dots \\ &= \frac{(-1)^n n!}{x^{n+1}} \left(\ln x - \sum_{k=1}^n \frac{1}{k} \right) \end{aligned}$$

Question.

$$y = x^n \ln x$$

Answer.

$$y_n = n! \left(\ln x + \sum_{k=1}^n \frac{1}{k} \right)$$

Proof by induction.

Question. If $y = [\ln(x + \sqrt{1+x^2})]^2$, then prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$.

Answer. We know,

$$(1+x^2)y_2 + xy_1 = 2$$

Using Leibniz Theorem,

$$(1+x^2)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2}2y_n + xy_{n+1} + ny_n = 0$$

$$\therefore (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$$

Question. If $y = e^{m \sin^{-1} x}$, then prove the following,

1. $(1-x^2)y_2 - xy_1 = m^2y$
2. $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

Answer.

$$\ln y = m \sin^{-1} x$$

$$(1-x^2)(y_1)^2 = m^2y^2$$

$$\implies (-2x)(y_1)^2 + (1-x^2)(2y_1y_2) = m^2(2yy_1)$$

$$\therefore (1-x^2)y_2 - xy_1 = m^2y$$

Using Leibniz Theorem,

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - xy_{n+1} - ny_n = m^2y$$

$$\therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$$

Question. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

Answer.

$$y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$$

$$\implies \left(y^{\frac{1}{m}} - y^{\frac{-1}{m}}\right)y_1 = 2my$$

$$(x^2-1)(y_1)^2 = m^2y^2 \quad (\text{Squaring})$$

$$\therefore (x^2-1)y_2 + xy_1 = m^2y$$

Using Leibniz Theorem,

$$(x^2-1)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2}(2)y_n + xy_{n+1} + ny_n = m^2y_n$$

$$\therefore (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$$

Question. If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0$.

Answer.

$$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

$$\implies (1-x^2)y_1 - xy = 1$$

$$\implies (1-x^2)y_2 - 3xy_1 = y$$

Using Leibniz Theorem,

$$(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - 3(xy_{n+1} + ny_n) = y_n$$

$$\therefore (1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0$$

Question. If $x = \sin\left(\frac{\ln y}{a}\right)$, prove that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.

Answer.

$$\begin{aligned}x &= \sin\left(\frac{\ln y}{a}\right) \\ \sin^{-1} x &= \frac{\ln y}{a} \\ \implies (1-x^2)(y_1)^2 &= a^2 y^2 \\ \therefore (1-x^2)y_2 - xy_1 &= a^2 y\end{aligned}$$

Using Leibniz Theorem,

$$\begin{aligned}(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - xy_{n+1} - ny_n &= a^2 y_n \\ \therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n &= 0\end{aligned}$$

1.7.6 If $y = (\sin^{-1} x)^2$, prove that,

Question.

$$(1-x^2)y_2 - xy_1 - 2 = 0$$

Answer.

$$\begin{aligned}y &= (\sin^{-1} x)^2 \\ \implies (1-x^2)(y_1)^2 &= y(\sin^{-1} x)^2 = 4y \\ \therefore (1-x^2)y_2 - xy_1 - 2 &= 0\end{aligned}$$

Question.

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$$

Answer. Using Leibniz Theorem,

$$\begin{aligned}(1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - xy_{n+1} - ny_n &= 0 \\ \therefore (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n &= 0\end{aligned}$$

Question. Deduce that $\lim_{x \rightarrow 0} \frac{y_{n+2}}{y_n} = n^2$ and find $y_n(0)$.

Answer. Put $x = 0$,

$$\lim_{x \rightarrow 0} \frac{y_{n+2}}{y_n} = n^2$$

Clearly,

$$y(0) = 0 \wedge y_1(0) = 0$$

Also,

$$y_2 = 2$$

Hence,

$$y_3(0) = y_5(0) = \dots = y_1(0) = 0$$

Moreover,

$$\begin{aligned}y_4 &= 2^2 y_2 = 2^2 \times 2 \\ y_6 &= 4^2 y_4 = 4^2 \times (2^2 \times 2) \\ \therefore y_{2n} &= 2 \prod_{n=1}^n (2n-2)^2\end{aligned}$$

Thus, we can say,

$$y_n = \begin{cases} 0 & , \text{ if } n \text{ is odd} \\ 2 \prod_{n=1}^n (2n-2)^2 & , \text{ if } n \text{ is even} \end{cases}$$

1.7.7 Expand the following functions in power of x by Maclaurin's Theorem OR write the series expansion of function about the point $x = 0$

Question.

$$a^x$$

Answer.

$$f(x) = a^x = \sum_{n=0}^{\infty} \frac{(x \ln a)^n}{n!}$$

Question.

$$e^{ax} \cos bx$$

Answer.

$$f(x) = e^{ax} \cos bx = \sum_{n=0}^{\infty} \frac{x^n f^{(n)}(x)}{n!} = \sum_{n=0}^{\infty} e^{ax} \frac{(\sqrt{a^2 + b^2} x)^n}{n!} \cos\left(bx + n \tan^{-1}\left(\frac{b}{a}\right)\right)$$

Question.

$$\ln(2x + 4)$$

Answer.

$$f(x) = \ln(2x + 4) = \ln 4 + \sum_{n=1}^{\infty} \frac{(-2x)^n}{(4)^{n+1}}$$

Question.

$$\cos(3x + 4)$$

Answer.

$$f(x) = \cos(3x + 2) = \sum_{n=0}^{\infty} \frac{x^n \cos\left(2 + n\frac{\pi}{2}\right)}{n!}$$

1.7.8 Using Taylor's Theorem

Question. e^x in power of $x = 1$.

Answer.

$$f(x) = e^x = e \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$$

Question. $4x^2 + 7x + 5$ in power of $x + 2$.

Answer.

$$\begin{aligned} f(x) &= 4x^2 + 7x + 5 \implies f(-2) = 7 \\ f'(x) &= 8x + 7 \implies f'(-2) = -9 \\ f''(x) &= 8 \implies f''(-2) = 8 \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= 4x^2 + 7x + 5 \\ &= 7 + (-9) \frac{(x+2)}{1} + (8) \frac{(x+2)^2}{2} \\ \therefore f(x) &= 4(x+2)^2 - 9(x+2) + 7 \end{aligned}$$

Question. Expand $\tan x$ in powers of $x - \frac{\pi}{4}$ upto first four terms.

Answer.

$$f(x) = \tan x = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

Question. Show that $\sin\left(\frac{\pi}{4} + x\right) = \frac{1}{\sqrt{2}} \left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)$.

Answer. We can write the Maclaurin Series for the function,

$$f(x) = \sin\left(x + \frac{\pi}{4}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \sin\left(\frac{\pi}{4} + n\frac{\pi}{2}\right) = \frac{1}{\sqrt{2}} \sum_{n=0}^{\infty} (-1)^{\lfloor \frac{n}{2} \rfloor} \frac{x^n}{n!}$$

Question.

$$\ln(1 - x + x^2) = \ln\left(\frac{1 + x^3}{1 + x}\right) = -x + \frac{x^2}{2} + \frac{2x^3}{3} + \dots$$

Answer. First,

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$$

Thus,^a

$$\begin{aligned} f(x) &= \ln(1 - x + x^2) \\ &= \ln(1 + x^3) - \ln(1 + x) \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n} - \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n} \\ \therefore f(x) &= -x + \frac{x^2}{2} + \frac{2x^3}{3} + \dots \end{aligned}$$

^afind answer in sigma notation

1.7.9 Question 9

Question. If $f(x) = x^3 + 2x^2 - 5x + 11$ find the values of $f\left(\frac{9}{10}\right)$, using Taylor Series for $f(x + h)$.

Answer. Clearly,

$$\begin{aligned} f(0.9) &= f(1 - 0.1) \\ &= f(1) + \frac{(-0.1)}{1!} f'(1) + \frac{(-0.1)^2}{2!} f''(1) + \frac{(-0.1)^3}{3!} f'''(1) \\ &= 9 - 0.2 + 0.05 - 0.001 \\ &= 8.849 \end{aligned}$$

Question. If $f(x) = x^3 - 2x + 5$, then find the value of $f(2.001)$. Find the approximate change in the value of $f(x)$ when x changes from 2 to 2.001.

Answer.

$$\begin{aligned} f(2.001) &= f(2 + 0.001) \\ &= f(2) + \frac{(0.001)}{1!} f'(2) + \frac{(0.001)^2}{2!} f''(2) + \frac{(0.001)^3}{3!} f'''(2) \\ &= 9.010006001 \end{aligned}$$

Thus, the approximate change is,

$$f(2.001) - f(2) = 0.010006001$$

Question. Show that $f\left(\frac{x^2}{1+x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x}{1+x}\right)^n f^{(n)}(x)$.

Answer. We know,

$$\frac{x^2}{1+x} = x - \frac{x}{1+x}$$

Thus, using Taylor's Theorem for $f(x - \frac{x}{1+x})$, we get,

$$f\left(x - \frac{x}{1+x}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{x}{1+x}\right)^n f^{(n)}(x)$$

1.8 Taylor's Theorem ²

If $f, f'', \dots, f^{(n)}$ all exist and are continuous on an interval (a, b) .

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad (\xi \in (a, x))$$

Here, $R_n(x) = \frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)$ is the remainder term. It denotes the error in the expansion of the function. It can also be written as,

$$\frac{(x-a)^n}{n!}f^{(n)}(a) + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(a + \theta(x-a)), \quad (\theta \in (0, 1])$$

If $\lim_{n \rightarrow \infty} R_n(x) \rightarrow 0$, then, Taylor's theorem becomes Taylor Series.

The Taylor Series expansion of f about $x = a$ is,

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!} f^{(n)}(a)$$

1.8.1 Maclaurin Series ³

If $a = 0$, then Taylor Series is known as Maclaurin Series.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0)$$

$$f(x+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(x) \quad (\text{Can be used to calculate errors})$$

Example. Find the Taylor Series Expansion of $\sin x$ about the point $x = \frac{\pi}{2}$.

Solution.

$$\begin{aligned} \sin x &= 1 - \frac{(x - \frac{\pi}{2})^2}{2!} + \frac{(x - \frac{\pi}{2})^4}{4!} - \cdots \\ &= \sum_{n=0}^{\infty} \frac{(x - \frac{\pi}{2})^{2n}}{(2n)!} \end{aligned}$$

Example. If $f(x) = 4x^5 + 3x^2 + 9$, then find the expansion about $x = 1$. Also, evaluate $f(1.01)$.

