Solution of question 9.3.8

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Question: Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that

(a) all the five cards are spades?
(b) only 3 cards are spades?
(c) none is a spade?

Solution: Let us define:

Parameter	Value	Description
n	5	number of cards drawn
p	$\frac{1}{4}$	drawing a spade card
q	$\frac{3}{4}$	drawing any other card
X	{0,1,2,3,4,5}	number of spade cards drawn
μ	$\frac{5}{4}$	mean of the distribution
σ^2	$\frac{15}{16}$	variance of the distribution

$$\mu = np \tag{1}$$
$$= 5\left(\frac{1}{4}\right) \tag{2}$$
$$= \frac{5}{4} \tag{3}$$
$$\sigma^2 = npq \tag{4}$$
$$= 5\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) \tag{5}$$
$$= \frac{15}{16} \tag{6}$$

Method 1: Gaussian Distribution

For the random variable X, the gaussian distribution function is defined as:

$$\Pr(X = x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{7}$$

The central limit theorm states that we can take a random variable Z such that,

$$Z = \frac{X-\mu}{\sigma} \tag{8}$$

Now, Z is a random variable with $\mu = 0$ and $\sigma^2 = 1$. Hence, the gaussian distribution function changes to:

$$\Pr(Z = x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{9}$$

(a) If we consider all cards to be spades,

$$X = 5 \tag{10}$$
$$Z = \frac{5-\frac{5}{4}}{\sqrt{\frac{15}{16}}} \tag{11}$$
$$= \sqrt{15} \tag{12}$$

Substituting values in (9),

$$\Pr\left(x = \sqrt{15}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{15}{2}} \tag{13}$$
$$= 0.0001245 \tag{14}$$

(b) If we consider 3 cards to be spades,

$$X = 3 \tag{15}$$
$$Z = \frac{3-\frac{5}{4}}{\sqrt{\frac{15}{16}}} \tag{16}$$
$$= \frac{7}{\sqrt{15}} \tag{17}$$

Substituting values in (9),

$$\Pr\left(x = \frac{7}{\sqrt{15}}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{49}{30}} \tag{18}$$
$$= 0.044 \tag{19}$$

(c) If we consider 0 cards to be spades,

$$X = 0 \tag{20}$$
$$Z = \frac{5}{\sqrt{3}} \tag{21}$$

Substituting values in (9),

$$\Pr\left(x = \sqrt{15}\right) = \frac{1}{\sqrt{2\pi}}e^{-\frac{5}{6}} \tag{22}$$
$$= 0.0978 \tag{23}$$

Method 2: Binomial Distribution

The pmf is given by

$$\Pr(X = r) = {}^{n}C_{r}p^{r}(1-p)^{n-r} \tag{24}$$

(a) If we consider all cards to be spades,

$$\Pr(X = 5) = {}^{5}C_{5}\left(\frac{1}{4}\right)^{5}\left(\frac{3}{4}\right)^{0} \tag{25}$$
$$= \frac{1}{1024} \tag{26}$$
$$= 0.00098 \tag{27}$$

(b) If we consider 3 cards to be spades,

$$\Pr(X = 3) = {}^{5}C_{5}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{2} \tag{28}$$
$$= \frac{45}{512} \tag{29}$$
$$= 0.08789 \tag{30}$$

(c) If we consider 0 cards to be spades,

$$\Pr(X = 0) = {}^{5}C_{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{5} \tag{31}$$
$$= \frac{243}{1024} \tag{32}$$
$$= 0.23730 \tag{33}$$