Basics of Electronics and Communication Engineering (FCE0106)

Srijan Mahajan (2023UCM2326)
Prof. Navneeta

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Chapter 1

Signals and Systems

A signal is defined as any physical quantity that varies with time, space, or any other independent variable or variables.
The types of signals are:

- **Continuous Time:** Represented as $x(t)$.
- **Discrete Time:** Represented as $x[n]$.

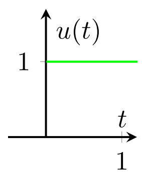
A system is an entity that processes a set of input signals to yield another set of output signals.

1.1 Elementary Functions

1.1.1 Unit Step Function

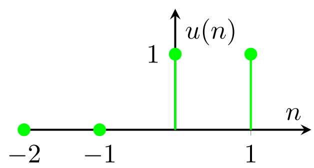
$$u(t) = \begin{cases} 1 & ,t > 0 \\ \frac{1}{2} & ,t = 0 \\ 0 & ,t < 0 \end{cases}$$

It is continuous for all t, except $t = 0$.



In discrete time, it is defined as,

$$u(n) = \begin{cases} 1 & ,n \geq 0 \\ 0 & ,n < 0 \end{cases}$$

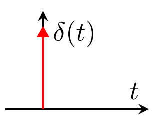


1.1.2 Unit Impulse Function

It is also called the Dirac Delta Function.

$$\delta(t) = 0 \ (t \neq 0) \wedge \int_{-\infty}^{\infty} \delta(t) \, \mathrm{d}t = 1$$

The Delta function has many properties which is useful in analysis of functions.



Theorem 1 (Sifting Property).

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,\mathrm{d}t = x(t)\Big|_0 = x(0)$$

Proof. Since the function is non-zero only at $t = 0$, we can say,

$$\int_{-\infty}^{\infty} x(t)\delta(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} x(0)\delta(t)\,\mathrm{d}t = x(0)$$

□

Theorem 2 (Another form).

$$\int_{t_1}^{t_2} x(t)\delta(t-t_0)\,\mathrm{d}t = \begin{cases} x(t_0) & , t_1 < t_0 < t_2 \\ 0 & , \text{otherwise} \end{cases}$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau$$

Theorem 3 (Scaling Property).

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

From this it follows that δ is an even function.

Theorem 4 (Sampling Property).

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

Theorem 5 (Differentiation Property).

$$\int_{-\infty}^{\infty} x(t)\delta'(t)\,\mathrm{d}t = -x'(0)$$

Theorem 6 (Amplitude Reversal).

$$t\delta'(t) = -\delta(t)$$

Theorem 7 (Derivative of Impulse Functions).

$$\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} = \delta'(t) = 0 \; (t \neq 0)$$

Where,

$$\int_{-\infty}^{\infty} \delta'(t) = 0$$

1.1.3 Discrete Time Unit Impulse Function

$$\delta(n) = \begin{cases} 1 & , n = 0 \\ 0 & , n \neq 0 \end{cases}$$

Theorem 8.

$$\delta(kn) = \delta(n)$$

Theorem 9.

$$\delta(n) = u(n) - u(n-1)$$

Or,

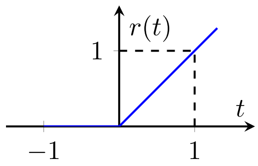
$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) = \sum_{k=-\infty}^{\infty} \delta(k)$$

Theorem 10.

$$x(n)\delta(n-k) = x(k)\delta(n-k)$$

1.1.4 Unit Ramp Function

$$r(t) = tu(t) = \begin{cases} t & ,t \geq 0 \\ 0 & ,t < 0 \end{cases}$$

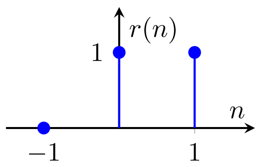


Theorem 11.

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = \delta(t) \ \wedge \ \frac{\mathrm{d}r(t)}{\mathrm{d}t} = u(t)$$

In discrete time it is defined as

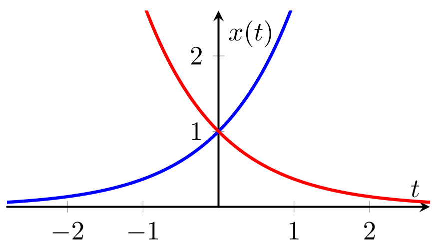
$$r(n) = nu(n)$$



1.1.5 Real Exponential Signals

$$x(t) = Ce^{at}$$

Similar results for discrete time exponential signals.



