Lecture 5. Spectral Geometry

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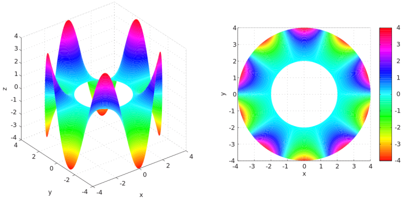

Figure: Harmonics on the annulus.

In Lec. 5 and 6, we take a deeper theoretical look at differential operators and their diffusion processes on manifolds. Through spectral analysis, we obtain nonlinear generalizations of Fourier and wavelet theory.

While direct applications are in signal processing, studying inverse problems also leads to geometric learning algorithms as in Lec. 3.

Table of Contents

In this Section and the next, we outline the spectral theory for general linear operators. A background in basic functional analysis is assumed.

Recall the spectral theorem for compact self-adjoint operators:

Theorem (Riesz-Schauder)

For any compact self-adjoint operator A on a real or complex Hilbert space H, there exists an orthonormal basis of H consisting of eigenvectors of A.
In particular, the complement of the kernel $N(A)$ admits a finite or countably infinite basis v_n of eigenvectors, with corresponding eigenvalues $\lambda_n \to 0$ for the latter case.

Issue 1. Noncompact operators may not have any eigenvectors. Eigenspaces need to be replaced by near-invariant subspaces, the decomposition into which is expressed as an integral in the limit.

Example. The operator $Af(x) = xf(x)$ on $L^2(0,1)$ has no eigenvalues, but $Af \simeq \lambda f$ on the invariant subspace of functions supported on $[\lambda, \lambda + \epsilon]$.

Issue 2. Unbounded operators are usually defined only on a dense subset of H. Domain issues need to be carefully examined when defining (self-)adjointness.

Example. The Laplacian on open $U \subset \mathbb{R}^n$ is defined only on the Sobolev space $W_2^2(U) \subsetneq L^2(U)$. Note $C_0^\infty(U)$ is dense in $L^2(U)$.

Defn.

- Let A be a (possibly unbounded) linear operator on the Hilbert space H with domain $D(A)$ dense in H.
- A is *closed* if its graph $\Gamma(A)$ is a closed subspace of $H \times H$.
- A' is an *extension* of A if $D(A) \subset D(A')$ and $A'|_{D(A)} = A$.
- A is *closable* if A has a closed extension. Its *closure* $\overline{A}$ is the smallest such extension.
- Consider the subspace

$$D^* := \{v \in H : \exists h \in H \text{ s.t. } (Ax, v) = (x, h)\ \forall x \in D(A)\}$$

The adjoint of A is the unique operator A^* with domain D^* and mapping $A^* v = h$.

Exercise. Show that $Af(x) = x^{-\alpha} f(x)$, $\alpha > 1/2$ defined on the subspace of $L^2[0, 1]$ of functions identically zero on some $[0, \epsilon]$ is unbounded and not closed.

Prop.

- A^* is closed for any A;
- A is closable iff $\overline{D(A^*)} = H$;
- If A is closable, then $A^{**} = \overline{A}$ and $(\overline{A})^* = A^*$.

Exercise. Show that the operator $Af(x) = x^{-\alpha} f(x)$ defined in the previous Exercise is closable.

Exercise. Prove the Proposition. Hint: let $V : (u, v) \mapsto (v, -u)$ in $H \times H$. Show that $\Gamma(A^*) = \overline{V\Gamma(A)}^{\perp}$ and $\overline{\Gamma(A)} = \Gamma(A^{**})$.

Exercise. Let $H = L^2(\mathbb{R})$. Define $Af(x) = (f, c)f_0(x)$ on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ for some $0 \neq c \in \mathbb{R}$ and $0 \neq f_0 \in H$. Show that $\overline{D(A)} = H$ and $f_0 \perp D(A^*)$, thus A is not closable.

Defn.

- A is *symmetric* if $A \subset A^*$
- A is *self-adjoint* if $A = A^*$
- A is *essentially self-adjoint* if $\overline{A}$ is self-adjoint.