Solution.

$$\begin{aligned} f(x) &= f(1) + \frac{(x-1)}{1!}f'(1) + \frac{(x-1)^2}{2!}f''(1) \cdots + \frac{(x-1)^n}{n!}f^{(n)}(1) \\ &= 16 + \frac{(x-1)}{1}(26) + \frac{(x-1)^2}{2!}(86) + \frac{(x-1)^3}{3!}(240) + \frac{(x-1)^4}{4!}(480) + \frac{(x-1)^5}{5!}(480) \end{aligned}$$

1.9 Hyperbolic Functions

These functions are called hyperbolic functions because $(\cosh x, \sinh x)$ satisfy the equation of a hyperbola⁴. The hyperbolic functions are-

- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\tanh x = \frac{\sinh x}{\cosh x}$
- $\operatorname{csch} x = \frac{1}{\sinh x}$
- $\operatorname{sech} x = \frac{1}{\cosh x}$
- $\operatorname{coth} x = \frac{1}{\tanh x}$

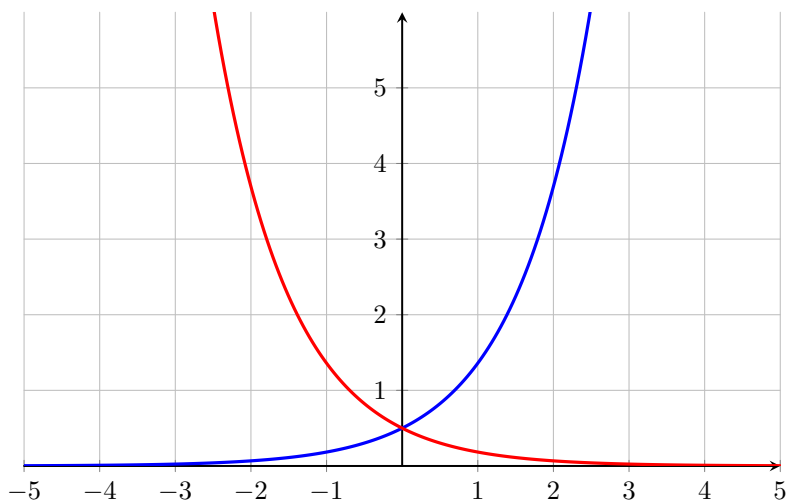
²By Brook Taylor

³By Colin Maclaurin

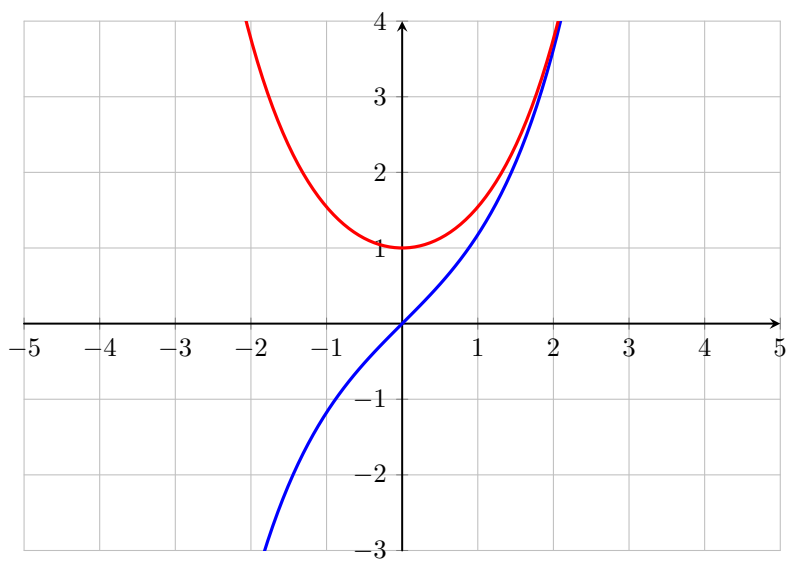
⁴Hence, $\cosh^2 x - \sinh^2 x = 1$

1.9.1 Graphs

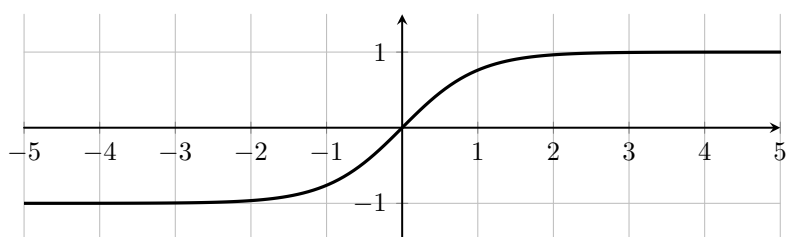
First, we observe, Now, we see,



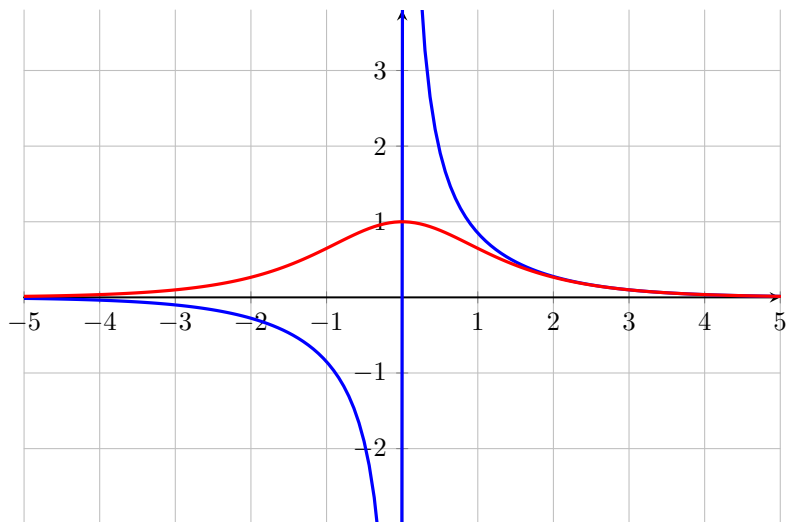
Graphs of $\frac{e^x}{2}$ and $\frac{e^{-x}}{2}$



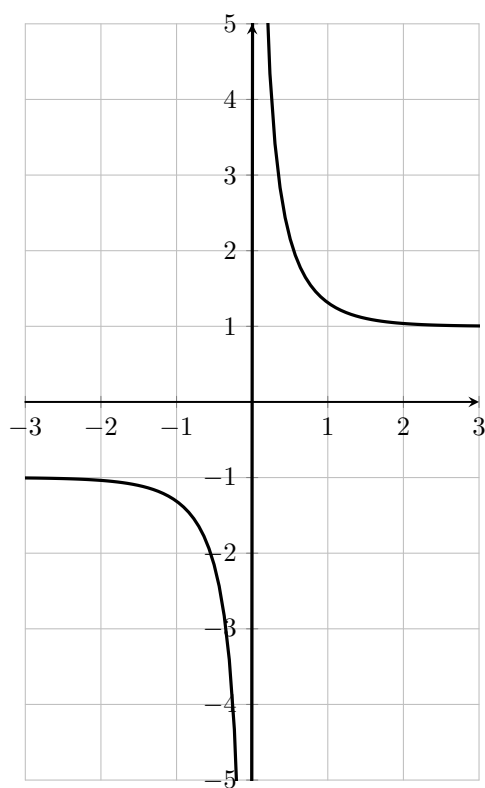
Graphs of $\sinh x$ and $\cosh x$



Graph of $\tanh x$



Graphs of $\operatorname{csch} x$ and $\operatorname{sech} x$



Graph of $\operatorname{coth} x$

1.9.2 Identities of Conversion

$$\sin(ix) = i \sinh x$$

$$\cos(ix) = \cosh x$$

$$\tan(ix) = i \tanh x$$

For any identity $f(\sin x, \cos x)$, $f(i \sinh x, \cosh x)$ is also valid.

1.9.3 Period

Hyperbolic functions have no period in \mathbb{R} , however in \mathbb{C} ,

- $\sinh x$ and $\cosh x$ have period $2i\pi$
- $\tanh x$ has period $i\pi$

1.9.4 Inverse Hyperbolic Functions

$$\sinh(x + iy) = u + iv \implies x + iy = \sinh^{-1}(u + iv)$$

Also, for the general solution,

$$\text{Sinh}^{-1}(u + iv) = 2n\pi i + x + iy$$

Example. Show that $\sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$.

Solution. Let $\sinh^{-1} z = w$,

$$\implies z = \sinh w \wedge \cosh w = \sqrt{z^2 + 1}$$

Also,

$$\begin{aligned}\sinh w + \cosh w &= e^w \\ \implies w &= \ln(z + \sqrt{z^2 + 1}) \\ \therefore \sinh^{-1} z &= \ln(z + \sqrt{z^2 + 1})\end{aligned}$$

Similarly,

$$\begin{aligned}\cosh^{-1} z &= \ln(z + \sqrt{z^2 - 1}) \\ \tanh^{-1} z &= \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)\end{aligned}$$

1.9.5 More Identities of Conversion

$$\sinh^{-1} x = -i \sin^{-1} ix$$

$$\cosh^{-1} x = -i \cos^{-1} x$$

$$\tanh^{-1} x = -i \tan^{-1} ix$$

Chapter 2

Functions of Several Variables

First, we say that the real line \mathbb{R} is both open and closed.

Example. $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given as,

$$f(x, y) = xy$$

Then the domain and range of f is?

Solution. Clearly,

$$\text{Dom}(f) = \mathbb{R}^2 \wedge \text{Ran}(f) = \mathbb{R}$$

Example. $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given as,

$$f(x, y) = \frac{1}{x^2 + y^2}$$

Then the domain and range of f is?

Solution. Clearly,

$$x^2 + y^2 > 0 \implies \frac{1}{x^2 + y^2} < \infty$$

Thus,

$$\text{Dom}(f) = \mathbb{R}^2 - (0, 0) \wedge \text{Ran}(f) = \mathbb{R}^+$$

Example. $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given as,

$$f(x, y) = \frac{1}{\sqrt{4 - x^2 - y^2}}$$

Then the domain and range of f is?

Solution. Clearly,

$$4 - x^2 - y^2 > 0 \implies x^2 + y^2 < 4$$

Thus,

$$\text{Dom}(f) = \{(x, y) | x^2 + y^2 < 4\} \wedge \text{Ran}(f) = [\frac{1}{2}, \infty)$$

2.1 Limit in Higher Dimensions

The limit of a function exists if on approaching the point from all directions from all possible paths, the same value is obtained.

Let there be $f : S \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, where, f may or may not be defined at (a, b) , then,

$$\lim_{(x, y) \rightarrow (a, b)} f(x, y) = L$$

$$\forall \epsilon > 0 \exists \delta > 0 (\begin{matrix} \sqrt{(x-a)^2 + (y-b)^2} < \delta \\ |x-a| < \delta, |y-b| < \delta \\ |x-a| + |y-b| < \delta \end{matrix}) \implies |f(x, y) - L| < \epsilon$$

Example. Find the limit of

$$f(x, y) = \frac{xy}{x^2 + y^2}$$

Solution. Let $y = mx$ be the path on which $(0, 0)$ lies.

$$f(x, mx) = \frac{m}{1 + m^2}$$

Since the value of the function depends upon the path,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2 + y^2} \text{ Does not exist}$$

2.2 Continuity

Consider a function f such that,

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

Then f is continuous at a point (a, b) , if $f(x, y)$ is defined at (a, b) and $L = f(a, b)$.

