

The Gap Number of the T -Tetromino

Robert Hochberg¹

Department of Mathematics, University of Dallas, Irving, TX

Abstract

A famous result of D. Walkup states that the only rectangles that may be tiled by the T -Tetromino are those in which both sides are a multiple of four. In this paper we examine the rest of the rectangles, asking how many T -tetrominos may be placed into those rectangles without overlap, or, equivalently, what is the least number of gaps that need to be present. We introduce a new technique for exploring such tilings, enabling us to answer this question for all rectangles, up to a small additive constant. We also show that there is some number G such that if both sides of the rectangle are at least 12, then no more than G gaps will be required. We prove that G is either 5, 6, 7 or 9.

Keywords: tiling, tetromino, Bellman-Ford

2010 MSC: 05-04, 05C22

1. Introduction and Definitions

A well-known result of D. Walkup [1] is that the T -tetromino can tile a rectangle if and only if both sides of that rectangle are multiples of 4. This result has been extended in several ways. Mike Reid [2] generalized the
 5 T -tetromino to a class of $(8n - 4)$ -ominoes, each of which tiles a rectangle

Email address: hochberg@udallas.edu (Robert Hochberg)

¹972-721-5307 (w), 252-258-5284 (c)

if and only if both sides are multiples of 4, and other authors [3, 4] have enumerated tilings of $4m \times 4n$ rectangles with T -tetrominos.

Another direction of inquiry asks about those rectangles for which both sides are not multiples of 4. For odd $n = 2k + 1$, it seems possible that $k^2 + k$ T -tetrominos could be packed into the $n \times n$ square so that a single 1×1 square remains uncovered. Wayne Goddard [5] showed that this was not possible. Zhan [6] generalized this result to all $m \times n$ rectangles with $mn \equiv 1 \pmod{4}$, showing again that it is impossible to leave just a single uncovered 1×1 square.

An uncovered 1×1 square in a tiling may be thought of as a special 1×1 tile, a *monomino*, employed as a second type of allowed tile. We will thus be interested in the following question: Given a rectangle, what is the greatest number of T -tetrominos that may be placed therein when tiled with T -tetrominos and monominos? Or, equivalently, what is the least number of monominos needed in such a tiling? Let us define $M(m, n)$ to be this number, the *gap number*. That is, $M(m, n)$ = the least number of monominos needed in a tiling of the $m \times n$ rectangle by T -tetrominos and monominos. We observe that $M(m, n) \equiv mn \pmod{4}$.

We will show that $M(m, n)$ is unbounded as m and n range over the positive integers. It remains unbounded even if we fix m to be 11. On the other hand, if m and n are both at least 12, then $M(m, n) \leq 9$. Thus, $G = \limsup_{m, n \rightarrow \infty} M(m, n)$ exists, and is at most 9. We call G the *gap cap* of the T -tetromino.

Outline of the paper. In Section 2 we investigate rectangles of odd widths $w \leq 11$, for which $M(w, n)$ increases without bound as n increases. Here we introduce the *fringe digraph* and show how the Bellman-Ford algorithm can be used to find lower bounds on the number of gaps needed, exactly

matching constructive upper bounds. Section 3 looks at even widths $w \leq 12$ and shows that for these widths, $M(w, n)$ remains bounded. Section 4 looks
35 at rectangles of widths $w = 13$ and $w = 15$, and Section 5 shows that the gap cap of the T -tetromino is at most 9.

2. Rectangles of odd width ≤ 11

We consider rectangles of widths 1, 3, 5, 7, 9 and 11 in this section. We will cover width three in some detail, and then the others more quickly.

40 2.1. Width 3

Theorem 1.

$$M(3, n) = \begin{cases} 5 & \text{if } n = 3 \\ \lfloor n/3 \rfloor & \text{if } n \equiv 0 \pmod{3}, n \neq 3 \\ \lfloor n/3 \rfloor + 3 & \text{if } n \equiv 1 \pmod{3} \\ \lfloor n/3 \rfloor + 2 & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

The following lemma does most of the work for us and will have an analogue for width 9.

Lemma 2. *In any tiling of a $3 \times n$ rectangle by T -tetrominos and monominoes, each 3×3 box must contain at least one monomino.*

45 *Proof of lemma:.* Given a 3×3 box consider how the center square is covered. If it is covered by a monomino, then we're done. If not, then it is covered by a T -tetromino in some way, as shown (up to symmetry) in Figure 1, where the dot represents the center square of some 3×3 box.

