Question ST 33.2023

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Question: Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion. Then $E\left({W_4}^2 \middle| W_2 = 2\right)$ in integer equals

Solution:

Parameter	Description
μ_x	Mean of x
$Var(x)$	Variance of x
$Cov(x,y)$	Covariance between x and y
σ_x	Standard deviation of x
ρ	Co-Relation coefficiant
$E(x)$	Expectation of x

In standard brownian motion,

$$W_i \sim N(0,i) \tag{1}$$

$$Cov\left(W_i, W_j\right) = min(i,j) \tag{2}$$

Now, we know that,

$$E\left(Y^2 \middle| X\right) = Var(Y|X) + (E(Y|X))^2 \tag{3}$$

These are defined as:

$$E(Y|X=x) = \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x-\mu_X) \tag{4}$$

$$Var(Y|X=x) = \left(1-\rho^2\right){\sigma_Y}^2 \tag{5}$$

The Co-Relation Coefficiant is given by:

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} \tag{6}$$

In our case,

$$Y = W_4 \tag{7}$$

$$X = W_2 \tag{8}$$

$$x = 2 \tag{9}$$

Hence, we get that,

$$\mu_X = \mu_Y = 0 \tag{10}$$

$$\sigma_X = \sqrt{Var(X)} \tag{11}$$

$$= \sqrt{2} \tag{12}$$

$$\sigma_Y = \sqrt{Var(Y)} \tag{13}$$

$$= \sqrt{4} \tag{14}$$

$$= 2 \tag{15}$$

$$\rho = \frac{min(2,4)}{\sqrt{2*4}} \tag{16}$$

$$= \frac{2}{\sqrt{8}} \tag{17}$$

$$= \frac{1}{\sqrt{2}} \tag{18}$$

Substituting the values in above equations,

$$E(Y|X=2) = \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \cdot 2 \tag{19}$$

$$= 2 \tag{20}$$

$$Var(Y|X=2) = \left(1-\frac{1}{2}\right)(2)^2 \tag{21}$$

$$= \frac{1}{2} \cdot 4 \tag{22}$$

$$= 2 \tag{23}$$

$$\tag{24}$$

Substituting these values in (3),

$$E\left(Y^2 \middle| X=2\right) = 2 + (2)^2 \tag{25}$$

$$= 6 \tag{26}$$

Hence, the answer of this question is:

$$E\left({W_4}^2 \middle| W_2 = 2\right) = 6 \tag{27}$$