Question ST 33.2023

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Question: Let $\{W_t\}_{t\geq 0}$ be a standard Brownian motion. Then $E\left(W_4{}^2 \middle| W_2 = 2\right)$ in integer equals

Solution:

Parameter	Description
μ_x	Mean of x
$Var(x)$	Variance of x
$Cov(x, y)$	Covariance between x and y
σ_x	Standard deviation of x
ρ	Co-Relation coefficiant
$E(x)$	Expectation of x

In standard brownian motion,

$$W_i \sim N(0, i) \tag{1}$$

$$Cov\left(W_i, W_j\right) = \min(i, j) \tag{2}$$

Now, we know that,

$$E\left(Y^2 \middle| X\right) = Var(Y|X) + (E(Y|X))^2 \tag{3}$$

X and Y can be represented as:

$$X = \sigma_X Z_1 + \mu_X \tag{4}$$

$$Y = \sigma_Y\left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y \tag{5}$$

where Z_1 and Z_2 are normal distributions.

$$Z_1, Z_2 \sim N(0, 1) \tag{6}$$

Writing the above equations in matrix form,

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y\rho & \sigma_Y\sqrt{1-\rho^2} \end{bmatrix}\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} \tag{7}$$

This can be represented as,

$$\mathbf{x} = A\mathbf{z} + \boldsymbol{\mu} \tag{8}$$

Taking expectation both sides,

$$E(\mathbf{x}) = E(A\mathbf{z} + \boldsymbol{\mu}) \tag{9}$$

$$= AE(\mathbf{z}) + E(\boldsymbol{\mu}) \tag{10}$$

$$= \boldsymbol{\mu} \tag{11}$$

We know that covariance matrix for X and Y is given by:

$$\boldsymbol{\sigma} = E\left((\mathbf{x}-\mu)(\mathbf{x}-\mu)^T\right) \tag{12}$$

$$= E\left((A\mathbf{z})(A\mathbf{z})^T\right) \tag{13}$$

$$= E\left(A\mathbf{z}\mathbf{z}^T A^T\right) \tag{14}$$

$$= AE\left(\mathbf{z}\mathbf{z}^T\right)A^T \tag{15}$$

Multiplying $\mathbf{z}$ and $\mathbf{z}^T$ we get,

$$\mathbf{z}\mathbf{z}^T = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}\begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} Z_1{}^2 & Z_1Z_2 \\ Z_1Z_2 & Z_2{}^2 \end{bmatrix} \tag{17}$$

We know that,

$$Var(Z_1) = E\left((Z_1-\mu)^2\right) \tag{18}$$

$$E\left(Z_1^2\right) = 1 \tag{19}$$

Same goes for Z_2 as Z_1 and Z_2 are both normal distributions.
Taking expectation both sides in equation (17),

$$E\left(\mathbf{z}\mathbf{z}^T\right) = \begin{bmatrix} E\left(Z_1^2\right) & E(Z_1Z_2) \\ E(Z_1Z_2) & E\left(Z_2^2\right) \end{bmatrix} \tag{20}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{21}$$

Hence,

$$\boldsymbol{\sigma} = AA^T \tag{22}$$

$$= \begin{bmatrix} \sigma_X & 0 \\ \sigma_Y\rho & \sigma_Y\sqrt{1-\rho^2} \end{bmatrix}\begin{bmatrix} \sigma_X & \sigma_Y\rho \\ 0 & \sigma_Y\sqrt{1-\rho^2} \end{bmatrix} \tag{23}$$

$$= \begin{bmatrix} (\sigma_X)^2 & \sigma_X\sigma_Y\rho \\ \sigma_X\sigma_Y\rho & (\sigma_Y)^2 \end{bmatrix} \tag{24}$$

The Co-Relation Coefficiant is given by:

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)\,Var(Y)}} \tag{25}$$

Substituting value of Z_1 in Y,

$$Y = \sigma_Y\rho\left(\frac{X-\mu_X}{\sigma_X}\right) + \sigma_Y\sqrt{1-\rho^2}Z_2 + \mu_Y \tag{26}$$

This an equation of Y in terms of X. All the terms except Z_2 are constants. Taking expectation and variance on both sides,

$$E(Y|X=x) = \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x-\mu_X) + \sigma_Y\sqrt{1-\rho^2}E(Z_2) \tag{27}$$

$$Var(Y|X=x) = \left(1-\rho^2\right)\sigma_Y{}^2 Var(Z_2) \tag{28}$$

Variance of constants terms is 0. Z_2 is a normal distribution so,

$$E(Z_2) = 0 \tag{29}$$

$$Var(Z_2) = 1 \tag{30}$$

By substituting these values in above equations,

$$E(Y|X=x) = \mu_Y + \rho\left(\frac{\sigma_Y}{\sigma_X}\right)(x-\mu_X) \tag{31}$$

$$Var(Y|X=x) = \left(1-\rho^2\right)\sigma_Y{}^2 \tag{32}$$

In our case,

$$Y = W_4 \tag{33}$$

$$X = W_2 \tag{34}$$

$$x = 2 \tag{35}$$

Hence, we get that,

$$\mu_X = \mu_Y = 0 \tag{36}$$

$$\sigma_X = \sqrt{2} \tag{37}$$

$$\sigma_Y = 2 \tag{38}$$

$$\rho = \frac{2}{\sqrt{8}} \tag{39}$$

$$= \frac{1}{\sqrt{2}} \tag{40}$$

Substituting the values in above equations,

$$E(Y|X=2) = \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \cdot 2 \tag{41}$$

$$= 2 \tag{42}$$

$$Var(Y|X=2) = \left(1-\frac{1}{2}\right)(2)^2 \tag{43}$$

$$= \frac{1}{2} \cdot 4 \tag{44}$$

$$= 2 \tag{45}$$

$$\tag{46}$$

Substituting these values in (3),

$$E\left(Y^2\middle|X=2\right) = 2 + (2)^2 \tag{47}$$

$$= 6 \tag{48}$$

Hence, the answer of this question is:

$$E\left(W_4{}^2\middle|W_2=2\right) = 6 \tag{49}$$