In cases a and b, it is impossible to cover both of the squares marked
50 with "X" without using a monomino which will lie within that 3×3 box. In cases c and d, the square marked with an "X" can not be covered with a T -tetromino. In case e, the square below the dot may be covered with a

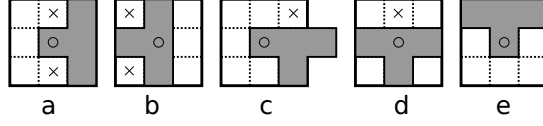


Figure 1: Every 3×3 box in any tiling of a $3 \times n$ rectangle must have a gap.

T -tetromino in only one way, up to symmetry, leaving two squares in that box that cannot both be tiled by T -tetrominos. ■

55 *Proof of Theorem.:* The cases for $n \leq 5$ are shown in Figure 2. In what follows, we assume $n \geq 6$. Let $n = 3k + r$ where $0 \leq r \leq 2$. Then within a $3 \times n$ rectangle we can find k disjoint 3×3 blocks, each of which contains a monomino, so that

$$M(3, n) \geq k = \lfloor n/3 \rfloor. \quad (1)$$

For $r = 0$, the 3×6 and 3×9 rectangles in Figure 2 may be combined
60 to show that $M(3, n) \leq \lfloor n/3 \rfloor$, finishing the proof for the case where $n \equiv 0 \pmod{3}$.

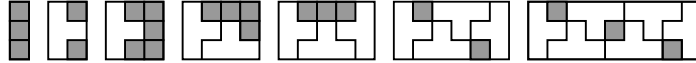


Figure 2: Optimal $3 \times n$ tilings for $n = 1, 2, 3, 4, 5, 6, 9$.

Now suppose $r = 1$. Since the $3 \times (3k + 1)$ rectangle contains three more squares than the $3 \times 3k$ rectangle, and recalling that $M(m, n) \equiv mn \pmod{4}$ we have that the number of gaps required for the $3 \times (3k + 1)$ rectangle must
65 equal $k + 3 + 4t$ for some integer t . By (1), t can't be negative, giving $M(3, n) \geq k + 3$. This number of gaps can be realized by simply appending a column of three monominos to an optimally-tiled $3 \times 3k$ rectangle, finishing the proof when $n \equiv 1 \pmod{3}$. If $r = 2$, then the number of gaps must be $k + 6 + 4t$ for some integer t . Again, by (1), $t \geq -1$, giving $M(3, n) \geq k + 2$.

70 This bound may be achieved by appending to the optimally-tiled $3 \times 3k$ rectangle two columns containing one T -tetromino and two monominoes. ■

2.2. Rectangles of Width 9

We treat width 9 next because it is resolved in much the same way as width 3. We have a similar lemma:

75 **Lemma 3.** *In any tiling of a $9 \times n$ rectangle by T -tetrominos and monominoes, each 9×17 box must contain at least one monomino.*

The proof of this lemma was carried out by exhaustive computer search, where we discovered that it is possible to completely cover by T -tetrominos alone sixteen consecutive columns in a width-9 strip, but not seventeen.

80 Thus $M(9, n) \geq \lfloor n/17 \rfloor$.

On the other hand, we have the tiling fragment shown in Figure 3 with which we may periodically cover arbitrarily many columns with only one monomino every seventeen columns. We have therefore proven:

Theorem 4. $M(9, n) = \lfloor n/17 \rfloor + e_9(n)$ for some bounded function $e_9(n)$. ■

85 We conjecture that $e_9(n) \leq 6$ for $n \geq 2$.

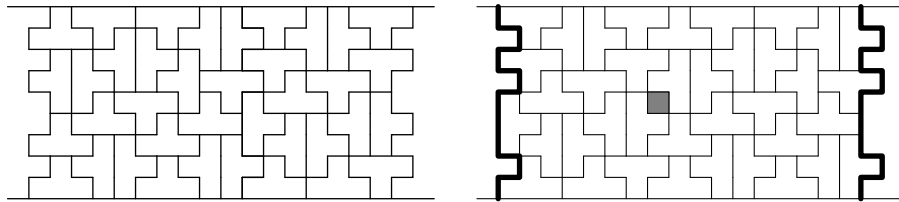


Figure 3: Left: Sixteen columns of a width-9 strip tiled without gaps. Right: A 9×17 cylinder with one monomino.

