Question ST 33.2023

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Question: Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion. Then $E\left(W_4{}^2 \middle| W_2 = 2\right)$ in integer equals
Solution:

Parameter	Description
μ_x	Mean of x
$Var(x)$	Variance of x
$Cov(x, y)$	Covariance between x and y
σ_x	Standard deviation of x
ρ	Co-Relation coefficiant
$E(x)$	Expectation of x

In standard brownian motion,

$$W_i \sim N(0, i) \tag{1}$$

$$Cov\left(W_i, W_j\right) = \min(i, j) \tag{2}$$

Now, we know that,

$$E\left(Y^2 \middle| X\right) = Var(Y|X) + (E(Y|X))^2 \tag{3}$$

X and Y can be represented as:

$$X = \sigma_x Z_1 + \mu_x \tag{4}$$

$$Y = \sigma_y\left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y \tag{5}$$

where Z_1 and Z_2 are normal distributions.

$$Z_1, Z_2 \sim N(0, 1) \tag{6}$$

Substituting value of Z_1 in Y,

$$Y = \sigma_Y \rho\left(\frac{x - \mu_X}{\sigma_x}\right) + \sigma_Y \sqrt{1-\rho^2} Z_2 + \mu_Y \tag{7}$$

This an equation of Y in terms of x. All the terms except Z_2 are constants. Taking expectation and variance on both sides,

$$E(Y|X = x) = \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x - \mu_X) + \sigma_Y \sqrt{1-\rho^2} E(Z_2) \tag{8}$$

$$Var(Y|X = x) = \left(1-\rho^2\right) \sigma_Y{}^2 Var(Z_2) \tag{9}$$

Variance of constants terms is 0. Z_2 is a normal distribution so,

$$E(Z_2) = 0 \tag{10}$$

$$Var(Z_2) = 1 \tag{11}$$

By substituting these values in above equations,

$$E(Y|X = x) = \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x - \mu_X) \tag{12}$$

$$Var(Y|X = x) = \left(1-\rho^2\right) \sigma_Y{}^2 \tag{13}$$

The Co-Relation Coefficiant is given by:

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}} \tag{14}$$

In our case,

$$Y = W_4 \tag{15}$$

$$X = W_2 \tag{16}$$

$$x = 2 \tag{17}$$

Hence, we get that,

$$\mu_X = \mu_Y = 0 \tag{18}$$

$$\sigma_X = \sqrt{Var(X)} \tag{19}$$

$$= \sqrt{2} \tag{20}$$

$$\sigma_Y = \sqrt{Var(Y)} \tag{21}$$

$$= \sqrt{4} \tag{22}$$

$$= 2 \tag{23}$$

$$\rho = \frac{\min(2, 4)}{\sqrt{2 \times 4}} \tag{24}$$

$$= \frac{2}{\sqrt{8}} \tag{25}$$

$$= \frac{1}{\sqrt{2}} \tag{26}$$

Substituting the values in above equations,

$$E(Y|X = 2) = \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \cdot 2 \tag{27}$$

$$= 2 \tag{28}$$

$$Var(Y|X = 2) = \left(1 - \frac{1}{2}\right)(2)^2 \tag{29}$$

$$= \frac{1}{2} \cdot 4 \tag{30}$$

$$= 2 \tag{31}$$

$$\tag{32}$$

Substituting these values in (3),

$$E\left(Y^2 \middle| X = 2\right) = 2 + (2)^2 \tag{33}$$

$$= 6 \tag{34}$$

Hence, the answer of this question is:

$$E\left({W_4}^2 \middle| W_2 = 2\right) = 6 \tag{35}$$