Prop.

- If A is symmetric, it is closable and $\overline{A}$ is symmetric.
- If $D(A) = H$ then A is self-adjoint and bounded.
- If $R(A) = H$ then A is self-adjoint and A^{-1} exists and is bounded.

Exercise. Prove the Proposition. Hint: if $D(A) = H$ and $(Ax, y) = (x, Ay)$ $\forall x, y \in H$, then A is bounded.

Exercise. Let $H = L^2[0,1]$ and $A = i \cdot d/dx$ on the boundary conditioned space $D_\gamma(A) = \{f \in H : f' \in H \text{ and } f(0) = f(1)e^{i\gamma}\}$, $\gamma \in \mathbb{R}$. Show that A is self-adjoint on $D_\gamma(A)$.

Table of Contents

Defn. A family of projections $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ is a *spectral family* if:

- $(E_\lambda x, x) \leq (E_\mu x, x)$ if $\lambda < \mu$
- E_λ is right continuous w.r.t. the strong operator topology
- $\lim_{\lambda\to-\infty} E_\lambda = 0$ and $\lim_{\lambda\to+\infty} E_\lambda = I_H$

Prop.
For every $x, y \in H$, the function $\lambda \mapsto (E_\lambda x, y)$ is of bounded variation: for any $\lambda_0 < \cdots < \lambda_n$,

$$\sum_{j=1}^{n} \left|(E_{\lambda_j} x, y) - (E_{\lambda_{j-1}} x, y)\right| \leq \|x\| \, \|y\|$$

Thus, for continuous f the Stieltjes integral $\int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, y)$ is well-defined.

Exercise. Prove the Proposition.

Prop.
Given a spectral family E_λ and real-valued continuous function f, define the operator A by

$$(Ax, y) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, y) \quad \forall x \in D(A),\ y \in H$$

on the domain

$$D(A) = \left\{ x \in H : \|Ax\|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 d(E_\lambda x, x) < \infty \right\}$$

Then $\overline{D(A)} = H$, $D(A) = D(A^*)$, A is self-adjoint, and $A(E_\mu - E_\lambda)$ extends $(E_\mu - E_\lambda)A$.

Exercise. Prove the Proposition. Hint: for any $x \in H$, consider $(E_\mu - E_\lambda)x$ for $\mu \gg 0$ and $\lambda \ll 0$.

Theorem (von Neumann)

Every self-adjoint operator A on Hilbert space H has a unique spectral representation, i.e. a spectral family $\{E_\lambda\}_{\lambda=-\infty}^{\infty}$ so that:

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

Functional Calculus. For a continuous complex function f, define $f(A) := \int f(\lambda) dE_\lambda$ on $D_f = \{x : \int |f(\lambda)|^2 d(E_\lambda x, x) < \infty\}$.

- The Cayley transform provides a 1-1 correspondence between self-adjoint operators and unitary transformations:

$$A \mapsto U_A := \int \frac{\lambda - i}{\lambda + i} dE_\lambda \,, \quad D(U_A) = H$$

- The resolvent $R_z := (A - zI_H)^{-1}$ is equal to $\int (\lambda - z)^{-1} dE_\lambda$

Defn.

- The *resolvent set* $\rho(A) = \{z \in \mathbb{C} : (A - zI_H)^{-1} : H \to D(A) \text{ exists and is bounded}\}$
- The *spectrum* $\sigma(A) = \mathbb{C} \setminus \rho(A)$
- The *point spectrum* $\sigma_p(A) = \{\lambda \in \sigma(A) : \exists u,\ Au = \lambda u\}$
- The *discrete spectrum* $\sigma_d(A) = \{\lambda \in \sigma_p(A) : \dim N(A - \lambda I_H) < \infty \text{ and } \lambda \text{ is isolated in } \sigma(A)\}$

Prop. Suppose $A = A^*$ with spectral family $\{E_\lambda\}$. Then:

- $\sigma(A)$ is closed and real
- $\mu \in \sigma(A)$ iff $E_{\mu+\epsilon} \neq E_{\mu-\epsilon}$ for $\forall \epsilon > 0$
- $\mu \in \sigma_p(A)$ iff $E_\mu \neq E_{\mu-0}$ for $\forall \epsilon > 0$

Exercise. Prove the Proposition.