2.3. Rectangles of Widths 1, 5, 7 and 11, and the Fringe Digraph

We ask the reader to verify that $M(1, n) = n$.

The left side of Figure 4 shows a fragment for tiling width-5 rectangles periodically, using one monomino for each five columns, giving $M(5, n) \leq n/5 + C$ for some constant C . The second fragment gives $M(7, n) \leq n/7 + C$. Unfortunately, the method used for widths 3 and 9 will not work to establish corresponding lower bounds for $M(5, n)$ and $M(7, n)$. When tiling a strip of width 5, it is possible to cover six consecutive columns without a monomino, but not seven, which would give the lower bound $M(5, n) \geq n/7$. And for a strip of width 7, we may tile up to eight columns without a monomino, but not nine, giving the lower bound $M(7, n) \geq n/9$. (See the right side of Figure 4.) This leaves a gap between our upper and lower bounds.

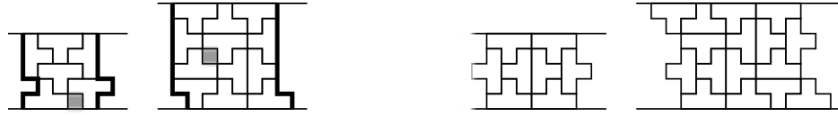


Figure 4: Widths 5 and 7. Left: 5×5 and 7×7 cylinders with one monomino. Right: It is possible to tile many columns without monominoes.

If a greedy tiler were to try to tile a rectangle of width 5 or 7 using one of the fragments given on the right in Figure 4, he would discover that many monominoes would be needed nearby, “making up” for the monomino-free columns. The case-by-case analysis required to quantify this rigorously quickly becomes onerous. We therefore turn again to the computer, and make use of the *fringe digraph*, which we now describe. Suppose we tile a rectangle with T -tetrominos and monominoes, starting from the left side of the rectangle, and placing tiles one-by-one so that we always cover the lowest of the left-most untiled squares. Then since our tiles both fit within a 2×3 box, our tiling can never have more than three partially-tiled columns,

called the *fringe*. For example, the tiling of the 3×9 rectangle shown in Figure 2 yields the sequence of fringes shown in Figure 5.

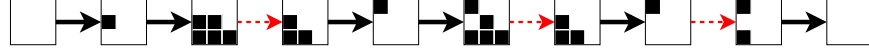


Figure 5: The sequence of fringes in our 3×9 tiling from Figure 2.

When we begin tiling a rectangle the fringe is empty, and this gives us the first node in the fringe digraph. The rest of the digraph is constructed recursively: Each time we discover a new fringe we create a corresponding node N in the digraph, and consider all ways to place a tile that covers the lowest of its left-most open squares. If it yields a previously-seen fringe, we find the node C corresponding to that fringe, and add a directed edge from N to C . If it gives a new fringe, we recursively visit that fringe, and then add the arc from N to that new node. Let F_w denote the fringe digraph for width w . F_3 is shown in Figure 6.

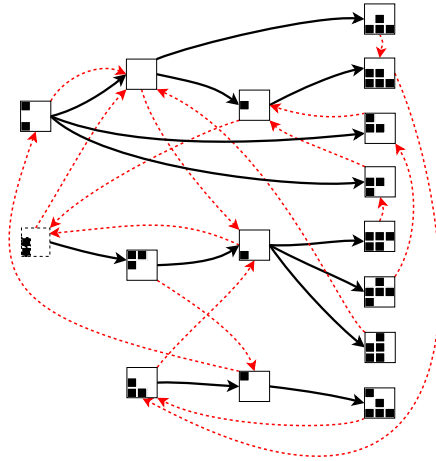


Figure 6: The fringe digraph F_3 .

We distinguish between those edges that correspond to adding a T -tetromino and those that correspond to adding a monomino. Let us call

them T -edges and M -edges respectively. In Figure 6, the M -edges are thinner and dotted. Not surprisingly, this digraph contains much information. For example, we can get a new proof that $M(3, n) \leq n/3 + C$ by considering the 3-cycle on the three lowest nodes of that figure: That cycle contains
125 two T -edges and one M -edge, accounting for $4+4+1=9$ squares, or three columns, containing a single monomino.

