Bijections between graphs and tilings with Walkup structure

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1 Introduction

Tilings and tessellation of the plane with different structures have a variety of relations and applications from the sphere of Group Theory to the Topology, from the Enumerative Combinatorics to the Probability Theory. In the last decade, a number of new methods have been introduced in the field, which have facilitated the solution of some hard problems. For instance, the simple combinatorial method and algebraic method, firstly used by Michael Reed. A structural approach has also been developed by Walkup and in the current research our main task is to examine the structure of the tilings, obtained by special class of figures.
The coverings with some polyominoes, introduced by Golomb in the emergence of the field of combinatorial geometry, are examined by the structural method. The considered figures are more specifically T-tetrominoes by Walkup and the generalization - stepwise n^2-omino by Delchev.

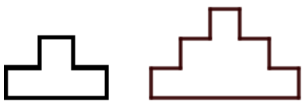


Figure 1: tetromino and stepwise n^2-omino (in the case n=3)
The figure stepwise $n^2 - mino$ is formed by n^2 unit squares. We do not make distinction between the four possible orientations.

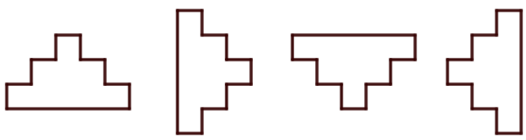
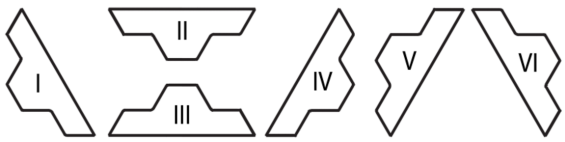


Figure 2: Orientations of the n^2-omino

A *tiling* of a rectangular region Γ with stepwise n^2-ominoes is an arrangement of stepwise n^2-ominoes in which every square of Γ is covered exactly once. An individual stepwise n^2-omino in such a tiling is called a tile. Let T_Γ denote the set of all possible tilings of Γ by stepwise n^2-minoes.
The main results for T-tetrominoes are obtained by D.W.Walkup [3]. He proved that if a rectangle can be tiled with T-tetrominoes, its sides are divisible by 4. Also, he proved the existence of a rigid structure for every rectangular region which has a valid tiling.
The generalization of the tetromino - the stepwise n^2-mino has also been examined. In 2003 Delchev proved the existence of similar structure to the Walkup structure for rectangular regions coverable by n^2-minoes.
In the beginning of the century, Michael Korn and Igor Pak proved a connection between tilings of rectangular regions with T-tetrominoes and chain graphs. They also found the number of tilings of a region, which is the Tutte polynomial on a specific type of subgraphs. In the conclusion of their research they left some interesting questions, which are not only mathematical but philosophical. One of them is connected with finding a new figure of the T-tetromino type and examine the enumeration of the tilings constructed by it and a connection to the Tutte polynomial.
The main aim of the current project is to answer this question.
In the research, we are interested in combinatorial structures formed by unit triangles and to be more specific, our examination is strongly connected with the following figure, which we shortly call polyiamond:



A subject of interest is the affine coordinate system. First, we prove an existence of rigid structure for every tiling of the plane with this type of polyiamond. Second, we consider some properties of the valid tilings as well as bijection between these tilings and chain graphs. Then we prove that only regions with sides divisible by 6 are coverable by polyiamonds and find a connection between the number of tilings and a specific graph, constructed after a chain graph.

2 Stepwise n^2-minoes

2.1 Definitions

For convenience let us use the following definitions, which are legacy from the Walkup's work, extended for arbitrary n.

Definition 1. The set of all valid tilings of the rectangular region Γ by T-tetrominoes is T_Γ.

Definition 2. A segment is every line segment forming the border of a cell in 1^{st} quadrant.

Definition 3. A border segment or border is a segment that lies on the border of two stepwise n^2-ominoes for every tiling of the quadrant.

Also, we use the following definitions of points in lemma proved in [2].

Definition 4. A point (i, j) is of type A or A-point if $i \equiv 0(\text{mod } n), j \equiv 0(\text{mod } n), i+j \equiv 0(\text{mod } 2n)$.

Definition 5. A point (i, j) is of type B or B-point if $i \equiv 0(\text{mod } n), j \equiv 0(\text{mod } n), i+j \equiv n(\text{mod } 2n)$.

