Solution of question 9.3.8

Gagan Singla - EE22BTECH11021

Question: Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?
(b) only 3 cards are spades?
(c) none is a spade?

Solution: Let us define:

Parameter	Value	Description
n	5	number of cards drawn
p	$\frac{1}{4}$	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
$\mu = np$	$\frac{5}{4}$	mean of the distribution
$\sigma^2 = npq$	$\frac{15}{16}$	variance of the distribution
Y	{0,1,2,3,4,5}	Number of spade cards drawn

(i) Gaussian Distribution
The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{1}$$

If we consider all cards to be spades,

$$Y = 5 \tag{2}$$

Substituting values in (1),

$$p_Y(5) = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{\left(5-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{3}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{15}{2}} \tag{4}$$

$$= 0.0001245 \tag{5}$$

If we consider 3 cards to be spades,

$$Y = 3 \tag{6}$$

Substituting values in (1),

$$p_Y(3) = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{\left(3-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{7}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{49}{30}} \tag{8}$$

$$= 0.044 \tag{9}$$

If we consider 0 cards to be spades,

$$Y = 0 \tag{10}$$

Substituting values in (1),

$$p_Y(0) = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{\left(0-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{11}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{5}{6}} \tag{12}$$

$$= 0.0978 \tag{13}$$

(ii) Solving using Q function
Consider a gaussian random variable Z,

$$Z \sim N(\mu, \sigma) \tag{14}$$

$$\sim N\left(\frac{5}{4}, \frac{\sqrt{15}}{4}\right) \tag{15}$$

Due to continuity correction $\Pr(Y = x)$ can be approximated using gaussian distribution as

$$p_Z(x) \approx \Pr(x - 0.5 < Z < x + 0.5) \tag{16}$$

$$\approx \Pr(Z < x + 0.5) - \Pr(Z < x - 0.5) \tag{17}$$

$$\approx F_Z(x + 0.5) - F_Z(x - 0.5) \tag{18}$$

CDF of Z is defined as:

$$F_Z(x) = \Pr(Z < x) \tag{19}$$

$$= \Pr\left(\frac{Z-\mu}{\sigma} < \frac{x-\mu}{\sigma}\right) \tag{20}$$

$$\implies \frac{Z-\mu}{\sigma} \sim N(0,1) \tag{21}$$

$$= 1 - \Pr\left(\frac{Z-\mu}{\sigma} > \frac{x-\mu}{\sigma}\right) \tag{22}$$

$$= \begin{cases} 1 - Q\left(\frac{x-\mu}{\sigma}\right) & x \geq \mu \\ Q\left(\frac{\mu-x}{\sigma}\right) & x < \mu \end{cases} \tag{23}$$

Then probability in terms of Q funtion is

$$\implies p_Z(x) \approx Q\left(\frac{(x-0.5)-\mu}{\sigma}\right) - Q\left(\frac{(x+0.5)-\mu}{\sigma}\right) \tag{24}$$

The Gaussian approximation for $\Pr(Y = 5)$ is

$$p_Z(5) \approx Q\left(\frac{4.5-1.25}{0.9375}\right) - Q\left(\frac{5.5-1.25}{0.9375}\right) \tag{25}$$

$$\approx Q(3.356) - Q(4.389) \tag{26}$$

$$\approx 0.0003888 \tag{27}$$

The Gaussian approximation for $\Pr(Y = 3)$ is

$$p_Y(3) \approx Q\left(\frac{2.5-1.25}{0.9375}\right) - Q\left(\frac{3.5-1.25}{0.9375}\right) \tag{28}$$

$$\approx Q(1.2909) - Q(2.3237) \tag{29}$$

$$\approx 0.08828 \tag{30}$$

The Gaussian approximation for $\Pr(Y = 0)$ is

$$p_Z(0) \approx Q\left(\frac{-0.5-1.25}{0.9375}\right) - Q\left(\frac{0.5-1.25}{0.9375}\right) \tag{31}$$

$$\approx (1 - Q(1.8073)) - (1 - Q(0.7745)) \tag{32}$$

$$= Q(0.7745) - Q(1.8073) \tag{33}$$

$$\approx 0.1839 \tag{34}$$

(iii) Gaussian vs Binomial vs Q-function Comparison

Y	Gaussian	Q-function	Binomial
0	0.0978	0.1839	0.2373
3	0.044	0.08828	0.08789
5	0.0001245	0.0003888	0.00098

(iv) Binomial vs Gaussian Graph