Lecture 6. Wavelets and Multiresolution

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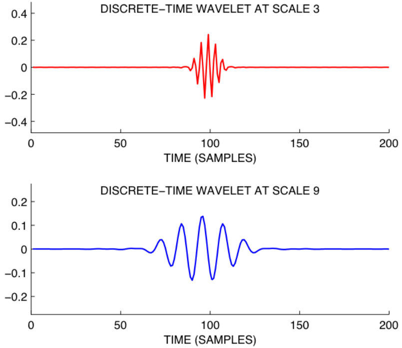


Figure: Wavelets at different scales.

- Wavelet theory is an analogue of Fourier theory using wavelet functions which are localized in time/space, generated from a single wavelet ψ by translations and dilations.

- Fix a *mother* (prototype) wavelet $\psi \in L^2(\mathbb{R})$
- For any signal $f \in L^2(\mathbb{R})$, define the wavelet coefficients

$$W[f](s,a) := \langle f, \psi_{s,a} \rangle, \quad \psi_{s,a}(t) := \frac{1}{s}\psi\left(\frac{t-a}{s}\right)$$

- If ψ satisfies the admissibility condition

$$\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega = c_\psi < \infty$$

  we can retrieve f from its transform via the inversion law:

$$f(t) = \frac{1}{c_\psi} \int_0^\infty \int_{-\infty}^\infty W[f](s,a)\psi_{s,a}(t) \frac{da\, ds}{s}$$

- In this case $\hat{\psi}(0) = \int \psi = 0$ and $\int f = 0$

- Assume ψ is real & even
- Define the *continuous* wavelet transform (CWT) operator:

$$T_s f(a) := W[f](s,a)$$

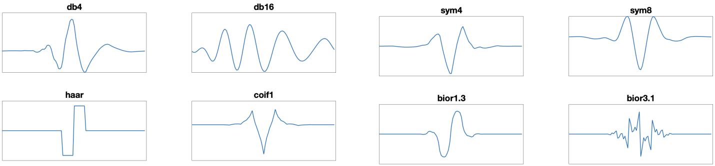
- Then $T_s f = \psi_{s,0} * f$, $\psi_{s,a} = T_s \delta_a$ and

$$T_s f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \hat{\psi}(s\omega) \hat{f}(\omega) d\omega$$

- T_s acts on the Fourier transform by multiplying the scaled band-pass filter $\hat{\psi}(s\omega)$. In this expression, the scale factor only appears in the frequency domain.
- The *discrete* WT is given by convolution with the family $\psi_{j,k}(t) = \alpha^{-j/2}\psi\left(\alpha^{-j}t - \beta k\right)$, $j,k \in \mathbb{Z}$

Exercise. Show the formal inversion law and verify the above properties.

Examples



- Haar wavelet: $\psi(t) = 1_{[0,1/2)}(t) - 1_{[1/2,1)}(t)$
- sinc wavelet: $\psi(t) = (\sin(2\pi t) - \sin(\pi t))/\pi t$
- 2D Mexican hat: $\psi(t) = \dfrac{1}{\pi\sigma^4}\left(1 - \dfrac{x^2+y^2}{2\sigma^2}\right) e^{-\frac{x^2+y^2}{2\sigma^2}}$
- Daubechies family: continuous, orthogonal, compact support

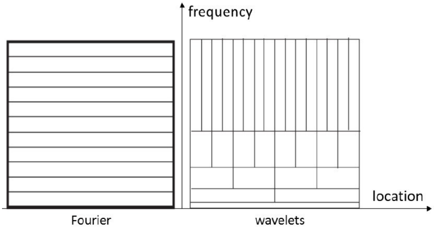


Figure: Comparison of Fourier and wavelet bases.

- While Fourier modes are globally distributed, wavelets are localized (translation a) in time/space (decay of ψ) and frequency (decay of $\hat{\psi}$).
- Wavelets allow more efficient representations of signals whose primary information lies in localized singularities, such as edges in images or step discontinuities in time series.