$a > 0$ and $a < 0$

1.1.6 Complex Exponential Signals

$$x(t) = e^{j\omega_0 t}$$

It has time period $\frac{2\pi}{|\omega_0|}$.

1.1.7 Signum Function

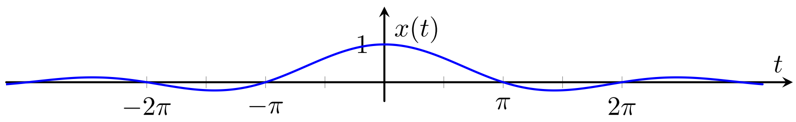
$$sgn(t) = \begin{cases} 1 & ,t > 0 \\ 0 & ,t = 0 \\ -1 & ,t < 0 \end{cases} = 2u(t) - 1$$

1.1.8 Sampling Function

$$Sa(t) = \frac{\sin t}{t}$$

To make the function continuous at $t = 0$, we define $Sa(0) = 1$.

$$sinc(t) = \frac{\sin \pi t}{\pi t} = Sa(\pi t)$$



1.2 Periodic and Aperiodic Signals

A signal is said to be periodic iff there exits T such that,

$$x(t+T) = x(t)\ \forall t$$

Similarly for a discrete time signal,

$$x(n+N) = x(n)\ \forall n$$

Theorem 12.

$$\int_{a}^{a+T} x(t)\,\mathrm{d}t = \int_{b}^{b+T} x(t)\,\mathrm{d}t\ \ \forall a, b$$

Theorem 13. A sum of M periodic continuous time signals is periodic iff,

$$\frac{T}{T_i} = n_i \quad 1 \le i \le M\ \ \wedge n_i \in \mathbb{Z}$$

1.3 Energy and Power Signals

The energy of a signal is given as,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t\ \text{ OR } \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The power of a signal is given as,

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} |x(t)|^2\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\,\mathrm{d}t\ \text{ OR } \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

A signal is said to be an energy signal if E_x is finite and $P_x = 0$. A signal is said to be a power signal[1] if P_x is finite and $E_x = \infty$.

A signal cannot be an energy signal and power signal at the same time. But it is possible for a signal to be neither an energy signal or a power signal.

[1] It is not practically possible to have a true power signal. All finite periodic signals are power signals.

1.4 Even and Odd Signals

A signal is said to even when,

$$x(-t) = x(t)$$

A signal is said to be odd when,

$$x(-t) = -x(t)$$

Any given signal can be broken down into it's even and odd components.

$$x(t) = \mathcal{E}\{x(t)\} + \mathcal{O}\{x(t)\}$$

Where,

$$\mathcal{E}\{x(t)\} = \frac{x(t) + x(-t)}{2} \wedge \mathcal{O}\{x(t)\} = \frac{x(t) - x(-t)}{2}$$

Theorem 14 (Multiplications)**.**

$$\begin{aligned} \text{Odd} \times \text{Odd} &= \text{Even} \\ \text{Odd} \times \text{Even} &= \text{Odd} \\ \text{Even} \times \text{Even} &= \text{Even} \end{aligned}$$

Theorem 15 (Derivatives)**.**

$$\begin{aligned} \frac{\mathrm{d}(\text{Even})}{\mathrm{d}t} &= \text{Odd} \\ \frac{\mathrm{d}(\text{Odd})}{\mathrm{d}t} &= \text{Even} \end{aligned}$$

1.5 Convolution

1.5.1 Continuous Time Convolution

A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.

$$y(t) = (x * h)(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

It is commutative, associative and distributive.

Theorem 16 (Time Shifting)**.**

$$x(t-t_1) * h(t-t_2) = t(t-t_1-t_2)$$

Theorem 17 (Width Property)**.** If the duration of x and h are finite and equal to W_x and W_h, then the duration of the convolution is $W_x + W_h$.