Definition 6. A point (i, j) is of type Cx if $i \equiv 0(\text{mod } n), j \not\equiv 0(\text{mod } n), j \not\equiv \pm 1(\text{mod } 2n)$.

Definition 7. A point (i, j) is of type Cx if $i \equiv 0(\text{mod } n), j \not\equiv 0(\text{mod } n), i \not\equiv \pm 1(\text{mod } 2n)$

Definition 8. A X-group assigned to a B-point with coodinates (pn, kn) is a set of all Cx-points with coordinates $(kn, pn + i), i \in 2, \ldots\ldots, 2n - 2$.

Definition 9. A Y-group assigned to a B-point with coodinates (pn, kn) is a set of all Cy-points with coordinates $(kn + 1, pn), i \in 2, \ldots\ldots, 2n - 2$.

Definition 10. An outer cornerless point is a point that doesn't lie on a vertex or inside polyomino for every tiling of the quadrant.

Definition 11. An inner cornerless point is a point that lies inside a polyomino for every tiling of the quadrant.

3 Main results for stepwise n^2-mino

In order to examine tilings of the quadrant with stepwise n^2-minoes, we will use the following lemma by Delchev [2]:

Lemma 1. Every tiling of the first quadrant satisfies the following rules:
(i) Every segment with vertex at A-point is a cut.
(ii) Every B-point is outer cornerless point.
(iii) Both the X- and the Y-group assigned to an arbitrary B-point are consisting of cornerless points. Exactly one of them is consisting only of inner points and exactly one - of outer.

For convenience, we will divide the section in two parts. In the first subsection we prove a bijection between the tilings of rectangular regions with sizes $2nk \times 2nl$ and $4k \times 4l$ and in the second subsection - bijection between tilings of rectangular regions and chain graphs.

3.0.1 Tilings of rectangular regions

D.W.Walkup [3] proved the following theorem for T-tetrominoes:

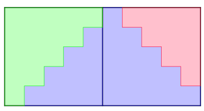
Theorem 2. If a rectangle $a \times b$ can be validly tiled by tetrominoes, then a and b are divisible by 4.

Later, K. Delchev made the straightforward generalization:

Theorem 3. If a rectangle $a \times b$ can be tiled by n^2-ominoes, a and b are divisible by $2n$.

We use it in the proof of Lemma 4. Also we generalize the M. Korn and I. Pak lemma for tetrominoes, which we use in the proof for the existence of bijection between the tilings with tetrominoes and stepwise n^2-minoes, to:

Theorem 4. In any tiling of a rectangle by n^2-ominoes, each tile contains $\frac{n(n-1)}{2}$ squares from one block and $\frac{n(n+1)}{2}$ squares from an adjacent block. Similarly, each block contains $\frac{n(n-1)}{2}$ squares from one tile and $\frac{n(n+1)}{2}$ squares from another tile.



The theorem follows of the Walkup structure for stepwise n^2-minoes, proved in [2].
Let us examine the tiling of $2nk \times 2nl$ and $4k \times 4l$ regions. Firstly, we define the following lemma:

Lemma 5. For rectangular regions with sizes $2nk \times 2nl$ and $4k \times 4l$, $|T_{\Gamma_{n^2}}| = |T_{\Gamma_4}|$.

Proof. Let's begin with the structure described in Walkup's original paper. Objects of our examination are the squares with size 2×2 with even coordinates. In figure 3 the border segments are indicated in heavy lines and the outer cornerless points are marked with blue dots.

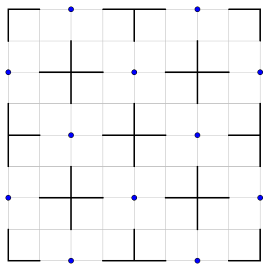
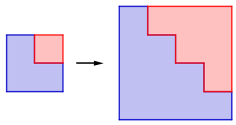


Figure 3: Walkup structure for tetrominoes

With P denote the set of all blocks 2×2 with even coordinates and with Q the set of all blocks $n \times n$ with coordinates (an, bn).



Observe the antiblocks with their bottom left vertex. We have a function for the coordinates of the blocks from both sets - $f : P \to Q$. Denote the number of blocks in P as V_P and in Q as V_Q.
Because the number of elements in the sets is equal, then $V_P = V_Q$ and the function is injective. What remains to be shown is that the function is also surjective.
Assume that the statement is false. Then there are elements in Q that do not have corresponding elements in P. In order for these elements to exist, a n^2-mino must be positioned in an invalid way in antiblocks considering Theorem 3. This is contradiction with Theorem 3 and our assumption. The function is also surjective. $\square$

After we proved the lemma, let us consider the bijection between tilings and chain graphs.

