2D Distribution functions

Functions used by Matlab for fitting optical depth

Definitions

Z_0 = overall offset due to background contributions

m_x & m_y = slope of linear background contributions

x_0 & y_0 = position of cloud peak in optical depth image

ζ = fugacity [fugacity = 1 when there is a condensate]

Note: Thomas-Fermi distributions use heaviside function [$\Theta(x)$] to truncate the condensate wavefunction where $V(\vec{r}) = \mu$

Thermal portions are described by the polylogarithm function

$$g_s(z) = \sum_{k=1} \frac{z^k}{k^s}$$

- Background contributions
- Thermal portion
- Condensate distribution
- Condensate truncation

- Pure gaussian (non-interacting condensate, no thermal portion)

$$\boxed{Z = Z_0 + m_x(x-x_0) + m_y(y-y_0) + |A_{BEC}|\, e^{-\frac{(x-x_0)^2}{2\sigma_x^2} - \frac{(y-y_{off})^2}{2\sigma_y^2}}}$$

- Bimodal gaussian (non-interacting condensate with surrounding thermal pedestal)

$$\boxed{Z = Z_0 + m_x(x-x_0) + m_y(y-y_0) + |A_{0D}| \frac{1}{g_2(\zeta)} g_2\left(\zeta e^{-\frac{(x-x_0)^2}{2\sigma_x^2} - \frac{(y-y_{off})^2}{2\sigma_y^2}}\right) + |A_{BEC}|\, e^{-\frac{(x-x_0)^2}{2\sigma_x^2} - \frac{(y-y_{off})^2}{2\sigma_y^2}}}$$

- Pure Thomas-Fermi (interaction dominated, no thermal portion)

$$\boxed{Z = Z_0 + m_x(x-x_0) + m_y(y-y_0) + |A_{BEC}|\left(1 - \frac{(x-x_0)^2}{\sigma_x^2} - \frac{(y-y_0)^2}{\sigma_y^2}\right) \Theta\left(1 - \frac{(x-x_0)^2}{\sigma_x^2} - \frac{(y-y_0)^2}{\sigma_y^2}\right)}$$

- Bimodal Thomas-Fermi (interaction dominated with surrounding thermal pedestal)

$$\boxed{\begin{aligned} Z = {} & Z_0 + m_x(x-x_0) + m_y(y-y_0) + |A_{0D}| \frac{1}{g_2(\zeta)} g_2\left(\zeta e^{-\frac{(x-x_0)^2}{2\sigma_x^2} - \frac{(y-y_{off})^2}{2\sigma_y^2}}\right) \\ & + |A_{BEC}|\left(1 - \frac{(x-x_0)^2}{\sigma_x^2} - \frac{(y-y_0)^2}{\sigma_y^2}\right) \Theta\left(1 - \frac{(x-x_0)^2}{\sigma_x^2} - \frac{(y-y_0)^2}{\sigma_y^2}\right) \end{aligned}}$$

Thermal distribution derivation

Neutral analysis density fitting derivations - 10.17.13 - Jim Aman

Excited atoms (thermal portion) using semi-classical approximation. The following is taken from Pethick and Smith 2nd ed. With equation numbers referencing their equations.

Consider number density of atoms per phase space volume $[(2\pi\hbar)^3]$ integrated from $\varepsilon > 0$ thus

2.45 $$n_{ex}(\vec{r}) = \int \frac{d\vec{p}}{(2\pi\hbar)^3} \frac{1}{\exp[(\varepsilon_p(\vec{r}) - \mu)/k_BT - 1]}$$

here we are considering particles with classical free particle energy

$$\varepsilon_p(\vec{r}) = \frac{p^2}{2m} + V(\vec{r})$$

Remember

$$\lambda_T = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}$$

now define

$$x = \frac{p^2}{2mk_BT} \quad \text{and} \quad z(\vec{r}) = e^{(\mu - V(\vec{r}))/k_BT} = \mathfrak{z} e^{-V(\vec{r})/k_BT}$$ 2.46

where $\mathfrak{z} = e^{\mu/k_BT}$ is the fugacity

then we see

$$p = \sqrt{2mxk_BT} \quad \text{and} \quad \frac{dp}{dx} = \frac{1}{2}(2mxk_BT)^{-1/2}(2mk_BT) = \sqrt{\frac{mk_BT}{2x}}$$

so plugging in

$$n_{ex}(\vec{r}) = \int_0^\infty \frac{4\pi p^2}{(2\pi\hbar)^3} \frac{1}{\exp[(p^2/2m + V(\vec{r}) - \mu)/k_BT] - 1} dp$$

$$= \int_0^\infty \frac{4\pi}{(2\pi\hbar)^3} 2mk_BTx \sqrt{\frac{mk_BT}{2x}} \frac{1}{z^{-1}e^x - 1} dx$$

$$= \int_0^\infty \frac{\sqrt{2}(mk_BT)^{3/2}}{2\pi^2\hbar^3} \frac{\sqrt{x}}{z^{-1}e^x - 1} dx$$