The fringe digraph F_w can also be used to find lower bounds for $M(w, n)$. Suppose, for example, we put weights on the edges of F_3 : -1 on T -edges and 2 on M -edges. It is not hard to verify that every cycle in the re-
130 sulting weighted digraph has non-negative weight. In particular, any cycle starting and ending at the node corresponding to an empty fringe will have non-negative weight, and these are the cycles that correspond to tilings of rectangles. But this weight can be non-negative only if there is at least one M -edge for every two T -edges, that is, if there is at least one monomino for
135 every three columns.

We ran this same analysis for the fringe digraphs F_5, F_7, F_9 and F_{11} using the Bellman-Ford algorithm to discover negative-weight cycles. The results are shown in Table 7. Let -1 and m be the weights assigned to each T -edge and M -edge respectively. Then the absence of negative-weight cycles
140 implies that any cycle containing τ T -edges and μ M -edges must satisfy $-\tau + m\mu \geq 0$, or equivalently, $\tau \leq \mu m$. Such a cycle would correspond to covering $4\tau + \mu$ squares of the rectangle, or $(4\tau + \mu)/w$ columns if the rectangle had width w . That is, the ratio of columns to monominos is $(4\tau + \mu)/\mu w$. Combined with the previous inequality, we have

$$\frac{\# \text{ columns}}{\# \text{ monominos}} = \frac{4\tau + \mu}{\mu w} \leq \frac{4m\mu + \mu}{\mu w} = \frac{4m + 1}{w}. \quad (2)$$

145 By selecting appropriate (that is, the smallest) values for m so that F_w

Fringe Digraph F_w	m	t	$\frac{4m+1}{w}$	# nodes in F_w
F_3	2	-1	3	16
F_5	6	-1	5	182
F_7	12	-1	7	1757
F_9	38	-1	17	15,496
F_{11}	96	-1	35	129,500

Figure 7: If we assign weight m to the M -edges and weight t to the T -edges of the fringe digraphs, then there will be no negative-weight cycles, so that there must be at least one monomino for every $(4m + 1)/w$ columns.

has no negative-weight cycles, as shown in the table, we obtain our desired lower bounds: In any tiling of a width-5 rectangle, there must be at least one monomino for every five columns, and for width-7, at least one monomino for every seven columns.

150 Finally, we consider rectangles of width 11. As the table proves, in any tiling of an $11 \times n$ rectangle we will find at least $n/35$ monominos. The tiling fragment at the top of Figure 8 shows that this bound is obtainable. This bound surprised us a bit, after having first found that 42 columns could be tiled without any gaps, as shown at the bottom of that figure.

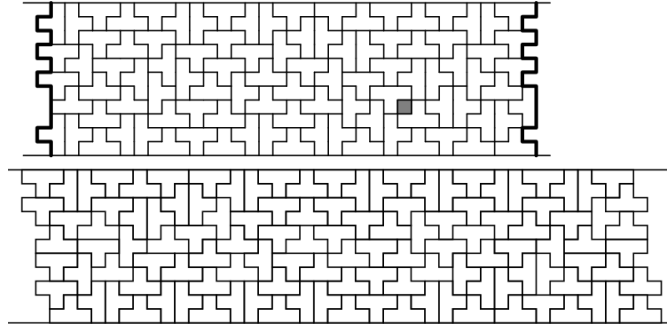


Figure 8: Top: A tiling fragment of width 11 that may tile periodically with one monomino in every 35 columns. Bottom: 42 columns tiled without any monominos.

155 We therefore have the following theorem:

Theorem 5. $M(5, n) = \lfloor n/5 \rfloor + e_5(n)$, $M(7, n) = \lfloor n/7 \rfloor + e_7(n)$, $M(11, n) = \lfloor n/35 \rfloor + e_{11}(n)$ for some bounded functions $e_5(n)$, $e_7(n)$ and $e_{11}(n)$. ■

We conjecture that $e_5(n) \leq 4$ for $n > 15$, $e_7(n) \leq 4$ for $n > 10$ and $e_{11}(n) \leq 7$ for $n > 1$.

160 3. Rectangles of Even Width at most 12

The situation is very different for even-width rectangles. Here we will see that for even w , $M(w, n)$ remains bounded as n grows. Walkup's theorem tells us that $M(w, n)$ cannot be zero unless both w and n are multiples of 4. Together with the observation that $M(w, n) \equiv wn \pmod{4}$, this gives the
165 value of $M(w, n)$ for most rectangles of even width w .