3.0.2 Chain graphs

We define an antiblock as a $n \times n$ with (an, bn) coordinates in its corners.
For a $2nk \times 2nl$ rectangle, let V be the set of points which have odd coordinates. Say that a directed graph on the vertices V is a chain graph if it satisfies the following properties:

- every edge connects vertices that are $n/2$ units apart (either vertically or horizontally)
- every vertex has indegree 1 and outdegree 1
- every white antiblock contained in the rectangle borders exactly two edges of the graph, and these edges are non-adjacent.

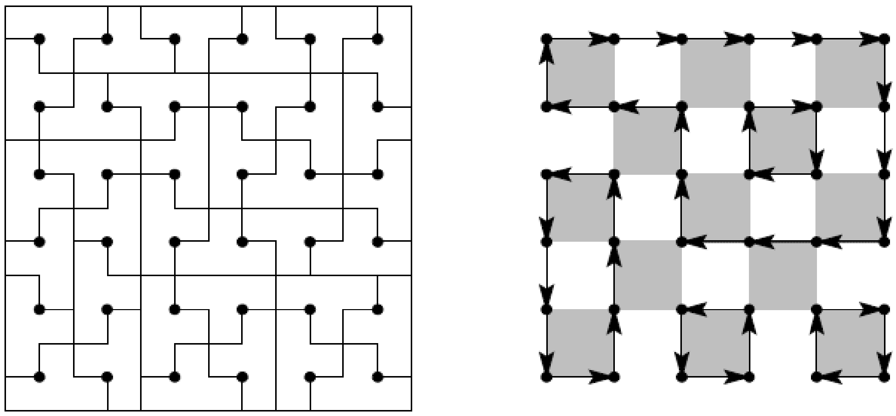


Figure 4: In the case with tetrominoes [1]

The bijection between the tilings of rectangular regions and chain graphs for tetrominoes was proved by Korn and Pak [1] . We will prove similar bijection but for the generalization of the tetromino - n^2-omino.

Lemma 6. For any $2nk \times 2nl$ rectangle, exists a bijection between the set of its tilings and the set of its chain graphs.

Proof. We struct the chain graph over a tiling of region Γ. In order to have a valid tiling of Γ, for each edge of the chain graph only one equivalent tile and setting for it must exist.
The edges of the graph have indegree 1 and outdegree 1 and the number of them is equal to the number of the blocks. Because of that the sum of the areas of the unit elements is equal to the area of the rectangular region. We won't have a bijection only if some

n^2- ominoes are overlapping.
Consider the edges of the chain graph. For each directed edge there is an equivalent n^2-omino. It is sufficient to prove that the directed edges do not allow overlapping. Let us examine the possible positions of the directed edges.

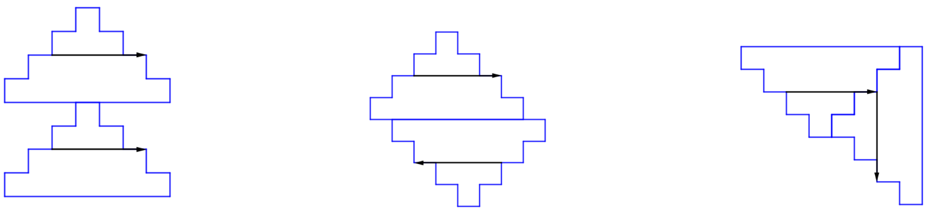


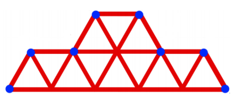
Figure 5: Positions of n^2-ominoes

We could notice that the edges do not allow overlapping of n^2-ominoes therefore there exists a bijection between the tilings and the chain graphs of a rectangular region. $\square$

4 Polyiamonds

4.1 Introduction

In the work of Michael Korn and Igor Pak [1], the authors pose a question connected with finding other figures, whose tilings have similar structure to the Walkup structure. Except for the polyiominoes, suitable figures for this purpose are the polyiamonds - figures made up by n unit equilateral triangles that are connected at their edges.
The tiling properties of polyiominoes and polyiamonds are an object of computational geometry research which aims to find the polyiominoes, tiling the plane isohedrally and by translations alone. Fukuda and Hiroshi [4] use different algorithms to create polyiominoes and polyiamonds that are fundamental domains for isohedral tilings having different symmetry groups. We use the following polyiamond generated by them and examine its properties and tilings:



In this section we will use affine coordinate system. The center of the coordinate system would be the fixed point O and the vectors e_1 and e_2 intersect at a $60°$ angle.
The positive part of the plane is defined as the points $a * e_1 + b * e_2$ where a and b are non-negative integers.