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FFT

Recall the 2-point Fast Fourier Transform. The discrete transform of size N input x_n is computed in $O(N \log N)$ time as:

$$
\begin{aligned}
DFT[N; x_n](k) &:= \sum_{n=0}^{N-1} x_n e^{-\frac{2\pi i}{N} nk} \\
&= \sum_{n=0}^{N/2-1} x_{2n} e^{-\frac{2\pi i}{N/2} nk} + e^{-\frac{2\pi i}{N} k} \sum_{n=0}^{N/2-1} x_{2n+1} e^{-\frac{2\pi i}{N/2} nk} \\
&= DFT[N/2; x_{2n}]_k + e^{-\frac{2\pi i}{N} k} DFT[N/2; x_{2n+1}]_k \quad k < N/2 \\
&= (\qquad " \qquad) - e^{-\frac{2\pi i}{N} k} (\qquad " \qquad) \quad k \geq N/2
\end{aligned}
$$

The DFT is recursively derived from the even/odd DFTs of half length – the information at half resolution (sampling) is reused.

MRA

- The analogous Fast Wavelet Transform is intimately related to multiresolution analysis – a decomposition of the signal space into wavelet spaces of discretized scale
- An MRA of $L^2(\mathbb{R})$ is an increasing sequence of closed linear subspaces V_j ($j \in \mathbb{Z}$) such that:

1. $v(x) \in V_j \Leftrightarrow v(2x) \in V_{j+1}$
2. $v(x) \in V_j \Leftrightarrow v(x+1) \in V_j$
3. $\overline{\cup V_j} = L^2(\mathbb{R})$, $\cap V_j = 0$
4. $\exists \phi \in V_0$ (*father wavelet*) s.t. $\{\phi(x-k) : k \in \mathbb{Z}\}$ is a frame for V_0 and $\hat{\phi}(0) = 1$

A *frame/Riesz basis* is a generalization of orthonormal basis: a spanning set e_k of V satisfying $m\|x\|^2 \leq \sum |\langle x, e_k \rangle|^2 \leq M\|x\|^2 \ \forall x \in V$.

- The projection operator $\Pi[V_j]$ provides approximations at resolution 2^j so that $\lim_{j\to\infty} \Pi[V_j]f = f$
- Let:

$$V_{j+1} = V_j \bigoplus^{\perp} W_j$$

  The detail space W_j contains the information required to move to higher resolutions, $\bigoplus_j^{\perp} W_j = L^2(\mathbb{R})$
- ψ is a mother wavelet corresponding to ϕ if $\{\psi(x-k) : k \in \mathbb{Z}\}$ is a frame for W_0
- Wavelet frames: $V_j = \langle \phi_{j,k}(t) := 2^{j/2}\phi(2^j t - k) : k \in \mathbb{Z}\rangle$ and $W_j = \langle \psi_{j,k} : k \in \mathbb{Z}\rangle$
- The Haar wavelet has father $1_{[0,1]}(t)$

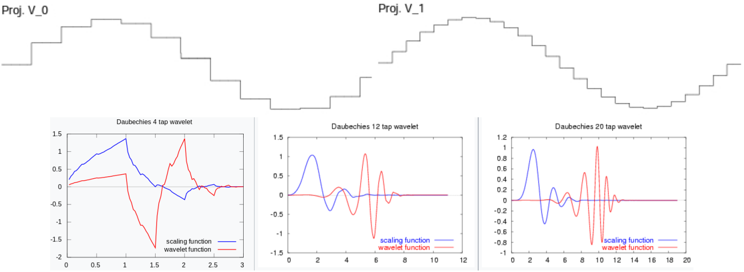

Figure: (1) The sine function at different resolutions of the Haar basis. (2) Mother-father pairs for the Daubechies family.

- Some wavelets do not admit a $\phi - \psi$ pair and has no MRA
- Orthonormality of $\phi_{j,k}$ and $\psi_{j,k}$ is generally too strict and *biorthogonal* wavelets in dual MRA sequences are used

Prop.