Theorem 6. *For even $w \leq 12$ and any $n \geq 2$, the value of $M(w, n)$ is the least positive integer k allowed by Walkup's theorem, such that $k \equiv wn \pmod{4}$, with only the following exceptions: $M(10, n) = 6$ for $n = 3, 7, 11, 15, 19$ and 23 .*

170 3.1. Proof of Theorem 6

The proof for width 2 is left to the reader. Rectangles of width 4, 8 and 12 require 0 or 4 monominos for length at least 2. (We note that $M(n, 1) = n$.) Figure 9 shows optimally-tiled rectangles of lengths 2, 3, 4 and 5, and to these may be concatenated any number of the $w \times 4$, monomino-free
175 rectangles, to obtain the theorem for widths 4, 8 and 12.

Rectangles of width 6 require two monominos when n is odd, and four when n is even. The four rectangles shown in Figure 10 have lengths in each of the congruence classes $\pmod{4}$ and central regions of width 4 that may be deleted or repeated to obtain tilings of $6 \times n$ rectangles for all $n \geq 2$,
180 matching the bounds of the theorem.

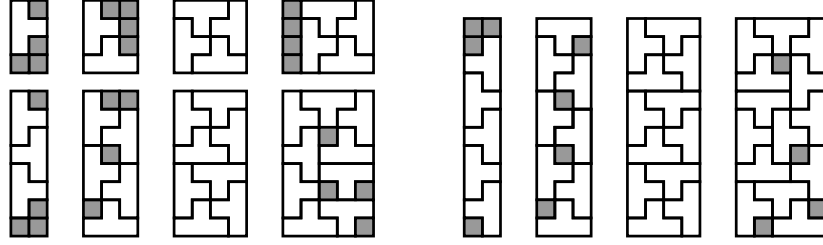


Figure 9: Rectangles of width 4, 8 and 12 require 0 or 4 monominoes.

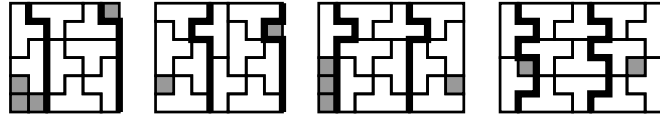


Figure 10: Rectangles of width 6 require 2 or 4 monominoes. The central section of each rectangle may be deleted or replicated, to obtain $6 \times n$ rectangles for all $n \geq 2$.

Rectangles of width 10 are handled in a similar fashion to those of width 6. The two rectangles on the left of Figure 11 show that $M(10, k) = 4$ for all even $k \geq 2$, again making use of monomino-free central sections of width 4 that may be deleted or repeated. For odd $k \geq 25$ Figure 11 shows 10×25 and 10×27 rectangles, each of which may be extended by any number of copies of the four-column unit shown. The odd values of k from 3 to 23 can then be verified by computer, which we have done. This completes the proof of Theorem 6. ■

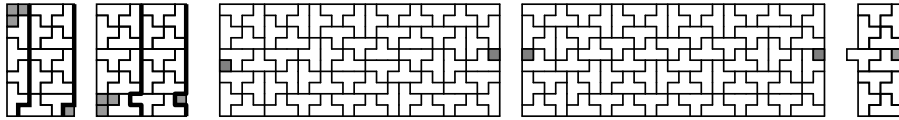


Figure 11: The central section of the left two figures may be deleted or repeated, showing that $M(10, k) = 4$ for all even $k \geq 2$. The two large rectangles show $M(10, 25) = M(10, 27) = 2$, and in each of those two rectangles, the rightmost monomino may be replaced by the fragment on the right to show that $M(10, k) = 2$ for all odd $k \geq 25$.

4. Rectangles of Width 13 and Width 15

190 After having found the rate of growth of $M(w, n)$ for the small, odd m discussed above, it came as a great surprise to find that for width 13, it was possible to tile arbitrarily many columns without needing a monomino. The tiling fragment on the left in Figure 12 shows how a periodic tiling of a strip of width 13 may be obtained with T -tetrominos only. We call this
195 fragment a *cylinder* of size 13×16 , and it is the smallest odd-width cylinder tileable by T -tetrominos alone. A 15×16 cylinder is shown on the right in that Figure.