We would cover the positive part of the plane (quadrant) with the following six orientations of a polyiamond:

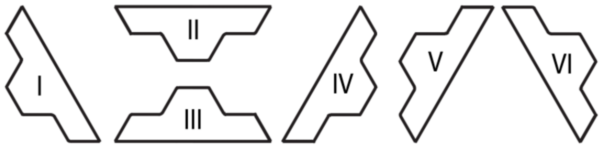


Figure 6: Orientations of the polyiamond

Polyiamond of orientation I will be denoted by polyiamond I, polyiamond of orientation II - polyiamond II, for the other orientations - analogically.

5 Definitions

We use the following definitions:

Definition 12. A point is of type A or $A-$point if its coordinates are of type $(4m+2, 6l+2)$ or $(4m, 6l+4)$.

Definition 13. A point of type B or $B-$point of the coordinate system if it is at distance greater than 1 of all adjacent $A-$points.

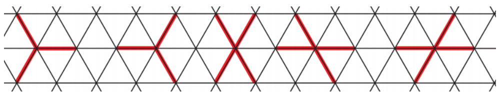
Definition 14. A border segment is a segment, which cannot be inside a polyiamond and can be only on its borders in a valid tiling of the region.

5.1 Main results for polyiamonds

5.1.1 The structure of tilings

Firstly, we examine the properties of the $A-$points and the $B-$points in valid tilings of the quadrant. We will prove the following lemma:

Lemma 7. All $B-$points are not a vertex of an acute angle of polyiamond in a valid tiling of the quadrant. All $A-$points are part of group of border segments of the following type (or after adding more border segments that consist the $A-$point):



Proof. With A_n we mark that the statement is true in the region between the lines:

$$x + 2y - 6n = 0$$

$$x + 2y = 0$$

In order to prove that A_n is true, we use induction on two statements - $B(n)$ and $C(n)$.

With $B(n)$ we mark the statement: If a point X with coordinated $(4k + 2, 6p + 2)$ or $4k, 6p+4$ is $A-$point, then point Y with coordinates $(4k-2, 6p+4)$ or $(4k-4, 6p-6)$ is also $A-$point.
$C(n)$ denote the statement: If the points $F(4k+1, 6p+1)$, $G(4k, 6p+2)$, $H(4k-1, 6p+2)$, $I(4k-2, 6p+2)$ and $J(4k-2, 6p+1)$ or $F(4k-1, 6p+3)$, $G(4k-2, 6p+4)$, $H(4k-3, 6p+4)$, $I(4k-4, 6p+4)$ and $J(4k-4, 6p+3)$ are $B-$points, then the points $E(4k-3, 6p+3)$, $D(4k-4, 6p+4)$, $C(4k-5, 6p+4)$, $B(4k-6, 6p+4)$ and $A(4k-6, 6p+3)$ or $E(4k-5, 6p+5)$, $D(4k-6, 6p+6)$, $C(4k-7, 6p+6)$, $B(4k-8, 6p+6)$ and $A(4k-8, 6p+4)$ are also $B-$points.
A base of the induction is A_1, which is obviously true:
On the graphics we will mark the border segments in a valid tiling of the quadrant in

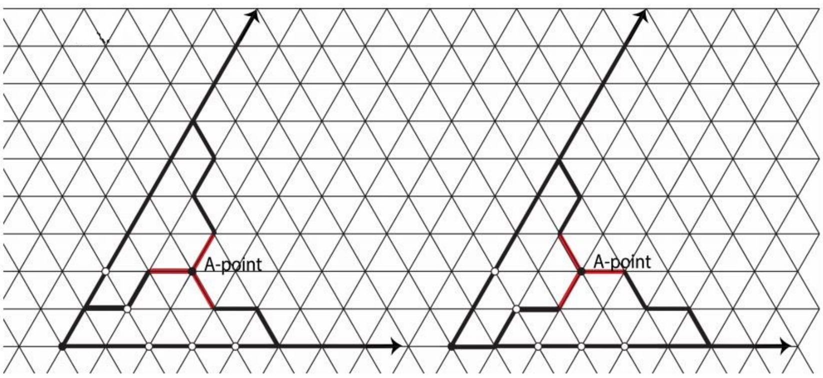


Figure 7: Base of induction

red, the $A-$points in black and the $B-$points - white.
An induction is used for the proof of $B(n)$ and $C(n)$ and we will show that statements $B(n+1)$ and $C(n+1)$ are true.