Theorem 18 (Differentiation Property)**.**

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}x(t) *\right) h(t) = x(t) * \left(\frac{\mathrm{d}}{\mathrm{d}t}h(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}y(t)$$

Theorem 19 (Time Scaling Property)**.**

$$x(at) * h(at) = \frac{1}{|a|}y(at)$$

Theorem 20 (Even and Odd)**.**

$$\text{Odd} * \text{Odd} = \text{Even}$$
$$\text{Odd} * \text{Even} = \text{Odd}$$
$$\text{Even} * \text{Even} = \text{Even}$$

Theorem 21 (Area Property)**.**

$$\int_{-\infty}^{\infty} y(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} x(t)\,\mathrm{d}t \int_{-\infty}^{\infty} h(t)\,\mathrm{d}t$$

1.5.2 Discrete Time Convolution

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

It is also commutative, associative and distributive. Similar properties like Time Shifting, Width Property, Sum Property[2] are similarly valid.

1.6 Continuous-Time Fourier Series

The Fourier Series allows us to represent any periodic signal as the sum of harmonically related sinusoidal functions. Any periodic signal can be expressed as a Fourier Series if it satisfies the Dirichlet conditions,

- if it is discontinuous, there are a finite number of discontinuities in the period T
- it has a finite average value over the period T
- it has a finite number of positive and negative maxima in the period T

1.6.1 Trigonometric Fourier Series

Any function x satisfying Dirichlet conditions can be expressed as,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

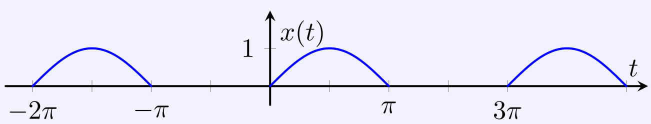
Where the coefficients are given as,

$$a_0 = \frac{1}{T}\int_0^T x(t)\,\mathrm{d}t \;\wedge\; a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)\,\mathrm{d}t \;\wedge\; b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)\,\mathrm{d}t$$

The integral may be carried over any full period.

- For any even function, $b_n = 0$
- For any odd function, $a_0 = 0 \wedge a_n = 0$

Example. Find the trigonometric Fourier Series for half-wave rectified sine wave.



Solution. Clearly, $T = 2\pi \implies \omega_0 = 1$. Moreover, x can be represented as,

$$x(t) = \begin{cases} \sin t & , 0 < t < \pi \\ 0 & , \pi < t < 2\pi \end{cases}$$

[2]Analogous to Area Property

Now, calculating the Fourier coefficients,

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \sin t \,\mathrm{d}t = \frac{1}{2\pi}\int_0^{\pi} \sin t \,\mathrm{d}t = \frac{1}{\pi}$$

$$\begin{aligned} a_n &= \frac{2}{2\pi}\int_0^{\pi} \sin t \cos(nt)\,\mathrm{d}t \\ &= \frac{\cos(\pi n)+1}{\pi(1-n^2)} \\ &= \begin{cases} 0 & , n = 3, 5, \ldots \\ \frac{2}{\pi(1-n^2)} & , n = 2, 4, \ldots \end{cases} \end{aligned}$$

Clearly, at $n = 1$,

$$\lim_{n\to 1} \frac{\cos(\pi n)+1}{\pi(1-n^2)} = 0 \implies a_1 = 0$$

$$\begin{aligned} b_n &= \frac{2}{2\pi}\int_0^{\pi} \sin t \sin(nt)\,\mathrm{d}t \\ &= \frac{\sin(n\pi)}{\pi(1-n^2)} \\ &= 0 \quad n \neq 1 \end{aligned}$$

Clearly, at $n = 1$,

$$\lim_{n\to 1} \frac{\sin(n\pi)}{\pi(1-n^2)} = 0 \implies b_1 = \frac{1}{2}$$

Now, for the line spectrum,

$$c_0 = \frac{1}{\pi} \wedge c_1 = \frac{1}{2} \wedge c_n = |a_n| \; (\forall n \geq 2)$$

