RANDOM NUMBERS

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CONTENTS

***Abstract*—This manual provides solutions to the Assignment on Random Numbers**

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/1.1.c
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.1.c
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/1.2.py
```

Download the above files and execute the following commands to produce Fig.1.2

```
$ python3 1.2.py
```

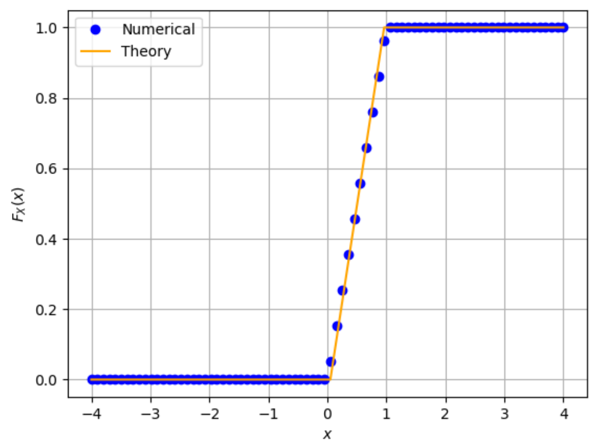

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: Given U is a uniform Random Variable

$$p_U(x) = 1 \text{ for } 0 < x < 1 \tag{1.2}$$

$$F_U(x) = \int_{-\infty}^{\infty} p_U(x)dx \tag{1.3}$$

$$\boxed{\implies F_U(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}} \tag{1.4}$$

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/1.3.py
```

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.
Solution: Download the following files and execute the C program.

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/1.4.c
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 1.4.c
$ ./a.out
```

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution:

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.8}$$

$$\implies \text{var}[U] = E\left[U^2\right] - E[U]^2 \tag{1.9}$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.10}$$

$$E[U] = \int_0^1 x \tag{1.11}$$

$$\implies \boxed{E[U] = \frac{1}{2}} \tag{1.12}$$

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.13}$$

$$E\left[U^2\right] = \int_0^1 x^2 dF_U(x) \tag{1.14}$$

$$\implies E\left[U^2\right] = \frac{1}{3} \tag{1.15}$$

$$\implies \boxed{\text{var}[U] = \frac{1}{12}} \tag{1.16}$$

Theoretical values

$$E(X) = 0.5 \tag{1.17}$$

$$Var(X) = 0.08333 \tag{1.18}$$

Numerical values calculated in C program

$$E(X) = 0.500007 \tag{1.19}$$

$$Var(X) = 0.083307 \tag{1.20}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: Download the following files and execute the C program.

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/2.1.c
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.1.c
$ ./a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted using the code below

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/2.2.py
```

run the following code to get the graph

```
$ python3 2.2.py
```

Some of the properties of CDF

a) $\lim_{x\to\infty} F_X(x) = 1$
b) $F_X(x)$ is non decreasing function.
c) Symmetric about one point.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted using the code below

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/2.3.py
```

Download the above files and execute the following commands to produce the PDF

```
$ python3 2.3.py
```

Some of the properties of the PDF:

a) Symmetric about $x = \mu$ in this case
b) Decreasing function for $x > \mu$ and increasing for $x < \mu$ and attains maximum at $x = \mu$
c) Area under the curve is unity.
d) the PDF takes non negative values

2.4 Find the mean and variance of X by writing a C program.
Solution: Download the following files and execute the C program.

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/2.4.c
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 2.4.c
$ ./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x\infty, \quad (2.3)$$

repeat the above exercise theoretically.
Solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x)\, dx \quad (2.4)$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

Taking $\frac{x^2}{2} = t$,

$$E[X] = -\int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t)\, dt \quad (2.6)$$

$$E[X] = 0 \quad (2.7)$$

To calculate variance,

$$var[X] = E\left[(X - E[X])^2\right] \quad (2.8)$$

$$var[X] = E\left[X^2\right] \quad (2.9)$$

$$var[X] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \quad (2.10)$$

$$var[X] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.11)$$

We know that,

$$\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) = \sqrt{2\pi} \quad (2.12)$$

$$var[X] = 1 \quad (2.13)$$

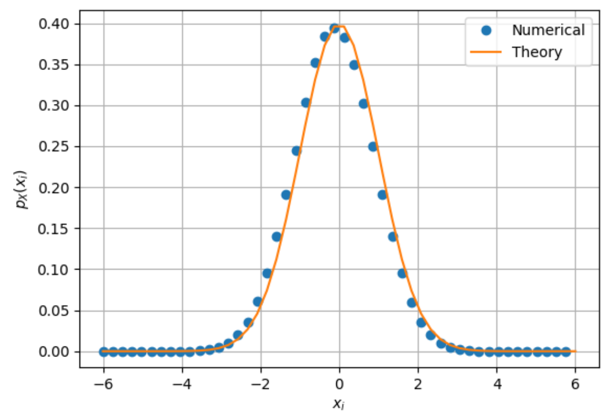

Fig. 2.5: PDF of X

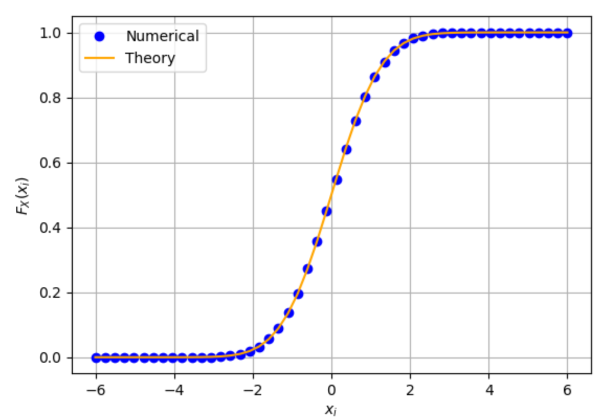

Fig. 2.5: CDF of X

The Python plot for the CDF and PDF of X are