Example. Discuss the continuity of

$$f(x, y) = \begin{cases} x^2 + 2y & , (x, y) \neq (1, 2) \\ 0 & , (x, y) = (1, 2) \end{cases}$$

Solution. Here,

$$\lim_{(x,y) \rightarrow (1,2)} f(x, y) = 5$$

Thus, the function is discontinuous.

However, it is a removal point of discontinuity since it can be made continuous by changing the function definition.

Example. Show that f is continuous at $(0, 0)$,

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

Solution. Suppose $\epsilon > 0$. Now choose $\delta = 2\epsilon$.

Let, $\sqrt{|x|^2 + |y|^2} < \delta$. Check,

$$\begin{aligned} |f(x, y) - f(0, 0)| &= \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \\ &\leq \frac{1}{2} \sqrt{x^2 + y^2} \quad (\because |xy| \leq \frac{x^2 + y^2}{2}) \\ &\leq \frac{\delta}{2} = \epsilon \end{aligned}$$

Now, if the repeated limit exists, then the limit may or may not exist, however if the limit exists, then the repeated limit must exist, i.e.,

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L \implies \lim_{x \rightarrow a} \left(\lim_{y \rightarrow b} f(x, y) \right) = \lim_{y \rightarrow b} \left(\lim_{x \rightarrow a} f(x, y) \right) = L$$

2.3 Partial Derivatives

The partial derivative of a function f w.r.t x at the point (a, b) is,

$$f_x(a, b) = \left. \frac{\partial f}{\partial x} \right|_{(a,b)} = \frac{\partial f(a, b)}{\partial x} = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h}$$

It is possible that a function is discontinuous at some point but its partial derivative exists.¹

¹For the mixed partial derivatives to be equal, the first partial derivatives must exist and be continuous.

Example. Find the partial derivative at $(0, 0)$,

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

Solution.

$$\frac{\partial f(0, 0)}{\partial x} = \lim_{h \rightarrow 0} \frac{f(h, 0) - 0}{h} = 0$$

Hence, clearly the function is discontinuous at $(0, 0)$ but the partial derivative exists.

Example. Given,

$$f(x, y) = 7x^2y^3 + 9xy^2 + 8xy + 20e^x \cos y$$

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$.

Solution.

$$f_x(x, y) = \frac{\partial f}{\partial x} = 14xy^3 + 9y^2 + 8y + 20e^x \cos y$$

$$f_y(x, y) = \frac{\partial f}{\partial y} = 21x^2y^2 + 18xy + 8x - 20e^x \sin y$$

$$f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = 14y^3 + 20e^x \cos y$$

$$f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = 42x^2y + 18x - 20e^x \cos y$$

$$f_{xy}(x, y) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = 42xy^2 + 18y + 8 - 20e^x \sin y$$

$$f_{yx}(x, y) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = 42xy^2 + 18y + 8 - 20e^x \sin y$$

Example.

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Show that, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$

Solution.

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \implies \frac{\partial^2 f}{\partial x^2} = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = \frac{z^2 + x^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \wedge \frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

2.4 Composition of Functions

Consider a function $z = f(x, y)$ and $x = g(t)$ and $y = h(t)$, then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

Example.

$$z = e^{xy^2} \wedge x = t \cos t \wedge y = t \sin t$$

Calculate $\frac{dz}{dt} \Big|_{t=\frac{\pi}{2}}$.

Solution. Clearly,

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\ &= y^2 e^{xy^2} [\cos t - t \sin t] + 2xy e^{xy^2} [\sin t + t \cos t] \\ \therefore \frac{dz}{dt} \Big|_{t=\frac{\pi}{2}} &= -\frac{\pi^3}{8}\end{aligned}$$

Theorem 11 (Euler's Theorem for Homogeneous Functions). If $f(x, y)$ is a homogeneous function of degree n , then,

$$\begin{aligned}x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} &= n f \\ x^2 \frac{\partial^2 f}{\partial x^2} + 2xy \frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} &= n(n-1)f\end{aligned}$$

Example. Given,

$$\ln u = \frac{x^3 + y^3}{3x + 4y}$$

Then find the value of $x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$.

Solution. Let $z = \ln u$. Using Euler's Theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$$

Now,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} \wedge \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y}$$

Example. Given,

$$u = \sin^{-1} \left(\frac{x + 2y + 3z}{x^8 + y^8 + z^8} \right)$$

Then prove that,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = -7 \tan u$$

Solution. Let $f = \sin u$. Using Euler's Theorem,

$$\begin{aligned}x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z} &= -7f \\ \implies x \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + y \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + z \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} &= -7 \sin u \\ \therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} &= -7 \tan u \quad \left(\frac{\partial f}{\partial u} = \cos u \right)\end{aligned}$$

2.5 Jacobian

Consider two functions $u(x, y)$ and $v(x, y)$, then the Jacobian² of u, v w.r.t. x, y is,

$$\mathbb{J}(x, y) = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Moreover, composition of Jacobian works as, if $u(r, s)$ and $v(r, s)$ where $r = f(x, y)$ and $s = g(x, y)$,

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \frac{\partial(r, s)}{\partial(x, y)}$$

²Jo niche hai voh operator mein jaega, matlab w.r.t. whom

Example (Polar Coordinates). If $x = r \cos \theta$ and $y = r \sin \theta$, then find $\mathbb{J}(r, \theta)$.

Solution.

$$\mathbb{J}(r, \theta) = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

Example (Cylindrical Coordinates). If $x = r \cos \theta$, $y = r \sin \theta$ and $z = z$, then find $\mathbb{J}(r, \theta, z)$.

Solution.

$$\mathbb{J}(r, \theta, z) = r$$

Example (Spherical Coordinates). If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$ where $\theta \in [0, 2\pi] \wedge \phi \in [0, \pi]$, then find $\mathbb{J}(r, \theta, \phi)$.

Solution.

$$\mathbb{J}(r, \theta, \phi) = r^2 \sin \theta$$

2.6 Explicit and Implicit Functions

A function f is said to be explicit if one variable can be represented in terms of other variables. If f is implicit, such that $f(x, y) = c$, then $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

Theorem 12 (Taylor's Theorem).

$$f(x, y) = f(a, b) + (x - a) \frac{\partial f(a, b)}{\partial x} + (y - b) \frac{\partial f(a, b)}{\partial y} + \frac{1}{2!} \left[(x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right]^2 f(a, b) + \dots + R_n$$

Where,

$$R_n = \frac{1}{(n+1)!} \left[(x - a) \frac{\partial}{\partial x} + (y - b) \frac{\partial}{\partial y} \right]^{n+1} f(a + \theta(x - a), b + \phi(y - b)) \quad (\theta, \phi \in (0, 1))$$

Example.

$$f(x, y) = x^3 + x^2y + xy^2 + y^2$$

Express f in powers of $(x - 1)$ and $(y + 2)$.

Solution. Here, $a = 1, b = -2$, Thus,

$$f(1, -2) = 7 \implies f_x(1, -2) = 3 \wedge f_y(1, -2) = -7 \implies f_{xx}(1, -2) = 2 \wedge f_{yy}(1, -2) = 4 \wedge f_{xy}(1, -2) = -2$$

Also,

$$f_{xxy} = 6 \wedge f_{yyy} = 0 \wedge f_{xxy} = 2 \wedge f_{yyx} = 2$$

$$f(x, y) = 7 + 3(x - 1) + (-7)(y + 2) + \frac{1}{2!} [2(x - 1)^2 + 4(y + 2)^2 + (-2)(x - 1)(y + 2)] +$$

$$\frac{1}{3!} [6(x - 1)^3 + 3(2)(x - 1)^2(y + 2) + 3(2)(x - 1)(y + 2)^2]$$

$$\therefore f(x, y) = 7 + 3(x - 1) - 7(y + 2) + (x - 1)^2 + 2(y + 2)^2 - (x - 1)(y + 2) + (x - 1)^3 + (x - 1)^2(y + 2) + (x - 1)(y + 2)^2$$

2.7 Maxima and Minima of Two Variables

A function $f(x, y)$ is said to obtain minimum value of f at point (a, b) , if there exists a neighborhood (N) of (a, b) such that $f(a, b) < f(x, y)$ for all $(x, y) \in N$.

A function $f(x, y)$ is said to obtain maximum value of f at point (a, b) , if there exists a neighborhood (N) of (a, b) such that $f(a, b) > f(x, y)$ for all $(x, y) \in N$.

2.7.1 Critical Point

Any point (a, b) is said to be a critical point of f if $\begin{cases} \frac{\partial f(a,b)}{\partial x} = 0 \\ \frac{\partial f(a,b)}{\partial y} = 0 \end{cases}$

$$A = \frac{\partial^2 f}{\partial x^2} \wedge B = \frac{\partial^2 f}{\partial x \partial y} \wedge C = \frac{\partial^2 f}{\partial y^2}$$

$$\begin{cases} AC - B^2 > 0 & , \begin{cases} A > 0 & , \text{minima} \\ A < 0 & , \text{maxima} \end{cases} \\ AC - B^2 < 0 & , \text{neither max nor min, i.e. saddle point} \\ AC - B^2 = 0 & , \text{Further investigation} \end{cases}$$

Example.

$$f(x, y) = (x - y)^2$$

Solution.

$$\frac{\partial f}{\partial x} = 0 \wedge \frac{\partial f}{\partial y} = 0 \implies x = y$$

Critical points are $S = \{x \in \mathbb{R} | (x, x)\}$.

$$f_{xx} = 2 \wedge f_{xy} = -2 \wedge f_{yy} = 2 \implies AC - B^2 = 0$$

Clearly, at S is set of minima.

Example.

$$f(x, y) = x^4 + y^4 - 2x^2 - 2y^2 + 4xy$$

Solution.

$$f_x = 4x^3 - 4x + 4y = 0$$

$$f_y = 4y^3 - 4y + 4x = 0$$

Thus, $x = -y$ and $(x, y) = \{(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})\}$.

$$f_{xx} = 12x^2 - 4 \wedge f_{xy} = 4 \wedge f_{yy} = 12y^2 - 4$$

At $(0, 0)$, $AC - B^2 = 0$

At $(\sqrt{2}, -\sqrt{2})$, $AC - B^2 > 0 \wedge A > 0$

At $(-\sqrt{2}, \sqrt{2})$, $AC - B^2 > 0 \wedge A > 0$

Thus, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ are points of minima.