$B(n) \rightarrow B(n+1)$

Let us consider points A, B, C, D and E. Assume that they are not in an acute angle of polyiamond in a valid tiling of the quadrant. We examine all cases and prove that this statement is false.

$\mathbf{C(n)} \rightarrow \mathbf{C(n+1)}$

Assume that the lemma is true for the region between the lines $x + 2y - 6n = 0$ and $x + 2y = 0$ and for the points on the line $x + 2y - 6n = 0$ from the point with coordinates $(6n, 0)$ to point N. Let us examine the next $A-$point on this line - point M.

Suppose that the border segments defined in the lemma are different. We consider all tilings of the quadrant and show that this assumption is false for every valid tiling. All cases are shown in the Appendix of the research.

We performed all induction steps - $B(n+1)$ and $C(n+1)$ and proved the induction hypothesis. [1]

□

5.1.2 Regions, which can be validly tiled with polyiamonds

Obviously, a figure coverable by our specific polyiamond (which consists of 12 unit triangles) must have a number of unit triangles divisible by 12. Since every figure can be divided to equilateral triangles, which have n^2 unit triangles, we will consider their tilings. n^2 is divisible to 12 when n is divisible 6.

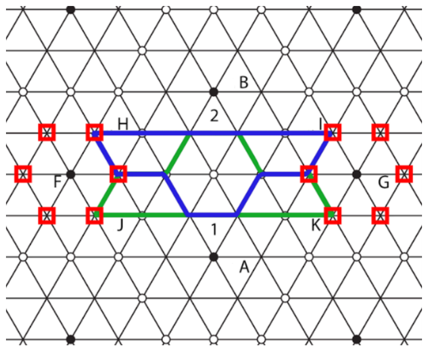
We will examine some properties of the regions with valid tilings. Let us consider a polyiamond which does not have $A-$points in the circumference. We will prove that in a valid tiling this is not possible.

Assume that polyiamonds of these type exist in a valid tiling. They will have vertices on distance 1 from $A-$point. Denote them as $C-$points, in the graphics are shown as tiny squares.

Without loss of generality, we will consider polyiamond with these properties. Points F and G are $A-$points. Suppose that one of the $C-$points around F and one of the $C-$points around G are vertices in the acute angles of polyiamond in a valid tiling of a region. The only two combinations of $C-$point at distance exactly 5 are points H and I and point J and K. (We are looking for distance with length 5 because the length of the side of the polyiamond is 5.)

On the graphics, one can see a blue polyiamond with vertices H and I. We will closely observe the covering of triangle 1. If the border segments from point A contain or , the polyiamond covering triangle 1 has a vertex which in $B-$point in an acute angle. This is contradiction. Analogically, we examine the combination of points J and K and again obtain contradiction.

[1] All performed induction are considered in the Appendix of the research, which could be found here https://goo.gl/U1F0p5



As a result, we obtain that if both vertices of acute angles of polyiamond are $C-$points, then there is a valid tiling of the region.
For convenience, we will number the vertices of the polyiamonds.
Suppose that in a valid tiling of the quadrant, vertices 3 and 4 are $A-$points. We will

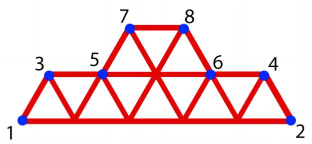


Figure 8: Numbered verticies of a polyiamond

examine all cases in which that is possible.
We observe that in every case, the polyiamond has $B-$point vertex in an acute angle,