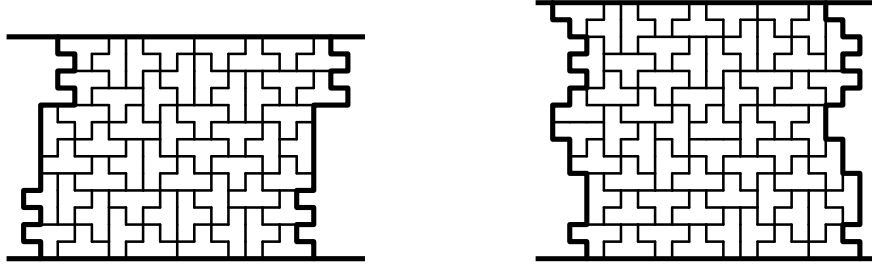


Figure 12: Cylinders of widths 13 and 15.

Walkup's theorem says that $M(w, n)$ can not be 0 for these widths, and Zhan's theorem [6] says that it also can not be 1.

200 **Theorem 7.** *For $w = 13$ and $w = 15$ and any $n \geq 2$, the value of $M(w, n)$ is the least positive integer $k \geq 2$ such that $k \equiv wn \pmod{4}$, with only the following exceptions: $M(13, 3) = 7$, $M(13, n) = 6$ for $n = 14, 18, 22, 30, 34, 38$, $M(15, 5) = 7$ and $M(15, n) = 6$ for $n = 10, 14, 18, 22, 26$.*

Proof of Theorem. Figure 13 shows four rectangles of width 13. These rectangles all share a common boundary, as shown, allowing the left and right
205 parts to be mixed and matched to create sixteen different rectangles, each

of which is tiled with the number of monominos guaranteed by the theorem. The lengths of these sixteen rectangles lie in the sixteen different congruence classes (mod 16), as shown in the middle column of the table in Figure 14. Furthermore, this boundary is shared by the 13×16 cylinder shown in Figure 12, allowing us to extend those rectangles to obtain all $13 \times n$ rectangles for $n \geq 39$, and some smaller ones. The few remaining, smaller rectangles may be checked by computer, which we have done.

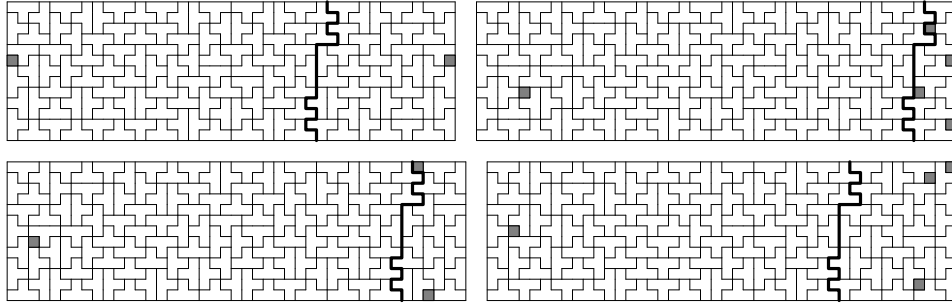


Figure 13: Rectangles of width 13 which may be mixed and matched at the border, together with the cylinder in Figure 12, to obtain all optimally-tiled $13 \times n$ rectangles for $n \geq 39$.

In the same fashion, we may mix and match the right and left parts of the rectangles shown in Figure 15 to obtain rectangles with the lengths shown in Figure 14 for width 15, hitting all sixteen congruence classes (mod 16) and tiled with the number of monominos guaranteed by the theorem. Again, the boundary shown is shared by the 15×16 cylinder shown in Figure 12, yielding optimally-tiled rectangles for all lengths at least 37. The few remaining, smaller rectangles may be checked by computer, and we have done that. This finishes the proof of Theorem 7. ■

mod 16	Width 13 start	Width 15 start
0	48	16
1	33	33
2	50	50
3	35	19
4	52	20
5	37	37
6	54	38
7	39	23
8	40	24
9	41	25
10	42	42
11	43	27
12	44	28
13	45	29
14	46	46
15	47	15

Figure 14: Lengths that may be obtained for rectangles of width 13 and 15 by mixing and matching the right and left halves of the rectangle shown in Figures 13 and 15.

5. Rectangles with Both Sides at Least 12

We may now find an upper bound on the gap cap ($= \limsup_{m,n \rightarrow \infty} M(m,n)$) of the T -tetromino.