For $(0, 0)$, along x ,

$$y = 0$$

2.8 Tutorial-3

2.8.1 Determine the domain and range of the following functions

Question.

$$f(x, y) = \frac{1}{x - y}$$

Answer. Here,

$$x - y \neq 0 \implies x \neq y$$

Thus, the domain is,

$$\text{Dom}(f) = \mathbb{R}^2 - \{x, y \in \mathbb{R} | x = y\}$$

Question.

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

Answer. Here,

$$1 - x^2 - y^2 - z^2 \geq 0 \implies x^2 + y^2 + z^2 \leq 1$$

Thus,

$$\text{Dom}(f) = \{x, y, z \in \mathbb{R} | x^2 + y^2 + z^2 \leq 1\}$$

Question.

$$f(x, y) = \sqrt{16 - 4x^2 - y^2}$$

Answer. Here,

$$16 - 4x^2 - y^2 \geq 0 \implies 4x^2 + y^2 \leq 16$$

Thus,

$$\text{Dom}(f) = \{x, y \in \mathbb{R} | 4x^2 + y^2 \leq 16\}$$

Question.

$$f(x, y) = \frac{\sqrt{4 - x^2}}{y^2 + 3}$$

Answer. Here,

$$4 - x^2 \geq 0 \implies x^2 \leq 4 \implies x \in [-2, 2]$$

Thus,

$$\text{Dom}(f) = \{x, y \in \mathbb{R} | x \in [-2, 2]\}$$

2.8.2 Examine if the limits of the following functions $f(x, y)$ as $(x, y) \rightarrow (0, 0)$ exist

Question.

$$f(x, y) = \begin{cases} \frac{x^2 + y^2}{x^2 - y^2} & , x \neq y \\ 0 & , x = y \end{cases}$$

Answer. Let the path of choice be $y = mx$,

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{1 - m^2}{1 + m^2}$$

Since the limit depends on the path, $f(x, y)$ is not continuous at $(0, 0)$.

Question.

$$f(x, y) = xy \left(\frac{x^2 - y^2}{x^2 + y^2} \right)$$

Answer. Suppose $\epsilon > 0$. Now choose $\delta = \sqrt{2\epsilon}$. Let $\sqrt{x^2 + y^2} < \delta$. Check,

$$\begin{aligned} |f(x, y)| &= \left| xy \frac{x^2 - y^2}{x^2 + y^2} \right| \\ &< |xy| \quad \left(\because \frac{x^2 - y^2}{x^2 + y^2} < 1 \right) \\ &< \frac{1}{2}(x^2 + y^2) \\ &< \frac{\delta^2}{2} = \epsilon \end{aligned}$$

Question.

$$f(x, y) = \begin{cases} x \sin \frac{1}{y} + y \sin \frac{1}{x} & , xy \neq 0 \\ 0 & , xy = 0 \end{cases}$$

Answer. Suppose $\epsilon > 0$. Now choose $\delta = \epsilon$.

Let $|x| + |y| < \delta$. Check,

$$\begin{aligned} |f(x, y) - f(0, 0)| &= \left| x \sin \frac{1}{y} + y \sin \frac{1}{x} - 0 \right| \\ &\leq \left| x \sin \frac{1}{y} \right| + \left| y \sin \frac{1}{x} \right| \\ &\leq |x| + |y| \\ &< \delta = \epsilon \end{aligned}$$

Question.

$$f(x, y) = \frac{\sin(xy)}{x^2 + y^2}$$

Answer. Let the path of choice be $y = mx$,

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{m}{1 + m^2}$$

Since the limit depends on the path, $f(x, y)$ is not continuous at $(0, 0)$.

2.8.3 Examine the continuity of the following functions at $(0, 0)$

Question.

$$f(x, y) = \begin{cases} \frac{xy^3}{x^2 + y^6} & , (x, y) \neq (0, 0) \\ 0 & , \text{otherwise} \end{cases}$$

Answer. Let the path of choice be $y^3 = mx$, thus,

$$\lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{m}{1 + m^2}$$

Since the limit depends on the path, $f(x, y)$ is not continuous at $(0, 0)$.

Question.

$$f(x, y) = \begin{cases} x^2 + y^2 & , x^2 + y^2 \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

Answer. Suppose $\epsilon > 0$. Now choose, $\delta = \sqrt{\frac{\epsilon}{2}}$. Let $|x| < \delta \wedge |y| < \delta$. Check,

$$\begin{aligned} |f(x, y)| &= |x^2 + y^2| \\ &\leq |x|^2 + |y|^2 \\ &\leq 2\delta^2 \\ &= \epsilon \end{aligned}$$

Question.

$$f(x, y) = \begin{cases} \frac{\sin^2(x-y)}{|x|+|y|} & , (x, y) \neq (0, 0) \\ 0 & , \text{otherwise} \end{cases}$$

Answer. Suppose $\epsilon > 0$, Now choose $\delta = \epsilon$. Let $|x| + |y| < \delta$. Check,

$$\begin{aligned} |f(x, y)| &= \left| \frac{\sin^2(x-y)}{|x|+|y|} \right| \\ &\leq \left| \frac{|x-y|^2}{|x|+|y|} \right| \\ &\leq |x| + |y| \quad (\because |x-y|^2 \leq (|x|+|y|)^2) \\ &< \delta = \epsilon \end{aligned}$$

Question. Find the partial derivative of the function $f(x, y, z) = \ln(x^2 + 2y + z^3)$.

Answer. W.r.t. x ,

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + 2y + z^3}$$

W.r.t. y ,

$$\frac{\partial f}{\partial y} = \frac{2}{x^2 + 2y + z^3}$$

W.r.t. z ,

$$\frac{\partial f}{\partial z} = \frac{3z^2}{x^2 + 2y + z^3}$$

Question. Show that the function $u = \ln(\sqrt{x^2 + y^2})$ satisfy the Laplace equation $\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\right)$.

Answer. Clearly,

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} \implies \frac{\partial^2 u}{\partial x^2} = \frac{y^2 - x^2}{x^2 + y^2}$$

By symmetry,

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 - y^2}{x^2 + y^2}$$

Thus,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

2.8.4 Find the first order partial derivatives of the following functions at the given points

Question.

$$f(x, y) = (3x^3 + 2xy)^3 \quad (2, 0)$$

Answer. W.r.t. x ,

$$\frac{\partial f}{\partial x} = 3(3x^3 + 2xy)^2(9x^2 + 2y) \implies \frac{\partial f}{\partial x} \Big|_{(2,0)} = 2592$$

W.r.t y ,

$$\frac{\partial f}{\partial y} = 3(3x^3 + 2xy)^2(2x) \implies \frac{\partial f}{\partial y} \Big|_{(2,0)} = 288$$

Question.

$$f(x, y) = \left(\frac{4x^2 + 2y^2}{xy} \right) \quad (\sqrt{3}, \sqrt{2})$$

Answer. Simplifying,

$$f(x, y) = 2 \cdot \left(\frac{2x}{y} + \frac{y}{x} \right)$$

W.r.t. x ,

$$\frac{\partial f}{\partial x} = 2 \left(\frac{2}{y} - \frac{y}{x^2} \right) \Rightarrow \frac{\partial f}{\partial x} \Big|_{(\sqrt{3}, \sqrt{2})} = \frac{4\sqrt{2}}{3}$$

W.r.t. y ,

$$\frac{\partial f}{\partial y} = 2 \left(\frac{-2x}{y^2} + \frac{1}{x} \right) \Rightarrow \frac{\partial f}{\partial y} \Big|_{(\sqrt{3}, \sqrt{2})} = -\frac{4\sqrt{3}}{3}$$

2.8.5 Calculate the partial derivatives of the following functions at $(0, 0)$

Question.

$$f(x, y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y} & , xy \neq 0 \\ 0 & , xy = 0 \end{cases}$$

Answer. W.r.t. x ,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= 0 \end{aligned}$$

Similarly for y .

Question.

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & , x^2 + y^2 \neq 0 \\ 0 & , x = y = 0 \end{cases}$$

Answer. W.r.t. x ,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= 0 \end{aligned}$$

Similarly for y .

Question.

$$f(x, y) = \begin{cases} \frac{x^6 - 2y^4}{x^2 + y^2} & , x^2 + y^2 \neq 0 \\ 0 & , x = y = 0 \end{cases}$$

Answer. W.r.t. x ,

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^6}{h^2} \\ &= 0 \end{aligned}$$

W.r.t. y ,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} \\ &= \lim_{k \rightarrow 0} \frac{-2k^4}{k^2} \\ &= 0 \end{aligned}$$

Question.

$$f(x, y) = \begin{cases} y \sin \frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

Answer. W.r.t. x ,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= \text{Not defined}\end{aligned}$$

W.r.t. y ,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \lim_{k \rightarrow 0} \frac{f(0, k) - f(0, 0)}{k} \\ &= 0\end{aligned}$$

Question.

$$f(x, y) = \begin{cases} \frac{x^3 + y^3}{x - y} & , x \neq y \\ 0 & , x = y \end{cases}$$

Answer.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= 0\end{aligned}$$

Similarly for y .

Question.

$$f(x, y) = \sqrt{|xy|}$$

Answer.

$$\begin{aligned}\frac{\partial f}{\partial x} &= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} \\ &= 0\end{aligned}$$

Similarly for y .

Question. If $F(x, y) = x^4 y^2 \sin^{-1} \frac{y}{x}$, then show that $x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} = 6F$.

Answer. Checking the condition for homogeneity,

$$F(\lambda x, \lambda y) = \lambda^6 F(x, y)$$

Thus, using Euler's Theorem for Homogeneous Functions,

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} = 6F$$

Question. If $F = \tan^{-1} \left(\frac{x^3 + y^3}{x - y} \right)$, then prove that,

1. $x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} = \sin 2F$
2. $x^2 \frac{\partial^2 F}{\partial x^2} + 2xy \frac{\partial^2 F}{\partial x \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} = 2 \cos 3F \sin F$

Answer. Let $\tan F = z = \frac{x^3 + y^3}{x - y}$, here, checking the condition for homogeneity for z ,

$$z = y^2 \frac{\left(\frac{x}{y} \right)^3 + 1}{\left(\frac{x}{y} \right) - 1}$$

Using Euler's Theorem for Homogeneous Functions on z ,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$$

Moreover,

$$\frac{\partial F}{\partial x} = \frac{1}{\sec^2 F} \frac{\partial z}{\partial x} \wedge \frac{\partial F}{\partial y} = \frac{1}{\sec^2 F} \frac{\partial z}{\partial y}$$

Replacing,

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} = 2 \tan F \sec^2 F = \sin 2F$$

Differentiating w.r.t. x and y , then multiplying with x and y respectively we get,

$$\begin{aligned} x \frac{\partial F}{\partial x} + x^2 \frac{\partial^2 F}{\partial x^2} + xy \frac{\partial^2 F}{\partial x \partial y} &= 2 \cos 2F \cdot x \frac{\partial F}{\partial x} \\ y \frac{\partial F}{\partial y} + y^2 \frac{\partial^2 F}{\partial y^2} + xy \frac{\partial^2 F}{\partial x \partial y} &= 2 \cos 2F \cdot y \frac{\partial F}{\partial y} \end{aligned}$$

