

Question ST 33.2023

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Question: Let $\{W_t\}_{t \geq 0}$ be a standard Brownian motion. Then $E(W_4^2 | W_2 = 2)$ in integer equals

Solution:

Parameter	Description
μ_x	Mean of x
$Var(x)$	Variance of x
$Cov(x, y)$	Covariance between x and y
σ_x	Standard deviation of x
ρ	Co-Relation coefficient
$E(x)$	Expectation of x

In standard brownian motion,

$$W_i \sim N(0, i) \quad (1)$$

$$Cov(W_i, W_j) = \min(i, j) \quad (2)$$

Now, we know that,

$$E(Y^2 | X) = Var(Y | X) + (E(Y | X))^2 \quad (3)$$

X and Y can be represented as:

$$X = \sigma_X Z_1 + \mu_X \quad (4)$$

$$Y = \sigma_Y (\rho Z_1 + \sqrt{1 - \rho^2} Z_2) + \mu_Y \quad (5)$$

where Z_1 and Z_2 are normal distributions.

$$Z_1, Z_2 \sim N(0, 1) \quad (6)$$

Writing the above equations in matrix form,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y \rho & \sigma_Y \sqrt{1 - \rho^2} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \quad (7)$$

This can be represented as,

$$\mathbf{x} = \mathbf{A}\mathbf{z} + \boldsymbol{\mu}$$

Taking expectation both sides,

$$E(\mathbf{x}) = E(\mathbf{A}\mathbf{z} + \boldsymbol{\mu})$$

$$= \mathbf{A}E(\mathbf{z}) + E(\boldsymbol{\mu})$$

$$= \boldsymbol{\mu}$$

We know that covariance matrix for X and Y is given by:

$$\sigma_z = E((\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T) \quad (12)$$

$$= E(\mathbf{A}\mathbf{z}(\mathbf{A}\mathbf{z})^T) \quad (13)$$

$$= E(\mathbf{A}\mathbf{z}\mathbf{z}^T\mathbf{A}^T) \quad (14)$$

$$= \mathbf{A}E(\mathbf{z}\mathbf{z}^T)\mathbf{A}^T \quad (15)$$

Multiplying \mathbf{z} and \mathbf{z}^T we get,

$$\mathbf{z}\mathbf{z}^T = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \quad (16)$$

$$= \begin{bmatrix} Z_1^2 & Z_1 Z_2 \\ Z_1 Z_2 & Z_2^2 \end{bmatrix} \quad (17)$$

We know that,

$$Var(Z_1) = E((Z_1 - \mu)^2) \quad (18)$$

$$E(Z_1^2) = 1 \quad (19)$$

Same goes for Z_2 as Z_1 and Z_2 are both normal distributions.

Taking expectation both sides in equation (17),

$$E(\mathbf{z}\mathbf{z}^T) = \begin{bmatrix} E(Z_1^2) & E(Z_1 Z_2) \\ E(Z_1 Z_2) & E(Z_2^2) \end{bmatrix} \quad (20)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (21)$$

Hence,

$$\sigma_z = \mathbf{A}\mathbf{A}^T \quad (22)$$

$$= \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y \rho & \sigma_Y \sqrt{1 - \rho^2} \end{bmatrix} \begin{bmatrix} \sigma_X & \sigma_Y \rho \\ 0 & \sigma_Y \sqrt{1 - \rho^2} \end{bmatrix} \quad (23)$$

$$= \begin{bmatrix} (\sigma_X)^2 & \sigma_X \sigma_Y \rho \\ \sigma_X \sigma_Y \rho & (\sigma_Y)^2 \end{bmatrix} \quad (24)$$

(8) The Co-Relation Coefficient is given by:

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}} \quad (25)$$

(9) Substituting value of Z_1 in Y,

$$Y = \sigma_Y \rho \left(\frac{X - \mu_X}{\sigma_X} \right) + \sigma_Y \sqrt{1 - \rho^2} Z_2 + \mu_Y \quad (26)$$

This an equation of Y in terms of X . All the terms except Z_2 are constants. Taking expectation on both sides,

$$E(Y|X = x) = E\left(\sigma_Y \rho \left(\frac{x - \mu_X}{\sigma_X}\right) + \sigma_Y \sqrt{1 - \rho^2} Z_2 + \mu_Y\right) \quad (27)$$

$$= E\left(\sigma_Y \rho \left(\frac{x - \mu_X}{\sigma_X}\right) + \mu_Y\right) + E\left(\sigma_Y \sqrt{1 - \rho^2} Z_2\right) \quad (28)$$

$$= \mu_Y + \rho \left(\frac{\sigma_Y}{\sigma_X}\right) (x - \mu_X) + \sigma_Y \sqrt{1 - \rho^2} E(Z_2) \quad (29)$$

Now for variance,

$$Var(Y|X = x) = Var\left(\sigma_Y \rho \left(\frac{x - \mu_X}{\sigma_X}\right) + \sigma_Y \sqrt{1 - \rho^2} Z_2 + \mu_Y\right) \quad (30)$$

$$= Var\left(\sigma_Y \rho \left(\frac{x - \mu_X}{\sigma_X}\right) + \mu_Y\right) + Var\left(\sigma_Y \sqrt{1 - \rho^2} Z_2\right) \quad (31)$$

Variance of constants terms is 0.

$$Var(Y|X = x) = Var\left(\sigma_Y \sqrt{1 - \rho^2} Z_2\right) \quad (32)$$

$$= (1 - \rho^2) \sigma_Y^2 Var(Z_2) \quad (33)$$

Z_2 is a normal distribution so,

$$E(Z_2) = 0 \quad (34)$$

$$Var(Z_2) = 1 \quad (35)$$

By substituting these values in above equations,

$$E(Y|X = x) = \mu_Y + \rho \left(\frac{\sigma_Y}{\sigma_X}\right) (x - \mu_X) \quad (36)$$

$$Var(Y|X = x) = (1 - \rho^2) \sigma_Y^2 \quad (37)$$

In our case,

$$Y = W_4 \quad (38)$$

$$X = W_2 \quad (39)$$

$$x = 2 \quad (40)$$

Hence, we get that,

$$\mu_X = \mu_Y = 0 \quad (41)$$

$$\sigma_X = \sqrt{2} \quad (42)$$

$$\sigma_Y = 2 \quad (43)$$

$$\rho = \frac{2}{\sqrt{8}} \quad (44)$$

$$= \frac{1}{\sqrt{2}} \quad (45)$$

Substituting the values in above equations,

$$E(Y|X = 2) = \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \cdot 2 \quad (46)$$

$$= 2 \quad (47)$$

$$Var(Y|X = 2) = \left(1 - \frac{1}{2}\right) (2)^2 \quad (48)$$

$$= \frac{1}{2} \cdot 4 \quad (49)$$

$$= 2 \quad (50)$$

Substituting these values in (3),

$$E(Y^2|X = 2) = 2 + (2)^2 \quad (51)$$

$$= 6 \quad (52)$$

$$E(W_4^2|W_2 = 2) = 6 \quad (53)$$