Prefix Sum of Multiplicative Functions

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1 Prefix Sum of Multiplicative Functions

Given a *multiplicative function* $f(n)$ where:

1. $f(p)$ is a polynomial of low degree.
2. $f(p^k)$ is easy to calculate.

The problem is to calculate $\sum_{i=1}^{N} f(n)$ in $o(n)$ time complexity.

Table 1: Frequently Used Notations in Section 1

Notation	Meaning
p	Aribtrary prime
p_k	The k-th smallest prime, specially $p_0 = 1$
$\text{lpf}(n)$	$\min\{p : p \mid n\}$, least prime factor of n, specially $\text{lpf}(1) = 1$
$\text{plp}(n)$	$\max\{c : \text{lpf}(n)^c \mid n\}$, power of $\text{lpf}(n)$ for n
$[P]$	Iverson bracket, 1 if statement P is true and 0 otherwise
$f(n)$	Function given in problem definition
$F_k(n)$	$\sum_{i=2}^{n}[p_k \le \text{lpf}(i)]f(i)$, prefix sum of $f(i)$ that $\text{lpf}(i) \ge p_k$
$F_P(n)$	$\sum_{2\le p\le n} f(p)$, prefix sum of $f(p)$ that $p \le n$

1.1 Min25 Sieve

Min25 Sieve solves this problem in $O(n^{3/4}/\log(n))$ time complexity and $O(\sqrt{n})$ space complexity. Here we define:

- $F_k(n) = \sum_{i=2}^{n}[p_k \le \text{lpf}(i)]f(i)$, the prefix sum of $f(i)$ that $\text{lpf}(i) \ge p_k$.
- $F_P(n) = \sum_{2\le p\le n} f(p)$, the prefix sum of $f(p)$ that $p \le n$.

It's straightforward that $\sum_{i=1}^{N} f(n) = F_1(n) + f(1)$ because $\text{lpf}(i) \ge 2$ for all $2 \le i \le n$. The next step is to calculate $F_k(n)$. Denote $\text{lpf}(i)$ and $\text{plp}(i)$ by p_j and c, we need to transform $F_k(n)$ to a form that is easy to calculate:

$$
\begin{aligned}
F_k(n) &= \sum_{i=2}^{n}[p_k \le \text{lpf}(i)]f(i) && \text{Definition of } F_k(n) \\
&= \sum_{i=2}^{n}[p_k \le p_j]f(i/p_j^c)\cdot f(p_j^c) && \text{Multiplicative function} \\
&= \sum_{j\ge k}\sum_{c\ge 1} f(p_j^c)\cdot \sum_{i=2}^{n}[\text{lpf}(i) = p_j \land \text{plp}(i) = c]f(i/p_j^c) && \text{Group by } p_j \text{ and } c \\
&= \sum_{j\ge k}\sum_{c\ge 1} f(p_j^c)\cdot \sum_{i=1}^{\lfloor n/p^c\rfloor}[i = 1 \lor \text{lpf}(i) > p_j]f(i) && \text{Exclude } p_j^{c+1} \mid i \text{ by } \text{lpf}(i/p_j^c)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j \ge k} \sum_{c \ge 1} f(p_j^c) \cdot \left(1 + F_{j+1}\left(\lfloor n/p_j^c \rfloor\right)\right) && \text{Definition of } F_k(n) \\
&= \sum_{\substack{j \ge k \\ p_j^2 \le n}} \sum_{c \ge 1} f(p_j^c) \cdot \left([c > 1] + F_{j+1}(\lfloor n/p_j^c \rfloor)\right) + F_P(n) - F_P(p_{k-1}) && \text{Take out } f(p) \\
&= \sum_{\substack{j \ge k \\ p_j^2 \le n}} \sum_{c \ge 1} f(p_j^c) \cdot F_{j+1}(\lfloor n/p_j^c \rfloor) + \sum_{\substack{j \ge k \\ p_j^2 \le n}} \sum_{c \ge 2} f(p_j^c) + F_P(n) - F_P(p_{k-1}) && \text{Take out } [c > 1] \\
&= \sum_{\substack{j \ge k \\ p_j^2 \le n}} \sum_{\substack{c \ge 1 \\ p_j^{c+1} \le n}} \left(f(p_j^c) \cdot F_{j+1}(\lfloor n/p_j^c \rfloor) + f(p_j^{c+1})\right) + F_P(n) - F_P(p_{k-1}) && F_{j+1}(\lfloor n/p_j^c \rfloor) = 0 \text{ for max } c
\end{aligned}
$$

By this formula,