Assignment I

CT100/G/22166/24 June Kimani Probability & Statistics II

1. i) Discrete random variable - a variable that takes a countable number of distinct values like 0, 1, 2, it has a probability mass function $P(X=x)$ for each possible x

ii) Continuous random variable - it takes values in an interval meaning the values are not countable. It has a probability density function or are given by the intergrals $P(a<X<b)$

ii) Expectation and moments.
Expectation which is also mean of a random variable X is the average value you would expect if you repeat an experiment many times.
Moments are numbers that describe the shape of a probability distribution, how it leans (skewness), and how tall or flat it is (kurtosis).

2.
$$f(x) \geq 0 \text{ for all } x$$
$$P(a \leq x \leq b) = \int_a^b f(x)\,dx = 1$$
$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad / \quad \Sigma P(x=x) = 1$$

3.

x	0	1	2	3	4	5
$P=(X=x)$	2c	0.1	c	0.2	4c	0.1

i): Find the value of c

$$\Sigma_x P(x=x) = 1$$
$$= 2c + 0.1 + c + 0.2 + 4c + 0.1 = 1$$
$$= 2c + c + 4c + 0.1 + 0.2 + 0.1 = 1$$
$$= 7c + 0.4 = 1$$
$$7c = 1 - 0.4$$
$$\frac{7c}{7} = \frac{0.6}{7}$$
$$c = 0.0857$$

ii) Compute $P(0 \geq x < 5)$.

$x = 0, 1, 2, 3, 4, 5$

$P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= 7c$

$c = 0.0857$

$= 7(0.0857) + 0.3 = 0.6 + 0.3$

$= 0.9$

$P(0 \geq x < 5) = 0.9$

4. Continuous random variable x has pdf given

$$f(x) = \begin{cases} k(1+x), & 4 \leq x \leq 7 \\ 0 & \text{otherwise} \end{cases}$$

Find constant k and hence compute $p(x > 5)$

$\int_a^b f(x)\,dx = \int_4^7 f(x)\,dx = 1$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_4^7 k(1+x)\,dx = 1$

$\int (1+x)\,dx = x + \frac{x^2}{2}$

$1 = k\left[x + \frac{x^2}{2}\right]_4^7$

at $x = 7$

$7 + \frac{7^2}{2} = 7 + \frac{49}{2}$

$= 7 + 24.5$

$= 31.5$

at $x = 4$

$4 + \frac{4^2}{2} = 4 + \frac{16}{2} = 12$

$= 31.5 - 12 = 19.5$

$1 = k \times 19.5$

$\frac{1}{19.5} = \frac{\cancel{19.5}k}{\cancel{19.5}}$ $k = \frac{1}{19.5}$

$k = 0.05128$

Compute $P(x > 5)$

$\int_5^7 k(1+x)\,dx$

$= k\left[x + \frac{x^2}{2}\right]_5^7$

at $x = 7 = 31.5$

at $x = 5$

$= 5 + \frac{5^2}{2}$

$= 5 + \frac{25}{2}$

$= 17.5$

$= 31.5 - 17.5$

$= 14$

$P(x > 5) = k \times 14$

$= 0.05128 \times 14$

$= 0.71795$

5. Random variable x has pdf

$$f(x) = \begin{cases} x^3 + 0.75 & 0 \leq x \leq 1 \\ 0 & \text{Otherwise} \end{cases}$$

Determine probability density function of $y = x^2$, using change of variable technique.

$f(x) = x^3 + 0.75$

~~Pdf~~ $= \quad y = x^2$

$f_y(y) = f_x(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$

$y = g(x)$

$x = g^{-1}(y)$

$X = 0 \leq x \leq 1$

$x = g^{-1}(y) = \sqrt{y}$

$\frac{dx}{dy} = \frac{d}{dy}\sqrt{y} = y^{1/2} = \frac{1}{2} y^{-1/2}$

$= \frac{1}{2\sqrt{y}}$

$f(x)(x) = x^3 + 0.75$

$= (\sqrt{y}^3) + 0.75 = y^{3/2} + 0.75$

$= ((y^{3/2}) + 0.75) \times \frac{1}{2\sqrt{y}}$

y'

