1

Solution of question 9.3.8

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Question: Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

- (a) all the five cards are spades?
- (b) only 3 cards are spades?
- (c) none is a spade?

Solution: Let us define:

Parameter	Value	Description
n	5	number of cards drawn
p	<u>1</u>	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
$\mu = np$	<u>5</u>	mean of the distribution
$\sigma^2 = npq$	15 16	variance of the distribution
Y	{0,1,2,3,4,5}	Number of spade cards drawn

(i) Gaussian Distribution

The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
 $(x \in Y)$ (1)

(a) If we consider all cards to be spades,

$$Y = 5 \tag{2}$$

Substituting values in (1),

$$p_Y(5) = \frac{1}{\sqrt{2\pi \left(\frac{15}{16}\right)}} e^{-\frac{\left(5 - \frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}}$$
(3)

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{15}{2}} \tag{4}$$

$$= 0.0001245$$
 (5)

(b) If we consider 3 cards to be spades,

$$Y = 3 \tag{6}$$

Substituting values in (1),

$$p_Y(3) = \frac{1}{\sqrt{2\pi \left(\frac{15}{16}\right)}} e^{-\frac{\left(3-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{7}$$

$$=\frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}}e^{-\frac{49}{30}}\tag{8}$$

$$= 0.044$$
 (9)

(c) If we consider 0 cards to be spades,

$$Y = 0 \tag{10}$$

Substituting values in (1),

$$p_Y(0) = \frac{1}{\sqrt{2\pi \left(\frac{15}{16}\right)}} e^{-\frac{\left(0 - \frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}}$$
(11)

$$= \frac{1}{\sqrt{2\pi \left(\frac{15}{16}\right)}} e^{-\frac{5}{6}} \tag{12}$$

$$= 0.0978$$
 (13)

(ii) Solving using Q function Consider a gaussian random variable Z,

$$Z \sim N(\mu, \sigma)$$
 (14)

$$\sim N\left(\frac{5}{4}, \frac{\sqrt{15}}{4}\right) \tag{15}$$

Due to continuity correction Pr(Y = x) can be approximated using gaussian distribution as

$$p_Z(x) \approx \Pr(x - 0.5 < Z < x + 0.5)$$
 (16)

$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5)$$

(17)

$$\approx F_Z(x+0.5) - F_Z(x-0.5) \tag{18}$$

CDF of Z is defined as:

$$F_Z(x) = \Pr(Z < x)$$

$$= \Pr\left(\frac{Z - \mu}{\sigma} < \frac{x - \mu}{\sigma}\right)$$
(20)

$$\Rightarrow \frac{Z - \mu}{\sigma} \sim N(0, 1)$$

$$= 1 - \Pr\left(\frac{Z - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right)$$

$$= \begin{cases} 1 - Q\left(\frac{x - \mu}{\sigma}\right) & x \ge \mu \\ Q\left(\frac{\mu - x}{\sigma}\right) & x < \mu \end{cases}$$
(23)

Then probability in terms of Q funtion is

$$\implies p_{Z}(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{24}$$

a) The Gaussian approximation for Pr(Y = 5) is

$$p_{Z}(5) \approx Q\left(\frac{4.5 - 1.25}{0.9375}\right) - Q\left(\frac{5.5 - 1.25}{0.9375}\right)$$

$$\approx Q(3.356) - Q(4.389) \tag{26}$$

$$\approx 0.0003888 \tag{27}$$

b) The Gaussian approximation for Pr(Y = 3) is

$$p_{Y}(3) \approx Q\left(\frac{2.5 - 1.25}{0.9375}\right) - Q\left(\frac{3.5 - 1.25}{0.9375}\right)$$

$$\approx Q(1.2909) - Q(2.3237) \tag{29}$$

$$\approx 0.08828 \tag{30}$$

c) The Gaussian approximation for Pr(Y = 0) is

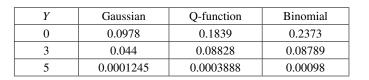
$$p_Z(0) \approx Q\left(\frac{-0.5 - 1.25}{0.9375}\right) - Q\left(\frac{0.5 - 1.25}{0.9375}\right)$$

$$\approx (1 - Q(1.8073)) - (1 - Q(0.7745))$$

$$= Q(0.7745) - Q(1.8073)$$

$$\approx 0.1839$$
(34)

(iii) Gaussian vs Binomial vs Q-function Comparison



(iv) Binomial vs Gaussian Graph