Theorem (Friedrichs extension)

Let A be a symmetric nonnegative operator on Hilbert space H. Then A has a self-adjoint extension A_F minimal in the sense that its corresponding quadratic form $Q_F(x, y) = (A_F x, y)$ has smallest domain.

The extension is obtained by considering the quadratic form $Q_A(x, y) := (Ax, y)$ on $D(A)$ and forming its metric completion H_Q with respect to the inner product $(x, y)_Q := Q(x, y) + (x, y)$. The extended form Q_F is closed and can be used to find the associated self-adjoint operator.

Table of Contents

Defn.

- Use multi-index notation: for $\alpha = (\alpha_1, \cdots, \alpha_n)$,

$$\alpha! = \alpha_1! \cdots \alpha_n!, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \quad \text{where } D_k = \frac{1}{i}\frac{\partial}{\partial x_k}$$

- We start with a region $\Omega \subset \mathbb{R}^n$.
- A partial differential operator (PDO) A of order m is an operator on $L^2(\Omega)$ with domain $C_0^\infty(\Omega)$ of the form:

$$Au = \sum_{|\alpha| \le m} a_\alpha D^\alpha u, \quad a_\alpha \in C^\infty(\Omega)$$

- The *symbol* of A is: $a(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$
- A is *elliptic* if a is invertible for all $x \in \Omega$ and $\xi \in \mathbb{R}^n \setminus \{0\}$

Assumptions

1. a_α is real for $|\alpha| = m$, so m is even and $a \geq 0$
2. A is *formally* self-adjoint: $A \stackrel{f}{=} A^* := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \overline{a_\alpha}$
3. A has a *divergence form*: for $\phi_{\alpha\beta}$ real, symmetric (and uniformly bounded for $|\alpha|, |\beta| < m/2$),
$$A = \sum_{|\alpha|=|\beta| \leq m/2} (-1)^{|\alpha|} D^\alpha (\phi_{\alpha\beta} D^\beta)$$
4. A is *uniformly elliptic*: for some $\nu > 0$,
$$(-1)^{m/2} \sum_{|\alpha|=|\beta|=m/2} \phi_{\alpha\beta} \xi^{\alpha+\beta} \geq \nu \sum_{|\alpha|=m/2} \xi^{2\alpha}$$

Exercise. Check the assumptions for the Laplacian. Show formal self-adjointness directly using the divergence theorem.

Sobolev Spaces

- Let the L^p-based (here $p < \infty$) Sobolev space of order k:

$$W_p^k(\Omega) := \left\{ f \in L^p(\Omega) : \|f\|_{k,p}^p := \int_\Omega \sum_{|\alpha| \leq k} |D^\alpha f|^p dx < \infty \right\}$$

  where derivatives are taken in the weak sense.

- For $p = 2$, this becomes a Hilbert space with inner product

$$(f, g)_{k,2} = \sum\nolimits_{|\alpha| \leq k} (D^\alpha f, D^\alpha g)$$

- $W_2^k(\Omega)$ also has a natural representation via fast-decaying Fourier transforms: the Sobolev norm is equivalent to

$$\|f\|_{FT,k}^2 := \int_\Omega (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 d\xi$$

- Denote the closure of $C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{k,2}$ by $\mathring{W}_2^k(\Omega)$
- $\mathring{W}_2^k(\Omega) \subseteq W_2^k(\Omega)$ with equality iff $\Omega = \mathbb{R}^n$ a.e.
- The idea is that $\mathring{W}_2^k(\Omega)$ consists of functions with derivatives vanishing up to order $k-1$ on the boundary $\partial\Omega$ which we assume Lipschitz.
- This is formalized by the *trace operator*: for $k=1$,

$$\mathcal{T}_p^1 : W_p^1(\Omega) \to L^p(\partial\Omega), \quad \mathring{W}_p^1(\Omega) = \ker \mathcal{T}_p^1$$

  an extension of 'restriction to $\partial\Omega$' to Sobolev spaces
- Defining $\mathcal{T}_p^k$, $k > 1$ involves fractional Sobolev spaces on $\partial\Omega$.