Adding the equations,

$$x^2 \frac{\partial^2 F}{\partial x^2} + 2xy \frac{\partial^2 F}{\partial x \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} = \sin 4F - \sin 2F = 2 \cos 3F \sin F$$

Question. If $F = \sin^{-1} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$, then prove that,

1. $x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} = \frac{1}{2} \tan F$
2. $x^2 \frac{\partial^2 F}{\partial x^2} + 2xy \frac{\partial^2 F}{\partial x \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} = -\frac{\sin F \cos 2F}{4 \cos^3 F}$

Answer. Let $\sin F = z = \frac{x+y}{\sqrt{x}+\sqrt{y}}$, here, checking the condition for homogeneity for z ,

$$z = y^{\frac{1}{2}} \left(\frac{x+y}{\sqrt{x}+\sqrt{y}} \right)$$

Using Euler's Theorem for Homogeneous Functions on z ,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{z}{2}$$

Moreover,

$$\frac{\partial F}{\partial x} = \frac{1}{\cos F} \frac{\partial z}{\partial x} \wedge \frac{\partial F}{\partial y} = \frac{1}{\cos F} \frac{\partial z}{\partial y}$$

Replacing,

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} = \frac{1}{2} \tan F$$

Differentiating w.r.t. x and y , then multiplying with x and y respectively we get,

$$\begin{aligned} x \frac{\partial F}{\partial x} + x^2 \frac{\partial^2 F}{\partial x^2} + xy \frac{\partial^2 F}{\partial x \partial y} &= \frac{1}{2} \sec^2 F \cdot x \frac{\partial F}{\partial x} \\ y \frac{\partial F}{\partial y} + y^2 \frac{\partial^2 F}{\partial y^2} + xy \frac{\partial^2 F}{\partial x \partial y} &= \frac{1}{2} \sec^2 F \cdot y \frac{\partial F}{\partial y} \end{aligned}$$

Adding the equations,

$$x^2 \frac{\partial^2 F}{\partial x^2} + 2xy \frac{\partial^2 F}{\partial x \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} = -\frac{\sin F \cos 2F}{4 \cos^3 F}$$

Question. If $u = \tan^{-1} \frac{y^2}{x}$, then prove that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin^2 u \sin 2u$.

Answer. Let $\tan u = z = \frac{y^2}{x}$, here, checking the condition for homogeneity for z , we get $n = 1$, Using Euler's Theorem

for Homogeneous Functions on z ,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

Moreover,

$$\frac{\partial u}{\partial x} = \frac{1}{\sec^2 u} \frac{\partial z}{\partial x} \wedge \frac{\partial u}{\partial y} = \frac{1}{\sec^2 u} \frac{\partial z}{\partial y}$$

Replacing,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{\sin 2u}{2}$$

Differentiating w.r.t. x and y , then multiplying with x and y respectively we get,

$$\begin{aligned} x \frac{\partial u}{\partial x} + x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 f}{\partial x \partial y} &= \cos 2u \cdot x \frac{\partial u}{\partial x} \\ y \frac{\partial u}{\partial y} + y^2 \frac{\partial^2 f}{\partial y^2} + xy \frac{\partial^2 f}{\partial x \partial y} &= \cos 2u \cdot y \frac{\partial u}{\partial y} \end{aligned}$$

Adding the equations,

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\sin^2 u \sin 2u$$

Question. If $u = u\left(\frac{x-y}{xy}, \frac{z-x}{zx}\right)$, then show that $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$.

Answer. Let $\frac{x-y}{xy} = a$ and $\frac{z-x}{zx} = b$, thus,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial a} \frac{\partial a}{\partial x} + \frac{\partial u}{\partial b} \frac{\partial b}{\partial x} = \frac{1}{x^2} \left(\frac{\partial u}{\partial a} - \frac{\partial u}{\partial b} \right) \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial a} \frac{\partial a}{\partial y} + \frac{\partial u}{\partial b} \frac{\partial b}{\partial y} = -\frac{1}{y^2} \frac{\partial u}{\partial a} \\ \frac{\partial u}{\partial z} &= \frac{\partial u}{\partial a} \frac{\partial a}{\partial z} + \frac{\partial u}{\partial b} \frac{\partial b}{\partial z} = \frac{1}{z^2} \frac{\partial u}{\partial b} \end{aligned}$$

Adding,

$$x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} + z^2 \frac{\partial u}{\partial z} = 0$$

Question. If $z = f(x, y)$ and $x = e^u \cos v$, $y = e^u \sin v$, prove that $x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \frac{\partial z}{\partial y}$ and $\left(\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = e^{-2u} \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right]$.

Answer. Clearly,

$$\begin{aligned} \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} \end{aligned}$$

Multiplying and adding,

$$x \frac{\partial z}{\partial v} + y \frac{\partial z}{\partial u} = e^{2u} \frac{\partial z}{\partial y}$$

Squaring and adding,

$$\left(\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = e^{-2u} \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right]$$

Question. Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ for the function $f(x, y)$ defined by

$$f(x, y) = \begin{cases} x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right) & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

Answer.

2.8.6 Find the Jacobian of the following functions

Question.

$$u(x, y, z) = \frac{yz}{x} \wedge v(x, y, z) = \frac{zx}{y} \wedge w(x, y, z) = \frac{xy}{z}$$

Answer. The Jacobian is given by,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix} = 4$$

Question.

$$u(r, \theta) = r \cos 2\theta \wedge v(r, \theta) = r \sin 2\theta$$

Answer. The Jacobian is given by,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \cos 2\theta & -2r \sin \theta \\ \sin 2\theta & 2r \cos \theta \end{vmatrix} = 2r \cos \theta$$

Question. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

Answer. First,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = -2(x - y)(y - z)(z - x)$$

Thus,

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{1}{\frac{\partial(u, v, w)}{\partial(x, y, z)}} = \frac{-1}{2(x - y)(y - z)(z - x)}$$

Question. If $x = u(1 - v)$, $y = uv$, prove that $\mathbb{J}\mathbb{J}' = 1$.

Answer. First calculating $\mathbb{J} = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} = u$. Moreover,

$$u = x + y \wedge v = \frac{y}{x + y}$$

Thus,

$$\mathbb{J}' = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{1}{x + y} & \frac{1}{(x + y)^2} \\ \frac{-y}{(x + y)^2} & \frac{x}{(x + y)^2} \end{vmatrix} = \frac{1}{x + y} = \frac{1}{u}$$

Hence, we can say that for the given set of variables,

$$\mathbb{J}\mathbb{J}' = 1$$

Question. If $u = x^2 - 2y^2$, $v = 2x^2 - y^2$, where $x = r \cos \theta$, $y = r \sin \theta$, show that $\frac{\partial(u, v)}{\partial(r, \theta)} = 6r^2 \sin 2\theta$.

Answer.

$$\begin{aligned} \frac{\partial(u, v)}{\partial(r, \theta)} &= \frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(r, \theta)} \\ &= \begin{vmatrix} 2x & -4y \\ 4x & -2y \end{vmatrix} r \\ &= r(12xy) \\ &= 6r^3 \sin 2\theta \end{aligned}$$

Question. If $u = x\sqrt{1-y^2} + y\sqrt{1-x^2}$, $v = \sin^{-1} x + \sin^{-1} y$, then show that u and v are functionally related and find the relationship.

Answer. Clearly,

$$\sin^{-1} u = v$$

2.8.7 Find f_u , f_v and total derivative for the functions

Question.

$$f(x, y) = e^x \cos y, x = ue^v, y = u + v - 1$$

Answer. Clearly,

$$\begin{aligned} f_u &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = e^x (e^v \cos y - \sin y) \\ f_v &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} = e^x (ue^v \cos y - \sin y) \end{aligned}$$

The total derivative is,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = e^x \cos y dx - e^x \sin y dy$$

Question.

$$f(x, y) = xy, x = e^{2u} \cos v, v = e^u \sin v$$

Answer. Clearly,

$$\begin{aligned} f_u &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = 3xy \\ f_v &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} = xy(\cos 2v) \end{aligned}$$

The total derivative is,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = y dx + x dy$$

Question.

$$f(x, y) = x \sinh y + y \cosh x, x = u^2 + 1, y = v^2$$

Answer. Clearly,

$$\begin{aligned} f_u &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} = 2u(\sinh y + y \sinh x) \\ f_v &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} = 2v(x \cosh y + \cosh x) \end{aligned}$$

The total derivative is,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = (\sinh y + y \sinh x) dx + (x \cosh y + \cosh x) dy$$

2.8.8 Using change of variable, prove the following

Question. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Answer. Let $r = e^{y-z}$, $s = e^{z-x}$ and $t = e^{x-y}$, thus,

$$\begin{aligned}\frac{\partial u}{\partial x} &= -s \frac{\partial u}{\partial s} + t \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial y} &= r \frac{\partial u}{\partial r} - t \frac{\partial u}{\partial t} \\ \frac{\partial u}{\partial z} &= -r \frac{\partial u}{\partial r} + s \frac{\partial u}{\partial s}\end{aligned}$$

Adding, we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Question. Transform the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ into polar coordinates.

Answer. Here, $x = r \cos \theta$ and $y = r \sin \theta$,

$$x^2 + y^2 = r^2 \quad \wedge \quad \theta = \tan^{-1} \frac{y}{x}$$

Hence,

$$\begin{aligned}\frac{\partial r}{\partial x} &= \cos \theta \quad \wedge \quad \frac{\partial r}{\partial y} = \sin \theta \quad \wedge \quad \frac{\partial \theta}{\partial x} = \frac{-\sin \theta}{r} \quad \wedge \quad \frac{\partial \theta}{\partial y} = \frac{\cos \theta}{r} \\ \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x} \\ \therefore \frac{\partial}{\partial x} &= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial y} \\ \therefore \frac{\partial}{\partial y} &= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}\end{aligned}$$

Using the above operators, we get,

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \cos^2 \theta \frac{\partial^2 u}{\partial r^2} - 2 \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial u}{\partial r} \\ \frac{\partial^2 u}{\partial y^2} &= \sin \theta \frac{\partial}{\partial r} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \sin^2 \theta \frac{\partial^2 u}{\partial r^2} + 2 \frac{\cos \theta \sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{\cos \theta \sin \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial u}{\partial r}\end{aligned}$$

Finally on adding we get,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0$$

Question. If $x - y = 2e^\theta \cos \phi$ and $x + y = 2ie^\theta \sin \phi$, then show that $\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}$.

