COVERING A RECTANGLE WITH T-TETROMINOES

D. W. WALKUP, Boeing Scientific Research Laboratories, Seattle, Washington

The principal objective of this note is to prove the following:

THEOREM 1. *A necessary and sufficient condition for an $a \times b$ rectangle to be dissectable into T-tetrominoes is that a and b be integral multiples of* 4.

A T-tetromino is a plane figure constructed from four unit squares arranged in the form of a T; it has a boundary of length ten with six outside corners and two inside corners. Since the 4×4 rectangle can be dissected into four T-tetrominoes, the sufficiency of the condition in Theorem 1 is clear. Certainly it is necessary that a and b be integers. A result of the above type is asked for in an editorial note to the solution of Elementary Problem E1543, this MONTHLY 70 (1963) 760–761.

Let the cartesian plane be marked off into unit squares by the lines with integral coordinates. Let R be a rectangle bounded by the axes and the lines $y=a$ and $x=b$. If R can be dissected into T-tetrominoes, then, by a simple repetition scheme, so can the entire quadrant between the positive axes. In Lemma 1 certain properties of dissections of the quadrant will be established, from which Theorem 1 will follow. For convenience we will use some special terminology:

DEFINITION 1. *A* **segment** *is a line segment of length* 1 *forming the edge of a unit square of the quadrant. A segment is a* **cut segment** (*or* **cut**) *if in every dissection of the quadrant it is one of the ten boundary segments of some T-tetromino. A point with nonnegative integer coordinates is a* **cornerless** *point if it does not lie at any one of the six outside corners of any T-tetromino in any dissection of the quadrant. A point is a type-A point if its coordinates are nonnegative and congruent modulo* 4 *to* $(0, 0)$ *or* $(2, 2)$; *it is a type-B point if its coordinates are nonnegative and congruent modulo* 4 *to* $(0, 2)$ *or* $(2, 0)$. *A* **translate** *of a point, segment, or T-tetromino is another point, segment, or T-tetromino in the quadrant obtained from the first by a displacement of $2k$ in y and $-2k$ in x, where k may be any positive or negative integer.*

Note that every segment on the axes is a cut segment and any translate of a type-A or type-B point is again a point of the same type.

LEMMA 1. *Every type-B point is cornerless and each of the* 2, 3, *or* 4 *segments incident on a type-A point is a cut.*

Proof. For each nonnegative integer λ let $P(\lambda)$ be the proposition that the lemma holds for all type-A and type-B points on or below the line $x+y=4\lambda$. $P(0)$ is true as the two segments incident on the origin are necessarily cuts. We shall demonstrate that $P(\lambda)$ implies $P(\lambda+1)$ using, for clarity, Figure 1, which shows the situation for $\lambda=2$. Certain cut segments and cornerless points required by $P(\lambda)$ are indicated in Figure 1 by heavy lines and dots.

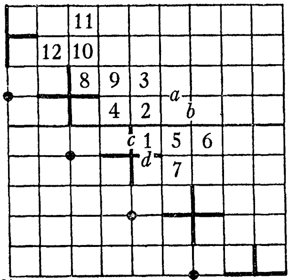

FIG. 1

If a dissection of the quadrant contained the T-tetromino 1-2-3-4 (Figure 1), it would also contain the T-tetromino 8-10-11-12 and hence, by induction, all upward translates of 1-2-3-4, since no other arrangement could cover squares 8 and 9. But, because of the relative position of the y-axis, not all upward translates of 1-2-3-4 can be in a dissection. Therefore, neither 1-2-3-4 nor, by symmetry, 1-5-6-7 can be in any dissection of the quadrant. The four remaining T-tetrominoes containing square 1 which do not overlap the cut segments c and d have segments a and b as edges. It follows that a, b, and their translates are all cut segments so that the diagrammed conditions of Figure 2 hold.

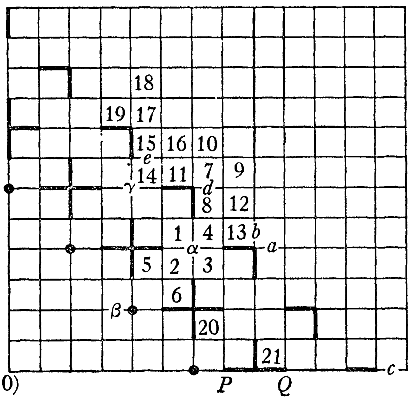

FIG. 2

We wish to prove the type-B point α is cornerless. (See Figure 2.) Suppose, to the contrary, that there is a dissection containing a T-tetromino having α at an outside corner. In this tetromino the unit square forming the outside corner at α cannot be 1, 2, or 3 because of the nearby cut segments. Consequently, either 1-2-3-5 or 1-2-3-6 is a tetromino of the supposed dissection. But no tetromino could then contain square 6 or 5 respectively, as β is a cornerless point. This proves that α is cornerless. By the same or similar arguments all translates of α, either inside the quadrant or on the axes, are cornerless.

We shall have shown $P(\lambda+1)$ true and consequently proven the lemma if we show that segments a, b, (Figure 2) and their translates are cuts. By symmetry, we need only show a and its translates are cuts. Since the border segment c must be a cut, it is sufficient to show that if some segment a is a cut so also is its adjacent translate d. Suppose, for the contradiction, that a is a cut and a dissection exists involving a tetromino containing 7 and 8. This tetromino cannot contain 4 since α is cornerless. It must contain two of the three squares 9, 10 and 11. If it contained 9 there would be no way of assigning 12 and 13. The tetromino must be 7-8-10-11. The squares 14 and 15 must be parts of different tetrominoes because γ is cornerless. But the only way to fill 15 and 16 then is to include 15-17-18-19 in the supposed dissection. It follows that all upward translates of 7-8-10-11 would be in the dissection. The absurdity of this is apparent at the y axis. We conclude d is a cut segment if a is. Thus the lemma is proven.

Proof of Theorem 1. Only the necessity remains to be proven. For which values of b does there exist a dissection of the quadrant which is simultaneously a dissection of the rectangle R? We cannot have b congruent to 2 modulo 4 as this puts a corner of R at a type-B point which, by Lemma 1, is cornerless. Moreover, b cannot be congruent to 3 or 1 modulo 4. To see this, note that there is no way to cover square 20 or 21 if the bounding line $x=b$ stands at P or Q, respectively, in Figure 2. Consequently b, and by symmetry a, must be congruent to 0 modulo 4.

Through use of some additional construction, Lemma 1 may be made to yield further information about dissections of a rectangle.

DEFINITION 2. *Let R be an $a\times b$ rectangle dissected into T-tetrominoes. A* **block** *of R is a 2×2 square of R whose vertices have even coordinates. A* **chain** *of R is any minimal subset of R which is both a union of blocks and a union of T-tetrominoes.*

THEOREM 2. *Let R be an $a\times b$ rectangle dissected into T-tetrominoes. Color every other block of R in checkerboard fashion. Every T-tetromino of the dissection will have three unit squares in one block and one unit square in an adjacent block. Every chain consists of an even number of blocks which may be cyclically ordered in such a way that the blocks are alternately colored and uncolored and the T-tetrominoes of the chain contain three unit squares of one block and one unit square of the succeeding block.*

Proof. The three-and-one coloration of T-tetrominoes follows from Lemma 1 and, say, inspection of Figure 2. Note that the type-A and type-B points are the corners of the blocks of R. The rest of the theorem is straightforward.