1.6.2 Exponential Fourier Series

Any function x satisfying Dirichlet conditions can be expressed as,

$$x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}$$

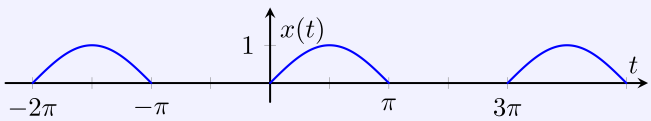
Where X_n is given by,

$$X_n = \frac{1}{T}\int_0^{T} x(t)e^{-jn\omega_0 t}\,\mathrm{d}t$$

Relation between Trigonometric Fourier Series and Exponential Fourier Series

$$X_n = \frac{a_n - jb_n}{2} \wedge X_{-n} = \frac{a_n + jb_n}{2}$$

Example. Find the Exponential Fourier Series for half-wave rectified sine wave.



Solution. Clearly, $T = 2\pi \implies \omega_0 = 1$. Moreover, x can be represented as,

$$x(t) = \begin{cases} \sin t & , 0 < t < \pi \\ 0 & , \pi < t < 2\pi \end{cases}$$

Now, calculating the Fourier coefficients,

$$\begin{aligned} X_n &= \frac{1}{2\pi}\int_0^T \sin t e^{-jnt}\,\mathrm{d}t \\ &= \frac{e^{-jn\pi}+1}{2\pi(1-n^2)} \\ &= \begin{cases} \frac{1}{\pi(1-n^2)} & , n = 0, \pm 2, \pm 4, \ldots \\ 0 & , n = \pm 3, \pm 5, \ldots \end{cases} \end{aligned}$$

Clearly, at $n = \pm 1$,

$$\lim_{n\to\pm 1} \frac{e^{-jn\pi}+1}{2\pi(1-n^2)} = \frac{\mp j}{4}$$

Theorem 22 (Time Shifting).

$$x(t-t_0) \leftrightarrow e^{-jn\omega_0 t_0} X_n$$

Theorem 23 (Frequency Shifting).

$$e^{jm\omega_0 t} x(t) \leftrightarrow X_{n-m}$$

Theorem 24 (Time Reversal).

$$x(-t) \leftrightarrow X_{-n}$$

Theorem 25 (Time Scaling).

$$x(at) \leftrightarrow X_n$$

Theorem 26 (Periodic Convolution). The periodic convolution of two periodic signals with same period is defined by,

$$x(t) \circledast y(t) = \frac{1}{T}\int_0^T x(\tau)y(t-\tau)\,\mathrm{d}\tau$$

Then, we can say that,

$$x(t) \circledast y(t) \leftrightarrow X_n Y_n$$

Theorem 27 (Multiplication).

$$x(t)y(t) \leftrightarrow \sum_{k=-\infty}^{\infty} X_k Y_{n-k}$$

Theorem 28 (Differentiation).

$$\frac{\mathrm{d}^m x(t)}{\mathrm{d}t^m} \leftrightarrow (jn\omega_0)^m X_n$$

Theorem 29 (Integration).

$$\int_{-\infty}^{t} x(t)\,\mathrm{d}t \leftrightarrow \frac{1}{jn\omega_0} X_n$$

Theorem 30 (Parseval's Theorem for Power Signals). If $x(t) \leftrightarrow X_n$, then,

$$\frac{1}{T}\int_0^T |x(t)|^2\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} |X_n|^2 = X_0 + 2\sum_{n=1}^{\infty} |X_n|^2$$

The theorem states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Proof.

$$\begin{aligned}
\frac{1}{T}\int_0^T |x(t)|^2 \,\mathrm{d}t &= \int_0^T x(t)x^*(t)\,\mathrm{d}t \\
&= \int_0^T x(t)\left(\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}\right)^* \mathrm{d}t \\
&= \int_0^T x(t)\left(\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right) \mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} X_n^* \left(\frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,\mathrm{d}t\right) \\
&= \sum_{n=-\infty}^{\infty} X_n X_n^* \\
&= \sum_{n=-\infty}^{\infty} |X_n|^2
\end{aligned}$$

□

1.7 Discrete-Time Fourier Series

Any function x can be expressed as,

$$x(n) = \sum_{k=k_0}^{k_0+N-1} X_k e^{jk\omega_0 n} = \sum_{k=<N>} X_k e^{jk\omega_0 n}$$