2.47 $$= \frac{2}{\sqrt{\pi}\lambda_T^3} \int_0^\infty dx \frac{\sqrt{x}}{z^{-1}e^x - 1}$$

↳ this integral is of the form

Note - Demarco's thesis has a footnote about doing these integrals (pg. 237). Here it is taken as an identity

$$\int_0^\infty dx \frac{x^{\gamma-1}}{z^{-1}e^{x}-1} = \sum_{n=1}^\infty \int_0^\infty dx\, x^{\gamma-1} e^{-nx} z^n$$

$$= \Gamma(\gamma)\, g_\gamma[z]$$

↳ Polylog

$$g_\gamma[z] = \sum_{n=1}^\infty \frac{z^n}{n^\gamma}$$

2.50 $$\boxed{n_{ex}(\vec{r}) = \frac{g_{3/2}[z(\vec{r})]}{\lambda_T^3} = \frac{1}{\lambda_T^3} g_{3/2}[\mathfrak{z} e^{-V(r)/k_BT}]}$$

For harmonic trap $V(r) = \frac{m}{2}(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)$

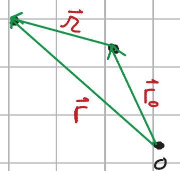
$$= \frac{1}{\lambda_T^3} g_{3/2}\left[\mathfrak{z} \exp\left[-\frac{m\omega_x^2x^2}{2k_BT}\right] \exp\left[-\frac{m\omega_y^2y^2}{2k_BT}\right] \exp\left[-\frac{m\omega_z^2z^2}{2k_BT}\right]\right]$$

this gives us the initial conditions of the thermal portion (excited, non-condensed atoms) in the gas. Once we remove the trap the atoms will begin to expand ballistically according to the equations of motion

2.53 $$\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m} \quad \text{and} \quad \frac{d\vec{p}}{dt} = 0$$

*Note the following derivation deviates from Pethick and Smith and can be found in "Making, probing, and understanding Bose-Einstein Condensates" Ketterle et. al (1999)

Thus, an atom at position $\vec{r}$ is a distance $\vec{\varrho}=\frac{\vec{p}t}{m}=(\vec{r}-\vec{r}_0)$ from its starting position $\vec{r}_0$



$$\vec{\varrho}=\vec{r}-\vec{r}_0=\frac{\vec{p}t}{m} \Rightarrow \vec{r}_0=\vec{r}-\frac{\vec{p}t}{m}$$

then the integral over the Bose distribution becomes

$$n(\vec{r},t)=\int\frac{d^3\vec{r}'\,d^3\vec{p}}{(2\pi\hbar)^3}\,\frac{1}{\exp\left[\left(\frac{p^2}{2m}+V(\vec{r}')-\mu\right)\frac{1}{k_BT}\right]-1}\,\delta^3\left(\vec{r}-\frac{\vec{p}t}{m}-\vec{r}'\right)$$

$$=\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\,\frac{1}{\exp\left[\left(\frac{p^2}{2m}+V\left(\vec{r}-\frac{\vec{p}t}{m}\right)-\mu\right)\frac{1}{k_BT}\right]-1}$$

Plugging in the harmonic potential we find that free expansion from a harmonic trap amounts to rescaling the spatial coordinates

35 $$n_{th,tof}(\vec{r},t)=\frac{1}{\lambda_T^3}\left[\prod_i\left(\frac{1}{1+\omega_i^2t^2}\right)\right]g_{3/2}\left[\mathcal{z}\exp\left\{\sum_i\frac{-m\omega_i^2r_i^2}{2k_BT}\,\frac{1}{(1+\omega_i^2t^2)}\right\}\right]$$

*Also see DeMarco's PhD thesis - 5.2.3

In the classical limit $(\mathcal{z}(\vec{r})\ll 1)$ we note that $g_{3/2}[\mathcal{z}(\vec{r})]\approx\mathcal{z}(\vec{r})$, thus comparing to Pethick and Smith's derivation of free expansion from the Maxwell-Boltzmann distribution we obtain similar results

From here on roughly following Natali's thesis ch. 6

Finally, we must also consider the column density along one direction since we are taking absorption images of the atoms. Using Beer's law we can relate the total absorption to the number density times the absorption cross section such that

$$OD=\ln\left(\frac{\text{Atoms}}{\text{Background}}\right)=\sigma_{abs}\int_{-\infty}^{\infty}n_{th,tof}(\vec{r},t)\,dz$$

