Solution of question 9.3.8

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Question: Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?
(b) only 3 cards are spades?
(c) none is a spade?

Solution: Let us define:

Parameter	Value	Description
n	5	number of cards drawn
p	$\frac{1}{4}$	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
$\mu = np$	$\frac{5}{4}$	mean of the distribution
$\sigma^2 = npq$	$\frac{15}{16}$	variance of the distribution
Y	{0,1,2,3,4,5}	Number of spade cards drawn

(i) Gaussian Distribution
The gaussian distribution function is defined as:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{1}$$

(a) If we consider all cards to be spades,

$$Y = 5 \tag{2}$$

Substituting values in (1),

$$p_Y(5) = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{\left(5-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{3}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{15}{2}} \tag{4}$$

$$= 0.0001245 \tag{5}$$

(b) If we consider 3 cards to be spades,

$$Y = 3 \tag{6}$$

Substituting values in (1),

$$p_Y(3) = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{\left(3-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{7}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{49}{30}} \tag{8}$$

$$= 0.044 \tag{9}$$

(c) If we consider 0 cards to be spades,

$$Y = 0 \tag{10}$$

Substituting values in (1),

$$p_Y(0) = \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{\left(0-\frac{5}{4}\right)^2}{2\left(\frac{15}{16}\right)}} \tag{11}$$

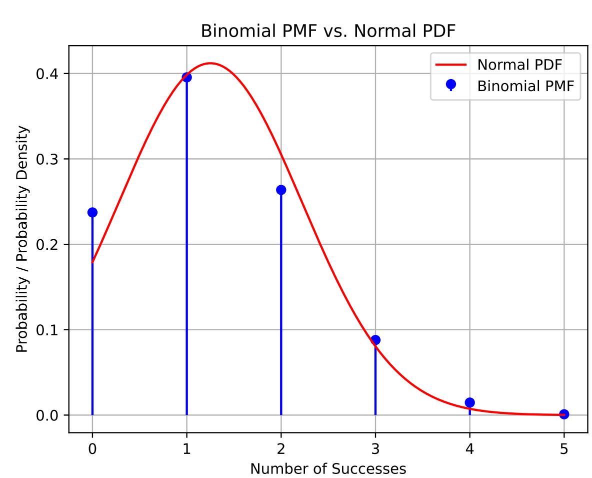
$$= \frac{1}{\sqrt{2\pi\left(\frac{15}{16}\right)}} e^{-\frac{5}{6}} \tag{12}$$

$$= 0.0978 \tag{13}$$

(ii) Gaussian vs Binomial Comparison

Y	Gaussian	Binomial
0	0.0978	0.2373
3	0.044	0.08789
5	0.0001245	0.00098

(iii) Binomial vs Gaussian Graph



(iv) Solving using Q function
Q function is defined

$$Q(x) = \int_{x}^{\infty} f(x)\, dx \tag{14}$$

then CDF of Y is:

$$\begin{aligned}\Pr(Y < x) &= \int_{-\infty}^{x} f(x)\, dx && (15)\\ &= 1 - \int_{x}^{\infty} f(x)\, dx && (16)\\ &= 1 - Q(x) && (17)\end{aligned}$$

and for finding $\Pr\left(Z = \frac{X-\mu}{\sigma}\right)$ Using approximation,

$$\Pr\left(Z = \frac{Y-\mu}{\sigma}\right) \approx \Pr\left(\frac{Y+0.5-\mu}{\sigma} < Z < \frac{Y-0.5-\mu}{\sigma}\right) \tag{18}$$

$$\approx \Pr\left(Z < \frac{Y+0.5-\mu}{\sigma}\right) - \Pr\left(Z < \frac{Y-0.5-\mu}{\sigma}\right) \tag{19}$$

$$\approx Q\left(\frac{Y-0.5-\mu}{\sigma}\right) - Q\left(\frac{Y+0.5-\mu}{\sigma}\right) \tag{20}$$

a)

$$Y = 5 \tag{21}$$

$$\Pr(Z = 3.872) \approx Q(3.356) - Q(4.389) \tag{22}$$

$$\approx 0.0003888 \tag{23}$$

b)

$$Y = 3 \tag{24}$$

$$\Pr(Z = 1.8073) \approx Q(1.2909) - Q(2.3237) \tag{25}$$

$$\approx 0.08828 \tag{26}$$

c)

$$Y = 0 \tag{27}$$

$$\Pr(Z = -1.2909) \approx Q(-1.8073) - Q(-0.7745) \tag{28}$$

$$\approx 0.1839 \tag{29}$$