$= \frac{y^{3/2}}{2\sqrt{y}\, y^{1/2}} + \frac{0.75}{2\sqrt{y}} = \frac{y}{2} + \frac{0.375}{\sqrt{y}}$

$$y = x^2 = f_y(y) = \begin{cases} \frac{y}{2} + \frac{0.375}{\sqrt{y}} & 0 \leq y \leq \\ 0 & \text{otherwise} \end{cases}$$

6. Random variable x has p.d.f.

$$f(x) = \begin{cases} 3(1-x)^2, & 0 \leq x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Find

i) Variance of X

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X] = \int_0^1 x f(x)\,dx \quad , \quad E[X^2] = \int_0^1 x^2 f(x)\,dx.$$

E [X]

$$x f(x) = x \cdot 3(1-x)^2 = 3x(1-2x+x^2)$$

$$= 3x - 6x^2 + 3x^3$$

$$E[X] = \int_0^1 (3x - 6x^2 + 3x^3)\,dx$$

$$\left[\frac{3x^2}{2} - 2x^3 + \frac{3x^4}{4}\right]_0^1$$

at $x = 1$

$$= \frac{3(1)^2}{2} - 2(1)^3 + \frac{3(1)^4}{4}$$

$$= 3/2 - 2 + 3/4 = \frac{1}{4}$$

at $x = 0 = 0$

$$E[X] = \frac{1}{4} = 0.25$$

$$E[X^2] = \int_0^1 x^2 fx\,dx$$

$$x^2 fx = x^2 \cdot 3(1-x)^2$$

$$= 3x^2(1-2x+x^2).$$

$$= 3x^2 - 6x^3 + 3x^4$$

$$= \int_0^1 (3x^2 - 6x^3 + 3x^4)\,dx.$$

$$\cancel{3x^2}\left[x^3 - \frac{3x^4}{2} + \frac{3x^5}{5}\right]_0^1$$

at $x = 1$

$$= (1)^3 - \frac{3(1)^4}{2} + \frac{3(1)^5}{5}$$

$$= 1 - 3/2 + 3/5 = \frac{1}{10}$$

at $x = 0 = 0$

$$E[X^2] = \frac{1}{10} = 0.1$$

$$Var(x) = E[X^2] - (E[x])^2$$

$$= 0.1 - (0.25)^2$$

$$= 0.1 - 0.0625$$

$$= 0.0375$$

ii Median of pdf

$P(X \leq m) = 0.5 \Rightarrow F(m) = 0.5$

$f(x) = \int_0^x 3(1-t)^2 dt = \cancel{3(1-2x^2+}$

$3\int_0^x (1-2t+t^2)dt = 3\left[t - t^2 + \frac{t^3}{3}\right] = 3(x - x^2 + \frac{x^3}{3})$