- refinement equation: $\phi(x) = \sum_k 2h_k\phi(2x-k)$ for a normalized 'scaling filter' h
- In the spectral domain, $\hat{\phi}(\omega) = DtFT[h](\omega/2) \cdot \hat{\phi}(\omega/2)$
- $V_j = \langle \phi_{j,k}(t) := 2^{j/2}\phi(2^j t - k) | k \in \mathbb{Z} \rangle$
- $\phi_{j,\ell} = \sum_k \sqrt{2} h_k \phi_{j+1,k+2\ell}$

Similarly, $\psi(x) = \sum_k 2g_k\phi(2x-k)$ and $\psi_{j,\ell} = \sum_k \sqrt{2} g_k \phi_{j+1,k+2\ell}$.

These structural equations form the basis of fast information extraction (halving resolution) and reconstruction (doubling resolution) without the need to evaluate integrals at every step.

- Denote the wavelet decomposition coefficients:

$$\Pi[V_{j+1}]f = \sum_k \lambda_{j+1,k}\phi_{j+1,k}$$
$$= \Pi[V_j]f + \Pi[W_j]f = \sum_k \lambda_{j,k}\phi_{j,k} + \sum_k \gamma_{j,k}\psi_{j,k}$$

- Using the structure equations, we derive:

Theorem (Fast Wavelet Transform)

The DWT $\gamma_{j,k}$ at resolution j can be retrieved from $j+1$ via the recursive relations

$$\lambda_{j,\ell} = \langle \Pi[V_j]f, \phi_{j,\ell}\rangle = \sum_k \sqrt{2}h_k\lambda_{j+1,k+2\ell} \quad \textit{and}$$
$$\gamma_{j,\ell} = \langle \Pi[W_j]f, \psi_{j,\ell}\rangle = \sum_k \sqrt{2}g_k\lambda_{j+1,k+2\ell}$$

Exercise. Verity the Proposition and FWT. Derive the Inverse FWT.

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- Let $\mathcal{M}^t$ be a compact oriented Riemannian manifold with Laplace-Beltrami operator $\Delta_{\mathcal{M}}$
- Suppose $K_s(\cdot,\cdot)$ is a smooth kernel in $C^\infty(\mathbb{R}^+ \times \mathcal{M} \times \mathcal{M})$ with associated $L^2(\mathcal{M})$ operator

$$T_s f(x) = \int_{\mathcal{M}} K_s(x,y) f(y) \, \mathrm{vol}_{\mathcal{M}}(y)$$

- K_s is a manifold wavelet if $T_s 1 = T_s^* 1 = 0$ and

$$\int_0^\infty \| T_s f \|^2 \frac{ds}{s} = c_K \, \| (I - \Pi_1) f \|^2 \quad \forall f \in L^2(\mathcal{M})$$

where Π_1 is the projection onto the constant functions and $c_K > 0$.

- Define the Schwartz space (algebra)

$$\mathcal{S}(\mathbb{R}^n) := \{g \in C^\infty(\mathbb{R}^n, \mathbb{C}) : \| |x|^\alpha D^\beta g \|_\infty < \infty \ \forall \alpha, \beta \in \mathbb{N}^n\}$$

- $\mathcal{S}(\mathbb{R}^n)$ consists of functions of rapid decay e.g. $|x|^\alpha e^{-c\|x\|^2}$
- $C_0^\infty \subset \mathcal{S} \overset{dense}{\subset} L^p \ \ (p < \infty)$

Lemma

Let T be a positive self-adjoint operator on Hilbert space H. For $g \in s \cdot \mathcal{S}(\mathbb{R}^+)$ and $c = \int_0^\infty |g(s)|^2 ds/s$,

$$\lim_{\epsilon \to 0^+,\, N \to \infty} \int_\epsilon^N |g(sT)|^2 \frac{ds}{s} = c(I_H - \Pi)$$

in the strong operator topology, where Π is projection onto $\ker T$.