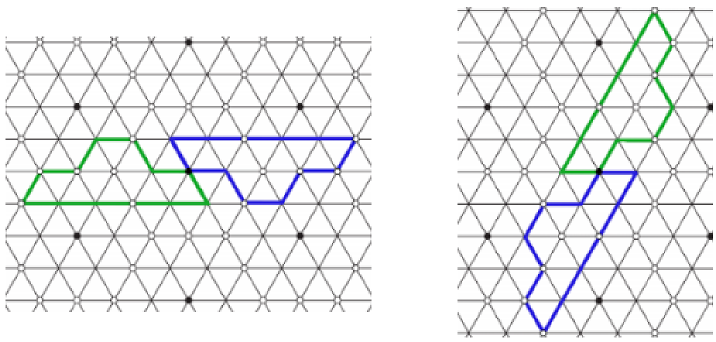
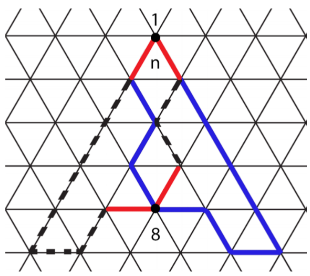


Figure 9

which is contradiction.
In conclusion, if we have an $A-$point in vertices 3 or/and 4, there is no valid taling of the quadrant. Vertices 5 and 6 obviously can not be $A-$points because a border segment would exist in the inner part of the polyiamond.
From the lemma which we proved in the previous section, we obtain that if vertex 1 is an

$A-$point, then vertex 8 is an $A-$point too. This is true because the difference between the coordinates is $(2, 2)$. The analogical statement is true for vertices 2 and 7.
We proved that in valid tiling of the quadrant every polyiamond has $A-$point in vertices 1 and 8 or vertices 2 and 7.
What remains to be shown is the relation between the different combination of border segments from $A-$points and the tilings of the quadrant.
Obviously, every tiling has only one corresponding combination of border segments or we can call it a *structure*. As we could see, this structure is similar to the Walkup structure.
It remains to be checked that for every *structure* there is only one tiling.
Let us consider a polyiamond and the $A-$points in its vertices. These $A-$points are forming border segments from the following type:



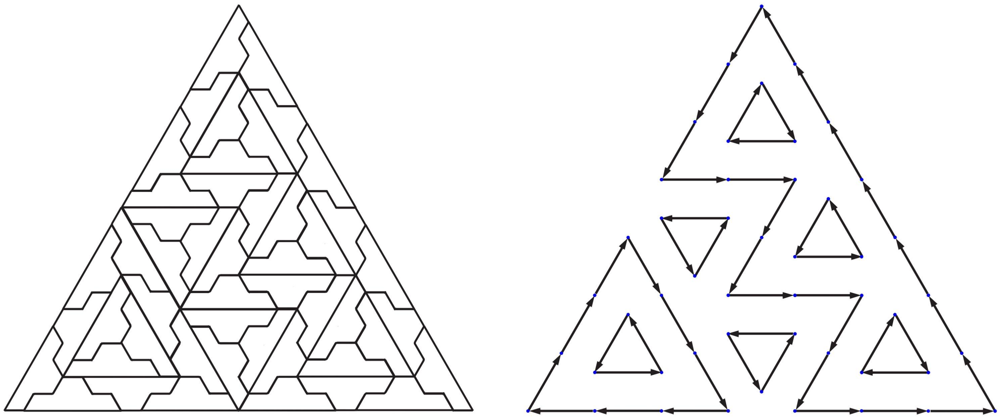
Suppose that we can have tha same border segments but they are formed by different polyiamond. Let us cover the unit triangle 1. In order to save the border segments from the vertex in the acute angle of the first polyiamond in rotation IV (in a dotted line), we should cover triangle 1 with polyiamond in rotation I. Actually, this is a contradiction because the second polyiamond will intersect the border segments formed by the first one.

5.2 Bijection between the tilings of a region with a polyiamond and chain graphs

A chain graph is a directed graph with vertices $B-$points, which after dividing the figure to rhombs, are in the center of the rhombs. The graphs satisfies the following properties:

- every edge connect vertices that are 3 unit apart (either vertically or horizontally)
- every vertex has an indegree and outdegree 1
- the angle between the edges is $180°$ or $60°$

- point on a perpendicular distance $\sqrt{2}/2$ of an edge is not a part of an edge of the graph



Let us prove the following theorem

Theorem 8. For every triangular region $6k \times 6k$, the number of tilings with polyiamonds is equal to the number of chain graphs.

The proof is based on bijection between the two sets. For every polyiamond, draw one directed edge from its vertex with number 6 to vertex with number 3 or from vertex with number 5 to vertex with number 4.