Exercise. Prove $\|f\|_{k,p} \sim \|f\|_p + \sum_{|\alpha|=k} \|D^\alpha f\|_p$ and $\|\cdot\|_{k,2} \sim \|\cdot\|_{FT,k}$

Exercise. Show that $\mathring{W}_2^1(0,1)$ consists of functions in $W_2^1(0,1)$ satisfying Dirichlet boundary conditions $\lim_{x\to 0+} f(x) = \lim_{x\to 1-} f(x) = 0$.

Theorem (Spectral theorem for elliptic differential operators)

Let A be an elliptic PDO on $C_0^\infty(\Omega)$ satisfying 1 $\sim$ 4. Then there exists a self-adjoint extension A_F with domain

$$D(A_F) = \mathring{W}_2^{m/2}(\Omega) \cap W_2^2(\Omega)$$

The spectrum is discrete, $\sigma(A_F) = \sigma_d(A_F) = \{\lambda_j\}_{j=1}^\infty$, with one accumulation point $+\infty$. The corresponding eigenfunctions $\{\psi_j\}_{j=1}^\infty$ form an orthonormal basis of $L^2(\Omega)$ and

$$A_F f \overset{L^2}{=} \sum_{j=1}^\infty \lambda_j (f, \psi_j)\psi_j \quad \forall f \in D(A_F)$$

The idea is to find some $A_\mu := A + \mu I_H \geq 0$ and form its Friedrichs extension. Then $R((A_\mu)_F^{-1}) \subset \mathring{W}_2^{m/2}(\Omega)$ is compactly embedded in $L^2(\Omega)$ and we apply Riesz-Schauder.

Boundary Conditions

In practice, the extension issue manifests in the form of imposing suitable boundary conditions due to the $\mathring{W}_2^{m/2}(\Omega)$ term.

1. Let p be the 1D momentum operator $-i\hbar\frac{d}{dx}$ on $L^2[0,1]$.
   - On the domain $C^\infty[0,1]$, $\sigma(p) = \mathbb{C}$
   - Imposing $f(0) = f(1) = 0$, p has no eigenvalues
   - Imposing $f(0) = f(1)e^{i\gamma}$, p becomes self-adjoint
2. The Laplacian on $L^2(\mathbb{R}^n)$ is *essentially self-adjoint*, i.e. $\bar{\Delta} = \Delta_F$ is the unique self-adjoint extension of Δ.
3. The Laplacian on $L^2[a,b]$ becomes essentially self-adjoint with the conditions:

$$\cos\theta f(a) + \sin\theta f'(a) = \cos\tau f(b) + \sin\tau f'(b) = 0$$

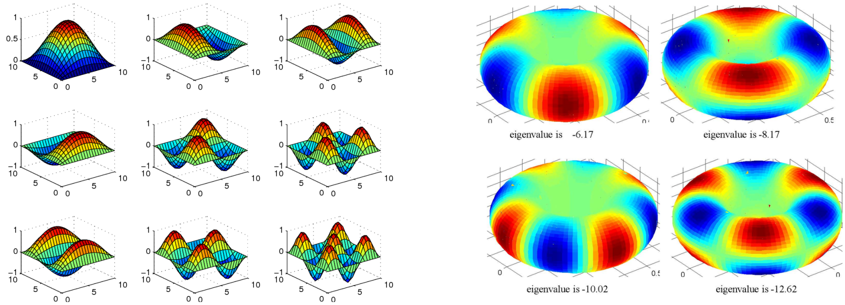

Figure: Eigenanalysis on the rectangle with suitable boundary conditions is equivalent to the torus.

