Solution of question 9.3.8

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Question: Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?
(b) only 3 cards are spades?
(c) none is a spade?

Solution: Let us define:

Parameter	Value	Description
n	5	number of cards drawn
p	$\frac{1}{4}$	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
μ	$\frac{5}{4}$	mean of the distribution
σ^2	$\frac{15}{16}$	variance of the distribution

$$\mu = np = 5\left(\frac{1}{4}\right) = \frac{5}{4} \tag{1}$$

$$\sigma^2 = npq = 5\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) = \frac{15}{16} \tag{2}$$

(i) Gaussian Distribution
Lets define a random variable Y which represents the number of spade cards drawn.

$$Y = \{0, 1, 2, 3, 4, 5\} \tag{3}$$

The gaussian distribution function is defined as:

$$P_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{4}$$

The central limit theorm states that we can take a random variable Z such that,

$$Z \approx \frac{Y-\mu}{\sigma} \tag{5}$$

Now, Z is a random variable with $\mathcal{N}(0, 1)$. Hence, the gaussian distribution function changes to:

$$P_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{6}$$

(a) If we consider all cards to be spades,

$$Y = 5 \tag{7}$$

$$Z \approx \frac{5-\frac{5}{4}}{\sqrt{\frac{15}{16}}} \approx \sqrt{15} \tag{8}$$

Substituting values in (6),

$$P_Z\left(\sqrt{15}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{15}{2}} \tag{9}$$

$$= 0.0001245 \tag{10}$$

(b) If we consider 3 cards to be spades,

$$Y = 3 \tag{11}$$

$$Z \approx \frac{3-\frac{5}{4}}{\sqrt{\frac{15}{16}}} \approx \frac{7}{\sqrt{15}} \tag{12}$$

Substituting values in (6),

$$P_Z\left(\frac{7}{\sqrt{15}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{49}{30}} \tag{13}$$

$$= 0.044 \tag{14}$$

(c) If we consider 0 cards to be spades,

$$Y = 0 \tag{15}$$

$$Z \approx -\frac{5}{\sqrt{15}} \tag{16}$$

Substituting values in (6),

$$P_Z\left(-\frac{5}{\sqrt{15}}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{5}{6}} \tag{17}$$

$$= 0.0978 \tag{18}$$

(ii) Binomial Distribution
Lets define a random variable X which represents the number of spade cards drawn.

$$X = \{0, 1, 2, 3, 4, 5\} \tag{19}$$

The pmf is given by

$$P_X(r) = {}^nC_r p^r (1-p)^{n-r} \tag{20}$$

(a) If we consider all cards to be spades,

$$P_X(5) = 0.00098 \tag{21}$$

(b) If we consider 3 cards to be spades,

$$P_X(3) = 0.08789 \tag{22}$$

(c) If we consider 0 cards to be spades,

$$P_X(0) = 0.23730 \tag{23}$$

(iii) Binomial vs Gaussian Graph