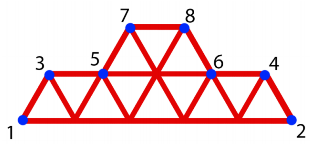
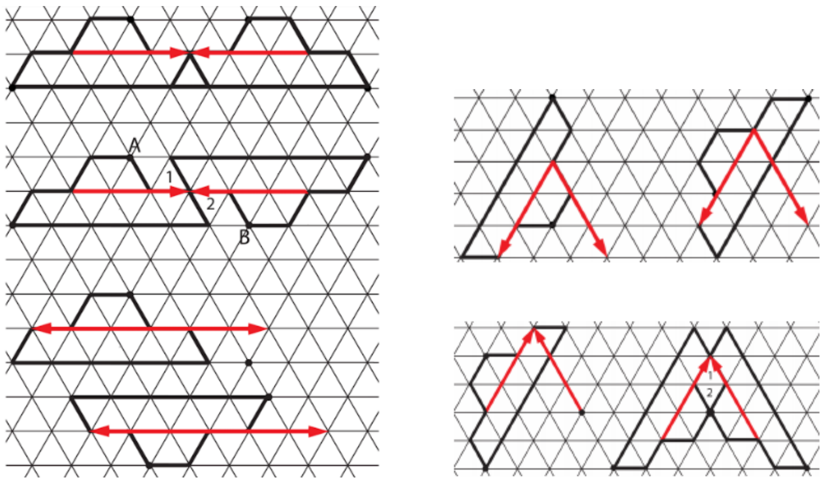


Figure 10: Numbered vertices of a polyiamond

For every triangle $6k \times 6k$, we will prove that the map $\phi(t)$ is bijection between the set of tilings of this triangle and the set of its chain graphs. Let us first show that $\phi(t)$ is a chain graph.

Obviously, the edges of $\phi(t)$ are with length 3. For the second restriction, we suppose that a vertex of $\phi(t)$ does not satisfies the properties of a chain graph. With the following examination of all cases, we show the contradiction: In all figures, observe that if we use the tiles corresponding to the edges, they will overlap, which is a contradiction.

For the third restriction, suppose that the angle obtained by the intersection of the edges



(a) (b)

is 120° - not 180° or 60°.

Consider the polyiamond corresponding to edge 1. What remains to be shown is that the polyiamonds corresponding to edge A intersect these corresponding to edge 1. With the following two figures we will show the possible positions of these polyiamonds and observe that in both cases overlapping occurs.

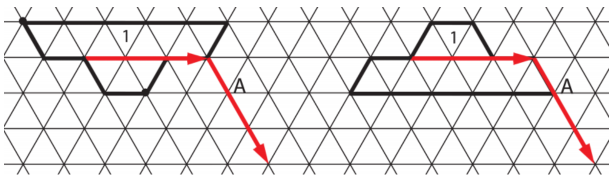


Figure 11: Edges - 120°

Let us consider the vertices of the map $\phi(t)$. From the definition for the map, we know that the coordinates of the vertices of the chain graph are the same as the coordinates of the vertices of the map.

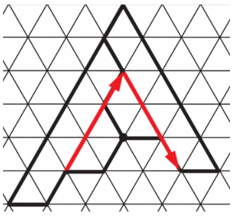
Thus, the map $\phi(t)$ satisfies the properties to be a chain graph.

It is obvious that for every edge exists a corresponding polyiamond - the map $\phi(t)$ is injective.

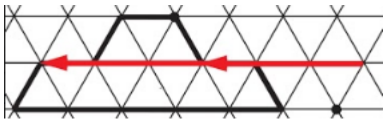
What remains to be shown is that every chain graph corresponds to one map $\phi(t)$ for some tiling.

As we just observed, for each edge there is only one polyiamond placement. So any graph will yield a collection of polyiamond placements. It remains to be checked that if they do not overlap and cover the surface of the whole figure, which we are tiling.

Because the number of edges is equal to the number of polyiamonds, the sum of their areas is equal to the area of the whole figure.
Let us show that they do not overlap.
The following two positions of edges of the chain graph are the only positions satisfing the definition. We could clearly observe that no overlapping occurs.



(a) Edges intersect in 60°



(b) Edges intersect in 180°

5.3 Enumeration of tilings

In this section, an object of our examination will be the number of tilings of certain figure. In the research of Korn and Pak [1], the tilings of rectangular regions with T-tetrominoes is connected with the Tutte polynomial - $T(3,3)$. We will show that there is no strong connection between the tilings of regions with the Walkup structure and this polynomial.