Table of Contents

Definitions

- Let $(\mathcal{M}, g)$ be a Riemannian manifold (w/o boundary) and E, F Hermitian vector bundles over $\mathcal{M}$.
- $\mathrm{Op}(E, F)$ is the space of linear operators between smooth sections $A : C^\infty(E) \to C^\infty(F)$
- A linear map $A \in \mathrm{Op}(E, F)$ is a PDO of order $\leq m$ if:

  (1) A is *local*, i.e. $\operatorname{supp} Af \subset \operatorname{supp} f$
  (2) A is locally a PDO of order $\leq m$ w.r.t. trivializations

- The *adjoint representation* of $f \in C^\infty(\mathcal{M})$ is:

$$\mathrm{ad}(f) : \mathrm{Op}(E, F) \to \mathrm{Op}(E, F), \quad T \mapsto [T, f]$$

  where f is also understood as $C^\infty(\mathcal{M})$-module multiplication.

- Alternatively: setting $\mathrm{PDO}^{-1}(E,F) = \{0\}$,

$$\mathrm{PDO}^{\leq m} := \{T \in \mathrm{Op} : \mathrm{ad}(f)T \in \mathrm{PDO}^{\leq m-1} \ \ \forall f\}$$

$$\mathrm{PDO}^{m} := \mathrm{PDO}^{\leq m} \setminus \mathrm{PDO}^{\leq m-1}, \quad \mathrm{PDO} := \cup_m \mathrm{PDO}^m$$

- Let $A \in \mathrm{PDO}^m(E,F)$. Its formal adjoint is the unique operator $A^* \in \mathrm{PDO}^m(F,E)$ satisfying, for all $e \in C_0^\infty(E)$ and $f \in C_0^\infty(F)$,

$$\int_{\mathcal{M}} (Ae, f)_F \, d\,\mathrm{vol}_{\mathcal{M}} = \int_{\mathcal{M}} (e, A^* f)_G \, d\,\mathrm{vol}_{\mathcal{M}}$$

Exercise. Prove that the two definitions provided are equivalent, in particular $\mathrm{PDO}^{\leq 0}(E,F) \cong \mathrm{Hom}(E,F)$. Hint: check that $\mathrm{ad}(\cdot)A$ is a derivation.

- For $A \in \mathrm{PDO}^{\leq m}$, define:

$$\sigma(A)(f_1, \cdots, f_m) = \frac{1}{m!}\,\mathrm{ad}(f_1)\cdots\mathrm{ad}(f_m)A \in \mathrm{PDO}^{\leq 0}(E, F)$$

- $\sigma(A)(f_1, \cdots, f_m)(p)$ depends only on $df_i(p)$ for $p \in \mathcal{M}$
- Fixing p and $\xi \in T_p^*\mathcal{M}$, we define the *symbol* $\sigma(A)$ as:

$$(p, \xi) \in T^*\mathcal{M} \;\mapsto\; \sigma(A)(\xi, \cdots, \xi)(p) \in \mathrm{Hom}(E_p, F_p)$$

- In other words, $\sigma(A)$ is a smooth section of the bundle $\mathrm{Hom}(\pi^*E, \pi^*F) \to T^*\mathcal{M}$, where $\pi : T^*\mathcal{M} \overset{proj}{\to} \mathcal{M}$
- A is *elliptic* if $\sigma(A)(p, \xi)$ is a fiber isomorphism for $\forall p,\ \xi \neq 0$

Exercise. Show that $\sigma(A)$ is equal to the original definition where E, F is the trivial line bundle over $U \subset \mathbb{R}^n$.

Exercise. Prove $\sigma(A^*) = (-1)^m \sigma(A)$, i.e. A elliptic $\Rightarrow$ A^* elliptic.