Theorem

For $g \in s \cdot \mathcal{S}(\mathbb{R}^+)$, let K_s be the kernel (spectral wavelet) associated to $T_s = g(-s^2\Delta_{\mathcal{M}})$. Then K_s is a manifold wavelet.

Proof

- Denote the harmonics by (λ_k, u_k), $\lambda_0 = 0$, $u_0 = \mathrm{vol}(\mathcal{M})^{-1/2}$
- u_k are orthogonal w.r.t. $\langle f, g\rangle_{\mathcal{M}} = \int fg \,\mathrm{vol}_{\mathcal{M}}$
- Writing $K_{\sqrt{s}}(x, y) = \sum_{k,\ell} a_{k\ell} u_k(x) u_\ell(y)$,

$$g(-s\Delta_{\mathcal{M}})u_j = \int_{\mathcal{M}} K_{\sqrt{s}}(\cdot, y) u_j(y) \,\mathrm{vol}_{\mathcal{M}}(y) = \sum_{k,\ell} a_{k\ell} u_k \delta_{j\ell}$$

yields $a_{k\ell} = \delta_{k\ell} g(s\lambda_k)$

- Thus $K_{\sqrt{s}}(x,y) = \sum_k g(s\lambda_k) u_k(x) u_k(y)$
- $\int K_{\sqrt{s}}(x,y)\,\mathrm{vol}_{\mathcal{M}}(x) = \int K_{\sqrt{s}}(x,y)\,\mathrm{vol}_{\mathcal{M}}(y) = 0$ follows from $\langle u_k, 1\rangle = 0$ $(k \geq 1)$ and $g(0) = 0$
- The 2nd condition follows from

$$\int_0^\infty \|g(-s\Delta_{\mathcal{M}})f\|^2 \frac{ds}{s} = \left\langle \int_0^\infty |g|^2(-s\Delta_{\mathcal{M}}) f \frac{ds}{s}, f \right\rangle_{\mathcal{M}},$$

applying the Lemma and substituting s^2. $\blacksquare$

Remark

- In the language of distributions, $K_s(x,y) = T_s\delta_y(x) = g(-s^2\Delta_{\mathcal{M}})\delta_y(x)$
- Analogously on $\mathbb{R}$, the wavelet $\psi \stackrel{f}{=} g(-d^2/dx^2)\delta$ defined as the inverse CFT of $g(\omega^2)$ is admissible
- Spectral wavelets implement scaling in the spectral domain, same as $\widehat{T_s f}(\omega) = \hat{\psi}(s\omega)\hat{f}(\omega)$ on $\mathbb{R}$:

Proposition

Let the manifold FT of $f \in L^2(\mathbb{R})$ be $\hat{f}(k) = \langle f, u_k\rangle$. Then:

$$\widehat{T_s f}(k) = g(s^2\lambda_k)\hat{f}(k) \quad \forall s > 0,\ k = 0, 1, \cdots$$

Theorem (Inverse manifold WT)

Any $f \in (I - P_1)L^2(\mathcal{M})$ may be reconstructed from $T_s f$ via

$$\int_0^\infty T_s^* T_s f \frac{ds}{s} = c_K f$$

Proof

- Define $\mathcal{H} = (I - P_1)L^2(\mathcal{M})$ and $\mathcal{K} = L^2(\mathbb{R}^+, \mathcal{H}, dt/t)$
- Let the bounded operator $U : \mathcal{H} \to \mathcal{K}$ be $Uf := (T_s f)_{s>0}$
- Check $U^* : \mathcal{K} \to \mathcal{H}$ is $U^*(h_s)(x) = \int_0^\infty T_s^* h_s(x) dt/t$
- $\|Uf\|_{\mathcal{K}} = c_K \|f\|_{\mathcal{H}}^2$ and U is a scaled isometry
- Thus $\langle U^* Uf, h\rangle_{\mathcal{H}} = \langle Uf, Uh\rangle_{\mathcal{K}} = c_K \langle f, h\rangle_{\mathcal{H}}$ by polarization and $U^* U = c_K$. ■

Schwartz wavelets

g only needs moderate decay to yield the L^2 theory. However, imposing Schwartz-type conditions gives wavelets adapted to the study of other function spaces.