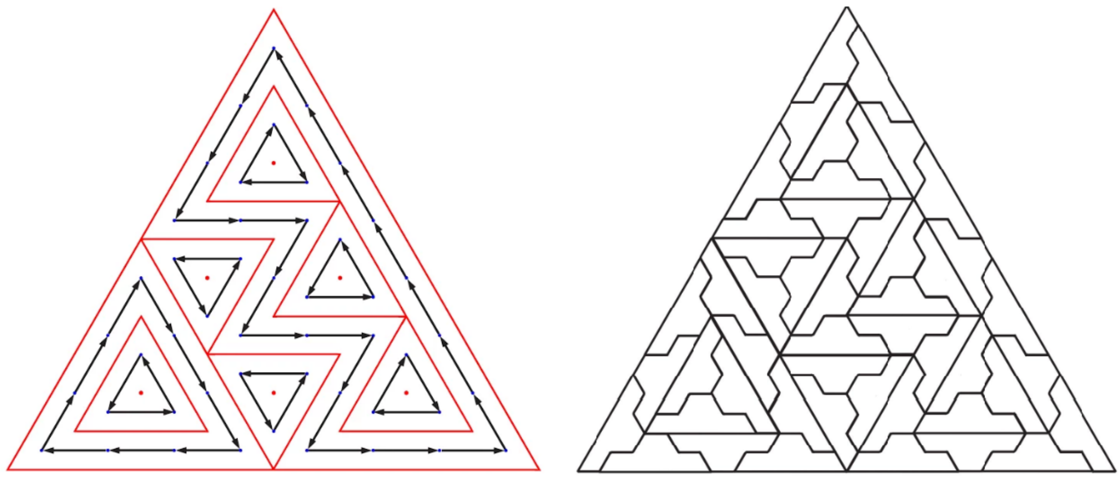
$$T(G;x,y) = \sum_{H \subset G} (x-1)^{c(H)-c(G)} (y-1)^{c(H)+|E(H)|-|V(G)|}$$

With C_Γ we will denote the set of chain graphs and with K - a chain graph of the region Γ. Also, H will be the graph constructed over all vertices $A-$points of the *structure* of Γ. The edges of H will be 6 units apart and won't instersect the edges of K. The set of these type of graphs is G_Γ. Denote the number of connected components of graph H with $c(H)$.
For the number of tilings of triangle region with polyiamonds, we will define the following lemma:

Theorem 9. For every $H \subset G_\Gamma$, there are exactly $2^{c(H)-1}$tilings of Γ. The number of all tilings of a region is equal to:

$$|T_\Gamma| = \sum_{H \subset G} 2^{c(H)-1}$$

First, we should prove that for any spanning subgraph $H \subset G_\Gamma$, the corresponding undirected chain graph has $c(H) - 1$ cycles. Because every edge of the chain graph has two directions and the cycles have to orientations, the number of valid chain graphs will be $2^{c(H)-1}$.



We have proven the bijection between chain graphs and tilings and as a result, the number of elements in the sets is equal.
Consider the connected components in an undirected chain graph. If their number is k, then the figure is divided to $k + 1$ simple coverable zones. Each zone has just one connected component of H. Suppose that there are more than one connected components in a zone. Then they must be divided by an edge of K but these edges will divide the zone in two or more zones too, which is a contradiction. So we have $k + 1$ connected components of H and $k = c(H) - 1$.

6 Conclusion

In the current research we examined T-tetrominoes, stepwise n^2-minoes and polyiamonds. We proved a bijection between the set of tiliings of a region with stepwise n^2-ominoes and the set of its chain graphs. In the second part, we showed that every tiling, constructed by number of specific polyiamonds, has a structure similar to the Walkup property but also considerably different and more complex.. In the near future, we plan to find more figures, generated by Fukuda and Hiroshi, which are similar to the T-tetromino, and prove that their tilings have set structure of points and border segments such as the Walkup property. It is also interesting to consider, which regions, coverable by our specific polyiamond, are completable. A region is *completable* if more polyiamonds can be added to the outside to form a triangle.

References

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[3] Walkup, D. Covering a rectangle with T-tetrominoes. Amer. Math. Monthly, 72:986–988, 1965.

[4] Fukuda, Hiroshi, et al. "Polyominoes and Polyiamonds as Fundamental Domains of Isohedral Tilings with Rotational Symmetry." Symmetry 3.4 (2011): 828-851.