```
https://github.com/Killomanny/Assignments/
    tree/main/Assignmets2.0/codes/2.5_1.py
https://github.com/Killomanny/Assignments/
    tree/main/Assignmets2.0/codes/2.5_2.py
```

3 From uniform to other

3.1 Generate samples of

$$V = -2\ln(1-U) \tag{3.1}$$

and plot its CDF.
Solution: Download the following files and execute the C program.

```
wget https://github.com/Killomanny/
    Assignments/tree/main/Assignmets2.0/
    codes/3.1.c
wget https://github.com/Killomanny/
    Assignments/tree/main/Assignmets2.0/
    codes/source.h
```

Download the above files and execute the following commands

```
$ gcc 3.1.c -lm
$ ./a.out
```

The CDF of V is plotted in Fig. 3.1 using the code below

```
wget https://github.com/Killomanny/
    Assignments/tree/main/Assignmets2.0/
    codes/3.1pyth.py
```

Download the above files and execute the following commands to produce Fig.3.1

```
$ python3 3.1pyth.py
```

3.2 Find a theoretical expression for $F_V(x)$.
Solution: If Y = g(X), we know that $F_Y(y) = F_X(g^{-1}(y))$, here

$$V = -2\ln(1-U) \tag{3.2}$$

$$1-U = e^{\frac{-V}{2}} \tag{3.3}$$

$$U = 1-e^{\frac{-V}{2}} \tag{3.4}$$

$$F_V(x) = F_U(1-e^{\frac{-x}{2}}) \tag{3.5}$$

$$\implies F_V(x) = \begin{cases} 0 & x < 0 \\ 1-e^{\frac{-x}{2}} & x \geq 0 \end{cases} \tag{3.6}$$

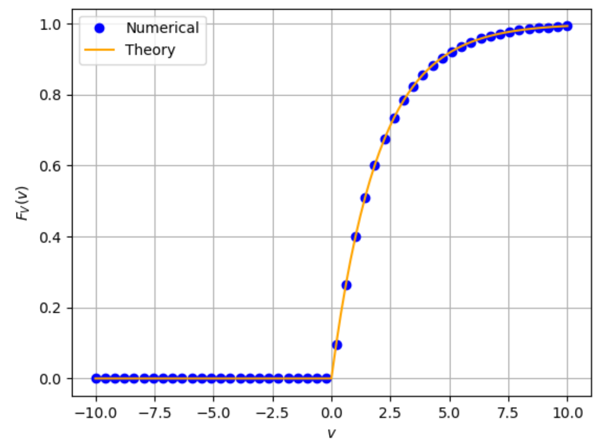

Fig. 3.1: The CDF of V

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:
Download and run the following C code to generate tri.dat file.

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/4.1.c
```