Answer. Solving,

$$x = e^\theta e^{i\phi} \quad \wedge \quad y = e^\theta e^{-i\phi}$$

Thus,

$$\frac{\partial x}{\partial \theta} = x \quad \wedge \quad \frac{\partial y}{\partial \theta} = y \quad \wedge \quad \frac{\partial x}{\partial \phi} = ix \quad \wedge \quad \frac{\partial y}{\partial \phi} = iy$$

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta} \\ \therefore \frac{\partial}{\partial \theta} &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \\ \frac{\partial u}{\partial \phi} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \phi} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \phi} \\ \therefore \frac{\partial}{\partial \phi} &= i \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)\end{aligned}$$

Using the above operators, we get,

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \\ \frac{\partial^2 u}{\partial \phi^2} &= -x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} - y^2 \frac{\partial^2 u}{\partial y^2} - x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y}\end{aligned}$$

Adding,

$$\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}$$

2.8.9 Find $\frac{dy}{dx}$ for the following implicit functions

Question.

$$y^4 + y^2 - 5y - x^2 + 4 = 0$$

at $(1, 1)$.

Answer. Differentiating w.r.t. x and y ,

$$\begin{aligned}f_x &= -2x \\ f_y &= 4y^3 + 2y - 5\end{aligned}$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x}{4y^3 + 2y - 5} \\ \therefore \frac{dy}{dx} \Big|_{(1,1)} &= 2\end{aligned}$$

Question.

$$xe^y + \sin(xy) + y - \ln 2 = 0$$

at $(0, \ln 2)$.

Answer. Differentiating w.r.t. x and y ,

$$\begin{aligned}f_x &= e^y + y \cos(xy) \\ f_y &= xe^y + x \cos(xy) + 1\end{aligned}$$

Thus,

$$\begin{aligned}\frac{dy}{dx} &= -\frac{e^y + y \cos(xy)}{xe^y + x \cos(xy) + 1} \\ \therefore \frac{dy}{dx} \Big|_{(0, \ln 2)} &= -(2 + \ln 2)\end{aligned}$$

Question. Let $f(x, y) = 3x^2 + 4y^2$. Use the differential of f at the point $(2, 1)$ to estimate the values $f(1.93, -1.05)$ and $f(2.07, -0.99)$.

Answer. Here

Question. Find the quadratic Taylor's polynomial approximation of $e^{-x^2-2y^2}$ near $(0, 0)$.

Answer.

Question. Using Taylor's formula, find quadratic and cubic approximations $e^x \sin y$ at origin. Estimate the error in approximations if $|x| \leq 0.1$, $|y| \leq 0.2$.

Answer.

Chapter 3

Multiple Integrals

3.1 Integration

Consider a function f bounded and defined on $[a, b]$. Then we can define the integral in the following way,

$$\int_a^b f(x) \, dx$$

We define Partitions as,

$$P = \{a = x_0, x_1, \dots, x_n = b\}$$

We can define the lower sum as,

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \wedge \Delta x_i = x_i - x_{i-1}$$

Thus, the lower sum of f w.r.t. partition P ,

$$L(P, f) = \sum_{i=1}^n m_i \Delta x_i$$

We can also define the upper sum as,

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

Thus, the upper sum of f w.r.t. partition P ,

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i$$

The lower integral is defined as,

$$\int_a^b f(x) \, dx = \sup_P L(P, f)$$

The upper integral is defined as,

$$\int_a^b f(x) \, dx = \inf_P U(P, f)$$

Hence, we can define a relation as,

$$\int_a^b f(x) \, dx \leq \int_a^b f(x) \, dx \leq \int_a^b f(x) \, dx$$

Theorem 13 (Riemann Integrable). A function f is said to be Riemann Integrable if,

$$\int_a^b f(x) \, dx = \int_a^b f(x) \, dx = \int_a^b f(x) \, dx$$

Example. Show that $\int_0^1 x^2 dx$ is Riemann integrable.

Solution.

$$x_i = \frac{i}{n} \wedge \Delta x_i = \frac{1}{n}$$

The partitions are,

$$P = \left\{ \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} \right\}$$

$$m_i = \frac{(i-1)^2}{n^2} \wedge M_i = \frac{i^2}{n^2}$$

Thus, the lower sum is,

$$L(P, f) = \left(1 - \frac{1}{n}\right) \left(2 - \frac{1}{n}\right) \frac{1}{6} \implies \sup L(P, f) = \frac{1}{3}$$

Also, the upper sum is,

$$U(P, f) = \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right) \frac{1}{6} \implies \inf U(P, f) = \frac{1}{3}$$

Clearly,

$$\int_0^1 x^2 dx = \int_0^1 x^2 dx = \int_0^1 x^2 dx = \frac{1}{3}$$

Example.

$$f(x) = \begin{cases} 0 & , x \in \mathbb{Q} \cap [0, 1] \\ 1 & , x \in \mathbb{Q}^c \cap [0, 1] \end{cases}$$

Solution. Clearly the partitions are,

$$P = \left\{ \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} \right\}$$

Also,

$$m_i = 0 \wedge M_i = 1$$

Thus, obviously, $L(P, f) \neq U(P, f)$ and hence the function is not Riemann Integrable.

If a function is not Riemann integrable, it does not mean that the integral does not exist.

3.2 Double Integrals

The double integral when $x \in [a, b]$ and $y \in [c, d]$ is defined as,

$$\int_c^d \int_a^b f(x, y) dx dy$$

The integral $\int_c^d \int_a^b f(x, y) dx dy$ and $\int_a^b \int_c^d f(x, y) dy dx$ are equal if f is continuous.

3.2.1 X-Regular and Y-Regular

Consider the integral,

$$\iint_{\Omega} f(x, y) dx dy$$

Where Ω is the region over which we are integrating.

The interval Ω is said to be x -regular if $\Omega = \{(x, y) : g(y) \leq x \leq h(y), c \leq y \leq d\}$ which means the integral is,

$$\int_c^d \int_{g(y)}^{h(y)} f(x, y) dx dy$$

The interval Ω is said to be y -regular if $\Omega = \{(x, y) : g(x) \leq y \leq h(x), a \leq x \leq b\}$ which means the integral is,

$$\int_a^b \int_{g(x)}^{h(x)} f(x, y) \, dy \, dx$$

If $f(x, y) = 1$, then $\iint_{\omega} f(x, y) \, dA$ gives area.

Example. Evaluate the integral where ω is bounded by the triangle $y = 0$, $x = 1$ and $y = x$.

$$\iint_{\Omega} x + y + xy \, dA$$

Solution. The region is defined by

$$x \in [0, 1] \wedge 0 \leq y \leq x$$

Thus, the integral is setup as,

$$\int_0^1 \int_0^x x + y + xy \, dy \, dx$$

It can also be written as,

$$y \in [0, 1] \wedge y \leq x \leq 1$$

Hence,

$$\int_0^1 \int_y^1 x + y + xy \, dx \, dy$$

So, we can say that the region Ω is both x -regular and y -regular.

Example. Evaluate

$$\iint_{\Omega} 2 + 4x \, dA$$

Where Ω is bounded by $y = x$ and $y = x^2$.

Solution. The region is defined by,

$$x \in [0, 1] \wedge x^2 \leq y \leq x$$

Thus, the integral is,

$$I = \int_0^1 \int_{x^2}^x 2 + 4x \, dy \, dx = \frac{2}{3}$$

Example. Find the area bounded by $y = 2x^2$ and $y^2 = 4x$.

Solution. The region is defined by,

$$x \in [0, 1] \wedge 2x^2 \leq y \leq 2\sqrt{x}$$

Thus, the integral is,

$$I = \int_0^1 \int_{2x^2}^{2\sqrt{x}} dy \, dx = \frac{2}{3}$$

Example. Find the volume of the solid $z = x^2 + y^2$ over the bounded region $y = x$, $x = 0$ and $x + y = 2$.

Solution. Clearly the region is described by,

$$x \in [0, 1] \wedge x \leq y \leq 2 - x$$

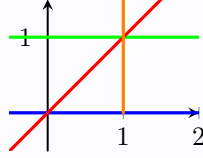
Thus, the volume is given by the following double integral,

$$V = \int_0^1 \int_x^{2-x} x^2 + y^2 \, dy \, dx = \frac{4}{3}$$

Example. Evaluate

$$\int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy$$

Solution. We can interchange the limits,



$$\begin{aligned} \int_0^1 \int_y^1 \frac{\sin x}{x} \, dx \, dy &= \int_0^1 \int_0^x \frac{\sin x}{x} \, dy \, dx \\ &= 1 - \cos 1 \end{aligned}$$

Example. Evaluate

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx$$

Solution. We can convert the integral into the following,

$$\begin{aligned} \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx &= \int_0^\infty \int_a^b e^{-yx} \, dy \, dx \\ &= \int_a^b \int_0^\infty e^{-yx} \, dx \, dy \\ &= \ln \frac{b}{a} \end{aligned}$$

3.3 Change of Variables

Consider the integral,

$$\iint f(x, y) \, dx \, dy$$

Now, if $x = p_1(u, v)$ and $y = p_2(u, v)$, then the integral can be expressed as,

$$\iint g(u, v) \mathbb{J}(u, v) \quad \left(\mathbb{J}(u, v) = \frac{\partial(x, y)}{\partial(u, v)} \right)$$

Theorem 14 (Gaussian Integral). The function $f(x) = e^{-x^2}$ is called the Gaussian function. It does not have an elementary indefinite integral. But the total area under the area can be calculated.

$$\int_0^\infty e^{-x^2} \, dx$$

Proof. Let,

$$\begin{aligned}
 I &= \int_0^{\infty} e^{-x^2} dx \\
 I &= \int_0^{\infty} e^{-y^2} dy \\
 \Rightarrow I^2 &= \left(\int_0^{\infty} e^{-x^2} dx \right) \left(\int_0^{\infty} e^{-y^2} dy \right)
 \end{aligned}$$

Here, since the variables are independent, then the integral can be resolved into a double integral,

$$I^2 = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

Now, changing into polar coordinates, i.e.,

$$x = r \cos \theta \wedge y = r \sin \theta \wedge \mathbb{J}(u, v) = r$$

The limits are now changed to,

$$r \in (0, \infty) \wedge \theta \text{ in } \left(0, \frac{\pi}{2}\right)$$

$$I^2 = \int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2} \cdot r dr d\theta$$

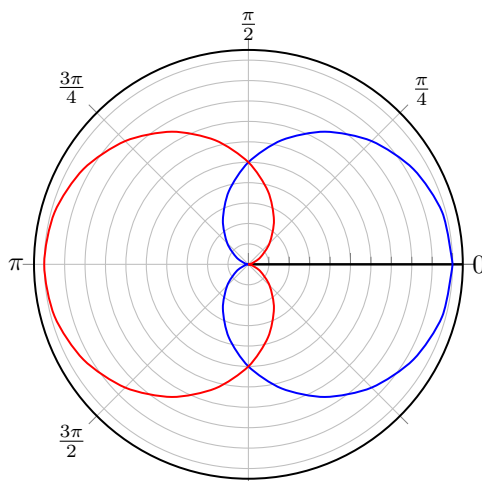
$$I^2 = \frac{\pi}{4}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}$$

□

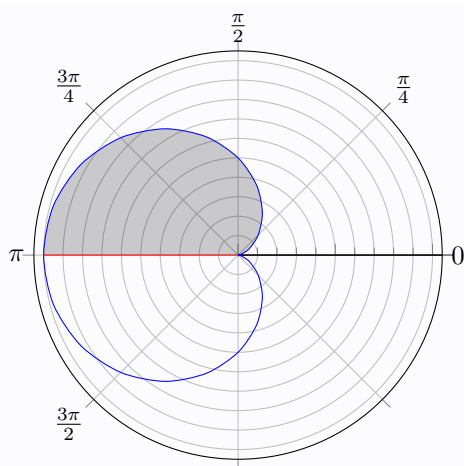
3.3.1 Integrals using Polar Coordinate Systems

Cardioid



$$r = 1 + \cos \theta \text{ and } r = 1 - \cos \theta$$

Example. Evaluate $\iint 3y dA$ above the x-axis and below the cardioid $r = 1 - \cos \theta$.