Examples

- The Laplace-Beltrami operator Δ_g is a 2nd order PDO on $\mathcal{M}$. It is formally self-adjoint and elliptic: $\sigma(-\Delta_g)(p,\xi) = \|\xi\|_g^2$
- The exterior derivative $d^k : \Omega^k(\mathcal{M}) \to \Omega^{k+1}(\mathcal{M})$ is a 1st order PDO with principal symbol $\sigma(d)(p,\xi) = \wedge\xi$
- The Laplace-de Rham operator $\Delta_{dR} = (d+d^*)^2$ is also formally self-adjoint elliptic.
- Forms $\Delta_{dR}\,\omega = 0$ are called harmonic. By Hodge theory,

$$\text{harmonic } k\text{-forms } \ker\left(\Delta_{dR} : \Omega^k(\mathcal{M}) \to \Omega^k(\mathcal{M})\right)$$

$$\overset{1-1}{\leftrightarrow} \ \text{cohomology classes } H^k_{dR}(\mathcal{M}) = \ker d^k \,/\, \operatorname{im} d^{k-1}$$

Exercise. Using the Hodge star $u \wedge *v \equiv (u,v)\,\mathrm{vol}$, show $d^* = \pm * d *$ and $\int(\Delta f, g)\,\mathrm{vol} = \int(df, dg)\,\mathrm{vol}$. Thus Δ_{dR} is self-adjoint and positive.

- Let $\nabla : C^\infty(E) \to C^\infty(T^*\mathcal{M} \otimes E)$ be a connection on E:

$$\nabla(fu) = df \otimes u + f\nabla u \quad \forall f \in C^\infty(M),\ u \in C^\infty(E)$$

  Then $\sigma(d)(p,\xi) = \xi\otimes$.
- The associated covariant Laplacian $\Delta_\nabla := \nabla^*\nabla$ on $C^\infty(E)$ is elliptic with $\sigma(-\Delta_\nabla)(p,\xi) = \|\xi\|_g^2$

Theorem (Weitzenböck identity)

Let ∇ be the Levi-Civita connection on a compact manifold $\mathcal{M}$. Regarding the Ricci tensor as an element of $\mathrm{End}(T^*\mathcal{M})$ *via duality, we have on* $\Omega^1(\mathcal{M})$,

$$\Delta_{dR} = \Delta_\nabla + \mathrm{Ric}$$

Remark

Positive Ricci curvature is a strong global constraint.

Corollary

If $\mathcal{M}$ is compact with positive Ricci curvature, $H^1_{dR}(\mathcal{M}) = 0$.

- $\mathcal{M}$ also has finite fundamental group
- If $\mathcal{M}$ is complete with Ricci curvature bounded below by $(n-1)K > 0$, then $diam(\mathcal{M}) \leq \pi/\sqrt{K}$ thus $\mathcal{M}$ is compact.

Theorem (Hamilton)

If $\mathcal{M}$ is a closed 3-manifold of positive Ricci curvature, the normalized Ricci flow converges, thus $\mathcal{M}$ admits spherical geometry. If $\pi_1(\mathcal{M}) = 0$, $\mathcal{M}$ is diffeomorphic to S^3.

Exercise. Prove the Corollary.

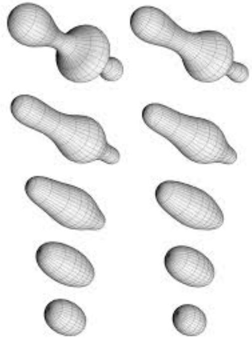

Figure: Ricci flow on surfaces smooth out curved areas. For 3-manifolds, the flow is complicated by the formation of singularities.

Table of Contents

Sobolev Spaces

- Let g be a Riemannian metric on $\mathcal{M}$. Let the vector bundle E on $\mathcal{M}$ have a metric h and a compatible connection ∇, so that

$$\nabla^m : C^\infty(E) \to C^\infty((T^*\mathcal{M})^{\otimes m} \otimes E)$$

and all related spaces have induced norms.

- For any $k \geq 0$, $1 \leq p < \infty$, and $\forall u \in C_0^\infty(E)$,

$$\|u\|_{k,p} := \Big(\sum_{m \leq k} \int_{\mathcal{M}} \|\nabla^m u\|^p \, d\,\mathrm{vol}_g \Big)^{1/p}$$

- Define $W_p^k(\mathcal{M})$ as the completion of $C_0^\infty(E)$ w.r.t. $\|\cdot\|_{k,p}$

Exercise. Show that $\|\cdot\|_{k,p}$ is well-defined up to equivalence, so that $W_p^k(\mathcal{M})$ is independent of the data (g, h, ∇).