Check

$$=\frac{\sigma_{abs}}{\lambda_T^3}\left[\prod_i\frac{1}{(1+\omega_i^2t^2)}\right]\int_{-\infty}^{\infty}dz\;g_{3/2}\left[\mathcal{z}\exp\left\{\sum_i\frac{-r_i^2}{2\sigma_i^2}\right\}\right]\quad\text{where } \sigma_i^2=\frac{k_BT}{m\omega_i^2}(1+\omega_i^2t^2)$$

Expanding the polylog into its series representation then

$$\int_{-\infty}^{\infty}dz\;g_{3/2}\left[\mathcal{z}\exp\left\{\sum_i\frac{-r_i^2}{2\sigma_i^2}\right\}\right]=\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{\mathcal{z}^n}{n^{3/2}}\exp\left(\frac{-x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}-\frac{z^2}{2\sigma_z^2}\right)^n dz$$

Let $\exp\left[\frac{-x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right]=\rho$ thus

$$=\int_{-\infty}^{\infty}dz\;\mathcal{z}\rho\exp\left[\frac{-z^2}{2\sigma_z^2}\right]+\frac{\mathcal{z}^2}{2^{3/2}}\rho^2\exp\left[\frac{-z^2}{2\sigma_z^2}\right]^2+\frac{\mathcal{z}^3}{3^{3/2}}\rho^3\exp\left[\frac{-z^2}{2\sigma_z^2}\right]^3+\dots$$

$$\Rightarrow\int_{-\infty}^{\infty}dz\sum_{n=1}^{\infty}e^{-z^2/2\sigma_z^2}=\frac{\sqrt{2\pi}\,\sigma_z}{n^{1/2}}$$

$$=\sqrt{2\pi}\,\sigma_z\sum_{n=1}^{\infty}\frac{\mathcal{z}^n\rho^n}{n^2}$$

Plugging back into the expression for optical depth

$$OD(x,y) = \frac{\sigma_{abs}}{\lambda_T^3} \frac{\sqrt{2\pi}\,\sigma_z}{(1+\omega_x^2 t^2)(1+\omega_y^2 t^2)(1+\omega_z^2 t^2)} g_2\left[\mathfrak{z}\exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right]\right] \quad \text{where } \sigma_i^2 = \frac{k_B T}{m\omega_i^2}(1+\omega_i^2 t^2)$$

to remove the dependence on trap frequencies we can find the peak OD where $x=y=0$

$$OD(0,0) = \frac{\sqrt{2\pi}}{\lambda_T^3} \frac{\sigma_{abs}\,\sigma_z}{(1+\omega_x^2 x^2)(1+\omega_y^2 y^2)(1+\omega_z^2 z^2)} g_2[\mathfrak{z}]$$

When a condensate is present $\mathfrak{z}=1$ [See Naval's doctorate thesis Section 6.3 for discussion on fitting fugacity]

therefore

$$\boxed{OD(x,y) = \frac{OD(0,0)}{g_2[\mathfrak{z}]} g_2\left[\mathfrak{z}\exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2}\right]\right]}$$

=> Optical depth relation to thermal portion

in the limit that $t >> \omega_x^{-1}, \omega_y^{-1}, \omega_z^{-1}$ then $\sigma_i^2 = \frac{k_B T}{m} t^2$

thus the temperature along each axis is given by

$$\boxed{T_i = \frac{m\sigma_i^2}{k_B t^2}}$$

=> Thermal temperature along each axis

We can also get the total number of atoms by remembering the boson normalization requirement

$$N = \int_{-\infty}^{\infty} d\vec{r}\; n(\vec{r})$$

$$= \int_{-\infty}^{\infty} d^3\vec{r}\; n(\vec{r},t) = \int d^3\vec{r}\; \frac{\sigma_{abs}}{\lambda_T^3} \frac{1}{(1+\omega_x^2 t^2)(1+\omega_y^2 t^2)(1+\omega_z^2 t^2)} \sum_{n=1}^{\infty} \frac{\mathfrak{z}^n}{n^{3/2}} \exp\left[-\frac{x^2}{2\sigma_x^2} - \frac{y^2}{2\sigma_y^2} - \frac{z^2}{2\sigma_z^2}\right]^n$$

This integral is the same thing we had for optical depth but now over all three axes so

$$N = \frac{(2\pi)^{3/2}}{\lambda_T^3} \frac{\sigma_x \sigma_y \sigma_z}{(1+\omega_x^2 t^2)(1+\omega_y^2 t^2)(1+\omega_z^2 t^2)} g_3[\mathfrak{z}]$$

Plugging in the expression for peak OD then

$$\boxed{N = 2\pi \frac{\sigma_x \sigma_y}{\sigma_{abs}} OD(0,0) \frac{g_3[\mathfrak{z}]}{g_2[\mathfrak{z}]}}$$

=> Thermal total number

*Note: Peak OD here is referred to the thermal portion of the cloud so peak OD is the same as the amplitude of the thermal portion.