4.2 Find the CDF of T.
Solution:

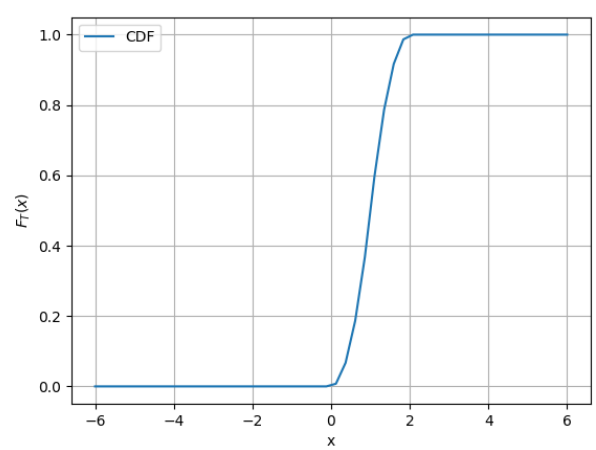

Fig. 4.2: CDF of T

The following code plots the CDF of T

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/4.2.py
```

4.3 Find the PDF of T. **Solution:** The following

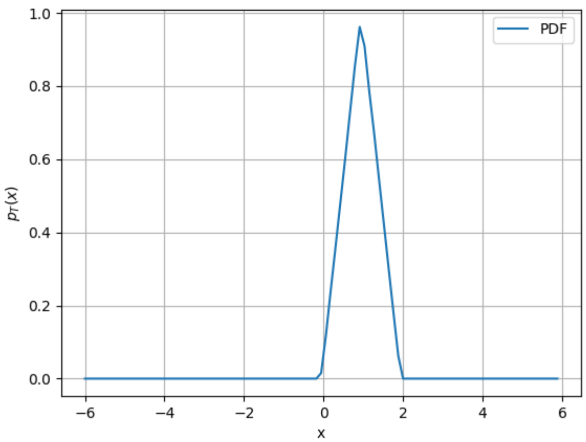

Fig. 4.3: PDF of T

code plots the PDF of T

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/4.3.py
```

4.4 Find the theoritical expressions for PDF and CDF of T.
Solution:

$$p_T(x) = p_{U_1+U_2}(x) = p_{U_1}(x) * p_{U_2}(x) \tag{4.2}$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x-\tau) \tag{4.3}$$

$$p_T(x) = \int_{0}^{1} p_{U_2}(x-\tau) \tag{4.4}$$

$$p_T(x) = \begin{cases} 0 & x \le 0 \\ \int_0^x 1 d\tau & 0 < x < 1 \\ \int_{x-1}^1 1 d\tau & 1 \le x < 2 \\ 0 & x > 2 \end{cases} \tag{4.5}$$

$$p_T(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 2-x & 1 \le x < 2 \\ 0 & x > 2 \end{cases} \tag{4.6}$$

Expression for CDF can be obtained by integrating $p_T(x)$ w.r.t. X

$$F_T(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \le x < 2 \\ 1 & x > 2 \end{cases} \tag{4.7}$$

4.5 Verify the results through a plot.
Solution:
PDF and CDF plotted by the Python codes

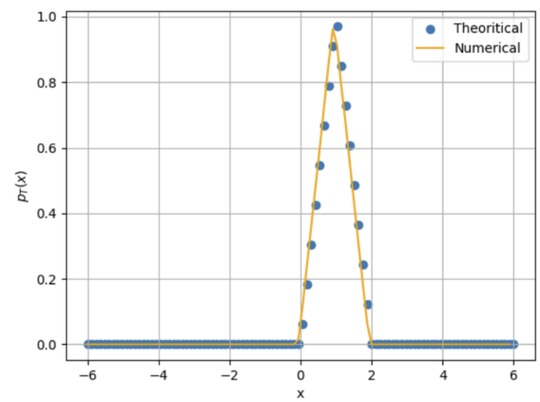

Fig. 4.5: Theoretical PDF of T

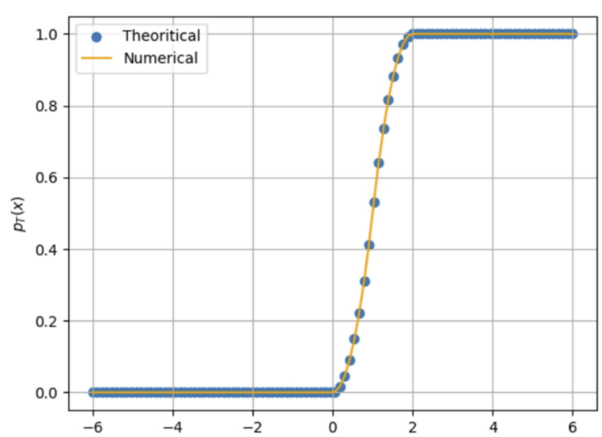

Fig. 4.5: The CDF of T

```
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/4.5pdf.py
wget https://github.com/Killomanny/
    Assignments/blob/main/Assignmets2.0/
    codes/4.5cdf.py
```