Remark

- $W^k_p(\mathcal{M})$ may also be defined normally as the $\|\cdot\|_{k,p}$-closure of the space of smooth sections $u \in C^\infty(E)$ with sufficiently integrable derivatives of order $\leq m$
- $\cdots$ or as distributions with weak derivatives in $L^2(E)$.
- In general, the above spaces are different and have bad analytic properties. *Unless*:
- $\mathcal{M}$ has bounded geometry if $\mathcal{M}$ has injectivity radius bounded below and $(\nabla^m R)_{m\geq 0}$ is uniformly bounded in tensor norm.
- In particular, compact $\Rightarrow$ bounded geometry.

These conditions rule out pathologies at both small & large scales, and appear naturally in noncompact geometric analysis.

Hodge Theory

- A linear map $F : V \to W$ is Fredholm if: $\operatorname{im} F$ is closed and $\operatorname{ind} F := \dim\ker F - \dim\operatorname{coker} F$ is well-defined.
- Fix $A \in PDO^m(E, F)$. A determines a bounded operator

$$A_k : W_p^{m+k}(E) \to W_p^k(F)$$

- Then: A_k is Fredholm, $\ker A_k = \ker A$, $\operatorname{coker} A_k \cong \ker A^*$
- The *analytical index* $\operatorname{ind} A := \operatorname{ind} A_m$ can be calculated in terms of *topological* invariants (Atiyah-Singer index theorem)

Theorem (Abstract Hodge decomposition)

Let $A : C^\infty(E) \to C^\infty(E)$ be a formally self-adjoint and elliptic PDO of order m. Regarding $A : W_2^m(E) \to L^2(E)$,

$$L^2(E) = \ker A \oplus \operatorname{im} A$$

Corollary (on real manifolds)

Any differential form ω has a unique representation $d\alpha + d^*\beta + \gamma$, where γ is harmonic. In particular, any de Rham cohomology class has a unique harmonic representative.

Corollary (on complex projective varieties)

The Dolbeault cohomology $\mathcal{H}^{p,q}(\mathcal{M}) = \ker \Delta_{\bar\partial}|\Omega^{p,q}(\mathcal{M})$. If $\mathcal{M}$ is Kähler, $\Delta_{dR} = 2\Delta_{\bar\partial}$ and $H^m_{dR}(\mathcal{M}) \otimes \mathbb{C} \cong \oplus_{p+q=m}\mathcal{H}^{p,q}(\mathcal{M})$.

Define the Hodge numbers $h^{p,q} := \dim \mathcal{H}^{p,q}$. Then $h^{p,q} = h^{q,p}$ (complex conjugate) and $h^{p,q} = h^{n-p,n-q}$ (Serre duality).

Exercise. Prove the Corollaries. Draw the *Hodge diamond* $(h^{p,q})$ for genus g projective curves.

Exercise. Prove that the odd Betti numbers of Kähler manifolds are even. Show that the Hopf surface is not Kähler.

Theorem (Spectral theorem for elliptic operators on manifolds)

Let A be a formally self-adjoint elliptic PDO on a Hermitian bundle E on a compact manifold $\mathcal{M}$. Regard $A : W_2^m(E) \to L^2(E)$. Then $\sigma(A) = \{\lambda_j\}_{j=1}^{\infty}$ is discrete with one accumulation point $+\infty$. The corresponding eigenfunctions $\{\psi_j\}_{j=1}^{\infty}$ form an orthonormal basis,

$$L^2(E) = \bigoplus_j \ker(\lambda_j - A) \quad \textit{and}$$

$$Af \overset{L^2}{=} \sum_{j=1}^{\infty} \lambda_j (f, \psi_j)\psi_j \quad \forall f \in W_2^m(E)$$

The theorem also holds when $\mathcal{M}$ has boundary if Dirichlet conditions $f|_{\partial\mathcal{M}} = 0$ are imposed.

Exercise. Show that if $\mathcal{M}$ is compact with boundary, $\sigma(-\Delta) > 0$.