[3] Where X_k is given by,

$$X_k = \frac{1}{N}\sum_{n=<N>} x(n)e^{-jk\omega_0 n}$$

Theorem 31 (Time Shifting).

$$x(n-n_0) \leftrightarrow e^{-jk\omega_0 n_0} X_k$$

Theorem 32 (Frequency Shifting).

$$e^{jM\omega_0 n} x(n) \leftrightarrow X_{k-M}$$

Theorem 33 (Time Reversal).

$$x(-n) \leftrightarrow X_{-k}$$

Theorem 34 (Periodic Convolution). The periodic convolution of two periodic signals with same period is defined by,

$$x(n) \circledast y(n) = \sum_{r=<N>} x(r)y(n-r)$$

Then, we can say that,

$$x(n) \circledast y(n) \leftrightarrow N X_k Y_k$$

Theorem 35 (Multiplication).

$$x(n)y(n) \leftrightarrow \sum_{r=<N>} X_r Y_{k-r}$$

[3]Here, $\sum_{k=<N>}$ represents sum over any range of consecutive k's of length N.

Theorem 36 (First Difference).

$$x(n) - x(n-1) \leftrightarrow (1 - e^{-jk\omega_0})X_k$$

Theorem 37 (Running Sum/Accumulation).

$$\sum_{k=-\infty}^{n} x(k) \leftrightarrow \frac{X_k}{1 - e^{-jk\omega_0}}$$

Theorem 38 (Parseval's Relation). If $x(n) \leftrightarrow X_k$, then,

$$\frac{1}{N} \sum_{n=<N>} |x(n)|^2 = \sum_{k=<N>} |X_k|^2$$

The theorem states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Proof.

$$\begin{aligned}
\frac{1}{N} \sum_{n=<N>} |x(n)|^2 &= \frac{1}{N} \sum_{n=<N>} x(n)x^*(n) \\
&= \frac{1}{N} \sum_{n=<N>} x(n) \left(\sum_{k=<N>} X_k e^{jk\omega_0 n} \right)^* \\
&= \sum_{k=<N>} X_k^* \left(\frac{1}{N} \sum_{n=<N>} x(n) e^{-jk\omega_0 n} \right) \\
&= \sum_{k=<N>} X_k X_k^* \\
&= \sum_{k=<N>} |X_k|^2
\end{aligned}$$

□

1.8 Continuous Time Fourier Transform

The Fourier Transform of x is given by,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, \mathrm{d}t$$

And the inverse Fourier Transform of X is given by,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, \mathrm{d}\omega$$

Theorem 39 (Time Shifting).

$$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(\omega)$$

Theorem 40 (Frequency Shifting).

$$x(t) e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$$

Theorem 41 (Time and Frequency Scaling).

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Theorem 42 (Area under $x(t)$).

$$\int_{-\infty}^{\infty} x(t)\,\mathrm{d}t = X(0)$$

Theorem 43 (Area under $X(\omega)$).

$$\int_{-\infty}^{\infty} X(\omega)\,\mathrm{d}\omega = 2\pi x(0)$$

Theorem 44 (Differentiation in Time Domain).

$$\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n} \leftrightarrow (j\omega)^n X(\omega)$$

Theorem 45 (Integration in Time Domain).

$$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau \leftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Theorem 46 (Differentiation in Frequency Domain).

$$t^n x(t) \leftrightarrow j^n \frac{\mathrm{d}^n X(\omega)}{\mathrm{d}\omega^n}$$

Theorem 47 (Convolution Property).

$$x(t) * y(t) \leftrightarrow X(\omega)Y(\omega)$$

Theorem 48.

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi}\left[X(\omega) * Y(\omega)\right]$$

Theorem 49 (Duality).

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Theorem 50 (Parseval's Relation).

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,\mathrm{d}\omega$$

1.8.1 Fourier Transform of Elementary Functions

Example (D.C. Value).

$$x(t) = A_0$$

Solution. Let there be a function $X(\omega) = A_0\delta(\omega)$, which is the Fourier Transform of $x(t)$,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,\mathrm{d}\omega$$

$$\therefore x(t) = \frac{A_0}{2\pi}$$

Thus,

$$\mathcal{F}\left\{\frac{A_0}{2\pi}\right\} = A_0\delta(\omega) \implies \mathcal{F}\{A_0\} = 2\pi A_0\delta(\omega)$$

Example (Impulse Function).

$$x(t) = \delta(\omega)$$

Solution.