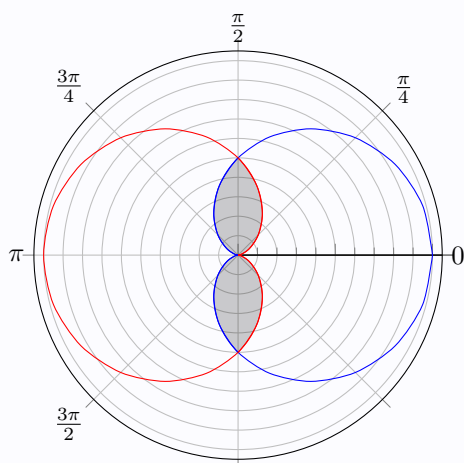
Solution. Here,

$$r \in (0, 1 - \cos \theta) \wedge \theta \in (0, \pi)$$

Thus, the integral can be written as,

$$\begin{aligned} I &= \int_0^\pi \int_0^{1-\cos \theta} 3r \sin \theta \cdot r \, dr \, d\theta \\ &= 4 \end{aligned}$$

Example. Find the area common the cardioids, $r = 1 - \cos \theta$ and $r = 1 + \cos \theta$.



Solution. By symmetry,

$$\begin{aligned} I &= 4 \int_0^{\pi/2} \int_0^{1-\cos \theta} r \, dr \, d\theta \\ &= \frac{3\pi}{4} - 4 \end{aligned}$$

3.4 Triple Integrals

Theorem 15. The conversion of Cartesian coordinates to spherical coordinates can be done by,

$$\begin{aligned} x &= r \cos \theta \sin \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \phi \end{aligned}$$

And the Jacobian is given by,

$$\mathbb{J}(r\theta, \phi) = r^2 \sin \phi$$

And the angles lie between,

$$\theta \in [0, 2\pi] \wedge \phi \in [0, \pi]$$

Example. Evaluate $\iiint_D x^2 + y^2 + z^2 \, dx \, dy \, dz$ where $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.

Solution. Converting to polar coordinates,

$$r \in [0, 1] \wedge \theta \in [0, 2\pi] \wedge \phi \in [0, \pi]$$

We get,

$$\begin{aligned} I &= \int_0^\phi \int_0^{2\pi} \int_0^1 r^2 \cdot r^2 \sin \phi \, dr \, d\theta \, d\phi \\ &= \left(\int_0^1 r^4 \, dr \right) \left(\int_0^{2\pi} d\theta \right) \left(\int_0^\pi \sin \phi \, d\phi \right) \\ &= \frac{4\pi}{5} \end{aligned}$$

Example (Not sure). Evaluate the integral $\iint_D xy\sqrt{1-x-y} \, dx \, dy$, where D is the region bounded by $x = 0$, $y = 0$ and $x + y = 1$ using the transform $x + y = u$ and $xy = v$.

Solution. Using the transformation of variables, we get,

$$u(1-v) = 0 \wedge uv = 0 \wedge u = 1$$

$$u \in [0, 1] \wedge v \in [0, 1]$$

The Jacobian is,

$$\mathbb{J}(u, v) = u$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^1 \int_0^1 v\sqrt{1-u} \cdot u \, du \, dv \\ &= \frac{2}{15} \end{aligned}$$

Example. Find the volume of the solid surrounded by the surface $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} + \left(\frac{z}{c}\right)^{\frac{2}{3}} = 1$.

Solution. First, choose,

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} = u \wedge \left(\frac{y}{b}\right)^{\frac{2}{3}} = v \wedge \left(\frac{z}{c}\right)^{\frac{2}{3}} = w$$

Thus, the curve is,

$$S : u^2 + v^2 + w^2 = 1 \quad (\text{A unit sphere})$$

The Jacobian is,

$$\mathbb{J}(u, v, w) = 27abc \cdot u^2 v^2 w^2$$

The integral now is,

$$I = \iiint_S 27abc \cdot u^2 v^2 w^2 \, dV$$

Now, converting to spherical coordinates,

$$u = r \cos \theta \sin \phi \wedge v = r \sin \theta \sin \phi \wedge w = r \cos \phi$$

The limits are

$$r \in [0, 1] \wedge \theta \in [0, 2\pi] \wedge \phi \in [0, \pi]$$

Thus, the final integral is,

$$\begin{aligned} I &= 27abc \int_0^{2\pi} \int_0^\pi \int_0^1 (r^3 \sin^2 \phi \cos \phi \sin \theta \cos \theta)^2 \cdot r^2 \sin \phi \, dr \, d\phi \, d\theta \\ &= 27abc \left(\int_0^1 r^8 \, dr \right) \left(\int_0^\pi \sin^5 \phi \cos^2 \phi \, d\phi \right) \left(\int_0^{2\pi} \sin^2 \theta \cos^2 \theta \, d\theta \right) \\ &= \frac{4\pi abc}{35} \end{aligned}$$

3.5 Beta Function

The function is defined as,

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx \quad m, n \geq 1$$

Theorem 16 (Symmetry).

$$\beta(m, n) = \beta(n, m)$$

Theorem 17.

$$\beta(m, n) = 2 \int_0^{\frac{\pi}{2}} \sin^{2m-1} \theta \cos^{2n-1} \theta \, d\theta$$

3.6 Gamma Function

This is an extension to the factorial,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx = (n-1)!$$

3.7 Tutorial-6

3.7.1 Evaluate the double integrals

Question. $\iint_R x^2 \, dA$, where R is the region bounded by $y = x^2$, $y = x + 2$.

Answer. The region described is,

$$x \in [-1, 2] \wedge x^2 \leq y \leq x + 2$$

Thus, the integral is,

$$\begin{aligned} I &= \int_{-1}^2 \int_{x^2}^{x+2} x^2 \, dy \, dx \\ &= \frac{163}{60} \end{aligned}$$

Question. $\iint_R (x^2 + y^2) \, dA$, where $R : 0 \leq y \leq \sqrt{1 - x^2}$, $0 \leq x \leq 1$.

Answer. Clearly,

$$\begin{aligned} I &= \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) \, dy \, dx \\ &= \frac{\pi}{8} \end{aligned}$$

Question. $\iint_R a^2 - x^2 - y^2 \, dA$, where R is the region $x^2 + y^2 \leq a^2$.

Answer. The region described is,

$$x \in [-1, 1] \wedge -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2} \quad (\because \text{Upper and lower parts of the disc})$$

Thus,

$$\begin{aligned} I &= \int_{-1}^1 \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a^2 - x^2 - y^2) \, dy \, dx \\ &= \text{Something} \end{aligned}$$

The better way is to change into polar coordinates, since the region described is a complete circle,

$$r \in [0, a] \wedge \theta \in [0, 2\pi]$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^a \int_0^{2\pi} (a^2 - r^2) \cdot r \, d\theta \, dr \\ &= \left(\int_0^a a^2 r - r^3 \, dr \right) \left(\int_0^{2\pi} d\theta \right) \\ &= \frac{\pi a^4}{2} \end{aligned}$$

Question. $\iint_R xy \, dx \, dy$, where R is the region bounded by x -axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$.

Answer. The region described is,

$$y \in [0, a] \wedge 2\sqrt{ay} \leq x \leq 2a$$

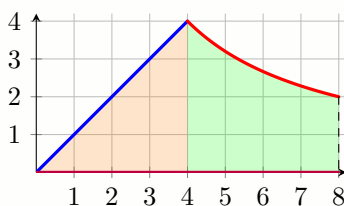
Thus, the integral is,

$$\begin{aligned} I &= \int_0^a \int_{2\sqrt{ay}}^{2a} xy \, dx \, dy \\ &= \frac{a^4}{3} \end{aligned}$$

Question. $\iint_R x^2 \, dx \, dy$, where R is the region in the first quadrant bounded by the lines $x = y$, $y = 0$, $x = 8$ and the curve $xy = 16$.

Answer. The region can be described in the following way,

$$(x \in [0, 4] \wedge 0 \leq y \leq x) \wedge \left(x \in [4, 8] \wedge 0 \leq y \leq \frac{16}{x} \right)$$

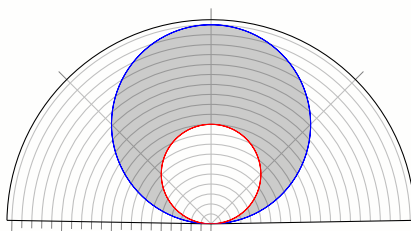


Thus, the integral is,

$$\begin{aligned} I &= \int_0^4 \int_0^x x^2 \, dy \, dx + \int_4^8 \int_0^{\frac{16}{x}} x^2 \, dy \, dx \\ &= 448 \end{aligned}$$

Question. $\iint r^3 \, dr \, d\theta$, over the area included between $r = 2 \sin \theta$ and $r = 4 \sin \theta$.