More on Δ

- Spectral geometry is the study of the relationship between geometry of manifolds and the spectra of PDOs, especially the Laplacian. For example,

Theorem (Lichnerowicz-Obata)

If $\mathcal{M}$ is closed with Ricci curvature bounded below by $(n-1)K$, then $\lambda_1 \geq nK$ with equality iff $\mathcal{M}$ is the n-sphere of radius $K^{-1/2}$.

- Δ on $\mathcal{M} = (S^1)^n$ recovers standard Fourier theory.
- The Schrödinger operator $-\Delta + V$ (potential) is also extensively studied in mathematical physics.

For geodesically complete *noncompact* $\mathcal{M}$ (e.g. bounded geometry), we only have essential self-adjointness. In particular:

Theorem

The continuous spectrum of $-\Delta$ on $\mathbb{R}^n$ is all of $\mathbb{R}_{\geq 0}$.

- The Fourier transform $\mathcal{F}$ on $L^2(\mathbb{R}^n)$ is unitary by Parseval
- $\mathcal{F}^{-1}\Delta\mathcal{F}$ is the multiplication operator $M_{\|\cdot\|^2}$
- Thus $\sigma(-\Delta) = \sigma(M_{\|\cdot\|^2}) = \text{ess ran}\,\|\cdot\|^2 = \mathbb{R}_{\geq 0}$

More generally, $\sigma(-\Delta) = \mathbb{R}_{\geq 0}$ for noncompact manifolds with certain symmetries or Ricci curvature bounds. Other results construct metrics with large gaps in $\sigma(-\Delta)$.

Exercise. Fill in the details of the above proof.

Appendix

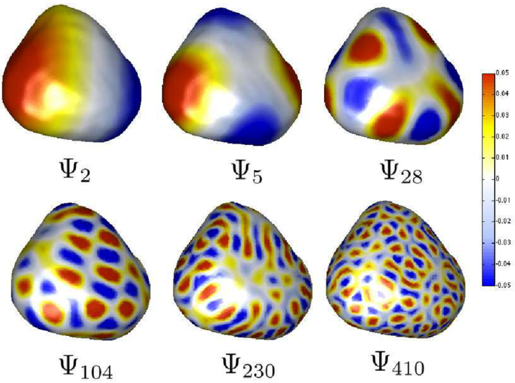

Figure: Eigenanalysis of brain amygdala structure.

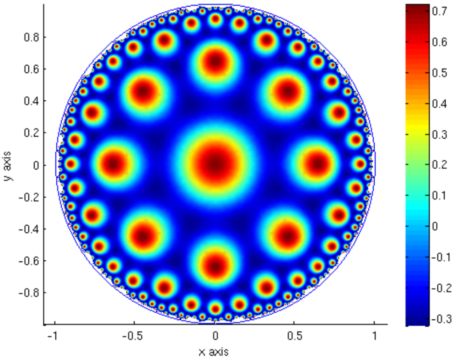

Figure: An eigenfunction on the Poincaré hyperbolic disk.

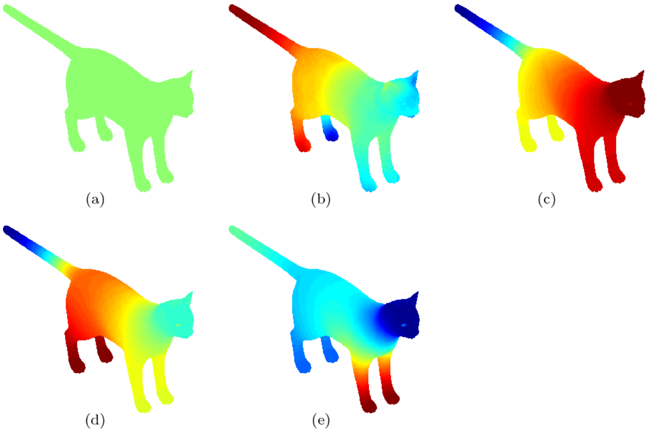

Figure: Eigenanalysis of Schrödinger's cat.