Solution of question 9.3.8

Gagan Singla - EE22BTECH11021

Question: Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?
(b) only 3 cards are spades?
(c) none is a spade?

Solution: Let us define:

Parameter	Value	Description
n	5	number of cards drawn
p	$\frac{1}{4}$	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
$\mu = np$	$\frac{5}{4}$	mean of the distribution
$\sigma^2 = npq$	$\frac{15}{16}$	variance of the distribution

(i) Gaussian Distribution

Lets define a random variable Y which represents the number of spade cards drawn.

$$Y = \{0, 1, 2, 3, 4, 5\} \tag{1}$$

The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{2}$$

The central limit theorm states that we can take a random variable Z such that,

$$Z \approx \frac{Y - \mu}{\sigma} \tag{3}$$

Now, Z is a random variable with $\mathcal{N}(0, 1)$. Hence, the gaussian distribution function changes to:

$$p_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad (x \in Z) \tag{4}$$

(a) If we consider all cards to be spades,

$$Y = 5 \tag{5}$$

$$Z \approx \frac{5 - \frac{5}{4}}{\sqrt{\frac{15}{16}}} \approx \sqrt{15} \tag{6}$$

Substituting values in (4),

$$p_Z\left(\sqrt{15}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{15}{2}} \tag{7}$$

$$= 0.0001245 \tag{8}$$

(b) If we consider 3 cards to be spades,

$$Y = 3 \tag{9}$$

$$Z \approx \frac{3 - \frac{5}{4}}{\sqrt{\frac{15}{16}}} \approx \frac{7}{\sqrt{15}} \tag{10}$$

Substituting values in (4),

$$p_Z\left(\frac{7}{\sqrt{15}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{49}{30}} \tag{11}$$

$$= 0.044 \tag{12}$$

(c) If we consider 0 cards to be spades,

$$Y = 0 \tag{13}$$

$$Z \approx -\frac{5}{\sqrt{15}} \tag{14}$$

Substituting values in (4),

$$p_Z\left(-\frac{5}{\sqrt{15}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{5}{6}} \tag{15}$$

$$= 0.0978 \tag{16}$$

(ii) Gaussian vs Binomial Comparison

Y	Gaussian	Binomial
0	0.0978	0.2373
3	0.044	0.08789
5	0.0001245	0.00098

(iii) Binomial vs Gaussian Graph