$= 3x - 3x^2 + x^3$

$F(m) = 0.5$

$F(m) = 3m - 3m^2 + m^3 = 0.5$

$0 = m^3 - 3m + 3m^2 - 0.5$

$m^3 - 3m + 3m^2 - 1 = (m-1)^3$

$(m-1)^3 + 0.5 = m^3 + 3m^2 - 3m - 1 + 0.5$

$= m^3 + 3m^2 - 3m - 0.5$

$(m-1)^3 + 0.5 = 0 = (m-1)^3 = -0.5$

$\sqrt[3]{(m-1)^3} = \sqrt[3]{-0.5}$

$m - 1 = \sqrt[3]{-0.5}$

$m = 1 - \sqrt[3]{0.5}$

$\sqrt[3]{0.5} = 0.7937$

$m = 1 - 0.7937$

$m = 0.2063$

$F(m) = 3m - 3m^2 + m^3$

$= 3(0.2063) - 3(0.2063)^2 + (0$

$= 0.6189 - 0.1277 + 0.008$

$= 0.4999 = 0.5$

$m = 1 - \sqrt[3]{0.5}$

$m = 0.2063.$

iii 60th percentile of the pdf

$P(X \leq p) = 0.6$

$F(p) = 0.6$

$F(x) = 3x - 3x^2 + x^3$ for $0 \leq x \leq 1$

$3p - 3p^2 + p^3 = 0.6$

$p^3 - 3p^2 + 3p - 0.6 = 0$

$(p-1)^3 = p^3 - 3p^2 + 3p - 1$

$p^3 - 3p^2 + 3p - 0.6 = (p-1)^3 + 0.4$

$(p-1)^3 + 0.4 = 0$

$\sqrt[3]{(p-1)^3} = \sqrt[3]{-0.4}$

$(p-1) = \sqrt[3]{-0.4}$

$p = 1 - \sqrt[3]{0.4}$

$\sqrt[3]{0.4} = 0.7368$

$1 - 0.7368 = 0.2632$

$P_{60} = 0.2632$

7. Suppose that a random variable X has the following pmf.

$$f(x) = \begin{cases} \frac{x^2+1}{18} & \text{for } x = 0,1,2,3,4 \\ 0 & \text{, elsewhere.} \end{cases}$$

Determine pmf of $Y = x^2$.

$P(X=x) = \frac{x^2+1}{18}$ for $x=0,1,2,3,4$	$P(X=x) = \frac{x^2+1}{35}$ for $x=0,1,2,3,4$	$x=3 \Rightarrow y=9$
$\sum_x P(X=x) = 1$	$x=0 = 1/35$	$x=4 \Rightarrow y=16.$
$x=0 = 0^2+1 = 1$	$x=1 = \frac{2}{35}$	$P(Y=0) = 1/35$
$x=1 = 2$	$x=2 = \frac{5}{35} = \frac{1}{7}$	$P(Y=1) = 2/35$
$x=2 = 5$	$x=3 = \frac{10}{35} = \frac{2}{7}$	$P(Y=4) = \frac{5}{35}$
$x=3 = 10$	$x=4 = \frac{17}{35}$	$P(Y=9) = \frac{10}{35}$
$x=4 = 17$	$= 1$	$P(Y=16) = \frac{17}{35}$
$= 1+2+5+10+17 = 35$	$Y = x^2$ for $x=0,1,2,3,4$	$= \frac{35}{35}$
$\sum_0^4 \frac{x^2+1}{18} = \frac{35}{18} > 1$	$x=0 \Rightarrow y=0$	$= 1$
	$x=1 \Rightarrow y=1$	
	$x=2 \Rightarrow y=4$	

8 P (defective) $= 0.1$

if sample of 6 pencs is taken & probability that it will contain.

i) No defective computer.

$n = 6$ $\quad x =$ no. defective comp.

$p = 0.1$

$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$P(X=0) = \binom{6}{0} (0.1)^0 (1-0.1)^{6-0}$

$= \binom{6}{0} (0.1)^0 (0.9)^6$

$= 1 \times 1 \times 0.9^6 = 0.9^6$

$= 0.5314$

ii) 5 or 6 defective computers:-

$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$

$x = 5$

$P(X=5) = \binom{6}{5} (0.1)^5 (1-0.1)^{6-5}$

$= \binom{6}{5} (0.1)^5 (0.9)^1 = 6(0.00001)(0.9)$

$= 0.000054$

$P(X=6)$

$= \binom{6}{6} (0.1)^6 (1-0.1)^{6-6}$

$= \binom{6}{6} (0.1)^6 (0.9)^0 = 1 \times 0.000001 \times 1$

$= 0.000001$

$= 0.000054 + 0.000001$

$= 0.000055$

iii) More than 2 defective computers.

$x > 2 = P(X > 2) = 1 - P(X \leq 2)$

$P(X > 2) = 3, 4, 5, 6$

$P = 1 - P(X \leq 2)$

$= P(0) + P(1) + P(2)$

$= 1 - 0.98411$

$= 0.0159$

$P(0) = 0.5314$

$P(X=1)$

$= \binom{6}{1} (0.1)^1 (0.9)^{6-1}$

$= 6(0.1)(0.59049)$

$= 0.354294$

$P(X=2)$

$= \binom{6}{2} (0.1)^2 (0.9)^{6-2}$

$= 15 (0.1)^2 (0.9)^4$

$= 0.098415$

$P(X \leq 2) = 0.5314 + 0.354294 + 0.098415$

$= 0.98411$