225 **Theorem 8.** *If m and n are both at least 12, then $M(m,n) \leq 9$.*

Proof: Theorem 6 suffices to prove our theorem for rectangles that have at least one side of length exactly 12. We therefore assume both m and n are at least 13, and write $m = 4l + r_1$ and $n = 4k + r_2$ where r_1 and r_2 are in $\{0, 13, 2, 15\}$. We cut strips with width r_1 and/or r_2 off the bottom and/or
230 right side of the rectangle, so that a $4l \times 4k$ block remains. This block requires no monominos, and the numbers of monominos needed by the strips of width 2, 13 or 15 are given by previous theorems. These numbers are shown inside

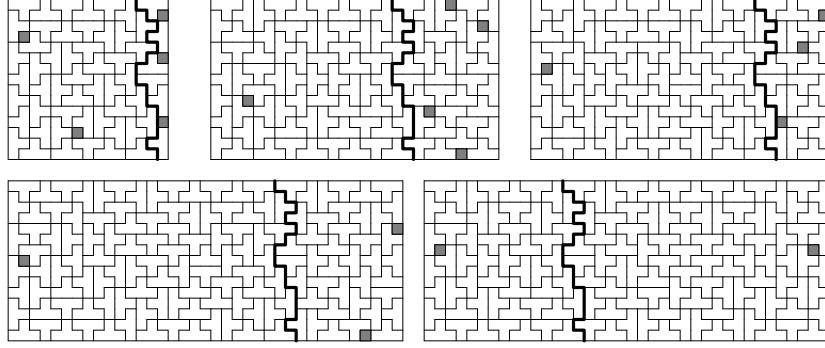


Figure 15: Rectangles of width 15 which may be mixed and matched at the border, together with the cylinder in Figure 12, to obtain all optimally-tiled $15 \times n$ rectangles for $n \geq 37$.

the rectangles in Figure 16. Note that the bottom rectangle in the third
column, the case where $m, n \equiv 2 \pmod{4}$, is different than the others. In
235 that case an L-shaped border may be removed from the right and bottom,
and it is easily shown that this may be tiled with just four monominoes. In
all cases, the total number of monominoes needed is not more than nine, and
the theorem is proved. ■

5.1. Discussion

240 We conjecture that the actual gap cap of the T -tetromino is five rather
than nine. There are some small rectangles of widths 13 and 15 that require
six monominoes, as listed in the statement of Theorem 7, so that the threshold
would have to be increased from 12 to 39 if this conjecture were true.

It would be nice to find the exact values of the functions $e_5(n)$, $e_7(n)$,
245 $e_9(n)$ and $e_{11}(n)$ mentioned in Section 2. Even better would be to find a
way to “turn the corner” in the $(4k+13) \times (4l+13)$ and $(4k+15) \times (4l+15)$
cases, and show that by combining the right and bottom rectangle slices,
only five monominoes are needed. This would reduce our bound on the gap
cap to seven. Perhaps doing something similar in the other cases as well

Polyomino	Gap Cap	Threshold
Monomino	0	1
Domino	1	1
<i>I</i> -tromino	2	3
<i>L</i> -tromino	2	4
<i>O</i> -tetromino	∞	
<i>L</i> -tetromino	4	4
<i>I</i> -tetromino	4	4

Figure 17: Gap caps and thresholds for some small polyominoes.

References

- [1] D. W. Walkup, Covering a rectangle with *T*-tetrominoes, Amer. Math. Monthly 72 (1965) 986–988.
- [2] M. Reid, Klarner systems and tiling boxes with polyominoes, J. Combin. Theory Ser. A 111 (1) (2005) 89–105. doi:10.1016/j.jcta.2004.10.010. URL <http://dx.doi.org/10.1016/j.jcta.2004.10.010>
- [3] M. Korn, I. Pak, Tilings of rectangles with T-tetrominoes, Theoret. Comput. Sci. 319 (1-3) (2004) 3–27. doi:10.1016/j.tcs.2004.02.023. URL <http://dx.doi.org/10.1016/j.tcs.2004.02.023>
- [4] C. Merino, On the number of tilings of the rectangular board with T-tetrominoes, Australas. J. Combin. 41 (2008) 107–114.
- [5] W. Goddard, Almost tilings with t-tetrominos, presented at the Cumberland Conference, write-up unavailable (May 2007).
- [6] S. Zhan, Tiling a deficient rectangle with t-tetrominoes, preprint (August 2012).