Question ST 33.2023

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Question: Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion. Then $E\left(W_4{}^2 \middle| W_2 = 2\right)$ in integer equals

Solution:

Parameter	Description
μ_x	Mean of x
$Var(x)$	Variance of x
$Cov(x, y)$	Covariance between x and y
σ_x	Standard deviation of x
ρ	Co-Relation coefficiant
$E(x)$	Expectation of x

In standard brownian motion,

$$W_i \sim N(0, i) \tag{1}$$

$$Cov\left(W_i, W_j\right) = \min(i, j) \tag{2}$$

Now, we know that,

$$E\left(Y^2 \middle| X\right) = Var(Y|X) + (E(Y|X))^2 \tag{3}$$

X and Y can be represented as:

$$X = \sigma_X Z_1 + \mu_X \tag{4}$$

$$Y = \sigma_Y\left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y \tag{5}$$

where Z_1 and Z_2 are normal distributions.

$$Z_1, Z_2 \sim N(0, 1) \tag{6}$$

Writing the above equations in matrix form,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y\rho & \sigma_Y\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \tag{7}$$

This can be represented as,

$$\mathbf{x} = A\mathbf{z} + \boldsymbol{\mu} \tag{8}$$

Taking expectation both sides,

$$E(\mathbf{x}) = E(A\mathbf{z} + \boldsymbol{\mu}) \tag{9}$$

$$= AE(\mathbf{z}) + E(\boldsymbol{\mu}) \tag{10}$$

$$= \boldsymbol{\mu} \tag{11}$$

We know that covariance matrix for X and Y is given by:

$$\sigma_{\mathbf{z}} = E\left((\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\right) \tag{12}$$

$$= E\left((A\mathbf{z})(A\mathbf{z})^T\right) \tag{13}$$

$$= E\left(A\mathbf{z}\mathbf{z}^T A^T\right) \tag{14}$$

$$= AE\left(\mathbf{z}\mathbf{z}^T\right)A^T \tag{15}$$

Multiplying $\mathbf{z}$ and $\mathbf{z}^T$ we get,

$$\mathbf{z}\mathbf{z}^T = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} Z_1{}^2 & Z_1 Z_2 \\ Z_1 Z_2 & Z_2{}^2 \end{bmatrix} \tag{17}$$

We know that,

$$Var(Z_1) = E\left((Z_1 - \mu)^2\right) \tag{18}$$

$$E\left(Z_1^2\right) = 1 \tag{19}$$

Same goes for Z_2 as Z_1 and Z_2 are both normal distributions.
Taking expectation both sides in equation (17),

$$E\left(\mathbf{z}\mathbf{z}^T\right) = \begin{bmatrix} E\left(Z_1^2\right) & E(Z_1 Z_2) \\ E(Z_1 Z_2) & E\left(Z_2^2\right) \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{21}$$

Hence,

$$\sigma_{\mathbf{z}} = AA^T \tag{22}$$

$$= \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y\rho & \sigma_Y\sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} \sigma_X & \sigma_Y\rho \\ 0 & \sigma_Y\sqrt{1-\rho^2} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} (\sigma_X)^2 & \sigma_X\sigma_Y\rho \\ \sigma_X\sigma_Y\rho & (\sigma_Y)^2 \end{bmatrix} \tag{24}$$

The Co-Relation Coefficiant is given by:

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)\,Var(Y)}} \tag{25}$$

Substituting value of Z_1 in Y,

$$Y = \sigma_Y\rho\left(\frac{X - \mu_X}{\sigma_X}\right) + \sigma_Y\sqrt{1-\rho^2} Z_2 + \mu_Y \tag{26}$$

This an equation of Y in terms of X. All the terms except Z_2 are constants. Taking expectation on both sides,

$$E(Y|X=x) = E\left(\sigma_Y\rho\left(\frac{x-\mu_X}{\sigma_X}\right) + \sigma_Y\sqrt{1-\rho^2}Z_2 + \mu_Y\right) \tag{27}$$

$$= E\left(\sigma_Y\rho\left(\frac{x-\mu_X}{\sigma_X}\right) + \mu_Y\right) + E\left(\sigma_Y\sqrt{1-\rho^2}Z_2\right) \tag{28}$$

$$= \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x-\mu_X) + \sigma_Y\sqrt{1-\rho^2}E(Z_2) \tag{29}$$

Now for variance,

$$Var(Y|X=x) = Var\left(\sigma_Y\rho\left(\frac{x-\mu_X}{\sigma_X}\right) + \sigma_Y\sqrt{1-\rho^2}Z_2 + \mu_Y\right) \tag{30}$$

$$= Var\left(\sigma_Y\rho\left(\frac{x-\mu_X}{\sigma_X}\right) + \mu_Y\right) + Var\left(\sigma_Y\sqrt{1-\rho^2}Z_2\right) \tag{31}$$

Variance of constants terms is 0.

$$Var(Y|X=x) = Var\left(\sigma_Y\sqrt{1-\rho^2}Z_2\right) \tag{32}$$

$$= \left(1-\rho^2\right)\sigma_Y{}^2 Var(Z_2) \tag{33}$$

Z_2 is a normal distribution so,

$$E(Z_2) = 0 \tag{34}$$

$$Var(Z_2) = 1 \tag{35}$$

By substituting these values in above equations,

$$E(Y|X=x) = \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x-\mu_X) \tag{36}$$

$$Var(Y|X=x) = \left(1-\rho^2\right)\sigma_Y{}^2 \tag{37}$$

In our case,

$$Y = W_4 \tag{38}$$

$$X = W_2 \tag{39}$$

$$x = 2 \tag{40}$$

Hence, we get that,

$$\mu_X = \mu_Y = 0 \tag{41}$$

$$\sigma_X = \sqrt{2} \tag{42}$$

$$\sigma_Y = 2 \tag{43}$$

$$\rho = \frac{2}{\sqrt{8}} \tag{44}$$

$$= \frac{1}{\sqrt{2}} \tag{45}$$

Substituting the values in above equations,

$$E(Y|X=2) = \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \cdot 2 \tag{46}$$

$$= 2 \tag{47}$$

$$Var(Y|X=2) = \left(1-\frac{1}{2}\right)(2)^2 \tag{48}$$

$$= \frac{1}{2} \cdot 4 \tag{49}$$

$$= 2 \tag{50}$$

Substituting these values in (3),

$$E\left(Y^2\middle| X=2\right) = 2 + (2)^2 \tag{51}$$

$$= 6 \tag{52}$$

$$E\left(W_4{}^2\middle| W_2=2\right) = 6 \tag{53}$$

Steps for Simulation:

1) Write functions to generate two normal distributions Z_1 and Z_2 using box muller method.
2) Assume values of constants,

$$\sigma_X = 0.5 \tag{54}$$

$$\sigma_Y = 0.8 \tag{55}$$

$$\rho = 0.5 \tag{56}$$

$$\mu_X = 1 \tag{57}$$

$$\mu_Y = 1.5 \tag{58}$$

3) Run the functions to generate Z_1 and Z_2 10000 times, and store all the outputs in a temporary array of size 2x10000.
4) Make three 2-D arrays for Matrices and Vectors using the above assumed values.
5) The dimension for matrix is 2x2 and the vector 1x2
6) For every iteration, assign values of one column of the temporary array to $\mathbf{z}$.
7) Mulitply the A matrix with $\mathbf{z}$ and add to $\boldsymbol{\mu}$.
8) The resultant gives you $\mathbf{x}$.