A manifold wavelet K_s on $\mathcal{M}^t$ is a Schwartz ($\mathcal{S}$-) wavelet if for any $X, Y \in \mathrm{PDO}(\mathcal{M}, \mathbb{R})$ with degree j, k resp. and $N \in \mathbb{N}_{\geq 0}$ there exists $C_{N,X,Y}$ such that:

$$s^{t+j+k} \left| \left(\frac{d^{\mathcal{M}}(x,y)}{s} \right)^N XY \left[K_s(x,y) \right] \right| \leq C_{N,X,Y}$$

for all $s > 0$, $x, y \in \mathcal{M}$.

Theorem

1. *The classical wavelet* $K_s(x,y) = \frac{1}{s^n}\psi(\frac{x-y}{s})$ *associated to* $g(-s^2\Delta)$ *on* $\mathbb{R}^n$ *where* $\hat{\psi}(\xi) = g(\|\xi\|^2)$ *is an* $\mathcal{S}$*-wavelet.*
2. *Spectral wavelets on compact manifolds are* $\mathcal{S}$*-wavelets.*

The following facilitates wavelet analysis of Hölder spaces.

Proposition

Let K_s *be a* $\mathcal{S}$*-wavelet on* $\mathcal{M}$ *and let* $f \in L^2(\mathcal{M})$. *Then* f *is* α*-Hölder continuous iff:*

$$\sup_{\mathcal{M}} |T_s f| \leq C s^\alpha \quad \forall s > 0$$

Exercise. Prove (1) of the Theorem.

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Diffusion Operators

- Let $(\mathcal{M}, d^{\mathcal{M}})$ be a complete Riemannian manifold
- Let L be a nonnegative essentially self-adjoint C^∞ (sub)elliptic order 2 PDO on $\mathcal{M}$ with spectral decomposition $\int \lambda dE_\lambda$
- Its diffusion semigroup $(P_s)_{s\geq 0}$ on $L^2(\mathcal{M})$ is:

$$P_s := e^{-sL} = \int_{-\infty}^{\infty} e^{-s\lambda} dE_\lambda$$

- Diffusion equation: $\frac{d}{ds} \circ P_s = LP_s$ (in the strong sense)
- Assume P_s is compact for $s > 0$, so $P = P_1$ has eigenfunctions $(\lambda_k, \xi_k)_{k\geq 0}$

Exercise. Show $\sigma(P) \subset [0,1]$ and $\sigma(P_s) = \{\lambda_k^s\}_{k\geq 0}$.

Lemma

- *(semigroup)* $P_0 = I$ *and* $P_s P_t = P_{s+t}$
- *(contraction)* $\|P_s\|_{L^2(\mathcal{M})} \leq 1$
- *(strong continuity) For* $f \in L^2(\mathcal{M})$, $s \mapsto P_s f$ *is continuous*
- *(self-adjointness)* $\int_{\mathcal{M}} (P_s f) g \operatorname{vol}_{\mathcal{M}} = \int_{\mathcal{M}} f (P_s g) \operatorname{vol}_{\mathcal{M}}$

Theorem

The diffusion semigroup $(P_s)_{s\geq 0}$ *has an associated symmetric heat kernel* $(H_s)_{s\geq 0}$ *which satisfies the Chapman-Kolmogorov equation:*

$$H_{t+s}(x,y) = \int_{\mathcal{M}} H_t(x,w) H_s(w,y) \operatorname{vol}_{\mathcal{M}}(w)$$

Exercise. Show the Lemma. Verify that $K_s(x,y) = \sum_k \lambda_k^s \xi_k(x) \xi_k(y)$.