Answer. Clearly,



$$\theta \in [0, \pi] \wedge 2 \sin \theta \leq r \leq 4 \sin \theta$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^\pi \int_{2 \sin \theta}^{4 \sin \theta} r^3 \, dr \, d\theta \\ &= \frac{90\pi}{4} \end{aligned}$$

3.7.2 Evaluate the following integrals by changing the order of integration if needed

Question.

$$\int_0^3 \int_{-y}^y x^2 + y^2 \, dx \, dy$$

Answer. Here the limits are,

$$y \in [0, 3] \wedge -y \leq x \leq y$$

These are equivalent to,

$$(x \in [-3, 0] \wedge -x \leq y \leq 3) \wedge (x \in [0, 3] \wedge x \leq y \leq 3)$$

Thus, the integral is,

$$\begin{aligned} I &= \int_{-3}^0 \int_{-x}^3 x^2 + y^2 \, dy \, dx + \int_0^3 \int_x^3 x^2 + y^2 \, dy \, dx \\ &= \frac{54}{3} \end{aligned}$$

Question.

$$\int_0^1 \int_2^{4-2x} dy \, dx$$

Answer. Here the limits are,

$$x \in [0, 1] \wedge 2 \leq y \leq 4 - 2x$$

These are equivalent to,

$$y \in [2, 4] \wedge 0 \leq x \leq \frac{4-y}{2}$$

Thus, the integral is,

$$\begin{aligned} I &= \int_2^4 \int_0^{\frac{4-y}{2}} dx \, dy \\ &= 1 \end{aligned}$$

Question.

$$\int_0^{\frac{3}{2}} \int_0^{9-4x^2} 16x \, dy \, dx$$

Answer. Here the limits are,

$$x \in [0, \frac{3}{2}] \wedge 0 \leq y \leq 9 - 4x^2$$

These are equivalent to,

$$y \in [0, 9] \wedge 0 \leq x \leq \frac{\sqrt{9-y}}{2}$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^9 \int_0^{\frac{\sqrt{9-y}}{2}} 16x \, dx \, dy \\ &= 81 \end{aligned}$$

Question.

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} \, dy \, dx$$

Answer. Here the limits are,

$$x \in [0, 2] \wedge 0 \leq y \leq 4 - x^2$$

These are equivalent to,

$$y \in [0, 4] \wedge 0 \leq x \leq \sqrt{4 - y}$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y} dx dy \\ &= \frac{e^8}{4} - \frac{1}{4} \end{aligned}$$

Question.

$$\int_{-a}^a \int_0^{\sqrt{a^2-y^2}} f(x, y) dx dy$$

Answer. Here the limits are,

$$y \in [-a, a] \wedge 0 \leq x \leq \sqrt{a^2 - y^2}$$

These are equivalent to,

$$x \in [0, a] \wedge -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2}$$

Thus, the integral is,

$$I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x, y) dy dx$$

Question.

$$\int_0^1 \int_{e^x}^e \frac{1}{\ln y} dy dx$$

Answer. Here the limits are,

$$x \in [0, 1] \wedge e^x \leq y \leq e$$

These are equivalent to,

$$y \in [1, e] \wedge 0 \leq x \leq \ln y$$

Thus, the integral is,

$$\begin{aligned} I &= \int_1^e \int_0^{\ln y} \frac{1}{\ln y} dx dy \\ &= e - 1 \end{aligned}$$

Question. By changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin px dx dy$, show that $\int_0^\infty \frac{\sin px}{x} dx = \frac{\pi}{2}$.

Answer. Let,

$$I = \int_0^\infty \int_0^\infty e^{-xy} \sin px dx dy$$

Changing the order of integration we get,

$$\begin{aligned} I &= \int_0^{\infty} \int_0^{\infty} e^{-xy} \sin px \, dy \, dx \\ &= \int_0^{\infty} \sin px \left. \frac{-1}{x} e^{-xy} \right|_0^{\infty} dx \\ \therefore I &= \int_0^{\infty} \frac{\sin px}{x} dx \end{aligned}$$

Integrating the original integral,

$$\begin{aligned} I &= \int_0^{\infty} \frac{e^{-yx}}{y^2 - p^2} (-y \sin px - p \cos px) \Big|_0^{\infty} dy \\ \therefore I &= \frac{\pi}{2} \end{aligned}$$

Thus, also,

$$\int_0^{\infty} \frac{\sin px}{x} = \frac{\pi}{2}$$

3.7.3 Find the volume of the following

Question. Region bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy plane.

Answer. Clearly the integral set-up is,

$$I = \iint_{\Omega} x^2 \, dA$$

The region Ω can be described as,

$$x \in [-2, 1] \wedge x \leq y \leq 2 - x^2$$

Integrating,

$$\begin{aligned} I &= \int_0^1 \int_x^{2-x^2} x^2 \, dy \, dx \\ &= \frac{63}{20} \end{aligned}$$

3.7.4 Use the given transformations to transform the integrals and evaluate them

Question. $u = 3x + 2y$, $v = x + 4y$ and $I = \iint_R (3x^2 + 14xy + 8y^2) \, dA$ where R is the region in the first quadrant bounded by the lines $y + \frac{3x}{2} = 1$, $y + \frac{3x}{2} = 3$, $y + \frac{x}{4} = 0$ and $y + \frac{x}{4} = 1$.

Answer. The area bound is a parallelogram,

$$x = \frac{4u - 2v}{10} \wedge y = \frac{3v - u}{10}$$

The limits are,

$$u \in [2, 6] \wedge v \in [0, 4]$$

The Jacobian is,

$$\mathbb{J}(u, v) = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{2}{5} & \frac{-1}{10} \\ \frac{-1}{10} & \frac{3}{10} \end{vmatrix} = \frac{1}{10}$$

Thus, the integral is,

$$\begin{aligned} I &= \int_2^6 \int_0^4 uv \cdot \frac{1}{10} dv du \\ &= 12.8 \end{aligned}$$

Question. $u = x + 2y$, $v = x - y$ and $I = \int_0^{\frac{2}{3}} \int_y^{2-2y} (x + 2y)e^{y-x} dA$.

Answer. The limits are,

$$u \in [0, 2] \wedge v \in [0, u]$$

The Jacobian is,

$$\mathbb{J}(u, v) = \frac{1}{J(x, y)} = \frac{1}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{-1}{3} \implies |\mathbb{J}(u, v)| = \frac{1}{3}$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^2 \int_0^u ue^{-v} \cdot \frac{1}{3} dv du \\ &= \frac{3e^{-2} + 1}{3} \end{aligned}$$

Question. $u = xy$, $v = x^2 - y^2$ and $I = \iint_R (x^2 + y^2) dA$, where R is the region bounded by $xy = 1$, $xy = 2$, $x^2 - y^2 = 1$ and $x^2 - y^2 = 2$.

Answer. The limits are,

$$u \in [1, 2] \wedge v \in [1, 2]$$

The Jacobian is,

$$\mathbb{J}(u, v) = \frac{1}{J(x, y)} = \frac{1}{\begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix}} = \frac{-1}{2(x^2 + y^2)} \implies |\mathbb{J}(u, v)| = \frac{1}{2(x^2 + y^2)}$$

The integral is,

$$\begin{aligned} I &= \int_1^2 \int_1^2 (x^2 + y^2) \cdot \frac{1}{2(x^2 + y^2)} dv du \\ &= \frac{1}{2} \end{aligned}$$

Question. $x = au$, $y = bv$, $z = cw$, and $I = \iiint_D dV$ where D is the ellipsoid : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Answer. The Jacobian is,

$$\mathbb{J}(u, v, w) = \frac{1}{\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}} = abc$$

The ellipsoid is converted to,

$$S : u^2 + v^2 + w^2 = 1$$

Which is a unit sphere. The integral is,

$$\begin{aligned} I &= \iiint_S abc dV \\ &= \frac{4abc}{3} \pi \end{aligned}$$

Question. $u = x$, $v = xy$, $w = 3z$ and $I = \iiint_D x^2y + 3xyz \, dV$ where $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x \leq 2, 0 \leq xy \leq 2, 0 \leq z \leq 1\}$.

Answer. The limits are,

$$u \in [1, 2] \wedge v \in [0, 2] \wedge w \in [0, 3]$$

The Jacobian is,

$$\mathbb{J}(u, v, w) = \frac{1}{3u}$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^3 \int_0^2 \int_1^2 v + \frac{vw}{u} \, du \, dv \, dw \\ &= 2 + 3 \ln 2 \end{aligned}$$

Question. Using appropriate transformation evaluate $\iint_R dA$, where R is the parallelogram with vertices $(1, 0)$, $(3, 1)$, $(2, 2)$ and $(0, 1)$.

Answer. The equations for the parallel lines are,

$$x + y = 1 \wedge x + y = 4 \wedge x - 2y = 1 \wedge x - 2y = -1$$

Thus, consider the transformation,

$$u = x + y \wedge v = x - 2y$$

Thus, the limits are,

$$u \in [1, 4] \wedge v \in [-1, 1]$$

The Jacobian is,

$$\mathbb{J}(u, v) = \frac{1}{\mathbb{J}(x, y)} = \frac{-1}{3} \implies |\mathbb{J}| = \frac{1}{3}$$

Thus, the integral is,

$$\begin{aligned} I &= \int_1^4 \int_{-1}^1 \frac{1}{3} \, dv \, du \\ &= 2 \end{aligned}$$

3.7.5 Evaluate the following volume integrals

Question. $\iiint_D (z^2x^2 + z^2y^2) \, dV$, where $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, -1 \leq z \leq 1\}$.

Answer. Converting to cylindrical coordinates,

$$r \in [0, 1] \wedge \theta \in [0, 2\pi] \wedge z \in [-1, 1]$$

Thus, the integral is,

$$\begin{aligned} I &= \int_{-1}^1 \int_0^{2\pi} \int_0^1 r^2 z^2 \cdot r \, dr \, d\theta \, dz \\ &= \frac{\pi}{3} \end{aligned}$$

Question. $\iiint_D xyz \, dV$, where $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq x^2 + y^2\}$.

Answer. Converting to cylindrical coordinates,

$$r \in [0, 1] \wedge \theta \in [0, 2\pi] \wedge z \in [0, r^2]$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 \int_0^{r^2} z r^2 \sin \theta \cos \theta \cdot r \, dz \, dr \, d\theta \\ &= 0 \end{aligned}$$

Question. $\iiint_D e^{(x^2+y^2+z^2)^{\frac{3}{2}}} \, dV$, where $D = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.

Answer. Converting to spherical coordinates,

$$r \in [0, 1] \wedge \theta \in [0, 2\pi] \wedge \phi \in [0, \pi]$$

Thus, the integral is,

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^\pi \int_0^1 e^{r^3} \cdot r^2 \sin \phi \, dr \, d\phi \, d\theta \\ &= \frac{4\pi(e-1)}{3} \end{aligned}$$

3.7.6 Using Beta and Gamma functions, evaluate the following

Question.

$$\int_0^\infty e^{-x^2} \, dx$$

Question.

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan x} \, dx$$

Question.

$$\int_0^1 x^m \left(\ln \frac{1}{x} \right)^n \, dx$$

Question.

$$\int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^6 \theta \, d\theta$$

Question.

$$\int_0^\infty \frac{x^c}{c^x} \, dx$$

Question.

$$\int_0^\infty a^{-bx^2} \, dx$$

Question.

$$\int_0^1 x^5 \left(\ln \frac{1}{x} \right)^3 \, dx$$

Question. Express $\int_0^1 x^m (1 - x^p)^n dx$ in terms of Beta function and hence evaluate the integral,

$$\int_0^1 x^{\frac{3}{2}} (1 - \sqrt{x})^{\sqrt{12}} dx$$

Question. Show that,

$$\beta(p, q) = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}} dy = \int_0^1 \frac{x^{p-1} + y^{q-1}}{(1+x)^{p+q}}$$