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} \, \mathrm{d}t$$

$$\therefore X(\omega) = 1$$

Thus,

$$\mathcal{F}\left\{\delta(t)\right\} = 1$$

Example (Exponential).

$$x(t) = e^{-at} u(t)$$

Solution.

$$\begin{aligned} X(\omega) &= \int_{0}^{\infty} e^{-at} e^{-j\omega t} \, \mathrm{d}t \\ &= \frac{1}{a + j\omega} \end{aligned}$$

Thus,

$$\mathcal{F}\left\{e^{-at} u(t)\right\} = \frac{1}{a + j\omega}$$

Example (Exponential).

$$x(t) = e^{-a|t|}$$

Solution.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{0} e^{at} e^{-j\omega t} \, \mathrm{d}t + \int_{0}^{\infty} e^{-at} e^{-j\omega t} \, \mathrm{d}t \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2} \end{aligned}$$

Thus,

$$\mathcal{F}\left\{e^{-a|t|}\right\} = \frac{2a}{a^2 + \omega^2}$$

Example (Signum Function).

$$x(t) = sgn(t)$$

Solution. Simplifying the function

$$x(t) = u(t) - u(-t) = \lim_{a \to 0} e^{-at} u(t) - e^{at} u(-t)$$

$$\begin{aligned} X(\omega) &= \lim_{a \to 0} \frac{1}{a + j\omega} - \frac{1}{a - j\omega} \\ &= \frac{2}{j\omega} \end{aligned}$$

Thus,

$$\mathcal{F}\left\{sgn(t)\right\} = \frac{2}{j\omega}$$

Example (Unit Step Function).

$$x(t) = u(t)$$

Solution.

$$x(t) = \frac{1}{2} + \frac{sgn(t)}{2}$$
$$X(\omega) = 2\pi\left(\frac{1}{2}\right)\delta(\omega) + \frac{1}{2}\frac{2}{j\omega}$$
$$= \frac{1}{j\omega} + \pi\delta(\omega)$$

Thus,

$$\mathcal{F}\left\{u(t)\right\} = \frac{1}{j\omega} + \pi\delta(\omega)$$

Example (Complex Exponential Signal).

$$x(t) = e^{j\omega_0 t}$$

Solution. Consider the D.C. value 1,

$$\mathcal{F}\left\{1\right\} = 2\pi\delta(\omega) \implies \mathcal{F}\left\{e^{j\omega_0 t}\right\} = 2\pi\delta(\omega - \omega_0) \quad \text{(Frequency Shifting)}$$

Example (Cosine Function).

$$x(t) = \cos\omega_0 t$$

Solution.

$$x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Thus,

$$\mathcal{F}\left\{\cos\omega_0 t\right\} = \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

Example (Sine Function).

$$x(t) = \cos\omega_0 t$$

Solution.

$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2}$$

Thus,

$$\mathcal{F}\left\{\sin\omega_0 t\right\} = \pi\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

Example (Rectangular Function).

$$x(t) = A\text{rect}\left(\frac{t}{\tau}\right)$$

Solution. Here, we use the method of differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = A\left[\delta(t + \frac{\tau}{2}) - \delta(t - \frac{\tau}{2})\right]$$

Now using the Differentiation Property,

$$(j\omega)X(\omega) = A\left[e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right] \implies X(\omega) = \frac{A}{\omega}2\sin\left(\omega\frac{\tau}{2}\right)$$

Thus,

$$\mathcal{F}\left\{A\text{rect}\left(\frac{t}{\tau}\right)\right\} = \frac{A}{\omega}2\sin\left(\omega\frac{\tau}{2}\right) = A\tau Sa\left(\omega\frac{\tau}{2}\right)$$

Example (Sampling Function).

$$x(t) = Sa(t) = \frac{\sin t}{t}$$

Solution. Using the Duality Property,

$$\mathcal{F}\left\{Sa(t)\right\} = \frac{\text{rect}\left(\frac{\omega}{2}\right)}{2}$$