Examples

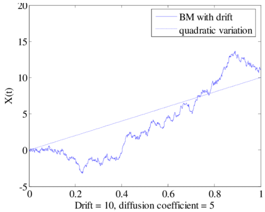

Figure: Brownian motion with positive drift and diffusion coefficient.

- Heat diffusion governed by the Laplacian
- Random walks induced by symmetric Markov chains
- p.d.f. of Brownian motion B_t, solutions of the SDE $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t$
- Time evolution of the quantum wavefunction via the Schrödinger equation

Eigenmap

- The diffusion metric in Lec. 3 has the more general form

$$D_s(x,y)^2 = \sum_k \lambda_k^s(\xi_k(x)-\xi_k(y))^2 = \left\| P_{s/2}\delta_x - P_{s/2}\delta_y \right\|_{L^2(\mathcal{M})}^2$$

  i.e. it measures the L^2-embedded distance between diffused point sources.

- By defining a high-pass filter $H^\epsilon = \{k : \lambda_k \geq \epsilon\}$, the Eigenmap algorithm produces the approximate isometry

$$\Gamma_s^\epsilon : (\mathcal{M}, D_s) \to (\mathbb{R}^{|H^\epsilon|}, \|\cdot\|_{euc})$$
$$x \mapsto (\lambda_k^{s/2}\xi_k(x) : k \in H^\epsilon)$$

Diffusion MRA

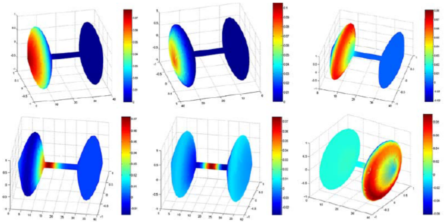

Figure: Localized wavelets on the dumbell manifold.

The classical MRA structure of wavelet pairs does not exist on manifolds. Coifman, Maggioni (2006) construct an MRA of $L^2(\mathcal{M})$ based on the time scaling inherent in diffusion processes.

Idea

- In MRA, $V_j = \langle \phi_{j,k} \rangle_{k \in \mathbb{Z}}$ was constructed from the base frame $V_0 = \langle \phi(t-k) \rangle_{k \in \mathbb{Z}}$ by applying the scaling semigroup R_s : $\nu(x) \mapsto s^{-1/2} \cdot \nu(s^{-1}x)$ at discrete $s_j = 2^{-j}$
- Similarly, we start with a frame Φ of local bump functions on 'dyadic cubes' at the finest level V_0
- The diffused family $P_{s_j}\Phi$ should form an approximate basis for a coarser (downsampled) space V_j
- Construct V_j by discarding eigenfunctions $\left\| P_{s_j} \xi_k \right\| = \lambda_k^{s_j} < \epsilon$ and taking only harmonics with high frequency content

Components

- precision $0 < \epsilon < 1$, discretized scales $(s_j)_{j \geq 0} \nearrow \infty$
- bump function family Φ constructed on dyadic cubes
- compact diffusion semigroup $(P_s)_{s \geq 0}$, eigenstates $(\lambda_k, \xi_k)_{k \geq 0}$
- high-pass band filters $H_j^\epsilon = \{k : \lambda_k^{s_j} \geq \epsilon\}$

Algorithm

- DMRA: $L^2(\mathcal{M}) = V_{-1} \supset V_0^\epsilon \supset V_1^\epsilon \supset \cdots$
- approximation spaces $V_j^\epsilon = \langle \xi_k : k \in H_j^\epsilon \rangle$
- detail spaces $V_j^\epsilon = V_{j+1}^\epsilon \bigoplus^\perp W_j^\epsilon$
- V_j is a compressed ϵ-approximation of $P_{s_j}\Phi$ or $\operatorname{im} P_{s_j}$
- apply ϵ-variants of Gram-Schmidt Orthogonalization to $P_{s_j}\Phi$ and $(P_{s_{j+1}} - P_{s_j})\Phi$ to obtain localized wavelets for V_j^ϵ, W_j^ϵ